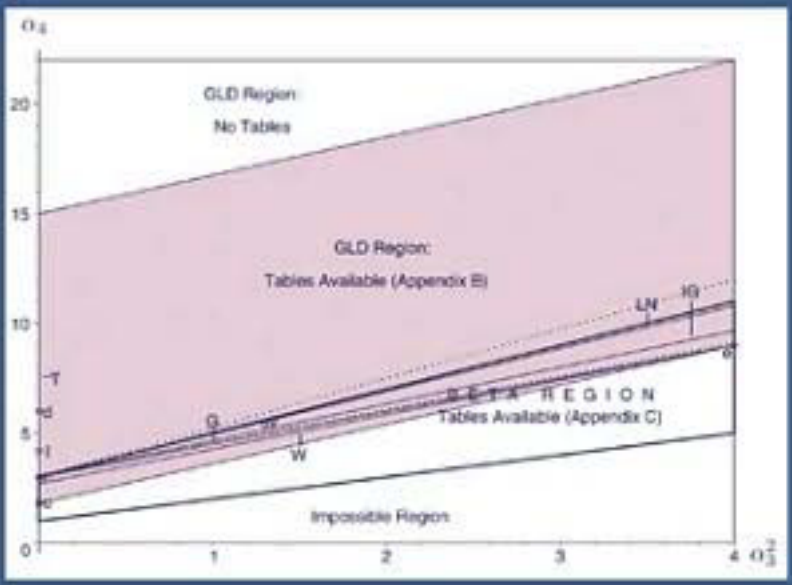
Handbook of Fitting Statistical Distributions with R



Zaven A. Karian
Edward J. Dudewicz

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

Handbook of Fitting Statistical Distributions with R

Handbook of Fitting Statistical Distributions with R

Zaven A. Karian
Edward J. Dudewicz


CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group an **informa** business

A CHAPMAN & HALL BOOK

MATLAB® is a trademark of The MathWorks, Inc. and is used with permission. The MathWorks does not warrant the accuracy of the text or exercises in this book. This book's use or discussion of MATLAB® software or related products does not constitute endorsement or sponsorship by The MathWorks of a particular pedagogical approach or particular use of the MATLAB® software.

Although all programs that accompany this book are believed to operate as documented, no guarantee is provided as to the performance or correctness of any program. Neither the authors nor the publisher guarantee the accuracy, sufficiency, or suitability of the software. There are no warranties, express or implied, including those of merchantability and fitness for a particular purpose, concerning the software. The user agrees to indemnify the publisher and authors from and against any demands, claims, actions or causes of action, damages losses, costs or expenses, including attorney's fees, court costs, penalties, or any other expenses incurred in connection with any claim of or action by a third party arising from or in any way involving or connected with the software or other materials in this book or any materials pertaining thereto.

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Chapman & Hall/CRC is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number: 978-1-58488-711-9 (Hardback)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (http://www.copyright.com/) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Library of Congress Cataloging-in-Publication Data

Karian, Zaven A.
Handbook of fitting statistical distributions with R / Zaven A. Karian, Edward J. Dudewicz.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-58488-711-9 (hardcover, includes cd-rom : alk. paper)
1. Distribution (Probability theory) 2. R (Computer program language) I. Dudewicz, Edward J. II. Title.

QA273.6.K374 2011
519.2'4--dc22 2010032085

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. A wide variety of references is listed. Reasonable efforts have been made to publish reliable data and information, but the authors and the publisher cannot assume responsibility for the validity of all materials or for the consequences of their use.

Although all programs that accompany this book are believed to operate as documented, no guarantee is provided as to the performance or correctness of any program. Neither the authors nor the publisher guarantee the accuracy, sufficiency, or suitability of the software. There are no warranties, express or implied, including those of merchantability and fitness for a particular purpose, concerning the software. The user agrees to indemnify the publisher and authors from and against any demands, claims, actions or causes of action, damages losses, costs or expenses, including attorney's fees, court costs, penalties, or any other expenses incurred in connection with any claim of or action by a third party arising from or in any way involving or connected with the software or other materials in this book or any materials pertaining thereto.

Preface

The fitting of statistical distributions to data often has its origins traced to the work of Karl Pearson in 1894, where he fitted a mixture of two normal distributions to data consisting of measurements of the ratio of forehead to body length of 1000 crabs, *Carcinus vulgaris*. (It turns out that crabs of genus *Carcinus* are of great interest today, over 115 years later, due to their invasive nature.)

Fitting of statistical distributions to data was a pioneering departure from previous methods, and its wide applicability led to the development of a large literature involving its use, methods, and computations over the 116 years since its beginnings.

The introduction of what is called the Generalized Lambda Distribution (GLD) in 1969 is a similar pioneering effort which also has departure from former methods, wide applicability, and development of a large literature involving its use. That development is detailed in the "Comments from GLD Pioneers" (which precedes Chapter 1 of this *Handbook*), where we are privileged to have comments from three of the key GLD Pioneers, giving their perspectives on the development of the GLD, which has continued over the 41 years 1969–2010. Their insights are of great value for those who continue to extend their work, and use them in ever new ways as shown in this *Handbook*. The base for the GLD, which has four parameters, is John Tukey's one-parameter "Tukey's Lambda" distribution which Dr. Tukey proposed in 1947.

The explosive growth of fitting distributions to data, coupled with the settling of some gaps in methods for use of the GLD, in particular extension to the EGLD that could match the mean, variance, skewness, and kurtosis of any dataset led us to publish our book *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods* (CRC Press, 2000). That book was broadly conceived — as a detailed exposition of the GLD, as a text, as a reference for researchers and practitioners — and it included the necessary tables, algorithms, and computer programs, with illustrations on a wide variety of datasets from real applications to aid the practitioner.

Comments from many users world-wide, and in many reviews, lead us to

believe we published the right book at the right time for many users. This was confirmed at the Symposium on Fitting Statistical Distributions to Data held in March 2006 at Auburn University, Auburn, Alabama, organized by one of us (Dr. Edward J. Dudewicz). Plenary Speakers included such well-known authors as Warren G. Gilchrist of Sheffield Hallam University, Sheffield, UK (*Statistical Modeling with Quantile Functions*, Chapman & Hall/CRC); Zaven A. Karian of Denison University, Granville, Ohio, USA (*Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods* (with E.J. Dudewicz), Chapman & Hall/CRC); and Haim Shore of Ben-Gurion University of the Negev, Be'er Sheva, Israel (*Response Modeling Methodology: Empirical Modeling for Engineering and Science*, World Scientific Publishing Co. Ltd.), among others; and, Dr. Karian gave a hands-on tutorial in a laboratory setting where every person was at a computer with the necessary software loaded, and participants actually determined fits associated with various datasets, some of which could be their own datasets. In particular, the broad geographical spread of the attendees (from such countries as Australia, England, Israel, as well as the United States), and from a spread of fields of interest, ranging from researchers in universities to practitioners in academia and industry, confirmed the high interest and use of the GLD.

There have also been other widely, and successfully, used methods of fitting distributions to data. While the scope of our 450+ page book *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods* (CRC Press, 2000) could not accommodate these, some (including Johnson's System, the Weibull Distributions, and the RMM System) were represented at the Auburn University Symposium. And the methodology of the GLD has moved forward, with new fitting methods, and new languages (in particular R).

In light of the above, we began preparing this *Handbook of Fitting Statistical Distributions with R* nearly five years ago. Due to the breadth of the field of Fitting Statistical Distributions, to achieve a cutting-edge coverage we have been joined by chapter authors who cover areas of their expertise in carefully defined chapters. This has also allowed us to have coverage that delves deeply into substantial applications that are at the cutting edge of their fields, and to have coverage that does not slight related fields, which play key roles in fitting in many applications, such as DOE (Design of Experiments), RNG (Random Number Generation), and Assessment of Fit Quality, among others. This coverage continues our tradition of giving the needed tables and computer programs, as well as applications to real datasets. The Chapter authors who join us in this effort are our first choices in their respective areas, which range from GLD methodology, to Johnson's System, to the RMM System, to other fitting systems, to applications in Agriculture, Reliability Estimation, Hurricanes/Typhoons/Cyclones, Hail Storms, Water Systems, Insurance and Inventory Management, and Materials Science, among others.

We thank those with whom we have worked for their pleasant cooperation

in preparing what we hope will be a *Handbook* that will be used widely and will advance the methodology, applications, and education of fitting statistical distributions. Special thanks are due to Dr. Kunio Shimizu, Professor, Department of Mathematics, Keio University, Yokohama, Japan, for editorial assistance with review and refereeing of materials for various chapters – this was vital in getting back to chapter authors in a timely fashion, and we are pleased Dr. Shimizu accepted this task, which he performed at the highest level in a very timely fashion.

We wish to express our appreciation to our Editors at CRC Press, Bob Stern and David Grubbs, whose rapid response to questions that arose, and generous leeway to choose the paths that would be most helpful for *Handbook* users, sped us on our way in this effort. We also wish to thank Jessica Vakili and Jim McGovern for their assitance in the production process of this book and Maggie Duffy for her suggestions that improved the book.

We are saddened by the untimely and unfortunate deaths of two of our chapter authors and colleagues, Weixing Cai (Chapter 21) and Igor Vajda (Chapter 23), while our work on this volume was in progress. We feel fortunate and grateful for having been able to collaborate with both. Weixing was a new Ph.D., having just finished his studies at Syracuse University and married, about to embark on a promising career in an Assistant Professor position, who died in a tragic accident. Igor was a seasoned researcher whose chapter is in our opinion stellar, combining theory and application in a way few papers or chapters do and few authors are capable of.

We welcome communications from users of this *Handbook*, and may include additional applications, methods, and references in subsequent editions.

Zaven A. Karian
Denison University
Granville, Ohio
karian@denison.edu

Edward J. Dudewicz
Syracuse University
Syracuse, New York
dudewicz@syr.edu

April 2010

The MathWorks, Inc
3 Apple Hill Drive
Natick, MA 01760-2098 USA
Tel: 508-647-7000
Fax: 508-647-7001
E-mail: infomathworks.com
Web: www.mathworks.com

About the Authors

Dr. Zaven A. Karian holds the Benjamin Barney Chair of Mathematics, and is Professor of Mathematics and Computer Science at Denison University in Ohio. He has been active as instructor, researcher, and consultant in mathematics, computer science, statistics, and simulation for over thirty-five years. He has taught workshops in these areas for a dozen educational institutions and national and international conferences (International Conference on Teaching Mathematics, Greece; Asian Technology Conference in Mathematics, Japan; Joint Meetings of the American Mathematical Society/Mathematical Association of America).

Dr. Karian has taught short courses of varying lengths for colleges and universities (Howard University, Washington, D.C.; The Ohio State University; State University of New York; and Lyndon State College, Vermont), for professional societies (Society for Computer Simulation, American Statistical Association, Mathematical Association of America (MAA), the Ohio Section of the MAA), and private and public foundations (Alfred P. Sloane Foundation, National Science Foundation). His consulting activities include Cooper Tire and Rubber Company, Computer Task Group, and Edward Kelcey and Associates (New Jersey), as well as over forty colleges and universities.

Dr. Karian is the author and co-author of ten texts, reference works, and book chapters and he has published over forty articles. He has served as editor for computer simulation of the *Journal of Applied Mathematics and Stochastic Analysis*, has been the Editor of the Classroom Resource Materials series of books for the MAA and he currently is the Editor of MAA Textbooks that is published by the Mathematical Association of America.

Dr. Karian holds the bachelor's degree from American International College in Massachusetts, master's degrees from the University of Illinois (Urbana-Champaign) and The Ohio State University, and his doctoral degree from The Ohio State University. He has been a Plenary Speaker on two occasions at the Asian Conference on Technology in Mathematics (Singapore, and Penang, Malaysia).

Dr. Karian has served on the International Program Committees of conferences (in Greece, Japan, People's Republic of China, the Czech Republic), the Board of Governors of the MAA, and the governing board of the Consortium of Mathematics and its Applications. He was a member of the Joint

MAA/Association for Computing Machinery (ACM) Committee on Retraining in Computer Science and he chaired the Task Force (of the MAA, ACM, and IEEE Computer Society) on Teaching Computer Science, the Subcommittee on Symbolic Computation of the MAA, and the Committee on Computing of the Ohio Section of the MAA.

Dr. Karian has been the Acting Director of the Computer Center, Denison University; Visiting Professor of Statistics, The Ohio State University; Chair of the Department of Mathematical Sciences, Denison University. He has been the recipient of the R.C. Good Fellowship of Denison University on three occasions. In 1999 he was given the Award for Distinguished College or University Teaching of Mathematics by the Ohio Section of the MAA.

Dr. Edward J. Dudewicz is Professor of Mathematics at Syracuse University, New York. Dr. Dudewicz is internationally recognized for his solution of the Heteroscedastic Selection Problem, his work on Fitting Statistical Distributions, his development of the Multivariate Heteroscedastic Method, and his solution of the Behrens-Fisher Problem. Dr. Dudewicz is one of a handful of scientists world-wide to be elected Fellows of all of the Institute of Mathematical Statistics, American Statistical Association, American Society for Quality, and AAAS. Dr. Dudewicz is Founding Editor of the *Basic References in Quality Control: Statistical Techniques* of the American Society for Quality, for which he received the ASQ Testimonial Award for "Leadership and Distinguished Service" from the Directors of the Society. He is also Founding Editor of the *American Journal of Mathematical and Management Sciences*, currently in its 30th volume. He has been a Fulbright Scholar. He has over 170 publications, including 11 books, one each translated into Arabic and into Indonesian. Dr. Dudewicz's most recent paper received referee reports with statements such as "This paper will be an important resource not only for the statistical community but also for researchers of every discipline." There are more than 700 citations to Dr. Dudewicz's works. As a teacher, Dr. Dudewicz's students describe him with such statements as "always available to help us on our homework and exams," " has a friendly and encouraging attitude," "He encouraged me to get into a cutting-edge, publishable research project ... now ... [published in] one of the premier journals," "encouraging, extremely helpful, knowledgeable, well organized and always well prepared ... a demanding teacher but loved by his students," "my inspiration; he never gets upset and generously walks you through your difficulty ... always there to show you the way ... ."

Dr. Dudewicz has been active as instructor, researcher, and consultant for over four decades. He has taught statistics and digital simulation at Syracuse University, Ohio State University, University of Rochester, University of Leuven, Belgium and National University of Comahue, Argentina and served as a staff member of the Instruction and Research Computer Center at Ohio State University, and as Head Statistician of New Methods Research, Inc. His con-

sulting activities include O. M. Scott and Sons Company, Ohio Bureau of Fiscal Review, Mead Paper Corporation, Blasland, Bouck, & Lee, Engineers & Geoscientists. Dr. Dudewicz is author, co-author, and editor of handbook chapters on statistical methods in the *Quality Control Handbook* and *Magnetic Resonance Imaging* and encyclopedia articles "Heteroscedasticty," Encyclopedia of Statistical Sciences, Wiley; and "Simulation Languages," Encyclopedia of Information Systems, Academic Press/Elsevier Science (USA). Dr. Dudewicz holds the bachelor's degree from the Massachusetts Institute of Technology and master's and doctoral degrees from Cornell University. He has been Visiting Scholar and Associate Professor at Stanford University, Visiting Professor at the University of Leuven, Belgium and at the Science University of Tokyo, Japan, Visiting Distinguished Professor at Clemson University, and Titular Professor at the National University of Comahue, Argentina while Fulbright Scholar to Argentina. His Editorial posts have included *Technometrics* (Management Committee), *Journal of Quality Technology* (Editorial Review Board), *Statistical Theory and Method Abstracts* (Editor, U.S.A.), *Statistics & Decisions* (Editor) and *American Journal of Mathematical and Management Sciences* (Founding Editor and Editor-in-Chief), and he also serves the journals *Information Systems Frontiers* (Executive Editorial Board) and *Journal of Statistical Theory and Practice* (Editorial Board). The *American Journal of Mathematical and Management Sciences* has been rated as one of the "top 30" journals. Dr. Dudewicz has served as President, Syracuse Chapter, American Statistical Association; Graduate Committee Chairman, Department of Statistics, Ohio State University; Chairman, University Statistics Council, Syracuse University; External Director, Advanced Simulation Project, National University of Comahue, Argentina; Awards Chairman, Chemical Division, American Society for Quality; and Founding Editor, Basic References in Quality Control: Statistical Techniques, American Society for Quality, for which he received the ASQ Testimonial Award for "Leadership and Distinguished Service."

Dr. Dudewicz's contributions in research include:

1. Solution of the Heteroscedastic Selection Problem. In 1975, Professor Dudewicz with co-author S.R. Dalal (one of the first students to enter the statistics Ph.D. program at The University of Rochester) published a solution of "The Heteroscedastic Selection Problem" given by Professor Robert E. Bechhofer of Cornell University in a paper in 1954 in the Annals of Mathematical Statistics which had stood for 21 years. This paper titled "Allocation of observations in ranking and selection with unknown variances," was a comprehensive 51-page analysis of the problem and its solution, including the tables needed to apply the solution. It was published in one of the premier statistics journals worldwide, *Sankhyā, The Indian Journal of Statistics*. There are now many published statistical procedures which follow the method established in this path-breaking 1975 paper, and they are often referred to as "Dudewicz-Dalal type" procedures. Google Scholar lists 76 citations of this paper. In Professor Dudewicz's election as a Fellow of AAAS,

the citation stated he was being "honored for research in mathematical statistics, particularly for solution of the heteroscedastic selection problem."

2. Fitting Statistical Distributions. One important statistical problem faced by researchers in virtually every field that utilizes statistical analysis is that of fitting a distribution to a set of data. Some notable milestones in this problem have been the development of the Pearson system (1895) and the Johnson system (1949). The Generalized Lambda Distribution (GLD) system was first broached in 1972 by Ramberg and Schmeiser, and system was further developed in a 1979 paper titled "A probability distribution and its uses in fitting data" by Professor Dudewicz and co-authors E.F. Mykytka, J.S. Ramberg, and P.R. Tadikamalla in the journal *Technometrics*. A precursor of this paper won the 1977 Shewell Award. Google Scholar lists 66 citations of this paper, from varied fields such as Medicine, Behavioral Science, Economics, and Management. This publication stood for two decades as the place to find the results, details, and tables needed to fit a GLD to a set of data. In the years since, Professor Dudewicz and co-authors have developed the GLD system further to satisfy such needs as being able to fit a distribution to any set of moments.

3. The Heteroscedastic Method. As noted above, in 1975 Dudewicz and Dalal solved the "Heteroscedastic Ranking and Selection Problem." They considered the general ranking and selection goal as well, and gave details for the cases of selecting the t best populations and of subset selection. Extending this idea for many multipopulation problems such as testing of hypotheses, multiple comparisons, estimation of ordered parameters, partitioning of a set of populations, confidence intervals, ANOVA and MANOVA and regression, he gave a general solution with the multivariate analogs of these problems. Professor Dudewicz and his graduate student T.A. Bishop in 1979 gave "The heteroscedastic method," in *Optimizing Methods in Statistics* (ed. J.S. Rustagi, Academic Press, New York). Professor Dudewicz's citation at election as Fellow of AAAS read, in part, "honored for ... development of the multivariate Heteroscedastic Method."

4. Solution of the Behrens-Fisher Problem. The problem of testing equality of two means when variances are not known has been called the most important problem of applied statistics by the famous statistician Henry Scheffé. The problem dates back to the early 1900s and is named after an astronomer (Behrens) and a statistician (Fisher). For many years no one could find an exact solution, i.e. a solution for which the Type I error probability was exactly the desired number (such as .01) for all values of the population variances. However, in 1950 Chapman gave one exact solution, and in 1974 Prokof'yev and Shishkin gave a second exact solution. In 1998 Professor Dudewicz and Mr. S.U. Ahmed gave a third exact solution in the paper "New exact and asymptotically optimal solution to the Behrens-Fisher problem, with tables," *American Journal of Mathematical and Management Sciences*. Their solution was stated to be asymptotically optimal, hence preferable to the other two solutions.

Dr. Dudewicz is one of a handful of scientists world-wide to be elected Fellows of all of the Institute of Mathematical Statistics, American Statistical Association, American Society for Quality, American Association for the Advancement of Science, and the New York Academy of Sciences. Recognitions of Dr. Dudewicz also include Research Award, Ohio State Chapter, Society of the Sigma Xi; Chancellor's Citation of Recognition, Syracuse University; Jacob Wolfowitz Prize for Theoretical Advances; Thomas L. Saaty Prize for Applied Advances (twice); Coauthor, Shewell Award paper; Jack Youden Prize for the best expository paper in *Technometrics*. In 1999 he was awarded the International Francqui Chair in Exact Sciences. He has received the Seal of Banares Hindu University (India). Dr. Dudewicz was invited jointly by the Director of Statistics, and the Chairman of the Department of Mathematics & Statistics, Auburn University, to organize and teach a one-week Symposium on Fitting Statistical Distributions to Data at Auburn University in 2006.

To

Susan, Steve, and Maya

and

To those loved in a special way,
Pat, Douglas, Margot, Robert, Lada, Connor, Dillon,
Carolyn, Jeff, Kenny, Kevin, Kimberly,
and to those who bring us close to Love,
especially Father William A.W., OP and Rev. Msgr. JRY

Comments from GLD Pioneers

As will be discussed in Chapter 8, fitting of statistical distributions to data often has its origins traced to work of Karl Pearson in 1894, when he fitted a mixture of two normal distributions to data consisting of measurements of the ratio of forehead to body length of 1000 crabs in *Carcinus vulgaris*. Crabs of genus *Carcinus* are of great interest today due to their invasive nature. (One question we do not know the answer to is if the crabs of the 1894 data might have been the subject of a Friday-night crab fest, or if they are nowhere nearly as tasty as the prized Blue Claw Crab.) Pearson's work was a watershed for fitting of statistical distributions to data; he was a true pioneer of distribution fitting. The aspects of his work that make it **pioneering** are its departure from former methods, its wide applicability, and the development of a large literature involving its use over 1894–1969, a period of 75 years.

The development of what is called the Generalized Lambda Distribution (GLD) in 1969 is a similar pioneering effort which also has departure from former methods, produced wider applicability, and promoted the development of a large literature involving its use. We are very pleased to have in this Chapter comments from three of the key GLD Pioneers, giving their perspectives on the development of the GLD, which continues over 1969–2010, a period of 41 years. Their insights are of great value for those who would build upon them, and use them in ever new ways as shown in this Handbook.

A. John S. Ramberg: Origins of the Ramberg, Schmeiser, Tukey (RST) Lambda Distribution

John S. Ramberg
114 Capricho Circle
Pagosa Springs, CO 81147
jsramberg@msn.com

The RST distribution resulted through a coincidence of serendipitous events that yielded the ideas as well as the wonderful students with whom I collaborated. I

will elaborate on this subsequently. It began at a University of Iowa Statistics Department informal seminar discussion in 1969 or 1970. At one of these sessions, Bob Hogg and Jon Cryer presented a summary of Jim Filliben's ASA presentation of his Ph.D. dissertation on percentile functions. Their summary included a discussion of an approximation that Jim's advisor, John Tukey, had given for the normal percentile function. After learning about this, I conjectured that Tukey's percentile function approximation might be generalized to yield approximations for many other distribution forms. I was enthusiastic about this, because of the inverse distribution function theorem. That is, I knew that a percentile function could be used to transform uniform zero-one random variates to the distribution specified by that percentile function.

I mentioned this conjecture to a simulation class. Bruce Schmeiser took up my challenge, and thus was the first to explore the possibilities of this distribution in detail for his M.S. thesis. I was later reminded that one of my Ph.D. students, Subhash Narula, collaborated with Bruce on his initial explorations. Both of these students have prospered in their academic careers, one in statistics and the other in simulation/operations research.

Bruce's work proceeded rapidly, without any major impediments. Through the contributions of his minor advisors, Fred Leone and John Liittschwager, a neat, well-written MS thesis resulted. I should comment that I sought the direct involvement of Fred and John, because I knew that similar consultation would have been helpful to me at the time that I was writing my MS thesis. Through the cooperation of these committee members, Bruce obtained superb advice from senior faculty who were more mature in this process than I was.

Following graduation, Bruce took a position with Ross Perot's Electronic Data Systems, and I edited his thesis into two papers for the *Communications of the Association of Computing Machinery*, one on the symmetric family and the other on the asymmetric family. These submissions received excellent reviews. The very positive comments from the reviewers and editor encouraged me to continue research on this subject. The acceptance of these papers was also instrumental in Bruce's admission to the Industrial and Systems Engineering Ph.D. Program at Georgia Tech.

A quick scan of these *ACM* papers will make it clear that I was not initially interested in the RST distribution as a model in itself, but rather as a viable approximation to distributions which had simulation applications. My background in experimental design influenced me on this point. I understood the importance of employing common random number sequences in Monte Carlo experiments to achieve blocking. The RST was essential in this process.

Following publication of our first two papers, Brian Joiner, a member of Joan Rosenblatt's engineering group at the National Bureau of Standards (NBS), wrote me that one of his colleagues at NBS had previously conjectured about generalizations of Tukey's percentile function. As far I as I was able to determine, his colleague did write a technical report on this conjecture, but did not follow up,

nor publish his ideas in the reviewed literature. Nevertheless, this led me to refer to our result as the lambda distribution, even though others had already termed it the RST distribution.

Later my interest turned to the application of the RST as an empirical model in its own right, and as a competitor to the Pearson System and the Johnson System of distributions, especially for simulation input representations. I collaborated with two Ph.D. students, Pandu Tadikamalla and Ed Mykytka on this development. Subsequently, I invited Ed Dudewicz to participate with us in the venture. Ed and I had both studied under Bob Bechhofer on Ranking and Selection at Cornell.

We tabled the values of the parameters of the RST in terms of the third and fourth standardized moments, facilitating its use as a model, and provided density plots that practitioners have found quite useful. We also illustrated the coverage of this distribution in comparison to that of other distribution systems through the use of the now famous third/fourth moment characterization diagram of Pearson. Of course, we were well aware of potential sampling variability problems of the third and fourth moments, but the simplicity of explaining this to researchers and practitioners employing the methods outweighed this disadvantage. I note that the major advantages of the RST are its simplicity and its use in the transformation of uniform variates to other distribution forms, but also acknowledge that it does not have a theoretical basis similar to that of the other two.

Many others have developed estimation methods that circumvent the downside of the method of moments. Following my move to Arizona, Ed Mykytka and I developed one based on percentiles. I should also remark that both Pandu and Ed went on to successful academic careers, as did another of my Ph.D. students who studied in this area, Mark Johnson. Incidentally, there was never a student, nor collaborator by the name of lambda!

The coincidence of serendipitous events that I mentioned at the outset, came about as follows. Fred Leone, who held a joint appointment in Statistics and Industrial and Management Engineering, recruited me to Iowa. I met him when he gave a seminar at Cornell. Through Fred's mentoring, I also received an appointment in the statistics department, headed by Bob Hogg. In fact, Bob was influential in my decision to join the Iowa faculty. He joined Fred and me for dinner on my first evening in Iowa City.

I had used his text in mathematical statistics, co-authored by Allen Craig, in my study for preliminary exams at Cornell, and I was struck by their ability to explain statistical results so clearly. An example was their proof of the central limit theorem. The opportunity to consult and collaborate with these two giants of the field was an obvious draw. Incidentally, my home department, Industrial and Management Engineering, had been founded by Ralph Barnes. It was under the direction of one of his former students, J. Wayne Deegan, who actually hired me.

When I joined the Iowa faculty in the Fall of 1967 my first teaching assignment was a course titled Stochastic Systems. I learned, to my surprise, that this meant digital computer simulation of stochastic systems, and not queueing systems as I had originally thought. While I had taken a number of courses in stochastic models and queueing, I had not had the wisdom to take any work in digital computer simulation. The Cornell OR group was well known for its path-breaking research in discrete event simulation, and thus it was assumed that I was knowledgeable. I had learned a bit through my association with graduate students specializing in simulation and job shop modeling. Also, my MS thesis was on Monte Carlo distribution sampling to compare two sequential ranking and selection procedures. Only at my thesis defense did I learn that my choice of the sum of uniform random variates (CLT) approach for normal random variates could have been improved upon.

Not wishing to confess my ignorance, I set out to learn the basics of digital computer simulation. Fortunately, two professors at Cornell, Bill Maxwell and Howard Krasnow, came to my rescue. Each supplied me with their excellent class notes. A scan of the simulation texts available at that time should make my dilemma clear. The existing texts were either on a specific language, or on statistical issues, or Monte Carlo. None of them embraced the whole of discrete event simulation methods, as do the excellent texts now available, such as the one by Averill Law and David Kelton.

I am indebted to John Liittschwager for making this class assignment as well as for his mentoring on thesis advising. My class attracted a stream of students from engineering, mathematics, statistics, and computer science. Typically, these students had excellent backgrounds in computing, thus complementing my own knowledge.

Teaching this course influenced by research immediately, as I learned the importance of the generation of random variates having specified distributions. More correctly I should term this transformation of random variates from a uniform zero one source to a specified distribution. I also studied the analysis of simulation output, learning that the work by Conway, Maxwell, and Miller was fundamental. When I first heard their seminars on this topic, I had regarded it as naïve. Clearly I had much to learn. My students, undergraduate and graduate, took pride in teaching me about simulation programming.

Modeling simulation input was a task for a person with my statistical education, and the conduct of simulation experiments meshed with my background in engineering statistics. The latter led to an invited paper in Tom Naylor's simulation proceedings. I presented this work on the same trip to the East coast, when I defended my Ph.D. thesis. While I was interested in the unique problems and solutions for specified distribution, such as the Box-Muller transformation, my engineering background pointed me toward seeking simple methods that could be easily understood by undergraduates in engineering and mathematics, and thus be RST applications.

My own research productivity, especially in simulation, was greatly diminished when I took a position at Arizona, and I have not contributed further to the development of the RST family. At Arizona my students did not have the fundamental backgrounds in computing and statistics that the ones at Iowa had. Second, I soon became department head and focused on the recruiting and development of young faculty. Incidentally, neither of the undergraduate programs in systems and industrial engineering, were accredited (ABET) when I arrived. The faculty lamented that their ABET reviewer had cited a major deficiency of these curricula was the absence of a course in simulation. They had addressed this issue, but the result was a course in Monte Carlo random variate generation, and not digital computer simulation. So one of my first tasks was to offer such a course, and make preparations for another ABET review, which was successful. In retrospect, my decision to become an administrator was not a wise one, as I soon became an example of the Peter Principle. My program prospered, through the support and mentoring of a wise dean of engineering, Dick Gallagher, it attained national prominence. Following my retirement as head, the program continued on this trajectory for a short time. However, with the decline of state funding of the University of Arizona, many prominent young faculty members departed for promising careers at other institutions.

The development of the RST distribution and more generally digital computer simulation has played a major role in my career. I am delighted to see that so many others have taken an interest in our results and accomplished additional research in this area.

B. Comments from Bruce Schmeiser: Origin of the Generalized Lambda Distribution

Bruce Schmeiser
School of Industrial Engineering
Purdue University
West Lafayette, IN 47907
bruce@ecn.purdue.edu

During a spring 1970 special-topics course at The University of Iowa, John Ramberg discussed Tukey's unpublished technical report describing the lambda distribution, most naturally expressed with its inverse cdf

$$F_X^{-1}(u) = u^\lambda - (1-u)^\lambda,$$

where $0 \le u \le 1$. Monte Carlo observations of X are obtained using uniformly distributed U. The distribution is symmetric about zero, so there is a positive constant b so that

$$F_Y^{-1}(u) = \mu + \sigma[u^\lambda - (1-u)^\lambda]/b,$$

yields Y with mean μ, standard deviation σ, and shape determined by λ. John Tukey suggested that the family of distributions could be useful for Monte Carlo studies of nonnormality.

John Ramberg wondered aloud whether anyone in the class was interested in studying the family further, including generalizing to asymmetric distributions. At that moment I modified my MS degree to include writing a thesis.

I started and finished the thesis during the summer of 1970, working mostly from afternoon until early morning, when I would sometimes take the operators of the university's main-frame computer to breakfast; they were good to me by letting me have direct access to the computer center's card reader and printer. Without that access, the usual one turn-around per day would have been insufficient for me to obtain the numerical results in my thesis. Sandwiched into the summer, I was dating my future wife; when I slept through a lunch date, she realized that I was a night person.

The thesis topic was straightforward. Following Tukey, the thesis defined the four-parameter Generalized Lambda Distribution (GLD) using the inverse cdf

$$F_Y^{-1}(u) = \lambda_1 + \sigma[u^{\lambda_3} - (1-u)^{\lambda_4}]/\lambda_2,$$

where λ_1 is any constant, λ_2 is a positive constant, and (λ_3, λ_4) is any pair of constants that yields a monotonically increasing function of u. The thesis contains many pages of tables mapping ($lambda_3$, $lambda_4$) to (α_3, $\alpha - 4$), the third and fourth standardized moments where

$$\alpha_k = E[(Y-\mu)^k/\sigma^k].$$

An emphasis of the thesis was approximating classical families of distributions. Random variate generation was in its infancy, with Johnk's 1964 *Metrika* paper (in German) being the first to transform $U(0,1)$ random numbers to the gamma family of distributions. Because Johnk's method is inefficient for many gamma shapes and, more importantly, because we didn't know that his paper existed, I created tables with appropriate values of (λ_1, λ_2, λ_3, λ_4) to approximate gamma distributions. The definition of *appropriate* was to match the first four moments; the method was to use an IMSL nonlinear-programming routine. Similar, but easier because of symmetry, were tables for Student's t distribution.

The usefulness of such tables vanished with the development of exact and efficient random-variate generation methods through the early 1980s. The only thesis approximation that retained some usefulness was Tukey's original normal approximation. Matching the first four moments, the thesis version of the standard normal approximation is

$$F_Z^{-1}(u) = [u^{0.135} - (1-u)^{0.135}]/0.1975.$$

The approximation, which truncates at a bit more than five standard deviations, gave rise to the biggest numerical frustration of the research. Having

obtained the GLD parameter values using nonlinear programming, I was trying to create a CALCOMP plot that compared the GLD approximation to the standard-normal density function. Plotting the normal density was easy, but I could not get the GLD approximation to appear. I assumed that the problem was with the CALCOMP plotting software or apparatus or with computing the GLD density using the inverse cdf. The solution, however, was to realize that, to within the accuracy of the plot, the approximation was indistinguishable from the standard normal density; the two curves were drawn on top of each other.

The GLD approximation to the exponential density was also quite good. The plot in the thesis showed visual error only at the mode, zero. Only later, when I was an assistant professor at Southern Methodist University in the late 1970s, did John Ramberg and I realize that the exponential distribution is a limiting case of the GLD family; the approximation error in the exponential plot could have been eliminated if I had better handled the numerical error in my computations.

In addition to John Ramberg's invaluable guidance, Fred Leone and John Liitschwager were memorably helpful. Professor Leone suggested the use of standardized moments and appropriate point estimators for those moments. Discussion with Professor Liitschwager was enlightening when I was frustrated with writing the thesis, my first archival document.

My impression is that I was the first student admitted to an Iowa Engineering graduate program who did not have an undergraduate engineering degree. My B.A. degree in the mathematical sciences—a combination of courses from the departments of mathematics, statistics, and computer science—meant that I was taking a FORTRAN course while my BSME roommate (Dan Gajewski) was taking a slide-rule course. In 1970, few engineering students, and few statistics students, had the computing background necessary for a numerically oriented thesis like mine.

The thesis experience was when I understood the thrill of research—knowing something that nobody else had ever known. Each day I had questions that I wanted to answer. And I understood the frustration of moving backward, realizing earlier errors. And I enjoyed the uncertainty of not knowing where the work would lead, and that anything was fair, mixing closed-form mathematics with computing.

John Ramberg's suggesting my thesis topic changed my life. Although I dropped out of the Iowa's Industrial and Management Engineering Ph.D. program (partly because I wasn't convinced that operations-research methods were used and partly because the program required at least one year of real-world experience), writing the MS thesis was the foundation for my enrolling in Georgia Tech's Industrial and Systems Engineering Ph.D. program in 1972 and my later entry into the community of researchers that is today the Simulation Society of Informs (The Institute of Operations Research and the Management Sciences).

C. Comments from Pandu R. Tadikamalla: Recollections at U of Iowa

Pandu R. Tadikamalla
Katz Graduate School of Business
University of Pittsburgh
Pittsburgh, PA 15260
pandu@katz.pitt.edu

I was very lucky to start working under Professor Ramberg as a graduate assistant from the very beginning of my arrival at the University of Iowa in September of 1971. Right from the beginning, I was hooked to do research in the Random Variate Generation area. At that time, there were not any "exact" algorithms for generating Gamma variates with non-integer values of the shape parameters. Early research work with Professor Ramberg resulted in two published papers in *AIIE Transactions* and *Journal of Statistical Computation and Simulation* on approximate methods for generating gamma variates. I was looking for a few more research "topics" so that I could complete my Ph.D dissertation. By that time Bruce had left the University of Iowa and Professor Ramberg and Professor Edward Dudewicz suggested that I could work on the lambda distribution.

Using the IMSL package and FORTRAN language, we developed extensive tables to fit the lambda distribution based on the first four moments of any data set and/or a distribution. The resulting paper was published in *Technometrics* and that paper also won the best paper award for that year from the ASQC. I continued that theme of fitting distributions to datasets after my graduation. Later, I teamed up with Professor Norman L. Johnson and developed the Tadikamalla-Johnson system of distributions. Several papers were published on this Tadikamalla-Johnson system of distributions in *Biometrika* and *Communications in Statistics*.

References

Filliben, J.J. (1969). *Simple and Robust Linear Estimation of the Location Parameter of a Symmetric Distribution,* Ph.D. Thesis, Princeton U., Princeton, N.J.

Hastings, C. Jr., Mosteller, F., Tukey, J.W., and Winsor, C.P. (1947). "Moments for small samples: A comparative study of order statistics," *Ann. of Mat. Statist.* 18, 3, 413–426.

Johnson, M.E., Wang, C., and Ramberg, J.S. (1984). "Generation of continuous multivariate distributions for statistical applications," *American Journal of Mathematical and Management Sciences,* Vol. 4, Nos. 3 and 4, pp. 225–248. (Thomas Saaty Prize)

Johnson, M.E. and Ramberg, J.S. (1977). "Elliptically Symmetric Distributions:

Characterizations and Random Variate Generation," Statistical Computing Section, *Proceedings of the American Statistical Association,* pp. 262–265.

Moberg, T.F., Ramberg, J.S. and Randles, R.H. (1978). "An adaptive M-estimator and its application to a selection problem," *Technometrics,* Vol. 22, No. 3, pp. 255–263.

Moberg, T.F., Ramberg, J.S. and Randles, R.H. (1980). "An adaptive multiple regression procedure based on M-estimators," *Technometrics*, Vol. 22, No. 2, pp. 213–224.

Mykytka, E.F. and Ramberg, J.S. (1979). "Fitting a Distribution to Data Using an Alternative to Moments," *Proceedings of the Winter Simulation Conference,* Vol. 2, pp. 361–374.

Ramberg, J.S. (1970). "Selection and Ranking Procedures: A Comment," *The Design of Computer Simulation Experiments,* pp. 161–164.

Ramberg, J.S. and Schmeiser, B.W. (1972). "An approximate method for generating symmetric random variables," *Communications of the Association for Computing Machinery,* Vol. 15, No. 11, pp. 987–990.

Ramberg, J.S. and Schmeiser, B.W. (1974). "An approximate method for generating asymmetric random variables," *Communications of the Association for Computing Machinery,* Vol. 17, No. 2, pp. 78–82.

Ramberg, J.S. (1975). "A Probability Distribution with Applications to Monte Carlo Simulation Studies," *Statistical Distributions in Scientific Work Vol. 2: Model Building and Model Selection,* Reidel, Boston, pp. 51–64.

Schmeiser, B.W. (1971). *A General Algorithm for Generating Random Variables,* M.S. Thesis, The University of Iowa, Iowa City, Iowa.

Contents

Preface v

About the Authors ix

Dedication xv

Comments from GLD Pioneers xvii

PART I: Overview 1

1 Fitting Statistical Distributions: An Overview 3
1.1 History and Background 4
1.2 The Organization of the Handbook 10
References for Chapter 1 16

PART II: The Generalized Lambda Distribution 19

2 The Generalized Lambda Family of Distributions 21
2.1 Definition of the Generalized Lambda Distributions 21
2.2 The Parameter Space of the GLD 23
2.3 Shapes of the GLD Density Functions 32
2.4 GLD Random Variate Generation 47
2.5 The Fitting Process 48
Problems for Chapter 2 49
References for Chapter 2 51

3 Fitting Distributions and Data with the GLD via the Method of Moments 53
3.1 The Moments of the GLD Distribution 54
3.2 The (α_3^2, α_4)-Space Covered by the GLD Family 59
3.3 Fitting the GLD through the Method of Moments 64

3.3.1 Fitting through Direct Computation . . . 65
3.3.2 Fitting by the Use of Tables . . . 74
3.3.3 Limitations of the Method of Moments . . . 75
3.4 GLD Approximations of Some Well-Known Distributions . . . 76
3.4.1 The Normal Distribution . . . 81
3.4.2 The Uniform Distribution . . . 82
3.4.3 The Student's t Distribution . . . 83
3.4.4 The Exponential Distribution . . . 85
3.4.5 The Chi-Square Distribution . . . 87
3.4.6 The Gamma Distribution . . . 88
3.4.7 The Weibull Distribution . . . 90
3.4.8 The Lognormal Distribution . . . 91
3.4.9 The Beta Distribution . . . 93
3.4.10 The Inverse Gaussian Distribution . . . 94
3.4.11 The Logistic Distribution . . . 96
3.4.12 The Largest Extreme Value Distribution . . . 98
3.4.13 The Extreme Value Distribution . . . 99
3.4.14 The Double Exponential Distribution . . . 100
3.4.15 The F-Distribution . . . 101
3.4.16 The Pareto Distribution . . . 103
3.5 Examples: GLD Fits of Data, Method of Moments . . . 105
3.5.1 Assessment of Goodness-of-Fit . . . 105
3.5.2 Example: Cadmium in Horse Kidneys . . . 109
3.5.3 Example: Brain (Left Thalamus) MRI Scan Data . . . 110
3.5.4 Example: Human Twin Data for Quantifying Genetic (vs. Environmental) Variance . . . 111
3.5.5 Example: Rainfall Distributions . . . 115
3.6 Moment-Based GLD Fit to Data from a Histogram . . . 116
3.7 The GLD and Design of Experiments . . . 120
Problems for Chapter 3 . . . 124
References for Chapter 3 . . . 125

4 The Extended GLD System, the EGLD: Fitting by the Method of Moments 129
4.1 The Beta Distribution and its Moments . . . 129
4.2 The Generalized Beta Distribution and its Moments . . . 134
4.3 Estimation of GBD($\beta_1, \beta_2, \beta_3, \beta_4$) Parameters . . . 138
4.4 GBD Approximations of Some Well-Known Distributions . . . 144

4.4.1 The Normal Distribution . . . 145
4.4.2 The Uniform Distribution . . . 146
4.4.3 The Student's t Distribution . . . 147
4.4.4 The Exponential Distribution . . . 148
4.4.5 The Chi-Square Distribution . . . 149
4.4.6 The Gamma Distribution . . . 150
4.4.7 The Weibull Distribution . . . 151
4.4.8 The Lognormal Distribution . . . 153
4.4.9 The Beta Distribution . . . 154
4.4.10 The Inverse Gaussian Distribution . . . 155
4.4.11 The Logistic Distribution . . . 155
4.4.12 The Largest Extreme Value Distribution . . . 155
4.4.13 The Extreme Value Distribution . . . 156
4.4.14 The Double Exponential Distribution . . . 156
4.4.15 The F-Distribution . . . 156
4.4.16 The Pareto Distribution . . . 156
4.5 Examples: GBD Fits of Data, Method of Moments . . . 156
4.5.1 Example: Fitting a GBD to Simulated Data from GBD$(3, 5, 0, -0.5)$ . . . 157
4.5.2 Example: Fitting a GBD to Data Simulated from GBD$(2, 7, 1, 4)$ . . . 158
4.5.3 Example: Cadmium in Horse Kidneys . . . 160
4.5.4 Example: Rainfall Data of Section 3.5.5 . . . 161
4.5.5 Example: Tree Stand Heights and Diameters in Forestry . . . 163
4.6 EGLD Random Variate Generation . . . 167
Problems for Chapter 4 . . . 167
References for Chapter 4 . . . 168

5 A Percentile-Based Approach to Fitting Distributions and Data with the GLD 171
5.1 The Use of Percentiles . . . 172
5.2 The (ρ_3, ρ_4)-Space of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ . . . 174
5.3 Estimation of GLD Parameters through a Method of Percentiles . 180
5.4 GLD Approximations of Some Well-Known Distributions . . . 186
5.4.1 The Normal Distribution . . . 186
5.4.2 The Uniform Distribution . . . 188
5.4.3 The Student's t Distribution . . . 188
5.4.4 The Exponential Distribution . . . 190
5.4.5 The Chi-Square Distribution . . . 192
5.4.6 The Gamma Distribution . . . 194
5.4.7 The Weibull Distribution . . . 196

5.4.8 The Lognormal Distribution . . . 197
5.4.9 The Beta Distribution . . . 199
5.4.10 The Inverse Gaussian Distribution . . . 201
5.4.11 The Logistic Distribution . . . 203
5.4.12 The Largest Extreme Value Distribution . . . 203
5.4.13 The Extreme Value Distribution . . . 205
5.4.14 The Double Exponential Distribution . . . 206
5.4.15 The F-Distribution . . . 207
5.4.16 The Pareto Distribution . . . 209
5.4.17 Summary of Distribution Approximations . . . 210
5.5 Comparison of the Moment and Percentile Methods . . . 211
5.6 Examples: GLD Fits of Data via the Method of Percentiles . . . 214
5.6.1 Example: Data from the Cauchy Distribution . . . 215
5.6.2 Data on Radiation in Soil Samples . . . 217
5.6.3 Data on Velocities within Galaxies . . . 218
5.6.4 Rainfall Data of Sections 3.5.5 and 4.5.4 . . . 219
5.7 Percentile-Based GLD Fit of Data from a Histogram . . . 222
Problems for Chapter 5 . . . 224
References for Chapter 5 . . . 225

6 Fitting Distributions and Data with the GLD through L-Moments 227
6.1 L-Moments . . . 227
6.2 The (τ_3, τ_4)-Space of the GLD . . . 229
6.3 Estimation of GLD Parameters through L-Moments . . . 232
6.4 Approximations of Some Well-Known Distributions . . . 239
6.4.1 The Normal Distribution . . . 240
6.4.2 The Uniform Distribution . . . 242
6.4.3 The Student's t Distribution . . . 242
6.4.4 The Exponential Distribution . . . 244
6.4.5 The Chi-Square Distribution . . . 245
6.4.6 The Gamma Distribution . . . 248
6.4.7 The Weibull Distribution . . . 249
6.4.8 The Lognormal Distribution . . . 250
6.4.9 The Beta Distribution . . . 251
6.4.10 The Inverse Gaussian Distribution . . . 253
6.4.11 The Logistic Distribution . . . 253
6.4.12 The Largest Extreme Value Distribution . . . 255

6.4.13 The Extreme Value Distribution . . . 256
6.4.14 The Double Exponential Distribution . . . 256
6.4.15 The F-Distribution . . . 258
6.4.16 The Pareto Distribution . . . 260
6.5 Examples of GLD Fits to Data via L-Moments . . . 261
6.5.1 Example: Cadmium Concentration in Horse Kidneys . . . 261
6.5.2 Example: Brain MRI Scan . . . 263
6.5.3 Example: Human Twin Data . . . 264
6.5.4 Example: Rainfall Distribution . . . 266
6.5.5 Example: Data Simulated from GBD(3, 5, 0, – 0.5) . . . 268
6.5.6 Example: Data Simulated from GBD(2, 7, 1, 4) . . . 269
6.5.7 Example: Tree Stand Heights and Diameters . . . 270
6.5.8 Example: Data from the Cauchy Distribution . . . 273
6.5.9 Example: Radiation in Soil Samples . . . 274
6.5.10 Example: Velocities within Galaxies . . . 275
6.6 Fitting Data Given by a Histogram . . . 275
References for Chapter 6 . . . 277

7 Fitting a Generalized Lambda Distribution Using a Percentile-KS (P-KS) Adequacy Criterion 279
7.1 Introduction . . . 280
7.2 The Generalized Lambda Distributions . . . 281
7.2.1 Definitions . . . 281
7.2.2 Existing Parameter Estimation Methods . . . 282
7.2.3 A New P-KS Method . . . 283
7.3 GLD Modeling of Data Coming from a GLD . . . 287
7.3.1 Results on the Choice of u . . . 287
7.3.2 Influence of the Sample Size . . . 291
7.4 Gaussian Data Approached by a GLD . . . 294
7.4.1 Confidence Intervals . . . 295
7.4.2 Modeling Adequacy. . . . 296
7.5 Comparison with the Method of Moments in Three Specific Cases . . . 299
7.5.1 Gaussian Data . . . 299
7.5.2 Uniform Data . . . 300
7.5.3 Student t Data . . . 300
7.6 Conclusions . . . 301
References for Chapter 7 . . . 303
Appendix for Chapter 7 . . . 305

8 Fitting Mixture Distributions Using a Mixture of Generalized Lambda Distributions with Computer Code 311
8.1 Brief Overview of the Generalized Lambda Distribution . . . 312
8.2 The Problem of Mixture Distributions . . . 314
8.3 Estimation of Parameters of a Mixture of Two GLDs . . . 315
8.4 Graphs of the Mixture Density of Two GLDs . . . 318
8.5 Fitting the Mixture of Two GLDs to Real Data . . . 319
8.5.1 Pearson's Data . . . 319
8.5.2 Cadmium in Horse Kidneys . . . 322
8.5.3 Exchange Rate Data for Japanese Yen . . . 323
8.6 Comparison with Normal Mixtures . . . 327
8.7 Conclusions and Research Problems Regarding the Mixture of Two GLDs . . . 332
References for Chapter 8 . . . 334
Appendix for Chapter 8 . . . 338

9 GLD–2: The Bivariate GLD Distribution 363
9.1 Overview . . . 364
9.2 Plackett's Method of Bivariate d.f. Construction: The GLD–2 . . . 366
9.3 Fitting the GLD–2 to Well-Known Bivariate Distributions . . . 376
9.3.1 The Bivariate Normal (BVN) Distribution . . . 377
9.3.2 Gumbel's Bivariate Exponential Type I (BVE) . . . 382
9.3.3 Bivariate Cauchy (BVC) . . . 383
9.3.4 Kibble's Bivariate Gamma (BVG) . . . 387
9.4 GLD–2 Fits: Distributions with Non-identical Marginals . . . 391
9.4.1 Bivariate Gamma BVG with Non-identical Marginals . . . 391
9.4.2 Bivariate with Normal and Cauchy Marginals . . . 392
9.4.3 Bivariate with Gamma and "Backwards Gamma" Marginals . . . 392
9.5 Fitting GLD–2 to Datasets . . . 396
9.5.1 Algorithm for Fitting the GLD–2 to Data . . . 396
9.5.2 Example: Human Twin Data of Section 3.5.4 . . . 403
9.5.3 Example: The Rainfall Distributions of Section 3.5.5 . . . 404
9.5.4 Example: The Tree Stand Data of Section 4.5.5 . . . 405
9.6 GLD–2 Random Variate Generation . . . 407
9.7 Conclusions and Research Problems Regarding the GLD–2 . . . 409
Problems for Chapter 9 . . . 412
References for Chapter 9 . . . 413

10 Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals 415
10.1 Introduction . . . 416
10.1.1 The Generalized Lambda Distribution . . . 416
10.1.2 Shape Functionals . . . 417
10.2 Description of Method . . . 418
10.2.1 Overview . . . 418
10.2.2 Theoretical Values of the Shape Functionals . . . 418
10.2.3 Optimization . . . 420
10.2.4 $\boldsymbol{u}$, $\boldsymbol{v}$ Selection . . . 420
10.2.5 Location and Scale Parameters . . . 421
10.3 Simulations . . . 422
10.3.1 Effect of Sample Size (RS Parameterization) . . . 422
10.3.2 Effect of Sample Size (FMKL Parameterization) . . . 424
10.3.3 Different Shapes . . . 425
10.3.4 Overall . . . 427
10.4 Example: Particulates . . . 430
10.5 Approximation . . . 431
10.6 Conclusion . . . 431
References for Chapter 10 . . . 432

11 Statistical Design of Experiments: A Short Review 433
11.1 Introduction to DOE . . . 434
11.1.1 Experiments . . . 435
11.1.2 Types of Experiments . . . 436
11.1.3 The Independent Variable (IV) . . . 436
11.1.4 Types of Causal (CV) or Independent (IV) Variables . . . 437
11.1.5 The Dependent Variable (DV) . . . 437
11.1.6 When to Use DOE . . . 438
11.1.7 Factors or Treatments . . . 438
11.1.8 Levels . . . 438
11.1.9 Regression Coefficients . . . 441
11.1.10 Residuals . . . 441
11.1.11 Optimality of Design . . . 441
11.1.12 Optimization . . . 442
11.1.13 Orthogonality . . . 442
11.2 Fundamentals of DOE . . . 446
11.2.1 Basic Principles . . . 446
11.2.2 Practical Considerations . . . 446
11.2.3 Designing . . . 447
11.2.4 Randomization . . . 451
11.2.5 Replication . . . 452

11.2.6 Blocking . . . 452
11.2.7 Degrees of Freedom . . . 453
11.2.8 Example: Full 2^3 Factorial . . . 453
11.2.9 Summary . . . 459
11.2.10 Example: Central Composite Designs (CCD) . . . 459
11.2.11 Taguchi Designs . . . 462
11.2.12 Example . . . 464
11.2.13 Summary of Taguchi Design . . . 468
11.2.14 Latin Hypercube Sampling . . . 469
11.3 Analysis Procedures . . . 471
11.3.1 How Many Runs? . . . 472
11.3.2 Other DOE Patterns and their Usage . . . 480
References for Chapter 11 . . . 482
Appendix for Chapter 11 . . . 486

PART III: Quantile Distribution Methods 501

12 Statistical Modeling Based on Quantile Distribution Functions 503
12.1 Distributions Formulated as Quantile Functions . . . 503
12.2 Describing and Analyzing Distributional Shape . . . 509
12.3 Model Construction . . . 517
12.4 Methods of Fitting Quantile Distributions: An Overview . . . 520
12.5 Minimization Methods of Fitting Quantile Distributions . . . 521
12.5.1 Rankits and Median Rankits . . . 521
12.5.2 Distributional Least Squares (DLS) . . . 524
12.5.3 Distributional Least Absolutes (DLA) . . . 526
12.6 Fitting Parametric Regression Models Based on Quantile Functions . . . 528
12.7 Validation . . . 533
12.8 Conclusion . . . 534
References for Chapter 12 . . . 535

13 Distribution Fitting with the Quantile Function of Response Modeling Methodology (RMM) 537
13.1 The General Approach to Fitting by Response Modeling Methodology (RMM) . . . 538
13.2 The Quantile Function of the RMM Model and its Estimation . . . 542
13.2.1 Derivation of the RMM Quantile Function . . . 542
13.2.2 Estimating RMM Quantile Function . . . 546
References for Chapter 13 . . . 556

14 Fitting GLDs and Mixture of GLDs to Data Using Quantile Matching Method 557
14.1 Introduction . . . 558
14.2 Methods . . . 559
14.3 Results . . . 561
14.3.1 Performance of Quantile Matching Estimation . . . 578
14.3.2 Quantile Matching Method for Mixture Data . . . 580
14.4 Conclusion . . . 582
References for Chapter 14 . . . 582

15 Fitting GLD to Data Using GLDEX 1.0.4 in R 585
15.1 Introduction . . . 586
15.2 Installation and Basic GLDEX Functions . . . 589
15.3 Fitting Examples . . . 590
15.4 Fitting Empirical Data . . . 601
15.5 Future Possible Improvements to `GLDEX 1.0.4` . . . 606
15.6 Conclusion . . . 607
References for Chapter 15 . . . 607

PART IV: Other Families of Distributions 609

16 Fitting Distributions and Data with the Johnson System via the Method of Moments 611
16.1 Components of the Johnson System . . . 611
16.2 The S_L Component . . . 613
16.3 The S_U Component . . . 618
16.4 The S_B Component . . . 621
16.5 Approximations of Some Well-Known Distributions . . . 624
16.5.1 The Normal Distribution . . . 625
16.5.2 The Uniform Distribution . . . 626
16.5.3 The Student's t Distribution . . . 627
16.5.4 The Exponential Distribution . . . 629
16.5.5 The Chi-Square Distribution . . . 630
16.5.6 The Gamma Distribution . . . 633
16.5.7 The Weibull Distribution . . . 634
16.5.8 The Lognormal Distribution . . . 635
16.5.9 The Beta Distribution . . . 635
16.5.10 The Inverse Gaussian Distribution . . . 638
16.5.11 The Logistic Distribution . . . 639
16.5.12 The Largest Extreme Value Distribution . . . 640
16.5.13 The Extreme Value Distribution . . . 641
16.5.14 The Double Exponential Distribution . . . 641

16.5.15 The F–Distribution 642
16.5.16 The Pareto Distribution 644
16.6 Examples of Johnson System Fits to Data 645
16.6.1 Example: Cadmium Concentration in Horse Kidneys . . . 645
16.6.2 Example: Brain MRI Scan 647
16.6.3 Example: Human Twin Data 648
16.6.4 Example: Rainfall Distribution 650
16.6.5 Example: Data Simulated from GBD(3, 5, 0, -0.5) 652
16.6.6 Example: Data Simulated from GBD(2, 7, 1, 4) 653
16.6.7 Example: Tree Stand Heights and Diameters 654
16.6.8 Example: Data from the Cauchy Distribution 657
16.6.9 Example: Radiation in Soil Samples 658
16.6.10 Example: Velocities within Galaxies 659
16.7 Fitting Data Given by a Histogram 660
References for Chapter 16 663

17 Fitting Distributions and Data with the Kappa Distribution through L-Moments and Percentiles 665
17.1 The Kappa Distribution 666
17.2 Estimation of Kappa Parameters via L-Moments 669
17.3 Estimation of Kappa Parameters via Percentiles 674
17.4 Approximations of Some Well-Known Distributions 679
17.4.1 The Normal Distribution 679
17.4.2 The Uniform Distribution 680
17.4.3 The Student's t Distribution 681
17.4.4 The Exponential Distribution 684
17.4.5 The Chi-Square Distribution 685
17.4.6 The Gamma Distribution 688
17.4.7 The Weibull Distribution 689
17.4.8 The Lognormal Distribution 690
17.4.9 The Beta Distribution 692
17.4.10 The Inverse Gaussian Distribution 694
17.4.11 The Logistic Distribution 695
17.4.12 The Largest Extreme Value Distribution 696
17.4.13 The Extreme Value Distribution 696
17.4.14 The Double Exponential Distribution 696
17.4.15 The F-Distribution 696
17.4.16 The Pareto Distribution 698
17.5 Examples of Kappa Distribution Fits to Data 699
17.5.1 Example: Cadmium Concentration in Horse Kidneys . . . 699
17.5.2 Example: Brain MRI Scan 701
17.5.3 Example: Human Twin Data 702

17.5.4 Example: Rainfall Distribution . . . 705
17.5.5 Example: Data Simulated from GBD(3, 5, 0, −0.5) . . . 708
17.5.6 Example: Data Simulated from GBD(2, 7, 1, 4) . . . 709
17.5.7 Example: Tree Stand Heights and Diameters . . . 711
17.5.8 Data from the Cauchy Distribution . . . 714
17.5.9 Example: Radiation in Soil Samples . . . 714
17.5.10 Velocities within Galaxies . . . 714
17.6 Fitting Data Given by a Histogram . . . 716
References for Chapter 17 . . . 718

18 Weighted Distributional L_α Estimates 719
18.1 Introduction . . . 720
18.1.1 Is the Normal Distribution Normal? . . . 721
18.1.2 Weighted L_α Regression . . . 722
18.2 Probability-Based Partially Adaptive Estimation . . . 724
18.2.1 Not-Necessarily Gaussian Error Distributions . . . 726
18.2.2 Estimation of the Parameters . . . 730
18.2.3 Probability-Based Distributional Regression . . . 732
18.3 Quantile-Based Partially Adaptive Estimation . . . 736
18.3.1 Quantile Models . . . 737
18.3.2 Quantile-Based Distributional Regression . . . 742
18.4 Controlled Random Search . . . 745
18.5 Goodness-of-Fit Assessment . . . 748
18.6 Empirical Examples . . . 752
18.6.1 Mayer's Data . . . 754
18.6.2 Martin Marietta Data . . . 754
18.6.3 Prostate Cancer Data . . . 757
18.6.4 Salinity Data . . . 760
18.6.5 Gaussian Data . . . 760
18.7 Conclusions and Future Research . . . 763
References for Chapter 18 . . . 764
Appendix for Chapter 18 . . . 771

19 A Multivariate Gamma Distribution for Linearly Related Proportional Outcomes 787
19.1 Introduction . . . 788
19.2 Definitions . . . 789
19.3 Basic Concepts . . . 793
19.4 The Fatal Shock Model . . . 802
19.5 Example . . . 807
19.6 Conclusion . . . 808
References for Chapter 19 . . . 808
Appendix for Chapter 19 . . . 811

PART V: The Generalized Bootstrap and Monte Carlo Methods **813**

20 The Generalized Bootstrap (GB) and Monte Carlo (MC) Methods **815**
20.1 The Generalized Bootstrap (GB) Method 816
20.2 Comparisons of the GB and BM Methods 823
Problems for Chapter 20 823
References for Chapter 20 824

21 The Generalized Bootstrap: A New Fitting Strategy and Simulation Study Showing Advantage over Bootstrap Percentile Methods **827**
21.1 Introduction 828
21.2 Algorithms for Three Methods of Distribution Fitting 831
21.2.1 Fitting the GLD through the Method of Moments (MOM) 831
21.2.2 EGLD: Method of Generalized Beta Distribution (MGBD) 831
21.2.3 Method of Percentiles (MOP) 832
21.3 Fitting Strategy and Simulation Study 833
21.4 Specific Examples 837
21.4.1 Sample 1: GBD, Not Covered 838
21.4.2 Sample 5: GBD, Covered 838
21.4.3 Sample 12: MOM, Not Covered 840
21.4.4 Sample 19: MOM, Covered 843
21.4.5 Sample 21: GBD, Covered 843
21.4.6 Sample 28: MOP, Not Covered 846
21.5 Conclusions 852
Acknowledgments 852
References for Chapter 21 852
Appendix for Chapter 21 854

22 Generalized Bootstrap Confidence Intervals for High Quantiles **877**
22.1 Introduction 878
22.1.1 High Quantile Estimation 878
22.1.2 The Bootstrap Method 879
22.2 Generalized Lambda Distribution and Generalized Bootstrap 881
22.3 Comparisons of Bootstrap Confidence Intervals for High Quantiles 882
22.3.1 Simulated Distributions 883

22.3.2 Choices of Quantile Levels and Sample Sizes 884
22.3.3 Criteria for Performance Evaluation 886
22.3.4 Simulation Algorithms 887
22.4 Simulation Results and Discussion 888
22.4.1 Performances for the Beta Distributions 890
22.4.2 Performances for the Gamma, Weibull, and Normal Distributions 890
22.5 Conclusions 897
References for Chapter 22 897
Appendix for Chapter 22 901

PART VI: Assessment of the Quality of Fits 915

23 Goodness-of-Fit Criteria Based on Observations Quantized by Hypothetical and Empirical Percentiles 917
23.1 Data and their Statistical Models 918
23.2 Assessment of Goodness-of-Fit 921
23.2.1 Special Distances, Divergences, and Disparities 922
23.2.2 Examples 927
23.3 Criteria of Goodness-of-Fit 931
23.3.1 Disparities, Divergences, and Metric Distances 931
23.3.2 Metricity and Robustness 937
23.4 Disparities Based on Partitions 939
23.4.1 Partitioning by Hypothetical Percentiles 943
23.4.2 Partitioning by Empirical Percentiles 943
23.5 Goodness-of-Fit Statistics Based on Spacings 947
23.5.1 Objectives of the Following Sections 947
23.5.2 Types of Statistics Studied 950
23.5.3 Structural Spacings Statistics 956
23.5.4 Organization of the Following Sections 958
23.6 Asymptotic Properties of Structural Statistics 958
23.6.1 Asymptotic Equivalence 959
23.6.2 Assumptions and Notations 960
23.6.3 Consistency under Hypothesis and Fixed Alternatives . . . 961
23.6.4 Asymptotic Normality under Local Alternatives 962
23.6.5 Asymptotic Normality under Fixed Alternatives 962
23.7 Asymptotic Properties of Power Spacings Statistics 964
23.7.1 Power Spacing Statistics 964
23.7.2 Consistency 968
23.7.3 Asymptotic Normality under Local Alternatives 969
23.7.4 Asymptotic Normality under Fixed Alternatives 971

23.7.5 Discussion . . . 974
23.8 The PODISTAT Program Package . . . 976
23.9 Proofs of Assertions . . . 982
23.9.1 Proofs for Structural Spacings Statistics . . . 982
23.9.2 Proofs for Power Spacings Statistics . . . 986
Acknowledgments . . . 991
References for Chapter 23 . . . 991

24 Evidential Support Continuum (ESC): A New Approach to Goodness-of-Fit Assessment, which Addresses Conceptual and Practical Challenges 995
24.1 Introduction . . . 996
24.2 Challenges to Effective G-O-F Testing . . . 997
24.2.1 Conceptual and Theoretical Challenges . . . 997
24.2.2 Practical Application Challenges . . . 1000
24.2.3 Addressing Goodness-of-Fit Challenges: A New Approach . 1001
24.3 Context . . . 1001
24.3.1 Data Description . . . 1001
24.3.2 Distributions Considered . . . 1002
24.4 The Method . . . 1003
24.4.1 Eight Pieces of Evidence . . . 1003
24.4.2 The Evidential Support Continuum (ESC) Method . . . 1003
24.4.3 Quantitative Evaluations . . . 1004
24.4.4 Graphical Evaluations . . . 1008
24.5 Results . . . 1011
24.5.1 The χ^2 Test . . . 1012
24.5.2 The K-S Test . . . 1013
24.5.3 Distribution Support to Spread of Data (Adequacy and Appropriateness . . . 1013
24.5.4 Fit of the Distribution p.d.f. to the Dataset Histogram (Main Body and Tails) . . . 1016
24.5.5 Fit of the Distribution c.d.f. to the Dataset e.d.f. (Main Body and Tails) . . . 1017
24.5.6 Constructing the ESC Diagram . . . 1020
24.6 Conclusions . . . 1021
24.6.1 ESCs Address Conceptual and Theoretical Challenges to Effective G-O-F . . . 1021
24.6.2 ESCs Address Practical Application Challenges to G-O-F . . . 1023
References for Chapter 24 . . . 1026
Appendix for Chapter 24 . . . 1029

25 Estimation of Sampling Distributions of the Overlapping Coefficient and Other Similarity Measures 1039
25.1 Introduction . . . 1040
25.2 Measures of Similarity . . . 1043
25.2.1 Overlap Coefficient δ . . . 1044
25.2.2 Matusita's Measure ρ . . . 1047
25.2.3 Morisita's Measure λ . . . 1047
25.2.4 MacArthur-Levins' Measure α^* . . . 1049
25.2.5 Common Properties . . . 1049
25.3 Measures of Similarity for Exponential Populations . . . 1050
25.4 Measures of Similarity for Normal Populations . . . 1053
25.4.1 The Equal Means Case . . . 1053
25.4.2 The Equal Variances Case . . . 1057
25.4.3 The General Case . . . 1057
25.5 Sampling Distributions: Exponential Populations Case . . . 1061
25.6 Sampling Distributions: Normal Populations Case . . . 1069
25.7 Conclusions . . . 1083
Acknowledgments . . . 1085
References for Chapter 25 . . . 1085

PART VII: Applications 1091

26 Fitting Statistical Distribution Functions to Small Datasets 1093
26.1 Introduction . . . 1094
26.2 Analysis Techniques . . . 1097
26.3 The Johnson Family of Distributions and the JFit and GAFit Tools . . . 1098
26.4 The Empirical Distribution Function (edf) . . . 1100
26.5 Goodness-of-Fit (GOF) . . . 1101
26.6 Example 1: The SST1 Dataset . . . 1103
26.6.1 Johnson Family Distributions that Minimize KS Scores . . . 1109
26.6.2 The Four-Parameter Generalized Lambda Distribution . . . 1110
26.7 Example 2: The SST2 Dataset . . . 1113
26.8 Example 3: The SST3 Dataset . . . 1118
26.9 Example 4: The SST4 Dataset . . . 1121
26.10 Example 5: The SST5 Dataset . . . 1123
26.11 Example 6: The SST47 Dataset . . . 1127
26.12 Example 7: The SST29 Dataset . . . 1129
26.13 Example 8: Solar Flux (Top 95% Amplitude) at 10.7 MHz . . . 1131

26.14 A Bimodal Fit to Data . . . 1132
26.15 Comparisons . . . 1135
26.16 Model Selection . . . 1136
26.17 Summary . . . 1138
References for Chapter 26 . . . 1138
Appendix for Chapter 26 . . . 1140

27 Mixed Truncated Random Variable Fitting with the GLD, and Applications in Insurance and Inventory Management 1145
27.1 Introduction . . . 1146
27.2 The Generalized Lambda Distribution and its Partial Moments . . 1148
27.3 Moments of Mixed Type of a Truncated Random Variable 1150
27.4 Applications . . . 1155
27.4.1 Optimum Deductible in Insurance Purchasing . . . 1155
27.4.2 Newsboy Model: Solution under Utility Maximization . . . 1161
27.5 Conclusion . . . 1167
Acknowledgments . . . 1167
References for Chapter 27 . . . 1167

28 Distributional Modeling of Pipeline Leakage Repair Costs for a Water Utility Company 1171
28.1 Introduction . . . 1172
28.2 Generalized Lambda Distributions . . . 1172
28.2.1 Basic Theory . . . 1173
28.2.2 Fitting Methods . . . 1173
28.3 Factors Influencing the Costs of Pipe Repairs . . . 1176
28.3.1 Continuous Data: Pipe Length and Age of Pipe . . . 1176
28.3.2 Discrete Data: Pipe Diameter, Pipe Location, and Month . . . 1177
28.3.3 GLD Fits on the Repair Costs in Relation to Pipe Diameter, Pipe Location, and Month . . . 1177
28.3.4 GLD Fits to Repair Costs on Pipelines with Diameter 300 in LOCA and LOCB Regions . . . 1185
28.3.5 Remarks . . . 1185
28.4 Rationally Setting Water Prices to Break Even: A Case of Pipe Repair Costs . . . 1185
28.4.1 Global Pipe Repair Cost Example . . . 1185
28.4.2 Individual Pipe Repair Cost Example . . . 1189
28.4.3 Other Considerations . . . 1200
28.5 Further Considerations in Choosing the GLD for Analysis . . . 1203
28.6 Conclusion . . . 1203
References for Chapter 28 . . . 1203

29 Use of the Generalized Lambda Distribution in Materials Science, with Examples in Fatigue Lifetime, Fracture Mechanics, Polycrystalline Calculations, and Pitting Corrosion 1207
29.1 Fatigue Lifetimes . . . 1208
29.1.1 Lifetime Distributions . . . 1209
29.1.2 Crack Initiation and Crack Propagation . . . 1211
29.2 Fracture Mechanics . . . 1217
29.2.1 Fracture of Adhesively Bonded Joints Composed of Pultruted Adherends . . . 1218
29.2.2 Modelling the Nuclear Reactor Pressure Vessel Steel Brittle Fracture . . . 1221
29.3 Extreme Values . . . 1227
29.3.1 Pitting Corrosion . . . 1229
29.3.2 Roughness . . . 1233
29.4 Conclusion . . . 1238
References for Chapter 29 . . . 1238

30 Fitting Statistical Distributions to Data in Hurricane Modeling 1245
30.1 Introduction . . . 1246
30.2 Hurricane Modeling Basics . . . 1247
30.3 Distributional Fitting at Individual Sites . . . 1250
30.4 Data Quality and Preparation Issues . . . 1252
30.5 Predictive Ability and Performance . . . 1255
30.6 Case Study of the New Orleans Levees: Wrath of Mother Nature or Ordinary Extreme Event . . . 1258
30.7 Conclusions . . . 1260
Acknowledgments . . . 1261
References for Chapter 30 . . . 1261

31 A Rainfall-Based Model for Predicting the Regional Incidence of Wheat Seed Infection by *Stagonospora nodorum* in New York 1263
31.1 Introduction . . . 1264
31.2 Materials and Methods . . . 1265
31.2.1 Probability Distribution for Seed Infection Incidence . . . 1265
31.2.2 Relating Mean Seed Infection to Rainfall . . . 1268
31.2.3 Binary Power Law . . . 1270
31.2.4 Model Verification and Validation . . . 1271
31.3 Results . . . 1272
31.3.1 Probability Distribution for Seed Infection Incidence . . . 1272
31.3.2 Relating Mean Seed Infection to Rainfall . . . 1272
31.3.3 Binary Power Law . . . 1272

31.3.4 Model Verification and Validation 1274
31.4 Discussion 1277
References for Chapter 31 1278

32 Reliability Estimation Using Univariate Dimension Reduction and Extended Generalized Lambda Distribution 1281
32.1 Introduction 1282
32.2 Calculation of Statistical Moments 1284
32.3 Estimating the Distribution of Performance Function 1286
32.3.1 Overview of GLD and GBD 1287
32.3.2 Estimation of EGLD Parameters 1289
32.3.3 Probability of Failure Estimation Using EGLD 1291
32.4 The UDR + EGLD Algorithm 1291
32.5 Example Problems 1292
32.5.1 Example 1: A Concave Function 1292
32.5.2 Example 2: A Non-Linear Function 1293
32.5.3 Example 3: Performance Function with Infinite MPP . . . 1294
32.5.4 Example 4: Vehicle Crash 1296
32.5.5 Example 5: I-Beam Design Problem 1297
32.5.6 Example 6: Checking for Formulation Invariance 1300
32.5.7 Effect of Probability Level 1300
32.6 Concluding Remarks 1301
Acknowledgments 1302
References for Chapter 32 1302

33 Statistical Analyses of Environmental Pressure Surrounding Atlantic Tropical Cyclones 1305
33.1 Introduction 1306
33.2 Historical Data Sources 1306
33.3 Distribution of Environmental Pressure 1308
33.4 Environmental Pressure for Specific Storms 1311
33.5 Relationship of Penv to Latitude, Time of Year, Pmin 1313
33.6 Extrapolation to the Full Historical Record 1315
33.7 Wind Field Impacts 1320
33.8 Conclusions 1323
References for Chapter 33 1323

34 Simulating Hail Storms Using Simultaneous Efficient Random Number Generators 1325
34.1 Simulation of a Hail Precipitation System 1326
34.2 Models Used 1331
34.3 Conclusions 1333

34.4 Tests of Simultaneous Generators 1334
34.5 Introduction of Several Generators in a Simulation 1335
34.6 Statistical Analysis 1337
References for Chapter 34 1339
Appendix for Chapter 34 1342

PART VIII: Appendices **1347**

A Programs and their Documentation 1349
A.1 General Computational Issues 1350
A.2 General Functions 1351
A.3 Functions for GLD Computations 1352
A.4 Functions for GBD Fits 1358
A.5 Functions for the Kappa Distribution 1360
A.6 Maple Programs for Johnson System Fits 1364
A.7 The Maple Code for the Bivariate GLD 1365
A.8 Content of the Attached CD 1365
A.9 The **R** Code of the Programs for GLD Fits 1367
A.10 **R** Code of the Programs for Kappa Distribution Fits 1387
A.11 Maple Code for Johnson System Fits 1397
A.12 Maple Code for Bivariate GLD (GLD–2) Fits 1404
B Table B–1 for GLD Fits: Method of Moments 1407
C Table C–1 for GBD Fits: Method of Moments 1429
D Tables D–1 through D–5 for GLD Fits: Method of Percentiles . . 1455
E Tables E–1 through E–5 for GLD Fits: Method of L-Moments . . 1511
F Table F–1 for Kappa Distribution Fits: Method of L-Moments . . 1557
G Table G–1 for Kappa Distribution Fits: Method of Percentiles . . 1589
H Table H–1 for Johnson System Fits in the S_U Region: Method of Moments 1611
I Table I–1 for Johnson System Fits in the S_B Region: Method of Moments 1627
J Table J–1 for p-Values Associated with Kolmogorov-Smirnov Statistics 1659
K Table K–1 Normal Distribution Percentiles 1661

PART I: Overview

Chapter 1 Fitting Statistical Distributions: An Overview

A sketch of this chapter appears in Section 1.2.

Chapter 1

Fitting Statistical Distributions: An Overview

Much of **modern human endeavor**, in wide-ranging fields that include science, technology, medicine, engineering, management, and virtually all of the areas that comprise human knowledge **involves the construction of statistical models** to describe the fundamental variables in those areas.[1] The most basic and widely used model, called the **probability distribution**, relates the values of the fundamental variables to their probability of occurrence. When the variable of interest can take on (subject to the precision of the measuring process) any value in an interval, the probability distribution is called **continuous**. This book deals with how to model (or, **how to fit) a continuous probability distribution** to data.

The area of fitting distributions to data has seen explosive growth in recent years.[2] Consequently, few individuals are well versed in the new results that have become available. In many cases these recent developments have solved old problems with the fitting process; they have also provided the practitioner with a confusing array of methods. Moreover, some of these methods, called "asymptotic" methods in the theoretical literature, were developed with very large sample sizes in mind and are ill-suited for many applications. With these facts in mind, in our book *Fitting Statistical Distributions: The Generalized Lambda Distribution and the Generalized Bootstrap Methods* we sought to

- Give the results, tables, algorithms, and computer programs needed to fit continuous probability distributions to data, using the Generalized Lambda Distribution (GLD) and the Generalized Bootstrap (GB) approaches;

[1] For example, in actuarial science, see Klugman, Panjer, and Willmot (1998), which while an "applied text" (p. ix), needed to assume "that the reader has a solid background in mathematical statistics" (p. ix).

[2] Recent works include Bowman and Azzalini (1997), Scott (1992), and Simonoff (1996). An overview is given by Müller (1997); see also Dudewicz, Carpenter, Mishra, Mulekar, Romeu, and Shimizu (2007, 2008).

- Bring together in one place the key results on GLD and GB fitting of continuous probability distributions, with emphasis on recent results that make these methods nearly universally applicable.

There are good reasons for using the GLD distribution and GB methods. GLD fits have been used successfully in many fields (e.g., the construction industry, atmospheric data, quality control, medical data, reliability). The inclusion of other methods was done in earlier work (see Part IV of Dudewicz and Karian (1985)), but not in Karian and Dudewicz (2000) in order to avoid the question: "Which method should be used in practice?" Our answer to this question has always been: Try the GLD and GB first and stop there if the results are acceptable. Since our work contributed to the refinement and (in some senses) completion of some aspects of the GLD and GB methods, whose proper exposition alone fills a book, this seemed and seems still a reasonable choice for our book, Karian and Dudewicz (2000).

However, the present work is a Handbook, and this allows us the space needed to answer the question: "What should be done if the results are not acceptable?" This is done with such systems as the Kappa System, the Johnson System, and others detailed in following chapters. It also allows us to delve into the many recent additions to the GLD and GB methods by authors worldwide, and to give more details of some application areas.

1.1 History and Background

The search for a method of fitting a continuous probability distribution to data is quite old. Pearson (1895) gave a four-parameter system of probability density functions, and fitted the parameters by what he called the "method of moments" (Pearson (1894)). It has been stated (Hald (1998), pp. 649–650) that

> Like Chebyshev, Fechner, Thiele, and Gram he [Pearson] felt the need for a collection of continuous distributions to choose among for describing the ... phenomena he was studying. He wanted a system embracing distributions with finite as well as infinite support and with skewness both to the right and the left.

The generalized lambda distribution has its origin in the one-parameter lambda distribution proposed by John Tukey[3] (1960). Tukey's lambda distribution was generalized, for the purpose of generating random variates for Monte Carlo simulation studies, to the four-parameter generalized lambda distribution, or GLD, by John Ramberg and Bruce Schmeiser (Ramberg and Schmeiser (1972, 1974)). Subsequently, a system, with the necessary tables, was developed for fitting a

[3]It has been pointed out that the Tukey Lambda was introduced in Hastings, Mosteller, Tukey, and Winsor (1947); however, the usual origin is stated as Tukey (1960).

wide variety of curve shapes by Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979). Since the early 1970s the GLD has been applied to fitting phenomena in many fields of endeavor with continuous probability density functions. In some cases problems arose, such as a need for tables for a range not yet published or computed, or a need for an extension to the part of moment-space not covered by the GLD; solutions to these problems are given in this Handbook.

We will cover the **applications** through a number of detailed examples. A few words about some important applications are appropriate in this introduction. In an early application of the GLD (at the time called the RS (or Ramberg-Schmeiser) distribution), Ricer (1980) dealt with **construction industry data**. His concern was to correct for the deviations from normality which occur in construction data, especially in expectancy pricing and in a competitive bidding environment, finding such quantities as the optimum markup.

In another important application area, **meteorology**, it is recognized that many variables have "... non-normality, [so] most climatologists have used empirical distributions ..." (Öztürk and Dale (1982), p. 995). As an alternative, fitting of solar radiation data with the GLD was successful due to the "flexibility and generality" (Öztürk and Dale (1982), p. 1003) of the GLD, which "could successfully be used to fit the wide variety of curve shapes observed." In many applications, this means that we can use the GLD to **describe data by using a single functional form** by specifying its four parameter values for each case, instead of giving the basic data (which is what the empirical distribution essentially does) for each case. The one functional form allows us to group cases that are similar, as opposed to being overburdened with a mass of numbers or graphs.

Each application area has its own concerns and goals. What they have in common is the need for a flexible, easy-to-work-with, complete system of continuous probability distributions—a need for which the GLD has been found to be quite suitable. We urge those who wish to look up some of the work in areas for which we have not given references to write the first paper applying the GLD, GB, or other system to those areas and send us a copy. Some of the more accessible references in several important areas are

- Modeling biological and physical phenomena, Silver (1977)
- The generation of random variables for Monte Carlo studies, Hogben (1963), Shapiro and Wilk (1965), Shapiro, Wilk, and Chen (1968), Karian and Dudewicz (1999)
- Sensitivity studies of statistical methods, Filliben (1969)
- Approximation of percentiles of distributions, Tukey (1960), Van Dyke (1961)
- Testing goodness-of-fit in logistic regression, Pregibon (1980)

- Modeling complex situations in applied research, such as quantal response in bioassay and economics, Mudholkar and Phatak (1984), Pregibon (1980)

- Quality management, engineering, control, and planning, Dudewicz (1999).

As a prelude, and for some a refresher, before defining the Generalized Lambda Distribution (GLD) family in Chapter 2 and other systems in subsequent chapters, we **review some basic notions from statistics**. For fuller details, see Karian and Dudewicz (1999) or Dudewicz and Mishra (1998).

If the variable X is involved in the model we are constructing, and we have no way of accurately predicting its value from other known variables, then X is usually called a **stochastic variable** or **random variable (r.v.)**.[4] If X can take on only a few discrete values (such as 0 or 1 for failure or success, or $0, 1, 2, 3, \ldots$ as the number of occurrences of some event of interest), then X is called a **discrete random variable**. If the outcome of interest X can take on values in a continuous range (such as all values greater than zero for an engine failure time), then X is called a **continuous random variable**.[5]

For any r.v. of interest, we wish to know: What are the chances of occurrence of the various possible values of the r.v.? For example, what is the probability that X is no larger than 3.2? Or, that X is no larger than 4.5 (or any other value)? Specifying $P(X \leq 3.2), P(X \leq 4.5)$, or in general $P(X \leq x)$, is **one way of specifying the chances** of occurrence of the various values that are possible. This is called giving the **distribution function (d.f.)** of X:

$$F_X(x) = P(X \leq x), \quad -\infty < x < +\infty. \tag{1.1.1}$$

As an example, if X has what is called the **standard normal or $N(0,1)$ distribution function**, then $F_X(x)$ is usually denoted by $\Phi(x)$. There is no simple formula for finding $\Phi(3.2)$, but $\Phi(3.2)$ can be approximated through numerical integration. Values for $\Phi(x)$ are given in Appendix K for various values of x. From the table in Appendix K, we find $\Phi(3.2) = .9993129$, i.e., the chance that X will be 3.2 or less is then 99.93129%. It is possible to plot the function $\Phi(x)$ by graphing a fine grid of points $(x, \Phi(x))$ using x-values included in Appendix K, and joining the points by a smooth curve. This is done in Figure 1.1–1. The plot, while specific to the standard normal d.f., has some properties in common

[4]Here we restrict ourselves to r.v.s that assume a single number, such as $X = 3.2$, called **univariate** r.v.s. The case when X may be a pair, such as $X = (T, H)$ with T=Temperature and H=Humidity, is also important (X is then called a **bivariate** r.v.) and is dealt with in Chapters 9 and 19.

[5]There are also **mixed** r.v.s that arise when X takes on some values with positive probability but is otherwise continuous. (E.g., at random choose to flip a fair coin, marked 3 on one side and 5 on the other side, or spin a spinner that points to a number on a circumference marked continuously from 0.0 to 1.0. Then with probability 0.25, $X = 3$; with probability 0.25, $X = 5$; and otherwise X is a number between 0 and 1.) We deal here only with the continuous part of X and its modeling.

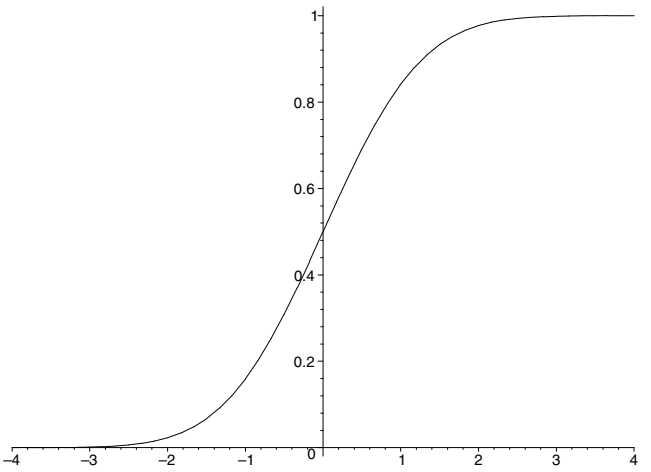


Figure 1.1–1. The d.f. of a $N(0,1)$ r.v.

with all plots of d.f.s of continuous r.v.s: the curve starts near a height of zero at the left (very small values of x), and increases continuously, approaching a height of 1.0 as x gets large.

A **second way of specifying the chances of occurrence of the various values of** X is to give what is called the **probability density function (p.d.f.)** of X. This is a function $f_X(x)$ that is ≥ 0 for all x, integrates to 1 over the range $-\infty < x < +\infty$, and such that for all x,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt. \tag{1.1.2}$$

Every d.f. $F_X(\cdot)$ has associated with it a unique[6] p.d.f. $f_X(\cdot)$, so we gain the same information by specifying either one of these functions. As an example, if X has the $N(0,1)$ d.f., then its p.d.f. is usually given the name $\phi(x)$ and has the simple expression

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < +\infty. \tag{1.1.3}$$

It is possible to plot the function $\phi(x)$ by graphing a fine grid of points $(x, \phi(x))$ using any x-value grid we wish and computing $\phi(x)$ from (1.1.3). This is done in Figure 1.1–2. The plot, while specific to the standard normal p.d.f., has some

[6]Actually, the p.d.f. is **unique up to sets of measure zero**, i.e. there is what is called an **equivalence class** of such functions. However, this is a notion not needed in our work (and those familiar with it are unlikely to need this review), so we do not dwell on it. One important point that comes from it, however, is that the p.d.f.s $f_1(x) = 1$ if $0 \leq x \leq 1$ (and $= 0$ otherwise), and $f_2(x) = 1$ if $0 < x < 1$ (and $= 0$ otherwise) are not different — they yield the same d.f., and both are called the **uniform p.d.f.** (on $(0,1)$).

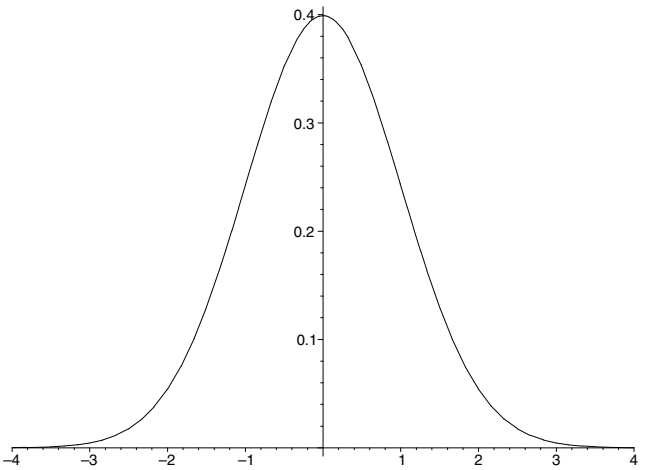


Figure 1.1–2. The p.d.f. of a $N(0,1)$ r.v.

properties in common with all plots of p.d.f.s: the curve is non-negative, the area under the whole curve is 1, and areas under the curve between values give us the probability of that range of values.

A **third way of specifying the chances of occurrence of the various values of** X is to give what is called the **inverse distribution function**, or **percentile function (p.f.)**, of X. This is the function $Q_X(y)$ which, for each y between 0 and 1, tells us the value of x such that $F_X(x) = y$:

$$Q_X(y) = (\text{The value of } x \text{ such that } F_X(x) = y), \quad 0 \leq y \leq 1. \tag{1.1.4}$$

For example, for the $N(0,1)$ d.f., we know from Appendix K that $\Phi(1.96) = 0.975$. Hence, $Q(0.975) = 1.96$ for the $N(0,1)$ distribution, i.e., the value that is (with probability .975) not exceeded is 1.96. Equivalently, with the $N(0,1)$ distribution one will find values of 1.96 or smaller 97.5% of the time. It is possible to plot the function $Q(y)$ by graphing a fine grid of points $(y, Q(y))$ using any y-value grid over the range 0 to 1. This is done in Figure 1.1–3 for the standard normal case, using numerical methods to converge on the root needed in (1.1.4). The plot, while specific to the standard normal case, has some properties in common with all plots of p.f.s: all "action" takes place for horizontal axis values between 0 and 1 (these represent probabilities of non-exceedance, and probabilities must be between 0 and 1), and the curve is increasing from the smallest value of X (in the limit as y tends to 0), to the largest value of X (in the limit as y tends to 1).

We see that there are three ways to specify the chances of occurrence of a r.v. (the d.f., the p.d.f., and the p.f.). For the $N(0,1)$ example it was not easy to specify the d.f. (we had to numerically calculate it; we may think of this as easy since virtually all statistics books have this table in them, nevertheless there is no simple formula); it was easy to specify the p.d.f.; and it was not easy to specify

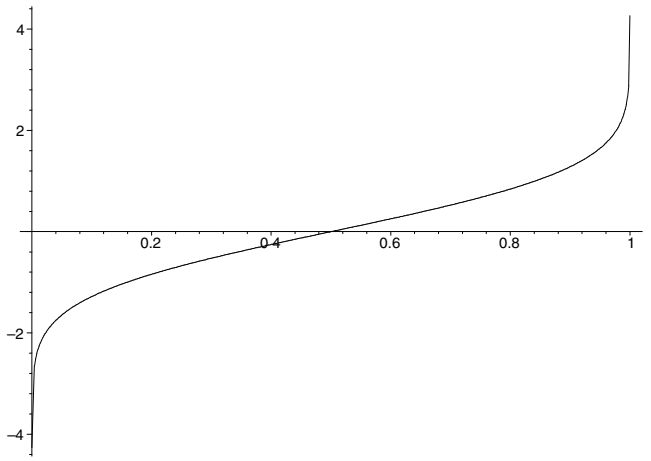


Figure 1.1–3. The p.f. of a $N(0,1)$ r.v.

the p.f. (numerical calculation was necessary; we may think of this as harder, as $Q(y)$ is given in most statistics books only for a few "holy" values of y, such as 0.90, 0.95, etc., but in fact it is no harder than finding the $N(0,1)$ d.f.). For other examples, it varies which of the d.f., p.d.f., and p.f. is easier to deal with. In particular, for the GLD we will see in Section 2.2 that the p.f. is very easy to obtain, the p.d.f. is also easy to find, and the d.f. needs numerical calculation.

For later reference, we give the p.d.f., d.f., and p.f. for a r.v. with the general normal distribution with mean μ and variance σ^2, $N(\mu, \sigma^2)$. The p.d.f. is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/(2\sigma^2)}, \tag{1.1.5}$$

the distribution function (d.f.) is

$$F(x) = P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right), \tag{1.1.6}$$

and the percentile function (p.f.) is

$$\begin{aligned} Q(y) &= (x \text{ such that } F(x) = y) \\ &= \mu + \sigma\Phi^{-1}(y). \end{aligned} \tag{1.1.7}$$

We should note that, in addition to $Q_X(y)$, there are several notations in common use for the p.f.; one usually finds the notation $F_X^{-1}(x)$. This notation, while more common in the literature, is often confused with $1/F_X(y)$, for which the same notation is used. For this reason, we will designate the p.f. by $Q_X(y)$ in most of this Handbook (the material is clearer with slightly different notations in some chapters, e.g. see Example 23.2.32 in Chapter 23).

1.2 The Organization of the Handbook

This Handbook is divided into eight parts, each consisting of material encompassing a theme. Along the way, we include a wide-ranging variety of applications, in detail, from a number of areas as illustrations in order to aid practitioners. We also include proofs of key results in order to aid those interested in the theoretical development, who may wish to teach a course in this area, or to do research in it. The book is divided into seven parts of 34 chapters and an eighth part consisting of appendices. For convenience, the chapters are numbered consecutively throughout the book, rather than within each part. The plan for the book is as follows:

PART I: Overview

Chapter 1: "Fitting Statistical Distributions: An Overview." This is the sole chapter in Part I. In a nutshell, Chapter 1 provides a historical context and gives a brief overview of the various families of distributions and methods that have been used in fitting of statistical distributions.

In greater detail, in Chapter 1 we note how statistical models and probability distributions are fundamental to virtually all areas of human knowledge. Since the probability distribution in any problem is usually unknown, the problem of fitting a statistical distribution to data arises as fundamental to modeling in virtually all areas.

Section 1.1 covers the history from Pearson's system in the 1890s, to the Generalized Lambda Distribution (GLD) in the 1970s to the present, with other systems that might be used (Kappa System, Johnson System, etc.) to be covered later in this Handbook. Section 1.1 then notes a broad array of application areas, and reviews some fundamental notions needed in fitting, such as the percentile function.

Section 1.2, the present section, gives an overview of the organization of this Handbook. It includes brief sketches of the seven Parts, 34 Chapters, and 11 Appendices of the Handbook.

PART II: The Generalized Lambda Distribution

Chaper 2: "The Generalized Lambda Family of Distributions." There are good reasons for using the GLD and EGLD distributions first when fitting distributions to data: they have been successfully used in many fields, include many different shapes and many well-known distributions, and can match any mean, variance, skewness, and kurtosis. Chapter 2 defines the basic four-parameter GLD family of distributions. The four parameters need to be specified in order to fit the data at hand. Generation of random variables from a fitted GLD (Section 2.4), and the fitting process for any family of distributions (Section 2.5) are also covered.

Chaper 3: "Fitting Distributions and Data with the GLD via the Method of Moments." Chapter 3 shows how to estimate the unknown GLD parameters with

the method of moments. This is how the parameters have usually been estimated in applications. (For many years the method of moments was the only method of fitting available — we will see in later chapters there are now other methods, and in many cases they yield even better fits, e.g. the percentile matching method of Chapter 5.)

Chapter 4: "The Extended GLD System, the EGLD: Fitting by the Method of Moments." As seen in Chapter 3 (see Figure 3.2–5), while the GLD moments cover a broad range of moment space, much broader than most other families of distributions. The next time someone shows you their "new" distribution family, ask them "Would you please show me on Figure 3.2–5 the range of moment space your new family covers?" Often it will turn out to be very small, even just a line or a point. Nevertheless, the GLD does not cover all of moment space, and in applications in the part the GLD does not cover this has been a problem. Chapter 4 develops the extended GLD (EGLD), by melding the generalized beta distribution family (GBD) with the GLD, to cover all possible moments, which solves this problem.

Chapter 5: "A Percentile-Based Approach to Fitting Distributions and Data with the GLD." Chapter 5 shows how to estimate the unknown GLD parameters using percentiles. Since moments do not always exist, and moment-based estimators can have high variability, the development of methods using percentiles allows for better fitting distributions in some applications. In fact, Karian and Dudewicz (2003, Section 7) showed a broad superiority of the percentile-based approach to the method of moments.

Chapter 6: "Fitting Distributions and Data with the GLD through L-Moments." This chapter describes a GLD parameter estimation scheme based on L-moments that requires the existence of only the first moment. This is of particular interest in applications where higher moments are believed not to exist, which arise in finance and other areas. One might wonder, "Why not just use the other methods even if the moments do not exist?" The answer is that the results are likely to be very bad, and so if higher-order moments do not exist, one should use a method that accounts for that fact.

Chapter 7: "Fitting a Generalized Lambda Distribution Using a Percentile-KS (P-KS) Adequacy Criterion." Chapter 7 presents a method for choosing parameter estimates that minimize the Kolmogorov-Smirnov goodness-of-fit statistic, using percentiles to make the computations feasible in time required, and including conditions ((7.2.15) and (7.2.16)) to assure all the observed data is included in the range where the fitted distribution has a positive density. Fully automated code is given in an appendix to this chapter. The results make this a method worthy of consideration in many applications.

Chapter 8: "Fitting Mixture Distributions Using a Mixture of Generalized Lambda Distributions with Computer Code." Fitting distributions with such

characteristics as two (or more) peaks in the density function by mixtures dates back to Karl Pearson's 1894 work with crab data. While Pearson used a mixture of two normal distributions, using mixtures of GLDs can provide similar fits with fewer parameters. For example, for a strongly skewed normal density using 16 parameters, a mixture of two GLDs with about half as many parameters does as well (measured by the overlapping coefficient) — see Figure 8.6–2 (b). Needed code is also provided, so this is a method that should be considered when the data has multiple peaks.

Chapter 9: "GLD-2: The Bivariate GLD Distribution." Data often has more than one characteristic of interest. If the characteristics of interest are independent one can model each one separately. If the characteristics of interest are not independent, one needs a bivariate distribution. Chapter 9 develops a bivariate version of the GLD (and of the EGLD). One example considered is (X,Y) where the components are rainfall in Rochester, New York and rainfall in Syracuse, New York (see Section 9.5.3). How to generate random variables from the fitted bivariate distribution is also considered (Section 9.6), and illustrated following equation (9.6.8); the needed computer code (written in Maple) is included in Section A.7 of Appendix A.

Chapter 10: "Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals." For distributions with heavy tails (heavier than the Gaussian distribution), this method of estimating shape first seems to do well enough to be one that should be considered. Note that the distribution called Gaussian, is also sometimes called the "normal" distribution. However, we have met researchers who take "normal" to mean to mean "usual." As the examples in this Handbook illustrate, the Gaussian distribution is not what one "usually" deals with in appliations.

Chapter 11: "Statistical Design of Experiments: A Short Review." Designed experiments (which means experiments where the data y have been taken at carefully chosen values of related variables x) are very useful in study of methods for fitting distributions to data — an example is given in Section 11.3.2, where Karian and Dudewicz (2003) used designs in showing that fitting via percentiles (as in Chapter 5) gives a better fit than fitting via moments (as in Chapter 3). We expect that this chapter will also be of broad use to Handbook users who need to decide how to run experiments.

PART III: Quantile Distribution Methods

Chapter 12: "Statistical Modeling Based on Quantile Distribution Functions." This chapter shows how statistical modeling can be approached using quantile functions instead of distribution functions and probability density functions. The coverage uses "rankits," defined in Section 12.5.1. The rankit is widely used in applications, but unknown to many theoretical statisticians, so you might find it interesting to study Section 12.5.1 and then discuss rankits with the next modeler

you encounter. Section 12.6 details gains in fitting regression models from the quantile function approach, while Section 12.7 discusses model validation. The results are promising enough to suggest this method as worthy of investigation for your next regression model.

Chapter 13: "Distribution Fitting with the Quantile Function of Response Modeling Methodology (RMM)." The coverage of Chapter 13 includes an example following equation (13.2.27) involving birth weight of twins. Study of the GLD model vs. the RMM model may be a good way to approach this method, for those not already familiar with it.

Chapter 14: "Fitting GLDs and Mixture of GLDs to Data Using Quantile Matching Method." Chapter 14 introduces a quantile matching method for fitting GLDs; this method also applies to fitting mixtures of GLDs. The results in real datasets are often convincing.

Chapter 15: "Fitting GLD to Data Using GLDEX 1.0.4 in R." GLDEX is an add-on package to R used to fit GLDs by a variety of methods. R is a freeware available from the site noted in line one of page one of Chapter 15. The last paragraph of Section 15.1 notes the website for an introduction to R. GLDEX can (Section 15.2) be installed using a drop down menu within the R interface. This makes substantial additional R-based software available to fit GLDs.

PART IV: Other Families of Distributions

Chapter 16: "Fitting Distributions and Data with the Johnson System via the Method of Moments." The Johnson system has four components, and has a unique distribution for each feasible (mean, variance, skewness, kurtosis) vector. One might wish to compare this to the (E)GLD fit one obtains to a dataset.

Chapter 17: "Fitting Distributions and Data with the Kappa Distribution through L-Moments and Percentiles." The Kappa distribution family was given in 1994 by Hosking. It seems to nicely complement the GLD distribution family in both L-moment space (see Figure 17.2–2), and percentile-space (see Figure 17.3–3). Therefore it would be natural to fit a Kappa as well as a GLD in applications.

Chapter 18: "Weighted Distributional L_α Estimates." Ordinary least squares (OLS) is built on the myth of a Gaussian error distribution. This chapter explores alternatives such as DR, which give results close to those of OLS when errors are Gaussian, but improved results in other situations. Appendix A.2 of this Chapter 18 gives the R code used in the examples. Section 18.6 conjectures that the EGLD would be a good point of departure. Perhaps try this and compare with OLS in your next regression.

Chapter 19: "A Multivariate Gamma Distribution for Linearly Related Proportional Outcomes." In Chapter 9 we developed and applied a bivariate GLD for

(X, Y). In Chapter 19 have Gamma distributions, and a linear relationship to an unknown Z. For applications, see Section 19.5. An Appendix has SAS code and R code.

PART V: The Generalized Bootstrap and Monte Carlo Methods

Chapter 20: "The Generalized Bootstrap (GB) and Monte Carlo (MC) Methods." Chapter 20 develops the Generalized Bootstrap method of fitting and contrasts it with the Bootstrap method. The latter has important drawbacks that the GB generalization addresses.

Chapter 21: "The Generalized Bootstrap: A New Fitting Strategy and Simulation Study Showing Advantage over Bootstrap Percentile Methods." Section 21.3 gives the Fitting Strategy for GB; this is a new strategy which takes advantage of fitting a GLD by Percentiles, by Moments, and by the EGLD. An example indicates advantage of GB over BM (see last paragraph of Section 21.3, and Section 21.5). MATLAB® code is given in an Appendix to Chapter 21.

Chapter 22: "Generalized Bootstrap Confidence Intervals for High Quantiles." High quantiles are important in many applications, e.g. in risk management. The Generalized Bootstrap (GB) performed well, while the Nonparametric Bootstrap (NPB) might totally fail to estimate high quantiles. Code for C and for R are given in an Appendix.

PART VI: Assessment of Quality of Fit

Chapter 23: "Goodness-of-Fit Criteria Based on Observations Quantized by Hypothetical and Empirical Percentiles." Chapter 23 gives new disparity statistics obtained from quantizing with empirical percentiles. Programs are given for evaluation (Section 23.8). Applications include horse kidney data (Example 23.8.5), where it is concluded that the mixture of two GLDs model from Chapter 8 (reprised at equation (23.2.34)) is not rejected at the 5% level of significance. This provides a new tool for goodness-of-fit evaluation.

Chapter 24: "Evidential Support Continuum (ESC): A New Approach to Goodness-of-Fit Assessment, which Addresses Conceptual and Practical Challenges." A new strategy called the evidential support continuum (ESC) is proposed for goodness-of-fit assessment, combining the results of multiple, and often contradictory, goodness-of-fit tests. Data from the field of exercise science (Section 24.3.1) is used to illustrate the ESC for 24 participants. Appendices A.1, A.2, and A.3 give the ESC for each participant, and give the conclusions on goodness-of-fit for that participant. Along the way, in Section 24.4.3.1 it is shown in detail how to obtain accurate p-values for the χ^2 test using simulation; this is an important result which is in no other books known to the authors of this Handbook.

Chapter 25: "Estimation of Sampling Distributions of the Overlapping Coefficient and Other Similarity Measures." The Overlapping Coefficient of two

probability density functions is the area that is under both of the curves. If it is 1 then the two densities are the same, while if it is 0 they allocate their probability to different sets of values. The Overlapping Coefficient is increasingly being used to compare densities because its value is easy to interpret (while values of other measures such as the Kullback-Leibler distance do not have an easy interpretation). The Overlapping Coefficient was apparently first defined by Czekanowski in 1909, and has been rediscovered under various names over the years, as it began to be used in more and more fields of application. Other similarity measures are reviewed, especially for exponential and normal populations, where the sampling distribution of the Overlapping Coefficient is studied and found to be able to be modeled by a beta distribution.

PART VII: Applications
Chapters 26 through 34 give applications, and methods used in those applications, in a broad set of fields. These are not routine applications, and have been chosen for such characteristics as innovative use of the methods of the previous chapters. (Some use other methods, not explored earlier in this Handbook, and some use modifications of methods.) The chapter authors are at leading companies, laboratories, and universities both in the U.S. and internationally, including Europe, Asia, Australia, and South America. (Since elsewhere in the Handbook we have representation from Africa, our only geographical expansion could come from the continent of Antarctica.)

Chapter 26: "Fitting Statistical Distribution Functions to Small Datasets."

Chapter 27: "Mixed Truncated Random Variable Fitting with the GLD, and Applications in Insurance and Inventory Management."

Chapter 28: "Distributional Modeling of Pipeline Leakage Repair Costs for a Water Utility Company."

Chapter 29: "Use of the Generalized Lambda Distribution in Materials Science with Examples in Fatigue Lifetime, Fracture Mechanics, Polycrystaline Calculations, and Pitting Corrosion."

Chapter 30: "Fitting Statistical Distributions to Data in Hurricane Modeling."

Chapter 31: "A Rainfall-Based Model for Predicting the Regional Incidence of Wheat Seed Infection by *Stagonospora nodorum* in New York."

Chapter 32: "Reliability Estimation Using Univariate Dimension Reduction and Extended Generalized Lambda Distribution."

Chapter 33: "Statistical Analyses of Environmental Pressure Surrounding Atlantic Tropical Cyclones."

Chapter 34: "Simulation of Hail Storms Using Simultaneous Efficient Random Number Generators."

PART VIII Appendices
The appendices give the tables and computer code needed for applications. Hence, although a person may write his or her own code if desired, the code we used will be available. The tables will be sufficient in many applications.

A. Programs and their Documentation

B. Table B–1 for GLD Fits: Method of Moments

C. Table C–1 for GBD Fits: Method of Moments

D. Tables D–1 through D–5 for GLD Fits: Method of Percentiles

E. Tables E–1 through E–5 for GLD Fits: Method of L-Moments

F. Table F–1 for Kappa Distribution Fits: Method of L-Moments

G. Table G–1 for Kappa Distribution Fits: Method of Percentiles

H. Table H–1 for Johnson System Fits in the S_U Region: Method of Moments

I. Table I–1 for Johnson System Fits in the S_B Region: Method of Moments

J. Table J–1 for p-Values Associated with Kolmogorov-Smirnov Statistics

K. Table K–1 Normal Distribution Probabilities

References for Chapter 1

Bowman, A. W. and Azzalini, A. (1997). *Applied Smoothing Techniques for Data Analysis, The Kernel Approach with S-Plus Illustrations*, Clarendon Press, Oxford.

Dudewicz, E. J. (1999). "Basic statistical methods," Chapter 44, *Juran's Quality Handbook, Fifth Edition* (edited by J. M. Juran, A. Blanton Godfrey, R. E. Hoogstoel, and E. G. Schilling), McGraw-Hill, New York.

Dudewicz, E. J., Carpenter, D. M., Mishra, S. N., Mulekar, M., Romeu, J. L., and Shimizu, K (eds.) (2007, 2008). "Fitting Statistical Distributions to Data," Volumes I & II, *American Series in Mathematical and Management Sciences,* Volumes 57 & 58, American Sciences Press, Inc., Syracuse, New York.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, John Wiley & Sons, New York.

Filliben, J. J. (1969). *Simple and Robust Linear Estimation of the Location Parameter of a Symmetric Distribution*, Ph.D. Thesis, Princeton University, New Jersey.

Hald, A. (1998). *A History of Mathematical Statistics from 1750 to 1930*, Wiley, New York.

Hastings, C., Mosteller, F., Tukey J. W., and Winsor, C. P. (1947). "Low moments for small samples: A comparative study of statistics," *Annals of Mathematical Statistics,* 18, 413–426.

Hogben, D. (1963). *Some Properties of Tukey's Test for Non-Additivity*, Ph.D. Thesis, Rutgers-The State University, New Jersey.

Karian, Z. A. and Dudewicz, E. J. (1999). *Modern Statistical, Systems, and GPSS Simulation*, Second Edition, CRC Press, Boca Raton, Florida.

Klugman, S. A., Panjer, H. H., and Willmot, G. E. (1998). *Loss Models, from Data to Decisions,* John Wiley & Sons, Inc., New York.

Mudholkar, G. D. and Phatak, M. V. (1984). "Quantile function models for quantal response analysis: An outline," *Topics in Applied Statistics–Proc. Stat. 81 Canada*, 621–627.

Müller, H.–G. (1997). "Density estimation (update)," article in *Encyclopedia of Statistical Sciences, Update Volume I* (S. Kotz, Editor-in-Chief), pp. 185–200, Wiley, New York.

Öztürk, A. and Dale, R. F. (1982). "A study of fitting the generalized lambda distribution to solar radiation data," *Journal of Applied Meteorology*, 12, 995–1004.

Öztürk, A. and Dale, R. F. (1985). "Least squares estimation of the parameters of the Generalized Lambda Distribution," *Technometrics*, 27, 81–84.

Pearson, K. (1894). "Contribution to the mathematical theory of evolution," *Philos. Trans. Royal Soc. London, Series A*, 185, 71–110. Reprinted in K. Pearson (1948).

Pearson, K. (1895). "Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material," *Philos. Trans. Royal Soc. London, Series A*, 186, 343–414. Reprinted in K. Pearson (1948).

Pregibon, D. (1980). "Goodness of link tests for generalized linear models," *Applied Statistics*, 29, 15–24.

Ramberg, J. S. and Schmeiser, B. W. (1972). "An approximate method for generating symmetric random variables," *Comm. ACM*, 15, 987–990.

Ramberg, J. S. and Schmeiser, B. W. (1974). "An approximate method for generating asymmetric random variables," *Comm. ACM*, 17, 78–82.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics*, 21, 201–214.

Ricer, T. L. (1980). *Accounting for Non–Normality and Sampling Error in Analysis of Variance of Construction Data*, M.S. Thesis (Adviser Richard E. Larew), Department of Civil Engineering, The Ohio State University, Columbus, Ohio, xi+183 pp.

Scott, D. W. (1992). *Multivariate Density Estimation*, Wiley, New York.

Shapiro, S. S. and Wilk, M. B. (1965). "An analysis of variance test for normality (complete samples)," *Biometrika*, 52, 591–611.

Shapiro, S. S., Wilk, M. B., and Chen, Hwei J. (1968). "A comparative study of various tests for normality," *Journal of the American Statistical Association*, 63, 1343–1372.

Silver, E. A. (1977). "A safety factor approximation based upon Tukey's Lambda distribution," *Operational Research Quarterly*, 28, 743–746.

Simonoff, J. S. (1996). *Smoothing Methods in Statistics*, Springer, New York.

Tukey, J. W. (1960). *The Practical Relationship Between the Common Transformations of Percentages of Counts and of Amounts*, Technical Report 36, Statistical Techniques Research Group, Princeton University.

Van Dyke, J. (1961). "Numerical investigation of the random variable $y = c(u^\lambda - (1-u)^\lambda)$," *Working Paper*, Statistical Engineering Laboratory, National Bureau of Standards, Washington, D.C.

PART II: The Generalized Lambda Distribution

Chapter 2	The Generalized Lambda Family of Distributions
Chapter 3	Fitting Distributions and Data with the GLD via the Method of Moments
Chapter 4	The Extended GLD System, the EGLD: Fitting by the Method of Moments
Chapter 5	A Percentile-Based Approach to Fitting Distributions and Data with the GLD
Chapter 6	Fitting Distributions and Data with the GLD through L-Moments
Chapter 7	Fitting a Generalized Lambda Distribution Using a Percentile-KS (P-KS) Adequacy Criterion
Chapter 8	Fitting Mixture Distributions Using a Mixture of Generalized Lambda Distributions with Computer Code
Chapter 9	GLD 2: The Bivariate GLD Distribution
Chapter 10	Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals
Chapter 11	Statistical Design of Experiments: A Short Review

A sketch of each of these chapters appears in Section 1.2.

Chapter 2

The Generalized Lambda Family of Distributions

As noted in Chapter 1, (Fitting Statistical Distributions: An Overview), "There are good reasons for using the GLD distribution and GB methods ... GLD fits have been used successfully in many fields ... Try the GLD and GB first and stop there if the results are acceptable." In this Chapter, and Chapters 3 through 10, we give details of the GLD approach, including extensions to all possible moments in Chapter 4, mixtures in Chapter 8, and bivariate versions in Chapter 9.

This Chapter consists of definition of the GLD (Section 2.1), the GLD parameter space (Section 2.2), shapes of GLD density functions (Section 2.3), GLD random variate generation (Section 2.4), a brief discussion of the fitting process in both the GLD case and for other distribution families (Section 2.5), and end with problems that can be used to test one's understanding in courses and seminars based on this Handbook.

2.1 Definition of the Generalized Lambda Distributions

The **generalized lambda distribution family with parameters $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$, $\boldsymbol{\lambda_3}$, $\boldsymbol{\lambda_4}$, GLD($\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$)**, is most easily specified in terms of its quantile or percentile function

$$Q(y) = Q(y; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \tag{2.1.1}$$

where $0 \leq y \leq 1$. The parameters λ_1 and λ_2 are, respectively, location and scale parameters, while λ_3 and λ_4 determine the skewness and kurtosis of the GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$). Recall that for the normal distribution there are also restrictions on (μ, σ^2), namely, $\sigma > 0$. The restrictions on $\lambda_1, \ldots, \lambda_4$ that yield a valid GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) distribution will be discussed in Section 2.2 and the impact

of λ_3 and λ_4 on the shape of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. will be considered in Section 2.3.

It is relatively easy **to find the probability density function from the percentile function of the GLD**, as we now show.

Theorem 2.1.2. *For the* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$*, the probability density function is*

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}, \quad \text{at } x = Q(y). \tag{2.1.3}$$

(Note that $Q(y)$ can be calculated from (2.1.1).)

Proof. Since $x = Q(y)$, we have $y = F(x)$. Differentiating with respect to x, we find

$$\frac{dy}{dx} = f(x)$$

or

$$f(x) = \frac{dy}{d(Q(y))} = \frac{1}{\dfrac{d(Q(y))}{dy}}\,. \tag{2.1.4}$$

Since we know the form of $Q(y)$ from (2.1.1), we find directly that

$$\frac{dQ(y)}{dy} = \frac{d}{dy}\left(\lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}\right) = \frac{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}{\lambda_2}, \tag{2.1.5}$$

from which the theorem follows using (2.1.5) in (2.1.4). ∎

In plotting the function $f(x)$ for a density such as the normal, where $f(x)$ is given as a specific function of x, we proceed by calculating $f(x)$ at a grid of x values, then plotting the pairs $(x, f(x))$ and connecting them with a smooth curve. **For the GLD family, plotting** $f(x)$ proceeds differently since (2.1.3) tells us the value of $f(x)$ at $x = Q(y)$. Thus, we take a grid of y values (such as .01, .02, .03, . . ., .99, that give us the 1%, 2%, 3%, . . ., 99% points), find x at each of those points from (2.1.1), and find $f(x)$ at that x from (2.1.3). Then, we plot the pairs $(x, f(x))$ and link them with a smooth curve.

As an example of plotting $f(x)$ for a GLD, consider the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with parameters

$$\lambda_1 = 0.0305, \quad \lambda_2 = 1.3673, \quad \lambda_3 = 0.004581, \quad \lambda_4 = 0.01020,$$

i.e., the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with (see (2.1.1))

$$Q(y) = 0.0305 + \left(y^{0.004581} - (1-y)^{0.01020}\right)/1.3673. \tag{2.1.6}$$

This GLD arose (see Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979)) as the fit to measurements of the coefficient of friction for a metal. For example, in the process noted in the above paragraph, we find that at $y = 0.25$,

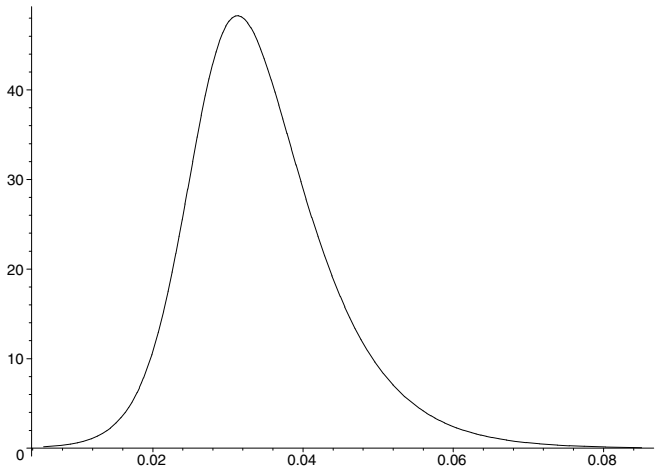


Figure 2.1–1. The p.d.f. of GLD$(0.0305, 1.3673, 0.004581, 0.01020)$.

Q(0.25)=0.028013029, from (2.1.6). Next, at $x = 0.028$, using (2.1.3) with the specified values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $f(0.028) = 43.0399612$. Hence, $(0.028, 43.04)$ will be one of the points on the graph of $f(x)$. Proceeding in this way for $y = 0.01, 0.02, \ldots, 0.99$, we obtain the graph of $f(x)$ given in Figure 2.1–1.

2.2 The Parameter Space of the GLD

We noted, following formula (2.1.1), that it does not always specify a valid distribution. The reason is that one cannot just write down any formula and be assured it will specify a distribution without checking the conditions needed for that fact to hold. In particular, **a function $f(x)$ is a probability density function if and only if it satisfies the conditions**

$$f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{2.2.1}$$

From (2.1.4) we see that for the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, conditions (2.2.1) are satisfied if and only if

$$\frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}} \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(Q(y))\, dQ(y) = 1. \tag{2.2.2}$$

Since from (2.1.4) we know that

$$f(Q(y))\, dQ(y) = dy,$$

and y is on the range $[0, 1]$, the second condition in (2.2.2) follows. Thus, for any $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the function $f(x)$ will integrate to 1. It remains to show that the first condition in (2.2.2) holds.

Since λ_1 does not enter into the first condition in (2.2.2), this parameter will be unrestricted, leading us to the following theorem.

Theorem 2.2.3. *The* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *specifies a valid distribution if and only if*

$$\frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}} \geq 0 \tag{2.2.4}$$

for all $y \in [0, 1]$.

The next theorem establishes the role of λ_1 **as a location parameter**.

Theorem 2.2.5. *If the random variable* X *is* GLD$(0, \lambda_2, \lambda_3, \lambda_4)$*, then the random variable* $X + \lambda_1$ *is* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Proof. If X is GLD$(0, \lambda_2, \lambda_3, \lambda_4)$, by (2.1.1) we have

$$Q_(y) = \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}\,.$$

Now

$$F_{X+\lambda_1}(x) = P[X + \lambda_1 \leq x] = P[X \leq x - \lambda_1] = F_X(x - \lambda_1), \tag{2.2.6}$$

hence $F_X(x - \lambda_1) = y$ also implies $F_{X+\lambda_1}(x) = y$, yielding

$$x - \lambda_1 = Q_X(y) = \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}\,, \quad x = Q_{X+\lambda_1}(y)\,, \tag{2.2.7}$$

whence

$$Q_{X+\lambda_1}(y) = x = \lambda_1 + Q_X(y) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}. \tag{2.2.8}$$

This proves that $X + \lambda_1$ is a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ random variable. ∎

Since $0 \leq y \leq 1$ in (2.2.4), we immediately have the following.

Corollary 2.2.9. *The* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *of (2.1.1) specifies a valid distribution if and only if*

$$g(y, \lambda_3, \lambda_4) \equiv \lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1} \tag{2.2.10}$$

has the same sign (positive or negative) for all y *in* $[0, 1]$*, as long as* λ_2 *takes that sign also. In particular, the* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *specifies a valid distribution if* $\lambda_2, \lambda_3, \lambda_4$ *all have the same sign.*

To determine the (λ_3, λ_4) **pairs that lead to a valid GLD**, we consider the following regions in (λ_3, λ_4)-space (also see Problem 2.9):

$$\begin{aligned}
\text{Region 1} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 \leq -1, \lambda_4 \geq 1\} && (2.2.11)\\
\text{Region 2} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 \geq 1, \lambda_4 \leq -1\} && (2.2.12)\\
\text{Region 3} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 \geq 0, \lambda_4 \geq 0\} && (2.2.13)\\
\text{Region 4} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 \leq 0, \lambda_4 \leq 0\} && (2.2.14)\\
\mathrm{V}_1 &= \{(\lambda_3, \lambda_4) \mid \lambda_3 < 0, 0 < \lambda_4 < 1\} && (2.2.15)\\
\mathrm{V}_2 &= \{(\lambda_3, \lambda_4) \mid 0 < \lambda_3 < 1, \lambda_4 < 0\} && (2.2.16)\\
\mathrm{V}_3 &= \{(\lambda_3, \lambda_4) \mid -1 < \lambda_3 < 0, \lambda_4 > 1\} && (2.2.17)\\
\mathrm{V}_4 &= \{(\lambda_3, \lambda_4) \mid \lambda_3 > 1, -1 < \lambda_4 < 0\} && (2.2.18)
\end{aligned}$$

The following lemma is a direct consequence of Corollary 2.2.9.

Lemma 2.2.19. *The* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is valid in Regions 3 and 4 specified in (2.2.13) and (2.2.14).*

Next, we consider the other Regions, starting with Regions V_1 and V_2.

Lemma 2.2.20. *The* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is not valid in Regions* V_1 *and* V_2*.*

Proof. By Corollary 2.2.9, the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is valid at $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ if and only if $g(y, \lambda_3, \lambda_4)$, as defined in (2.2.10), has the same sign for all y in $[0, 1]$, and λ_2 takes that same sign. In Region V_1 we have $\lambda_3 < 0$ and $0 < \lambda_4 < 1$. It is easy to see that

$$\lim_{y \to 0^+} g(y, \lambda_3, \lambda_4) = -\infty \qquad \text{and} \qquad \lim_{y \to 1^-} g(y, \lambda_3, \lambda_4) = +\infty,$$

so that $g(y, \lambda_3, \lambda_4)$ cannot keep the same sign over the interval $[0, 1]$, hence the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is not valid for (λ_3, λ_4) in V_1. The analysis for V_2 is similar (with λ_3 and λ_4 interchanged). ∎

Lemma 2.2.21. *The* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is valid in Regions 1 and 2 specified in (2.2.11) and (2.2.12).*

Proof. We will show that for (λ_3, λ_4) in Region 1, $g(y, \lambda_3, \lambda_4)$, (defined in (2.2.10)), is negative for all y in $[0, 1]$. We start by considering

$$\frac{\partial\, g(y, \lambda_3, \lambda_4)}{\partial\, \lambda_3} = y^{\lambda_3 - 1} + \lambda_3 \ln(y) y^{\lambda_3 - 1}.$$

Since this is positive ($\lambda_3 < 0$ and $\ln(y) < 0$), $g(y, \lambda_3, \lambda_4)$ increases as λ_3 increases and

$$\begin{aligned}
g(y, \lambda_3, \lambda_4) &\leq g(y, -1, \lambda_4)\\
&= \frac{-1}{y^2} + \lambda_4 (1 - y)^{\lambda_4 - 1} = h(y, \lambda_4). && (2.2.22)
\end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial\, h(y,\lambda_4)}{\partial\, \lambda_4} &= (1-y)^{\lambda_4-1} + \lambda_4(1-y)^{\lambda_4-1}\ln(1-y) \\ &= (1-y)^{\lambda_4-1}\left[1+\lambda_4\ln(1-y)\right]\end{aligned}$$

and $\dfrac{\partial\, h(y,\lambda_4)}{\partial\, \lambda_4} \geq 0$ if and only if

$$\lambda_4 \leq \frac{-1}{\ln(1-y)}. \tag{2.2.23}$$

Case 1: $-1/\ln(1-y) \leq 1$, equivalently, $1-e^{-1} \leq y \leq 1$.
By (2.2.23) $h(y,\lambda_4)$ increases with λ_4 and

$$g(y,\lambda_3,\lambda_4) \leq h(y,\lambda_4) \leq h(y,1) = \frac{-1}{y^2} + 1 \leq 0.$$

Case 2: $-1/\ln(1-y) > 1$, equivalently, $0 \leq y < 1-e^{-1}$.
In this case, $h(y,\lambda_4)$ will be largest when $\lambda_4 = -1/\ln(1-y)$. Therefore,

$$\begin{aligned}h(y,\lambda_4) &\leq h(y, \frac{-1}{\ln(1-y)}) \qquad (2.2.24)\\ &= -\frac{1}{y^2} - \left(\frac{1}{\ln(1-y)}\right)(1-y)^{\left(-1-\frac{1}{\ln(1-y)}\right)} = f(y).\end{aligned}$$

The derivative of $f(y)$, after some simplification, is

$$f'(y) = \frac{2(1-y)^2\ln^2(1-y) - y^3e^{-1}(1+\ln(1-y)}{y^3\ln^2(1-y)(1-y)^2}.$$

Since the denominator is non-negative, $f'(y)$ has the same sign as

$$\frac{2e(y-1)^2}{y}\left(\frac{\ln(1-y)}{y}\right)^2 - (1+\ln(1-y)).$$

To show that $f'(y) \geq 0$, it suffices to establish that

$$k(y) = \frac{2e(y-1)^2}{y}\left(\frac{\ln(1-y)}{y}\right)^2 \geq 1$$

because we would then have $\ln(1-y) < 0, \quad -(1+\ln(1-y)) > -1$, and $f'(y) = k(y) - (1+\ln(1-y)) > 0$. The first factor of $k(y)$, $2(y-1)^2$, decreases as y increases on $[0, 1-e^{-1}]$, the interval to which y is constrained. Therefore,

$$\frac{2e(y-1)^2}{y} \geq \frac{2e^{-1}}{y} \geq \frac{2e^{-1}}{1-e^{-1}} = \frac{2}{e-1} > 1. \tag{2.2.25}$$

The other factor of $k(y)$, $\ln^2(1-y)/y^2$, increases as y increases on $[0, 1-e^{-1}]$. Hence, its value must exceed

$$\lim_{y\to 0^+}\left(\frac{\ln(1-y)}{y}\right)^2 = 1.$$

This, together with (2.2.25), makes $k(y) > 1$, hence, $f'(y) > 0$. We now have

$$f(y) \le f(1-e^{-1}) = \frac{1-2e}{(1-e)^2} < 0,$$

yielding,

$$g(y, \lambda_3, \lambda_4) \le h(y, \lambda_4) \le f(y) < 0.$$

A similar argument, with λ_3 and λ_4 interchanged, gives $g(y, \lambda_3, \lambda_4) < 0$ for (λ_3, λ_4) from Region 2 and Corollary 2.2.9 is used to conclude the proof. ∎

The situation is quite different in Regions V_3 and V_4. We start by observing that the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is valid at **only some points of V_3**. For example, at $\lambda_3 = -1/2$, $\lambda_4 = 2$, the $g(y, \lambda_3, \lambda_4)$ of (2.2.10) is

$$g(y) = -\frac{1}{2}y^{-3/2} + 2(1-y),$$

for which $g'(y) = \frac{3}{4}y^{-5/2} - 2$, which is ≤ 0 if and only if $(3/4)y^{-5/2} \le 2$, i.e., if and only if $3/8 \le y^{5/2}$, or $y \ge (3/8)^{2/5} = .67548$. Noting that

$$\lim_{y\to 0^+} g(y) = -\infty \text{ and } g(1) = -\frac{1}{2},$$

we see that $g(y)$ increases as y increases from 0 to $(3/8)^{2/5}$, then decreases. Its maximum occurs at $y = (3/8)^{2/5}$, in which case

$$\begin{aligned} g\left(\left(\frac{3}{8}\right)^{2/5}\right) &= -\frac{1}{2}((3/8)^{2/5})^{-3/2} + 2(1-(3/8)^{2/5}) \\ &= -0.90064 + 0.64904 = -0.25160. \end{aligned}$$

Thus, $g(y)$ is negative for all y in $[0, 1]$, and the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (with $\lambda_2 < 0$) is valid at $(\lambda_3, \lambda_4) = (-1/2, 2)$.

This can be **contrasted with the point** $(\lambda_3, \lambda_4) = (-1/2, 1)$ where

$$g(y) = 1 - \frac{1}{2y^{3/2}}.$$

In this case

$$\lim_{y\to 0^+} g(y) = -\infty \text{ and } g(1) = \frac{1}{2},$$

which establishes $(\lambda_3, \lambda_4) = (-1/2, 1)$ as **a point of V_3 where the GLD is not valid**.

The following result, due to Karian, Dudewicz, and McDonald (1996), gives the **complete characterization of the valid points of Regions $\mathbf{V_3}$ and $\mathbf{V_4}$.**

Lemma 2.2.26. *A point in Region* V_3 *is valid if and only if*

$$\frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}(\lambda_4-1)^{\lambda_4-1} < \frac{-\lambda_3}{\lambda_4}. \tag{2.2.27}$$

Proof. Let $0 < y < 1$ and $-1 < \lambda < 0$ (think of λ as λ_3), and $f(\lambda) = \alpha > 1$ (think of α as λ_4, so we are considering Region V in the second quadrant). Let

$$G(y) = \lambda y^{\lambda-1} + \alpha(1-y)^{\alpha-1}. \tag{2.2.28}$$

Since the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is valid if $G(y)$ has constant sign for all y in $[0, 1]$, we examine the zeros of G. $G(y) = 0$ is equivalent to

$$\frac{-\lambda}{\alpha} = \frac{(1-y)^{\alpha-1}}{y^{\lambda-1}}. \tag{2.2.29}$$

Through the substitutions

$$\beta = \left(\frac{-\lambda}{\alpha}\right)^{1/(\alpha-1)} \quad \text{and} \quad \gamma = \frac{1-\lambda}{\alpha-1},$$

(2.2.29) can be simplified to

$$\beta = y^{\gamma}(1-y), \tag{2.2.30}$$

where $\beta,\ \gamma > 0$.

Differentiating with respect to y we obtain a relation for the critical points of $h(\lambda, y) = y^{\gamma}(1-y)$:

$$\frac{\partial\ h(\lambda, y)}{\partial\ y} = \gamma y^{\gamma-1}(1-y) - y^{\gamma}$$

which is zero if and only if $\gamma y^{\gamma-1} = y^{\gamma}(1+\gamma)$, i.e., if and only if y has the value $y_c = \gamma/(1+\gamma)$. At y_c, $h(\lambda, y)$ has a maximum since $h(\lambda, 0) = h(\lambda, 1) = 0$ and $h(\lambda, y) \geq 0$. This maximum is given by

$$h(\lambda, y_c) = y_c^{\gamma}(1-y_c) = \frac{y_c^{\gamma}}{(1+\gamma)} = \frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}}.$$

Since the difference of the two sides in (2.2.30) will go from positive at $y = 0$ to negative at $y = y_c$, G changes sign on $[0, 1]$ if and only if

$$\frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \geq \beta. \tag{2.2.31}$$

By restating this in terms of the λ_3 and λ_4 parameters of the GLD, we see that (λ_3, λ_4) fails to yield a valid GLD if and only if

$$\frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}(\lambda_4-1)^{\lambda_4-1} \geq \frac{-\lambda_3}{\lambda_4}. \quad \blacksquare \tag{2.2.32}$$

The following theorem, by summarizing the results of Lemmas 2.2.19, 2.2.20, 2.2.21. and 2.2.26, **completely characterizes the** (λ_3, λ_4) **pairs for which the GLD is valid**.

Theorem 2.2.33. *With a suitable* λ_2*, the* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is valid at points* (λ_3, λ_4) *if and only if* (λ_3, λ_4) *is in one of the unshaded regions depicted in Figure 2.2–1. The curved boundaries between the valid and non-valid regions are given by.*

$$\frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}(\lambda_4-1)^{\lambda_4-1} = \frac{-\lambda_3}{\lambda_4} \text{ (Region 5 in the second quadrant)}$$

and

$$\frac{(1-\lambda_4)^{1-\lambda_4}}{(\lambda_3-\lambda_4)^{\lambda_3-\lambda_4}}(\lambda_3-1)^{\lambda_3-1} = \frac{-\lambda_4}{\lambda_3} \text{ (Region 6 in the fourth quadrant).}$$

Figure 2.2–1 shows all the (λ_3, λ_4) points (the points in the unshaded region) for which a valid $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ exists. The shaded region consists of the points excluded by Lemma 2.2.20 and Theorem 2.2.33. Therefore, for (λ_3, λ_4) in the shaded region there will not exist a valid $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distribution.

Theorem 2.2.33 gives an algebraic characterization of the boundary between the (λ_3, λ_4) points of V_3 and V_4 that lead to a valid $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and those that do not. However, it is not clear from this algebraic formulation that the shape of the shaded region is as depicted in Figure 2.2–1. The next two theorems clarify this point.

Theorem 2.2.34. *If* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3^*, \lambda_4^*)$ *is valid for a point* $(\lambda_3^*, \lambda_4^*)$ *in* V_3 *and* (λ_3, λ_4^*)*, is a point with* $-1 \leq \lambda_3 \leq \lambda_3^*$*, then* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4^*)$ *is also valid for* (λ_3, λ_4^*).

Proof. We know from Corollary 2.2.9 that for $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3^*, \lambda_4^*)$ to be valid,

$$g(y, \lambda_3^*, \lambda_4^*) = \lambda_3^* y^{\lambda_3^*-1} + \lambda_4^*(1-y)^{\lambda_4^*-1} \tag{2.2.35}$$

must have the same sign for all y in $[0, 1]$. Since

$$\lim_{y\to 0^+} \lambda_3^* y^{\lambda_3^*-1} = -\infty \qquad \text{and} \qquad \lim_{y\to 0^+} \lambda_4^*(1-y)^{\lambda_4^*-1} = \lambda_4,$$

$g(y, \lambda_3^*, \lambda_4^*) < 0$ for some sufficiently small y. Hence, $g(y, \lambda_3^*, \lambda_4^*) \leq 0$ for all y in $[0, 1]$. We next observe that

$$\frac{\partial\, g(y, \lambda_3, \lambda_4)}{\partial\, \lambda_3} = y^{\lambda_3-1}(1 + \lambda_3 \ln y) \tag{2.2.36}$$

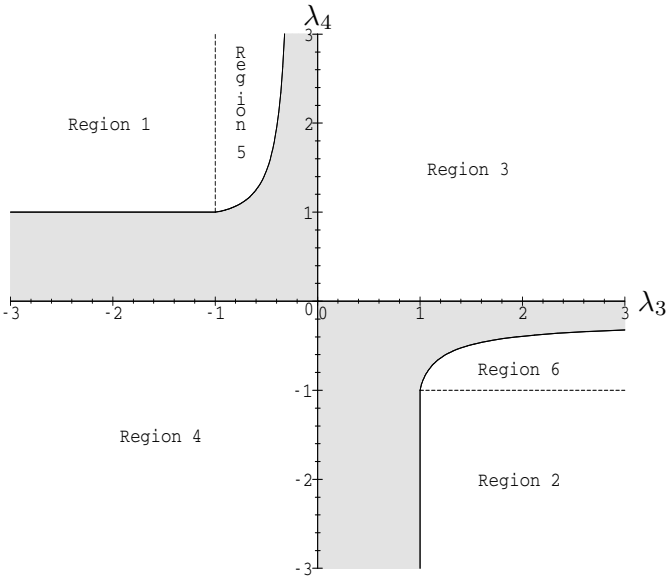


Figure 2.2–1. Regions 1, 2, 3, 4, 5, and 6 where the GLD is valid.

and the right-hand side of (2.2.36) is positive because both λ_3 and $\ln y$ are negative. Hence, for any y in $[0,1]$, $g(y,\lambda_3,\lambda_4)$ is an increasing function of λ_3 and

$$g(y,\lambda_3,\lambda_4^*) \le g(y,\lambda_3^*,\lambda_4^*) \le 0$$

for all y in $[0,1]$. By Corollary 2.2.9, GLD$(\lambda_1,\lambda_2,\lambda_3,\lambda_4^*)$ must be valid. ∎

Theorem 2.2.37. *Given $-1 < \lambda_3^* < 0$, there exists a $\lambda_4^* > 1$ such that the* GLD$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ *is not valid for points (λ_3,λ_4) with $\lambda_4 \le \lambda_4^*$ and it is valid for points (λ_3,λ_4) with $\lambda_4 > \lambda_4^*$.*

Proof. From (2.2.32), GLD$(\lambda_1,\lambda_2,\lambda_3^*,\lambda_4)$ is valid if and only if

$$\frac{(1-\lambda_3^*)^{1-\lambda_3^*}}{(\lambda_4-\lambda_3^*)^{\lambda_4-\lambda_3^*}}(\lambda_4-1)^{\lambda_4-1} < \frac{-\lambda_3^*}{\lambda_4}$$

which is equivalent to

$$-\frac{(1-\lambda_3^*)^{1-\lambda_3^*}}{\lambda_3^*} < \frac{(\lambda_4-\lambda_3^*)^{\lambda_4-\lambda_3^*}}{\lambda_4(\lambda_4-1)^{\lambda_4-1}} = h(\lambda_4). \tag{2.2.38}$$

Differentiating $h(\lambda_4)$ we have

$$h'(\lambda_4) = \frac{(\lambda_4-\lambda_3^*)^{\lambda_4-\lambda_3^*}}{\lambda_4(\lambda_4-1)^{\lambda_4-1}}\left[\lambda_4\ln(\lambda_4-\lambda_3^*) - \lambda_4\ln(\lambda_4-1)\right]. \tag{2.2.39}$$

The terms outside of the brackets in (2.2.39) are positive. The expression inside the brackets can be rewritten as

$$\lambda_4 \ln\left(\frac{\lambda_4 - \lambda_3^*}{\lambda_4 - 1}\right)$$

and since $\lambda_4 - \lambda_3^* > \lambda_4 - 1$, the bracketed part of (2.2.39) is also positive. Therefore, $h(\lambda_4)$ is a continuous and increasing function of λ_4 and it must attain all values between

$$h(1) = (1 - \lambda_3^*)^{\lambda_3^* - 1} \tag{2.2.40}$$

and

$$\lim_{\lambda_4 \to \infty} h(\lambda_4) = \lim_{\lambda_4 \to \infty} \frac{(\lambda_4 - \lambda_3^*)^{\lambda_4 - \lambda_3^*}}{\lambda_4 (\lambda_4 - 1)^{\lambda_4 - 1}} = \infty. \tag{2.2.41}$$

The limit in (2.2.41) is infinite because the degree (in λ_4) of the numerator is $\lambda_4 - \lambda_3^*$ which is larger than λ_4, the degree in the denominator. In particular, since

$$(1 - \lambda_3^*)^{\lambda_3^* - 1} < -\frac{(1 - \lambda_3^*)^{1 - \lambda_3^*}}{\lambda_3^*} < \infty,$$

$h(\lambda_4)$ must attain the value

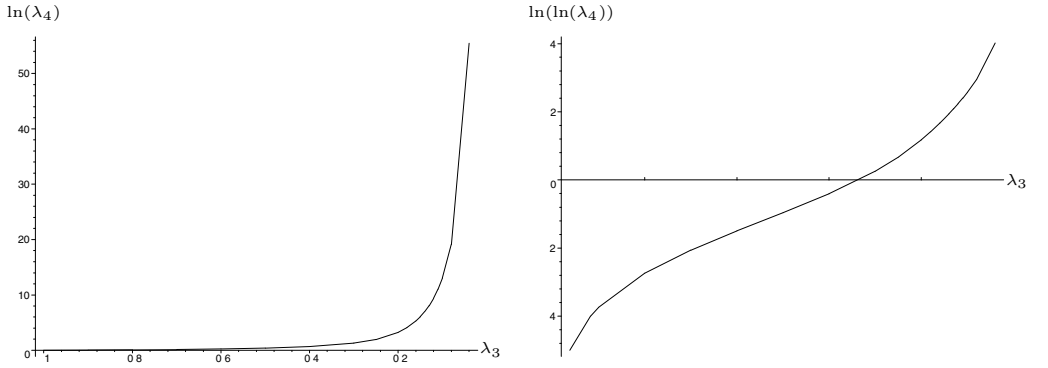
$$-\frac{(1 - \lambda_3^*)^{1 - \lambda_3^*}}{\lambda_3^*}$$

for some $\lambda_4 = \lambda_4^*$. Since $h(\lambda_4)$ increases with λ_4, when $\lambda_4 > \lambda_4^*$, the inequality of (2.2.38) holds and the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is valid at such points. Similarly when $\lambda_4 < \lambda_4^*$, the inequality of (2.2.38) cannot hold and the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is not valid at these points. ∎

Theorems 2.2.34 and 2.2.37 **justify the shape of the invalid shaded region of V_3 given in Figure 2.2–1, and of the valid Region 5**. Similar arguments, with λ_3 and λ_4 interchanged, establish analogous results for V_4 and Region 6. One of the consequences of Theorem 2.2.37 is that for a given λ_3 in $(-1, 0)$, there is one and only one λ_4 for which (λ_3, λ_4) is on the boundary curve specified in Theorem 2.2.33. Therefore, the boundary curve can be viewed as the graph of a function (say $\lambda_4 = B(\lambda_3)$) that specifies the points $(\lambda_3, B(\lambda_4))$ on the boundary.

As λ_3 approaches 0 from the left, $B(\lambda_3)$ grows at a surprisingly high rate — this is not apparent in Figure 2.2–1 because $\lambda_3 < -0.3$ in Figure 2.2–1. However, some direct (and difficult) computations yield the following values of $B(\lambda_3)$ for $\lambda_3 = -0.3,\ -0.2,\ -0.1,\ -0.08,$ and -0.04.

$$B(-0.3) \quad = \quad 3.6196669$$



(a) (b)

Figure 2.2–2. Growth of $\ln(\lambda_4)$ (a) and $\ln(\ln(\lambda_4))$ (b) as λ_3 increases to 0 on the boundary of valid and non-valid regions in V_3.

$$\begin{aligned} B(-0.2) &= 25.450660 \\ B(-0.1) &= 476523.97 \\ B(-0.08) &= 1.9935190 \times 10^8 \\ B(-0.04) &= 1.2580865 \times 10^{24}. \end{aligned}$$

It can be seen in Figure 2.2–2 (a) that even $\ln(B(\lambda_3))$ rises quite rapidly when λ_3 gets sufficiently close to 0. A significant, although somewhat more moderated, growth can be seen in Figure 2.2–2 (b) which gives the graph of $\ln(\ln(B(\lambda_3)))$.

2.3 Shapes of the GLD Density Functions

In this section we undertake **an investigation of the possible shapes of the GLD$(\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \lambda_4})$ p.d.f.s** by charting, over (λ_3, λ_4)-space, the regions where the p.d.f. has zero, one, or two relative extreme points (points where the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. has a relative maximum or relative minimum). This is followed by graphs of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s that illustrate shapes associated with various regions of (λ_3, λ_4)-space.

Lemma 2.3.1. *The relative extreme points of the* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *p.d.f. occur at values of* y *where*

$$g(y) = \frac{y^{\lambda_3 - 2}}{(1-y)^{\lambda_4 - 2}} = \frac{\lambda_4(\lambda_4 - 1)}{\lambda_3(\lambda_3 - 1)}. \tag{2.3.2}$$

Proof. It was shown in Section 2.1 (see equation (2.1.3)) that the p.d.f. of a

$\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ random variable is given by

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}} \tag{2.3.3}$$

where $x = Q(y)$, or equivalently, $y = F(x)$. Differentiating with respect to x,

$$\begin{aligned} f'(x) &= \frac{d}{dy}\left(\frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4^{\lambda_4 - 1}}\right)\frac{dy}{dx} \\ &= \frac{d}{dy}\left(\frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4^{\lambda_4 - 1}}\right) f(x) \\ &= -\lambda_2 \frac{(\lambda_3(\lambda_3 - 1)y^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1-y)^{\lambda_4 - 2})}{(\lambda_3 y^{\lambda_3 - 1} + \lambda_4^{\lambda_4 - 1})^2} f(x). \end{aligned} \tag{2.3.4}$$

The lemma is obtained by setting the parenthesized expression in the numerator of (2.3.4) to zero. ∎

Theorem 2.3.5. *The p.d.f. of the* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *has no relative extreme points in the following regions of* (λ_3, λ_4)*-space.*

$$\begin{aligned} E_{0,1} &= \{(\lambda_3, \lambda_4) \mid 0 < \lambda_3 < 1, \ \lambda_4 > 1\} \\ E_{0,2} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 > 1, \quad 0 < \lambda_4 < 1\} \\ E_{0,3} &= \{(\lambda_3, \lambda_4) \mid \lambda_3 < 0, \quad 0 < \lambda_4 < 1\} \\ E_{0,4} &= \{(\lambda_3, \lambda_4) \mid 0 < \lambda_3 < 1, \ \lambda_4 < 0\}. \end{aligned}$$

Proof. Before starting the proof we note that the GLD is valid for (λ_3, λ_4) in $E_{0,1}$ and $E_{0,2}$ but not valid in $E_{0,3}$ and $E_{0,4}$ (see Figure 2.2–1). The latter two regions are included for the sake of completeness.

The left-hand side of (2.3.2) is always positive. However, in all of the cases listed, the right-hand side is negative. ∎

Theorem 2.3.6. *The* $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *p.d.f. has a unique relative extremum when* $\lambda_3 > 2$ *and* $\lambda_4 > 2$.

Proof. Differentiating the $g(y)$ that was defined at (2.3.2) in Lemma 2.3.1,

$$\begin{aligned} g'(y) &= \frac{(\lambda_3 - 2)(1-y)^{\lambda_4 - 2} y^{\lambda_3 - 3} + (\lambda_4 - 2) y^{\lambda_3 - 2}(1-y)^{\lambda_4 - 3}}{(1-y)^{2(\lambda_4 - 2)}} \\ &= \left(\frac{y^{\lambda_3 - 3}(1-y)^{\lambda_4 - 3}}{(1-y)^{2(\lambda_4 - 2)}}\right)((\lambda_3 - 2)(1-y) + (\lambda_4 - 2)y). \end{aligned} \tag{2.3.7}$$

Hence, $g'(y)$ has the same sign as

$$h(y) = (\lambda_3 - 2)(1-y) + (\lambda_4 - 2)y. \tag{2.3.8}$$

When $\lambda_3 > 2$ and $\lambda_4 > 2$, $h(y)$, and consequently $g'(y)$, is positive. Moreover,

$$\lim_{y\to 0^+} g(y) = 0 \qquad \text{and} \qquad \lim_{y\to 1^-} g(y) = \infty,$$

making $g(y)$ a function that increases from 0 to ∞. Therefore, (2.3.2) must hold at exactly one value of y which is the only critical point of the p.d.f. of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Moreover, since the $f'(x)$ of (2.3.4) changes sign from positive to negative as y moves from 0 to 1, this unique extreme point is a relative maximum of the p.d.f. of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. ∎

Theorem 2.3.9. *The* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *p.d.f. has a unique relative extremum when* $\lambda_3 < 2$, $\lambda_4 < 2$, *and* (λ_3, λ_4) *is not in one of the regions excluded by Theorem 2.3.5.*

Proof. From (2.3.8), the constraints $\lambda_3 < 2$ and $\lambda_4 < 2$ make $h(y)$, and hence $g'(y)$, negative. Also,

$$\lim_{y\to 0^+} g(y) = \infty \qquad \text{and} \qquad \lim_{y\to 1^-} g(y) = 0.$$

Thus, $g(y)$ decreases from ∞ to 0, yielding a unique solution to (2.3.2). ∎

Lemma 2.3.10. *Let*

$$U(\lambda_3, \lambda_4) = \frac{(2-\lambda_3)^{\lambda_3-2}(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}{(\lambda_4-2)^{\lambda_4-2}}, \tag{2.3.11}$$

$$V(\lambda_3, \lambda_4) = \frac{\lambda_4(\lambda_4-1)}{\lambda_3(\lambda_3-1)}. \tag{2.3.12}$$

If $\lambda_4 > 2$, *and* $1 < \lambda_3 < 2$ *or* $\lambda_3 < 0$, *then the* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *p.d.f. has*

two relative extrema if $U(\lambda_3, \lambda_4) < V(\lambda_3, \lambda_4)$
one relative extremum if $U(\lambda_3, \lambda_4) = V(\lambda_3, \lambda_4)$
no relative extrema if $U(\lambda_3, \lambda_4) > V(\lambda_3, \lambda_4)$.

Proof. We have already established that $g'(y)$ has the same sign as $h(y)$ (as defined in (2.3.8)). $h(y)$ will be zero if and only if

$$y = y_0 = \frac{2-\lambda_3}{\lambda_4-\lambda_3}.$$

For the constraints specified in this lemma we see that $h(y)$ is negative, zero, or positive depending on whether y is less than, equal to, or greater than y_0, giving $g(y)$ a "parabolic shape" that opens up. The existence and number of solutions to

(2.3.2) will depend on how this parabolic-shaped curve intersects the horizontal line

$$y = \frac{\lambda_4(\lambda_4 - 1)}{\lambda_3(\lambda_3 - 1)} = V(\lambda_3, \lambda_4).$$

There are three possibilities:

$$g(y_0) \begin{cases} < & V(\lambda_3, \lambda_4) \text{ and (2.3.2) has two solutions} \\ = & V(\lambda_3, \lambda_4) \text{ and (2.3.2) has a unique solution} \\ > & V(\lambda_3, \lambda_4) \text{ and (2.3.2) has no solutions.} \end{cases}$$

The following direct computation of $g(y_0)$ completes the proof.

$$\begin{aligned} g(y_0) &= \left(\frac{2-\lambda_3}{\lambda_4-\lambda_3}\right)^{\lambda_3-2} \Big/ \left(1-\frac{2-\lambda_3}{\lambda_4-\lambda_3}\right)^{\lambda_4-2} \\ &= \frac{(2-\lambda_3)^{\lambda_3-2}}{(\lambda_4-\lambda_4)^{\lambda_4-\lambda_3}} \times \frac{(\lambda_4-\lambda_3)^{\lambda_4-2}}{(\lambda_4-2)^{\lambda_4-2}} \\ &= \frac{(2-\lambda_3)^{\lambda_3-2}(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}{(\lambda_4-2)^{\lambda_4-2}} = U(\lambda_3, \lambda_4). \blacksquare \end{aligned}$$

Lemma 2.3.13. *If $\lambda_4 > 2$ and $\lambda_3 < 0$, then $U(\lambda_3, \lambda_4)/V(\lambda_3, \lambda_4)$ is a decreasing function of λ_3.*

Proof. From

$$\frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} = \frac{\lambda_3(\lambda_3-1)(2-\lambda_3)^{\lambda_3-2}(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}}{\lambda_4(\lambda_4-1)(\lambda_4-2)^{\lambda_4-2}}$$

we have

$$\begin{aligned} \frac{\partial}{\partial\lambda_3} \frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} &= -\frac{(\lambda_4-2)^{2-\lambda_4}(\lambda_4-\lambda_3)^{\lambda_4-\lambda_3}(2-\lambda_3)^{\lambda_3-2}}{\lambda_4(\lambda_4-1)} \\ &\quad \times \left(\lambda_3^2 \ln\left(\frac{\lambda_4-\lambda_3}{2-\lambda_3}\right) - \lambda_3 \ln\left(\frac{\lambda_4-\lambda_3}{2-\lambda_3}\right) - 2\lambda_3 + 1\right). \end{aligned} \tag{2.3.14}$$

Except for the initial negative sign, all portions of this expression are positive. $\blacksquare$

Theorem 2.3.15. *For each $\lambda_4^* > 2$, there is one and only one λ_3^* between -1 and 0 for which*

$$\frac{(2-\lambda_3^*)^{\lambda_3^*-2}(\lambda_4^*-\lambda_3^*)^{\lambda_4^*-\lambda_3^*}}{(\lambda_4^*-2)^{\lambda_4^*-2}} = \frac{\lambda_4^*(\lambda_4^*-1)}{\lambda_3^*(\lambda_3^*-1)}$$

and the p.d.f. of the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3^*, \lambda_4^*)$ *has a unique relative extremum. At points* (λ_3, λ_4) *with* $\lambda_3 < \lambda_3^*$ *the p.d.f. has no relative extrema and at points with* $\lambda_3 > \lambda_3^*$ *the p.d.f. has exactly two relative extrema.*

Proof. We know from Lemma 2.3.13 that $U(\lambda_3, \lambda_4)/V(\lambda_3, \lambda_4)$ decreases with λ_3. We observe that

$$\begin{aligned} \lim_{\lambda_3 \to 0^-} \frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} &= \frac{U(0, \lambda_4)}{\lim_{\lambda_3 \to 0^-} V(\lambda_3, \lambda_4)} \\ &= \left(\frac{2^{-2} \lambda_4^{\lambda_4}}{(\lambda_4 - 2)^{\lambda_4 - 2}} \right) \Big/ \infty = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{\lambda_3 \to -1^+} \frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} &= \frac{U(-1, \lambda_4)}{V(-1, \lambda_4)} \\ &= \frac{2(\lambda_4 + 1)^{\lambda_4 + 1}}{27 \lambda_4 (\lambda_4 - 1)(\lambda_4 - 2)^{\lambda_4 - 2}}. \end{aligned} \tag{2.3.16}$$

The derivative of (2.3.16), with respect to λ_4, is

$$\frac{2(\lambda_4 + 1)(\lambda_4 - 2)^{2 - \lambda_4}}{27 \lambda_4 (\lambda_4 - 1)(\lambda_4 + 1)^{\lambda_4}}.$$

Since this derivative is positive, $U(-1, \lambda_4)/V(-1, \lambda_4)$ is an increasing function of λ_4 and $U(-1, \lambda_4)/V(-1, \lambda_4) \geq U(-1, 2)/V(-1, 2) = 1$, making $U(\lambda_3, \lambda_4) > V(\lambda_3, \lambda_4)$ at $\lambda_3 = -1$.

We now have $U(\lambda_3, \lambda_4)/V(\lambda_3, \lambda_4)$ decreasing from a number larger than 1 at $\lambda_3 = -1$ to 0 as $\lambda_3 \to 0^-$. This means that for a fixed $\lambda_4^* > 2$, the surfaces $U(\lambda_3, \lambda_4)$ and $V(\lambda_3, \lambda_4)$ cross exactly once for some λ_3^* between -1 and 0. Moreover, when $\lambda_3 < \lambda_3^*$, $U(\lambda_3, \lambda_4) > V(\lambda_3, \lambda_4)$ indicating, by Lemma 5, the absence of extreme points for the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.; and, when $\lambda_3 > \lambda_3^*$, $U(\lambda_3, \lambda_4) < V(\lambda_3, \lambda_4)$, indicating the presence of two relative extrema. ∎

Lemma 2.3.17. *If* $\lambda_4 > 2$ *and* $1 < \lambda_3 < 2$, *then* $U(\lambda_3, \lambda_4)/V(\lambda_3, \lambda_4)$ *is a decreasing function of* λ_4

Proof. From (2.3.11) and (2.3.12),

$$\frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} = \frac{\lambda_3 (\lambda_3 - 1)(2 - \lambda_3)^{\lambda_3 - 2} (\lambda_4 - \lambda_3)^{\lambda_4 - \lambda_3}}{\lambda_4 (\lambda_4 - 1)(\lambda_4 - 2)^{\lambda_4 - 2}}$$

and we have

$$\begin{aligned} \frac{\partial}{\partial \lambda_4} \frac{U(\lambda_3, \lambda_4)}{V(\lambda_3, \lambda_4)} = \frac{\lambda_3 (\lambda_3 - 1)(2 - \lambda_3)^{\lambda_3 - 2} (\lambda_4 - 2)^{2 - \lambda_4} (\lambda_4 - \lambda_3)^{\lambda_4 - \lambda_3}}{\lambda_4^2 (\lambda_4 - 1)^2} \times \\ \left[\lambda_4^2 \ln \left(\frac{\lambda_4 - \lambda_3}{\lambda_4 - 2} \right) - \lambda_4 \ln \left(\frac{\lambda_4 - \lambda_3}{\lambda_4 - 2} \right) - 2\lambda_4 + 1 \right]. \end{aligned} \tag{2.3.18}$$

It is clear that, with the exception of the expression in brackets, all terms in (2.3.18) are positive. Therefore, $\dfrac{\partial}{\partial\lambda_4}\dfrac{U(\lambda_3,\lambda_4)}{V(\lambda_3,\lambda_4)} < 0$ if and only if

$$\lambda_4^2 \ln\left(\frac{\lambda_4-\lambda_3}{\lambda_4-2}\right) - \lambda_4 \ln\left(\frac{\lambda_4-\lambda_3}{\lambda_4-2}\right) - 2\lambda_4 + 1 < 0. \tag{2.3.19}$$

The inequality in (2.3.19) can be written as

$$\ln\left(\frac{\lambda_4-\lambda_3}{\lambda_4-2}\right) < \frac{2\lambda_4-1}{\lambda_4^2-\lambda_4}$$

and is equivalent to

$$\frac{\lambda_4-\lambda_3}{\lambda_4-2} < e^{(2\lambda_4-1)/(\lambda_4^2-\lambda_4)}.$$

Since $(\lambda_4-\lambda_3)/(\lambda_4-2) < 1$ when $1 < \lambda_3 < 2$, it is sufficient for the proof of this lemma to show that

$$1 < e^{(2\lambda_4-1)/(\lambda_4^2-\lambda_4)} = a(\lambda_4) \tag{2.3.20}$$

when $\lambda_4 > 2$. We establish this by observing that

$$a'(\lambda_4) = -\frac{(2\lambda_4^2-2\lambda_4+1)e^{(2\lambda_4-1)/(\lambda_4^2-\lambda_4)}}{\lambda_4^2(\lambda_4-1)^2} < 0.$$

Hence

$$\begin{aligned} a(\lambda_4) &> \lim_{\lambda_4\to\infty} a(\lambda_4) \\ &= \lim_{\lambda_4\to\infty} e^{(2\lambda_4-1)/(\lambda_4^2-\lambda_4)} \\ &= 1. \end{aligned}$$

Having verified the inequality in (2.3.20), we have the proof of the lemma. ∎

Theorem 2.3.21. *If $\lambda_4 > 2$ and $1 < \lambda_3 < 2$, then the p.d.f. of the* GLD$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ *has exactly two relative extrema.*

Proof. From Lemma 2.3.17, we know that for the specified values of λ_3 and λ_4, $U(\lambda_3,\lambda_4)/V(\lambda_3,\lambda_4)$ is a decreasing function of λ_4. Hence,

$$\begin{aligned} \frac{U(\lambda_3,\lambda_4)}{V(\lambda_3,\lambda_4)} &< \lim_{\lambda_4\to 2^+} \frac{U(\lambda_3,\lambda_4)}{V(\lambda_3,\lambda_4)} \\ &= \frac{\lim_{\lambda_4\to 2^+} U(\lambda_3,\lambda_4)}{V(\lambda_3,2)} = \frac{1}{V(\lambda_3,2)} \\ &= \frac{\lambda_3(\lambda_3-1)}{2} \le 1. \end{aligned}$$

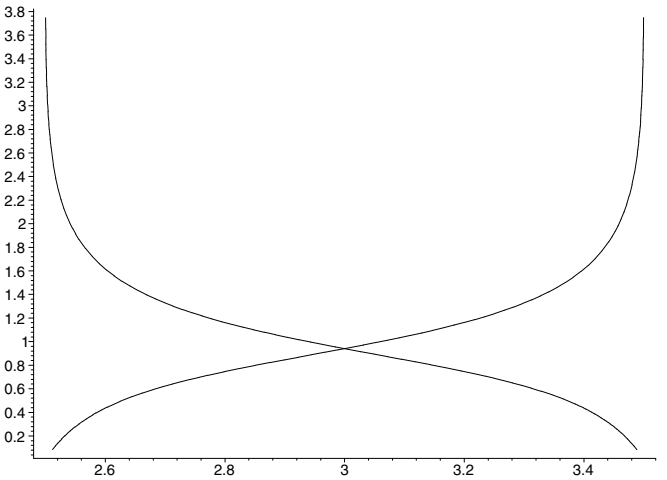


Figure 2.3–1. The GLD$(3, 2, 1.5, .5)$ and GLD$(3, 2, .5, 1.5)$ p.d.f.s; one is the symmetric image of the other about the line $x = 3$.

This makes $U(\lambda_3, \lambda_4) < V(\lambda_3, \lambda_4)$ and the theorem follows from Lemma 2.3.10. ∎

Symmetry and the presence (or absence) of tails are additional shape details of interest for GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ **distributions**. We can see from (2.1.3) that if $f(x) = f(Q(y))$, the p.d.f. of a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distribution can be viewed as a function k of y, λ_1, λ_2, λ_3, and λ_4, and that then

$$k(y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = k(1 - y, \lambda_1, \lambda_2, \lambda_4, \lambda_3).$$

If x in the domain of f corresponds to y via $x = Q(y)$, we must have, from (2.1.1),

$$x = Q(y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1 - y)^{\lambda_4}}{\lambda_2} = \lambda_1 + A.$$

The x-value that corresponds to $1 - y$ when λ_3 and λ_4 are interchanged is

$$Q(1 - y, \lambda_1, \lambda_2, \lambda_4, \lambda_3) = \lambda_1 + \frac{(1 - y)^{\lambda_4} - (y)^{\lambda_3}}{\lambda_2} = \lambda_1 - A.$$

Therefore, **the GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$ is the symmetric image, about the line $x = \lambda_1$, of the** GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ **p.d.f.** This is illustrated in Figure 2.3–1. These comments are summarized in the following theorem.

Theorem 2.3.22. *The p.d.f.s of* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *and* GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$ *are symmetric images of each other and the axis of symmetry is the line* $x = \lambda_1$.

Depending on the choice of λ_3 and λ_4, a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. may have all possible combinations of finite or infinite right and left tails. For p.d.f. *not* to

have an infinite right tail, its c.d.f. $F(x)$ must attain the value 1 at some finite point $x = x_0$. This means that $Q(1) = x_0$. Since $Q(y) = \lambda_1 + [y^{\lambda_3} - (1-y)^{\lambda_4}]/\lambda_2$ we see that Q is defined at $y = 1$ if and only if $\lambda_4 \geq 0$. Thus, a GLD p.d.f. has an infinite right tail if and only if $\lambda_4 < 0$. A similar argument shows that a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. will have an infinite left tail if and only if $\lambda_3 < 0$. A similar analysis can be applied for other choices of λ_3 and λ_4. For example, if $\lambda_3 = 0$ and $\lambda_4 > 0$,

$$\lim_{y \to 0^+} Q(y) = \lambda_1 \quad \text{and} \quad \lim_{y \to 1^-} Q(y) = Q(1) = \lambda_1 + 1/\lambda_2.$$

We formalize these results on the support of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ in the next theorem. We emphasize these results since, while their derivation is simple, **the results of Theorem 2.3.23 are easy to give incorrectly**. The results come from an early work, a 1971 M.S. thesis at the University of Iowa, for Regions 1, 2, 3, and 4, and are new for Regions 5 and 6.

Theorem 2.3.23. *The support of the* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *p.d.f. (i.e., points where the p.d.f. is positive) is as given in the following tables; the first for Regions 1, 2, 5, and 6 and the second for Regions 3 and 4 (the regions are shown in Figure 2.2–1).*

Region	λ_3	λ_4	Support
Region 1	$\lambda_3 < -1$	$\lambda_4 > 1$	$(-\infty,\ \lambda_1 + 1/\lambda_2]$
Region 2	$\lambda_3 > 1$	$\lambda_4 < -1$	$[\lambda_1 - 1/\lambda_2,\ \infty)$
Region 5	$-1 < \lambda_3 < 0$	$\lambda_4 > 1$	$(-\infty,\ \lambda_1 + 1/\lambda_2]$
Region 6	$\lambda_3 > 1$	$-1 < \lambda_4 < 0$	$[\lambda_1 - 1/\lambda_2,\ \infty)$

Region	λ_3	λ_4	Support
Region 3	$\lambda_3 > 0$	$\lambda_4 > 0$	$[\lambda_1 - 1/\lambda_2,\ \lambda_1 + 1/\lambda_2]$
	$\lambda_3 > 0$	$\lambda_4 = 0$	$[\lambda_1 - 1/\lambda_2,\ \lambda_1]$
	$\lambda_3 = 0$	$\lambda_4 > 0$	$[\lambda_1,\ \lambda_1 + 1/\lambda_2]$
Region 4	$\lambda_3 < 0$	$\lambda_4 < 0$	$(-\infty,\ \infty)$
	$\lambda_3 < 0$	$\lambda_4 = 0$	$(-\infty,\ \lambda_1]$
	$\lambda_3 = 0$	$\lambda_4 < 0$	$[\lambda_1,\ \infty)$

Theorems 2.3.5, 2.3.6, 2.3.9, 2.3.15, and 2.3.21, together with the symmetry properties of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ about the line $\lambda_3 = \lambda_4$, completely characterize the regions of (λ_3, λ_4)-space that give rise to GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s with 0, 1, or 2 relative extreme points. Figure 2.3–2 summarizes these

results. The region designated by an "X" in Figure 2.3–2 does not produce valid GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distributions. All other regions are labeled with the number of relative extreme points associated with the (λ_3, λ_4) points in that region.

Figures 2.3–3 through 2.3–9 show GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s with (λ_3, λ_4) taken from various regions of (λ_3, λ_4)-space. **Figure 2.3–3a** shows p.d.f.s with $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2.5$ and $\lambda_4 = 0.5, 0.75, 1.0, \ldots, 2.5$. The graph that rises to the highest point on the left (at $x = -1$) is the one corresponding to $\lambda_4 = 0.5$, the next highest corresponds to $\lambda_4 = 0.75$, and so on. From Theorem 2.3.23, we know that p.d.f.s with these values of λ_3 and λ_4 will not have infinite right or left tails. Moreover, we can observe from Figure 2.3–2 that there will be a transition from zero to one and eventually to two critical points as λ_4 moves from 0.5 to 2.25. It is not apparent in Figure 2.3–3a which of the graphs, if any, have critical points. As we look at the p.d.f.s with $\lambda_4 = 0.5, 0.75, 1.0$ in **Figure 2.3–3b** we see that, consistent with the foregoing analysis, these p.d.f.s do not have critical points whereas the p.d.f.s in **Figure 2.3–3c**, corresponding to $\lambda_4 = 1.25, 1.5, 1.75$, exhibit two critical points (this is perhaps most apparent for $\lambda_4 = 1.5$ and 1.75). The last set of the p.d.f.s from Figure 2.3–3a, those for $\lambda_4 = 2.0, 2.25, 2.5$, are shown, with considerable rescaling, in **Figure 2.3–3d** where the presence of a critical point is clearly visible.

Figure 2.3–4a depicts GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s for $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1.5$ and $\lambda_4 = 0.5, 0.75, 1.0, \ldots, 2.5$. As before, the graph that rises to the highest point on the left (at $x = -1$) is the one corresponding to $\lambda_4 = 0.5$, the next highest corresponds to $\lambda_4 = 0.75$, and so on.

It is clear from Figure 2.3–4a that the the first three p.d.f.s (corresponding to $\lambda_4 = 0.5, 0.75, 1.0$) do not exhibit critical points; perhaps with some difficulty, we can see that the next three p.d.f.s (corresponding to $\lambda_4 = 1.0, 1.25, 1.5$) have a single critical point (this is best observed on the right side of the graphs in Figure 2.3–4a where these curves "turn up" as x gets close to 1. The remaining three p.d.f.s are replotted in Figure 2.3–4b where one can more clearly see the two critical points of the graphs corresponding to $\lambda_4 = 2.25$ and 2.5.

Figure 2.3–5 gives GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s associated with $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0.5$ and $\lambda_4 = 0.25, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5$. In this case the p.d.f. corresponding to $\lambda_4 = 0.25$ is the one that rises to the highest point in the center of the graph, the next highest corresponding to $\lambda_4 = 0.5$, and so on. It is easy to see the transition from one to no critical points as λ_4 goes through the $\lambda_4 = 1$ barrier.

Figures 2.3–6 and 2.3–7a and 2.3–7b show some unusual p.d.f. shapes with (λ_3, λ_4) from the second quadrant. These are the kinds of sharp peaks that are often found in applications in **spectroscopy** that occur with spectra of **autoregressive processes**. The multiple peaks of spectra often look like the overlay of the several curves in Figure 2.3–6, indicating that it may be possible to model them as a **mixture of GLD random variables**. For an overview of the procedures used in spectroscopy, see Dudewicz, Mommaerts, and van der Meulen

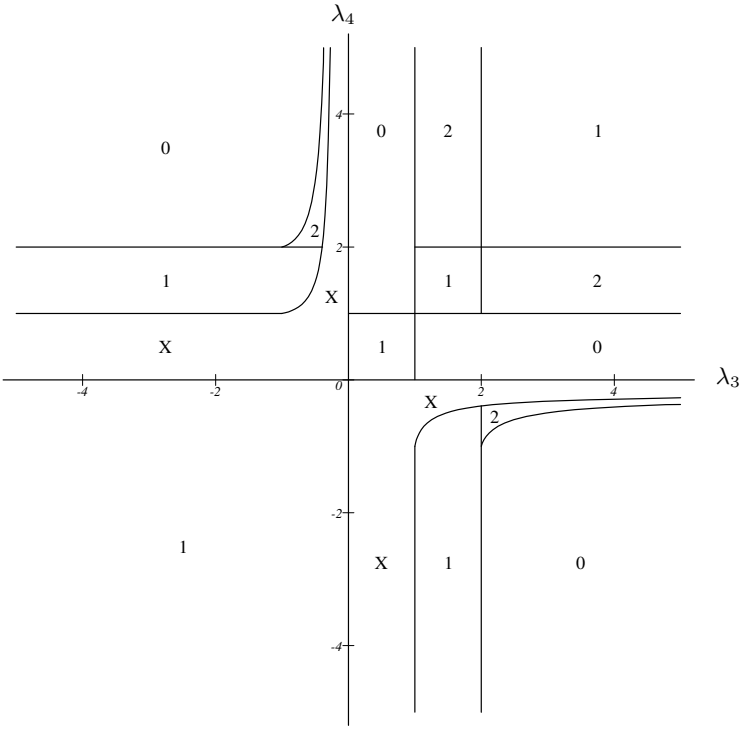


Figure 2.3–2. The number of critical points of the p.d.f.s of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distributions.

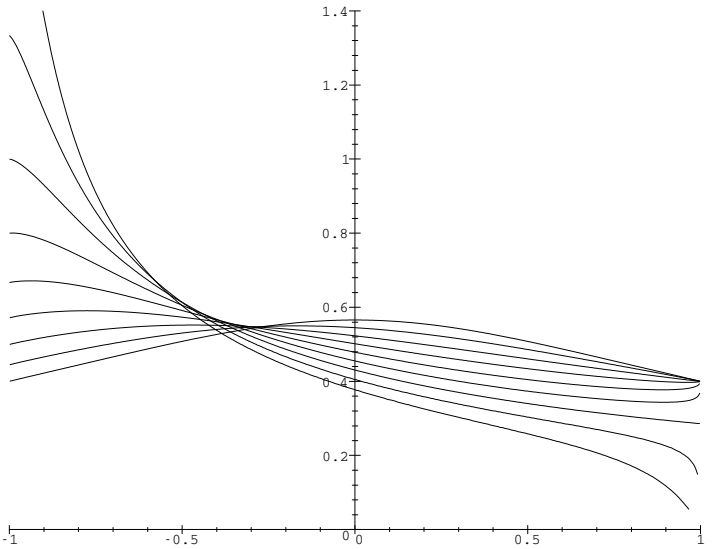


Figure 2.3–3a. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 2.5$ and $\lambda_4 = 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5$.

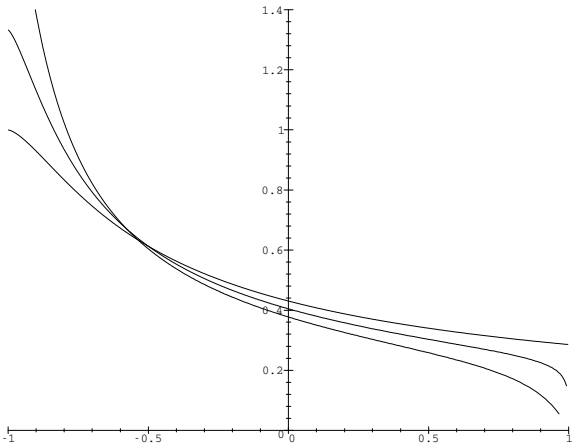


Figure 2.3–3b. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 2.5$ and $\lambda_4 = 0.5, 0.75, 1.0$.

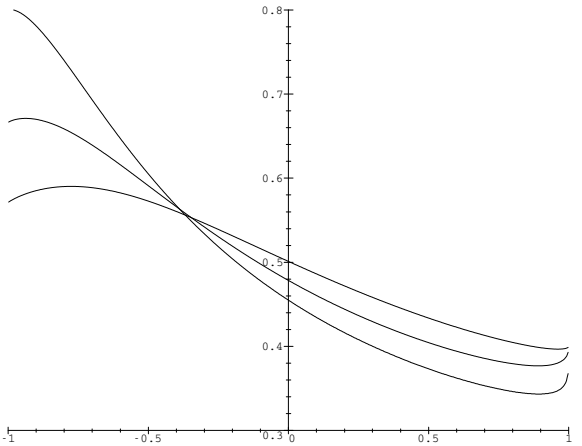


Figure 2.3–3c. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 2.5$ and $\lambda_4 = 1.25, 1.5, 1.75$.

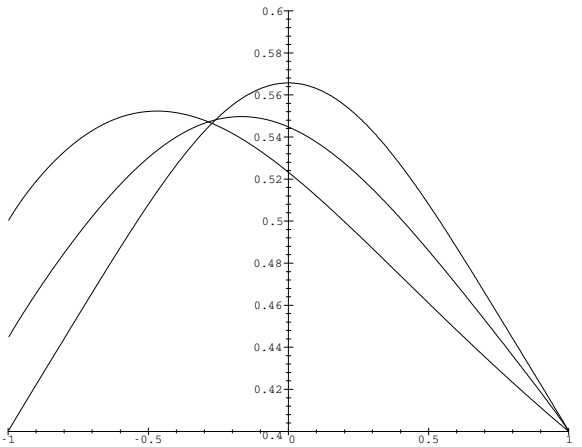


Figure 2.3–3d. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 2.5$ and $\lambda_4 = 2.0, 2.25, 2.5$.

(1991). Since spectroscopy procedures are not simple, there may be a potential for substantial advances through the use of mixtures of GLDs in this area. Recall that Theorem 2.3.23 indicates the presence of a left tail for these choices of (λ_3, λ_4). In Figure 2.3–6, $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -0.2$ and $\lambda_4 = 27, 30, 35, 50$. The highest curve (in the middle) corresponds to $\lambda_4 = 27$, the next highest corresponds to $\lambda_4 = 30$, and so on. The (λ_3, λ_4) points for all four p.d.f.s fall in the second quadrant region marked with a "2" in Figure 2.3–2.

The (λ_3, λ_4) points for the p.d.f.s of Figure 2.3–7a ($\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -0.5$ and $\lambda_4 = 2.91, 2.92, \ldots, 2.99$) cross the boundary between the two regions that give rise to two and then to zero critical points. The graphs are so packed together that it is difficult to distinguish among them. Figure 2.3–7b gives a magnification of the portion of the graphs where critical points seem to appear. The highest graph corresponds to $\lambda_4 = 2.91$, the next highest to $\lambda_4 = 2.92$, and so on. It is clear that there are two critical points when $\lambda_4 = 2.91$ and no critical points when $\lambda_4 = 2.99$. The point on the boundary where the transition occurs is (approximately) $(-0.5, 2.996)$.

Figures 2.3–8a and 2.3–8b show GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$, and $\lambda_4 = 1.0, 1.25, 1.5, 1.75, 2.0, 6.0, 20$. Figure 2.3–8a shows the first four of these with the p.d.f. with the highest point on the right corresponding to $\lambda_4 = 1.0$, the next highest to $\lambda_4 = 1.25$, etc. The critical point for the p.d.f. with $\lambda_4 = 1.25$ is clearly visible; the ones for $\lambda_4 = 1.5$ and 1.75 are more difficult to spot. Although the p.d.f.s in Figure 2.3–8b correspond to three distinct λ_4 values ($\lambda_4 = 2.0, 6.0, 20.0$), only two graphs are discernible; the graphs for $\lambda_4 = 6.0$ and 20 are so close that they cannot be distinguished.

The p.d.f.s depicted in Figure 2.3–9 have shapes that are frequently encountered in applications. For these p.d.f.s $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -0.25$ and $\lambda_4 = -0.1$, -0.2, -0.35, -0.5. The curve that rises highest corresponds to $\lambda_4 = -0.1$, the next highest to $\lambda_4 = -0.2$, etc. Since this places (λ_3, λ_4) in the third quadrant, by Theorem 1.4.23, they all have infinite left and right tails and by earlier results, each has a unique critical point.

We close this section with some notes on shape and related results from the literature. Freimer, Kollia, Mudholkar, and Lin (1988) used a slightly different parametrization of the GLD than the traditional one used at (2.1.1), but, in the main, their results are similar to those of this section. In their Section 3 they note

> The family ... is very rich in the variety of density and tail shapes. It contains unimodal, U-shaped, J-shaped and monotone p.d.f.s. These can be symmetric or asymmetric and their tails can be smooth, abrupt, or truncated, and long, medium or short ... properties ... relevant in ... modelling data and determining the methods of subsequent analysis.

They then classify the GLD with respect to tail shape and density shape.

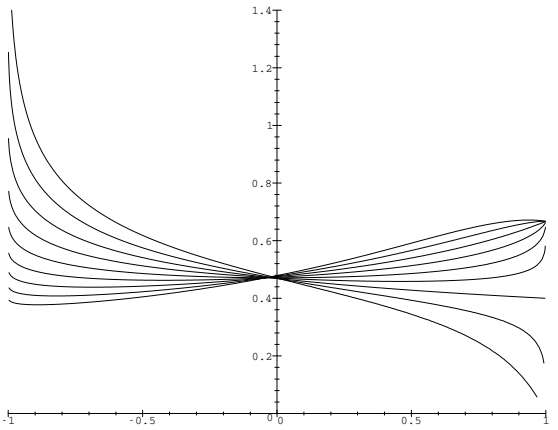


Figure 2.3–4a. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 1.5$ and $\lambda_4 = 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5$.

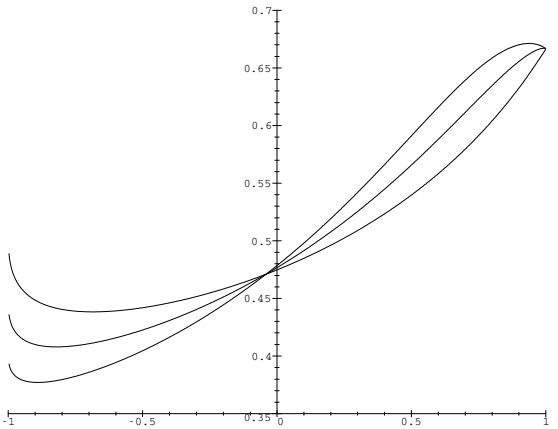


Figure 2.3–4b. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 1.5$ and $\lambda_4 = 2.0, 2.25, 2.5$.

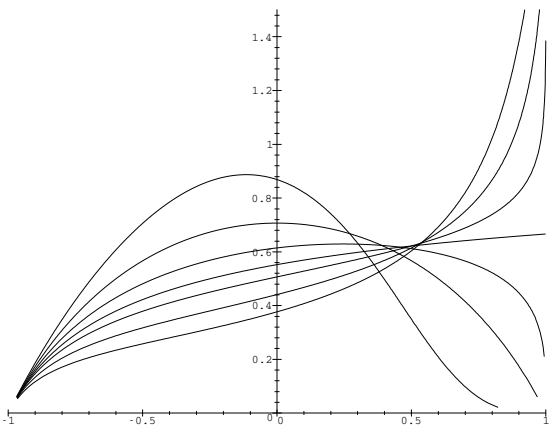


Figure 2.3–5. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = 0.5$ and $\lambda_4 = 0.25, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5$.

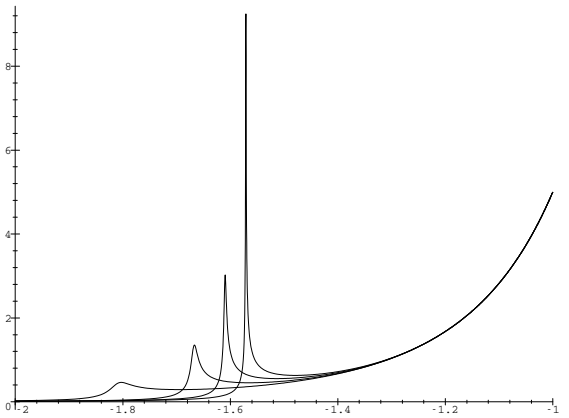


Figure 2.3–6. GLD$(0, -1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -0.2$ and $\lambda_4 = 27, 30, 35, 50$.

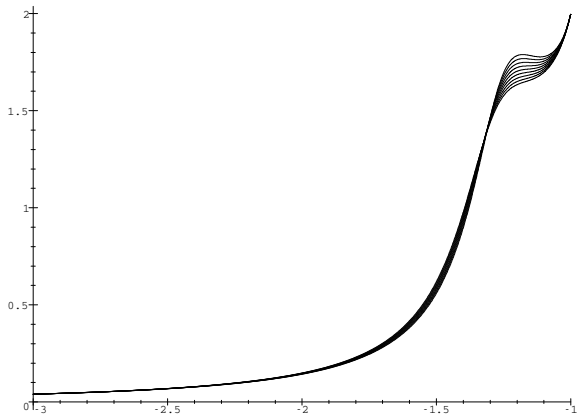


Figure 2.3–7a. GLD$(0, -1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -0.5$ and $\lambda_4 = 2.91, 2.92, 2.93, \ldots, 2.99$.

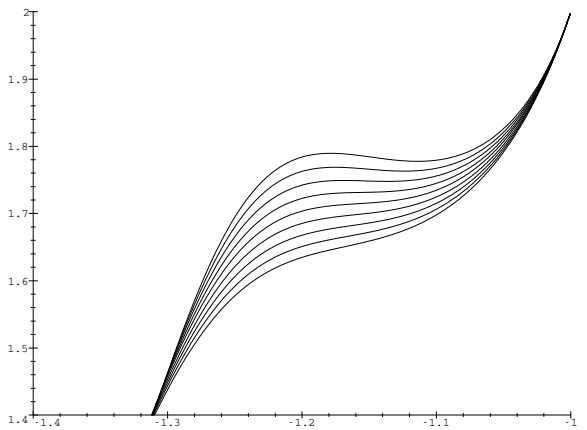


Figure 2.3–7b. GLD$(0, -1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -0.5$ and $\lambda_4 = 2.91, 2.92, 2.93, \ldots, 2.99$.

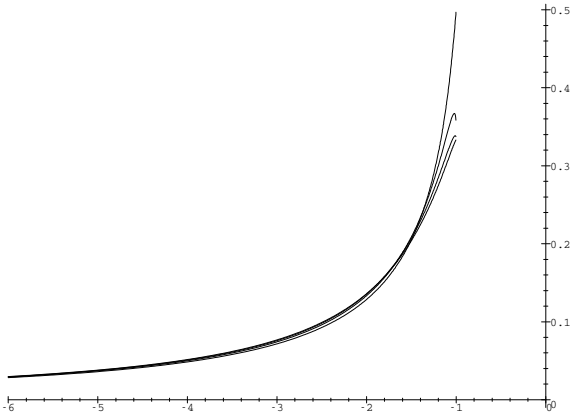


Figure 2.3–8a. GLD$(0, -1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -3$ and $\lambda_4 = 1.0, 1.25, 1.5, 1.75$.

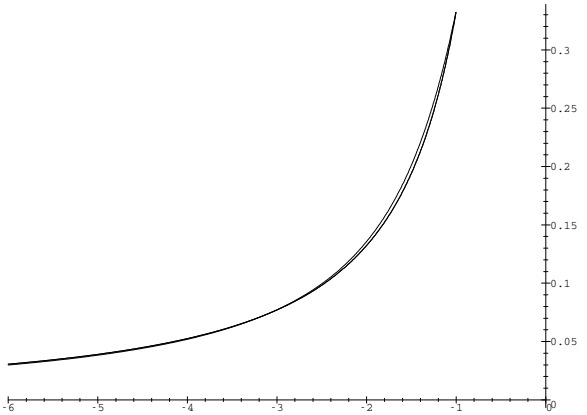


Figure 2.3–8b. GLD$(0, -1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -3$ and $\lambda_4 = 2.0, 6.0, 20$.

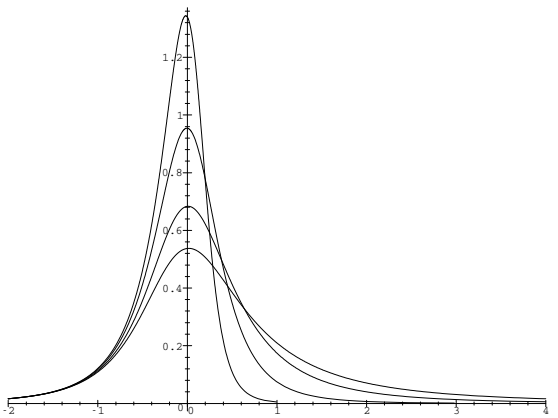


Figure 2.3–9. GLD$(0, 1, \lambda_3, \lambda_4)$ p.d.f.s with $\lambda_3 = -0.25$ and $\lambda_4 = -0.1, -0.2, -0.35, -0.5$.

Freimer, Kollia, Mudholkar, and Lin (1988) also (in their Section 4) **relate the GLD system to that of Karl Pearson**, adding to what was shown by Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979). As **similarities**, they note

- Both contain the uniform and exponential distributions;
- Both accommodate densities of varied shapes.

As **differences**, they note

- Only the GLD includes the logistic distribution;
- Only the Pearson includes the normal distribution (however, the GLD can come very close to the normal distribution);
- The Pearson covers all skewness and all kurtosis values, the GLD does not. This is a problem in some applications and it is solved by the Extended GLD developed in our Chapter 4.

2.4 GLD Random Variate Generation

As we noted in the Introduction to this Chapter, one of the important applications of the GLD has been the **generation of r.v.s for Monte Carlo studies**. This important application arises due to the confluence of several key results, which we will now state.

Theorem 2.4.1. *If $Q_X(\cdot)$ is the percentile function of a random variable X, and U is a uniform random variable on $(0,1)$, then $Q_X(U)$ has the same d.f. as does X.*

This result is key in simulation and Monte Carlo studies (for a proof, see p. 156 of Karian and Dudewicz (1999)), as it allows easy generation of a stream of random variables from any distribution for which the percentile function is readily available. This follows from

Corollary 2.4.2. *If $U_1, U_2, \ldots$ are independent uniform random variables on $(0,1)$, then*

$$X_1 = Q_X(U_1),\ X_2 = Q_X(U_2),\ \ldots \tag{2.4.3}$$

are independent random variables each with the same d.f. as X.

For a proof, also see p. 156 of Karian and Dudewicz (1999). The percentile function is not available in a closed (or easy-to-work-with) form for many of the most important distributions, such as the normal distribution. However, the GLD is (see (2.1.1)) **defined** by its p.f., which is a simple-to-calculate expression.

Thus, **r.v.s for a simulation study can easily be generated from any distribution that can be modeled by a GLD.**

Example. Suppose we have modeled an important r.v. by an approximate standard normal distribution X. We will show in Section 3.4.1 that a close fit to the standard normal is available via the GLD with

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0.1975, 0.1349, 0.1349) \tag{2.4.4}$$

and this GLD has p.f. (see (2.1.1))

$$\begin{aligned} Q(y) &= (y^{0.1349} - (1-y)^{0.1349})/0.1975 \\ &= 5.06329(y^{0.1349} - (1-y)^{0.1349}). \end{aligned} \tag{2.4.5}$$

Thus, if $U_1, U_2, \ldots$ are independent uniform r.v.s on $(0, 1)$, then

$$Q(U_1),\ Q(U_2),\ \ldots \tag{2.4.6}$$

are independent and (approximately) $N(0, 1)$ r.v.s for the simulation study at hand.

There are a number of good sources of independent uniform r.v.s on $(0, 1)$. For example (see p. 137 of Karian and Dudewicz (1999)), the generator called URN41 has a period of approximately 5×10^{18} (see Karian and Dudewicz (1999), pp. 132–133 for the period, Figure 3.5–8 for its FORTRAN code, Figure 3.5–9 for its C code, and Appendix G (p. 493) for its first 100 numbers), has passed extensive testing, and yields

$$U_1 = 0.67460162,\ U_2 = 0.15152637,\ \ldots\ . \tag{2.4.7}$$

Thus, using these in (1.5.5) (see (1.5.6)), we find the approximate normal r.v.s

$$\begin{aligned} X_1 &= 5.06329((0.67460162)^{0.1349} - (1-0.67460162)^{0.1349}) = 0.44975078 \\ X_2 &= 5.06329((0.15152637)^{0.1349} - (1-0.15126372)^{0.1349}) = -1.026919958. \end{aligned}$$

We can continue the stream by generating additional uniform r.v.s on $(0, 1)$, U_3, U_4, ... and evaluate $Q(U_3)$, $Q(U_4)$, ...

Since we will see in Section 3.2 (especially Figure 3.2–5) that **the GLD covers a broad space of distributions** and can model many different shapes well, it follows that **the GLD is very useful for modeling input to simulation and Monte Carlo studies: one can change the distribution being used by simply altering the lambda vector.**

2.5 The Fitting Process

Given a random sample x_1, x_2, $x_3, \ldots x_n$, the basic problem in fitting a statistical distribution to this data is that of approximating the distribution from which the

sample was obtained. If it is known, because of theoretical considerations, that the distribution is of a certain type (e.g., a gamma distribution with unknown parameters), then through moment matching, or some other means, one can determine a specific distribution that fits the data. This, however, is generally not the case and, in the absence of any knowledge regarding the distribution, it makes sense to appeal to a flexible family of distributions and choose a specific member of that family.

By a **flexible** family we mean one whose members can assume a large variety of shapes: skewness in either direction, tails that are truncated or extend to infinity on either or both sides, bell-shaped distributions as well as inverted bell-shaped ones. A second desirable quality for family of distributions to be suitable for fitting is for the family to be able to represent a **wide range of distributional characteristics** such as moments (or combination of moments) or percentiles (or combinations of percentiles). A third desirable feature would be for the distributions in the family to have **convenient, preferably closed-form, expressions for at least one of their p.d.f., c.d.f., and quantile function**.

The desirability features of flexible shapes, wide-ranging representation of distributional features, and conveniently expressible p.d.f., c.d.f., or quantile function, sets a high threshold for the choice of a suitable family of distributions, and the GLD family meets that threshold. This Handbook considers, in some detail, several other families of distributions that can be used effectively for fitting: the generalized beta distribution (GBD), the kappa distribution, the Johnson system and the generalized gamma distribution.

Having chosen a particular family of distributions, the next step in the fitting process is to pick a specific member of that family as the distribution that "best" approximates the distribution from which the data was extracted. Here too, there are a number of choices. The traditional and frequently used method of choosing a specific family member is to match the moments of the sample with their corresponding distribution moments and determine parameter values that produce a specific family member. This approach, referred to as the method of moments, is discussed in connection with the GLD, the GBD, and Johnson system in Chapters 3, 4, and 16, respectively. Choosing a specific family member can also be achieved through matching of certain percentiles (Chapter 5 does this for the GLD and Chapter 17 for the kappa distribution). A third approach is to determine specific parameter values by matching L-moments and Chapters 6 and 17 do this for the GLD and the kappa distribution, respectively.

Problems for Chapter 2

2.1. Suppose that the d.f. of X is $F_X(x) = 0$ if $x \leq 1, = (x^4 - 1)/255$ if $1 < x < 4$, and $= 1$ if $x \geq 4$.

a. Find the p.d.f. of X, say $f_X(x)$, and graph it. Also graph the d.f. of X.

b. Find the p.f. of X, say $Q_X(y)$, and graph it. Find $Q_X(.95)$.

2.2. Suppose that the p.d.f. of X is $f_X(x) = 5e^{-5x}$ when x is positive (and $= 0$ otherwise).

a. Graph the p.d.f. of X. Find and graph the d.f. of X. Find $F_X(5.3)$.

b. Find the p.f. of X and graph it. Find $Q_X(.75)$.

2.3. Suppose the p.f. of X is $Q(y) = y^2$ (for y between 0 and 1).

a. Find $f(x)$, the p.d.f. of X, and graph it. (Note that your answer should not have any y variables in it — they need to be eliminated.)

b. Find $F(x)$, the d.f. of X, and graph it.

c. Using $Q(y)$, at what x is the probability below x equal to 0.9?

d. Using $F(x)$, at what x is the probability below x equal to 0.9?

2.4. Identify the p.f. $Q(y) = y^2$ as a member of GLD family, i.e., what are the values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (see (2.1.1))? Then, use Theorem 2.1.2 to find $f(x)$. Do your results agree with those in Theorem 2.3.23 as to when the p.d.f. is positive? Does Corollary 2.2.9 say this is a valid distribution case of the GLD?

2.5. Suppose that X has a valid GLD distribution of the form (2.1.1), i.e., with p.f.

$$Q(y) = \lambda_1 + (y^{\lambda_3} - (1-y)^{\lambda_4})/\lambda_2.$$

In their work on GLDs, Freimer, Kollia, Mudholkar, and Lin (1988) took the p.f. to be

$$Q^*(y) = a_1 + (y^{a_3} - 1)/(a_2 a_3) - ((1-y)^{a_4} - 1)/(a_2 a_4).$$

Suppose that $Q(y) = Q^*(y)$ for all y between 0 and 1. Then what is the relationship between the vectors $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and (a_1, a_2, a_3, a_4)? If you cannot find a relationship that makes $Q(y) = Q^*(y)$ for all y, then in what sense is $Q^*(y)$ a "GLD", or is it simply another similar, but different, family of p.f.s?

2.6. Joiner and Rosenblatt (1971) used the p.f.

$$Q_1(y) = b_1 + y^{b_3}/b_2 - (1-y)^{b_4}/b_5.$$

Can you find relationships that make $Q(y) = Q_1(y)$? That make $Q_1(y) = Q^*(y)$ (for Q, see (2.1.1); for Q^*, see Problem 2.5)?

2.7. Hogben (1963) and Shapiro and Wilk (1965) did sampling studies (Monte Carlo experiments) using the p.f.

$$Q_2(y) = y^{c_3}/c_2 - (1-y)^{c_4}.$$

Relate this to the p.f.s Q, Q^*, and Q_1 (of (2.1.1), Problem 2.5, and Problem 2.6, respectively).

2.8. Gilchrist (1997) states the result: If $R(y)$ and $S(y)$ are each individually valid percentile functions (that may involve various parameters), then

$$T(y) = A + B(R(y) + S(y))$$

is also a valid percentile function. Prove this result.

2.9. In (2.2.11) through (2.2.18) we investigated the validity of the GLD in 8 regions of (λ_3, λ_4)-space. But the region $\lambda_4 = 1$ with $-1 < \lambda_3 < 0$ and the region $\lambda_3 = 1$ with $-1 < \lambda_4 < 0$ were not considered. For each of these two regions, determine if the GLD is valid in that region. If it is, plot some of the GLDs that are associated with that region.

References for Chapter 2

Dudewicz, E. J., Mommaerts, W., and van der Meulen, E. C. (1991). "Maximum entropy methods in modern spectroscopy: A review and an empiric entropy approach," *The Frontiers of Statistical Scientific Theory & Industrial Applications* (Chief Editors A. Öztürk and E. C. van der Meulen), American Sciences Press, Inc., Columbus, Ohio, 115–160.

Freimer, M., Kollia, G., Mudholkar, G. S., and Lin, C. T. (1988). "A study of the Generalized Tukey Lambda family," *Communications in Statistics–Theory and Methods*, 17, 3547–3567.

Gilchrist, W. (1997). "Modelling with quantile distribution functions," *Journal of Applied Statistics*, 24, 113–122.

Hogben, D. (1963). *Some Properties of Tukey's Test for Non-Additivity*, Ph.D. Thesis, Rutgers-The State University, New Jersey.

Joiner, B. L. and Rosenblatt, J. R. (1971). "Some properties of the range in samples from Tukey's symmetric Lambda distributions," *Journal of the American Statistical Association*, 66, 384–399.

Karian, Z. A. and Dudewicz, E. J. (1999). "Fitting the Generalized Lambda Distribution to data: A method based on percentiles," *Communications in Statistics: Simulation and Computation*, 28(3), 793–819.

Karian, Z. A., Dudewicz, E. J. and McDonald, P. (1996). "The extended generalized lambda distribution system for fitting distributions to data: History,

completion of theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics: Simulation and Computation*, 25(3), 611–642.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics,*21, 201–214.

Shapiro, S. S. and Wilk, M. B. (1965). "An analysis of variance test for normality (complete samples)," *Biometrika*, 52, 591–611.

Chapter 3

Fitting Distributions and Data with the GLD via the Method of Moments

In most practical applications, when constructing a statistical model we do not know the appropriate probability distribution (or do not know it fully). **If the appropriate probability distribution is** fully **known** (e.g., if it is known that X follows the normal distribution with mean 5 feet and standard deviation 6 inches), **then this distribution should be used** in the model. However, if a variable such as the height of females in a certain population is stated in the literature to be normal in distribution with population mean 5 feet and standard deviation 6 inches, **very often these are only estimates obtained from a sample**. Note that the normal distribution for height would be **inappropriate formally** since it gives $P(X < 0) > 0$, when we know that X cannot be negative. This need not be a reason to reject the normal model since for this model

$$P(X < 0) = \Phi(-10) = 0.7619855 \times 10^{-23}$$

(National Bureau of Standards (1953)), and a model that comes close to the true value of zero for the probability is **acceptable as long as it is used intelligently** by realizing that we have a close approximation to the true model, and do not make claims such as "persons of negative height will occur once each $1/(0.7619855 \times 10^{-23}) = 1.3 \times 10^{23}$ people." While such a claim is absurd, and no reasonable person would think a model bad because an unreasonable person could use it to make such a statement, the adage "What is so uncommon as common sense?" is one we have found to have much truth to it.

Even if we know, by theoretical derivation from reasonable assumptions, for example, that X is normal in distribution, we will often not know its parameters. These will need to be estimated from whatever data is available. Suppose that we have a set of data $X_1, X_2, \ldots, X_n$ that are independent and identically distributed

random variables. If the data is normally distributed, i.e., if X_i is $N(\mu, \sigma^2)$ for $i = 1, 2, \ldots, n$, then the mean μ and the variance σ^2 are usually estimated, respectively, by

$$\hat{\mu} \equiv \bar{X} = \frac{X_1 + \ldots + X_n}{n}$$

and

$$\hat{\sigma}^2 \equiv s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1).$$

Note that a random variable X that is $N(\mu, \sigma^2)$ has

$$\begin{cases} (\text{measure of center}) = E(X) = \mu, \\ (\text{measure of variability}) = E(X-\mu)^2 = \sigma^2, \\ (\text{measure of skewness}) = E(X-\mu)^3/\sigma^3 = 0, \\ (\text{measure of kurtosis}) = E(X-\mu)^4/\sigma^4 = 3. \end{cases}$$

If a random variable Y has a distribution other than the normal, we might attempt to approximate it by a random variable X that is $N(\mu, \sigma^2)$ for some μ and σ^2. We can do this successfully for center $E(Y)$ and variability $\text{Var}(Y) = E(Y - E(Y))^2$ by choosing μ and σ^2 in such a way as to match the center and variability of Y with the same center and variability for X. However, after that is done there are no free parameters in the distribution of X, and (unless the skewness and kurtosis of Y are, respectively, 0 and 3) we will not be able to match them in X. Hence, the normal family of distributions cannot be used to match data successfully unless the data is symmetric (so its skewness is 0) and has tail weight similar to that of the normal (so that its kurtosis is near 3). For this reason, families of distributions with additional parameters are often used, allowing us to match more than the center and the variability of Y.

In order to find a moment-based GLD fit to a given dataset $X_1, X_2, \ldots, X_n$, we determine the first four moments ($\bar{X}$, and the second, third, and fourth central moments) of $X_1, X_2, \ldots, X_n$, set these equal to their GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ counterparts, and solve the resulting equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

In Section 3.1 we consider the first four moments of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distributions and in Section 3.2 we determine the possible values that these moments can attain. Fitting a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ through the method of moments is developed in Section 3.3. Applications of these results for approximating some well-known distributions, and for fitting a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to a dataset, are developed in Sections 3.4 and 3.5, respectively.

3.1 The Moments of the GLD Distribution

In this section we develop expressions for the moments of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ random variables. We start by setting $\lambda_1 = 0$ to simplify this task; next, we

obtain the non-central moments of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$; and finally, we derive the central GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ moments.

Theorem 3.1.1. *If X is a* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *random variable, then $Z = X - \lambda_1$ is* GLD$(0, \lambda_2, \lambda_3, \lambda_4)$.

Proof. Since X is GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$,

$$Q_X(y) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2},$$

and

$$F_{X-\lambda_1}(x) = P[X - \lambda_1 \leq x] = P[X \leq x + \lambda_1] = F_X(x + \lambda_1). \tag{3.1.2}$$

If we set $F_X(x + \lambda_1) = y$, we obtain

$$x + \lambda_1 = Q_X(y) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \quad x = Q_{X-\lambda_1}(y). \tag{3.1.3}$$

From (3.1.2) we also have $F_{X-\lambda_1}(x) = y$ which with (3.1.3) yields

$$Q_{X-\lambda_1}(y) = x = \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2},$$

proving that $X - \lambda_1$ is GLD$(0, \lambda_2, \lambda_3, \lambda_4)$. ∎

Having established λ_1 as a location parameter, we now determine the non-central moments (when they exist) of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Theorem 3.1.4. *If Z is* GLD$(0, \lambda_2, \lambda_3, \lambda_4)$, *then $E(Z^k)$, the expected value of Z^k, is given by*

$$E(Z^k) = \frac{1}{\lambda_2^k} \sum_{i=0}^{k} \left[\binom{k}{i} (-1)^i \beta(\lambda_3(k-i) + 1,\ \lambda_4 i + 1)) \right] \tag{3.1.5}$$

where $\beta(a, b)$ is the beta function defined by

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx. \tag{3.1.6}$$

Proof.

$$\begin{aligned} E(Z^k) &= \int_{-\infty}^{\infty} z^k f(z)\, dz = \int_0^1 (Q(y))^k\, dy \qquad (3.1.7) \\ &= \int_0^1 \left(\frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2} \right)^k dy = \frac{1}{\lambda_2^k} \int_0^1 \left(y^{\lambda_3} - (1-y)^{\lambda_4} \right)^k dy. \end{aligned}$$

By the binomial theorem,

$$\left(y^{\lambda_3} - (1-y)^{\lambda_4}\right)^k = \sum_{i=0}^{k} \left[\binom{k}{i} (y^{\lambda_3})^{k-i} (-(1-y)^{\lambda_4})^i \right]. \tag{3.1.8}$$

Using (3.1.8) in the last expression of (3.1.7), we get

$$\begin{aligned} E(Z^k) &= \frac{1}{\lambda_2^k} \sum_{i=0}^{k} \left[\binom{k}{i} (-1)^i \int_0^1 y^{\lambda_3(k-i)} (1-y)^{\lambda_4 i}\, dy \right] \\ &= \frac{1}{\lambda_2^k} \sum_{i=0}^{k} \left[\binom{k}{i} (-1)^i \beta(\lambda_3(k-i)+1, \lambda_4 i + 1) \right], \end{aligned}$$

completing the proof of the theorem. ∎

Before continuing with our investigation of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ moments, we note three properties of the beta function that will be useful in our subsequent work.

Properties of the beta function

1. The integral in (3.1.6) that defines the beta function will converge if and only if a and b are positive (this can be verified by choosing c from the $(0,1)$ interval and considering the integral over the subintervals $(0,c)$ and $(c,1)$).
2. When a and b are positive, $\beta(a,b) = \beta(b,a)$. Using the substitution $y = 1-x$ in the integral for $\beta(a,b)$ will transform it to the integral for $\beta(b,a)$.
3. By direct evaluation of the integral in (3.1.6), it can be determined that for $u > -1$,

$$\beta(u+1, 1) = \beta(1, u+1) = \frac{1}{u+1}. \tag{3.1.9}$$

The first of these observations, along with (3.1.5) of Theorem 3.1.4, helps us determine the conditions under which the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ moments exist.

Corollary 3.1.10. *The* k-th GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *moment exists if and only if* $\lambda_3 > -1/k$ *and* $\lambda_4 > -1/k$.

Proof. From Theorem 3.1.1, $E(X^k)$ will exist if and only if $E(Z^k) = E((X - \lambda_1)^k)$ exists, which, by Theorem 3.1.4, will exist if and only if

$$\lambda_3(k-i) + 1 > 0 \text{ and } \lambda_4 i + 1 > 0, \quad \text{for } i = 0, 1, \ldots, k.$$

This condition will prevail if and only if $\lambda_3 > -1/k$ and $\lambda_4 > -1/k$. ∎

Since, ultimately, we are going to be interested in the first four moments of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, we will **need to impose the condition** $\lambda_3 > -1/4$

and $\lambda_4 > -1/4$ throughout the remainder of this chapter. The next theorem gives an explicit formulation of the first four centralized GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ moments.

Theorem 3.1.11. *If X is* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *with $\lambda_3 > -1/4$ and $\lambda_4 > -1/4$, then its first four moments, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (mean, variance, skewness, and kurtosis, respectively), are given by*

$$\alpha_1 = \mu = E(X) = \lambda_1 + \frac{A}{\lambda_2}, \tag{3.1.12}$$

$$\alpha_2 = \sigma^2 = E\left[(X-\mu)^2\right] = \frac{B - A^2}{\lambda_2^2}, \tag{3.1.13}$$

$$\alpha_3 = E(X - E(X))^3/\sigma^3 = \frac{C - 3AB + 2A^3}{\lambda_2^3 \sigma^3}, \tag{3.1.14}$$

$$\alpha_4 = E(X - E(X))^4/\sigma^4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \sigma^4}, \tag{3.1.15}$$

where

$$A = \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4}, \tag{3.1.16}$$

$$B = \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3,\, 1+\lambda_4), \tag{3.1.17}$$

$$\begin{aligned} C = {} & \frac{1}{1+3\lambda_3} - \frac{1}{1+3\lambda_4} - 3\beta(1+2\lambda_3,\, 1+\lambda_4) \\ & + 3\beta(1+\lambda_3,\, 1+2\lambda_4), \end{aligned} \tag{3.1.18}$$

$$\begin{aligned} D = {} & \frac{1}{1+4\lambda_3} + \frac{1}{1+4\lambda_4} - 4\beta(1+3\lambda_3,\, 1+\lambda_4) \\ & + 6\beta(1+2\lambda_3,\, 1+2\lambda_4) - 4\beta(1+\lambda_3,\, 1+3\lambda_4). \end{aligned} \tag{3.1.19}$$

Proof. Let Z be a GLD$(0, \lambda_2, \lambda_3, \lambda_4)$ random variable. By Theorem 3.1.1,

$$E(X^k) = E((Z + \lambda_1)^k).$$

We first express $E(Z^i)$, for $i = 1$, 2, 3, and 4, in terms of A, B, C, and D. To do this for $E(Z)$, we use Theorem 3.1.4 to obtain

$$E(Z) = \frac{1}{\lambda_2}\left(\beta(\lambda_3 + 1, 1) - \beta(1, \lambda_4 + 1)\right),$$

and from (3.1.9) we get

$$E(Z) = \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3 + 1} - \frac{1}{\lambda_4 + 1}\right) = \frac{A}{\lambda_2}. \tag{3.1.20}$$

For $E(Z^2)$ we again use Theorem 3.1.4 and the simplification allowed by (3.1.9) to get

$$\begin{aligned} E(Z^2) &= \frac{1}{\lambda_2^2}\left(\beta(2\lambda_3+1,1)-\beta(\lambda_3+1,\lambda_4+1)+\beta(1,2\lambda_4+1)\right) \\ &= \frac{1}{\lambda_2^2}\left(\frac{1}{2\lambda_3+1}-\frac{1}{2\lambda_4+1}-2\beta(\lambda_3+1,\lambda_4+1)\right)=\frac{B}{\lambda_2^2}. \end{aligned} \tag{3.1.21}$$

Similar arguments, with somewhat more complicated algebraic manipulations, for $E(Z^3)$ and $E(Z^4)$ produce

$$E(Z^3) = \frac{C}{\lambda_2^3} \tag{3.1.22}$$

$$E(Z^4) = \frac{D}{\lambda_2^4}. \tag{3.1.23}$$

We now use (3.1.20) to derive (3.1.12):

$$\alpha_1 = E(X) = E(Z+\lambda_1) = \lambda_1 + E(Z) = \lambda_1 + \frac{A}{\lambda_2}.$$

Next, we consider (3.1.13):

$$\begin{aligned} \alpha_2 &= E(X^2)-\alpha_1^2 = E((Z+\lambda_1)^2)-\alpha_1^2 \\ &= E(Z^2)+2\lambda_1 E(Z)+\lambda_1^2-\alpha_1^2. \end{aligned} \tag{3.1.24}$$

Substituting A/λ_2 for $E(Z)$ and $\lambda_1 + A/\lambda_2$ for α_1 in (3.1.24) and using (3.1.21), we get

$$\alpha_2 = E(Z^2) - \frac{A^2}{\lambda_2^2} = \frac{B-A^2}{\lambda_2^2}.$$

The derivations of (3.1.14) and (3.1.15) are similar but algebraically more involved. ∎

Corollary 3.1.25. *If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the first four moments of* GLD(λ_1, $\lambda_2, \lambda_3, \lambda_4$)*, then the first four moments of* GLD($\lambda_1, \lambda_2, \lambda_4, \lambda_3$) *will be*

$$\alpha_1 - \frac{2A}{\lambda_2}, \qquad \alpha_2, \qquad -\alpha_3, \qquad \alpha_4. \tag{3.1.26}$$

Proof. The exchange of λ_3 and λ_4 in the expressions for A, B, C, and D in (3.1.16) through (3.1.20) changes the signs of A and C and leaves B and D intact.

Thus, from (3.1.12), the first moment of GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$ will be $\lambda_1 - A/\lambda_2 = \alpha_1 - 2A/\lambda_2$.

Since B and A^2 are not affected by the exchange of λ_3 and λ_4, from (3.1.13), α_2 will be the second moment of GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$. C, A, and A^3 of (3.1.14) all change signs when λ_3 and λ_4 are switched, making $-\alpha_3$ the third moment of GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$. Since D, AC, A^2B, and A^4 of (3.1.15) are not affected by the exchange of λ_3 and λ_4, GLD$(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$ will have α_4 for its fourth moment. ∎

3.2 The (α_3^2, α_4)-Space Covered by the GLD Family

If a random variable Y has a distribution other than the GLD, we might try to approximate it by a random variable X that is GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for some λ_1, λ_2, λ_3, λ_4. Suppose that the first four moments of Y are $\alpha_1 = \mu$, $\alpha_2 = \sigma^2$, α_3, and α_4. If we can choose λ_3, λ_4 so that a GLD$(0, 1, \lambda_3, \lambda_4)$ has third and fourth moments α_3 and α_4, then we can let λ_1 and λ_2 be solutions of the equations

$$\mu = \lambda_1 + \frac{A}{\lambda_2}, \quad \sigma^2 = \frac{B - A^2}{\lambda_2^2}. \tag{3.2.1}$$

It follows from Theorem 3.1.11 that the resulting λ_1, λ_2, λ_3, λ_4 specify a GLD with the desired first four moments. Here, we note that A, B, C, D are functions only of λ_3, λ_4, and that (3.2.1) can be solved for any μ and any $\sigma^2 > 0$. We have, therefore, established the following consequence of Theorem 3.1.11.

Corollary 3.2.2. *The* GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *can match any first two moments* μ *and* σ^2*, and some third and fourth moments* α_3 *and* α_4*.*

The larger the set of (α_3, α_4) that the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ can generate, the more useful the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family will be in fitting a broad range of datasets and approximating a variety of other random variables. So we next consider the spectrum of values that α_3 and α_4 can attain. From Corollary 3.1.25, we know that if (α_3, α_4) can be attained, then so can $(-\alpha_3, \alpha_4)$ (by switching λ_3 and λ_4), allowing us to consider the (α_3^2, α_4)-space associated with the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Figures 3.2–1, 3.2–2, and 3.2–3 show the (α_3^2, α_4) contour plots for (λ_3, λ_4) from Regions 3, 4, and 5 and 6, respectively (recall that these regions were defined in Section 2.2 and illustrated in Figure 2.2–1). The curves in Figure 3.2–1 are associated with a sequence of values of λ_4, with λ_3 ranging on the interval $(0, 15)$ for each of these choices of λ_4. For example, the curve labeled "0.02" is obtained by plotting the (α_3^2, α_4) pairs when λ_4 is set to 0.02 and λ_3 is taken from the interval $(0, 15)$. All the curves of Figure 3.2–1 are obtained in a similar manner

with $\lambda_4 = 0.02, 0.07, 0.12, \ldots, 0.52, 0.6, 0.7, \ldots, 1.0$ and λ_3 from the interval $(0, 15)$.

The construction of Figure 3.2–2 is similar to that of Figure 3.2–1, with (λ_3, λ_4) taken from Region 4. Note that in this case we must have

$$-1/4 < \lambda_3 < 0 \quad \text{and} \quad -1/4 < \lambda_4 < 0$$

since otherwise (see Corollary 3.1.10) α_3 or α_4 or both may not exist. Some of the λ_4 values associated with these curves are given in Figure 3.2–2; the other values of λ_4 are $-0.0125, -0.0250, \ldots, -0.075$. In Figure 3.2–3 the roles of λ_3 and λ_4 are reversed in the sense that λ_3 is fixed (to the values shown in the figure) and λ_4 is allowed to range upward (in the case of Region 5) from the boundary that defines the region (see Theorem 2.2.33).

Figure 3.2–4 gives a comprehensive view of the connection between the GLD (α_3^2, α_4)-space and the regions of (λ_3, λ_4). The area marked "Impossible Region" is where $\alpha_4 \leq 1 + \alpha_3^2$, an impossibility since the inequality

$$\alpha_4 > 1 + \alpha_3^2 \tag{3.2.3}$$

holds for all distributions (it is less well-known than the inequality $E(X^2) \geq \mu^2$, which follows from $\text{Var}(X) = E(X^2) - \mu^2 \geq 0$). Moreover, since $\alpha_4 > 1 + \alpha_3^2$ always holds, α_4 necessarily exceeds 1. This result, given in some classical books, is well-known in the field of distribution fitting but is not covered in most texts on probability and statistics. Moreover, we are not aware of any texts that give a simple proof. For a brief indication of how an advanced proof may proceed, see Kendall and Stuart (1969), p. 92, Exercise 3.19. We now state and prove this result.

Theorem 3.2.4. *For any r.v. X for which these moments exist, $\alpha_4 > 1 + \alpha_3^2$. (Note that equality is not possible for any continuous distribution.)*

Proof. Since

$$\alpha_i = \frac{E(X - E(X))^i}{\sigma_X^i} = E\left(\frac{X - E(X)}{\sigma_X}\right)^i,$$

we can assume without loss of generality that $E(X) = 0$ and $\sigma_X = 1$, so that $E(X^2) = 1$ (because α_i involves only $X^* = (X - E(X))/\sigma_X$, which has $E(X^*) = 0$ and $E(X^{*^2}) = 1$). Now the Schwarz inequality (see, for example, Dudewicz and Mishra (1988), p. 240, Theorem 5.3.23) says that for any r.v.s, X and Y, for which the expectation exists, $(E(XY))^2 \leq E(X^2)E(Y^2)$. If we take the two r.v.s in the Schwarz inequality to be X and $X^2 - 1$, then

$$\begin{aligned}
\Big(E(X(X^2-1))\Big)^2 &\leq E(X^2)E((X^2-1)^2) \\
\Big(E(X^3 - X)\Big)^2 &\leq 1 \cdot E(X^4 - 2X^2 + 1) \\
\Big(E(X^3) - 0\Big)^2 &\leq E(X^4) - 2 + 1 \\
\alpha_3^2 &\leq \alpha_4 - 1.
\end{aligned}$$

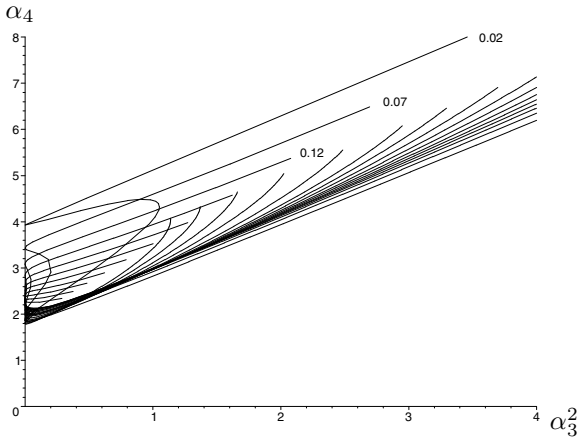


Figure 3.2–1. (α_3^2, α_4)-space generated by (λ_3, λ_4) from Region 3 (see (2.2.13)).

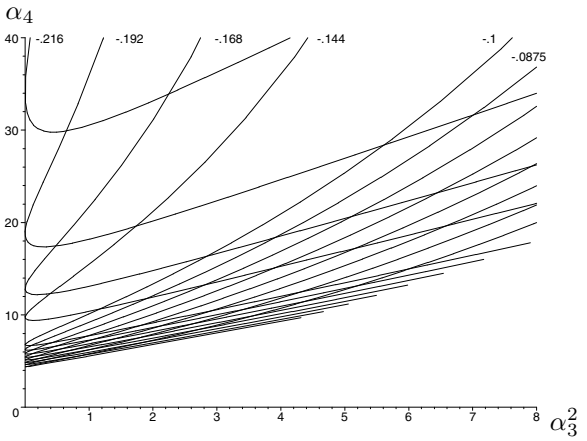


Figure 3.2–2. (α_3^2, α_4)-space generated by (λ_3, λ_4) from Region 4 (see (2.2.14)).

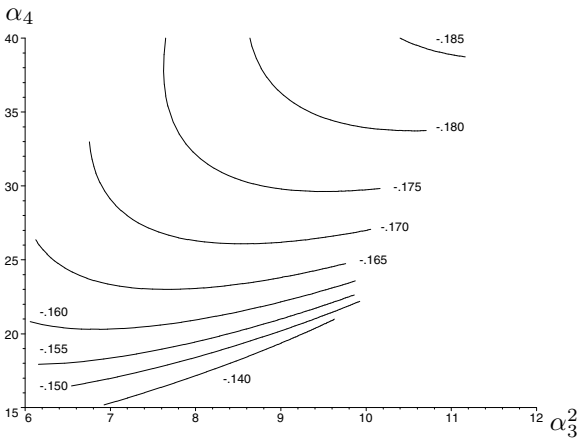


Figure 3.2–3. (α_3^2, α_4)-space generated by (λ_3, λ_4) from Regions 5 and 6 (defined in Theorem 2.2.33).

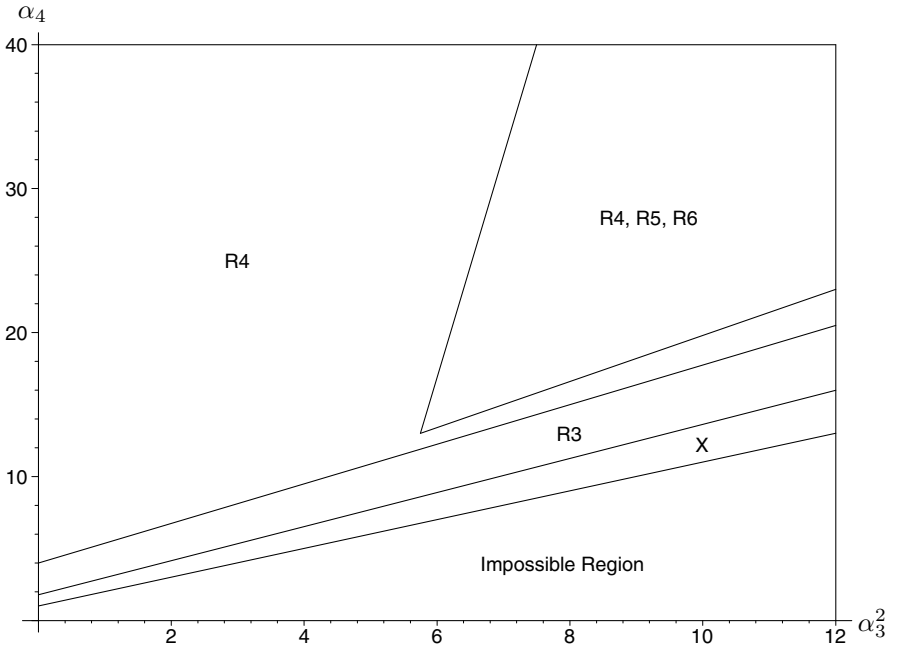


Figure 3.2–4. Regions of (α_3^2, α_4) moment-space that the GLD can attain (R3, R4, R5, R6); cannot attain (X); impossible for any distribution.

The proof will be complete when we show that equality is not possible for any continuous distribution.

The Schwarz inequality also asserts that equality occurs if and only if for some constant, a, we have $Y = aX$. In our case, this implies that equality holds if and only if $X^2 - 1 = aX$ or

$$X = \frac{a \pm \sqrt{a^2 + 4}}{2}.$$

Thus, we can have equality if and only if X is a r.v. that takes on only the two values

$$\frac{a - \sqrt{a^2 + 4}}{2} \quad \text{and} \quad \frac{a + \sqrt{a^2 + 4}}{2}.$$

Suppose these values are attained with probabilities p and $1 - p$, respectively $(0 \le p \le 1)$. Then $E(X) = 0$ implies

$$p = +\frac{1}{2}\left(1 + \frac{a}{\sqrt{a^2 + 4}}\right). \tag{3.2.5}$$

Since $0 \le p \le 1$, $-1 \le a/\sqrt{a^2 + 4} \le 1$. And $E(X^2) = 1$ also implies (3.2.5). So $\alpha_3^2 = \alpha_4 - 1$ if and only if for some a

$$P\left(X = \frac{a - \sqrt{a^2 + 4}}{2}\right) = \frac{1}{2}\left(1 + \frac{a}{\sqrt{a^2 + 4}}\right)$$

and

$$P\left(X = \frac{a + \sqrt{a^2+4}}{2}\right) = \frac{1}{2}\left(1 - \frac{a}{\sqrt{a^2+4}}\right).$$

For general a, $(\alpha_3, \alpha_4) = (a, a^2 + 1)$. When $a = 0$, $P(X = -1) = 0.5 = P(X = +1)$ and $(\alpha_3, \alpha_4) = (0, 1)$. ∎

Immediately above the Impossible Region in Figure 3.2–4 there is a narrow "sliver" marked by "X." The GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ does not produce (α_3^2, α_4) in this region. We saw in Figures 3.2–1, 3.2–2, and 3.2–3 that Regions 3, 4, 5, and 6 cannot yield points in area X. That Regions 1 and 2 also cannot follows from the fact that $\lambda_3 \leq -1$ (Region 1) and $\lambda_4 \leq -1$ (Region 2) violate the conditions $\lambda_3 > -0.25$ and $\lambda_4 > -0.25$ needed for the third and fourth moments to exist (see Corollary 3.1.10). Other distributions (e.g., the beta distribution) do have their (α_3^2, α_4) in this area. In Chapter 4 we give an extension of the GLD that covers this portion of (α_3^2, α_4)-space. (While area X of Figure 3.2–4 may look "small," it is important in a variety of applications.)

The remaining portions of Figure 3.2–4 are marked with "R3," designating that (λ_3, λ_4) has to be chosen from Region 3 for this portion of (α_3^2, α_4)-space, and with "R4, R5, R6," designating that (λ_3, λ_4) is to be chosen from one of Regions 4, 5, or 6 to generate (α_3^2, α_4) in this area. The boundaries between the various portions of Figure 3.2–4 are drawn reasonably accurately, except for the boundaries that enclose the "R4, R5, R6" area; these are rough approximations obtained through numeric computations from the curves in Figure 3.2–3.

The (α_3^2, α_4)-space covered by the GLD already includes the moment combinations of such distributions as the uniform, Student's t, normal, Weibull, gamma, lognormal, exponential, and some beta distributions, among others. Thus, it is a rich class in terms of moment coverage. To put this in context, we show in **Figure 3.2–5** the (α_3^2, α_4) pairs associated with a number of well-known distributions. **The shaded region is the region covered by the (α_3^2, α_4) pairs of the GLD**

In Figure 3.2–5, the lines that are designated by "W," "L-N," "G," and "S" (the latter refers to the line defined by $\alpha_3^2 = 0$) show the (α_3^2, α_4) pairs for the Weibull, lognormal, gamma, and Student's t distributions, respectively. The area designated by **"B E T A R E G I O N" shows the (α_3^2, α_4) points that can be produced by the beta distribution. This region extends from the Impossible Region to slightly beyond the line marked for the lognormal distribution**. The point designated by a small square with label "u" and located at $(0, 1.8)$ represents the uniform distribution; the point at $(0, 3)$ labeled with "n" represents the normal distribution, $N(\mu, \sigma^2)$; and the point located at $(4, 9)$ and labeled with "e" gives the (α_3^2, α_4) point associated with the exponential distribution.

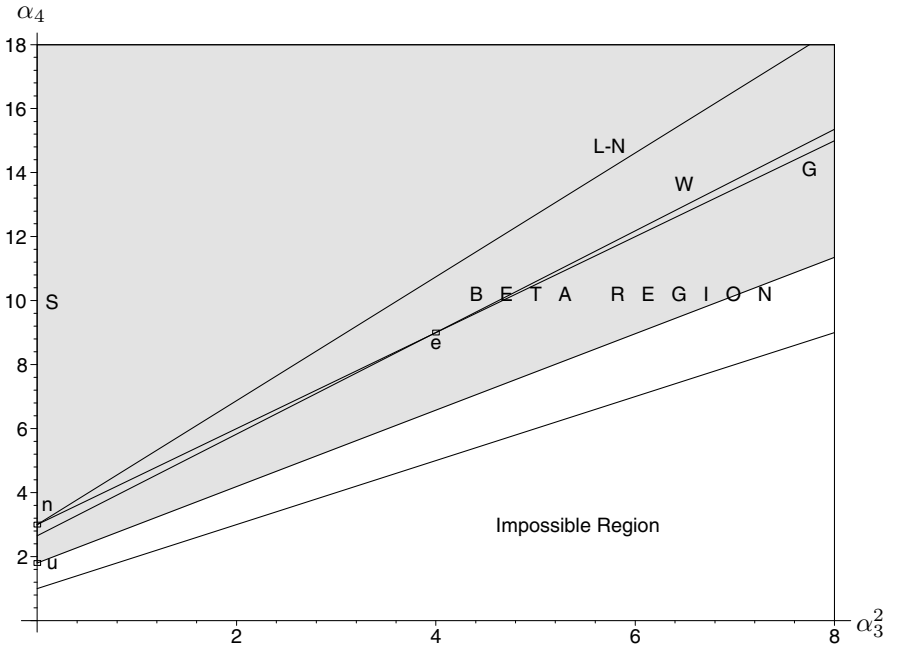


Figure 3.2–5. (α_3^2, α_4) points of some distributions (the **shaded region** consists of GLD (α_3^2, α_4) points).

3.3 Fitting the GLD through the Method of Moments

As stated at the beginning of the chapter, our intention is to fit a GLD to a dataset by equating $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$, the sample statistics corresponding to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and solving the equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. For a dataset $X_1, X_2, \ldots, X_n$, the **sample moments** corresponding to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are denoted $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ and are defined by

$$\hat{\alpha}_1 = \overline{X} = \sum_{i=1}^{n} X_i/n, \tag{3.3.1}$$

$$\hat{\alpha}_2 = \hat{\sigma}^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2/n, \tag{3.3.2}$$

$$\hat{\alpha}_3 = \sum_{i=1}^{n} (X_i - \overline{X})^3/(n\hat{\sigma}^3), \tag{3.3.3}$$

$$\hat{\alpha}_4 = \sum_{i=1}^{n} (X_i - \overline{X})^4/(n\hat{\sigma}^4). \tag{3.3.4}$$

These are not the maximum likelihood estimators (those would have some ns replaced by $n-1$), but correspond to method-of-moments estimators.

Solving the system of equations

$$\alpha_i = \hat{\alpha}_i \qquad \text{for } i = 1, 2, 3, 4 \tag{3.3.5}$$

for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is simplified somewhat by observing that A, B, C, D of (3.1.16) through (3.1.19) are free of λ_1 and λ_2, and λ_2 drops out of (3.1.14) and (3.1.15) because (see (3.1.13)) $\lambda_2^i \sigma^i = (B - A^2)^{i/2}$ for $i = 3$ and 4. Thus, α_3 and α_4 depend only on λ_3 and λ_4. Hence, if λ_3 and λ_4 can be obtained by solving the subsystem

$$\alpha_3 = \hat{\alpha}_3 \quad \text{and} \quad \alpha_4 = \hat{\alpha}_4 \tag{3.3.6}$$

of two equations in the two variables λ_3 and λ_4, then using (3.1.13) and (3.1.12) successively will yield λ_2 and λ_1.

Unfortunately, (3.3.6) is complex enough to prevent exact solutions, forcing us to appeal to numerical methods to obtain approximate solutions. Algorithms for finding numerical solutions to systems of equations such as (3.3.6) are generally designed to "search" for a solution by checking if an initial set of values ($\lambda_3 = \lambda_3^*$, $\lambda_4 = \lambda_4^*$ in the case of (3.3.6)) can be considered an approximate solution. This determination is made by checking if

$$\max(|\alpha_3 - \hat{\alpha}_3|, |\alpha_4 - \hat{\alpha}_4|) < \epsilon, \tag{3.3.7}$$

when $\lambda_3 = \lambda_3^*$ and $\lambda_4 = \lambda_4^*$. The positive number ϵ represents the accuracy associated with the approximation; if it is determined that the initial set of values $\lambda_3 = \lambda_3^*$, $\lambda_4 = \lambda_4^*$ does not provide a sufficiently accurate solution, the algorithm searches for a better choice of λ_3 and λ_4 and iterates this process until a suitable solution is discovered (i.e., one that satisfies (3.3.7)). In algorithms of this type there is no assurance that the algorithm will terminate successfully nor that greater accuracy will be attained in successive iterations. Therefore, such searching algorithms are designed to terminate (unsuccessfully) if (3.3.7) is not satisfied after a fixed number of iterations.

If table values of approximate solutions to (3.3.11) are readily available, an alternate grid-based algorithm can be used. In this case, the algorithm first conducts a "table lookup" to determine values of λ_{3a}, λ_{3b}, λ_{4a}, λ_{4b}, so that the desired solution (or at least some solution) satisfies $\lambda_{3a} \le \lambda_3 \le \lambda_{3b}$ and $\lambda_{4a} \le \lambda_4 \le \lambda_{4b}$. Next, an $n \times n$ grid is constructed on the rectangle $[\lambda_{3a}, \lambda_{3b}] \times [\lambda_{4a}, \lambda_{4b}]$ and for every point of that grid, $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$ is evaluated. The algorithm returns the (λ_3, λ_4) point on the grid that produces the smallest $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$. Karian and Dudewicz (2007) describe how this seemingly inefficient algorithm can be improved by repeatedly "zooming in" and reinitializing the grid.

3.3.1 Fitting through Direct Computation

The outcome of searching algorithms (success or failure, and in the former case a particular solution) depends on the equations themselves, the ϵ of (3.3.7), the

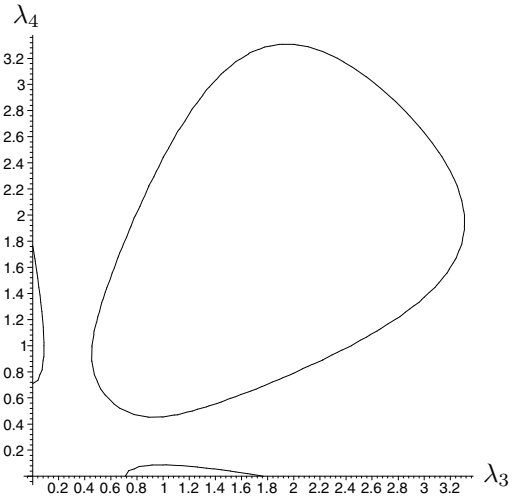


Figure 3.3–1. The contour curve for $\alpha_4 = 2$.

maximum number of iterations allowed, and the initial starting point for the search. Such algorithms usually have built-in specifications for ϵ and the maximal number of iterations, leaving the **choice of the starting point as the only real option for the user**. To get some insight into where to look for solutions (i.e., how to choose a starting point), consider a specific case where

$$\hat{\alpha}_1 = 0, \quad \hat{\alpha}_2 = 1, \quad \hat{\alpha}_3^2 = 0.025, \quad \hat{\alpha}_4 = 2. \tag{3.3.8}$$

It is clear from Figures 3.2–1, 3.2–2, and 3.2–3 that the only region of (λ_3, λ_4)-space that can produce a solution is Region 3 (this can also be observed from Figure 3.2–4). The equation $\alpha_4 = 2$, represented by a curve in (λ_3, λ_4)-space, is shown in Figure 3.3–1. The (λ_3, λ_4) points on this curve satisfy the second equation; hence, they represent **potential** solutions to the fourth equation in (3.3.8). The actual solutions will also have to be on the curve specified by $\alpha_3^2 = 0.025$. We see from the intersection of the curves of Figure 3.3–1 with the contour curves of $\alpha_3^2 = 0.025$ (shown in Figure 3.3–2), that there seem to be four solutions with (λ_3, λ_4) roughly

$$(0.03, 0.75), \ (0.8, 0.5), \ (0.95, 0.1), \ (3.25, 2.25). \tag{3.3.9}$$

There are also four additional solutions,

$$(0.75, 0.03), \ (0.5, 0.8). \ (0.1, 0.95), \ (2.25, 3.25),$$

when λ_3 and λ_4 are exchanged. The symmetry of solutions about the $\lambda_3 = \lambda_4$ line is due to the presence of two values of $\hat{\alpha}_3 = \pm\sqrt{0.025}$ (the rough solutions of (3.3.9) are associated with $\hat{\alpha}_3 = +\sqrt{0.025}$ and the other four with $\hat{\alpha}_3 = -\sqrt{0.025}$).

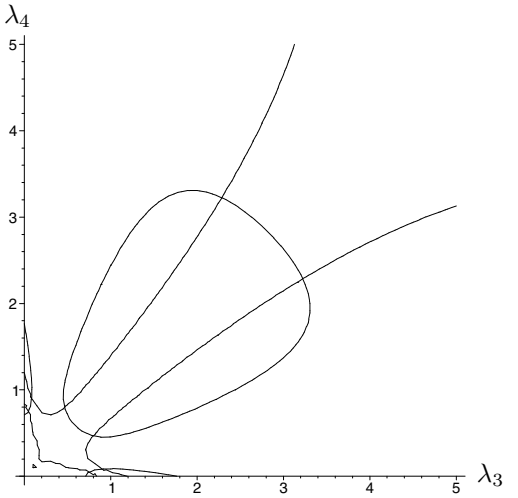


Figure 3.3–2. The contour curves for $\alpha_3^2 = 0.025$ and $\alpha_4 = 2$.

In practice, of course, we have either $\hat{\alpha}_3 = +\sqrt{0.025}$ or $\hat{\alpha}_3 = -\sqrt{0.025}$ and then find four solutions for the appropriate case.

The **R** procedure `FindLambdasM`, that is described in Appendix A, was devised specifically to produce solutions to (3.3.5). This program uses the table lookup and grid-based search described in the previous section. The only argument of `FindLambdasM` is the vector $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$ and it returns the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the fitted distribution. If such a solution cannot be found with $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) < 10^{-5}$, then `FindLambdasM` returns `0 0 0 0`. More detailed information about `FindLambdasM`, as well as the other programs included with this book is given in Appendix A.

For the illustration at hand, the use of

```
> A <- c(0, 1, sqrt(0.025), 2)
> FindLambdasM(A)
```

produces

```
[1] 2.562450e-01 4.992390e-01 8.014867e-01 4.640761e-01 2.478080e-08
```

which corresponds to the second $(\lambda_3, \lambda_4) = (0.8, 0.5)$ point of (3.3.9). We designate this fit by

$$\mathrm{GLD}_2(0.2562, 0.4992, 0.8015, 0.4641)$$

and note that $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 2.5 \times 10^{-8}$.

To search for the other solutions we use the **R** program `RefineSearchGLDM`. This program needs 5 arguments: the vector $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$, the minimum and maximum values, represented as a vector, of λ_3 to be used in the search for a solution, a similar vector to be used for λ_4, the number of grid partitions to used during the search, and the number of iterations where these partitions are to

be applied. `RefineSearchGLDM` returns the the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the fit with the smallest value of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$ and the value of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$.

To find a solution that is close to the first $(\lambda_3, \lambda_4) = (0.03, 0.75)$ point given in (3.3.9), we use

```
> RefineSearchGLDM(A,c(0,0.1),c(0.5,1),50,4)
```

and get

```
[1] -1.332727e+00  2.919003e-01  2.981683e-02  7.181446e-01  2.548631e-07
```

indicating the fit

$$\text{GLD}_1(-1.3327, 0.2919, 0.02982, 0.7181)$$

and $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 2.5 \times 10^{-7}$. Similar searches for the remaining, third and fourth, (λ_3, λ_4) points of (3.3.9) give the fits

$$\text{GLD}_3(1.1896, 0.34639, 0.9384, 0.07762)$$

and

$$\text{GLD}_4(0.1267, 0.5357, 3.2257, 2.2835).$$

The p.d.f.s of these GLD fits are shown in Figure 3.3–3 where the GLD_i p.d.f. is labeled with (i). In GLD_2, GLD_3 and GLD_4, $\lambda_3 > \lambda_4$ but in GLD_1, $\lambda_3 < \lambda_4$.

Note that **had the original $\hat{\alpha}_3^2$ and $\hat{\alpha}_4$ been 0.5 and 2**, respectively, we would easily determine from Figures 3.2–1, 3.2–2, and 3.2–3 (or, with some difficulty from Figure 3.2–4) that (3.3.6) would not have any solutions. This can be seen even more convincingly in Figure 3.3–4 which shows that the contour curves of the two equations do not intersect when (λ_3, λ_4) is in Region 3.

Figure 3.3–5 shows a family of **contour curves for Region 3 for** α_4 with values

$$1.825, 1.85, 1.9, 2, 2.1, 2.25, 2.5, 2.75, 3, 3.5, 4.$$

The curve associated with $\alpha_4 = 1.825$ consists of the innermost oval and the lowest branches along the λ_3 and λ_4 axes, the curve for $\alpha_4 = 1.85$ consists of the next larger oval and the next higher branches along the axes, and so on. Figure 3.3–6 gives **contour curves for Region 3 for** α_3^2 with values

$$0.005, 0.01, 0.015, 0.025, 0.05, 0.1, 0.2, 0.3, 0.5, 0.75, 1, 1.25, 1.5, 2.$$

The curve closest to the line $\lambda_3 = \lambda_4$ (not shown) and on either side of this line is associated with $\alpha_3^2 = 0.005$ and subsequent curves moving away from $\lambda_3 = \lambda_4$ represent increasing values of α_3^2. The "dense" set of curves in the lower left corner are branches of the curves in the larger portion of Figure 3.3–6. The one farthest from the origin is associated with $\alpha_3^2 = 0.005$ and is actually connected with the rest of the curve for $\alpha_3^2 = 0.005$. As α_3^2 increases the curves become

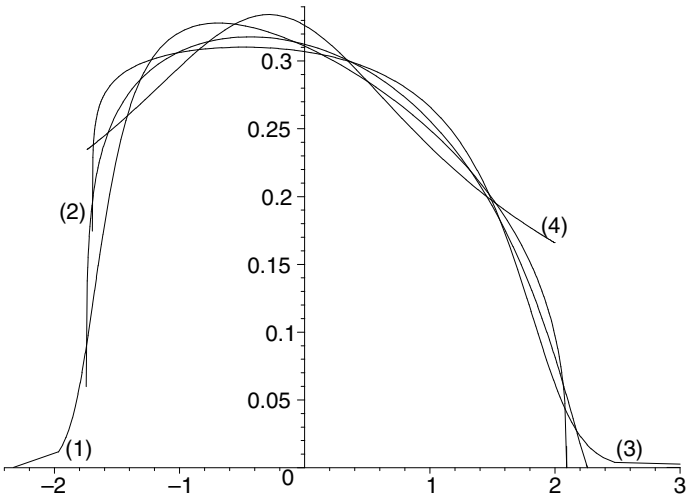


Figure 3.3–3. Four GLD fits for the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ specified by (3.3.8).

disconnected and move closer to the origin. These two families of curves (the α_4 curves of Figure 3.3–5 and the α_3^2 curves of Figure 3.3–6) are shown in Figure 3.3–7 and provide a rough guide for determining initial searching points when Region 3 solutions are sought.

The **contour plots for Region 4** with α_4 taking values

$$6,\ 6.5,\ 7,\ 8,\ 9,\ 10,\ 12,\ 14,\ 17,\ 20,\ 25,\ 30,\ 45,\ 65$$

and α_3^2 taking values

$$0.2,\ 0.4,\ 0.7,\ 1,\ 1.5,\ 2,\ 3,\ 5,\ 7,\ 10,\ 13,\ 17$$

are given in Figure 3.3–8. The curves that are open near the origin are associated with α_4 and those that are open away from the origin are the curves for α_3^2. In both cases, the smallest values (of α_4 or α_3^2) produce curves that are closest to $\lambda_3 = \lambda_4$ with the curves moving away from this line with increasing values. Figure 3.3–8 not only provides starting points for the search of solutions to (3.3.6), but it also indicates that, if a solution exists, it will be unique, except for the interchange of λ_3 and λ_4.

It seems from Figure 3.3–8 that there should be a unique (up to symmetry) GLD fit with (λ_3, λ_4) from Region 4 for the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ specified in (3.3.8). Moreover, the (λ_3, λ_4) associated with this fit should be close to the origin. Several attempts made through `RefineSearchGLDM` to find a solution in Region 4 failed (i.e., gave large values of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_3 - \alpha_3|)$), but gave an indication that if there were to be a solution, it would have to be close to the origin. Eventually,

```
> RefineSearchGLDM(A, c(−0.0002, 0), c(−0.0002, 0), 100, 4)
```

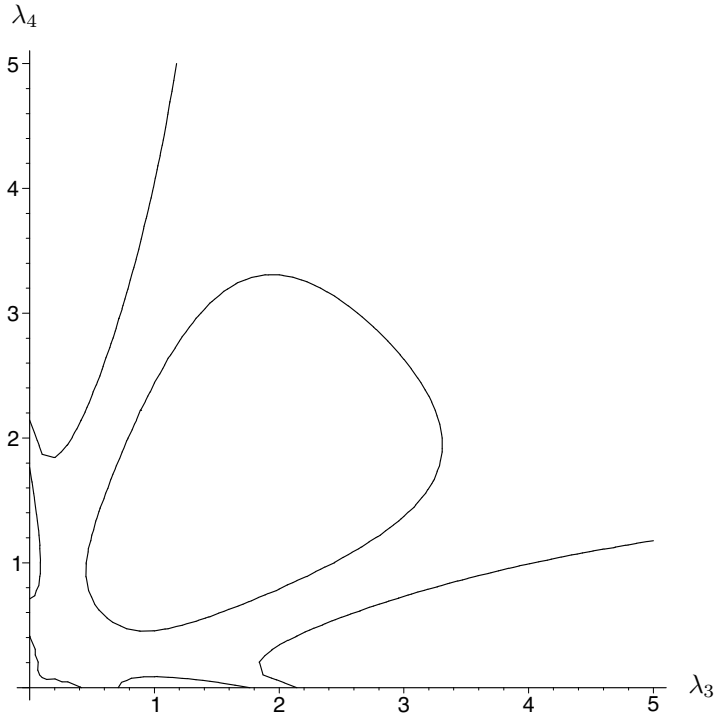


Figure 3.3–4. The contour curves for and $\alpha_3^2 = 0.5$ and $\alpha_4 = 2$.

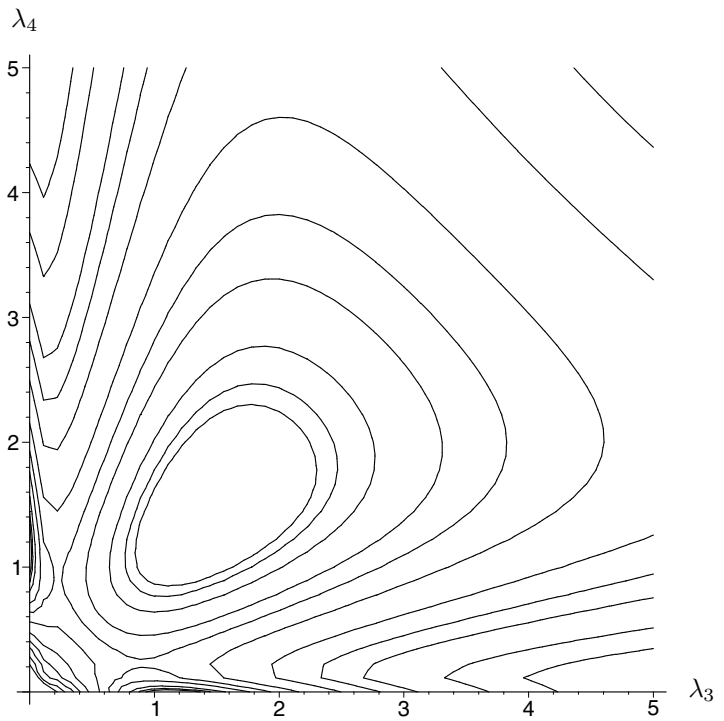


Figure 3.3–5. Contour curves of α_4 with (λ_3, λ_4) from Region 3.

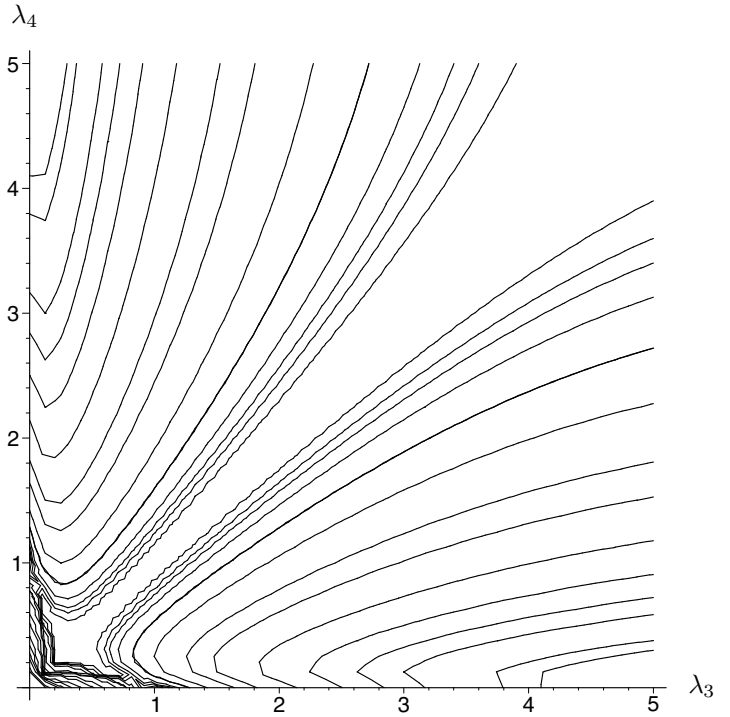


Figure 3.3–6. Contour curves of α_3^2 with (λ_3, λ_4) from Region 3.

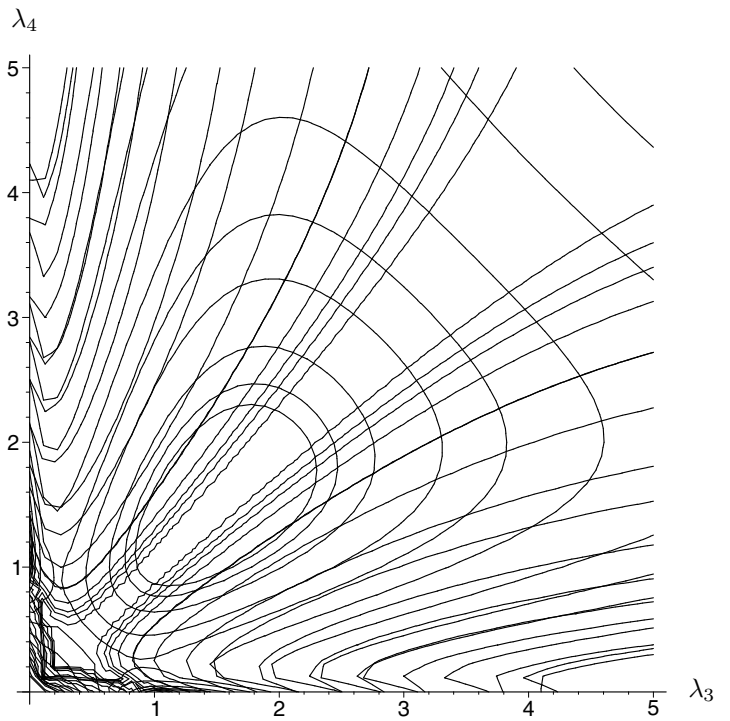


Figure 3.3–7. Contour curves of α_3^2 and α_4 with (λ_3, λ_4) from Region 3.

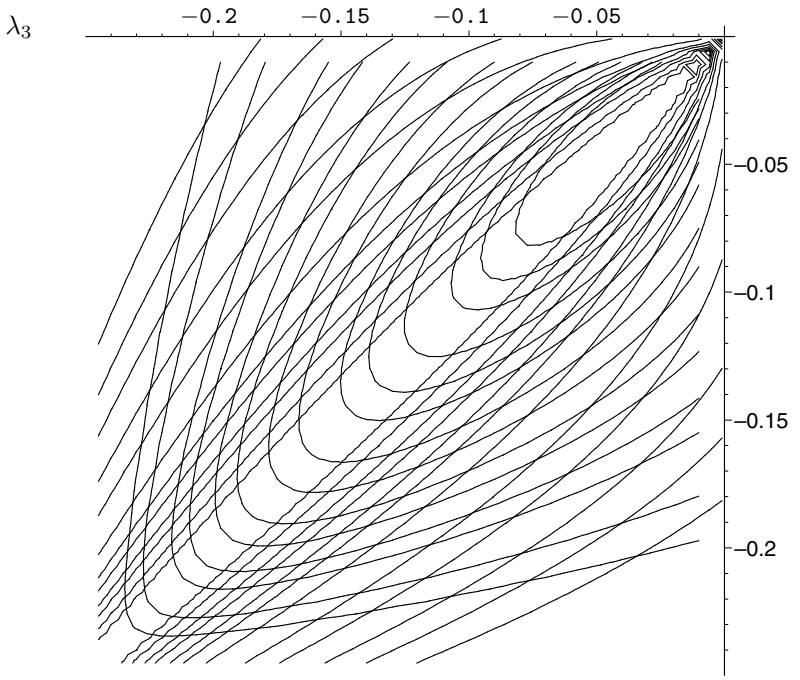


Figure 3.3–8. Contour curves of α_3^2 and α_4 with (λ_3, λ_4) from Region 4.

yielded a fifth fit,

$$\mathrm{GLD}_5(0.07325, -0.0001710, -0.0001005, -0.00008796)$$

with $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_3 - \alpha_3|) = 2.1 \times 10^{-3}$.

When search algorithms such as the one implemented in `FindLambdasM` fail, they fail because the surfaces associated with the equations have sharp corners or points where differentiability fails. This is not the case here. `FindLambdasM` and `RefineSearchGLDM` have difficulty finding a solution in Region 4 because of the proximity of the solution to the origin. When α_3 and α_4, particularly α_4, are calculated with (λ_3, λ_4) near the origin, unless *very* high levels of computational precision are used, the computational errors at intermediate levels could get magnified throughout the search path of the algorithm. This phenomenon is illustrated in Figures 3.3–9 and 3.3–10. In Figure 3.3–9 the surface α_4 is plotted using 25 digits of precision for all computations (this is well beyond the precision allowed by most hardware-based floating point operations). We can see that the surface is smooth and well-behaved. In Figure 3.3–10 the same surface is plotted with only 10 digits of precision, a rather common level of precision in most computing environments. It is clear that substantial errors are produced when (λ_3, λ_4) is near the origin. If `FindLambdasM` is unable to obtain a solution within the specified number of iterations and error tolerance, ϵ, it gives the approximate

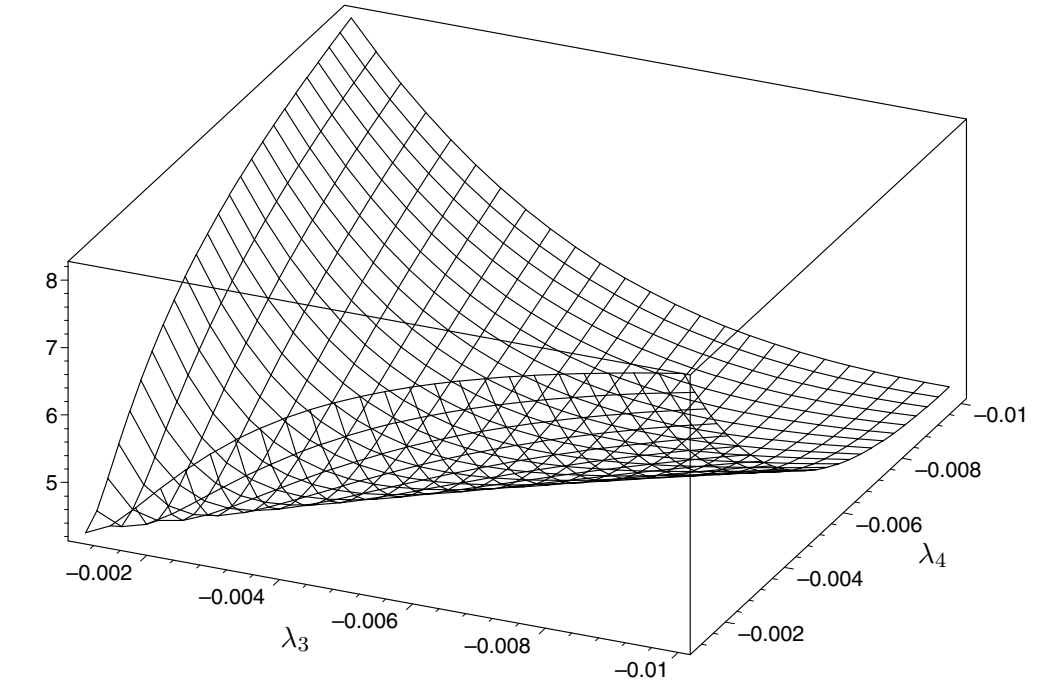


Figure 3.3–9. The surface α_4, plotted with high precision computation, near the origin in Region 4.

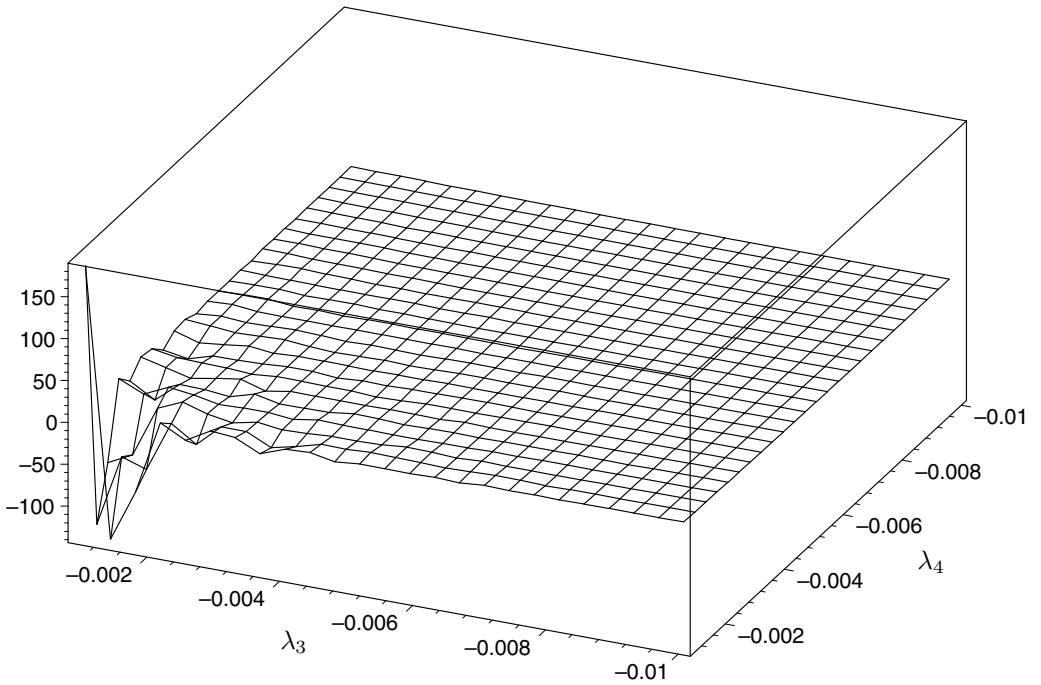


Figure 3.3–10. The surface α_4, plotted with ordinary precision computation, near the origin in Region 4.

values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that it has computed, along with an appropriate warning.

A final word of caution: when `FindLambdasM` returns $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, there is no assurance that the GLD associated with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is valid. Thus, one needs to check that (λ_3, λ_4) is in one of the valid regions of Figure 2.2–1.

3.3.2 Fitting by the Use of Tables

Some readers may not have sufficient expertise in programming or adequate programming support to use the type of analysis that was illustrated in Section 3.3.1. For this reason, a number of investigators have provided tables for the estimation of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The first of these was given by Ramberg and Schmeiser (1974); more comprehensive tables (in the sense of coverage of (α_3^2, α_4)-space) were provided subsequently by Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979); Cooley (1991) used greater computational precision to improve previous tables; and Dudewicz and Karian (1996) provide the most accurate and comprehensive tables to date. The latter are given in Appendix B. To capture as much precision as possible within the table of Appendix B, the notation a^b is used for the entries of the table to mean $a \times 10^{-b}$. For example, an entry of 0.1417^2 represents 0.001417.

Unless some simplifications are used, tabulated results for determining λ_1, λ_2, λ_3, λ_4, from $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ would require a "four-dimensional" display, a decidedly impractical undertaking. To make the tabulation manageable, we first use not $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$ but $(0, 1, |\hat{\alpha}_3|, \hat{\alpha}_4)$ and obtain a solution $(\lambda_1(0,1), \lambda_2(0,1), \lambda_3, \lambda_4)$ to (3.3.5). Note that interchanging λ_3 and λ_4 would change the signs of A and C in (3.1.12) through (3.1.15), changing the sign of α_3 and necessitating a sign change for $\lambda_1(0,1)$. Therefore, when $\hat{\alpha}_3 < 0$, we interchange λ_3 and λ_4 and change the sign of $\lambda_1(0,1)$. Next, we obtain the solution to (3.3.5) associated with $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ by setting

$$\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1 \quad \text{and} \quad \lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2}.$$

We summarize this process in the GLD–M algorithm below.

Algorithm GLD–M: Fitting a GLD distribution to data by the method of moments.

GLD–M–1. Use (3.3.1) through (3.3.4) to compute $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$;

GLD–M–2. Find the entry point in a table of Appendix B closest to $(|\hat{\alpha}_3|, \hat{\alpha}_4)$;

GLD–M–3. Using $(|\hat{\alpha}_3|, \hat{\alpha}_4)$ from Step GLD–M–2 extract $\lambda_1(0,1), \lambda_2(0,1), \lambda_3$, and λ_4 from the table;

GLD–M–4. If $\hat{\alpha}_3 < 0$, interchange λ_3 and λ_4 and change the sign of $\lambda_1(0,1)$;

GLD–M–5. Compute $\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1$ and $\lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2}$.

To illustrate the use of Algorithm GLD–M and the table of Appendix B suppose that $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ have been computed to have values

$$\hat{\alpha}_1 = 2, \quad \hat{\alpha}_2 = 3, \quad \hat{\alpha}_3 = -\sqrt{0.025}, \quad \hat{\alpha}_4 = 2. \tag{3.3.10}$$

Note that $\hat{\alpha}_4$ has been taken to be the same as, and $\hat{\alpha}_3$ has been taken to be the negative of, the previous values from (3.3.8) used in the `FindLambdasM` procedure. Step GLD–M–1 is taken care of since $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ is given. For Step GLD–M–2, we observe that $\hat{\alpha}_3 = -0.15811$; hence, the closest point to $(|\hat{\alpha}_3|, \hat{\alpha}_4)$ in the Table of Appendix B is $(0.15, 2.0)$, giving us

$$\lambda_1(0,1) = -1.3231, \quad \lambda_2(0,1) = 0.2934, \quad \lambda_3 = 0.03145, \quad \lambda_4 = 0.7203.$$

The instructions on the use of the table in Appendix B indicate that a superscript of b in a table entry designates a factor of 10^{-b}. In this case, an entry of 0.3145^1 for λ_3 indicates a value of $0.3145 \times 10^{-1} = 0.03145$. Since $\alpha_3 < 0$, Step GLD–M–4 readjusts these to

$$\lambda_1(0,1) = 1.3231, \quad \lambda_2(0,1) = 0.2934, \quad \lambda_3 = 0.7203, \quad \lambda_4 = 0.03145.$$

With the computations in Step GLD–M–5 we get

$$\lambda_1 = 4.2917, \quad \lambda_2 = 0.1694, \quad \lambda_3 = 0.7203, \quad \lambda_4 = 0.03145.$$

3.3.3 Limitations of the Method of Moments

The wide applicability of the methods developed in Sections 3.3.1 and 3.3.2 will be apparent when we use GLD distributions to approximate a number of commonly encountered distributions (Section 3.4) and when we fit GLD distributions to several datasets (Section 3.5). However, it is worth keeping in mind that most methods have limitations and we discuss the limitations associated with fitting a GLD through the method of moments here.

Algorithm GLD–M, through the table of Appendix B, enables us to fit a GLD when (α_3^2, α_4) is confined by

$$1.8(\hat{\alpha}_3^2 + 1) \leq \hat{\alpha}_4 \leq 1.8\hat{\alpha}_3^2 + 15. \tag{3.3.11}$$

The upper restriction $\hat{\alpha}_4 \leq 1.8\hat{\alpha}_3^2 + 15$ is forced on us by limitations of table space and difficulties associated with computations when this restriction is removed. We can see from Figures 3.2–1 through 3.2–3 that the GLD is capable of generating distributions with (α_3^2, α_4) beyond this constraint (see also the shaded region of possible (α_3^2, α_4) pairs in Figure 3.2–5). If needed, it is quite likely that, perhaps with some difficulty, we would be able to find a suitable fit in this region.

The lower restriction of (3.3.11), $1.8(\hat{\alpha}_3^2 + 1) \leq \hat{\alpha}_4$, is based on computational results that are depicted in Figures 3.2–1 through 3.2–3 and thus represent an

approximation of the true boundary. Recall (Theorem 3.2.4) that for all distributions we must have $\alpha_4 > 1+\alpha_3^2$. Thus, while the upper restriction of (3.3.11) may be overcome through greater computational effort, the lower restriction eliminates the possibility of fitting a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ when

$$1 + \hat{\alpha}_3^2 < \hat{\alpha}_4 < 1.8(\hat{\alpha}_3^2 + 1).$$

While analyses of actual data by Wilcox (1990), Pearson and Please (1975), and Micceri (1989), indicate that values of $|\hat{\alpha}_3|$ up to 4 and values of $\hat{\alpha}_4$ up to 50 are realistic, it is most common for data to produce (α_3^2, α_4) with

$$1 + \hat{\alpha}_3^2 < \hat{\alpha}_4 \leq 1.8\hat{\alpha}_3^2 + 15,$$

making the lower constraint of (3.3.11) a more serious limitation. In Chapter 4 we develop the EGLD system, the Extended GLD, to address this problem.

A different problem arises when we try to find a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ approximation to a distribution when (some of) the first four moments of the distribution do not exist. This type of fitting problem will also arise in a less obvious form if we encounter data that is a random sample from such a distribution. We address this difficulty in Chapter 5 by devising a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fitting method that depends on percentiles rather than moments.

In terms of a preference between the two approaches discussed in Sections 3.3.1 (direct computation) and 3.3.2 (use of tables), we note that the unavailability of the proper computing environment and, to a lesser extent, simplicity are the principle advantages of using tables. There are, however, two disadvantages:

1. Because of length limitations, existing tables provide at most one solution even when multiple solutions may exist.

2. Results obtained through Algorithm GLD–M, because of their dependence on tables, are less accurate than estimations of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by direct computation. We know, for example, from the GLD_1 fit of Section 3.3.1 that 0.7181 and 0.02982 would be more precise values for λ_3 and λ_4, respectively.

If one has access to a computational system that can provide solutions to equations like (3.3.5), it is possible to use the tables of Appendix B to determine a good starting point for the search so that an accurate solution may be obtained with considerable efficiency.

Of course, the ultimate criterion is **goodness of fit** of the fitted distribution to the true (unknown) distribution. Methods of assessing this are discussed in Section 3.5.1.

3.4 GLD Approximations of Some Well-Known Distributions

In Section 2.3 we saw the large variety of shapes that the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. can attain. For the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to be useful for fitting distributions to

data, it should be able to provide good fits to many of the distributions the data may come from. In this section we see that the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fits well many of the most important distributions.

We apply five checks on each occasion where we fit a GLD to a distribution. The **first check** considers the closeness of $\hat{f}(x)$, the approximating GLD p.d.f., to $f(x)$, the p.d.f. of the distribution being approximated. The proximity of $\hat{f}(x)$ to $f(x)$ is determined by approximating $\sup|\hat{f}(x) - f(x)|$. The p.d.f. of the distribution we will be approximating, $f(x)$, is available to us; therefore, there is no difficulty with computing $f(x)$. To compute $\hat{f}(x)$,

- we take 2249 equispaced points $y_i = i/2250$, for $i = 1, 2, \ldots, 2249$ from the interval $(0, 1)$;
- using (2.1.1) we compute $x_i = Q(y_i)$ for $i = 1, 2, \ldots, 2249$;
- using (2.1.3) we compute $\hat{f}(x_i)$ for $i = 1, 2, \ldots, 2249$.

We now use the approximation

$$\sup|\hat{f}(x) - f(x)| \approx \max_{1 \le i \le 2249} |\hat{f}(x_i) - f(x_i)|.$$

In actual practice we found that using 2249 points does not limit the accuracy of the $\sup|\hat{f}(x) - f(x)|$ that we compute. We have obtained essentially the same (i.e., within 10^{-4}) values for $\sup|\hat{f}(x_i) - f(x_i)|$ when the 2249 points have been increased to 4999 points.

For a **second check**, we look at the proximity of the approximating and approximated d.f.s, $\hat{F}(x)$ and $F(x)$, respectively. While p.d.f. differences are less easy to interpret, differences in d.f.s have an immediate meaning in the probability assigned to easy-to-interpret events. To check the closeness of $\hat{F}(x)$ to $F(x)$, we follow the same idea (and use the same points x_i) as in the computation of $\hat{f}(x)$ and use the approximation

$$\sup|\hat{F}(x) - F(x)| \approx \max_{1 \le i \le 2249} |\hat{F}(x_i) - F(x_i)|.$$

Again, in practice, $\sup|\hat{F}(x) - F(x)|$ does not change much (less than 10^{-5}) when a much larger number of points is used.

To get a quantitative measure of the quality of approximations we also consider the "distances" between $g(x)$, the p.d.f. of the distribution being approximated, and $f(x)$, the p.d.f. of the approximating GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. There is a large body of literature involving evaluation of the estimate $f(x)$ of a p.d.f. $g(x)$, through **nonnegative divergence** or **pseudodistance** measures designated by $D(f, g)$, which rely on the L_p-norm, which we now define.

Definition 3.4.1. *The L_p norm of an integrable function f over R is*

$$||f||_p = \left(\int_R |f(x)|^p \, dx \right)^{1/p}$$

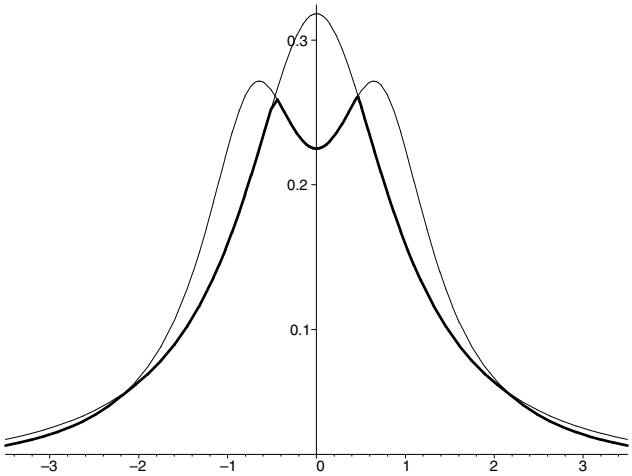


Figure 3.4–1. Relationship between the overlapping coefficient $\Delta(f, g)$ and the L_1 norm, $||f - g||_1$ of any two p.d.f.s $f(x)$ and $g(x)$.

and the L_p distance between to integrable functions f and g over R is $||f - g||_p$.

The L_p distance is a commonly used form of $D(f, g)$ (see Györfi, Liese, Vajda, and van der Meulen (1998) for details). We will concentrate on the cases $p = 1$ and $p = 2$.

The case $p = 1$ has a natural interpretation in terms of probability. The **overlapping coefficient**, $\Delta(f, g)$, of any two p.d.f.s, $f(x)$ and $g(x)$, is defined as the area that is under **both** p.d.f.s and above the horizontal axis; equivalently, $\Delta(f, g)$ is the area above the horizontal axis and below the function $\min(f(x), g(x))$. (For an introductory discussion of the history and literature of this subject, see Mishra, Shah, and Lefante (1986) and for examples, see Dudewicz and Mishra (1988).) The relationship of the L_1 distance, $||f - g||_1$, to $\Delta(f, g)$ is shown in Figure 3.4–1 where $||f - g||_1$ is the area between the two illustrated p.d.f.s and $\Delta(f, g)$ is the area under $\min(f(x), g(x))$ which is shown in heavy print. It can be seen from Figure 3.4–1 that

$$\Delta(f, g) + ||f - g||_1 = \int_{-\infty}^{\infty} \max(f(x), g(x))\, dx. \tag{3.4.2}$$

Hence, the $||f-g||_1$ and $\Delta(f, g)$ are two sides of the same coin: $||f-g||_1$ measures the difference, while $\Delta(f, g)$ measures the commonality, of two p.d.f.s

In Point 6 of Section 3.5.1 we will discuss some of the many statistical tests of the hypothesis that given data comes from a specified d.f., $G(x)$. Many of these tests rely on the sample (empirical) d.f. of the data. However, a number of the tests have an interpretation based on the closeness of an $f(x)$, the estimating p.d.f., to $g(x)$, the p.d.f. being estimated (see Dudewicz and van der Meulen (1981) for such an interpretation with the entropy test). Since it is of interest

to consider distance measures between p.d.f.s $f(x)$ and $g(x)$, based on $f(x)$ and $g(x)$ themselves (and not their d.f.s), we begin with the definition of L_1 and L_2 norms.

When there is complete agreement between the functions f and g,

$$||f - g||_1 = ||f - g||_2 = 0.$$

Moreover, because they are p.d.f.s, the integrals of f and g are equal to 1 and

$$||f - g||_1 \leq 2.$$

The integrations that lead to $||f - g||_1$ and $||f - g||_2$ must be done numerically because we do not have a closed-form expression for $f(x)$, the p.d.f. of the approximating GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The results, therefore, will be numerical approximations. The following algorithm will produce approximate L_1 and L_2 distances.

Algorithm L_1L_2: Approximations to L_1 and L_2 Distances.

L_1L_2–1. Input
- n, a positive integer ≥ 3.
- $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the parameters of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit with p.d.f. $f(x)$.
- $g(x)$, the p.d.f. of the distribution being fitted.

L_1L_2–2. Compute values $p_i = i/n$ for $i = 1, 2, \ldots, n-1$.

L_1L_2–3. Compute the $n-1$ percentile points π_p using $\pi_i = Q(p_i)$ for $i = 1, 2, \ldots, n-1$ (see (2.1.1)).

L_1L_2–4. Compute the $n-1$ y-coordinates of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f. using

$$y_i = \frac{\lambda_2}{\lambda_3 p_i^{\lambda_3 - 1} + \lambda_4 (1 - p_i)^{\lambda_4 - 1}},$$

for $i = 1, 2, \ldots, n-1$ (see (2.1.3)). The points (π_i, y_i) are on the graph of f.

L_1L_2–5. Compute the $n-1$ values on $g(x)$, the function being fitted, by $Y_i = g(\pi_i)$ for $i = 1, 2, \ldots, n-1$, making sure that when pi_i is outside of the support of $g(x)$, Y_i is assigned a value of zero. The points (π_i, Y_i) are on the graph of g.

L_1L_2–6. Let $\Delta_i = \pi_{i+1} - \pi_i$ for $i = 1, 2, \ldots, n-2$.

L_1L_2–7. Compute the sums

$$S_1 = \sum_{i=1}^{n-2} \Delta_i |y_i - Y_i| \quad \text{and} \quad S_2 = \sum_{i=1}^{n-2} \Delta_i (y_i - Y_i)^2.$$

L_1L_2–8. Compute

$$T_1 = \int_{-\infty}^{\pi_1} g(x)\,dx + \int_{\pi_{n-1}}^{\infty} g(x)\,dx$$

and

$$T_2 = \int_{-\infty}^{\pi_1} g^2(x)\,dx + \int_{\pi_{n-1}}^{\infty} g^2(x)\,dx.$$

L_1L_2–9. $S_1 + T_1$ approximates the L_1 distance between f and g and $\sqrt{S_2 + T_2}$ approximates the L_2 distance between f and g.

In Algorithm L_1L_2, S_1 and S_2 are Riemann sums for the integrals

$$\int_{\pi_1}^{\pi_{n-1}} |f(x) - g(x)|\,dx \quad \text{and} \quad \int_{\pi_1}^{\pi_{n-1}} (f(x) - g(x))^2\,dx, \tag{3.4.3}$$

respectively. Therefore, S_1 and S_2 converge to the values of these definite integrals as $n \to \infty$. Since the accumulated probability of $f(x)$ on the intervals $(-\infty, \pi_1)$ and (π_{n-1}, ∞) is $2/n$ and $f(x)$ and $g(x)$ are non-negative,

$$\begin{aligned} \int_{-\infty}^{\pi_1} & |f(x) - g(x)|\,dx + \int_{\pi_{n-1}}^{\infty} |f(x) - g(x)|\,dx \\ &\leq \int_{-\infty}^{\pi_1} (f(x) + g(x))\,dx + \int_{\pi_{n-1}}^{\infty} (f(x) + g(x))\,dx \\ &\leq \frac{2}{n} + T_1. \end{aligned} \tag{3.4.4}$$

Therefore,

$$\Big|\; ||f - g||_1 - (S_1 + T_1) \;\Big| \leq \frac{2}{n} \tag{3.4.5}$$

and

$$||f - g||_1 = \lim_{n\to\infty} (S_1 + T_1)\,. \tag{3.4.6}$$

It can be similarly established that

$$||f - g||_2 = \lim_{n\to\infty} \sqrt{S_2 + T_2}, \tag{3.4.7}$$

justifying the conclusions in Step L_1L_2–9.

We use the L_1 and L_2 distance measures, $||\hat{f} - f||_1$ and $||\hat{f} - f||_2$, as our **third and fourth checks**, respectively. For a **fifth check** we consider the closeness of the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ of the fitted GLD to the $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of the distribution that is approximated. Since in all cases `FindLambdasM` will be used to obtain the approximating GLD, we are assured, subject to a warning message from `FindLambdasM`, that both $|\hat{\alpha}_3 - \alpha_3|$ and $|\hat{\alpha}_4 - \alpha_4|$ are less than 10^{-5}. From

(3.1.12) and (3.1.13) we know that there are no difficulties associated with the computations of $\hat{\alpha}_1$ and $\hat{\alpha}_2$; therefore, we expect $|\hat{\alpha}_1 - \alpha_1|$ and $|\hat{\alpha}_2 - \alpha_2|$ to be no larger than 10^{-4}. Since the use of `FindLambdasM` takes care of this check, we do not explicitly mention it as we look for fits to distributions in the following sections.

3.4.1 The Normal Distribution

The normal distribution, with mean μ and variance σ^2 ($\sigma > 0$), $N(\mu, \sigma^2)$, has p.d.f.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty.$$

Since all normal distributions can be obtained by a location and scale adjustment to $N(0,1)$, we consider a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit to $N(0,1)$ for which

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = 0, \quad \alpha_4 = 3.$$

Appendix B suggests the existence of a solution with (λ_3, λ_4) near $(0.13, 0.13)$. Using `FindLambdasM` we obtain the approximation

$$\text{GLD}(-2.2 \times 10^{-9}, 0.1975, 0.1349, 0.1349).$$

For our **first check** of the fit, we observe that the graphs of the $N(0,1)$ and the fitted GLD p.d.f.s, given in Figure 3.4–2 (a), show the two p.d.f.s to be "nearly identical" (the $N(0,1)$ p.d.f. is slightly higher at the center). Specifically, we compute

$$\sup|\hat{f}(x) - f(x)| = 0.002812,$$

where $f(x)$ and $\hat{f}(x)$ are the p.d.f.s of the $N(0,1)$ and the fitted distributions, respectively. As a **second check** of the fit, we observe that the graphs of the $N(0,1)$ and the fitted d.f.s, given in Figure 3.4–2 (b), cannot be distinguished. Specifically,

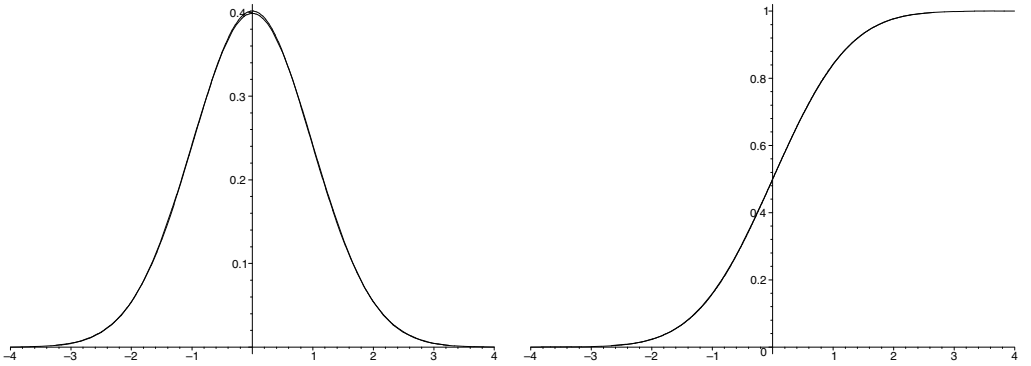
$$\sup|\hat{F}(x) - F(x)| = 0.001087,$$

where $F(x)$ and $\hat{F}(x)$ are the d.f.s of the $N(0,1)$ and the fitted distributions, respectively. This means that the probability that X is at most x differs from its approximation by no more than 0.001087 at any x.

For our **third and fourth checks** we note that the L_1 and L_2 distances for this approximation are

$$||\hat{f} - f||_1 = 0.006650 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.003231.$$

Since (λ_3, λ_4) is from Region 3, GLD$(-2.2 \times 10^{-9}, 0.1975, 0.1349, 0.1349)$ has finite support (see Theorem 2.3.23) in the form of the interval $[-5.06, 5.06]$. This may or may not be desirable — see the discussion at the beginning of this chapter.



(a) (b)

Figure 3.4–2. $N(0,1)$ with its fitted GLD; the GLD p.d.f. rises higher at the center (a). The two d.f.s are so close that they appear as a single curve (b).

3.4.2 The Uniform Distribution

The continuous uniform distribution on the interval (a, b) with $a < b$ has p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

For simplicity we consider the uniform distribution on the interval $(0, 1)$ for which

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{12}, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{9}{5}.$$

For this distribution, `FindLambdasM` yields

$$\text{GLD}(0.5, 2.0000, 1.0000, 1.0000).$$

In this case, the fit is perfect because using these values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, in (2.1.1) gives $Q(y) = y$ where $Q(y)$ is the inverse distribution function of the fitted GLD. Therefore, we must also have $F(y) = y$, matching the distribution function of the uniform distribution on the $(0, 1)$ interval. Hence, the results of our **checks** are:

$$\sup|\hat{f}(x) - f(x)| = 0, \quad \sup|\hat{F}(x) - F(x)| = 0, \quad ||\hat{f} - f||_1 = 0, \quad ||\hat{f} - f||_2 = 0.$$

For the general uniform distribution on (a, b) we have

$$\alpha_1 = \frac{a+b}{2}, \quad \alpha_2 = \frac{(b-a)^2}{12}, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{9}{5}.$$

Since α_3 and α_4 do not change, to fit the uniform distribution on (a, b) we need only adjust λ_1 and λ_2.

If X is a uniform random variable on $[a, b]$, then its distribution function is $F(x) = (x-a)/(b-a)$ (with $a \le x \le b$) and its quantile function is $Q_X(y) = a+(b-a)y$ (with $0 \le y \le 1$). Setting $Q_X(y)$ equal to the GLD quantile function gives

$$\lambda_1 + \left[y^{\lambda_3} - (1-y)^{\lambda_4}\right]/\lambda_2 = a + (b-a)y.$$

If we let $\lambda_3 = \lambda_4 = 1$, this reduces to

$$(\lambda_1 - 1/\lambda_2) + (2/\lambda_2)y = a + (b-a)y,$$

from which we get $\lambda_1 = a+1/\lambda_2$ and $\lambda_2 = 2/(b-a)$, equivalently, $\lambda_1 = (a+b)/2$ and $\lambda_2 = 2/(b-a)$. We thus have the exact GLD fit GLD$((a+b)/2,\ 2/(b-a),\ 1,\ 1)$. Note that when $a = 0$ and $b = 1$, this leads to the exact fit GLD$(1/2,\ 2,\ 1,\ 1)$ which we discovered earlier.

Setting $\lambda_3 = \lambda_4 = 1$ is not our only option. As another possibility, we can take $\lambda_3 = 1$ and $\lambda_4 = 0$ to get, by equating quantile functions,

$$\lambda_1 + (y-1)/\lambda_2 = a + (b-a)y,$$

and eventually, $\lambda_1 = b$ and $\lambda_2 = 1/(b-a)$. This leads to a second exact fit GLD$(b,\ 1/(b-a),\ 1,\ 0)$ (or, GLD$(1, 1, 1, 0)$ in the case of the uniform distribution on $[0, 1]$). Setting $\lambda_3 = 0$ and $\lambda_4 = 1$ produces yet a third GLD representation of the uniform distribution as GLD$(a,\ 1/(b-a),\ 0,\ 1)$ (or, GLD$(0, 1, 0, 1)$ in the case of the uniform distribution on $[0, 1]$).

3.4.3 The Student's t Distribution

The Student's t distribution with ν degrees of freedom, $t(\nu)$, has p.d.f.

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{(\nu+1)/2}}, \qquad -\infty < x < \infty.$$

The specification of the $t(\nu)$ p.d.f. uses the gamma function, $\Gamma(t)$, which for $t > 0$ is defined by

$$\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy.$$

Some properties of $\Gamma(t)$ will be developed in Section 3.1 (for a more detailed discussion see Artin (1964)).

The existence of the i-th moment of $t(\nu)$ depends on the relative sizes of i and ν. For the i-th moment to exist, the integral

$$\int_{-\infty}^{\infty} \frac{x^i\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{(\nu+1)/2}}\,dx$$

must converge. The power of x in the integrand is $i - \nu - 1$. Therefore, the integral will converge if and only if $i - \nu - 1 < -1$ or $i < \nu$.

Since we need the first four moments of a distribution to apply the method of moments, we can only consider t distributions with $\nu \geq 5$. (In Chapter 5 we develop other methods that can be used for approximating t distributions with small ν.) We expect that the limiting case $\nu = 5$ may be the one where a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit will be most difficult.

The first four moments of $t(\nu)$ are

$$\alpha_1 = 0, \quad \alpha_2 = \frac{\nu}{\nu - 2}, \quad \alpha_3 = 0, \quad \alpha_4 = 3\,\frac{\nu - 2}{\nu - 4} \tag{3.4.8}$$

which, when $\nu = 5$, become

$$\alpha_1 = 0, \quad \alpha_2 = \frac{5}{3}, \quad \alpha_3 = 0, \quad \alpha_4 = 9.$$

To fit $t(5)$, we appeal to `FindLambdasM` and obtain

$$\text{GLD}(0.1641 \times 10^{-9}, -0.2481, -0.1359, -0.1359).$$

For our **first check** we observe the p.d.f.s of this GLD and $t(5)$, shown in Figure 3.4–3 (a) (the one that rises higher at the center is the GLD p.d.f.), and compute

$$\sup |\hat{f}(x) - f(x)| = 0.03581.$$

Our **second check** leads us to consider the graphs of the d.f.s of $t(5)$ and its fitted distribution. These are shown in Figure 3.4–3 (b) (the $t(5)$ d.f. rises slightly higher on the left and is slightly lower on the right) and yield

$$\sup |\hat{F}(x) - F(x)| = 0.01488.$$

For our **third and fourth checks** we obtain

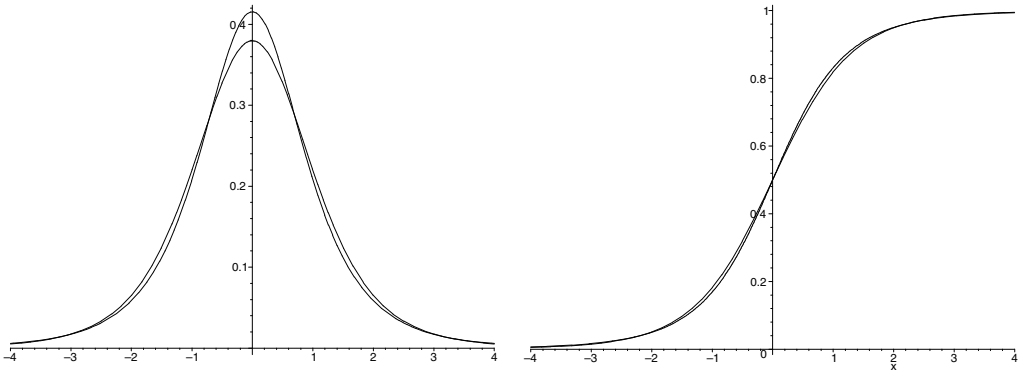
$$||\hat{f} - f||_1 = 0.006650 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.003231.$$

While this seems to be a reasonably good fit, it does not look as good as the $N(0, 1)$ fit. Also, unlike the $N(0, 1)$ fit, the support of this GLD fit is $(-\infty, \infty)$ — as it is for $t(5)$.

As ν gets large, the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fits for $t(\nu)$ get better. For $\nu = 6, 10,$ and 30 we get, respectively, the following fits.

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2.4 \times 10^{-10}, -0.1376, -0.08020, -0.08020) \text{ with}$$
$$\sup |\hat{f}(x) - f(x)| = 0.02311, \text{ and } \sup |\hat{F}(x) - F(x)| = 0.009513$$
$$||\hat{f} - f||_1 = 0.04362 \text{ and } ||\hat{f} - f||_2 = 0.02251;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.0002229, 0.02340, 0.01479, 0.01479), \text{ with}$$



(a) (b)

Figure 3.4–3. The p.d.f. and d.f. of $t(5)$ with the fitted GLD. The GLD p.d.f. rises higher at the center (a). The d.f. rises higher on the left and is lower on the right (b)).

$$\sup|\hat{f}(x) - f(x)| = 0.01039 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.004165$$
$$||\hat{f} - f||_1 = 0.02104 \text{ and } ||\hat{f} - f||_2 = 0.01065;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (3.4 \times 10^{-9}, 0.1452, 0.09701, 0.09701), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.004544 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.001766$$
$$||\hat{f} - f||_1 = 0.01009 \text{ and } ||\hat{f} - f||_2 = 0.005089.$$

3.4.4 The Exponential Distribution

The exponential distribution with parameter $\theta > 0$ has p.d.f.

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

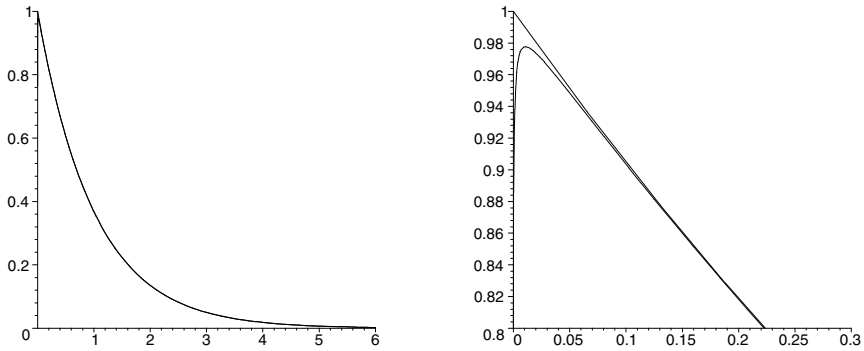
For this distribution

$$\alpha_1 = \theta, \quad \alpha_2 = \theta^2, \quad \alpha_3 = 2, \quad \alpha_4 = 9.$$

We can see that α_3 and α_4 do not change because θ is a scale parameter. Therefore, if a specific exponential distribution, say with $\theta = 1$, can be fitted, then other exponential distributions can be fitted using the λ_3 and λ_4 from the fit obtained for $\theta = 1$.

For $\theta = 1$,

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 2, \quad \alpha_4 = 9.$$



(a) (b)

Figure 3.4–4. (a) The exponential p.d.f. with $\theta = 1$ and its fitted GLD. (b) Zoomed-in version of (a).

Since $(\alpha_3, \alpha_4) = (2, 9)$ is an entry point in the table of Appendix B, we can use the table values without the concern about interpolation errors. This produces, after the adjustments mandated in Step GLD–M–5 of Algorithm GLD–M,

$$\text{GLD}(1.7240 \times 10^{-4}, 9.2756 \times 10^{-5}, 1.1840 \times 10^{-8}, 9.2766 \times 10^{-5}).$$

We consider the exponential p.d.f., with $\theta = 1$, and its fitted GLD p.d.f., plotted together in Figure 3.4–4(a) and note that the two p.d.f.s seem identical. The explanation for the surprising result

$$\sup |\hat{f}(x) - f(x)| = 0.8608$$

is that, although not evident from Figure 3.4–4 (a), the GLD p.d.f. has support $[-10781, 10781]$ and for negative values close to 0, it assumes values near 0.8608 where the exponential p.d.f. is zero. This yields a large difference over a small range for the p.d.f.s, as shown in Figure 3.4–4 (b) (though the d.f.s are very close).

When $\theta = 3$, the fitted p.d.f. has

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (3.1572 \times 10^{-4}, 3.0919 \times 10^{-5}, 1.1840 \times 10^{-8}, 9.2766 \times 10^{-5}),$$

which leads to the observation that λ_3 approaches 0 as do λ_2 and λ_4 but with the condition $\lambda_4 = \theta\lambda_2$. If we set $\lambda_3 = 0$, $\lambda_4 = \theta\lambda_2$ and then take the limit of the GLD quantile function $Q(y)$ as $\lambda_2 \to 0$, we get

$$\lim_{\lambda_2 \to 0} Q(y) = \lim_{\lambda_2 \to 0} \left(\lambda_1 + \frac{y^0 - (1-y)^{\theta\lambda_4}}{\lambda_2} \right) = \lambda_1 - \theta \ln(1-y),$$

which (when $\lambda_1 = 0$) becomes the quantile function of the exponential distribution with parameter θ. Therefore, the exponential distribution can be realized as a limiting case of GLD$(0, \lambda_2, 0, \theta\lambda_2)$, as $\lambda_2 \to 0$. This allows us, at least theoretically, to make

$$\sup |\hat{f}(x) - f(x)|, \quad \sup |\hat{F}(x) - F(x)|, \quad ||\hat{f} - f||_1, \quad ||\hat{f} - f||_2,$$

all arbitrarily close to zero.

3.4.5 The Chi-Square Distribution

The p.d.f. of the $\chi^2(\nu)$ distribution with ν degrees of freedom is given by

$$f(x) = \begin{cases} \dfrac{x^{\nu/2-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The first four moments of $\chi^2(\nu)$ are

$$\alpha_1 = \nu, \quad \alpha_2 = 2\nu, \quad \alpha_3 = 2\frac{\sqrt{2}}{\sqrt{\nu}}, \quad \alpha_4 = 3 + \frac{12}{\nu}.$$

We first illustrate the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit for $\nu = 5$ where `FindLambdasM` is used to obtain

$$\text{GLD}(2.6050, 0.01756, 0.009469, 0.05422)$$

with support $[\lambda_1 - 1/\lambda_2, \lambda_1 + 1/\lambda_2] = [-54.35, 59.56]$ (Theorem 2.3.23).

The **first check** regarding the closeness of the $\chi^2(5)$ and the fitted p.d.f.s is shown in Figure 3.4–5 (the GLD p.d.f. rises higher at the center). We can see from Figure 3.4–5 that this GLD fits $\chi^2(5)$ reasonably well, and has long but finite left and right tails. To complete this check, we determine

$$\sup |\hat{f}(x) - f(x)| = 0.02115.$$

For our **second check**, we obtain

$$\sup |\hat{F}(x) - F(x)| = 0.01357$$

but we do not illustrate the $\chi^2(5)$ and the fitted d.f.s because the two graphs cannot be visually distinguished.

The computations associated with the **third and fourth checks** yield

$$||\hat{f} - f||_1 = 0.05879 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.02444.$$

When we try to fit $\chi^2(1)$ by the methods of this chapter, we run into difficulties. For $\nu = 1$,

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 2.8284, \quad \alpha_4 = 15,$$

placing (α_3^2, α_4) well outside the range of the table of Appendix B and our computational capability. When $\nu = 2$, $\chi^2(2)$ is the same as the exponential distribution with $\theta = 2$ and fitting this distribution was covered in Section 3.4.4. There are no difficulties associated with fitting $\chi^2(\nu)$ distributions with $\nu \geq 3$. For $\nu = 3$,

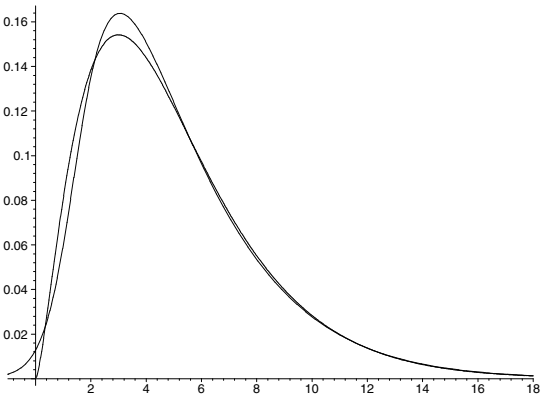


Figure 3.4–5. The $\chi^2(5)$ p.d.f. with its fitted GLD (the GLD p.d.f. rises higher at the center).

$\nu = 10$, and $\nu = 30$, we find, respectively,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.8596, 0.009543, 0.002058, 0.02300), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.05833 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.01589$$
$$||\hat{f} - f||_1 = 0.05934 \text{ and } ||\hat{f} - f||_2 = 0.04086;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (7.1747, 0.02168, 0.02520, 0.09388), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.007793 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.01182$$
$$||\hat{f} - f||_1 = 0.05188 \text{ and } ||\hat{f} - f||_2 = 0.01478;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (26.4479, 0.01896, 0.05578, 0.1366), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.002693 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.007649$$
$$||\hat{f} - f||_1 = 0.03959 \text{ and } ||\hat{f} - f||_2 = 0.007589.$$

We know that for large ν, the closeness of $\chi^2(\nu)$ to $N(\nu, 2\nu)$ would produce a reasonably good fit.

3.4.6 The Gamma Distribution

The p.d.f. of the Gamma distribution, $\Gamma(\alpha, \theta)$, with parameters $\alpha > 0$ and $\theta > 0$ is given by

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

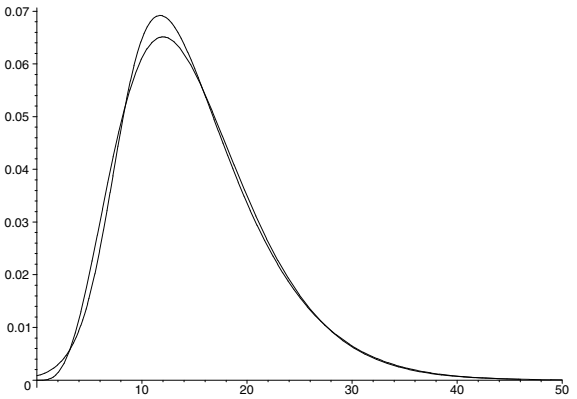


Figure 3.4–6. The gamma p.d.f. with $\alpha = 5$ and $\theta = 3$ and its fitted GLD (the GLD p.d.f. rises higher at the center).

If $\alpha = 1$, this is the exponential distribution; if $\alpha = \nu/2$ and $\theta = 2$, it is the $\chi^2(\nu)$. The $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ for $\Gamma(\alpha, \theta)$ are given by

$$\alpha_1 = \alpha\theta, \quad \alpha_2 = \alpha\theta^2, \quad \alpha_3 = \frac{2}{\sqrt{\alpha}}, \quad \alpha_4 = 3 + \frac{6}{\alpha}.$$

For the purpose of this illustration we take $\alpha = 5$ and $\theta = 3$ for which

$$\alpha_1 = 15, \quad \alpha_2 = 45, \quad \alpha_3 = 2\frac{\sqrt{5}}{5}, \quad \alpha_4 = \frac{21}{5}.$$

The GLD fit that results from `FindLambdasM` is

$$\text{GLD}(10.7621, 0.01445, 0.02520, 0.09388).$$

For the **first check**, we consider the two p.d.f.s shown in Figure 3.4–6 (the curve that rises higher in the middle is the GLD p.d.f.). This seems to be a reasonable fit for which

$$\sup |\hat{f}(x) - f(x)| = 0.005195.$$

The fit is somewhat deceptive in that its support is, by Theorem 2.3.23, the interval $[-58.43, 79.96]$. For the **second check**, we cannot distinguish the graphs of the d.f.s of $\Gamma(5, 3)$ and its fitted GLD and

$$\sup |\hat{F}(x) - F(x)| = 0.01182.$$

For our **third and fourth** checks we note that

$$||\hat{f} - f||_1 = 0.05188 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.01207.$$

3.4.7 The Weibull Distribution

A Weibull random variable with parameters $\alpha > 0$ and $\beta > 0$ has p.d.f.

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

An excellent reference on the Weibull distribution, its multivariate generalizations, as well as reliability-related distributions is Harter (1993), which contains several hundred references on the Weibull distribution alone.

For a Weibull r.v., X, it can be easily established that

$$E(X^i) = \alpha^{-i/\beta} \Gamma\left(\frac{i+\beta}{\beta}\right),$$

leading us to

$$\begin{aligned}
\alpha_1 &= \alpha^{-\beta^{-1}} \Gamma\left(\frac{\beta+1}{\beta}\right) \\
\alpha_2 &= -\alpha^{-2\,\beta^{-1}} \left(-\Gamma\left(\frac{\beta+2}{\beta}\right) + \left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^2\right) \\
\alpha_2^{3/2}\alpha_3 &= -\alpha^{-3\,\beta^{-1}} \left(-\Gamma\left(\frac{3+\beta}{\beta}\right) + 3\,\Gamma\left(\frac{\beta+2}{\beta}\right)\Gamma\left(\frac{\beta+1}{\beta}\right) - 2\left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^3\right) \\
\alpha_2^2\alpha_4 &= \alpha^{-4\,\beta^{-1}} \left(\Gamma\left(\frac{4+\beta}{\beta}\right) - 4\,\Gamma\left(\frac{3+\beta}{\beta}\right)\Gamma\left(\frac{\beta+1}{\beta}\right)\right. \\
&\quad \left. +6\,\Gamma\left(\frac{\beta+2}{\beta}\right)\left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^2 - 3\left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^4\right).
\end{aligned}$$

Note that α_3 and α_4 do not depend on the parameter α. If we take $\alpha = 1$ and $\beta = 5$, the moments of the Weibull distribution will be

$$\alpha_1 = 0.91816, \quad \alpha_2 = 0.04423, \quad \alpha_3 = -0.2541, \quad \alpha_4 = 2.8802.$$

For this distribution, `FindLambdasM` provides the GLD approximation

$$\text{GLD}(0.9935, 1.0491, 0.2121, 0.1061).$$

The support of the fitted GLD is the interval $[0.0404, 1.947]$. For the **first check** we see in Figure 3.4–7 that the Weibull and the fitted p.d.f.s are "nearly identical" and calculate

$$\sup |\hat{f}(x) - f(x)| = 0.03537.$$

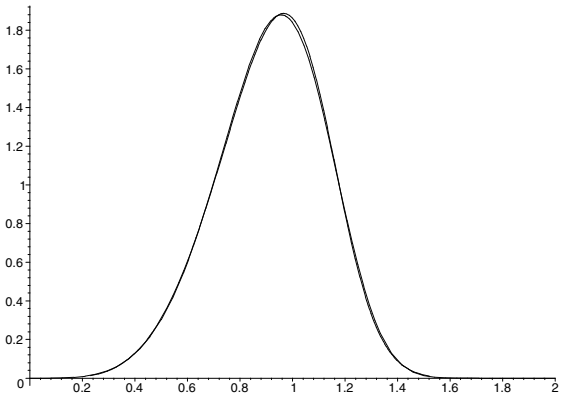


Figure 3.4–7. The Weibull p.d.f. with $\alpha = 1$ and $\beta = 5$ and its fitted GLD (the GLD p.d.f. is to the right near its peak).

As our **second check** we note that the graphs of the Weibull and the fitted distributions appear to be identical and

$$\sup |\hat{F}(x) - F(x)| = 0.002765.$$

For our **third and fourth checks** we obtain the following L_1 and L_2 measures.

$$||\hat{f} - f||_1 = 0.01477 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.01678.$$

3.4.8 The Lognormal Distribution

The p.d.f. of the lognormal distribution with parameters μ and $\sigma > 0$ is

$$f(x) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(\ln(x) - \mu)^2}{2\sigma^2}\right], & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The moments of the lognormal distributions are

$$\begin{aligned} \alpha_1 &= e^{\mu + \sigma^2/2}, \\ \alpha_2 &= (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \\ \alpha_3 &= \sqrt{e^{\sigma^2} - 1}\left(e^{\sigma^2} + 2\right), \\ \alpha_4 &= e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 3. \end{aligned}$$

In the special case when $\mu = 0$ and $\sigma = 1/3$, the moments, obtained by direct computation, are

$$\alpha_1 = 1.0571, \quad \alpha_2 = 0.1313, \quad \alpha_3 = 1.0687, \quad \alpha_4 = 5.0974.$$

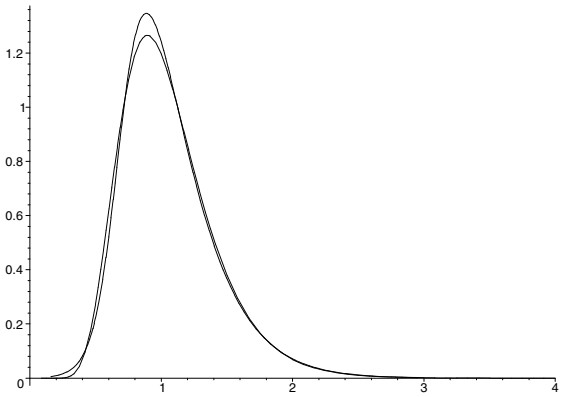


Figure 3.4–8. The lognormal p.d.f. with $\mu = 0$ and $\sigma = 1/3$ and its fitted GLD (the GLD p.d.f. rises higher at the center).

With the starting point of $(0.01, 0.03)$ indicated by Appendix B, we get, using `FindLambdasM`, the GLD fit

$$\text{GLD}(0.8451, 0.1085, 0.01017, 0.03422).$$

The support of this fit is $[-8.37, 10.06]$. For our **first check**, we plot the p.d.f.s of the lognormal with $\mu = 0$ and $\sigma = 1/3$ and the fitted GLD. This is shown in Figure 3.4–8 (the graph that rises higher at the center is the p.d.f. of $\text{GLD}(0.8451, 0.1085, 0.01017, 0.03422)$. We also compute

$$\sup |\hat{f}(x) - f(x)| = 0.09535.$$

For our **second check** we note that the d.f.s of the two distributions are virtually identical and

$$\sup |\hat{F}(x) - F(x)| = 0.01235.$$

For our **third and fourth checks** we note that

$$||\hat{f} - f||_1 = 0.05118 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.05217.$$

The choice of parameters in this, as well as previous distributions, has been quite arbitrary. Generally, GLD fits can be obtained with most, but not all, choices of parameters. In the case of the lognormal distribution, had we chosen $\mu = 0$ and $\sigma = 1$, the resulting moments would have been

$$\alpha_1 = 1.6487, \quad \alpha_2 = 4.6708, \quad \alpha_3 = 6.1849, \quad \alpha_4 = 113.9364,$$

making the search for a solution, if indeed one exists, quite difficult. In general, for (α_3^2, α_4) to be within the range of computation, we must have $0 < \sigma \leq 0.55$. When σ is small (e.g., $\sigma = 0.1$) the hazard rate of the lognormal is increasing;

when σ is moderate (e.g., $\sigma = 0.5$) it increases and then slowly decreases; when σ is large (e.g., $\sigma = 1.0$) it is decreasing. The latter case arises when $X = \ln(Y)$ and Y is $N(0,1)$, making X lognormal with $\mu = 0$ and $\sigma = 1$. Thus, it is possible to fit the GLD in the ranges of most use in reliability applications (see, for example, Nelson (1982), p. 35).

3.4.9 The Beta Distribution

A random variable has the beta distribution if for parameters β_3, $\beta_4 > -1$, it has p.d.f.

$$f(x) = \begin{cases} \dfrac{x^{\beta_3}(1-x)^{\beta_4}}{\beta(\beta_3+1, \beta_4+1)}, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The notation of β_3 and β_4 for the parameters of the beta distribution is used for reasons that will become clear in Chapter 4, when we consider a generalization of this distribution. If $\beta_3 = \beta_4 = 0$, this is the uniform distribution on $(0,1)$. The moments of the beta distribution (these will be derived in Section 3.1) are

$$\begin{aligned} \alpha_1 &= \frac{\beta_3+1}{\beta_3+\beta_4+2}, \\ \alpha_2 &= \frac{(\beta_3+1)(\beta_4+1)}{(\beta_3+\beta_4+2)^2(\beta_3+\beta_4+3)}, \\ \alpha_3 &= \frac{2(\beta_4-\beta_3)\sqrt{\beta_3+\beta_4+3}}{(\beta_3+\beta_4+4)\sqrt{(\beta_3+1)(\beta_4+1)}}, \\ \alpha_4 &= \frac{3(\beta_3+\beta_4+3)\left(\beta_3\beta_4(\beta_3+\beta_4+2)+3\beta_3^2+5\beta_3+3\beta_4^2+5\beta_4+4\right)}{(\beta_3+\beta_4+5)(\beta_3+\beta_4+4)(\beta_3+1)(\beta_4+1)}. \end{aligned}$$

If we take the specific beta distribution with $\beta_3 = \beta_4 = 1$, we get

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{20}, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{15}{7}$$

and using `FindLambdasM`, we obtain the fit

$$\text{GLD}(0.5000, 1.9693, 0.4495, 0.4495).$$

Our **first check** indicates that this is an excellent fit (with support the interval $[-.0078, 1.0078]$) to the chosen beta distribution, as can be seen from Figure 3.4–9 where the *two* p.d.f.s are indistinguishable. Moreover,

$$\sup |\hat{f}(x) - f(x)| = 0.04717.$$

The quality of this fit is confirmed by our **second check** where the d.f.s of the beta distribution with parameters $\beta_3 = \beta_4 = 1$ and its fitted GLD are virtually identical and

$$\sup |\hat{F}(x) - F(x)| = 0.0003842.$$

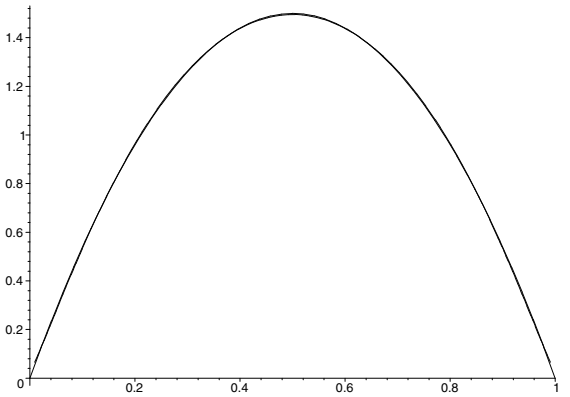


Figure 3.4–9. The beta p.d.f. with $\beta_3 = \beta_4 = 1$ and its fitted GLD (the *two* p.d.f.s are nearly indistinguishable).

For our **third and fourth checks** we have

$$||\hat{f} - f||_1 = 0.004495 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.01057.$$

In spite of this good fit, in some cases the GLD is unable to provide a good approximation to beta distributions because the (α_3^2, α_4) points of these distributions are outside the range of the GLD (they lie in the region marked X in Figure 3.2–4). Consider, for example, the beta distribution with $\beta_3 = -1/2$, $\beta_4 = 1$, for which

$$\alpha_1 = \frac{1}{5}, \qquad \alpha_2 = \frac{8}{175}, \qquad \alpha_3 = \frac{\sqrt{14}}{3}, \qquad \alpha_4 = \frac{42}{11}.$$

In this case (α_3^2, α_4) lies below the region that the GLD moments cover (see (3.3.11) and Figure 3.2–5). Only some of the (α_3^2, α_4) points in the BETA REGION of Figure 3.2–5 are in the GLD (α_3^2, α_4) region that was given in (3.3.11). This motivates (in Chapter 4) extending the GLD to cover the portion of (α_3^2, α_4)-space below the region described in (3.3.11).

3.4.10 The Inverse Gaussian Distribution

The p.d.f. of the inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$ is given by

$$f(x) = \begin{cases} \sqrt{\dfrac{\lambda}{2\pi x^3}} \exp\left[-\dfrac{\lambda(x-\mu)^2}{2\mu^2 x}\right], & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This distribution has applications to problems associated with the times required to cover fixed distances in linear Brownian motion. The reader may wish to

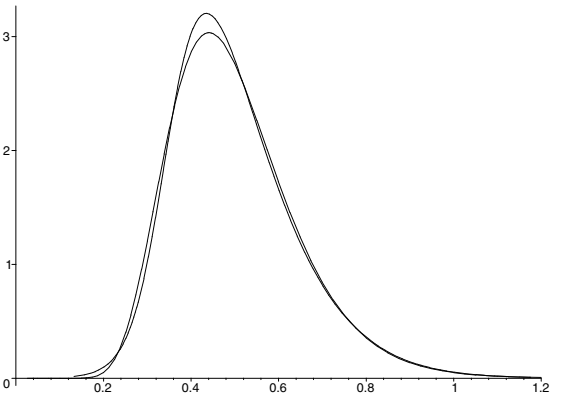


Figure 3.4–10. The inverse Gaussian p.d.f. ($\mu = 0.5$ and $\lambda = 6$) and its fitted GLD (the GLD p.d.f. rises higher at the center).

consult Govindarajulu (1987), p. 611. The moments of this distribution are

$$\alpha_1 = \mu, \quad \alpha_2 = \frac{\mu^3}{\lambda}, \quad \alpha_3 = 3\sqrt{\frac{\mu}{\lambda}}, \quad \alpha_4 = 3 + \frac{15\mu}{\lambda}.$$

We first consider the special case of $\mu = 0.5$ and $\lambda = 6$ with

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{48}, \quad \alpha_3 = \frac{\sqrt{3}}{2}, \quad \alpha_4 = \frac{17}{4}.$$

Use of `FindLambdasM` gives the fit

$$\text{GLD}(0.4164, 0.6002, 0.02454, 0.08009).$$

We note that the support of this GLD is $[-1.25, 2.08]$ and proceed to our **first check** by considering the graphs of the p.d.f.s of this inverse Gaussian distribution and its fitted GLD (given in Figure 3.4–10) and determine that

$$\sup |\hat{f}(x) - f(x)| = 0.1900.$$

For our **second check**, we note that it is impossible to graphically distinguish the d.f. of this inverse Gaussian distribution from that of its fitted GLD and

$$\sup |\hat{F}(x) - F(x)| = 0.01053.$$

The L_1 and L_2 measures for the **third and fourth checks**, are

$$||\hat{f} - f||_1 = 0.04623 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.07151.$$

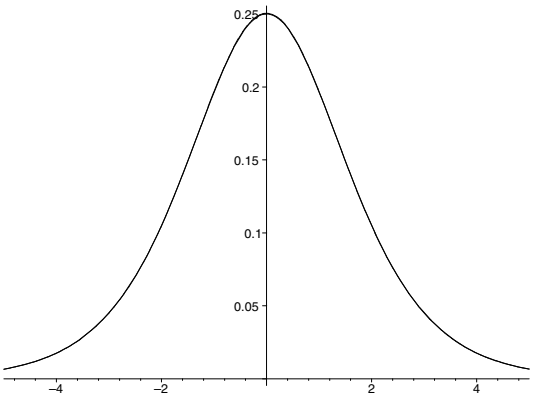


Figure 3.4–11. The logistic p.d.f. ($\mu = 0$ and $\sigma = 1$) and its fitted GLD. The two p.d.f.s cannot be distinguished.

3.4.11 The Logistic Distribution

The p.d.f. of the logistic distribution, with parameters μ and $\sigma > 0$, is given by

$$f(x) = \frac{e^{-(x-\mu)/\sigma}}{\sigma\left(1 + e^{-(x-\mu)/\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty.$$

For this distribution,

$$\alpha_1 = \mu, \quad \alpha_2 = \frac{\pi^2\sigma^2}{3}, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{21}{5}.$$

The α_3 and α_4 do not depend on the parameters of the logistic distribution and $(\alpha_3, \alpha_4) = (0, 4.2)$ is an entry in Table B–1 of Appendix B.

In the special case of $\mu = 0$ and $\sigma = 1$, we use Table B–1 of Appendix B to obtain the fit

$$\text{GLD}(-7.1 \times 10^{-6}, -0.0003246, -0.0003244, -0.0003244)$$

and note that the support of the fitted distribution is $(-\infty, \infty)$. Next, we observe that the p.d.f.s of this logistic distribution and its fitted GLD are graphically indistinguishable, as shown in Figure 3.4–11. Moreover,

$$\sup|\hat{f}(x) - f(x)| = 0.00008419.$$

This takes care of our **first check**. For our **second check**, we consider the d.f.s of the two distributions and observe that they look identical and

$$\sup|\hat{F}(x) - F(x)| = 0.00006202.$$

For our **third and fourth checks** we note that

$$||\hat{f} - f||_1 = 0.0002744 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.0001077.$$

By all of our criteria, this is an unusually good approximation and when we look at the λ_1, λ_2, λ_3, λ_4 of this approximation we see that λ_1 is very close to zero and λ_2, λ_3, λ_4, are also close to zero and are almost equal to each other. This prompts us to consider distributions of type GLD(0, λ, λ, λ) that have quantile function

$$Q(y) = \frac{y^\lambda - (1-y)^\lambda}{\lambda}.$$

It can be shown that

$$\lim_{\lambda \to 0} Q(y) = \ln\left(\frac{y}{1-y}\right)$$

converges to the quantile function of the logistic distribution with $\mu = 0$ and $\sigma = 1$, indicating that arbitrarily good approximations to the logistic with $\mu = 0$ and $\sigma = 1$ are attainable through the GLD.

Investigating GLD approximations to logistic distributions with other values of μ and σ, we conjecture that GLD(μ, λ/σ, λ, λ) should provide a good approximation to the general logistic distribution, when λ is small. To establish this result, we note that the quantile function of this GLD is given by

$$Q(y) = \mu + \sigma \frac{y^\lambda - (1-y)^\lambda}{\lambda}$$

and

$$\lim_{\lambda \to 0} Q(y) = \mu + \sigma \ln(y) - \sigma \ln(1-y) = \mu + \sigma \ln\left(\frac{y}{1-y}\right). \tag{3.4.9}$$

To obtain the d.f. of the GLD(μ, λ/σ, λ, λ) (in the limit as $\lambda \to 0$) distribution, we set the last expression in (3.4.9) to x

$$x = \mu + \sigma \ln\left(\frac{y}{1-y}\right),$$

and rearrange to obtain

$$\frac{x-\mu}{\sigma} = \ln\left(\frac{y}{1-y}\right),$$

or

$$e^{(x-\mu)/\sigma} = \frac{y}{1-y}.$$

Solving this equation for y we can represent the d.f. of GLD(μ, λ/σ, λ, λ) (in the limit as $\lambda \to 0$) by

$$F(x) = \frac{1}{1 + e^{-(x-\mu)/\sigma}}.$$

Now differentiating with respect to x gives a p.d.f that is identical to the p.d.f. of the logistic distribution with parameters μ and σ. We conclude that **arbitrarily**

good approximations to the logistic distributions can be attained by considering the GLD$(\mu, \lambda/\sigma, \lambda, \lambda)$ **when** λ **is sufficiently small**.

The logistic is used as a life distribution (see Nelson (1982)), and is not a member of the Pearson family (see Freimer, Kollia, Mudholkar, and Lin (1988), p. 3560). Since the **loglogistic** distribution deals with $Y = e^X$ where X is logistic, by taking logarithms, we can reduce the loglogistic to the logistic.

3.4.12 The Largest Extreme Value Distribution

The largest extreme value distribution, with parameters μ and $\sigma > 0$, has p.d.f.

$$f(x) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \exp\left[-e^{-(x-\mu)/\sigma}\right], \quad \text{for} \quad -\infty < x < \infty.$$

This distribution can be used to describe human life (see Nelson (1982), p. 40) and has moments

$$\alpha_1 = \mu + \gamma\sigma, \quad \alpha_2 = \frac{1}{6}\pi^2\sigma^2, \quad \alpha_3 = \sqrt{1.29857}, \quad \alpha_4 = 5.4,$$

where $\gamma \approx 0.57722$ is Euler's constant.

Since α_3 and α_4 are independent of the parameters of the distribution, we can consider fitting a specific distribution, knowing that fits for other distributions of this family can be obtained in a similar manner. If we set $\mu = 0$ and $\sigma = 1$,

$$\alpha_1 = .5772, \quad \alpha_2 = 1.6449, \quad \alpha_3 = 1.1395, \quad \alpha_4 = 5.4.$$

Using `FindLambdasM`, we obtain the fit

$$\text{GLD}(-0.1859, 0.02109, 0.006701, 0.02284)$$

and note that the support of this GLD is $[-47.647, 47.275]$. We compare the p.d.f. of this distribution and that of its fitted GLD. This comparison, illustrated in Figure 3.4–12, yields

$$\sup|\hat{f}(x) - f(x)| = 0.02216$$

for our **first check**. Next, we compare the d.f.s of the extreme value distribution with $\mu = 0$ and $\sigma = 1$ to observe that the d.f.s cannot be visually distinguished and

$$\sup|\hat{F}(x) - F(x)| = 0.01004,$$

completing our **second check**. The **third and fourth checks** indicate

$$||\hat{f} - f||_1 = 0.04093 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.02284.$$

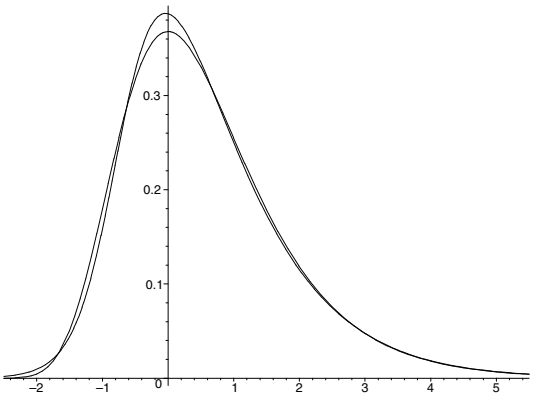


Figure 3.4–12. The largest extreme value p.d.f. ($\mu = 0$ and $\sigma = 1$) and its fitted GLD (the GLD p.d.f. rises higher at the center).

3.4.13 The Extreme Value Distribution

This distribution, also called the **smallest extreme value distribution**, is important in some applications. For example, Weibull (1951) reported that the strength of a certain material follows an extreme value distribution with $\mu = 108\text{kg/cm}^2$ and $\sigma = 9.27\text{kg/cm}^2$ (also see Nelson (1982), p. 41).

Also, if X is Weibull, then $\ln(X)$ has an extreme value distribution, making this relationship similar to the one between the lognormal and the normal distributions. It is also known (see Kendall and Stuart (1969), pp. 85, 335, 344) that if X has the extreme value distribution then the p.d.f. of $(X - \mu)/\sigma$ is $f(x) = e^{-x-e^{-x}}$. The extreme value distribution is also used in life and failure data analysis, "weakest link" situations, temperature minima, rainfall in droughts, human mortality of the aged, etc., and often represents the first failure (which fails the unit).

The extreme value distribution has parameters μ and $\sigma > 0$ and p.d.f.

$$f(x) = \frac{1}{\sigma} e^{(x-\mu)/\sigma} \exp\left[-e^{(x-\mu)/\sigma}\right], \quad \text{for} \quad -\infty < x < \infty.$$

If Y is extreme value with parameters μ_Y and σ_Y, **then $X = -Y$ is largest extreme value** with parameters $\mu_X = -\mu_Y$ and $\sigma_X = \sigma_Y$. This follows since

$$P(Y \le y) = 1 - e^{-e^{(y-\mu_Y)/\sigma_Y}}.$$

Hence,

$$\begin{aligned} P(X \le x) &= P(-Y \le x) = P(Y \ge -x) = 1 - P(Y \le -x) \\ &= 1 - \left(1 - e^{-e^{(-x-\mu_Y)/\sigma_Y}}\right) = e^{-e^{-(x+\mu_Y)/\sigma_Y}} = e^{-e^{-(x-\mu_X)/\sigma_X}}. \end{aligned}$$

The moments of the two distributions have the relation

$$\alpha_1(X) = -\alpha_1(Y), \quad \alpha_2(X) = \alpha_2(Y), \quad \alpha_3(X) = -\alpha_3(Y), \quad \alpha_4(X) = \alpha_4(Y).$$

Thus, **to fit a largest extreme value with moments** $(5, 16, 0.5, 9.2)$ **is the same as fitting an extreme value with moments** $(-5, 16, -0.5, 9.2)$**.** As we have already considered fitting a largest extreme value distribution, Section 3.4.12 has the details for fitting an extreme value distribution.

The moments of the distribution are

$$\alpha_1 = \mu - \gamma\sigma, \quad \alpha_2 = \frac{1}{6}\pi^2\sigma^2, \quad \alpha_3 = -\sqrt{1.29857}, \quad \alpha_4 = 5.4,$$

where $\gamma \approx 0.57722$ is Euler's constant.

3.4.14 The Double Exponential Distribution

The double exponential distribution with parameter $\lambda > 0$ has p.d.f.

$$f(x) = \frac{e^{-|x|/\lambda}}{2\lambda} \quad \text{for} \quad -\infty < x < \infty.$$

The moments of this distribution are

$$\alpha_1 = 0, \quad \alpha_2 = 2\lambda^2, \quad \alpha_3 = 0, \quad \alpha_4 = 6.$$

Since α_3 and α_4 are constants, we will be able to find GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fits to this distribution if we can find a fit when $\lambda = 1$ and the moments are

$$\alpha_1 = 0, \quad \alpha_2 = 2, \quad \alpha_3 = 0, \quad \alpha_4 = 6.$$

In this situation, `FindLambdasM` provides the fit

$$\text{GLD}(2.8 \times 10^{-10}, -0.1192, -0.08020, -0.08020).$$

Note that the support of this GLD is $(-\infty, \infty)$. We now compare the p.d.f.s of the double exponential distribution with $\lambda = 1$ to that of the fitted GLD. The graphs of these p.d.f.s are given in Figure 3.4–13 (a) where the graph of the double exponential p.d.f. rises higher in the center. Our observation that

$$\sup |\hat{f}(x) - f(x)| = 0.1485$$

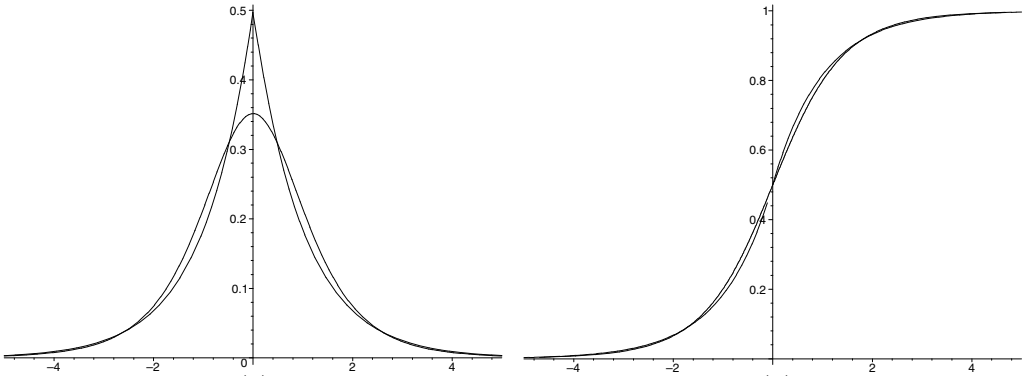
completes the **first check**. For the **second check**, we compute

$$\sup |\hat{F}(x) - F(x)| = 0.02871$$

and obtain a plot of the d.f.s of the two distributions, shown in Figure 3.4–12 (b). Note that in contrast to most of the fits provided so far, there is an observable difference between the two d.f.s (the double exponential d.f. is lower immediately to the left of the center and higher immediately to the right of the center).

For our **third and fourth checks** we note that

$$||\hat{f} - f||_1 = 0.1292 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.081727151.$$



(a) (b)

Figure 3.4–13. The p.d.f.s (a) and d.f.s (b) of the double exponential ($\lambda = 1$) and its fitted GLD.

3.4.15 The F-Distribution

The p.d.f. of the F distribution, with $\nu_1 > 0$ and $\nu_2 > 0$ degrees of freedom, $F(\nu_1, \nu_2)$, is given by

$$f(x) = \frac{\Gamma\left((\nu_1+\nu_2)/2\right)(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \times \frac{x^{(\nu_1-2)/2}}{\left(1+x\frac{\nu_1}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}$$

when $x > 0$ and $f(x) = 0$ when $x \leq 0$.

The power of x in the p.d.f. of the F distribution is $-1 - \nu_2/2$. Therefore for the i-th moment to exist, we must have $i - 1 - \nu_2/2 < -1$ or $\nu_2 > 2i$. This immediately restricts the moment-based GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fits to those F distributions with $\nu_2 > 8$.

The moments of the F distribution (when $\nu_2 > 8$), are

$$\begin{aligned}
\alpha_1 &= \frac{\nu_2}{\nu_2 - 2}, \\
\alpha_2 &= 2\,\frac{{\nu_2}^2\,(\nu_1+\nu_2-2)}{\nu_1\,(\nu_2-4)\,(\nu_2-2)^2}, \\
\alpha_3 &= \frac{2\sqrt{2}\,(\nu_2-2+2\nu_1)}{\nu_1\,(\nu_2-6)\,\sqrt{\frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)}}}, \\
\alpha_4 &= 3\,\frac{(\nu_1{\nu_2}^2+4\,{\nu_2}^2+8\,\nu_1\nu_2+{\nu_1}^2\nu_2-16\,\nu_2+10\,{\nu_1}^2-20\,\nu_1+16)\,(\nu_2-4)}{\nu_1\,(\nu_1+\nu_2-2)\,(\nu_2-6)\,(\nu_2-8)}.
\end{aligned}$$

If we set $\nu_2 = 10$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ will exist. However, if we also choose $\nu_1 = 5$, then we have $(\alpha_3^2, \alpha_4) = (14.9538, 53.8615)$, taking us outside the range of the

table in Appendix B as well as outside our range of computation. α_3^2 and α_4 both decrease with increasing ν_1, but when $\nu_2 = 10$, this does not help since

$$\lim_{\nu_1 \to \infty} \alpha_3^2 = 12, \qquad \lim_{\nu_1 \to \infty} \alpha_4 = 45,$$

and (α_3^2, α_4) stays out of range for all $\nu_1 > 0$.

More generally, by considering α_3^2 as a function of ν_1 and ν_2 and taking the derivative of $\alpha_3^2(\nu_1, \nu_2)$ with respect to ν_2, we can determine that $\alpha_3^2(\nu_1, \nu_2)$ decreases with increasing ν_2 when $\nu_2 > 8$. This implies that for $8 \leq \nu_2 \leq 15$,

$$\alpha_3^2(\nu_1, \nu_2) \geq \alpha_3^2(\nu 1, 15) = \frac{88(2\nu_1 + 13)^2}{81(\nu_1(\nu_1 + 13)}.$$

This last expression is a decreasing function of ν_1. Hence,

$$\alpha_3^2(\nu_1, \nu_2) \geq \lim_{\nu_1 \to \infty} \frac{88(2\nu_1 + 13)^2}{81(\nu_1(\nu_1 + 13)} \approx 4.35,$$

placing (α_3^2, α_4) outside of our computational range whenever $\nu_2 \leq 15$.

With similar analyses, we can determine that in order to have $\alpha_3^2 \leq 4$ when $\nu_2 = 16$, we must have $\nu_1 \geq 28$; when $\nu_2 = 20$, we must have $\nu_1 \geq 6.26$; and when $\nu_2 = 30$, we must have $\nu_1 \geq 3.52$.

It is, therefore, possible to obtain fits for large ν_2. For example, if we let $\nu_2 = 25$ and $\nu_1 = 6$, we get

$$\alpha_1 = 1.0869, \quad \alpha_2 = 0.5439, \quad \alpha_3 = 1.8101, \quad \alpha_4 = 9.1986$$

and the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit

$$\text{GLD}(0.6457, -0.06973, -0.01100, -0.04020)$$

by using the entries of Appendix B. We observe that the support of this fit is $(-\infty, \infty)$. For the **first check**, we compare the p.d.f. of $F(6, 25)$ with that of its GLD fit (see Figure 3.4–14) and compute

$$\sup |\hat{f}(x) - f(x)| = 0.1361.$$

For the **second check**, we note that d.f.s of $F(6, 25)$ and the fitted GLD cannot be visually distinguished and

$$\sup |\hat{F}(x) - F(x)| = 0.02684.$$

The L_1 and L_2 distances for this approximation are

$$||\hat{f} - f||_1 = 0.09612 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.08323.$$

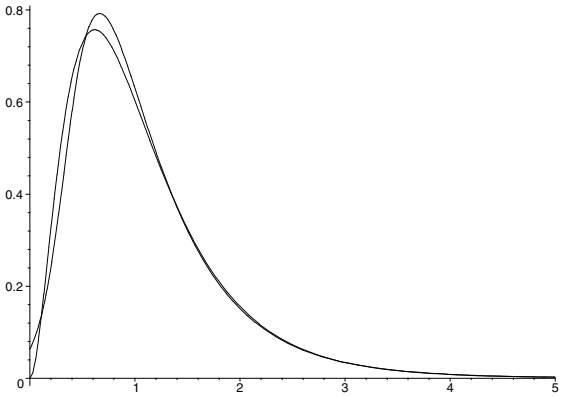


Figure 3.4–14. The p.d.f.s of $F(6, 25)$ and its fitted GLD (the one that rises higher at the center is the p.d.f. of the fitted GLD).

In similar ways, we can get approximations for some other F-distributions. Specifically, for $(\nu_1, \nu_2) = (6, 12)$ and $(6, 16)$, we have, respectively,

$$\begin{aligned}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.7192, -0.2705, -0.05458, -0.1581), \text{ with} \\ \sup|\hat{f}(x) - f(x)| = 0.21138 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.04733 \\ ||\hat{f} - f||_1 = 0.2121 \text{ and } ||\hat{f} - f||_2 = 0.14566;\end{aligned}$$

$$\begin{aligned}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.6193, -0.1731, -0.02495, -0.1041), \text{ with} \\ \sup|\hat{f}(x) - f(x)| = 0.1347 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.02428 \\ ||\hat{f} - f||_1 = 0.08817 \text{ and } ||\hat{f} - f||_2 = 0.07911.\end{aligned}$$

3.4.16 The Pareto Distribution

The Pareto distribution, with parameters $\beta > 0$ and $\lambda > 0$, has p.d.f.

$$f(x) = \begin{cases} \dfrac{\beta\lambda^\beta}{x^{\beta+1}}, & \text{if } x > \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The moments of the Pareto distributions are

$$\begin{aligned}\alpha_1 &= \frac{\beta\,\lambda}{\beta - 1}, \\ \alpha_2 &= \frac{\beta\,\lambda^2}{(\beta-1)^2\,(\beta-2)},\end{aligned}$$

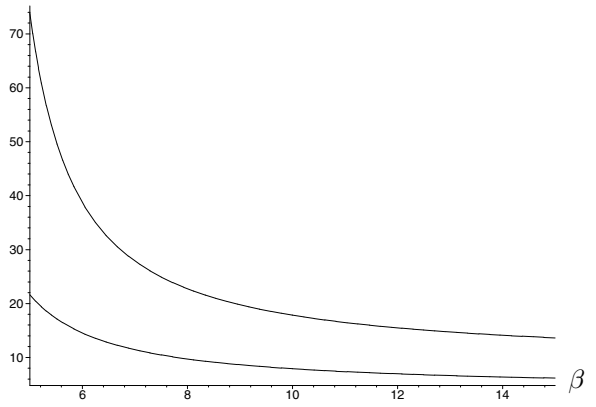


Figure 3.4–15. The α_3^2 and α_4 of the Pareto distribution as functions of β (α_4 is the higher curve).

$$\alpha_3 = \frac{2\,(\beta+1)}{(\beta-3)\sqrt{\frac{\beta}{\beta-2}}},$$

$$\alpha_4 = 3\,\frac{(3\,\beta^2+\beta+2)\,(\beta-2)}{\beta\,(\beta-3)\,(\beta-4)}.$$

We see that α_3 and α_4 depend only on the single parameter β and for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to exist, we must have $\beta > 4$. For almost all reasonable choices of β, α_3 and α_4 are out of range for us to be able to compute $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. For example, when $\beta = 5$, $\alpha_3 = 4.6476$ and $\alpha_4 = 73.8000$. As β increases both α_3 and α_4 decrease. However,

$$\lim_{\beta\to\infty} \alpha_3 = 2 \quad \text{and} \quad \lim_{\beta\to\infty} \alpha_4 = 9$$

so β has to get quite large for α_3 and α_4 to come near our tabulated range. This can be seen from the graphs of α_3^2 and α_4 given in Figure 3.4–15 where α_4 is the higher of the two curves.

It is worthwhile to point out that despite the difficulties associated with finding fits for the Pareto distribution, this distribution can be realized as a special case of the GLD. In fact, for all $\beta > 4$, the Pareto distribution with parameters λ and β is identical to

$$\text{GLD}\left(\lambda, \frac{-1}{\lambda}, 0, \frac{-1}{\beta}\right).$$

This can be seen by comparing the quantile function of the Pareto with parameters λ and β with that of the GLD with parameters λ, $-1/\lambda$, 0, $-1/\beta$; both produce the quantile function

$$Q(y) = \lambda(1-y)^{-1/\beta}.$$

3.5 Examples: GLD Fits of Data, Method of Moments

We have already seen details of the shapes of GLD distributions (Section 2.4), a GLD fitted to an actual dataset (the fit to measurements of the coefficient of friction of a metal at (2.1.6)), and fits to many of the most important distributions encountered in applications in various areas.

In this section we consider **fits of the GLD to actual datasets**. A number of examples will be given, **from a variety of fields**, in order to illustrate a spectrum of nuances that arise in the fitting process. In this section, we will fit datasets for which the estimated (α_3^2, α_4) pair (denoted by $(\hat{\alpha}_3^2, \hat{\alpha}_4)$) is in the region covered by the GLD that was given in Figure 3.2–4. The reasons for this restriction are twofold: first, as we saw in Section 3.4.9, when we attempt to fit distributions that have (skewness^2, kurtosis) that is not in the area covered by the GLD (wide though this is) the fit will usually not be excellent; second, in Chapter 4 we will extend the GLD to an EGLD that covers the (α_3^2, α_4) points not covered by the GLD so it makes sense to consider such examples in Chapter 4.

It is true that there is variability of sampling in the estimates of (skewness^2, kurtosis): even if the true (α_3^2, α_4) point is in the GLD region, the estimate from the data might not be or if the true point is not in the GLD region, the estimate from the data might be. Thus, one may, in applications, end up fitting by a model that cannot cover the true distribution very well. For this reason, when we take up these additional examples in Chapter 4, after we extend the GLD to the EGLD, we will attempt to fit both a GLD and an EGLD and compare the two fits.

We should also note, as we will see in detail in Chapter 4, that the method of extension of the GLD is such that there is a zone of overlap of the (α_3^2, α_4) points of the two models. This means that we in fact have a zone where both model types will fit the data well, so "wrong zone" (skewness^2, kurtosis) should be less of a problem in applications when $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ is "near" the boundary.

In the examples we discuss we have both real datasets from the literature, and datasets with simulated (Monte Carlo) data. While with a real dataset we can assess how well the model fits the data, we cannot assess how well it fits the underlying true distribution (as that distribution is not known). With simulated datasets we can do both, which is why we have included them.

3.5.1 Assessment of Goodness-of-Fit

There are a number of aspects of "goodness-of-fit," a topic on which many papers and books have been written that are relevant to the subject area of this book. We will cover them mainly in this section. **The situation** we have **is as follows:**

S–1. We are seeking to model a phenomenon X of interest in some area of research.

S–2. There are certain distributions $F_1, F_2, \ldots, F_k$ that we wish to include, in some form, among those we wish to consider to describe X. These may be ones that it is believed X truly follows (e.g., they are derived from assumptions X is believed to obey), or they may simply be ones that yielded reasonable approximations in the previous studies.

S–3. We have available data on the phenomenon X in some form, for example,

1. Independent observations $X_1, X_2, \ldots, X_n$ on the phenomenon.
2. A histogram of the distribution of X based on data (but the data itself is not available to us).
3. The sample moments $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ based on data (but we do not have the data).
4. Some other form of information based on data is available.

S–4. We are considering using a family of distributions $G(\omega)$ to fit the phenomenon's p.d.f./d.f./p.f., where ω is the vector of parameters, which (when chosen) picks a particular member of the family $G(\omega)$. For example, if G is the GLD family then $\omega = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

We **make the following points** regarding the circumstances described in S–1 through S–4.

Point 1. In most cases, **the F_is of S–2 will merely be approximations to a true (and unknown) distribution** of X. If they are truly the distributions that X follows, and have one or more unknown parameters, then one should act as though X is in the family $F_i(\omega)$ and use techniques which optimally choose ω for that family.

Point 2. In light of Point 1 **it is not necessarily required that the fitted G distribution have all of the characteristics of** F_i of S–2. For example, in the first paragraph of this chapter we noted that often the support (range of possible values) of F_i may be ones we know cannot occur; in such a setting we certainly do not want to require that G have the same support as F_i.

Point 3. In light of S–1, we will **want the family $G(\omega)$ to be one that can come reasonably close to including** $F_1, F_2, \ldots, F_k$ when ω is chosen appropriately. For this reason, in Section 3.4 we described how close the GLD family can come to various widely used distributions, using a fit based on matching the first four moments of the GLD to the first four moments of the particular F_i. We might also fit with the EGLD of Chapter 4, or with a Method of Percentiles as in Chapter 5, especially if we did not find the fit to F_i satisfactory. **If F_i is**

fitted with G_0, then the question arises: Is G_0 a good approximation to F_i? Some ways to approach this question include

- Assess the adequacy and appropriateness of the support of G_0.
- Compare the p.d.f.s, d.f.s, and p.f.s of G_0 with those of F_i.
- Compare G_0 with F_i on other measures that "matter" for the application under consideration; for example, moments, probabilities of key events, hazard functions, and so on.

These are not simple questions for which there is a simple formula that can provide a correct answer — they are some of the most difficult questions that arise in modeling, and usually require serious interaction between the statistician and the subject matter specialists — but it is essential that they be carefully and fully considered if we are to have confidence in the model.

Point 4. With data of any of the forms described in S–3, we can use the Method of Moments to fit a GLD. In case of S–3–2, the moments are approximated using the midpoint assumption; with S–3–1 and S–3–2, the Method of Percentiles (presented in Chapter 5) can be used to fit a GLD; in case of S–3–2, the percentiles needed are approximated (an analog of case S–3–3 is to have available certain sample percentiles). What can be done with situation S–3–4 depends on the specifics of the information available.

Point 5. After we fit a distribution G to the data the question will arise: **Is this a good fit to the true underlying distribution?** Since the true distribution is unknown, this is a difficult question. In case S–3–3 we can reasonably ask that G have values close to the specified sample moments (or, if sample percentiles were given, to the sample percentiles). In case S–3–2 we should **overplot the histogram of the data and p.d.f. g of the d.f. G** and examine key shape elements. In case S–3–1 we can **overplot the d.f. G and the empiric d.f. of the data** and examine their closeness.

Point 6. There are many **statistical tests of the hypothesis** where the data, $X_1, X_2, \ldots, X_n$, come from the distribution G. These are in addition to **the eyeball test**, which should always be used (and in which many experimenters place the most faith). Based on the theorem that follows we assert: The hypothesis that the data come from G is **equivalent to the hypothesis that $G(X_1), G(X_2), \ldots, G(X_n)$ are independent uniform r.v.s on $(0, 1)$.**

Theorem 3.5.1. *If a r.v. Y has continuous d.f. $H(y)$, then the r.v. $Z = H(Y)$ has the uniform distribution on $(0, 1)$.*

There are many statistical tests for this hypothesis such as the Kolmogorov–Smirnov D, Cramér–von Mises W^2, Kuiper V, Watson U^2, Anderson–Darling A^2, log–statistic Q, χ^2, entropy, as well as other tests of uniformity. Some references

are Dudewicz and van der Meulen (1981) (where the entropy test is developed, and which has references to the literature) and Shapiro (1980) (which, while oriented to testing normality, has excellent comments on goodness-of-fit testing in general).

Of the goodness-of-fit tests, the **oldest is the chi-square test** proposed by Pearson (1900). The idea of the test is to divide the range of the distribution into k **cells** and compare the **observed number** in each cell to the number that would be **expected** if the assumed distribution is true. Under certain construction of the cells, the resulting **test statistic has (approximately) a chi-square distribution with degrees of freedom** $k - 1 - t$ where t is the number of parameters estimated (e.g., $t = 4$ for the GLD). Drawbacks of the test are

- low power (ability to reject an incorrect fit) vs. other tests;
- loss of information when the data are grouped into cells;
- arbitrariness of the choice of the cells.

For reference on the chi-square test, see Moore (1977). We illustrate the chi-square test in the examples that we consider below.

We should note **why some experimenters place the most faith in the eyeball test**, though they may also do a formal test of the hypothesis:

- the chi-square test itself is approximate (and assumes use of the method of maximum likelihood, rather than the method of moments);
- for any test, failure to reject does not mean the hypothesis is true; it could, for example, be that the sample size is too small to detect differences that exist between the true and hypothesized models;
- for any test, rejection does not mean that the hypothesized model is inappropriate for the purpose intended.

However, the eyeball test requires experience. Therefore, **we recommend starting with a hypothesis test and** if we then wish to go against the conclusion of the test based on examination of the overplot of hypothesis and data, **make the case for reversal**. This is analogous to a lawyer making a case at an appellate level. As Gibbons (1997, p. 81) states, "the investigator hopes to be able to accept the null hypothesis, even when it appears to be only nearly true." The eyeball test is not a license to follow a personal whim, and it is important to guard against any lack of rigor in so important a decision as choice of distributional model for the phenomenon under study. The eyeball test is sometimes called a **graphical test** (see Ricer (1980), p. 18). If the plot is of the e.d.f. and the fitted G, we might look for criss-crossing of the two rather than one staying consistently on one side of the other. We should always be aware that there is subjectivity to the analysis.

The **Kolmogorov–Smirnov** or KS test is based on the largest difference (in absolute value) between the e.d.f. and its hypothesized counterpart. This can be interpreted as a quantification of the eyeball test. Only the biggest deviation is used; if one curve is uniformly on one side of the other but does not reach very far away at any one point, the test will not reject; for this reason, the test is sometimes viewed as lacking in power. In each of the examples that follow, if we are able to obtain a fit, we give the KS statistic for that fit. Since it is more informative to have the p-values for such tests, the p-values (obtained from the table in Appendix H) are also given. A good comparison of the KS and chi-square tests is given by Gibbons (1997, pp. 80–82). The use of the KS statistic to give a confidence interval for the true d.f. (see Gibbons (1997), Section 3.2) is also of some interest.

There is a large literature on quantification of eyeball tests; one article, with references, worth considering is Gan, Koehler, and Thompson (1991).

3.5.2 Example: Cadmium in Horse Kidneys

Elinder, Jönsson, Piscator and Rahnster (1981) investigated histopathological changes due to different levels of metal concentrations in horse kidneys. We list below part of their data dealing with the average cadmium concentrations in the kidney cortex of horses.

11.9	16.7	23.4	25.8	25.9	27.5	28.5	31.1
32.5	35.4	38.3	38.5	41.8	42.9	50.7	52.3
52.5	52.6	54.5	54.7	56.6	56.7	58.0	60.8
61.8	62.3	62.5	62.6	63.0	67.7	68.5	69.7
73.1	76.0	76.9	77.7	78.2	80.3	93.7	101.0
104.5	105.4	107.0					

For this data, using the program `FitGLDM` we find

$$\hat{\alpha}_1 = 57.2442, \quad \hat{\alpha}_2 = 576.0741, \quad \hat{\alpha}_3 = 0.2546, \quad \hat{\alpha}_4 = 2.5257$$

and obtain the GLD fit

$$\text{GLD}(41.7897, 0.01134, 0.09853, 0.3606)$$

whose support is $[-46.355, 129.935]$.

Figure 3.5–1 (a) shows the p.d.f. of the fitted GLD with a histogram of the data; Figure 3.5–1 (b) shows the e.d.f. of the data with the d.f. of the fitted GLD. These figures indicate a reasonably good fit which is substantiated with a chi-square test. The test uses the classes

$$(-\infty, 30), \quad [30, 50), \quad [50, 60), \quad [60, 70), \quad [70, 85), \quad [85, \infty)$$

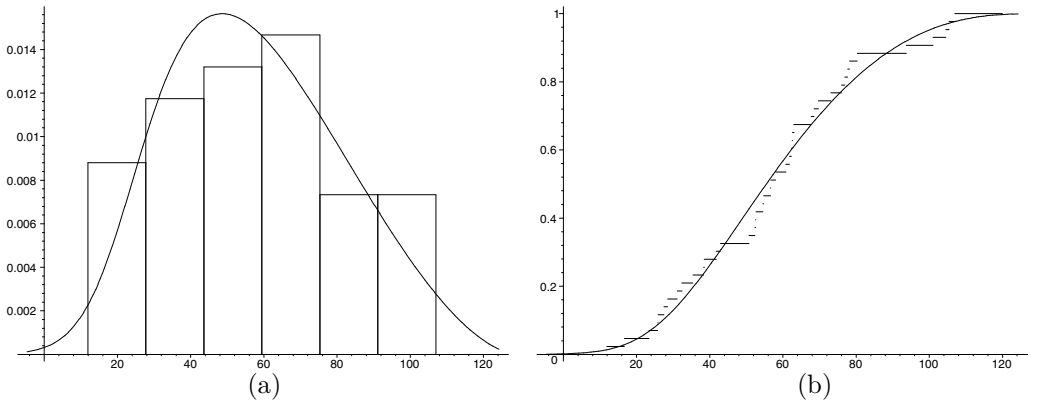


Figure 3.5–1. Histogram of cadmium concentrations and fitted p.d.f. (a); the e.d.f. of the data with the d.f. of its fitted GLD (b).

whose respective frequencies are

$$7, \quad 7, \quad 9, \quad 9, \quad 6, \quad 5.$$

Calculation of expected frequencies yields

$$5.5633, \quad 12.3032, \quad 6.5387, \quad 5.8075, \quad 6.6775, \quad 6.1097$$

and the chi-square statistic and the p-value that result are

$$5.6087 \qquad \text{and} \qquad 0.01787,$$

respectively. The Kolmogorov-Smirnov statistic for this GLD fit is $KS = 0.1025$ and $KS\sqrt{n} = 0.1025\sqrt{43} = 0.6721$. From the table in Appendix J we see that the p-value for this test is 0.76.

3.5.3 Example: Brain (Left Thalamus) MRI Scan Data

Dudewicz, Levy, Lienhart, and Wehrli (1989) give data on the brain tissue MRI scan parameter, AD. It should be noted that the term "parameter" is used differently in brain scan studies — it is used to designate what we would term random variables. In the cited study the authors show that AD^{-2} has a normal distribution while AD does not, and report the following 23 observations associated with scans of the left thalamus.

108.7	107.0	110.3	110.0	113.6	99.2	109.8	104.5
108.1	107.2	112.0	115.5	108.4	107.4	113.4	101.2
98.4	100.9	100.0	107.1	108.7	102.5	103.3	

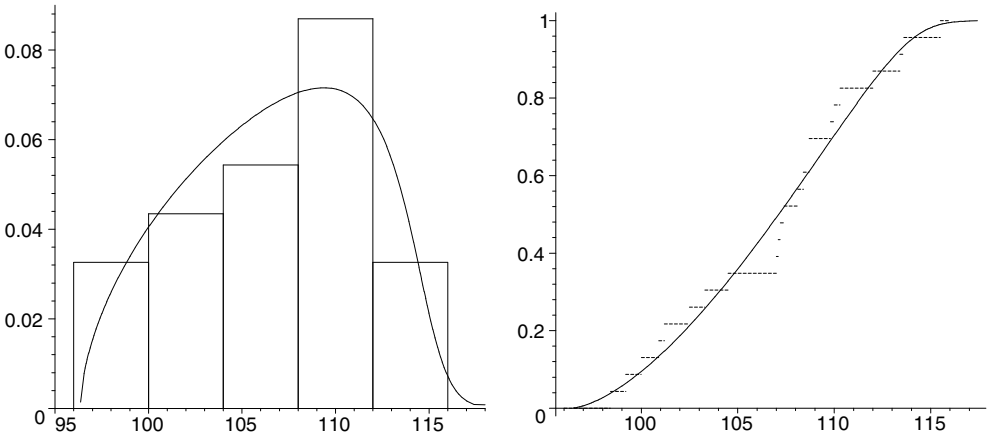


Figure 3.5–2. Histogram of AD and GLD_1 and GLD_2 p.d.f.s, designated by (1) and (2) in (a); e.d.f.s of AD with the d.f.s of GLD_1 and GLD_2 in (b).

The computations through `FitGLDM` indicate

$$\hat{\alpha}_1 = 106.8349, \quad \hat{\alpha}_2 = 22.2988, \quad \hat{\alpha}_3 = -0.1615, \quad \hat{\alpha}_4 = 2.1061$$

and provide the fit

$$\text{GLD}(112.1465, 0.06332, 0.6316, 0.05350)$$

with support $[96.353, 127.940]$.

Figure 3.5–2 (a) shows the p.d.f.s of this fit and a histogram of the data; Figure 3.5–1 (b) shows the e.d.f. of the data with the d.f. of the GLD fit.

With the small sample size of this example, we are not able to perform a chi-square test but we can partition the data into classes, such as

$$(-\infty, 103), \quad [103, 108), \quad [108, 111), \quad [111, \infty),$$

whose respective frequencies are

$$6, \quad 6, \quad 7, \quad 4,$$

and calculate the chi-square statistics for this to obtain 1.4715. For this fit $KS = 0.1438$, $KS\sqrt{n} = 0.1438\sqrt{23} = 0.6896$, which indicates (from Appendix J) a p-value of 0.73.

3.5.4 Example: Human Twin Data for Quantifying Genetic (vs. Environmental) Variance

There is variability in most human characteristics: people differ in height, weight, and other variables. One question of interest in fields such as health, clothing

manufacture, etc. (see Dudewicz, Chen, and Taneja (1989), pp. 223–228) is how much of the variability is due to genetic makeup, and how much is due to environmental influences. There is little dispute that some of the variability is from each source; there is a great controversy about how much of the variability should be attributed to each source. One approach to quantifying this variability is through so-called "twin studies," where human twins are studied in various scenarios. In such studies, normality of the variables is often assumed when this assumption may not always be valid (see Williams and Zhou (1998)). Some references to this interesting field include Christian, Carmelli, Castelli, Fabsitz, Grim, Meaney, Norton, Reed, Williams, and Wood (1990), and Christian, Kang, and Norton (1974). Interesting datasets in this area come from the Indiana Twin Study. We focus on one dataset[1] which is given in sorted form in Table 3.5–3.

From `FitGLDM`, we determine the sample moments of X to be

$$\hat{\alpha}_1 = 5.48585, \quad \hat{\alpha}_2 = 1.3082, \quad \hat{\alpha}_3 = -0.04608, \quad \hat{\alpha}_4 = 2.7332$$

and obtain the fit

$$\text{GLD}(5.5872, 0.2266, 0.2089, 0.1762).$$

To get a sense of the quality of this fit we look at the histogram of X with the superimposed graph of the fitted p.d.f. (Figure 3.5–4 (a)) as well as the e.d.f. of X with the superimposed d.f. of the GLD fit (Figure 3.5–4 (b)). Next, we apply the chi-square test with the 8 classes

$$(-\infty, 4), \quad [4, 4.5), \quad [4.5, 5), \quad [5, 5.5), \quad [5.5, 6), \quad [6, 6.5), \quad [6.5, 7), \quad [7, \infty)$$

for which the observed frequencies, respectively, are

$$12, \quad 15, \quad 12, \quad 21, \quad 25, \quad 11, \quad 18, \quad 9.$$

The chi-square test yields expected frequencies of

$$12.6118,\ 12.0610,\ 16.8281,\ 20.0075,\ 20.2773,\ 17.3429,\ 12.2937,\ 11.5777$$

for these 8 classes, giving us

$$8.8226 \qquad \text{and} \qquad 0.03174$$

for the chi-square statistic and p-value for this test. We also obtain $KS = 0.03978$ and $KS\sqrt{n} = 0.03978\sqrt{123} = 0.4412$ for a p-value of 0.99.

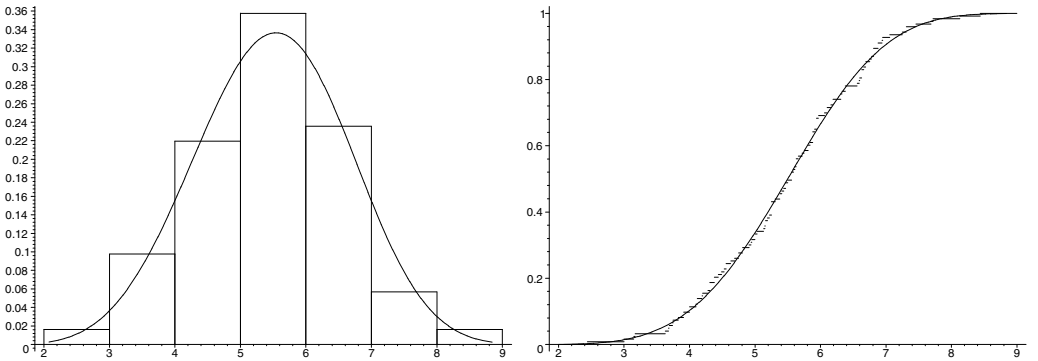
With a similar analysis of Y, using the same classes, we find

$$\hat{\alpha}_1 = 5.3666, \quad \hat{\alpha}_2 = 1.2033, \quad \hat{\alpha}_3 = -0.01219, \quad \hat{\alpha}_4 = 2.7665,$$

[1]This data comes from the Ph.D. thesis of Dr. Cynthia Moore, under the supervision of Dr. Joseph C. Christian, Department of Medical and Molecular Genetics, Indiana University School of Medicine, kindly shared with us by Dr. Moore. The data collection was supported by the National Institutes of Health Individual Research Fellowship Grant–"Twin Studies in Human Development." PHS-5-F32-HD06869, 1987–1990.

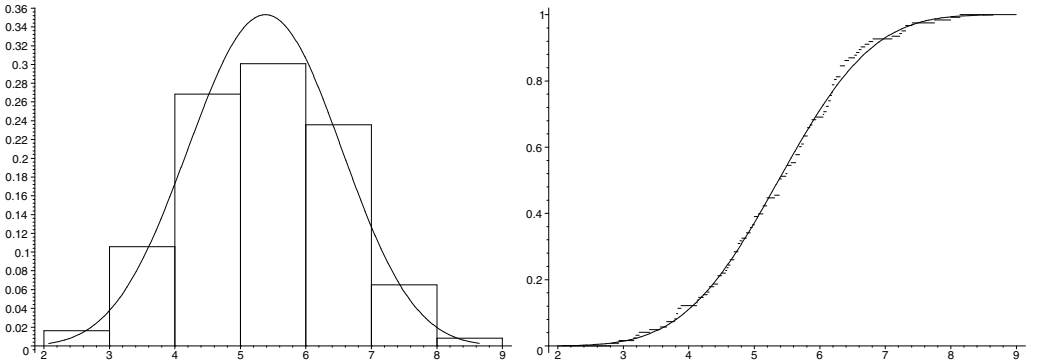
Table 3.5–3. Birth Weights of Twins.

Twin 1	Twin 2	Twin 1	Twin 2	Twin 1	Twin 2	Twin 1	Twin 2
2.44	2.81	3.00	3.78	3.15	2.93	3.17	4.13
3.63	3.19	3.68	5.38	3.69	3.56	3.74	3.24
3.75	3.16	3.83	3.83	3.91	3.81	3.91	4.60
4.00	3.66	4.00	4.28	4.10	5.00	4.12	4.75
4.12	6.31	4.19	4.31	4.20	4.75	4.28	4.50
4.31	3.66	4.31	3.88	4.31	4.69	4.38	3.40
4.38	5.00	4.44	5.13	4.49	4.15	4.53	3.83
4.56	4.31	4.56	5.38	4.63	4.12	4.69	4.63
4.75	4.56	4.75	4.63	4.81	4.38	4.81	4.44
4.91	5.13	4.94	4.78	4.95	4.22	5.00	5.38
5.00	6.16	5.03	4.94	5.13	3.81	5.15	5.00
5.16	5.75	5.16	6.69	5.19	4.94	5.22	4.75
5.25	5.38	5.25	5.63	5.25	5.81	5.25	6.10
5.25	6.25	5.31	4.69	5.38	5.69	5.38	6.81
5.41	5.69	5.44	5.75	5.47	5.13	5.47	6.75
5.50	5.38	5.56	4.44	5.56	5.48	5.56	6.31
5.59	6.22	5.61	6.18	5.63	4.69	5.63	4.88
5.63	4.97	5.66	5.38	5.72	5.06	5.75	5.63
5.81	5.19	5.81	5.94	5.84	5.56	5.88	4.88
5.88	5.69	5.88	5.88	5.88	5.88	5.91	5.50
5.94	4.81	5.94	5.41	5.94	5.75	5.94	6.31
5.97	5.63	6.06	5.31	6.10	5.19	6.10	6.13
6.16	5.85	6.19	4.44	6.19	5.50	6.31	5.81
6.31	6.10	6.33	8.14	6.38	5.19	6.38	6.05
6.56	6.38	6.59	6.16	6.60	6.53	6.63	6.19
6.63	6.19	6.63	6.38	6.66	7.10	6.69	5.81
6.69	6.13	6.75	6.56	6.78	6.22	6.81	6.19
6.81	6.60	6.81	7.41	6.88	6.06	6.88	6.63
6.94	5.50	6.95	5.72	7.06	7.31	7.25	8.00
7.31	4.58	7.31	7.31	7.46	7.22	7.69	7.25
7.72	6.44	8.13	7.75	8.44	6.31		



(a) (b)

Figure 3.5–4. The histogram of X and its fitted p.d.f. (a); the empirical d.f. of X and the d.f. of its fitted GLD (b).



(a) (b)

Figure 3.5–5. The histogram of Y and its fitted p.d.f. (a); the empirical d.f. of Y and the d.f. of its fitted GLD (b).

$$\text{GLD}(5.3904, 0.2293, 0.1884, 0.1807).$$

The comparative p.d.f. and d.f. plots for Y are given in Figures 3.5–5 (a) and (b). When we apply the chi-square test we obtain observed frequencies of

$$15, \quad 11, \quad 19, \quad 19, \quad 21, \quad 22, \quad 7, \quad 9,$$

expected frequencies of

$$13.5765,\ 13.4925,\ 18.6505,\ 21.4160,\ 20.4945,\ 16.2641,\ 10.5722,\ 8.5335,$$

and chi-square statistic and p-value of 4.1567 and 0.2450, respectively. In this case, $KS = 0.04824$ and $KS\sqrt{n} = 0.04824\sqrt{123} = 0.5350$ for a p-value of 0.94.

3.5.5 Example: Rainfall Distributions

Statistical modeling plays an essential role in the study of rainfall and the relationships between rainfall at multiple sites (Shimizu (1993)). Lognormal distributions have been used extensively in this work, and in univariate cases have worked well; however, with multiple sites the many rejections of lognormality (e.g., see the Rs in "Table 2. Test for lognormality" on p. 168 of Shimizu (1993)) indicate a need for more modeling flexibility. While the data for Shimizu's studies was from sites in Japan and is no longer available, similar data for U.S. sites is readily available from the U.S. National Oceanic and Atmospheric Administration. From that data, shown in Table 3.5–6 is data from the period May 1998 to October 1988 in

Table 3.5–6. Rainfall (in inches) at Rochester, N.Y. (X) and Syracuse, N.Y., (Y), from May to October of 1998, on on days when both sites had positive rainfall.

X	Y	X	Y	X	Y
.03	.05	.11	.07	.07	.04
.09	.14	.01	.14	.69	.72
.62	.60	.08	.21	.61	.36
.08	.09	.50	.54	.23	.18
1.27	.22	.03	1.26	.09	.10
1.20	.21	1.65	.94	1.87	.17
.15	.05	2.61	.94	.85	.37
1.35	.48	.15	.28	.31	.38
.16	.48	.27	.31	.06	.04
.23	1.09	1.38	.42	.74	1.79
2.51	1.25	.13	.11	.10	.16
.24	1.05	.76	.36	.03	.42
.29	.08	.02	.17	.07	.02
.17	.19	.47	.04	.41	.79
.03	.37	.15	.18	.01	.09
.02	.19	.19	.22		

Rochester, New York and Syracuse, New York. This data is for the 47 days in which both cities had positive rainfall measured. (The study of Shimizu (1993) looks at all 4 combinations of "rain, no rain" possibilities; however, for this example's purpose we concentrate on only the case of rain at both sites.)

The moments of X are

$$\hat{\alpha}_1 = 0.4913, \quad \hat{\alpha}_2 = 0.4074, \quad \hat{\alpha}_3 = 1.8321, \quad \hat{\alpha}_4 = 5.7347,$$

and those of Y are

$$\hat{\alpha}_1 = .3906, \quad \hat{\alpha}_2 = 0.1533, \quad \hat{\alpha}_3 = 1.6163, \quad \hat{\alpha}_4 = 5.2245.$$

In both cases, $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ is located outside of the region of the table of Appendix B and outside of the region for reliable computations. We will return to this example in Chapters 4 and 5 and fit GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distributions to both X and Y via the EGLD and through a percentile-based method.

3.6 Moment-Based GLD Fit to Data from a Histogram

In the examples of Section 3.5 the actual data on the phenomenon of interest, $X_1, X_2, \ldots, X_n$, was available to us. We acted assuming that these were independent and identically distributed observations — an assumption that should be verified. In many cases the data are given in the form of a **histogram**. While it would in general be preferable to have the actual data (going to a histogram involves a certain loss of information, such as the distribution of the observations within the classes), it nevertheless is possible to proceed with fitting of a distribution using data in the form of a histogram.

One could **seek the original data** from the authors of the histogram, but this is often difficult: the author may have moved, or may not be able to release the data without a time-consuming approval process, or may not have retained the basic data. As an example of use of histograms, one may look in virtually any scientific journal or newspaper. For example, Dahl-Jensen, Mosegaard, Gundestrup, Clow, Johnsen, Hansen, and Balling (1998) in their Figure 2 give six histograms, each based on 2000 Monte Carlo observations. As they use between 5 and 15 classes in each histogram, they are easily able to meet the rules for an informative histogram: there should be **no more than two classes with a frequency less than** 5**, and classes should be of equal width**. While histograms **with classes of unequal width** can be used, they are often misused. In order not to be misleading, **the heights of the bars must be adjusted so the area of the bar is proportional to the frequency**. Thus, if class 1 has a width of 2.5 and 150 observations, and its height is 150 in the frequency histogram, and if class 2 has width 5 and also 150 observations, then its height must be 75. Below we give two examples of the use of a histogram to fit a distribution.

Sometimes the histogram data is presented to us in the form of a table such as Table 3.6–1, which comes from Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979, p. 207). The classes have equal widths (assuming the lowest class is 0.010 to 0.015 and the highest class is 0.060 to 0.065), but there are more than 2 classes with fewer than 5 observations in them, so we combine the two lowest and the two highest classes. Now there are no classes with fewer than 5 observations in them. Classes with low frequencies have highly variable estimates

Table 3.6–1. Observed Coefficient of Friction Frequencies.

Coefficient of Friction	Observed Frequency
less than 0.015	1
0.015 – 0.020	9
0.020 – 0.025	30
0.025 – 0.030	44
0.030 – 0.035	58
0.035 – 0.040	45
0.040 – 0.045	29
0.045 – 0.050	17
0.050 – 0.055	9
0.055 – 0.060	4
0.060 or more	4
Total	250

of true p.d.f. heights, due to inadequate amount of observation, and it is preferable to avoid them in order to avoid a bias in the fitting.

We use the mid-point assumption that every data point is located at the center of its class interval; for the first class, we assume that its entries are located at 0.0125 and for the last class we assume that the entries have the value 0.0625. With these assumptions, we can compute approximate moments for this data and obtain (from `FitGLDM`)

$$\hat{\alpha}_1 = 0.03448, \quad \hat{\alpha}_2 = 9.2380 \times 10^{-5}, \quad \hat{\alpha}_3 = 0.5374, \quad \hat{\alpha}_4 = 3.2130,$$

and the GLD fit

$$\mathrm{GLD}_1(0.02889, 18.1935, 0.05744, 0.1850).$$

We note that the original data are in fact given in Hahn and Shapiro (1967), and from the original data (not a histogram) Hahn and Shapiro computed the sample moments as

$$\hat{\alpha}_1 = .0345, \quad \hat{\alpha}_2 = 0.00009604, \quad \hat{\alpha}_3 = .87, \quad \hat{\alpha}_4 = 4.92. \tag{3.6.1}$$

We now also fit a GLD using (3.6.1) to illustrate the change that results from having the original data vs. only the histogram of the data. The GLD_2 associated with the moments given in (3.6.1) is given by

$$\mathrm{GLD}_2(0.03031, 1.5771, 0.005174, 0.01190).$$

The histogram of the data with the fitted GLD_1 and GLD_2 p.d.f.s is given in Figure 3.6–2 (a) (the GLD_2 p.d.f. rises higher at the center). The e.d.f of the

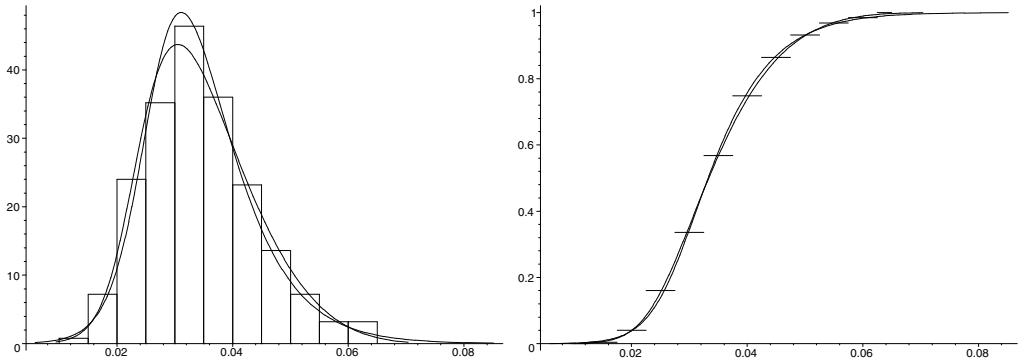


Figure 3.6–2. Histogram of coefficients of friction data and its fitted p.d.f. (a). Empirical d.f. of coefficients of friction and the d.f. of its fitted GLD (b).

data together with the two fitted d.f.s is given in Figure 3.6–2 (b) (the GLD_2 d.f. extends farther to the left).

We do a chi-square test using the classes

$$(-\infty,\, 0.025),\quad [0.025,\, 0.030),\quad [0.030,\, 0.035),\quad [0.035,\, 0.040),$$
$$[0.040,\, 0.045),\quad [0.045,\, 0.050),\quad [0.050,\, \infty)$$

for which the observed frequencies are, respectively,

$$40,\quad 44,\quad 58,\quad 45,\quad 29,\quad 17,\quad 17.$$

We get the following expected frequencies:

$$\begin{array}{lcl} \text{GLD}_1 & : & 38.8184,\ 50.0488,\ 52.7344,\ 42.9687,\ 29.6631,\ 18.2495,\ 17.5171 \\ \text{GLD}_2 & : & 347900,\ 51.3916,\ 58.3496,\ 44.7998,\ 27.8931,\ 15.6860,\ 17.0898. \end{array}$$

These give the following chi-square statistics and p-values for the two fits:

$$\begin{array}{lcl} \text{GLD}_1 & : & \chi^2 \text{ statistic } = 1.5045, \quad p-\text{value } = 0.8048 \\ \text{GLD}_2 & : & \chi^2 \text{ statistic } = 2.0008, \quad p-\text{value } = 0.3677. \end{array}$$

For the Kolmogorov-Smirnov tests for these two fits, we have

$$\begin{array}{lcl} KS_1 = 0.1279 & : & KS\sqrt{n} = 0.1279\sqrt{250} = 2.0223, \quad p-\text{value } \approx 0 \\ KS_2 = 0.1288 & : & KS\sqrt{n} = 0.1288\sqrt{250} = 2.0365, \quad p-\text{value } \approx 0. \end{array}$$

We conclude that at least in this example (with a relatively large sample size, and a well-constructed histogram), the effect of the lack of the original data has been negligible: both GLD_1 and GLD_2 models fit the data quite well (the

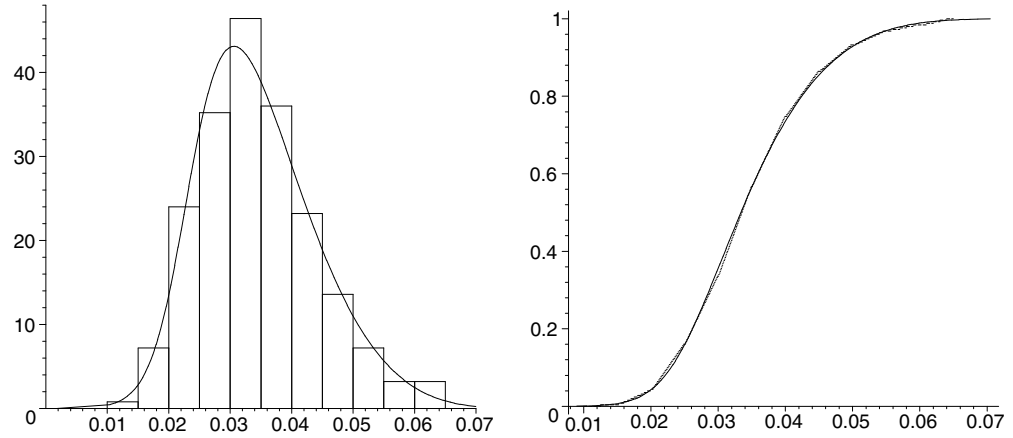


Figure 3.6–3. The histogram of coefficients of friction data and the GLD_3 p.d.f. (a). The e.d.f. of coefficients of friction and the d.f. of GLD_3 (b).

very poor results from the Kolmogorov-Smirnov test are a consequence of our assumption that all data points are located at the mid-points of their intervals).

As an alternative to the mid-point assumption we consider now the, equally reasonable, assumption that the data in a given interval is spread uniformly throughout that interval. This produces

$$\hat{\alpha}_1 = 0.03448, \quad \hat{\alpha}_2 = 9.43 \times 10^{-5}, \quad \hat{\alpha}_3 = 0.5289, \quad \hat{\alpha}_4 = 3.1886,$$

and the GLD fit

$$\mathrm{GLD}_3(0.02895,\ 18.2454,\ 0.05964,\ 0.1865),$$

with support $[-0.0259, 0.0838]$.

A chi-square test, using the same intervals as before, gives expected frequencies of

$$39.7949,\ 49.3164,\ 52.2461,\ 42.6025,\ 29.7241,\ 18.4326,\ 17.8833,$$

and the chi-square statistic and its associated p-value

$$1.5154 \quad \text{and} \quad 0.4687,$$

respectively. For this fit we also have $KS = 0.02386$ and $KS\sqrt{n} = 0.02386\sqrt{250} = 0.3773$ that gives a p-value of 0.999. Figure 3.6–3 (a) shows the GLD_3 p.d.f. with the data histogram and Figure 3.6–3 (b) gives the e.d.f. of the data with the d.f. of the GLD_3. We can see that GLD_3 is also a good fit, with its chi-square p-value about the same as that for GLD_1 and a much better KS p-value.

3.7 The GLD and Design of Experiments

In many studies including, but not limited to, simulation studies and theoretical studies of sensitivity of results to distributions, there is a distribution one thinks of as the "best fit" to the true underlying distribution. For example, in the coefficient of friction data of the example in Section 3.6, the **underlying "true" state of nature is estimated to have moments**

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.03448, 9.238 \times 10^{-5}, 0.5374, 3.2130) \tag{3.7.1}$$

and the GLD p.f.

$$Q(y) = 0.02889 + (y^{0.05744} - (1-y)^{0.1850})/18.1935. \tag{3.7.2}$$

However, as these are only estimates from the data, very often it will be desired to **re-do the analysis (e.g., re-run the simulation program, or refine the theoretical analysis) with a changed distribution** (a distribution in the neighborhood of the best estimate, but varied as much as seems reasonable in the setting using the best available engineering information).

This type of analysis is relatively simple to **perform using the GLD and notions of statistical design of experiments** (e.g., see Section 6.4.2 of Karian and Dudewicz (1999a)). This approach is superior to the "grab bag of distributions" approach sometimes used, wherein one haphazardly adds alternative distributions to the set used. To vary the distribution from the fitted GLD in (3.7.2), we vary the parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, but these do not have intrinsic value to us (as we may not easily understand the meaning of a change of 0.5 units in λ_4, for example). So, instead we **vary the** $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$**, fit a new GLD for each desired set of** α**-values, and run our study with the new fitted GLD.**

In the coefficient of friction example, suppose we are comfortable with the fitted mean (0.03448) and variance (9.2380×10^{-5}) and do not believe that perturbations of reasonable size in these values will impact the variables of interest to us. However, we believe that the shape parameters (skewness α_3 and kurtosis α_4) may have an impact when varied within reasonable bounds. Further, suppose that we have reason to believe that α_3 may vary from 0.5374 by up to ± 0.2, and α_4 may vary from 3.2130 by up to ± 0.5. That reason can either be based on **previous studies** in the area, or on **theoretical considerations**, or on **statistical reasons for which we might take these bounds to be** ± 2 **standard deviations (estimated) of the estimates** in (3.7.1). Note here that one needs to estimate (and take twice the square root of) the variances

$$\mathrm{Var}(\hat{\alpha}_1) = \frac{\alpha_2}{n}, \tag{3.7.3}$$

$$\mathrm{Var}(\hat{\alpha}_2) = \frac{(n-1)^2}{n^3}\mu_4 - \frac{(n-1)(n-3)}{n^3}\alpha_2^2 \approx \frac{\mu_4 - \alpha_2^2}{n}, \tag{3.7.4}$$

$$\begin{aligned}\mathrm{Var}(\hat{\alpha}_3) &\approx (\alpha_2^2\mu_6 - 3\alpha_2\mu_3\mu_5 - 6\alpha_2^3\mu_4 \\ &\quad +2.25\mu_3^2\mu_4 + 8.75\alpha_2^2\mu_3^2 + 9\alpha_2^5)/(n\alpha_2^5), \qquad (3.7.5)\\ \mathrm{Var}(\hat{\alpha}_4) &\approx (\alpha_2^2\mu_8 - 4\alpha_2\mu_4\mu_6 - 8\alpha_2^2\mu_3\mu_5 + 4\mu_4^3 \\ &\quad -\alpha_2^2\mu_4^2 + 16\alpha_2\mu_3^2\mu_4 + 16\alpha_2^3\mu_3^2)/(n\alpha_2^6). \qquad (3.7.6)\end{aligned}$$

In the above we already know how to estimate α_2 by $\hat{\alpha}_2$. The new expressions μ_3, μ_4, μ_5, μ_6, and μ_8 are related to the $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that we are familiar with by the equations

$$\mu_3 = \alpha_2^{1.5}\alpha_3, \quad \mu_4 = \alpha_2^2\alpha_4, \quad \mu_5 = \alpha_2^{2.5}\alpha_5, \quad \mu_6 = \alpha_2^3\alpha_6, \quad \mu_8 = \alpha_2^4\alpha_8. \qquad (3.7.7)$$

Hence, they can be estimated by replacing right-hand side terms by their estimates. The approximations in (3.7.5) and (3.7.6) are up to order $O(n^{-3/2})$. A good reference is Cramér (1946, pp. 354, 357). In this example, we get

$$\begin{aligned}\alpha_5 &= 4.5553,\\ \alpha_6 &= 17.0073,\\ \alpha_8 &= 114.2131,\\ \mu_3 &= 4.7716 \times 10^{-7},\\ \mu_4 &= 2.7142 \times 10^{-8},\\ \mu_5 &= 3.7365 \times 10^{-10},\\ \mu_6 &= 1.3408 \times 10^{-11},\\ \mu_8 &= 8.3182 \times 10^{-15},\\ \mathrm{Var}(\alpha_3) &= 0.01600,\\ \mathrm{Var}(\alpha_4) &= 0.07150,\\ \sqrt{\mathrm{Var}(\alpha_3)} &= 0.1265,\\ \sqrt{\mathrm{Var}(\alpha_4)} &= 0.2674.\end{aligned}$$

Hence, $2\sqrt{\mathrm{Var}(\alpha_3)} = 0.253$ and $2\sqrt{\mathrm{Var}(\alpha_4)} = 0.536$. A study that varies α_3 by up to ± 0.25, and α_4 by up to ± 0.54, is reasonable. We use ± 0.2 and ± 0.5 for simplicity below.

We now choose the (α_3, α_4) for our experiments using what is called a **Central Composite Design in two variables** (α_3 and α_4), with a multiplier of 1.5. The sample point $(0.5374, 3.2130)$ is at the center of our experiments, and is called a **Center Point**. We go out from this Center Point to a distance $\pm 1.5d_1 = \pm 0.2$ on α_3, and $\pm 1.5d_2 = \pm 0.5$ on α_4, so we have $d_1 = .1333$, $d_2 = .3333$.
This sets the **Star Points** as

$$(0.5374 - 0.2,\ 3.2130) \quad = \quad (0.3374, 3.2130)$$

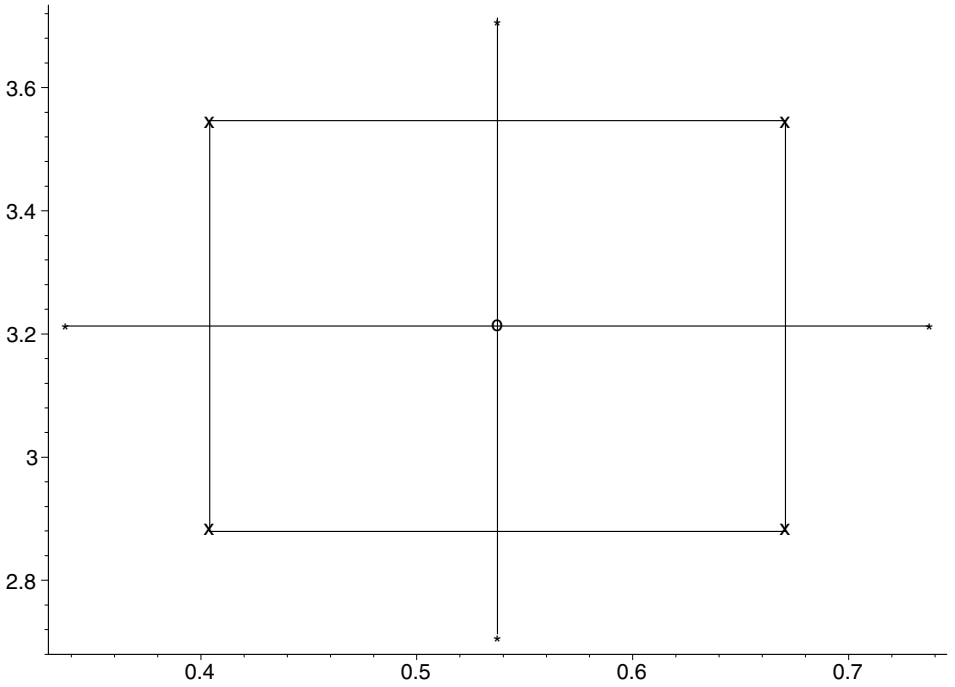


Figure 3.7–1. The design points for the sensitivity study of the coefficient of friction example.

$$\begin{aligned}(0.5374 + .02,\ 3.2130) &= (0.7374, 3.2130)\\ (0.5374,\ 3.2130 - 0.5) &= (0.5374, 2.7130)\\ (0.5374,\ 3.2130 + 0.5) &= (0.5374, 3.7130).\end{aligned} \tag{3.7.8}$$

The **Factorial Points** are taken as

$$\begin{aligned}(0.5374 - d_1, 3.2130 - d_2) &= (0.4041, 2.8797)\\ (0.5374 - d_1, 3.2130 + d_2) &= (0.4041, 3.5463)\\ (0.5374 + d_1, 3.2130 - d_2) &= (0.6707, 2.8797)\\ (0.5374 + d_1, 3.2130 + d_2) &= (0.6707, 3.5463).\end{aligned} \tag{3.7.9}$$

This is illustrated on coordinate axes in Figure 3.7–1. Also see Figure 6.4–2 of Karian and Dudewicz (1999b) for the general case and see Dudewicz and Karian (1985, pp. 189, 196, 206, 233) for details for more than two variables; that case would arise if we varied all of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and there are ways to cut the number of points used from the full $2^k + 2k + 1$ with k variables without losing much information in a precise sense. This is desirable since when $k = 2$ we have only 9 points, but this would, if not reduced, become 25 with $k = 4$ and grow rapidly with k.

In Figure 3.7–1 note that the factorial points lie on the corners of a box (shown by "x"), and the star points (shown by "*") lie on perpendiculars from the center point and go out a distance ±0.2 (horizontally), ±0.5 (vertically) from the center of the design area. One should also be careful that the specified variations continue to fall in the possible region of Figure 3.2–4; if they move outside the GLD area, one may need to utilize the EGLD of Chapter 4.

We now fit a GLD for each of the 9 points (we already have this for the center point, so there are 8 additional points to be fitted). This process yields the following (α_3, α_4) and their associated GLDs (the GLDs associated with star points are designated with the superscript "*," those associated with factorial points are similarly marked with an "x," and the one associated with the center point is designated by "o").

$$
\begin{aligned}
(\alpha_3, \alpha_4) &= (0.5374, 3.2130) \quad , \quad \mathrm{GLD}^o(0.02902, 18.1399, 0.05850, 0.1825);\\
(\alpha_3, \alpha_4) &= (0.3374, 3.2130) \quad , \quad \mathrm{GLD}^*(0.03136, 17.0249, 0.07594, 0.1412);\\
(\alpha_3, \alpha_4) &= (0.7374, 3.2130) \quad , \quad \mathrm{GLD}^*(0.02595, 19.0831, 0.03251, 0.2410);\\
(\alpha_3, \alpha_4) &= (0.5374, 2.7130) \quad , \quad \mathrm{GLD}^*(0.02547, 23.6743, 0.04238, 0.3404);\\
(\alpha_3, \alpha_4) &= (0.5374, 3.7130) \quad , \quad \mathrm{GLD}^*(0.03080, 10.8659, 0.04175, 0.08697);\\
(\alpha_3, \alpha_4) &= (0.4041, 2.8797) \quad , \quad \mathrm{GLD}^x(0.02895, 22.5726, 0.07766, 0.2452);\\
(\alpha_3, \alpha_4) &= (0.4041, 3.5463) \quad , \quad \mathrm{GLD}^x(0.03160, 11.9256, 0.05164, 0.09109);\\
(\alpha_3, \alpha_4) &= (0.6707, 2.8797) \quad , \quad \mathrm{GLD}^x(0.02450, 21.8367, 0.02642, 0.3223);\\
(\alpha_3, \alpha_4) &= (0.6707, 3.5463) \quad , \quad \mathrm{GLD}^x(0.02881, 14.7763, 0.04340, 0.1434).
\end{aligned}
$$

The graphs of these p.d.f.s are shown in Figure 3.7–2, where the fit associated with the center point is shown by diamond-shaped points. From Figure 3.7–2 we can observe the spread of the distributions chosen for the sensitivity study about the distribution associated with the center point.

As a final point, note that if we are interested in an output of an experiment (simulation or other experiment), we can now fit a metamodel via regression. That will show the effect of changes in α_3 and α_4 from the center on the output, and is a useful way of summarizing the information in the experiments run at the design points. For a detailed example, including SAS code and contour graphs for interpretation, see Section 8.3 of Karian and Dudewicz (1999a).

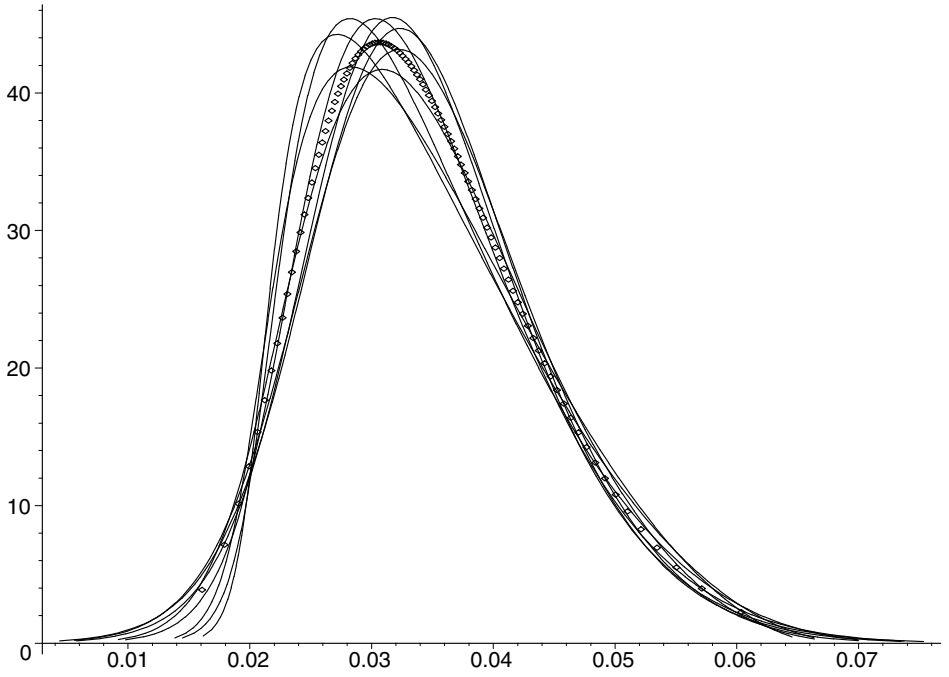


Figure 3.7–2. The p.d.f.s associated with the design points for the sensitivity study of the coefficient of friction.

Problems for Chapter 3

3.1. For the GLD studied in (2.1.6), find the value of $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and identify (α_3^2, α_4) in Figure 3.2–4.

3.2. For the two GLDs of Figure 2.3–1, find the first four moments, and identify the distributions in the plot of Figure 3.2–4.

3.3. For the 9 GLDs plotted in Figure 2.3–3a, find the first four moments, and identify their points in Figure 3.2–4.

3.4. For the 9 GLDs plotted in Figure 2.3–4a, find the first four moments, and identify their points in Figure 3.2–4.

3.5. For the 7 GLDs plotted in Figure 2.3–5, find the first four moments, and identify their points in Figure 3.2–4.

3.6. For the 4 GLDs plotted in Figure 2.3–6, find the first four moments, and identify their points in Figure 3.2–4.

3.7. For the 9 GLDs plotted in Figure 2.3–7a, find the first four moments, and identify their points in Figure 3.2–4.

3.8. For the 4 GLDs plotted in Figure 2.3–8a, find the first four moments, and identify their points in Figure 3.2–4.

3.9. For the 3 GLDs plotted in Figure 2.3–8b, find the first four moments, and identify their points in Figure 3.2–4.

3.10. For the 4 GLDs plotted in Figure 2.3–9, find the first four moments, and identify their points in Figure 3.2–4.

3.11. In Chapter 2 (e.g., see Figures 2.3–6, 2.3–7a, 2.3–8a) plots were given of GLDs with parameters in Regions 1 and 5 of Figure 2.2–1, but no plots for Regions 2 and 6. Construct similar plots for Regions 2 and 6, and find their (α_3^2, α_4) points (if they exist). (Hint: See Theorem 2.3.22.)

3.12. Ramberg (1975) developed the model

$$Q(y) = \lambda_1 + (\lambda_4 y^{\lambda_3} - (1-y)^{\lambda_3})/\lambda_2,$$

and gave a set of tables. Fit some examples with the GLD and with the above model and compare. Which do you prefer, and why?

3.13. In Figure 3.5–4 (b) we plotted the e.d.f. and the fitted GLD d.f. on the same axes. Plot a 95% confidence band on the same axes to quantify the quality of the fit. (The confidence band uses the e.d.f. data to yield a band in which the true d.f. will, with high confidence, lie. See Gan, Koehler, and Thompson (1991).)

References for Chapter 3

Artin, E. (1964). *The Gamma Function*, Holt Rinehart, and Winston, New York.

Christian, J. C., Carmelli, D., Castelli, W. P., Fabsitz, R., Grim, C. E., Meaney, F. J., Norton, J. A. Jr., Reed, T., Williams, C. J., and Wood, P. D. (1990). "High density lipoprotein cholesterol: A 16–year longitudinal study in aging male twins," *Arteriosclerosis*, 10, 1020–1025.

Christian, J. C., Kang, K. W., and Norton, J. A. Jr. (1974). "Choice of an estimate of genetic variance from twin data," *American Journal of Human Genetics*, 26, 154–161.

Cooley, C. A. (1991). *The Generalized Lambda Distribution: Applications and Parameter Estimation*, Honors Project, Denison University. Available from: Doan Library, Denison University, Granville, Ohio 43023.

Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press, Princeton, New Jersey.

Dahl-Jensen, D., Mosegaard, K., Gundestrup, N., Clow, G. D., Johnsen, S. J., Hansen, A. W., and Balling, N. (1998). "Past temperatures from the Greenland ice sheet," *Science*, 282, 268–271.

Dudewicz, E. J., Chen, P., and Taneja, B. K. (1989). *Modern Elementary Probability and Statistics, with Statistical Programming in SAS, MINITAB & BMDP* (Second Printing), American Sciences Press, Inc., Columbus, Ohio.

Dudewicz, E. J. and Karian, Z. A. (1985). *Modern Design and Analysis of Discrete-Event Computer Simulations*, IEEE Computer Society, New York.

Dudewicz, E. J. and Karian, Z. A. (1996). "The extended generalized lambda distribution (EGLD) system for fitting distributions to data with moments, II: Tables," *The American Journal of Mathematical and Management Sciences,* 16, 271–332.

Dudewicz, E. J., Levy, G. C., Lienhart, J. L., and Wehrli, F. (1989). "Statistical analysis of magnetic resonance imaging data in the normal brain data, screening normality, discrimination, variability), and implications for expert statistical programming for ESS^{TM} (the Expert Statistical System)," *American Journal of Mathematical and Management Sciences*, 9, 299–359.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, John Wiley & Sons, New York.

Dudewicz, E. J. and van der Meulen, E. C. (1987). "The empiric entropy, a new approach to nonparametric entropy estimation," *New Perspectives in Theoretical and Applied Statistics* (eds. M. L. Puri, J. P. Vilaplana, and W. Wertz), John Wiley & Sons, Inc., New York, 207–227.

Elinder, C. G., Jönsson, L., Piscator, M., and Rahnster, B (1981). "Histopathological changes in relation to cadmium concentration in horse kidneys," *Environmental Research*, 26, 1–21.

Freimer, M., Kollia, G., Mudholkar, G. S., and Lin, C. T. (1988). "A study of the Generalized Tukey Lambda family," *Communications in Statistics–Theory and Methods*, 17, 3547–3567.

Gan, F. F., Koehler, K. J., and Thompson, J. C. (1991). "Probability plots and distribution curves for assessing the fit of probability models," *The American Statistician*, 45 (1), 14–21.

Gibbons, J. D. (1997). *Nonparametric Methods for Quantitative Analysis (Third Edition)*, American Sciences Press, Inc., Columbus, Ohio.

Govindarajulu, Z. (1987). *The Sequential Statistical Analysis of Hypothesis Testing, Point and Interval Estimation, and Decision Theory*, American Sciences Press, Inc., Columbus, Ohio.

Györfi, L., Liese, F., Vajda, I., and van der Meulen, E. C. (1998). "Distribution estimates consistent in χ^2–divergence," *Statistics*, 32, 31–57.

Hahn, G. J. and Shapiro, S. S. (1967). *Statistical Models in Engineering*, John Wiley & Sons, New York.

Harter, H. Leon (1993). The Cronological Annotated Bibliography of Order Statistics, Volumes I–VIII, American Sciences Press, Inc., Columbus, Ohio.

Karian, Z. A. and Dudewicz, E. J. (1999a). "Fitting the Generalized Lambda Distribution to data: A method based on percentiles," *Communications in Statistics: Simulation and Computation*, 28(3), 793–819.

Karian, Z. A. and Dudewicz, E. J. (1999b). *Modern Statistical, Systems, and GPSS Simulation*, Second Edition, CRC Press, Boca Raton, Florida.

Karian, Z. A. and Dudewicz, E. J. (2007). "Computational issues in fitting statistical distributions to data," *American Journal of Mathematical and Management Sciences*, 27, 319–349.

Kendall, M. G. and Stuart, A. (1969). *The Advanced Theory of Statistics, Volume I, Distribution Theory* (Third Edition), Hafner Publishing Company, New York, and Charles Griffen & Company Limited, London.

Micceri, T. (1989), "The unicorn, the normal curve, and other improbable creatures," *Psych. Bull.*, 105, 156–166.

Mishra, S. N., Shah, A. K., and Lefante, J. J. (1986). "Overlapping coefficient: The generalized t approach," *Communications in Statisitcs,* 15, 123–128.

Moore, D. S. (1977). "Generalized inverses, Wald's method, and the construction of chi-squared tests of fit," *Journal of the American Statistical Association*, 72, 131–137.

National Bureau of Standards (1953). *Tables of Normal Probability Functions, Applied Mathematics Series 23*, Issued June 5, 1953 (a reissue of Mathematical Table 14, with corrections), U.S. Government Printing Office, Washington, D.C. (available from National Technical Information Service, Springfield, Virginia 22151, Document No. PB–175 967).

Nelson, W. (1982). *Applied Life Data Analysis*, John Wiley & Sons, Inc., New York.

Ortega, J. M. and Rheinboldt, W. C. (1970). *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York.

Pearson, E. S. and Pease, N. W. (1975). "Relation between the shape of population distribution and the robustness of four simple statistics," *Biometrika*, 62, 223–241.

Pearson, K. (1900). "On the theory of contingency and its relation to association and normal correlation," *Philosophy Magazine*, 50, 157–175.

Ramberg, J. S. (1975). "A probability distribution with applications to Monte Carlo simulation studies," *Statistical Distributions in Scientific Work, Volume 2–Model Building and Model Selection* (ed. G. P. Patil, S. Kotz, and J. K. Ord), D. Reidel Publishing Company, Dordrecht-Holland, 51–64.

Ramberg, J. S. and Schmeiser, B. W. (1974). "An approximate method for generating asymmetric random variables," *Comm. ACM*, 17, 78–82.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics*, 21, 201–214.

Ricer, T. L. (1980). *Accounting for Non–Normality and Sampling Error in Analysis of Variance of Construction Data*, M.S. Thesis (Adviser Richard E. Larew),

Department of Civil Engineering, The Ohio State University, Columbus, Ohio, xi+183 pp.

Shapiro, S. S. (1980). *How to Test Normality and Other Distributional Assumptions*, Vol. 3 of The ASQC Basic References in Quality Control: Statistical Techniques (Edward J. Dudewicz, Editor), American Society for Quality, Milwaukee, Wisconsin.

Shimizu, K. (1993). "A bivariate mixed lognormal distribution with an analysis of rainfall data," *Journal of Applied Meteorology*, 32, 161–171.

Weibull, W. (1951). "A statistical distribution function of wide applicability," *Journal of Applied Mechanics*, 18, 293–297.

Wilcox, R. R. (1990), "Comparing the means of two independent groups," *Biometrical Journal*, 7, 771–780.

Williams, C. J. and Zhou, L. (1998). "A comparison of tests for detecting genetic variance from human twin data based on absolute intra-twin differences," *Communications in Statistics–Simulation*, 27, 51–65.

Chapter 4

The Extended GLD System, the EGLD: Fitting by the Method of Moments

In this chapter we address one of the two limitations of the GLD (when fitted via the method of moments) discussed in Section 3.3.3: the inability to provide fits for (α_3^2, α_4) in the region $1 + \alpha_3^2 < \alpha_4 \leq 1.8(\alpha_3^2 + 1)$. We do this by developing and appending a Generalized Beta Distribution (GBD) that covers this region not covered by the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and using the method of moments to fit the resulting Extended GLD (the EGLD) distributions to data.

The method described in the following sections is due to Karian, Dudewicz and McDonald (1996). In Section 4.1 we define the beta distribution and derive its moments. Next, in Section 4.2, we develop the GBD family of distributions as a generalization of the beta distribution and in Section 4.3, we use the method of moments to estimate the parameters of the GBD/EGLD. Illustrations for approximating known distributions and for fitting the EGLD to a dataset follow in Sections 4.4 and 4.5, respectively.

4.1 The Beta Distribution and its Moments

The beta distribution, discussed in many statistics texts (see, for example, Dudewicz and Mishra (1988), p. 137), with parameters $\beta_3 > -1$ and $\beta_4 > -1$, is defined through its p.d.f. by

$$f(x) = \begin{cases} \dfrac{x^{\beta_3}(1-x)^{\beta_4}}{\beta(\beta_3+1, \beta_4+1)}, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1.1}$$

where $\beta(a, b)$ is the beta function in (3.1.6).

Before proceeding to determine the moments of a beta random variable, we establish several useful results related to beta and gamma functions. For $a > 0$, the gamma function encountered in Section 3.4 is defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt. \tag{4.1.2}$$

Two of the properties of the gamma function are given in Lemmas 4.1.3 and 4.1.4 and its most basic connection to the beta function is given in Theorem 4.1.5. The moments of the beta distribution are derived in Theorem 4.1.9, its Corollary 4.1.15, and Theorem 4.1.16.

Lemma 4.1.3. *For $a > 0$,*

$$\Gamma(a+1) = a\Gamma(a).$$

Proof. Integrating $\Gamma(a+1)$ by parts gives

$$\begin{aligned} \Gamma(a+1) &= \int_0^\infty t^a e^{-t}\, dt = \left[-t^a e^{-t}\right]_{t=0}^{t=\infty} + \int_0^\infty e^{-t}\, d(t^a) \\ &= 0 + \int_0^\infty a e^{-t} t^{a-1}\, dt = a\Gamma(a). \quad \blacksquare \end{aligned}$$

Lemma 4.1.4. *For $a > 0$,*

$$\Gamma(a) = 2\int_0^\infty u^{2a-1} e^{-u^2}\, du.$$

Proof. The proof is a direct consequence of making the substitution $t = u^2$ in (4.1.2). $\blacksquare$

Theorem 4.1.5. *For a and b positive,*

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proof. We start by considering $\Gamma(a)\Gamma(b)$ which, by Lemma 4.1.4, can be written as

$$\begin{aligned} \Gamma(a)\Gamma(b) &= \left(2\int_0^\infty u^{2a-1} e^{-u^2}\, du\right)\left(2\int_0^\infty v^{2b-1} e^{-v^2}\, dv\right) \\ &= 4\int_0^\infty \int_0^\infty u^{2a-1} v^{2b-1} e^{-(u^2+v^2)}\, du\, dv. \end{aligned}$$

Using the substitution $u = r\cos\theta$, $v = r\sin\theta$ in the above integral, we get

$$\begin{aligned}\Gamma(a)\Gamma(b) &= 4\int_0^\infty \int_0^{\frac{\pi}{2}} r^{2a-1}\cos^{2a-1}\theta r^{2b-1}\sin^{2b-1}\theta e^{-r^2} r\,dr\,d\theta \\ &= 4\int_0^\infty r^{2a+2b-1}e^{-r^2}\,dr \int_0^{\frac{\pi}{2}} \cos^{2a-1}\theta\sin^{2b-1}\theta\,d\theta. \qquad (4.1.6)\end{aligned}$$

We now evaluate each of the two integrals in (4.1.6) separately. In the first integral we use the substitution $x = r^2$ to get

$$\begin{aligned}\int_0^\infty r^{2a+2b-1}e^{-r^2}\,dr &= \int_0^\infty \frac{x^{a+b-\frac{1}{2}}e^{-x}}{2\sqrt{x}}\,dx \\ &= \int_0^\infty \frac{x^{a+b-1}}{2}e^{-x}\,dx = \frac{1}{2}\Gamma(a+b). \qquad (4.1.7)\end{aligned}$$

The second integral of (4.1.6), through the substitution $y = \sin^2\theta$ (noting that this makes $dy = 2\sin\theta\cos\theta d\theta$), yields

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2a-1}\theta\sin^{2b-1}\theta\,d\theta &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{2a-2}\theta\sin^{2b-2}\theta(2\sin\theta\cos\theta)\,d\theta \\ &= \frac{1}{2}\int_0^1 (1-y)^{a-1}y^{b-1}\,dy = \frac{1}{2}\beta(b,a) \\ &= \frac{1}{2}\beta(a,b). \qquad (4.1.8)\end{aligned}$$

The proof of the theorem is completed by substituting the results of (4.1.7) and (4.1.8) in (4.1.6). ∎

Theorem 4.1.9. *If X is a beta random variable with parameters β_3 and β_4 and if $B_i = \beta_3 + \beta_4 + i$, $i = 1, 2, \ldots$, then $E(X^k)$, the k-th moment of X, is given by*

$$E(X^k) = \frac{\beta(\beta_3 + k + 1, \beta_4 + 1)}{\beta(\beta_3 + 1, \beta_4 + 1)} \qquad (4.1.10)$$

and also by the recursive relation

$$E(X^k) = \left(\frac{\beta_3 + k}{B_{k+1}}\right) E(X^{k-1}) \text{ for } k = 1, 2, 3, \ldots, \quad E(X^0) = 1. \qquad (4.1.11)$$

Proof. By definition

$$\begin{aligned}E(X^k) &= \int_0^1 \frac{x^k x^{\beta_3}(1-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)}\,dx \\ &= \frac{\beta(\beta_3+k+1,\beta_4+1)}{\beta(\beta_3+1,\beta_4+1)}\int_0^1 \frac{x^{\beta_3+k}(1-x)^{\beta_4}}{\beta(\beta_3+k+1,\beta_4+1)}\,dx. \qquad (4.1.12)\end{aligned}$$

Since the integral in (4.1.12) is the integral of the beta p.d.f. with parameters $\beta_3 + k$ and β_4, its value must be 1. This establishes the part of the theorem given in (4.1.10).

For the second part of the theorem, we consider $E(X^k)/E(X^{k-1})$, which, from (4.1.10), can be written as

$$\frac{E(X^k)}{E(X^{k-1})} = \frac{\beta(\beta_3 + k + 1, \beta_4 + 1)}{\beta(\beta_3 + k, \beta_4 + 1)}. \tag{4.1.13}$$

Applying Theorem 4.1.5 to the numerator and denominator of the ratio in (4.1.13), we have

$$\frac{E(X^k)}{E(X^{k-1})} = \frac{\Gamma(\beta_3 + k + 1)\Gamma(\beta_4 + 1)}{\Gamma(\beta_3 + \beta_4 + k + 2)} \times \frac{\Gamma(\beta_3 + \beta_4 + k + 1)}{\Gamma(\beta_3 + k)\Gamma(\beta_4 + 1)}. \tag{4.1.14}$$

After canceling $\Gamma(\beta_4 + 1)$, and using Lemma 4.1.3 on $\Gamma(\beta_3 + k + 1)$ and $\Gamma(\beta_3 + \beta_4 + k + 2)$ in (4.1.14), we obtain

$$\frac{E(X^k)}{E(X^{k-1})} = \frac{\beta_3 + k}{\beta_3 + \beta_4 + k + 1} = \frac{\beta_3 + k}{B_{k+1}},$$

which completes the proof of the theorem by establishing (4.1.11). ∎

Corollary 4.1.15. *If X is a beta random variable with parameters β_3 and β_4 and we let $B_i = \beta_3 + \beta_4 + i$, $i = 1, 2, \ldots$, then*

$$E(X^k) = \frac{(\beta_3 + 1)(\beta_3 + 2)\cdots(\beta_3 + k)}{B_2 B_3 \cdots B_{k+1}}.$$

Proof. In this proof by mathematical induction, we first observe that when $k = 1$ the result follows from the substitution of $k = 1$ in (4.1.11). Now assume that the Corollary holds for $k - 1$, i.e., assume that

$$E(X^{k-1}) = \frac{(\beta_3 + 1)(\beta_3 + 2)\cdots(\beta_3 + k - 1)}{B_2 B_3 \cdots B_k}$$

and use (4.1.12) to express $E(X^k)$ by

$$\begin{aligned} E(X^k) &= \left(\frac{\beta_3 + k}{B_{k+1}}\right) E(X^{k-1}) \\ &= \left(\frac{\beta_3 + k}{B_{k+1}}\right) \frac{(\beta_3 + 1)(\beta_3 + 2)\cdots(\beta_3 + k - 1)}{B_2 B_3 \cdots B_k} \\ &= \frac{(\beta_3 + 1)(\beta_3 + 2)\cdots(\beta_3 + k)}{B_2 B_3 \cdots B_{k+1}}. \end{aligned}$$ ∎

Theorem 4.1.16. *If X is a beta random variable with parameters β_3 and β_4 and $B_i = \beta_3 + \beta_4 + i,\ i = 1, 2, \ldots,$ then*

$$\alpha_1 = E(X) = \mu = \frac{\beta_3 + 1}{B_2}, \tag{4.1.17}$$

$$\alpha_2 = E\left[(X-\mu)^2\right] = \sigma^2 = \frac{(\beta_3+1)(\beta_4+1)}{B_2^2 B_3}, \tag{4.1.18}$$

$$\sigma^3\alpha_3 = E\left[(X-\mu)^3\right] = 2\frac{(\beta_3+1)(\beta_4+1)(\beta_4-\beta_3)}{B_2^3 B_3 B_4}, \tag{4.1.19}$$

$$\begin{aligned}\sigma^4\alpha_4 &= E\left[(X-\mu)^4\right] = 3\left(\frac{(\beta_3+1)(\beta_4+1)}{B_2^4 B_3 B_4 B_5}\right)\\ &\quad\times\left(\beta_3\beta_4 B_2 + 3\beta_3^2 + 3\beta_4^2 + 5\beta_3 + 5\beta_4 + 4\right).\end{aligned} \tag{4.1.20}$$

Proof. Setting $k = 1$ in Corollary 4.1.15 gives the first assertion. To obtain (4.1.18), the second assertion, we note that

$$\sigma^2 = E(X^2) - [E(X)]^2$$

and by Corollary 4.1.15,

$$\sigma^2 = \frac{(\beta_3+1)(\beta_3+2)}{B_2 B_3} - \left(\frac{\beta_3+1}{B_2}\right)^2,$$

which can be simplified to (4.1.18).

In (4.1.19),

$$\begin{aligned}E\left[(X-\mu)^3\right] &= E\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right]\\ &= E(X^3) - 3\mu E(X^2) + 2\mu^3.\end{aligned}$$

Using Corollary 4.1.15, we get

$$\begin{aligned}E\ \left[(X-\mu)^3)\right] &= \frac{(\beta_3+1)(\beta_3+2)(\beta_3+3)}{B_2 B_3 B_4} - 3\frac{(\beta_3+1)^2(\beta_3+2)}{B_2^2 B_3}\\ &\quad +2\left(\frac{\beta_3+1}{B_2}\right)^3 = \left(\frac{\beta_3+1}{B_2^3 B_3 B_4}\right)\\ &\quad\times\left((\beta_3+2)(\beta_3+3)B_2^2 - 3(\beta_3+1)(\beta_3+2)B_2 B_4 + 2(\beta_3+1)^2 B_3 B_4\right)\\ &= 2\frac{(\beta_3+1)(\beta_4+1)(\beta_4-\beta_3)}{B_2^3 B_3 B_4}.\end{aligned}$$

For the proof of (4.1.20), we note that

$$\begin{aligned} E\left[(X-\mu)^4\right] &= E\left[X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4\right] \\ &= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4. \end{aligned}$$

From Corollary 4.1.15, we have

$$\begin{aligned} E\left[(X-\mu)^4\right] &= \frac{(\beta_3+1)(\beta_3+2)(\beta_3+3)(\beta_3+4)}{B_2B_3B_4B_5} \\ &\quad -4\left(\frac{\beta_3+1}{B_2}\right)\left(\frac{(\beta_3+1)(\beta_3+2)(\beta_3+3)}{B_2B_3B_4}\right) \\ &\quad +6\left(\frac{\beta_3+1}{B_2}\right)^2\left(\frac{(\beta_3+1)(\beta_3+2)}{B_2B_3}\right) - 3\left(\frac{\beta_3+1}{B_2}\right)^4 \\ &= \left(\frac{\beta_3+1}{B_2^4B_3B_4B_5}\right) \\ &\quad \times\Big[(\beta_3+2)(\beta_3+3)(\beta_3+4)B_2^3 - 4(\beta_3+1)(\beta_3+2)(\beta_3+3)B_2^2B_5 \\ &\quad + 6(\beta_3+1)^2(\beta_3+2)B_2B_4B_5 - (\beta_3+1)^3B_3B_4B_5\Big]. \end{aligned}$$

Through cumbersome but straightforward algebraic manipulations the last expression in the brackets can be equated to

$$3\left(\beta_3\beta_4B_2 + 3\beta_3^2 + 3\beta_4^2 + 5\beta_3 + 5\beta_4 + 4\right),$$

completing the proof of (4.1.20). ∎

4.2 The Generalized Beta Distribution and its Moments

To be able to use the beta distribution for fitting datasets, we need to make the distribution more flexible so that its support is not confined to the interval $(0, 1)$. This can be done through the introduction of location and scale parameters, β_1 and β_2, respectively. The transformation given in the following definition establishes a generalized beta random variable.

Definition 4.2.1. *If X is a beta random variable with parameters β_3 and β_4, for $\beta_2 > 0$, and any β_1, the random variable $Y = \beta_1 + \beta_2 X$ is said to have a Generalized Beta Distribution,* $\mathrm{GBD}(\beta_1, \beta_2, \beta_3, \beta_4)$.

We now proceed to establish the p.d.f. of a $\mathrm{GBD}(\beta_1, \beta_2, \beta_3, \beta_4)$ random variable.

Theorem 4.2.2. *The p.d.f. of the* GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ *random variable is*

$$f(x) = \begin{cases} \dfrac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1 \le x \le \beta_1+\beta_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let Y be a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ random variable. From Definition 4.2.1, $Y = \beta_1 + \beta_2 X$ where X is a beta random variable with parameters β_3 and β_4. The distribution function (d.f.) of Y is

$$F_Y(y) \;=\; P(Y \le y) = P(\beta_1 + \beta_2 X \le y) = P\left(X \le \frac{y-\beta_1}{\beta_2}\right).$$

Since X is a beta random variable, from (4.1.1) we have

$$F_Y(y) \;=\; \int_0^{(y-\beta_1)/\beta_2} \frac{x^{\beta_3}(1-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)}\,dx. \tag{4.2.3}$$

Using the substitution $u = \beta_1 + \beta_2 x$ in (4.2.3), we get

$$F_Y(y) = \int_{\beta_1}^{y} \frac{(u-\beta_1)^{\beta_3}(\beta_1+\beta_2-u)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}\,du,$$

which, through differentiation gives us the p.d.f. as stated. ∎

In order to fit a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ to a dataset using the method of moments, we will need to match $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ (as defined in (3.3.1) through (3.3.4)) to their GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ counterparts, α_1, α_2, α_3, α_4. These $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, for GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ distributions, are given in the following theorem.

Theorem 4.2.4. *Let Y be a* GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ *random variable with mean μ_Y and variance σ_Y^2. Then*

$$\alpha_1 \;=\; \mu_Y = \beta_1 + \frac{\beta_2(\beta_3+1)}{B_2}, \tag{4.2.5}$$

$$\alpha_2 \;=\; E\left[(Y-\mu_Y)^2\right] = \sigma_Y^2 = \frac{\beta_2^2(\beta_3+1)(\beta_4+1)}{B_2^2 B_3}, \tag{4.2.6}$$

$$\alpha_3 \;=\; \frac{E\left[(Y-\mu_Y)^3\right]}{\sigma_Y^3} = \frac{2(\beta_4-\beta_3)\sqrt{B_3}}{B_4\sqrt{(\beta_3+1)(\beta_4+1)}}, \tag{4.2.7}$$

$$\alpha_4 \;=\; \frac{E\left[(Y-\mu_Y)^4\right]}{\sigma_Y^4} = \frac{3B_3\left(\beta_3\beta_4 B_2 + 3\beta_3^2 + 5\beta_3 + 3\beta_4^2 + 5\beta_4 + 4\right)}{B_4 B_5(\beta_3+1)(\beta_4+1)}, \tag{4.2.8}$$

where $B_i = \beta_3 + \beta_4 + i$ for $i = 1, \ldots$.

Proof. We know from Definition 4.2.1 that $Y = \beta_1 + \beta_2 X$ where X has a beta distribution with parameters β_3 and β_4. From (4.1.17),

$$\alpha_1 = \mu_Y = E(Y) = \beta_1 + \beta_2 E(X) = \beta_1 + \frac{\beta_2(\beta_3+1)}{B_2}.$$

To prove the theorem for α_2, α_3, and α_4, we note that for $i = 2, 3, 4$,

$$\begin{aligned} \alpha_i &= E\left[(Y-\mu_Y)^i\right] = E\left[\left(\beta_1 + \beta_2 X - \beta_1 - \frac{\beta_2(\beta_3+1)}{B_2}\right)^i\right] \\ &= \beta_2^i E\left[\left(X - \frac{\beta_3+1}{B_2}\right)^i\right] = \beta_2^i E\left[(X-\mu_X)^i\right] \end{aligned} \tag{4.2.9}$$

where μ_X denotes the mean of the beta random variable X. We see from (4.2.9) that $\alpha_2 = \beta_2^2 \sigma_X^2$ where σ_X^2 is the variance of X and (4.2.6) follows from (4.1.18).

To prove the results for α_3 and α_4 we divide by the appropriate power of σ_Y and use (4.1.19) or (4.1.20). ∎

We now proceed to show that the (α_3^2, α_4)-space of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ distributions contains the region

$$1 + \alpha_3^2 < \alpha_4 < 1.8(1 + \alpha_3^2)$$

that the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ did not cover. The contour plots of the (α_3^2, α_4)-points of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ depicted in Figure 4.2–1 give a preliminary indication that the desired region is covered. The curves of Figure 4.2–1 correspond to values of $\beta_3 = -.99, -.9, -.7, -.5, -.25, 0, .25, .75, 1.5, 2.5, 4, 6$ and $-1 < \beta_4 < \infty$. The lines shown in heavy print are the boundary lines $\alpha_4 = 1 + \alpha_3^2$ and $\alpha_4 = 3 + 2\alpha_3^2$. The "turning points" of the curves occur when $\beta_3 = \beta_4$ and we can see from (4.2.7) that at these points $\alpha_3 = 0$ and from (4.2.8) that

$$\alpha_4 = 3\frac{2\beta_3+3}{2\beta_3+5} \times \frac{2\beta_3^3 + 8\beta_3^2 + 10\beta_3 + 4}{(2\beta_3+4)(2\beta_3+1)^2} = 3\frac{2\beta_3+3}{2\beta_3+5}.$$

More specifically, when $\beta_3 = \beta_4 = -1$, the turning point is at $\alpha_4 = 1$ and when $\beta_3 = \beta_4 = 0$, the turning point is at $\alpha_4 = 1.8$. Moreover, when $\beta_3 = \beta_4$,

$$\lim_{\beta_3 = \beta_4 \to \infty} \alpha_4 = 3.$$

The following theorem gives the (α_3^2, α_4)-space of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ distributions.

Theorem 4.2.10. *The* (α_3^2, α_4)*-space covered by the* GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ *is exactly*

$$1 + \alpha_3^2 < \alpha_4 < 3 + 2\alpha_3^2. \tag{4.2.11}$$

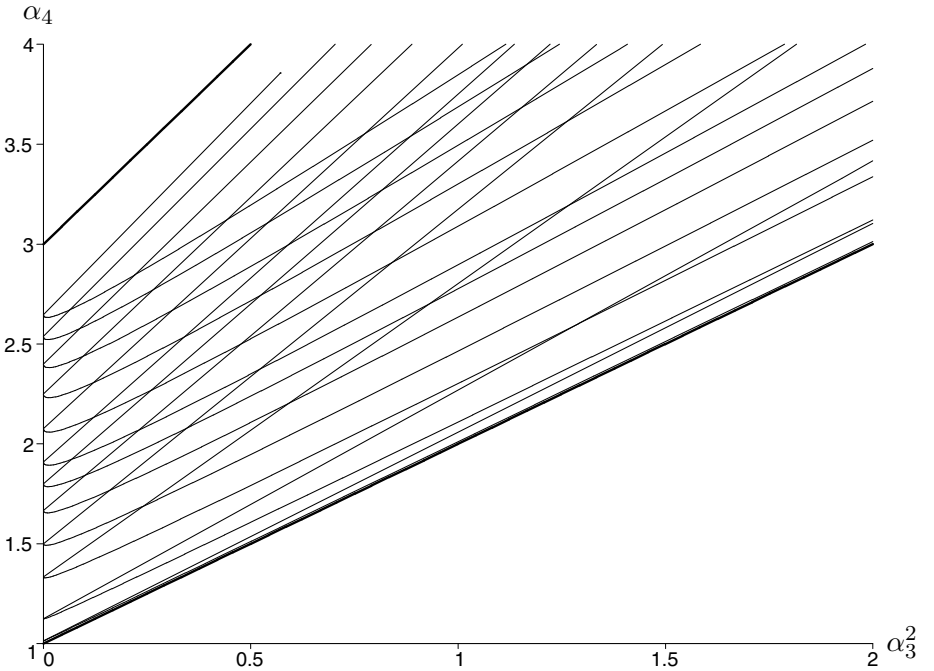


Figure 4.2–1. Contour plots of the (α_3^2, α_4) for $\beta_3 = -.99, -.9, -.7, -.5, -.25, 0, .25, .75, 1.5, 2.5, 4, 6$ and for $-1 < \beta_4 < \infty$.

Proof. As has been previously noted, the left inequality holds for all distributions. From (4.2.7) of Theorem 4.2.4,

$$3 + 2\alpha_3^2 = \frac{8(\beta_4 - \beta_3)^2 B_3}{B_4^2(\beta_3 + 1)(\beta_4 + 1)} + 3. \tag{4.2.12}$$

If we consider the right side of (4.2.11) as a fraction N/D, then

$$\begin{aligned} D &= B_4^2(\beta_3 + 1)(\beta_4 + 1), \qquad (4.2.13)\\ N &= 8(\beta_4 - \beta_3)^2 B_3 + 3B_4^2(\beta_3 + 1)(\beta_4 + 1) \\ &= 3\left[(B_3 + 1)(B_2 + 2)(\beta_3 + 1)(\beta_4 + 1) + \frac{8}{3}(\beta_4 - \beta_3)^2 B_3\right] \\ &= 3B_3\left[(B_2 + 2)(\beta_3 + 1)(\beta_4 + 1) + \frac{8}{3}(\beta_4 - \beta_3)^2\right] + R \end{aligned}$$

where

$$R = 3B_3(B_2 + 1)(\beta_3 + 1)(\beta_4 + 1) > 0. \tag{4.2.14}$$

Therefore,

$$\begin{aligned} N &> 3B_3\left[B_2(\beta_3+1)(\beta_4+1)+2(\beta_3+1)(\beta_4+1)+\frac{8}{3}(\beta_4-\beta_3)^2\right] \\ &= 3B_3\left[B_2\beta_3\beta_4+3\beta_3^2+3\beta_4^2+5(\beta_3+\beta_4)+4+S\right], \end{aligned}$$

with

$$S=\frac{2}{3}\beta_3^2+\frac{2}{3}\beta_4^2-\frac{4}{3}\beta_3\beta_4=\frac{2}{3}(\beta_4-\beta_3)^2\geq 0.$$

This implies

$$N>3B_3\left[B_2\beta_3\beta_4+3\beta_3^2+3\beta_4^2+5(\beta_3+\beta_4)+4\right]$$

and from (4.2.13) and (4.2.14),

$$3+2\alpha_3^2=\frac{N}{D}>\frac{3B_3\left[B_2\beta_3\beta_4+3\beta_3^2+3\beta_4^2+5(\beta_3+\beta_4)+4\right]}{B_4^2(\beta_3+1)(\beta_4+1)}=\alpha_4.$$

Since α_3^2 and α_4 are continuous functions of β_3 and β_4, the (α_3^2,α_4)-space covered by the GBD$(\beta_1,\beta_2,\beta_3,\beta_4)$ is the entire space specified by the theorem. We have now shown that the (α_3^2,α_4) of every GBD$(\beta_1,\beta_2,\beta_3,\beta_4)$ satisfies (4.2.11). To complete the proof we need to show that for every pair $(\alpha_3^{2*},\alpha_4^*)$ satisfying (4.2.11), there is a vector $(\beta_1^*,\beta_2^*,\beta_3^*,\beta_4^*)$ such that the GBD$(\beta_1^*,\beta_2^*,\beta_3^*,\beta_4^*)$ has $\alpha_3^2=\alpha_3^{2*}$ and $\alpha_4=\alpha_4^*$. This would show that the GBD$(\beta_1,\beta_2,\beta_3,\beta_4)$ "covers" the space described by (4.2.11). This is established numerically in Section 4.3. ∎

4.3 Estimation of GBD$(\beta_1,\beta_2,\beta_3,\beta_4)$ Parameters

To use the method of moments to estimate GBD$(\beta_1,\beta_2,\beta_3,\beta_4)$ parameters, we need to solve the system of equations

$$\alpha_i=\hat{\alpha}_i \quad \text{for } i=1,\,2,\,3,\,4$$

for $\beta_1,\beta_2,\beta_3,\beta_4$, where $\hat{\alpha}_i$ are specified in (3.3.1) through (3.3.4) and α_i are given in Theorem 4.2.4. This task is simplified somewhat by the fact that the subsystem of equations for $i=3$ and 4 depends only on β_3 and β_4. Thus, if β_3 and β_4 could be obtained from $\alpha_3=\hat{\alpha}_3$ and $\alpha_4=\hat{\alpha}_4$, then $\alpha_2=\hat{\alpha}_2$ would yield β_2 and subsequently, $\alpha_1=\hat{\alpha}_1$ would give us β_1.

The complexity of the third and fourth equations makes it impossible to find closed form solutions; therefore, as in the case of the GLD$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ distributions, we appeal to numeric methods based on search algorithms.

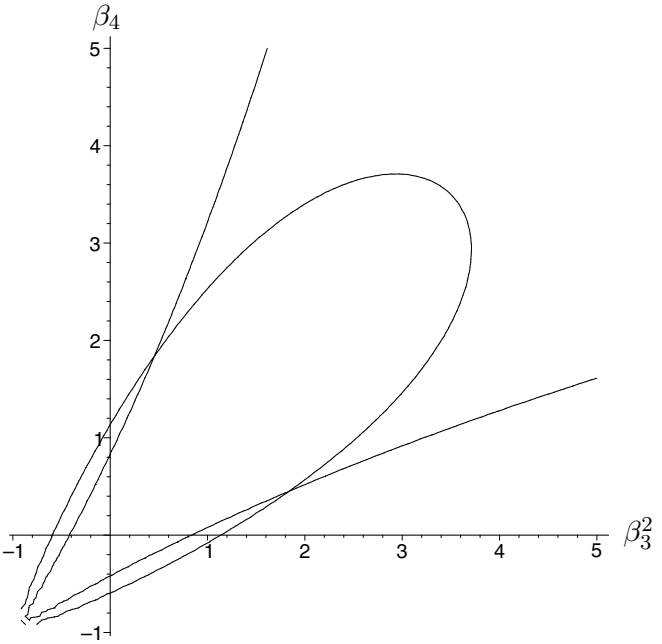


Figure 4.3–1. The contour curves for $\alpha_3^2 = 0.25$ and $\alpha_4 = 2.5$.

To develop some insight about where to begin the search for a solution we start by considering a specific case by letting $\hat{\alpha}_3^2 = 0.25$ and $\hat{\alpha}_4 = 2.5$. Figure 4.3–1 shows the contour curves for this choice of $\hat{\alpha}_3^2$ and $\hat{\alpha}_4$ (the "oval-shaped" curve is for $\hat{\alpha}_4 = 2.5$). The intersection of these two curves, roughly at $(2, 0.5)$ and $(0.5, 2)$, indicates the existence of solutions with (β_3, β_4) near these points when $(\hat{\alpha}_3, \hat{\alpha}_4) = (\pm 0.5, 2.5)$. For $\hat{\alpha}_3 = +\sqrt{0.25} = 0.5$ we have the solution near $(0.5, 2)$ and for $\hat{\alpha}_3 = -0.5$ we have the solution near $(2, 0.5)$.

A more precise solution can be obtained through the use of the **R** program `FindBetasM`. This program, discussed in some detail in Appendix A, is similar to the `FindLambdasM` program of Section 3.3. It takes the vector of (α_1, α_2, α_3, α_4) as its only argument and returns the vector of values of β_1, β_2, β_3, β_4. If the search for a solution fails, `FindBetasM` returns the vector `0 0 0 0`.

Suppose that for some $\hat{\alpha}_1$ and $\hat{\alpha}_2$ (say, $\hat{\alpha}_1 = 0$ and $\hat{\alpha}_2 = 1$), $\hat{\alpha}_3 = 0.5$ and $\alpha_4 = 2.5$, we want to obtain a solution for $\beta_1, \beta_2, \beta_3, \beta_4$. The command we use is

```
> FindBetasM(c(0, 1, 0.5, 2.5));
```

and the result from this command is

```
[1] − 1.6439073, 4.8592432, 0.4498795, 1.8358348
```

indicating the GBD($\beta_1, \beta_2, \beta_3, \beta_4$) fit

$$\text{GBD}(-1.6439, 4.8592, 0.4499, 1.8358). \tag{4.3.1}$$

Figures 4.3–2 and 4.3–3 give α_3^2 and α_4 contour curves, respectively. The curves in Figure 4.3–2 are for $\alpha_3^2 = 0.01, 0.05, 0.1, 0.2, 0.4, 1, 2, 3, 4, 5,$ and 6. Each curve has two "branches" arranged symmetrically about the line $\beta_3 = \beta_4$. The innermost pair (those on either side of $\beta_3 = \beta_4$ and closest to it) is associated with $\alpha_3^2 = 0.01$, the next pair away from $\beta_3 = \beta_4$ is associated with $\alpha_3^2 = 0.05$, and so on. The contour curves given in Figure 4.3–3 have $\alpha_4 = 1.3, 1.6, 1.8, 1.9, 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.75, 3.5,$ and 4.5. The innermost loop (hardly visible, near the point $(-1, -1)$) is for $\alpha_4 = 1.3$, the next is for $\alpha_4 = 1.6$, and so on.

To provide a graphic guide for solutions to $\hat{\alpha}_3 = \alpha_3$ and $\hat{\alpha}_4 = \alpha_4$, Figure 4.3–4 superimposes the sets of contours from Figures 4.3–2 and 4.3–3. The curves given in solid lines are the contours for α_4 and those in dotted lines are the contours for α_3^2.

Those who do not have access to a computing system that (through **R** or some other software) can provide solutions to $\alpha_3 = \hat{\alpha}_3$ and $\alpha_4 = \hat{\alpha}_4$ can consult Dudewicz and Karian (1996) for extensive tabled values of (β_3, β_4) for (α_3, α_4) that cover the region of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ stipulated in Theorem 4.2.10. We include the table from Dudewicz and Karian (1996) in Appendix C and give an algorithm for using this table to obtain approximate solutions of β_3 and β_4.

Algorithm GBD–M: Fitting a GBD distribution to data by the method of moments.

GBD–M–1. Use (3.3.1) through (3.3.4) to compute $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$;

GBD–M–2. Find the entry point in a table of Appendix C closest to $(|\hat{\alpha}_3|, \hat{\alpha}_4)$;

GBD–M–3. Using $(|\hat{\alpha}_3|, \hat{\alpha}_4)$ from Step GBD–M–2 extract β_3 and β_4 from the table;

GBD–M–4. If $\hat{\alpha}_3 < 0$, interchange β_3 and β_4;

GBD–M–5. Substitute β_3 and β_4 (obtained in GBD–M–3 or GBD–M–4) and $\hat{\alpha}_2$, for $\beta_3, \beta_4,$ and α_2, respectively, in (4.2.6) and solve it for β_2. Equivalently, compute β_2 from

$$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{\frac{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2}{(\beta_3 + 1)(\beta_4 + 1)}}.$$

GBD–M–6. Substitute β_2, β_3 and β_4 (obtained in GBD–M–3 or GBD–M–4 and GBD–M–5) and $\hat{\alpha}_1$, for $\beta_2, \beta_3, \beta_4,$ and α_1, respectively, in (4.2.5) and solve it for β_1. Equivalently, compute β_1 from

$$\beta_1 = \hat{\alpha}_1 - \frac{\beta_2(\beta_3 + 1)}{\beta_3 + \beta_4 + 2}.$$

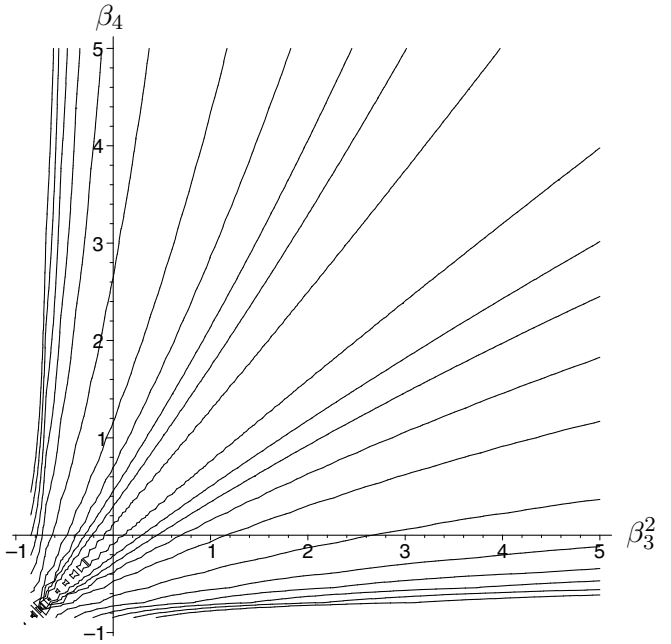


Figure 4.3–2. Contour curves for $\alpha_3^2 = .01, .05, .1, .2, .4, 1, 2, 3, 4, 5, 6$.

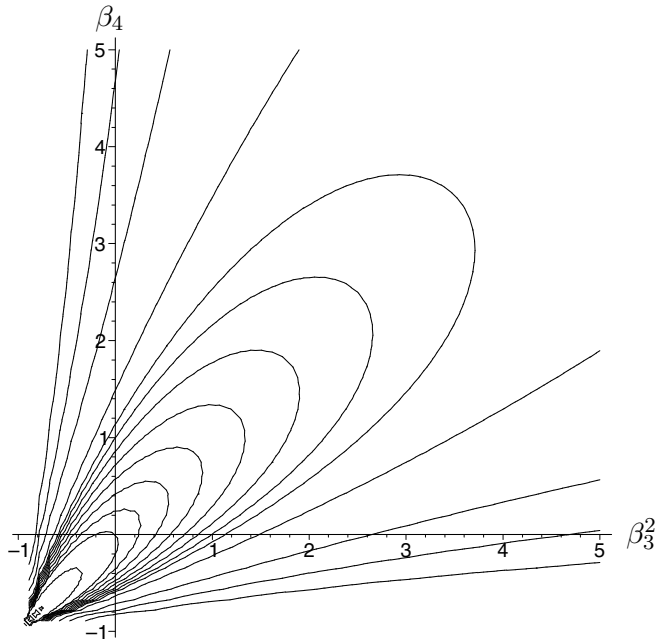


Figure 4.3–3. Contour curves for $\alpha_4 = 1.3, 1.6, 1.8, 1.9, 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.75, 3.5, 4.5, 6$.

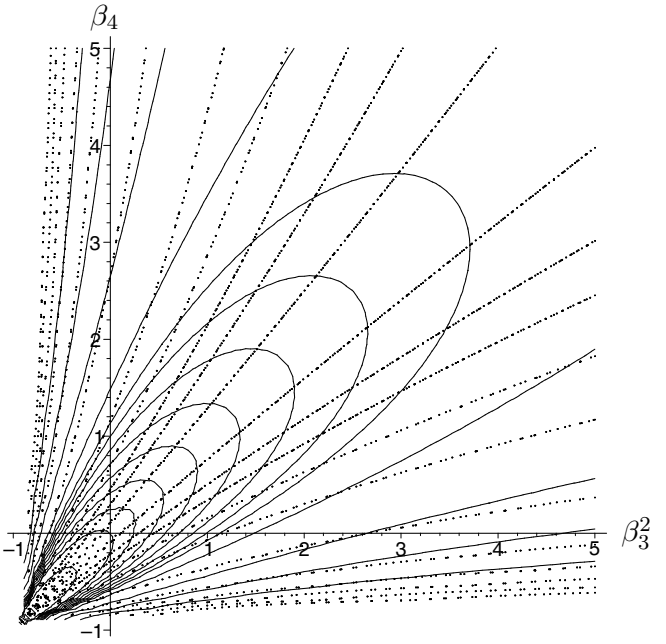


Figure 4.3–4. The contour curves for α_3^2 and α_4.

To illustrate the use of Algorithm GBD–M and the table of Appendix C, consider the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ that were discussed earlier in the second paragraph of this section:

$$\hat{\alpha}_1 = 0, \quad \hat{\alpha}_2 = 1, \quad \hat{\alpha}_3 = 0.5, \quad \hat{\alpha}_4 = 2.5.$$

The entry point to the table of Appendix C is $\hat{\alpha}_3 = 0.5$ and $\hat{\alpha}_4 = 2.49$ and this produces $\beta_3 = 0.4253$ and $\beta_4 = 1.7781$. Next, following Step GBD–M–5, we compute $\beta_2 = 4.8186$. Finally, we follow Step GBD–M–6 and compute $\beta_1 = -1.6339$. As expected, this fit,

$$\text{GBD}(-1.6339, 4.8186, 0.4253, 1.7781), \tag{4.3.2}$$

obtained from Algorithm GBD–M and Appendix C, has $\beta_1, \beta_2, \beta_3, \beta_4$ close to the more accurate fit in (4.3.1) that was obtained through direct computation. For the fit in (4.3.2), we have

$$\begin{aligned} |\hat{\alpha}_1 - \alpha_1| &= 3.6 \times 10^{-6}, \quad |\hat{\alpha}_2 - \alpha_2| = 1.7 \times 10^{-5}, \\ |\hat{\alpha}_3 - \alpha_3| &= 2.2 \times 10^{-5}, \quad |\hat{\alpha}_4 - \alpha_4| = 1.0 \times 10^{-2}, \end{aligned}$$

and for the fit in (4.3.1), we have

$$\begin{aligned} |\hat{\alpha}_1 - \alpha_1| &= 1.7 \times 10^{-5}, \quad |\hat{\alpha}_2 - \alpha_2| = 3.7 \times 10^{-5}, \\ |\hat{\alpha}_3 - \alpha_3| &= 4.1 \times 10^{-6}, \quad |\hat{\alpha}_4 - \alpha_4| = 1.6 \times 10^{-5}. \end{aligned}$$

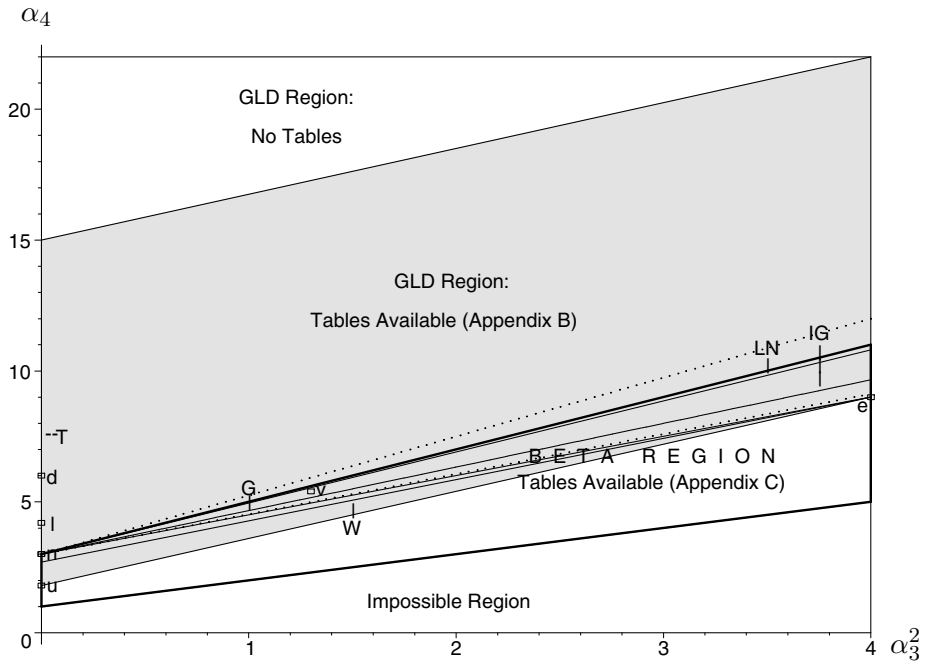


Figure 4.3–5. The (α_3^2, α_4)-space covered by the EGLD system.

However, if we use the full accuracy provided by `FindBetasM`, instead of the truncations shown in (4.3.1), we obtain

$$\begin{aligned} |\hat{\alpha}_1 - \alpha_1| &= 1.8 \times 10^{-8}, & |\hat{\alpha}_2 - \alpha_2| &= 1.8 \times 10^{-9}, \\ |\hat{\alpha}_3 - \alpha_3| &= 2.1 \times 10^{-8}, & |\hat{\alpha}_4 - \alpha_4| &= 4.1 \times 10^{-9}. \end{aligned}$$

We now combine the results of this chapter with those of Chapter 3 to form the **system of distributions whose (α_3^2, α_4)-space (actually $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$-space since any α_1 and α_2 are possible) covers the entire range of possibilities**.

Definition 4.3.4. *The Extended Generalized Lambda Distribution (*EGLD*) system consists of the* GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) *and* GBD($\beta_1, \beta_2, \beta_3, \beta_4$) *families of distributions.*

Appendices B and C provide tabled solutions for some, but not all, of the (α_3^2, α_4) pairs arising from EGLD distributions. Figure 4.3–5 shows the (α_3^2, α_4)-space (restricted to $\alpha_3^2 \leq 4$ and $\alpha_4 \leq 22$) and various components of this space. The GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) covers the shaded region, where $\alpha_3^2 \leq 4$ and $-1.9\alpha_3^2 + \alpha_4 \leq 15$ and solutions are available from the table of Appendix B. It also covers the region above the shaded area where tabled solutions are not available and computations become increasingly difficult as (α_3^2, α_4) moves away from the shaded area.

The portion of (α_3^2, α_4)-space covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ is enclosed between two thick lines and marked "B E T A R E G I O N" in Figure 4.3–5. The GLD and GBD regions overlap and within this intersection both GLD and GBD fits are possible. This possibility has benefits as discussed in Section 3.5 and reduces the impact of the variability in (α_3^2, α_4). The table of Appendix C provides approximate solutions for some of the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$, with $\alpha_3^2 \leq 4$. When the (α_3^2, α_4) within this region gets close to the boundaries, computations become difficult, making it impossible to extend the table of Appendix C to (α_3^2, α_4) that are close to the boundaries. This is especially true for the upper boundary $(\alpha_4 = 3 + 2\alpha_3^2)$ where at least one of β_3 or β_4 gets very large when (α_3^2, α_4) gets close to the boundary. However, when (α_3^2, α_4) is close to the upper boundary we are also in the GLD region and have in Appendix B the table needed to fit a GLD. This, coupled with the fact that there are no computational problems in the non-overlap GBD region, means we have no problem fitting an EGLD anywhere in the combined regions.

The portion of Figure 4.3–5, bounded by the dotted lines, that straddles the GLD and GBD regions represents the (α_3^2, α_4) points of the F-distribution with $\alpha_3^2 < 4$. The lines that show the (α_3^2, α_4) points of a number of distributions considered in Section 3.4 are also shown in Figure 4.3–5 (some of these lines are actually curves when extended beyond $\alpha_3^2 = 4$). The lines of (α_3^2, α_4) points of the gamma, inverse Gaussian, lognormal, Student's t, and Weibull distributions are labeled with "G," "IG," "LN," "T," and "W," respectively. Of the distributions considered in Section 3.4, the double exponential, exponential, logistic, normal, uniform, largest extreme value (hence, also the extreme value) have single points for their (α_3^2, α_4). These are designated in Figure 4.3–5, respectively, by "d," "e," "l," "n," "u," and "v." Only two of the distributions considered in Section 3.4 are not designated in Figure 4.3–5. The chi-square distribution is a special case of the gamma, hence its (α_3^2, α_4) points are included among those of the gamma distribution, and the Pareto distribution does not have any (α_3^2, α_4) points that are within the α_3^2 and α_4 limitations of Figure 4.3–5.

4.4 GBD Approximations of Some Well-Known Distributions

In this section we attempt to fit a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ to the distributions already considered in Section 3.4, where GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fits were found. The **R** program, `FindBetasM`, is specifically designed for this purpose and its only argument is the vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the distribution being approximated. The output of `FindBetasM` is the vector $(\beta_1, \beta_2, \beta_3, \beta_4)$ that specifies the GBD distribution. In all cases, `FindBetasM` makes sure that $\max|\hat{\alpha}_i - \alpha_i| \leq 10^{-5}$ and if this is not possible, it returns (0, 0, 0, 0) as an indication that a fit with this constraint could not be attained. Whenever this occurs, if we can make a reasonable guess (from

the tables of Appendix C) regarding the location of (β_3, β_4), we can conduct a search for a solution in a specific region of (β_3, β_4)-space. This can be done through the program `RefineSeachGBD`, which uses five arguments: the vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, a vector indicating the minimum and maximum values of β_3 for the search, a similar vector for β_4, a positive integer indicating the grid partitions to be used (the larger this argument, the finer the grid), a positive integer indicating the number of times to "zoom in" during the grid search. `RefineSearchGBD` returns β_1, β_2, β_3 and β_4 with the minimum value of $\max(|\hat{\alpha}_3 - \alpha_3|, \hat{\alpha}_4 - \alpha_4|)$ as well as the value of this $\max(|\hat{\alpha}_3 - \alpha_3|, \hat{\alpha}_4 - \alpha_4|)$.

For each of the distributions we approximate, we check the $\alpha_4 < 3 + 2\alpha_3^2$ constraint and the closeness of the approximating p.d.f. to the p.d.f. being approximated. This is done by checking the support of the fitted EGLD, comparing the original and the fitted p.d.f.s graphically, and by computing $\sup|\hat{f}(x) - f(x)|$ where $\hat{f}(x)$ is the p.d.f. of the approximating distribution and $f(x)$ is the p.d.f. of the distribution being approximated. As was done in Section 3.4, we also evaluate $\sup|\hat{F}(x) - F(x)|$ where $F(x)$ and $\hat{F}(x)$ are the d.f.s of the approximating and approximated distributions, respectively. The "discrepancies" between $\hat{f}(x)$ and $f(x)$ as measured by the L_1 and L_2 distances $||\hat{f} - f||_1$ and $||\hat{f} - f||_2$ are also provided.

In many cases, the (α_3^2, α_4) points are located within the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ region but near the $\alpha_4 = 3+2\alpha_3^2$ boundary, either making it impossible to compute $\beta_1, \beta_2, \beta_3, \beta_4$ or producing fits with very large β_3 or β_4 or both.

4.4.1 The Normal Distribution

The first four moments of $N(\mu, \sigma^2)$, given in Section 3.4.1, are

$$\alpha_1 = \mu, \qquad \alpha_2 = \sigma^2, \qquad \alpha_3 = 0, \qquad \alpha_4 = 3.$$

Since these satisfy $\alpha_4 = 3 + 2\alpha_3^2$, this (α_3^2, α_4) is located on the boundary of the (α_3^2, α_4) pairs that are attainable by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. We saw earlier that when $\beta_3 = \beta_4$ (a necessary condition for symmetric distributions such as $N(0, 1)$),

$$\alpha_4 = 3\frac{2\beta_3 + 3}{2\beta_3 + 5}.$$

Therefore, for large β_3, and $\beta_4 = \beta_3$, a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ will have its (α_3^2, α_4) close to $(0, 3)$, the (α_3^2, α_4) of $N(0, 1)$. Using `FindBetasM` with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 3)$ gives the fit

$$\text{GBD}(-1414.214,\ 2828.428,\ 999998.848,\ 999998.848).$$

The graph of the p.d.f. of the approximating GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ is indistinguishable from that of the $N(0, 1)$ distribution, giving a visual indication that this fit may be even better than the rather good GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit that

was obtained in Section 3.4.1 (see Figure 3.4–1, where the $N(0,1)$ and the fitted GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ p.d.f.s are *almost* indistinguishable). The support of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation is $[-1414.214,\ 1414.214]$, contrasted with the support of $[-5.06,\ 5.06]$ for the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit; which one is preferable depends on the application on hand, as discussed at the beginning of Chapter 3.

For this fit we have

$$\sup|\hat{f}(x) - f(x)| = 1.117 \times 10^{-7} \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 3.400 \times 10^{-8}.$$

The L_1 and L_2 norms of the difference between $f(x)$ and $\hat{f}(x)$ are, respectively

$$||\hat{f} - f||_1 = 1.1666 \times 10^{-7} \quad \text{and} \quad ||\hat{f} - f||_2 = 9.6436 \times 10^{-8}.$$

It is interesting to note that although the $(\alpha_3^2, \alpha_4) = (0, 3)$ could not be attained by a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$, the approximation that was obtained has $(\alpha_3^2, \alpha_4) = (0,\ 2.999997000)$. Even though, by all the measures we have used, this GBD approximation to $N(0,1)$ is quite good, it is possible to obtain better approximations by letting β_3 and β_4 get even larger. For example, if could use the **R** program `RefineSearchGBD` to search for a fits with larger β_3 and β_4. `RefineSearchGBD` works in a manner similar to `RefineSearchGLDM` described in the previous chapter: its five arguments consist of the vector $(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)$, a vector specifying the minimum and maximum values for the search of β_3, a similar vector for β_4, a positive integer indicating the grid to be used, and another positive integer specifying the number of repeated applications the grid search. Detailed information about `RefineSearchGBD` is given in Appendix A.

When `RefineSearchGBD` is used through the command

```
> RefineSearchGBD(c(0,1,0,3), c(1000000,100000000), c(1000000,100000000), 50, 4)
```

we obtain β_1, β_2, β_3, β_4, and $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$, respectively, as

```
[1] -1.414213e+04  2.828426e+04  9.999987e+07  9.999987e+07  3.000004e-08.
```

4.4.2 The Uniform Distribution

The moments of the uniform distribution on the interval (a, b) are, as noted in Section 3.4.2,

$$\alpha_1 = \frac{a+b}{2}, \qquad \alpha_2 = \frac{(b-a)^2}{12}, \qquad \alpha_3 = 0, \qquad \alpha_4 = \frac{9}{5}.$$

Since

$$\alpha_4 = \frac{9}{5} < 3 = 3 + 2\alpha_3^2$$

for all uniform distributions, we should expect GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximations that are reasonably good. In the specific case of the uniform distribution on the $(0, 1)$ interval, we search via `FindBetasM` to obtain the fit

$$\text{GBD}(0.0005664,\ 0.9994,\ -0.002370,\ -0.001343).$$

Since, in this case, β_3 and β_4 are close to 0, we cannot rely that the restriction $\max|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4| < 10^{-5}$ holds. Following the command `> U <- c(1/2, 1/48, 0, 9/5)` we appeal to `RefineSearchGBD` with

$$> \texttt{RefineSearchGBD}(\texttt{U}, \texttt{c}(-0.00001, 0.00001), \texttt{c}(-0.00001, 0.00001), 50, 4)$$

to obtain

$$[1]\ 0.00000\text{e}+00\ \ 1.00000\text{e}+00\ \ -4.76456\text{e}-22\ \ -4.76456\text{e}-22\ \ 0.00000\text{e}+00$$

for β_1, β_2, β_3, β_4 and $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$, respectively. There is a clear indication that there may exist a perfect fit with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0, 1, 0, 0)$. This can easily be confirmed by substituting these values in the p.d.f. of the GBD given in Theorem 4.2.2. In general, for the uniform distribution on the interval (a, b), the fit (again, this will be an exact fit) will be

$$\text{GBD}(a, b - a, 0, 0).$$

4.4.3 The Student's t Distribution

We recall from the discussion of $t(\nu)$ (the t distribution with ν degrees of freedom) in Section 3.4.3 that for the i-th moment of $t(\nu)$ to exist, ν must exceed i. Whenever the first 4 moments of $t(\nu)$ exist (i.e., when $\nu \geq 5$), these moments, as given in Section 3.4.3, will be

$$\alpha_1 = 0, \qquad \alpha_2 = \frac{\nu}{\nu - 2}, \qquad \alpha_3 = 0, \qquad \alpha_4 = 3\frac{\nu - 2}{\nu - 4}.$$

For all $t(\nu)$ distributions,

$$\alpha_4 = 3\frac{\nu - 2}{\nu - 4} > 3 = 3 + 2\alpha_3^2,$$

locating (α_3^2, α_4) outside of the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. This makes it impossible to find a moment-based GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation to $t(\nu)$, particularly when ν is small. For large ν, the (α_3^2, α_4) of $t(\nu)$ will come close to the boundary $\alpha_4 = 3 + 2\alpha_3^2$ and we may be able to find a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$, such as the one for $N(0, 1)$ in Section 4.4.1, that provides a reasonable approximation to such a $t(\nu)$.

4.4.4 The Exponential Distribution

The exponential distribution with parameters α and θ, defined in Section 3.4.4, has

$$\alpha_1 = \theta, \qquad \alpha_2 = \theta^2, \qquad \alpha_3 = 2, \qquad \alpha_4 = 9$$

and

$$\alpha_4 = 9 < 11 = 3 + 2\alpha_3^2.$$

It is clear from the table of Appendix C that a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation of the exponential distribution must have large β_4 and β_3 near zero. For $\theta = 1$, we set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 2, 9)$ and use it in `FindBetasM` to find the approximation

$$\text{GBD}(9.8095 \times 10^{-5},\ 29432.45,\ -2.6412 \times 10^{-4},\ 29425.57).$$

We again have a small β_3, indicating the possibility of a large $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$, which turns out to be about 6×10^{-4}. The indication from this fit is that we should be looking for GBD fits with large β_2 and β_4 and β_1 and β_3 close to zero. We note that substituting $\beta_1 = \beta_3 = 0$ and $\beta_2 = \beta_4$ in the expression for the p.d.f. of the GBD (Theorem 4.2.2) and then taking the limit as $\beta_4 \to \infty$, converts the GBD p.d.f. to e^{-x}, the p.d.f. of the exponential distribution with $\theta = 1$. Thus, we can obtain as precise a fit to the exponential distribution with $\theta = 1$ as we wish by letting $\beta_1 = \beta_3 = 0$ and $\beta_2 = \beta_4$ get sufficiently large. In this sense, we have a perfect fit "in limit."

For the general exponential distribution, we still need to set $\beta_1 = \beta_3 = 0$. However, when we look at the ratio β_2/β_4 (obtained from the expression for β_2 in Step GBD–M–5 of Algorithm GBD–M) we get

$$\frac{\beta_2}{\beta_4} = \frac{(\beta_4 + 2)\sqrt{\theta^2(\beta_4 + 3)}}{\beta_4\sqrt{\beta_4 + 1}}.$$

The limit, as $\beta_4 \to \infty$ of this expression is θ. This allows us to obtain a perfect "in limit" fit for the general exponential by taking $\beta_1 = \beta_3 = 0$, $\beta_2 = \theta r$ and $\beta_4 = r$ and letting r get large.

As an example, consider the exponential distribution with $\theta = 3$. If we set $\beta_1 = \beta_3 = 0$, $\beta_2 = 3 \times 10^6$ and $\beta_4 = 10^6$ we get an approximation with

$$\max\left(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|\right) < 4.8 \times 10^{-5},$$

$$\sup|\hat{f}(x) - f(x)| = 3.7 \times 10^{-5} \qquad \sup|\hat{F}(x) - F(x)| = 4.0 \times 10^{-6},$$

$$||\hat{f} - f||_1 = 7.4 \times 10^{-5} \quad \text{and} \quad ||\hat{f} - f||_2 = 3.7 \times 10^{-5}.$$

4.4.5 The Chi-Square Distribution

As noted in Section 3.4.5, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ for $\chi^2(\nu)$, the chi-square distribution with ν degrees of freedom, are

$$\alpha_1 = \nu, \qquad \alpha_2 = 2\nu, \qquad \alpha_3 = \frac{2\sqrt{2}}{\sqrt{\nu}}, \qquad \alpha_4 = 3 + \frac{12}{\nu}.$$

Since

$$\alpha_4 = 3 + \frac{12}{\nu} < 3 + \frac{16}{\nu} = 3 + 2\alpha_3^2,$$

the (α_3^2, α_4) pairs of the $\chi^2(\nu)$ are within the region covered by the GBD with (α_3^2, α_4) approaching the boundary of this region as $\nu \to \infty$. The contrast worthy of note between the $t(\nu)$ and $\chi^2(\nu)$ distributions is that the (α_3^2, α_4) pairs approach the boundary from opposite sides; from inside the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ in the case of $\chi^2(\nu)$, and from outside this region for $t(\nu)$.

To fit a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ to a specific $\chi^2(\nu)$ distribution, say for $\nu = 5$, we use `FindBetasM` to obtain the fit

$$\text{GBD}(8.1347 \times 10^{-4},\ 6.0000 \times 10^4,\ 1.4989,\ 2.9989 \times 10^4).$$

Having already observed that for a good fit β_4 needs to be large, we explore that possibility by letting `SC` represent the $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of this distribution and issuing the command

$$> \texttt{RefineSearchGBD(CS, c(1, 2), c(1000000, 100000000), 50, 4)}$$

to get

$$[1]\ \ \texttt{-8.750011e-08}\ \ \texttt{1.999998e+08}\ \ \texttt{1.500000}\ \ \texttt{9.999987e+07}\ \ \texttt{3.360004e-07}$$

which indicates that $\beta_3 = 1.5$, β_4 is large and for this specific choice of β_3 and β_4, $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 3.36 \times 10^{-7}$.

With similar efforts for other values of ν, we conclude that the $\chi^2(\nu)$ distribution can be approximated by GBD$(0, 2r, \nu/2 - 1, r)$, when r is large. In fact this GBD provides a perfect fit "in limit." It is easiest to observe this when ν is even, say $\nu = 6$. Substituting $\beta_1 = 0$, $\beta_2 = 2r$, $\beta_3 = 2$ and $\beta_4 = r$ in the GBD p.d.f. gives

$$\frac{1}{16} x^2 (r+1)(r+2)(r+3) r^{-r-3} \left(r - \frac{x}{2}\right)^r.$$

The limit as $r \to \infty$ of $(r+1)(r+2)(r+3)r^{-3}$ is 1 and when we remove this part from the expression for the p.d.f. we are left with

$$\frac{1}{16} x^2 \left(1 - \frac{x}{2r}\right),$$

whose limit as $r \to \infty$ is

$$\frac{x^2}{16}e^{-x/2},$$

or the p.d.f. of the $\chi^2(6)$ distribution.

For the specific case of the $\chi^2(5)$ distribution with the

$$\text{GBD}(8.1347 \times 10^{-4},\ 6.0000 \times 10^4,\ 1.4989,\ 2.9989 \times 10^4).$$

approximation that we got in our initial attempt, we have

$$\max\left(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|\right) < 4.8 \times 10^{-5},$$

$$\sup|\hat{f}(x) - f(x)| = 3.6 \times 10^{-5} \qquad \sup|\hat{F}(x) - F(x)| = 1.7 \times 10^{-4},$$

$$||\hat{f} - f||_1 = 7.7 \times 10^{-5} \quad \text{and} \quad ||\hat{f} - f||_2 = 3.0 \times 10^{-5}.$$

4.4.6 The Gamma Distribution

The gamma distribution with parameters α and θ, defined in Section 3.4.6, has

$$\alpha_1 = \theta\alpha, \qquad \alpha_2 = \theta^2\alpha, \qquad \alpha_3 = \frac{2}{\sqrt{\alpha}}, \qquad \alpha_4 = 3 + \frac{6}{\alpha}.$$

Since

$$\alpha_4 = 3 + \frac{6}{\alpha} < 3 + \frac{8}{\alpha} = 3 + 2\alpha_3^2,$$

$\text{GBD}(\beta_1, \beta_2, \beta_3, \beta_4)$ approximations should be available for all gamma distributions. For the specific case where $\alpha = 5$ and $\theta = 3$, considered in Section 3.4.6, we have

$$\alpha_1 = 15, \qquad \alpha_2 = 45, \qquad \alpha_3 = 2\frac{\sqrt{5}}{5}, \qquad \alpha_4 = \frac{21}{5}.$$

The table of Appendix C indicates a β_3 in the vicinity of 3 and a large β_4. Using `FindBetasM`, we obtain the approximation

$$\text{GBD}(0.0041716,\ 88302.79,\ 3.996201,\ 29414.08).$$

When a more detailed search is conducted by letting `G` be the $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of this distribution and issuing the command

```
> RefineSearchGBD(G, c(3, 5), c(1000000, 10000000), 50, 4)
```

we get

```
[1] 1.469977e − 05  2.999834e + 07  3.999987e + 00  9.999424e + 06  3.398393e − 07
```

indicating that for $\beta_3 \approx 4$ and $\beta_4 \approx 10^7$, $\max\left(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|\right) = 3.4 \times 10^{-7}$. The observation that is almost forced upon us is that β_1 should be 0, β_3 should be 4 and we should have $\beta_2 = 3\beta_4$ as both get large.

By varying and fitting other gamma distributions we observe the more general principle: The GBD approximation of the gamma distribution with parameters α and θ is GBD$(0,\, r\theta,\, \alpha-1,\, r)$, as $r \to \infty$. To verify this observation, we substitute $\beta_1 = 0$, $\beta_2 = r\theta$, $\beta_3 = 4$ and $\beta_4 = r$ in the GBD p.d.f. given in Theorem 4.2.2 to obtain

$$\frac{x^{\alpha-1}(r\theta - x)^r}{\beta(\alpha,\, r+1)(r\theta)^{\alpha+r}}.$$

Converting the β function to a Γ function (Theorem 4.1.5) we have

$$\frac{x^{\alpha-1}(r\theta - x)^r \Gamma(\alpha + r + 1)}{\Gamma(\alpha)\Gamma(r+1)(r\theta)^{\alpha+r}},$$

which after some rearrangement yields

$$\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^{\alpha}}\right)\left(\frac{r\theta - x}{r\theta}\right)^r\left(\frac{\Gamma(\alpha + r + 1)}{r^{\alpha}\Gamma(r+1)}\right)$$

The limit as $r \to \infty$ of the expression in the second set of large parentheses raised to the r power is $e^{-x/\theta}$ (this expression simplifies to $(1 - x/(r\theta))^r$. The limit of the expression in the third set of parentheses is 1. This is bit harder to justify but if one were to assume α and θ to be positive integers, then the gamma functions could be written as factorials and, following some cancelations, taking the limit as $r \to \infty$ would give 1. We now observe that "in limit," we have reduced the GBD p.d.f. to to the p.d.f. of the gamma distribution with parameters α and θ.

If we use the better of the two GBD fits discussed earlier,

$$\text{GBD}(1.5 \times 10^{-5},\ 2.9998 \times 10^{7},\ 4.0000,\ 10^{7}),$$

we find that the graphs of the p.d.f.s of the approximation and of the gamma distribution with $\alpha = 5$ and $\theta = 3$ are virtually identical. Moreover,

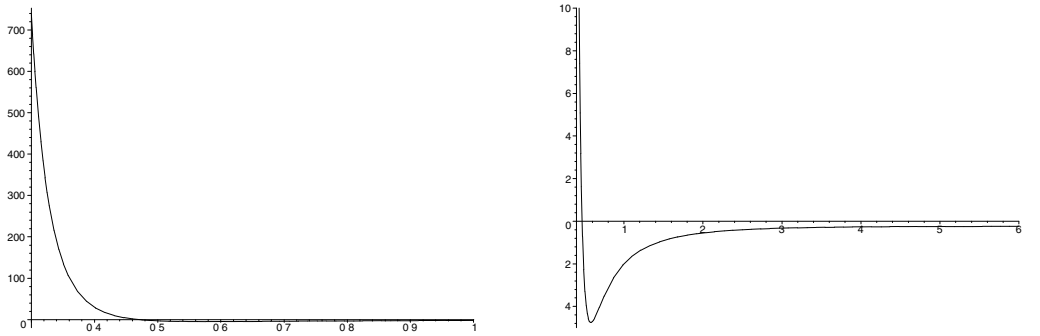
$$\max\left(|\hat{\alpha}_3 - \alpha_3|,\, |\hat{\alpha}_4 - \alpha_4|\right) = 3.4 \times 10^{-7},$$

$$\sup|\hat{f}(x) - f(x)| = 0.0001230 \qquad \sup|\hat{F}(x) - F(x)| = 0.0002363,$$

$$||\hat{f} - f||_1 = 0.0008456 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.0002107.$$

4.4.7 The Weibull Distribution

The $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ for the Weibull distribution with parameters α and β are given in Section 3.4.7. It is difficult to determine analytically the values of α, if there are any, for which $\alpha_4 < 3 + 2\alpha_3^2$. From the graph of $\alpha_4 - 3 - 2\alpha_3^2$ for $0 < \alpha \leq 1$ given in Figure 4.4–1 (a), we see that $\alpha_4 > 3 + 2\alpha_3^2$ for small values of α and the inequality may be reversed when α is closer to 1 than to 0. The latter assertion is substantiated in Figure 4.4–1 (b) where the same graph is given for $0.4 \leq \alpha \leq 6$. An approximation of the specific value α_0 of α where $\alpha_4 = 3 + 2\alpha_3^2$



(a) (b)

Figure 4.4–1. Graph of $\alpha_4 - 3 - 2\alpha_3^2$ for $0 \le \alpha \le 1$ (a) and for $0.4 \le \alpha \le 6$ (b).

can now be found (using Newton's Method or the bisection method), giving us $\alpha_0 = 0.4709331608$. Thus, we conclude that a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation to a Weibull distribution is likely to be possible if and only if $\alpha > \alpha_0 = 0.47$.

For the Weibull distribution considered in Section 3.4.7, with $\alpha = 1$ and $\beta = 5$, a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ should be attainable. The moments of this distribution (from Section 3.4.7) are

$$\alpha_1 = 0.91816, \qquad \alpha_2 = 0.04423, \qquad \alpha_3 = -0.2541, \qquad \alpha_4 = 2.8803$$

and the table of Appendix C indicates a possible solution when β_3 is somewhat larger than 7 and β_4 is somewhat larger than 5. Note that in this case $\alpha_3 < 0$ and once β_3 and β_4 are determined, they must be interchanged.

From `FindBetasM` we get the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation

$$\text{GBD}(-0.5796,\ 2.2701,\ 15.5955,\ 7.5567)$$

which seems to be quite good as can be seen in Figure 4.4–2 where the p.d.f.s of the distribution, its GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation, and its GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ approximation (from Section 3.4.7) are plotted. At the point where the three curves peak, the lowest of the three curves is the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ approximation, the middle one is the Weibull distribution, and the highest curve is the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, approximation. For this GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ fit, we have

$$\sup|\hat{f}(x) - f(x)| = 0.03200 \qquad \sup|\hat{F}(x) - F(x)| = 0.002916$$

and the L_1 and L_2 norms of the difference of $f(x)$ and $\hat{f}(x)$

$$||\hat{f} - f||_1 = 0.01150 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.01710.$$

We also note that the supports of the GBD and the GLD approximations are $[-0.580, 1.690]$ and $[.0806, 3.8934]$, respectively.

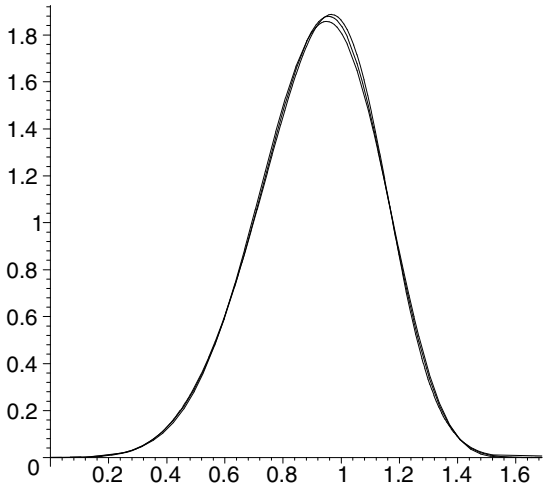


Figure 4.4–2. The p.d.f.s of the Weibull distribution with $\alpha = 1$, $\beta = 5$, its GBD and GLD approximations.

4.4.8 The Lognormal Distribution

Closed form expressions for the first four moments of the lognormal distribution were given in Section 3.4.8. In that section, where the lognormal distribution was defined, we considered $\mu = 0$ and $\sigma = 1/3$, which yielded

$$\alpha_1 = 1.0571, \qquad \alpha_2 = 0.1313, \qquad \alpha_3 = 1.0687, \qquad \alpha_4 = 5.0974.$$

Indeed, $\alpha_4 = 5.0974 < 5.2842 = 3 + 2\alpha_3^2$, placing (α_3^2, α_4) within the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. However, (α_3^2, α_4) is close to the boundary and computations for $\beta_1, \beta_2, \beta_3, \beta_4$ produce only rough approximations with very large values of β_2 and β_4 (sometimes in excess of 10^{100}, which is beyond the capability of **R**) or produce errors. For example, when

```
> FindBetasM(c(1.0571, .1313, 1.0687, 4.8);
```

is used (with a considerably smaller $\alpha_4 = 4.8$), `0 0 0 0` is returned, indicating that `FindBetasM` was not able to find a solution. When we let `LN` be the $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of this lognormal distribution and give the command

```
RefineSearchGBD(LN, c(0, 5), c(100000, 1000000), 50, 5),
```

```
[1] 4.308728e − 01 2.019651e + 05 1.986740e + 00  9.632523e + 05  8.855382e − 02
```

is returned, indicating $\beta_3 = 1.986740$, $\beta_4 = 963252.3$, and the rather unsatisfactory $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 0.0855382$.

Figure 4.4–3 shows the p.d.f.s of the lognormal distribution and its approximating GBD. For this approximation we have

$$\sup|\hat{f}(x) - f(x)| = 0.1357 \qquad \sup|\hat{F}(x) - F(x)| = 0.01587$$

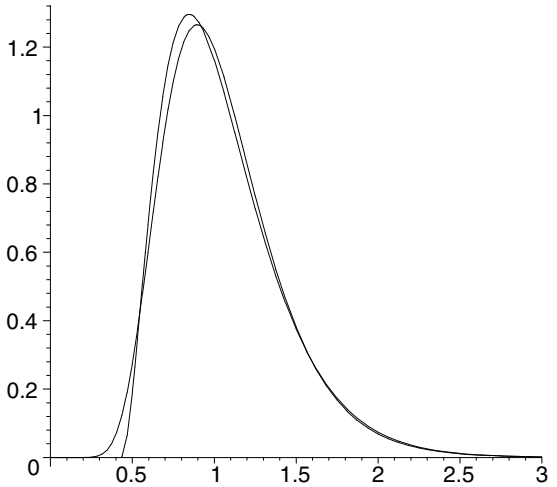


Figure 4.4–3. The p.d.f.s of the lognormal distribution ($\mu = 0$ and $\sigma = 1/3$) with its approximating GBD distribution.

and the L_1 and L_2 norms of the difference of $f(x)$ and $\hat{f}(x)$

$$||\hat{f} - f||_1 = 0.06238 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.07060.$$

Investigations of other choices of μ and σ reveal that the situation encountered above does not change appreciably as long as σ stays small (i.e., (α_3^2, α_4) remains in the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ but close enough to the boundary to make computations unreliable). Moreover, as σ gets large (α_3 and α_4 do not depend on μ), (α_3^2, α_4) moves well into the region covered by the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. For example, when $\mu = 0$ and $\sigma = 1$,

$$\alpha_1 = 1.6487, \qquad \alpha_2 = 4.6708, \qquad \alpha_3 = 6.1849, \qquad \alpha_4 = 113.9364$$

and when $\mu = 0$ and $\sigma = 2$,

$$\alpha_1 = 7.3890, \qquad \alpha_2 = 2926.3598, \qquad \alpha_3 = 414.3593, \qquad \alpha_4 = 9.2206 \times 10^6,$$

and in both cases, (α_3^2, α_4) is well outside of the region covered by the GBD.

4.4.9 The Beta Distribution

The moments of the beta distribution derived in Section 4.1 are given in equations (4.1.17) through (4.1.20). It was also established through the definition of the GBD (see Definition 4.2.1) that if X is beta with parameters β_3 and β_4, then X is also GBD with parameters $\beta_1 = 0$, $\beta_2 = 1$, β_3, β_4. Therefore, the GBD$(0, 1, \beta_3, \beta_4)$ is an exact fit for the beta distribution with parameters β_3 and β_4.

4.4.10 The Inverse Gaussian Distribution

The definition of this distribution and its moments are given in Section 3.4.10. The inverse Gaussian distribution with parameters μ and λ has moments

$$\alpha_1 = \mu, \quad \alpha_2 = \frac{\mu^3}{\lambda}, \quad \alpha_3 = 3\sqrt{\frac{\mu}{\lambda}}, \quad \alpha_4 = 3 + \frac{15\mu}{\lambda}.$$

In this case

$$3 + 2\alpha_2^2 = \frac{3\lambda + 18\mu}{\lambda} > \frac{3\lambda + 15\mu}{\lambda} = \alpha_4,$$

placing (α_3^2, α_4) inside the space covered by the GBD for all $\mu > 0$ and $\lambda > 0$. However, as can be seen from the table of Appendix C, (α_3^2, α_4) pairs obtained from an inverse Gaussian distribution are beyond the entries of the table, and generally, beyond computation range for `FindBetasM`.

For the specific case of $\mu = 1/2$ and $\lambda = 6$, that was considered in Section 3.4.10, we have

$$\alpha_1 = 0.5000, \qquad \alpha_2 = 0.02083, \qquad \alpha_3 = 0.86603, \qquad \alpha_4 = 4.2500.$$

Although this (α_3, α_4) point is inside the range covered by the GBD, β_4 gets very large as α_4 approaches 4.25. When $(\alpha_3, \alpha_4) = (0.86603, 4.12499)$, we get $(\beta_3, \beta_4) = (4.3332, 712487.7862)$ and when α_4 is increased by 0.000005 so that $(\alpha_3, \alpha_4) = (0.86603, 4.124995)$, we get $(\beta_3, \beta_4) = (4.3333, 1.4250 \times 10^8)$. Furthermore, for this (β_3, β_4), we have $|\hat{\alpha}_3 - \alpha_3| = 10^{-10}$ and $|\hat{\alpha}_4 - \alpha_4| = 0.1250$.

4.4.11 The Logistic Distribution

We saw in Section 3.4.11, where this distribution is defined, that $(\alpha_3^2, \alpha_4) = (0, 4.2)$ for this distribution. This is outside the (α_3^2, α_4)-space covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$; therefore, GBD cannot be fitted to a logistic distribution through the method of moments.

4.4.12 The Largest Extreme Value Distribution

Section 3.4.12 gives the definition and moments of this distribution, for which $(\alpha_3^2, \alpha_4) = (1.1395, 5.4)$. Since

$$3 + 2\alpha_3^2 = 5.59714 > 5.4 = \alpha_4,$$

the (α_3^2, α_4) of the largest extreme value distribution is inside the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. However, as was the case with the inverse Gaussian, this point is in the region that is outside our range of computation.

4.4.13 The Extreme Value Distribution

This distribution, defined in Section 3.4.13, has the same (α_3^2, α_4) as the largest extreme value distribution, producing the same circumstances discussed in Section 4.4.12.

4.4.14 The Double Exponential Distribution

Section 3.4.14 defines this distribution and gives its moments. For all values of the parameter $\lambda > 0$ of this distribution we have $\alpha_3 = 0$ and $\alpha_4 = 6$. Since (α_3^2, α_4) is outside the space covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$, it is not possible to fit a GBD to a double exponential distribution through the method of moments.

4.4.15 The F-Distribution

This distribution and its moments are given in Section 3.4.15. It is clear from Figure 4.3–5 that the (α_3^2, α_4)-space of the F-distribution overlaps the space covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. But in almost all cases, even when (α_3^2, α_4) is within the intersection of the F-distribution and GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ regions, the (α_3^2, α_4) obtained from an F-distribution, such as the $(\alpha_3^2, \alpha_4) = (1.8101, 9.1986)$ of Section 3.4.15, is beyond the computation range for the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$.

4.4.16 The Pareto Distribution

Section 3.4.16 gives the definition and the moments of the Pareto distribution. The analysis of α_3 and α_4 in Section 3.4.16 makes it clear that (α_3^2, α_4) produced from this distribution cannot be in the region covered by the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$. Therefore, it is not possible to fit a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ to a Pareto distribution through the method of moments.

4.5 Examples: GBD Fits of Data, Method of Moments

In this section we illustrate the use of the EGLD in fitting datasets. For the first two examples, we use data that is simulated from GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ distributions. In one case, we are able to obtain a GBD fit but no GLD fits; in the other case, we obtain GLD and GBD fits. In the next two examples in Sections 4.5.3 and 4.5.4, we reconsider two examples from Chapter 3. For the data on cadmium concentrations in horse kidneys, a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ fit is obtained, giving us two fits for this data. The rainfall data of Section 3.5.4 did not yield any fits through the GLD but we are now able to find a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ fit in the EGLD system for both X and Y components of this data. In Section 4.5.5 we introduce an example from forestry on the stand heights and diameters of trees.

4.5.1 Example: Fitting a GBD to Simulated Data from GBD(3, 5, 0, −0.5)

The following data, considered by Karian, Dudewicz and McDonald (1996), was generated from the GBD(3, 5, 0, −0.5) distribution.

7.88	6.77	7.98	6.73	6.79	6.99	3.25	4.10	7.25	5.74
5.03	5.22	7.99	7.41	7.21	7.81	7.87	7.73	8.00	5.75
6.81	7.79	7.42	6.17	7.96	4.63	7.43	7.77	3.55	4.31
8.00	6.82	7.72	7.90	7.43	6.96	6.71	4.50	4.88	6.03
6.06	4.56	4.06	4.47	6.21	3.87	6.46	3.29	5.99	7.59
7.96	7.59	7.13	5.13	7.89	7.24	7.00	8.00	6.65	4.52
7.98	7.84	7.98	8.00	7.95	5.20	7.85	7.95	6.79	6.84
5.19	5.72	7.99	7.33	7.34	3.09	7.41	5.60	4.76	5.03

The $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ for this data is

$$\hat{\alpha}_1 = 6.4975, \quad \hat{\alpha}_2 = 2.0426, \quad \hat{\alpha}_3 = -0.7560, \quad \hat{\alpha}_4 = 2.3536.$$

The point $(\hat{\alpha}_3^2, \hat{\alpha}_4) = (0.5715, 2.3536)$ is in the region covered by both the GLD, and GBD. However, as can be seen from the table of Appendix B, this $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ is outside the computation range for the GLD. To obtain a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ fit, we use `FitGBD` to obtain

$$\text{GBD}(3.0550, 4.9616, 0.08261, -0.5222),$$

with support [3.055, 8.017] that covers the data range.

A histogram of the data with the p.d.f.s of GBD(3, 5, 0, −0.5), the distribution from which the data was generated, and the fitted GBD are shown in Figure 4.5–1 (a) (the fitted p.d.f. is slightly lower near the point 3). Figure 4.5–1 (b) gives the empirical d.f. together with the d.f.s of the two GBDs, which cannot be visually distinguished.

If we partition this data into the 7 classes

$$(-\infty, 4.5),\ [4.5, 5.5),\ [5.5, 6.5),\ [6.5, 7.0),\ [7.0, 7.5),\ [7.5, 7.9),\ [7.9, \infty),$$

we observe frequencies of

$$9, \quad 12, \quad 10, \quad 11, \quad 12, \quad 12, \quad 14.$$

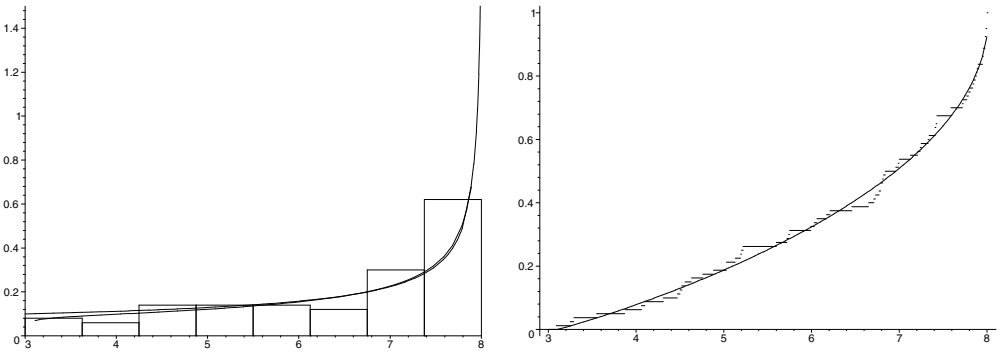
The expected frequencies from the fitted distribution,

$$10.6410,\ 9.7212,\ 12.4789,\ 8.0792,\ 10.7142,\ 14.4024,\ 13.9173,$$

give a chi-square statistic and p-value of

$$2.8911 \quad \text{and} \quad 0.2356,$$

respectively. From the Kolmogorov-Smirnov statistic for this fit we get $KS = 0.06889$ and $KS\sqrt{n} = 0.06889\sqrt{80} = 0.6162$ and a p-value of 0.84.



(a) (b)

Figure 4.5–1. Histogram of data generated from GBD$(3, 5, 0, -0.5)$ and the p.d.f.s of the original and fitted GBDs (a); the e.d.f. of the data with the d.f. of its fitted GBD (b).

4.5.2 Example: Fitting a GBD to Data Simulated from GBD$(2, 7, 1, 4)$

The following data, from Karian, Dudewicz and McDonald (1996), is generated from the GBD(2,7,1,4) distribution.

3.88	5.31	3.26	3.65	6.78	3.31	6.09	3.42	3.45	4.83
4.01	4.37	3.34	4.20	2.23	3.17	3.20	3.12	5.50	5.45
4.28	4.77	4.49	3.74	4.59	3.64	3.16	4.66	6.44	2.04
3.87	3.37	2.35	4.29	6.25	2.77	3.73	7.28	4.22	4.48
3.54	3.79	3.78	2.28	4.09	5.02	3.37	3.00	2.75	4.12
4.86	4.17	5.94	5.14	3.29	3.79	3.59	4.79	3.18	4.84

For this data

$$\hat{\alpha}_1 = 4.1053, \quad \hat{\alpha}_2 = 1.2495, \quad \hat{\alpha}_3 = 0.6828, \quad \hat{\alpha}_4 = 3.2998.$$

The point $(\hat{\alpha}_3^2, \hat{\alpha}_4) = (0.4663, 3.2998)$ is in the region where the GBD and GLD overlap. When we use `FitGLDM` we get the GLD fit

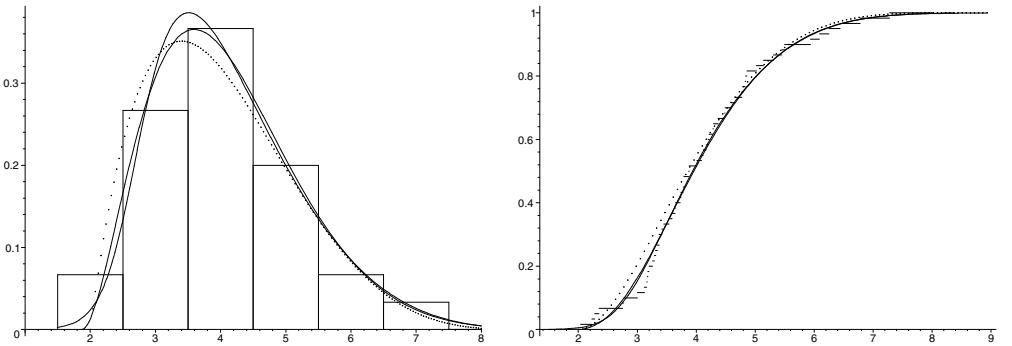
$$\text{GLD}(3.2841, 0.1539, 0.04245, 0.2007),$$

with support $[-3.212, 9.978]$ that covers the data range. When we use `FitGBD` we get the GBD fit

$$\text{GBD}(1.8696, 10.4885, 1.9341, 9.8320),$$

that has support [1.870, 12.358], which also covers the data range.

Figure 4.5–2 (a) shows a histogram of the data with three p.d.f.s: one for the distribution from which the data was generated (shown as a dotted line), one for



(a) (b)

Figure 4.5–2. Histogram of data generated from GBD$(2, 7, 1, 4)$ and the p.d.f.s of the original (dotted line) and fitted GLD and GBD (a); the e.d.f. of the data with the d.f.s of the original (dotted line) and its fitted GLD and GBD (b).

the GLD fit (the one that rises higher near the center), and one for the GBD fit. Figure 4.5–2 (b) shows the empirical d.f. of the data with the d.f.s of the three p.d.f.s depicted in Figure 4.5–2 (a). The d.f. of the distribution from which the data was generated is shown as a dotted curve and the other two d.f.s cannot be distinguished.

To apply a chi-square test, the data is partitioned into the 6 intervals

$$(-\infty, 3), \quad [3, 3.5), \quad [3.5, 4.0), \quad [4.0, 4.5), \quad [4.5, 5.0), \quad [5.0, \infty),$$

producing the observed frequencies

$$6, \quad 14, \quad 11, \quad 11, \quad 7, \quad 11.$$

The expected frequencies for the GLD fit,

$$9.1915, \quad 10.9071, \quad 11.1862, \quad 9.4086, \quad 7.1169, \quad 12.1896,$$

yield a chi-square statistic and p-value of

$$2.3754 \quad \text{and} \quad 0.1233,$$

respectively. The Kolmogorov-Smirnov test for this GLD fit gives $KS = 0.07649$ and $KS\sqrt{n} = 0.07649\sqrt{60} = 0.5925$ for a p-value of 0.87.

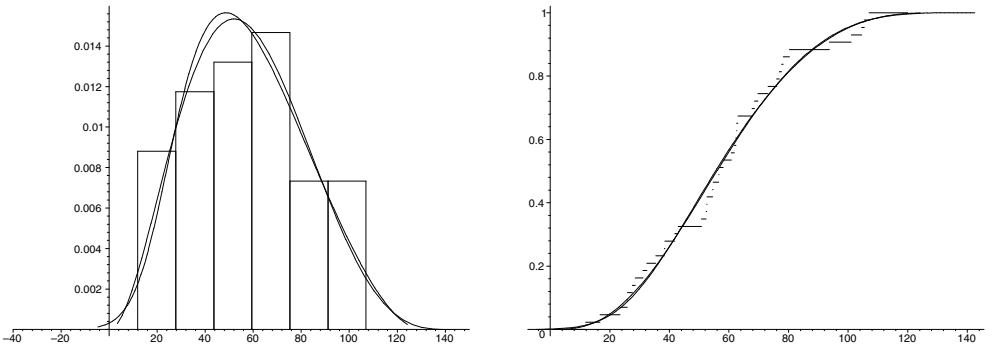
The GBD fit gives expected frequencies of

$$9.8047, \quad 10.2981, \quad 10.7895, \quad 9.4861, \quad 7.3604, \quad 12.2612,$$

producing a chi-square statistic and p-value of

$$3.2002 \quad \text{and} \quad 0.07363,$$

respectively. The Kolmogorov-Smirnov test for this GLD fit gives $KS = 0.0850$ and $KS\sqrt{n} = 0.0850\sqrt{60} = 0.5929$ for a p-value of 0.78.



(a) (b)

Figure 4.5–3. Histogram of the cadmium data and the p.d.f.s of the fitted GLD and GBD (a); the e.d.f. of the data with the d.f.s of the fitted GLD and GBD (b).

4.5.3 Example: Cadmium in Horse Kidneys

In this section we reconsider the data of Section 3.5.2 and fit it with a GBD. Recall that in Section 3.5.2 we gave the moments,

$$\hat{\alpha}_1 = 57.2442, \quad \hat{\alpha}_2 = 576.0741, \quad \hat{\alpha}_3 = 0.2546, \quad \hat{\alpha}_4 = 2.5257,$$

of the data and the GLD fit

$$\text{GLD}(41.7897, 0.01134, 0.09853, 2.3606).$$

To fit a GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ to this data we use `FitGBD` to obtain

$$\text{GBD}(-0.1827, 144.3947, 2.0502, 3.6193),$$

with support $[-0.183, 144.212]$ that covers the data range.

The p.d.f.s of the two fits together with a histogram of the data are shown in Figure 4.5–3 (a) (the GLD p.d.f. rises higher near the center). The e.d.f. of the data and the d.f.s of the fitted GLD and GBD are shown in Figure 4.5–3 (b).

Next, we partition the data into the 6 classes

$$(-\infty, 30), \quad [30, 50), \quad [50, 60), \quad [60, 70), \quad [70, 85), \quad [85, \infty)$$

and obtain the following frequencies in these classes

$$7, \quad 7, \quad 9, \quad 9, \quad 6, \quad 5.$$

From the expected frequencies for the GBD fit (those for the GLD fit are given in Section 3.5.2)

$$5.8577, \quad 11.7590, \quad 6.5332, \quad 5.9843, \quad 6.9084, \quad 5.9574,$$

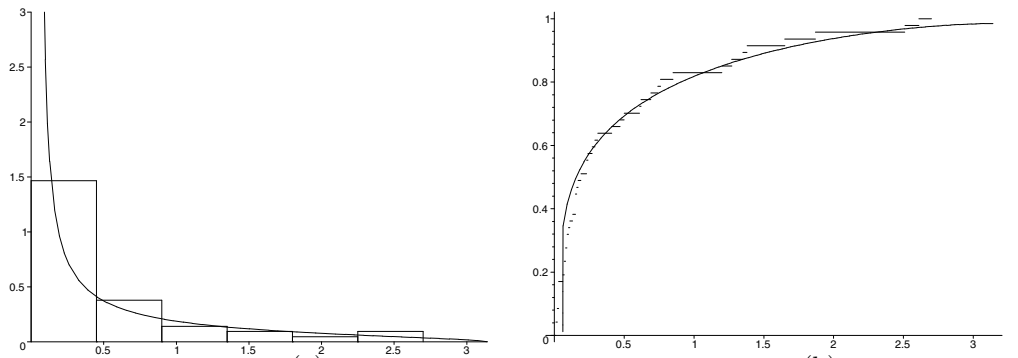


Figure 4.5–4. Histogram of X and the p.d.f. of the fitted GBD (a); the e.d.f. of X with the d.f. of its fitted GBD (b).

we compute the chi-square statistic and corresponding p-value to obtain

$$4.8731 \quad \text{and} \quad 0.02728,$$

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.09612$ and $KS\sqrt{n} = 0.09612\sqrt{43} = 0.6303$, giving a p-value of 0.82.

For comparison, we note that in Section 3.5.2 a chi-square statistic of 5.6095 with a p-value of 0.01786 and a KS statistic of 0.1025 with a p-value of 0.76 were obtained for the GLD fit.

4.5.4 Example: Rainfall Data of Section 3.5.5

Table 3.5–6 of Section 3.5.5 gives the Rainfall (in inches) at Rochester (X) and Syracuse (Y), New York, from May to October of 1998. In Section 3.5.5, after computing the sample moments for X and Y, we were not able to find a GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) fit to either X or Y. We now attempt to fit a GBD(β_1, β_2, β_3 β_4), to X and Y. We start with the moments of X (from Section 3.5.5),

$$\hat{\alpha}_1 = 0.4913, \quad \hat{\alpha}_2 = 0.4074, \quad \hat{\alpha}_3 = 1.8321, \quad \hat{\alpha}_4 = 5.7347.$$

We observe that $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ lies in the region covered by the GBD and use `FitGBD` to obtain the fit

$$\text{GBD}(0.06170, 3.0764, -0.7499, 0.5408),$$

with support [0.0617, 3.138] that does not extend far enough to the left to cover the entire data range.

Figure 4.5–4 (a) shows a histogram of X and its fitted GBD and Figure 4.5–4 (b) shows the e.d.f. of X with the d.f. of its fitted GBD.

We partition the data into the classes

$$(-\infty, 0.07), \quad [0.07, 0.1), \quad [0.1, 0.2), \quad [0.2, 0.45), \quad [0.45, 1.0), \quad [1.0, \infty)$$

and get the observed expected frequencies

$$9, \quad 6, \quad 9, \quad 7, \quad 8, \quad 8.$$

From the fitted GBD, the expected frequencies

$$12.3401, \quad 5.6345, \quad 6.7382, \quad 7.0311, \quad 7.0502, \quad 8.0888$$

are obtained. Because of the very sharp rise in the graph of the fitted GBD for values near 0.06 (see Figure 4.5–4 (a)) the computation of the first expected frequency can cause some difficulty.

These observed and expected frequencies lead, respectively, to the following chi-square statistic and p-value

$$1.7190 \quad \text{and} \quad 0.1898.$$

For this fit, $KS = 0.1915$ and $KS\sqrt{n} = 0.1915\sqrt{47} = 1.3129$, producing a p-value of 0.065.

The moments of Y are

$$\hat{\alpha}_1 = 0.3906, \quad \hat{\alpha}_2 = 0.1533, \quad \hat{\alpha}_3 = 1.6164, \quad \hat{\alpha}_4 = 5.2245.$$

`FitGBD` produces the fit

$$\text{GBD}(0.07209, 2.1729, -0.5816, 1.4355),$$

with support $[0.0721, 2.245]$. As was the case for X, this support does not extend far enough to the left to cover the entire data range. Thus, the EGLD, through its GBD part, has succeeded in completing the GLD coverage.

A histogram of Y and the fitted p.d.f. are given in Figure 4.5–5 (a) and the e.d.f. of Y and the d.f. of the fitted GBD are given in Figure 4.5.–5 (b).

When Y is partitioned into the classes

$$(-\infty, 0.08), \ [0.08, 0.15), \ [0.15, 0.22), \ [0.22, 0.38), \ [0.38, 0.73), \ [0.73, \infty),$$

we get observed frequencies of

$$7, \quad 7, \quad 9, \quad 8, \quad 8, \quad 8.$$

The expected frequencies from the fitted GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$

$$6.9517, \quad 10.9348, \quad 5.1887, \quad 7.3122, \quad 8.5232, \quad 8.0677,$$

produce the chi-square statistic and p-value of

$$4.3131 \quad \text{and} \quad 0.03782,$$

respectively. For this fit, $KS = 0.1489$, $KS\sqrt{n} = 0.1489\sqrt{47} = 1.0208$, and a p-value of 0.27.

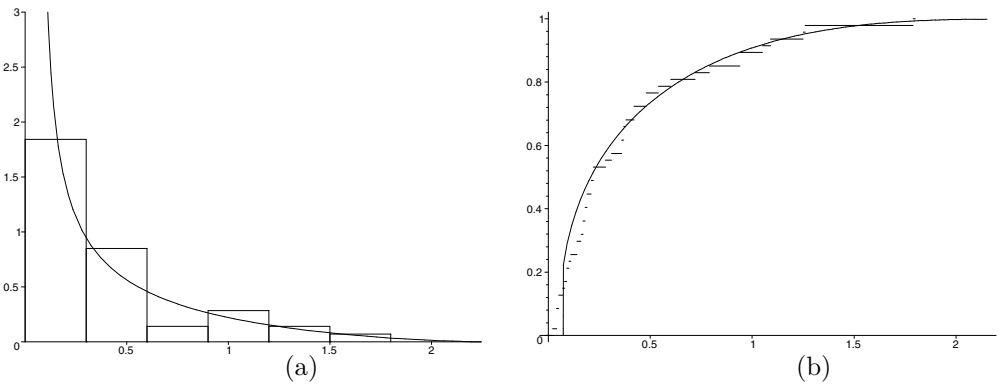


Figure 4.5–5. Histogram of Y and the p.d.f. of the fitted GBD (a); the e.d.f. of Y with the d.f. of its fitted GBD (b).

4.5.5 Example: Tree Stand Heights and Diameters in Forestry

The estimation of tree stand volume is an important problem in forestry. Such estimations are based on measurements of tree diameters and heights (see Schreuder and Hafley (1977)). Unpublished studies of the $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ of the two variables DBH (diameter, in inches, at breast height) and H (tree height, in feet) have shown that for the DBH variable, about 2% of stands have $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ in the GLD but not the GBD region, about 70% of stands have $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ in the overlapping region, and the other 28% have $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ pairs in the GBD region only.

For the H variable, these three percentages are 0%, 20%, and 80%, respectively.

Table 4.5–6 gives DBH and H values for tree stands of 0.1 acres of Douglas fir in Idaho. (We are grateful for the communication of this data by Professor Lianjun Zhang of the Faculty of Forestry, SUNY College of Environmental Science and Forestry, Syracuse, New York, and Professor James A. Moore, Director of the Inland Tree Nutrition Cooperative (IFTNC), of the Department of Forest Resources, University of Idaho, and for permission to quote this example and data from their studies of tree growth in the Northwest, USA.)

For the DBH data in Table 4.5–6, we have

$$\hat{\alpha}_1 = 6.7404, \quad \hat{\alpha}_2 = 6.6721, \quad \hat{\alpha}_3 = 0.4544, \quad \hat{\alpha}_4 = 2.7450.$$

The $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ lies in the overlap region of the GLD and GBD and we can invoke `FitGLDM` and `FitGBD` to obtain the fits

$$\text{GLD}(4.7744, 0.08911, 0.06257, 0.3056)$$

and

$$\text{GBD}(1.3522, 16.7462, 1.6296, 4.5429),$$

Table 4.5–6. Tree diameters DBH (in inches), and heights H (in feet).

DBH	H	DBH	H	DBH	H	DBH	H	DBH	H
3.30	28.0	5.90	43.0	3.80	30.0	7.00	64.0	5.80	50.0
4.40	48.0	4.60	50.0	8.20	70.0	3.30	30.0	10.20	75.0
11.30	86.0	4.80	34.0	9.10	72.0	4.40	44.0	5.60	30.0
8.50	73.0	8.20	63.0	4.70	52.0	8.00	64.0	8.20	66.0
7.70	64.0	10.00	78.0	4.80	40.0	7.10	60.0	5.50	44.0
12.80	90.0	8.90	78.0	9.80	80.0	7.10	60.0	6.10	64.0
10.00	76.0	8.10	71.0	2.20	14.0	7.30	47.0	10.80	74.0
5.60	60.0	6.40	47.0	4.30	49.0	9.50	80.0	7.70	72.0
7.40	67.0	3.60	48.0	5.50	52.0	6.50	59.0	10.30	82.0
4.90	62.0	5.80	62.0	14.80	82.0	6.40	66.0	4.90	49.0
10.00	80.0	3.40	35.0	6.30	42.0	8.90	85.0	8.10	78.0
10.30	78.0	3.00	28.0	2.40	19.0	4.70	30.0	4.10	34.0
4.00	31.0	4.20	26.0	10.70	75.0	4.00	35.0	2.50	18.0
10.20	80.0	3.50	30.0	9.30	78.0	8.60	68.0	9.10	80.0
8.10	68.0	6.90	67.0	5.50	50.0	5.80	58.0	5.50	39.0
10.40	94.0	4.40	45.0	4.70	55.0	4.70	55.0	4.50	55.0
4.90	55.0	3.00	24.0	10.30	80.0	5.30	40.0	8.80	80.0
7.80	66.0	6.50	40.0	7.20	68.0	7.20	67.0		

respectively. The support of the GLD fit, the interval $[-6.45, 16.00]$, covers the spread of the DBH data $[2.2, 14.8]$ but allows the possibility of values, including negative ones, well below the minimum of the data. The support of the GBD fit, $[1.35, 18.10]$, provides a better coverage of the spread of the data.

A histogram of DBH together with the p.d.f.s of the GLD (rising higher at the center) and GBD fits is shown in Figure 4.5–7 (a) and the e.d.f. of DBH with the d.f.s of the two fitted distributions is shown in Figure 4.5–7 (b).

To perform a chi-square test for these two fits, we partition the data for DBH into the classes

$$(-\infty, 3.75), \quad [3.75, 4.5), \quad [4.5, 5.0), \quad [5.0, 6.0), \quad [6.0, 7.0),$$
$$[7.0, 8.0), \quad [8.0, 9.0), \quad [9.0, 10.25), \quad [10.25, \infty)$$

and obtain the observed frequencies

$$10, \quad 9, \quad 11, \quad 11, \quad 7, \quad 10, \quad 12, \quad 10, \quad 9,$$

and expected frequencies of

$$10.0864,\ 8.4738,\ 6.5326,\ 13.6000,\ 12.7848,\ 11.0069,\ 8.8806,\ 8.1155,\ 9.5195$$

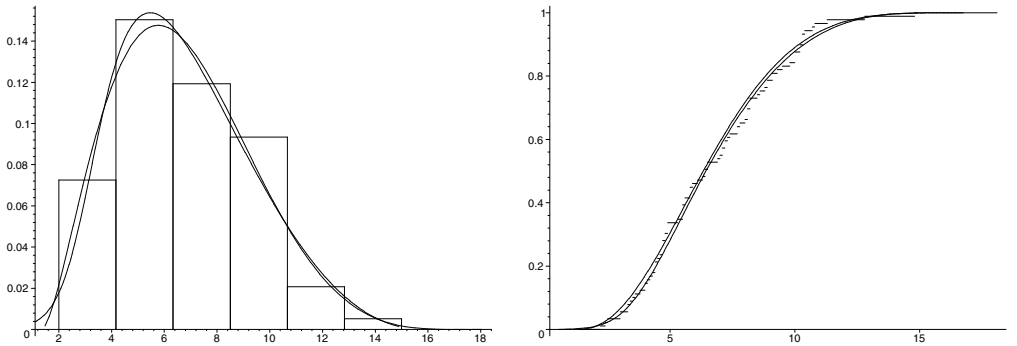


Figure 4.5–7. Histogram of DBH and the p.d.f.s of the fitted GLD and GBD (a); the e.d.f. of DBH with the d.f.s of its fitted GLD and GBD (b).

and

$$10.9161,\ 8.1165,\ 6.1359,\ 13.0145,\ 12.7265,\ 11.2992,\ 9.2219,\ 8.3249,\ 9.2445$$

for the GLD and GBD fits, respectively. The chi-square statistic and p-value for the GLD fit are

$$7.8569 \quad \text{and} \quad 0.09696,$$

and for the GBD fit these are

$$8.2471 \quad \text{and} \quad 0.08293.$$

For the GLD fit we have $KS = 0.07018$ and $KS\sqrt{n} = 0.07018\sqrt{89} = 0.6621$ with a p-value of 0.77. For the GBD fit we have $KS = 0.06844$ and $KS\sqrt{n} = 0.06844\sqrt{89} = 0.6457$ with a p-value of 0.80.

For the H data of Table 4.5–6, we have

$$\hat{\alpha}_1 = 57.1348, \quad \hat{\alpha}_2 = 365.3751, \quad \hat{\alpha}_3 = -0.2825, \quad \hat{\alpha}_4 = 2.1046,$$

and again, $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ is in the GLD and GBD overlap region. `FitGLDM` yields the fit

$$\text{GLD}(82.0498, 0.01442, 0.6212, 0.02459)$$

and `FitGBD` yields the fit

$$\text{GBD}(11.3851, 77.9226, 0.7781, 0.2504).$$

The support of the GLD fit, [12.68, 151.42], covers the spread of the data, which is $[14.0, 94.0]$. However, the GBD fit, with support [11.39, 89.31] does not cover the spread of the data (see Problem 4.1).

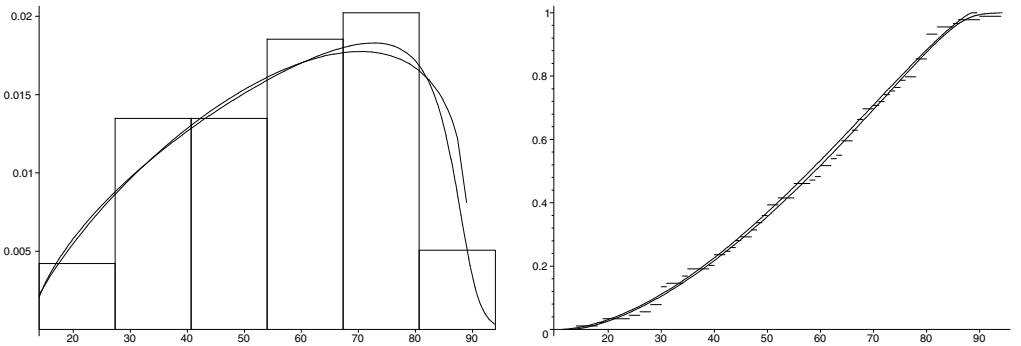


Figure 4.5–8. Histogram of H and the p.d.f.s of the fitted GLD and GBD (a); the e.d.f. of H with the d.f.s of its fitted GLD and GBD (b).

The p.d.f.s of the fitted GLD and GBD and a histogram of H are shown in Figure 4.5–8 (a) (the GLD p.d.f. rises higher near the center) and the fitted d.f.s and the e.d.f. of H are shown in Figure 4.5–8 (b).

To perform a chi-square test the data is partitioned into the classes

$$(-\infty, 30), \quad [30, 37.5), \quad [37.5, 47.5), \quad [47.5, 55), \quad [55, 62.5),$$
$$[62.5, 67.5), \quad [67.5, 75), \quad [75, 80), \quad [80, \infty),$$

with observed frequencies of

$$7, \quad 10, \quad 11, \quad 9, \quad 11, \quad 11, \quad 9, \quad 8, \quad 13.$$

The expected frequencies of the GLD fit are

$$9.3662,\ 7.2608,\ 11.8326,\ 10.2410,\ 11.2074,\ 7.8919,\ 12.1650,\ 7.9312,\ 11.1038,$$

leading to the chi-square statistic and p-value of

$$4.2159 \quad \text{and} \quad 0.3776.$$

The Kolmogorov-Smirnov statistic for this fit is $KS = 0.0570$ and $KS\sqrt{n} = 0.4783$, giving a p-value of 0.98.

For the GBD fit we get expected frequencies of

$$8.9212,\ 7.2814,\ 12.0189,\ 10.3900,\ 11.2446,\ 7.8001,\ 11.8147,\ 7.6531,\ 11.5859$$

and a chi-square statistic and p-value of

$$3.9952 \quad \text{and} \quad 0.4067,$$

respectively. The Kolmogorov-Smirnov statistic for the GBD fit is $KS = 0.06276$ and $KS\sqrt{n} = 0.06276\sqrt{89} = 0.59210$, giving a p-value of 0.875.

This example is continued in Section 5.5.4, where we consider the most important aspect of this application: the fitting of a bivariate GLD–2 distribution to the pair (DBH, H).

4.6 EGLD Random Variate Generation

We saw in Section 2.3 that the GLD is very useful in simulation and Monte Carlo studies as it can model a wide range of distribution shapes with a simple model from which it is easy to generate data. It is also easy to change the distribution (e.g., for sensitivity studies) by simply changing the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The EGLD completes the coverage of moment-space, as we saw in Section 4.3, especially Figure 4.3–5. To complete this gain for simulation and Monte Carlo studies, we now specify **how to generate the GBD part of the EGLD**.

Since (see the proof of Theorem 4.2.2) a GBD r.v., Y, can be represented as

$$Y = \beta_1 + \beta_2 X, \tag{4.6.1}$$

where X has the beta distribution of (4.1.1), **to generate values of Y we can generate values of X, say X_1, X_2, ... and then calculate**

$$Y_1 = \beta_1 + \beta_2 X_1,\ Y_2 = \beta_1 + \beta_2 X_2,\ \ldots . \tag{4.6.2}$$

Three methods of generating X_1, X_2, ... are given in Karian and Dudewicz (1999), Section 4.6.8. Their Method 1 deals with the case where β_3 and β_4 are both from the set of values -0.5, 0, 0.5, 1, 1.5, 2, 2.5, ...; Method 2 uses the p.f. of a beta r.v., with the incomplete beta function and Newton-Raphson iteration; Method 3 works when β_3 and β_4 are positive. For other methods, some of them exact and fast, see Tadikamalla (1984, pp. 207–208).

Problems for Chapter 4

4.1. In the forestry example of Section 4.5.5, we found a GBD fit to the H (height) data that had support [11.4, 89.3]. However, the data ranged from a minimum of 18.0 to a maximum of 94.0. Thus, as noted in the example, the fit had the undesirable property that it could not "explain" all of the observed data. One way to handle such problems is to **constrain the search** for a fit to GBDs that satisfy the condition:

$$\beta_1 \le 14 \quad \text{and} \quad \beta_1 + \beta_2 \ge 94 \tag{$*$}$$

since then **the support of the fitted GBD will be at least that of the data**.

Schreuder, H. T. and Hafley, W. L. (1977). "A useful bivariate distribution for describing stand structure of tree heights and diameters," *Biometrics*, 33, 471–478.

Tadikamalla, P. R. (1984). *Modern Digital Simulation, Vol. I*, American Sciences Press, Inc., Columbus, Ohio.

Chapter 5

A Percentile-Based Approach to Fitting Distributions and Data with the GLD

In this chapter we consider a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fitting process that is based exclusively on percentiles. The concept and name "percentile" (also, "quartile" and "decile") are due to Galton (1875) (see Hald (1998, pp. 602, 604)), who in his 1875 paper proposed to characterize a distribution by its location (median), and its dispersion (half the inter-quartile range). These are basically (5.1.2) and half of a special case of (5.1.3) below. However, these do not include the two shape measures (5.1.4) and (5.1.5). Mykytka (1979) initially used percentiles for the estimation of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ parameters through methods that relied on a mixture of moments and percentiles. The percentile-based approach that we will describe was proposed in Karian and Dudewicz (1999). It fits a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distribution to a given dataset by specifying four percentile-based sample statistics and equating them to their corresponding GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ statistics. The resulting equations are then solved for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, with the constraint that the resulting GLD be a valid distribution.

Percentile-based fits provide good fits, as was shown by Karian and Dudewicz (2003). To make the percentile approach an acceptable alternative to the method of moments and to provide the necessary computational support for the use of percentile-based fits, Dudewicz and Karian (1999) give extensive tables for estimating the parameters of the fitted GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distribution. These tables are reproduced in Appendix D.

There are three principal **advantages to the use of percentiles:**

1. There is a large class of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ distributions that have fewer than four moments and these distributions are excluded from consideration when one uses parameter estimation methods that require moments. On the occasions when moments do not exist or may be out of table

range, percentiles can still be used to estimate parameters and obtain GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) fits.

2. The equations associated with the percentile method that we will consider are simpler and the computational techniques required for solving them provide greater accuracy.

3. The relatively large variability of sample moments of orders 3 and 4 can make it difficult to obtain accurate GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) fits through the method of moments.

5.1 The Use of Percentiles

For a given dataset, $X_1, X_2, \ldots, X_n$, let $\hat{\pi}_p$ denote the $(100p)$th percentile of the data. $\hat{\pi}_p$ is computed by first writing $(n+1)p$ as $r + (a/b)$, where r is a positive integer and a/b is a proper fraction, possibly zero. If $Y_1, Y_2, \ldots, Y_n$ are the order statistics of $X_1, X_2, \ldots, X_n$, then $\hat{\pi}_p$ can be obtained from

$$\hat{\pi}_p = Y_r + \frac{a}{b}\left(Y_{r+1} - Y_r\right). \tag{5.1.1}$$

This definition of the **$(100p)$th data percentile** differs from the usual definition. Consider, for example, $p = 0.5$ where the sample median is usually defined as $M_n = Y_k$ if $n = 2k+1$ for some integer k and $M_n = (Y_k + Y_{k+1})/2$ if $n = 2k$ for some integer k. By contrast, the sample quantile of order 0.5 is usually defined as $Z_{0.5} = Y_{[0.5n]+1}$ where $[0.5n]$ denotes the largest integer less than $0.5n$. Since the sample quantile is defined as a function of a single order statistic, it is mathematically somewhat simpler. However, the sample median is a better estimate of the population median. The $\hat{\pi}_p$ that we have defined is the generalization of the definition of the sample median to the case $p \neq 0.5$ as described in Hogg and Tanis (1997), p. 25.

The sample statistics that we will use are defined by

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{5.1.2}$$

$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{5.1.3}$$

$$\hat{\rho}_3 = \frac{\hat{\pi}_{0.5} - \hat{\pi}_u}{\hat{\pi}_{1-u} - \hat{\pi}_{0.5}}, \tag{5.1.4}$$

$$\hat{\rho}_4 = \frac{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}{\hat{\rho}_2}, \tag{5.1.5}$$

where u is an arbitrary number between 0 and 1/4. These statistics have the following interpretations (where for ease of discussion we momentarily assume $u = 0.1$).

1. $\hat{\rho}_1$ is the **sample median**;
2. $\hat{\rho}_2$ is the **inter-decile range**, i.e., the range between the 10th percentile and 90th percentile;
3. $\hat{\rho}_3$ is the **left-right tail-weight ratio**, a measure of relative tail weights of the left tail to the right tail (distance from median to the 10th percentile in the numerator and distance from 90th percentile to the median in the denominator);
4. $\hat{\rho}_4$ is the **tail-weight factor** or the ratio of the inter-quartile range to the inter-decile range, which cannot exceed 1 and measures the tail weight (values that are close to 1 indicate the distribution is not greatly spread out in its tails, while values close to 0 indicate the distribution has long tails).

In the case of $N(\mu, \sigma^2)$, the normal distribution with mean μ and variance σ^2, we have

$$\hat{\rho}_1 = \mu, \qquad \hat{\rho}_2 = 2.56\sigma, \qquad \hat{\rho}_3 = 1, \qquad \hat{\rho}_4 = 1.36/2.56 = 0.53.$$

This indicates, respectively, that the median of $N(\mu, \sigma^2)$ is μ, the middle 80% of the probability is in the range of about two-and-a-half standard deviations from the median, left and right tail weights are equal, and the inter-quartile range is 53% of the inter-decile range.

From the definition of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ inverse distribution function (2.1.1), we now define ρ_1, ρ_2, ρ_3, ρ_4, the GLD counterparts of $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$, as

$$\rho_1 = Q\left(\frac{1}{2}\right) = \lambda_1 + \frac{(\frac{1}{2})^{\lambda_3} - (\frac{1}{2})^{\lambda_4}}{\lambda_2}, \tag{5.1.6}$$

$$\rho_2 = Q(1-u) - Q(u) = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\lambda_2}, \tag{5.1.7}$$

$$\rho_3 = \frac{Q(\frac{1}{2}) - Q(u)}{Q(1-u) - Q(\frac{1}{2})} = \frac{(1-u)^{\lambda_4} - u^{\lambda_3} + (\frac{1}{2})^{\lambda_3} - (\frac{1}{2})^{\lambda_4}}{(1-u)^{\lambda_3} - u^{\lambda_4} + (\frac{1}{2})^{\lambda_4} - (\frac{1}{2})^{\lambda_3}}, \tag{5.1.8}$$

$$\rho_4 = \frac{Q(\frac{3}{4}) - Q(\frac{1}{4})}{\rho_2} = \frac{(\frac{3}{4})^{\lambda_3} - (\frac{1}{4})^{\lambda_4} + (\frac{3}{4})^{\lambda_4} - (\frac{1}{4})^{\lambda_3}}{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}. \tag{5.1.9}$$

The following are direct consequences of the definitions of ρ_1, ρ_2, ρ_3, ρ_4:

1. Since λ_1 may assume any real value, we can see from (5.1.6) that this is also true for ρ_1.
2. Since $0 < u < 1/4$, we have $u < 1 - u$ and from (5.1.7) we see that $\rho_2 \geq 0$.
3. The numerator and denominator of ρ_3 in (5.1.8) are both positive; therefore, $\rho_3 \geq 0$.

4. In (5.1.9), because of the restriction on u, the denominator of ρ_4 must be greater than or equal to its numerator, confining ρ_4 to the unit interval.

In summary, the definitions of ρ_1, ρ_2, ρ_3, ρ_4 lead to the restrictions:

$$-\infty < \rho_1 < \infty, \quad \rho_2 \geq 0, \quad \rho_3 \geq 0, \quad 0 \leq \rho_4 \leq 1. \tag{5.1.10}$$

If we consider $\rho_3 = \rho_3(u, \lambda_3, \lambda_4)$ and $\rho_4 = \rho_4(u, \lambda_3, \lambda_4)$ as functions of u, λ_3 and λ_4, we see that

$$\rho_3(u, \lambda_3, \lambda_4) = \frac{1}{\rho_3(u, \lambda_4, \lambda_3)} \tag{5.1.11}$$

and

$$\rho_4(u, \lambda_3, \lambda_4) = \rho_4(u, \lambda_4, \lambda_3). \tag{5.1.12}$$

A given set of values for ρ_1, ρ_2, ρ_3, ρ_4, subject to the restrictions given in (5.1.10), can be attained from *some* inverse distribution function but it is not clear that it can be attained from a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ inverse distribution. To develop some insight into **the possible** ρ_1, ρ_2, ρ_3, ρ_4 **that are attainable from** GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ **distributions**, we let $f(u, \lambda_3, \lambda_4)$ be the denominator of ρ_4 given in (5.1.9), i.e.,

$$f(u, \lambda_3, \lambda_4) = (1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}.$$

Differentiating $f(u, \lambda_3, \lambda_4)$ with respect to u and simplifying we get

$$\frac{\partial}{\partial u} f(u, \lambda_3, \lambda_4) = \\ -\lambda_3 \left((1-u)^{\lambda_3 - 1} + u^{\lambda_3 - 1} \right) - \lambda_4 \left((1-u)^{\lambda_4 - 1} + u^{\lambda_4 - 1} \right). \tag{5.1.13}$$

This derivative will be negative if both λ_3 and λ_4 are positive, making $f(u, \lambda_3, \lambda_4)$ a decreasing function of u when $\lambda_3 \geq 0$ and $\lambda_4 \geq 0$. This makes ρ_4 an increasing function of u. Moreover, ρ_4 will attain a value of 1 if $u = 1/4$, regardless of the values of λ_3 and λ_4. Figure 5.1–1 gives a more precise view of the impact of u on the surfaces $\rho(u, \lambda_3, \lambda_4)$. Three such surfaces (from the lowest to the highest) for $u = 0.01,\ 0.15$, and 0.23, respectively, are shown in Figure 5.1–1.

5.2 The (ρ_3, ρ_4)-Space of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

The fitting of a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to a given dataset $X_1, X_2, \ldots, X_n$ is done by solving the system of equations $\hat{\rho}_i = \rho_i$ $(i = 1, 2, 3, 4)$ for λ_1, λ_2, λ_3, λ_4. The definitions of $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$ in (5.1.2) through (5.1.5) may have seemed strange or arbitrary to this point. However, we now observe the main advantage of these definitions: the subsystem $\hat{\rho}_3 = \rho_3$ and $\hat{\rho}_4 = \rho_4$ involves only λ_3 and λ_4, allowing

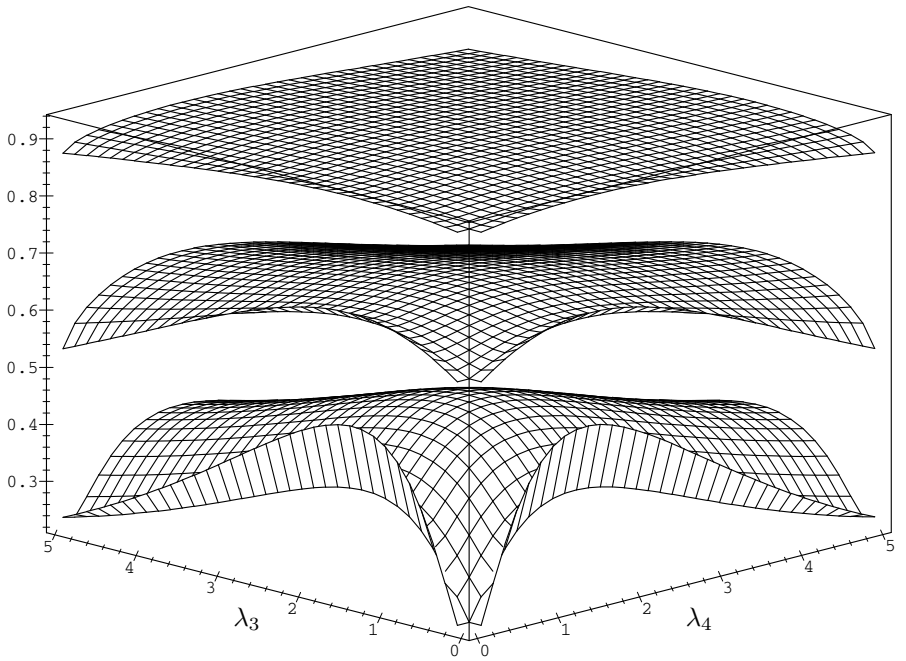


Figure 5.1–1. The surfaces (in rising order) $\rho_4(0.01, \lambda_3, \lambda_4)$, $\rho_4(0.15, \lambda_3, \lambda_4)$, and $\rho_4(0.23, \lambda_3, \lambda_4)$.

us to first solve this subsystem for λ_3 and λ_4 and use these values of λ_3 and λ_4 in $\hat{\rho}_2 = \rho_2$ to obtain λ_2 from

$$\lambda_2 = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\hat{\rho}_2}, \tag{5.2.1}$$

and finally, using the values of $\lambda_2,\ \lambda_3,$ and λ_4 in $\hat{\rho}_1 = \rho_1$ to obtain

$$\lambda_1 = \hat{\rho}_1 - \frac{(1/2)^{\lambda_3} - (1/2)^{\lambda_4}}{\lambda_2}. \tag{5.2.2}$$

As we consider solving the system $\hat{\rho}_3 = \rho_3,\ \hat{\rho}_4 = \rho_4$, it becomes necessary to give u a specific value. For a particular u we must have $(n+1)u \geq 1$ to be able to compute $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$, and eventually $\hat{\rho}_2$, $\hat{\rho}_3$, and $\hat{\rho}_4$. If u is too small, say $u = 0.01$, then our method will be restricted to large samples ($n \geq 99$ for the $u = 0.01$ case). Unfortunately, we cannot resolve this problem by choosing u close to $1/4$ because then the denominator of ρ_4 will be close to its numerator, making ρ_4 close to 1 and rendering it useless as a measure of "tail weight." We have found $u = 0.1$ to be a good compromise: it accommodates all $n \geq 9$ and allows ρ_4 to function as a reasonable measure of tail weight. (For a more detailed discussion of the influence of u on the confidence intervals of various percentiles,

see Fournier, Rupin, Bigerelle, Najjar, Iost, and Wilcox (2007).) **Throughout the rest of this chapter we take** $u = 0.1$.

Figures 5.2–1, 5.2–2, and 5.2–3 show contour plots of (ρ_3, ρ_4) for (λ_3, λ_4) from Regions 1, 2, 5 and 6, Region 3, and Region 4, respectively (recall that the (λ_3, λ_4) regions were discussed in Section 2.2 and are illustrated in Figure 2.2–1 of that section). Half of the contour curves of Figures 5.2–1 through 5.2–3 are obtained by holding λ_3 fixed and varying λ_4; the other half are generated by reversing the roles of λ_3 and λ_4.

It is clear from Figures 5.2–1 through 5.2–3 that there are constraints on (ρ_3, ρ_4). In some cases (e.g., when $(\rho_3, \rho_4) = (0.1, 0.7)$), the equations $\hat{\rho}_3 = \rho_3$ and $\hat{\rho}_4 = \rho_4$ have no solutions; in other cases (e.g., when $(\rho_3, \rho_4) = (0.3, 0.45)$), there seem to be multiple solutions. We can also see that the bulk of (ρ_3, ρ_4)-space is covered by (λ_3, λ_4) chosen from Regions 3 and 4 and very little of (ρ_3, ρ_4)-space is covered by (λ_3, λ_4) from the other regions. This prompts us to give preference to numerical searches for solutions from Regions 3 and 4.

The reasons for the existence of multiple solutions can be understood by considering the solutions associated with a fixed $(\hat{\rho}_3^*, \hat{\rho}_4^*)$. Any solution to $\rho_3 = \hat{\rho}_3^*$ and $\rho_4 = \hat{\rho}_4^*$ must be located simultaneously on these two surfaces. For purposes of illustration let us suppose that $(\hat{\rho}_3^*, \hat{\rho}_4^*) = (0.4, 0.5)$. Since a solution must be located on the intersection of the surface $\rho_4 = 0.5$ and the plane at 0.5 (Figure 5.2–4 illustrates this for (λ_3, λ_4) from Region 3), all points of the intersection of the surface and the plane in Figure 5.2–4 are **possible** solutions. Similar intersection curves will also arise from considering the surface $\rho_3 = 0.4$ and the horizontal plane at 0.4. The actual solutions will be located at the points where these curves themselves intersect. Figure 5.2–5 shows the curves associated with $(\hat{\rho}_3^*, \hat{\rho}_4^*) = (0.4, 0.5)$ (the curves drawn in heavy lines are associated with $\rho_4 = 0.5$ and the more lightly drawn curves are for $\rho_3 = 0.4$).

Figure 5.2–5 indicates the presence of solutions near the points $(\lambda_3, \lambda_4) = (3, 21)$ and $(\lambda_3, \lambda_4) = (6, 1)$. Moreover, there seems to be a possibility of another solution in the lower right portion of the graph. When that portion of the graph is magnified (Figure 5.2–6), it becomes clear that there is indeed a third solution near $(\lambda_3, \lambda_4) = (40, 0.2)$. We can also observe in Figure 5.2–4 that the surface ρ_4 has a "crown" when (λ_3, λ_4) is near $(0, 0)$. It is possible that because of the limited resolution of the figure, we are not able to see the intersection of the surface ρ_4 and the horizontal plane at 0.5. There is, in fact, a fourth solution near the origin. Another possible solution suggested by Figure 5.2–5 is one near $(0.1, 40)$; the λ_4 of such a solution, if it exists, will have to be larger than 60. It is possible that there may be additional solutions that were not part of this analysis (Regions 1, 2, 4, 5, and 6).

The location and the number of solutions change as $(\hat{\rho}_3^*, \hat{\rho}_4^*)$ changes. Figure 5.2–7 shows the intersections of the "solution curves" from Region 3 for $\rho_3 = 0.15, 0.2, \ldots, 0.6$ and $\rho_4 = 0.1, 0.15, \ldots, 0.6$. As in Figures 5.2–5 and 5.2–6, the curves associated with ρ_4 are shown with heavy lines and those associated with

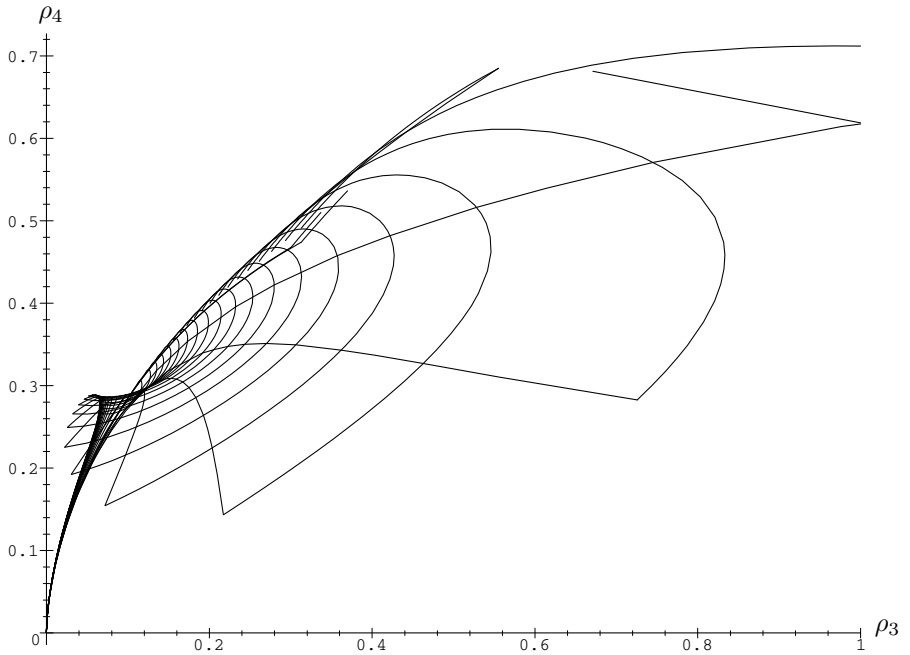


Figure 5.2–1. (ρ_3, ρ_4) generated by (λ_3, λ_4) from Regions 1, 2, 5, and 6 (see Figure 2.2–1).

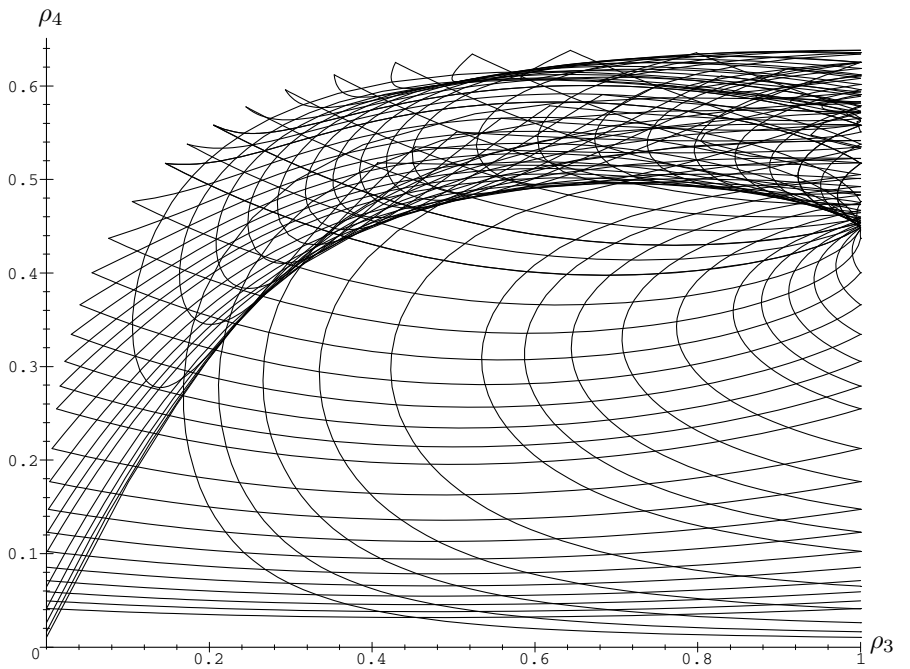


Figure 5.2–2. (ρ_3, ρ_4) generated by (λ_3, λ_4) from Region 3 (see Figure 2.2–1).

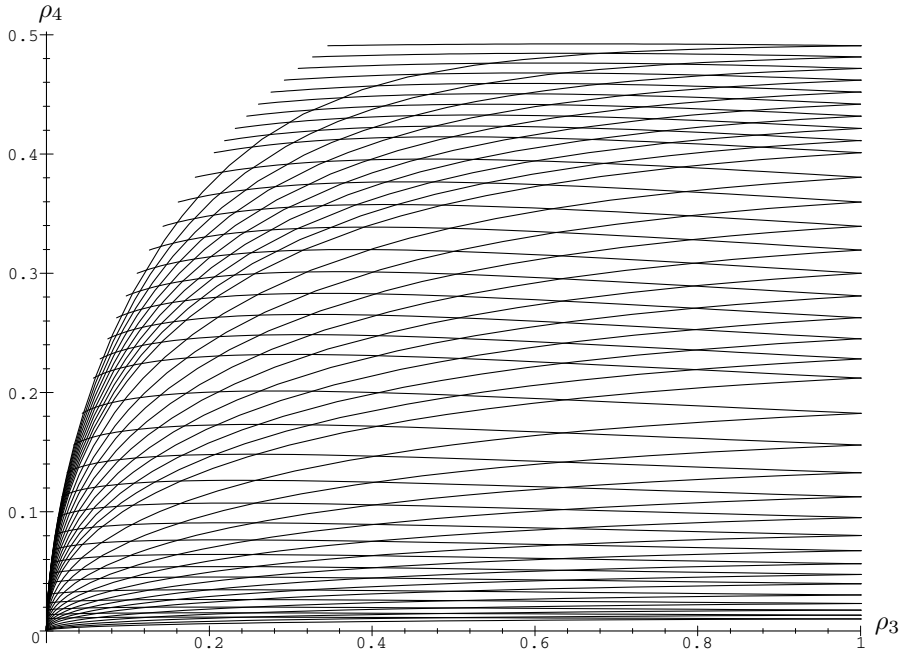


Figure 5.2–3. (ρ_3, ρ_4) generated by (λ_3, λ_4) from Region 4 (see Figure 2.2–1).

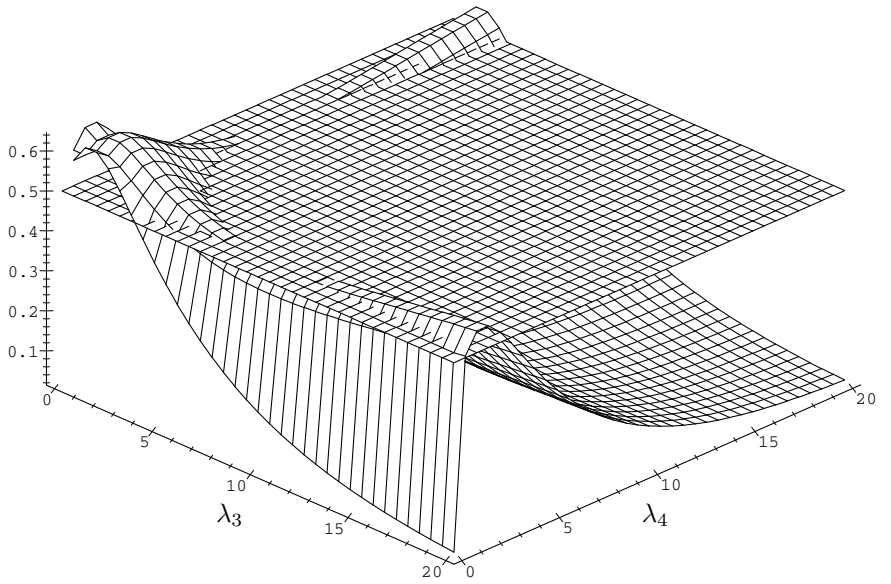


Figure 5.2–4. The surface ρ_4 with horizontal plane at .5 (Region 3).

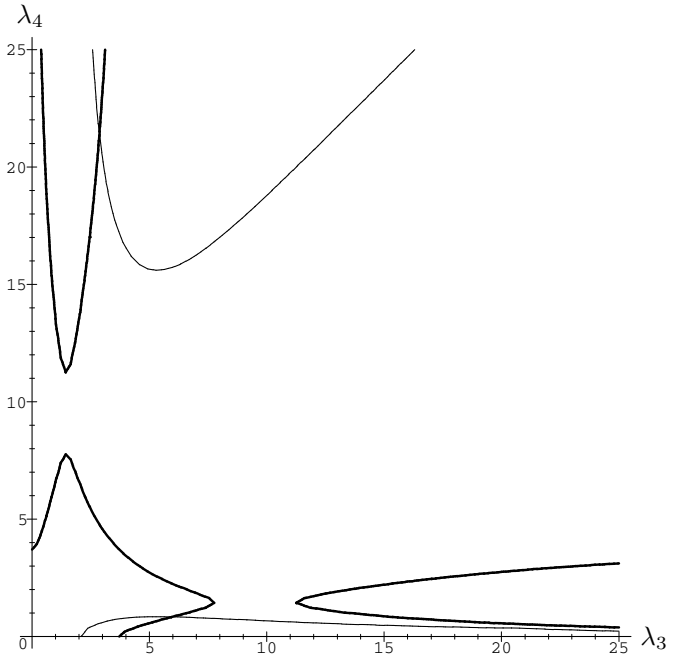


Figure 5.2–5. Solutions to $\rho_3 = .4$, $\rho_4 = .5$ at intersection points (Region 3).

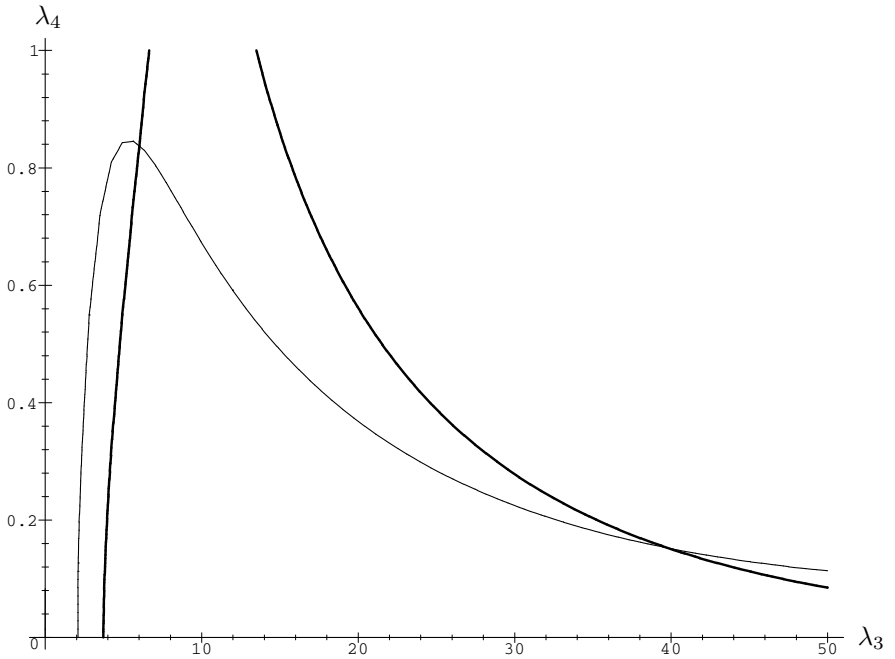


Figure 5.2–6. Solutions to $\rho_3 = 0.4$, and $\rho_4 = 0.5$ at intersection points (Region 3).

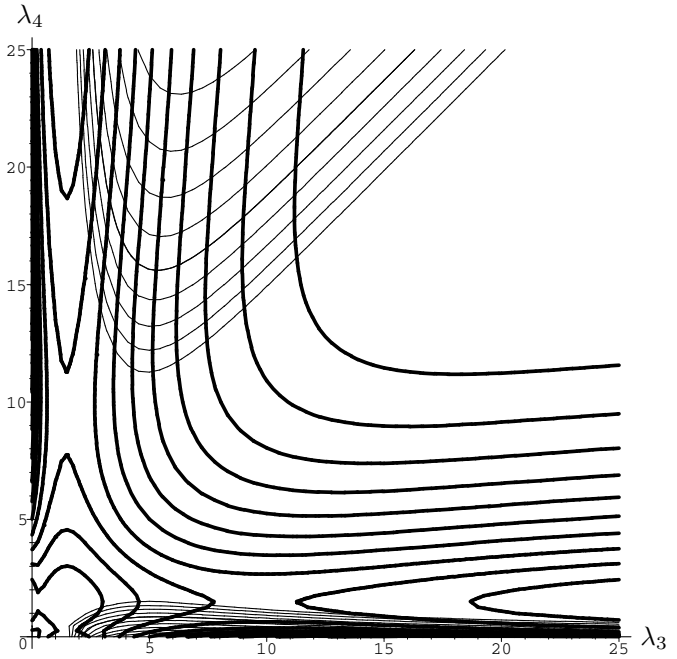


Figure 5.2–7. Intersections of ρ_3 and ρ_4 curves (Region 3).

ρ_3 with more lightly drawn lines.

5.3 Estimation of GLD Parameters through a Method of Percentiles

As was the case with the equations of Chapters 3 and 4, $\hat{\rho}_3 = \rho_3$ and $\hat{\rho}_4 = \rho_4$ cannot be solved in closed form. We use the program `FindLambdasP`, written in **R** and similar to the grid-based searching programs of Chapters 3 and 4. to obtain approximate solutions to these equations.

Solutions for a given $(\hat{\rho}_3, \hat{\rho}_4)$ can be found in various regions of (λ_3, λ_4)-space (depicted in Figure 2.2–1). Depending on the precise values of $\hat{\rho}_3$ and $\hat{\rho}_4$, as many as four solutions may exist in just one region (this occurs in Region 3). The following are more detailed observations that can be made regarding the presence of solutions in various regions of (λ_3, λ_4)-space.

1. It is clear from (5.1.12) that the ρ_4 curves are symmetric with respect to the line $\lambda_3 = \lambda_4$. We can also see from (5.1.11) that symmetric images of the ρ_3 curves (about the line $\lambda_3 = \lambda_4$) would be added in Figure 5.2–7 if ρ_3 were allowed to exceed 1. When $\hat{\rho}_3 > 1$, we can obtain solutions by replacing $\hat{\rho}_3$ with $1/\hat{\rho}_3$ and exchanging the λ_3 and λ_4 that result from a solution associated with $(1/\hat{\rho}_3, \hat{\rho}_4)$.

2. For a substantial portion of $(\hat{\rho}_3, \hat{\rho}_4)$ with $0 < \hat{\rho}_3 < 1$ and $0 < \hat{\rho}_4 < 0.625$, there are solutions from Region 3.

   (a) Table D–1 of Appendix D gives solutions from Region 3 for a large subset of $(\hat{\rho}_3, \hat{\rho}_4)$ with this constraint.

   (b) For a subportion of the (ρ_3, ρ_4)-space with $0.1 < \rho_4 < 0.625$ and ρ_3 relatively close to ρ_4, there is another solution in Region 3. Table D–2 of Appendix D gives such solutions.

   (c) In the region $3.8 \leq \rho_3 \leq 1$ and $0.54 \leq \rho_4 < 0.625$, depending on the value of ρ_4, there may also be a solution (λ_3, λ_4) near $(0, 0)$; as ρ_3 and ρ_4 get large (i.e., as ρ_3 gets closer to 1 and ρ_4 gets closer to 0.625), these solutions gradually move away from the origin. Such solutions are tabulated in Table D–3 of Appendix D.

   (d) In yet a smaller region within $0.66 \leq \rho_3 \leq 1$ and $0.586 \leq \rho_4 \leq 0.625$, it may be possible to obtain an additional solution. Such solutions are given in Table D–4 of Appendix D.

   (e) It is also possible that for certain $(\hat{\rho}_3, \hat{\rho}_4)$ there may be yet another solution in Region 3. Such a solution, on the rare occasions that it exists, will have a small λ_3 (generally $\lambda_3 < 0.2$) and a considerably larger λ_4 (generally $\lambda_4 > 60$). These solutions are not tabulated.

3. When $0 < \hat{\rho}_3 < 1$ and $0 < \hat{\rho}_4 < 0.48$, there is a unique solution from Region 4. In contrast to the Region 3 solutions, these will produce GLD distributions with infinite left and right tails (see Section 2.3 for a comprehensive discussion of GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ shapes). Solutions from Region 4 are given in Table D–5 of Appendix D.

4. Solutions from Regions 1, 2, 5, and 6 are also possible. But, as can be observed from Figure 5.2–1, these are associated with a very limited set of $(\hat{\rho}_3, \hat{\rho}_4)$. Such solutions are not tabulated.

These observations are summarized in Figure 5.3–1 which shows the (ρ_3, ρ_4) pairs for which solutions are tabulated in Tables D–1, D–2, D–3, D–4, and D–5 of Appendix D. These regions are designated in Figure 5.3–1 by T1, T2, T3, T4, and T5, respectively. Boundaries are designated for T1, T2, T3, and T5, but not for T4.

1. The region for Table D–4 is not designated by its own boundary since it represents a rather small neighborhood of the point $(0.8, 0.6)$.

2. **The boundary of the region for Table D–2**, except for the small portion on the right where $\rho_3 = 1$, **is depicted with two thick curves**.

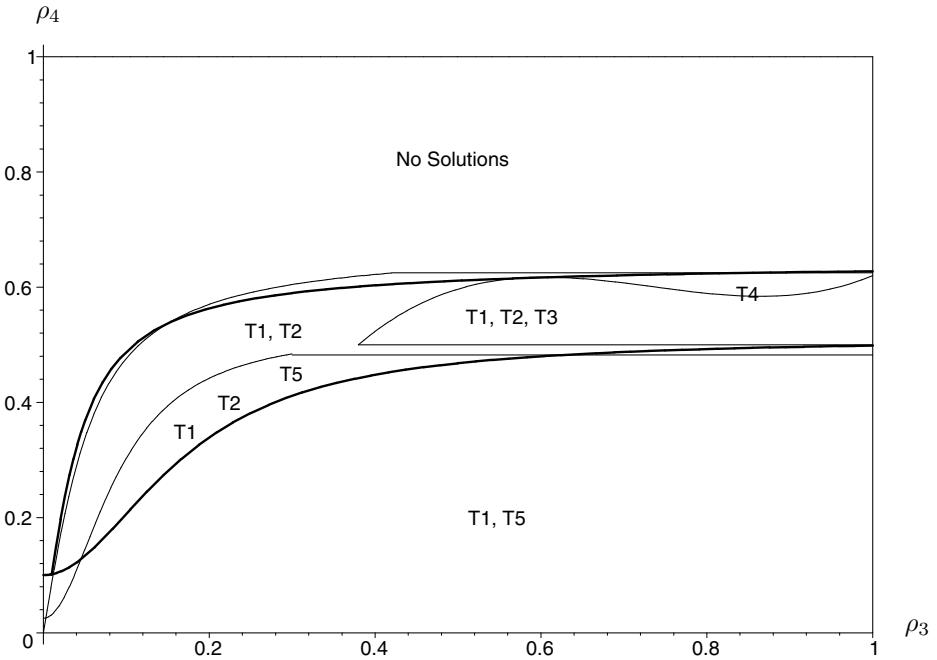


Figure 5.3–1. (ρ_3, ρ_4)-space covered by Tables 1, 2, and 3 (denoted (by T1, T2, and T3, respectively). The $N(0,1)$ and Cauchy points are $(1, .526)$ and $(1, .325)$, respectively.

3. We can see from Figure 5.3–1 that most of the time when solutions exist there will be at least two solutions. Moreover, on some occasions (e.g., when $\rho_3 = 0.9$ and $\rho_4 = 0.61$), Tables 1, 2, 3, and 4 will provide four distinct solutions. On somewhat rare occasions (e.g., near the point that has $\rho_3 = 0.4$ and $\rho_4 = 0.6$), only a single solution will be given in Table 1.

4. As can be observed from Figures 5.2–1, 5.2–2, and 5.2–3, solutions will not exist in any GLD parameter Region (1, 2, 3, 4, 5, or 6) in the area marked "No Solutions" in Figure 5.3–1.

5. The curves in Figure 5.3–1 are obtained by smoothing the "edge points" of the tables; consequently, the points on the curves represent approximations of these edge points. **The "upper" boundaries of the regions for Tables 1 and 5 are shown by lightly drawn curves** (the higher curve is the boundary for the region of Table 1). On the right, these two regions are bounded by $\rho_3 = 1$ and below they are bounded by $\rho_4 = 0$.

6. The boundary of Table 3 consists of the line $\rho_3 = 1$, a lightly drawn horizontal line at about $\rho_4 = 0.5$, and a lightly drawn curved upper portion.

With the availability of the tables of Appendix D, the algorithm below shows how to obtain numerical values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for a GLD fit.

Algorithm GLD–P: Fitting a GLD distribution to data, percentile method.

GLD–P–1. Use (5.1.2) through (5.1.5) to compute $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$.

GLD–P–2. Find the entry point in one or more of the tables of Appendix D closest to $(\hat{\rho}_3, \hat{\rho}_4)$; if $\hat{\rho}_3 > 1$, use $(1/\hat{\rho}_3, \hat{\rho}_4)$ instead of $(\hat{\rho}_3, \hat{\rho}_4)$.

GLD–P–3. Using the entry point from Step GLD–P–2, extract $\hat{\lambda}_3$ and $\hat{\lambda}_4$; if $\hat{\rho}_3 > 1$, interchange $\hat{\lambda}_3$ and $\hat{\lambda}_4$.

GLD–P–4. Use $\hat{\lambda}_3$ for λ_3 and $\hat{\lambda}_4$ for λ_4 in (5.1.7) to determine $\hat{\lambda}_2$.

GLD–P–5. Use $\hat{\lambda}_2$ for λ_2, $\hat{\lambda}_3$ for λ_3, and $\hat{\lambda}_4$ for λ_4 in (5.1.6) to obtain $\hat{\lambda}_1$.

To estimate $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ with greater accuracy than would be possible through Algorithm GLD–P, we can use `FindLambdasP`, the searching program mentioned earlier. `FindLambdasP` has a single vector argument whose components are ρ_1, ρ_2, ρ_3, ρ_4 and whose output is a vector of vectors, each subvector specifying the λ_1, λ_2, λ_3, λ_4 of a fit. Almost always `FindLambdasP` will return a multiplicity of fits for a specified ρ_1, ρ_2, ρ_3, ρ_4, sometimes there may be as many as four fits. It is also possible, although not common, for a fit to appear more than once in the fits provided by `FindLambdasP`. There are two reasons for this redundancy: First, during a search, the program can "migrate" into regions outside of its intended boundaries and rediscover a $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ set essentially identical to one already obtained and second, in addition to the searches of the T1 through T5 regions of Figure 5.3–1, other searches are conducted near the origin where there generally are potential computational difficulties.

If a specific solution is expected near a given (ρ_3, ρ_4) point, the program `RefineSearchGLDP` can be used to look for a solution near this point. The purpose and use of `RefineSearchGLDP` is similar to those of `RefineSearchGLDM` `RefineSearchGBD`, described in earlier chapters. With this program, the tables of Appendix D can be used to obtain reasonable starting points for a search that ultimately is likely to estimate $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ quite accurately. Full details about `FindLambdasP` and `RefineSearcgGLDP`, as well as other programs, are provided in Appendix A.

The two examples that illustrate this below use the program `FindLambdasP` to find an approximate solution.

Example 1. Suppose we want to find percentile-based fits when

$$\hat{\rho}_1 = 0, \quad \hat{\rho}_2 = 1, \quad \hat{\rho}_3 = 0.3, \quad \hat{\rho}_4 = 0.45.$$

From the location of $(0.3, 0.45)$ in Figure 5.3–1, we should expect three solutions to arise from entries of Tables D–1, D–2, and D–5 of Appendix D. These solutions, extracted from the tables, have, respectively,

$$(\lambda_3, \lambda_4) = (3.5362, 22.3488),\ (6.97905, 0.528742),\ (-0.0089, -0.2455).$$

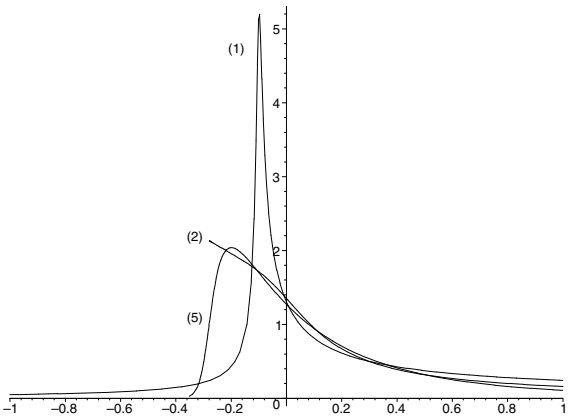


Figure 5.3–2. The GLD_1, GLD_2, and GLD_5 p.d.f.s of Example 1, marked by "(1)," "(2)," and "(5)," respectively.

The more precisely computed estimators of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ can now be obtained through

```
> FindLambdasP(c(0, 1, 0.3, 0.45))
```

The three fits that result are, in order,

$$\text{GLD}_1(-0.1072484,\ 0.782925,\ 3.574022,\ 22.14595),$$

$$\text{GLD}_2(0.6190899,\ 1.113640,\ 7.217482,\ 0.5225044),$$

and

$$\text{GLD}_5(-0.2281918,\ -0.854285,\ -0.01319106,\ -0.2679865).$$

The fact that the supports of GLD_1, GLD_2, and GLD_3 are, respectively,

$$[-1.38, 1.17], \quad [-0.28, 1.52], \quad \text{and} \quad (-\infty, \infty),$$

gives us an indication that we are likely to have three rather diverse fits to choose from for our application. This is even more apparent from Figure 5.3–2 which shows the GLD_1, GLD_2, and GLD_5 p.d.f.s, designated by "(1)," "(2)," and "(5)," respectively. Because the computations of `FindLambdasP` guarantee that $|\hat{\rho}_3 - \rho_3|$ and $|\hat{\rho}_4 - \rho_4|$ are less than 10^{-4} and $\hat{\rho}_1$ and $\hat{\rho}_2$ can be easily obtained from (5.1.6) and (5.1.7), we can be certain that for all three fits ($\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$) are very close to (0, 1, 0.3, 0.45). Actually, in this case, $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|) < 10^{-6}$ for all three fits.

Example 2. Let us consider

$$\hat{\rho}_1 = 0, \quad \hat{\rho}_2 = 1, \quad \hat{\rho}_3 = 0.9, \quad \hat{\rho}_4 = 0.61.$$

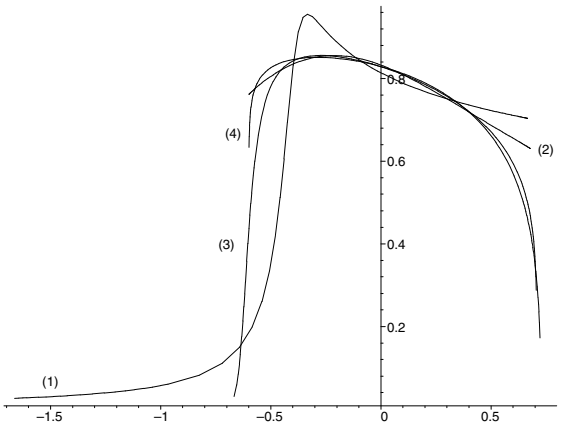


Figure 5.3–3. The GLD_1, GLD_2, GLD_3, and GLD_4 p.d.f.s of Example 2, marked by "(1)," "(2)," "(3)," and "(4)," respectively.

We can see from Figure 5.3–1 that four solutions may be attained from entries of Tables D–1, D–2, D–3, and D–4 with

$$(\lambda_3,\, \lambda_4) \;=\; (1.2099,\, 32.4923),\; (2.48046,\, 2.05430),\; (0.01419,\, 0.84334),\; (0.91871, 0.67829).$$

Since there are entries for $(\rho_3, \rho_4) = (0.9, 0.61)$, the exact values of our $(\hat{\rho}_3, \hat{\rho}_4)$, we can obtain accurate estimators of (λ_3, λ_4) either through `FindLambdasP` or through Algorithm GLD–P. `FindLambdasP` provides the fits

$$\mathrm{GLD}_1(-0.5076702,\; 0.8513396,\; 1.21023,\; 32.46875),$$

$$\mathrm{GLD}_2(0.03939029,\; 1.563265,\; 2.480447,\; 2.054294),$$

$$\mathrm{GLD}_3(0.06359343,\; 1.508428,\; 0.9187062,\; 0.6782812),$$

and

$$\mathrm{GLD}_4(-0.5394768.\; 0.8022563,\; 0.0142295,\; 0.8432495).$$

As in Example 1, the supports of these fits

$$[-1.68, 0.67], \quad [-0.60, 0.68], \quad [-1.79, 0.71], \quad \text{and} \quad [-0.60, 0.73]$$

for GLD_1, GLD_2, GLD_3, and GLD_4, respectively, indicate the possibility of a variety of p.d.f. shapes. This is substantiated in Figure 5.3–3 where the GLD_1, GLD_2, GLD_3, and GLD_4 p.d.f.s are shown (designated, respectively, by "(1)," "(2)," "(3)," "(4)").

We are again sure that $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|) < 10^{-4}$ and closer inspection leads us to conclude that the largest error, 1.7×10^{-5}, is associated with GLD_4. To obtain a GLD_4 fit with a smaller error, we can use

```
> RefineSearchGLDP(c(0,1,0.9,0.61),c(0.01,0.02),c(0.8,0.9),150,4)
```

to get the fit

$$\text{GLD}(-0.5395897,\ 0.8021997,\ 0.01419289,\ 0.8433399),$$

with $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|) < 2.1 \times 10^{-7}$.

5.4 GLD Approximations of Some Well-Known Distributions

In this section we use the percentile-based method described in Sections 5.1 through 5.3 to fit GLD distributions to the important distributions encountered in applications that were considered in Sections 3.4 and 4.4. It follows from the discussion of Section 5.3, and particularly from Figure 5.3–1, that in most cases we will be able to approximate a given distribution by several GLD distributions. Of course, this does not mean that all, or even any, of the approximations will be good ones. In the following sections as we consider the distributions of Sections 3.4 and 4.4, we will discover that, generally, the percentile method produces three approximations, of which one is clearly superior to the others. In our first example we approximate the $N(0,1)$ distribution, obtain three fits, and give all the details associated with each fit. In subsequent sections we concentrate, on the best approximation.

5.4.1 The Normal Distribution

The percentile statistics, ρ_1, ρ_2, ρ_3, ρ_4, for $N(\mu, \sigma^2)$ are

$$\rho_1 = \mu, \quad \rho_2 = 2.5631\sigma, \quad \rho_3 = 1, \quad \rho_4 = 0.5263.$$

Locating (ρ_3, ρ_4) in Figure 5.3–1, we see that three approximations are available from Tables D–1, D–2, and D–3. Through the use of `FindLambdasP`, we get the three approximations GLD_1, GLD_2, and GLD_3 to $N(0,1)$, associated with Tables D–1, D–2, and D–3, respectively. These are

$$\text{GLD}_1(-0.8584,\ 0.3967,\ 1.5539,\ 15.4774),$$

$$\text{GLD}_2(-7.7 \times 10^{-12},\ 0.5456,\ 3.3897,\ 3.3897),$$

$$\text{GLD}_3(-2.0 \times 10^{-6},\ 0.2143,\ 0.1488,\ 0.1488),$$

with respective supports

$$[-3.38, 1.66], \quad [-1.83, 1.83], \quad \text{and} \quad [-4.67, 4.67].$$

GLD_1 is asymmetric (even though it has $\rho_3 = 1$ and hence, as measured by the left-right tail-weight is ρ_3-symmetric) because $\lambda_3 \neq \lambda_4$; therefore, it may not

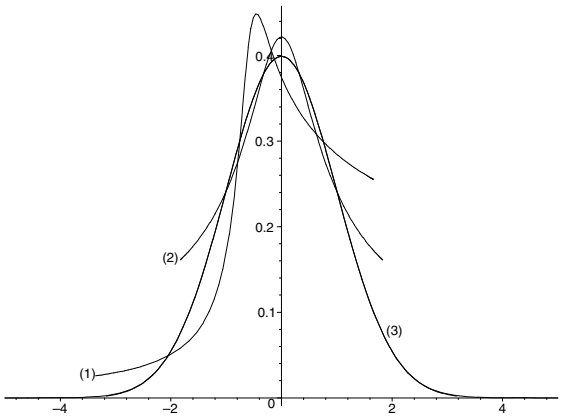


Figure 5.4–1. The GLD_1, GLD_2, GLD_3 fits to $N(0,1)$ marked by "(1)," "(2)," and "(3)," respectively; the $N(0,1)$ and GLD_3 p.d.f.s cannot be distinguished.

be a suitable fit for $N(0,1)$, depending on why $N(0,1)$ is chosen in a particular application. GLD_2 is symmetric but its support is much too confined to be a suitable fit for $N(0,1)$. GLD_1, GLD_2, GLD_3, and the $N(0,1)$ p.d.f.s are shown in Figure 5.4–1 and the GLDs are marked by "(1)," "(2)," and "(3)." The $N(0,1)$ p.d.f. cannot be seen as a distinct curve because it coincides (visually) with GLD_3.

The support of the moment-based GLD approximation of $N(0,1)$ obtained in Section 3.4.1 was $[-5.06, 5.06]$ and that of the GBD fit obtained in Section 4.4.1 was $[-10119, 10119]$. By comparison, the support of GLD_3 is more limited; however, this may not be a problem in many applications (see discussion at the beginning of Chapter 3) because the $N(0,1)$ tail probability outside of the GLD_3 support is approximately 3×10^{-6}. We complete our **first check** for the $N(0,1)$ fits by noting that

$$\begin{aligned} \sup|\hat{f}_1(x) - f(x)| &= 0.1548, \\ \sup|\hat{f}_2(x) - f(x)| &= 0.08637, \\ \sup|\hat{f}_3(x) - f(x)| &= 0.0006141, \end{aligned}$$

where $\hat{f}_i(x)$ are the GLD_i p.d.f.s and $f(x)$ is the p.d.f. of $N(0,1)$. (See Section 3.4 for an explanation of how $\sup|\hat{f}(x) - f(x)|$ and $\sup|\hat{F}(x) - F(x)|$ are computed.)

There are perceptible differences between the graphs of the GLD_1 and GLD_2 d.f.s and the d.f. of $N(0,1)$. However, the graphs of the $N(0,1)$ and GLD_3 d.f.s appear to be identical and to complete our **second check**, we note that

$$\begin{aligned} \sup|\hat{F}_1(x) - F(x)| &= 0.04796, \\ \sup|\hat{F}_2(x) - F(x)| &= 0.03338, \\ \sup|\hat{F}_3(x) - F(x)| &= 0.0005066, \end{aligned}$$

where $\hat{F}_i(x)$ are the GLD_i d.f.s and $F(x)$ is the d.f. of $N(0,1)$.

For our **third and fourth checks**, we determine the L_1 and L_2 distances between $\hat{f}_i$ and f

$$\begin{aligned} ||\hat{f}_1 - f||_1 &= 0.2662 \quad \text{and} \quad ||\hat{f}_1 - f||_2 = 0.1335 \\ ||\hat{f}_2 - f||_1 &= 0.1682 \quad \text{and} \quad ||\hat{f}_2 - f||_2 = 0.08736 \\ ||\hat{f}_3 - f||_1 &= 0.002202 \quad \text{and} \quad ||\hat{f}_3 - f||_2 = 0.0009305. \end{aligned}$$

As indicated in the introductory paragraph of this section, we obtained several approximations, with one of the fits, GLD_3, clearly superior to the others.

5.4.2 The Uniform Distribution

The values of ρ_1, ρ_2, ρ_3, ρ_4 for the uniform distribution on the interval (a, b) are

$$\rho_1 = \frac{1}{2}(a+b), \quad \rho_2 = \frac{4}{5}(a+b), \quad \rho_3 = 1, \quad \rho_4 = \frac{5}{8}.$$

When we use `FindLambdasP`, we get the perfect fit

$$\text{GLD}(0.5,\ 2,\ 1.0000,\ 1.0000)$$

to the uniform distribution on (0, 1). This is not surprising since, as was pointed out in Section 3.4.2, the uniform dsitribution is a special case of the GLD as well as the GBD (see Section 4.4.2 for the latter assertion).

5.4.3 The Student's t Distribution

Because of symmetry, $t(\nu)$, the Student's t distribution with ν degrees of freedom, has $\rho_3 = 1$. When $\nu = 1$, $\rho_4 = 0.3249$ and as ν gets large, ρ_4 gets close to 0.5263, the ρ_4 of $N(0,1)$. To fit $t(1)$, we first determine ρ_1, ρ_2, ρ_3, ρ_4 to obtain

$$\rho_1 = 0, \quad \rho_2 = 6.1554, \quad \rho_3 = 1, \quad \rho_4 = 0.3249.$$

Next, we locate $(\rho_3, \rho_4) = (1, 0.32)$ in Figure 5.3–1 and find out that fits are available through Tables D–1 and D–5. Doing the required computations, we are led to the fits (associated, respectively, with Tables D–1 and D–5),

$$\text{GLD}_1(1.1 \times 10^{-11},\ 0.1698,\ 6.1599,\ 6.1599),$$

$$\text{GLD}_5(6.6 \times 10^{-9},\ -2.0681,\ -0.8728,\ -0.8728).$$

Both GLD_1 and GLD_5 are symmetric and GLD_5 has $(-\infty, \infty)$ for its support. The p.d.f. plots of $t(1)$, GLD_1, and GLD_5 are shown in Figure 5.4–2 where the p.d.f.s of $t(1)$ and GLD_5 appear almost identical. To complete our **first check** for the GLD_5 fit, we note that

$$\sup |\hat{f}(x) - f(x)| = 0.005167$$

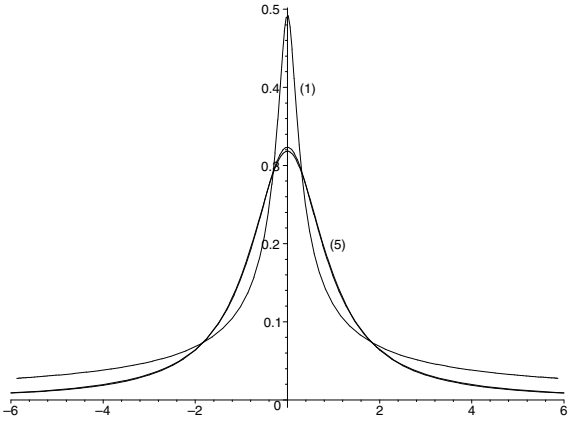


Figure 5.4–2. The GLD_1 and GLD_5 fits to $t(1)$ marked by "(1)" and "(5)," respectively; the $t(1)$ and GLD_5 p.d.f.s are almost identical.

(for the GLD_1 fit this is 0.1743). The d.f. of $t(1)$ cannot be graphically distinguished from the d.f. of the GLD_5 and

$$\sup |\hat{F}(x) - F(x)| = 0.002307$$

(for the GLD_1 this figure is 0.05086). For our **third and fourth checks**, we note that for GLD_5

$$||\hat{f} - f||_1 = 0.01330 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.004831.$$

We cannot compare this method of fitting $t(\nu)$ with previous ones from Chapters 2 and 3 because when $\nu \leq 4$, the requisite moments do not exist (see Section 3.4.3).

For $\nu = 5$, we were able to obtain the fit

$$\text{GLD}_m(1.6 \times 10^{-10},\ -0.2481,\ -0.1359,\ -0.1359)$$

for $t(5)$ through the method of moments in Section 3.4.3. For $t(5)$, the percentile-based method of this chapter produces

$$\rho_1 = 0, \quad \rho_2 = 2.9518, \quad \rho_3 = 1, \quad \rho_4 = 0.4924$$

and the fit (actually, the best of three possible fits)

$$\text{GLD}_p(1.4 \times 10^{-5},\ -0.06464,\ -0.04130,\ -0.04130).$$

The p.d.f.s of $t(5)$ and GLD_p are visually indistinguishable but there is a perceptible distinction between the p.d.f.s of $t(5)$ and GLD_m (see Figure 3.4–2), indicating that GLD_p is a superior fit. For the GLD_p, we have

$$\sup |\hat{f}(x) - f(x)| = 0.001401 \quad \text{and} \quad \sup |\hat{F}(x) - F(x)| = 0.001549,$$

compared to 0.03581 and 0.01488, respectively, for the GLD_m. The L_1 and L_2 distances for the GLD_p approximation to $t(5)$ are

$$||\hat{f} - f||_1 = 0.007294 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.002244,$$

with comparable figures of 0.006650 and 0.003231, repectively for GLD_m.

The "best" fits for $\nu = 6$, 10, and 30 yield, respectively,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -0.01352, -0.008764, -0.008764), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.001326, \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.001396,$$
$$||\hat{f} - f||_1 = 0.006558 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.002080;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4.5 \times 10^{-5}, 0.08238, 0.05492, 0.05493), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.001089 \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.001041,$$
$$||\hat{f} - f||_1 = 0.004820 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.001664;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (8.5 \times 10^{-9}, 0.1721, 0.1179, 0.1179), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.0007979 \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.0006904.$$
$$||\hat{f} - f||_1 = 0.003077 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.001704.$$

5.4.4 The Exponential Distribution

The d.f. of the exponential distribution with parameter θ is

$$F(x) = \int_0^x f(t)\,dt = 1 - e^{-x/\theta} \quad \text{for } x \geq 0 \tag{5.4.1}$$

and 0 for $x < 0$, where $f(t)$ is the p.d.f. of the distribution (see Section 3.4.4). The quantile function, Q, of this distribution is given by

$$Q(x) = -\theta \ln(1 - x).$$

Since for $0 < p < 1$, π_p, the $100p$-th percentile, is characterized by $\pi_p = Q(p)$, we can easily compute $\pi_{0.1}$, $\pi_{0.25}$, $\pi_{0.5}$, $\pi_{0.75}$, and $\pi_{0.9}$ to obtain the ρ_1, ρ_2, ρ_3, ρ_4 of the exponential distribution with parameter θ. These are

$$\rho_1 = \theta \ln 2, \quad \rho_2 = 2\theta \ln 3, \quad \rho_3 = \frac{\ln 9}{\ln 5} - 1 = 0.3652, \quad \rho_4 = \frac{1}{2}.$$

For all values of θ, (ρ_3, ρ_4), rounded to two decimals, is (0.37,0.50) and Figure 5.3–1 indicates the possibility of solutions from Tables D–1 and D–2. When $\theta = 3$, these lead, through the use of `FindLambdasP`, to the two fits

$$\text{GLD}_1(1.0498, 0.1250, 2.9578, 22.5624),$$

$$\text{GLD}_2(5.0180, 0.1967, 5.6153, 0.7407),$$

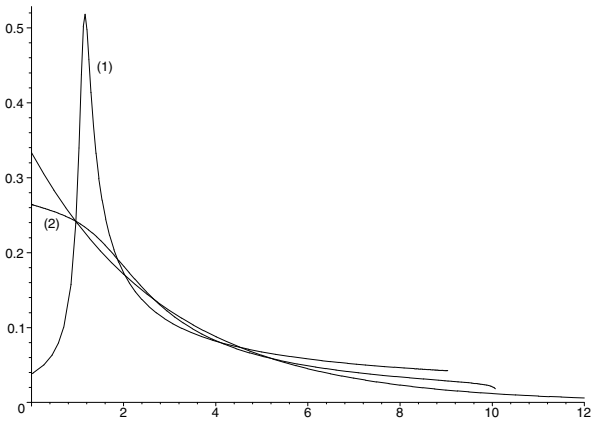


Figure 5.4–3. The p.d.f.s of the exponential distribution (with $\theta = 3$), GLD_1 marked by "(1)," and GLD_2 marked by "(2)".

with respective supports

$$[-6.95, 9.05] \qquad \text{and} \qquad [-0.066, 10.10].$$

Figure 5.4–3 shows the p.d.f.s of GLD_1, GLD_2, and the exponential distribution with $\theta = 3$. It seems from Figure 5.4–3 that both fits have significant differences from the exponential, with GLD_2 (marked by "(2)") being the better of the two fits. To complete our **first check** for GLD_2 we note that

$$\sup |\hat{f}(x) - f(x)| = 0.2655.$$

For our **second check** we compare the d.f.s of the GLD_2 and exponential ($\theta = 3$) distributions and observe that although they are visibly distinct, they are close to each other and

$$\sup |\hat{F}(x) - F(x)| = 0.03449.$$

The results associated with our **third and fourth checks** are

$$||\hat{f} - f||_1 = 0.1470 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.08147.$$

In Section 3.4.4 the reasonably good fit

$$\text{GLD}_m(0.02100, -0.0003603, -0.4072 \times 10^{-5}, -0.001076)$$

with support $(-\infty, \infty)$ was obtained through the method of moments, and in Section 4.4.4 a very good fit that could not be visually distinguished from the exponential was obtained through the GBD. Figure 5.4–4 provides a comparison of the GLD_m and GLD_2 fits by showing their p.d.f.s with the p.d.f. of the exponential distribution with $\theta = 3$.

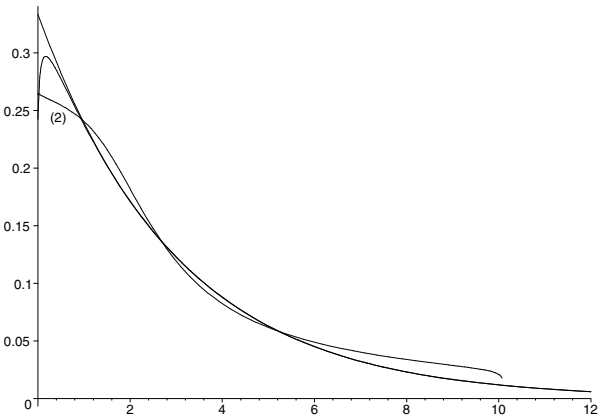


Figure 5.4–4. The p.d.f.s of the exponential distribution (with $\theta = 3$), GLD_2, and the moment-based GLD fit of Section 3.4.4.

Since we know, from Section 3.4.4, that the GLD(0, λ_2, 0, $\theta\lambda_2$) approaches the exponential distribution as $\lambda_2 \to 0$, we attempt to obtain a better fit through

```
> R <- c(3*log(2), 6*log(3), log(9)/log(5)-1, 1/2)
> RefineSearchGLDP(R, c(0.000000001,0.0001), c(0.000000005,0.1), 50, 4).
```

The result is

$$\text{GLD}(2.9673 \times 10^{-5},\ 3.3331 \times 10^{-5},\ 5 \times 10^{-9},\ 10^{-6}).$$

5.4.5 The Chi-Square Distribution

In Sections 3.4.5 and 4.4.5 we considered $\chi^2(5)$, the chi-square distribution with $\nu = 5$ degrees of freedom. For this distribution,

$$\rho_1 = 4.3515, \quad \rho_2 = 7.6260, \quad \rho_3 = 0.5611, \quad \rho_4 = 0.5181$$

and from Figure 5.3–1 we see that there are three solutions associated with Tables D–1, D–2, and D–3. Using `FindLambdasP` we get the percentile-based GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) approximations:

$$\text{GLD}_1(2.7300, 0.1198, 2.3638, 18.7648),$$

$$\text{GLD}_2(6.5087, 0.1775, 6.0038, 1.3273),$$

$$\text{GLD}_3(2.4772, 0.03448, 0.01867, 0.1163).$$

When the graphs of the p.d.f.s of GLD_1, GLD_2, and GLD_3 are compared with that of the $\chi^2(5)$ p.d.f., it becomes obvious that GLD_3 is the superior fit.

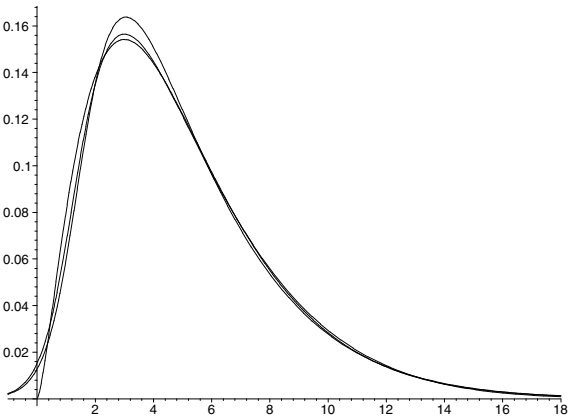


Figure 5.4–5. The $\chi^2(5)$ p.d.f. with its moment-based GLD fit (highest rising curve) and percentile-based GLD fit (next highest).

The GBD in Section 4.4.5 led to a very good fit (graphically indistinguishable from $\chi^2(5)$) and the method of moments produced the reasonably good fit

$$\text{GLD}_m(2.6040, 0.01756, 0.009469, 0.05422)$$

in Section 3.4.5 (see Figure 3.4–4). Figure 5.4–5 shows the p.d.f.s of the $\chi^2(5)$, GLD_3, and the GLD_m fit obtained in Section 3.4.5. The highest rising curve is the moment-based p.d.f., the next highest is the GLD_3 p.d.f., and the lowest curve is the p.d.f. of $\chi^2(5)$. GLD_3 seems to be the better of the two fits; however, both fitted distributions have supports that extend below zero. To complete our **first check** we note that the support of GLD_3 is $[-26.53, 31.48]$ and

$$\sup|\hat{f}(x) - f(x)| = 0.01538.$$

For our **second check** we observe that the d.f.s of GLD_3 and $\chi^2(5)$ are indistinguishable and

$$\sup|\hat{F}(x) - F(x)| = 0.01207.$$

Our **third and fourth checks** yield

$$||\hat{f} - f||_1 = 0.03824 \text{ and } ||\hat{f} - f||_2 = 0.01667.$$

For other values of ν we continue to obtain multiple and distinct GLD fits. The best of these fits for $\nu = 1$, 3, 10, and 30 are, respectively,

$$\begin{gathered}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2.6435, 0.3706, 5.1334, 0.2524), \text{ with} \\ \sup|\hat{f}(x) - f(x)| = 4.1942 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.07715, \\ ||\hat{f} - f||_1 = 0.2879 \text{ and } ||\hat{f} - f||_2 = 0.5567;\end{gathered}$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.7445, 0.02524, 0.004756, 0.06524), \text{ with}$$

$$\sup|\hat{f}(x) - f(x)| = 0.06369 \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.01786,$$
$$||\hat{f} - f||_1 = 0.04136 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.03286;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (7.1811, 0.03341, 0.04262, 0.1541), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.004836, \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.008063,$$
$$||\hat{f} - f||_1 = 0.03098 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.008194;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (26.8432, 0.02402, 0.07997, 0.1743), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.001235 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.004341$$
$$||\hat{f} - f||_1 = 0.02081 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.003376.$$

5.4.6 The Gamma Distribution

The d.f. of $\Gamma(\alpha, \theta)$, the gamma distribution with parameters α and θ can be expressed as

$$F(x) = \int_0^x f(t)\,dt = 1 - \frac{\Gamma_{x/\theta}(\alpha)}{\Gamma(\alpha)} \quad \text{for } x \geq 0, \tag{5.4.2}$$

(and 0 if $x < 0$) where $f(t)$ is the p.d.f. of the gamma distribution (see Section 3.4.6), and $\Gamma_b(a)$ is the **incomplete gamma function**

$$\Gamma_b(a) = \int_0^b t^{a-1}e^{-t}\,dt.$$

For a discussion of the incomplete gamma function, see Abramowitz and Stegun (1964).

While the values of $\pi_{0.1}$, $\pi_{0.25}$, $\pi_{0.5}$, $\pi_{0.75}$, and $\pi_{0.9}$ depend on both α and θ, ρ_3 and ρ_4 are functions of only α. For the specific case of $\alpha = 5$ and $\theta = 3$ considered in Sections 3.4.6 and 4.4.6, the values of ρ_1, ρ_2, ρ_3, ρ_4 are

$$\rho_1 = 14.0127, \quad \rho_2 = 16.6830, \quad \rho_3 = 0.6736, \quad \rho_4 = 0.5225.$$

We can see from Figure 5.3–1 that for $(\rho_3, \rho_4) = (0.67, 0.52)$ three fits can be obtained through Tables D–1, D–2, and D–3. Through `FindLambdasP` these produce, respectively, the following approximations to the gamma distribution with $\alpha = 5$ and $\theta = 3$:

$$\text{GLD}_1(9.9665, 0.0571, 2.1130, 17.3939),$$

$$\text{GLD}_2(17.48460, 0.0819, 5.6513, 1.7157),$$

$$\text{GLD}_3(10.7732, 0.02224, 0.04257, 0.1538).$$

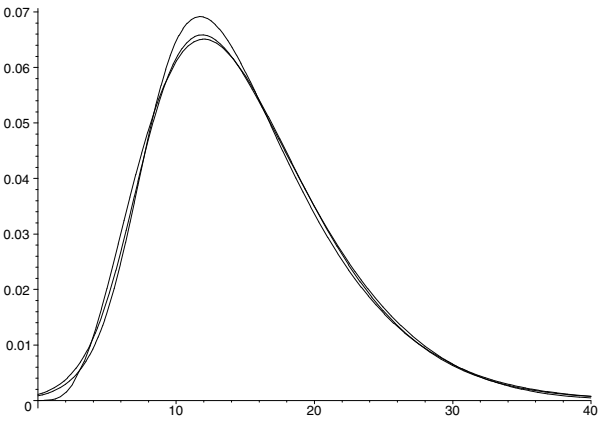


Figure 5.4–6. The p.d.f.s of GLD_m (the highest curve), GLD_3 (the next highest), and the gamma distributions (with $\alpha = 5$, $\theta = 3$).

Upon inspection, we find that GLD_3, with support $[-34.07, 55.61]$, is the best of these fits with

$$\sup |\hat{f}(x) - f(x)| = 0.003227.$$

This takes care of our **first check**. For our **second check** we note that the graphs of the d.f.s of GLD_3 and the gamma distribution with $\alpha = 5$ and $\theta = 3$ look identical and

$$\sup |\hat{F}(x) - F(x)| = 0.008064.$$

For our **third and fourth checks** we have

$$||\hat{f} - f||_1 = 0.03100 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.006694.$$

In Section 3.4.6 a reasonably good fit

$$\mathrm{GLD}_m(10.7621,\, 0.01445,\, 0.02520,\, 0.09388)$$

was obtained when the $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ parameters were estimated via the method of moments.

In Section 4.4.6, through the GBD approximation, the very good fit

$$\mathrm{GBD}(1.5 \times 10^{-5},\, 2.9998 \times 10^{7},\, 4.0000,\, 10^{7})$$

was obtained. Moreover, it was established there that perfect "in limit" GBD approximations are available for all gamma distributions. The p.d.f. of this GBD fit was visually indistinguishable from that of the gamma p.d.f. with $\alpha = 5$ and $\theta = 3$. Figure 5.4–6 shows the p.d.f. of the gamma distribution with $\alpha = 5$ and $\theta = 3$, along with the two approximations GLD_m and GLD_3. The curve that rises highest is the GLD_m p.d.f., the next highest is the GLD_3 p.d.f., and the lowest one is the p.d.f. of the gamma distribution. Of the two approximations, GLD_3 may be superior in many applications.

5.4.7 The Weibull Distribution

The cumulative distribution function for the Weibull distribution with parameters α and β is expressible in closed form as

$$F(x) = \int_0^x f(t)\,dt = 1 - e^{-\alpha x^\beta} \quad \text{for } x \geq 0, \tag{5.4.3}$$

(and 0 if $x < 0$), where $f(t)$ is the Weibull p.d.f. (see Section 3.4.7). From (5.4.3) and the fact that $p = F(\pi_p)$, where π_p is the p-th percentile of this distribution, we have

$$p = 1 - e^{-\alpha \pi_p^\beta}$$

and solving this for π_p we get

$$\pi_p = \left[-\frac{\ln(1-p)}{\alpha}\right]^{1/\beta}. \tag{5.4.4}$$

This allows us to derive $\pi_{0.1},\ \pi_{0.25},\ \pi_{0.5},\ \pi_{0.75}$, and $\pi_{0.9}$ and subsequently obtain

$$\rho_1 = \left(\frac{\ln 2}{\alpha}\right)^{1/\beta}, \tag{5.4.5}$$

$$\rho_2 = \left(\frac{\ln 10}{\alpha}\right)^{1/\beta} - \left(\frac{\ln(10/9)}{\alpha}\right)^{1/\beta}, \tag{5.4.6}$$

$$\rho_3 = \frac{(\ln 2)^{1/\beta} - (\ln(10/9))^{1/\beta}}{(\ln 10)^{1/\beta} - (\ln 2)^{1/\beta}}, \tag{5.4.7}$$

$$\rho_4 = \frac{(2\ln 2)^{1/\beta} - (\ln(4/3))^{1/\beta}}{(\ln 10)^{1/\beta} - (\ln(10/9))^{1/\beta}}. \tag{5.4.8}$$

We can see that as in the situation with the gamma distribution ρ_1 and ρ_2 depend on both distribution parameters, but ρ_3 and ρ_4 depend only on β.

When we take $\alpha = 1$ and $\beta = 5$, as was done in Sections 3.4.7 and 4.4.7, we obtain

$$\rho_1 = 0.9293, \quad \rho_2 = 0.5439, \quad \rho_3 = 1.1567, \quad \rho_4 = 0.5296.$$

Note that since $\rho_3 > 1$ we need to use $1/\rho_3 = 0.8645$ and exchange λ_3 and λ_4. The three solutions available from Tables D–1, D–2, and D–3 and obtained through `FindLambdasP` are, respectively,

$$\text{GLD}_1(1.0938,\ 1.8256,\ 16.3596,\ 1.7350),$$

$$\text{GLD}_2(0.8858,\ 2.5734,\ 2.5935,\ 4.2156),$$

$$\text{GLD}_3(0.9823,\ 1.0492,\ 0.2031,\ 0.1136).$$

Visual inspection leads us to discard GLD_1 and GLD_2 in favor of GLD_3, whose p.d.f. cannot be distinguished from that of the Weibull p.d.f. with $\alpha = 1$ and $\beta = 5$. To complete our **first check** we observe that the support of GLD_3 is $[0.029, 1.935]$ and

$$\sup |\hat{f}(x) - f(x)| = 0.01919.$$

For our **second check** we note that the graphs of the d.f.s of GLD_3 and the Weibull distribution with $\alpha = 1$ and $\beta = 5$ appear to be identical and

$$\sup |\hat{F}(x) - F(x)| = 0.001934.$$

The **third and fourth checks** yield

$$||\hat{f} - f||_1 = 0.008105 \text{ and } ||\hat{f} - f||_2 = 0.008482.$$

The moment-based GLD and the GBD approximations obtained in Sections 3.4.7 and 4.4.7 were also good fits to this Weibull distribution (see Figure 4.4–3).

5.4.8 The Lognormal Distribution

The lognormal random variable, W, is defined by $W = e^X$ where X is $N(\mu, \sigma^2)$ (i.e., X is normally distributed with mean μ and variance σ^2). Since $X = \sigma Z + \mu$, where Z is $N(0, 1)$, we can define W by

$$W = e^{\sigma Z + \mu} = e^{\mu} e^{\sigma Z}.$$

For $0 \leq p \leq 1$, π_p, the p-th percentile of W has, by definition, the property

$$p = P(W \leq \pi_p) = P(e^{\mu} e^{\sigma Z} \leq \pi_p) = P(Z \leq \frac{\ln \pi_p - \mu}{\sigma}).$$

If we define z_p to be the $100p$-th percentile of Z (i.e., $p = P(Z \leq z_p)$), we would have

$$z_p = \frac{\ln \pi_p - \mu}{\sigma}$$

and

$$\pi_p = e^{\sigma z_p + \mu}. \tag{5.4.9}$$

Since tabled values of z_p are commonly available, the $\pi_{0.1}$, $\pi_{0.25}$, $\pi_{0.5}$, $\pi_{0.75}$, and $\pi_{0.9}$ needed can be obtained from the corresponding values of z_p and equations (5.1.2) through (5.1.5) can be used to obtain the following formulas for ρ_1, ρ_2, ρ_3, ρ_4:

$$\rho_1 = e^{\mu}, \tag{5.4.10}$$

$$\rho_2 = e^{\sigma z_{0.9}+\mu} - e^{-\sigma z_{0.9}+\mu}, \tag{5.4.11}$$

$$\rho_3 = \frac{e^{\sigma z_{0.5}+\mu} - e^{-\sigma z_{0.9}+\mu}}{e^{\sigma z_{0.9}+\mu} - e^{\sigma z_{0.5}+\mu}} = \frac{e^{\sigma z_{0.5}} - e^{-\sigma z_{0.9}}}{e^{\sigma z_{0.9}} - e^{\sigma z_{0.5}}}, \tag{5.4.12}$$

$$\rho_4 = \frac{e^{\sigma z_{0.75}+\mu} - e^{-\sigma z_{0.75}+\mu}}{e^{\sigma z_{0.9}+\mu} - e^{-\sigma z_{0.9}+\mu}} = \frac{e^{\sigma z_{0.75}} - e^{-\sigma z_{0.75}}}{e^{\sigma z_{0.9}} - e^{-\sigma z_{0.9}}}. \tag{5.4.13}$$

We see from (5.4.12) and (5.4.13) that the expressions for ρ_3 and ρ_4 do not involve μ.

We now consider, as we did in Sections 3.4.8 and 4.4.8, the specific lognormal distribution that has $\mu = 0$ and $\sigma = 1/3$. For these values of μ and σ,

$$\rho_1 = 1, \quad \rho_2 = 0.8806, \quad \rho_3 = 0.6523, \quad \rho_4 = 0.5149$$

and we are led to the following three solutions from Tables D–1, D–2, and D–3, respectively:

$$\text{GLD}_1(0.8001, 1.0811, 2.2104, 17.0428),$$

$$\text{GLD}_2(1.2074, 1.5215, 6.2031, 1.6028),$$

$$\text{GLD}_3(0.8394, , 0.2926, 0.02930, 0.1002).$$

By looking at the graphs of the GLD_1 and GLD_2 we determine that they are not suitable approximations to the lognormal distribution under consideration, but GLD_3 provides a very good fit. Computation yields $[-2.57, 4.24]$ as the support of this fit and

$$\sup |\hat{f}(x) - f(x)| = 0.05408$$

completing our **first check**. For our **second check** we note that the graphs of the d.f.s of GLD_3 and the lognormal distribution under consideration appear identical and

$$\sup |\hat{F}(x) - F(x)| = 0.006991.$$

For our **third and fourth checks** we obtain

$$||\hat{f} - f||_1 = 0.02776 \text{ and } ||\hat{f} - f||_2 = 0.02529.$$

In Section 4.4.8 we found out that the GBD does not provide approximations to lognormal distributions and in Section 3.4.8 we found that

$$\text{GLD}_m(0.8451, 0.1085, 0.01017, 0.03422)$$

was a reasonably good fit for the lognormal with $\mu = 0$ and $\sigma = 1/3$ (see Figure 3.4–7). The p.d.f.s of the lognormal with $\mu = 0$ and $\sigma = 1/3$, the GLD_m, and the GLD_3 are shown in Figure 5.4–7. The one that rises highest is the GLD_m p.d.f., the next highest is the GLD_3 p.d.f., and the lowest one is the p.d.f. of the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$. Of these two approximations, GLD_3 appears to be the better one.

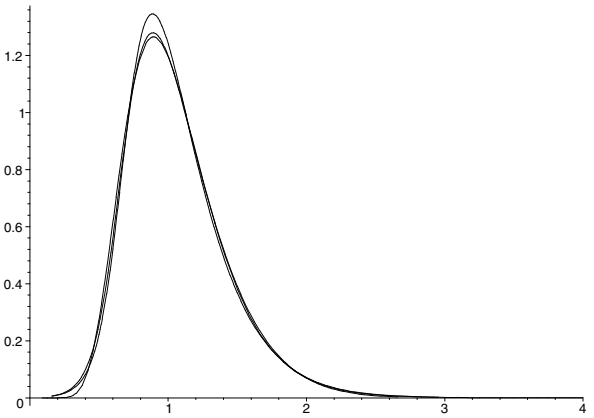


Figure 5.4–7. The p.d.f.s of GLD_m (the highest curve), GLD_3 (the next highest), and the lognormal distributions ($\mu = 0$, $\sigma = 1/3$).

5.4.9 The Beta Distribution

The percentiles, and hence ρ_1, ρ_2, ρ_3, ρ_4, of the beta distribution require numerical integration even when the parameters, β_3 and β_4, are specified. (For the definition of the beta distribution, see Section 3.4.9.) We know from the development of the $\text{GBD}(\beta_1, \beta_2, \beta_3, \beta_4)$ in Chapter 4 that the $\text{GBD}(\beta_1, \beta_2, \beta_3, \beta_4)$ is a generalization of the beta distribution and, consequently, provides perfect fits for the beta distribution. In Section 3.4.9 we found out that, in general, the (α_3^2, α_4) of the beta distributions lies outside of the range covered by the $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ moments, making it impossible to fit a beta distribution through the method of moments. We will see in Section 5.4.17 that the (ρ_3, ρ_4) points of the beta distribution are within the (ρ_3, ρ_4)-space of the $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, allowing us to approximate beta distributions with GLDs through our percentile-based method.

In the specific case considered in Section 3.4.9 where $\beta_3 = \beta_4 = 1$, the distribution function turns out to be $F(x) = 6x - 6x^2$ for $0 \le x \le 1$. Computing ρ_1, ρ_2, ρ_3, ρ_4 we have

$$\rho_1 = \frac{1}{2}, \qquad \rho_2 = 0.6084, \qquad \rho_3 = 1, \qquad \rho_4 = 0.5708.$$

From `FindLambdasP`, we are led to the following four approximations.

$$\text{GLD}_1(0.2231,\ 1.5067,\ 1.2610,\ 22.2492),$$

$$\text{GLD}_2(0.5000,\ 2.4331,\ 2.8374,\ 2.8374),$$

$$\text{GLD}_3(0.5000,\ 1.9443,\ 0.4398,\ 0.4398),$$

GLD_1, GLD_2 and GLD_3 are approximations associated with Tables D–1, D–2, and D–3 of Appendix D. It seems, from Table D–4, that there may exist an additional GLD approximation and when we specifically look for this approximation through

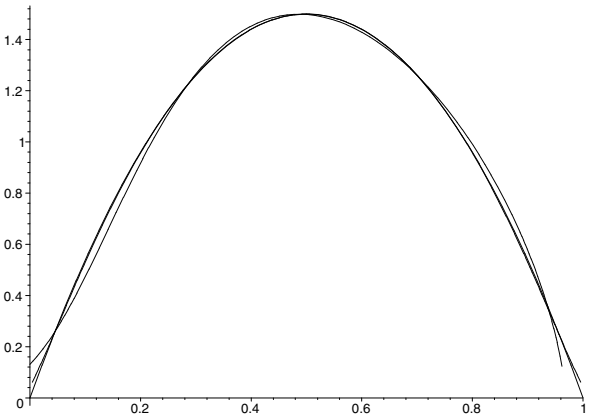


Figure 5.4–8. The p.d.f.s of GLD_3 (highest at 0 and 1), GLD_2 (next highest), and the beta distributions ($\beta_3 = \beta_4 = 1$).

```
> Rhos <- c(0.5, 0.6084, 1, 0.5708)
> RefineSearchGLDP(Rhos, c(0,0.4),c(0,1),50,4)
```

we find

$$\text{GLD}_4(0.3819,\ 1.7030,\ 0.2054,\ 0.5859).$$

GLD_1 and GLD_2 are not good fits but GLD_3 and GLD_4 are, with GLD_3 somewhat superior to GLD_4. Figure 5.4–8 shows the p.d.f.s of GLD_3, GLD_4 and the beta distribution with $\beta_3 = \beta_4 = 1$. The GLD_4 p.d.f. is higher at the endpoints near 0 and 1, GLD_3 gets closer to the x-axis at these points and the beta p.d.f. reaches the x-axis at these points.

The supports of GLD_3 and GLD_4 are

$$[-0.014, 1.014] \qquad \text{and} \qquad [-0.20, 0.97],$$

respectively, and the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of GLD_3 are very close to those of the moment-based $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ approximation obtained in Section 3.4.9. To complete our **first check** we note that

$$\sup|\hat{f}_3(x) - f(x)| = 0.04528 \quad \text{and} \quad \sup|\hat{f}_4(x) - f(x)| = 0.1917,$$

where $f_3(x)$ and $f_4(x)$ are the p.d.f.s of GLD_3 and GLD_4, respectively. The graphs of the d.f.s of GLD_3, GLD_4 and the beta distribution with $\beta_3 = \beta_4 = 1$ appear to be identical and

$$\sup|\hat{F}_3(x) - F(x)| = 0.001038 \quad \text{and} \quad \sup|\hat{F}_4(x) - F(x)| = 0.008390,$$

where $F_3(x)$ and $F_4(x)$ are the d.f.s of GLD_3 and GLD_4, respectively. This takes care of our **second check**.

The **third and fourth checks** for GLD_3 and GLD_4 produce

$$||\hat{f}_3 - f||_1 = 0.004589 \quad \text{and} \quad ||\hat{f}_3 - f||_2 = 0.008567$$

and

$$||\hat{f}_4 - f||_1 = 0.03297 \quad \text{and} \quad ||\hat{f}_4 - f||_2 = 0.04436.$$

Percentile-based fits do not always produce good approximations for beta distributions. For example, when we take $\beta_3 = -0.5$ and $\beta_4 = 1$, we get

$$\rho_1 = 0.1206, \quad \rho_2 = 0.5274, \quad \rho_3 = 0.2824, \quad \rho_4 = 0.5364$$

and the approximations

$$\text{GLD}_1(0.03023, 1.4662, 2.9159, 30.7896)$$

and

$$\text{GLD}_2(0.4299, 2.2685, 3.4343, 0.3323799).$$

Of these, GLD_2 is the better approximation with

$$\sup|\hat{f}(x) - f(x)| = 41.5647, \quad \sup|\hat{F}(x) - F(x)| = 0.07202,$$

and

$$||\hat{f} - f||_1 = 0.2137, \quad ||\hat{f} - f||_2 = 1.2829.$$

5.4.10 The Inverse Gaussian Distribution

This distribution, whose p.d.f. is given in Section 3.4.10, does not have a closed form d.f. or percentile function. We will see in Section 5.4.17 that the (ρ_3, ρ_4) points of the inverse Gaussian distribution are within the (ρ_3, ρ_4)-space of the GLD, making it possible to fit any inverse Gaussian distribution.

To find a percentile-based GLD approximation to the inverse Gaussian we specify its parameters μ and λ and use numeric methods to obtain the needed percentiles. The distribution that we considered in Section 3.4.10 had $\mu = 0.5$ and $\lambda = 6$; for this distribution, we obtain

$$\pi_{.1} = 0.3339, \ \pi_{.25} = 0.3962, \ \pi_{.5} = 0.4801, \ \pi_{.75} = 0.5821, \ \pi_{.9} = 0.6916$$

from which we obtain

$$\rho_1 = 0.4801, \quad \rho_2 = 0.3578, \quad \rho_3 = 0.6915, \quad \rho_4 = 0.5197.$$

Figure 5.3–1 suggests the presence of three solutions associated with Tables D–1, D–2, and D–3 of Appendix D. Through `FindLambdasP` we obtain the fits

$$\text{GLD}_1(0.3934, 2.6871, 2.1018, 16.9307),$$

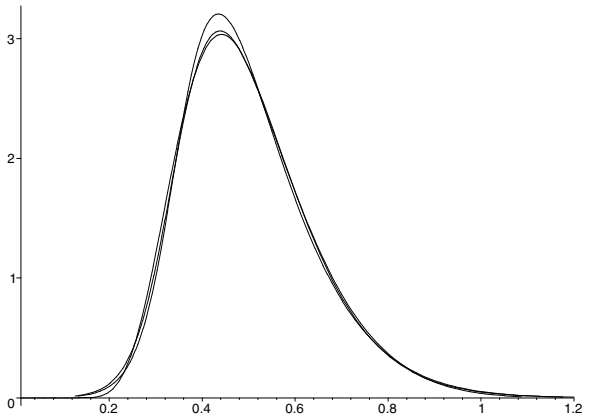


Figure 5.4–9. The p.d.f.s of GLD_m (highest at center), GLD_3 (lowest at center), and the inverse Gaussian ($\mu = 0.5$, $\lambda = 6$).

$$\text{GLD}_2(0.5523,\, 3.7976,\, 5.7440,\, 1.7714),$$

$$\text{GLD}_3(0.4176,\, 0.9573,\, 0.04301,\, 0.1348).$$

Graphic inspection of the three fits reveals that GLD_3 is the best of the three fits. Figure 5.4–9 shows the p.d.f. of the inverse Gaussian distribution with the p.d.f.s of GLD_3 and GLD_m, where GLD_m is the moment-based fit

$$\text{GLD}_m(0.4164, 0.6002, 0.02454, 0.08009)$$

that was obtained in Section 3.4.10. The GLD_m p.d.f. is the highest of the three curves near the center and the GLD_3 is the lowest.

To complete our **first check** we note that the supports of GLD_3 and GLD_m are $[-0.63, 1.46]$ and $[-1.25, 2.08]$, respectively, and for GLD_3

$$\sup |\hat{f}(x) - f(x)| = 0.1139.$$

For our **second check** we observe that the d.f.s of GLD_3 and the lognormal with $\mu = 0.5$ and $\lambda = 6$ are visually indistinguishable and

$$\sup |\hat{F}(x) - F(x)| = 0.006377.$$

The L_1 and L_2 distances, our **third and fourth checks**, are

$$||\hat{f} - f||_1 = 0.02602 \;\text{ and }\; ||\hat{f} - f||_2 = 0.03558.$$

5.4.11 The Logistic Distribution

The p.d.f. of this distribution is given in Section 3.4.11 and the following d.f. is obtained from it through integration:

$$F(x) = \frac{1}{1 + e^{-(x-\mu)/\sigma}} \quad \text{for} \quad -\infty < x < \infty.$$

The percentiles needed for the computation of ρ_1, ρ_2, ρ_3, ρ_4 are

$$\pi_{.1} = \mu - \sigma \ln 9,\ \pi_{.25} = \mu - \sigma \ln 3,\ \pi_{.5} = \mu,\ \pi_{.75} = \mu + \sigma \ln 3,\ \pi_{.9} = \mu + \sigma \ln 9,$$

and

$$\rho_1 = \mu, \qquad \rho_2 = 2\sigma \ln 9, \qquad \rho_3 = 1, \qquad \rho_4 = \frac{1}{2}.$$

Since ρ_3 and ρ_4 are both independent of the distribution's parameters, if we can fit a specific logistic distribution, we should be able to fit any logistic distribution. For the specific case of $\mu = 0$ and $\sigma = 1$,

$$\rho_1 = 0, \qquad \rho_2 = 4.3944, \qquad \rho_3 = 1, \qquad \rho_4 = 0.5.$$

Figure 5.3–1 indicates possible approximations from regions T1, T2, and T5 and from `FindLambdasP` the approximations

$$\text{GLD}_1(-1.1573,\ 0.2464,\ 1.8094,\ 12.3677),$$

$$\text{GLD}_2(2.4 \times 10^{-14},\ 0.3078,\ 3.7094,\ 3.7094),$$

$$\text{GLD}_5(0,\ -1.1799 \times 10^{-8},\ -1.1799 \times 10^{-8},\ -1.1799 \times 10^{-8})$$

are obtained.

The distribution being fitted is symmetric with support the entire real line. GLD_1 is asymmetric with support $[-5.22, 2.90]$, GLD_2 is symmetric with support $[-3.25, 3.25]$ and GLD_5 is symmetric with infinite support in both directions. The higher quality of the GLD_5 approximation is also substantiated visually because the graph of the p.d.f. of the fitted distribution and that of GLD_5 look identical and $\sup |\hat{f}(x) - f(x)|$, $\sup |\hat{F}(x) - F(x)|$, $||\hat{f} - f||_1$, and $||\hat{f} - f||_2$ are all less than 10^{-6}.

It isn't surprising that GLD_5 is an excellent approximation to this logistic distribution because we established in Section 3.4.11 that the d.f of $\text{GLD}(\mu, r/\sigma, r, r)$ converges to the d.f of the logistic distribution with parameters μ and σ as $r \to 0$.

5.4.12 The Largest Extreme Value Distribution

The largest extreme value distribution, whose p.d.f. is given in Section 3.4.12, has, for $\sigma > 0$,

$$F(x) = e^{-e^{-\frac{x-\mu}{\sigma}}} \quad \text{for} \quad -\infty < x < \infty$$

for its d.f. From $F(x)$ we can derive

$$\pi_{.1} = \mu - \sigma \ln(\ln 10), \tag{5.4.14}$$

$$\pi_{.25} = \mu - \sigma \ln(\ln 4), \tag{5.4.15}$$

$$\pi_{.5} = \mu - \sigma \ln(\ln 2), \tag{5.4.16}$$

$$\pi_{.75} = \mu - \sigma \ln(\ln 4/3), \tag{5.4.17}$$

$$\pi_{.9} = \mu - \sigma \ln(\ln 10/9), \tag{5.4.18}$$

from which we obtain

$$\rho_1 = \mu - \sigma \ln(\ln 2), \tag{5.4.19}$$

$$\rho_2 = \sigma \left(\ln(\ln 10) - \ln(\ln(10/9))\right), \tag{5.4.20}$$

$$\rho_3 = \frac{\ln(\ln 10) - \ln(\ln 2)}{\ln(\ln 2) - \ln(\ln 10/9)} \approx 0.6373, \tag{5.4.21}$$

$$\rho_4 = \frac{\ln 2 + \ln(\ln 2) - \ln(\ln 4/3)}{\ln(\ln 10) - \ln(\ln 10/9)} \approx 0.5098. \tag{5.4.22}$$

For the specific case of $\mu = 0$ and $\sigma = 1$ considered in Section 3.4.12, we have

$$\rho_1 = 0.3665, \quad \rho_2 = 3.0844, \quad \rho_3 = 0.6373, \quad \rho_4 = 0.5098$$

and we see from Figure 5.3–1 there should be three fits associated with Tables D–1, D–2, and D–3 of Appendix D. Through `FindLambdasP` we obtain the fits

$$\text{GLD}_1(-0.3024, 0.3081, 2.2785, 16.8688),$$

$$\text{GLD}_2(1.1586, 0.4276, 6.6360, 1.5197),$$

$$\text{GLD}_3(-0.1758, 0.05658, 0.01958, 0.06517).$$

Graphic inspection of these fits reveals that GLD_3 is the best of the three fits. Figure 5.4–10 shows the p.d.f. of the largest extreme value distribution with the p.d.f.s of GLD_3 and GLD_m, where GLD_m is the moment-based fit

$$\text{GLD}_m(-0.1859, 0.02109, 0.006701, 0.02284)$$

that was obtained in Section 3.4.12. The p.d.f. of GLD_m rises highest near the center and the p.d.f. of GLD_3 is the lowest near the center.

The observation that GLD_3 has support $[-17.79, 17.44]$ and

$$\sup |\hat{f}(x) - f(x)| = 0.01396,$$

completes our **first check.** For our **second check** we note that the d.f.s of GLD_3 and the distribution being fitted are visually indistinguishable and

$$\sup |\hat{F}(x) - F(x)| = 0.006280.$$

Our **third and fourth checks** yield

$$||\hat{f} - f||_1 = 0.02391, \quad \text{and} \quad ||\hat{f} - f||_2 = 0.011798.$$

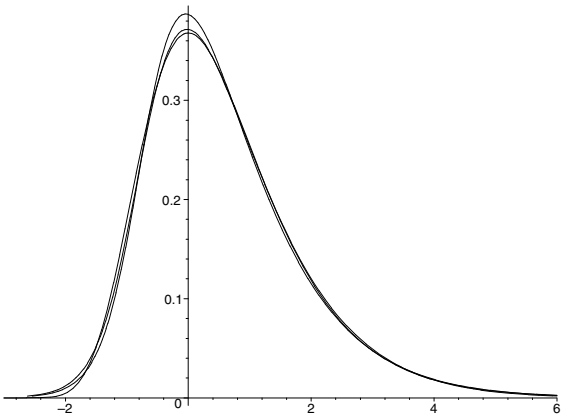


Figure 5.4–10. The p.d.f.s of GLD_m (highest at the center), GLD_3 (lowest at the center), and the largest extreme value distribution ($\mu = 0$, $\sigma = 1$).

5.4.13 The Extreme Value Distribution

The p.d.f. of this distribution is given in Section 3.4.13 and its d.f. is

$$F(x) = 1 - e^{-e^{(x-\mu)/\sigma}} \quad \text{for} \quad -\infty < x < \infty.$$

The percentiles needed for the computation of ρ_1, ρ_2, ρ_3, ρ_4 are

$$\pi_{.1} = \mu + \sigma \ln(\ln(10/9)), \tag{5.4.23}$$

$$\pi_{.25} = \mu + \sigma \ln(\ln(4/3)), \tag{5.4.24}$$

$$\pi_{.5} = \mu + \sigma \ln(\ln 2), \tag{5.4.25}$$

$$\pi_{.75} = \mu + \sigma \ln(\ln 4), \tag{5.4.26}$$

$$\pi_{.9} = \mu + \sigma \ln(\ln 10), \tag{5.4.27}$$

and

$$\rho_1 = \mu + \sigma \ln(\ln 2), \tag{5.4.28}$$

$$\rho_2 = \sigma \left(\ln(\ln 10) - \ln(\ln 10/9)\right), \tag{5.4.29}$$

$$\rho_3 = \frac{\ln(\ln 2) - \ln(\ln 10/9)}{\ln(\ln 10) - \ln(\ln 2)} \approx 1.5692, \tag{5.4.30}$$

$$\rho_4 = \frac{\ln 2 + \ln(\ln 2) - \ln(\ln 4/3)}{\ln(\ln 10) - \ln(\ln 10/9)} \approx 0.5098. \tag{5.4.31}$$

Because of the close relationship of this distribution with the largest extreme value distribution, we obtain the same ρ_2 and ρ_4 that we did in Section 5.4.12

and the ρ_3 that we get is the reciprocal of the ρ_3 of the largest extreme value distribution (see (5.4.21) and (5.4.22)). Recall that (ρ_3, ρ_4) and $(1/\rho_3, \rho_4)$ lead to GLD fits with their λ_3 and λ_4 interchanged (see (5.1.11) of Section 5.1). Therefore, we can get a percentile-based GLD fit for the extreme value distribution whenever we are able to fit the largest extreme value distribution with the same parameters and since ρ_3 and ρ_4 are constants, percentile-based fits are available for all possible parameter values of the distribution.

5.4.14 The Double Exponential Distribution

The double exponential distribution, whose p.d.f. is given in Section 3.4.14, has d.f.

$$F(x) = \begin{cases} 1 - \frac{1}{2}e^{-x/\lambda} & \text{if } x \geq 0 \\ \frac{1}{2}e^{x/\lambda} & \text{if } x < 0. \end{cases}$$

The percentiles

$$\pi_{.1} = -\lambda \ln 5,\ \pi_{.25} = -\lambda \ln 2,\ \pi_{.5} = 0,\ \pi_{.75} = \lambda \ln 1,\ \pi_{.9} = \lambda \ln 5$$

lead to

$$\rho_1 = 0, \quad \rho_2 = 2\lambda \ln 5, \quad \rho_3 = 1, \quad \rho_4 = \ln 2 / \ln 5.$$

Since ρ_3 and ρ_4 are both independent of λ if we can fit a specific double exponential, then we should be able to fit all double exponential distributions. Figure 5.3–1 indicates two possible fits from Tables D–1 and D–5 of Appendix D for all possible values of the parameter λ. In the specific case of $\lambda = 1$ considered in Section 3.4.14, from `FindLambdasP` we obtain the two fits

$$\text{GLD}_1(2.8 \times 10^{-11},\ 0.3833,\ 4.5849,\ 4.5849),$$

$$\text{GLD}_5(9.1 \times 10^{-7},\ -0.7626,\ -0.3552,\ -0.3552).$$

The support of the distribution being fitted is $(-\infty, \infty)$ whereas the supports of GLD_1 and GLD_5 are, respectively, $[-2.61, 2.61]$ and $(-\infty, \infty)$. Also, visual inspection indicates that GLD_5 is the better fit. In Section 3.4.14 we obtained the moment-based fit

$$\text{GLD}_m(2.8 \times 10^{-10},\ -0.1192,\ -0.08020,\ -0.08020).$$

Figure 5.4–11 shows the p.d.f.s of the double exponential (with $\lambda = 1$), GLD_5, and GLD_m. The double exponential p.d.f. rises to a sharp point at 0 and the higher rising curve at the center is the p.d.f. of GLD_5.

The computation

$$\sup |\hat{f}(x) - f(x)| = 0.08034$$

completes our **first check** (note that for GLD_m this figure was 0.1457).

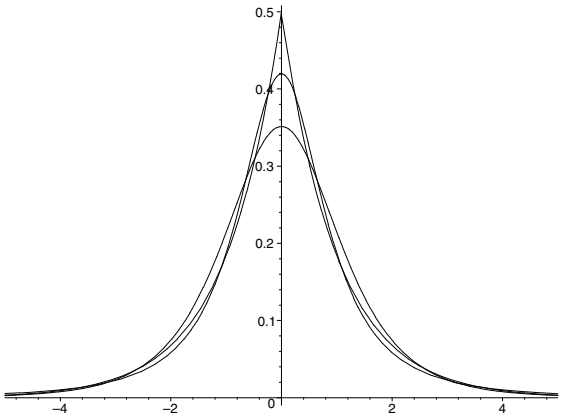


Figure 5.4–11. The p.d.f.s of GLD_m (lowest at center), GLD_5, and the double exponential with $\lambda = 1$ (highest at center).

For our **second check** we observe that the d.f.s of the double exponential and GLD_5 appear to be identical and

$$\sup |\hat{f}(x) - f(x)| = 0.01046.$$

For GLD_m this figure was 0.02871. Our **third and fourth checks** yield

$$||\hat{f} - f||_1 = 0.08864 \quad \text{and} \quad ||\hat{f} - f||_2 = 0.05534.$$

The corresponding figures for the GLD_m approximation were 0.1292 and 0.08173, respectively.

5.4.15 The F-Distribution

The F-distribution whose p.d.f. is given in Section 3.4.15 does not have a closed-form representation of its d.f. or its percentile function. Thus, we have to use numeric methods to obtain ρ_1, ρ_2, ρ_3, ρ_4 in specific instances.

When $\nu_1 = 6$ and $\nu_2 = 25$, the case considered in Section 3.4.15, we obtain

$$\rho_1 = 0.9158, \quad \rho_2 = 1.6688, \quad \rho_3 = 0.5058, \quad \rho_4 = 0.5031.$$

The three possible approximations suggested by Tables D–1, D–2, and D–3 of Appendix D, respectively, lead to

$$\text{GLD}_1 = (0.6080, 0.5372, 2.5962, 18.7786),$$

$$\text{GLD}_2 = (1.4933, 0.7809, 6.7632, 1.1197),$$

$$\text{GLD}_3 = (0.5289, 0.02891, 0.003098, 0.01949).$$

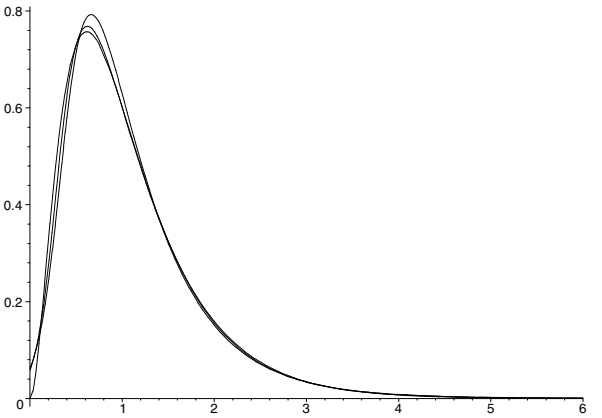


Figure 5.4–12. The p.d.f.s of GLD_m (highest at center), GLD_3, and $F(6, 25)$ (lowest at center).

Visual inspection indicates that GLD_3, with support $[-34.13, 35.19]$, is the best of these three fits. Figure 5.4–12 shows the $F(6, 25)$, GLD_3, and GLD_m p.d.f.s, where GLD_m is the approximation

$$\text{GLD}_m = (0.6457, -0.06973, -0.01100, -0.04020)$$

that was obtained in Section 3.4.15. The curve that rises the highest at the center is the GLD_m p.d.f., the next highest is the GLD_3 p.d.f., and the lowest is the p.d.f. of $F(6, 25)$.

To complete our **first check**, we note that for GLD_3,

$$\sup |\hat{f}(x) - f(x)| = 0.06549.$$

For our **second check**, we observe that the d.f.s of the $F(6, 25)$ and GLD_3 distributions cannot be visually distinguished and

$$\sup |\hat{F}(x) - F(x)| = 0.01154.$$

The **third and fourth checks** give

$$||\hat{f} - f||_1 = 0.03629 \text{ and } ||\hat{f} - f||_2 = 0.03493.$$

Our discussion of the F-distribution in Section 3.4.15 showed that moment-based fits were possible only for relatively large values of ν_2. The percentile-based approach of this chapter turns out to be far more flexible, allowing us to obtain fits for a variety of choices of ν_1 and ν_2. For example, for $(\nu_1, \nu_2) = (2, 4)$, $(4, 6)$, $(6, 12)$ and $(6, 16)$ we get, respectively, the following fits:

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.0001655, -0.5002, -4.0779 \times 10^{-5}, -0.5001)$, with

$$\sup|\hat{f}(x) - f(x)| = 0.01291 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.0002522,$$
$$||\hat{f} - f||_1 = 0.0003389 \text{ and } ||\hat{f} - f||_2 = 0.001360;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.3243, -0.3425, -0.02450, -0.2968), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.1250, \text{ and } \sup|\hat{F}(x) - F(x)| = 0.01388,$$
$$||\hat{f} - f||_1 = 0.03576 \text{ and } ||\hat{f} - f||_2 = 0.04522;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.4963, -0.1270, -0.01241, -0.09145), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.06950 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.01199,$$
$$||\hat{f} - f||_1 = 0.1124 \text{ and } ||\hat{f} - f||_2 = 0.05923;$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.5123, -0.05754, -0.005869, -0.04000), \text{ with}$$
$$\sup|\hat{f}(x) - f(x)| = 0.06772 \text{ and } \sup|\hat{F}(x) - F(x)| = 0.01193,$$
$$||\hat{f} - f||_1 = 0.03614 \text{ and } ||\hat{f} - f||_2 = 0.03595.$$

The approximations for $F(4,6)$, $F(6,12)$, and $F(6,16)$ were not provided by `FindLambdasP`; they were searched for by using `RefineSearchGLDP`. Moreover, in the case of $F(6,16)$, $\max(|\hat{\rho}_3, -\rho_3|, |\hat{\rho}_4, -\rho_4|) = 0.0001469$, which exceeds the threshold of 10^{-4}.

5.4.16 The Pareto Distribution

From the p.d.f. of the Pareto distribution given in Section 3.4.16, through integration we obtain the d.f.

$$F(x) = 1 - \left(\frac{\lambda}{x}\right)^{\beta} \quad \text{for} \quad x > \lambda$$

and $F(x) = 0$ if $x \leq \lambda$, and the quantile function

$$Q(x) = \frac{\lambda}{(1-x)^{1/\beta}}.$$

From Q we can compute

$$\rho_1 = \lambda 2^{1/\beta}, \tag{5.4.32}$$
$$\rho_2 = \lambda\left(10^{1/\beta} - (10/9)^{1/\beta}\right), \tag{5.4.33}$$
$$\rho_3 = \frac{1 - (5/9)^{1/\beta}}{5^{1/\beta} - 1}, \tag{5.4.34}$$
$$\rho_4 = \frac{2^{1/\beta} - (2/3)^{1/\beta}}{5^{1/\beta} - (5/9)^{1/\beta}}. \tag{5.4.35}$$

In Sections 3.4.16 we determined that Pareto distributions with parameters λ and β can be representd by GLD(λ, $-1/\lambda$, 0, $-1/\beta$) and, in fact, when $\lambda = 1$ and $\beta = 5$,

```
> Rhos <- c(1.1487, 0.5636, 0.2921, 0.4618)
> RefineSearchGLDP(Rhos, c(-0.00001,0), c(-0.1,-0.3), 50, 4)
```

produces $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of

```
[1]  1.000002e+00 -1.000037e+00 -1.280000e-11 -2.000067e-01.
```

which puts the approximating GLD close to GLD(1, -1, 0, -0.2).

5.4.17 Summary of Distribution Approximations

It should be clear from Sections 5.4.1 through 5.4.16 that the use of percentiles allows us to approximate a variety of distributions, in many cases yielding better approximations than the moment-based methods of Chapters 3 and 4. It also seems that the (ρ_3, ρ_4) points of the GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) cover a large enough area to provide flexibility in approximating distributions (e.g., the beta, F, and some Student's t distributions) that could not be fitted with the GBD($\beta_1, \beta_2, \beta_3, \beta_4$) or with the GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) when moments are used. Figure 5.4–13 charts the location of the (ρ_3, ρ_4) points for the distributions considered in Sections 5.4.1 through 5.4.16. The (ρ_3, ρ_4) points associated with these distributions consist of a single point, a curve, or a region in (ρ_3, ρ_4)-space.

The (ρ_3, ρ_4) points of the uniform, normal, logistic, Cauchy (or $t(1)$, the Student's t distribution with one degree of freedom), double exponential, exponential, and largest extreme value (hence also extreme value) distributions are marked with small rectangles and labeled with "u," "n," "l," "d," "e," "c," and "v," respectively. The first five of these have $\rho_3 = 1$ and are located at the right edge of Figure 5.4–13; the last two are more centrally located.

The (ρ_3, ρ_4) points of Student's t, gamma (this includes the chi-square as a special case), Weibull, lognormal, inverse Gaussian, and Pareto distributions are represented by curves that are labeled with "T," "G," "W," "LN," "IG," and "P," respectively. The curves for the gamma and Weibull intersect at "e," the (ρ_3, ρ_4) point of the exponential distribution. The lognormal and inverse Gaussian curves are very close to each other; the higher of the two curves that the label "LN, IG" points to is the curve for the lognormal, the lower one is for the inverse Gaussian. With the exception of the curve for the Student's t and Pareto distributions, all curves extend from the vicinity of $(0, 0)$ to a point where $\rho_3 = 1$. The curve for the Pareto distribution also starts near $(0, 0)$ but extends only to the (ρ_3, ρ_4) point of the exponential distribution. The curve, actually straight line, of the t distribution connects the points associated with the Cauchy and normal distributions.

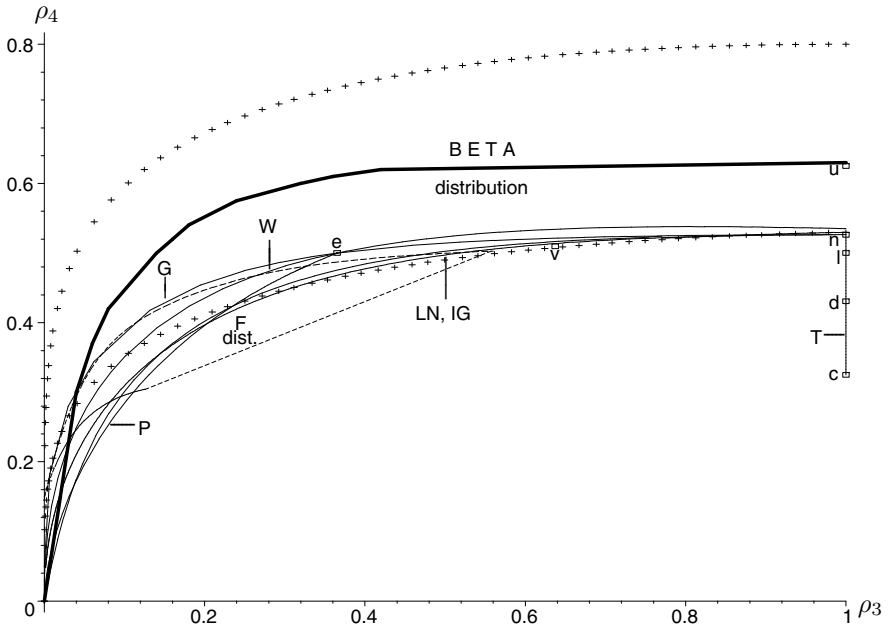


Figure 5.4–13. The (ρ_3, ρ_4) points associated with the distributions considered in Sections 5.4.1 through 5.4.16.

The region enclosed by the two curves marked with "+" represents the area covered by the beta distribution. The region enclosed by the dashed lines and marked "F dist." is the area covered by the F-distribution. Except for the region of the F-distribution, the points, curves, and regions of Figure 5.4–13 provide an accurate representation of the (ρ_3, ρ_4) points for each distribution. The (ρ_3, ρ_4) points of the F-distribution are much harder to compute and the region marked "F dist." is only a reasonable approximation of the true (ρ_3, ρ_4)-space of the F-distribution.

The thick curve in Figure 5.4–13 that goes through the beta region is the boundary that separates the points that are within the range of the tables of Appendix D (the points below the curve) from those that are outside this range (the points above the curve). Points that are above this curve are generally also outside of computation range of `FindLambdasP`.

5.5 Comparison of the Moment and Percentile Methods

As was noted in Section 5.1, a percentile-based method of fitting the GLD is important when moments do not exist (or are out of table range), and to avoid the possibly relatively large variability of sample moments of orders 3 and 4. Indeed, the use of sample percentiles in estimation of population quantities such as the

mean, variance, skewness, and kurtosis has been popular in statistics for some time due to its robustness. In terms of the sample mean and sample median, one often finds such statements as "... the sample median is less affected by extreme values than the mean. With particularly small sample sizes, the sample median is often a better indicator of central tendency than the mean." (Gibbons (1997), p. 97).

Even for higher-order moments such as variance, skewness, and kurtosis, it has been found that better tail weight classification can be found using percentiles rather that the sample kurtosis (Hogg (1972)). In light of a substantial literature along these lines, it is natural to expect the percentile-based method to compare favorably with its moment-based counterpart in the context of fitting a GLD.

We are currently investigating ways in which the relative goodness of these two methods can be quantified. The methods we are studying involve comparisons of the Mean Squared Errors (MSEs) of the estimators of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ obtained by the two methods, as well as comparing the resulting fitted distributions (using chi-squared discrepancies and Kolmogorov-Smirnov distances, for example). While detailed results are not available, preliminary indications are that the percentile approach, while not uniformly superior to the moment method, does better especially with datasets where moments are misled by high variability of sample moments.

One quantitative comparison may be of some interest. It is known that the sample mean $\bar{X}$ is asymptotically normal with center the population mean and variance σ^2/n, based on a random sample of size n from a population with finite variance σ^2. This follows from the classical Central Limit Theorem (see Theorem 6.3.2 of Dudewicz and Mishra (1988)). It is also known that the 0.5 quantile $Z_{0.5}$ is, in the same setting, asymptotically normal with center the population median $\xi_{0.5}$ and variance

$$\left(4f^2(\xi_{0.5})n\right)^{-1},$$

where f is the population probability density function, which is assumed to be continuous and positive at $\xi_{0.5}$ (see, for example, Theorem 7.4.21 of Dudewicz and Mishra (1988)). In the case of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ it is known that

$$\sigma^2 = \frac{B - A^2}{\lambda_2^2} \tag{5.5.1}$$

with

$$A = \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4} \tag{5.5.2}$$

$$B = \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3, 1+\lambda_4), \tag{5.5.3}$$

where β designates the beta function. It follows from the definition of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ that

$$f(\xi_{0.5}) = \lambda_2 \left(\lambda_3(0.5)^{\lambda_3 - 1} + \lambda_4(0.5)^{\lambda_4 - 1}\right)^{-1}.$$

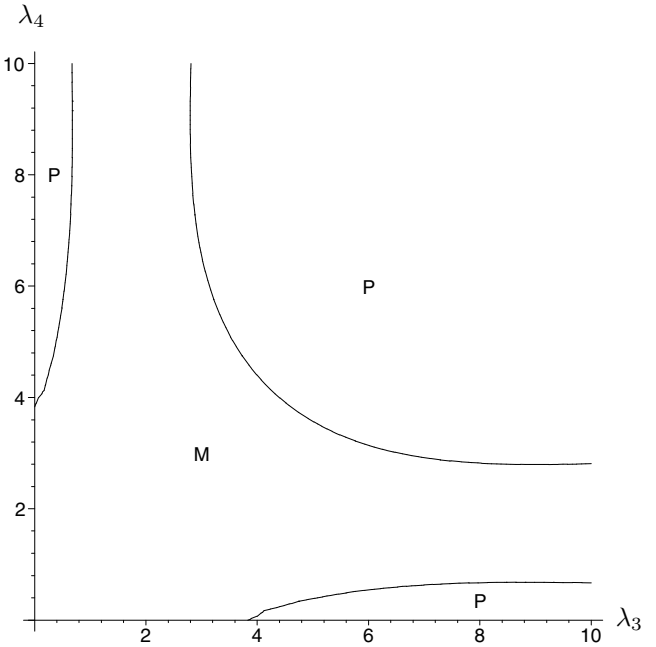


Figure 5.5–1. First quadrant regions of (λ_3, λ_4)-space where variance of $\bar{X}$ is greater than the variance of $Z_{0.5}$ (marked by "P") and less than the variance of $Z_{0.5}$ (marked by "M").

Therefore, the asymptotic distribution of $Z_{0.5}$ has a smaller variance than the asymptotic distribution of $\bar{X}$ if and only if

$$\left(\frac{\lambda_3}{2^{\lambda_3}} + \frac{\lambda_4}{2^{\lambda_4}}\right)^2 < \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3, 1+\lambda_4) - \left(\frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4}\right)^2.$$

In Figures 5.5–1 and 5.5–2 the regions of (λ_3, λ_4)-space where this inequality holds are marked by "P" (indicating a lower asymptotic variance, and a potential preference, for the percentile method) and the regions where the reverse inequality holds are marked by "M" (indicating a potential preference for the moment method).

Since, for both the moment and percentile methods, tabled solutions have so far been confined to the first and third quadrants of (λ_3, λ_4)-space, these are the only (λ_3, λ_4) sets for which we show the status of the inequality. Moreover, in the third quadrant of (λ_3, λ_4)-space, the GLD has its first four moments if and only if $-0.25 \leq \lambda_3, \lambda_4 \leq 0$. Thus, a comparison of the two methods is irrelevant outside of this square. We can see that on a relatively large portion of this square the asymptotic distribution of $Z_{0.5}$ has smaller variance than the asymptotic distribution of $\bar{X}$.

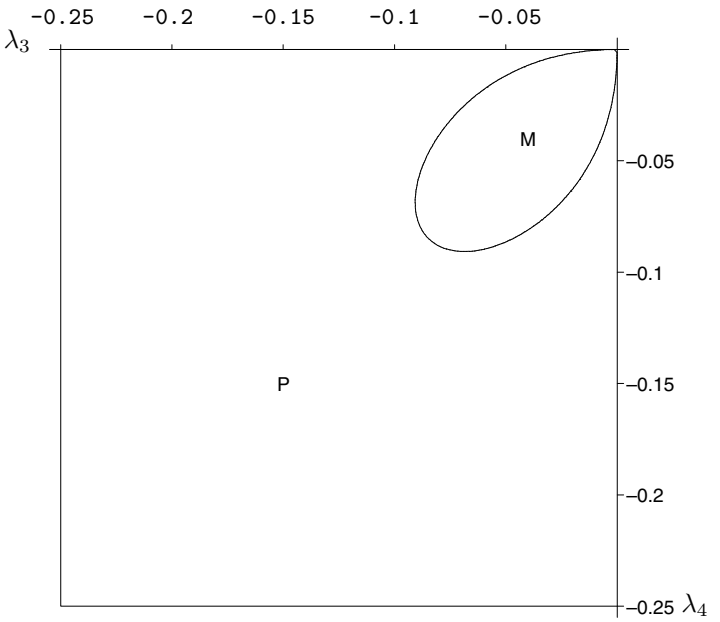


Figure 5.5–2. Third quadrant regions of (λ_3, λ_4)-space where variance of $\bar{X}$ is greater than the variance of $Z_{0.5}$ (marked by "P") and less than the variance of $Z_{0.5}$ (marked by "M").

5.6 Examples: GLD Fits of Data via the Method of Percentiles

In this section we apply the method of this chapter to various datasets. In the first application, in Section 5.6.1, we consider data generated from the Cauchy distribution. Since the $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of the Cauchy distribution do not exist, we do not expect that data generated from the Cauchy could have a fit, much less a good one, through the method of moments. In Sections 5.6.2 and 5.6.3 we introduce data from some scientific studies involving measurements of radiation in soil samples and velocities within galaxies. In Section 5.6.4 we return to the Rainfall data of Sections 3.5.5 (where no GLD fit could be obtained through the use of moments) and 4.5.4 (where a GBD fit was obtained).

In all cases, the GLD fits of the following subsections are obtained through the **R** program `FitGLDP`. This program requires a single argument that is the vector of the data values to which a fit is desired. The program can fail to find a fit (in which case it returns `0 0 0 0`) or it supplies one or more fits as a vector of vectors, each component of which is a set of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that represents a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ fit. There could be occasions where essentially the same fit is given twice.

5.6.1 Example: Data from the Cauchy Distribution

The data for this example (listed below) comes from Karian and Dudewicz (1999). It is Example 2 of that paper and is generated from the Cauchy distribution.

1.99	–.424	5.61	–3.13	–2.24	–.014	–3.32	–.837	–1.98	–.120
7.81	–3.13	1.20	1.54	–.594	1.05	.192	–3.83	–.522	.605
.427	.276	.784	–1.30	.542	–.159	–1.66	–2.46	–1.81	–.412
–9.67	6.61	–.598	–3.42	.036	.851	–1.34	–1.22	–1.47	–.592
–.311	3.85	–4.92	–.112	4.22	1.89	–.382	1.20	3.21	–.648
–.523	–.882	.306	–.882	–.635	13.2	.463	–2.60	.281	1.00
–.336	–1.69	–.484	–1.68	–.131	–.166	–.266	.511	–.198	1.55
–1.03	2.15	.495	6.37	–.714	–1.35	–1.55	–4.79	4.36	–1.53
–1.51	–.140	–1.10	–1.87	.095	48.4	–.998	–4.05	–37.9	–.368
5.25	1.09	.274	.684	–.105	20.6	.311	.621	3.28	1.56

We first attempt to obtain fits by using the moment-based methods of Chapters 3 and 4 and compute $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ to get

$$\hat{\alpha}_1 = .3464 \quad \hat{\alpha}_2 = 49.4908 \quad \hat{\alpha}_3 = 1.8671 \quad \hat{\alpha}_4 = 31.3916.$$

We note that these computations as well as subsequent ones yield slightly different results from those given in Karian and Dudewicz (1999) because the computations in Karian and Dudewicz (1999) are based on the simulated data prior to its truncation to three digits. The (α_3^2, α_4) point that we have is outside the range of the tables in Appendices B and C and also beyond our range of computation, making it impossible to obtain a GLD fit by the methods discussed in Chapters 3 and 4.

To obtain a percentile-based fit, we compute $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$:

$$\hat{\rho}_1 = -0.1820 \quad \hat{\rho}_2 = 7.2600 \quad \hat{\rho}_3 = 0.6632 \quad \hat{\rho}_4 = 0.2981$$

and obtain two fits from `FindLambdasP` based on entries from Tables D–1 and D–5, respectively, of Appendix D.

$$\text{GLD}_1(-0.3848,\ 0.1260,\ 5.2456,\ 10.2631),$$

$$\text{GLD}_5(-0.2830,\ -2.4471,\ -0.9008,\ -1.0802).$$

GLD_5 turns out to be the superior fit. (This is not surprising since the (λ_3, λ_4) for GLD_5 is from Region 5, assuring us that the support of the resulting fit will be $(-\infty, \infty)$; by contrast, the support of GLD_1 is $[-8.3, 7.6]$.)

Although most of the data is concentrated on the interval $[-6, 6]$, the range of the data is $[-37.9, 48.4]$. A histogram on $[-37.9, 48.4]$ would be so compressed that its main features would not be visible. A slightly distorted histogram of the data (when 8 of the 100 observations outside of the interval $[-6, 6]$ are ignored) and the GLD_1 and GLD_5 p.d.f.s are shown in Figure 5.6–1 (a) (the p.d.f. of GLD_1

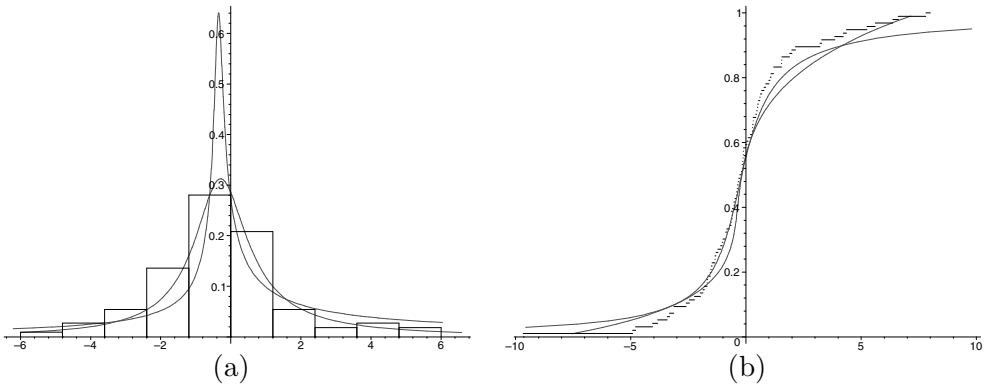


Figure 5.6–1. Histogram of data generated from the Cauchy distribution and the p.d.f.s of the fitted GLD_1 and GLD_5 (a); the e.d.f. of the data with the d.f.s of the fitted GLD_1 and GLD_2 (b).

rises higher at the center). Figure 5.6–1 (b) shows the e.d.f. of the data with d.f.s of GLD_1 and GLD_5.

When the data is partitioned into the intervals

$$(-\infty, -3], \quad (-3, -1.5], \quad (-1.5, -.7], \quad (-.7, -.4], \quad (-.4, 0],$$
$$(0, .4], \quad (.4, .7], \quad (.7, 1.5], \quad (1.5, 3], \quad (3, \infty),$$

we obtain observed frequencies of

$$10, \quad 12, \quad 12, \quad 10, \quad 14, \quad 8, \quad 8, \quad 7, \quad 6, \quad 13.$$

and the expected frequencies for these intervals that result from GLD_1 are (note that one of these expected frequencies is less than 5)

$$10.2541, \quad 7.1461, \quad 9.3966, \quad 10.8406, \quad 18.5661,$$
$$8.1297, \quad 4.0926, \quad 7.5987, \quad 8.9770, \quad 14.9985.$$

These lead to the chi-square goodness-of-fit statistic and corresponding p-value of

$$10.2462 \quad \text{and} \quad 0.06855,$$

respectively. The KS statistic for this fit is $KS = 0.09521$ with $KS\sqrt{n} = 0.09521\sqrt{100} = 0.9521$, which leads to a p-value of 0.33.

For GLD_5, the expected frequencies are

$$10.2644, \quad 9.7906, \quad 14.5336, \quad 8.6288, \quad 12.2154,$$
$$9.8865, \quad 5.4195, \quad 8.9350, \quad 7.4653, \quad 12.8607,$$

and the resulting chi-square statistic and p-value are

$$3.7225 \quad \text{and} \quad 0.5900.$$

The KS statistic in this case is 0.04651, which results in $KS\sqrt{n} = 0.4651$ and p-value of 0.98. Both of these results justify our earlier observation that GLD_5 is the better of the two fits.

5.6.2 Data on Radiation in Soil Samples

Florida gypsum and phosphate mine tailings produce radiation in the form of radon 222. A monitoring of these mines in Polk County, reported by Horton (1979) (also see McClave, Dietrich, and Sincich (1997, p. 38)) gave data on these exhalation rates, part of which follows (in increasing order).

178.99	205.84	357.17	393.55	538.37	558.33	599.84
752.89	878.56	880.84	961.40	1096.43	1150.94	1322.76
1426.57	1480.04	1489.86	1572.69	1698.39	1709.79	1774.77
1830.78	1888.22	1977.97	2055.20	2315.52	2367.40	2617.57
2758.84	2770.23	2796.42	2996.49	3017.48	3750.83	3764.96
4132.28	5402.35	6815.69	9139.21	11968.23		

For this data

$$\hat{\alpha}_1 = 2384.8422, \quad \hat{\alpha}_2 = 5.5198 \times 10^6, \quad \hat{\alpha}_3 = 2.3811, \quad \hat{\alpha}_4 = 9.1440$$

and moment-based GLD and EGLD fits are not possible. However, the $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$ for this data are

$$\hat{\rho}_1 = 1742.2800, \quad \hat{\rho}_2 = 4867.3110, \quad \hat{\rho}_3 = 0.3776, \quad \hat{\rho}_4 = 0.3881$$

and `FindLambdasP` can be used to obtain the following two percentile-based fits, associated with Tables D–1 and D–5 of Appendix D, respectively.

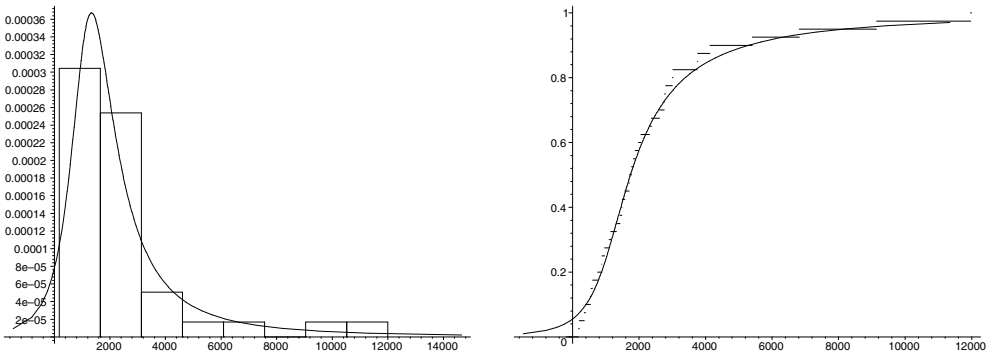
$$\text{GLD}_1(1377.6122, 0.0001678, 4.0305, 17.2300),$$

$$\text{GLD}_5(1314.9610, -0.0008317, -0.2609, -0.6357).$$

The support of GLD_1, $[-4582.25, 7337.47]$, indicates that the fitted GLD_1 distribution extends too far to the left and not far enough to the right. By contrast, the GLD_5 distribution extends indefinitely in both directions and its support is $(-\infty, \infty)$. Visual inspection indicates that GLD_5, in spite of its support, is the better of the two fits. Figure 5.6–2 (a) shows a histogram of the data with the p.d.f. of GLD_5 and Figure 5.6–2 (b) shows the e.d.f. of the data with the d.f. of GLD_5.

To get a chi-square goodness-of-fit statistic for the GLD_5 fit, we partition the data into the intervals

$$[0, 600), \quad [600, 1250), \quad [1250, 1700), \quad [1700, 2300), \quad [2300, 3400), \quad [3400, \infty)$$



(a) (b)

Figure 5.6–2. Histogram of radiation data and the p.d.f. of GLD_5 (a); the e.d.f. of the data with the d.f. of GLD_5 (b).

and obtain the observed frequencies

$$7, \quad 6, \quad 6, \quad 6, \quad 8, \quad 7$$

and expected frequencies

$$5.3512,\ 7.7003,\ 6.4009,\ 4.6083, 6.1627,\ 7.9462.$$

The chi-square statistic and its associated p-value for this fit are

$$1.5990 \quad \text{and} \quad 0.2060.$$

The KS statistic for the GLD_5 fit is $KS = 0.07116$ and $KS\sqrt{n} = 0.07116\sqrt{40} = 0.4501$, yielding a p-value of 0.99.

5.6.3 Data on Velocities within Galaxies

In astronomy, the cluster named A1775 is believed to consist of two clusters that are in close proximity. Oegerle, Hill, and Fitchett (1995) gave velocity observations (in kilometers per second) from A1775 (see also McClave, Dietrich, and Sincich (1997, p. 85)) which include the following velocity observations (in increasing order).

18499	18792	18933	19026	19111	19111	19130
19179	19225	19404	19408	19595	19595	19619
19673	19740	19807	19866	20210	20210	20875
21911	21993	22192	22193	22417	22417	22426
22513	22625	22647	22682	22738	22738	22744
22779	22781	22796	22809	22922	22922	23017
23059	23121	23220	23261	23303	23303	23408
23432	24909					

The $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$ for this data are

$$\hat{\rho}_1 = 22417.0000, \quad \hat{\rho}_2 = 4183.6000, \quad \hat{\rho}_3 = 3.7671, \quad \hat{\rho}_4 = 0.7682.$$

The (ρ_3, ρ_4) for this data is not covered by any of the tables of Appendix D and if we locate this point in Figure 5.4–13, we see that it is located outside the region covered by the GLD tables but within the region of the generalized beta distribution of the EGLD discussed in Chapter 4.

The $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$ for this data,

$$\hat{\alpha}_1 = 21456.59, \quad \hat{\alpha}_2 = 303844, \quad \hat{\alpha}_3 = -0.2639, \quad \hat{\alpha}_4 = 1.5473,$$

places (α_3^2, α_4) outside of the GLD range but within the EGLD system covered by the GBD which gives the fit

$$\text{GBD}(18682.12, 4877.3018, -0.4766, -0.6033).$$

The principal purpose of this example is to show that there are datasets for which percentile-based fits are not possible. Having established this, we note that the support for the fitted GBD is $[18682, 23559]$ but the data ranges over the interval $[18499, 24909]$. The method described in Problems 4.1 through 4.3 may well produce a better fit by guaranteeing that the support of the fitted GBD covers the data range.

5.6.4 Rainfall Data of Sections 3.5.5 and 4.5.4

Rainfall (in inches) at Rochester (X) and Syracuse (Y), New York was given in Table 3.5–6 of Section 3.5.5 where we were not able to find a GLD fit to either X or Y by using moments. In Section 4.5.4 we found GBD fits for both X and Y. Here we reconsider this data with the view of developing percentile-based GLD fits to both X and Y.

We compute the $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$ for X (rainfall in Rochester) to get

$$\hat{\rho}_1 = 0.1900, \quad \hat{\rho}_2 = 1.4060, \quad \hat{\rho}_3 = 0.1302, \quad \hat{\rho}_4 = 0.4339$$

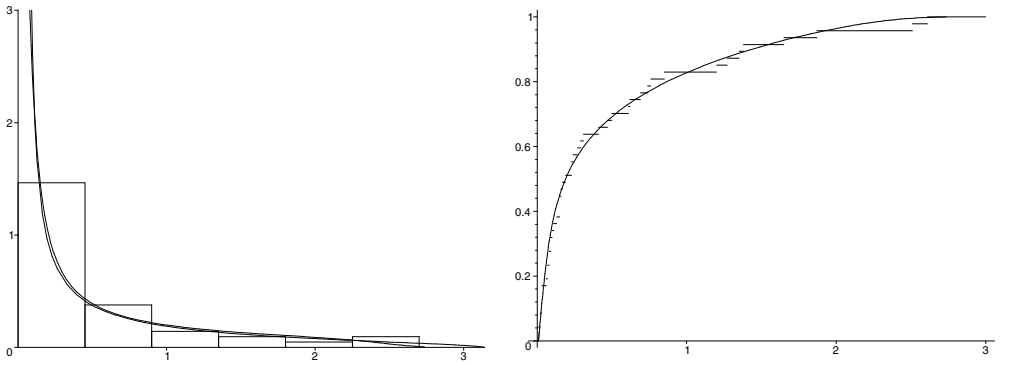
and, using `FitGLDP`, obtain the two fits

$$\text{GLD}_1(0.08299, 0.4698, 4.3140, 34.6818),$$

$$\text{GLD}_2(1.7685, 0.5682, 4.9391, 0.1055),$$

from the (λ_3, λ_4)-spaces associated with Tables D–1 and D–2 of Appendix D. The supports of GLD_1 and GLD_2 are, respectively, $[-2.04, 2.21]$ and $[0.0085, 3.53]$. The support of GLD_1 is ill-suited for this data and visual inspection of the GLD_1 p.d.f confirms the view that GLD_2 is the better of the two fits. Figure 5.6–3 (a) shows a histogram of X with the p.d.f. of GLD_2 and the p.d.f of the EGLD fit

$$\text{GBD}(0.06170, 3.0764, -0.7499, 0.5408),$$



(a) (b)

Figure 5.6–3. Histogram of rainfall data (X) and the p.d.f.s of the fitted GLD_2 and GBD (a); the e.d.f. of the data with the d.f. of the fitted GLD_2 (b).

obtained in Section 4.5.4. The two p.d.f.s are almost indistinguishable. Figure 5.6–3 (b) shows the e.d.f. of X with the d.f. of GLD_2 (the d.f. of the GBD is not included as it is indistinguishable from that of the GLD_2 distribution).

To check the quality of the GLD_2 fit, we partition X into the intervals

$$(-\infty, 0.07), \quad [0.07, 0.1), \quad [0.1, 0.2), \quad [0.2, 0.45), \quad [0.45, 1.0), \quad [1.0, \infty)$$

(the same intervals used in Section 4.5.4) and obtain observed frequencies of

$$9, \quad 6, \quad 9, \quad 7, \quad 8, \quad 8$$

and expected frequencies of

$$12.8889, \quad 3.9444, \quad 7.1678, \quad 7.5653, \quad 7.3533, \quad 8.0808.$$

This leads to a chi-square statistic and corresponding p-value of

$$2.8132 \quad \text{and} \quad 0.09349,$$

respectively. For the GLD_2 fit we have $KS = 0.08274$, $KS\sqrt{n} = 0.08274\sqrt{47} = 0.5672$, yielding a p-value of 0.90.

For comparison, we note that the chi-square statistic and p-value associated with the GBD fit of Section 4.5.4 were

$$1.7190 \quad \text{and} \quad 0.1898,$$

respectively. The KS ststistic for the GBD fit is 0.1915, for a p-value of 0.15.

The $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$ for Y are

$$\hat{\rho}_1 = 0.2200, \quad \hat{\rho}_2 = 1.0100, \quad \hat{\rho}_3 = 0.2053, \quad \hat{\rho}_4 = 0.3663.$$

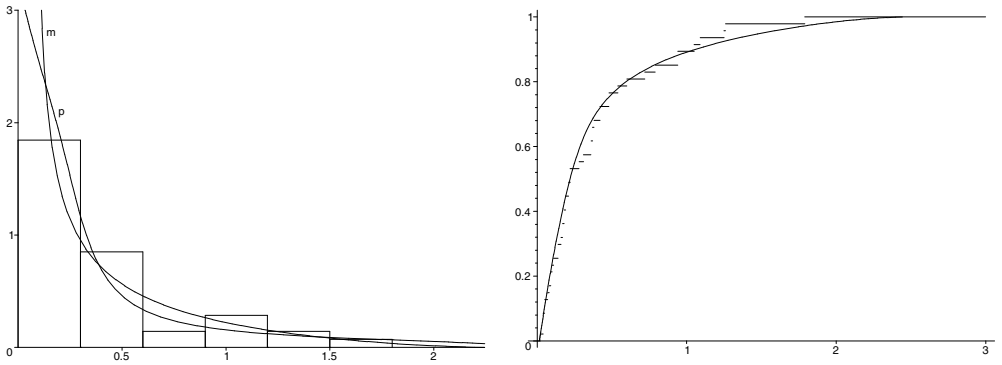


Figure 5.6–4. Histogram of rainfall data (Y) and the p.d.f.s of the fitted GLD_2 and GBD (a); the e.d.f. of the data with the d.f. of the fitted GLD_2 (b).

Appendix D indicates the possibilities of fits associated with Tables D–1, D–2, and D–5. These fits, obtained through `FindGLDP`, are

$$\text{GLD}_1(0.1674,\ 0.6762,\ 4.8129,\ 23.8840),$$

$$\text{GLD}_2(1.3381,\ 0.7557,\ 9.6291,\ 0.2409),$$

$$\text{GLD}_5(0.08251,\ -4.0872,\ -0.08545,\ -0.6986).$$

The supports of GLD_1, GLD_2 and GLD_5 are, respectively,

$$[-1.31, 0.84], \qquad [.015, 2.09], \qquad (-\infty, \infty).$$

The most reasonable of these is the support of GLD_2 and it can also be confirmed visually that the GLD_2 p.d.f. is the most suitable of the three fitted GLDs.

Figure 5.6–4 (a) shows a histogram of Y with the p.d.f. of GLD_2 and the p.d.f. of the EGLD fit

$$\text{GBD}(0.07209,\ 2.1729,\ -0.5816,\ 1.4355)$$

that was obtained in Section 4.5.4. The moment-based GBD p.d.f. is marked with "m" and the p.d.f. of GLD_2 is marked with "p." Figure 5.6–4 (b) depicts the e.d.f. of Y with the d.f. of GLD_2 (the d.f. of the GBD fit is not included as it cannot be distinguished from that of the GLD_2).

To obtain a chi-square statistic for the GLD_2 fit we partition Y into the intervals

$$(-\infty, 0.08),\ \ [0.08, 0.15),\ \ [0.15, 0.22),\ \ [0.22, 0.38),\ \ [0.38, 0.73),\ \ [0.73, \infty)$$

(the same intervals that were used in Section 4.4.5) and determine the observed frequencies

$$7, \quad 7, \quad 9, \quad 8, \quad 8, \quad 8$$

and expected frequencies

$$8.8871, \quad 8.0552, \quad 6.5577, \quad 9.0284, \quad 6.9056, \quad 7.5661.$$

From these we obtain the chi-square statistic and p-value

$$1.1764 \quad \text{and} \quad 0.1841,$$

respectively. Also, for this fit, $KS = 0.1016$ and $KS\sqrt{n} = 0.1016\sqrt{47} = 0.6965$, giving a p-value of 0.72.

For comparison we note that the chi-square statistic and p-value associated with the GBD fit of Section 4.5.4 were

$$4.3131 \quad \text{and} \quad 0.03782,$$

respectively. For the GBD fit, $KS = 0.1489$ and $KS\sqrt{n} = 0.1489\sqrt{47} = 1.0208$ with a p-value of 0.27.

5.7 Percentile-Based GLD Fit of Data from a Histogram

In the examples of Section 5.6 the actual data $X_1, X_2, \ldots, X_n$ on the phenomenon of interest was available to us. We assumed that these were independent and identically distributed observations, an assumption that should be tested or otherwise verified. However, in many cases, such as that described in Section 3.6 where the method of moments was used, the data are given in the form of a **histogram**. In such situations, the key is to estimate the percentiles from the histogram. The two alternatives considered in Section 3.6 were the mid-point assumption (all points in an interval are located at the center of the interval) and uniform spread assumtion (all points in an interval are uniformly spread out throughout the interval). We consider percentile-beased GLD fits now under each of these assumptions.

With the mid-point assumption, we have

$$\hat{\rho}_1 = 0.03250, \quad \hat{\rho}_2 = 0.02500, \quad \hat{\rho}_3 = 0.6667, \quad \hat{\rho}_4 = 0.6000.$$

There are two possible fits for the $(\hat{\rho}_3, \hat{\rho}_4)$ that we have (GLD_{m1} and GLD_{m2}, from Tables D–1 and D–2, respectively, of Appendix D). Using `FitGLDP` we obtain

$$\text{GLD}_{m1}(0.02329,\ 34.7366,\ 1.6444,\ 28.4009),$$

$$\text{GLD}_{m2}(0.03629,\ 61.9834,\ 3.0126,\ 1.4785).$$

with respective supports $[-0.0055, 0.052]$ and $[0.0202, 0.052]$. Neither of these fits covers the range of the data and both seem to be ill-suited.

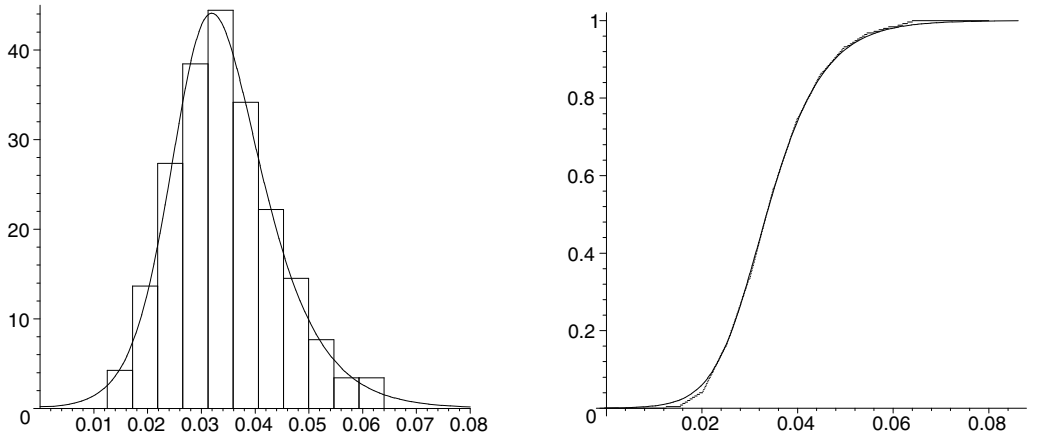


Figure 5.7–1. Histogram of coefficients of friction and the p.d.f.s of the fitted GLD_5 and GLD_m (a); the e.d.f. of the data with the d.f. of the fitted GLD_5 and GLD_m (b).

When we assume that the data is uniformly spread within its interval, we get

$$\hat{\rho}_1 = 0.03352, \quad \hat{\rho}_2 = 0.02531, \quad \hat{\rho}_3 = 0.7786, \quad \hat{\rho}_4 = 0.5009,$$

leading (through `FitGLDP`) to the fits

$$\mathrm{GLD}_{u1}(0.02773,\, 39.9909,\, 2.1117,\, 14.3900),$$

$$\mathrm{GLD}_{u3}(0.03134,\, 0.8063,\, 0.003404,\, 0.005942).$$

Since the supports of these fits are [0.0027, 0.053] and [−1.21, 1.27], respectively, we see that GLD_{u1} does not cover the data range but GLD_{u3} does and it is a better fit. Figure 5.7–1 (a) shows the histogram of the data with the p.d.f. of GLD_{u3} and Figure 5.7–1 (b) shows the e.d.f. of the data with the d.f. of GLD_{u3}

To obtain a chi-square statistic for the GLD_{u3} fit, we partition the data (under the uniform spread assumption) into the intervals

$$(-\infty,\, 0.025),\ \ [0.025,\, 0.03),\ \ [0.03,\, 0.035),\ \ [0.035,\, 0.04),$$
$$[0.04,\, 0.045),\ \ [0.045,\, 0.05),\ \ [0.05,\, \infty)$$

and obtain observed frequencies of

$$40, \quad 44, \quad 58, \quad 45, \quad 29, \quad 17, \quad 17$$

and expected frequencies of

$$40.6494,\ \ 45.8984,\ \ 54.0771,\ \ 44.6777,\ \ 29.1748,\ \ 16.9067,\ \ 18.6157.$$

The chi-square statistic and its associated p-value for this fit are 0.5176 and 0.7720, respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.0225$ and $KS\sqrt{n} = 0.0225\sqrt{250} = 0.3557$, giving a p-value in excess of 0.999.

Problems for Chapter 5

5.1. In Section 5.4.1 we fitted the GLD to the $N(0,1)$ p.d.f. using Algorithm GLD–P of Section 5.3 with $u = 0.1$. Recall that $u = 0.1$ accommodates sample sizes as small as 9 whereas $u = 0.01$ requires sample sizes of at least 99 and, in general, u can be any number strictly between 0 and 0.25. The quality of the fit that we obtained, as measured by the L_1 norm (see Section 5.4.1), was 0.002. The fit and its L_1 norm varies as u varies, hence the L_1 norm of the fit is a function of u, say $L_1(u)$.

a. Use Algorithm GLD–P to find GLD fits to $N(0,1)$ with $u = 0.01, 0.05, 0.1, 0.2, 0.24$. When several fits are available choose what seems to be the best of the available fits, as was done in Section 5.4.1.

b. Find $L_1(0.01)$, $L_1(0.05)$, $L_1(0.1)$, $L_1(0.2)$, $L_1(0.24)$ for the fits obtained in part a.

c. Graph $L_1(u)$ of u in the interval $(0, 0.25)$. What u do you predict will yield the L_1-best GLD approximation to $N(0,1)$? Test your prediction by finding the fit and its L_1 norm for that u value.

5.2. Repeat Problem 5.1 for $U(0,1)$, the uniform distribution on the interval $(0,1)$.

5.3. Repeat Problem 5.1 for $t(1)$, the Student's t distribution with $\nu = 1$.

5.4. Repeat Problem 5.1 for the exponential distribution with $\theta = 3$.

5.5. Repeat Problem 5.1 for $\chi^2(5)$, the chi-square distribution with $\nu = 5$.

5.6. Repeat Problem 5.1 for $\Gamma(5,3)$, the gamma distribution with $\alpha = 5$ and $\theta = 3$.

5.7. Repeat Problem 5.1 for the Weibull distribution with $\alpha = 1$ and $\beta = 5$.

5.8. Repeat Problem 5.1 for the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$.

5.9. Repeat Problem 5.1 for the beta distribution with $\beta_3 = \beta_4 = 1$.

5.10. Repeat Problem 5.1 for the inverse Gaussian distribution with $\mu = 0.5$ and $\lambda = 6$.

5.11. Repeat Problem 5.1 for the logistic distribution with $\mu = 0$ and $\sigma = 1$.

5.12. Repeat Problem 5.1 for the largest extreme value distribution with $\mu = 0$ and $\sigma = 1$.

5.13. Repeat Problem 5.1 for the double exponential distribution with $\lambda = 1$.

5.14. Repeat Problem 5.1 for $F(6,25)$, the F-distribution with $\nu_1 = 6$ and $\nu_2 = 25$.

5.15. Repeat Problem 5.1 for the Pareto distribution with $\beta = 5$ and $\lambda = 1$.

5.16. For each of Problems 5.1 through 5.15, find the value of u that achieves

$$\min_u \max_g \; L_1(u, g)$$

where the maximum is taken over all g in the class of distributions under consideration.

5.17. In Section 5.6.1 two fits GLD_1 and GLD_5 were obtained to data that was generated from the Cauchy distribution. By plotting the p.d.f.s of GLD_1 and GLD_5 with the p.d.f. of the Cauchy distribution, make an initial assessment of the "closeness" of the Cauchy p.d.f. to the GLD_1 and GLD_5 p.d.f.s. Next, substantiate your observation quantitatively by computing

$$\sup_x |\hat{f}_1(x) - f(x)| \quad \text{and} \quad \sup_x |\hat{f}_5(x) - f(x)|.$$

Now do graphic and quantitative assessments of the d.f.s of the Cauchy and the fitted GLDs.

5.18. In addition to introducing the terms and concepts of "percentile," "quartile," and "decile" (see the introduction to this chapter), Galton went on to develop "regression" (see Section 10.2 of Dudewicz, Chen, and Taneja (1989)). While regression often uses the assumption of normal distribution for its residuals, a GLD or EGLD distribution for the residuals is also a possibility. Develop the details and in an example contrast the results with those of normal regression. You can use simulated data from a non-normal distribution such as the Cauchy, which has no mean or higher order moments but can be fitted well using the GLD as seen in Section 5.6.1.

5.19. In Example 3.5.4 we fitted each of X and Y from birth weight of twins with a GLD. The fit to Y was good (p-value of 0.24), but that to X was surprisingly poor (p-value of 0.03). Use the method of this chapter to fit X and Y using percentiles, and compare the quality of the fits with those obtained in Chapter 3.

References for Chapter 5

Abramowitz, M. and Stegun, I. A. (Editors) (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards Applied Mathematics Series • 55, U.S. Government Printing Office, Washington, D.C. 20402.

Dudewicz, E. J., Chen, P., and Taneja, B. K. (1989). *Modern Elementary Probability and Statistics, with Statistical Programming in SAS, MINITAB, & BMDP* (Second Printing), American Sciences Press, Inc., Columbus, Ohio.

Dudewicz, E. J. and Karian, Z. A. (1999). "Fitting the Generalized Lambda Distribution (GLD) system by a method of percentiles, II: Tables," *American Journal of Mathematical and Management Sciences*, 19 (1 & 2), 1–73.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, John Wiley & Sons, New York.

Dudewicz, E. J. and van der Meulen, E. C. (1987). "The empiric entropy, a new approach to nonparametric entropy estimation," *New Perspectives in Theoretical and Applied Statistics* (eds. M. L. Puri, J. P. Vilaplana, and W. Wertz), John Wiley & Sons, Inc., New York, 207–227.

Fournier, B., Rupin, N., Bigerelle, M., Najar, D., Iost, A., and Wilcox, R. (2007). "Estimating the parameters of a generalized lambda distribution," *Computational Statistics and Data Analysis*, 251, 2813–2835.

Galton, F. (1875). "Statistics by intercomparison, with remarks on the law of frequency of error," *Philosophical Magazine*, 49, 33–46.

Gibbons, J. D. (1997). *Nonparametric Methods for Quantitative Analysis (Third Edition)*, American Sciences Press, Inc., Columbus, Ohio.

Hald, A. (1998). *A History of Mathematical Statistics from 1750 to 1930*, Wiley, New York.

Hogg, R. V. (1972). "More light on the kurtosis and related statistics," *Journal of the American Statistical Association,* 67, 422–424.

Hogg, R. V. and Tanis, E. A. (1997). *Probability and Statistical Inference (Fifth Edition)*, Prentice Hall, Upper Saddle River, New Jersey.

Horton, T. R. (1979). "A preliminary radiological assessment of radon exhalation from phosphate gypsum piles and inactive uranium mill tailings piles," *EPA–520/5–79–004*, Environmental Protection Agency, Washington, D.C.

Karian, Z. A. and Dudewicz, E. J. (1999). "Fitting the Generalized Lambda Distribution to data: A method based on percentiles," *Communications in Statistics: Simulation and Computation*, 28(3), 793–819.

Karian, Z. A. and Dudewicz, E. J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society,* 2, 171–187.

McClave, U. T., Dietrich, F. H., II, and Sincich, T. (1997). *Statistics*, Seventh Edition, Prentice Hall, Upper Saddle River, New Jersey.

Mishra, S. N., Shah, A. K., and Lefante, J. J. (1986). "Overlapping coefficient: The generalized t–approach," *Communications in Statistics*, 15, 123–128.

Mykytka, E. F. (1979). "Fitting a distribution to data using an alternative to moments," *IEEE Proceedings, 1979 Winter Simulation Conference*, 361–374.

Oegerle, W. R., Hill, J. M., and Fitchett, M. J. (1995). "Observations of high dispersion clusters of galaxies: Constraints on cold dark matter," *The Astronomical Journal*, 110(1).

Chapter 6

Fitting Distributions and Data with the GLD through L-Moments

L-moments, defined as linear combinations of expectations of order statistics, are useful in fitting distributions because they specify location, scale, and shape (symmetry and kurtosis) attributes like ordinary moments do. An advantage of L-moments is that they exist whenever the underlying random variable has a finite mean, enabling the use of L-moments when ordinary moments are not available. A difficulty in fitting a GLD through the method of moments is the complexity of the equations that need to be solved to determine the GLD parameters. Here too, the use of L-moments provides an advantage since the equations associated with the determination of the GLD parameters, although not trivial, are simpler than the ones associated with fitting a GLD through moments.

Following the definition of L-moments in Section 6.1, this chapter considers the L-moment ratios that the GLD can assume (Section 6.2). Subsequent sections explore the GLD parameter estimation through L-moments (Section 6.3) and the application of the estimation methods to approximating some well-known distributions (Section 6.4) and fitting a GLD to data (Section 6.5).

The use of L-moment for fitting GLD distributions was considered by Petersen (2001) and subsequently by Karian and Dudewicz (2003) who concluded that, although it provided good fits, it was not as good as the percentile method when the quality of fits was measured by the L^2-norm. Karvanen and Nuutinen (2008) have also considered the use of the L-moments in connection with the GLD.

6.1 L-Moments

One way of defining the L-moments of a random variable, X, is through the probability weighted moments of X which are defined (see Greenwood, Landwehr,

Matalas, and Wallis (1979)) by

$$M_{k,r,s} = E\left[X^k (F(X))^r (1 - F(X))^s\right] \tag{6.1.1}$$

where $F(X)$ is the distribution function of X. If we set

$$\beta_j = M_{1,j,0} = E\left[X(F(X))^j\right], \tag{6.1.2}$$

then the L-moments of X, Λ_i, can be defined as the linear combinations

$$\Lambda_1 = \beta_0, \qquad \Lambda_i = \sum_{j=0}^{i-1} p_{i,j}\beta_j \text{ for } i = 2, 3, \ldots \tag{6.1.3}$$

where

$$p_{i,j} = (-1)^{i-1-j}\binom{i-1}{j}\binom{i+j-1}{j} = \frac{(-1)^{i-1-j}(i+j-1)!}{(j!)^2(i-j-1)!}. \tag{6.1.4}$$

In the literature L-moments are designated by λ_1, $\lambda_2, \ldots$; however, to avoid confusion with the λ_1, λ_2, λ_3, λ_4 of the GLD, we choose to designate L-moments by Λ_1, $\Lambda_2, \ldots$ throughout this text.

The first four L-moments, the ones that will be useful in fitting GLD distributions, can now be derived from (6.1.2), (6.1.3), and (6.1.4)

$$\Lambda_1 = \beta_0 \tag{6.1.5}$$
$$\Lambda_2 = 2\beta_1 - \beta_0 \tag{6.1.6}$$
$$\Lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0 \tag{6.1.7}$$
$$\Lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0. \tag{6.1.8}$$

A more direct and equivalent way of defining L-moments (Hosking (1989)) is through the order statistics of a random sample from X. If $X_{1:n} \le X_{2:n} \le \cdots X_{n:n}$ designate the order statistics of a random sample of size n, the L-moments of X can be defined by

$$\Lambda_i = \left(\frac{1}{i}\right)\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}E[X_{i-k:i}], \quad \text{for} \quad i = 1, 2, \ldots \tag{6.1.9}$$

Since the expected values of order statistics can be obtained from

$$E[X_{i:n}] = \frac{n!}{(i-1)!(n-i)!}\int x\left\{F(x)\right\}^{i-1}\left\{1 - F(x)\right\}^{n-i}\, dF(x) \tag{6.1.10}$$

(see David (1981)), we can express the first four L-moments by

$$\Lambda_1 = E[X] = \int_0^1 x(F)\, dF \tag{6.1.11}$$

$$\Lambda_2 = \frac{1}{2}E[X_{2:2} - X_{1:2}] = \int_0^1 x(F)\{2F-1\}\,dF \tag{6.1.12}$$

$$\Lambda_3 = \frac{1}{3}E[X_{3:3} - 2X_{2:3} + X_{1:3}] = \int_0^1 x(F)\{6F^2 - 6F + 1\}\,dF \tag{6.1.13}$$

$$\begin{aligned}\Lambda_4 &= \frac{1}{4}E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}] \\ &= \int_0^1 x(F)\{20F^3 - 30F^2 + 12F - 1\}\,dF\,. \end{aligned} \tag{6.1.14}$$

It is clear that $\Lambda_1 = \mu$ is the mean of X and therefore is a measure of location. To see that $\Lambda_2 = (1/2)E[X_{2:2} - E_{1:2}]$ is a measure of scale, consider a random sample of size 2 (i.e., $n = 2$). The smaller Λ_2 is, the closer $E[X_{2:2}]$ will be to $E[X_{1:2}]$, indicating a smaller dispersion and a large Λ_2 implies a large dispersion. To interpret Λ_3, we consider $n = 3$ and from

$$3\Lambda_3 = E[X_{3:3} - 2X_{2:3} + X_{1:3}] = E[X_{3:3} - X_{2:3}] - E[X_{2:3} - X_{1:3}]$$

we see that the closer the median (or $E[X_{2:3}]$) is to the minimum (or $E[X_{1:3}]$) the larger Λ_3 will be, indicating a large positive skewness; if the median is close to the maximum, then a smaller Λ_3 will result, indicating negative skewness.

For $n = 4$, from $\Lambda_4 = (1/4)E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]$ we observe that if $E[X_{3:4}]$ and $E[X_{2:4}]$ are close to each other and away from $E[X_{4:4}]$ and $E[X_{1:4}]$ then we would have a sharply peaked distribution that is common among heavy-tailed distributions; this would also indicate a large Λ_4. If, on the other hand, $E[X_{3:4}]$ and $E[X_{2:4}]$ are away from the center and close to $E[X_{4:4}]$ and $E[X_{1:4}]$, respectively, then a "flatter" distribution would be indicated (note that it is possible for Λ_4 to be negative). For a more detailed discussion of the interpretation of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ see Hosking (1990).

Since Λ_3 and Λ_4, as well as higher L-moments, are not independent of scale, we define the moment ratios

$$\tau_r = \Lambda_r/\Lambda_2 \text{ for } r = 3, 4, \ldots$$

and use τ_3 and τ_4 as measures of skewness and kurtosis, respectively. Λ_1, Λ_2, τ_3, and τ_4 are referred to, respectively, as the L-location, L-scale, L-skewness, and L-kurtosis of X.

It is worth noting that the existence of the mean implies the existence of the expectation of all order statistics (David (1981)) and hence the existence of all L-moments. In addition, Hosking (1989) shows that if a random variable, X, has a finite mean then its distribution can be characterized by its L-moments.

6.2 The (τ_3, τ_4)-Space of the GLD

Recall that the GLD, defined through its quantile function $Q(y)$ by

$$Q(y) = Q(y;\, \lambda_1,\, \lambda_2,\, \lambda_3,\, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \quad 0 \le y \le 1,$$

has p.d.f.

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}} \quad \text{for} \quad x = Q(y).$$

If we apply the transformation $x = Q(y)$, or equivalently $y = Q^{-1}(x) = F(x)$ to the integrals in 6.1.11 through 6.1.14, we can express the first four GLD L-moments by

$$\Lambda_1 = \lambda_1 + \frac{1}{\lambda_2(\lambda_3+1)} - \frac{1}{\lambda_2(\lambda_4+1)}, \tag{6.2.1}$$

$$\Lambda_2 = \frac{\lambda_3}{\lambda_2(\lambda_3+1)(\lambda_3+2)} + \frac{\lambda_4}{\lambda_2(\lambda_4+1)(\lambda_4+2)}, \tag{6.2.2}$$

$$\Lambda_3 = \frac{\lambda_3(\lambda_3-1)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)} - \frac{\lambda_4(\lambda_4-1)}{\lambda_2(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}, \tag{6.2.3}$$

$$\Lambda_4 = \frac{\lambda_3(\lambda_3-1)(\lambda_3-2)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)} + \frac{\lambda_4(\lambda_4-1)(\lambda_4-2)}{\lambda_2(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}. \tag{6.2.4}$$

From these we can derive expressions for τ_4 and τ_4 which, although a bit complicated, have the advantage of being independent of λ_1 and λ_2. The following theorem, stated here without proof, gives constraints on τ_3 and τ_4. The theorem incorporates several results that are established by Hosking (1989).

Theorem 6.2.5. *Let X be a nondegenerate random variable with finite mean. Then*

$$|\tau_r| < 1 \quad \textit{for} \quad r \geq 3 \qquad \textit{and} \qquad \frac{5}{4}\tau_3^2 - \frac{1}{4} \leq \tau_4.$$

The appropriateness of the use of L-moments for obtaining GLD fits to data depends, in part, on the extent to which the (τ_3, τ_4) points of the GLD family cover the region described in Theorem 6.2.5. Graphic renditions of the (τ_3, τ_4) points with (λ_3, λ_4) from various valid GLD regions can give us a sense of the span of the (τ_3, τ_4) points associated with specified (λ_3, λ_4) regions. Figure 6.2–1 (a) shows the (τ_3, τ_4) points for (λ_3, λ_4) from the first quadrant. More specifically, the 24 curves in Figure 6.2–1 (a) are obtained by setting

$$\lambda_3 = 0.05, 0.1, 0.2, 0.4, 0.6, 1, 1.5, 2, 5, 8, 12, 17.5, 30, 40, 55, 75, 100, 130, 165, 205, 250, 300, 360, 430$$

and letting λ_4 range continuously over the interval $[0, 430]$. As λ_3 gets larger, the curves move higher (i.e., closer to $(\tau_3, \tau_4) = (0, 1)$) and the curves that "loop" in the lower left portion of Figure 6.2–1 (a) are connected to the smaller values of λ_3. At first glance it seems that the (τ_3, τ_4) points are always above the horizontal axis but closer inspection reveals that τ_4 can become negative near the origin.

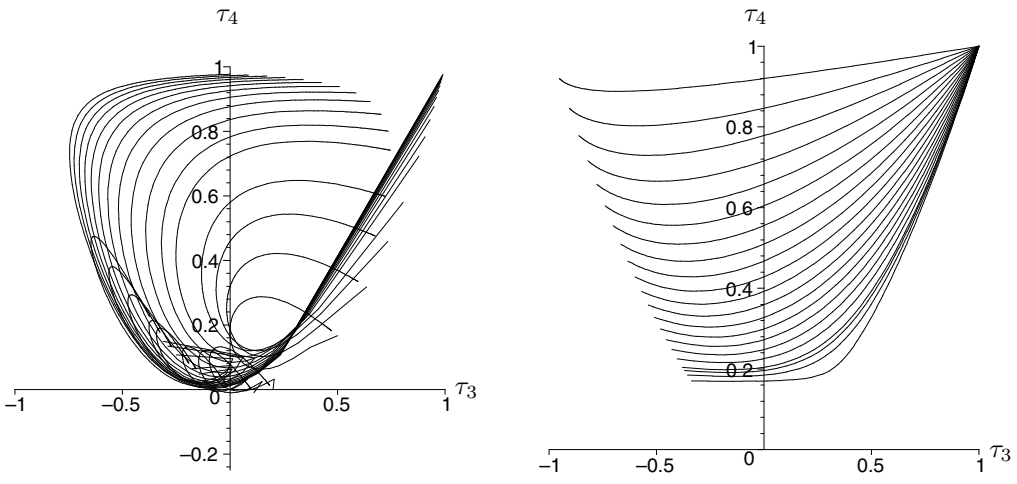


Figure 6.2–1. (τ_3, τ_4) points for (λ_3, λ_4) from the first quadrant (a); (τ_3, τ_4) points for (λ_3, λ_4) from the third quadrant (b).

This can be substantiated by considering τ_4 when $1 < \lambda_3, \lambda_4 < 2$. We know from (6.2.2) that Λ_2 is always positive (recall that for the GLD to be valid, λ_2 has to have the same sign as λ_3 and λ_4) and from (6.2.4) that Λ_4 is negative for $1 < \lambda_3, \lambda_4 < 2$. This would make $\tau_4 = \Lambda_4/\Lambda_2$ negative. What remains clear from Figure 6.2–1 (a) is that, even when negative, τ_4 stays well above $-1/4$. When we use standard analytical methods to determine the (λ_3, λ_4) that minimizes τ_4, we find that this minimum is attained when

$$\lambda_3 = \lambda_4 = \sqrt{6} - 1 = 1.4495.$$

Substitution of this value for λ_3 and λ_4 in the expression for τ_4, yields

$$20\sqrt{6} - 49 = -0.01021$$

as the smallest value of τ_4 that can be attained by a GLD distribution.

Figure 6.2–1 (b) shows the (τ_3, τ_4) points when (λ_3, λ_4) is in the third quadrant. The 21 curves are obtained by setting

$$\begin{aligned}\lambda_3 \quad = \quad & -.95, -.90, -.85, -.80, -.75, -.70, -.65, -.60, -.55, -.50, -.45, \\ & -.40, -.35, -.30, -.25, -.20, -.15, -.10, -.08, -.05, -.01\end{aligned}$$

and letting λ_4 range continuously over the interval $[-1, 0]$. As λ_3 moves from -0.01 to -0.95, the curves move higher on the τ_4-axis and as λ_4 moves from -1 to 0, the points on the curve move from right to left.

The other valid regions of the GLD do not contribute any additional coverage of the (τ_3, τ_4)-space of the GLD and the full sense of the portion of the possible (τ_3, τ_4)-space that is covered by the GLD is given in Figure 6.2–2, which shows

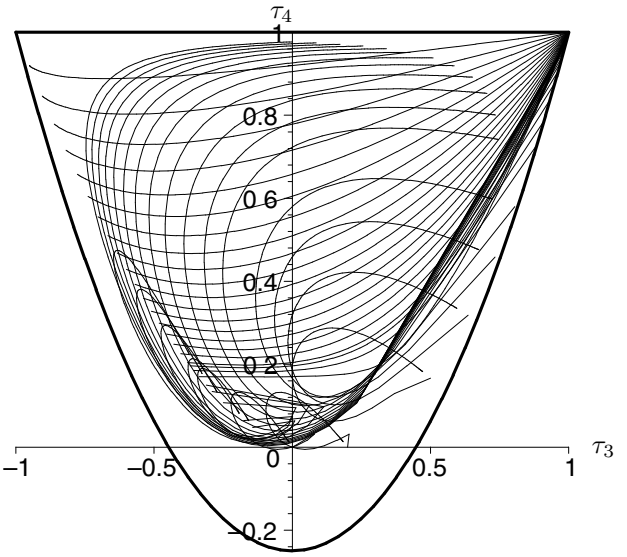


Figure 6.2–2. Portion of the (τ_3, τ_4)-space covered by the GLD.

the curves of Figures 6.2–1 (a) and Figures 6.2–1 (b) together with the boundary (drawn in thick line) of the (τ_3, τ_4)-space given in Theorem 6.2.5. There is a clear indication that only in rare cases the GLD will provide distributions whose (τ_3, τ_4) is below the horizontal axis (i.e., has $\tau_4 < 0$) and will provide a multiplicity of distributions for most other (τ_3, τ_4).

It is not known if the portion of the (τ_3, τ_4)-space, mostly below the τ_3-axis, not covered by the GLD can actually be attained by any distribution. Theorem 6.2.5 restricts (τ_3, τ_4) to the region bounded by the thick lines of Figure 6.2–2; it does not necessarily follow that the entirety of this restricted region is attainable. Hosking and Wallis (1997) display the (τ_3, τ_4) points for a variety of distributions (normal, uniform, exponential, Gumbel, generalized Pareto, generalized extreme value, generalized logistic, lognormal, and Pearson type III) and only the generalized Pareto distribution has (τ_3, τ_4) points that are below the τ_3-axis (the Pareto distribution will be more fully explored in Section 6.4.16); however, even in this case, (τ_3, τ_4) is in the region covered by the GLD (see Hosking and Wallis (1997), Figure 2.5, page 25).

6.3 Estimation of GLD Parameters through *L*-Moments

To fit a GLD distribution through its L-moments to data, we need to obtain the L-moments of the data, determine the data $\hat{\tau}_3$ and $\hat{\tau}_4$, set these equal to the GLD τ_3 and τ_4 (which, as noted earlier, are functions of only λ_3 and λ_4) and solve the resulting equations for λ_3 and λ_4. Once λ_3 and λ_4 are determined, λ_2 can be

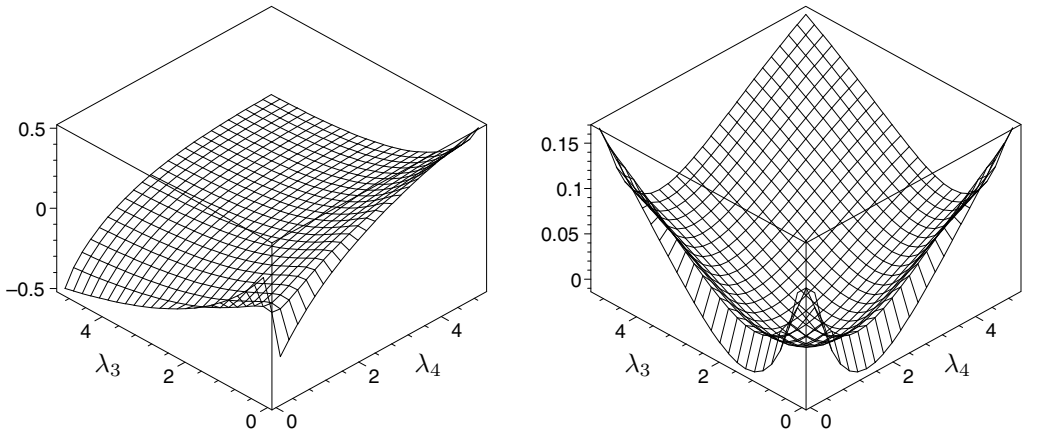


Figure 6.3–1. The τ_3 surface for (λ_3, λ_4) from the first quadrant (a); the τ_4 surface for (λ_3, λ_4) from the first quadrant (b).

obtained from (6.2.2) and, eventually, λ_1 can be computed from (6.2.1).

In a manner analogous to the definition of $\Lambda_1, \Lambda_2, \ldots$ in (6.1.3), we define the L-moments of a sample, by first taking the order statistics of the sample: $x_1 \le x_2 \le \cdots \le x_n$ and then determining the sample L-moments, $\ell_1, \ell_2, \ldots$ by

$$\ell_1 = \bar{x}, \qquad \ell_i = \sum_{j=0}^{i-1} p_{i,j} b_j \quad \text{for} \quad i = 2, 3, \ldots \tag{6.3.1}$$

where $p_{i,j}$ is as defined in (6.1.4) and

$$b_j = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{(i-1)(i-2)\cdots(i-j)}{(n-1)(n-2)\cdots(n-j)} x_i, \quad \text{for} \quad j = 0, 1, \ldots, n-1. \tag{6.3.2}$$

If we let $\hat{\tau}_3 = \ell_3/\ell_2$ and $\hat{\tau}_4 = \ell_4/\ell_2$, then the main task in estimating the GLD parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, is obtaining solutions of the system of equations

$$\tau_3 = \hat{\tau}_3, \quad \tau_4 = \hat{\tau}_4 \tag{6.3.3}$$

for λ_3 and λ_4 (as noted above, once this is done, λ_2 and λ_1 can be determined easily). To gain some insight into solutions of (6.3.3), we first look at the surfaces τ_3 and τ_4 as functions of λ_3 and λ_4. Figure 6.3–1 (a) shows the τ_3 surface and Figure 6.3–1 (b) the τ_4 surface, both for (λ_3, λ_4) in the first quadrant.

There is an indication of some difficulty near the origin since both surfaces seem to rise rather sharply as (λ_3, λ_4) approaches $(0, 0)$. However, this is not likely to be a problem since $|\tau_i| < 1$ for $i \ge 3$. The τ_4 surface indicates that we need to be aware of the possibility of multiple solutions for (λ_3, λ_4) in the first quadrant. Moreover, some of the solutions are likely to be close to one of the axes $\lambda_3 = 0$ or $\lambda_4 = 0$.

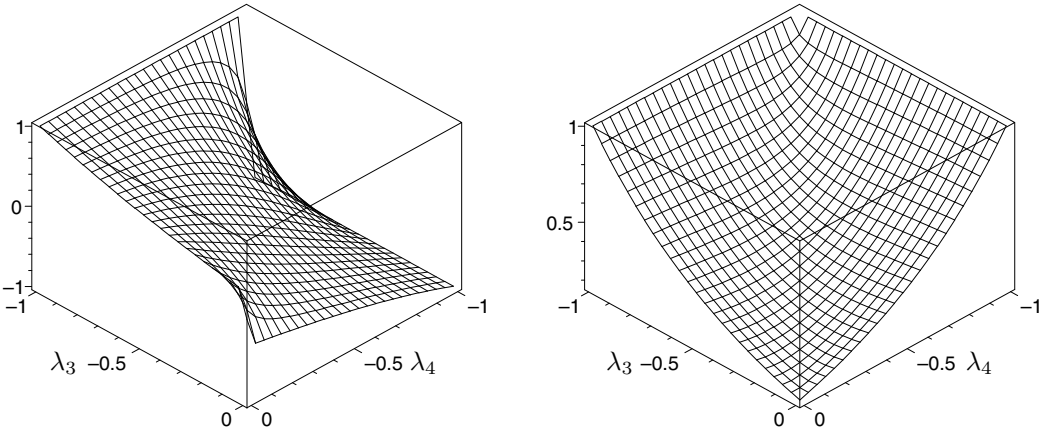


Figure 6.3–2. The τ_3 surface for (λ_3, λ_4) from the third quadrant (a); the τ_4 surface for (λ_3, λ_4) from the third quadrant (b).

By contrast the τ_3 and τ_4 surfaces are much smoother when (λ_3, λ_4) is restricted to the third quadrant. In this case recall that we must have $\lambda_3 > -1$ and $\lambda_4 > -1$ in order for the mean of the GLD to exist. Figures 6.3–2 (a) and 6.3–2 (b) illustrate, respectively, the τ_3 and τ_4 surfaces when (λ_3, λ_4) is restricted to the third quadrant. To have a sense about the possible location of solutions to (6.3.3) for specific values of $\hat{\tau}_3$ and $\hat{\tau}_4$, we consider contour curves for specified values of τ_3 and τ_4. Figure 6.3–3 (a) gives the contour curves for

$$\tau_3 = 0,\ 0.05,\ 0.1,\ 0.2,\ 0.3$$

when $0 \le \lambda_3,\ \lambda_4 \le 5$. We know from (6.2.3) that exchanging λ_3 and λ_4 would reverse the sign of τ_3. Therefore, in Figure 6.3–3 (a), curves corresponding to negative values τ_3 would be located symmetrically across the $\lambda_3 = \lambda_4$ line that results when $\tau_3 = 0$. In Figure 6.3–3 (a) the curves below the $\lambda_3 = \lambda_4$ moving toward the λ_3-axis are associated with $\tau_3 = 0.05$, 0.1, 0.2, and 0.3, in that order. The curve connecting $(\lambda_3, \lambda_4) = (1, 0)$ to $(\lambda_3, \lambda_4) = (0, 1)$ is part of the $\tau_3 = 0$ contour and the small segments below this curve are associated with $\tau_3 = 0.05$, 0.1, 0.2, and 0.3.

The τ_4 values used in Figure 6.3–3 (b) are

$$\tau_4 = -0.01,\ -0.005,\ 0,\ 0.005,\ 0.01,\ 0.02,\ 0.035,\ 0.05,\ 0.075,\ 0.1,\ 0.15.$$

The concentric oval curves in the center correspond to increasing values of τ_4 from the innermost to the outer curves. The segments along the axes are part of the contours of the same τ_4 values with the curves closest to the axes corresponding to the smallest values of τ_4. Although in Figure 6.3–3 (a), and particularly in Figure 6.3–3 (b), some sharp corners are observed, the contours are, in fact,

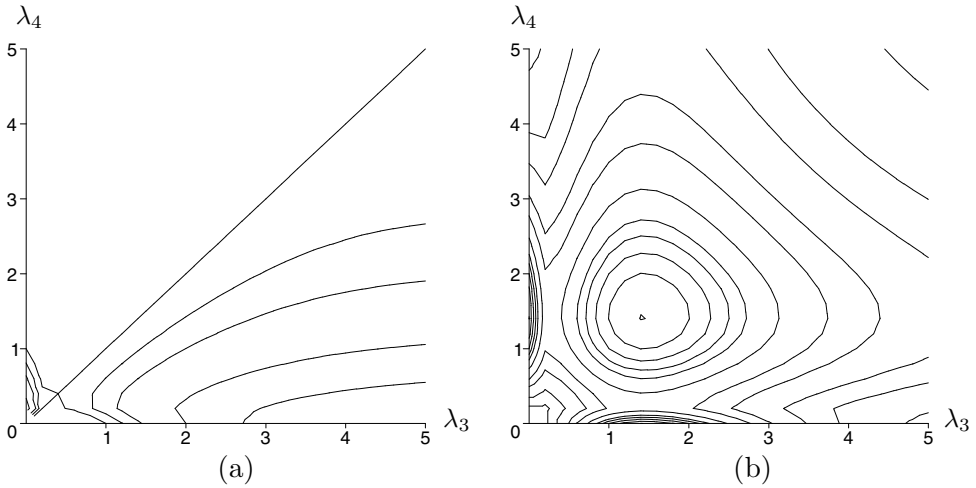


Figure 6.3–3. Contour curves for $\tau_3 = 0, .05, .1, .2, .3$ and (λ_3, λ_4) from the first quadrant (a); contour curves for $\tau_4 = -.01, -.005, 0, .005, .01, .02$ .035, .05, .075, .1, .15 and (λ_3, λ_4) from the first quadrant (b).

smooth. The sharp corners are artifacts due to limitations of computational and graphic resolution.

Solutions to (6.3.3) will be at intersection points of τ_3 and τ_4 contours. In Figure 6.3–4 (a) we show both sets of contours (including the ones corresponding to negative τ_3) for (λ_3, λ_4) in the first quadrant. From the τ_3 and τ_4 surfaces for (λ_3, λ_4) in the third quadrant (Figures 6.3–2 (a) and (b)), we expect that the contour curves for the third quadrant of (λ_3, λ_4)-space to be considerably simpler. This is indeed the case as shown in Figure 6.3–4 (b).

This figure shows τ_3 contours for

$$\tau_3 = 0, \pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, \pm 0.5, \pm 0.6, \pm 0.7, \pm 0.8, \pm 0.9$$

as curves that connect the origin to $(-1, -1)$ with the $\tau_3 = 0$ curve represented by $\lambda_3 = \lambda_4$. From this central $\lambda_3 = \lambda_4$ line the curves move outward as the magnitude of τ_3 increases. The curves corresponding to positive values of τ_3 are below the line $\lambda_3 = \lambda_4$ and those with negative values of τ_3 are above this line. The τ_4 contours of Figure 6.3–4 (b) use

$$\tau_4 = 0.2, 0.25, 0.3, 0.35, 0.425, 0.5, 0.6, 0.7, 0.8, 0.95.$$

as curves that connect the origin to $(-1, -1)$ with the $\tau_3 = 0$ curve represented by $\lambda_3 = \lambda_4$. From this central $\lambda_3 = \lambda_4$ line the curves move outward as the magnitude of τ_3 increases. The curves corresponding to positive values of τ_3 are below the line $\lambda_3 = \lambda_4$ and those with negative values of τ_3 are above this line. The τ_4 contours of Figure 6.3–4 (b) use

$$\tau_4 = 0.2, 0.25, 0.3, 0.35, 0.425, 0.5, 0.6, 0.7, 0.8, 0.95.$$

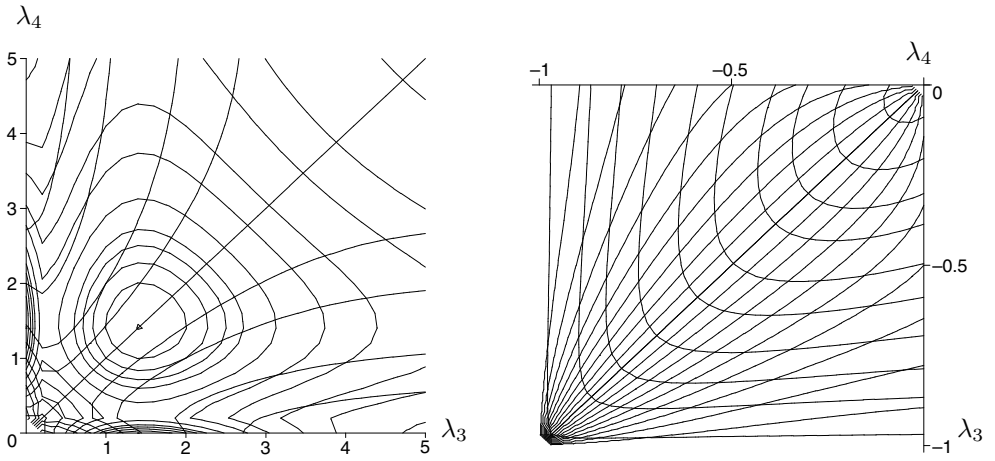


Figure 6.3–4. Contour curves for $\tau_3 = 0, .05, .1, .2, .3$ and (λ_3, λ_4) from the first quadrant (a); contour curves for $\tau_4 = -.005, -.01, 0, .005, .01, .02, .035, .05, .075, .1, .15$ and (λ_3, λ_4) from the first quadrant (b).

These curves connect points on one axis to the same point on the other axis and the curves move away from the origin as τ_4 increases.

Figures 6.3–4 (a) and (b) help us understand where solutions to (6.3.3) are likely to be. However, more careful analysis will be required in a specific case. Suppose that we have $\hat{\tau}_3 = 0.1$ and $\hat{\tau}_4 = 0.01$. We consider the superimposed contours for this specification, depicted in Figure 6.3–5, and observe several solutions. The thick lines, the τ_3 contour at $\tau_3 = 0.1$, intersect the $\tau_4 = 0.01$ contour at three points. From Figure 6.3–5 we can estimate these points (equivalently the solutions to the simultaneous equations $\{\tau_3 = 0.1,\ \tau_4 = 0.01\}$) as

$$(\lambda_3, \lambda_4) = (1.3, 0.1),\ (1.4, 0.6),\ (2.7, 1.4).$$

In this case there are no solutions with (λ_3, λ_4) in the third quadrant.

Graphic solutions, such as the ones illustrated above, generally lack precision. Far more accurate solutions can be obtained through computations that can guarantee that

$$\max\left(|\tau_3 - \hat{\tau}_3|,\ |\tau_4 - \hat{\tau}_4|\right)$$

is small, say less than 10^{-5}. Computations that lead to solutions to (6.3.3) will have to rely on some searching scheme and, regardless of the specific scheme used, will focus the search to a specific region of (λ_3, λ_4)-space and discover a single solution in that region. To obtain multiple solutions, we use Figures 6.3–4 (a) and (b) as guides and divide the first and third quadrants of (λ_3, λ_4)-space into the 5 rectangles, $R_1, \ldots, R_5$, given below and conduct searches in each rectangle.

$$\mathrm{R}_1\text{: } 0.00001 \le \lambda_3 \le 1, \qquad 0.00001 \le \lambda_4 \le 1;$$

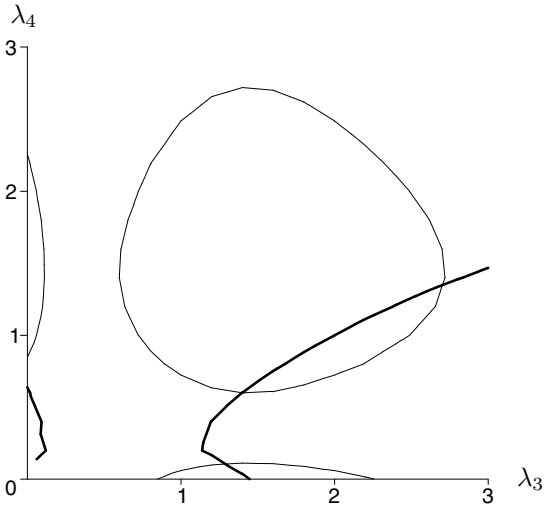


Figure 6.3–5. Contours for $\tau_3 = 0.1$ (thick curves) and $\tau_4 = 0.01$.

R_2: $1 \le \lambda_3 \le 15$,	$0 \le \lambda_4 \le 0.25$;
R_3: $1 \le \lambda_3 \le 15$,	$0.25 \le \lambda_4 \le 1$;
R_4: $1 \le \lambda_3 \le 15$,	$1 \le \lambda_4 \le 15$;
R_5: $-0.999 \le \lambda_3 \le 0.00001$,	$-0.999 \le \lambda_4 \le 0.00001$.

All 5 rectangles are either below the line $\lambda_3 = \lambda_4$ or they straddle it. There is no need to have symmetrically arranged rectangles above this line since solutions in those regions will correspond to $(-\tau_3, \tau_4)$ with λ_3 and λ_4 reversed.

To enable practitioners to obtain GLD fits using L-moments, five tables, one for each rectangle, are provided in Appendix E for that purpose. The (τ_3, τ_4) points for each table are associated with the (λ_3, λ_4) points of a corresponding rectangle. To assist in using the tables of Appendix E, Figure 6.3–6 indicates the table choices, T1 through T5, that would be appropriate for a particular (τ_3, τ_4). The (τ_3, τ_4) points corresponding to the first rectangle form a triangle depicted with thick lines and marked by "T1"; for the second rectangle we have a region bounded by dotted lines and marked "T2"; for the third rectangle the (τ_3, τ_4) region, marked "T3," is bounded by lines with regular thickness; for the fourth rectangle, the (τ_3, τ_4) region is shaded and marked "T4"; and for the fifth rectangle we have the large region marked "T5" that is enclosed by lines with ordinary thickness.

The tables of Appendix E are arranged in columns headed by A, B, λ_3, and λ_4. The entries of columns A and B have values

$$A = \frac{\lambda_3}{(\lambda_3+1)(\lambda_3+2)} + \frac{\lambda_4}{(\lambda_4+1)(\lambda_4+2)}$$
$$B = \frac{1}{\lambda_3+1} - \frac{1}{\lambda_4+1}$$

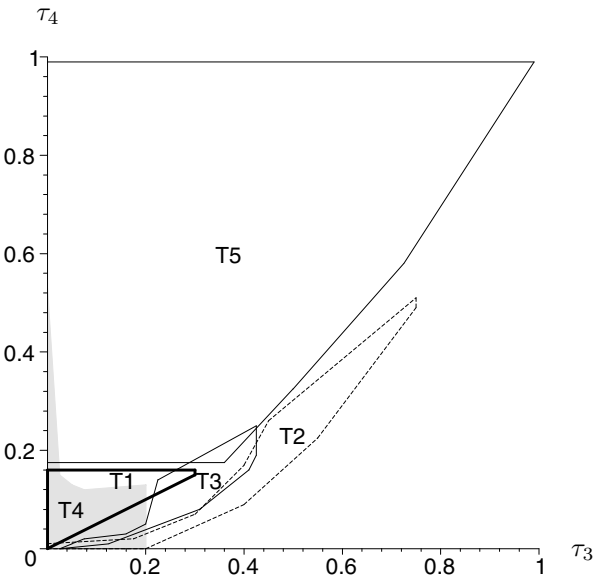


Figure 6.3–6. Table choices for various (τ_3, τ_4) combinations.

and their purpose, as can be seen from (6.2.1) and (6.2.2), is to assist in the computation of λ_1 and λ_2. Table-based L-moment GLD fits, for a given dataset, can be determined through the use of the following algorithm.

Algorithm GLD–L: Fitting GLD distributions to data through L-moments.

1. GLD–L. Use (6.1.4), (6.3.1) and (6.3.2) to compute ℓ_1, ℓ_2, ℓ_3, ℓ_4 and then $\hat{\tau}_3 = \ell_3/\ell_2$ and $\hat{\tau}_4 = \ell_4/\ell_2$;

2. GLD–L. Use Figure 6.3–6 to determine which tables of Appendix E to use and for each such table, execute the following three steps;

3. GLD–L. Find the closest entry to $(|\hat{\tau}_3|, \hat{\tau}_4)$ in the table and extract the values of A, B, λ_3 and λ_4 from that table;

4. GLD–L. If $\hat{\tau}_3 < 0$ then exchange λ_3 and λ_4;

5. GLD–L. Set $\lambda_2 = A/\ell_2$ and then set $\lambda_1 = \ell_1 - B/\lambda_2$.

To illustrate the use of Algorithm GLD–L, we consider the fit that was depicted in Figure 6.3–5 by assuming that we have

$$\ell_1 = 0, \quad \ell_2 = 1, \quad \ell_3 = 0.1, \quad \ell_4 = 0.01.$$

From the first step of Algorithm GLD–L we have $(\hat{\tau}_3, \hat{\tau}_4) = (0.1, 0.01)$ and Figure 6.3–6 indicates that potentially, Tables E–1 through E–4 may provide solutions.

Consulting Table E–1 we see that there is no solution in this case. However, Tables E–2, E–3, and E–4 each provide, in that order,

$$\begin{aligned}(A,\, B,\, \lambda_3,\, \lambda_4) &= (0.1950,\, -0.5177,\, 1.3051,\, 0.0510)\\ (A,\, B,\, \lambda_3,\, \lambda_4) &= (0.3157,\, -0.2083,\, 1.3979,\, 0.5992)\\ (A,\, B,\, \lambda_3,\, \lambda_4) &= (0.3265,\, -0.1560,\, 2.7138,\, 1.3513).\end{aligned}$$

The $(\lambda_3,\, \lambda_4)$ values are certainly close to the observations that we made from Figure 6.3–5. Executing the last step of Algorithm GLD–L, we can now determine all of the GLD parameters associated with the given $\ell_1,\, \ell_2,\, \ell_3,\, \ell_4$ to get

$$\begin{aligned}(\lambda_1,\, \lambda_2,\, \lambda_3,\, \lambda_4) &= (2.6549,\, 0.1950,\, 1.3051,\, 0.0510)\\ (\lambda_1,\, \lambda_2,\, \lambda_3,\, \lambda_4) &= (0.6598,\, 0.3157,\, 1.3979,\, 0.5992)\\ (\lambda_1,\, \lambda_2,\, \lambda_3,\, \lambda_4) &= (0.4778,\, 0.3265,\, 2.7138,\, 1.3513).\end{aligned}$$

The software on the CD that accompanies this book contains programs, written in **R**, that for a specified dataset, will produce all suitable fits. GLD fits whose support does not cover the data range are considered unsuitable and are not included in the output of the search program. The program is invoked by the **R** command `FitGLDL(Data)` where `Data` is a vector that represents the data. On rare occasions `FitGLDL` will fail to produce a valid fit. Specific uses of the program `FitGLDL` and how to deal with situations where the program may miss a fit will be illustrated in Sections 6.5.7 and 6.5.10, respectively.

6.4 Approximations of Some Well-Known Distributions

In this section we follow the pattern of previous chapters and fit a GLD, through the use of L-moments, to some well-known distributions. When the quantile function, $x(F)$ or $Q(y)$ of a distribution is known, equations (6.1.11) through (6.1.14) allow us to obtain $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$, by integration. For the distributions that we consider in subsequent subsections, the same ones considered in previous chapters, this will be the case for the uniform, exponential, Weibull, logistic, double exponential and Pareto distributions. In other situations (the normal, lognormal largest extreme value), although quantile functions are not available, expressions for $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are available in the literature (see Appendix A of Hosking and Wallis (1997)). In the remaining cases (the t, chi-square, gamma, lognormal, beta, inverse Gaussian, F distributions) we resort to numeric calculations to approximate $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$.

Regardless of how $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$, and hence $(\tau_3,\, \tau_4)$, are obtained, to find an approximation to a distribution we will need to determine the GLD $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. This can be done by using the **R** program `FindLambdasL(LMoms)` where `LMoms`

stands for the vector of L-moments of the distribution to be approximated. `FindLambdasL` may not find any fits or it may find several distinct fits. In the former case, it returns `0 0 0 0` and in the latter case, it returns a vector of vectors, each subvector representing a fit. It is also possible that some of the multiple fits found by `FindLmbdasL` may be almost identical. More details on the use of this and other programs are given in Appendix A.

6.4.1 The Normal Distribution

The $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ of the normal distribution, $N(\mu, \sigma^2)$, are given (Hosking and Wallis (1997), page 193) by

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\mu, \frac{\sigma}{\sqrt{\pi}}, 0, \frac{3\sigma(10\arctan(\sqrt{2}) - 3\,\pi)}{\pi^{3/2}}\right).$$

From this we get

$$(\tau_3,\, \tau_4) = \left(0,\, \frac{30}{\pi}\arctan\sqrt{2} - 9\right) = (0,\, 0.1226),$$

which indicates that τ_3 and τ_4 are independent of the distribution parameters, μ and σ. Since, in this case, solutions to (6.3.3) will not be influenced by μ and σ, we set $\mu = 0$ and $\sigma = 1$.

First we attempt to find GLD approximations through the tabled values of Appendix E. Figure 6.3–6 indicates possible solutions from Tables E–1 and E–4 and, indeed, these tables provide

$$(A,\, B,\, \lambda_3, \lambda_4) = (0.1220,\, 0,\, 0.1511,\, 0.1511)$$

and

$$(A,\, B,\, \lambda_3, \lambda_4) = (0.2602,\, 0,\, 4.2126,\, 4.2126),$$

respectively, giving rise to the two approximations

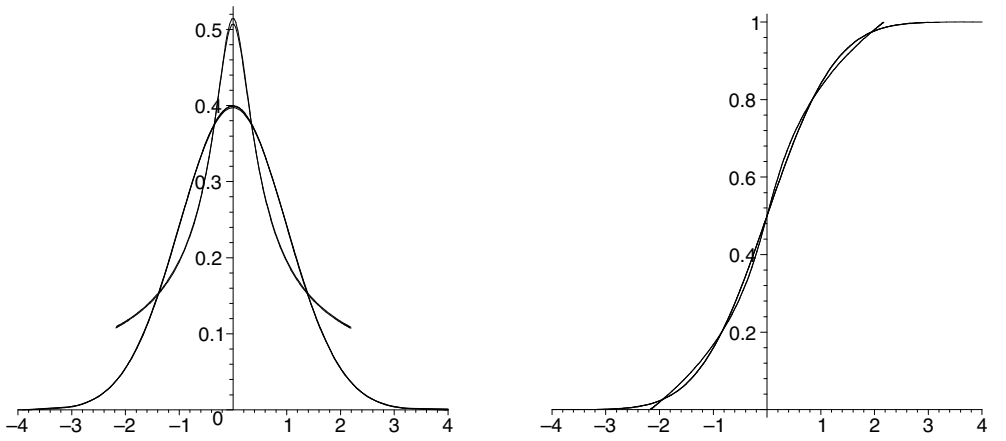
$$\begin{aligned} &\mathrm{GLD}_1(0,\, 0.2162,\, 0.1511,\, 0.1511)\\ &\mathrm{GLD}_2(0, 0.4612,\, 4.2126,\, 4.2126), \end{aligned}$$

with respective supports $[-4.6245,\, 4.6245]$ and $[-2.1683,\, 2.1683]$. Based on the support alone, it seems that GLD_2 is not a good approximation.

It is also possible to obtain approximations through direct computation by appealing to the **R** program `FindLambdasL(LMom)`, where `LMom` is the vector of L-moments. In this case,

```
FindLambdasL(c(0, 0.5642, 0, 0.06917))
```

produces the approximations



(a) (b)

Figure 6.4–1. The $N(0,\,1)$, GLD_1, GLD_3 p.d.f.s that peak near $(0,\,0.4)$ with GLD_2 and GLD_4 p.d.f.s (a); c.d.f.s of the same distributions (b).

$$\text{GLD}_3(0,\,0.2053,\,0.1416,\,0.1416)$$
$$\text{GLD}_4(-0.1414\times 10^{-5},\,0.4588,\,4.2557,\,4.2557),$$

that are more precise versions of GLD_1 and GLD_2. The supports of GLD_3 and GLD_4 are $[-4.8704, 4.8704]$ and $[-2.1794, 2.1794]$, respectively. Figure 6.4–1 (a) shows five density functions: the p.d.f.s of GLD_1 through GLD_4 and the $N(0,\,1)$ density. The $N(0,\,1)$, GLD_1 and GLD_3 seem to be a single curve that peaks at about $(0,\,0.4)$ and the GLD_2 and GLD_4 p.d.f.s have a slight separation and peak near $(0,\,0.5)$. Figure 6.4–1 (b) depicts the c.d.f.s associated with the same five distributions, with the same clustering pattern. To get a more quantitative sense of the quality of these approximations, we let $f(x)$ and $F(x)$ denote the p.d.f. and c.d.f. of $N(0,\,1)$ and $\hat{f}_i(x)$ and $\hat{F}_i(x)$ represent, respectively, the p.d.f. and c.d.f. of GLD_i, for $i = 1, \ldots, 4$. We then have

$$\begin{aligned}
&\sup|\hat{f}_1(x) - f(x)| = 0.001663 \quad \text{and} \quad \sup|\hat{F}_1(x) - F(x)| = 0.0009463;\\
&\sup|\hat{f}_2(x) - f(x)| = 0.1160 \quad \text{and} \quad \sup|\hat{F}_2(x) - F(x)| = 0.02327;\\
&\sup|\hat{f}_3(x) - f(x)| = 0.0009736 \quad \text{and} \quad \sup|\hat{F}_3(x) - F(x)| = 0.0003815;\\
&\sup|\hat{f}_4(x) - f(x)| = 0.1085 \quad \text{and} \quad \sup|\hat{F}_4(x) - F(x)| = 0.02183.
\end{aligned}$$

The L_1 and L_2 distances between the $N(0, 1)$ p.d.f. and the p.d.f.s of these four approximations are:

$$\begin{aligned}
&||f - \hat{f}_1||_1 = 0.005688 \qquad ||f - \hat{f}_1||_2 = 0.002411;\\
&||f - \hat{f}_2||_1 = 0.1914, \qquad ||f - \hat{f}_2||_2 = 0.1044;\\
&||f - \hat{f}_3||_1 = 0.003488, \qquad ||f - \hat{f}_3||_2 = 0.001472
\end{aligned}$$

$$||f - \hat{f}_4||_1 = 0.1960, \qquad ||f - \hat{f}_4||_2 = 0.1075.$$

Two things are clear: GLD_2 and GLD_4, or more generally approximations from that region of (λ_3, λ_4)-space, do not provide good fits and approximations obtained through computation rather than use of tables are superior, but not significantly superior.

6.4.2 The Uniform Distribution

The quantile function of the uniform distribution on the interval $[a, b]$ is readily available as $Q(y) = a + y(b - a)$ from which, using (6.1.11) through (6.1.14), we can obtain

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\frac{a+b}{2}, \frac{1}{6}, 0, 0\right).$$

Since τ_3 and τ_4 are both zero and are independent of the parameters of the uniform distribution, we consider only the case $a = 0$ and $b = 1$ that gives rise to approximations $\text{GLD}_1(0, 1, 0, 1)$, $\text{GLD}_2(0.5, 2, 1, 1)$, and $\text{GLD}_3(1, 1, 1, 0)$ all of which are the same and are identical to the uniform distribution. This can be seen by substituting the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ specified by these GLDs into the expression for the GLD quantile function, $Q(y)$, and obtaining, in each case, $Q(y) = y$. We therefore have a perfect fit to the uniform distribution, just as we had with the GBD approximation and the GLD approximations through ordinary moments and percentiles.

6.4.3 The Student's t Distribution

The Student's t distribution with parameter ν, designated by $t(\nu)$, is defined in Section 3.4.3. In other chapters we have considered approximations to $t(\nu)$ with $\nu = 1$, 5, 6, 10, and 30. Since the first r moments of $t(\nu)$ exist only when $r < \nu$, it is impossible to approximate $t(1)$ through L-moments. However, L-moment based GLD fits to $t(\nu)$ should be available when $\nu > 1$. Because the L-moments of $t(\nu)$ are not available in closed form, we use numerical methods to approximate $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ and use these to obtain a fit for $t(\nu)$.

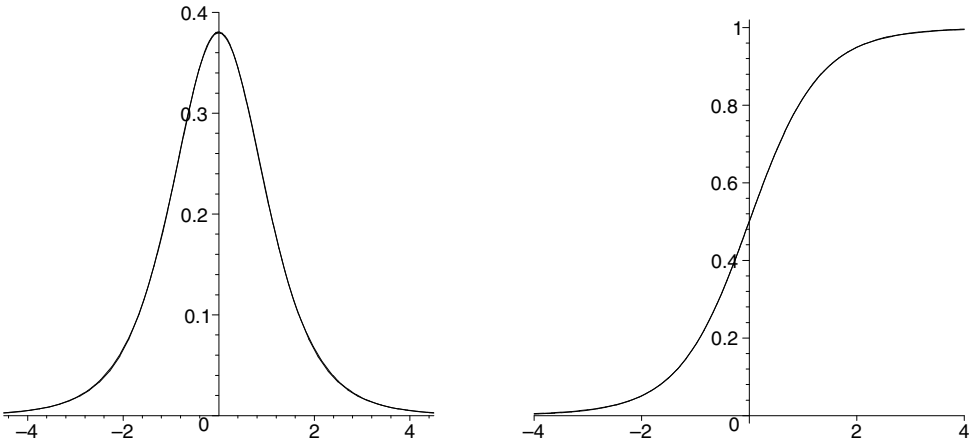
For $\nu = 5$, we obtain

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.6839, 0, 0.1259)$$

from which we get, using `FindLambdasL`, the two fits

$$\text{GLD}_1(0, -0.07607, -0.04831, -0.04831)$$
$$\text{GLD}_2(-3.4611 \times 10^{-5}, 0.3369, 5.3044, 5.3048).$$

The supports of GLD_1 and GLD_2 are $(-\infty, \infty)$, and $[-2.97, 2.97]$, respectively, giving a clear indication that GLD_2 is not suitable. Visually, the p.d.f.s of GLD_1 and $t(5)$ cannot be distinguished, as seen from the depiction of these two graphs



(a) (b)

Figure 6.4–2. The $t(5)$ and GLD_1 p.d.f.s (a); the $t(5)$ and GLD_1 c.d.f.s (b).

in Figure 6.4–2 (a). The same is true for the c.d.f.s of these distributions that are shown in Figure 6.4–2 (b). For the GLD_1 approximation of $t(5)$, we have

$$\sup|\hat{f}(x) - f(x)| = 0.001322, \qquad \sup|\hat{F}(x) - F(x)| = 0.001135$$

and

$$||\hat{f} - f||_1 = 0.007532, \qquad ||\hat{f} - f||_2 = 0.002399.$$

These figures indicate that, by these measures, this approximation is clearly superior to the one obtained through the method of moments and slightly better than the approximation by the method of percentiles (GBD fits are not available for $t(\nu)$).

For $\nu = 6$, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0,\ 0.6555,\ 0,\ 0.1093)$$

and the two fits

$$\text{GLD}_1(0,\ -3.3630 \times 10^{-4},\ -2.2037 \times 10^{-4},\ -2.2037 \times 10^{-4})$$
$$\text{GLD}_2(-6.8501 \times 10^{-6},\ 0.36318,\ 5.0012,\ 5.0014),$$

with respective supports of $(-\infty, \infty)$ and $[-2.75, 2.75]$. Again, GLD_2 is not suitable and for GLD_1 we get

$$\sup|\hat{f}(x) - f(x)| = 0.002051, \qquad \sup|\hat{F}(x) - F(x)| = 0.001816$$

and

$$||\hat{f} - f||_1 = 0.01007, \qquad ||\hat{f} - f||_2 = 0.003328,$$

making the GLD_1 approximation to $t(6)$ superior to the one obtained through moments and almost as good as the approximation through percentiles.

When $\nu = 10$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0,\ 0.6183,\ 0,\ 0.09394)$$

and two fits are obtained. The better of these two approximations is

$$\text{GLD}(0,\ 0.06657,\ 0.04391, 0.04391)$$

with support $[-15.02,\ 15.02]$. For this approximation to $t(10)$,

$$\sup|\hat{f}(x) - f(x)| = 0.001610, \qquad \sup|\hat{F}(x) - F(x)| = 0.0005227$$

and

$$||\hat{f} - f||_1 = 0.006006, \qquad ||\hat{f} - f||_2 = 0.002287,$$

making this GLD approximation superior to the one obtained through moments and comparable to the approximation through percentiles.

For $t(30)$, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0,\ 0.5812,\ 0,\ 0.07681)$$

for which two fits are obtained. The better of these two approximations is

$$\text{GLD}(0,\ 0.1592, 0.1081, 0.1081)$$

and has support $[-6.28, 6.28]$. For this approximation,

$$\sup|\hat{f}(x) - f(x)| = 0.001303, \qquad \sup|\hat{F}(x) - F(x)| = 0.0005036$$

and

$$||\hat{f} - f||_1 = 0.004672, \qquad ||\hat{f} - f||_2 = 0.001940,$$

making this GLD approximation to $t(30)$ superior to the one obtained through moments and comparable to the approximation through percentiles.

6.4.4 The Exponential Distribution

The exponential distribution, defined in Section 3.4.4, has a single positive parameter, θ, and an easily derived quantile function from which we obtain

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\theta,\ \frac{\theta}{2},\ \frac{\theta}{6},\ \frac{\theta}{12}\right).$$

Since $(\tau_3,\ \tau_4) = (1/3,\ 1/6)$ and τ_3 and τ_4 are independent of θ, we set, as we have done on other occasions, $\theta = 1$. Figure 6.3–6 indicates that an approximation

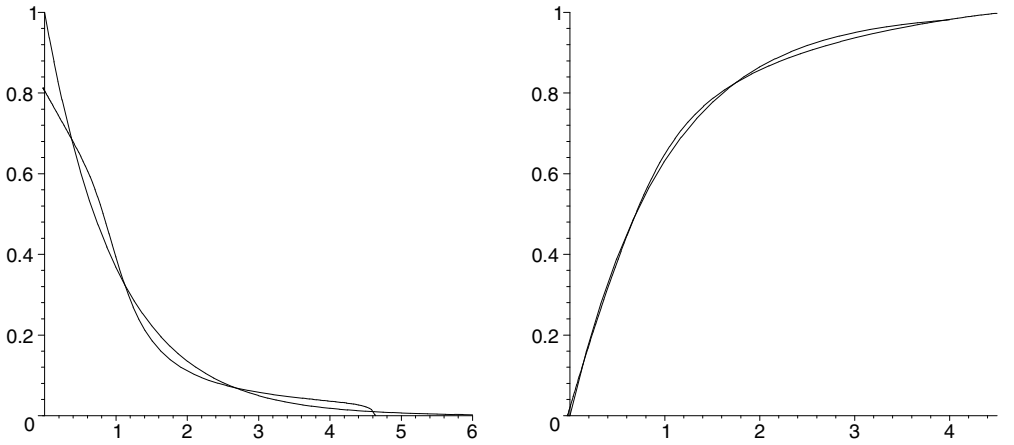


Figure 6.4–3. The exponential ($\theta = 1$) and GLD_1 p.d.f.s (a); the exponential and GLD_1 c.d.f.s (b).

should be available from Table E–3 of Appendix E and possibly another one from Table E–5. Using `FindLambdasL` we obtain the approximation

$$\text{GLD}(2.3067,\ 0.4290,\ 9.6400,\ 0.5278)$$

with support $[-0.024, 4.64]$. The graphs of the p.d.f. of this approximation and that of the exponential distribution with $\theta = 1$, shown in Figure 6.4–3 (a), imply that this fit is a poor one. The same conclusion can be drawn from Figure 6.4–3 (b), which shows the c.d.f. of the approximation with the c.d.f. of the exponential distribution with $\theta = 1$. This perception is reinforced by

$$\sup|\hat{f}(x) - f(x)| = 0.8182, \qquad \sup|\hat{F}(x) - F(x)| = 0.01967$$

and

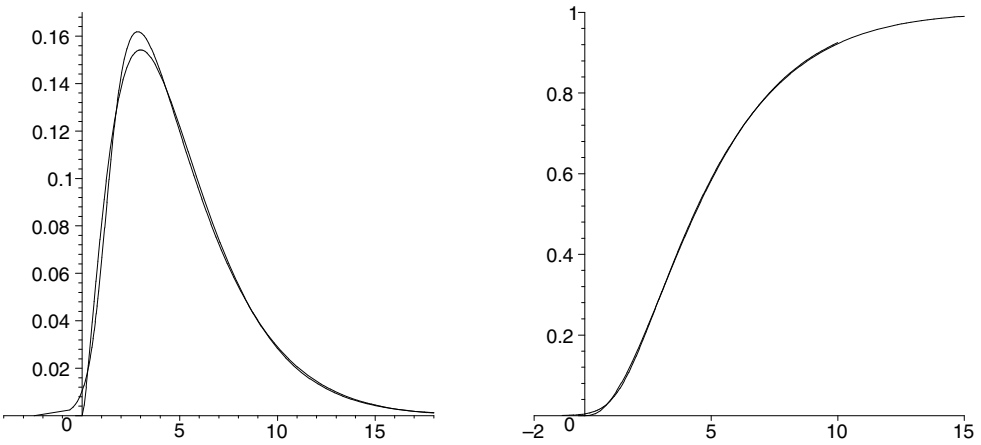
$$||\hat{f} - f||_1 = 0.1517, \qquad ||\hat{f} - f||_2 = 0.1515.$$

(From Figure 6.4–3 (a), it seems that $\sup|\hat{f}(x) - f(x)|$ should be much smaller than 0.8128; however, this value is attained near $x = -0.24$ when the p.d.f. of the exponential distribution is zero.)

In spite of the difficulty of finding a good GLD fit through L-moments, note that the exponential distribution is a limiting case of the $\text{GLD}(0,\ \lambda_2,\ 0,\ \theta\lambda_2)$ as $\lambda_2 \to 0$ (see Section 3.4.4).

6.4.5 The Chi-Square Distribution

The $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ of the chi-square distribution with parameter ν, designated by $\chi^2(\nu)$ and defined in Section 3.4.5, is not available in closed form and must be



(a) (b)

Figure 6.4–4. The $\chi^2(5)$ p.d.f. and the GLD p.d.f. (higher at the peak) (a); the $\chi^2(5)$ and GLD c.d.f.s (b).

calculated numerically. As was done in Sections 3.4.5, 4.4.5, and 5.4.5, we begin by considering the $\nu = 5$ case for which

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (5, 1.6977, 0.3553, 0.2330).$$

The only approximation available in this situation is

$$\text{GLD}(2.3327, 0.03011, 0.01348, 0.1033)$$

which has support $[-30.88, 35.54]$. The p.d.f.s of the $\chi^2(5)$ and this approximating GLD are shown in Figure 6.4–4 (a) and the c.d.f.s of these distributions are given in Figure 6.4–4 (b). In Figure 6.4–4 (a), the curve that is higher at the peak is the GLD density.

We can see from Figure 6.4–4 (a) and (b) that this fit is a moderately good one, a view that is supported by the computations that lead to

$$\sup|\hat{f}(x) - f(x)| = 0.01639, \qquad \sup|\hat{F}(x) - F(x)| = 0.08264$$

and

$$||\hat{f} - f||_1 = 0.04608, \qquad ||\hat{f} - f||_2 = 0.01805.$$

These values indicate that this approximation to $\chi^2(5)$ is better than the one obtained through moments, comparable to the one based on percentiles and not as good as the GBD approximation.

Other $\chi^2(\nu)$ distributions considered in previous chapters are for $\nu = 3$, 10, and 30. When $\nu = 3$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (3, 1.2732, 0.3459, 0.1902).$$

The approximation provided by `FindLambdasL` is

$$\text{GLD}_1(5.8494,\ 0.1748,\ 11.5299,\ 0.7302).$$

From Figure 6.3–6, it seems that there should be at least another solution near the origin and, after several attempts, the program `RefineSearchGLDL` yields

$$\text{GLD}_2(0.6870,\ 0.02116,\ 0.003180,\ 0.05498).$$

The supports of GLD_1 and GLD_2 are, respectively, $[0.13,\ 11.57]$ and $[-46.57,\ 47.94]$. GLD_2, is a moderately good approximation while GLD_1, at least by comparison to GLD_2, is not a good approximation. For GLD_1 we have

$$\sup|\hat{f}(x) - f(x)| = 0.1301, \qquad \sup|\hat{F}(x) - F(x)| = 0.01383$$

and

$$||\hat{f} - f||_1 = 0.1273, \qquad ||\hat{f} - f||_2 = 0.05832$$

and for GLD_2,

$$\sup|\hat{f}(x) - f(x)| = 0.04644, \qquad \sup|\hat{F}(x) - F(x)| = 0.008893$$

and

$$||\hat{f} - f||_1 = 0.04230, \qquad ||\hat{f} - f||_2 = 0.02966.$$

The GLD_2 approximation seems to be slightly superior to the one obtained through moments and percentiles.

For $\chi^2(10)$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (10, 2.4609, 0.3617, 0.3184)$$

and the single GLD approximation to $\chi^2(10)$ through L-moments is

$$\text{GLD}_1(6.9132,\ 0.03001,\ 0.03305, 0.1426)$$

with support $[-26.29, 40.11]$. From

$$\sup|\hat{f}(x) - f(x)| = 0.005710, \qquad \sup|\hat{F}(x) - F(x)| = 0.007031$$

and

$$||\hat{f} - f||_1 = 0.04134, \qquad ||\hat{f} - f||_2 = 0.01078.$$

we can see that we have a reasonably good approximation to $\chi^2(10)$. This approximation is slightly better than the one obtained through moments and comparable to the one percentile-based approximation.

When $\nu = 30$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (30, 4.3338, 0.3655, .5405).$$

The two approximations available for $\chi^2(30)$ are

$$\text{GLD}_1(26.3841, 0.02228, 0.06705, 0.1674)$$
$$\text{GLD}_2(35.5156, 0.05330, 13.1810, 1.7435)$$

with respective supports of $[-18.4935, 71.2616]$ and $[16.7538, 54.2775]$. For GLD_1,

$$\sup|\hat{f}(x) - f(x)| = 0.001722, \qquad \sup|\hat{F}(x) - F(x)| = 0.004598$$

and

$$||\hat{f} - f||_1 = 0.03018, \qquad ||\hat{f} - f||_2 = 0.04861,$$

indicating that GLD_1 is a reasonably good approximation but GLD_2 is not. The $\sup|\hat{f}(x) - f(x)|$, $\sup|\hat{F}(x) - F(x)|$ and $||\hat{f} - f||_1$ of GLD_1 are superior to the corresponding values for the fit based on ordinary moments but not as good as the ones for the corresponding values of the percentile-based fit.

6.4.6 The Gamma Distribution

The gamma distribution with positive parameters α and θ is defined in Section 3.4.6. As was done in Sections 3.4.6, 4.4.6, and 4.4.6, we choose the specific gamma distribution with $\alpha = 5$ and $\theta = 3$. For this gamma distribution,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (15, 3.6914, 0.5415, 0.4765)$$

and we find the single approximation

$$\text{GLD}(10.3637, 0.02018, 0.03327, 0.1439)$$

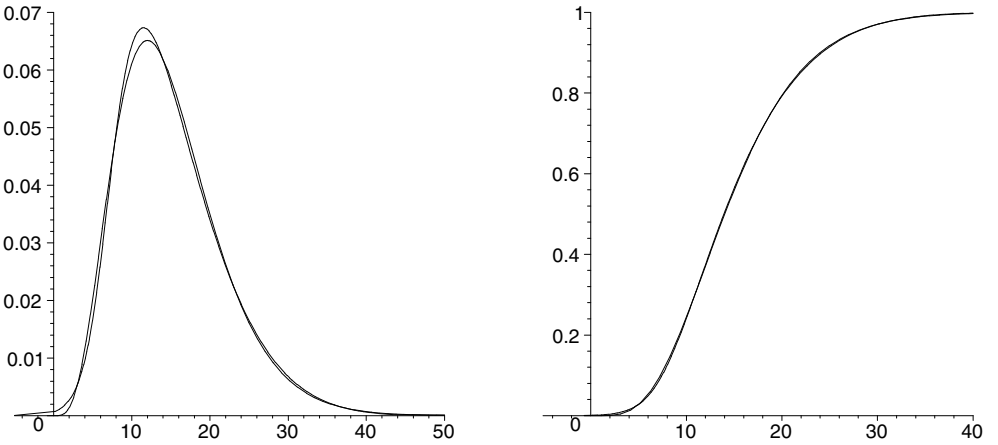
that has support $[-39.1971, 59.9246]$.

Figure 6.4–5 (a) shows the p.d.f. of the approximating GLD distribution and the p.d.f. of the gamma distribution with $\alpha = 5$ and $\theta = 3$ (the GLD p.d.f. rises higher at the peak). Figure 6.4–5 (b) shows the c.d.f.s of the same distributions. From these graphs and

$$\sup|\hat{f}(x) - f(x)| = 0.003787, \qquad \sup|\hat{F}(x) - F(x)| = 0.006961,$$

$$||\hat{f} - f||_1 = 0.04122, \qquad ||\hat{f} - f||_2 = 0.008753,$$

it seems that this is a reasonable approximation. By the $\sup|\hat{f}(x) - f(x)|$ and $\sup|\hat{F}(x) - F(x)|$ measures, this approximation to the gamma distribution with $\alpha = 5$ and $\theta = 3$, is somewhat better than the one obtained through ordinary moments, comparable to the one based on percentiles, and inferior to the GBD approximation.



(a) (b)

Figure 6.4–5. The gamma ($\alpha = 5$, $\theta = 3$) and its approximating GLD p.d.f. (higher at the peak) (a); the gamma ($\alpha = 5$, $\theta = 3$) and its approximating c.d.f.s (b).

6.4.7 The Weibull Distribution

The Weibull distribution, defined in Section 3.4.7, has positive parameters α and β. When we consider this distribution with $\alpha = 1$ and $\beta = 5$, as was done in Sections 3.4.7, 4.4.7, and 5.4.7, we have the distribution function

$$F(x) = 1 - e^{-x^5}, \quad \text{for} \quad x \geq 0$$

and quantile function

$$Q(y) = \left(\ln\left(\frac{1}{1-y}\right)\right)^{\frac{1}{5}}.$$

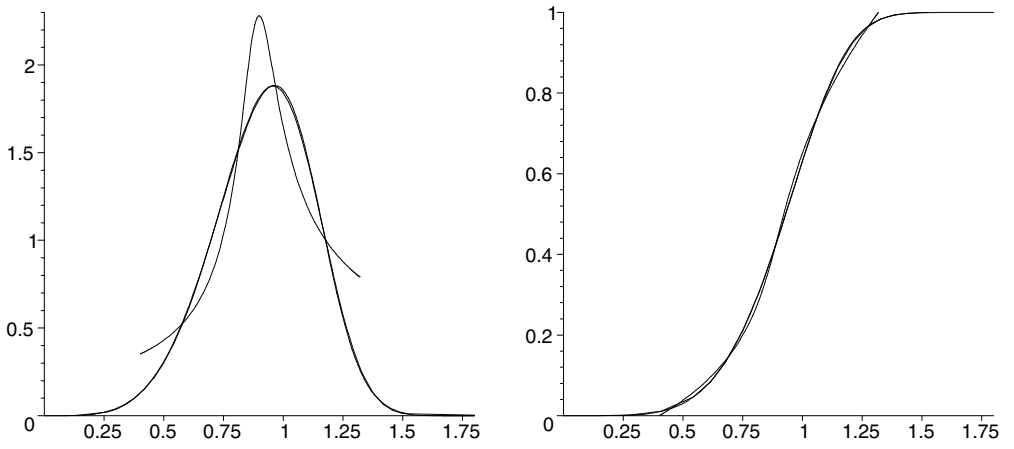
Using $Q(y)$ in (6.1.11) through (6.1.14) leads to

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.9182,\ 0.1189,\ -0.005664,\ 0.01361),$$

which, through `FindLambdasL`, gives the two approximations

$$\text{GLD}_1(0.9901,\ 1.0577,\ 0.2116,\ 0.1095)$$
$$\text{GLD}_2(0.8596,\ 2.1811,\ 2.7507,\ 6.2057).$$

The supports of GLD_1 and GLD_2 are $[0.0445, 1.9355]$ and $[0.4013, 1.3179]$, respectively, and the inappropriateness of GLD_2 is clear from Figure 6.4–6 (a), which shows the GLD_1 and GLD_2 p.d.f.s with the p.d.f. of the Weibull distribution with $\alpha = 1$ and $\beta = 5$. In both, the density plots of Figure 6.4–6 (a) and the distribution plots of Figure 6.4–6 (b), GLD_1 and the Weibull distribution are indistinguishable, whereas the GLD_2 curves are visibly apart from the Weibull distribution.



(a) (b)

Figure 6.4–6. The Weibull ($\alpha = 1$, $\beta = 5$) and its approximating GLD_1 and GLD_2 p.d.f.s (a); the Weibull ($\alpha = 1$, $\beta = 5$) and its approximating c.d.f.s (b).

For GLD_1, we compute

$$\sup|\hat{f}(x) - f(x)| = 0.02451 \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.001815$$

and

$$||\hat{f} - f||_1 = 0.01170, \qquad ||\hat{f} - f||_2 = 0.01251,$$

to confirm that we have a good fit, comparable to the ones obtained from the GBD and the GLD through ordinary moments and percentiles.

6.4.8 The Lognormal Distribution

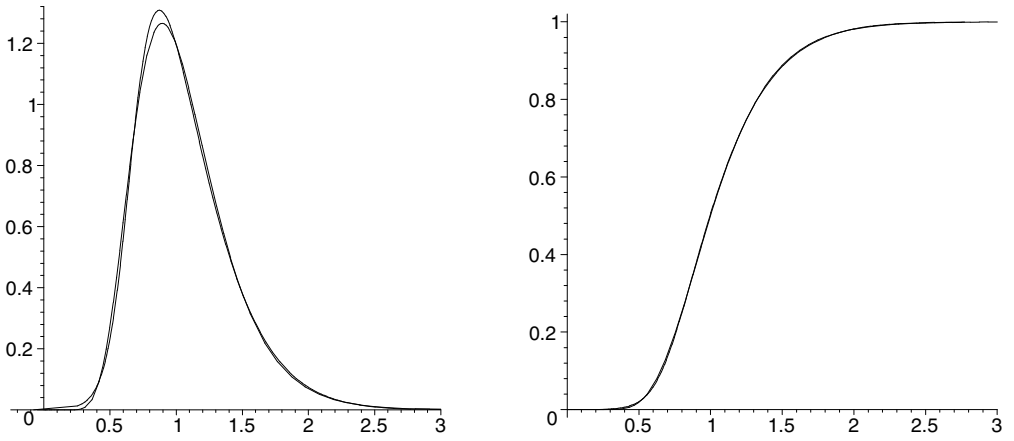
The lognormal distribution is defined, through its parameters μ and $\sigma > 0$, in Section 3.4.8. Approximations to the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$ are considered in other chapters and here we attempt to approximate this same distribution through L-moments. The $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ for the lognormal with $\mu = 0$ and $\sigma = 1/3$ are not available in closed form and numerical computations produce

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.0571, 0.1970, 0.03188, 0.02821).$$

By invoking the program `FindLambdasL` we get

$$\text{GLD}(0.8216, 0.2459, 0.02169, 0.08595)$$

as the sole GLD approximation to this lognormal distribution. The support for this GLD is $[-3.2488, 4.8920]$. Figure 6.4–7 (a) and Figure 6.4–7 (b) that show, respectively, the p.d.f.s and c.d.f.s of this GLD with their counterparts from the



(a) (b)

Figure 6.4–7. The p.d.f. of the lognormal with $\mu = 0$, $\sigma = 1/3$ (lower at the center) with its approximating GLD (a); the c.d.f. of the lognormal with $\mu = 0$, $\sigma = 1/3$ with its approximating GLD (b).

lognormal distribution, indicate that we should have a fairly good fit. In Figure 6.4–7 (a), the p.d.f. that rises higher at the center and also higher near the origin is the GLD; the two c.d.f.s of Figure 6.4–7 (b) are indistinguishable.

Through numeric computations we can determine that

$$\sup|\hat{f}(x) - f(x)| = 0.06338 \qquad \sup|\hat{F}(x) - F(x)| = 0.006638$$

and

$$||\hat{f} - f||_1 = 0.03808, \qquad ||\hat{f} - f||_2 = 0.03530,$$

confirming the view that this GLD provides a reasonably good fit that is comparable to the GLD approximations obtained through ordinary moments and percentiles.

6.4.9 The Beta Distribution

Section 3.4.9 defines the beta distribution in terms of its two parameters $\beta_3 > -1$ and $\beta_4 > -1$. As in Sections 3.4.9, 4.4.9, and 5.4.9, here we consider the two cases $\beta_3 = \beta_4 = 1$ and $\beta_3 = -1/2$, $\beta_4 = 1$. In both cases we must resort to numerical methods to determine the L-moments, $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$.

When $\beta_3 = \beta_4 = 1$, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.5,\ 0.1286,\ 0,\ 0.007193)$$

which, through `FindLambdasL`, leads to the two approximations

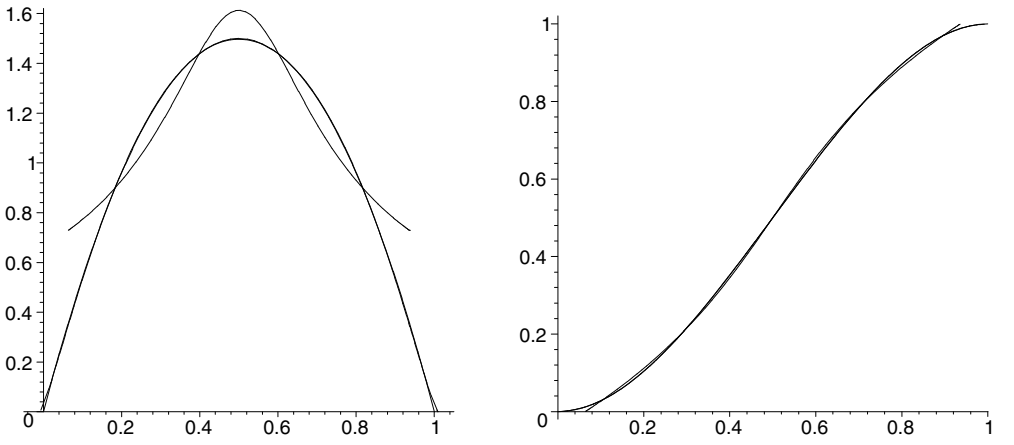


Figure 6.4–8. The p.d.f. of the beta distribution ($\beta_3 = \beta_4 = 1$) and the p.d.f.s of GLD_1 and GLD_2 (a); the c.d.f. of the beta distribution ($\beta_3 = \beta_4 = 1$) and the c.d.f.s of GLD_1 and GLD_2 (b).

$$\text{GLD}_1(0.5000,\ 1.9646,\ 0.4476,\ 0.4476)$$
$$\text{GLD}_2(0.5000,\ 2.2935,\ 3.1449,\ 3.1449)$$

with respective supports $[-0.0089,\ 1.0089]$ and $[0.064,\ 0.94]$. Both supports are close to the support of the beta distribution which is $(0, 1)$, tempting us to believe that we may have two good approximations. This is definitely not the case, as shown in Figure 6.4–8 (a) where the GLD_1, GLD_2, and the beta p.d.f.s are plotted. In this figure the GLD_1 and beta p.d.f.s are almost identical and the GLD_2 p.d.f. rises higher at the center on a smaller support. A careful viewing of Figure 6.4–8 (a) will show small discrepancies between the GLD_1 and beta p.d.f.s near the origin and near the point $(1, 0)$. The c.d.f.s of the same distributions are shown in Figure 6.4–8 (b) where the GLD_1 and beta c.d.f.s seem identical. The results

$$\sup|\hat{f}(x) - f(x)| = 0.02914 \qquad \sup|\hat{F}(x) - F(x)| = 0.0004337,$$
$$||\hat{f} - f||_1 = 0.004102, \qquad ||\hat{f} - f||_2 = 0.005693,$$

for GLD_1 confirm the earlier observation that GLD_1 is a good approximation that is almost as good as the approximations obtained through ordinary moments and percentiles.

When $\beta_3 = -1/2$ and $\beta_4 = 1$, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.2,\ 0.1125,\ 0.03653,\ 0.01023)$$

and, through `FindLambdasL`, we get the approximation

$$\text{GLD}(0.4577,\ 2.1368,\ 3.8883,\ 0.3240)$$

that has support $[-0.010, 0.93]$. For this rather poor fit we have

$$\sup|\hat{f}(x) - f(x)| = 34.5350 \quad \text{and} \quad \sup|\hat{F}(x) - F(x)| = 0.06506$$

and

$$||\hat{f} - f||_1 = 0.2248, \qquad ||\hat{f} - f||_2 = 1.2663.$$

6.4.10 The Inverse Gaussian Distribution

The inverse Gaussian distribution, defined in Section 3.4.10, has two positive parameters μ and λ. As was done in Sections 3.4.10, 4.4.10, and 5.4.10, we consider the specific distribution with $\mu = 0.5$ and $\lambda = 6$. Since the $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ for this distribution are not available in closed form, we use numeric computation to obtain the following L-moments

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.5,\ 0.07944,\ 0.01097,\ 0.01066).$$

This leads us, through `FindLambdasL`, to the single L-moment approximation

$$\text{GLD}(0.4098,\ 0.8561,\ 0.03402,\ 0.1237)$$

with support $[-0.7566, 1.5762]$. At first glance, it seems that this support extends too far to the left and not enough to the right (the inverse Gaussian distribution has support $(0, \infty)$). However, the "extra" cumulative probability that is introduced on the left by this GLD is about 10^{-4} and on the right this approximation is "deficient" by a cumulative probability of less than 10^{-5}.

Figure 6.4–9 (a) shows the inverse Gaussian p.d.f. with the p.d.f. of the approximating GLD and Figure 6.4–9 (b) shows the c.d.f.s of these same distributions. Visually, the approximation seems reasonably good and this perception is reinforced by the calculations that yield

$$\sup|\hat{f}(x) - f(x)| = 0.1340, \qquad \sup|\hat{F}(x) - F(x)| = 0.006077,$$

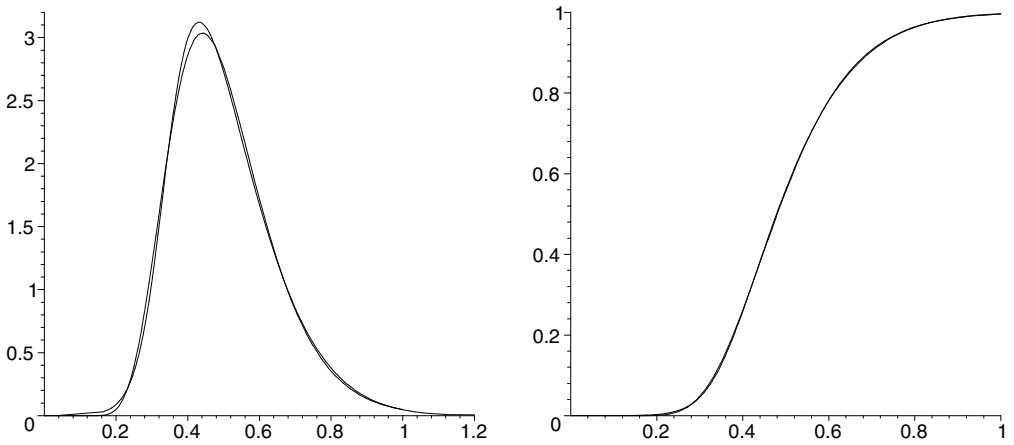
$$||\hat{f} - f||_1 = 0.03596, \qquad ||\hat{f} - f||_2 = 0.05064.$$

These figures are smaller than similar ones obtained for the GLD fit using ordinary moments and are comparable to the ones for percentile-based GLD approximation (GBD did not provide an approximation in this case).

6.4.11 The Logistic Distribution

The logistic distribution with parameters μ and $\sigma > 0$ is defined in Section 3.4–11. The quantile function for this distribution,

$$Q(y) = \mu + \sigma \ln\left(\frac{y}{1-y}\right),$$



(a) (b)

Figure 6.4–9. Inverse Gaussian ($\mu = 0.5$, $\lambda = 6$) p.d.f. with its approximating GLD p.d.f. (the GLD rises higher at its peak) (a); The inverse Gaussian ($\mu = 0.5$, $\lambda = 6$) c.d.f. with the d.f. of its approximating GLD (b).

can be obtained through integration and the application of (6.1.11) through (6.1.14) gives

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\mu,\ \sigma,\ 0,\ \frac{\sigma}{6}\right).$$

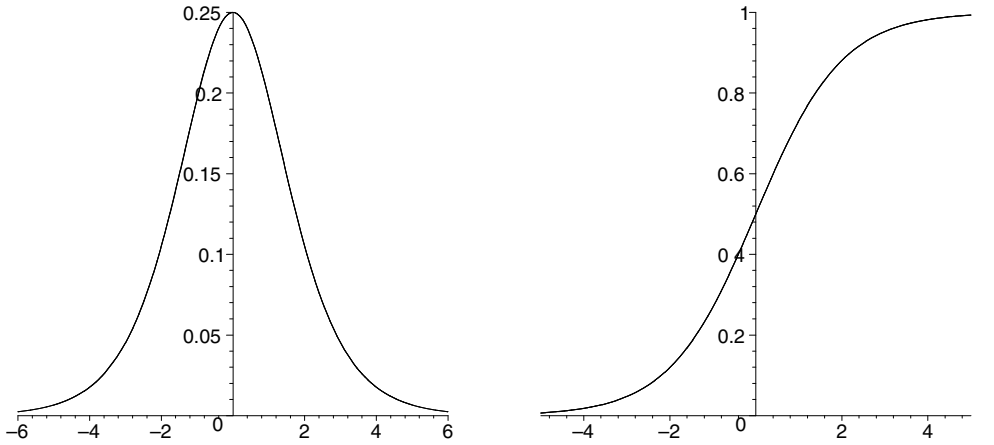
Since $\tau_3 = 0$ and $\tau_4 = 1/6$ are independent of the distribution parameters, we consider, as we have done on other occasions, the logistic distribution with $\mu = 0$ and $\sigma = 1$. For this distribution, with $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0,\ 1,\ 0,\ 1/6)$, `FindLambdasL` provides the four approximations

$$\begin{aligned}
&\mathrm{GLD}_1(0.0000,\ 1.7057 \times 10^{-6},\ 1.7057 \times 10^{-6},\ 1.7057 \times 10^{-6})\\
&\mathrm{GLD}_2(-0.8239,\ 0.2186,\ 2.8784,\ 11.8679)\\
&\mathrm{GLD}_3(0.0000,\ -1.7140 \times 10^{-6},\ -1.7140 \times 10^{-6},\ -1.7140 \times 10^{-6})\\
&\mathrm{GLD}_4(0.0000,\ -3.6696 \times 10^{-7},\ -3.6696 \times 10^{-7},\ -3.6696 \times 10^{-7})
\end{aligned}$$

that have supports of $[-586277.,\ 586277.]$, $[-5.40,\ 3.75]$, $(-\infty,\ \infty)$, and $(-\infty,\ \infty)$, respectively. Since the lack of symmetry and the support of GLD_2 makes it unsuitable, we consider only GLD_1, GLD_3, and GLD_4. Figure 6.4–10 (a) shows the p.d.f.s of the logistic distribution ($\mu = 0$ and $\sigma = 1$) with the p.d.f.s of GLD_1, GLD_3, and GLD_4. These plots are indistinguishable as are the graphs of the c.d.f.s of these four distributions shown in Figure 6.4–10 (b).

For GLD_1, GLD_3, and GLD_4 we have $\sup|\hat{f}(x) - f(x)|$, $\sup|\hat{F}(x) - F(x)|$, $||\hat{f} - f||_1$, and $||\hat{f} - f||_2$ are all less than 10^{-4}, confirming that GLD_1, GLD_3, and GLD_4 are good approximations.

Of course, all of this is to be expected because we showed in Section 3.4.11 that the d.f.s of $\mathrm{GLD}(\mu,\ r/\sigma,\ r, r)$ converge to the d.f. of the logistic distribution



(a) (b)

Figure 6.4–10. The logistic ($\mu = 0$, $\sigma = 1$), GLD_1, GLD_3 and GLD_4 p.d.f.s (a); the logistic ($\mu = 0$, $\sigma = 1$), GLD_1, GLD_3 and GLD_4 c.d.f.s (b).

with parameters μ and σ.

6.4.12 The Largest Extreme Value Distribution

The largest extreme value distribution, with parameters μ and $\sigma > 0$, is defined in Section 3.4.12. Hosking and Wallis (1997), in their Appendix A, give

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\mu + \sigma\gamma,\ \sigma \ln 2,\ \sigma \ln\left(\frac{9}{8}\right),\ \sigma(16 \ln 2 - 10 \ln 3)\right)$$

where $\gamma \approx 0.57722$ is Euler's constant. From

$$\tau_3 = \Lambda_3/\Lambda_2 = \frac{\ln(9/8)}{\ln 2} = 0.1699 \quad \text{and} \quad \tau_4 = \Lambda_4/\Lambda_2 = \frac{16 \ln 2 - 10 \ln 3}{\ln 2} = 0.1504,$$

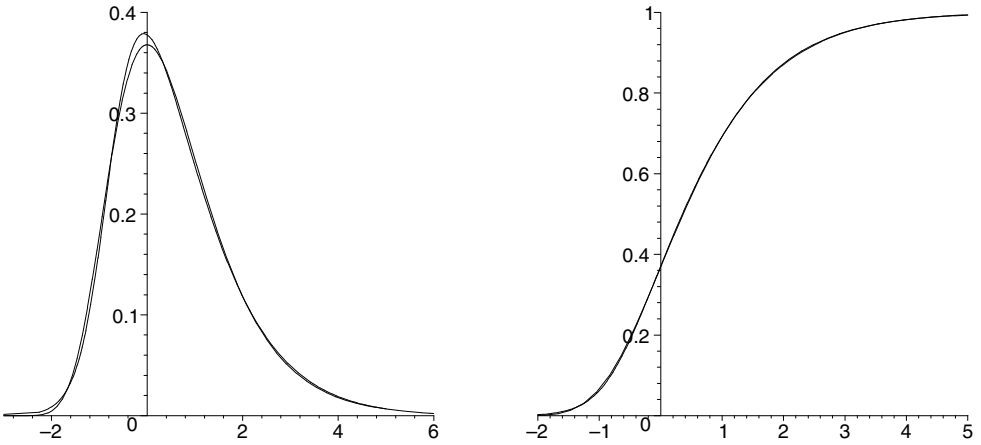
we can see that τ_3 and τ_4 are independent of the distribution parameters and, as was done in other considerations of this distribution, we set $\mu = 0$ and $\sigma = 1$ for which

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.5772,\ 0.6931,\ 0.1178,\ 0.1042).$$

Using `FindLambdasL`, we obtain the single GLD approximation

$$\text{GLD}(-0.2360,\ 0.04949,\ 0.01525,\ 0.05850)$$

with support $[-20.48,\ 20.01]$. Figure 6.4–11 (a) shows the p.d.f. of the largest extreme value distribution with $\mu = 0$ and $\sigma = 1$ with the p.d.f. of its approximating GLD (the p.d.f. that rises higher at its peak is the GLD) and Figure 6.4–11 (b) shows the c.d.f.s of these same distributions. For this GLD approximation we



(a) (b)

Figure 6.4–11. The largest extreme value ($\mu = 0$, $\sigma = 1$) p.d.f. (lower at its peak) and its GLD approximation (a); the c.d.f.s of the same distributions (b).

have

$$\sup|\hat{f}(x) - f(x)| = 0.01580 \qquad \sup|\hat{F}(x) - F(x)| = 0.005961$$

and

$$||\hat{f} - f||_1 = 0.03303 \qquad ||\hat{f} - f||_2 = 0.01671,$$

which indicates that this GLD fit is better than the one based on ordinary moments and is comparable to the one obtained through percentiles (the GBD did not provide an approximation for the largest extreme value distribution).

6.4.13 The Extreme Value Distribution

As was pointed out in Section 3.4.13, the distribution of an extreme value random variable, X, is closely related to the distribution of a largest extreme value random variable, Y. The L-moments of X, $\Lambda_i(X)$, are related to the L-moments of Y, $\Lambda_i(Y)$, through

$$\Lambda_1(Y) = -\Lambda_1(X), \quad \Lambda_2(Y) = \Lambda_2(X), \quad \Lambda_3(Y) = -\Lambda_3(X), \quad \Lambda_4(Y) = \Lambda_4(X).$$

Thus, finding an approximation to an extreme value distribution is equivalent to finding one for the corresponding largest extreme value distribution, interchanging λ_3 and λ_4 and adjusting λ_2 and λ_1 (see Algorithm GLD-L of Section 6.3).

6.4.14 The Double Exponential Distribution

The double exponential distribution, defined in Section 3.4.14, has a single positive parameter, λ. The quantile function, $Q(y)$, for this distribution is given

by

$$Q(y) = \begin{cases} \lambda \ln(2y), & \text{if } y \le 1/2 \\ -\lambda \ln(2-2y), & \text{if } y > 1/2. \end{cases} \tag{6.4.1}$$

Performing the integrations prescribed in (6.1.11) through (6.1.14), we determine that

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(0,\ \frac{3}{4}\lambda,\ 0,\ \frac{17}{96}\lambda\right)$$

and observe that $\tau_3 = 0$ and $\tau_4 = 17/72$ are independent of the distribution parameter λ. When, as has been done elsewhere in this text, we set $\lambda = 1$, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0,\ 0.75,\ 0,\ 0.1771)$$

which, through `FindLambdasL`, leads us to the two GLD approximations

$$\text{GLD}_1(7.1903 \times 10^{-7},\ 0.2782,\ 6.2655,\ 6.2655)$$
$$\text{GLD}_2(0.0000,\ -0.3080,\ -0.1742,\ -0.1742).$$

The supports of GLD_1 and GLD_2, respectively, are $[-3.59,\ 3.59]$ and $(-\infty,\ \infty)$. The support of the double exponential distribution is $(-\infty,\ \infty)$ and the restriction imposed by the GLD_1 support is significant because the cumulative double exponential probability not covered by GLD_1 is about 0.0275. This limitation of GLD_1 becomes visible in Figure 6.4–12 (a) which shows the double exponential p.d.f. with the p.d.f.s of GLD_1 and GLD_2. The curve with the truncated support and rising highest at the center is the p.d.f. of GLD_1, the one that is lowest at the center is the p.d.f. of GLD_2.

Computations show that for GLD_1

$$\sup|\hat{f}(x) - f(x)| = 0.3561, \qquad \sup|\hat{F}(x) - F(x)| = 0.04480,$$

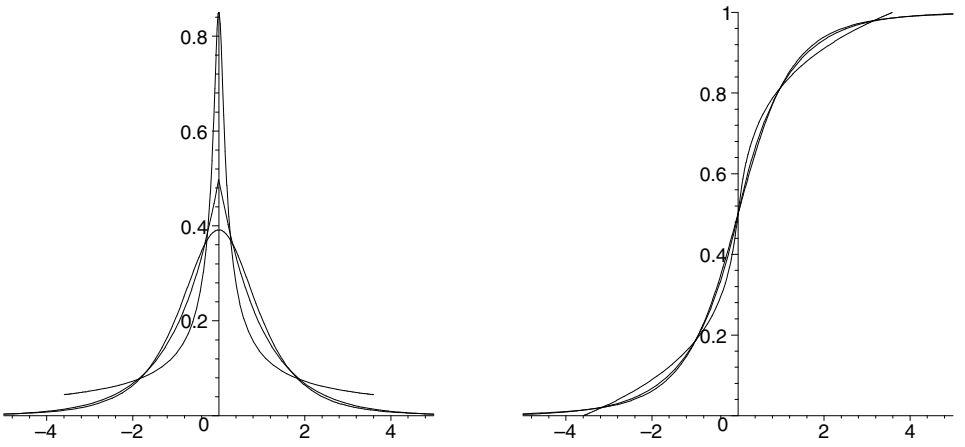
$$||\hat{f} - f||_1 = 0.3221, \qquad ||\hat{f} - f||_2 = 0.1796.$$

For GLD_2 these figures are

$$\sup|\hat{f}(x) - f(x)| = 0.1081 \ \text{ and } \ \sup|\hat{F}(x) - F(x)| = 0.01476,$$

$$||\hat{f} - f||_1 = 0.09425, \qquad ||\hat{f} - f||_2 = 0.05558.$$

Neither approximation seems to be particularly good and the figures for GLD_2 indicate that it is the better of the two approximations, somewhat better than the fit than that obtained through ordinary moments and comparable with the one obtained by percentiles.



(a) (b)

Figure 6.4–12. The double exponential ($\lambda = 1$), GLD_1 and GLD_2 p.d.f.s (the GLD_1 is highest and GLD_2 is lowest at the center) (a); the double exponential ($\lambda = 1$), GLD_1 and GLD_2 c.d.f.s (b).

6.4.15 The F-Distribution

The F distribution, $F(\nu_1, \nu_2)$, with parameters ν_1 and ν_2 is defined in Section 3.4.15. It is shown there that for the i-th moment of $F(\nu_1, \nu_2)$ to exist, we must have $\nu_2 > 2i$. Thus, L-moment approximations to $F(\nu_1, \nu_2)$ must be restricted to the situations where $\nu_2 > 2$. Approximations to $F(6, 25)$ were obtained in Sections 3.4.15 and 5.4.15. Therefore, we start with this distribution for which, through numeric computation, we have

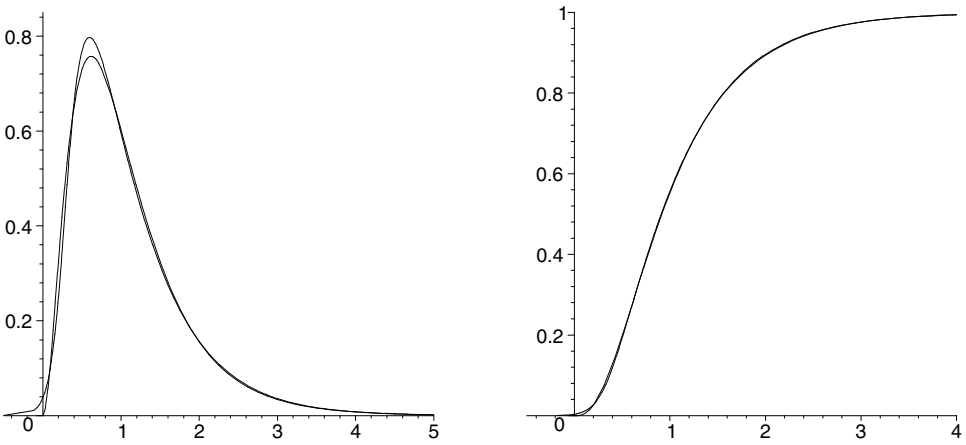
$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.0870, 0.3806, 0.09666, 0.06319).$$

In this case, `FindLambdasL` does not give a satisfactory result (i.e., the λ_1, λ_2, λ_3, λ_4 returned by `FindLambdasL` has a $\max(|\tau_3 - \hat{\tau}_3|, |\tau_3 - \hat{\tau}_3|)$ that is too large). However, Table E–1 of Appendix E indicates that there should be a solution near $(\lambda_3, \lambda_4) = (0.0027, 0.0217)$ and searching in that vicinity directly, through the **R** program `RefineSearchGLDL`, we find the approximation

$$\text{GLD}(0.5046, 0.002944, 0.0002642, 0.001982)$$

(for more details on how to use `RefineSearchGLDL` consult Appendix E).

The support of this GLD approximation is $[-339.1865, 340.1957]$, which extends far beyond the support of any F distribution on the left. The cumulative GLD probability to the left of the support of $F(6, 25)$ (which is $(0, \infty)$) is about 0.035, which is large enough to be of concern. Figure 6.4–13 (a) shows the p.d.f.s of $F(6, 25)$ and its approximating GLD where the extension of the GLD to the



(a) (b)

Figure 6.4–13. The p.d.f.s of $F(6, 25)$ and its GLD (rises higher at its peak) approximation (a); the c.d.f.s of these distributions (b).

left of the vertical axis is clearly visible (the GLD p.d.f. rises higher at its peak). Figure 6.4–13 (b) shows the c.d.f.s of these distributions.

For this approximation to $F(6, 25)$, we have

$$\sup|\hat{f}(x) - f(x)| = 0.07822, \qquad \sup|\hat{F}(x) - F(x)| = 0.008434,$$

$$||\hat{f} - f||_1 = 0.04478, \qquad ||\hat{f} - f||_2 = 0.03903.$$

which, by these measures, indicates a superior fit than the one obtained through ordinary moments and comparable to the one from percentiles.

Section 5.4.16 also considered the F distributions with $(\nu_1, \nu_2) = (2, 4)$, $(4, 6)$, $(6, 12)$, $(6, 16)$. We summarize the approximations to the F distribution for these ν_1 and ν_2 combinations below.

- $(\nu_1, \nu_2) = (2, 4)$:

  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (2, 1.3254, 0.7920, 0.5634)$,
  GLD$(0.007812, -0.4938, -0.0003223, -0.49600)$,
  $\sup|\hat{f}(x) - f(x)| = 0.8483$ and $\sup|\hat{F}(x) - F(x)| = 0.005673$,
  $||\hat{f} - f||_1 = 0.1478, \qquad ||\hat{f} - f||_2 = 0.3467.$

- $(\nu_1, \nu_2) = (4, 6)$:

  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.5, 0.7711, 0.3503, 0.2317)$,
  GLD$(0.3120, -0.3621, -0.02179, -0.3115)$,
  $\sup|\hat{f}(x) - f(x)| = 0.1078$ and $\sup|\hat{F}(x) - F(x)| = 0.007909$,
  $||\hat{f} - f||_1 = 0.0399, \qquad ||\hat{f} - f||_2 = 0.04361.$

- $(\nu_1, \nu_2) = (6, 12)$:
  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.2, 0.4699, 0.1300, 0.09643)$,
  GLD(0.4764, −0.1539, −0.01279, −0.1106),
  $\sup|\hat{f}(x) - f(x)| = 0.05986$ and $\sup|\hat{F}(x) - F(x)| = 0.008558$,
  $||\hat{f} - f||_1 = 0.04356$, $||\hat{f} - f||_2 = 0.05537$.

- $(\nu_1, \nu_2) = (6, 16)$:
  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.1429, 0.4246, 0.1223, 0.07883)$,
  GLD(0.4893, −0.08243, −0.007094, −0.05751),
  $\sup|\hat{f}(x) - f(x)| = 0.08054$ and $\sup|\hat{F}(x) - F(x)| = 0.008515$,
  $||\hat{f} - f||_1 = 0.04416$, $||\hat{f} - f||_2 = 0.03925$.

6.4.16 The Pareto Distribution

The Pareto distribution with positive parameters β and λ is defined in Section 3.4.16. From the quantile function of the Pareto distribution, given by

$$Q(y) = \frac{\lambda}{(1-y)^{1/\beta}}, \tag{6.4.2}$$

we can use (6.1.11) through (6.1.14) to obtain

$$\begin{aligned}
\Lambda_1 &= \frac{\lambda\beta}{\beta - 1}, \\
\Lambda_2 &= \frac{\lambda\beta}{(\beta-1)(2\beta-1)}, \\
\Lambda_3 &= \frac{\lambda\beta(\beta+1)}{(\beta-1)(2\beta-1)(3\beta-1)}, \\
\Lambda_4 &= \frac{\lambda\beta(\beta+1)(2\beta+1)}{(\beta-1)(2\beta-1)(3\beta-1)(4\beta-1)}.
\end{aligned}$$

To find an approximation to the Pareto distribution with $\lambda = 1$ and $\beta = 5$ through L-moments, we first establish that for this distribution

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \left(\frac{5}{4}, \frac{5}{36}, \frac{5}{84}, \frac{55}{1596}\right) = (1.25, 0.1389, 0.05952, 0.034461)$$

and $(\tau_3, \tau_4) = (0.4286, 0.2481)$. We see from Figure 6.3–6 that this point is located on the border of Tables E–3 and E–5 of Appendix E and for this reason, `FindLambdasL` does not provide any solutions. But when `RefineSearchGLDL` is used with specifications to search near the borders of Tables E–3 and E–5, the following two distinct approximations are obtained

$$\begin{aligned}
&\text{GLD}_1(1.8937, 1.1118, 13.6787, 0.2758) \\
&\text{GLD}_2(1.0000, -1.00000, -5.1920 \times 10^{-12}, -0.2000).
\end{aligned}$$

The supports of GLD_1 and GLD_2 are, respectively, [0.9943, 2.7931] and $(-\infty, \infty)$ and, at first glance, neither approximation seems to match the support of the Pareto distribution under consideration, which is $(1, \infty)$.

What is intriguing about GLD_2 is its unusual set of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. It seems that if exact computations were possible, the GLD_2 parameters might well be $(1, -1, 0, -1/5)$ (note that a GLD with these parameters has support $[\lambda_1, \infty) = [1, \infty)$). This is consistent with the result established in Section 2.4.16 that all Pareto distributions are included in the GLD family. More specifically, the GLD family member that corresponds to the Pareto distribution with parameters λ and β is

$$\text{GLD}\left(\lambda, \frac{-1}{\lambda}, 0, \frac{-1}{\beta}\right).$$

6.5 Examples of GLD Fits to Data via L-Moments

This section uses the L-moment parameter estimation method of Section 3 of this chapter to fit GLD distributions to the datasets that have been considered in previous chapters. One of the advantages of L-moments that will become apparent is the ability of L-moments to find fits when other fitting methods, such as the method of moments, could not produce a fit. The fits to a specific dataset is obtained through the **R** program `FitGLDL`, which takes for its only argument the vector consisting of the data and returns, as a vector of vectors, one or more GLD fits (in case `FitGLDL` cannot find a fit, it returns `0 0 0 0`).

6.5.1 Example: Cadmium Concentration in Horse Kidneys

This data, introduced in Section 3.5.2, has L-moments

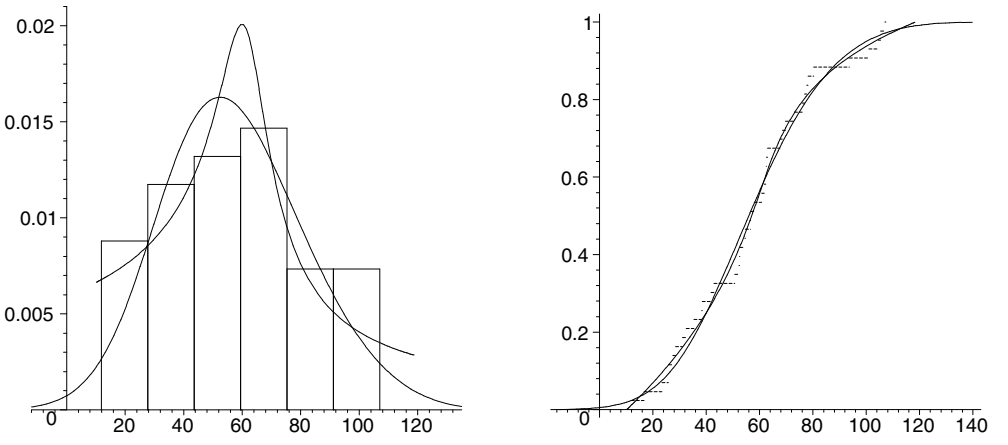
$$\ell_1 = 57.2442, \quad \ell_2 = 13.8771, \quad \ell_3 = 0.6352, \quad \ell_4 = 1.6543.$$

The data L-moments, $\ell_1, \ell_2, \ell_3, \ell_4$, (see (6.3.1) for the definition of ℓ_i) can be found by using the **R** program `FindLMoms(Data)` where `Data` is the vector representing the data of this example. The **R** program `FitGLDL(Data)` gives the two L-moment based fits

$$\text{GLD}_1(49.8471, 0.008438, 0.1049, 0.1868)$$
$$\text{GLD}_2(64.2907, 0.01848, 6.4389, 2.7786)$$

with respective supports of $[-68.67, 168.37]$ and $[10.18, , 118.40]$, both of which cover the data range. (Note that the use of `FindLMoms` is not necessary since `FitGLDL(Data)` automatically prints out the L-moments of `Data`.)

A histogram of the data, together with the p.d.f.s of GLD_1 and GLD_2 is shown in Figure 6.5–1 (a), where the curve that rises higher is the p.d.f. of GLD_2. Figure 6.5–1 (b) gives the e.d.f. of the data with the d.f.s of GLD_1 and GLD_2.



(a) (b)

Figure 6.5–1. Histogram of cadmium concentration and GLD_1 and GLD_2 (higher at center) p.d.f.s (a); the data e.d.f. with the d.f.s of GLD_1 and GLD_2 (b).

Next, we employ the chi-square and Kolmogorov-Smirnov tests to get a clearer sense of the quality of these seemingly good fits. As was done in Section 3.5.2, for the chi-square test, we use the intervals

$$(-\infty, 30), \quad [30, 50), \quad [50, 60), \quad [60, 70), \quad [70, 85), \quad [85, \infty)$$

whose observed frequencies are

$$7, \quad 7, \quad 9, \quad 9, \quad 6, \quad 5.$$

In the case of the GLD_1 fit we get the expected frequencies

$$5.5695, \quad 11.7925, \quad 6.9191, \quad 6.1775, \quad 6.7231, \quad 5.8183$$

which give a chi-square statistic and associated p-value of

$$4.4234 \quad \text{and} \quad 0.03545,$$

respectively. The Kolmogorov-Smirnov statistic for GLD_1 is $KS_1 = 0.09232$ and since the sample size for this data is 43, $\sqrt{n}KS_1 = 0.6054$. From Appendix J, we find that the p-value of the Kolmogorov-Smirnov test is 0.86.

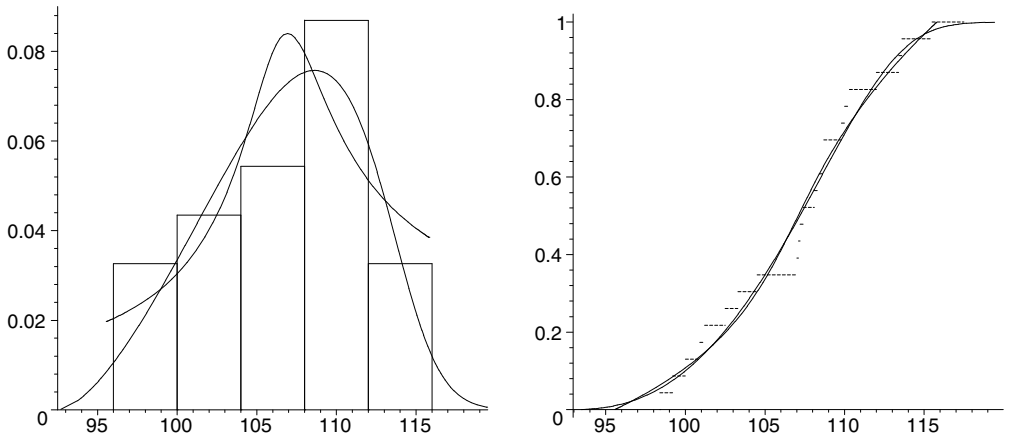
For the GLD_2 fit we get, using the same class intervals as before, the expected frequencies

$$6.5104, \quad 9.7900, \quad 7.7311, \quad 7.3586, \quad 5.7112, \quad 5.8987,$$

yielding a chi-square statistic and p-value of

$$1.5578 \quad \text{and} \quad 0.2120,$$

respectively. In this case, the Kolmogorov-Smirnov statistic, $KS_2 = 0.06649$ and $\sqrt{n}KS_2 = 0.4360$, giving a Kolmogorov-Smirnov test p-value of 0.99.



(a) (b)

Figure 6.5–2. Histogram of MRI scans the GLD_1 and GLD_2 (higher at center) p.d.f.s (a); the data e.d.f. with the d.f.s of GLD_1 and GLD_2 (b).

6.5.2 Example: Brain MRI Scan

This data, that is given in Section 3.5.3, has

$$\ell_1 = 106.8348, \quad \ell_2 = 2.7988, \quad \ell_3 = -0.1542, \quad \ell_4 = 0.2401.$$

Using the program `FitGLDL(Data)`, we get the two fits

$$\begin{aligned} &GLD_1(110.1128, 0.05672, 0.3810, 0.09888) \\ &GLD_2(105.6770, 0.09878, 2.5638, 5.0159). \end{aligned}$$

The supports of GLD_1 and GLD_2 are [92.48258, 127.7431] and [95.5536, 115.8003], respectively, and both supports cover the data range.

A data histogram, together with the GLD_1 and GLD_2 p.d.f.s, is shown in Figure 6.5–2 (a) (the curve that rises higher is the GLD_2 p.d.f.) and Figure 6.5–2 (b) gives the e.d.f. of this data with the d.f.s of GLD_1 and GLD_2.

To apply the chi-square test, we choose, as was done in Section 6.5.3, the class intervals

$$(-\infty, 103), \quad [103, 108) \quad [108, 111), \quad [111, \infty)$$

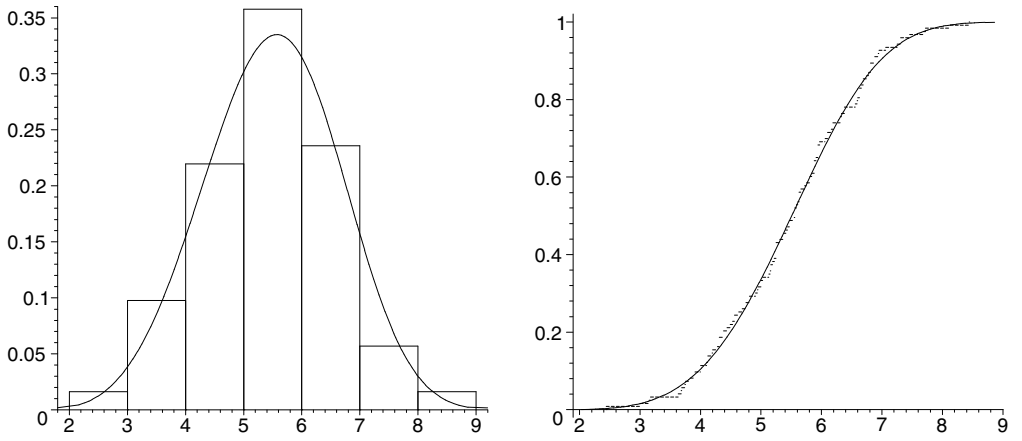
that have frequencies of

$$6, \quad 6, \quad 7, \quad 4.$$

In the case of the GLD_1 fit we get the expected frequencies

$$5.2867, \quad 7.6062, \quad 5.0914, \quad 5.0156$$

and the chi-square statistic 1.3565. For this fit, we have a Kolmogorov-Smirnov statistic of $KS_1 = 0.13835$, and with the sample size of $n = 23$, $\sqrt{n}KS_1 = 0.6635$.



(a) (b)

Figure 6.5–3. Histogram of birth weights X and fitted GLD p.d.f. (a); e.d.f. of X with the d.f. of the fitted distribution (b).

From Appendix J, we determine that the p-value of this Kolmogorov-Smirnov test is 0.77.

For the GLD_2 fit we get, using the same class intervals, the expected frequencies

$$5.08793, \quad 8.1583, \quad 4.6602, \quad 5.0935,$$

yielding a chi-square statistic of 2.14340. In this case, the Kolmogorov-Smirnov statistic, KS_2, is 0.1458, $\sqrt{n}KS_2 = 0.70$, and the p-value of this test is 0.71.

6.5.3 Example: Human Twin Data

This data, introduced in Section 3.5.4, consists of birth weights (X, Y) of twins, where X is the weight of the first-born and Y is that of the second-born. For X,

$$\ell_1 = 5.4859, \quad \ell_2 = 0.6535, \quad \ell_3 = -0.009286, \quad \ell_4 = 0.07140,$$

and through `FitGLDL` we obtain the single fit

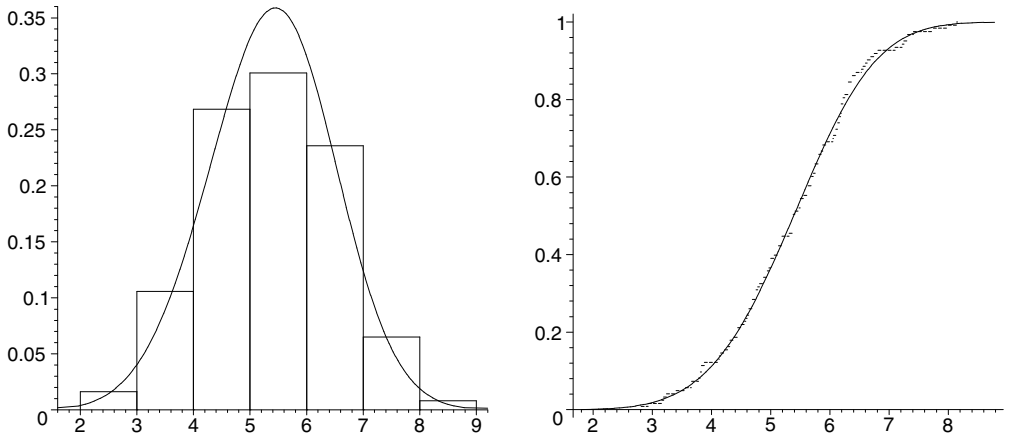
$$\text{GLD}(5.6319, 0.2225, 0.2130, 0.1670)$$

whose support [1.1379, 10.13292] covers the data range.

From the histogram of X with the p.d.f. of the fitted GLD, shown in Figure 6.5–3 (a), this fit seems to be a reasonably good one (Figure 6.5–3 (b) shows the d.f. of the GLD with the e.d.f. of X).

For a chi-square test we use the class intervals

$$(-\infty, 4), \ [4, 4.5), \ [4.5, 5), \ [5, 5.4), \ [5.4, 5.8), [5.8, 6.2), \ [6.2, 6.8), \ [6.8, \infty)$$



(a) (b)

Figure 6.5–4. Histogram of birth weights Y and fitted GLD p.d.f. (a); e.d.f. of Y with the d.f. of the fitted distribution (b).

with frequencies

$$12, \quad 15, \quad 12, \quad 17, \quad 16, \quad 19, \quad 16, \quad 16,$$

to obtain the expected frequencies

$$12.9006,\ 11.9345,\ 16.5959,\ 15.7109,\ 16.4040,\ 15.4116,\ 18.0908,\ 15.9517$$

and chi-square statistic and p-value, respectively, of

$$3.3160 \quad \text{and} \quad 0.3454.$$

The Kolmogorov-Smirnov statistic for this fit is $KS = 0.03952$ and $\sqrt{n}KS = 0.4383$, where $n = 123$ is the sample size of X. From Appendix J, the p-value for this test is 0.99.

For the birth weights of the second-born twins, Y, we have

$$\ell_1 = 5.3666, \quad \ell_2 = 0.6251, \quad \ell_3 = -0.01047, \quad \ell_4 = 0.07480,$$

that leads to the GLD fit

$$\text{GLD}(5.4932,\ 0.1948,\ 0.1674,\ 0.1347),$$

whose support $[0.3603,\ 10.6261]$ covers the data range. A histogram of Y with the p.d.f. of the fitted GLD is shown in Figure 6.5–4 (a) and the e.d.f. of Y with the d.f. of the GLD is shown in Figure 6.5–4 (b). These figures indicate a good fit.

For the chi-square test, we use the same intervals that were used for X to get observed frequencies of

$$15, \quad 11, \quad 19, \quad 17, \quad 16, \quad 19, \quad 16, \quad 10$$

and expected frequencies of

$$13.7844,\ 12.8713,\ 18.2029,\ 17.1481,\ 17.3945,\ 15.4623,\ 16.3875,\ 11.7490.$$

The chi-square statistic and p-value, respectively, for this test are

$$1.6062 \quad \text{and} \quad 0.6580.$$

The Kolmogorov-Smirnov statistic for this fit is $KS = 0.04440$, with $\sqrt{n}KS = 0.4924$ and a p-value of 0.97.

6.5.4 Example: Rainfall Distribution

The data for this example consists of rainfall measurements (in inches) in Rochester, NY (designated by X) and in Syracuse, NY (designated by Y). Section 3.5.5 has a listing of X and Y. For X, we have

$$\ell_1 = 0.4913, \quad \ell_2 = 0.3105, \quad \ell_3 = 0.1495, \quad \ell_4 = 0.06617,$$

and, through `FitGLDL`, the GLD fit

$$\text{GLD}(2.0337,\ 0.4916,\ 6.7089,\ 0.1262).$$

The support of this GLD is $[-0.00052, 4.07]$ and it covers the data range. Figures 6.5–5 (a) and (b), that show the a histogram of X with the p.d.f. of its fitted GLD and the e.d.f. of X with the d.f. of the GLD, respectively, indicate that we should have a good fit.

To perform a chi-square test, we use, as was done in Section 4.5.4, the intervals

$$(-\infty, 0.06), \quad [0.06, 0.1), \quad [0.1, 0.2), \quad [0.2, 0.45), \quad [0.45, 1), \quad [1, \infty),$$

with observed frequencies

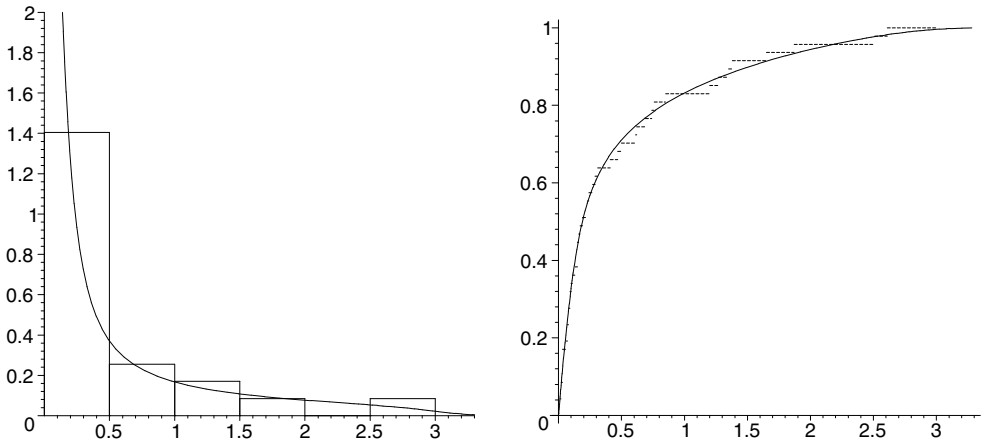
$$8, \quad 7, \quad 9, \quad 7, \quad 8, \quad 8$$

to get the expected frequencies

$$9.9944, \quad 5.4045, \quad 8.7645, \quad 8.3126, \quad 6.6144, \quad 7.9096$$

and chi-square statistic and p-value of

$$1.3739 \quad \text{and} \quad 0.2411,$$



(a) (b)

Figure 6.5–5. Histogram of rainfall X and fitted GLD p.d.f. (a); e.d.f. of X with the d.f. of the fitted distribution (b).

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.05730$ and $\sqrt{n}KS = \sqrt{47}KS = 0.3928$. From Appendix J we determine that the p-value of this test is 0.99.

For the Syracuse rainfall data, Y, the L-moments are

$$\ell_1 = 0.3906, \quad \ell_2 = 0.2008, \quad \ell_3 = 0.07778, \quad \ell_4 = 0.03654,$$

and through `FitGLDL` we obtain the fit

$$\text{GLD}(1.0161,\ 0.9983,\ 8.1400,\ 0.3626),$$

whose support $[0.01447, 2.0178]$ covers the range of Y. Figure 6.5–6 (a) shows a histogram of Y with the p.d.f. of this fitted GLD and Figure 6.5–6 (b) shows the e.d.f. of Y with the d.f. of the GLD.

For our chi-square test we take the intervals

$$(-\infty,\ 0.08),\quad [0.08,\ 0.15),\quad [0.15,\ 0.23),\quad [0.23,\ 0.38),\quad [0.38,\ 0.73),\quad [0.73,\ \infty)$$

and obtain the observed frequencies

$$7,\quad 7,\quad 11,\quad 6,\quad 8,\quad 8,$$

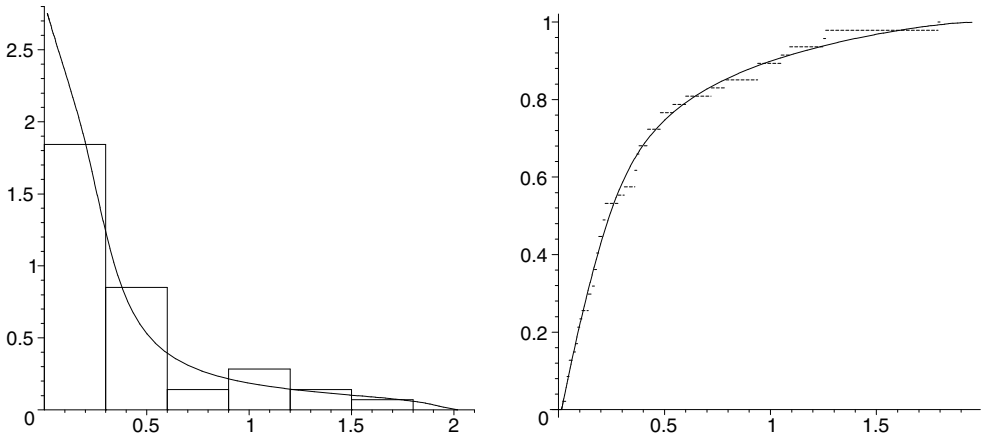
expected frequencies

$$7.9992,\quad 7.5116,\quad 7.1688,\quad 8.6318,\quad 7.9913,\quad 7.6973,$$

and chi-square statistic and p-value of

$$3.0215 \quad \text{and} \quad 0.08217.$$

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.07398$ with $\sqrt{n}KS = 0.5072$, leading to the p-value of 0.96.



(a) (b)

Figure 6.5–6. Histogram of rainfall Y and fitted GLD p.d.f. (a); e.d.f. of Y with the d.f. of the fitted distribution (b).

6.5.5 Example: Data Simulated from GBD(3, 5, 0, − 0.5)

This data, obtained through simulation from the generalized beta distribution with parameters 3, 5, 0, and −0.5, was introduced in Section 4.5.1. For this data we have

$$\ell_1 = 6.4975, \quad \ell_2 = 0.7984, \quad \ell_3 = -0.1903, \quad \ell_4 = 0.01103,$$

and, from `FitGLDL`, we obtain the GLD fit

$$\text{GLD}(3.2847,\ 0.21142, 0.01137, 2.2308)$$

with support $[-1.45, 8.01]$ that covers the data range. Figure 6.5–7 (a) shows a histogram of the data with the p.d.f. of the fitted GLD and Figure 6.5–7 (b) shows the e.d.f. of the data with the d.f. of the GLD.

For a chi-square test of this fit, we use, as in Section 4.5.1, the class intervals

$$(-\infty,\ 4.5),\ [4.5,\ 5.5),\ [5.5,\ 6.5),\ [6.5,\ 7.0),\ [7.0,\ 7.5),\ [7.5,\ 7.9),\ [7.9,\ \infty)$$

and obtain the observed frequencies

$$9, \quad 12, \quad 10, \quad 11, \quad 12, \quad 12, \quad 14$$

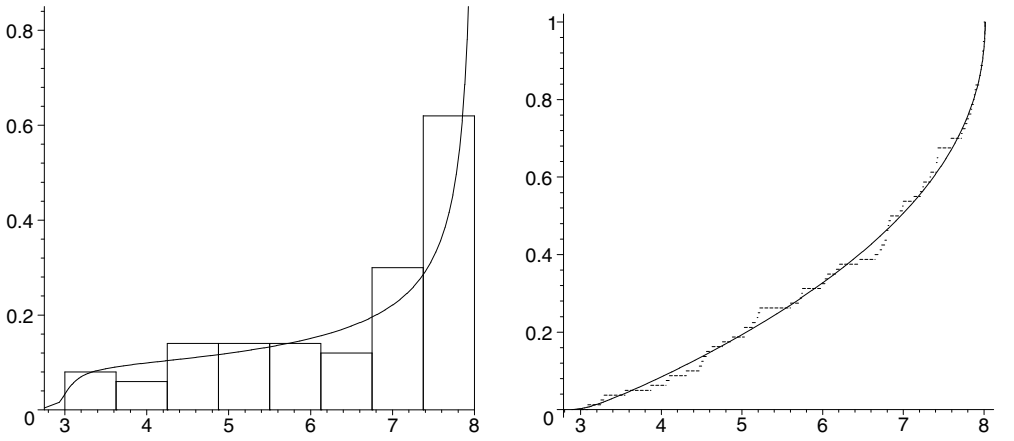
and the expected frequencies

$$10.9247, \quad 9.5923, \quad 12.1518, \quad 7.8586, \quad 10.5041, \quad 14.5172, \quad 14.4513,$$

yielding the chi-square statistic and p-value of

$$3.2438 \quad \text{and} \quad 0.1975,$$

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.06601$, making $\sqrt{n}KS = \sqrt{80}KS = 0.5904$ for a p-value of 0.87.



(a) (b)

Figure 6.5–7. Histogram of GBD$(3, 5, 0, -0.5)$ data and the p.d.f. of its fitted GLD (a); e.d.f. of the GBD$(3, 5, 0, -0.5)$ data and the d.f. of the fitted GLD (b).

6.5.6 Example: Data Simulated from GBD(2, 7, 1, 4)

The data for this example was generated from the GBD(2, 7, 1, 4) distribution and was introduced in Section 4.5.6. The L-moments of this data are

$$\ell_1 = 4.1053, \quad \ell_2 = 0.6276, \quad \ell_3 = 0.08993, \quad \ell_4 = 0.09960.$$

From `FitGLDL` we get two GLD fits

$$\text{GLD}(3.5251, 0.02989, 0.01042, 0.02844)$$

with support $[-28.86, 35.91]$. Figure 6.5–8 (a) shows a histogram of the data with this GLD p.d.f. and Figure 6.5–8 (b) shows the e.d.f. of the data with the d.f. the fitted GLD.

To perform a chi-square test for this fit, we choose (as in Section 4.5.2) the class intervals

$$(-\infty, 3.0), \quad [3.0, 3.5), \quad [3.5, 4.0), \quad [4.0, 4.5), \quad [4.5, 5.0), \quad [5.0, \infty)$$

and get the observed frequencies

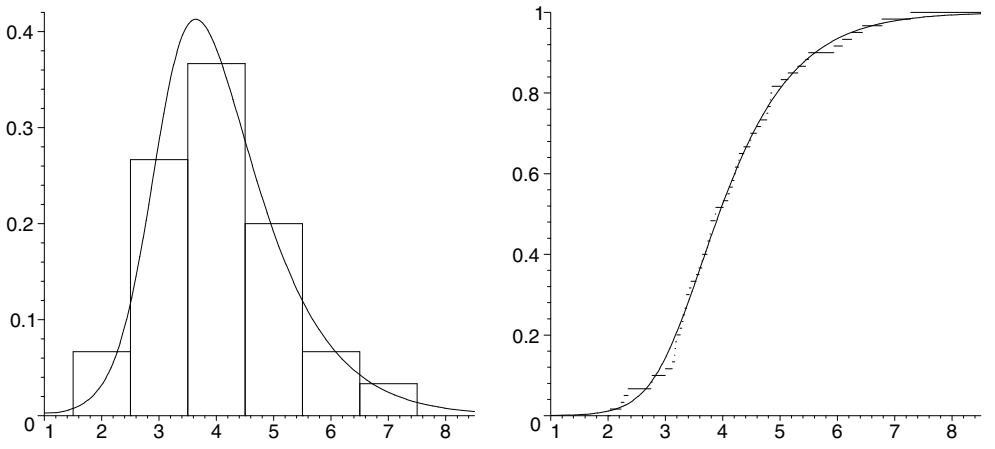
$$6, \quad 14, \quad 11, \quad 11, \quad 7, \quad 11$$

and the expected frequencies

$$8.5684, \quad 10.7460, \quad 12.1362, \quad 10.0877, \quad 7.1185, \quad 11.3431.$$

This chi-square test gives the chi-square statistic and associated p-value of

$$1.9565 \quad \text{and} \quad 0.1619,$$



(a) (b)

Figure 6.5–8. Histogram of GBD$(2, 7, 1, 4)$ data and the GLD_1 and GLD_2 (higher at its peak) p.d.f.s (a); e.d.f. of the GBD$(2, 7, 1, 4)$ data and the d.f. of the fitted GLD (b).

respectively. The Kolmogorov-Smirnov statistic for the GLD_1 fit is $KS_1 = 0.06296$ and, since the sample size is $n = 60$, we have $\sqrt{n}KS_1 = 0.4877$ and a p-value of 0.97.

6.5.7 Example: Tree Stand Heights and Diameters

This data, taken from forestry and discussed in greater detail in Section 4.5.5, consists of pairs (DBH, H) where DBH stands for the diameters (in inches) of trees at breast height and H stands for the heights of trees (in feet). For DBH, we have

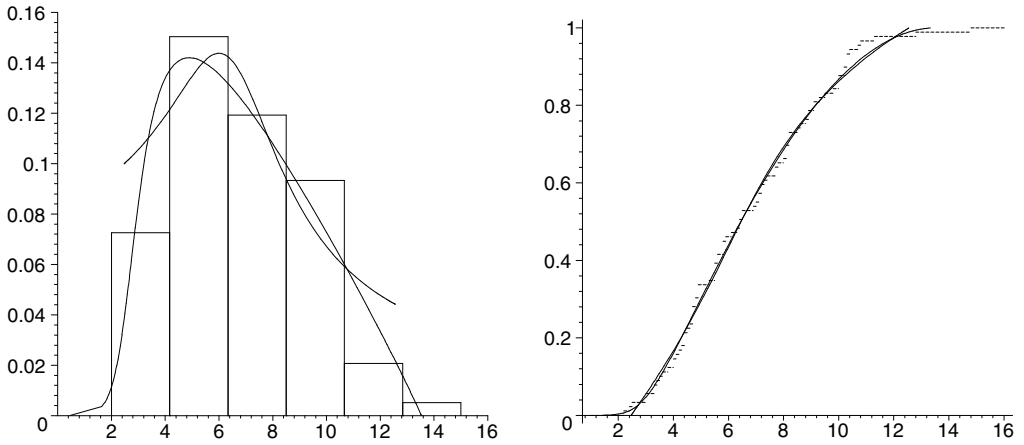
$$\ell_1 = 6.7404, \quad \ell_2 = 1.4780, \quad \ell_3 = 0.1298, \quad \ell_4 = 0.08624$$

and when `FitGLDL` is invoked, the two fits

$$\text{GLD}_1(3.7908, 0.1027, 0.03624, 0.5102)$$
$$\text{GLD}_2(7.5103, 0.1982, 4.4825, 1.9849).$$

are obtained. The minimum and maximum values of DBH are 2.2 and 14.8, respectively. The GLD_1 support, $[-5.95, 13.53]$, fails to cover this span on the right while GLD_2 support $[2.47, 12.55]$ fails to cover it at both extremes.

Figure 6.5–9 (a) shows a histogram of DBH with the p.d.f.s of GLD_1 and GLD_2 and, as we would expect, the GLD_2 p.d.f. is the one with the more limited support. Figure 6.5–9 (b) shows the d.f.s of GLD_1 and GLD_2 with the e.d.f. of DBH. If one could ignore the fact that GLD_1 does not cover the data range, one may consider GLD_1 a reasonable fit; however, GLD_2 seems to be a poor fit.



(a) (b)

Figure 6.5–9. Histogram of DBH and the p.d.f.s of GLD_1 and GLD_2 (with the smaller support) (a); e.d.f. of DBH and the d.f.s of GLD_1 and GLD_2 (b).

If, as in Section 4.5.5, the class intervals

$$(-\infty,\ 3.75),\ [3.75,\ 4.5),\ [4.5,\ 5.0),\ [5.0,\ 6.0),\ [6.0,\ 7.0),$$
$$[7.0,\ 8.0),\ [8.0,\ 9.0),\ [9.0,\ 10.25),\ [10.25,\ \infty)$$

are used for a chi-square test regarding the GLD_1 fit to DBH, we get the observed frequencies

$$10,\ \ 9,\ \ 11,\ \ 11,\ \ 7,\ \ 10,\ \ 12,\ \ 10,\ \ 9,$$

the expected frequencies

$$11.4039,\ 9.0472,\ 6.3069,\ 12.4097,\ 11.5409,\ 10.3013,\ 8.8607,\ 8.8564,\ 10.2730,$$

and chi-square statistic and $p$value of

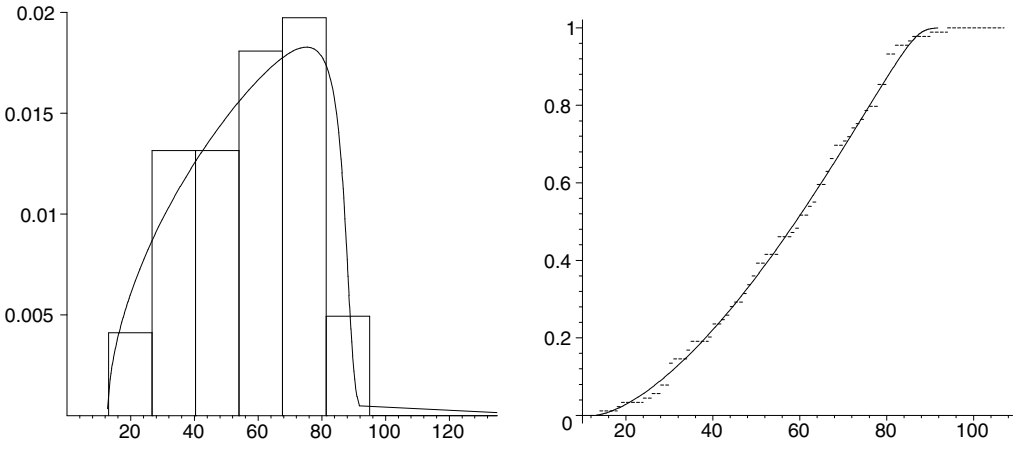
$$7.0387, \quad \text{and} \quad 0.1339,$$

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS_1 = 0.05063$ with $\sqrt{n}KS_1 = \sqrt{89}KS_1 = 0.4776$ and a p-value of 0.98.

If the judgment of the quality of this fit were being made on the basis of the chi-square and Kolmogorov-Smirnov p-values alone, we would find GLD_1 to be a rather good fit, in spite of its inappropriateness due to the limitation of its support.

A similar analysis for GLD_2, using the same class intervals that were used in connection with GLD_1, gives the expected frequencies

$$12.2369,\ 8.0681,\ 5.8173,\ 12.5034,\ 12.4374,\ 10.6035,\ 8.3765,\ 8.1097,\ 10.8473$$



(a) (b)

Figure 6.5–10. Histogram of H and the p.d.f. of its fitted GLD (a); the e.d.f. of H and the d.f. of the fitted GLD (b).

and chi-square statistic and p-value of

$$10.0488 \quad \text{and} \quad 0.03961,$$

respectively. For the GLD_2 fit, the Kolmogorov-Smirnov statistic is $KS_2 = 0.05696$ with $\sqrt{n}KS_2 = 0.5374$, yielding a p-value of 0.93.

To obtain L-moment fits for H, we note that

$$\ell_1 = 57.1348, \quad \ell_2 = 11.0570, \quad \ell_3 = -0.8635, \quad \ell_4 = 0.3934,$$

and we use `FitGLDL` to get the two fits

$$\text{GLD}_1(83.7797, 0.01406, 0.6394, 0.01570)$$
$$\text{GLD}_2(52.5943, 0.02786, 1.9015, 3.5836).$$

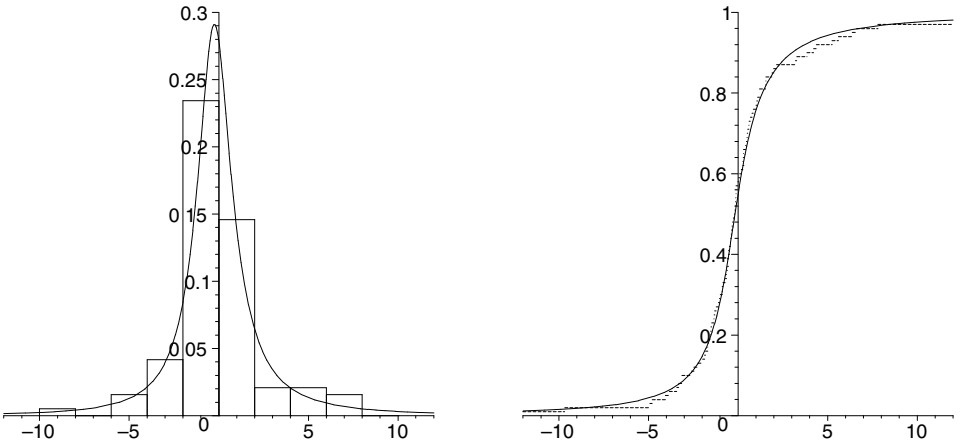
The support of GLD_1 is [12.64, 154.91] and it covers the data range. Since the support of GLD_2, [16.69, 88.49], does not cover the data range, we confine our attention to GLD_1. Figure 6.5–10 (a) shows a histogram of H with the p.d.f. of the fitted GLD_1 and Figure 6.5–10 (b) shows the e.d.f. of H with the d.f. of the GLD_1.

As was done in Section 4.5.5, we use the class intervals

$$(-\infty, 30), \quad [30, 37.5), \quad [37.5, 47.5), \quad [47.5, 55), \quad [55, 62.5),$$
$$[62.5, 67.5), \quad [67.5, 75), \quad [75, 80), \quad [80, \infty),$$

for a chi-square test. From the observed frequencies

$$7, \quad 10, \quad 11, \quad 9, \quad 11, \quad 11, \quad 9, \quad 8, \quad 13,$$



(a) (b)

Figure 6.5–11. Histogram of the Cauchy distribution data the p.d.f. of its fitted GLD (a); the e.d.f. of the data and the d.f. of the fitted GLD (b).

and the expected frequencies

9.6868, 7.2458, 11.6688, 10.0349, 10.9686, 7.7464, 12.0664, 8.0736, 11.5087

associated with these intervals we get the chi-square statistic and p-value of

$$4.2770 \quad \text{and} \quad 0.3698,$$

respectively. The Kolmogorov-Smirnov statistics for the GLD_1 fit is $KS = 0.06190$ and from $\sqrt{n}KS = 0.5840$ we get the p-value of 0.88.

6.5.8 Example: Data from the Cauchy Distribution

The data for this example, introduced in Section 5.6.1 and generated from a Cauchy distribution, has

$$\ell_1 = 0.3464, \quad \ell_2 = 2.3658, \quad \ell_3 = 0.4343, \quad \ell_4 = 1.3295.$$

Through `FitGLDL`, the fit

$$\text{GLD}(-0.2490, -1.1749, -0.6030 - 0.6893),$$

with support $(-\infty, \infty)$ is obtained. Figure 6.5–11 (a) shows a histogram of the data with the p.d.f. of the fitted GLD and Figure 6.5–11 (b) show the e.d.f. of the data with the d.f. of the GLD.

To do a chi-square test, we use the class intervals

$$(-\infty, -3.0), \ [-3.0, -1.5), \ [-1.5, -0.7), \ [-0.7, -0.4), \ [-0.4, 0.0),$$
$$[0.0, 0.4), \ [0.4, 0.7), \ [0.7, 1.5), \ [1.5, 3.0), \ [3.0, \infty)$$

that were used in Section 5.6.1. From the observed frequencies

$$10, \quad 12, \quad 12, \quad 10, \quad 14, \quad 8, \quad 8, \quad 7, \quad 6, \quad 13$$

and expected frequencies

$$8.9059,\ 11.1285,\ 15.1306,\ 8.1470,\ 11.5135,\ 10.1219,\ 6.0089,\ 10.4363,\ 8.5875,\ 10.0199$$

we get the chi-square statistic and p-value of

$$5.7108 \quad \text{and} \quad 0.3354,$$

respectively. The Kolmogorov-Smirnov statistic for this fit is $KS = 0.05145$ and the sample size n is 100, giving us $\sqrt{n}KS = 0.5145$ that indicates a p-value of 0.95.

6.5.9 Example: Radiation in Soil Samples

This data, introduced in Section 5.6.2, has

$$\ell_1 = 2384.8422, \quad \ell_2 = 1112.3770, \quad \ell_3 = 450.6757, \quad \ell_4 = 343.1844$$

and from `FitGLDL`, we get the fit

$$\text{GLD}(998.0320, -0.0003188, -0.07266, -0.3423)$$

that has support $(-\infty, \infty)$. Figures 6.5–12 (a) and (b) show, respectively, a histogram of the data with the p.d.f. of the fitted distribution and the e.d.f. of the data with the d.f. of the fitted GLD.

To perform a chi-square test, we use the class intervals

$$(-\infty, 700), \ [700, 1200), \ [1200, 1700), \ [1700, 2300), \ [2300, 3400), \ [3400, \infty),$$

the ones that were used in Section 5.6, to obtain the observed frequencies

$$7, \quad 6, \quad 6, \quad 6, \quad 8, \quad 7$$

and expected frequencies

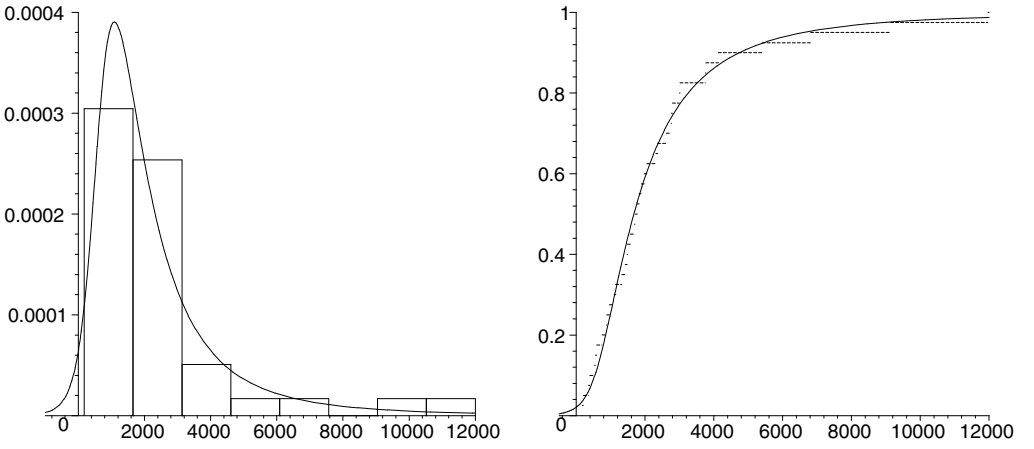
$$5.8213,\ 7.4440,\ 7.0323,\ 6.0771,\ 6.2126,\ 7.4126.$$

From these frequencies we get the chi-square statistic and p-value of

$$1.2085 \quad \text{and} \quad 0.2716,$$

respectively. The size of this sample is $n = 40$ and the Kolmogorov-Smirnov statistic for this test is $KS = 0.06630$, giving us $\sqrt{n}KS = 0.4193$ for a p-value of 0.99.

Although the chi-square p-value is good and the p-value of the Kolmogorov-Smirnov test is very good, we should note that the accumulated probability below zero for the fitted GLD is 0.021.



(a) (b)

Figure 6.5–12. Histogram of radiation in soil samples and the p.d.f. of its fitted GLD (a); the e.d.f. of the data and the d.f. of the fitted GLD (b).

6.5.10 Example: Velocities within Galaxies

This data, described in Section 5.6.3, has

$$\ell_1 = 21456.5882, \quad \ell_2 = 981.1686, \quad \ell_3 = -108.7784, \ell_4 = -68.8118.$$

When `FitGLDL` is invoked, `0, 0, 0, 0` is returned. When we look at the τ_3 and τ_4 of this data we find that $(\tau_3, \tau_4) = (-0.1109, -0.07013)$, an indication that an L-moment based GLD fit is not possible (recall that it was shown in Section 6.2 that the smallest value of τ_4 that could be represented by the GLD family is -0.01021).

6.6 Fitting Data Given by a Histogram

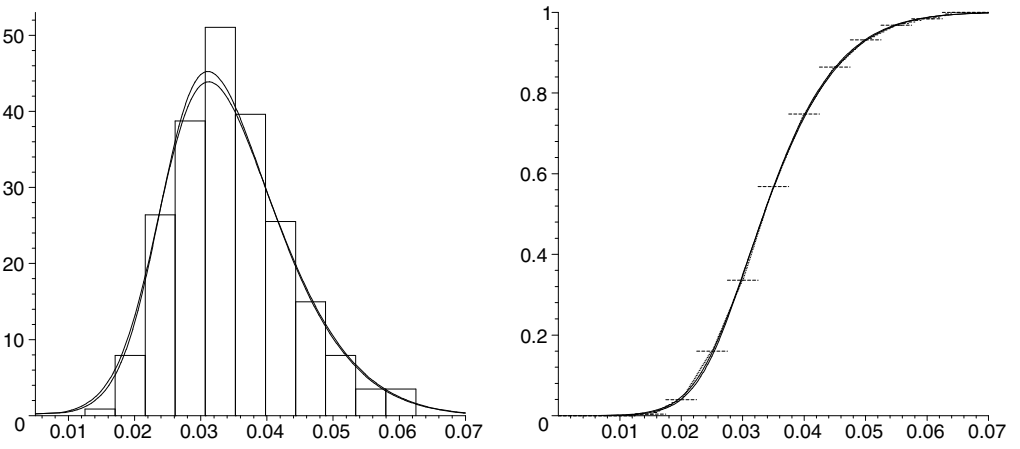
This data of coefficients of friction given in Section 3.6 (see Table 3.6–1) is in terms of frequencies of observations in 11 non-overlapping intervals. We have already considered two different options on how to interpret this data. Case 1: assume, as we did in Section 3.6, that all data within an interval is located at the center of the interval and Case 2: assume, as we did in Section 5.7, that the data within an interval is uniformly spread throughout the interval.

With the assumption of Case 1, we have

$$\ell_1 = 0.03448, \quad \ell_2 = 0.005328, \quad \ell_3 = 0.0005443, \quad \ell_4 = 0.0007216$$

and the fit

$$\mathrm{GLD}_1(0.02990, 13.5995, 0.04696, 0.1200),$$



(a) (b)

Figure 6.6–1. Histogram of coefficients of friction and the p.d.f.s of its fitted GLDs (a); the e.d.f. of the data and the d.f.s of the fitted GLDs (b).

whereas with the assumption of Case 2, we get

$$\ell_1 = 0.03448, \quad \ell_2 = 0.005453, \quad \ell_3 = 0.0005196, \quad \ell_4 = 0.0007312$$

and the fit

$$\mathrm{GLD}_2(0.03000, 14.0019, 0.05143, 0.1256).$$

In both cases, the supports of these fits cover the data spread.

Figure 6.6–1 (a) shows the histogram of the data with the p.d.f.s of GLD_1 and GLD_2 (the GLD_1 p.d.f. is higher at its peak). Figure 6.6–1 (b) shows the two e.d.f.s (under the Case 1 and Case 2 assumptions) with the d.f.s of GLD_1 and GLD_2. It is hard to see the e.d.f. for Case 2 since it agrees very closely with the GLD_1 and GLD_2 d.f.s

To apply the chi-square test for both fits, we choose the class intervals

$$(-\infty, 0.025), \quad [0.025, 0.03), \quad [0.03, 0.035), \quad [0.035, 0.04),$$
$$[0.04, 0.045), \quad [0.045, 0.05), \quad [0.05, \infty),$$

as was done in Section 3.6. For both GLD_1 and GLD2 this choice produces the observed frequencies

$$40, \quad 44, \quad 58, \quad 45, \quad 29, \quad 17, \quad 17.$$

(Note that the slight difference in the data, actually the interpretation of the data, could lead to different observed frequencies but that does not happen here.) The expected frequencies for GLD_1 are

$$37.1094, \ 49.5605, \ 55.1758, \ 44.3115, \ 29.6021, \ 17.3950, \ 16.8457,$$

leading to the chi-square statistic and associated p-value of

$$1.0269 \quad \text{and} \quad 0.5984,$$

respectively. For the GLD_2 fit, the expected frequencies are

$$39.0625,\ 48.3398,\ 53.5889,\ 43.8232,\ 29.7852,\ 17.8528,\ 17.5476,$$

with chi-square statistic and p-value of

$$0.8853 \quad \text{and} \quad 0.6423,$$

respectively.

For the GLD_1 fit, the Kolmogorov-Smirnov statistic is $KS_1 = 0.1240$ and since the sample size is $n = 250$, $\sqrt{n}KS_1 = 1.9606$ for a p-value of less than 0.005. For GLD_2, we have $KS_2 = 0.01752$, $\sqrt{n}KS_2 = 0.2770$ and a p-value of grater than 0.99.

References for Chapter 6

David, H. A. (1981). *Order Statistics*, John Wiley & Sons, Inc., new York.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C., and Wallis, J. R. (1979). "Probability weighted moments: Definition and relation to parameters of several distributions expressible in inverse form," *Water Resources Research*, 17, pp. 1049–1054.

Hosking, J. R. M. (1989). "Some Theoretical Results Concerning L-Moments," *Research Report RC14492*, IBM Research Division, Yorktown Heights, N.Y.

Hosking, J. R. M. (1990). "L-Moments: Analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society, Series B*, 52, pp. 105–124.

Hosking, J. R. M. and Wallis, J.R. (1997). *Regional Frequency Analysis*, Cambridge University Press, Cambridge, U.K.

Karian, Z. A. and Dudewicz, E. J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society,* 2, 171–187.

Karvanen, J. and Nuutinen, A. (2008). "Characterizing the generalized lambda distribution by L-moments," *Comput. Statistics & Data Analysis*, 52, 1971–1983.

Petersen, A. (2001). Personal communication based on his Master's Thesis, Frekvensanalyse av Hydrologisk og Meteorologisk Torke i Danmark (in Norwegian), Hovedoppgave ved Institutt For Geofysikk, University of Oslo, Norway.

Chapter 7

Fitting a Generalized Lambda Distribution Using a Percentile-KS (P-KS) Adequacy Criterion

B. Fournier
CEA, DEN/DANS/DMN/SRMA/LA2M,
Bat. 453, 91191 Gif sur Yvette Cedex, France
benjamin.fournier@cea.fr

N. Rupin
EDF R&D, Départ. Matériaux et Mécanique des composants,
avenue des Renardiéres, Ecuelles, 77818 Moret-sur-Loing Cedex, France

M. Bigerelle
Laboratoire Roberval, FRE 2833, UTC/CNRS,
Centre de Recherches de Royallieu, BP20529, 60205 Compiégne France

D. Najjar
LMPGM Laboratoire de Métallurgie Physique et Génie des Matériaux,
CNRS UMR 8517, Equipe Caractérisation et Propriétés de la Périsurface,
ENSAM, 8 Boulevard Louis XIV, 59046 Lille Cedex, France

A. Iost
LMPGM Laboratoire de Métallurgie Physique et Génie des Matériaux,
CNRS UMR 8517, Equipe Caractérisation et Propriétés de la Périsurface,
ENSAM, 8 Boulevard Louis XIV, 59046 Lille Cedex, France

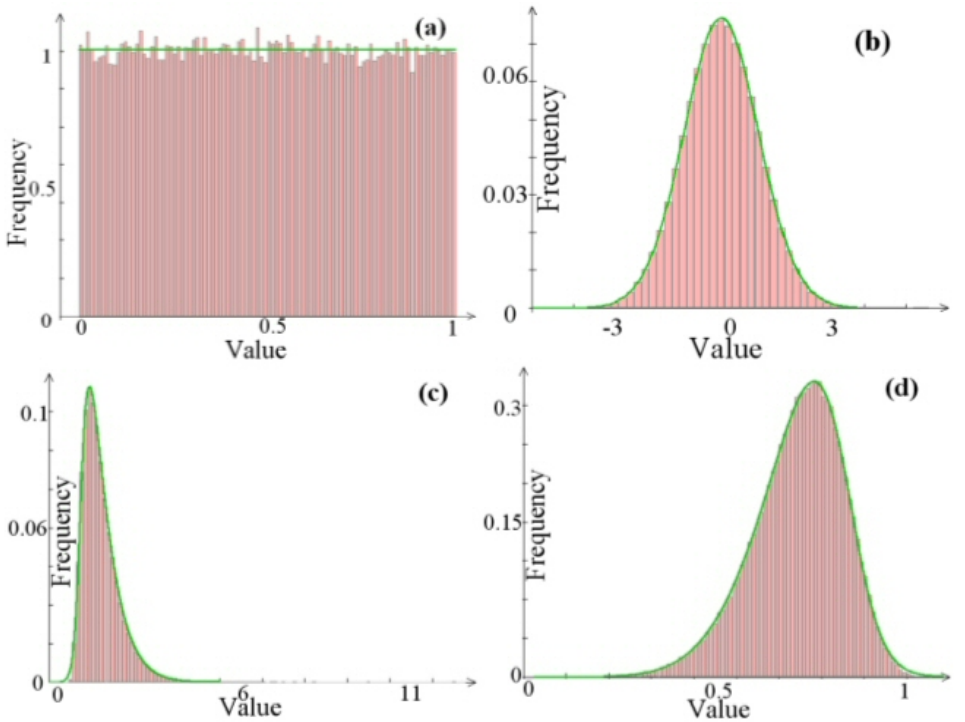


Figure 7.1–1. GLD modeling results: (a) uniform distribution, (b) standard Gaussian distribution, (c) a lognormal distribution, (d) a right-skewed Weibull distribution.

7.1 Introduction.

The family of four-parameter Generalized Lambda Distributions (GLD) is known for its high flexibility. It provides an accurate approximation of most of the usual statistical distributions (e.g., Gaussian, uniform, lognormal, Weibull) as illustrated in Figure 7.1–1 and detailed by Karian and Dudewicz (see Karian and Dudewicz (2000)). Generalized lambda distributions are used in many fields where precise data modeling is required such as finance (Tarsitano (2004) and Corrado (2001)), corrosion (Najjar, Bigerelle, Lefebvre, and Iost, (2003)), meteorology (Öztürk and Dale (1982)), fatigue of materials (Bigerelle, Najjar, Fournier, Rupin, and Iost, (2005)), independent component analysis (Eriksson, Karvanen, and Koivunen, (2000) and Karvanen, Eriksson, and Koivunen, (2002)), statistical process control (Pal (2005), Fournier, Rupin, Bigerelle, Najjar, and Iost (2006) and Negiz and Cinar (1997)), simulation of queue systems (Dengiz (1988)), or for generating random numbers (Stengos and Wu (2004) and Wilcox (2002)).

GLDs are often used to model empirical data and several methods for estimating its parameters are available. Generally, these estimators use complex minimization techniques. Several estimates of the parameters of a GLD have been

devised such as the method of moments, the method of percentiles, the "least square" method, and the goodness-of-fit methods (see Karian and Dudewicz (2000), King and MacGillivray (1999), Lakhany and Mausser (2000), Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979) and Öztürk and Dale (1985)). A thorough study of the method of moments and the method of percentiles is provided in Karian and Dudewicz (2003), where it is concluded that the percentile method performs best over a wide range of situations. As highlighted by Karian and Dudewicz, several sets of parameters can always be found using the method of moments or the method of percentiles. To choose between these minimal goodness-of-fit tests or an expert decision must be carried out. The "starship method," developed by King and MacGillivray (1999), is based on the minimization (in a four dimensional space 4D) of a goodness-of-it criterion and thus can be fully automated. One of the main advantages of an automated and relatively fast fitting method is that it allows thorough studies such as the influence of the sample size on the determination of parameters. However, such automatic methods based on goodness-of-fit become impractical with large sample sizes, due to high execution time.

The fitting method presented here is fully automated and ensures a given level of adequacy (the code is given in the appendix at the end of this chapter). In addition, as the minimization space is reduced to a 2D space, the computation time remains reasonable. The present chapter starts by a brief summary of the definition of GLDs. The usual fitting methods are briefly described and the new estimator is then detailed. Results previously obtained with this method (see Fournier, Rupin, Bigerelle, Najjar, and Iost (2007)) on the effect of sample size are summarized. Finally, three examples taken from Karian and Dudewicz (2000) are studied using the new method and compared with the results obtained with the method of moments.

7.2 The Generalized Lambda Distributions

Lambda distributions were introduced by Tukey (see Tukey (1962)), which were subsequently generalized to GLDs, as a family of four-parameter statistical distributions (Filliben (1969) and (1975), Joiner and Rosenblatt (1971), Ramberg and Schmeiser (1972) and (1974)). This section summarizes its central features, reviews extant estimation techniques, and then introduces a new estimation method.

7.2.1 Definitions

Here, GLDs are defined and some of their properties are briefly reviewed. For more details, see Karian and Dudewicz (2000). A Generalized Lambda Distribution, denoted by $\mathrm{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, can be described in terms of a percentile function, Q, the inverse of the cdf (cumulative distribution function), F by

$$Q(y) = Q(y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \tag{7.2.1}$$

where $y \in [0, 1]$, and λ_1 and λ_2 are, respectively, location, scale parameters, and λ_3 and λ_4 are, respectively, related to skewness and the kurtosis. The probability density function (pdf) of the GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) is given by

$$f(x) = f(Q(y)) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}. \tag{7.2.2}$$

Equation (7.2.2) defines a pdf if and only if f meets the following conditions:

$$\forall x \in D \quad f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = 1, \tag{7.2.3}$$

with D the domain of definition of f. These conditions lead to the specification of six regions of the (λ_3, λ_4) space in which equation (7.2.2) defines a valid pdf (see Karian and Dudewicz (2000)).

$$\text{Region 1} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \leq -1, \lambda_4 \geq 1\}, \tag{7.2.4}$$

$$\text{Region 2} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \geq 1, \lambda_4 \leq -1\}, \tag{7.2.5}$$

$$\text{Region 3} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \geq 0, \lambda_4 \geq 0\}, \tag{7.2.6}$$

$$\text{Region 4} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \leq 0, \lambda_4 \leq 0\}, \tag{7.2.7}$$

$$\text{Region 5} = \left\{(\lambda_3, \lambda_4) \mid -1 < \lambda_3 < 0, \lambda_4 > 1, \frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4 - \lambda_3)^{\lambda_4 - \lambda_3}} (\lambda_4 - 1)^{\lambda_4 - 1} < \frac{-\lambda_3}{\lambda_4} \right\} \tag{7.2.8}$$

$$\text{Region 6} = \left\{(\lambda_3, \lambda_4) \mid \lambda_3 > 1, -1 < \lambda_4 < 0, \frac{(1-\lambda_4)^{1-\lambda_4}}{(\lambda_3 - \lambda_4)^{\lambda_3 - \lambda_4}} (\lambda_3 - 1)^{\lambda_3 - 1} < \frac{-\lambda_4}{\lambda_3} \right\}. \tag{7.2.9}$$

In addition, the GLD defined by a set of parameters included in these 6 valid regions, can exhibit either bounded (they are defined on a finite support $[a, b]$) or unbounded densities either on both sides (they are defined on an infinite support $]-\infty, +\infty[$) or on one side (they are defined on $[a, +\infty[$ or $]-\infty, b]$. (These latter cases will not be tackled in this chapter). These cases and the associated restrictions on the parameters are summarized in Table 7.2–1.

7.2.2 Existing Parameter Estimation Methods

As previously noted, several estimates of the parameters of a GLD have been devised. A thorough study of the method of moments and the method of percentiles is provided in Karian and Dudewicz (2003). Using a L^2 Norm the authors conclude that the *method of percentiles is superior to the method of moments over a*

Table 7.2–1. Definition domains of the GLD for various combinations of λ_3 and λ_4 (from Karian and Dudewicz (2000)).

λ_3	λ_4	Support
$\lambda_3 < -1$	$\lambda_4 > 1$	$]-\infty, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 1$	$\lambda_4 < -1$	$[\lambda_1 - 1/\lambda_2, \infty[$
$-1 < \lambda_3 < 0$	$\lambda_4 > 1$	$]-\infty, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 1$	$-1 < \lambda_4 < 0$	$[\lambda_1 - 1/\lambda_2, \infty[$
$\lambda_3 > 0$	$\lambda_4 > 0$	$[\lambda_1 - 1/\lambda_2, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 0$	$\lambda_4 = 0$	$[\lambda_1 - 1/\lambda_2, \lambda_1]$
$\lambda_3 = 0$	$\lambda_4 > 0$	$[\lambda_1, \lambda_1 + 1/\lambda_2]$
$\lambda_3 < 0$	$\lambda_4 < 0$	$]-\infty, \infty[$
$\lambda_3 < 0$	$\lambda_4 = 0$	$]-\infty, \lambda_1]$
$\lambda_3 = 0$	$\lambda_4 < 0$	$[\lambda_1, \infty[$

broad range of (α_3, α_4)*-space*, (with α_3 and α_4 being respectively the third and fourth moments).

Regarding the goodness-of-fit method of estimation, the "starship" method, proposed by King & MacGillivray (1999), which minimizes a goodness-of-fit criterion on a 4D (four dimensional) space, can be used. (A direct minimization of a four-variable $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ function is carried out). This method can be tremendously time consuming (especially for large sample sizes). Indeed, due to the numerous local minima that appear in practice, a very precise meshing of the 4D space must be used. To reduce the computation time, Lakhany & Mausser (2000) proposed an interesting iterative method. Instead of calculating the goodness-of-fit estimator on a full 4D grid, they used successive simplices from random starting points until the goodness-of-fit test stops rejecting the model. This results in an acceptable estimate of the parameters, but not necessarily the values that provide the best fit. And compared to the estimates that provide the best fit, bias and standard errors can be relatively high.

7.2.3 A New P-KS Method

In essence, there are two types of estimation strategies when dealing with a GLD. The first type estimates the parameters with something like the method of moments or the percentile method, and then checks on the results are performed with a goodness-of-fit test. This results in a quick estimate of the parameters but it is possible that the goodness-of-fit test will reject. The second type uses a goodness-of-fit criterion. More precisely, the strategy is to find the set of parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ that give the lowest value of, for example, the Kolmogorov-Smirnov estimator E_{KS} defined by

$$E_{KS} = \max_{x \in D} \mid \hat{F}_n(x) - F(x) \mid, \tag{7.2.10}$$

where D and F have the same definitions as in Section 7.2.1, and $\hat{F}_n$ is the empirical distribution function (edf). However, this approach requires expensive computation times, particularly when the sample size is large. In order to deal with this problem, the authors resorted to a new method that combines the two types of estimation methods just described.

Briefly, to reduce execution time, rather than minimizing E_{KS} as a function of four parameters, minimization is performed on a two-dimensional grid only. More precisely, the (λ_3, λ_4) space is discretized by a 2D square grid of $N \times N$ couples of values. Therefore, one must define the support $[\lambda_3^{min}, \lambda_4^{min}] \times [\lambda_3^{max}, \lambda_4^{max}]$ and the step s of the grid. Many techniques exist to optimize the grid in order to minimize the computation time, but they require a certain amount of *a priori* knowledge about the GLD. King & MacGillivray (1999) present some guidelines to build such an efficient grid. Nonetheless, the values of λ_3 and λ_4 are not sufficient to calculate E_{KS}: a set of (λ_1, λ_2) parameters must be associated with each set of the (λ_3, λ_4) parameters of the grid. The strategy here, to avoid a discretization of a full 4D space, only one pair of (λ_1, λ_2) parameters will be associated with each point of the grid, thanks to equations (7.2.11) and (7.2.12). As detailed below, these equations give respectively the values of λ_2 and λ_1 as the solutions of $\hat{\rho}_2 = \rho_2$ and $\hat{\rho}_1 = \rho_1$ (the expressions of ρ_1 and ρ_2 are given in Karian and Dudewicz (2000))

$$\lambda_2 = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\hat{\rho}_2}, \tag{7.2.11}$$

$$\lambda_1 = \hat{\rho}_1 - \frac{\left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}}{\lambda_2}, \tag{7.2.12}$$

with

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{7.2.13}$$

$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{7.2.14}$$

where u is chosen by the investigator such that $u \in]0, 0.25[$ and $\hat{\pi}_p$ is an estimate of the p^{th} quantile. These last four equations are directly taken from the percentiles method detailed in Karian and Dudewicz (2000). Instead of using the percentiles estimates, one could have derived the same method (a Kolmogorov-Smirnov distance minimized on a 2D grid) using the equations (2.2.1) Karian and Dudewicz (2000) that link λ_1 and λ_2 to the first two moments of the GLD $\hat{\mu}$ and $\hat{\sigma}^2$. However, as noted by Karian and Dudewicz, the additional degree of freedom offered by the parameter u enables better fit in various cases. Naturally, the choice of u influences the final set of parameters obtained.

For each point (i, j) of the grid, E_{KS} is calculated between the empirical data and the GLD with parameters $(\lambda_1(i,j), \lambda_2(i,j), \lambda_3(i,j), \lambda_4(i,j))$. The minimal value of E_{KS} is determined over the whole grid and then a simplex is started from the grid point where this minimal value occurs (see Nelder and Mead (1965)). This minimization method is used because it does not require the computation of complex derivatives and also to ensure some consistency with the work done by previous authors (Ramberg, Dudewicz, Tadikamalla, and E. Mykytka (1979) and Öztürk and Dale (1985)). Figure 7.2–2 presents the response surface obtained for a series of 10^4 standard Gaussian values. Notice that the surface is very smooth and that in this particular case, there is apparently a unique minimum in the range of (λ_3, λ_4) values tested.

From Section 7.2.1, several considerations limit the range of admissible values of the GLD parameters. An additional issue is that the support of a GLD can be either bounded (it is defined on a finite support $[a, b] \subset \mathbb{R}$) or unbounded (it is defined on $\mathbb{R}$). Each case corresponds to different domains of (λ_3, λ_4) values as pointed out in Table 7.2–1. The choice between bounded or unbounded GLD in practical modeling will be discussed in Section 7.4. Here, if a bounded version is specified, the algorithm is restricted to explore only the values of λ_3 and λ_4 corresponding to a bounded GLD, which, from Table 7.2–1, leads to the conditions $\lambda_3 > 0$ and $\lambda_4 > 0$. Moreover, to ensure that the modeling will be defined at least on all the empirical data (of the dataset S) the two following conditions are imposed:

$$\lambda_1 - \frac{1}{\lambda_2} \leq \min(S), \tag{7.2.15}$$

$$\lambda_1 + \frac{1}{\lambda_2} \geq \max(S). \tag{7.2.16}$$

These conditions can of course be relaxed, however, it is consistent with goodness-of-fit tests like the χ^2 test. Indeed, the value of the test statistic is infinite when the theoretical frequency is null and not the observed frequency. Therefore, the adequacy would systematically be rejected in the case where either equation (7.2.15) or (7.2.16) would not be satisfied.

It should be noted that this new estimator minimizes E_{KS} given that two equations ($\hat{\rho}_2 = \rho_2$ and $\hat{\rho}_1 = \rho_1$) of the percentile method are satisfied (the algorithm for obtaining E_{KS} is depicted in Figure 7.2–3). However, the resulting estimate is not necessarily optimal in terms of minimizing the Kolmogorov-Smirnov distance because the explored values of λ_1 and λ_2 are constrained by these two equations. But the adequacy obtained with this 2D method was compared with the starship method using a very fine 4D meshing in several simulations. In all cases, the values of the four parameters were only slightly different, and the differences in E_{KS} never exceeded two percent in the case of the 4D method. Additionally, when plotting the two GLD distributions obtained both by a 2D and

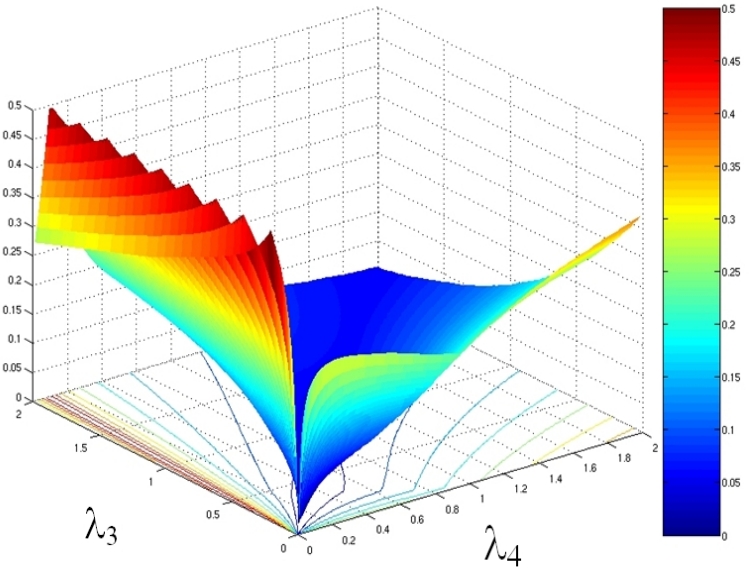


Figure 7.2–2. Values of the E_{KS} distance in the (λ_3, λ_4) space obtained for the modeling of 10^4 standard Gaussian data.

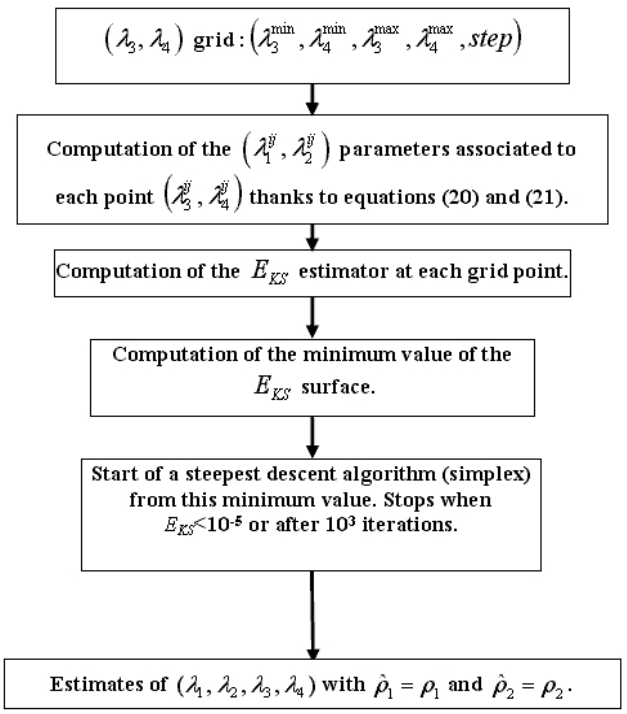


Figure 7.2–3. Flowchart illustrating the algorithm proposed for the new method of calculation of the GLD parameters.

a 4D minimization, no visible difference was seen. Therefore, these two methods appear to lead to very comparable GLDs, but the new 2D method is faster.

7.3 GLD Modeling of Data Coming from a GLD

In this section, the main results obtained in Fournier, Rupin, Bigerelle, Najjar, and A. Iost (2007) concerning the sampling distribution of the proposed estimator is studied. We begin with the situation where data are generated from a GLD.

7.3.1 Results on the Choice of u

When using the percentile method, the choice for u is usually 0.1 which, Karian and Dudewicz (2000) say, is a *"good compromise."* Indeed, for small sample sizes, u must not be too small in order to allow the calculation of $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$ involved in equation (7.2.14), and thus must satisfy $(n+1)u \geq 1$ (where n is the sample size). Nevertheless, for large sample sizes this leads to a wide range of possible values of u:

$$\frac{1}{n+1} \leq u < 0.25. \tag{7.3.1}$$

Evidently, a comprehensive study regarding the choice of u, has not been published, and so this issue is addressed here using simulations studies based on 10^4 replications and sample sizes n. We begin with the situation where data are generated from GLD$(0, 0.19, 0.14, 0.14)$ (these specific values of the GLD parameters were chosen to study a symmetric distribution close to the standard Gaussian distribution (see Figure 7.3–1)). Of course, the generality of the results are limited, but this choice was made to illustrate tendencies in cases where the actual GLD parameters are *not too close* to the boundaries of the intervals given by (equations (7.2.4) through (7.2.9)). Here, n is taken to be 10^4. Results on other sample sizes are reported in Section 7.3.2.

For the four estimators of λ_1, λ_2, λ_3, λ_4, the value of u has very little influence on their mean value. Moreover, it does not affect much the width of the 95% confidence interval for the parameters λ_3 and λ_4. Figure 7.3–2 shows a plot the .95 confidence intervals (the .025 and .975 quantiles of the sampling distribution) and the mean values of the four estimators of the first two parameters for different values of u. The confidence interval of the location parameter λ_1 is found to widen substantially as the value of u decreases. In the case of the scale parameter λ_2, estimated with relatively high precision, the 95% confidence interval width is slightly smaller for small values of u. More specifically, the lower .025 quantile of the sampling distribution, when estimating λ_2, is higher which has a direct and strong effect on the support of the obtained GLD as shown in Figure 7.3–3 a).

Indeed, as shown in Figure 7.3–3 a), the 95% confidence interval is much thinner for low values of u. This can be understood since, the lower u is, the

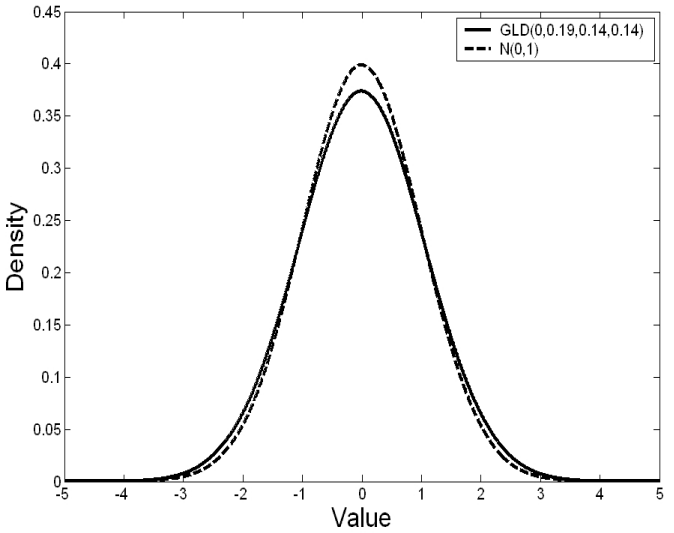


Figure 7.3–1. The pdf of the GLD$(0, 0.19, 0.14, 0.14)$ compared to the standard Gaussian one.

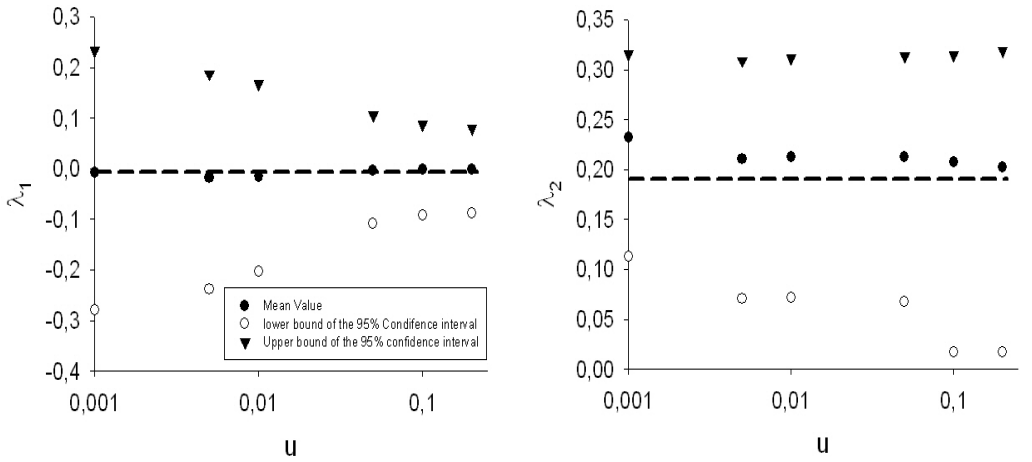


Figure 7.3–2. 95% confidence interval and mean values of a) λ_1 and b) λ_2 for 10^4 datasets of 10^4 values coming from GLD$(0, 0.19, 0.14, 0.14)$ and approached by a GLD using different values of u (the theoretical GLD parameters are indicated by the horizontal dotted lines).

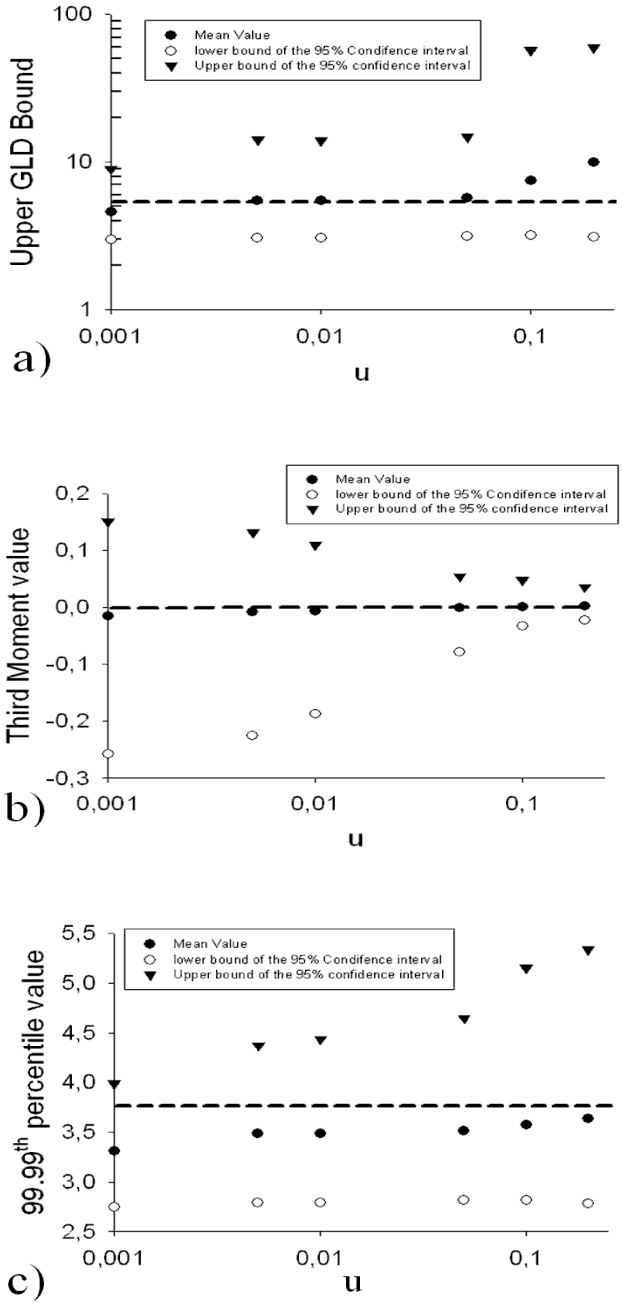


Figure 7.3–3. 95% confidence interval of (a) the upper bound of the GLD support (as defined in Table 7.2–1), b) the third moment of the modeled GLD, and c) the 99.99th percentile of the modeled GLD, for 10^4 datasets of 10^4 values coming from GLD$(0, 0.19, 0.14, 0.14)$ and approached by a GLD using different values of u (the corresponding theoretical values of the GLD$(0, 0.19, 0.14, 0.14)$ are indicated by horizontal dotted lines.

more weight is given to the tails in the fitting process. That is why the precision on the bounds of the distribution is, generally speaking, better for low values of u (because $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$ give information about extreme percentiles that are close to the actual bounds of the distribution. Whatever the accuracy of their estimation, these percentiles constrain the bounds and prevent them from being too far apart).

The accuracy of the estimated moments and percentiles is also an interesting indication of the modeling ability of a GLD.

- Concerning the first two moments and the fourth, the choice for u was found to have no significant influence. However, as illustrated by Figure 7.3–3, the skewness is slightly underestimated. These results show that using very small values for u significantly decreases the modeling accuracy as far as properties of symmetry are concerned. Evidently, this result comes from the fact that when u is very small, the estimates of the percentiles $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$ are extreme ones, so they have a relatively large standard error (due to the fact they are estimated on a little amount of data). So in this case, a value of u sufficiently high should be used to ensure that the corresponding percentiles can be estimated with relatively high precision.

- As for estimating the percentiles, u was found to have no significant influence on central percentiles (such as the 50th or 90th). For low values of u, extreme percentiles (e.g., the 99.99th) are significantly underestimated, whereas, as u increases, the bias decreases, but the confidence interval is larger.

- No significant influence of parameter u on the confidence interval of the E_{KS} value was found.

These results illustrate the difficulty in choosing u. There is no absolute best choice since the accuracy of the result depends on the final goal of the modeling process. The main guidelines are:

- The use of low u values (e.g. ≤ 0.01) is desirable if the focus is on the bounds. However, low values of u result in a widening of some confidence intervals (e.g. for λ_1 and the third moment).

- If we need to approach the theoretical distribution from which the empirical data come while minimizing the noise due to the estimation, it seems more appropriate to use values of u larger than 0.02 (in the present simulation, this limit will decrease if the sample size increases), since in the range $0.02 \leq u < 0.25$ the confidence intervals of the location parameter λ_1 and the third moment decrease significantly.

It is thus possible to optimize the value of u for a given criterion (e.g. the accuracy of the predicted bounds) but it may be at the expense of other properties.

Of course, the choice for u would benefit from a more comprehensive study, including the effect of the sample size. The next section reports results for a range of sample sizes that help provide perspective.

7.3.2 Influence of the Sample Size

Here, the value of u will be fixed at 0.1, as done in Karian and Dudewicz (2000). Simulations are performed as previously described, only now a wide range of sample sizes are used. In each case the values of the four GLD parameters and the level of adequacy (the value of E_{KS}) are recorded. Here, $10 \leq n \leq 10^7$ is considered.

Figure 7.3–4 presents the mean values of the estimates of the GLD parameters and the 95% confidence intervals of their sampling distributions. As indicated in Figure 7.3–4 the estimation of the GLD parameters, based on the new method, is approximately unbiased for $n \geq 10^3$, meaning that $(|E(\hat{\lambda}_i) - \lambda_i| \leq 10^{-3})$. Additionally, for all four parameters, the sampling distribution of the estimators are almost symmetrically distributed when $n \geq 2.10^3$. For $n < 10^3$ the four estimated parameters are biased and their distribution is more or less skewed. This bias may partly be explained by the various constraints to which the parameters are submitted (Karian and Dudewicz (2000)). The influence of the bounds of the admissibility domains for the parameters is particularly visible for the last three parameters. Indeed, in Figures 7.3–4 b) through 7.3–4 d), the distributions of these parameters are strictly positive. Theoretically, λ_3 and λ_4 could be either positive or negative. Nevertheless, as the data is known to come from a bounded distribution, λ_3 and λ_4 were restricted to positive values and equations (7.2.15) and (7.2.16) hold.

These conditions and that of Section 7.2.1 strictly limit the range of admissible GLD parameters and may also be responsible for the bias on the parameters estimated on a low number of empirical data.

Based on results shown in Figure 7.3–4 a) and 7.3–4 b), the upper bound of the GLD support, which can be calculated thanks to Table 7.2–1, as a function of the sample size, is shown in Figure 7.3–5. As expected, for small sample sizes, the confidence interval is very wide and non-symmetrical. That is, the precision of the estimate is relatively poor.

7.3.2.1 Adequacy

Figure 7.3–6 presents the E_{KS} distance as a function of the sample size and also shows the .025 and .975 quantiles of the sampling distribution. The decrease is almost proportional to $1/\sqrt{n}$, with an excellent correlation, as shown in Table 7.3–7. The width of the 95% confidence interval decreases in the same way.

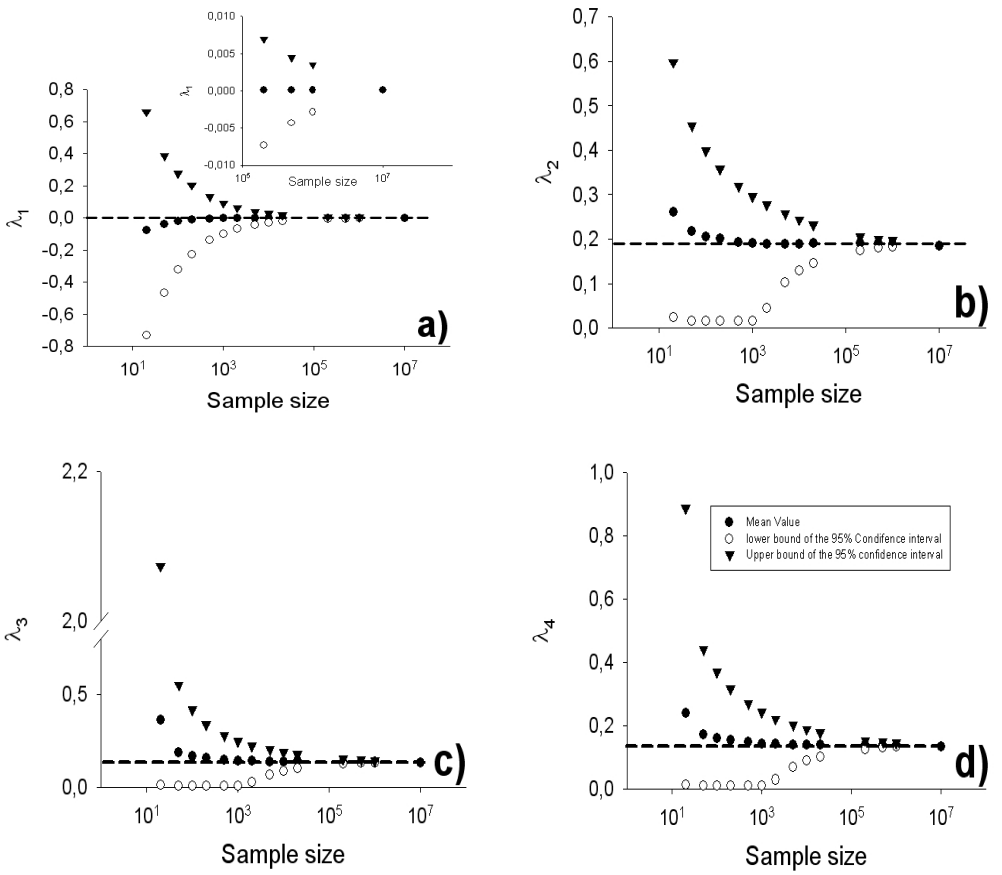


Figure 7.3–4. Influence of the sample size on the 95% confidence interval and on the mean value of the estimates of the four GLD parameters for various sample sizes and when sampling coming from GLD$(0, 0.19, 0.14, 0.14)$. (Actual values are indicated by the horizontal dotted line).

7.3.2.2 Moments and Percentile Modeling

Detailed results can be found in Fournier, Rupin, Bigerelle, Najjar, and Iost (2007). The main result is that the estimates of both moments and percentiles are reasonably unbiased for $n \geq 10^3$ and the difference between the .975 and .025 quantiles of the sampling distributions decreases approximately like $1/\sqrt{n}$ for $n \geq 2.10^3$.

7.3.2.3 Some Limits and Perspectives

As indicated by simulations, GLD estimates are biased but consistent when data are generated from a GLD model. A partial explanation for this bias lays in the

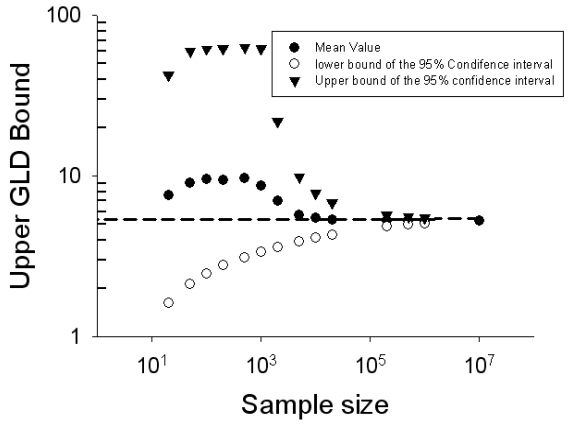


Figure 7.3–5. Influence of the sample size on the 95% confidence interval and on the mean value of the upper bound of the GLD support calculated on samples coming from $GLD(0, 0.19, 0.14, 0.14)$ and approached by a GLD (the theoretical value is indicated by the horizontal dotted line).

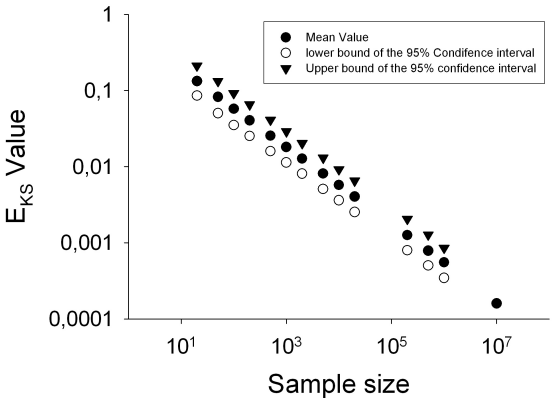


Figure 7.3–6. Influence of the sample size on the 95% confidence interval and on the mean value of the E_{KS} distance calculated between the samples coming from $GLD(0, 0.19, 0.14, 0.14)$ and their GLD modeling.

Table 7.3–7. Coefficients of the power relationship: $E_{KS} = \gamma_{i,1}.n^{\gamma_{i,2}}$, and the corresponding correlation.

	$\gamma_{i,1}$	$\gamma_{i,2}$	R
mean value	0.62	-0.51	$\simeq 1$
2.5th percentile	0.42	-0.53	0.9995
97.5th percentile	0.97	-0.51	$\simeq 1$

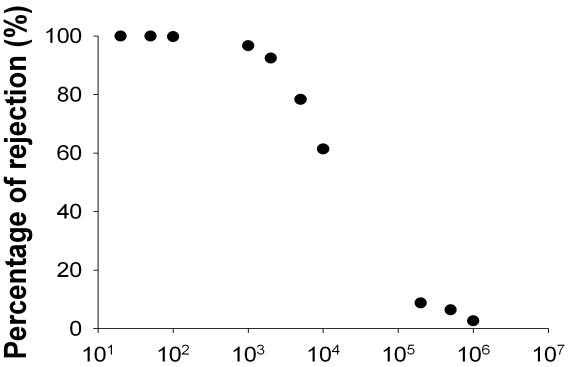


Figure 7.3–8. Percentage of rejected GLD modeling when compared to the *"true"* GLD from which the samples are selected.

various constraints on the estimates used in the algorithm. However, the intrinsically high versatility of GLDs is also partly responsible (e.g. several different sets of parameters can define results in GLDs whose shapes will be close to each other, Karian and Dudewicz (2000)). Bootstrap methods might reduce bias to a reasonable degree, but the extent to which this is true, and how it affects the standard errors of the estimates, remains to be determined.

The new method minimizes the distance between an empirical dataset and the GLD model. The obtained E_{KS} distances were plotted in Figure 7.3–6 and the Kolmogorov-Smirnov goodness-of-fit test never rejected the adequacy (between the empirical dataset and the modeling) at the 5% confidence level. Of course, because the true underlying distribution is known (GLD$(0, 0.19, 0.14, 0.14)$), it is possible to test the modeling adequacy with this true distribution.

For each set of GLD estimates, a large number (10^6) of data coming from the modeled GLD were compared to the estimated fit using a two-sample Kolmogorov-Smirnov test. The results of the Kolmogorov-Smirnov test at a 5% confidence level are plotted in Figure 7.3–8 and indicate that, for small sample sizes, the goodness-of-fit test never rejected.

7.4 Gaussian Data Approached by a GLD

Let us now study the ability of the GLD to approach data that does not correspond to a consistent case.

When dealing with data coming from a distribution that actually does not (i.e., the GLD cannot converge statistically towards this distribution) the following question arises: are GLDs able to accurately model such data? And, of course, up to what sample size? To answer these questions, exactly the same simulation as in Section 7.3.2 will be carried out except that the empirical data will come from a standard Gaussian distribution. There is no denying that the following

results will only answer the questions in the particular case of Gaussian data and that, when dealing with other types of data, the same simulations should be carried out again for such specific data (as done, for small sample sizes by King and MacGillivray (see King, and MacGillivray (1999)). Nevertheless, due to the wide applicability of Gaussian modeling, the following results still bear some generality.

As mentioned above, the GLD may have either a finite support (they are bounded both on the left and on the right) or an infinite support (they are unbounded). When dealing with experimental data different goals might well be looming ahead. On the one hand, if there are theoretical reasons or a sufficient experimental background to assert that the data are not bounded it may be better to model the data with an unbounded GLD (which corresponds to $\lambda_3 \leq 0$ and $\lambda_4 \leq 0$). On the other hand, if the data are known to come from a distribution that is bounded because of some physical limit (e.g., the depth of corrosion pits appearing on a steel sheet is limited on the left either by 0 or by the resolution of the measurement device and on the right by the thickness of the sheet), the modeling must also be bounded (for the GLD, this corresponds to $\lambda_3 > 0$ and $\lambda_4 > 0$). Additionally, when no theoretical consideration allows this kind of choice, and as the data available is always of finite values, bounded modeling with bounds far apart (but larger of course) from the experimental ones is often a good compromise.

So here, each sample is modeled by both a bounded and an unbounded GLD by the respective constraints ($\lambda_3 > 0$, $\lambda_4 > 0$) and ($\lambda_3 \leq 0$, $\lambda_4 \leq 0$). It must be noticed that, at least for small sample sizes, there is no rule so as to know which modeling is the more adequate (according to the Kolmogorov-Smirnov criterion). Indeed, as will be shown below, for small sample sizes if no constraint is put on the algorithm (the whole range of valid parameters is tested) the result may be either a bounded or an unbounded GLD. Consequently, to study the properties of these two possibilities separately, the previous constraints have been used.

7.4.1 Confidence Intervals

Figure 7.4–1 shows results on the size of the confidence intervals, again meaning the .025 and .975 quantiles of the sampling distribution of the four GLD estimators. As expected, the results are very similar to those presented in Figure 7.3–4. It is interesting to note, however, that here the values of the four estimators stabilize around the values $(0, 0.21, 0.142, 0.142)$ for sample sizes of up to 10^5. However, the values of the last three estimators decrease again for larger sample sizes (see the details inserted in Figure 7.4–1). This feature stems from equations (7.2.15) and (7.2.16). When $\lambda_2 = 0.21$, which is the value around which the distribution is centered for $n \leq 10^5$, the GLD is defined on $[-4.76, 4.76]$. As the sample size increases, the probability of generating very large or very small values (i.e., outside of the previous interval) in a given dataset increases, and thus

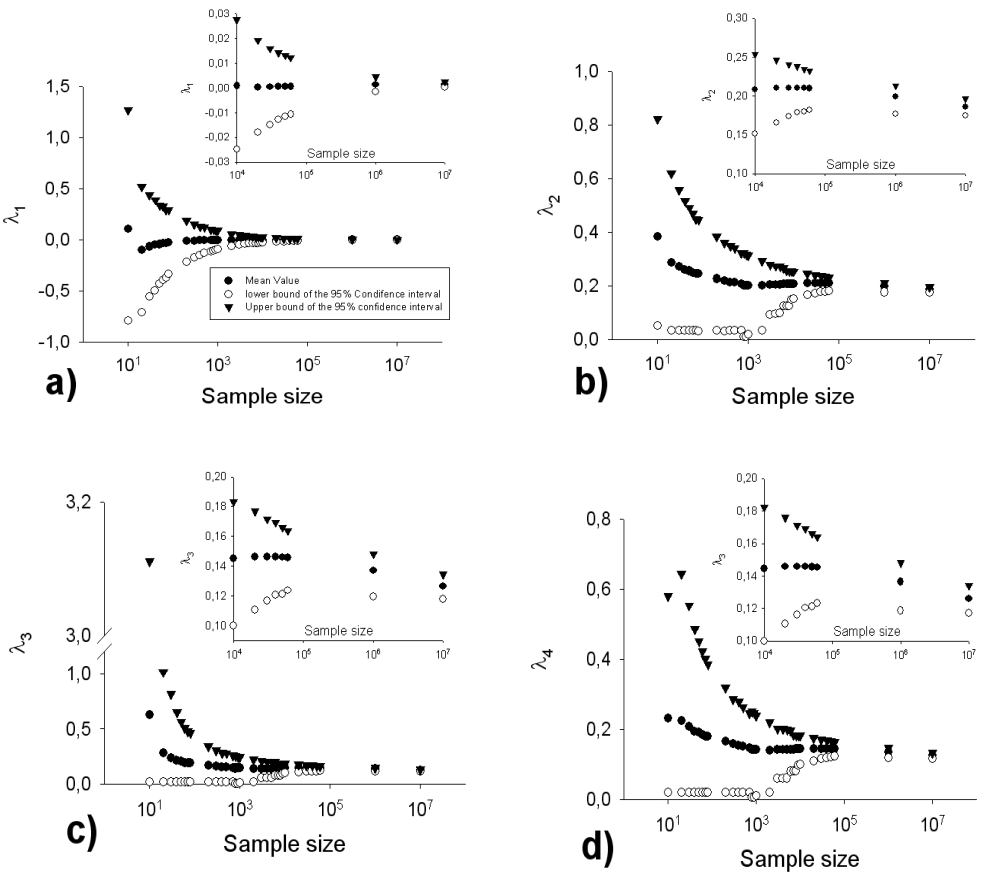


Figure 7.4–1. The sample size versus the .025 and .975 quantiles of the sampling distribution and the mean value of the four GLD estimators calculated on samples coming from $N(0,1)$ and based on a bounded GLD.

the bounds of the GLD must become wider. This implies that λ_2 must decrease (c.f. equations (7.2.15) and (7.2.16)). As λ_2 is linked to λ_3 and λ_4 by equation (7.2.11) it also implies a decrease in these two parameters.

Results on the estimates obtained in the unbounded case are plotted in Figure 7.4–2. First notice that the confidence interval for the last three parameters is very narrow for $n \geq 10^3$. The location parameter has a very similar dependence on the sample size as in the bounded case and also tends to 0 (or a slightly positive value). Also, the mean values continue to change up to $n = 10^6$.

7.4.2 Modeling Adequacy

Figure 7.4–3 shows that the mean value of E_{KS}, and the .025 and .975 quantiles of its sampling distribution for both bounded and the unbounded models, up to

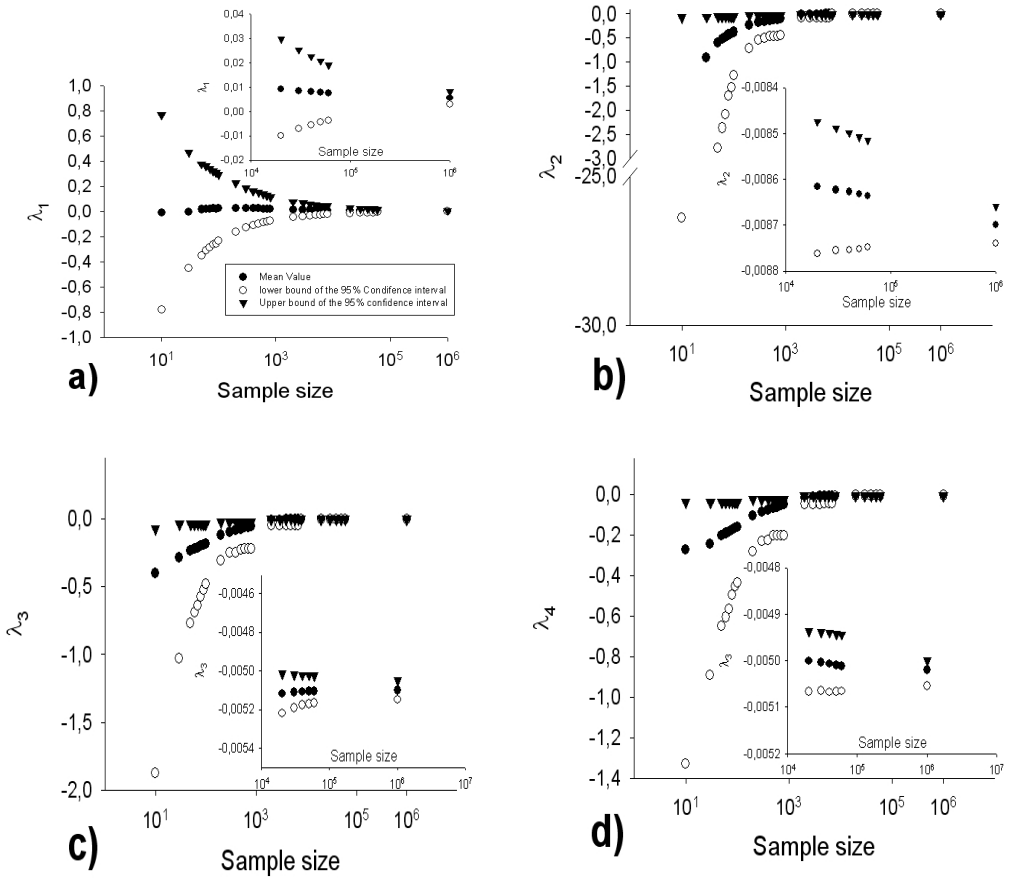


Figure 7.4–2. Influence of the sample size on the 95% confidence interval and on the mean value of the four GLD estimators sample sizes of 10^4 from $N(0,1)$, using an unbounded GLD.

approximately $n = 200$. For larger sample sizes, the distance between the unbounded modeling and the Gaussian sample stops decreasing with $E_{KS} = 0.01$, whereas the bounded GLD still continues to improve its accuracy up to $n = 10^6$. Nevertheless, for larger sample sizes the E_{KS} distance also ceases to decrease. These results indicate that for $n > 200$ the unbounded GLD is less adequate than the bounded one. Indeed, for sample sizes lower than 10^6, the GLD modeling is closer to the sample (in terms of E_{KS}) than the Gaussian one. Therefore, if there is no theoretical reason for the data to actually be Gaussian the use of a usual goodness-of-fit criterion suggests that a bounded GLD modeling is more satisfactory. This result is both interesting and risky. Indeed, it is an illustration of the high flexibility of GLDs since, even in a non-consistent case, GLDs are able to fit in more accurately with the data than their *"true"* distribution. Nevertheless, this can be considered as misleading since it could lead to the wrong conclusion: "the

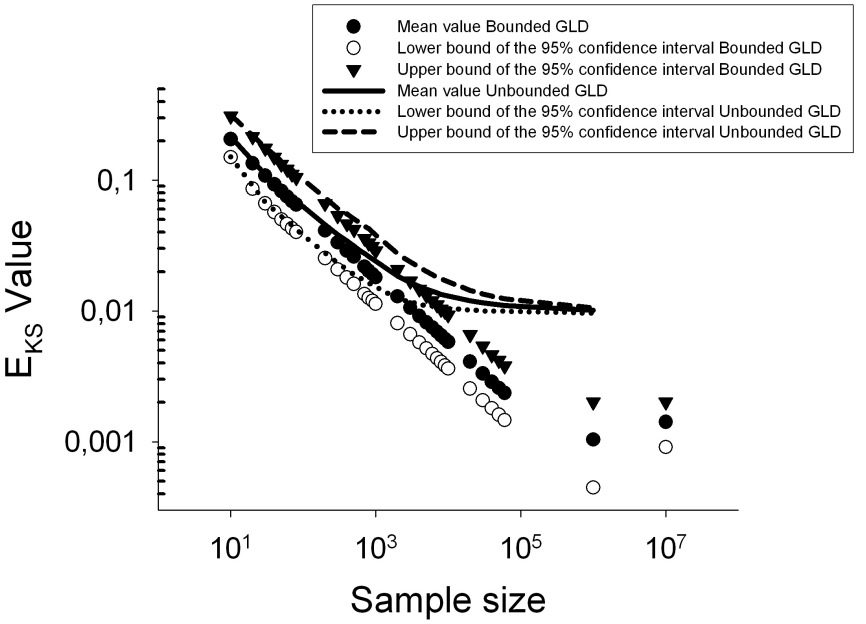


Figure 7.4–3. Influence of the sample size on the 95% confidence interval and on the mean value of the E_{KS} distance obtained for both a bounded and an unbounded modeling of samples coming from $N(0,1)$.

data is GLD and not Gaussian." However this apparent drawback may partly come from the fact that the Kolmogorov-Smirnov distance (chosen because of its generality and ease of use) is not the best measure to test normality. It is possible that the use of other tests (such as the Shapiro-Wilk test for example) or criteria allow to better distinguish and reject the GLD assumption.

It is noted that when applying the Kolmogorov-Smirnov test at the 5% level, Figure 7.4–4 shows that neither the Gaussian nor the bounded GLD is ever rejected until $n = 10^6$. On the contrary, the unbounded GLD is rejected in more and more cases as soon as $n = 6 \times 10^3$ and is always rejected for $n \geq 3 \times 10^4$. This result shows that, as this is not a consistent case, neither the bounded nor the unbounded GLD are able to tend towards a perfect Gaussian law. Nevertheless, the bounded GLD remains a very good approximation of the Gaussian distribution up to $n = 10^6$ since a usual goodness-of-fit test is not able to reject the adequacy (at the 5% confidence level). So, for $n \leq 10^6$ both the Gaussian (which is also never rejected by the two-sample test) and the bounded GLD modeling would be acceptable. For larger sizes ($n > 10^6$) the bounded GLD modeling is more and more often rejected. Indeed, as for such a large dataset the probability to observe random data outside $[-4.76, 4.76]$ is not negligible anymore, the GLD parameters have to accommodate to still fulfill conditions (7.2.15) and (7.2.16). Practically speaking the GLD support must be widened and this results in a loss of accuracy in modeling data in bulk.

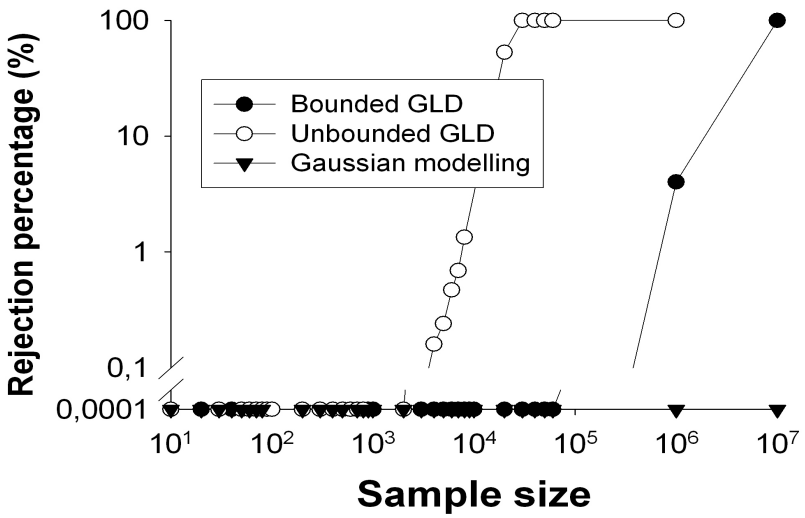


Figure 7.4–4. Influence of the sample size on the percentage of GLD modeling rejected by the Kolmogorov-Smirnov test at the 5% confidence level.

7.5 Comparison with the Method of Moments in Three Specific Cases

In order to illustrate the results obtained with this new P-KS method, in comparison with the well-known method of moments, three cases, treated in Karian and Dudewicz (2000) are reported here. Samples of sizes 10^2, 10^3, 10^4, 10^5 and 10^6 data coming from a standard Gaussian distribution, a uniform distribution and a Student t distribution are fitted with both the method of moments and the new method presented here. For these two methods, a grid of 100×100 set of λ_3, λ_4 values are taken within the range $[0, 1]$ for bounded GLDs and within the range $[-1, 0]$, for unbounded ones. Then the simplex minimization is performed to obtain the best set of parameters. For the method of moments a Kolmogorov-Smirnov test is performed with the final set of parameters.

7.5.1 Gaussian Data

The results of Table 7.5–1 show that, for a given set of random Gaussian numbers, whatever its size, the new P-KS method gives a higher adequacy to the sample of data. If this difference is very limited for small sample sizes it becomes much more significant for larger sample sizes, since the P-value strongly decreases for the method of moments, whereas it remains close to 0.9 for the new P-KS method. Here, only bounded GLD are considered. For very large sample sizes the new P-KS method converges towards a different set of parameters than both the method of moments and the method of percentiles (Karian and Dudewicz (2000)).

Table 7.5–1. GLD fits of 10^2 to 10^6 random Gaussian data obtained with either the the method of moments or the new P-KS method. P is the p-value of the Kolmogorov-Smirnov test and $H = 0$ when the null hypothesis cannot be rejected and 1 when it can be rejected at the 5% confidence level.

	λ_1	λ_2	λ_3	λ_4	E_{KS}	P	H
10^2 data							
Method of moments	0.4551	0.2895	0.2779	0.1106	0.0448	0.9876	0
New P-KS method	0.1855	0.0114	0.007	0.0056	0.0390	0.9978	0
10^3 data							
Method of moments	-0.0344	0.2346	0.1639	0.1598	0.0184	0.9385	0
New P-KS method	-0.0314	0.2168	0.1480	0.1480	0.0130	0.9965	0
10^4 data							
Method of moments	0.0031	0.1823	0.1223	0.1237	0.0062	0.9593	0
New P-KS method	0.0024	0.1521	0.1005	0.1009	0.0051	0.9950	0
10^5 data							
Method of moments	0.0254	0.1928	0.1340	0.1286	0.0040	0.2310	0
New P-KS method	0.0071	0.2244	0.1578	0.1577	0.0024	0.8256	0
10^6 data							
Method of moments	-0.0035	0.1979	0.1348	0.1354	0.0012	0.3308	0
New P-KS method	0.0008	0.2058	0.1423	0.1415	0.0007	0.8778	0

7.5.2 Uniform Data

Here again, only bounded GLD are considered. The new P-KS method cannot be said to give a systematic better adequacy, more specifically at small sample size. However, as soon as the sample size exceeds 1000, both the method of moments and the new P-KS method give very similar results and they seem to converge towards the same set of parameters (as can be seen in Table 7.5–2). This result was expected since the uniform distribution belongs to the GLD family.

7.5.3 Student t Data

In the case of the Student t distribution, the parameter θ was taken equal to 5, as in Karian and Dudewicz (2000). As the tails of this distribution are much heavier than the Gaussian ones, both bounded and unbounded GLD will be considered. The results are presented in Table 7.5–3, where (B) designates a bounded fit and (U) an unbounded one. These results show that, systematically, unbounded GLDs offer a better fit than bounded ones. And the new P-KS method always lead to a better adequacy. Moreover, the method of moments and the new P-KS method lead to strictly different set of parameters for the GLD. In the case of the method of moment for bounded GLD, as constraints (7.2.15) and (7.2.16) are

Table 7.5–2. GLD fits of 10^2 to 10^6 random uniform data obtained with either the method of moments or the new P-KS method. P is the p-value of the Kolmogorov-Smirnov test and $H = 0$ when the null hypothesis cannot be rejected and 1 when it can be rejected at the 5% confidence level.

	λ_1	λ_2	λ_3	λ_4	E_{KS}	P	H
10^2 data							
Method of moments	0.7748	1.1541	0.5581	0.0824	0.0501	0.9629	0
New P-KS method	0.5098	1.7272	0.460	0.4480	0.0720	0.6378	0
10^3 data							
Method of moments	0.4941	1.9778	0.9766	0.9534	0.0204	0.8754	0
New P-KS method	0.4929	1.9658	0.9089	0.9126	0.0178	0.9527	0
10^4 data							
Method of moments	0.5004	1.9991	1.0110	0.9910	0.0093	0.6096	0
New P-KS method	0.4997	1.9951	1.0107	0.9980	0.0078	0.9950	0
10^5 data							
Method of moments	0.4997	1.9950	0.9836	0.9988	0.0028	0.6700	0
New P-KS method	0.5001	1.9977	0.9860	1.0010	0.0024	0.8477	0
10^6 data							
Method of moments	0.5000	2.0000	1.0026	1.0026	0.0006	0.9527	0
New P-KS method	0.5001	1.9996	0.9998	0.9999	0.0007	0.9009	0

also applied, no fit is found for large sample sizes, because, the initial grid is not detailed enough since λ_3 and λ_4 would have very small values.

7.6 Conclusions

The new approach proposed here provides a quick and completely automated estimate of the GLD parameters while ensuring a relatively good fit to the data (which is not the case with usual percentile or moment-based methods for which an *a posteriori* goodness-of-fit test must be performed). First, the new method is based in part on the percentile method and extensive simulations provide some information relevant to choosing the quantile to be used, u. These partial results indicate that there is no trivial rule for choosing u. Depending on the final modeling goal, either a large or a small value may be more suitable.

Second, the sampling distribution of the new estimator was studied and compared to extant techniques. In particular, the standard error of the new estimator was found to compare favorably. The estimates are reasonably unbiased with a sample size that exceeds 10^3 and it was found that the difference between the .025 and .975 quantiles of the sampling distribution are proportional to $1/\sqrt{n}$.

Third, both bounded and unbounded GLDs were used to model Gaussian

Table 7.5–3. GLD fits of 10^2 to 10^6 random Student t data obtained with either the the method of moments or the new P-KS method. P is the p-value of Kolmogorov-Smirnov test and $H = 0$ when the null hypothesis cannot be rejected and 1 when it can be rejected at the 5% confidence level.

	λ_1	λ_2	λ_3	λ_4	E_{KS}	P	H
10^2 data							
Method of moments (B)	0.2009	0.0001	0.0001	0.0001	0.054	0.93	0
Method of moments (U)	0.2045	0.1203	-0.0576	-0.0783	0.053	0.94	0
New P-KS method (B)	0.0728	0.0280	0.0165	0.0177	0.058	0.88	0
New P-KS method (U)	0.1605	-1.5705	-0.5868	-0.4968	0.052	0.95	0
10^3 data							
Method of moments (B)	0.0322	0.0002	0.0001	0.0001	0.041	0.19	0
Method of moments (U)	0.0047	0.0736	-0.0439	-0.0481	0.026	0.60	0
New P-KS method (B)	-0.0482	0.0348	0.0213	0.0226	0.031	0.40	0
New P-KS method (U)	0.0068	-0.7290	-0.3370	-0.3120	0.021	0.84	0
10^4 data							
Method of moments (B)	-0.4346	0.0001	0.00001	0.00008	0.059	0	1
Method of moments (U)	-0.0258	0.2035	-0.118	-0.1134	0.011	0.38	0
New P-KS method (B)	-0.0341	0.0277	0.0176	0.0189	0.013	0.21	0
New P-KS method (U)	-0.0093	-0.0931	-0.0570	-0.0580	0.005	0.99	0
10^5 data							
Method of moments (B)			NO FIT FOUND				
Method of moments (U)	-0.0045	0.2165	-0.1226	-0.1212	0.012	0	1
New P-KS method (B)	-0.0005	0.0151	0.0101	0.0102	0.056	0.03	1
New P-KS method (U)	0.0039	-0.0391	-0.0208	-0.0209	-0.003	0.66	0
10^6 data							
Method of moments (B)			NO FIT FOUND				
Method of moments (U)	0.0134	0.2622	-0.1404	-0.1428	0.018	0	1
New P-KS method (B)	-0.0004	0.0029	0.002	0.002	0.039	0	1
New P-KS method (U)	0.0024	-0.0791	-0.0503	-0.0500	0.001	0.46	0

data and were found valuable for sample sizes up to $n = 200$. However, for larger sample sizes, the bounded GLD definitely leads to a better fit. The bounded GLD is found to give a very accurate approximation of Gaussian data for sample sizes as large as $n = 10^6$. For larger sample sizes the GLD bounds must be widened in order to model the whole range of empirical data, which eventually leads to a loss of global adequacy.

Finally, data coming from three different types of distributions were fitted by a GLD using both the well-known method of moments and the new P-KS method. For Gaussian data the new P-KS method systematically performs better. For uniform data, both the method of moments and the new P-KS method lead to very comparable results. For Student t data, however, the new P-KS method definitely offers much better adequacy and lead to very distinct sets of parameters than the method of moments.

References for Chapter 7

Bigerelle, M., Najjar, D., Fournier, B., Rupin, N., and Iost, A. (2005). "Application of lambda distributions and bootstrap analysis to the prediction of fatigue lifetime and confidence intervals," *International Journal of Fatigue,* 28, 223–236.

Corrado, C.J. (2001). "Option pricing based on the generalized lambda distribution," *J. Fut. Mark.,* 21, 213–236.

Dengiz, B. (1988). "The generalized lambda distribution in simulation of M/M/1 queue systems," *J. Fac. Eng. Arch.,* Gazi Univ., 3, 161–171.

Eriksson, J., Karvanen, J., and Koivunen, V. (2000). "Source distribution adaptive maximum likelihood estimation of ICA model," *ICA2000 Proceedings of the Second International Workshop on Independent Component Analysis and Blind Signal Separation.*

Filliben, J.J. (1969). "Simple and Robust Linear Estimation of the Location Parameter of a Symmetric Distribution," PhD thesis, Princeton University, Princeton, NJ.

Filliben, J.J. (1975). "The probability plot correlation coefficient test for normality," *Technometrics,* 17, 111.

Fournier, B., Rupin, N., Bigerelle, M., Najjar, D., and Iost, A. (2006). "Application of the generalized lambda distributions in a statistical process control methodology," *Journal of Statistical Process Control*, 16, 1087–1098.

Fournier, B., Rupin, N., Bigerelle, M., Najjar, D., and Iost, A. (2007). "Estimating the parameters of a generalized lambda distribution," *Computational Statistics & Data Analysis,* 51, 2813–2835.

Joiner, B.L. and Rosenblatt, J. R. (1971). "Some properties of the range in samples from Tukey's symmetric lambda distributions," *J. Amer. Stat. Assoc.,* 66, 394.

Karian, Z.A. and Dudewicz, E.J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* CRC Press, Boca Raton, FL.

Karian, Z.A. and Dudewicz, E.J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society,* 2, 171–187.

Karvanen, J., Eriksson, J., and Koivunen, V. (2002). "Adaptive score functions for maximum likelihood ICA," *The Journal of VLSI Signal Processing,* 32, 83–92.

King, R. and MacGillivray, H. (1999). "A starship estimation method for the generalized lambda distributions," *Australian and New Zealand Journal of Statistics,* 41, 353–374.

Lakhany, A. and Mausser, H. (2000). "Estimating the parameters of the gener-

alized lambda distribution," *Algo Research Quarterly,* 3, 47–58.

Najjar, D., Bigerelle, M., Lefebvre, C., and Iost, A. (2003). "A new approach to predict the pit depth extreme value of a localized corrosion process," *ISIJ*, 4, 720–725.

Negiz, A. and Cinar, A. (1997). "Statistical monitoring of multivariate dynamic processes with state-space models," *AIChE Journal*, 43, 2002–2020.

Nelder, J. and Mead, R. (1965). "A simplex method for function minimization," *Computer Journal,* 7, 308–313.

Öztürk, A. and Dale, R.F. (1982). "A study of fitting the generalized lambda distribution to solar radiation data," *Journal of Applied Meteorology,* 21, 995–1004.

Öztürk, A. and Dale, R.F. (1985). "Least squares estimation of the parameters of the generalized lambda distribution," *Technometrics,* 27, 81–84.

Pal, S. (2005). "Evaluation of non-normal process capability indices using generalized lambda distributions," *Quality Engineering*, 77–85.

Ramberg, J.S., Dudewicz, E.J., Tadikamalla, P., and Mykytka, E. (1979). "A probability distribution and its use in fitting data," *Technometrics,* 21, 201–214.

Ramberg, J.S. and Schmeiser, B.W. (1972). "An approximate method for generating symmetric random variables," *Comm. ACM,* 15, 987–990.

Ramberg, J.S. and Schmeiser, B.W. (1974). "An approximate method for generating asymmetric random variables," *Comm. ACM,* 17, 78–82.

Stengos, T. and Wu, X. (2004). "Information-theoretic distribution tests with application to symmetry and normality," *Journal of Econometrics,* to be published.

Tarsitano, A. (2004). "Fitting the Generalized Lambda Distribution to Income Data," *COMPSTAT 2004 – Proceedings in Computational Statistics,* 16th Symposium Held in Prague, Czech Republic.

Tukey. J.W. (1962). "The future of data analysis," *Ann. Math. Stat.,* 33, 1–67.

Wilcox, R.R. (2002). "Comparing the variances of two independent groups," *British Journal of Mathematical and Statistical Psychology,* 55, 169-175.

Appendix for Chapter 7

This appendix reports the source code for the new P-KS method in MATLAB® language. Comments are given to explain each coding step. The initial step of the 2D square grid discretization of the λ_3, λ_4 space can of course be optimzed such as the minimization procedure. The present code example is thus aimed at being illustrative and is not an optimized procedure.

A.1 Main procedure: `FitLambda`

```
function Result=FitLambda (Data,u,l3min,l3max,l4min,l4max,bornegauche,
    bornedroite,step,fichier)

%-----------------------------------------------------------------------
%*******PARAMETERS DESCRIPTION*****************************************
% Data=series of data we want to approximate by a GLD
% u = parameter coming from the percentile method 0<u<0.25;
% l3min : minimum value of lambda3 for the initial grid
% l4min : minimum value of lambda4 for the initial grid
% l3max : maximum value of lambda3 for the initial grid
% l4max : maximum value of lambda4 for the initial grid
% bornegauche : left bound of the GLD
% bornedroite : right bound of the GLD
%%%% if one wants the GLD to be unbounded (e.g. on both sides), then
%%%% bornedroite = Inf and bornegauche = -Inf
%%%% else if one wants the GLD to bounded (e.g.) on the right side,
%%%% then bornegauche = -Inf and bornedroite = 3 (e.g) : it wlll ensure
%%%% that the maximum value on which the GLD is definite is lower than
%%%% bornedroite.
% step : number of steps in the initial grid of lambda3/lambda4
% fichier : file name in which the results are written.
%-----------------------------------------------------------------------

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%********DEFAULT VALUES***************************************************
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Here is an exemple of values that can be used to test the program : just
% remove the "%" at the begining of lines and type FitLambda () in MatLab.

 %Data=randn (1000,1)
 % u=0.1;
 % l3min=-2;
 % l4min=-2;
 % l3max=2;
 % l4max=2;
 % step=30;
 % fichier='test.txt';
 % bornegauche=-1000;
 % bornedroite=1000;

 file=fopen (fichier,'w');
 fprintf (file,'% s \t % s \t % s \t % s \t % s \t % s \t % s \t % s
    \t % s \t % s \t % s \t % s \t % s \n','L1','L2','L3','L4',
    'kolavant','kolapres','H','P','K','m1','m2','m3','m4');

% number of values of the GLD generated to compare with
```

```
% the experimental data
NbLambda=max (5000,2*length (Data));
if NbLambda>10000000
NbLambda=10000000;
end

% Percentiles calculation
Data=sort (Data);
Rho1=CalculRho (Data,0.5);
Rho2=CalculRho (Data,1-u)-CalculRho (Data,u);

% Calculation of the Kolmogorov-Smirnov (2 samples) distance between
% the experimental data and the GLD defined at the grid values.
test=0;
step=(l3max-l3min)/step;
iter=1;
Mini=1;
while test==0 & iter<3;
    i=0;
    j=0;
    for x=l3min:step:l3max
        i=i+1;
        xx (i)=x;
        j=0;
       for y=l4min:step:l4max
           j=j+1;
           yy (j)=y;
           xy=[x,y];
           Haut (i,j)=Hauteur (xy,Rho1,Rho2,Data,NbLambda,bornegauche,bornedroite,u);
       end
    end

    % Find the minimum on the grid
    [minvalbycol,minindinumlgn]=min (Haut);
    [minvaltot,minnumcol]=min (minvalbycol);
    Mini=Haut (minindinumlgn (minnumcol), minnumcol);
    kolavant=Mini;
    % if there is a valid minimum, save the position of this minimum
    % in l3ini and l4ini
    if Mini~=1
        l3ini=xx (minindinumlgn (minnumcol));
        l4ini=yy (minnumcol);
        test=1;
    else
        % if there is no valid minimum, a possibility is to restart the
        % calculation on a finer grid.
        %////////////////////////////////////////////////////////////////
        % Here it would also be possible to extend the grid bounds, or to
        % relocate the whole grid... Many posibilities are available, just
        % like for the way of defining the initial grid. Here only a very simple
        % square grid is used, but there are more "clever" choices to make,
        % if one has more information on the expected GLD...
        %////////////////////////////////////////////////////////////////

        step=step/10;
        iter=iter+1;
    end
end

% if, after 3 rescaling operation on the initial grid, no valid minimum is
% found, the procedure fails and one must restart with either a more
```

```
% extended initial grid, or a grid located somewhere else in the
% (lambda3,lambda4) space.
if Mini==1
    Result=0;
    return

else

% If there was a valid minimum on the grid, a simplex minimization is
% launched from this minimum, to improve the adequacy.
    l3l4=[l3ini,l4ini];
     options=optimset ('DISPLAY','off','TOLFUN',10^-15,'TOLX',10^-15,'MAXITER',5000);
    [Coord,kolapres]=
    fminsearch(@Hauteur,l3l4,options,Rho1,Rho2,Data,NbLambda,bornegauche,bornedroite,u);
% L1, L2, L3 and L4 are the four GLD parameters obtained with this
% procedure.
% y and x can be used to plot the GLD found.
% H, P, K are the results of the 2samples Kolmogorov-Smirnov test between
% (see MatLab documentation for definition)
% the final GLD and the experimental data
% m1 to m4 are the four moments of the initial series of data.

% All these values are stored in the file "fichier", however, if one wants
% them to appear on MatLab desktop, just remove the ";" at the line ends.
L2=((1-u)^Coord(1)-u^Coord(2)+(1-u)^Coord(1)-u^Coord(2))/Rho2;
L1=Rho1-(0.5^Coord(1)-0.5^Coord(2))/L2;
L3=Coord (1);
L4=Coord (2);
y=[0,linspace (10^-10,9.99999999*10^-1,floor (NbLambda)),1];
x=GenererLambda (NbLambda,L1,L2,L3,L4);
[H,P,K]=kstest2 (x,Data);
m1=mean (Data);
m2=std (Data);
m3=skewness (Data);
m4=kurtosis (Data);

% Savind the results.
fprintf(file,'% g \t % g \t % g \t % g \t % g \t % g \t % g \t % g \t % g \t % g \t
     % g \t% g \t % g \n ',L1,L2,L3,L4,kolavant,kolapres,H,P,K,m1,m2,m3,m4);
Result=1;
fclose (file)

% End of the procedure.

end
}
```

A.2 Secondary Procedures

- Percentile

  ```
  function result=percentile (X,L1,L2,L3,L4)

  result=L1+(X.^L3-(1-X).^L4)./L2;
  ```

- GenererLambda

  ```
  function Result=GenererLambda (N,L1,L2,L3,L4)
  y=[0,linspace (10^-10,9.99999999*10^-1,floor (N)),1];
  L=length (y);
  ```

```
Result=L1+(y.^L3-(1-y).^L4)/L2;
```

- CalculRho

```
function Rho=CalculRho (Data,p)

% Calculation of the pth percentile as defined in Karian and Dudewicz (2000)

Taille = length (Data (:,1));
x=p*(Taille-1)+1.;
r=floor (x);
fraction=x-r;
if r>1
    r=r-1;
end
Rho=Data (r,1)+fraction*(Data (r+1,1)-Data (r,1));
```

- Hauteur

```
function Kolmo=Hauteur(xy,Rho1,Rho2,Data,NbLambda,bornegauche,bornedroite,u)

% Procedure used to calculate the 2 samples Kolmogorov-Smirnov distance,
% while taking into account the intervals of definition of valid GLD and the
% constraints on the bounds.

%Values of lambda3 and lambda4
x=xy (1);
y=xy (2);

lambda2=((1-u)^x-u^y+(1-u)^x-u^y)/Rho2;
lambda1=Rho1-(0.5^x-0.5^y)/lambda2;

% Quantities necessary to test the validity of the GLD (cf Karian and
% Dudewicz, 2000)
CV3=((1-x)^(1-x))/((y-x)^(y-x))*(y-1)^(y-1);
CV4=((1-y)^(1-y))/((x-y)^(x-y))*(x-1)^(x-1);
Kolmo=1;

yGLD=rand (NbLambda,1);
% Attention la fonction GenererLambda ne les gnre pas alatoirement:
% rgulirement rpartis!!

if (x<0 & y>0 & y<1) | (x>0 & x<1 & y<0) | (x>-1 & x<0 & y>1 &
    CV3>=-x/y) | (x>1 & y>-1 & y<0 & CV4>=-y/x)

    Kolmo=1;

else

    if x>0 & y>0 & bornegauche~=-Inf & bornedroite~=Inf
        if lambda1-1/lambda2<=Data (1) & lambda1+1/lambda2>=Data (end) &
           lambda1-1/lambda2>=bornegauche & lambda1+1/lambda2<=bornedroite
            DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
            [H,P,K]=kstest2 (Data,DataLambda);
            Kolmo=K;
        else
            Kolmo=1;
        end
```

```
    elseif x>0 & y==0 & bornegauche~=-Inf & bornedroite~=Inf
        if lambda1<=Data (1) & lambda1+1/lambda2>=Data (end) &
           lambda1>=bornegauche & lambda1+1/lambda2<=bornedroite
            DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
            [H,P,K]=kstest2 (Data,DataLambda);
            Kolmo=K;
        else
            Kolmo=1;
        end

    elseif x==0 & y>0 & bornegauche~=-Inf & bornedroite~=Inf
        if lambda1-1/lambda2<=Data (1) & lambda1>=Data (end) &
           lambda1-1/lambda2>=bornegauche & lambda1<=bornedroite
            DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
            [H,P,K]=kstest2 (Data,DataLambda);
            Kolmo=K;
        else
            Kolmo=1;
        end
    elseif x<0 & y<0 & bornegauche==-Inf & bornedroite==Inf
            DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
            [H,P,K]=kstest2 (Data,DataLambda);
            Kolmo=K;
    elseif x<0 & y==0 & bornegauche==-Inf & bornedroite~=Inf
            if lambda1+1/lambda2>=Data (end) & lambda1+1/lambda2<=bornedroite
                DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
                [H,P,K]=kstest2 (Data,DataLambda);
                Kolmo=K;
            else
                K=1;
            end
    elseif x==0 & y<0 & bornedroite==Inf & bornegauche~=-Inf
        if lambda1-1/lambda2<=Data (1) & lambda1-1/lambda2>=bornegauche
             DataLambda=GenererLambda_alea (yGLD,lambda1,lambda2,x,y);
            [H,P,K]=kstest2 (Data,DataLambda);
            Kolmo=K;
        else
            K=1;
        end
    end

Kolmo;

end
```

Chapter 8

Fitting Mixture Distributions Using a Mixture of Generalized Lambda Distributions with Computer Code

Wei Ning
Department of Mathematics and Statistics,
Bowling Green State University, Bowling Green, OH 43403, USA
wning@bgsu.edu

Yunchuan Gao
Mapleridge Capital Corporation,
5000 Yonge Street, Toronto, ON M2N 7E9, CANADA
yunchuangao@gmail.com

Edward J. Dudewicz
Department of Mathematics,
Syracuse University, Syracuse, NY 13244, USA
dudewicz@syr.edu

In this chapter, we will discuss the problems of fitting data with the mixture of two generalized lambda distributions (GLDs) and its application. Finite mixture distributions have provided an approach to the statistical modeling of a wide variety of random phenomena. Most of the work has been done using mixtures of normal distributions. The most striking property of a mixture distribution is that the shape of the density is extremely flexible. Because of its flexibility, it has been applied to many fields such as: biology, genetics, medicine, and the social

sciences. In these applications, finite mixture distributions underpin a variety of techniques in major areas of statistics including: cluster analysis, discriminant analysis, image analysis and survival analysis, in addition to their more direct role in data analysis and inference of providing descriptive models for distributions. Mixture distributions also play an important role in neural networks.

One of the first, perhaps even the first, analysis involving the use of mixture models was undertaken by Karl Pearson (1894), who fitted a mixture of two normal probability density functions to data provided by Weldon (1892) and (1893), which consisted of measurements on the ratio of forehead length to body length of 1000 crabs. Since then, much work has been done in this field.

Throughout this chapter, we assume that the reader is familiar with the definition and the basic properties of the GLD family of distributions that was developed in Chapters 2 and 3. Therefore, we start our discussion by giving a brief overview of mixture distributions. Then we propose the Newton-Krylov method to estimate the parameters in a mixture of two generalized lambda distributions. Graphs are provided in the following section to show the shape of the mixture of two generalized lambda distributions with different parameters. In Section 7.5, we apply the method to three real datasets: the first is the data of crabs which was used by Karl Pearson in 1894 to study the mixture of two normal distributions, the second is data of cadmium in horse kidneys studied by Ellinder, Jönson, Piscator, and Rahnster (1981), and the third is the data of exchange rate of Japanese Yen as compared to the U.S. dollar from 1862 to 2003, which was studied by Nadarajah and Kotz (2004).

By estimating a mixture of two GLDs, we show that our method can provide a good fit for these datasets. The flexibility of the family of GLD mixtures is compared to the flexibility of the family of normal mixtures in Section 8.6 by calculating Kullback-Leibler (KL) distance between two mixture distributions, as well as the overlapping coefficient (δ). The results show that a mixture of GLDs performs as well as the mixture of normal distributions. Moreover, we can use the mixture distribution of GLDs with fewer components compared to the mixture of normal distributions, which often require more components. Discussion is provided in Section 8.7 and some future research questions are raised.

8.1 Brief Overview of the Generalized Lambda Distribution

The development of the GLD is part of the search for a method of fitting a continuous probability distribution to data. It was Pearson in 1895 who gave a four-parameter system of probability density functions, and fit the parameters by the method of moments (MM). John Tukey (1960) proposed a one-parameter lambda distribution. Tukey's lambda was generalized, for the purpose of generating random variables for Monte Carlo simulation studies, to the four parameters

GLD, by John Ramberg and Bruce Schmeiser (Ramberg and Schmeiser (1972, 1974)). Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979) developed a four-parameter system with the necessary tables for fitting a wide variety of curve shapes. Since the early 1970s, the GLD has been applied to fitting phenomena in many fields of endeavor with continuous probability density functions (pdf). In this section, we briefly introduce the GLD and its properties. For more details of the GLD, see Karian and Dudewicz (2000).

The GLD family with four parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which we will denoted by $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, has a pdf

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}, \quad \text{at } x = Q(y),$$

where $Q(y)$ is the percentile function defined as

$$Q(y) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2},$$

where $0 \le y \le 1$. λ_1 and λ_2 are location and scale parameters, respectively. λ_3, λ_4 determine the skewness and kurtosis of the distribution. Not every choice of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ leads to a valid distribution, but we have the following theorem.

Theorem 8.1.1 *The* $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *specifies a valid distribution, if and only if,*

$$g(y, \lambda_3, \lambda_4) \equiv \lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1} \tag{8.1.1}$$

has the same sign for all y *in* $[0, 1]$, *as long as* λ_2 *takes that sign also. In particular, the* $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *specifies a valid distribution if* $\lambda_2, \lambda_3, \lambda_4$ *all have the same sign.*

The first four moments of $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are given in the following theorem.

Theorem 8.1.2 *If* X *is* $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *with* $\lambda_3 > -1/4$ *and* $\lambda_4 > -1/4$, *then its first four moments,* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *are given by*

$$\begin{aligned}
\alpha_1 &= \mu = E(X) = \lambda_1 + \frac{A}{\lambda_2}, \\
\alpha_2 &= \sigma^2 = E[(X-\mu)^2] = \frac{B - A^2}{\lambda_2^2}, \\
\alpha_3 &= E(X - E(X))^3/\sigma^3 = \frac{C - 3AB + 2A^3}{\lambda_3^3 \sigma^3}, \\
\alpha_4 &= E(X - E(X))^4/\sigma^4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \sigma^4}
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4},\\
B &= \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3, 1+\lambda_4),\\
C &= \frac{1}{1+3\lambda_3} - \frac{1}{1+3\lambda_4} - 3\beta(1+2\lambda_3, 1+\lambda_4) + 3\beta(1+\lambda_3, 1+2\lambda_4),\\
D &= \frac{1}{1+4\lambda_3} + \frac{1}{1+4\lambda_4} - 4\beta(1+3\lambda_3, 1+\lambda_4) + 6\beta(1+2\lambda_3, 1+2\lambda_4)\\
&- 4\beta(1+\lambda_3, 1+3\lambda_4).
\end{aligned}$$

8.2 The Problem of Mixture Distributions

The first study of mixture models goes back to Karl Pearson (1894). Pearson used the mixture of two normal distributions with different means and variances in proportions p_1 and p_2 to model data consisting of measurements on the ratio of forehead to body length of 1000 crabs. Let $\boldsymbol{X_1, X_2, ..., X_n}$ denote a random sample of size n, where $\boldsymbol{X_j}$ is a p-dimensional random vector with a density function $f(\boldsymbol{x_j})$ on $\boldsymbol{R}^p$. We suppose that the density $f(\boldsymbol{x_j})$ can be written in the form

$$f(\boldsymbol{x_j}, \boldsymbol{\theta}) = \sum_{i=1}^{c} p_i f_i(\boldsymbol{x_j}, \boldsymbol{\theta_i}),\ j = 1, 2, ..., n, \tag{8.2.1}$$

where $\boldsymbol{\theta}$= $(p_1, p_2, ..., p_{c-1}, \boldsymbol{h'})'$, $\boldsymbol{h}$=$(\theta_1, \theta_2, ..., \theta_c)'$, and the p_i are nonnegative proportions that satisfy $\sum_{i=1}^{c} p_i = 1$. $f(\boldsymbol{x_j}, \boldsymbol{\theta})$ is called the **density of a finite mixture distribution**, $p_1, p_2, ..., p_c$ are called the **mixing weights**, and $f_i(\boldsymbol{x_j}, \theta_i)$ is called the i-th **component density of the mixture**.

The literature in this field is quite extensive. Titterington, Smith, and Makov (1985), McLachlan and Basford (1988) and Lindsay (1995) provided extensive discussions about the background of mixture models. Hartigan (1985) gave an important insight into the irregularity problems of finite mixture models. In his paper, he proved that the likelihood ratio test (LRT) statistic will be unbounded in probability if the parameter space associated with it is unbounded and the sample size is large.

Neyman and Scott (1966) studied the use of $C(\alpha)$ test (Neyman, (1959)), which is a test based on the residual for a regression of a score function for parameters of interest on the score function for the nuisance parameters against the alternative of any mixture. Gupta and Miyawolei (1978) proposed a method of moments estimate for the mixture of two uniform distributions. This was extended to the mixture of an arbitrary number of uniform distributions by Gupta,

Taneja, and Janardan (1986). Lindsay (1989) proposed a moment-based procedure to estimate the parameters of a mixture distribution with known number of components. Chen and Traylor (1995) discussed irregular models which include finite mixture models. Ghosh and Sen (1985), Bickel and Chernoff (1993), Chernoff and Lander (1995), Lemdani and Pons (1999), Ducunha-Castelle and Gassiat (1999), and Chen and Chen (2001) studied the asymptotic distributions of the LRT in mixture models.

It should be noted that the asymptotic null distribution of the LRT is complex and hard to use in practice. Chen, Chen, and Kalbfleisch (2001) proposed a modified LRT procedure for homogeneity in finite mixture models and showed the modified LRT has χ^2- type of null limit distribution.

However, all the procedures proposed above normally involve establishing the identifiability of the parameters. Ning, Gupta, Yu, and Zhang (2009) propose a moment-based test for the case of location shift in a more general setting without identifying the parameters. They show that the test statistic has chi-square distribution in normal distribution cases, otherwise it has an asymptotic standard normal distribution. Ning, Zhang, and Yu (2009) apply a similar method to a specific distribution family: the natural exponential family with quadratic variance functions (NEF-QVF). They also show that the test statistic follows a simple asymptotic standard normal distribution.

To estimate the parameters, a variety of approaches have been used: graphical methods, MM, Minimum-distance methods, maximum likelihood estimates (MLE), Bayesian approach, and so on. Pearson (1894) used the MM to estimate the mixture of two normal distributions (1000 crabs data), and some development work is due to Lindsay and Basak (1993) and Furman and Lindsay (1994). They also developed the system of moment equations in the special case of two normal components with a common covariance matrix. Ning, Gao, and Dudewicz (2008) discussed the fitting mixture of two GLD distributions, and also compared it with the mixture of two normal distributions.

8.3 Estimation of Parameters of a Mixture of Two GLDs

The density function of the mixture of two GLDs is

$$f(x, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = pf(x, \boldsymbol{\theta_1}) + (1-p)f(x, \boldsymbol{\theta_2}) \tag{8.3.1}$$

where

$$\begin{aligned} f(x, \boldsymbol{\theta_1}) &= \frac{\lambda_2}{\lambda_3 y^{(\lambda_3 - 1)} + \lambda_4 (1-y)^{\lambda_4 - 1}} \\ f(x, \boldsymbol{\theta_2}) &= \frac{\lambda_2'}{\lambda_3' y^{(\lambda_3' - 1)} + \lambda_4' (1-y)^{\lambda_4' - 1}} \end{aligned}$$

with $\boldsymbol{\theta_1}= (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\boldsymbol{\theta_2}= (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4)$, and p is the mixing weight. We will find MLE of all nine parameters in this mixture model. The likelihood function is

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n} f(x_j, \boldsymbol{\theta})$$

where $\boldsymbol{\theta}= (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, p)$, and the log-likelihood function is

$$\ln L(\boldsymbol{\theta}) = \sum_{j=1}^{n} \ln(pf(x, \boldsymbol{\theta_1}) + (1-p)f(x, \boldsymbol{\theta_2})).$$

We obtain MLEs from the equations

$$\begin{aligned}
\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \lambda_i} &= \sum_{j=1}^{n} \frac{1}{pf(x_j, \boldsymbol{\theta_1}) + (1-p)f(x_j, \boldsymbol{\theta_2})} p \frac{\partial f(x_j, \boldsymbol{\theta_1})}{\partial \lambda_i} = 0,\ i = 1, 2, 3, 4, \\
\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \lambda'_i} &= \sum_{j=1}^{n} \frac{1}{pf(x_j, \boldsymbol{\theta_1}) + (1-p)f(x_j, \boldsymbol{\theta_2})} (1-p) \frac{\partial f(x_j, \boldsymbol{\theta_2})}{\partial \lambda'_i} = 0,\ i = 1, 2, 3, 4, \\
\frac{\partial \ln L(\boldsymbol{\theta})}{\partial p} &= \sum_{j=1}^{n} \frac{1}{pf(x_j, \boldsymbol{\theta_1}) + (1-p)f(x_j, \boldsymbol{\theta_2})} (f(x_j, \boldsymbol{\theta_1}) - f(x_j, \boldsymbol{\theta_2})) = 0
\end{aligned}$$

The partial derivatives in the above equations are:

$$\begin{aligned}
\frac{\partial f(x, \boldsymbol{\theta_1})}{\partial \lambda_1} &= -\lambda_2^2 \frac{\lambda_3(\lambda_3 - 1)y^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1-y)^{\lambda_4 - 2}}{\left[\lambda_3 y^{\lambda_3 - 1} + \lambda_4(1-y)^{\lambda_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_1})}{\partial \lambda_2} &= \frac{\left[y^{\lambda_3} - (1-y)^{\lambda_4}\right]\left[\lambda_3(\lambda_3 - 1)y^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1-y)^{\lambda_4 - 2}\right]}{\left[\lambda_3 y^{\lambda_3 - 1} + \lambda_4(1-y)^{\lambda_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_1})}{\partial \lambda_3} &= \lambda_2 y^{\lambda_3} \ln y \frac{\lambda_3(\lambda_3 - 1)y^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1-y)^{\lambda_4 - 2}}{\left[\lambda_3 y^{\lambda_3 - 1} + \lambda_4(1-y)^{\lambda_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_1})}{\partial \lambda_4} &= \lambda_2 (1-y)^{\lambda_4} \ln(1-y) \frac{\lambda_3(\lambda_3 - 1)y^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1-y)^{\lambda_4 - 2}}{\left[\lambda_3 y^{\lambda_3 - 1} + \lambda_4(1-y)^{\lambda_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_2})}{\partial \lambda'_1} &= -\lambda_2'^2 \frac{\lambda'_3(\lambda'_3 - 1)y^{\lambda'_3 - 2} - \lambda'_4(\lambda'_4 - 1)(1-y)^{\lambda'_4 - 2}}{\left[\lambda'_3 y^{\lambda'_3 - 1} + \lambda'_4(1-y)^{\lambda'_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_2})}{\partial \lambda'_2} &= \frac{\left[y^{\lambda'_3} - (1-y)^{\lambda'_4}\right]\left[\lambda'_3(\lambda'_3 - 1)y^{\lambda'_3 - 2} - \lambda'_4(\lambda'_4 - 1)(1-y)^{\lambda'_4 - 2}\right]}{\left[\lambda'_3 y^{\lambda'_3 - 1} + \lambda'_4(1-y)^{\lambda'_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_2})}{\partial \lambda'_3} &= \lambda'_2 y^{\lambda'_3} \ln y \frac{\lambda'_3(\lambda'_3 - 1)y^{\lambda'_3 - 2} - \lambda'_4(\lambda'_4 - 1)(1-y)^{\lambda'_4 - 2}}{\left[\lambda'_3 y^{\lambda'_3 - 1} + \lambda'_4(1-y)^{\lambda'_4 - 1}\right]^3}, \\
\frac{\partial f(x, \boldsymbol{\theta_2})}{\partial \lambda'_4} &= \lambda'_2 (1-y)^{\lambda'_4} \ln(1-y) \frac{\lambda'_3(\lambda'_3 - 1)y^{\lambda'_3 - 2} - \lambda'_4(\lambda'_4 - 1)(1-y)^{\lambda'_4 - 2}}{\left[\lambda'_3 y^{\lambda'_3 - 1} + \lambda'_4(1-y)^{\lambda'_4 - 1}\right]^3}.
\end{aligned}$$

We use the Newton-Krylov method (Kelley (1995), Knoll and Keyes (2004)) to solve the nonlinear equations

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \lambda_i} = \frac{\partial \ln L(\boldsymbol{\theta})}{\partial \lambda_i'} = \frac{\partial \ln L(\boldsymbol{\theta})}{\partial p} = 0, \tag{8.3.2}$$

for $i = 1, 2, 3, 4$, to obtain the MLEs of the parameters. The Newton-Krylov method is a Jacobian free method. The linear system at each step of Newton's method is solved by an iterative method. Instead of using the full Jacobian matrix, we compute the product of the Jacobian with a vector. An important feature of this method is the fact that the overall nonlinear convergence of the method is not directly affected by the approximations made in the preconditioning compared to the classic Newton method (Knoll and Keyes (2004)).

Algorithm for solving the nonlinear MLE equations

I Load data set $\{x_i, i = 1, 2, ..., n\}$;

II Set maximum iteration number = 100; error tolerance = 1.0E-3 for Newton-Krylov method;

III Provide initial guess $(p^{(0)}, \lambda_3^{(0)}, \lambda_4^{(0)}, \lambda_3{}'^{(0)}, \lambda_4{}'^{(0)})$ for Newton-Krylov algorithm. Set $j = 0$, then calculate the initial values of $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_1{}'^{(0)}, \lambda_2{}'^{(0)}$ by using percentile method.

IV Use Theorem 8.1.2 to check the validity of all the λ values. If not valid, go to (III) to choose different initial values. If all values are valid, go to next step.

V For each x_i, solve for y_i^j and $f^j(x_i, \boldsymbol{\theta_1})$, $f^j(x_i, \boldsymbol{\theta_2})$;

VI Apply Newton-Krylov method to the nonlinear equations (8.3.2);

VII Calculate the Jacobian matrix J of (8.3.2);

VIII Use the Krylov method to solve the linear system arising from Newton's method;

IX Get next set of $(p^{j+1}, \boldsymbol{\theta_1}^{j+1}, \boldsymbol{\theta_2}^{j+1})$;

X If error is less than tolerance or maximum iteration is reached, stop the Newton-Krylov algorithm; otherwise, $j = j + 1$, go to (VI).

Remark 1: In Step (III), in order to choose good initial values for the last two parameters for each GLD, Karian and Dudewicz (2000, Page 18) indicate six valid regions for λ_3 and λ_4 which will lead to a valid GLD. It is better do some searching in each of these regions to choose the best values among all the regions. However, this might be time consuming. To avoid spending a large amount of time calculating, in our program, we choose values of $\lambda_3 = -10, \lambda_4 = -10$ and $\lambda_3' = -10, \lambda_4' = -10$ to start the iteration with step length 0.01 for each iteration, that is, initial values from the first and third quadrant, which are two of the six valid regions of the GLD. The problem of how to choose the initial values over all six valid regions while keeping searching time relatively short, needs further exploration.

Remark 2: In Step (III), we use the percentile method to calculate the initial values. Compared to the method of moments, the percentile method can still be used when the moments of GLD distributions do not exist. Also, the equations we use in the percentile method are simpler, and the computational techniques required for solving them provide greater accuracy. The third and fourth sample moments may make it difficult to obtain accurate fits by the method of moments, which is due to the large variabilities.

8.4 Graphs of the Mixture Density of Two GLDs

In this section, we first show two graphs of mixture densities of two GLDs with different combinations of parameters, then we compare them with the normal mixtures.

Remark: In Figure 8.4–1 we show two mixtures of GLDs with different parameters and proportions. We also want to know if any of these GLDs are close to some normal mixture. Actually, from p. 67 of Karian and Dudewicz (2000), we know that $\text{GLD}(0, 0.1975, 0.1349, 0.1349)$ is the one that is close to $N(0, 1)$. Since changing λ_1 only changes the mean (see equation (2.2.1) on p. 48 of Karian and Dudewicz (2000)), the above mixtures of GLDs are close to the mixture of $N(0.3, 1)$ and $N(4, 1)$, on the left, and the mixture of $N(0.06, 1)$ and $N(0.2, 1)$, on the right in Figure 8.4–1. We plot the normal mixtures with the GLD mixtures on the same graph for these two situations, which indicates that both of them are very close to each other and almost overlap on each other on graphs. We also list KL distances and overlapping coefficients, which we will define and explore in more details later in Section 8.6. All the values that are obtained support our claim that these two types of mixtures are close to each other (this is verified by the fact that, in this case, the KL distance is close to 0 and the overlapping coefficient is close to 1).

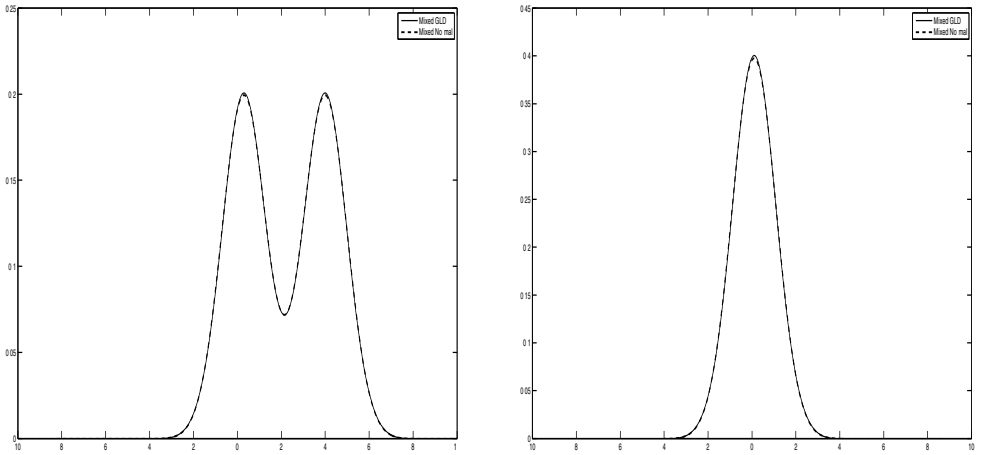


Figure 8.4–1. Mixture densities of two normal and mixture densities of two GLDs. two GLDs. Left: $(1/2)\text{GLD}(0.3, 0.1975, 0.1349, 10.1349) + (1/2)$ $\text{GLD}(4.0, 0.1975, 0.1349, 10.1349)$ and $(1/2)N(0.3, 1) + (1/2)N(4.0, 1)$ with KL distance= $1.9664e-4$, δ=0.9942; Right: $(2/3)\text{GLD}(0.06, 0.1975, 0.1349, 10.1349)$ $+(1/3)\text{GLD}(0.2, 0.1975, 0.1349, 10.1349)$ and $(2/3)N(0.06, 1) + (1/3)N(0.2, 1)$ KL distance= $3.3041e-4$, δ=0.9843.

8.5 Fitting the Mixture of Two GLDs to Real Data

8.5.1 Pearson's Data

The mixture of two normal distributions was first studied by Karl Pearson by fitting data consisting of measurements on the ratio of forehead length to body length of 1000 crabs. Pearson divided the data into 29 intervals with interval length 0.003 and counted the frequency for each interval. Pearson (1894) summarizes these results in a table on page 85 without listing all the data. Therefore, we use the midpoints for the intervals to replace the data, as shown in Table 8.5–1.

By using the algorithm introduced in Section 8.4, the density function of the mixture of two GLDs we use to fit the data is

$$f(x, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = pf(x, \boldsymbol{\theta_1}) + (1-p)f(x, \boldsymbol{\theta_2}),$$

where $\boldsymbol{\theta_1}$= [0.6415, 13.2180, 0.135, 0.205], $\boldsymbol{\theta_2}$= [0.6564, 11.3280, 0.550, 0.150], $p = 0.802$, and

$$\begin{aligned} f(x, \boldsymbol{\theta_1}) &= \frac{13.218}{0.135y^{-0.865} + 0.205(1-y)^{-0.795}}, \\ f(x, \boldsymbol{\theta_2}) &= \frac{11.328}{0.55y^{-0.45} + 0.15(1-y)^{-0.85}}. \end{aligned}$$

Table 8.5–1. Measurements of the ratio of the forehead to body of 1000 crabs.

Midpoint	Frequency	Midpoint	Frequency
0.5815	1	0.5855	3
0.5895	5	0.5935	2
0.5975	7	0.6015	10
0.6055	13	0.6095	19
0.6135	20	0.6175	25
0.6215	40	0.6255	31
0.6295	60	0.6335	62
0.6375	54	0.6415	74
0.6455	84	0.6495	86
0.6535	96	0.6575	85
0.6615	75	0.6655	47
0.6695	43	0.6735	24
0.6775	19	0.6815	9
0.6855	5	0.6895	0
0.6935	1		

For the same data, Pearson (1894) used the normal mixture

$$\begin{aligned} f(x, \theta_1, \theta_2) &= 0.4145N(-3.517, 4.4685) \\ &+ 0.5855N(2.490, 3.1145). \end{aligned}$$

We plot Pearson's normal mixture by a dotted curve and the GLD mixture by a solid curve on the same graph in Figure 8.5–2. We also use **R** package **GLDEX** (Su, 2007) to obtain the q-q plot for the mixture of GLDs, which is also shown in Figure 8.5–2. We also compare the theoretical values of four moments of the normal mixture and the GLD mixture with those of the empirical data in Table 8.5–3.

From the q-q plot, we can see that the mixture of GLDs is a good fit to the data. The theoretical values of four moments of the GLD mixture are very close to the values of the empirical data except the skewness is a little different from the true one. Compared to those values of the normal mixture, the GLD mixture seems to be a better fit.

Remark: Su (2007) designed the **R** package **GLDEX** and provided the estimations of the mixture of two GLDs by using the partition ML fitting scheme and the ML solutions for the Pearson data is

$$\begin{aligned} \boldsymbol{\theta_1} &= [0.6415, -16.2898, -0.2009, -1.9243e-18], \\ \boldsymbol{\theta_2} &= [0.6455, 19.6777, 1.3814e-15, 0.3232], \\ p &= 0.426. \end{aligned}$$

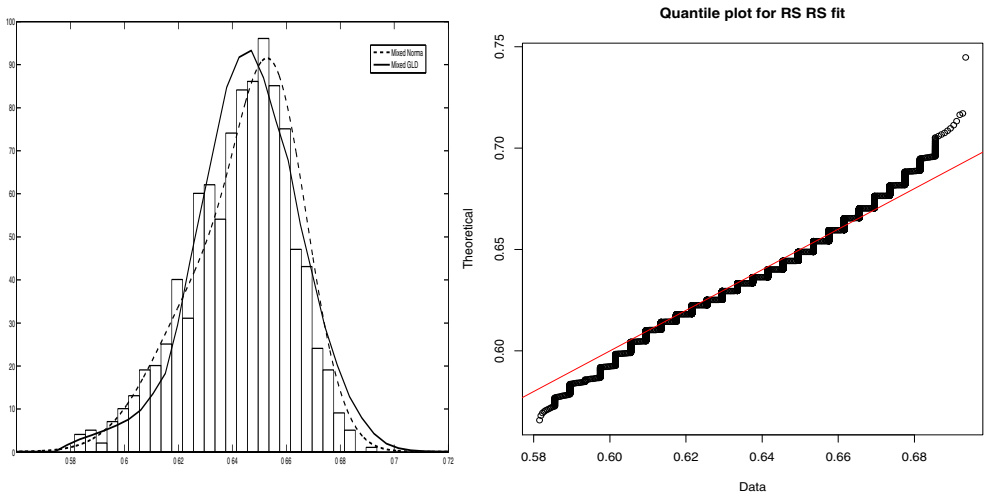


Figure 8.5–2. Histogram of Pearson data and approximated mixture of two GLDs with q-q plot.

Table 8.5–3. Comparison of four moments

	Mean	Variance	Skewness	Kurtosis
Data	0.644696	0.000363	-0.497589	3.055038
GLDs	0.643633	0.000452	-0.220658	3.361925
Normals	-0.0001582	4.7692	-0.4981	3.0473

And

$$\begin{aligned} \boldsymbol{\theta_1} &= [0.62872205, -8.60218836, -0.07459090, -0.06747053], \\ \boldsymbol{\theta_2} &= [0.65570055, -7.02777408, -0.06938910, -0.04185058], \\ p &= 0.28135331, \end{aligned}$$

for the ML solutions. To make a comparison with our solutions, we give, in Figure 8.5–4, the q-q plots for both estimations. We also list the theoretical moments for all three types of estimations in Table 8.5–5.

From the q-q plots of all three types of estimations, we can see that our estimated distribution can control the tail weights much better. From the comparison of four theoretical moments, the partition ML estimated distribution is very far from the data on both skewness and kurtosis in comparison to ours. The ML estimated distribution is also farther away from the data on kurtosis in comparison to our estimations. All three estimated distributions have means and variances very close to the ones of the data.

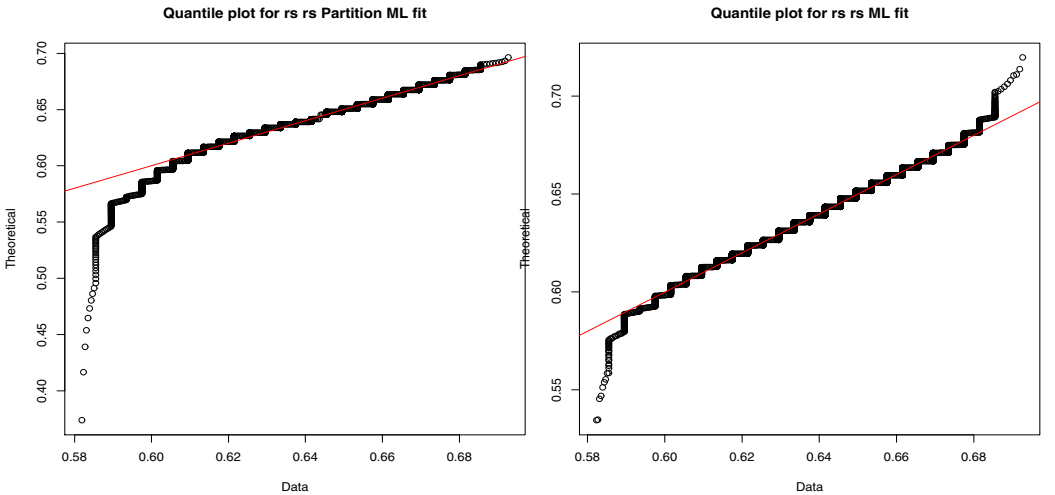


Figure 8.5–4. Left: q-q plot for the partition ML fitting scheme. Right: q-q plot for the ML solution.

Table 8.5–5. Comparison of four moments

	Mean	Variance	Skewness	Kurtosis	log-L
Data	0.644696	0.000363	-0.497589	3.055038	
Authors' Estimation	0.643633	0.000452	-0.220658	3.361925	2532
The Partition ML	0.644346	0.000472	-2.311851	30.451236	2650
The ML Estimation	0.644682	0.000382	-0.605247	4.224451	2561

In the last column in the above table, we also list the log-likelihood values obtained by these three different methods. We can see that the partition maximum likelihood has the highest value among all three methods. However, it also provides the worst match to the moments of the data. Although our method has the smallest log-likelihood value, it does not significantly differ from those of the other methods, and it has closer moments. We should mention that when fitting data by different ways and examining the goodness-of-fits, it is better to check both aspects: the log-likelihood values and first four moments obtained from different methods to decide which is better, instead of checking only one.

8.5.2 Cadmium in Horse Kidneys

This data is taken from Ellinder, Jönson, Piscator, and Rahnster (1981), which deals with histopathological changes due to different levels of metal concentrations in horse kidneys. Here we take part of their data dealing with the average cadmium concentrations in the kidney cortex of horses.

11.9	16.7	23.4	25.8	25.9	27.5	28.5	31.1	32.5	35.4
38.3	38.5	41.8	42.9	50.7	52.3	52.5	52.6	54.5	54.7
56.6	56.7	58.0	60.8	61.8	62.3	62.5	62.6	63.0	67.7
68.5	69.7	73.1	76.0	76.9	77.7	78.2	80.3	93.7	101.0
104.5	105.4	107.0							

Karian and Dudewicz (2000) use a single GLD(41.7897, 0.01134, 0.09853, 0.3606) to fit the data by using the method of moments. Now we use the algorithm to find estimates of the parameters in the mixture of two GLDs models for this data.

From the algorithm, the density function of the estimated mixture of GLDs is

$$f(x, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = pf(x, \boldsymbol{\theta_1}) + (1-p)f(x, \boldsymbol{\theta_2}),$$

where $\boldsymbol{\theta_1}$= [57.8233, 0.0076, 0.1432, 0.1356], $\boldsymbol{\theta_2}$= [56.2136, 0.0193, 0.4601, 0.4838], $p = 0.4000$ and

$$\begin{aligned} f(x, \boldsymbol{\theta_1}) &= \frac{0.0076}{0.1432y^{-0.8568} + 0.1356(1-y)^{-0.8644}}, \\ f(x, \boldsymbol{\theta_2}) &= \frac{0.0193}{0.4601y^{-0.5399} + 0.4838(1-y)^{-0.5162}}. \end{aligned}$$

Figure 8.5–6 shows the histogram of the data and the pdf of the fitted mixture distribution, and the Figure 8.5–7 shows the q-q plots for the single GLD and the mixture of GLDs. We also calculate the theoretical values of four moments for both fitted distributions in Table 8.5–8.

From the histogram and q-q plots, we can see that both the single GLD and the mixture of GLDs provide good fits. The comparison of the values of four moments indicates that the single GLD performs better than the mixture of two GLDs since the moment values of the single GLD are closer to the true values of the empirical data. However, the Kolmogorov-Smirnov resample test provided by the **R** package **GLDEX** (Su, 2007) shows that 922 times out of 1000, the p-value does not reject the null hypothesis, which indicates that the mixture of two GLDs provides a satisfactory fit.

8.5.3 Exchange Rate Data for Japanese Yen

Table 8.5–9 lists the exchange rate (ER) data for Japanese Yen as compared to the U.S. dollar from 1862 to 2003, which was studied by Nadarajah and Kotz (2004). They fit the transformed data (by log) with the standard Laplace and skew Laplace-normal distributions. Ali, Woo, and Pal (2008) fit the same transformed data with the double generalized Pareto and the skewed double generalized Pareto distributions, which they claimed was better than Nadarajah and Kotz's fit. Now, we consider fitting the mixture of two GLDs to the transformed data, then compare it with the other fits.

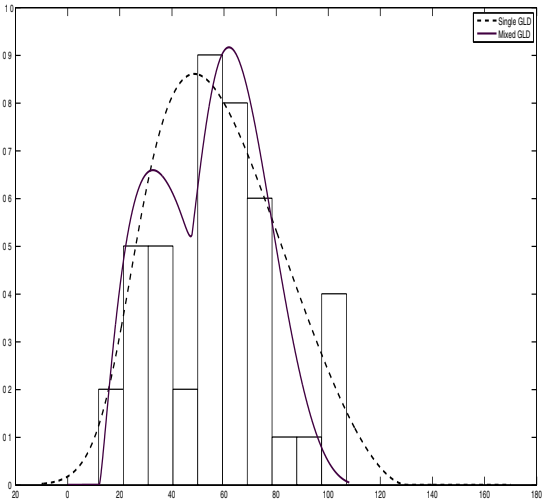


Figure 8.5–6. Histogram of Horse Kidney data and approximated two-GLD mixture.

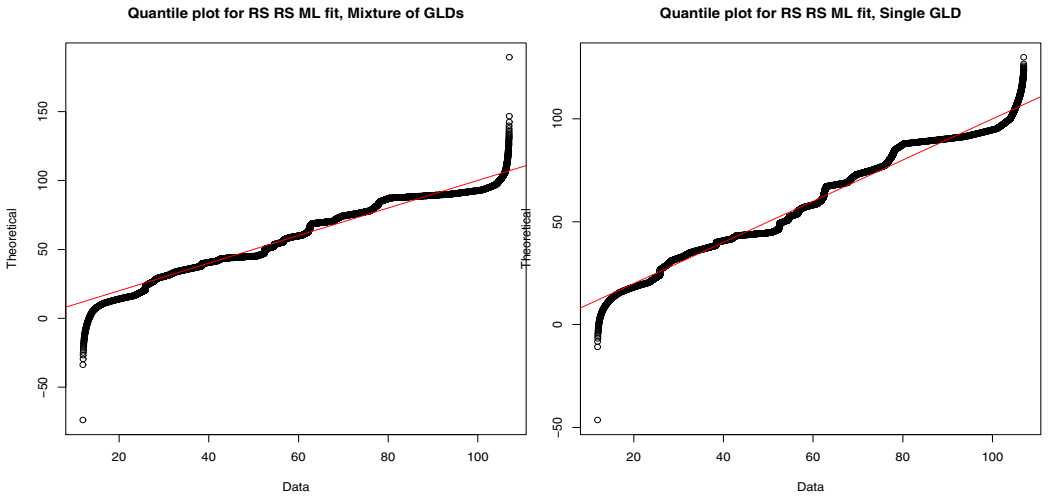


Figure 8.5–7. q-q plots of approximated mixture GLDs and single GLD.

Table 8.5–8. Comparison of four moments.

	Mean	Variance	Skewness	Kurtosis
Data	57.2325581	574.9808004	0.2511626	2.5212375
Single GLD	57.2605646	576.6131795	0.2545435	2.5257027
Two GLDs	56.8894479	611.8345689	-0.0144806	2.6169442

Table 8.5–9. Exchange rate data for Japanese Yen.

Year	ER	Year	ER	Year	ER	Year	ER
1862	0.5982	1897	2.0566	1933	3.244	1968	360
1863	0.5415	1898	2.0253	1934	3.481	1969	360
1864	0.8996	1899	2.0228	1935	3.476	1970	360
1865	0.9205	1900	2.0305	1936	3.545	1971	315.01
1866	0.9184	1901	2.0253	1937	3.441	1972	301.66
1867	0.9064	1902	2.0000	1938	3.7	1973	280.27
1868	0.9403	1903	2.0382	1939	4.264	1974	301.02
1869	0.9189	1904	2.0356	1940	4.309	1975	305.16
1870	0.9027	1905	2.0126	1941	4.305	1976	293.08
1871	0.9116	1906	2.0305	1942	4.3	1977	239.98
1872	0.9084	1907	2.0305	1943	4.29	1978	194.3
1873	0.9709	1908	2.0202	1944	4.29	1979	240.3
1874	0.9709	1909	2.0202	1945	15	1980	203.1
1875	1.0000	1910	2.0279	1946	15	1981	219.8
1876	0.9804	1911	2.0279	1947	50	1982	234.7
1877	1.0417	1912	2.0177	1948	270	1983	231.7
1878	1.0811	1913	2.0279	1949	360	1984	251.6
1879	1.0929	1914	2.04	1950	360	1985	200.25
1880	1.105	1915	2.01	1951	360	1986	157.473
1881	1.1081	1916	1.985	1952	360	1987	121.012
1882	1.1332	1917	1.965	1953	360	1988	124.931
1883	1.0959	1918	1.918	1954	360	1989	143.85
1884	1.1364	1919	2.005	1955	360	1990	135.4
1885	1.2085	1920	1.99	1956	360	1991	124.8
1886	1.2659	1921	2.073	1957	360	1992	124.8
1887	1.303	1922	2.046	1958	360	1993	111.8
1888	1.3158	1923	2.162	1959	360	1994	99.7
1889	1.2699	1924	2.581	1960	360	1995	103.35
1890	1.1976	1925	2.299	1961	360	1996	115.9
1891	1.2987	1926	2.0408	1962	360	1997	130.61
1892	1.4493	1927	2.1468	1963	360	1998	113.2
1893	1.8018	1928	2.1825	1964	360	1999	102.21
1894	2.0833	1929	2.0367	1965	360	2000	114.27
1895	1.9277	1930	2.0182	1966	360	2001	131.63
1896	1.937	1931	2.852	1967	360	2002	118.74
		1932	4.878			2003	107.31

For convenience, we take a log transformation of the original data, then use a mixture of GLDs to fit the transformed data. From the algorithm, the density function of the estimated mixture of GLDs is

$$f(x, \boldsymbol{\theta_1}, \boldsymbol{\theta_2}) = pf(x, \boldsymbol{\theta_1}) + (1-p)f(x, \boldsymbol{\theta_2}),$$

where $\boldsymbol{\theta_1}$= [0.2659, 5.2374, 0.5724, −0.0001], $\boldsymbol{\theta_2}$= [2.4184, 4.8586, −0.0040, 1.7860], $p = 0.5967$ and

$$\begin{aligned} f(x, \boldsymbol{\theta_1}) &= \frac{5.2374}{0.5724y^{-0.4276} + (-0.0001)(1-y)^{-1.0001}}, \\ f(x, \boldsymbol{\theta_2}) &= \frac{4.8586}{(-0.004)y^{-1.004} + 1.786(1-y)^{0.786}}. \end{aligned}$$

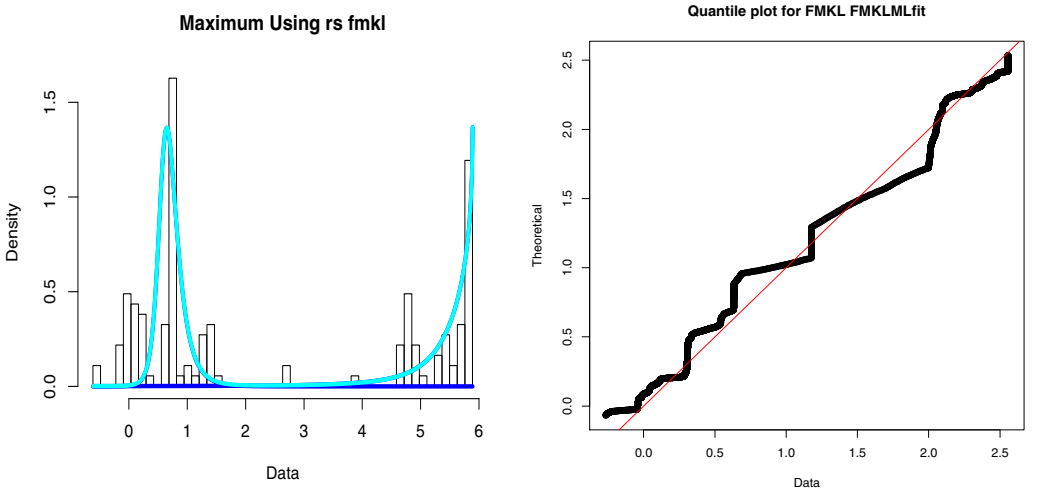


Figure 8.5–10. Left: Histogram of transformed Yen data and approximated mixture of two GLDs with q-q plot on the right.

Figure 8.5–10 shows a histogram of the data and the q-q plot of the fitted mixture distribution. From the q-q plot, we can see that the estimated mixture of two GLDs fits the data well. We also compare the theoretical values of four moments of the mixture of two GLDs with those of the empirical data in Table 8.5–11, which reveals that the values are very close.

We also obtained $-\ln L = 257.8513$. Nadarajah and Kotz (2003) fitted the transformed data using the standard Laplace distribution and skew Laplace-normal distribution by the method of maximum likelihood and obtained $-\ln L = 374.2$ for the standard Laplace distribution and $-\ln L = 299.9$ for the skew-normal distribution. Ali, Woo, and Pal (2008) also fitted the same data using the double generalized Pareto distribution and the skewed double generalized Pareto

Table 8.5–11. Comparison of four moments

	Mean	Variance	Skewness	Kurtosis
Data	1.101794	1.099263	0.356127	1.303753
Two GLDs	1.121942	0.979535	0.342064	1.294751

distribution, and they obtained $-\ln L$ values are 370.0703 and 297.0618, respectively. We evaluate the goodness-of-fit of these models with ours by the Akaike information criterion (AIC, Akaike, 1973), which selects the best model with the smallest AIC value by calculating

$$AIC = -2\ln L + 2p,$$

where p is the number of free parameters in the model. The AIC values for Nadarajah and Kotz's models are 754.4 and 605.8, and Ali, Woo, and Pal's models are 748.1406 and 602.124. The AIC value of our model is 531.7026. This indicates that our model provides a better fit than the other models.

8.6 Comparison with Normal Mixtures

Marron and Wand (1992) demonstrated that the family of normal mixtures is a very broad one. They represent this by giving 15 examples of the univariate normal mixture density with various combinations of components. In this section, we will show that the family of GLD mixtures is also a broad one by approaching several normal mixtures from those 15 examples. Even more, we will show that we may have some mixtures of GLDs with fewer components to provide a similar quality of fits as the normal ones with more components. To measure the similarity of the behaviors of two types of mixture models, Kullback and Leibler (1951) defined a distance function between the "true" probability distribution and a "target" probability distribution.

Definition 8.6.1 (Kullback-Leibler (KL) distance). *Let $f_1(x)$ be a true density function for a variable X and $f_2(x)$ be a target density function, which is used for X in practice. The KL distance is defined as*

$$I(1:2) = \int_E f_1(x) \ln \frac{f_1(x)}{f_2(x)} dx \tag{8.6.1}$$

where E is a Borel set on which $f_1(x)$ and $f_2(x)$ are well defined.

Theorem 8.6.2 *$I(1:2)$ is almost positive definite; that is, $I(1:2) \geq 0$, with equality if and only if $f_1(x) = f_2(x)$ for $x \in E$ except on some zero-measure set.*

The smaller the KL distance, the closer the target density function is to the true density function. Specifically, if the target distribution we use to fit the

data in practice equals the true distribution, then the KL distance is 0. We also calculate the overlapping coefficient $\delta(f_1(x), f_2(x))$ which is defined as

Definition 8.6.3. *(Overlapping Coefficient). If $f_1(x)$ and $f_2(x)$ are two density functions, then overlapping coefficient $\delta(f_1, f_2)$ is defined as*

$$\delta(f_1, f_2) = \int \min(f_1(x), f_2(x))dx. \tag{8.6.2}$$

The overlapping coefficient measures the amount of similarity between two distributions. The larger the coefficient, the greater the similarity of these two functions. In fact, $\delta(f_1, f_2)$ is the area that is under both the pdf's. Use of $\delta(f_1, f_2)$ in statistics dates back to at least 1963, in nonparametric statistics to at least 1975, and it is currently widely used in biology, economics, and statistics (see Mulekar, Gonzales, and Aryal (2008) and Dudewicz and Mishra (1988)).

To compare the mixture of GLD models with the mixture of normal models, we choose four normal mixtures from fifteen normal mixture models in Marron and Wand (1992), and simulate a mixture of three normal distributions. We estimated the parameters of the mixture of two GLDs for each case.

(1) Bimodal density

$$\begin{aligned} f_1(x) &= \frac{1}{2}N(-1, (2/3)^2) + \frac{1}{2}N(1, (2/3)^2), \\ f_2(x) &= \frac{1}{2}\text{GLD}(-1.071, 0.1, 0.037, 0.048), \\ &+ \frac{1}{2}\text{GLD}(1.055, 0.132, 0.064, 0.049), \\ I(1:2) &= 0.0039, \ \ \delta(f_1, f_2) = 0.9014. \end{aligned}$$

See Figure 8.6–1 (a).

(2) Outlier density

$$\begin{aligned} f_1(x) &= \frac{1}{10}N(0, 1) + \frac{9}{10}N(1, (1/10)^2), \\ f_2(x) &= \frac{1}{10}\text{GLD}(-0.2, 0.32, 0.1449, 0.35), \\ &+ \frac{9}{10}\text{GLD}(1.085, 1.2, 0.1649, 0.0225), \\ I(1:2) &= 0.0293, \ \ \delta(f_1, f_2) = 0.8391. \end{aligned}$$

See Figure 8.6–1 (b).

(3) Kurtotic Unimodal density

$$f_1(x) = \frac{2}{3}N(0, 1) + \frac{1}{3}N(1, (1/10)^2),$$

$$\begin{aligned} f_2(x) &= \frac{2}{3}\text{GLD}(0.1, 0.1975, 0.1449, 0.2349), \\ &+ \frac{1}{3}\text{GLD}(1.0, 0.1975, 0.230, 0.02), \\ I(1:2) &= 0.0901, \ \ \delta(f_1, f_2) = 0.9540. \end{aligned}$$

See Figure 8.6–2 (a)

(4) Strongly Skewed density

$$\begin{aligned} f_1(x) &= \sum_{i=0}^{7} \frac{1}{8} N(3(2/3)^i - 1, (2/3)^{2i}), \\ f_2(x) &= \frac{4}{5}\text{GLD}(-0.68, -0.65, -0.03, -0.6), \\ &+ \frac{1}{5}\text{GLD}(-0.85, -0.9, -0.03, -0.4), \\ I(1:2) &= 0.0106, \ \ \delta(f_1, f_2) = 0.9473. \end{aligned}$$

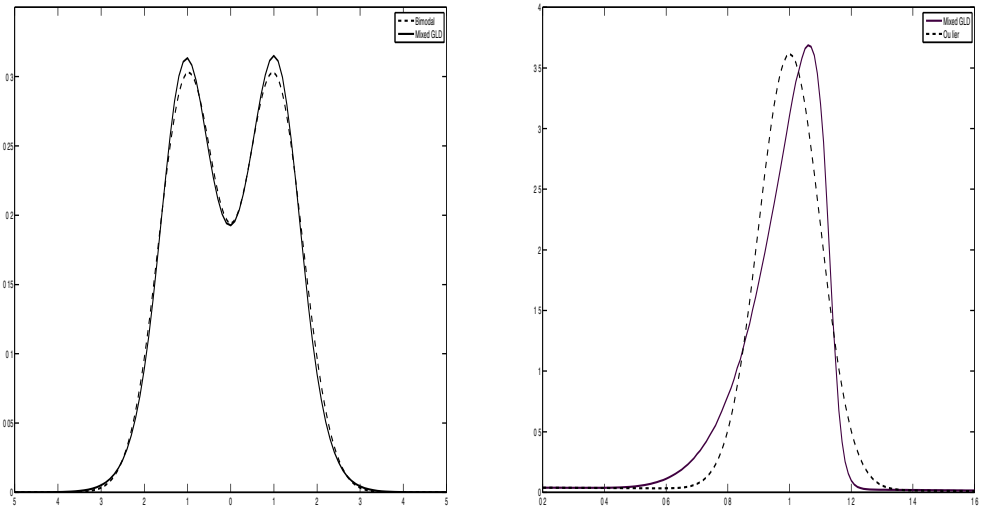
See Figure 8.6–2 (b).

(5) Now we consider the following mixture of three normal distributions (the true model) and the estimated mixture of GLDs. The comparison of the density curves of these two types of mixtures is shown in Figure 8.6–3 (notice the high quality of these fits in both cases). We also compare the density curves of a single GLD and the normal mixture.

$$\begin{aligned} f_1(x) &= \frac{1}{3}N(-1, 1) + \frac{1}{3}N(0, 1) + \frac{1}{3}N(1, 1), \\ f_2(x) &= \frac{9}{10}\text{GLD}(0.223, 0.181, 0.199, 0.142), \\ &+ \frac{1}{10}\text{GLD}(0.1095, -1.9951, -0.94, -1), \\ I(1:2) &= 0.013, \ \ \delta(f_1, f_2) = 0.9579. \end{aligned}$$

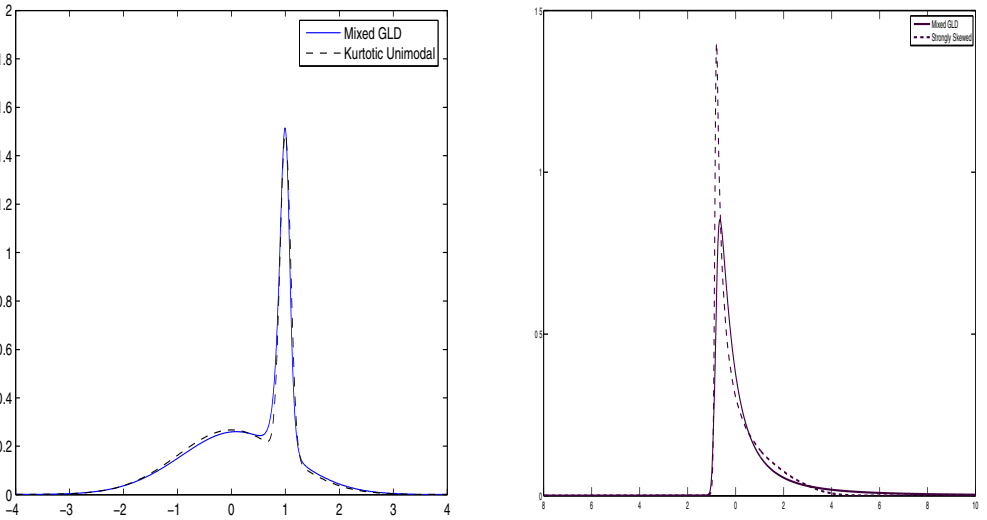
Remarks:

(1) In the above, we compare the mixture of GLDs and normal distributions by using the KL distance and overlapping coefficient. To find the mixture of GLDs in each case, we do not follow the steps in the Newton-Krylov algorithm used in Section 8.6 since no real data is available. We have developed a MATLAB® program to calculate the KL distance and the overlapping coefficient for each case with given normal mixtures starting with valid initial values of GLDs. We stop the iteration when the KL distance is relatively small and the overlapping coefficient is relatively large. The above results indicate that the GLD mixture can approximate the normal mixture well.



(a) (b)

Figure 8.6–1. Left: Bimodal density and approximated GLD mixture; right: Outlier density and approximated GLD mixture.



(a) (b)

Figure 8.6–2. Left: Kurtotic Unimodal density and approximated GLD mixture; right: Strongly Skewed density and approximated GLD mixture.

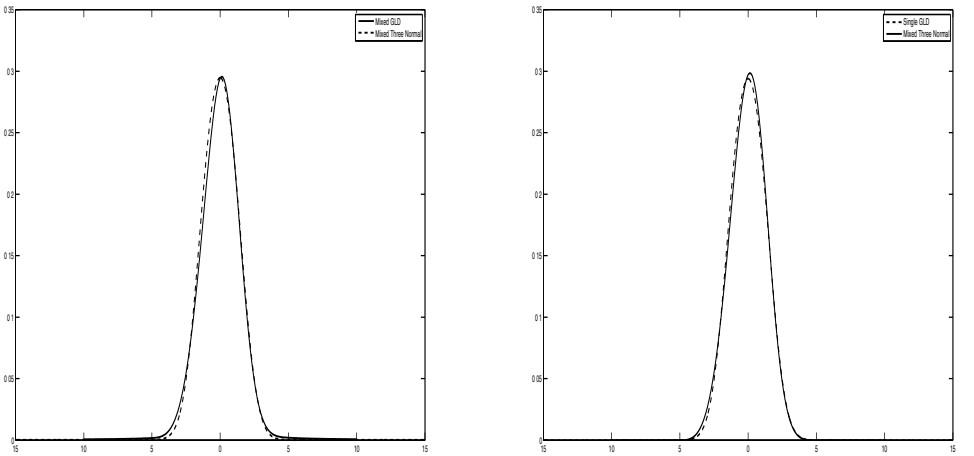


Figure 8.6–3. Left: Mixture of two GLDs and Normal mixture; Right: Single GLD and Normal mixture.

In some cases, we can use fewer components of GLDs to approach the normal mixture with more components. For instance, in the strongly skewed density model, we use the mixture of two GLDs to approach the mixture of eight normal distributions. The KL distance is 0.0106 and the overlapping coefficient is 0.9473, which indicates that we do have a good approximation. For the other cases, our KL distances are also relatively small, and the overlapping coefficients are close to 1.

(2) We also use a single GLD, GLD$(0.2204573, 0.1795412, 0.1980730, 0.1417218)$, to approximate the three normal mixtures above. We sketch the density curves of this single GLD and the normal mixture on the right graph, which is shown in Figure 8.6–3. The graphs in Figure 8.6–3 indicate that both two GLDs and the single GLD can approximate the mixture of three normal distributions quite well. In addition, we calculate the KL distance and the overlapping coefficient δ to provide a more quantitative assessment of these two approximations. For the single GLD, KL=0.011, and $\delta = 0.8349$; for the mixture of two GLDs, KL=0.013, and $\delta = 0.9579$. We can see the KL distances for both approximations are close to each other, but the δ value of the mixture of two GLDs is much better than that of the single GLD. The point we are trying to make here is: the choice of a better approximation depends on the situation we have or the degree of accuracy we want to achieve. In this case, for example, we will obtain a better approximation by choosing the mixture of two GLDs. However, the price we pay is that we need to estimate more parameters, and it may cause us more difficulties.

(3) The other issue we should note here is the computation time. In our program, we actually did exhaustive searches for the estimations of all the nine parameters, in order to make the KL distance relatively small, and the overlapping coefficient is relatively large at the same time. During the computation process, we found that the values of the KL distance and overlapping coefficient δ are very sensitive to the values of λs, that is, a small change of λs will provide some significant changes of KL and δ. We take the step length to be 0.01 for each λ in each iteration. The readers can apply our algorithm to find estimations by using different criteria of KL and δ, for instance, $KL < 0.001$ and $\delta > 0.99$, if interested. A listing of the actual code, in C, is given in the Appendix at the end of this chapter.

8.7 Conclusions and Research Problems Regarding the Mixture of Two GLDs

In this chapter, we discuss the mixture distribution of two GLDs and use the Newton-Krylov algorithm to estimate the parameters in the mixture model for given data. Several graphs of the density curves of the mixtures of two GLDs with a different combination of parameters are provided to show the change of the shapes. To make a comparison with the normal family, the KL distance and δ are used to measure the similarity of behaviors between these two families.

Using the GLD family has several advantages compared to the other distributions because of its flexibility and generality. A GLD has four parameters with the location and scale determined by λ_1 and λ_2, and the skewness and kurtosis is determined by λ_3 and λ_4. This makes the GLD family more flexible, while fitting a data, compared to the other distribution families. For instance, the normal family of distributions cannot be used to match data successfully unless the data is symmetric and has tail weight similar to that of the normal. In addition, the GLD family of distributions with additional parameters (λ_3, λ_4) allow us to match more than the center and variability of the data.

We also know that the GLD family of distributions can approximate many well-known families of distribution very well (Karian and Dudewicz (2000)). This indicates that the GLD family performs almost as well as these distributions while fitting a data with fewer restrictions on the assumptions of the distribution of the data.

Due to its high generality and flexibility, the mixture of GLDs also shows good properties. One of which is using fewer components when fitting a data, compared to the mixture of normal distributions. In Section 8.6, for the comparison (4), we use the mixture of only two GLDs to approach the strongly skewed density, while using the mixture of eight normal distributions to approach the same density and both perform about equally well (KL distance=0.0106 and $\delta = 0.9473$). In comparison (5), we simulate a mixture of three normal distributions, and use

a mixture of two GLDs to approach it with KL distance 0.013 and overlapping coefficient 0.9579.

In the density function $f(x)$ of a GLD defined in Section 8.1, y is defined to include 0 and 1. It should be noted that when y is 0 and $(\lambda_3 - 1) < 0$, or y is 1 and $(\lambda_4 - 1) < 0$, the expression in Theorem 8.1.2 be undefined due to a zero denominator. To avoid this situation in our program, for a given set of λs, we solve $Q(y)$. Then we check whether $y = 0$ and $(\lambda_3 - 1) < 0$ or $y = 1$ and $(\lambda_4 - 1) < 0$ or both hold. If one of them holds, or both of them hold, which means these λs will not lead to a valid distribution, we skip these λs and go to next step to choose a new set of λs.

The limitation of our algorithm is that the estimation time may be longer, especially for large data. This is due to the searching of the initial values. We chose appropriate initial values of λ_3 and λ_4 in the valid regions (Karian and Dudewicz (2000)), then obtained λ_1 and λ_2 by using a percentile method. If all the values specify a valid GLD, we use the Newton-Krylov algorithm to solve the non-linear equations; otherwise, we change the initial values of λ_3 and λ_4 by taking the step length 0.01. Actually, we do an exhaustive search here until we reach the maximum number of iterations or the error gets below the preset tolerance level. Both, the maximum number of iterations and the error tolerance level impact computation time.

To find a mixture of two GLDs which can approach a normal mixture with a reasonably small KL distance and large overlapping coefficient, we set up intervals from which we take initial values of all the parameters, since we cannot use the percentile method without real data, and change the initial values by step length 0.01 if they specify an invalid GLD or not reach the requirement of KL distance and overlapping coefficient. Future research on this will involve narrowing the choices for the intervals for all the parameters, which may shorten the searching time and speed up the fitting procedure.

In this chapter we have focused on a mixture of two GLDs. We now note some issues related to such mixtures:

(1) **Testing the homogeneity of the mixture distribution.** This is a common question while dealing with mixture distributions. For given data, before estimation, we prefer to check if the data comes from a single distribution or a mixture of distributions because we are always looking for the simplest model to describe the data. In general, researchers have contributed to this field by creating many distributions, such as normal distributions, exponential distributions. The likelihood ratio test is the traditional method to test the hypothesis of homogeneity by either obtaining the exact null distribution or an asymptotic null distribution of the test statistic. However, due to the complexity of the GLD density function, it is difficult to derive the exact null distribution of the test statistic; therefore, we will seek its asymptotic distribution in future research. The other way to test such a

hypothesis is using an information approach with the Akaike Information Criterion (AIC) or Schwarz Information Criterion (SIC) to select the best model for a given data.

(2) **Accessing the number of the components in mixture models.** If we accept the alternative hypothesis of a mixture of GLDs, then the next question is to estimate the number of the component density functions in the mixture model. McLachlan and Peel (2000) proposed the LTR statistic and penalized likelihood criteria, such as AIC, BIC, and SIC to test the null hypothesis of the number of components being a given constant mainly on normal mixtures. We plan to follow their idea to derive the LRT statistic or penalized likelihood criteria for a mixture of GLDs.

(3) **Extending to the multivariate case.** We have obtained some results for the univariate case. However, in some situations, the data will get involved with higher dimensions, such as in the analysis of microarray gene expression data. Therefore, we plan to investigate the case of a mixture of multivariate GLDs in future research.

References for Chapter 8

Akaike, H. (1974). "A new look at statistical model identification," *IEEE Transactions on Automatic Control,* **19,** 716–723.

Ali, M.M., Woo, J., and Pal, M. (2008). "Some skew-symmetric reflected distributions," *American Journal of Mathematical and Management Sciences,* 28, Nos. 1&2, 41–59.

Bickel, P.J. and Chernoff, H. (1993). "Asymptotic distribution of the likehood ratio statistic in a prototypical nonregular problems," *Statistics and Probability: A Raghu Raj Bahadur Festschrift,* 83–96, New York: Wiley.

Chernoff, H. and Lander, E. (1995). "Asymptotic distribution of the likelihood ratio test that a mixture of two binomials is a single binomial," *Journal of Statistical Planning Inference* **43,** 19–40.

Chen, H. and Chen, J. (2001). "The likelihood ratio test for homogeneity in the finite mixture models," *The Canadian Journal of Statistics,* **29,** 201-215.

Chen, H., Chen, J., and Kalbfleisch, J.D. (2001). "A modified likelihood ratio test for homogeneity in the finite mixture models," *Journal of the Royal Statistical Society: Series B,* **63,** 19–29.

Chen, R. C. H., and Traylor, L. (1995). "Non-regular maximum likelihood problemes (with discussions)," *Journal of the Royal Statistical Society: Series B,* **57**, 3–44.

Dacunha-Castelle, D. and Gassisat, E. (1999). "Testing the order of a model using locally conic parameterization population mixtures and stationary ARMA processes," *The Annals of Statistics,* **27,** 1178–1209.

Dudewicz, E. J., and Mishra, S. N. (1988). *Modern mathematical Statistics*, John Wiley & Sons, New York.

Ellinder, C. G., Jönson, L., Piscator, M., and Rahnster, B. (1981). "Histopathological changes in relation to cadmium concentration in horse kidneys," *Environmental Research,* **26,** 1–21.

Furman, D. W. and Lindsay, B. G. (1994). "Measuring the relative effectiveness of moment estimators as starting values in maximizing likelihoods," *Computational Statistics and Data Analysis*, **17,** 493–507.

Ghosh, J. K. and Sen, P. K. (1985). "On the Asymptotic Performance of the Log Likelihood Ratio Statistic for the Mixture Model and Related Issues," in *Proc. Berkeley Conf. in Honor of J. Neyman and J. Kiefer*(eds. L. LeCam and R. A. Olshen), **2,** 789–806. Belmont: Wadsworth.

Gupta, A.K. and Miyawolei, T. (1978). "On a uniform mixture model," *Biometrical Journal,* **20,** 631–637.

Gupta, A.K., Taneja, V.S., and Janardan, K.G. (1986). "Estimation of an arbitary mixture of uniform models," *Metron,* **44,** 247–256.

Hartigan, J.A. (1985). "A Failure of Likelihood Asymptotic for Normal Mixtures," *Proc. Berkeley Conf. in Honor of J. Neyman and J. Kiefer,* **2,** 807–810. Belmont: Wadsworth.

Karian, Z.A. and Dudewicz, E.J. (2000). *F*itting Statistical Distributions, The Generalized Lambda Distribution and Generalized Bootstrap Methods, Boca Raton, FL: CRC Press.

Kelley, C. T. (1995). *Iterative Methods for Linear and Nonlinear Equations*, Society for Industrial and Applied Mathematics, Philadelphia.

Knoll, D. A. and Keyes, D. E. (2004). "Jacobian-free Newton-Krylov methods: A survey of approaches and applications," *Journal of Computational Physics,* **193(2)**.

Kullback, S. and Leibler, R. A. (1951). "On information and sufficiency," *Annals of the Institute of Statistical Mathematics.,* **22,** 79–86.

Lemdani, M. and Pons, O. (1999). "Likelihood ratio tests in contamination models," *Bernoulli,* **5,** 705–719.

Lindsay, B.G. (1989). "Moment matrices: Applications in mixtures," *Annals of Statistics,* **17,** 722–740.

Lindsay, B.G. (1995). *Mixture models: Theory, Geometry and Applications,* Hayward: Institute for Mathematical Statistics.

Lindsay, B.G. and Basak, P. (1993). "Multivariate normal mixtures: A fast

consistent method of moments," *Journal of the American Statistical Association*, **88,** 468–476.

Marron, J.S. and Wand, M.P. (1992). "Exact mean integrated squared error," *Annals of Statistics,* **20,** 712–736.

McLachlan, G.J. and Peel, D. (2000). *Finite Mixture Models,* John Wiley & Son, New York.

McLachlan, G.J. and Basford, K.E. (1988). *Mixture Models: Inference and Application to Clustering,* Dekker, New York. Dekker.

Mulekar, M.S., Gonzales, S., and Aryal, S. (2008). "Bias and variance estimates for the overlap of two exponential populations," *American Journal of Mathematical and Management Sciences,* **28,** 61–79.

Nadarajah, S. and Kotz, S. (2003). "Skewed distributions generated by the normal kernel," *Statistics and Probability Letters*, **65,** 269–277.

Nadarajah, S. and Kotz, S. (2004). "Skewed distributions generated by the Laplace kernel," *American Journal of Mathematical and Management Sciences*, **24**, 321–349.

Neyman, J. (1959). "Optimal aymtotic tests for composite hypotheses," in: Grenander, Ulf (Ed.), *Probability and Statistics: The Harald Cramer Volume,* John Wiley & Son, New York.

Neyman, J. and Scott, E.L. (1966). "On the use of $C(\alpha)$ optimal tests of composite hypotheses (with discussion)," *Bull. Inst. Int. Statist,* **41,** 477–497.

Ning, W., Gao, Y. C. and Dudewicz, E. J. (2008). "Fitting mixture distributions using generalized lambda distributions and comparison with normal mixture," *American Journal of Mathematical and Management Sciences*, **28**, Nos. 1&2, 81–99.

Ning, W., Gupta, A. K., Yu., C. and Zhang, S. G., (2009). "A moment-based test for homogeneity in finite mixture models," *Communication in Statistics-Theory and Methods,* **38**, 1371–1382.

Ning, W., Zhang, S. G. and Yu, C. (2009). "A moment-based test for the homogeneity in mixture natural exponential family with quadratic variance functions," *Statistics and Probability Letters,* **79**(6), 828–834.

Pearson, K. (1894). "Contributions to the mathematical theory of evolution," *Philosophical Transactions of the Royal Society of London A.* **185,** 71–110.

Pearson, K. (1895). "Contributions to the mathematical theory of evolution, II: Skew variation," *Philosophical Transactions of the Royal Society of London A.* **186,** 343–414.

Ramberg, J.S. and Schmeiser, B.W. (1972). "An approximate method for generating symmetric random variables," *Comm. ACM,* **15,** 987–990.

Ramberg, J.S. and Schmeiser, B.W. (1974). "An approximate method for generating symmetric random variables," *Comm. ACM,* **17,** 78–82.

Ramberg, J.S, Tadikamalla, P.R., Dudewicz, E.J., and Mykytka, E.F. (1979). "A probability distribution and its uses in fitting data," *Technometrics,* **21,** 201–214.

Su, S. (2007). "Fitting single and mixture of generalized lambda distributions to data via discretized and maximum likelihood methods: GLDEX in **R**," *Journal of Statistical Software,* **21,** 9.

Titterington, D.M., Smith, A.F., and Makov, U.E. (1985). *Statistical Analysis of Finite Mixture Distributions,* John Wiley & Son, New York.

Tukey, J. W. (1960). "The practical relationship between the common transformations of percentages of counts and of amounts," *Technical Report,* **36,** Statistical Techniques Research Group, Princeton University.

Weldon, W.F.R. (1892). "Certain Correlated Variations in *Carcinus vulgaris*," *Proceedings of the Royal Society of London,* **51,** 2–21.

Weldon, W.F.R. (1893). "Certain Correlated Variations in *Carcinus moenas*, *Proceedings of the Royal Society of London,* **54,** 318–329.

Appendix for Chapter 8: Code for Various Computations

In this appendix we list the actual code that was used for the computations that were carried out in this chapter. The Newton-Krylov method (program `nsoli.m`) is the central part of the collection of programs. It is a Jocobian-free method compared to the traditional Newton method and is used to solve the nonlinear equations to obtain MLEs while fitting the mixture of GLDs to the data. The `MLE.m` function is the program that sets up the nine nonlinear equations of partial derivatives of the log-likelihood function. This program is called by the program `nsoli.m`. The `GLD.m` program is used to compute the density function at a given value, which, in turn, is called by the program `MLE.m`. The program `Kullback.m` is used to calculate the Kullback distance between the mixture of GLDs and the mixture of normals. The Gaussian quadrature (program `QUAD1D.m`) is used to calculate the integral corresponding to KL distance, which is called by `Kullback.m`. The `GAULEG.m` program is used to compute points on the horizontal axis to evalute their weights for the Gaussian Quadrature one-dimensional integral, `QUAD1D.m`.

The last two sections of the appendix give examples of two uses of the programs. One to plot the single and mixed fits for the cadmium in horse kidneys data and the other for obtaining a GLD mixture fit for the exchange rate data.

A.1 Calculation of the KL Distance: `Kullback.m`

```
 function [out] = Kullback(p, LAMBDA)
%
% Computes the Kullback distance.
%
% inputs:
%      - p:
%      - LAMBDA: vector of dimension 8;
% calls:
%     - Interval, QUAD1D;
%
% September 2007
%
    [a, b] = Interval(LAMBDA);
%
    if (a < -50)
        a = -50;
    end
    if (b > 50)
        b = 50;
    end
%
% calls integral QUAD1D to computes KL distance;
```

```
    out = QUAD1D(a,b,p,LAMBDA);

%END of Kullback.
```

A.2 Integral Using Gaussian Quadrature: QUAD1D.m

```
function [out] = QUAD1D(X1, X2, p, LAMBDA)
%
% Computes the integral corresponding to KL distance using
% Gaussian Quadrature;
%
% [out] = QUAD1D(X1, X2, p, LAMBDA)
%
% inputs:
%      - X1, X2: interval of integral;
%      - p:
%      - LAMBDA: parameters of GLD with dim 8;
% outputs:
%      - out: integral result;
% calls:
%      - GAULEG (Gaussian Quadrature weights);
%      - FUNC: integrand;
%
% September 2007.

% points and weights of Gaussian Quadrature;
[X, W] =  GAULEG(X1, X2, 20);
%
out = 0.0;
%
for J = 1:20
   out = out + W(J) * FUNC(X(J), p, LAMBDA);
end

% ------------------------------------------------------------------
function [out] = FUNC(x, p, LAMBDA)
% integrand;
out = F1(x) * log(F1(x) / F2(x,p,LAMBDA(1),LAMBDA(2),LAMBDA(3),LAMBDA(4),...
      LAMBDA(5),LAMBDA(6),LAMBDA(7),LAMBDA(8)));
```

A.3 Gaussian Quadrature to Calculate One-Dimension Integral: - GAULEG.m

```
function [X, W] = GAULEG(X1, X2, N)
%
% Computes Gaussian Quadrature integral;
%
% [X, W] = GAULEG(X1, X2, N)
```

```
%
% input:
%     - [X1, X2]: integral interval;
%     - N: number of points and weights to evaluate;
% outputs:
%     - X: points on x-axis to evaluate;
%     - W: weights corresponding to X;
%
% Yunhuan Gao, May 2000.
%
% tolerance;
EPS = 3.0e-14;
%
M = (N+1) / 2.0;
XM = 0.5 * (X2 + X1);
XL = 0.5 * (X2 - X1);

for I =1:M
   Z = cos(pi * (I - 0.25) / (N + 0.5));
%
   flag = 0;
   while (flag == 0)
        P1 = 1.0;
        P2 = 0.0;
        for J = 1:N
            P3= P2;
            P2 = P1;
            P1 = ((2.0 * J - 1.0) * Z * P2 - (J - 1.0) * P3) / J;
        end
        PP = N * (Z * P1 - P2) / (Z * Z - 1.0);
        Z1 = Z;
        Z = Z1 - P1/PP;

        if (abs(Z-Z1)>EPS )
            flag = 0;
        else
            flag = 1;
        end
   end
%
   X(I) = XM - XL * Z;
   X(N + 1 - I) = XM + XL * Z;
   W(I) = 2.0 * XL / ((1.0 - Z * Z) * PP * PP);
   W(N + 1 - I) = W(I);
end
% END of GAULEG.
```

A.4 Compute GLD Density Function at x with Four Parameters of Distribution: `GLD.m`

```
function [out] =  GLD(x, lambda1, lambda2, lambda3, lambda4)
%
% GLD computes GLD density function at x;
%
% [out] =  GLD(x, lambda1, lambda2, lambda3, lambda4);
% inputs:
%      - x: evaluated value;
%      - lambda1,lambda2,lambda3,lambda : parameters of GLD;
% outputs:
%      - out: density function;
% calls:
%      - SignTest (test the sign of function g);
%      - Q;
%
% notice:
%      - for given parameters Lambdas', solve for x given y in [0,1];
%      - y(x) is monotonic;
%      - the program uses bisection root finding method;
%
% September 2007
%
% set up tolerance of root finding;
tol = 1e-4;
%
% [init0,init1] : interval for y;
init0 = 0.0;
init1 = 1.0;
%
out = 0;
%
maxiter = 500;
iter = 1;
%
try
    if (SignTest(lambda1,lambda2,lambda3,lambda4) == 0)
        msg = 'g does not have constant sign';
        error(msg);
    end

    if ((Q(init0, lambda1,lambda2,lambda3,lambda4)-x) ...
            * (Q(init1, lambda1,lambda2,lambda3,lambda4)-x) < 0)
        mid = (init0 + init1 ) / 2.0;
        err = Q(mid,lambda1,lambda2,lambda3,lambda4) - x;
        while(abs(err)>tol && iter<maxiter)
            if (sign(Q(init0,lambda1,lambda2,lambda3,lambda4)-x) ==
                    sign(Q(mid,lambda1,lambda2,lambda3,lambda4)-x))
```

```
                init0 = mid;
            else
                init1 = mid;
            end
            mid = (init0 + init1 ) / 2.0;
            err = Q(mid,lambda1,lambda2,lambda3,lambda4) - x;
            iter = iter + 1;
        end
%
        out = lambda2 / ( lambda3 * mid^(lambda3-1) +
              lambda4 * (1-mid)^(lambda4-1) );

    else
        msg = 'No solution x for give interval of y.';
        error(msg);
    end
%
catch
end
% END of GLD

% ------------------------------------------------------------
function [out] = SignTest(lambda1, lambda2, lambda3, lambda4)
%
% Test the sign of g() and lambda2;
%
out =1.0;
s = sign(lambda2);
%
for i = 0.01:0.01:1.0
   if (sign(lambda3*i^(lambda3-1)+lambda4*(1-i)^(lambda4-1)) == s)
       out = out*1;
   else
       out = out * 0;
   end
end
% END of SignTest;
%
function [out] = Q(y, lambda1, lambda2, lambda3, lambda4)
%
% function Q()
%
out = lambda1 + ( y^lambda3 - ( 1 - y)^lambda4 ) / lambda2;
```

A.5 Setup the Nonlinear Equations for MLE: `MLE.m`

```
function [out] = MLE(U)
%
```

```
% This function sets up the nonlinear equations for MLE
%
% INPUT:
% U is input variables arranged as
% U=[p, lambda1,lambda2,lambda3,lambda4,lambda5,lambda6,lambda7,lambda8]
%
% Read x's from data file Pearson.data
x = load('Pearson.data');
n = length(x);
%
p = U(1);
%
% For each x(i), calculate y(i) and GLD(i) (= f);
for i = 1 : n
   [y1(i), GLD1(i)] = f(x(i), U(2), U(3), U(4), U(5));
   [y2(i), GLD2(i)] = f(x(i), U(6), U(7), U(8), U(9));
end

%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Note that the output of MLE has 9 components;
MLE1 = 0; MLE2 = 0; MLE3 = 0; MLE4 = 0; MLE5 = 0; MLE6 = 0;
MLE7 = 0; MLE8 = 0; MLE9 = 0;
%
for i = 1 : n
   MLE1 = MLE1 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * p *
          df1([U(2),U(3),U(4),U(5)], y1(i));
   MLE2 = MLE2 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * p *
          df2([U(2),U(3),U(4),U(5)], y1(i));
   MLE3 = MLE3 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * p *
          df3([U(2),U(3),U(4),U(5)], y1(i));
   MLE4 = MLE4 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * p *
          df4([U(2),U(3),U(4),U(5)], y1(i));
%
   MLE5 = MLE5 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * (1-p) *
          df1([U(6),U(7),U(8),U(9)], y2(i));
   MLE6 = MLE6 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * (1-p) *
          df2([U(6),U(7),U(8),U(9)], y2(i));
   MLE7 = MLE7 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * (1-p) *
          df3([U(6),U(7),U(8),U(9)], y2(i));
   MLE8 = MLE8 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) * (1-p) *
          df4([U(6),U(7),U(8),U(9)], y2(i));
%
   MLE9 = MLE9 + 1.0 / (p * GLD1(i) + (1-p) * GLD2(i)) *
          ( GLD1(i) - GLD2(i) ) ;
end
%
out = [MLE1; MLE2; MLE3; MLE4; MLE5; MLE6; MLE7; MLE8; MLE9];
```

```
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% The formulae for partial derivatives
function [out] = df1(LAMBDA,y)
L1 = LAMBDA(1); L2 = LAMBDA(2); L3 = LAMBDA(3); L4 = LAMBDA(4);
out = - L2^2 * (L3 * (L3-1) * y^(L3-2) - L4 * (L4-1) * (1-y)^(L4-2)) ...
    / (L3*y^(L3-1)+L4*(1-y)^(L4-1))^3;
%out = - L2^2 * (L3 * (L3-1) * y^(L3-2) - L4 * (1-y)^(L4-2)) ...
%    / (L3*y^(L3-1)+L4*(1-y)^(L4-1))^3;
%
function [out] = df2(LAMBDA,y)
L1 = LAMBDA(1); L2 = LAMBDA(2); L3 = LAMBDA(3); L4 = LAMBDA(4);
out = ( y^(L3) - (1-y)^L4 ) * ( L3*(L3-1)*y^(L3-2) - ...
          L4*(L4-1)*(1-y)^(L4-2) ) / ( L3*y^(L3-1) + L4*(1-y)^(L4-1) )^3;
%out = ( y^(L3) - (1-y)^L4 ) * ( L3*(L3-1)*y^(L3-2) - ...
%          L4*(1-y)^(L4-2) ) / ( L3*y^(L3-1) + L4*(1-y)^(L4-1) )^3;
%
function [out] = df3(LAMBDA,y)
if (y == 0)
    out = 0;
else
L1 = LAMBDA(1); L2 = LAMBDA(2); L3 = LAMBDA(3); L4 = LAMBDA(4);
out = L2 * y ^(L3) * log(y) * ( L3*(L3-1)*y^(L3-2) - ...
         L4*(L4-1)*(1-y)^(L4-2) ) / ( L3*y^(L3-1)+L4*(1-y)^(L4-1) )^3;
%out = L2 * y ^(L3) * log(y) * ( L3*(L3-1)*y^(L3-2) - ...
%         L4*(1-y)^(L4-2) ) / ( L3*y^(L3-1)+L4*(1-y)^(L4-1) )^3;
end
%
function [out] = df4(LAMBDA,y)
L1 = LAMBDA(1); L2 = LAMBDA(2); L3 = LAMBDA(3); L4 = LAMBDA(4);
out = L2*(1-y)^(L4)*log(1-y)* ( L3*(L3-1)*y^(L3-2) - ...
          L4*(L4-1)*(1-y)^(L4-2) ) / ( L3*y^(L3-1) + L4*(1-y)^(L4-1) )^3;
%out = L2*(1-y)^(L4)*log(1-y)* ( L3*(L3-1)*y^(L3-2) - ...
%          L4*(1-y)^(L4-2) ) / ( L3*y^(L3-1) + L4*(1-y)^(L4-1) )^3;


function [a, b] = Interval(LAMBDA)
%
% Computes the range of Q() for y in [0,1];
%
% [a, b] = Interval(LAMBDA)
%
% inputs:
%      - LAMBDA: a vector of dim 8;
% outputs:
%      - [a,b]: interval;
%
% September 2007
%
```

```
y = 0.01:0.01:1.0;
%
for i = 1:100
    range(i) = Q(y(i),LAMBDA(1),LAMBDA(2),LAMBDA(3),LAMBDA(4));
end

a1 = min(range); b1 = max(range);
%
for i = 1:100
    range(i) = Q(y(i),LAMBDA(5),LAMBDA(6),LAMBDA(7),LAMBDA(8));
end
%
a2 = min(range); b2 = max(range);
%
% Combine intervals;
if ((b1<a2) || (b2<a1))
    a = 0; b = 0;
else
    a = max(a1,a2);
    b = min(b1,b2);
end
```

A.6 Solves Nonlinear Systems Using Newton-Krylov Method: nsoli.m

```
function [sol, it_hist, ierr, x_hist] = nsoli(x,f,tol, parms)
% Newton-Krylov solver, globally convergent
%        solver for nonlinear systems f(x) = 0
%
% Inexact-Newton-Armijo iteration
%
% Eisenstat-Walker forcing term
%
% Parabolic line search via three point interpolation.
%
% function [sol, it_hist, ierr, x_hist] = nsoli(x,f,tol,parms)
%
% inputs:
%        initial guess = x
%        nonlinear functions = f
%        tol = [atol, rtol] relative/absolute
%            error tolerances for the nonlinear iteration
%        parms = [maxit, maxitl, etamax, lmeth, restart_limit]
%            maxit = maxmium number of nonlinear iterations
%                default = 40
%            maxitl = maximum number of inner iterations before restart
%                in GMRES(m), m = maxitl
%                default = 40
```

```
%
%                For iterative methods other than GMRES(m) maxitl
%                is the upper bound on linear iterations.
%
%            |etamax| = Maximum error tolerance for residual in inner
%                iteration. The inner iteration terminates
%                when the relative linear residual is
%                smaller than eta*| F(x_c) |. eta is determined
%                by the modified Eisenstat-Walker formula if etamax > 0.
%                If etamax < 0, then eta = |etamax| for the entire
%                iteration.
%                default: etamax = .9
%
%            lmeth = choice of linear iterative method
%                    1 (GMRES), 2 GMRES(m),
%                    3 (BICGSTAB), 4 (TFQMR)
%                 default = 1 (GMRES, no restarts)
%
%            restart_limit = max number of restarts for GMRES if
%                    lmeth = 2
%                  default = 20
%
% output:
%        sol = solution
%        it_hist(maxit,3) = l2 norms of nonlinear residuals
%            for the iteration, number of function evaluations,
%            and number of steplength reductions
%        ierr = 0 upon successful termination
%        ierr = 1 if after maxit iterations
%             the termination criterion is not satisfied
%        ierr = 2 failure in the line search. The iteration
%             is terminated if too many steplength reductions
%             are taken.
%
%    x_hist = matrix of the entire interation history.
%             The columns are the nonlinear iterates. This
%             is useful for making movies, for example, but
%             can consume way too much storage. This is an
%             OPTIONAL argument. Storage is only allocated
%             if x_hist is in the output argument list.
%
% internal parameters:
%       debug = turns on/off iteration statistics display as
%               the iteration progresses
%
%       alpha = 1.d-4, parameter to measure sufficient decrease
%
%       sigma0 = .1, sigma1 = .5, safeguarding bounds for the linesearch
```

```
%
%       maxarm = 20, maximum number of steplength reductions before
%                    failure is reported

% Set the debug parameter; 1 turns display on, otherwise off.
%
debug = 0;
%
% Set internal parameters.
%
alpha = 1.d-4; sigma0 = .1; sigma1 = .5; maxarm = 20; gamma = .9;
%
% Initialize it_hist, ierr, x_hist, and set the default values of
% those iteration parameters which are optional inputs.
%
ierr = 0; maxit = 40; lmaxit = 40;
etamax = .9; it_histx = zeros(maxit,3);
lmeth = 1; restart_limit = 20;
if nargout == 4, x_hist = x; end
%
% Initialize parameters for the iterative methods.
% Check for optional inputs.
%
gmparms = [abs(etamax), lmaxit];
if nargin == 4
    maxit = parms(1); lmaxit = parms(2); etamax = parms(3);
    it_histx = zeros(maxit,3);
    gmparms = [abs(etamax), lmaxit];
    if length(parms)>= 4
       lmeth = parms(4);
    end
    if length(parms) == 5
       gmparms = [abs(etamax), lmaxit, parms(5), 1];
    end
end
%
rtol = tol(2); atol = tol(1); n = length(x); fnrm = 1; itc = 0;
%
% Evaluate f at the initial iterate,and
% compute the stop tolerance.
%
f0 = feval(f,x);
fnrm = norm(f0);
it_histx(itc+1,1) = fnrm; it_histx(itc+1,2) = 0; it_histx(itc+1,3) = 0;
fnrmo = 1;
stop_tol = atol + rtol*fnrm;
outstat(itc+1, :) = [itc fnrm 0 0 0];
%
```

```
% main iteration loop
%
while(fnrm > stop_tol & itc < maxit)
%
% Keep track of the ratio (rat = fnrm/frnmo)
% of successive residual norms and
% the iteration counter (itc).
%
    rat = fnrm/fnrmo;
    fnrmo = fnrm;
    itc = itc+1;
    [step, errstep, inner_it_count,inner_f_evals] = ...
         dkrylov(f0, f, x, gmparms, lmeth);

%
%   The line search starts here.
%
    xold = x;
    lambda = 1; lamm = 1; lamc = lambda; iarm = 0;
    xt = x + lambda*step;
    ft = feval(f,xt);
    nft = norm(ft); nf0 = norm(f0); ff0 = nf0*nf0;
    ffc = nft*nft; ffm = nft*nft;
    while nft >= (1 - alpha*lambda) * nf0;
%
%   Apply the three point parabolic model.
%
        if iarm == 0
            lambda = sigma1*lambda;
        else
            lambda = parab3p(lamc, lamm, ff0, ffc, ffm);
        end
%
% Update x; keep the books on lambda.
%
        xt = x+lambda*step;
        lamm = lamc;
        lamc = lambda;
%
% Keep the books on the function norms.
%
        ft = feval(f,xt);
        nft = norm(ft);
        ffm = ffc;
        ffc = nft*nft;
        iarm = iarm+1;
        if iarm > maxarm
            disp(' Armijo failure, too many reductions ');
```

```
            ierr = 2;
            disp(outstat)
            it_hist = it_histx(1:itc+1,:);
        if nargout == 4, x_hist = [x_hist,x]; end
            sol = xold;
            return;
        end
    end
    x = xt;
    f0 = ft;
%
%   End of line search.
%
    if nargout == 4, x_hist = [x_hist,x]; end
    fnrm = norm(f0);
    it_histx(itc+1,1) = fnrm;
%
%   How many function evaluations did this iteration require?
%
    it_histx(itc+1,2) = it_histx(itc,2)+inner_f_evals+iarm+1;
    if itc == 1, it_histx(itc+1,2) = it_histx(itc+1,2)+1; end;
    it_histx(itc+1,3) = iarm;
%
    rat = fnrm/fnrmo;
%
%   Adjust eta as per Eisenstat-Walker.
%
    if etamax > 0
        etaold = gmparms(1);
        etanew = gamma*rat*rat;
        if gamma*etaold*etaold > .1
            etanew = max(etanew,gamma*etaold*etaold);
        end
        gmparms(1) = min([etanew,etamax]);
        gmparms(1) = max(gmparms(1),.5*stop_tol/fnrm);
    end
%
    outstat(itc+1, :) = [itc fnrm inner_it_count rat iarm];
%
end
sol = x;
it_hist = it_histx(1:itc+1,:);
if debug == 1
    disp(outstat)
    it_hist = it_histx(1:itc+1,:);
end
%
% on failure, set the error flag
```

```
%
if fnrm > stop_tol, ierr = 1; end
%
%
function lambdap = parab3p(lambdac, lambdam, ff0, ffc, ffm)
% Apply three-point safeguarded parabolic model for a line search.
%
% function lambdap = parab3p(lambdac, lambdam, ff0, ffc, ffm)
%
% input:
%       lambdac = current steplength
%       lambdam = previous steplength
%       ff0 = value of \| F(x_c) \|^2
%       ffc = value of \| F(x_c + \lambdac d) \|^2
%       ffm = value of \| F(x_c + \lambdam d) \|^2
%
% output:
%       lambdap = new value of lambda given parabolic model
%
% internal parameters:
%   sigma0 = .1, sigma1 = .5, safeguarding bounds for the linesearch
%

%
% set internal parameters
%
sigma0 = .1; sigma1 = .5;
%
% compute coefficients of interpolation polynomial
%
% p(lambda) = ff0 + (c1 lambda + c2 lambda^2)/d1
%
% d1 = (lambdac - lambdam)*lambdac*lambdam < 0
%      so if c2 > 0 we have negative curvature and default to
%      lambdap = sigam1 * lambda
%
c2 = lambdam*(ffc-ff0)-lambdac*(ffm-ff0);
if c2 >= 0
    lambdap = sigma1*lambdac; return
end
c1 = lambdac*lambdac*(ffm-ff0)-lambdam*lambdam*(ffc-ff0);
lambdap = -c1*.5/c2;
if lambdap < sigma0*lambdac, lambdap = sigma0*lambdac; end
if lambdap > sigma1*lambdac, lambdap = sigma1*lambdac; end
%
%
function [step, errstep, total_iters, f_evals] = ...
    dkrylov(f0, f, x, params, lmeth)
```

```
% Krylov linear equation solver for use in nsoli
%
%
% function [step, errstep, total_iters, f_evals]
%                               = dkrylov(f0, f, x, params, lmeth)
%
% Input:  f0 = function at current point
%         f = nonlinear function
%              the format for f is  function fx = f(x)
%              Note that for Newton-GMRES we incorporate any
%              preconditioning into the function routine.
%         x = current point
%         params = vector to control iteration
%              params(1) = relative residual reduction factor
%              params(2) = max number of iterations
%              params(3) = max number of restarts for GMRES(m)
%              params(4) (Optional) = reorthogonalization method in GMRES
%                   1 -- Brown/Hindmarsh condition (default)
%                   2 -- Never reorthogonalize (not recommended)
%                   3 -- Always reorthogonalize (not cheap!)
%
%         lmeth = method choice
%              1 GMRES without restarts (default)
%              2 GMRES(m), m = params(2) and the maximum number
%                   of restarts is params(3)
%              3 Bi-CGSTAB
%              4 TFQMR
%
% Output: x = solution
%         errstep = vector of residual norms for the history of
%                 the iteration
%         total_iters = number of iterations
%
% initialization
%
lmaxit = params(2);
restart_limit = 20;
if length(params) >= 3
    restart_limit = params(3);
end
if lmeth == 1, restart_limit = 0; end
if length(params) == 3
%
% default reorthogonalization
%
     gmparms = [params(1), params(2), 1];
elseif length(params) == 4
%
```

```
% reorthogonalization method is params(4)
%
     gmparms = [params(1), params(2), params(4)];
else
     gmparms = [params(1), params(2)];
end
%
% linear iterative methods
%
if lmeth == 1 | lmeth == 2  % GMRES or GMRES(m)
%
% compute the step using a GMRES routine especially designed
% for this purpose
%
    [step, errstep, total_iters] = dgmres(f0, f, x, gmparms);
    kinn = 0;
%
%   restart at most restart_limit times
%
    while total_iters == lmaxit & ...
          errstep(total_iters) > gmparms(1)*norm(f0) & ...
          kinn < restart_limit
        kinn = kinn+1;
        [step, errstep, total_iters] = dgmres(f0, f, x, gmparms,step);
    end
    total_iters = total_iters+kinn*lmaxit;
    f_evals = total_iters+kinn;
%
% Bi-CGSTAB
%
elseif lmeth == 3
    [step, errstep, total_iters] = dcgstab(f0, f, x, gmparms);
    f_evals = 2*total_iters;
%
% TFQMR
%
elseif lmeth == 4
    [step, errstep, total_iters] = dtfqmr(f0, f, x, gmparms);
    f_evals = 2*total_iters;
else
    error(' lmeth error in fdkrylov')
end
%
%
function z = dirder(x,w,f,f0)
% Finite difference directional derivative
% Approximate f'(x) w
%
```

```
% function z = dirder(x,w,f,f0)
%
% inputs:
%           x, w = point and direction
%           f = function
%           f0 = f(x), in nonlinear iterations
%                f(x) has usually been computed
%                before the call to dirder
%
% Use a hardwired difference increment.
%
epsnew = 1.d-7;
%
n = length(x);
%
% scale the step
%
if norm(w) == 0
    z = zeros(n,1);
return
end
%
% Now scale the difference increment.
%
xs=(x'*w)/norm(w);
if xs ~= 0.d0
     epsnew=epsnew*max(abs(xs),1.d0)*sign(xs);
end
epsnew=epsnew/norm(w);
%
% del and f1 could share the same space if storage
% is more important than clarity.
%
del = x+epsnew*w;
f1 = feval(f,del);
z = (f1 - f0)/epsnew;
%
function [x, error, total_iters] = dgmres(f0, f, xc, params, xinit)
% GMRES linear equation solver for use in Newton-GMRES solver
%
% function [x, error, total_iters] = dgmres(f0, f, xc, params, xinit)
%
% Input:  f0 = function at current point
%         f = nonlinear function
%              the format for f is  function fx = f(x)
%              Note that for Newton-GMRES we incorporate any
%              preconditioning into the function routine.
%         xc = current point
```

```
%         params = two dimensional vector to control iteration
%              params(1) = relative residual reduction factor
%              params(2) = max number of iterations
%            params(3) (Optional) = reorthogonalization method
%                   1 -- Brown/Hindmarsh condition (default)
%                   2 -- Never reorthogonalize (not recommended)
%                   3 -- Always reorthogonalize (not cheap!)
%
%         xinit = initial iterate. xinit = 0 is the default. This
%              is a reasonable choice unless restarted GMRES
%              will be used as the linear solver.
%
% Output: x = solution
%         error = vector of residual norms for the history of
%                 the iteration
%         total_iters = number of iterations
%
% Requires givapp.m, dirder.m
%
% initialization
%
errtol = params(1);
kmax = params(2);
reorth = 1;
if length(params) == 3
    reorth = params(3);
end
%
% The right side of the linear equation for the step is -f0.
%
b = -f0;
n = length(b);
%
% Use zero vector as initial iterate for Newton step unless
% the calling routine has a better idea (useful for GMRES(m)).
%
x = zeros(n,1);
r = b;
if nargin == 5
    x = xinit;
    r = -dirder(xc, x, f, f0)-f0;
end
%
h = zeros(kmax);
v = zeros(n,kmax);
c = zeros(kmax+1,1);
s = zeros(kmax+1,1);
rho = norm(r);
```

```
g = rho*eye(kmax+1,1);
errtol = errtol*norm(b);
error = [];
%
% Test for termination on entry.
%
error = [error,rho];
total_iters = 0;
if(rho < errtol)
%   disp(' early termination ')
return
end
%
v(:,1) = r/rho;
beta = rho;
k = 0;
%
% GMRES iteration
%
while((rho > errtol) & (k < kmax))
    k = k+1;
%
%   Call directional derivative function.
%
    v(:,k+1) = dirder(xc, v(:,k), f, f0);
    normav = norm(v(:,k+1));
%
%   Modified Gram-Schmidt
%
    for j = 1:k
        h(j,k) = v(:,j)'*v(:,k+1);
        v(:,k+1) = v(:,k+1)-h(j,k)*v(:,j);
    end
    h(k+1,k) = norm(v(:,k+1));
    normav2 = h(k+1,k);
%
%   Reorthogonalize?
%
if  (reorth == 1 & normav + .001*normav2 == normav) | reorth ==  3
    for j = 1:k
        hr = v(:,j)'*v(:,k+1);
h(j,k) = h(j,k)+hr;
        v(:,k+1) = v(:,k+1)-hr*v(:,j);
    end
    h(k+1,k) = norm(v(:,k+1));
end
%
%   Watch out for happy breakdown.
```

```
%
    if(h(k+1,k) ~= 0)
    v(:,k+1) = v(:,k+1)/h(k+1,k);
    end
%
%   Form and store the information for the new Givens rotation.
%
    if k > 1
        h(1:k,k) = givapp(c(1:k-1),s(1:k-1),h(1:k,k),k-1);
    end
%
%   Don't divide by zero if solution has  been found.
%
    nu = norm(h(k:k+1,k));
    if nu ~= 0
%        c(k) = h(k,k)/nu;
        c(k) = conj(h(k,k)/nu);
        s(k) = -h(k+1,k)/nu;
        h(k,k) = c(k)*h(k,k)-s(k)*h(k+1,k);
        h(k+1,k) = 0;
        g(k:k+1) = givapp(c(k),s(k),g(k:k+1),1);
    end
%
%   Update the residual norm.
%
    rho = abs(g(k+1));
    error = [error,rho];
%
%   end of the main while loop
%
end
%
% At this point either k > kmax or rho < errtol.
% It's time to compute x and leave.
%
y = h(1:k,1:k)\g(1:k);
total_iters = k;
x = x + v(1:n,1:k)*y;
%
%
function vrot = givapp(c,s,vin,k)
%  Apply a sequence of k Givens rotations, used within gmres codes.
%
%  function vrot = givapp(c, s, vin, k)
%
vrot = vin;
for i = 1:k
    w1 = c(i)*vrot(i)-s(i)*vrot(i+1);
```

```
%
%   Here's a modest change that makes the code work in complex
%   arithmetic. Thanks to Howard Elman for this.
%
%    w2 = s(i)*vrot(i)+c(i)*vrot(i+1);
    w2 = s(i)*vrot(i)+conj(c(i))*vrot(i+1);
    vrot(i:i+1) = [w1,w2];
end
%
function [x, error, total_iters] = ...
                    dcgstab(f0, f, xc, params, xinit)
% Forward difference Bi-CGSTAB solver for use in nsoli
%
% function [x, error, total_iters]
%                    = dcgstab(f0, f, xc, params, xinit)
%
% Input:  f0 = function at current point
%         f = nonlinear function
%              the format for f is  function fx = f(x)
%              Note that for Newton-GMRES we incorporate any
%              preconditioning into the function routine.
%         xc = current point
%         params = two dimensional vector to control iteration
%              params(1) = relative residual reduction factor
%              params(2) = max number of iterations
%
%         xinit = initial iterate. xinit = 0 is the default. This
%              is a reasonable choice unless restarts are needed.
%
%
% Output: x = solution
%         error = vector of residual norms for the history of
%                 the iteration
%         total_iters = number of iterations
%
% Requires: dirder.m
%
% initialization
%
b = -f0;
n = length(b); errtol = params(1)*norm(b); kmax = params(2); error = [];
rho = zeros(kmax+1,1);
%
% Use zero vector as initial iterate for Newton step unless
% the calling routine has a better idea (useful for GMRES(m)).
%
x = zeros(n,1);
r = b;
```

```
if nargin == 5
    x = xinit;
    r = -dirder(xc, x, f, f0)-f0;
end
%
hatr0 = r;
k = 0; rho(1) = 1; alpha = 1; omega = 1;
v = zeros(n,1); p = zeros(n,1); rho(2) = hatr0'*r;
zeta = norm(r); error = [error,zeta];
%
% Bi-CGSTAB iteration
%
while((zeta > errtol) & (k < kmax))
    k = k+1;
    if omega == 0
       error('Bi-CGSTAB breakdown, omega = 0');
    end
    beta = (rho(k+1)/rho(k))*(alpha/omega);
    p = r+beta*(p - omega*v);
    v = dirder(xc,p,f,f0);
    tau = hatr0'*v;
    if tau == 0
        error('Bi-CGSTAB breakdown, tau = 0');
    end
    alpha = rho(k+1)/tau;
    s = r-alpha*v;
    t = dirder(xc,s,f,f0);
    tau = t'*t;
    if tau == 0
       error('Bi-CGSTAB breakdown, t = 0');
    end
    omega = t'*s/tau;
    rho(k+2) = -omega*(hatr0'*t);
    x = x+alpha*p+omega*s;
    r = s-omega*t;
    zeta = norm(r);
    total_iters = k;
    error = [error, zeta];
end
%
%

function [x, error, total_iters] = ...
                     dtfqmr(f0, f, xc, params, xinit)
% Forward difference TFQMR solver for use in nsoli
%
% function [x, error, total_iters]
%                     = dtfqmr(f0, f, xc, params, xinit)
```

```
%
% Input:  f0 = function at current point
%         f = nonlinear function
%              the format for f is  function fx = f(x)
%              Note that for Newton-GMRES we incorporate any
%              preconditioning into the function routine.
%         xc = current point
%         params = two dimensional vector to control iteration
%              params(1) = relative residual reduction factor
%              params(2) = max number of iterations
%
%         xinit = initial iterate. xinit = 0 is the default. This
%              is a reasonable choice unless restarts are needed.
%
% Output: x = solution
%         error = vector of residual norms for the history of
%                 the iteration
%         total_iters = number of iterations
%
% Requires: dirder.m
%
% initialization
%
b = -f0;
n = length(b); errtol = params(1)*norm(b); kmax = params(2); error = [];
x = zeros(n,1);
r = b;
if nargin == 5
    x = xinit;
    r = -dirder(xc, x, f, f0)-f0;
end
%
u = zeros(n,2); y = zeros(n,2); w = r; y(:,1) = r;
k = 0; d = zeros(n,1);
v = dirder(xc, y(:,1),f,f0);
u(:,1) = v;
theta = 0; eta = 0; tau = norm(r); error = [error,tau];
rho = tau*tau;
%
% TFQMR iteration
%
while( k < kmax)
    k = k+1;
    sigma = r'*v;
%
    if sigma == 0
        error('TFQMR breakdown, sigma = 0')
    end
```

```
%
    alpha = rho/sigma;
%
%
    for j = 1:2
%
%   Compute y2 and u2 only if you have to
%
        if j == 2
            y(:,2) = y(:,1)-alpha*v;
            u(:,2) = dirder(xc, y(:,2),f,f0);
        end
        m = 2*k-2+j;
        w = w-alpha*u(:,j);
        d = y(:,j)+(theta*theta*eta/alpha)*d;
        theta = norm(w)/tau; c = 1/sqrt(1+theta*theta);
        tau = tau*theta*c; eta = c*c*alpha;
        x = x+eta*d;
%
%   Try to terminate the iteration at each pass through the loop
%
        if tau*sqrt(m+1) <=  errtol
            error = [error, tau];
            total_iters = k;
            return
        end
    end
%
    if rho == 0
        error('TFQMR breakdown, rho = 0')
    end
%
    rhon = r'*w; beta = rhon/rho; rho = rhon;
    y(:,1) = w + beta*y(:,2);
    u(:,1) = dirder(xc, y(:,1),f,f0);
    v = u(:,1)+beta*(u(:,2)+beta*v);
    error = [error, tau];
    total_iters = k;
end
%
```

A.7 Example: Plot the GLD Fit of Horse Data

```
% This program plots the fitting of Horse data with single and two GLDs.
z = load('Horse.data');

% mixed GLD
%  [0.7;50;0.022;0.1349;0.1349;25;0.0674;0.4349;0.4349];
```

```
x = 0.1:0.06:108.1;

for i = 1:1801
    y(i) = 6.4*(0.7*GLD(x(i)/1.2,50,0.022,0.1349,0.2549) +
           0.3*GLD(x(i)/1.2,25,0.0674,0.4349,0.4349));
end
hist(z);
hold on;

% single GLD

u = -10:0.1:170;

for i = 1:1801
v(i) = GLD(u(i),41.7897, 0.01134, 0.09853, 0.3606);
end

h = plot(u,v*550,'r--',x,y*62,'b');
legend(h, 'Single GLD','Mixed GLD');
```

A.8 Example: GLD Mixture Fitting for the Exchange Rate Data

```
x=load(Yen.data);
% Load the Exchange rate data Yen.txt
U=[0.5, 0.1, 1,0.2,0.1,1,2,0.1,1 ] ;
% input the initial values of parameters MLE;
% Call the function MLE.m to solve the nonlinear equations

Out=[0.2659,5.2374,0.5724,-0.0001,2.4181,4.8656,-0.004,1.786,0.5967]
```

Chapter 9

GLD–2: The Bivariate GLD Distribution

We saw in Chapters 2–6 that the GLD and EGLD are very successful in practice in fitting a variety of datasets. Since data is often bivariate (the simplest multivariate setting), it is desirable to have a way of fitting a **bivariate GLD**, a distribution whose univariate components (not necessarily independent) have univariate GLD (or EGLD) distributions. A bivariate GLD, called GLD–2, was developed by Beckwith and Dudewicz (1996), and its essential features are described in this chapter; for full details, see Beckwith and Dudewicz (1996) and Karian and Dudewicz (1999). The algorithm developed by Beckwith and Dudewicz (1996) is based on the following procedure.

1. Fit univariate GLDs separately to the components X and Y of the bivariate random variable (X, Y).
2. Use the method R. L. Plackett (1965) proposed for generating bivariate distributions with specified marginals to develop a bivariate distribution with these marginals.
3. Optimize over the infinite set of distributions in the Plackett class to obtain a fit to the actual dataset.
4. Calculate a bivariate plot and various quantitative measures to visually and quantitatively assess the quality of the fit.

Following the brief overview of Section 9.1 this chapter describes Plackett's Method in Section 9.2 and its use to develop the GLD–2. Next, the GLD–2 is used to fit several well-known bivariate distributions with identical and non-identical margins (Sections 9.3 and 9.4), and datasets (Section 9.5). Section 9.6 gives a procedure for generating GLD–2 random variates. We conclude by describing some open problems related to the GLD–2 in Section 9.7, followed by a set of problems for this chapter that will allow students in courses using this book as a

text and other readers to test their understanding of the methods and concepts of the GLD–2.

9.1 Overview

Suppose the random variables X and Y have p.d.f.s $f(x)$ and $g(y)$ and d.f.s $F(x)$, and $G(y)$, respectively. The d.f., $H(x, y)$, and p.d.f., $h(x, y)$, formulas for the GLD–2 are, respectively:

$$H(x,y) = \begin{cases} \dfrac{S(x,y) - \sqrt{S^2(x,y) - 4\Psi(\Psi-1)F(x)G(y)}}{2(\Psi-1)} & (\Psi \neq 1) \\ F(x)G(y) & (\Psi = 1) \end{cases} \tag{9.1.1}$$

and

$$h(x,y) = \frac{\Psi f(x)g(y)\left[1 + (\Psi-1)(F(x)+G(y)-2F(x)G(y)\right]}{\left(S^2(x,y) - 4\Psi(\Psi-1)F(x)G(y)\right)^{3/2}}, \tag{9.1.2}$$

where Ψ is nonnegative and

$$S(x,y) = 1 + (F(x)+G(y))(\Psi-1).$$

Note that $\Psi = 1$ is the case when X and Y are independent random variables, in which case $H(x, y) = F(x)G(y)$ and $h(x, y) = f(x)g(y)$ for all (x, y).

To illustrate this method we give the final results of fitting the bivariate data which arose in a study of imaging in the brain (Dudewicz, Levy, Lienhart, and Wehrli (1989)). The issue was to fit the measured quantity called **AD** (the ^1H hydrogen density in a specific portion of brain tissue as related to the average density in the whole brain). In this example, the AD measurements are for the Cortical White Matter Right side (CWR) (designated by X) and the Cortical White Matter Left side (CWL) (designated by Y). Thus, the data are the pairs (X, Y) and the concern of the research was to determine if X and Y were related. The data, from Dudewicz, Levy, Lienhart, and Wehrli (1989, p. 337), is given in Table 9.1–1. The results of applying Algorithm GLD–2 of Section 9.3 to the data of Table 9.1–1, shown in Figures 9.1–2, 9.1–3, and 9.1–4, can be interpreted as follows:

1. Figure 9.1–2 shows the GLD p.d.f. that is fitted to X (together with the histogram for X) and the fitted d.f. (with the empiric distribution of X). Figure 9.1–3 does the same for Y. These figures indicate that the GLD fits the data well.

2. A two-dimensional scatterplot with median lines is shown in Figure 9.1–4. The counts a, b, c, and d specified in the algorithm of Section 9.3 are shown on the scatterplot.

Table 9.1–1. Brain tissue ^1H hydrogen density data.

X	Y	X	Y	X	Y
96.8	99.2	96.8	96.8	95.6	100.8
107.6	102.9	86.5	86.1	87.3	85.7
99.6	98.1	94.5	95.5	89.8	91.2
99.5	103.9	97.0	92.9	88.2	88.8
102.9	109.2	92.2	91.0	89.4	94.1
84.8	82.8	100.3	98.2	88.6	89.9
97.9	95.9	102.5	95.6	87.7	86.4
103.2	100.2	100.6	99.4	88.1	91.0

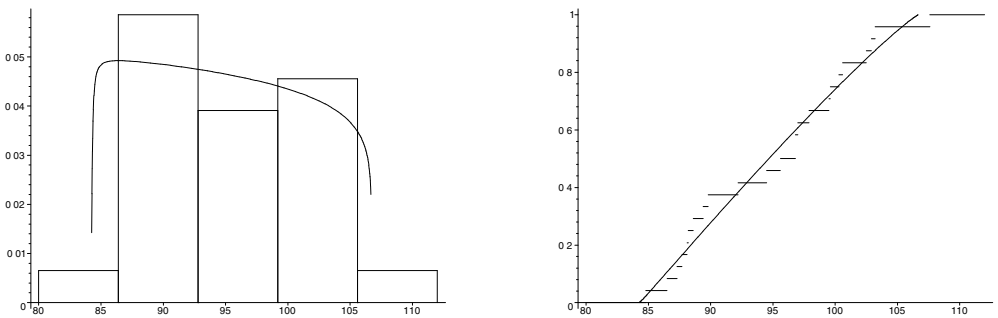


Figure 9.1–2. Histogram of X with its GLD p.d.f.; empirical distribution with its GLD d.f.

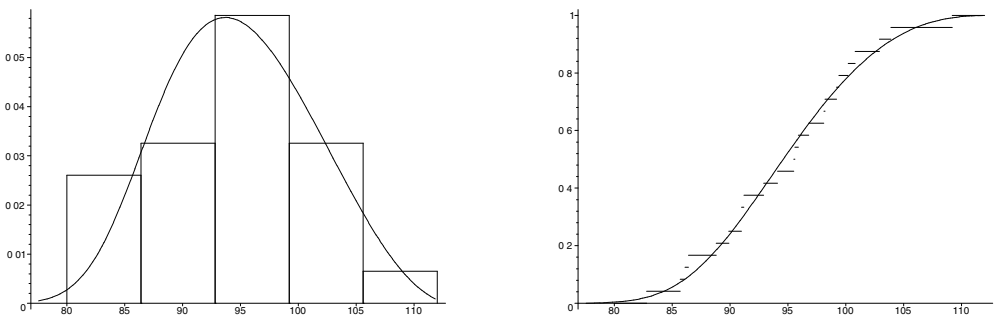


Figure 9.1–3. Histogram of Y with its GLD p.d.f.; empirical distribution with its GLD d.f.

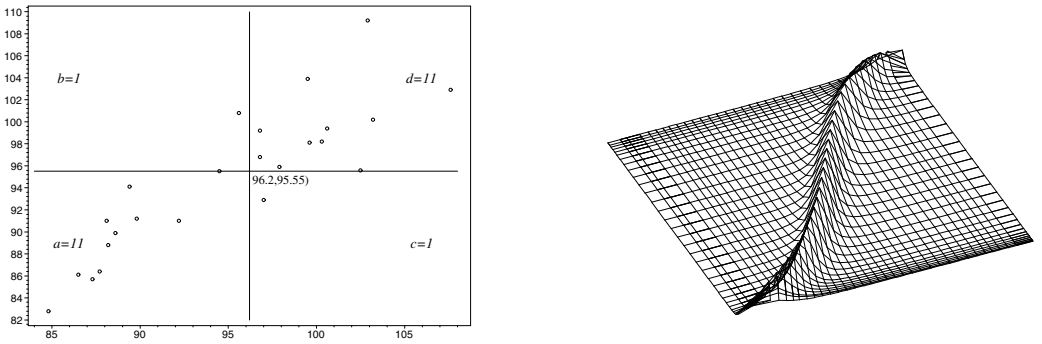


Figure 9.1–4. The (X, Y) scatterplot and the bivariate GLD–2 p.d.f. for (X, Y).

3. Figure 9.1–4 also shows the three-dimensional GLD–2 p.d.f. that is obtained by using (9.1.2). The "snake shape" of this surface indicates that the GLD–2 can produce a good fit even in the absence of symmetry, a property that is associated with some distributions such as the bivariate normal. For non-convex contours, see McLachlan (1992), page 41.

We see that X and Y are not independent. In fact, they are highly correlated, and so a tumor in one side might stand out on a visual X and Y comparison. Also, the fits are quite good for each variable. Some software that could be useful in this connection is given in the article by Beckwith and Dudewicz (1996); for additional details, also see Karian and Dudewicz (1999).

We conclude that the GLD–2 has, in this case, handled well a dataset that might have led researchers seeing the histogram of Figures 9.1–2 and 9.1–3 to wonder: How can we ever fit the pair (X, Y) since X is so non-normal, and the data contours are non-convex? With this motivation, we proceed to develop the GLD–2 in detail.

9.2 Plackett's Method of Bivariate d.f. Construction: The GLD–2

The goal here is to construct a family of bivariate distributions, with given marginals $F_X(x)$ and $G_Y(y)$, such that the family contains the bivariate distributions that correspond to the independent case as well as the upper and lower Fréchet bounds; i.e., the family includes the bivariate d.f.s

$$F(x)G(y), \tag{9.2.1}$$

$$H_0(x, y) = \max\{F(x) + G(y) - 1, 0\}, \tag{9.2.2}$$

and

$$H_1(x, y) = \min\{F(x), G(y)\}. \tag{9.2.3}$$

Fréchet considered convex mixtures of the upper and lower bounds,

$$H(x, y) = \lambda H_0(x, y) + (1 - \lambda)H_1(x, y) \ (0 \le \lambda \le 1), \tag{9.2.4}$$

and Morgenstern (later extended by Farlie and Gumbel, now called the **FGM family**; see Hutchinson and Lai (1990)) considered

$$H(x, y) = F(x)G(y)[1 + \alpha A(F)B(G)] \tag{9.2.5}$$

where $A(F(x))$ and $B(F(y))$ are functions of x and y that tend to zero as $F(x)$ and $G(y) \to 1$, respectively. Fréchet's mixture certainly contains the upper and lower bounds, and the FGM family contains the independent case, but neither contains all three. Another problem with the FGM family is that the constraints on $A(F)$ and $B(G)$ require that α be small, hence the family only models cases of weak dependence.

In his method, Plackett used the ideas of a two-by-two contingency table. Given a scatterplot of bivariate data, partition the x and y values with a horizontal line and a vertical line, dividing the data into four quadrants. Each quadrant has a probability. Suppose that the true bivariate d.f. of the pair (X, Y) is

$$H(x, y) = P(X \le x, Y \le y). \tag{9.2.6}$$

Then the probabilities that a pair (X, Y) falls into the four quadrants are $H(x, y)$ in the lower-left quadrant, $F(x) - H(x, y)$ in the upper-left quadrant, $G(y) - H(x, y)$ in the lower-right quadrant, and $1 - F(x) - G(y) + H(x, y)$ in the upper right. For example, the probability that (X, Y) is in the upper-left quadrant is

$$\begin{aligned} P((X, Y) \quad & \text{is in the upper} - \text{left quadrant}) \\ &= P(X \le x, Y > y) \\ &= P(X \le x, Y \le \infty) - P(X \le x, Y \le y) \\ &= F(x) - H(x, y), \end{aligned} \tag{9.2.7}$$

where $F(x)$ is the d.f. of X (and we also used $G(y)$ for d.f. of Y). We show the situation in Figure 9.2–1.

Plackett proposed a system that solves

$$\Psi = \frac{H(1 - F - G + H)}{(F - H)(G - H)}, \tag{9.2.8}$$

and showed that for all $\Psi \in [0, \infty)$ equation (9.2.8) has a unique root H which satisfies $H_0(x, y) \le H(x, y) \le H_1(x, y)$, a condition shown by Fréchet to be necessary for any bivariate distribution. This H is a valid bivariate distribution,

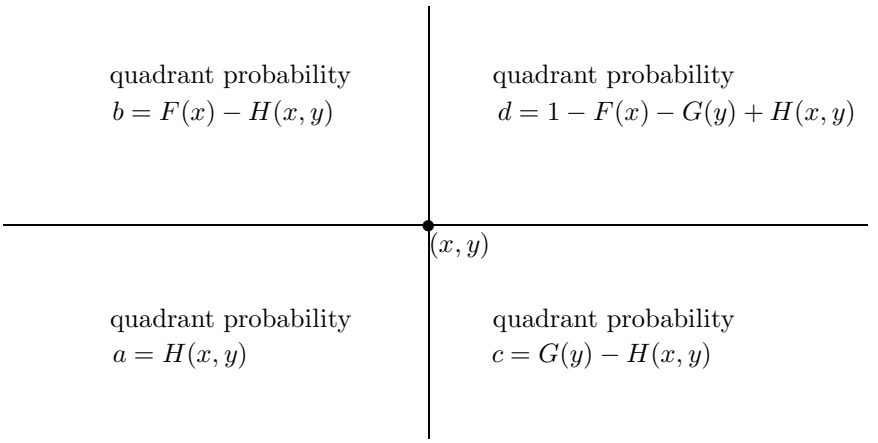


Figure 9.2–1. The probabilities that (X, Y) falls into the four quadrants determined by a point (x, y).

and its marginals are $F_X(x) = F(x)$ and $G_Y(y) = G(y)$. Further, it is possible to attain the independent joint distribution $F(x)G(y)$. Note that if we call the probabilities of the four quadrants a, b, c, d as in Figure 9.2–1, then Plackett is proposing a system where, for some value of the constant Ψ,

$$ad = \Psi bc \tag{9.2.9}$$

for all points (x, y) in the plane. We will see below that this in fact specifies a bivariate d.f. with marginals $F(x)$ and $G(y)$ (though this is not obvious at this point, and the proof will take some effort).

It is unlikely that H, the solution of (9.2.8), will be in a simple form; often numerical calculations will be needed. **To find the solution of Plackett's equation (9.2.8),** we follow the development given by Mardia (1967, 1970) with some additional details. First, rewrite (9.2.8) as a quadratic in H. From (9.2.8) we have the equivalent statements

$$\begin{aligned}
\Psi(F - H)(G - H) &= H(1 - F - G + H), \\
\Psi(FG - H(F + G) + H^2) &= H(1 - F - G) + H^2, \\
H^2(1 - \Psi) + H[1 + (F + G)(\Psi - 1)] - \Psi FG &= 0.
\end{aligned}$$

Next, solve the last equation for H and obtain

$$H = \frac{-(1 + (F + G)(\Psi - 1)) \pm \sqrt{(1 + (F + G)(\Psi - 1))^2 + 4(1 - \Psi)\Psi FG}}{2(1 - \Psi)}.$$

To simplify, let

$$S = 1 + (F + G)(\Psi - 1), \tag{9.2.10}$$

and obtain

$$H = \frac{-S \pm \sqrt{S^2 + 4(1-\Psi)\Psi FG}}{2(1-\Psi)} = \frac{S \pm \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)}.$$

For the independent case, let $\Psi = 1$ in (9.2.8); that is,

$$\begin{aligned} 1 &= \frac{H(1-F-G+H)}{(F-H)(G-H)}, \\ H(1-F-G+H) &= (F-H)(G-H), \\ H - HF - HG + H^2 &= FG - HG - HF + H^2, \\ H &= FG. \end{aligned} \tag{9.2.11}$$

For $\Psi \neq 1$, Mardia (1967, 1970) showed that **only one root will yield a valid d.f.:**

$$H = \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)}. \tag{9.2.12}$$

To see this, label the two roots of the quadratic as

$$\alpha = \frac{S + \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \text{ and } \beta = \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)},$$

or

$$\begin{aligned} \alpha &= \frac{1 + (F+G)(\Psi-1) + \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\ &= \frac{1}{2(\Psi-1)} + \frac{F+G}{2} + \frac{\sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \end{aligned} \tag{9.2.13}$$

and

$$\begin{aligned} \beta &= \frac{1 + (F+G)(\Psi-1) - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\ &= \frac{1}{2(\Psi-1)} + \frac{F+G}{2} - \frac{\sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)}. \end{aligned} \tag{9.2.14}$$

Then consider four cases: $\Psi = 0, 0 < \Psi < 1, \Psi = 1,$ and $\Psi > 1$. **If $\Psi > 1$,** then the root

$$\alpha > \frac{1}{2(\Psi-1)} + \frac{F+G}{2} > \frac{F+G}{2} > \min(F, G)$$

and hence does not satisfy the Fréchet upper bound (and thus is not a valid bivariate d.f.).

Noting that

$$\begin{aligned} \alpha\beta &= \left(\frac{S + \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)}\right)\left(\frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)}\right) \\ &= \frac{S^2 - S^2 + 4(\Psi-1)\Psi FG}{4(\Psi-1)^2} = \frac{\Psi FG}{\Psi-1} \end{aligned}$$

is negative **for all** $\Psi \in (0,1)$, we conclude that one of the roots α and β is negative (the other positive) in this case. Now for $\Psi \in (0,1)$, $-4(\Psi-1)\Psi FG$ is positive, so

$$\begin{aligned}
\beta &= \frac{1}{2(\Psi-1)} + \frac{F+G}{2} - \frac{\sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\
&\geq \frac{1}{2(\Psi-1)} + \frac{F+G}{2} - \frac{\sqrt{S^2}}{2(\Psi-1)} \\
&= \frac{1}{2(\Psi-1)} + \frac{F+G}{2} - \frac{1+F(\Psi-1)+G(\Psi-1)}{2(\Psi-1)} \\
&= \frac{1}{2(\Psi-1)} + \frac{F+G}{2} - \frac{1}{2(\Psi-1)} - \frac{F+G}{2} = 0.
\end{aligned}$$

Thus for $\Psi \in (0,1)$, we have $\beta \geq 0$, hence α is negative and therefore invalid. **If** $\Psi = 1$, we have $H = FG$, the independent case in (9.2.11) above. From (9.2.8) we see $H = \max(0, F+G-1)$ if and only if $\Psi = 0$, and $H = \min(F,G)$ if and only if $\Psi = \infty$. These observations together imply that for $\Psi \in [0,\infty)$, α **is not a valid distribution, and also that we can achieve the independent case and the Fréchet bounds.**

What remains to be shown then is that

$$\beta = H(x,y) = \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \tag{9.2.15}$$

is a valid bivariate d.f. There are (see p. 123 of Dudewicz and Mishra (1988), e.g.) four criteria to check:

1. $\lim_{x,y\to\infty} H(x,y) = 1$;
2. $\lim_{x\to-\infty} H(x,y) = \lim_{y\to-\infty} H(x,y) = 0$;
3. $\lim_{h\to0^+} H(x+h,y) = \lim_{h\to0^+} H(x,y+h) = H(x,y)$; and
4. $H(a,b) + H(a+h,b+k) - H(a+h,b) - H(a,b+k) \geq 0$, for any $h,k > 0$; that is, H assigns positive probability to all rectangles.

We would also like to verify that the given joint d.f. has the desired marginals; that is,

5. $\lim_{y\to\infty} H(x,y) = F(x)$ and $\lim_{x\to\infty} H(x,y) = G(y)$.

To establish criterion 1, note that $\lim_{x\to\infty} F(x) = \lim_{y\to\infty} G(y) = 1$, so by (9.2.11)

$$\lim_{x,y\to\infty} S = \lim_{F,G\to1} 1 + (F+G)(\Psi-1) = 2\Psi - 1,$$

hence, from (9.2.12), when $\Psi \neq 1$ (the case $\Psi = 1$ follows from (9.2.11))

$$\begin{aligned}\lim_{x,y\to\infty} H(x,y) &= \lim_{F,G\to 1} \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\ &= \frac{2\Psi - 1 - \sqrt{(2\Psi-1)^2 - 4(\Psi-1)\Psi}}{2(\Psi-1)} \\ &= \frac{2\Psi - 1 - \sqrt{4\Psi^2 - 4\Psi + 1 - 4\Psi^2 + 4\Psi}}{2(\Psi-1)} \\ &= \frac{2\Psi - 2}{2(\Psi-1)} = 1.\end{aligned}$$

Criterion 2 is similar: one notes that

$$\lim_{x\to-\infty} F(x) = 0,$$

hence

$$\lim_{x\to-\infty} S = \lim_{F\to 0} 1 + (F+G)(\Psi-1) = 1 + G(\Psi-1)$$

and note that $1 + G(\Psi - 1) \geq 0$, so

$$\begin{aligned}\lim_{x\to-\infty} H(x,y) &= \lim_{F\to 0} \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\ &= \frac{1 + G(\Psi-1) - \sqrt{[1+G(\Psi-1)]^2}}{2(\Psi-1)} \\ &= \frac{1 + G(\Psi-1) - |1+G(\Psi-1)|}{2(\Psi-1)} = 0;\end{aligned}$$

by symmetry, $\lim_{y\to-\infty} H(x,y) = 0$.

To verify criteria 3 and 4, it is sufficient to show that

$$h(x,y) \equiv \frac{\partial^2 H}{\partial F \partial G} \geq 0$$

for all real numbers x and y. Below at (9.2.19) we find that

$$h(x,y) = \frac{\Psi f g[1 + (\Psi-1)(F+G-2FG)]}{(S^2 - 4\Psi(\Psi-1)FG)^{3/2}},$$

which is (since the other terms are nonnegative) ≥ 0 if and only if

$$1 + (\Psi-1)(F+G-2FG) \geq 0. \tag{9.2.16}$$

Now consider three cases: $0 \leq \Psi < 1, \Psi = 1,$ and $\Psi > 1$.

Clearly (9.2.16) is satisfied if $\Psi = 1$. If $0 \leq \Psi < 1$, then note the left-hand side of (9.2.16) can be bounded as follows:

$$1 + 2FG - F - G \leq 1 + (\Psi-1)(F+G-2FG) < 1,$$

since using calculus we find $F+G-2FG$ achieves its minimum of 0 at $F=0, G=1$ or $F=G=0$ and its maximum of 1 at $F=0, G=1$ or $F=1, G=0$. It follows that for $0 \leq \Psi < 1$ we have

$$0 \leq 1 + (\Psi - 1)(F + G - 2FG) < 1.$$

In the case of $\Psi > 1, 1 + (\Psi - 1)(F + G - 2FG) > 1 + F + G - 2FG$. As $1 + F + G - 2FG$ achieves its minimum of 1 at $F = G = 0$ or $F = G = 1$ and its maximum of 2 at $F = 0, G = 1$ or $F = 1, G = 0$, when $\Psi > 1$ we have

$$1 + (\Psi - 1)(F + G - 2FG) \geq 1.$$

Thus, H is a valid bivariate d.f. for all $\Psi \geq 0$ and is given by

$$H = \begin{cases} \dfrac{S - \sqrt{S^2 - 4\Psi(\Psi-1)FG}}{2(\Psi-1)} & (\Psi \neq 1) \\ FG & (\Psi = 1). \end{cases} \tag{9.2.17}$$

To see that the marginals are as in 5, note that since $\lim_{x\to\infty} F(x) = 1$, by (9.2.10)

$$\lim_{x\to\infty} S = \lim_{F\to 1} 1 + (F+G)(\Psi - 1) = \Psi + G(\Psi - 1),$$

thus (see (9.2.14))

$$\begin{aligned} \lim_{x\to\infty} H(x,y) &= \lim_{F\to 1} H(x,y) = \lim_{F\to 1} \frac{S - \sqrt{S^2 - 4(\Psi-1)\Psi FG}}{2(\Psi-1)} \\ &= \frac{\Psi + G(\Psi-1) - \sqrt{[\Psi + G(\Psi-1)]^2 - 4(\Psi-1)\Psi G}}{2(\Psi-1)} \\ &= \frac{\Psi + G(\Psi-1) - \sqrt{\Psi^2 - 2G\Psi(\Psi-1) + G^2(\Psi-1)^2}}{2(\Psi-1)} \\ &= \frac{\Psi + G(\Psi-1) - \sqrt{[\Psi - G(\Psi-1)]^2}}{2(\Psi-1)} \\ &= \frac{\Psi + G(\Psi-1) - \Psi + G(\Psi-1)}{2(\Psi-1)} \\ &= \frac{2G(\Psi-1)}{2(\Psi-1)} = G \end{aligned}$$

and by symmetry $\lim_{y\to\infty} H(x,y) = F$.

Mardia also derived **the p.d.f. of** H as follows. Given a d.f. $H(x,y)$, the p.d.f. $h(x,y)$ is given by

$$\begin{aligned} h(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial F}\frac{dF}{dy}\right) = \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial F} f\right) \\ &= \frac{\partial}{\partial F}\left(\frac{\partial H}{\partial y}\right) f = \frac{\partial}{\partial F}\left(\frac{\partial H}{\partial G}\frac{dG}{dy}\right) = fg\frac{\partial^2 H}{\partial F \partial G}, \end{aligned} \tag{9.2.18}$$

with the derivative interchanges permissible due to the continuity of $H(x, y)$, $F(x)$, and $G(y)$. Making substitutions to simplify notation:

$$D = (S^2 - 4\Psi(\Psi - 1)FG)^{\frac{3}{2}}, p = \Psi,$$

and noting that (9.2.10) implies $\frac{\partial S}{\partial F} \equiv \frac{\partial S}{\partial G} = \Psi - 1 = p - 1$, from (9.2.14) when $\Psi \neq 1$, we find

$$\begin{aligned}
\frac{\partial H}{\partial F} &= \frac{p-1}{2(p-1)} - \frac{1}{2(p-1)} \\
&\quad \times \left[\frac{1}{2}(S^2 - 4p(p-1)FG)^{-\frac{1}{2}} \times (2S(p-1) - 4p(p-1)G)\right] \\
&= \frac{1}{2} - \frac{1}{2}(S^2 - 4p(p-1)FG)^{-\frac{1}{2}} \times (S - 2pG).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^2 H}{\partial F \partial G} &= -\frac{1}{2}\left[\begin{array}{ll} -\frac{1}{2} & (S^2 - 4p(p-1)FG)^{-\frac{3}{2}}(2S(p-1) - 4p(p-1)F) \\ & \times(S - 2pG) + (S^2 - 4p(p-1)FG)^{-\frac{1}{2}}((p-1) - 2p) \end{array}\right] \\
&= -\frac{1}{2}\left[\begin{array}{ll} -\frac{1}{2} & (S^2 - 4p(p-1)FG)^{-\frac{3}{2}}2(p-1)(S - 2pF) \\ & \times(S - 2pG) - (S^2 - 4p(p-1)FG)^{-\frac{1}{2}} \times (p+1) \end{array}\right] \\
&= \frac{-1}{2D}\left[(p-1)(S - 2pF)(S - 2pG) + (p+1)(S^2 - 4p(p-1)FG)\right] \\
&= \frac{p}{D}\left[S^2 - p(SF + SG) + SF + SG - 2(p-1)FG\right] \\
&= \frac{p}{D}\left[S^2 - (p-1)(SF + SG - 2FG)\right] \\
&= \frac{p}{D}\left[S(1 + (p-1)(F+G) - (p-1)(F+G)) - 2(p-1)FG)\right] \\
&= \frac{p}{D}\left[S - 2(p-1)FG)\right] \\
&= \frac{p}{D}\left[1 + (p-1)(F+G) - 2(p-1)FG)\right] \\
&= \frac{p}{D}\left[1 + (p-1)(F+G-2FG)\right] \\
&= \frac{\Psi\left[1 + (\Psi-1)(F+G-2FG)\right]}{(S^2 - 4\Psi(\Psi-1)FG)^{\frac{3}{2}}}.
\end{aligned}$$

Hence,

$$h = \frac{\Psi f g[1 + (\Psi-1)(F+G-2FG)]}{(S^2 - 4\Psi(\Psi-1)FG)^{\frac{3}{2}}}. \tag{9.2.19}$$

One sees from (9.2.14) and (9.2.10) that this also holds for $\Psi = 1$, i.e., for all Ψ.

The results derived above are summarized in the following theorem.

Theorem 9.2.20. *Let X be a r.v. with d.f. $F(x)$, and let Y be a r.v. with d.f. $G(y)$. Then for each value of $\Psi \neq 1$ from $[0, \infty)$ the bivariate function*

(9.2.15)

$$\beta = H(x,y) = \frac{S - \sqrt{S^2 - 4\Psi(\Psi-1)FG}}{2(\Psi-1)},$$

where $S = 1+(F+G)(\Psi-1)$ *(see (9.2.10)), is a bivariate d.f. with the marginals* $F(x)$ *and* $G(y)$*. When* $\Psi = 1$*, we have*

$$H(x,y) = F(x)G(y),$$

the case of independent components X *and* Y*. In all cases,* H *is the unique solution of the equation (see (9.2.8))*

$$\Psi = \frac{H(1-F-G+H)}{(F-H)(G-H)}.$$

When $\Psi = 0$*,*

$$H = \max(0, F+G-1),$$

and when $\Psi = \infty$*,*

$$H = \min(F,G),$$

showing that the Fréchet bounds can be attained. The p.d.f. of $H(x,y)$ *is (see (9.2.19))*

$$h = \frac{\Psi f g[1+(\Psi-1)(F+G-2FG)]}{(S^2-4\Psi(\Psi-1)FG)^{\frac{3}{2}}}.$$

Given then that H is a valid distribution, the question arises: **What value of Ψ is appropriate?** Plackett gives two examples of how one might answer this. In an example using the Bivariate Normal he equates the two d.f.s at the univariate medians, and given a contingency table proposes a consistent estimator Ψ^+. The standard (zero means and unit variances) **Bivariate Normal** is given by its p.d.f.

$$N(x,y|\rho) = \frac{\exp\left(\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}\right)}{2\pi\sqrt{1-\rho^2}}, \quad |\rho| \le 1; \tag{9.2.21}$$

the corresponding d.f. is given by

$$M(x,y|\rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{\exp\left(\frac{-(u^2-2\rho uv+v^2)}{2(1-\rho^2)}\right)}{2\pi\sqrt{1-\rho^2}} du\, dv. \tag{9.2.22}$$

Thus, at $x = y = 0$ (the "median vector") the d.f. equals

$$M(0,0|\rho) = \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\exp(\frac{-(u^2-2\rho uv+v^2)}{2(1-\rho^2)})}{2\pi\sqrt{1-\rho^2}} du\, dv = \frac{\cos^{-1}(-\rho)}{2\pi} \tag{9.2.23}$$

and since $x = y = 0$, $F = G = \frac{1}{2}$, and $S = 1 + (F + G)(\Psi - 1) = \Psi$ (see (9.2.14)) at the medians the d.f. is given by

$$\begin{aligned} H(0,0) &= \frac{S - \sqrt{S^2 - 4\Psi(\Psi-1)FG}}{2(\Psi-1)} = \frac{\Psi - \sqrt{\Psi^2 - \Psi^2 + \Psi}}{2(\Psi-1)} = \frac{\Psi - \sqrt{\Psi}}{2(\Psi-1)} \\ &= \frac{\sqrt{\Psi}}{2(1+\sqrt{\Psi})} \times \frac{(\Psi - \sqrt{\Psi})(1+\sqrt{\Psi})}{\sqrt{\Psi}(\Psi-1)} = \frac{\sqrt{\Psi}}{2(1+\sqrt{\Psi})}. \end{aligned} \tag{9.2.24}$$

Equating (9.2.23) and (9.2.24), we have

$$\frac{\cos^{-1}(-\rho)}{2\pi} = \frac{\sqrt{\Psi}}{2(1+\sqrt{\Psi})}.$$

We now solve for Ψ to obtain

$$\Psi = \left(\frac{\cos^{-1}(-\rho)}{\cos^{-1}(-\rho) - \pi} \right)^2. \tag{9.2.25}$$

Thus with the Ψ of (9.2.25) the Bivariate Normal and H will agree (at least) at this point.

Plackett also suggested **an estimator of** Ψ useful for fitting datasets: Divide the joint distribution into four quadrants, using lines $x = h$ and $y = k$ for some constants h, k. Then count the number of (x, y) points in each quadrant; let a, b, c, d denote the counts; that is,

$$a(x \le h, y \le k), b(x \le h, y > k), c(x > h, y \le k), \text{ and } d(x > h, y > k).$$

Then $\Psi^+ = \frac{ad}{bc}$ is Plackett's estimator. One may motivate this estimator as follows: Since $ad = bc$ for the d.f. $H(x, y)$ for any pair (x, y) (see (9.2.9)) by the derivation of the class of d.f.s H, it is always true that $\Psi = ad/(bc)$. It is then reasonable to estimate Ψ using the counts a, b, c, d from the data. Plackett states that Ψ^+ is asymptotically normal, and that the variance is given by $V(\Psi^+) = (\Psi)^2[1/a + 1/b + 1/c + 1/d]$ where a, b, c, and d are as given in Figure 9.2–1. Mardia (1967) showed that the variance is minimized if h and k are chosen as the respective medians.

With the above groundwork, we are in a position to develop a bivariate GLD in Section 9.3. There (9.2.17) and (9.2.19) are taken with f, g, F, and G chosen as univariate GLDs; to fit a known bivariate distribution, we fit the marginals $f(x)$ and $g(y)$ each with a univariate GLD, while for datasets with unknown distributions the GLDs can be fitted with some appropriate method and substituted. Additionally, we can specify any arbitrary pair of marginals in (9.2.17) and (9.2.19) to investigate the possible curve shapes. In the sections that follow we develop these methods and discuss several examples.

9.3 Fitting the GLD–2 to Well-Known Bivariate Distributions

We now introduce the new bivariate GLD–2. The d.f. is (see Theorem 9.2.20) the solution of Plackett's equation

$$\Psi = \frac{H(1-F-G+H)}{(F-H)(G-H)},$$

i.e.,

$$H(x,y) = \begin{cases} \dfrac{S - \sqrt{S^2 - 4\Psi(\Psi-1)FG}}{2(\Psi-1)} & (\Psi \neq 1) \\ FG & (\Psi = 1), \end{cases} \tag{9.3.1}$$

with

$$S = S(x,y) = 1 + (F(x) + G(y))(\Psi - 1),$$

where $F = F(x)$ and $G = G(y)$ are GLD marginal d.f.s and $\Psi \in [0, \infty)$ is a measure of association between the marginals. The p.d.f. $h(x,y)$ is

$$h(x,y) = \frac{\Psi f(x) g(y)[1 + (\Psi-1)(F(x) + G(y) - 2F(x)G(y))]}{(S^2(x,y) - 4\Psi(\Psi-1)F(x)G(y))^{\frac{3}{2}}},$$

where $f(x)$ and $g(x)$ are (possibly different) GLD marginal p.d.f.s.

In order to develop a bivariate GLD using Plackett's method Beckwith and Dudewicz (1996) proposed the following algorithm for approximating any specified p.d.f. $f(x,y)$:

Algorithm GLD–BV: Bivariate Approximation to $f(x,y)$ with a GLD–2.

GLD–BV–1. Specify the bivariate p.d.f. $f(x,y)$ to be fitted and its marginals $f_1(x)$ and $f_2(y)$.

GLD–BV–2. Fit both marginals with GLDs, using some suitable method (such as the Method of Moments, or if the moments do not exist the Method of Percentiles).

GLD–BV–3. Graph the marginals $f_1(x), f_2(y)$ and their GLD fits to verify the quality of the univariate fits.

GLD–BV–4. Evaluate the GLD–2 distribution numerically on a grid of $n \times n$ values of x and y for a broad range of Ψ values.

GLD–BV–5. Choose Ψ optimizing the fit with some criterion, yielding $\Psi = \Psi_0$.

GLD–BV–6. Graph the true distribution $f(x, y)$ and the GLD–2 with $\Psi = \Psi_0$ to visualize the quality of the fit.

In the remainder of this section we consider four bivariate distributions: the **Bivariate Normal**, **Gumbel's Bivariate Exponential**, the **Bivariate Cauchy**, and **Kibble's Bivariate Gamma**, each with univariate marginals of the same name. The marginals are fitted with the Method of Moments (except for the Cauchy, which has no moments, in which case the Method of Percentiles is used). Hutchinson and Lai (1990) is an excellent reference on bivariate distributions that is broad and encyclopedic.

9.3.1 The Bivariate Normal (BVN) Distribution

Here the standard BVN of (9.2.21) is used, hence we have $N(0, 1)$ marginals, correlation coefficient ρ, and moments $\mu = 0, \sigma^2 = 1, \alpha_3 = 0$, and $\alpha_4 = 3$. From Section 3.4.1 we find the fitted GLD has $\lambda = (0, 0.1975, 0.1349, 0.1349)$, and is a good univariate fit.

Having the univariate fits, the appropriate value of Ψ for each ρ is needed next. Use of Algorithm GLD–BV of Section 9.3 proceeds as follows:

- Set n, the number of grid values for each marginal.
- Calculate equally spaced x and y values, and the corresponding ρ value. Also calculate $f(x), f(y)$, and store these values in arrays.
- Find the value of Ψ (to three decimal places) which minimizes the L_1 error.
- Output the results for each $\rho = -0.9, -0.8, \cdots, 0.9$.

Since (see Beckwith and Dudewicz (1996)) there is a unique value of Ψ that minimizes the L_1 error, our program (the listing is in Appendix A) asks the user for N, the maximum number of iterations ($10 \leq N \leq 40$ is usually sufficient). The L_1 error is calculated for $\Psi = 0, 1, 2, \ldots, N$ (for brevity, we designate this succession of values of Ψ by $\Psi = 0(1)N$) and the least value Ψ_1 is kept. The initial error should be large for small Ψ, decrease near the minimum, and then increase again. So the error is calculated from $\Psi_1 - 1$ to $\Psi_1 + 1$ in increments of 0.001 until a three decimal place "best" $\Psi = \Psi_0$ is found.

At each iteration, four other measures of error are calculated: the smallest ratio and the largest ratio of the estimated $f(x, y)$ to the true $f(x, y)$, and the smallest and largest absolute differences of estimated and true $f(x, y)$. Also, the estimated volume of the two joint distributions is calculated to ensure accuracy of the L_1 error (if the volumes aren't near one, the process is suspect). The truncation required (at ± 4 in this case) will lead to error, as will the choice of Riemann partition size. These errors are summarized in Table 9.3–1 (excluding

Table 9.3–1. Optimal Ψ and Corresponding Error Estimates.

Distribution	h = GLD–2	Max $\lvert h - f \rvert$	Max h/f	Min h/f	L_1 Error
BVN, $\rho = -0.8$	$\Psi = 0.083$	0.0611	1772332.764	0.003	0.17694
BVN, $\rho = -0.5$	$\Psi = 0.274$	0.0135	8739.460	0.006	0.08661
BVN, $\rho = 0$	$\Psi = 1.000$	0.0023	1.101	0.330	0.00927
BVN, $\rho = 0.2$	$\Psi = 1.624$	0.0038	10.942	0.040	0.03159
BVN, $\rho = 0.5$	$\Psi = 3.653$	0.0119	8727.850	0.006	0.08661
BVN, $\rho = 0.9$	$\Psi = 28.395$	0.0721	7721885.406	0.003	0.22283
BVE, $\theta = 0$	$\Psi = 0.989$	0.3031	1.004	0.695	0.01919
BVE, $\theta = 0.5$	$\Psi = 0.477$	0.1705	130.895	0.666	0.08628
BVE, $\theta = 0.5$	$\Psi = 0.278$	0.0934	15793.256	0.723	0.10465
BVC of (5.3.4)	$\Psi = 1.553$	0.0512	8.093	0.097	0.32887
BVC, Independent	$\Psi = 1.000$	0.0041	1.084	0.958	0.02019
BVG, $\rho = 0.2$	$\Psi = 1.611$	0.0338	3.894	0.109	0.09931
BVG, $\rho = 0.4$	$\Psi = 2.605$	0.0458	62.180	0.053	0.12313
BVG, $\rho = 0.6$	$\Psi = 4.614$	0.0731	25062.656	0.038	0.15694
BVG, $\rho = 0.8$	$\Psi = 11.004$	0.1298	1204440.624	0.039	0.21297

$\min \lvert f - h \rvert$, which was $< 10^{-5}$ for all cases). See Figure 9.3–2[1] for a graph of $\Psi = 3(0.1)5$ (i.e., for $\Psi = 3, 3.1, 3.2, \ldots, 5$), with the corresponding errors, from fitting the Bivariate Normal with $\rho = 0.5$, and optimal $\Psi = 3.653$.

With this estimate of Ψ, a visual check is important. See Figures 9.3–3 through 9.3–8 for graphs of the GLD–2, the true BVN, and the error

$$f(x, y|\rho) - h(x, y|\Psi),$$

both as 3-D density graphs and 2-D contour graphs. The values of ρ used are -0.8, -0.5, $0, 0.2$, 0.5, and 0.9. The BVN seems to work well for most of these examples, the method is quite efficient, and most of the error seems due to the univariate fits. Good references on graphic visualization of bivariate distributions are the paper of Johnson, Wang, and Ramberg (1984) and the book of Johnson (1987).

[1]Some of the figures of this and the next two sections (specifically, Figures 9.3–2 through 9.3–12, 9.3–14, 9.3–15, 9.3–17 through 9.3–20, 9.4–2, 9.4–3, 9.4–5, and 9.5–1 through 9.5–6) are reprinted with permission from Beckwith and Dudewicz (1996), and are copyright ©1996 by American Sciences Press, Inc., 20 Cross Road, Syracuse, New York 13224-2104.

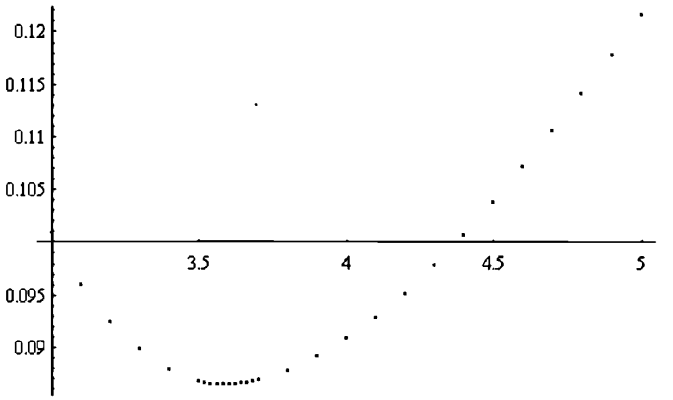


Figure 9.3–2. L_1 error (vertical axis) vs. Ψ (horizontal axis) for the BVN with $\rho = 0.5$.

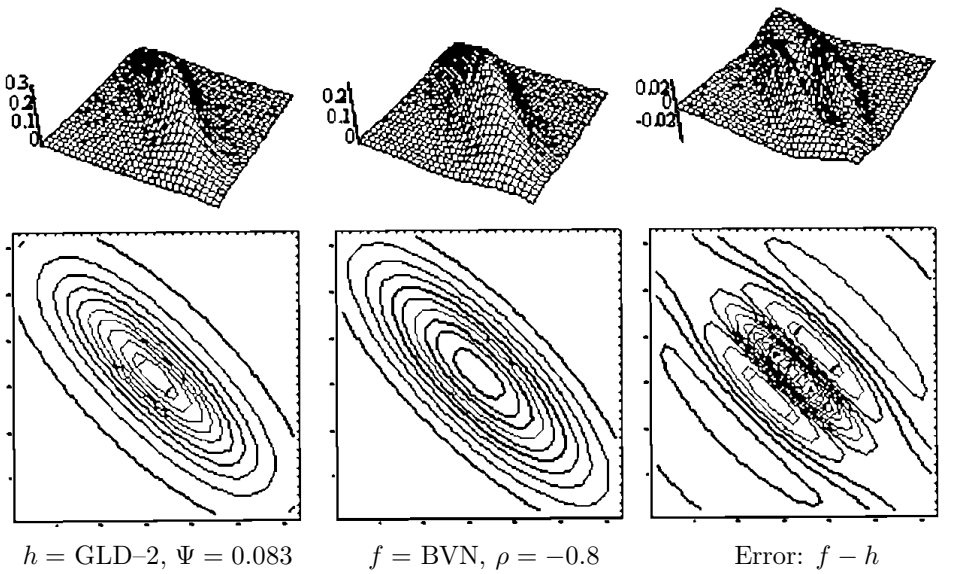
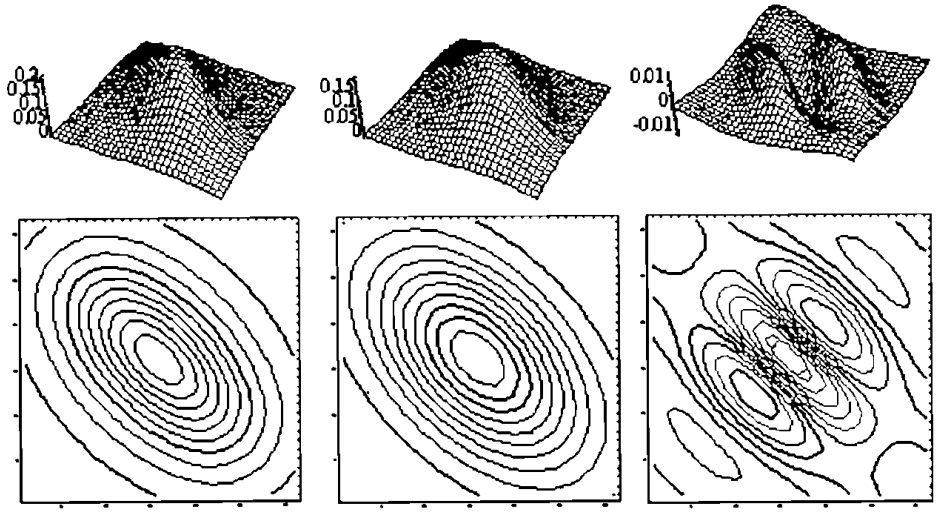
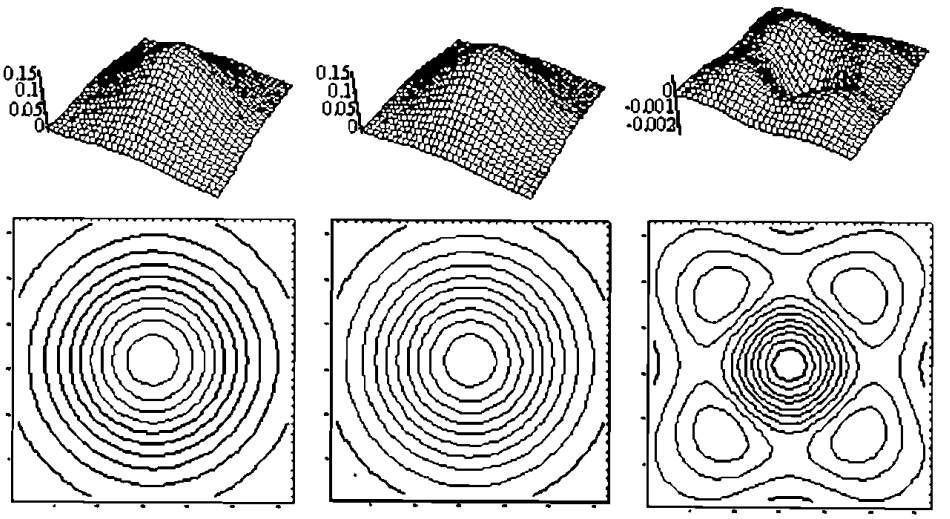


Figure 9.3–3. Bivariate normal ($\rho = -0.8$) and GLD–2.



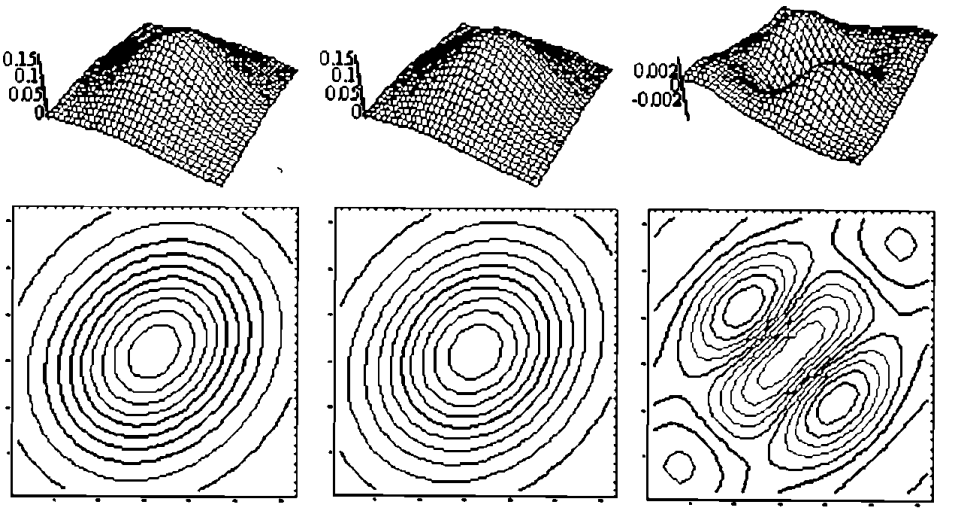
h = GLD–2, $\Psi = 0.274$ f = BVN, $\rho = -0.5$ Error: $f - h$

Figure 9.3–4. Bivariate normal ($\rho = -0.5$) and GLD–2.



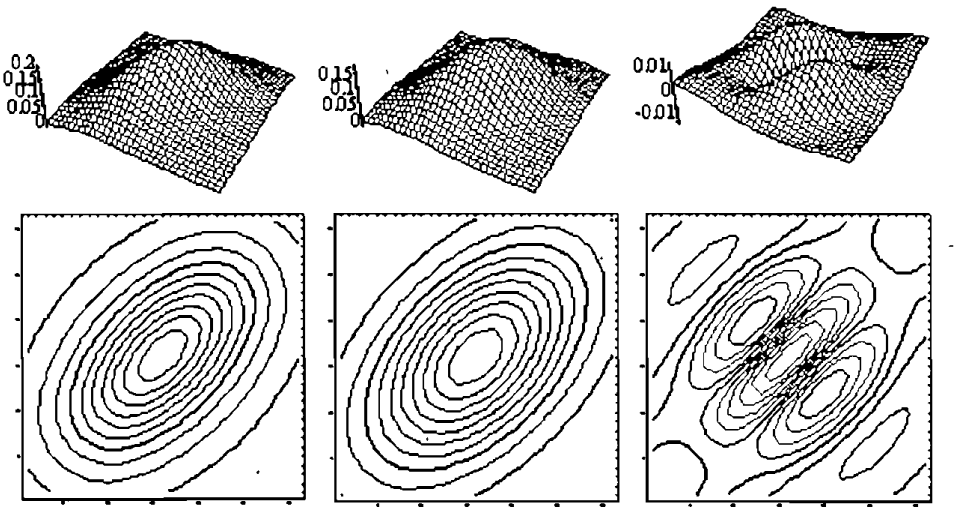
h = GLD–2, $\Psi = 1$ f = BVN, $\rho = 0$ Error: $f - h$

Figure 9.3–5. Bivariate normal ($\rho = 0$) and GLD–2.



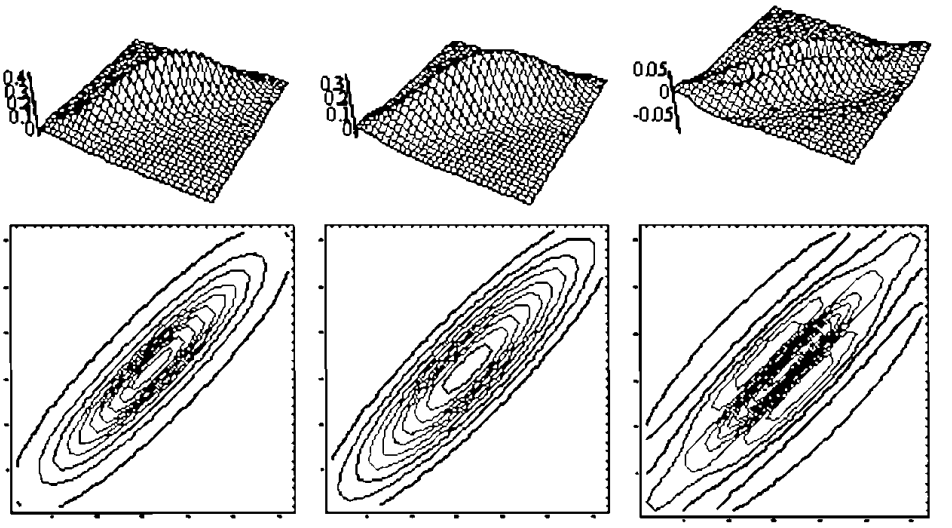
h = GLD–2, $\Psi = 1.624$ f = BVN, $\rho = 0.2$ Error: $f - h$

Figure 9.3–6. Bivariate normal ($\rho = 0.2$) and GLD–2.



h = GLD–2, $\Psi = 3.563$ f = BVN, $\rho = 0.5$ Error: $f - h$

Figure 9.3–7. Bivariate normal ($\rho = 0.5$) and GLD–2.



h = GLD–2, $\Psi = 28.395$ f = BVN, $\rho = 0.9$ Error: $f - h$

Figure 9.3–8. Bivariate normal ($\rho = 0.9$) and GLD–2.

9.3.2 Gumbel's Bivariate Exponential Type I (BVE)

Here the d.f. is

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)} \tag{9.3.2}$$

and the p.d.f. is

$$f(x, y) = [(1 + \theta x)(1 + \theta y) - \theta]e^{-(x+y+\theta xy)}, \tag{9.3.3}$$

where $0 \leq \theta \leq 1$ and $x, y \geq 0$ (see Hutchinson and Lai (1990)).

The marginals are each standard Exponential, that is Exp[1], hence

$$\mu = 1, \quad \sigma^2 = 1, \quad \alpha_3 = 2, \quad \alpha_4 = 9.$$

From the tables (also from Section 3.4.4),

$$\lambda(0, 1) = (-0.993, -0.001081, -0.00000407, -0.001076),$$

so by Step 5 of Algorithm GLD–M in Section 3.3.2, we have

$$\lambda_1(1, 1) = \lambda_1(0, 1)1 + 1 = -0.993 + 1 = 0.007$$

and

$$\lambda_2(1, 1) = \frac{\lambda_2(0, 1)}{1} = -0.001081.$$

Therefore,

$$\lambda = (0.007, -0.001081, -0.00000407, -0.001076).$$

The quality of the univariate exponential GLD fit was covered in Section 3.4.4 (e.g., see Figure 3.4–4). Eleven graphs of the BVE for $\theta = 0(0.1)1$ are shown in Figure 9.3–9.

Graphs of the optimal GLD–2, BVE, and the difference function for $\theta = 0, 0.5$, and 0.9 are shown in Figures 9.3–10 through 9.3–12.

9.3.3 Bivariate Cauchy (BVC)

Fitting the Bivariate Cauchy (BVC) reveals interesting facets of the GLD. The p.d.f. of the BVC is

$$f(x, y) = \frac{1}{2\pi}(1 + x^2 + y^2)^{(-3/2)}, \quad -\infty < x,\ y < \infty, \tag{9.3.4}$$

with marginals

$$f_1(x) = f_2(x) = \frac{1}{\pi(1 + x^2)} \quad -\infty < x,\ y < \infty. \tag{9.3.5}$$

Since the Cauchy distribution has no expected values, the Method of Moments cannot be used in fitting the GLD to the Cauchy. With the Method of Percentiles, noting that the Cauchy is $t(1)$, the Student's t with 1 degree of freedom, from Section 5.4.3 we have the fit with

$$\lambda = (0, -2.0676, -0.8727, -0.8727). \tag{9.3.6}$$

The percentile function for $t(1)$ is available in closed form: $F(x) = p$ where $0 < p < 1$ yields

$$p = \frac{1}{\pi}\left(\tan^{-1}(x) + \frac{\pi}{2}\right) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}, \tag{9.3.7}$$

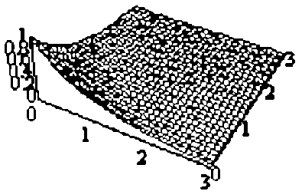
from which one finds

$$Q(p) = \tan\left(\pi(p - \frac{1}{2})\right). \tag{9.3.8}$$

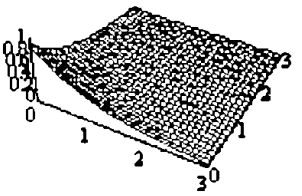
Using an alternative percentile method due to Mykytka (1979), Beckwith and Dudewicz (1996) obtain the fit with

$$\lambda = (0, -2.136833, -0.88871662, -0.88871662), \tag{9.3.9}$$

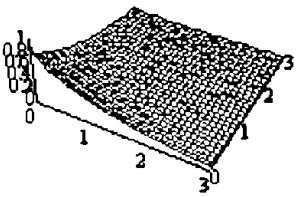
which is very close to the fit (9.3.6) of Section 5.4.3. Note that both fits are in Region 4 (see Figure 2.2–1). Hence both have support $(-\infty, \infty)$ and only the means ($\alpha_1 = 0$, in both cases) exist (see Theorem 2.3.23, Theorem 3.1.4, and Corollary 3.1.10). Figure 9.3–13 contains a graph of the univariate Cauchy p.d.f., its GLD fit via (9.3.9), and the difference of the two p.d.f.s.



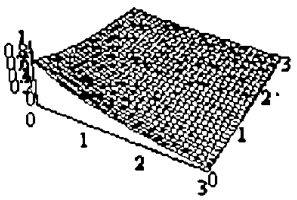
$\theta = 0$



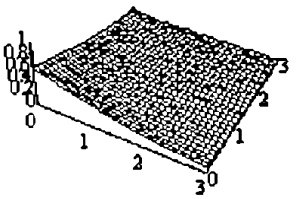
$\theta = 0.1$



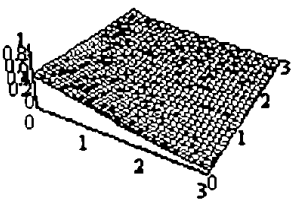
$\theta = 0.2$



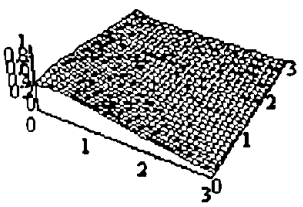
$\theta = 0.3$



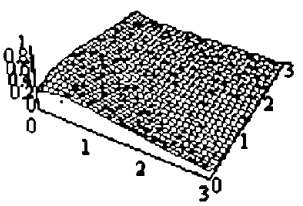
$\theta = 0.4$



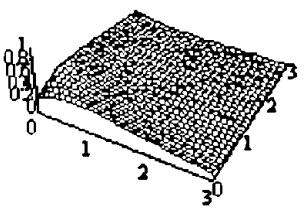
$\theta = 0.5$



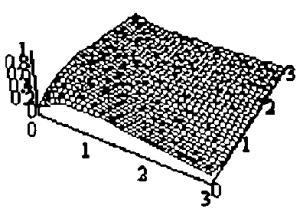
$\theta = 0.6$



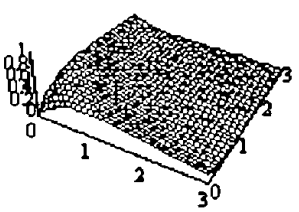
$\theta = 0.7$



$\theta = 0.8$

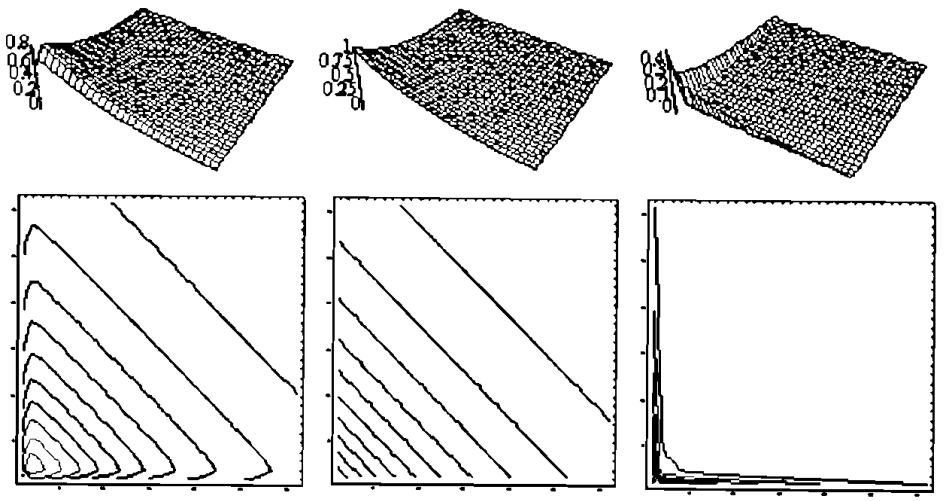


$\theta = 0.9$



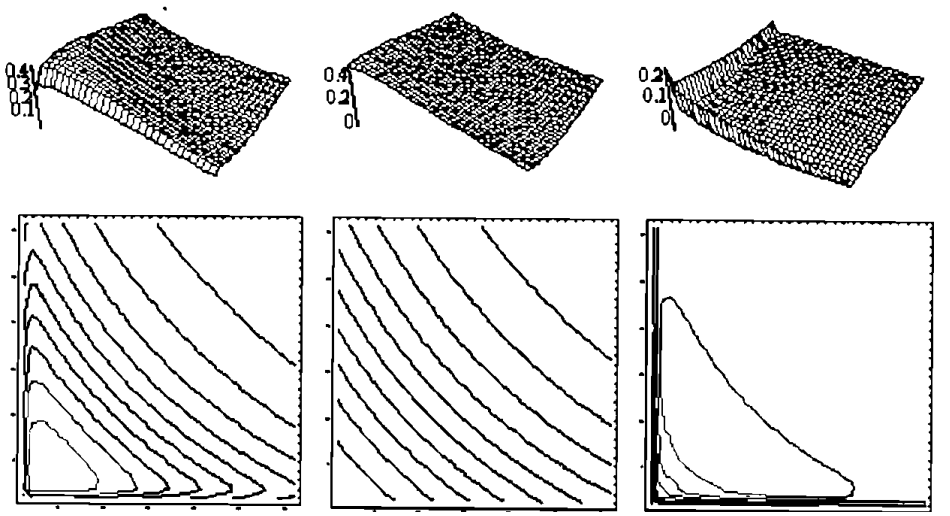
$\theta = 1$

Figure 9.3–9. Graphs of the bivariate exponential p.d.f.



h = GLD–2, $\Psi = 0.989$ f = BVE, $\theta = 0$ Error: $f - h$

Figure 9.3–10. Bivariate exponential ($\theta = 0$) and GLD–2.



h = GLD–2, $\Psi = 0.477$ f = BVE, $\theta = 0.5$ Error: $f - h$

Figure 9.3–11. Bivariate exponential ($\theta = 0.5$) and GLD–2.

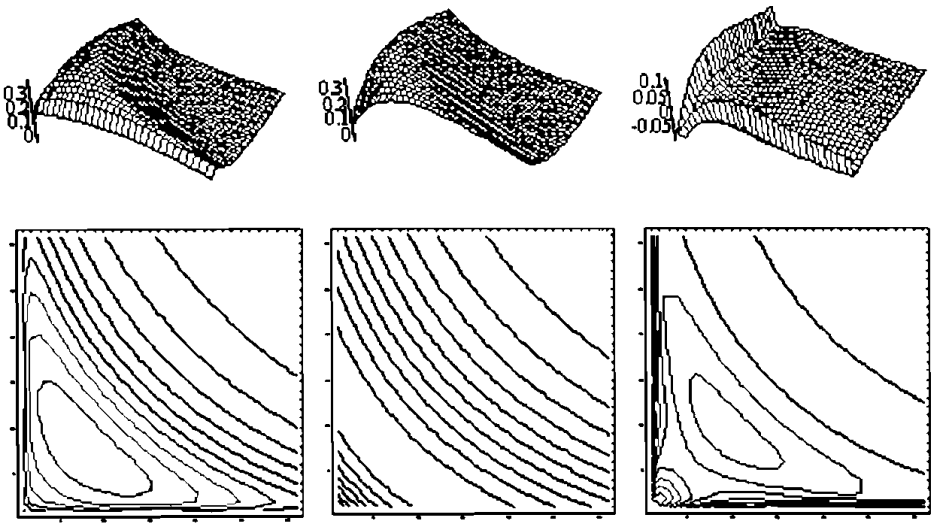


Figure 9.3–12. Bivariate exponential ($\theta = 0.9$) and GLD–2.

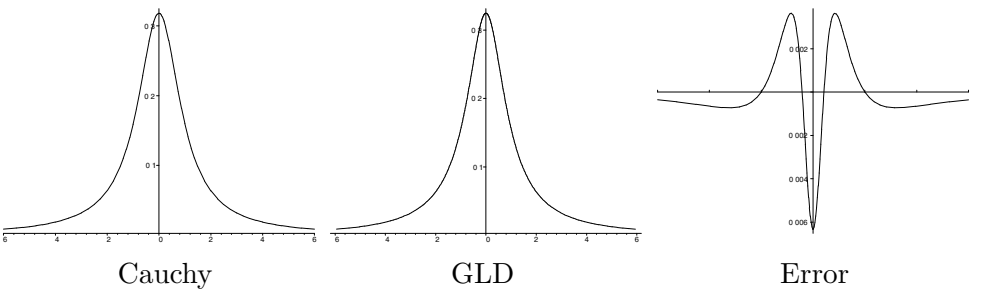


Figure 9.3–13. The univariate Cauchy p.d.f., its GLD approximation, and approximation error.

Note that in the bivariate Cauchy distribution function there is no coefficient of association for the marginals: there is only one distribution (this is a special (bivariate) case of the multivariate Cauchy), and it is **not** the independent joint distribution which has

$$f(x, y) = f_1(x) \cdot f_2(y).$$

Using Algorithm GLD–BV with f the BVC of (9.3.3) yields Figure 9.3–14. For f the independent joint distribution, we obtain Figure 9.3–15. For each, the smallest L_1 error occurred at or near $\Psi = 1$, which yields an independent bivariate p.d.f. The errors $\max \mid f - h \mid$ for the BVC and the independent joint distribution were very low, 0.0512 and 0.0041, respectively; the L_1 error using the BVC was 0.32887 (the largest of all cases) whereas the L_1 error for the independent joint distribution was 0.02019, indicating a very good fit. Note that we used $\Psi = 1$ in Figure 9.3–15 (due to theoretical implications of the symmetry, although Algorithm GLD–BV yields $\Psi = 1.5534$ for minimum L_1 error).

9.3.4 Kibble's Bivariate Gamma (BVG)

This distribution has p.d.f.

$$f(x, y) = \frac{f_\alpha(x) f_\alpha(y) \Gamma(\alpha)}{1 - \rho} \exp\left[-\frac{\rho(x + y)}{1 - \rho}\right] (xy\rho)^{-(a-1)/2} I_{\alpha-1}\left(\frac{2\sqrt{xy\rho}}{1 - \rho}\right) \tag{9.3.10}$$

where $x, y \geq 0,\ \alpha > 0,\ 0 \leq \rho < 1,\ f_\alpha(t) = \frac{t^{\alpha-1} e^{-t}}{\Gamma(\alpha)}$, and

$$I_k(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{k+2r}}{r! \Gamma(k + r + 1)}$$

is a modified Bessel function. Although the p.d.f. appears complicated, it is one of the simpler bivariate Gamma distributions as compared, e.g., with Jensen's (see Hutchison and Lai (1990)), and the convergence of the Bessel function is good for small values of ρ.

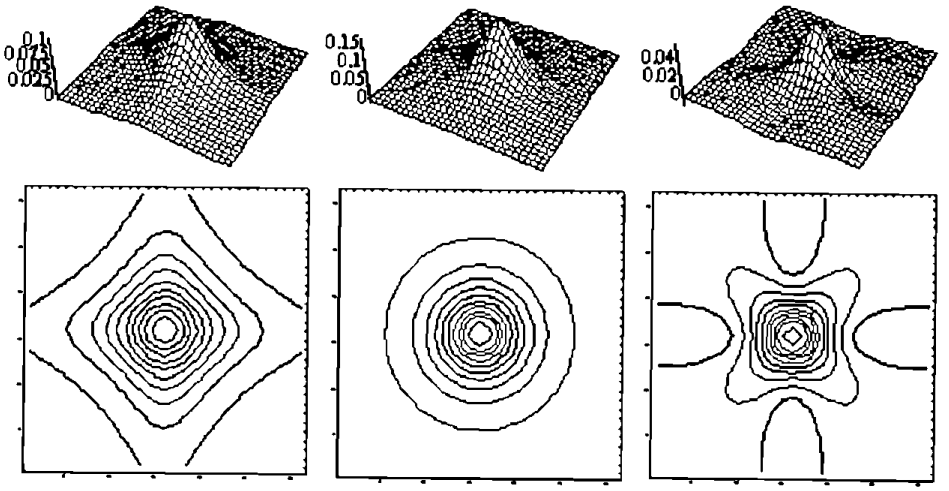
The marginals are each $\Gamma(\alpha, 1)$, for which with $\alpha = 2$ we have (see Section 3.4.6) $\mu = 2,\ \sigma^2 = 2,\ \alpha_3 = \sqrt{2}$, and $\alpha_4 = 6$. From the Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979) tables

$$\lambda(0, 1) = (-0.782, 0.0379, 0.005603, 0.0365),$$

so by Step 5 of Algorithm GLD-M of Section 3.3.2, $\lambda_1(2, 2) = \lambda_1(0, 1)\sqrt{2} + 2 = 0.8941,\ \lambda_2(2, 2) = \lambda_2(0, 1)/\sqrt{2} = 0.02680$, and

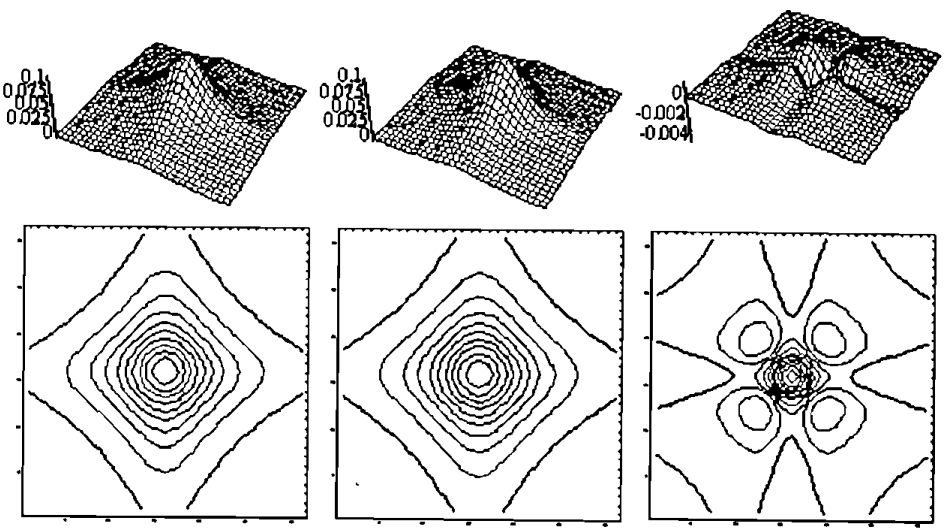
$$\lambda = (0.894085, 0.0267993, 0.005603, 0.0365).$$

Figure 9.3–16 contains graphs of $\Gamma(2, 1)$, along with its GLD fit. See Figures 9.3–17 through 9.3–20 for graphs of the GLD–2, Bivariate Gamma, and error.



h = GLD–2, $\Psi = 1$ f = BVC Error: $f - h$

Figure 9.3–14. Bivariate Cauchy of (9.3.4).



h = GLD–2, $\Psi = 1$ $f =$ Independent Cauchy Error: $f - h$

Figure 9.3–15. Bivariate independent Cauchy.

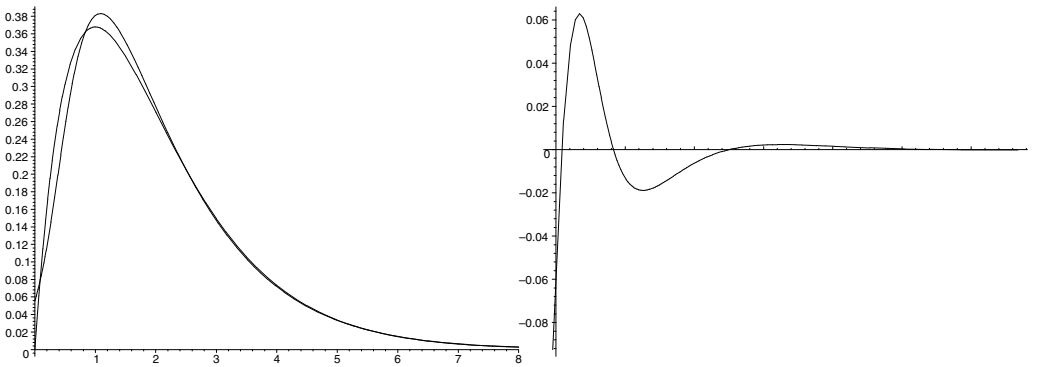
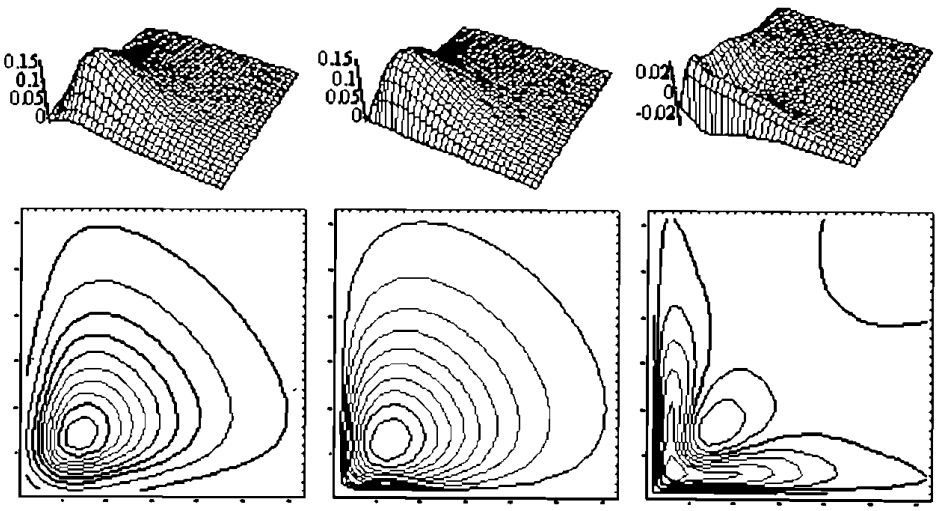
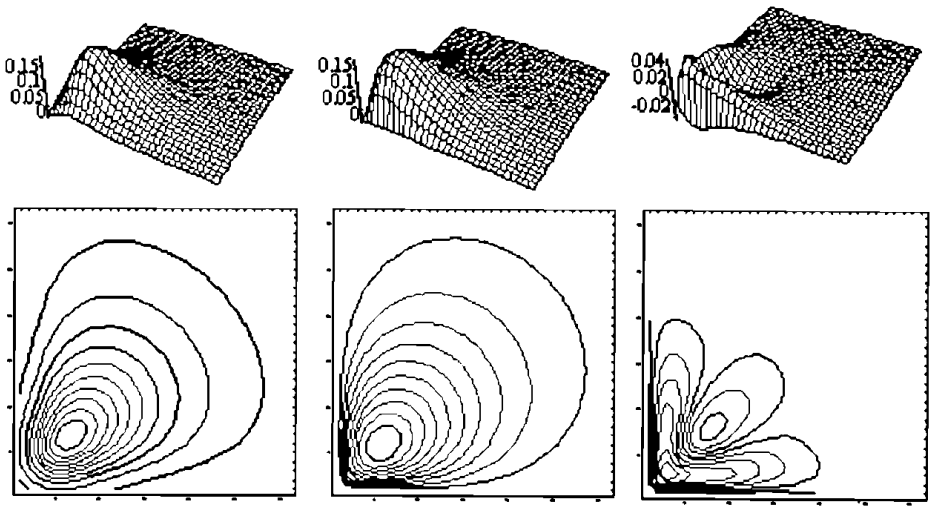


Figure 9.3–16. The $\Gamma(2,1)$ p.d.f. with its GLD approximation, and the error of the GLD approximation.



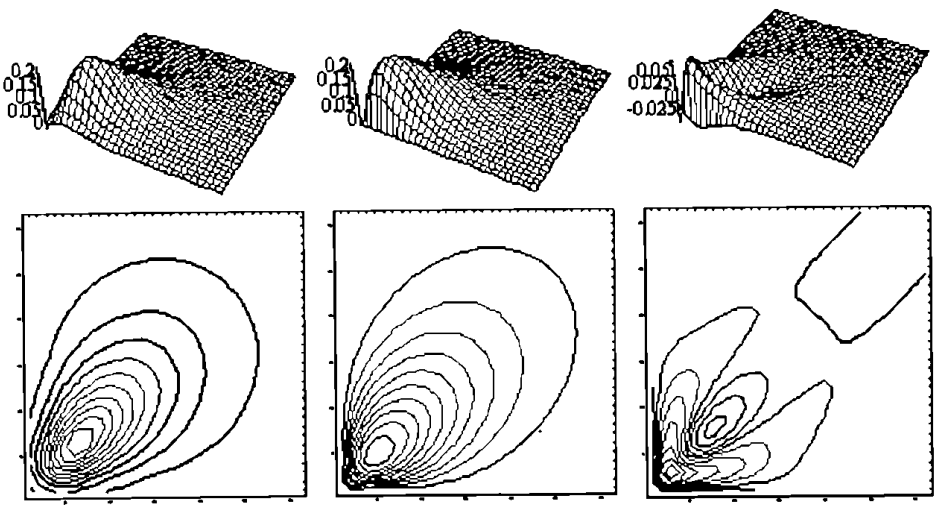
h = GLD–2, $\Psi = 1.611$ f = BVG, $\rho = 0.2$ Error: $f - h$

Figure 9.3–17. Bivariate gamma ($\rho = 0.2$) and GLD–2.



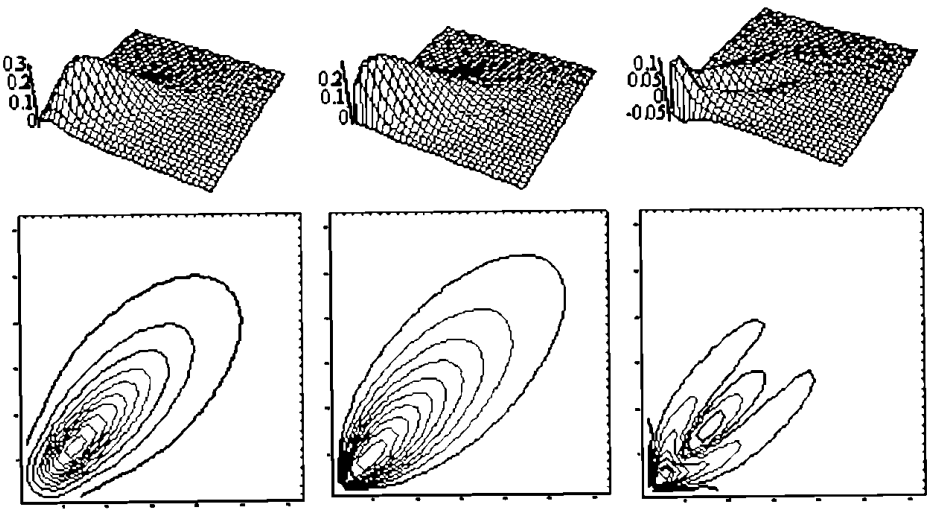
h = GLD–2, $\Psi = 2.605$ f = BVG, $\rho = 0.4$ Error: $f - h$

Figure 9.3–18. Bivariate gamma ($\rho = 0.4$) and GLD–2.



h = GLD–2, $\Psi = 4.614$ f = BVG, $\rho = 0.6$ Error: $f - h$

Figure 9.3–19. Bivariate gamma ($\rho = 0.6$) and GLD–2.



h = GLD–2, $\Psi = 11.004$ f = BVG, $\rho = 0.8$ Error: $f - h$

Figure 9.3–20. Bivariate gamma ($\rho = 0.8$) and GLD–2.

9.4 GLD–2 Fits: Distributions with Non-Identical Marginals

In the examples of Section 9.3, known bivariate distributions whose marginals are identical were considered. We now consider bivariate distributions with different marginals to examine the variety of shapes that can be achieved by the GLD–2.

9.4.1 Bivariate Gamma BVG with Non-Identical Marginals

First, consider X to be the $\Gamma(2, 1)$ r.v. and Y the $\Gamma(4, 1)$ r.v., both fitted by GLDs. $\Gamma(2, 1)$ was fitted in Section 9.3.4 via

$$\lambda = (0.894085, 0.0267993, 0.005603, 0.036)$$

and $\Gamma(4, 1)$ can be fitted by a univariate GLD with

$$\lambda = (2.656, 0.0421, 0.0194, 0.08235).$$

See Figure 9.4–1 for graphs of the univariate GLD fits, and Figure 9.4–2 for contour graphs of the fitted GLD–2 with $\Psi = 0.05(0.115)0.95$ (negative correlation) and $\Psi = 1(3)13$ (zero and positive correlation). The contour graphs in Figure 9.4–2 are truncated to $0 \leq x, y \leq 10$. (Note the contour heights between graphs are **not** the same.)

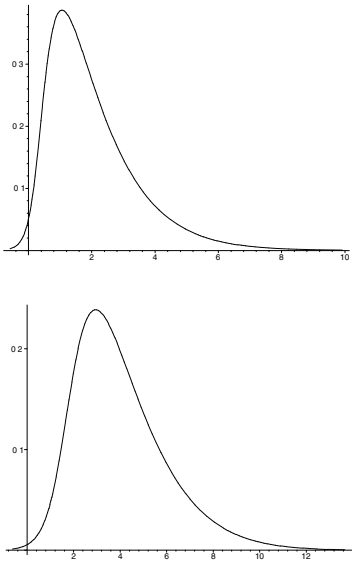


Figure 9.4–1. Univariate GLD fits to $\Gamma(2,1)$ (on the top) and $\Gamma(4,1)$ (on the bottom).

9.4.2 Bivariate with Normal and Cauchy Marginals

As our second example with non-identical marginals, suppose X is Cauchy and Y is $N(0,1)$. Each of these distributions was used previously in Section 9.3 in fitting their respective bivariate p.d.f.s. Figure 9.4–3 shows several contour graphs, again with $\Psi = 0.05(0.15)0.95$ and $\Psi = 1(3)13$.

9.4.3 Bivariate with Gamma and "Backwards Gamma" Marginals

The final example of the non-identical marginals case involves a non-standard marginal. To reverse the shape of one of the marginals, one can switch λ_3 and λ_4; this will skew the distribution exactly opposite to the original. In the case of Gamma, however, the p.d.f. is non-negative only for $x \geq 0$, whereas this new "Backwards Gamma" is defined for x both negative and positive. To create a p.d.f. more similar to $\Gamma(4,1)$ with a reversed shape, we add an offset of 5 to λ_1 to shift the p.d.f. The new parameters are then

$$\lambda = (7.656, 0.0421, 0.08235, 0.0194).$$

See Figure 9.4–4 for a graph of the GLD "Backwards Gamma" and Figure 9.4–5 for contour graphs of the GLD–2 for 12 values of Ψ.

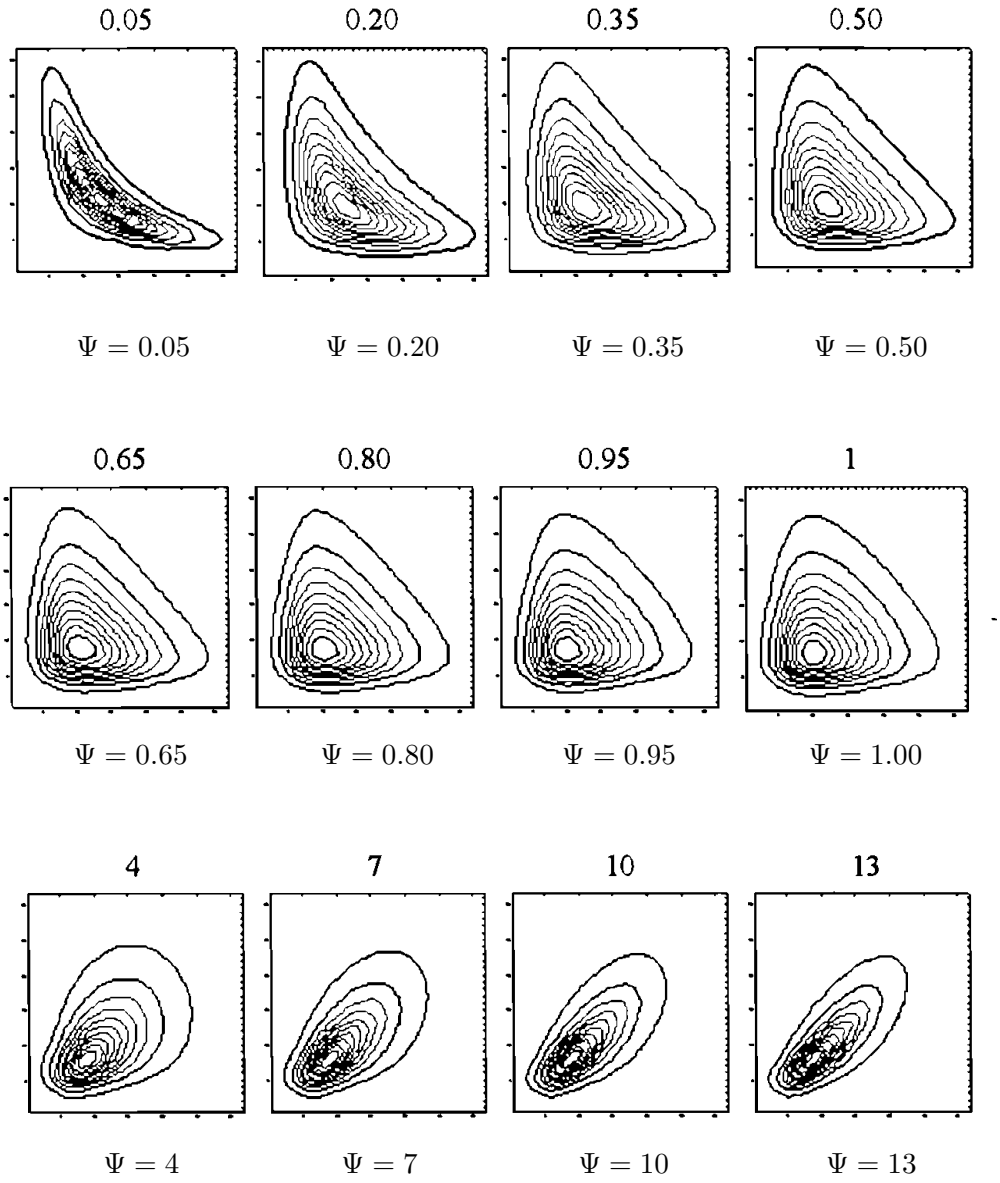
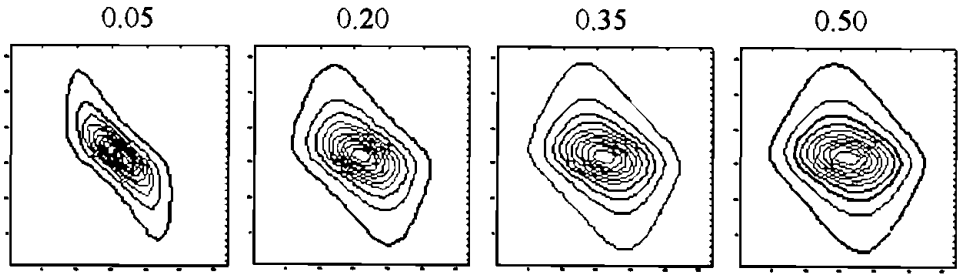
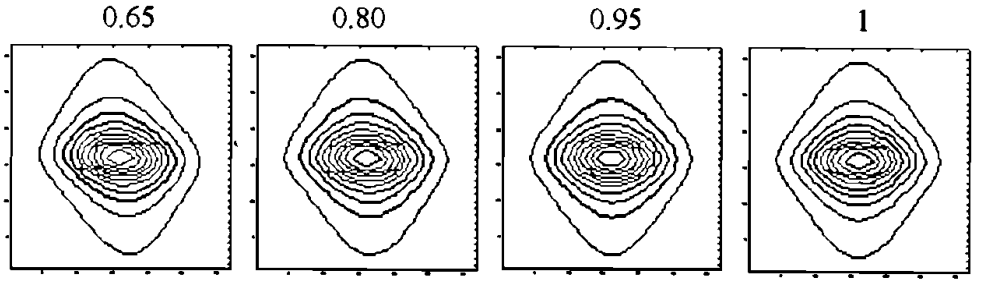


Figure 9.4–2. GLD–2 with different gamma marginals (X is $\Gamma(2,1)$ and Y is $\Gamma(4,1)$) for 12 values of Ψ (graphs show $0 < x, y < 6$).

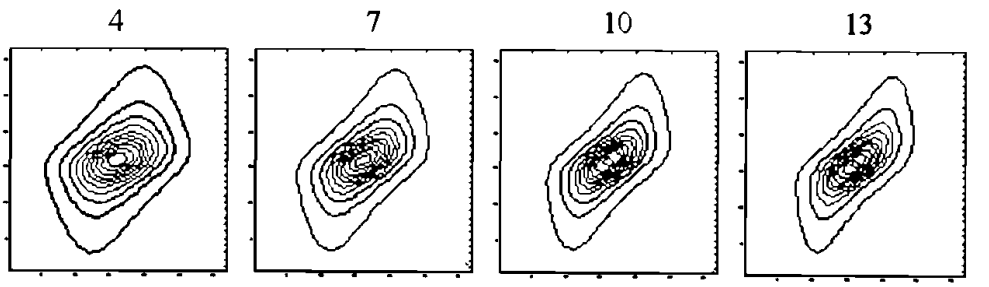


$\Psi = 0.05$ $\Psi = 0.20$ $\Psi = 0.35$ $\Psi = 0.50$

$\Psi = 0.05(0.15)0.95$ and $1(3)12$ (continued).



$\Psi = 0.65$ $\Psi = 0.80$ $\Psi = 0.95$ $\Psi = 1.00$



$\Psi = 4$ $\Psi = 7$ $\Psi = 10$ $\Psi = 13$

Figure 9.4–3. The GLD–2 for one Cauchy marginal (X) and one normal marginal (Y) for $\Psi = 0.05(0.15)0.95$ and $1(3)12$ (concluded).

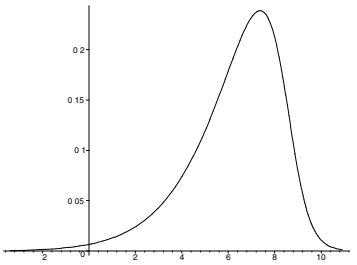


Figure 9.4–4. The univariate GLD for the "backwards $\Gamma(4, 1)$."

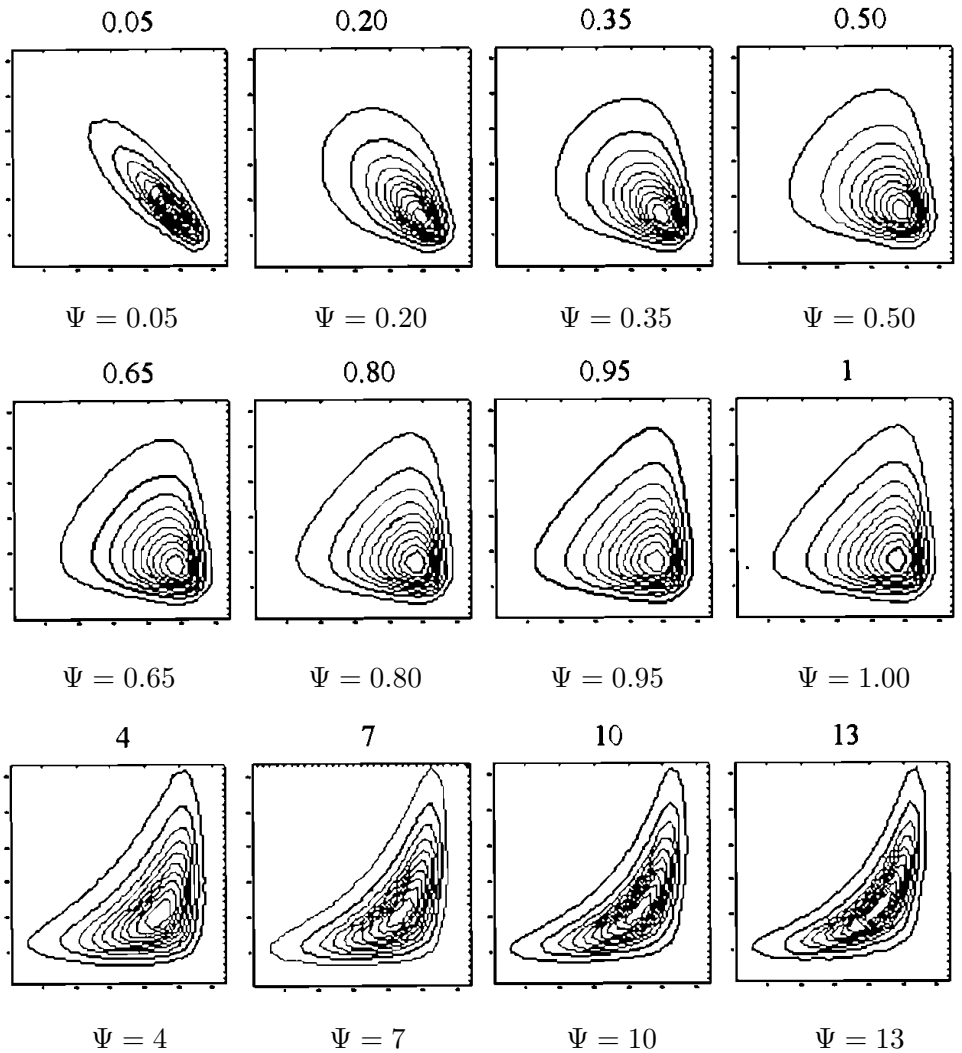


Figure 9.4–5. The GLD–2 with X as the $\Gamma(2, 1)$ and Y as the "backwards $\Gamma(4, 1)$" for $\Psi = 0.05(0.15)0.95$ and $1(3)12$.

9.5 Fitting GLD–2 to Datasets

To fit the GLD–2 to a dataset we begin, as in Algorithm GLD–BV in Section 9.3 for a known bivariate distribution, by approximating the marginals with GLDs. Since the true distributions are unknown, the sample moments or percentiles are used. In this section we develop an algorithm for fitting the GLD–2 to a dataset; next, we show the results of applying the algorithm to six datasets considered by Beckwith and Dudewicz (1996), and follow this up by giving the full details of fitting GLD–2 distributions in three examples.

9.5.1 Algorithm for Fitting the GLD–2 to Data

The success obtained in fitting known bivariate distributions leads us to expect good results in the more difficult case of bivariate datasets with unknown marginals and joint distributions. In Plackett (1965) a method of estimating Ψ was proposed for this situation: graph a scatterplot of the data, along with the two lines at $x = \tilde{x}$ and $y = \tilde{y}$, the respective medians; count the number of points in each of the four resulting quadrants, denoting these four counts in the sets $\{x \le \tilde{x}, y \le \tilde{y}\}$, $\{x \le \tilde{x}, y > \tilde{y}\}$, $\{x > \tilde{x}, y \le \tilde{y}\}$, and $\{x > \tilde{x}, y > \tilde{y}\}$, by a, b, c, and d, respectively; then use $\Psi^+ = (a \cdot d)/(b \cdot c)$ to estimate Ψ (this estimator is motivated by the relationship given in (9.2.9)).

Beckwith and Dudewicz (1996) proposed the following algorithm for fitting the GLD–2 to bivariate datasets:

Algorithm GLD–BVD: Fitting Bivariate Data with a GLD–2.

GLD–BVD–1. Given a bivariate dataset

$$Z_1, Z_2, \ldots, Z_n = (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n),$$

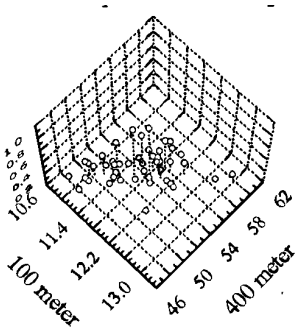
with unknown bivariate p.d.f $f(x, y)$, fit both marginals X and Y with GLDs, using some suitable method (such as the Method of Moments, or the Method of Percentiles).

GLD–BVD–2. Graph the marginal d.f.s $F(x)$ and $F(y)$ along with the empiric d.f.s (e.d.f.s) to verify the quality of the univariate fits, and perform other checks of the univariate fits as discussed in previous chapters.

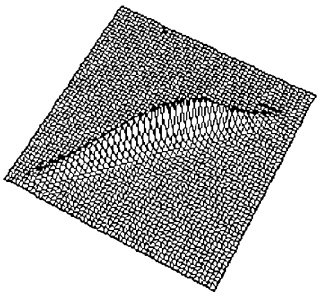
GLD–BVD–3. Graph a scatterplot of the (X,Y) data with the lines $x = \tilde{x}$ and $y = \tilde{y}$, i.e., at the respective medians.

GLD–BVD–4. Count the number of points in each of the four resulting quadrants, labeling them a, b, c, and d, as in Figure 9.2–1.

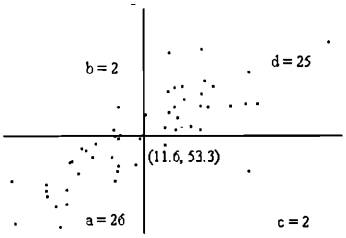
GLD–BVD–5. Calculate $\Psi^+ = (a \cdot d)/(b \cdot c)$, and use this value of Ψ in (9.3.1) with the GLD marginal d.f.s (or in $h(x, y)$ which follows (9.3.1)).



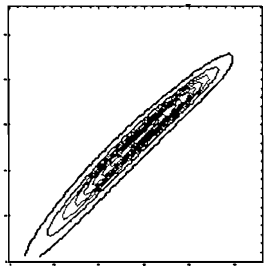
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 162.5$



2-D scatterplot with median lines

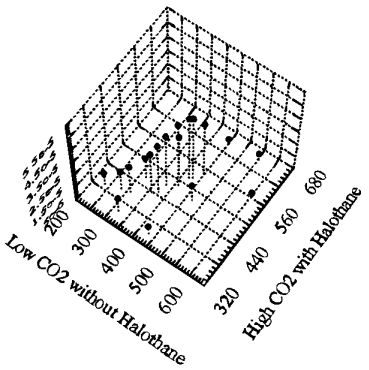


GLD–2 contour plot

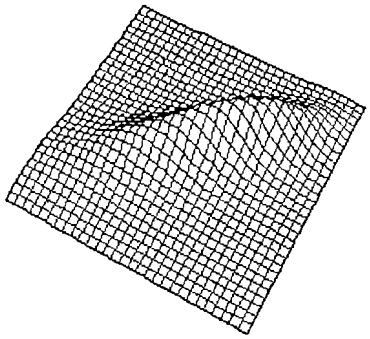
Figure 9.5–1. Track records (in seconds) for 55 countries, 100 meter vs. 400 meter.

GLD–BVD–6. Graph the resulting GLD–2 and a 3-D scatterplot using height values from the GLD–2 to visualize the quality of the fit, and perform quantitative checks of the fit as well.

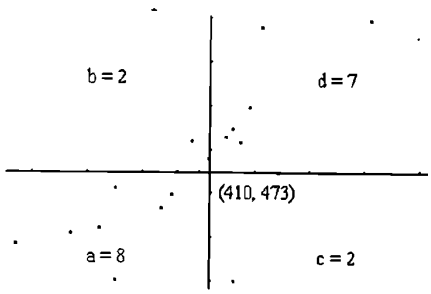
Algorithm GLD–BVD was applied by Beckwith and Dudewicz (1996) to three datasets involving imaging data in the normal brain (see Dudewicz, Levy, Lienhart, and Wehri (1989)) and three datasets from Johnson and Wichern's (1992) text. As it is instructive to see the wide range of applications, of p.d.f. shapes and contours, and of values of Ψ, we provide some of the details of these six applications and then give new examples with full details. For each of the datasets, the univariate GLD fits were calculated using the method of moments and the empiric d.f.s were graphed for a visual check of the fit. The resulting GLD p.d.f. was also graphed. Then the data were graphed in a bivariate scatterplot along with the horizontal and vertical lines at the respective medians to determine the values $a, b, c,$ and d. The respective counts were labeled in each quadrant, and the median lines intersect at $(\tilde{x}, \tilde{y})$. With the resulting Ψ^+, the data were graphed in a 3-D scatterplot with heights calculated from the GLD–2 p.d.f. Also graphed was the resulting GLD–2, both as a 3-D density plot and a 2-D contour plot. Figures 9.5–1 through 9.5–6 give the bivariate graphs for each of the six datasets.



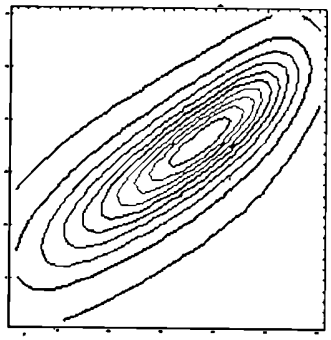
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 1$



2-D scatterplot with median lines



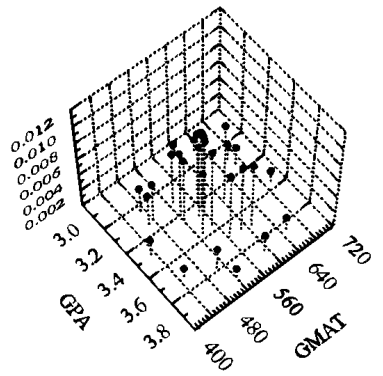
GLD–2 contour plot

Figure 9.5–2. Comparison of the effect of two levels of carbon dioxide in sleeping dogs.

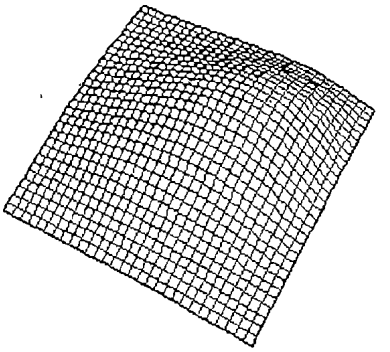
The *first dataset* is from national track records for 55 countries; the first variable is the time (in seconds) of the 100-meter run, and the second is the time (in seconds) of the 400-meter run. The computed sample correlation coefficient is $r = 0.834692$; this modestly high correlation results in a large Ψ^+ of 162.5. The *second dataset* compares the effect of two levels of carbon dioxide on an anesthetic, halothane, in sleeping dogs; here $r = 0.718082$ and $\Psi^+ = 14$. The *third dataset* compares the GPAs and GMAT scores for 31 students admitted to a graduate school of business; here $r = 0.00409619$, and (since this is close to zero), $\Psi^+ = 1.469$ is close to 1.

The fourth, fifth, and sixth datasets are Magnetic Resonance Imaging (MRI) brain scan data. Several measures were taken from MRI scans on approximately 40 patients over a period of time (the "Screened" data was used).

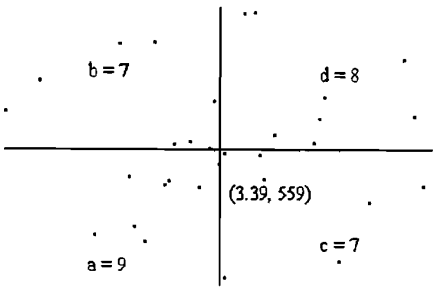
The *first MRI dataset* (our fourth dataset) is from the Putamen section, and the two variables are T2 left and right; here $r = 0.797282$ and $\Psi^+ = 64$. The *second MRI dataset* is AD data, with the first variable Cortical White Right,



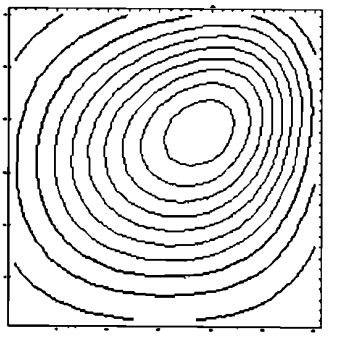
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 1.469$



2-D scatterplot with median lines

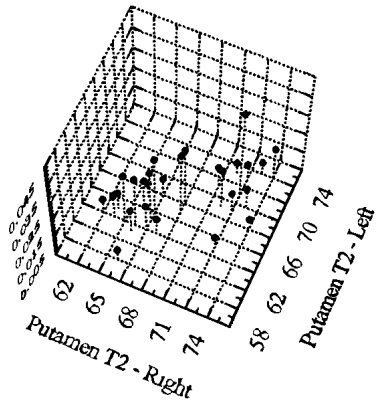


GLD–2 contour plot

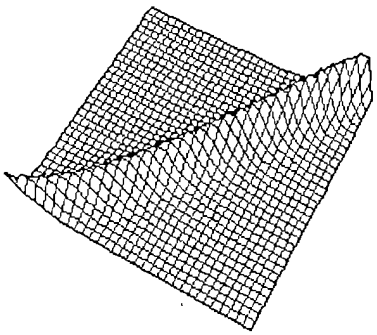
Figure 9.5–3. Comparison of GPA and GMAT scores of students admitted to graduate business schools.

and the second Caudate Right; here $r = 0.783512$ and $\Psi^+ = 23.333$. The *third MRI dataset* is Putamen Right, T2 vs. AD; here $r = -0.603856$ and (since r is negative and somewhat large in absolute value) $\Psi^+ = 0.0222$ is close to zero. It is difficult to judge the quality of the fits when the distributions are unknown, as we do not have the error values like $|h(x, y) - f(x, y)|$ that were available in the known-distribution case, or the known distribution, to compare with. Given this lack of information, however, the graphs presented here do seem to be evidence of fits that are reasonable.

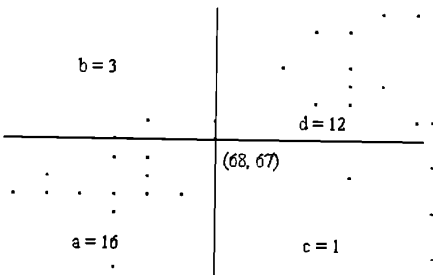
Up to this point in Chapter 9 we have emphasized **visual assessment of goodness-of-fit**. In the univariate case in Section 3.5.1, we noted that in addition to the "eyeball test," there were many statistical tests of the hypothesis that the data come from the fitted distribution, and that it was important to perform at least one of these, lest one's assessment be overly subjective. In the bivariate case, we lack the key result of Point 6 of Section 3.5.1 (Theorem 3.5.1) that one can transform to univariate uniformity. There is, in fact, no such transformation



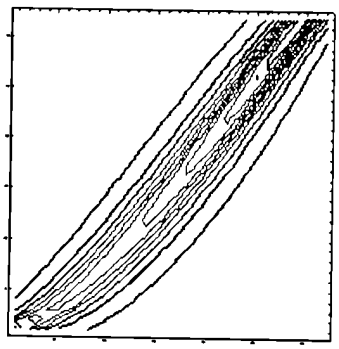
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 64$



2-D scatterplot with median lines



GLD–2 contour plot

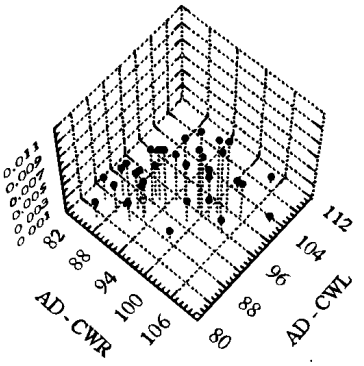
Figure 9.5–4. Comparison of the T2 left and T2 right variables from Putamen sections.

in two or more dimensions. This limits the number of quantitative assessments available; this is an important problem in theoretical statistics which is currently under investigation.

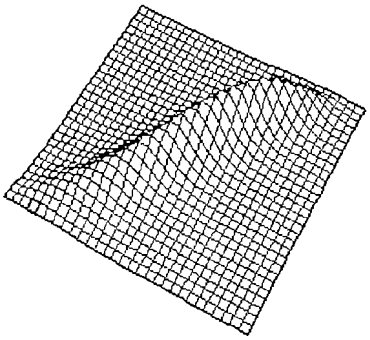
One test that is available is the **chi-square test of the hypothesis that the data** $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ **are independent observations from the fitted distribution** $H(x, y)$**.** To conduct this test, we divide the plane into k non-overlapping and exhaustive cells and obtain the probability that H assigns to each cell. For any rectangular area such as that shown in Figure 9.5–7, the probability can be found by computing (for $a < b,\ c < d$)

$$P(a < X \leq b, c < Y \leq d) = H(b, d) - H(a, d) - H(b, c) + H(a, c). \quad (9.5.1)$$

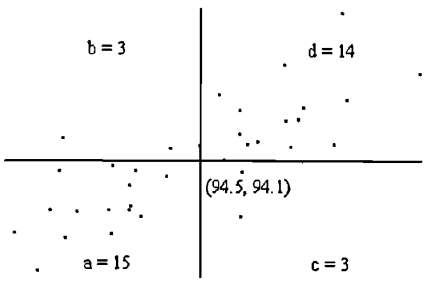
This formula holds even if a, b, c, d take on values such as $\pm\infty$, making it convenient to divide the plane into rectangular cells. As one desires the expected number of points in each cell to be at least 5, one can partition each of X and



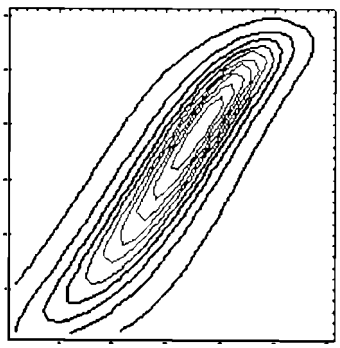
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 23.333$



2-D scatterplot with median lines



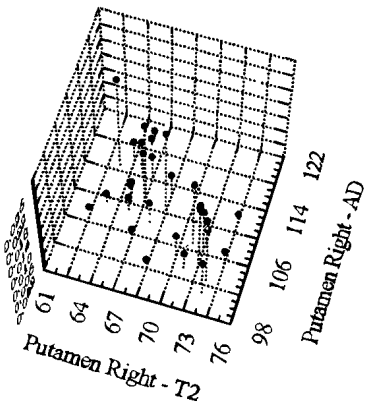
GLD–2 contour plot

Figure 9.5–5. AD of CWR and CWL variables in MRI data.

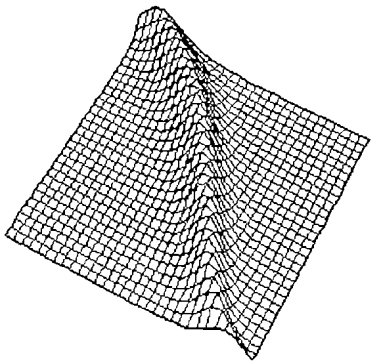
Y into cells, say 4 cells each. This yields $4 \times 4 = 16$ cells in the plane, some of which may have very small probability; such cells can be combined with adjacent cells in order to raise the expected number of points in the combined cell to 5 or higher.

We will return to assessment in the examples that follow and in the discussion of research problems in Section 9.6. We have already shown the GLD–2 fits to a number of datasets, along with visual evidence of goodness-of-fit, plus contours which the typical bivariate models in use today could not come close to duplicating (due to the existence in the data of non-elliptical, possibly even non-convex, contours).

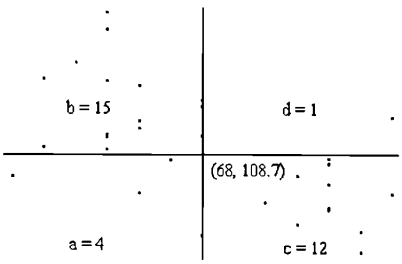
In two of the following three examples we examine in greater detail the quantitative assessment of goodness-of-fit via the chi-square test as discussed above.



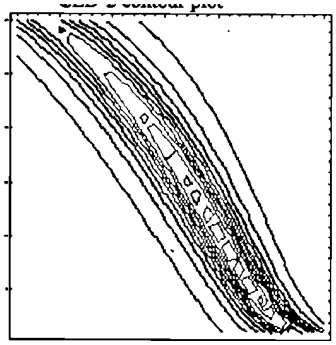
3-D scatterplot with GLD–2 heights



GLD–2, $\Psi = 0.022$



2-D scatterplot with median lines



GLD–2 contour plot

Figure 9.5–6. Putamen right, T2 vs. AD.

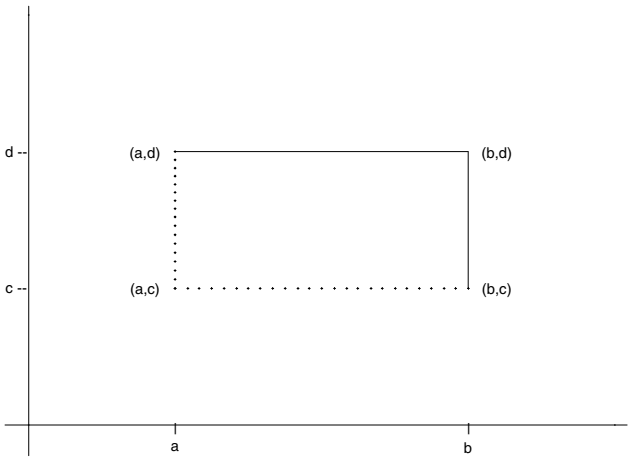


Figure 9.5–7. A rectangular area in the plane and its probability via (9.5.1). Left and lower boundaries are excluded.

9.5.2 Example: Human Twin Data of Section 3.5.4

This dataset of birth weights of twins was considered in Section 3.5.4 (see Table 3.5–3) where both of its components were fitted with GLDs through the method of moments. The fits for X and Y that were obtained were, respectively,

$$\text{GLD}_x(5.5872, 0.2266, 0.2089, 0.1762)$$

and

$$\text{GLD}_y(5.3904, 0.2293, 0.1884, 0.1807).$$

A histogram of X with the fitted GLD_x and a plot of the e.d.f. of X with the d.f. of GLD_x, given in Figure 3.5–4 indicate that GLD_x fits X reasonably well. Similar graphic comparisons in Figure 3.5–5 establish that GLD_y is also a reasonably good fit to Y. The qualities of the two fits were further substantiated through chi-square tests that yielded p-values of 0.03175 for GLD_x and 0.2450 for GLD_y.

The scatterplot of Figure 9.5–8 (a) shows

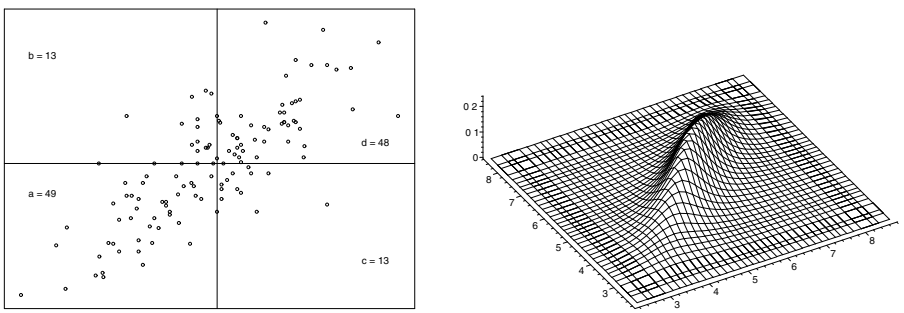
$$a = 49, \qquad b = 13, \qquad c = 13, \qquad d = 48$$

from which we obtain $\Psi^+ = 13.9172$. The GLD–2 that results from this is shown in Figure 9.5–8 (b).

To perform a chi-square goodness-of-fit test we partition the support of the GLD–2 into rectangular cells designated by

$$I_x \times I_y = \{(x, y) \mid x \in I_x \text{ and } y \in I_y\},$$

where I_x and I_y are intervals of finite or infinite length. Since the rectangular cells are of the type shown in Figure 9.5–7, we can use (9.5.1) to compute the probabilities, and hence the expected frequencies for each cell. Table 9.5–9 gives the cells used in this case as well as the observed and expected frequencies associated with each cell.



(a) (b)

Figure 9.5–8. The scatterplot of the birth weights of twins (a) and the fitted GLD–2 (b).

Table 9.5–9. The cells, expected frequencies, and observed frequencies for the twin data and its fitted GLD–2.

Cell	Expected Frequency	Observed Frequency
$(-\infty, 4] \times (-\infty, 4.2]$	7.5625	11
$(4, \infty) \times (-\infty, 4.2]$	10.7254	7
$(-\infty, 4.5] \times (4.2, 5]$	8.4201	9
$(4.5, 5.5] \times (4.2, 5]$	13.8320	13
$(5.5, \infty) \times (4.2, 5]$	5.1798	8
$(-\infty, 5.25] \times (5, 5.75]$	11.1951	8
$(5.25, 5.75] \times (5, 5.75]$	9.4963	9
$(5.75, \infty) \times (5, 5.75]$	11.3240	13
$(-\infty, 5.75], (5.75, 6.5]$	7.5109	8
$(5.75, 6.5] \times (5.75, 6.5]$	11.0697	9
$(6.5, \infty) \times (5.75, 6.5]$	7.5785	12
$(-\infty, 6.75] \times (6.5, \infty)$	9.3363	7
$(6.75, \infty) \times (6.5, \infty)$	9.7695	9

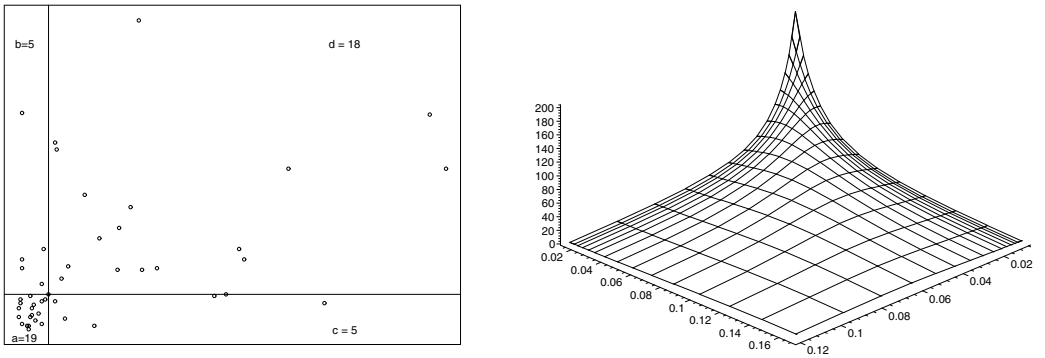
The chi-square statistic and p-value for this test are 9.3116 and 0.02542. Note that to compute the chi-square statistic and p-value we use the chi-square distribution with 3 degrees of freedom because we have 13 cells and 9 estimated parameters: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for each of the GLD_x and GLD_y fits and Ψ^+, hence degrees of freedom is $13 - 9 - 1 = 3$.

9.5.3 Example: The Rainfall Distributions of Section 3.5.5

The rainfall data of Section 3.5.5 (see Table 3.5–6) gives the rainfall (in inches) in Rochester, N.Y. (X) and Syracuse, N.Y. (Y) from May to October 1998, on days when both locations had positive rainfall. Our attempt, in Section 3.5.5, to fit a GLD to X and Y through the method of moments failed because the (α_3^2, α_4) points for both X and Y were outside the region covered by the GLD; in Section 4.5.4 we were able to obtain EGLD fits to both X and Y; and in Section 5.6.4 we used the percentile approach to obtain two GLD fits to X and three fits to Y.

Here we consider the bivariate (X, Y) and use the best of the GLD fits that were obtained in Section 5.6.4. For X and Y, respectively, these were

$$\text{GLD}_x(1.7684, 0.5682, 4.9390, 0.1055)$$



(a) (b)

Figure 9.5–10. The scatterplot of the rainfall data (a) and the fitted GLD–2 (b).

and

$$\mathrm{GLD}_y(1.3381, 0.7557, 9.6291, 0.2409).$$

Figure 5.6–3 (a) shows a histogram of X with the p.d.f.s of GLD_x and the GBD fit of Section 4.5.4. In Figure 5.6–3 (b) we see the e.d.f. of X with the d.f. of GLD_x. Figure 5.6–4 gives similar illustrations for Y. These are reasonably good fits, an assertion that is supported by p-values of 0.09342 and 0.2791 for GLD_x and GLD_y, respectively.

From the scatterplot of (X, Y) in Figure 9.5–10 (a) we see that

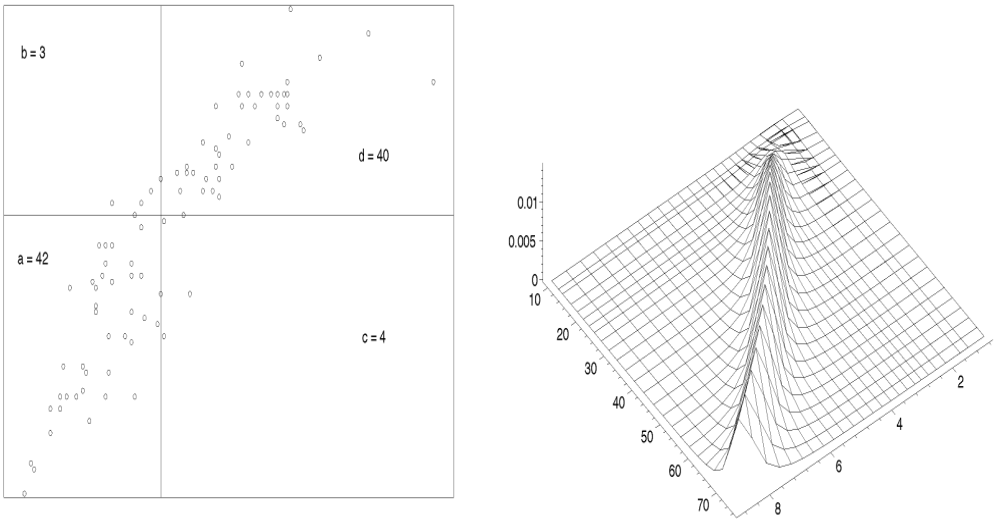
$$a = 19, \qquad b = 5, \qquad c = 5, \qquad d = 18$$

from which we obtain $\Psi^+ = 13.68$. The GLD–2 fit to this bivariate data is shown in Figure 9.5–10 (b).

A chi-square goodness-of-fit test is not available in this case because there are 47 data points and 9 estimated parameters. Hence, we would need to partition the support of the fitted GLD–2 into a minimum of 11 cells to have $11 - 9 - 1 = 1$ be positive, forcing some of the expected frequencies to fall below 5. In such situations the test statistic will not have an approximate chi-square distribution. One could obtain the null distribution by Monte Carlo methods and then complete a test (see Chapter 24 for how to do this). This would also be true if degrees of freedom were zero or negative.

9.5.4 Example: The Tree Stand Data of Section 4.5.5

Two variables, DBH representing the tree diameter at breast height (in inches) and H representing the tree height (in feet), were considered in Section 4.5.5, and GLD (moment-based) and GBD fits for both variables were obtained. To obtain



(a) (b)

Figure 9.5–11. The scatterplot of the tree stand data (a) and the fitted GLD–2 (b).

Table 9.5–12. The cells, expected frequencies, and observed frequencies for the tree stand data and its fitted GLD–2.

Cell	Expected Frequency	Observed Frequency
$(-\infty, \infty) \times (-\infty, 26]$	6.1395	5
$(-\infty, 4] \times (26, 39]$	6.0533	8
$(4, \infty) \times (26, 39]$	6.0464	5
$(\infty, 5] \times (39, 50]$	6.6154	8
$(5, \infty) \times (39, 50]$	6.8840	9
$(-\infty, 6] \times (50, 60]$	6.9502	8
$(6, \infty) \times (50, 60]$	7.3736	3
$(-\infty, 7.25] \times (60, 69]$	8.0411	8
$(7.25, \infty), (60, 69]$	6.0959	8
$(-\infty, 8.5) \times (69, 78]$	7.3022	5
$(8.5, \infty) \times (69, 78]$	7.2650	9
$(-\infty, 10.5] \times (78, \infty)$	6.5677	10
$(10.5, \infty) \times (78, \infty)$	7.6502	3

a GLD–2 for the bivariate (DBH, H), we use the GLD fits of Section 4.5.5 that were

$$\text{GLD}_{DBH}(4.7744, 0.08911, 0.06257, 0.3056)$$

and

$$\text{GLD}_{H}(82.0495, 0.01442, 0.6212, 0.02459)$$

for DBH and H, respectively. We note that p-values of 0.09696 and 0.3776 were computed for the GLD_{DBH} and GLD_{H} fits, respectively, in Section 4.5.5.

The (DBH, H) scatterplot is given in Figure 9.5–11 (a) with $a = 42$, $b = 3$, $c = 4$, and $d = 40$, giving us $\Psi^{+} = 140$. The GLD–2 fit to (DBH,H) is shown in Figure 9.5–11 (b).

With the notation used earlier, a partitioning of the plane into 13 cells and associated expected and observed frequencies are shown in Table 9.5–12. We have degrees of freedom $13-9-1 = 3$ (since there are 13 cells, 9 fitted parameters, and we lose 1 for the chi-square even if we fit no parameters). The chi-square statistic and p-value that result from this test are 11.0670 and 0.01137, respectively.

9.6 GLD–2 Random Variate Generation

The generation of r.v.s for Monte Carlo studies is one of the important applications of fitted distributions. In previous chapters we developed methods for generating GLD random variates (Section 2.4) and EGLD random variates (Section 3.6). Here we describe a method for generating random variates from a GLD–2. Design of experiments using the GLD–2 is also possible (generalizing Section 3.7).

Recall that the GLD–2 is specified by its distribution function $H(x, y)$ given in (9.1.1) and its p.d.f. $h(x, y)$ given in (9.1.2). These involve a univariate GLD $F(x)$, a univariate GLD $G(y)$, and a nonnegative constant Ψ. The $F(x)$, $G(y)$, and Ψ may be chosen to match a particular theoretical model distribution, as noted in Sections 9.3 and 9.4, or to fit a dataset through Algorithm GLD–BVD of Section 9.5. Since "bivariate inverse functions" are not available, the generation of r.v.s from a GLD–2 involves considerations that did not arise in our previous discussions of the univariate cases in Sections 2.4 and 4.6. The three theorems that follow enable us to overcome the non-existence of bivariate inverse functions.

Theorem 9.6.1. *If* $\mathbf{Z} = (X, Y)$ *is a r.v. with any bivariate distribution, then its distribution function* $H(x, y)$ *may be written as*

$$H(x, y) = F(x)G(y \,|\, x) \tag{9.6.2}$$

where $F(x)$ *and* $G(y)$ *are the distribution functions of* X *and* Y*, respectively, and*

$$G(y \,|\, x) = P(Y \le y \,|\, X \le x) \tag{9.6.3}$$

is the conditional distribution function of Y given that $X \le x$.

The proof of Theorem 9.6.1 is a simple application of the definitions of conditional probability and distribution function (in the bivariate case). The following is a direct consequence of Theorem 9.6.1.

Theorem 9.6.4. *Suppose $\mathbf{Z} = (X, Y)$ is a r.v. with a* GLD–2 *distribution, X_1 is generated from $F(x)$ through the method of Section 2.4, and Y_1 is generated from the p.d.f.*

$$g(y \,|\, x_1) = \frac{h(x_1, y)}{f(x_1)}. \tag{9.6.5}$$

Then the pair (X_1, Y_1) has the same distribution as (X, Y).

Since the $g(y \,|\, x_1)$ in (9.6.5) is the conditional p.d.f. of Y given that $X = x_1$, Theorem 9.6.4 provides us with a method of generating a GLD–2 r.v. (X, Y) if we can generate Y from the p.d.f. $g(y \,|\, x_1)$. This is non-trivial, since $g(y \,|\, x_1)$ is not a GLD distribution. However, the following method can be used because we have explicit expressions for both the numerator and denominator of (9.6.5).

Theorem 9.6.6. *Let $Y_1 = G^{-1}(U \,|\, x_1)$ where U is a uniform r.v. on $(0, 1)$. Then Y_1 has p.d.f. $g(y \,|\, x_1)$ described in (9.6.5) and is the solution of the equation*

$$G(y_1 \,|\, x_1) = U. \tag{9.6.7}$$

To obtain a solution to (9.6.7) numerical integration can be used to find y_1 such that

$$\int_{-\infty}^{y_1} \frac{h(x_1 \,|\, y)}{f(x_1)}\, dy = U,$$

i.e., such that

$$\int_{-\infty}^{y_1} h(x_1 \,|\, y)\, dy = U f(x_1). \tag{9.6.8}$$

Example: The program `GLD2RAND` for generating GLD–2 random variates (written in Maple) is included in Section A.7 of Appendix A. The first two parameters of the program are the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the two marginal GLDs, each entered as a list; the third parameter is the value of Ψ; and the fourth parameter is the number of observations that are to be generated.

For a specific illustration, we fit the GLD–2 to the bivariate normal with marginals that are $N(0, 1)$ and $N(100, 225)$ with $\rho = 0.5$. The GLD fits to $N(0, 1)$ and $N(100, 225)$ are, respectively,

$$\mathrm{GLD}_X(0, 0.1975, 0.1349, 0.1349)$$

and

$$\text{GLD}_Y(100, .01316, .1349, .1349).$$

GLD_X is the fit that was obtained in Section 3.4.1. We know from Table 9.3–1 that $\Psi = 3.653$ when $\rho = 0.5$. Thus, we invoke `GLD2RAND` with the following sequence of **R** commands:

```
> LambdaX <- c(0,0.1975,0.1349,0.1349)
> LambdaY <- c(100,.01316,.1349,.1349)
> GLD2RAND(LambdaX, LambdaY, 3.653, 200)
```

to generate 200 random observations from the GLD–2 that has GLD_X and GLD_Y for its marginals and $\Psi = 3.653$. In Table 9.6–1 we give the 200 (X, Y) pairs that are generated and note that when the means and variances of the X and Y components and the correlation coefficient are computed we get

$$\bar{X} = 0.02836, \quad \bar{Y} = 100.7915, \quad s_X^2 = 0.8657, \quad s_Y^2 = 236.6534, \quad r = 0.44642.$$

To obtain a quantitative assessment of the quality of the generator `GLD2RAND`, we perform a chi-square goodness-of-fit test on the data of Table 9.6–1 by partitioning the plane into 16 rectangles. These rectangular cells along with their observed and expected frequencies, shown in Table 9.6–2, produce a chi-square statistic of 9.7538 and a p-value of 0.1354.

9.7 Conclusions and Research Problems Regarding the GLD–2

The GLD is a versatile family for distribution fitting. In this Chapter 9 we have seen a bivariate version, the GLD–2. From the examples given, using a variety of marginals and correlations as well as a variety of datasets, it appears that the GLD–2 fits both some bivariate known distributions and some bivariate datasets well. We are currently reinvestigating an extension of the EGLD to a bivariate version, so that an even broader range of probability distributions arising in practice can be fitted.

Some historical, as well as research frontiers, comments may be useful to the reader seeking to put the material on bivariate distributions into perspective. **First**, since any bivariate d.f. $F(x, y)$ is a function of the marginals $F_1(x)$ and $F_2(y)$ called a **copula**, $F(x, y) = C(F_1(x), F_2(y))$, it seems unlikely that a bivariate GLD (by any generalization to two dimensions) would have a closed form for its d.f. since its univariate form is specified by its percentile function. This would also seem to apply to the multivariate (three or more variates) cases. Note that Johnson and Kotz (1973) proposed a non-closed-form multivariate extension of Tukey's original (ungeneralized) lambda distribution.

Second, the method R.L. Plackett (1965) proposed for generating bivariate distributions from specified marginals, used by Beckwith and Dudewicz (1996) to

Table 9.6–1. Random sample generated through `GLD2RAND`.

X	Y	X	Y	X	Y	X	Y	X	Y
–1.317	121.67	–0.660	110.23	0.527	114.91	–1.078	73.38	–0.592	109.72
1.372	120.35	–0.390	92.58	1.628	118.59	0.145	124.96	0.475	96.86
–1.493	87.13	–0.466	83.17	1.414	132.92	–0.254	101.03	1.725	113.71
–1.550	83.10	–0.483	82.95	1.147	112.55	2.232	107.89	1.303	96.01
1.275	104.53	–0.390	114.97	–0.405	90.74	–0.552	66.66	0.481	106.21
2.188	130.50	1.700	99.20	–1.136	97.54	–0.300	71.33	0.156	103.95
–0.130	108.99	–0.251	88.79	0.253	121.89	1.537	113.89	1.455	114.54
0.143	123.82	0.269	86.96	1.656	108.19	–1.990	76.31	–0.124	114.12
–1.237	59.44	–0.419	94.03	–0.598	92.88	0.144	89.77	–0.656	99.61
–0.035	106.33	0.389	101.10	0.322	122.00	0.704	87.40	0.054	91.55
–0.532	100.84	1.654	112.07	–0.333	102.42	1.407	115.35	1.115	106.33
0.758	129.15	–0.809	91.10	1.306	101.25	–0.640	90.60	–0.359	99.46
0.604	91.59	–1.186	77.78	0.477	109.26	–1.276	87.40	–0.702	125.75
–0.184	92.96	0.046	109.40	–0.494	100.24	–0.250	104.30	–0.526	96.55
–1.452	89.53	–1.965	77.31	0.659	138.29	0.962	122.57	–0.125	103.10
–0.298	107.37	–0.675	109.30	–0.726	101.03	–0.669	91.72	–0.882	93.42
–0.311	109.44	0.890	120.64	0.534	103.07	0.833	91.90	1.074	139.31
–0.060	60.69	–0.381	130.80	–1.348	94.48	1.124	91.01	0.455	111.76
0.549	114.00	0.510	100.17	0.877	119.61	–0.869	73.01	–0.711	106.25
–0.236	95.90	1.072	92.29	0.049	87.45	–0.805	84.26	–0.065	99.16
–1.336	94.24	–0.017	97.81	–0.030	84.06	–0.541	86.40	–0.512	100.54
0.308	93.00	–0.803	88.39	0.126	96.47	0.317	96.93	0.028	78.65
0.544	76.04	–0.760	104.93	1.124	101.78	–0.448	95.93	1.062	112.39
0.758	84.65	0.008	89.67	0.681	117.64	0.867	98.18	0.095	95.97
–0.246	108.54	–0.544	112.44	–0.469	89.67	0.121	86.74	–0.606	112.39
–1.029	99.20	–0.006	100.28	0.972	117.57	0.377	101.40	0.008	88.34
–2.832	95.78	0.013	70.34	0.442	81.08	–0.840	95.19	0.351	123.69
1.954	99.87	–0.683	109.03	–0.081	116.07	0.839	106.13	0.750	94.11
–0.600	120.54	2.124	116.90	–2.430	66.66	–0.367	98.26	0.969	115.48
–0.101	102.38	0.856	123.43	–1.025	80.91	–1.674	92.37	0.105	90.38
0.364	106.62	0.113	102.00	–0.050	116.07	0.474	93.99	–1.089	92.71
–0.467	105.80	0.135	116.27	–0.894	101.25	–0.143	93.91	–0.229	115.29
1.000	122.57	0.612	97.69	–1.343	89.09	1.000	116.62	–0.904	81.58
1.452	124.96	–1.465	80.39	1.542	110.47	0.676	121.78	–1.112	86.46
–1.011	72.21	0.139	107.97	–1.027	107.33	0.771	105.20	–0.997	92.79
–0.249	86.11	1.891	120.35	0.164	88.49	1.143	132.92	–0.797	85.46
–0.121	95.19	–0.502	119.79	0.737	91.33	–1.105	124.96	0.797	106.21
–0.014	86.11	–0.518	92.37	0.938	82.73	–0.122	87.98	1.957	117.12
–0.843	102.00	2.110	98.03	–0.460	95.51	–0.563	85.76	0.437	107.97
–1.342	94.20	–0.484	120.54	0.264	84.59	0.008	129.40	0.710	79.65

Table 9.6–2. The cells, expected frequencies, and observed frequencies for the data of Table 9.6–1.

Cell	Expected Frequency	Observed Frequency
$(-\infty, -1] \times (-\infty, 89.5]$	16.7091	14
$(-1, -0.25] \times (-\infty, 89.5]$	15.6534	12
$(-0.25, 0.25] \times (-\infty, 89.5]$	8.3016	11
$(0.25, \infty) \times (-\infty, 89.5]$	7.7286	8
$(-\infty, -0.7] \times (89.5, 100]$	13.9775	13
$(-0.7, 0] \times (89.5, 100]$	15.7682	20
$(0, 0.7] \times (89.5, 100]$	13.6189	12
$(0.7, \infty) \times (89.5, 100]$	8.2427	10
$(-\infty, -0.7] \times (100, 110.5]$	8.2427	6
$-0.7, 0] \times (100, 110.5]$	13.6189	19
$(0, 0.7] \times (100, 110.5]$	15.7682	12
$(0.7, \infty) \times (100, 110.5]$	13.9775	10
$(-\infty, -0.25] \times (110.5, \infty)$	7.7286	10
$(-0.25, 0.25] \times (110.5, \infty)$	8.3016	8
$(0.25, 1] \times (110.5, \infty)$	15.6534	18
$(1, \infty] \times (110.5, \infty)$	16.7091	17

yield the GLD–2, rarely generates a closed-form expression for d.f.s, but this is not a problem as computer calculation of the properties of the resulting distribution is readily available.

Third, Plackett (1981, p. 151) has noted that the method of Plackett (1965) for generating bivariate distributions was in fact introduced in 1913 in Pearson (1913) and Pearson and Heron (1913).

Fourth, due to the lack of a transformation from (X, Y) into a bivariate uniform distribution on the unit square (as noted in Section 9.5, in the paragraph before that containing equation (9.5.1)), a Kolmogorov-Smirnov type test based on the statistic

$$D_n = \sup_{(x,y)} |F_n(x, y) - H(x, y)| \tag{9.7.1}$$

does not reduce to testing bivariate uniformity; hence, critical values will be functions of the fitted H (when we act as if it is the true d.f. of (X, Y)) as well as of the sample size n. This means we would have to calculate the critical values anew in each application. (Above, $F_n(x, y)$ is the bivariate empiric d.f. .) In addition, the *sup* will require care to compute (but will be possible since it should occur at or just before a jump). While $F_n(x, y)$ is called the **empirical process** and

there is a great deal of research on it (e.g., see Csörgö and Szyszkowicz (1994) and its references; on p. 97 these authors seem to indicate computational methods will be needed for the critical values, and in any case they do not seem to have reduced their theory to practice). We are not aware of this concluding in a (non-asymptotic) goodness-of-fit test, even if, e.g., H is bivariate normal (in which case one could calculate the necessary tables, which might also depend on the correlation coefficient). There appear to be many interesting and important research problems in this area (and some of them might be approached via computer simulation in terms of the tables needed for implementation, or via algorithms to calculate the needed critical values anew each time one tests using D_n of (9.6.1)).

Fifth, from the graphs in Section 9.5, it will be clear that the GLD–2 can have non-elliptic contours, in fact even non-convex contours. This is of importance in applications, as it allows modeling of the (widespread) situations where these sorts of contours arise. Work on multivariate models is still in rapid development; some references one might consult for some of these aspects (elliptically contoured distributions, multivariate models with given marginals, etc.) include Dall'Aglio, Kotz, and Salinetti (1991), Anderson, Fang, and Olkin (1994), and Hayakawa, Aoshima, Shimizu, and Taneja (1995, 1996, 1997, 1998).

Problems for Chapter 9

9.1 Let X and Y be random variables with marginal p.d.f.s that are uniform on $(0, 1)$. Find and provide a contour plot of $H(x, y)$ with marginals the same as those of X and Y, but for which (9.2.9) holds for $\Psi = 4$.

9.2 Do Problem 9.1 with $\Psi = 0.25$.

9.3 Do Problem 9.1 for each of the following values of Ψ: 0.01, 0.1, 1, 10, 100, ∞.

9.4 Suppose (X, Y) has a bivariate d.f. $F(x, y)$ which is one discussed in Hutchinson and Lai (1990) but not considered in Section 9.3. Develop a GLD–2 fit $H(x, y)$, plot contours, and consider the error $f - h$. How good a fit to (X, Y) can be obtained using the GLD–2?

9.5 In Section 9.5.3, a GLD–2 fit was obtained to a set of data on rainfall (X, Y) at two cities. Test the hypothesis that the data comes from the fitted distribution. [Hint: Split the plane into 10 rectangles with frequencies of at least 4.0 per rectangle. Compute a chi-square discrepancy measure. To assess its significance, we cannot use a chi-square distribution, since the degrees of freedom would be $10-9-1 = 0$. However, one can find the approximate null distribution by Monte Carlo simulation. Do so and find the p-value of the computed test statistic. At level of significance 0.01 do you accept or reject the null hypothesis that the data $(X_1, Y_1), (X_2, Y_2), \ldots, (X_{47}, Y_{47})$ comes from the fitted GLD–2?]

9.6 Generate 25 points from the GLD–2 $h(x, y)$ fitted to the rainfall data of Section 9.5.3. Fit a GLD–2 $h^*(x, y)$ to this set of simulated data. Compare this GLD–2 to that with p.d.f. plotted in Figure 9.5–10 (b). In particular, find

$$\sup_{(x,y)} |h(x, y) - h^*(x, y)|,$$

and plot $h^*(x, y)$ and $e(x, y) = |h(x, y) - h^*(x, y)|$.

References for Chapter 9

Anderson, T. W., Fang, K. T., and Olkin, I. (Editors) (1994). *Multivariate Analysis and Its Applications*, Institute of Mathematical Statistics Lecture Notes–Monograph Series, Vol. 24, Institute of Mathematical Statistics, Hayward, California.

Beckwith, N. B. and Dudewicz, E. J. (1996). "A bivariate Generalized Lambda Distribution (GLD–2) using Plackett's method of construction: Distribution, examples and applications," *American Journal of Mathematical and Management Sciences*, 16, 333–393.

Csörgö, M. and Szyszkowicz, B. (1994). "Weighted multivariate empirical processes and continguous change-point analysis," *Change-point Problem*, Institute of Mathematical Statistics Lecture Notes–Monograph Series, Vol. 23, Carlstein, E., Müller, H.-G., and Siegmund, D., Editors, pp. 93–98.

Dall'Aglio, G., Kotz, S., and Salinetti, G. (Editors) (1991). *Advances in Probability Distributions with Given Marginals, Beyond the Copulas*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Dudewicz, E. J., Levy, G. C., Lienhart, J. L. and Wehrli, F. (1989). "Statistical analysis of magnetic resonance imaging data in the normal brain: Data, screening normality, discrimination, variability), and implications for expert statistical programming for ESS^{TM} (the Expert Statistical System)," *American Journal of Mathematical and Management Sciences*, 9, 299–359.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, John Wiley & Sons, New York.

Farlie, D. J. G. (1960). "The performance of some correlation coefficients for a general bivariate distribution," *Biometrika*, 47, 307–323.

Hayakawa, T., Aoshima, M., Shimizu, K., and Taneja, V. S. (Editors) (1995, 1996, 1997, 1998). *MSI–2000: Multivariate Statistical Analysis in Honor of Professor Minoru Siotani, Vol. I, II, III, IV*, American Sciences Press, Inc., Columbus, Ohio.

Hutchinson, T. P. and Lai, C. D. (1990). *Continuous Bivariate Distributions, Emphasizing Applications,* Rumsby Scientific Publishing, Adelaide, South Australia.

Johnson, M. E. (1987). *Multivariate Statistical Simulation.* Wiley, New York.

Johnson, M. E., Wang, C., and Ramberg, J. S. (1984). "Generation of continuous multivariate distributions for statistical applications," *American Journal of Mathematical and Management Sciences*, 4, 225–248.

Johnson, N. L. and Kotz, S. (1973). "Extended and multivariate Tukey lambda distributions," *Biometrika*, 60, 655–661.

Johnson, R. A. and Wichern, D. W. (1992). *Applied Multivariate Statistical Analysis*, Prentice Hall, Englewood Cliffs New Jersey.

Karian, Z. A. and Dudewicz, E. J. (1999). "Fitting the Generalized Lambda Distribution to data: A method based on percentiles," *Communications in Statistics: Simulation and Computation*, 28, 793–819.

McLachlan, G. J. (1992). "Cluster analysis and related techniques in medical research," *Statistical Methods in Medical Research*, 1, 27–48.

Mardia, K. V. (1967). "Some contributions to contingency-type distributions," *Biometrika*, 54, 235–249.

Mardia, K. V. (1970). *Families of Bivariate Distributions*, Hafner, Darien, Connecticut.

Mykytka, E. F. (1979). "Fitting distributions to data using an alternative to moments," *IEEE Proceedings of the 1979 Winter Simulation Conference*, 361–374.

Plackett, R. L. (1965). "A class of bivariate distributions," *Biometrica*, 60, 516–522.

Plackett, R. L. (1981). *The Analysis of Categorical Data (Second Edition)*, Macmillan Publishing Co., Inc., New York.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *tecjnometrics*, 21, 201–214.

Chapter 10

Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals[1]

Robert A. R. King
School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, NSW 2308, Australia
Robert.King@newcastle.edu.au

H. L. MacGillivray
School of Mathematical Sciences
Queensland University of Technology
GPO Box 2434, Brisbane Qld 4001, Australia
h.macgillivray@qut.edu.au

The generalized lambda distribution (GLD) is a family of distributions that can take on a very wide range of shapes within one distributional form. We present a fitting method for the GLD using location and scale-free shape functionals to fit the shape functions before fitting the location and scale parameters. Such functions provide an alternative to the moments as a "shape-first" approach, with the advantage of being defined for parameter values where moments are infinite. We investigate the performance of the method with a simulation study and illustrate its use to choose parameter values of the GLD to approximate other distributions.

[1]This chapter appeared as a paper, under the same title, in *American Journal of Mathematical and Management Sciences*, (Vol. 27, (2007), pp. 441–460). Copyright © 2007 by American Sciences Press, Inc., Syracuse, New York 13224–2104. Reprinted by permission.

10.1 Introduction

10.1.1 The Generalized Lambda Distribution

The generalized lambda distribution (GLD) is a family of distributions that can take on a very wide range of shapes within one distributional form. In this chapter we consider two parameterizations of the distribution, both defined by their quantile function. Firstly, Ramberg and Schmeiser's (1974) (referred to here as the RS parameterization), defined by:

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \tag{10.1.1}$$

where λ_1 is a location parameter, λ_2 is an inverse scale parameter, either strictly positive, or strictly negative, depending on values of the shape parameters, λ_3 and λ_4. The regions of the λ_3, λ_4 plane that produce a valid distribution are not simple, and were first given in full in Karian, Dudewicz, and McDonald (1996) (or see Karian and Dudewicz (2000, Section 1.3)). Secondly, we consider the Freimer, Mudholkar, Kollia, and Lin's (1988) parameterization (referred to as the FMKL parameterization), defined by

$$F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2}. \tag{10.1.2}$$

(This equation subscripts the lambda parameters differently than Freimer, Mudholkar, Kollia, and Lin (1988). We have chosen this to maintain consistency with the Ramberg parameterization, i.e., λ_1 location, λ_2 scale, and λ_3, λ_4 shape.) This parameterization produces a valid distribution for all values of λ_3, λ_4, at the cost of a slightly more complex formula and the need to apply the limit results that apply as λ_3 and λ_4 go to 0 or 1.

We present a fitting method based on location and scale-free shape functionals. Our method is based on a tailweight functional, and two alternative asymmetry functionals.

The method is illustrated with a real dataset and with approximation of distributions. Its sampling behavior is investigated and compared to moment estimates (as described in Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979), for the RS parameterization only), the method of Öztürk and Dale (1985) (for the RS parameterization only) and the starship method (King and MacGillivray 1999). The starship method, named as such by Owen (1988) because it "flies over the parameter space" is based on viewing the distribution as a transformation of the uniform. It estimates the parameters that give the best fit (as assessed by some goodness-of-fit method, in the examples in this chapter either the Kolmogorov-Smirnov (KS) or Anderson-Darling (AD)) between the data transformed by the distribution function and the uniform distribution.

10.1.2 Shape Functionals

We use the Ratio of Spread Functions (ROSF) as a tailweight measure, as introduced by MacGillivray and Balanda (1988). It is defined as

$$R_F(u,v) = \frac{S_F(u)}{S_F(v)} \quad \text{for} \quad \frac{1}{2} < v < u < 1, \tag{10.1.3}$$

where the spread function is

$$SF_(u) = F^{-1}(u) - F^{-1}(1-u). \tag{10.1.4}$$

The ratio of spread functions describes the position of probability mass in the tails of a distribution for any pair of u, v values.

The first of the two asymmetry functionals is the functional, introduced by David and Johnson (1956) (re-parameterized for $1/2 < u < 1$):

$$\gamma_F(u) = \frac{F^{-1}(1-u) + F^{-1}(u) - 2m_F}{F^{-1}(u) - F^{-1}(1-u)} \quad \text{for} \quad \frac{1}{2} < u < 1. \tag{10.1.5}$$

The functional is related to the asymmetry ordering, $\overset{<m}{\gamma^2}$ (MacGillivray (1986), Theorem 2.3). The numerator is twice the mean of the probability weights at u and $1-u$ less the median, so the larger the discrepancy in distance from the median of these two quantiles, the larger the functional becomes. The functional is scaled by the spread function at u, and is thus bounded (Theorem 10.3.6).

Theorem 10.3.6. *$\gamma_F(u)$ is bounded by -1 and 1.*

Proof:
Consider $A = F^{-1}(1-u) - m_F$ and $B = m_F - F^{-1}(u)$. We have

$$\gamma_F(u) = \frac{A-B}{S_F(u)},$$

where $S_F(u)$ is as defined in (10.1.4). Since $1-u > 1/2 > u$,

$$F^{-1}(1-u) \geq m_F \geq F^{-1}(u).$$

Thus, $A \geq 0$, $B \geq 0$, $S_F(u) \geq A$ and $S_F(u) \geq B$. Therefore, $A - B \leq A \leq S_F(u)$, so $\gamma_F(u) \leq 1$.

Similarly, $A - B \geq -B \geq -S_F(u)$, so $\gamma_F(u) \geq -1$. ∎

The second asymmetry functional we call the η functional is defined by

$$\eta_F(u,v) = \frac{F^{-1}(u) + F^{-1}(1-u) - F^{-1}(v) - F^{-1}(1-v)}{F^{-1}(v) - F^{-1}(1-v)} \text{ for } \frac{1}{2} < v < u < 1. \tag{10.1.7}$$

It appears in MacGillivray (1986), as Theorem 2.3(c), but is not named. The theorem referred to shows the link between this functional and another of the skewness orderings of MacGillivray (1986). It proves that

$$F \underset{star}{\overset{m}{\leq}}_{2} G \rightarrow \eta_F(u,v) \leq \eta_G(u,v), \tag{10.1.8}$$

where $\underset{star}{\overset{m}{\leq}}_{2}$ is an ordering, stronger than $\overset{m}{\underset{\gamma}{\leq}}_{2}$ above, defined in MacGillivray (1986). Because of the complexity of the shape behavior of the GLD, we considered these two functionals to investigate if a functional related to a weaker shape ordering (the γ functional) would perform better. King (1999) reports some regions of the λ_3, λ_4 plane where changes in the shape parameters are consonant with these shape orderings.

10.2 Description of Method

10.2.1 Overview

Our method calculates estimates $\hat{\lambda}_3$, $\hat{\lambda}_4$ of the shape parameters that minimize a measure of the difference between the sample values of the shape functionals detailed above and the theoretical value of the functionals in terms of the parameters of the GLD. In the case of approximation of another distribution, the "sample values" are replaced by the values of the functionals for the target distribution.

Using the $\hat{\lambda}_3$, $\hat{\lambda}_4$ so calculated and the quartiles and median, we calculate estimates $\hat{\lambda}_2$ and $\hat{\lambda}_1$ by equating the theoretical and sample values.

10.2.2 Theoretical Values of the Shape Functionals

The expressions for the theoretical values are obtained substituting the expressions of equations (10.1.1) and (10.1.2) into equations (10.1.3), (10.1.5), and (10.1.7). For both parameterizations, the location and scale-free nature of the measures mean that they are functions of only u, v, λ_3, λ_4.

10.2.2.1 Ramberg and Schmeiser Parameterization

$$\begin{aligned}
R_F(u,v) &= \frac{F^{-1}(u) - F^{-1}(1-u)}{F^{-1}(v) - F^{-1}(1-v)} && \text{(from (10.1.3))} \\
&= \frac{u^{\lambda_3} - (1-u)^{\lambda_3} + u^{\lambda_4} - (1-u)^{\lambda_4}}{v^{\lambda_3} - (1-v)^{\lambda_3} + v^{\lambda_4} - (1-v)^{\lambda_4}} && (10.2.1) \\
\gamma_F(u,v) &= \frac{F^{-1}(u) + F^{-1}(1-u) - 2m_F}{F^{-1}(u) - F^{-1}(1-u)} && \text{(from (10.1.5))} \\
&= \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + u^{\lambda_3} - (1-u)^{\lambda_4} - 2(0.5^{\lambda_3} - 0.5^{\lambda_4})}{(1-u)^{\lambda_3} - u^{\lambda_4} - u^{\lambda_3} + (1-u)^{\lambda_4}} && (10.2.2)
\end{aligned}$$

$$\eta_F(u,v) = \frac{F^{-1}(u)+F^{-1}(1-u)-F^{-1}(v)-F^{-1}(1-v)}{F^{-1}(v)-F^{-1}(1-v)} \qquad \text{(from (10.1.3))}$$

$$= \frac{(u^{\lambda_3}+(1-u)^{\lambda_3}-v^{\lambda_3}-(1-v)^{\lambda_3})-(u^{\lambda_4}+(1-u)^{\lambda_4}-v^{\lambda_4}-(1-v)^{\lambda_4})}{(v^{\lambda_3}-(1-v)^{\lambda_3})-((1-v)^{\lambda_4})-v^{\lambda_3})} \tag{10.2.3}$$

Of course, these expressions are only valid for the values of λ_3, λ_4 for which the Ramberg parameterization forms a proper statistical distribution (Karian and Dudewicz 2000, Section 1.3).

10.2.2.2 FMKL Parameterization

The FMKL results, including the limits as λ_3, λ_4 or both approach zero, are:

$$R_F(u,v) = \frac{F^{-1}(u)-F^{-1}(1-u)}{F^{-1}(v)-F^{-1}(1-v)}$$

$$= \begin{cases} \frac{(u^{\lambda_3}-(1-u)^{\lambda_3})/\lambda_3+(u^{\lambda_4}-(1-u)^{\lambda_4})/\lambda_4}{(v^{\lambda_3}-(1-v)^{\lambda_3})/\lambda_3+(v^{\lambda_4}-(1-v)^{\lambda_4})/\lambda_4} & 0<|\lambda_3|,\ |\lambda_4|<\infty \\ \frac{\ln(u/(1-u))+(u^{\lambda_4}-(1-u)^{\lambda_4})/\lambda_4}{\ln(v/(1-v))+(v^{\lambda_4}-(1-v)^{\lambda_4})/\lambda_4} & \lambda_3=0,\ 0<|\lambda_4|<\infty \\ \frac{(u^{\lambda_3}-(1-u)^{\lambda_3})/\lambda_3+\ln(u/(1-u)}{(v^{\lambda_3}-(1-v)^{\lambda_3})/\lambda_3+\ln(v/(1-v)} & 0<|\lambda_3|<\infty,\ \lambda_4=0 \\ \frac{\ln(u/(1-u))}{\ln(v/(1-v))} & \lambda_3=\lambda_4=0. \end{cases} \tag{10.2.4}$$

Case descriptions are omitted from equation 13 but repeat those of equation 12 in the same order.

$$\gamma_F(u,v) = \frac{F^{-1}(u)+F^{-1}(1-u)-2m_F}{F^{-1}(u)-F^{-1}(1-u)}$$

$$= \begin{cases} \frac{((1-u)^{\lambda_3}+u^{\lambda_3}-2(1/2)^{\lambda_3})/\lambda_3-(u^{\lambda_4}+(1-u)^{\lambda_4}-2(1/2)^{\lambda_4})/\lambda_4}{((1-u)^{\lambda_3}-u^{\lambda_3})/\lambda_3+((1-u)^{\lambda_4}-u^{\lambda_4})/\lambda_4} \\ \frac{\ln(4u(1-u))-((1-u)^{\lambda_4}-1)/\lambda_4-(u^{\lambda_4}-1)/\lambda_4-2((1/2)^{\lambda_4}-1)/\lambda_4}{\ln(u/(1-u))-((1-u)^{\lambda_4}-1)/\lambda_4+(u^{\lambda_4}-1)/\lambda_4} \\ \frac{-\ln(4u(1-u))+((1-u)^{\lambda_3}-1)/\lambda_3+(u^{\lambda_3}-1)/\lambda_3-2((1/2)^{\lambda_3}-1)/\lambda_3}{\ln(u/(1-u))-((1-u)^{\lambda_3}-1)/\lambda_3+(u^{\lambda_3}-1)/\lambda_3} \\ 0 \quad \text{(note that this case is symmetric).} \end{cases} \tag{10.2.5}$$

$$\eta_F(u,v) = \frac{F^{-1}(u)+F^{-1}(1-u)-F^{-1}(v)-F^{-1}(1-v)}{F^{-1}(v)-F^{-1}(1-v)}$$

$$= \begin{cases} \dfrac{(u^{\lambda_3}+(1-u)^{\lambda_3}-v^{\lambda_3}+(1-v)^{\lambda_3})/\lambda_3-(u^{\lambda_4}+(1-u)^{\lambda_4}-v^{\lambda_4}-(1-v)^{\lambda_4})/\lambda_4}{(v^{\lambda_3}-(1-v)^{\lambda_3})/\lambda_3-((1-v)^{\lambda_4}-v^{\lambda_4})/\lambda_4} & \\ & 0<|\lambda_3|,\ |\lambda_4|<\infty \\ \dfrac{\ln(\frac{u(1-u)}{v(1-v)})-(u^{\lambda_4}+(1-u)^{\lambda_4}-v^{\lambda_4}-(1-v)^{\lambda_4})/\lambda_4}{\ln(v/(1-v))-((1-v)^{\lambda_4}-v^{\lambda_4})/\lambda_4} & \lambda_3=0,\ 0<|\lambda_4|<\infty \\ \dfrac{(u^{\lambda_3}+(1-u)^{\lambda_3}-v^{\lambda_3}-(1-v)^{\lambda_3})/\lambda_3+\ln(\frac{v(1-v)}{u(1-u)})}{(v^{\lambda_3}-(1-v)^{\lambda_3})/\lambda_3+\ln(v/(1-v))} & 0<|\lambda_3|<\infty,\ \lambda_4=0 \\ 0 \quad \text{(note that this case is symmetric).} & \lambda_3=\lambda_4=0 \end{cases} \tag{10.2.6}$$

The limits for infinite values of λ_3 or λ_4 are not reported here, as they are not of concern for a fitting method.

10.2.3 Optimization

Since these expressions are functionals, rather than single-valued shape measures, there are a number of options for optimization. We chose to carry out the optimization separately for select pairs of quantiles, $F^{-1)}(u)$, $F^{-1}(v)$, with $1/2 < v < u < 1$ (Section 10.2.4 addresses the choice of u, v pairs), producing estimates $\hat{\lambda}_3(u,v)$, $\hat{\lambda}_4(u,v)$ that minimize (using a Nelder-Mead optimization[1]):

$$\left|\eta(u,v,\hat{\lambda}_3,\hat{\lambda}_4)-\eta(u,v,\text{data})\right|+\left|R(u,v,\hat{\lambda}_3,\hat{\lambda}_4)-R(u,v,\text{data})\right|, \tag{10.2.7}$$

or, when using γ as the skewness functional,

$$\begin{aligned} \frac{1}{2}\left|\gamma(u,\hat{\lambda}_3,\hat{\lambda}_4)-\gamma(u,\text{data})\right| \quad &+ \quad \frac{1}{2}\left|\gamma(v,\hat{\lambda}_3,\hat{\lambda}_4)-\gamma(v,\text{data})\right| \\ &+ \quad \left|R(u,v,\hat{\lambda}_3,\hat{\lambda}_4)-R(u,v,\text{data})\right|. \end{aligned} \tag{10.2.8}$$

The medians of $\hat{\lambda}_3(u,v)$ and $\hat{\lambda}_4(u,v)$ over all the u, v pairs then form $\hat{\lambda}_3$ and $\hat{\lambda}_4$. We use the median for its simplicity and robustness. The use of the L_1 norm improves the robustness of the method. An L_2 norm was tried for these minimizations and tended to produce worse results. A possible alternative approach would be to treat the functionals as such and apply a metric suitable for comparing continuous functions.

10.2.4 *u*, *v* Selection

To reduce the impact of the unbounded ROSF and improve speed, we took a subset of the available order statistics to obtain the $\hat{\lambda}_3(u,v)$, $\hat{\lambda}_4(u,v)$. This subset of depths is formed by including the following order statistics (with the quantiles marked with an asterisk calculated via linear interpolation between the ideal depths (see below) of the surrounding order statistics) and using all possible pairs from this restricted set.

[1] With either `fmins` in MATLAB or `optim` in **R**.

- All the data between the minimum and the first quartile* (inclusive).
- Every second order statistic between the first quartile* and the third octile.
- The median*.
- Every second order statistic between the fifth octile and the third quartile* (here "every second" is interpreted to ensure that if $X_{(i)}$ is included, $X_{(n+1-i)}$ is also included).
- All the data between the third quartile* and the maximum (inclusive).

The order statistics selected were linked to a depth using their ideal depths of Hoaglin, Mosteller, and Tukey (1982): $u = (i - 1/3)/(n + 1/3)$. This subset was then further reduced if there were ties. If ($x_{(i)} = x_{(i-1)}$ or $x_{(n+1-i)} = x_{(n+2-i)}$) and ($x_{(i-1)}$ and $x_{(n+2-i)}$ included) then $x_{(i)}$ and $x_{(n+1-i)}$ were excluded.

10.2.5 Location and Scale Parameters

$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are calculated using the estimated shape parameters and the median and inter-quartile range of the data.

10.2.5.1 Ramberg Parameterization

$$\hat{\lambda}_2 = \frac{\left(\frac{3}{4}\right)^{\hat{\lambda}_3} + \left(\frac{3}{4}\right)^{\hat{\lambda}_4} - \left(\frac{1}{4}\right)^{\hat{\lambda}_3} - \left(\frac{1}{4}\right)^{\hat{\lambda}_4}}{Q_3 - Q_1}, \tag{10.2.9}$$

$$\hat{\lambda}_1 = m - \frac{\left(\frac{1}{2}\right)^{\hat{\lambda}_3} - \left(\frac{1}{2}\right)^{\hat{\lambda}_4}}{\hat{\lambda}_2}. \tag{10.2.10}$$

10.2.5.2 Freimer Parameterization

$$\begin{aligned}\hat{\lambda}_2 &= \frac{\left(\left(\frac{3}{4}\right)^{\hat{\lambda}_3} - 1\right) - \left(\left(\frac{1}{4}\right)^{\hat{\lambda}_3} - 1\right)}{\hat{\lambda}_3(Q_3 - Q_1)} \\ &+ \frac{\left(\left(\frac{3}{4}\right)^{\hat{\lambda}_4} - 1\right) - \left(\left(\frac{1}{4}\right)^{\hat{\lambda}_4} - 1\right)}{\hat{\lambda}_4(Q_3 - Q_1)},\end{aligned} \tag{10.2.11}$$

$$\hat{\lambda}_1 = m - \frac{\left(\left(\frac{1}{2}\right)^{\hat{\lambda}_3} - 1\right)/\hat{\lambda}_3 - \left(\left(\frac{1}{2}\right)^{\hat{\lambda}_4} - 1\right)/\hat{\lambda}_4}{\hat{\lambda}_2}, \tag{10.2.12}$$

(or the appropriate limit equation in the cases where λ_3 or λ_4 were zero).

Table 10.3–1. Parameter values investigated.

sample sizes	param.	set name	λ_1	λ_2	λ_3	λ_4
$n = 25$ to 200	FMKL	`FMKL1`	0	1	0.1	0.1
	RS	`RS1`	0	0.1975	0.1349	0.1349
$n = 100$	FMKL	`FMKL2`	0 2	0.2	0.7	
	FMKL	`FMKL3`	0	2	0.4	0.4
	FMKL	`FMKL4`	0	1	0.1	0.2

10.3 Simulations

We investigate the sampling behavior of the method through a series of simulations. The parameter values used are shown in Table 10.3–1. The simulations investigate the effect of sample size for both parameterizations, and illustrate a range of shapes of the GLD.

We compare two shape functional methods (using the η or γ functionals) with a number of alternatives. For the FMKL parameterization, we compare to the starship method with two alternative "internal goodness-of-fit" measures. For RS parameterization, we compare to these starship methods and also the method of moments and Öztürk and Dale's method, as described in Section 10.1.1.

10.3.1 Effect of Sample Size (RS Parameterization)

This section investigates the effect of sample size for 6 estimation methods. These are two shape functional methods (based on the η or γ functionals), two starship methods (based on internal goodness-of-fit measures Kolmogorov-Smirnov (KS) or Anderson-Darling (AD)), the method of moments and Öztürk and Dale's methods by considering a range of sample sizes ($n = 25$ to 200) for one set of parameters. These are the parameters that approximate the standard normal. Figure 10.3–2 shows the bias (for each parameter) of the four methods against the log of the sample size.

The sampling distributions of $\hat{\lambda}_1$ are consistent in shape across the differing sample sizes. All appear to be close to symmetric and to be centered close to the true value of zero, with the exception of the method for small n. The distribution for Öztürk and Dale's method for $n = 25$ contains one apparent outlier at -6.38 (the estimates of all four parameters for this dataset are $(-6.385, 0.1268, 0.1685, 488.4)$).

The shape of the sampling distributions of $\hat{\lambda}_2$ seem to change more than those for $\hat{\lambda}_1$ with changing n. For lower values of n (especially $n = 25$, 50 and 75), the distributions appear to be a little asymmetric, with symmetry increasing for larger values of n. The estimates from the shape functional methods appear

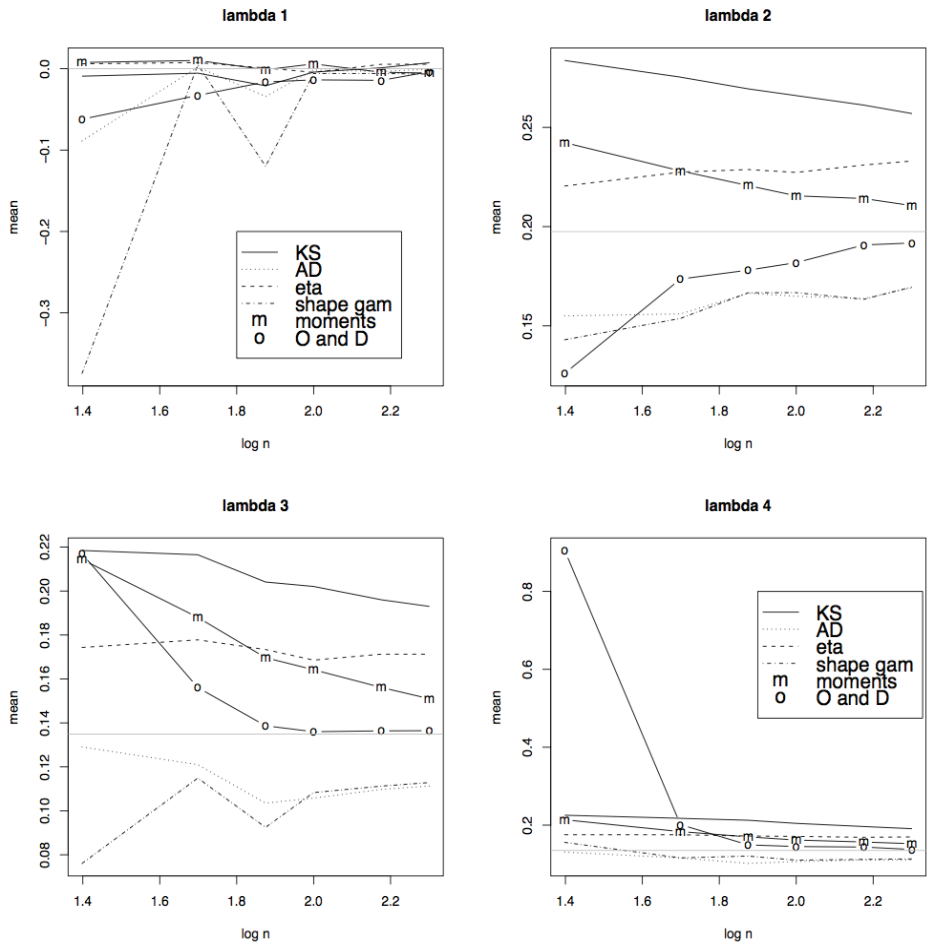


Figure 10.3–2. Bias: mean of sampling distribution for 6 methods, RS parameterization, against log(sample size) (true value marked by gray horizontal line). Based on 1000 repetitions.

to have sampling distributions that are skewed to the right, but with much lower variance (with the η method having the lowest variance). The starship and moment methods tend to over-estimate λ_2, while the shape functional methods tend to under-estimate.

For all the estimation methods, the sampling distributions of $\hat{\lambda}_3$ and $\hat{\lambda}_4$ are right skewed. This makes sense because their values ($\lambda_3 = \lambda_4 = 0.1349$) are close to the boundary of the valid region of λ_3 values for the RS parameterization.

The shape functional methods have lower variance than the others. The starship variance was lower than moments for smaller samples, and approximately equal for larger samples. The shape functional methods had a small underestimating bias, and the starship and moments had a small over-estimating bias. Öztürk and Dale's method had a low bias for λ_4, with larger samples ($n > 75$),

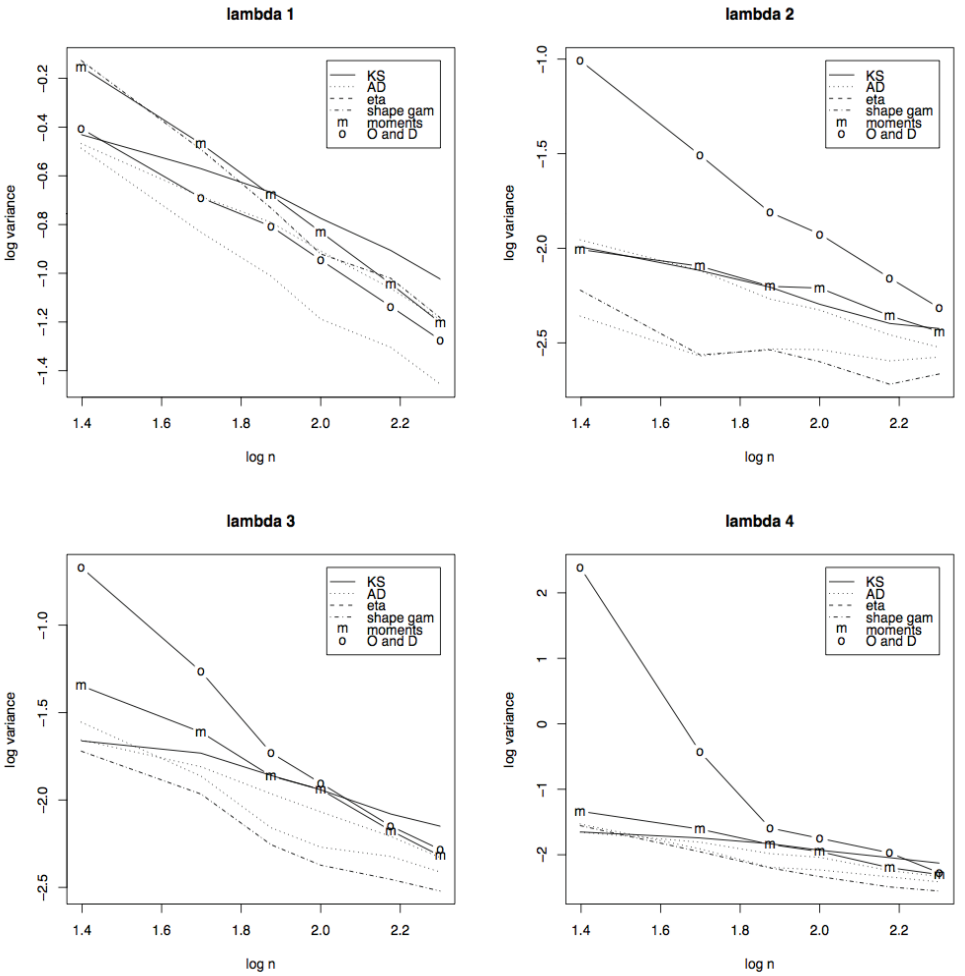


Figure 10.3–3. Sampling variance: log(variance) against log(sample size), RS parameterization.

but had the highest variance, and high bias for λ_4.

Plotting log S^2 against log n allows the estimation of the power of the relationship between sample size and variation, as well as a general view of the performance of the various methods. For the RS parameterization, there is not a consistent linear pattern in these plots, compared to the FMKL plots (Figure 10.3–3). We can say that overall, the shape functional methods tend to have lower sampling variance than the alternatives.

10.3.2 Effect of Sample Size (FMKL Parameterization)

The parameter values for the FMKL sample size investigation give a symmetric distribution with tails very slightly lighter than the normal.

Figure 10.3–4 shows the bias (for each parameter) of the four methods against the log of the sample size.

For $\hat{\lambda}_1$, the sampling distributions are symmetric, centered near the true value of zero. The shape functional estimators have higher variance than the starship. For $\hat{\lambda}_2$, the sampling distributions tend to be skewed right (note that since λ_2 is an inverse scale parameter, this indicates a skew towards estimated distributions that have lower spread), with the mean close to the true value of 1. The shape functional estimators have lower variance compared to the variance from the starship estimation (Figure 10.3–5).

For both $\hat{\lambda}_3$ and $\hat{\lambda}_4$ (for this case, the true values of both are 0.1), the sampling distributions are skewed for lower sample sizes, gaining symmetry as the sample size increases. The estimates tend to be biased, with both starship methods under-estimating λ_3 (and therefore estimating distributions with heavier tails than the truth) and both shape functional methods overestimating λ_4 (estimating distributions with lighter tails than the truth). The biases for both methods are reduced as sample size increases, with the shape functional method dropping to a lower bias.

We used simple linear regression to calculate the slopes of the lines in Figure 10.3–5. These slopes indicate the power of the relationship between sample size and sampling variance (the results are in Table 10.3–6). There is, of course, no guarantee that this relationship is linear, or that errors around it are normally distributed. Nevertheless, such estimates of slope are useful indications of the relationship and the lines appear to be close enough to straight to make this a reasonable exercise. They indicate that sampling variance is at least inversely proportional to sample size for all four methods.

10.3.3 Different Shapes

Here we present the results of simulations with a sample size of 100, for three different shaped distributions. The parameter values are given in Table 10.3–1 and their densities are plotted in Figure 10.3–7.

10.3.1 Set FMKL2

For this combination of parameters, the starship method outperformed the shape functional method (see Table 10.3–8). For the location and scale parameters, the performance of the methods was similar. However, with the shape parameters, the γ functional method had standard deviations markedly higher than the starship, while the η functional method had very large biases, particularly for λ_4, with the mean of the sampling distribution giving a distribution with a right tail that does not end as sharply as the original.

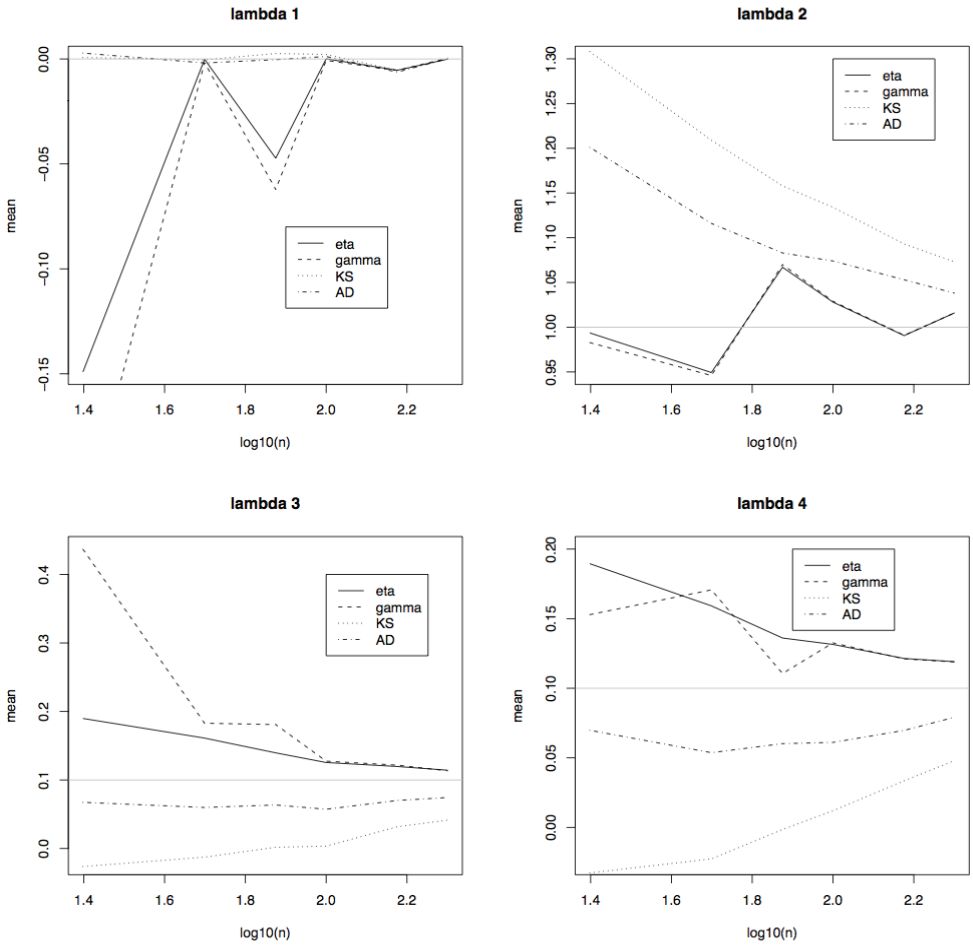


Figure 10.3–4. Bias: mean of sampling distribution for 4 methods, FMKL parameterization, against log(sample size) (true value marked by horizontal line).

10.3.2 Set FMKL3

For this set of parameters, which is a symmetric distribution with short tails, the starship and shape functional methods perform similarly (Table 10.3–9). The starship (A-D) method tends to have lower bias and variance, while the η has lower variance, but higher bias than the γ method.

10.3.3 Set FMKL4

For this set of parameters, which produce a distribution with some asymmetry and longer tails than set FMKL3, the shape functional methods perform better than the starship (Table 10.3–10). Variances are roughly comparable, but the

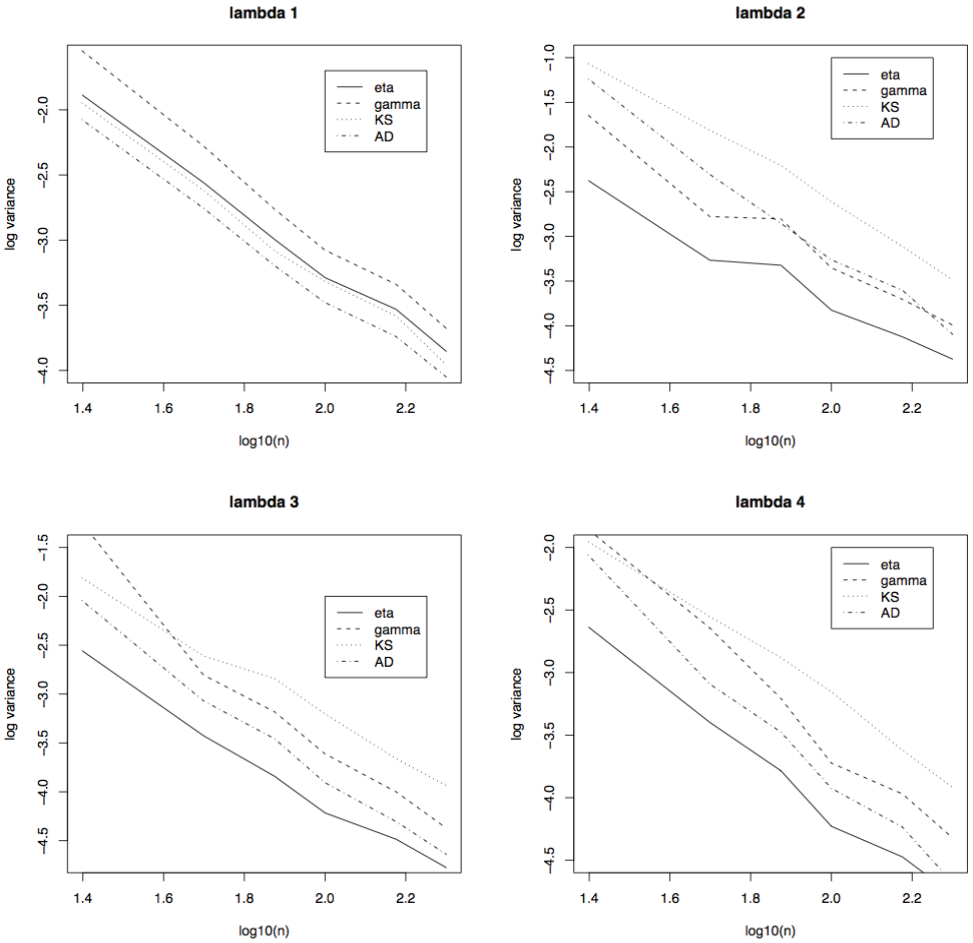


Figure 10.3–5. Sampling variance: log(variance) against the log(sample size), FMKL parameterization.

shape functional biases are lower.

10.3.4 Overall

Across the simulations, there is not a clearly best performing method. It does seem that very short tails are more problematic for the shape functional methods (they perform poorly for set FMKL3, and poorly for λ_4 in set FMKL2). Within the shape functional methods, η tends to have lower variance and bias than γ. Within the starship methods, the A-D version tends to outperform the K-S version. Sets FMKL1 and FMKL4 have tails with weight close to, or heavier than the normal. In these the shape functional methods performed well, and the η performed the best.

Table 10.3–6. Sample size versus sampling variance – simulation set FMKL1.

method		λ_1	λ_2	λ_3	λ_4
starship (K-S)	slope	−0.9463	−1.165	−1.012	−0.9428
	(S.E.)	0.03557	0.03477	0.03873	0.03367
starship (A-D)	slope	−0.9422	−1.340	−1.236	−1.223
	(S.E.)	0.03147	0.05504	0.04533	0.05473
shape functionals	slope	−0.9376	−0.9408	−1.0595	−1.0310
(η)	(S.E.)	0.03357	0.07488	0.05115	0.04384
shape functionals	slope	−1.018	−1.091	−1.441	−1.204
(γ)	(S.E.)	0.03872	0.09927	0.1324	0.06168

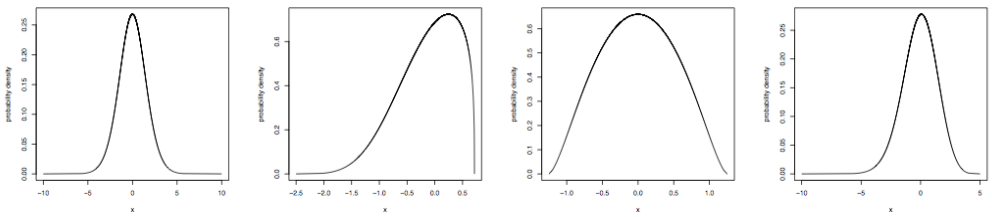


Figure 10.3–7. FMKL distributions investigated in this sections simulation study.

Table 10.3–8. Simulation results, set FMKL2.

method	mean	std. dev.
$\lambda_1(0)$		
starship (K-S)	−0.00292	0.07742
starship (A-D)	0.000995	0.07446
shape functionals (η)	−0.01736	0.07898
shape functionals (γ)	0.0009982	0.09722
$\lambda_2(2)$		
starship (K-S)	2.15	0.3762
starship (A-D)	2.115	0.3723
shape functionals (η)	2.627	0.4266
shape functionals (γ)	2.206	0.5842
$\lambda_3(0.2)$		
starship (K-S)	0.158	0.1404
starship (A-D)	0.171	0.1281
shape functionals (η)	0.009324	0.1544
shape functionals (γ)	0.1585	0.2068
$\lambda_4(0.7)$		
starship (K-S)	0.6402	0.1791
starship (A-D)	0.6789	0.1755
shape functionals (η)	0.3817	0.1139
shape functionals (γ)	0.6811	0.3467

Table 10.3–9. Simulation results, set FMKL3.

method	mean	std. dev.
$\lambda_1(0)$		
starship (K-S)	−0.00134	0.07827
starship (A-D)	−0.00071	0.07211
shape functionals (η)	0.000107	0.08126
shape functionals (γ)	0.00057	0.09653
$\lambda_2(2)$		
starship (K-S)	2.284	0.4714
starship (A-D)	2.145	0.3887
shape functionals (η)	2.243	0.5209
shape functionals (γ)	2.13	0.5481
$\lambda_3(0.4)$		
starship (K-S)	0.2946	0.1802
starship (A-D)	0.3617	0.1434
shape functionals (η)	0.3542	0.1692
shape functionals (γ)	0.4051	0.2626
$\lambda_4(0.4)$		
starship (K-S)	0.2868	0.1751
starship (A-D)	0.3578	0.1421
shape functionals (η)	0.3535	0.1686
shape functionals (γ)	0.4038	0.2616

Table 10.3–10. Simulation results, set FMKL4.

method	mean	std. dev.
$\lambda_1(0)$		
starship (K-S)	−0.0025	0.1879
starship (A-D)	−0.00131	0.1719
shape functionals (η)	−0.00156	0.1959
shape functionals (γ)	0.00254	0.2208
$\lambda_2(1)$		
starship (K-S)	1.123	0.2681
starship (A-D)	1.069	0.2101
shape functionals (η)	1.054	0.1727
shape functionals (γ)	1.041	0.2116
$\lambda_3(0.1)$		
starship (K-S)	0.01995	0.1997
starship (A-D)	0.06786	0.1506
shape functionals (η)	0.1049	0.1392
shape functionals (γ)	0.1137	0.181
$\lambda_4(0.2)$		
starship (K-S)	0.1146	0.191
starship (A-D)	0.1641	0.1412
shape functionals (η)	0.213	0.1138
shape functionals (γ)	0.2268	0.1712

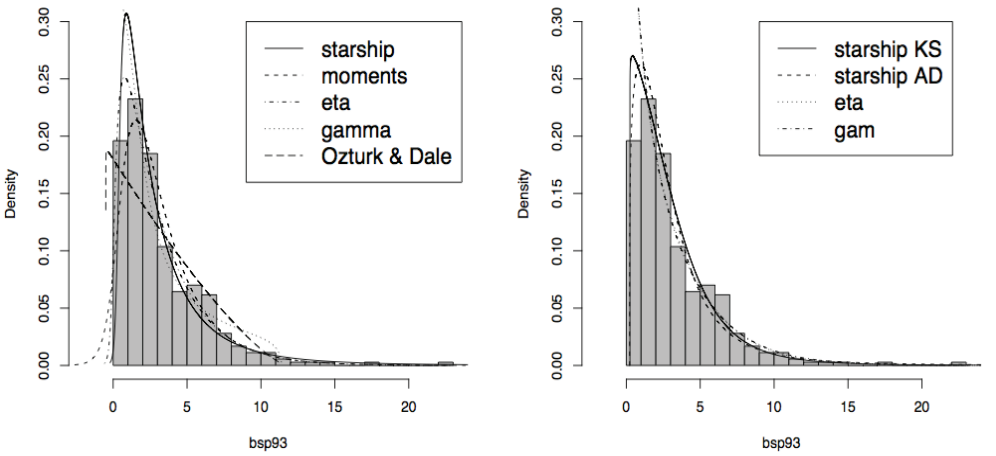


Figure 10.4–1. Histograms of BSP with fitted density functions: RS parameterization, left; FMKL parameterization, right.

10.4 Example: Particulates

To illustrate the method we present fits of both the parameterizations of the GLD to some environmental data. These are daily averages (for all of 1993) of half-hourly atmospheric measurements, β scattering of particulates (BSP, an indirect measure of particulates, measured in 10^{-5}m). The data was collected by the Queensland Department of the Environment (Australia) and kindly made available to the authors. Figure 10.4–1 shows histograms of the BSP data, together with fitted GLD densities. Five RS parameterization fits are shown on the left, 4 FMKL fits on the right.

The data is strongly skewed, and the different fitting methods have come to different conclusions about the behavior of the left tail. The shape functional method for the FMKL (either asymmetry measure) and the γ based method for the RS have fitted distributions with an infinite density at the left tail. The Öztürk and Dale method has fitted a distribution with a theoretical minimum $F^{-1}(0)$ around -12, but which has almost no mass below $F^{-1}(10^{-10}) = -0.506$. The remaining methods fit distributions with a zero density at their minimum. The sharpness of descent of the left tail ranges from the longest left tail in the moment fit (RS), through the η (RS), starship (RS) to the starship (FMKL) fits, which have an infinite derivative of the density at their minimum.

A good fit to this data needs to capture the strongly skewed nature and the distinctive shape of each tail. The values physically must be ≥ 0, all the observations are greater than 0.26×10^{-5}m and there doesn't seem to be a detection limit effect (the 10 smallest values are all distinct). The starship fits appear to perform best of all the candidates in this case. The shape functional methods do reasonably well, but all except the γ(RS) over-estimate the minimum value. The

Table 10.5–1. Parameter values to approximate well-known distributions.

distribution	λ_1	λ_2	λ_3	λ_4	method
$F_{3,5}$	0.71484	1.71709	0.736353	−0.6348276	η
χ_3	2.12500	0.65000	0.750000	−0.100000	star(AD)
χ_6	5.04115	0.43016	0.532361	0.083139	η
χ_{10}	9.04343	0.32854	0.416365	0.034754	η
double exp.	0	2.35313	−0.459314	−0.4593304	η
$N(0,1)$	0.0	1.43865	0.149657	0.149669	η
logistic(0,1)	-3×10^{-6}	1.0000	2.4×10^{-05}	-2.3×10^{-5}	η
t_5	0.0	1.557687	0.03565	−0.035625	η
t_{10}	0.0	1.44454	0.1	0.1	η
t_{30}	-10^{-6}	1.45702	0.119450	0.119471	η

γ(RS) is however, a poor fit to the rest of the distribution.

It appears that the starship fits performed best in the left tail because it demands a good fit across all quantiles in contrast to the "averaging effect" across quantiles of the shape functional method.

10.5 Approximation

We also investigated the shape functional method (using the η functional), as a way of calculating approximations of the generalized lambda (FMKL parameterization) to other distributions. In all but one case (χ^2_3), the shape functional approximation was closer than the starship approximation. Table 10.5–1 gives the parameter values of these approximations.

10.6 Conclusion

Fitting using the η asymmetry functional and the ratio of spread functions provides an alternative fitting method. It allows fitting of shape parameters before any estimation of scale and location, including in those areas of the parameter space where moments do not exist. Our simulation study shows that while the method does not seem to estimate as well as the starship (King and MacGillivray (1999)) for distributions with very short tails (i.e., with λ_3 or $\lambda_4 > 0.3$), it does well where the tails of the distribution are as heavy, or heavier than the normal. The example (particulate levels) and approximation study show some promising areas for the method.

We recommend the estimation method in cases where the tails of the data seem to meet the requirement of tailweight at least equal to the normal.

References for Chapter 10

David, F. N. and Johnson, N. L. (1956). "Some tests of significance with ordered variables," *Journal of the Royal Statistical Society,* series B 18, 1–20. cited by MacGillivray (MacGillivray 1986).

Freimer, M., Mudholkar, G. S., Kollia, G. and Lin, C. T. (1988). "A study of the generalized Tukey lambda family," *Communications in Statistics – Theory and Methods*, 17, 3547–3567.

Hoaglin, D. C., Mosteller, F. and Tukey, J. W. (1982). *Understanding Robust and Exploratory Data Analysis,* Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions to Data: The Generalized Lambda Distribution and the Generalized Bootstrap Methods,* CRC Press, Boca Raton, Florida.

Karian, Z. A., Dudewicz, E. J. and McDonald, P. (1996). "The extended generalized lambda distribution system for fitting distributions to data: History, completion of theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics – Computation and Simulation*, 25(3), 611–642.

King, R. A. R. (1999). *New Distributional Fitting Methods Applied to the Generalised λ Distribution*, PhD thesis, Queensland University of Technology, George St, Brisbane 4000, Queensland, Australia.

King, R. A. R. and MacGillivray, H. L. (1999). "A starship method for fitting the generalized λ distributions," *Australian and New Zealand Journal of Statistics*, 41(3), 353–374.

MacGillivray, H. L. (1986). "Skewness and asymmetry: Measures and orderings," *The Annals of Statistics*, 14(3), 994-1011.

MacGillivray, H. L. and Balanda, K. (1988). "The relationship between skewness and kurtosis," *Australian Journal of Statistics*, 30, 319–337.

Owen, D. B. (1988). "The starship," *Communications in Statistics – Computation and Simulation*, 17, 315–323.

Öztürk, A. and Dale, R. F. (1985). "Least squares estimation of the parameters of the generalized lambda distribution," *Technometrics*, 27, 81–84.

Ramberg, J. S. and Schmeiser, B. W. (1974). "An approximate method for generating asymmetric random variables," *Communications of the Association for Computing Machinery*, 17, 78-82.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J. and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics*, 21, 201–214.

Chapter 11

Statistical Design of Experiments: A Short Review

Allan T. Mense
Principal Engineering Fellow
Allan_T_Mense@raytheon.com

Jerry L. Alderman
Engineering Fellow
jerryalderman@comcast.net

William C. Thomas
Engineering Fellow
wcthomas1@raytheon.com

Raytheon Missile Systems
Tucson, Ariz. USA

The following three sections provide a short overview of statistical Design of Experiments (DOE).

There are a number of good texts on DOE and five of these, Montgomery (2009), Box, Hunter, and Hunter (2005), Mathews (2007), Anthony (2003), and Anderson and Whitcomb (2000), are listed in the references. The most recent edition of Montgomery (2009) is particularly recommended. It is assumed the reader is familiar with statistics at the undergraduate level, including a basic understanding of Analysis of Variance (ANOVA). Suitable texts are listed in the references and there are many others. Two good undergraduate statistics books that lean toward the engineering community are: Montgomery and Runger (2006)

and Hayter (2002). More advanced texts that are given in the references are Montgomery and Runger (2006), Hogg and Tanis (1997), Stuart and Ord (1994), Kendall and A. Stuart (1967), Papoulis and Pillai (2002), and Dudewicz and Mishra (1988).

11.1 Introduction to DOE

Statistical DOE is a disciplined technique for planning experiments and analyzing the collected experimental data. The most important reason for using DOE is in the planning stage for the experiments that need to be performed, including the order in which they should be carried out. Once the data is obtained from these experiments then the second part of DOE is to analyze the data, called **model building**. This analysis results in the production of a formula, called a **regression equation**, for each response variable (RV) of interest. A typical equation might have the form $y = f(x_1, x_2, \ldots, x_n) + \varepsilon$, where y is the dependent or response variable and the set $\{x_1, x_2, \ldots, x_n\}$ represents the causal or independent variables (IV) and ε is an error term due to random effects not included in $f(x_1, x_2, \ldots, x_n)$.

For example, y might be the output power of a radio transmitter, and the x_i the electrical part values, circuit board dimensions, etc. The term ε represents the random error that inevitably accompanies any measured response. This error may be due to myriad causes, e.g., random variations in parts, construction, dimensions, manufacturing processes, operating environment, and even the measurement process itself. These random effects occur whenever one deals with real products. Running an experiment with the same set of x-values seldom produces the same y-value unless the measuring instrumentation is very coarse and masks the variation. Hence, the inclusion of the error term ε.

In the vocabulary of designed experiments, the independent or causal variables are called **factors** and the values they are assigned are called **levels**. Independent variables that can vary continuously over a defined range, but are otherwise uncontrolled, are called covariates. Analysis of **covariates** is a separate subject and treated in depth in Chapter 15 of Weber and Skillings (2000).

The functional form $f(x)$ may be as complex as necessary to explain the response, but many times simple polynomials are adequate to describe the response within a small region of interest about a selected design point. It is also sometimes helpful to use some function of y as the response, e.g., $\text{logit}(y) = ln(y/(1-y))$. This is particularly useful when y is bounded, say between 0 and 1. While we will limit our discussions to situations in which there is a single response (RV) and multiple IVs, in practice there may be multiple RVs so y may be an array of different responses all dependent on the same set of IVs (see Tabachnick and Fidell (2007), and Gujarati (2002)). Summarizing, one finds that DOE helps in

- identifying relationships between cause and effect;

- providing an understanding of interactions among causative factors;
- determining the levels at which to set the controllable factors (product dimension, alternative material, alternative designs, etc.) in order to optimize reliability;
- minimizing experimental error (noise); and
- improving the robustness of the design or process to variation.

In applying DOE methods, care must be taken to avoid oversimplifying the response model (e.g., using a linear model to represent a nonlinear response) or applying a valid linear model outside the range for which it was defined. The selected model must have the mathematical flexibility to match the response over the desired range of the independent variables. This is something of an art and comes from practice. Examples will be discussed later.

As a rule, in medical and biological experiments the input variables are called **treatments** and the term factor or covariate is used in non-biological experiments. Independent variables that are not controllable or are not controlled are referred to as **random factors** or **covariates**. Covariates may play a significant role in determining the value of the response and must be recorded each time a response value is measured.

The analysis process requires setting levels for each independent variable and then taking data on the response. This includes a record of both the random and fixed factor level values. From this data one composes a mathematical equation for the expected value of a response. As stated earlier, this model can be simply written as $y = f(x_1, x_2, \ldots, x_n) + \varepsilon$ or more often as $y = \mu_y\{x_1, x_2, \ldots, x_n\} + \varepsilon$. The label y is the response and represents the deterministic (non-stochastic) term that gives a mean or expected value for the response y when the fixed factors, $\{x\}$, take on specific values. The term ε represents the random error and varies from experiment to experiment for the same fixed x values. Clearly the vector, ε will be orthogonal to the experimental array of x-vectors, $\{x\}$.

Model parameters $\{\beta, \varepsilon\}$ are usually denoted with Greek letters and the estimators of those parameters by Latin letters $\{b, e\}$. We will follow this convention. As a historical aside, the subject of "Designed Experiments" can be attributed to Sir Ronald A. Fisher who, in the 1920s and 30s was responsible for statistics and data analysis at the Rothamsted Agricultural Experimental Station near London, England.

11.1.1 Experiments

To obtain data, we perform experiments whenever possible or observe behavior without interfering with the process when experiments are not possible. An experiment is a carefully conceived and executed plan for data collection. Treatments or factors believed to influence the responses of interest are chosen using

subject matter experts (SMEs) and many times techniques such as brainstorming are used to ensure a thorough understanding of exactly which factors should or should not be included in the data taking and analysis. Simply using a rule such as "when in doubt include the factor" is not a best practice and requires far more data than can typically be collected. One should always have some causal reasons in mind for selecting factors. This is particularly true when the number of experiments is limited to, say, fewer than 30 experiments.

11.1.2 Types of Experiments

To obtain data one must run experiments (or simulations that have embedded in the simulation stochastic events). Types of designs are listed here according to the experimental objective they meet.

Comparative objective: You have what is called a **comparative design** if you have one or several factors under investigation, but the primary goal of your experiment is to make a conclusion about one *a priori* important factor in the presence of, and/or in spite of the existence of the other factors. The question of interest is whether that factor is significant.

Screening objective: The primary purpose of the experiment is to select or screen out the few important main effects from the many less important ones. These screening designs are also termed **main effects designs**.

Response Surface (method) objective: These experiments are designed to estimate interactions and, at a minimum, quadratic effects. Such models give us an idea of the (local) shape of the response surface we are investigating. For this reason, they are termed **response surface method (RSM) designs**. RSM designs are used to

- find improved or optimal process settings;
- troubleshoot process problems and weak points; and
- make a product or process more robust against external and non-controllable influences. Robust means it is relatively insensitive to these influences.
- Create meta models for parameter determination.

11.1.3 The Independent Variable (IV)

The independent or causal variables are usually designated by X or by listing the set of values $\{X_1, X_2, \ldots, X_k\}$. They are chosen by the experimenter as the variables most likely to causally affect the responses of interest, Y. In a designed experiment values or levels of the X_j variables are set to some specific value, say x_{ji}, and the experiment is performed, which leads to recording the response y_i. Designing experiments that help one to sort out the important few from the less

important many is the first goal of any DOE plan and is best done by covering as large a range of x-values as one can, but to do so in a systematic way while simultaneously staying within allowable ranges of the experimental factors.

11.1.4 Types of Causal (CV) or Independent (IV) Variables

Quantitative: Variables that can take on any value specified over some range of possible values, called **interval variables**. They may be discrete or continuous. They are called either **factors** or **treatments** in the language of DOE.

Qualitative or Categorical: These variables can be nonnumerical, such as High, Medium, or Low; they can be numerical but ordinal, such as 1st, 2nd, 3rd, etc.; or they can be specified by some scale whose intervals are not defined. They also are called either **factors** or **treatments** in the language of DOE.

11.1.5 The Dependent Variable (DV)

Before we go much further, the reader is reminded that response(s) of the experiments should be measured by test equipment, whose own errors of measurement do not swamp the response values being measured. This prerequisite may seem obvious, but many times cannot be verified when looking at historical data taken by other experimenters.

11.1.5.1 Types of Response or Dependent Variables

Continuous: Response measured using real numbers for which the regression equation is intended to predict (there is nothing to preclude complex numbers but no software is commercially available to treat such problems).

Binary: A response, such as a pass/fail, that can be represented as a 1 or a 0. This type of response data occurs in reliability analyses. The regression model used to produce an equation for the reliability (e.g., probability the response = 1) is typically a logit function or sometimes a probit function.

11.1.5.2 Nuisance or Noise Variable (NV)

These are variables that are random in nature and are uncontrollable by the experimenter. They interfere with the analysis of the response and their effect on the response needs to be separated, if possible, from the effects due to the factors of interest. Random choosing of which experiments to run or a process called "blocking" can be used to aid in removing the effects of these variables on the response.

11.1.6 When to Use DOE

When there are either no analytical formulas or when computer simulations take excessive time and resources, the use of DOE is a must in order to create models (also called **meta models**) for the response behavior. DOE can be applied to all phases of a products production cycle (design; development; test and verification; manufacturing and assembly) and also to process designs and implementation.

DOE can be used to develop a regression model for the responses of interest. It is the only technique that, if appropriately applied, will give the most accurate response equations with the least variance in the model coefficients, and do so with the fewest number of experiments (or simulations) and least amount of post-data analysis. There are Gaussian space-filling models also part of using DOE in simulations that can be used, but will not be discussed in this article (See Montgomery (2009)). A typical regression equation for two factors might take the form shown below.

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{12} X_1 X_2 + \beta_{11} X_x^2 + \beta_{22} X_2^2 + \cdots + \varepsilon$$

which is nonlinear in the regression variables (X_1 and X_2) but is still linear in the β coefficients. Due to the linearity in the coefficients, this form of a regression equation is called a **linear regression equation**. We use the data to determine the numerical values of the estimators for the β coefficients and ε. The above equation represents a surface above the $X_1 - X_2$ plane.

11.1.7 Factors or Treatments

These are the CVs or IVs we propose to investigate as to their effect on the response or DVs. When the factors are under our control, then designed experiments are possible and a reasonable amount of thinking goes into deciding the levels or values of the IVs used in the experiments.

11.1.8 Levels

Once factors are determined, then choosing the appropriate levels for each factor is the next step. Screening experiments typically use two levels for each factor. Reducing the number of causal factors is called "pruning" the regression equation. The screening experiments are called 2^k **full factorial** or 2^{k-r} **fractional factorial experiments**. In these experiments, one has k treatments (IV) and each treatment takes on one of two possible levels (hence the 2 in 2^k). If k is large and the experiments/resources are constrained, then one may not have the convenience of being able to run the full 2^k experiments. In which case there are fractional factorial experiments that are reduced in number, hence 2^{k-r}, so the number of experiments is reduced and thus more affordable and the ability to select the key causal factors is not too impaired. Before going into more detail, it is useful to try to visualize such experimental plans (shown in Figure 11.1–1).

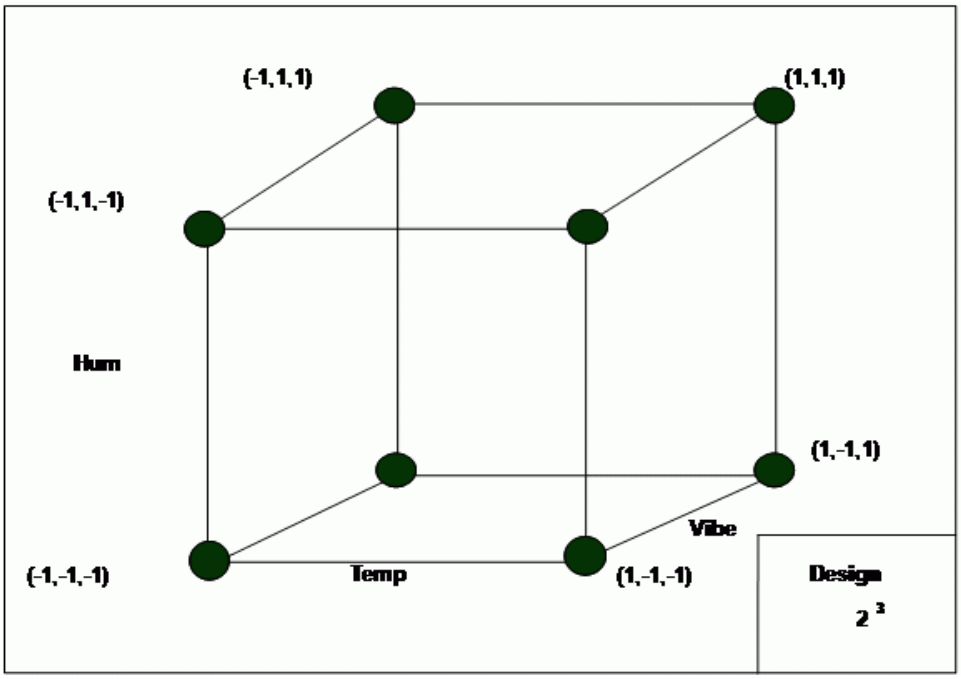


Figure 11.1–1. Full factorial experimental design for three factors with two levels for each factor.

One method of visualizing DOE designs is geometric, and this can be done rather easily for small numbers of factors, e.g., three. Shown in Figure 11.1–1 is a 2^3 screening design.

The values shown in parentheses (± 1 values) for Figure 11.1–1 are the IVs but in "coded" form. One of the useful features of standard DOE methods is to normalize all data such that the causal variables can only take on values between -1 and 1, which then allows us to view the largest β coefficients in the regression equation as multiplying the most important factors. These are the factors that most influence the response.

Once the range of actual (not coded) values for the IVs is narrowed to a region of experimental interest, one can look for optimization of the response. Optimization can have a wide range of meanings, but assuming some type of maximum or minimum is not uncommon. Clearly, to find such a region the mathematical model must have some set of even-power polynomials, e.g., quadratic in the factors, or else the solution will lean against one or more constraints or boundaries. To determine such optimum responses, one performs a series of what are called **response surface experiments**. The response surface methodology (RSM) is thoroughly discussed in Schmidt and Launsby (2003). The names central composite design (CCD), Box-Behnken, Taguchi, are associated with such experimental designs (see Figure 11.1–2). The Taguchi designs can be very useful in creating what are called **robust designs** in which the response of interest is not sensitively

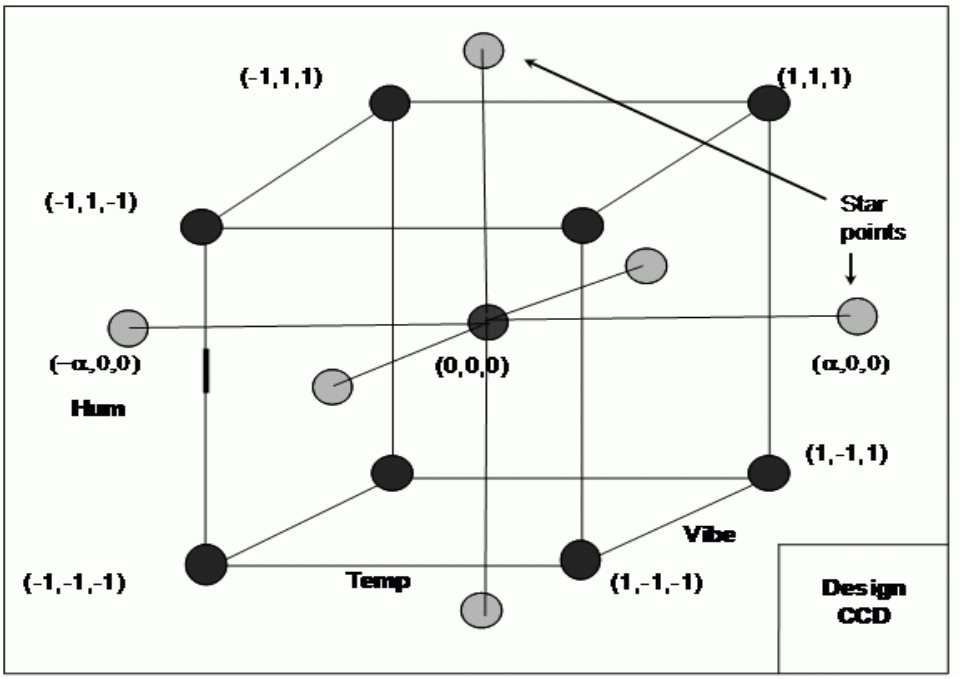


Figure 11.1–2. Central Composite Design (CCD). Note the star points and center points.

affected by extraneous (or noise) variables. In electrical engineering parlance we maximize the signal-to-noise ratio. This is discussed in detail in Montgomery (2009).

The reader should not confuse optimization of the dependent variable (response) with the use of optimal designs, which are special patterns of the independent variables to achieve special properties of the regression equation, e.g., the pattern to minimize variance of the regression coefficients, b_j, is called **D-optimality** (see section A.2 of the Appendix at the end of this chapter).

Determining the factors to include in an experiment (or simulation) is the most important step in the DOE process. It requires a good bit of time, usually with more than one SME at the table. Once treatments have been chosen, the levels that should be used for each treatment have to be explored. In a 2^k experiment, there are only two levels for each of the k treatments; in a 3^k experiment, there are three levels; and in CCD experiments, there can be five or more levels. Many custom designs are possible in which a different number of levels is used for each treatment. Fortunately, available software (Design Expert, JMP, and Minitab, SAS, SPSS, DOEPro^{TM}) can perform all the hard work in laying out the design and analyzing the results. However, one must be careful not to prescribe levels that either push the experiment into an operational domain it will never experience or that will catastrophically ruin the experiment. The exception being experiments in which destructive testing is the type of experiment being

analyzed, e.g., finding the always-fail and always-safe voltages of firing caps for explosives (once treatments have been decided then the experiments should be run on the same equipment set up by the same operators and performed in the same manner).

This data, which includes the values for the levels and any other outside observational measurements (called **covariate data**), is then analyzed using fairly sophisticated software as indicated above. One does not have to use the analysis techniques provided by the DOE software tool manufacturers if one has a specific preferred method. What is important is that the treatments and fixed levels for the treatments be laid out using a DOE methodology. As will be discussed later, the use of orthogonal solution techniques may provide better regression equation models than some techniques used by the software manufacturers.

11.1.9 Regression Coefficients

There are many ways of determining the estimators of the regression coefficients given the data. Some methods that are popular are 1) the method of least squares, 2) the maximum likelihood method and, as will be discussed below, 3) the orthogonal solutions method. The first two methods all make use of what are called the **model residuals** or the **distribution of these residuals** in determining the coefficients and the overall error in the regression equation. See Montgomery (2009) for details.

11.1.10 Residuals

Once a form for the regression equation (or equations) has been found, e.g., $y = \hat{y} + \varepsilon$ where the $\hat{y} = \mu_y$ term accounts for the functional form of the regression equation and represents the mean value that y would have given a fixed set of treatment levels, one turns his or her attention to the error term represented as ε. It can be used to determine how well the model fits the data. When rewritten as $e = y - \hat{y}$, e is called a **residual** and there is one residual value for each experiment performed. The sum of these residuals squared, called **SSE (sum of squared errors)**, is proportional to the variance estimator of y. So you can estimate the dispersion of y with reference to the equation that best fits the data. Looking at the residuals versus actual response values can be a very useful diagnostic in determining whether the variance is a constant or is in need of some form of data transformation.

11.1.11 Optimality of Design

By optimality in this sense one means the selection of a pattern of fixed factors and levels that achieves the maximization or minimization of some property in the design. For example, a D-optimal design is a set of treatment levels that will maximize the determinant of the $X'X$ matrix, which of course minimizes the

values in the $(X'X)^{-1}$ matrix, which will later be seen to minimize the variance of the regression coefficients, β_j, in the final regression equation. There is an alphabet soup of design patterns, e.g., A-optimality, D-optimality, G-optimality, I-optimality, etc. See Chapters 6, 11, and 12 of Montgomery (2009). Some designs must be generated by computer and are not simple designs, as will be shown in this short review.

11.1.12 Optimization

As stated earlier, the type of mathematical model used to fit the data determines the ease with which one might find an optimized response, i.e., a maximum or minimum of the response variable(s). Most DOE software has the ability to find optimal solutions. Discussion will follow some case studies shown later.

11.1.13 Orthogonality

Orthogonality between variables means there is a perfect nonassociation between those variables. We usually use this when speaking of causal variables (or IVs) in regression analysis of the DOE data. If two variables are orthogonal then knowing the value of one variable gives no information about the value of the other. Note that we did not use the word "correlation." Correlation is a measure of the linear relationship between two random variables and we are not dealing with pairs of random variables when we use fixed factors for a designed experiment. Correlation in dealing with random factors, covariates, etc., will be discussed later in this chapter.

Orthogonality can be very useful in statistical analyses; in the days before computers when such statistical greats as Fisher were performing the first DOEs, they sought out ways to separate causal variables without needing excessive computations. In orthogonal experimental designs, the random assignment of subjects, the choosing of levels of the IV, and good controls will allow one to attribute the changes in the DV unambiguously to various main effects and interactions. If all pairs of IVs in a set are independent of each other, then each IV simply adds to the prediction of the response (DV). When variables are correlated they have shared or overlapping variance. By constructing a 2^k experimental design, one automatically orthogonalizes the IVs as is evident from the design shown in Table 11.1–3 that is a full 2^3 factorial design.

If one treats each column (Temp, Hum, and Vibe) as a vector and performs a vector dot product, one finds a problem from the start. The vectors are not orthogonal, e.g., the Temp vector dotted with the Vibe vector does not equal zero. To handle this situation, one first nondimensionalizes these vectors. In the DOE business this is called **coding**. If AV stands for actual value, CV for coded value, LV for the lowest value of the actual variable and HV for the highest value of the actual variable, then one can obtain the coded values as seen in Table 11.1–4 by

Table 11.1–3. Actual experiment level values for three factors.

Uncoded	RF Gain			
Temp (F)	Humidity	Vibration (dB)	$< y >$	St. Dev.
0	0.1	3	6.030077	11.33534
0	0.1	30	23.74418	0.630825
0	0.9	3	7.692419	4.200711
0	0.9	30	32.9233	10.95717
100	0.1	3	105.5224	7.294723
100	0.1	30	135.848	10.47417
100	0.9	3	106.5215	5.705587
100	0.9	30	140.4063	11.22857

Table 11.1–4. Values from Table 11.1–3 in coded form.

Coded		
Temperature (F)	Humidity	Vibration (dB)
-1	-1	-1
-1	-1	1
-1	1	-1
-1	1	1
1	-1	-1
1	-1	1
1	1	-1
1	1	1

performing the following computation on each of the IV values in Table 11.1–3.

$$CV = \frac{AV - (LV + HV)/2}{(HV - LV)/2}$$

This coding uses mean centering of the variable and then scaling to limit the coded variable range between -1 to $+1$. The inverse calculation is performed using the formula

$$AV = \frac{LV + HV}{2} + \frac{HV - LV}{2} CV.$$

This procedure is not uncommon on many engineering analyses. It goes without saying, however, that experiments are run using the actual values of the variables. It makes no sense to think of actually using the coded values — but we have seen some engineers try it. Example: For the temperature variable value from row one of the Temp column in Table 11.1–3 (i.e., $AV = 0$), and using $LV = 0$, $HV = 100$, then $CV = (0(0 + 100)/2)/((100 - 0)/2) = -50/50 = -1$.

There is no absolute need to nondimensionalize the response, although it can ease the model-building process and the interpretation of the results. In most texts on regression theory, the actual values of the IVs are used. The IVs are coded most of the time in DOE work. We recommend the coded approach. See section 6.9 of Montgomery (2009) for an interesting discussion and counterexample.

$$\begin{aligned} & [-1,-1,-1,-1,+1,+1,+1,+1][-1,+1,-1,+1,-1,+1,-1,+1]^T \\ = \; & ((-1)(-1)+(-1)(+1)+(-1)(-1)+(-1)(+1) \\ + \; & (+1)(+1)+(+1)(-1)+(+1)(+1))=0, \end{aligned}$$

where the superscript T indicates that the transpose vector is to be used.

Using this set of vectors in the coded space we note two important properties. First, all physical dimensions are removed and each variable (vector) has a value between -1 and $+1$. Second, the vectors are orthogonal. The meaning of orthogonality is clear as can be seen by taking the dot (or inner) product of any two vectors. Example: The dot product of Temp with Vibe is shown below. The dot product is zero, which means these two vectors are orthogonal. This can be shown to be true for any pair of vectors in Table 11.1–4. Orthogonality is a special property of this 2^k design. There are obviously many possible nonorthogonal designs. This clever choice of designs allows for useful interpretation. First, if one obtains a regression equation in terms of these vectors and then decides to drop one of the variables because it does not significantly affect the response, the coefficients for the remaining IVs are unaffected. Only the fit of the mean response to the data changes, i.e., the residuals, become larger. This will become clearer in later sections of this chapter.

As stated earlier, almost all DOE applications depend heavily on the use of orthogonal experiments for design and for regression analysis of collected experimental data, yet few tools or texts directly address the fundamental role of orthogonality. Here we will show that

- any dataset can be transformed into an orthogonal dataset;
- any regression analysis built on an orthogonal dataset is extremal in the sense of the principle of least squares; and
- viewed in this way it will be clear that, in a fundamental sense, the concept of orthogonality is as fundamental as the principle of least squares since regression models built on an orthogonal basis will always satisfy the principle of least squares and any least-squares solution will produce an error term that is orthogonal to the data matrix.

It will be said of two vectors $\vec{A}$ and $\vec{B}$ that they are orthogonal if the scalar product of $\vec{A}$ and $\vec{B}$ is zero: $\vec{A}\cdot\vec{B}=0$ (the sum of the products of corresponding

components = 0). Note that in matrix notation this could also be written as $\vec{A}^T\vec{B} = 0$

If $\vec{A}$ and $\vec{B}$ are not orthogonal and are not parallel, two orthogonal vectors can be constructed from them using $\vec{A}$ and $\vec{B}_1$ where

$$\vec{B}_1 = \vec{B} - (\vec{A} \cdot \vec{B})/(\vec{A} \cdot \vec{A}).$$

This process can be extended to include any number of vectors and is known as a Gram-Schmidt orthogonalization procedure (see Boas (2008)). The Gram-Schmidt orthogonalization procedure is relatively intuitive and is easy to visualize in three or fewer dimensions, but there are more efficient ways to transform any given data set into an orthogonal one. Among these are Givens Rotations (Golub and Van Loan (1996)), Householder transformations (Householder (1958), and the QR decomposition (Golub and Van Loan (1996)) that will be discussed below.

Let $\boldsymbol{X}$ represent the data matrix and $\boldsymbol{y}$ the response vector and assume that we are seeking the coefficients beta of a model such that $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ is a vector of random errors often assumed to be normal but seldom are for small datasets.

In the traditional approach we would seek coefficients β such that the sum of the squares of the error terms $\boldsymbol{\varepsilon^T\varepsilon}$ was minimized; this is the principle of least squares. Here, however, we will just observe that, without loss of generality, we can seek regression coefficients β such that $\boldsymbol{X^T\varepsilon} = 0$. That is the error vector is orthogonal to the data matrix. If this were not so, the error term could itself be regressed against this same data matrix and this process could continue until the resultant error term was either orthogonal to $\boldsymbol{X}$ or identically zero. Using this property the left multiplication of the regression equation with $\boldsymbol{X^T}$ results in $\boldsymbol{X^Ty} = \boldsymbol{X^TX\beta}$ and this leads immediately to $\boldsymbol{\beta} = \boldsymbol{(X^TX)^{-1}X^Ty}$, which is the same equation for the coefficients β that follows from the principle of least squares.

Thus, we have seen that the simple argument that the error term must be orthogonal to the data matrix implies a solution that is the same as that found from an application of the principle of least squares. Since the converse is also true, the astute reader will observe that the process of stepwise regression (see Chapter 9, Montgomery, Peck, and Vining (2006)) is actually an implementation of a stepwise Gram-Schmidt orthogonalization process. In the special case that the columns of the data matrix, $\boldsymbol{X}$, are themselves orthogonal unit vectors the general solution for the regression coefficients, $\boldsymbol{\beta}$ becomes $\boldsymbol{\beta} = \boldsymbol{X^Ty}$ and the coefficient of each column of $\boldsymbol{X}$ is just the scalar product of that column of $\boldsymbol{X}$ with the response vector, $\boldsymbol{y}$.

As stated above, it is always possible to transform the data so as to make the columns of the data matrix into orthogonal unit vectors. The "QR" decomposition of a data matrix $\boldsymbol{X}$ transforms it into the product of a Hermitian matrix $\boldsymbol{Q}$ and a lower triangular matrix $\boldsymbol{R}$ such that $\boldsymbol{X} = \boldsymbol{Q \cdot R}$ where $\boldsymbol{Q^T \cdot Q}$= 1 and $\boldsymbol{R}$ is lower triangular. This transforms the regression equation into $\boldsymbol{y} = \boldsymbol{Q \cdot \beta_N}$ where

$\boldsymbol{\beta_N} = \boldsymbol{R\beta}$. It follows immediately that $\boldsymbol{\beta}_N = \boldsymbol{Q^T y}$ and $\boldsymbol{\beta}$ is determined from the relation $\boldsymbol{\beta}_N = \boldsymbol{R\beta}$. This equation can always be solved by back substitution because the matrix $\boldsymbol{R}$ is lower triangular. A good reference for QR decomposition on the Internet is the Web site:

http://tutorial.math.lamar.edu/Classes/LinAlg/QRDecomposition.aspx.

The key to fully understanding the methods of the DOE lies in the recognition that the data is collected in such a pattern that the data matrix is orthogonal. When it is constructed in this manner, no further data transformation is required. Keys for accomplishing this lie in the (clever) use of Hadamard matrices (Hadamard (1893), Horadam (2007)), or of Box-Behnken, Plackett-Burman, CCD designs (Box and Behnken (1960), Plackett and Burman (1946), Box and Wilson (1951)) and Taguchi designs (Taguchi and S. Konishi (1987)).

The takeaway from this section is that one need not begin with an orthogonal design matrix to find an appropriate regression equation that has orthogonal regressors. However, the orthogonal regressors will be linear combinations of the original regressors and thus less interpretable than in their raw form. If one takes the regression equation using these orthogonal regressors, prunes it to obtain the best fit to the data, and then transforms the equation back into the original regressors, one has a useful mathematical model. However, the equation loses the orthogonal property by which each regressor is independent of all other regressors and thus loses some of the convenience in interpretation. Clearly, there are many ways to find good metamodels from the data. Much of this will be demonstrated in the case studies that follow.

11.2 Fundamentals of DOE

11.2.1 Basic Principles

The purpose of this section is to take the reader through a complete DOE, discuss decisions and analyze the results. Outputs from several commercial software codes will be demonstrated. A helpful discussion of topics in this section is available in Keppel and Wickens (2004), which also includes an entire section on ANOVA.

11.2.2 Practical Considerations

Before going through the mathematics involved in DOE one must clearly have goals in mind for the experiments or observations that will need to be done. For a medical trial the goal is to have a better patient outcome; for an industrial process one may want a higher product yield or higher product reliability; for a weapons system one may want a smaller miss distance for the warhead of the weapon. A set of experimental runs (called the **test plan**) cannot be laid out until all the goals, i.e., responses are established and it is ensured they are measurable.

Given the desired responses or DVs, one then needs to establish a set of possible IVs. As mentioned in Chapter 1 the idea of simply throwing all possible IVs plus interactions between different IVs into the model and then taking enough data to calculate the regression coefficients that would multiply these variables and their interactions would require enormous time and resources. This philosophy is not a good practice and one seldom has the funds necessary to carry out such comprehensive studies. This is not a procedure that should be carried out alone. This is one situation in which collecting ideas on what variables may be the most important, how they should be measured, and with what acceptable measurement errors, is a difficult job. It is possibly the most important job of the DOE planner and requires general management consensus to secure the necessary funding.

In the examples to be presented it may seem that the goals and the variable selection process are easy or straightforward. They are not. In particular one needs to decide which covariates need to be recorded and how this will be done without unduly affecting the experiment's outcomes. Decide ahead of time how experiments are to be randomized and blocked. Discussions of these topics will follow. The methodology will be discussed and demonstrated step by step in the next few sections of this chapter. A useful text for performing this procedure is Schmidt and Launsby (2003) and while this is the least mathematically rigorous of the recommended books on DOE it is probably the clearest exposition on the practical application of the DOE methodology. Much of the next section will be borrowed from this text.

Planning can best be done using a flowchart: Figures 11.2–1 and 11.2–2 show a flowchart adapted by permission from Air Academy Associates (Schmidt and Launsby (2003)) that takes the experimenter through the steps.

11.2.3 Designing

Once the goals have been determined, measurable responses (and covariates) identified, and number and types of factors listed, one is ready to decide on the order in which the experiments, including replicates, need to be performed. Most DOE software will generate a random testing table. Randomize the sets of IVs to use to reduce the effects of the (unknown) extraneous or noise variables. Replicate, i.e., set up and run, an experiment again using the same IV values to gain information on the variability and pure (model independent) error.

In performing multivariate regression, which is what the resultant analysis procedure will be once the data has been collected from the DOE, we draw upon various computer software programs to perform the hard math. In a nutshell, however, we are still trying to find the coefficients in some linear (in the coefficients) equation for each response variable.

A very general representation of what is called a "saturated 2^k design" is

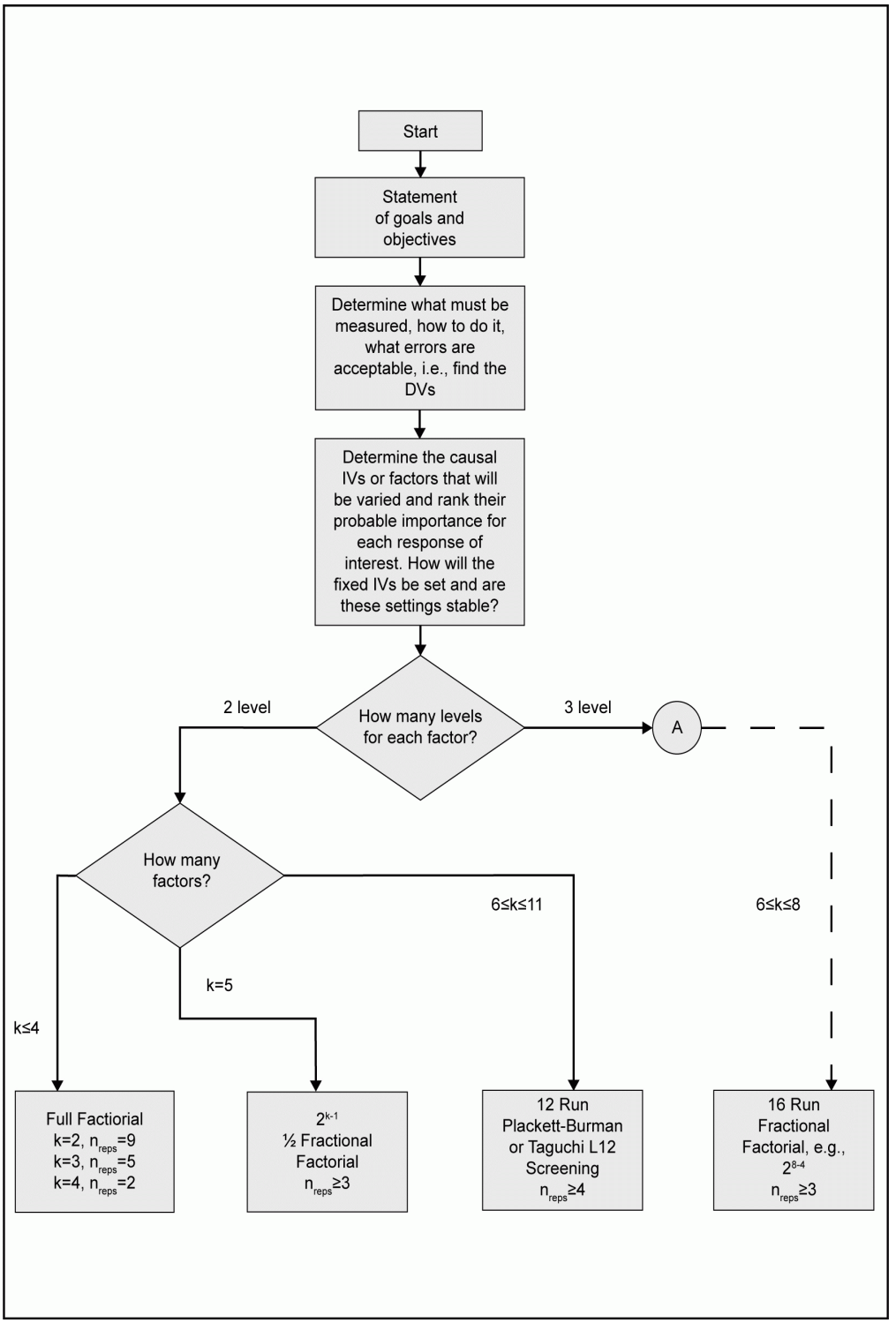


Figure 11.2–1. Planning steps for a designed experiment. (Air Academy Associates)

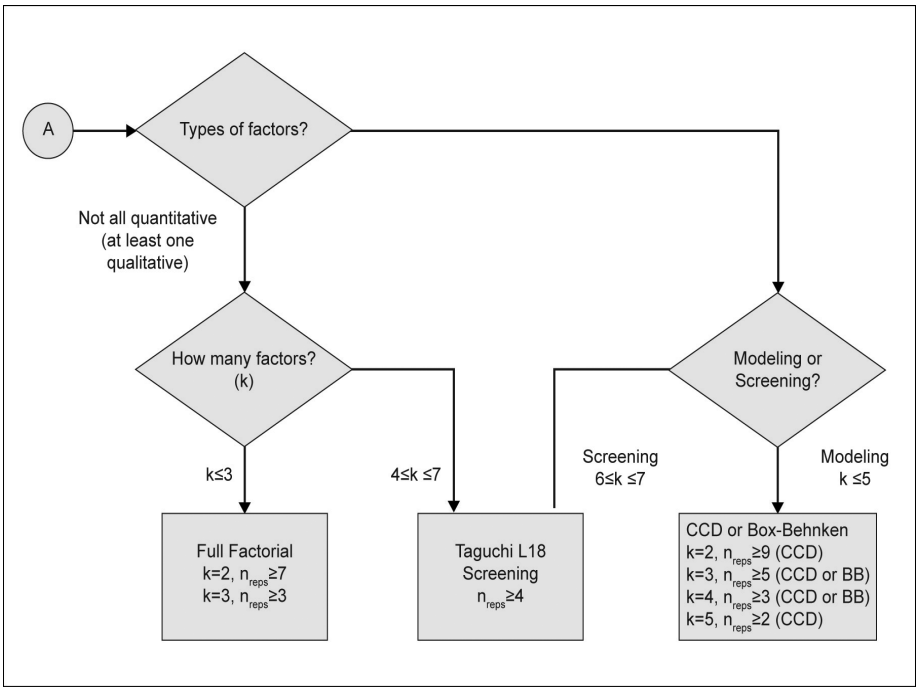


Figure 11.2–2. Completion of flowchart from Figure 11.2–1 showing how to deal with categorical or qualitative factors, as well as quantitative factors. (Air Academy Associates)

shown below.

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j + \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \sum_{l=j+1}^{k} \beta_{ijl}\, x_i x_j x_l + \cdots + \beta_{12\cdots k}\, x_1 x_2 \cdots x_k + \varepsilon$$

By saturated one means there are as many terms as there are data points. Thus, this saturated model has all the main factors plus all possible interactions up through the k-factor interaction. There are 2^k coefficients to evaluate in this model. This model will fit the average response perfectly at each of the 2^k treatment combinations of the full factorial design.

When using smaller-than-saturated designs (which is the usual case and usually the desired case), Montgomery (2009) and others recommend the use of the **principle of hierarchy** in the design. This principle says if there is an interaction term, say β_{123}, then all lower-order interactions should be included (e.g., β_1, β_2, β_3, β_{12}, β_{13}, β_{23}) in the model. This inclusion means that even if the ANOVA analysis shows some of the lower-order coefficients are not statistically significant, they should be included in the model. To the best of the authors'

knowledge, this is a rule of thumb and is not rigorously required. Sometimes when there are no replicates of the corner points it may be more useful to leave the lower-order (statistically not significant) terms out of the model and allow extra data points for calculating the error variance. It is worthy of note that when higher-order interactions are included in the model, there are often problems giving useful physical interpretation of the regression equation. One must realize, however, that one is constructing a mathematical model and not a physical model, and when viewed in that light there is no particular reason to have an interpretation of the main factors alone. But it should be noted that there appears to be a nonlinearity occurring, shown by an interaction.

For a 2^k design it is easy to show that

$$b_0 = \frac{1}{N}\sum_{i=1}^{N} y_i = \bar{y}, \quad b_j = \frac{1}{N}\sum_{i=1}^{N} x_{ij}y_i = \frac{1}{2}\left(\bar{y}_{j=+1} - \bar{y}_{j=-1}\right),$$

where the mean values in the parentheses are based on$N/2$ observations. Rather than performing calculations for each term separately, one can solve for what is needed using matrix methods only because the model is linear in the regression coefficients. This exercise in matrix algebra can be shown (see Montgomery (2009)) to have the formulation $\vec{y} = \vec{X}\vec{\beta} + \vec{\varepsilon}$, where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \vec{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{k1} \\ 1 & x_{12} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n} & \cdots & x_{kn} \end{bmatrix}, \quad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$

The index n refers to the number of runs including replicates and k is the number of factors or IVs. This is called a **fixed-effects model** as all the values in the X matrix are selected and held constant by the experimenter. There are random effects models where all the x-values and levels are randomly selected and mixed-effects models that have both fixed and random effects. See Montgomery (2009) and Weber and Skillings (2000).

A simple way to think of this problem is that a vector y can be represented as a weighted sum (weights are $beta_0$ through β_k) of vectors $(1, x_1, \ldots, x_n)$ plus the vector ε that is orthogonal to the vector space defined by $(1, x_1, \ldots, x_n)$ and represents the error.

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix} + \cdots + \beta_k \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kn} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

The formal solution to this problem (assuming $X'X$ has an inverse) is given by the expression $\vec{b} = (X'X)^{-1}X'\vec{y}$, where we now use $\vec{b}$ instead of $\vec{\beta}$ because

we are actually solving for estimators of β using the data (stored in X) and the estimator for the mean value of y given a set of x-values is given by $\vec{\hat{y}} = \vec{\mu}_{y|x} = (X(X'X)^{-1}X'\vec{y} = H\vec{y}$. The matrix H is called the **hat matrix** as it takes $\vec{y}$ and converts it to $\vec{\hat{y}}$. We see immediately that ε must be perpendicular to all the vectors in X, otherwise ε could be represented by some linear combination of x vectors and there would be zero error. Since the estimators $\{b\}$ are calculated from fixed numbers $\{x\}$ and the vector $\vec{y}$, one can calculate variances, $(\text{Var}(b_j))$, and covariances, $(\text{Cov}(b_j, b_k))$. An estimate for the matrix $\text{Var}(y) = \text{Var}(\varepsilon) = \sigma^2 I$, which is a matrix. For reference the expressions are shown below.

$$SSE = \vec{y}'\vec{y} - \vec{\beta}'X'\vec{y} \quad \text{and} \quad \hat{\sigma}^2 = \frac{SSE}{n-(k+1)} \text{ is an unbiased estimator of } \sigma^2,$$

$$E[\vec{b}] = E[(X'X)^{-1}X'\vec{y}] = E[(X'X)^{-1}X'(X\vec{\beta} + \vec{\varepsilon})] = E[(X'X)^{-1}(X'X)\vec{\beta}] = \vec{\beta}$$

$$\text{Cov}(\vec{\beta} = E\left[(\vec{b} - E[\vec{b}])(\vec{b} - E[\vec{b}])'\right] = \sigma^2(X'X)^{-1} \approx \hat{\sigma}^2(X'X)^{-1}.$$

The regression coefficients have confidence intervals given by

$$Pr\left\{b_j - t_{\alpha/2,n-k-1}\sqrt{\hat{\sigma}^2 c_{jj}} \le \beta_j \le b_j + t_{\alpha/2,n-k-1}\sqrt{\hat{\sigma}^2 c_{jj}}\right\} = 1 - \alpha.$$

The value c_{jj} is the value of the jth diagonal element of the $(X'X)^{-1}$ matrix and use has been made of the t-distribution since we are using the sample data to estimate the variance. See Montgomery (2009) for derivations.

Since we are estimating a range of equally probable values for β_j and using a sum of terms to find the estimators b_j we can call upon the Central Limit Theorem (CLT; see Section A.2 of the Appendix at the end of this chapter) and assume a normal distribution for the error associated with calculating the regression coefficients. Obviously, if the lower bound in the above equation is < 0 and the upper bound is > 0 then the coefficient β_j is said to be statistically no different than zero (at the given confidence level $1 - \alpha$) and we remove it and its regressor data from the regression analysis. This is called **pruning the model**.

If we have an orthogonal design the $(X'X)$ matrix is diagonal as is its inverse. In fact, if we use coded variables for the x-values in an orthogonal design, then each diagonal element of $(X'X)^{-1}$ is equal to $1/n$ where n is the number of data points or experimental runs. The matrix math will not be repeated here. The mathematics is presented in all its glory in Chapter 10 of Montgomery (2009) as well as in regression texts such as Draper and Smith (1998) and Montgomery, Peck, and Vining (2006). Fortunately, computer software does all the hard work and provides this information for the user. More will be said for this analysis in Section 3 of this chapter.

11.2.4 Randomization

Randomization is one of the allies of the experimenter as it allows the analysis process to filter out so-called noise or unknown nuisance variables that might

interfere with understanding the causal effects. It is important to randomize the experiments if time is available and the cost of doing so is not extreme (Montgomery (2009), page 12). Randomization in a factorial design means that all the different combinations of levels being tested for all the factors of interest are chosen in a random manner. This helps reinforce the critical assumption of independence of responses and that trends or systematic noise factors (not included in the design matrix) will become part of the error term by being equally spread over all the experimental conditions (Kirk (2002)). Replication is not always convenient to perform and so many experimenters simply use repetitions instead of replications. This occurs typically in test measurement stages of a product during an assembly process. Repetitions actually measure the variation of the measuring process or instrumentation and do not correctly measure the variability of the product being measured.

11.2.5 Replication

Replication is not repetition. There is a difference between replication and simple repetition of an experiment (see pages 12 and 13 in Montgomery (2009)). By replication we mean an independent repeat of each factor combination. Repetition means to set up the experiment with a given set of IVs and then simply press the button again and again and record the responses as though each run was independent of the previous runs. In general such a repetition procedure leads to correlation of responses and that affects the experimental error. The two most important reasons for replication are that (1) it allows the experimenter to estimate the random error in the experiments, which one needs to know in order to separate out real effects from noise, and (2) if the mean ($\bar{y}$) is used to estimate the true mean response at one of the IV levels, replication helps to obtain a more precise estimate of that mean. For example, if σ^2 is the variance of an individual observation and there are n replicates, the variance of the sample mean is σ^2/n.

11.2.6 Blocking

Blocking is one of the final aids to the experimenter. It is a technique used to improve the precision in estimating differences between factors of interest. See Montgomery (2009), page 13. It helps to reduce so-called known nuisance factors that influence the response of interest but are not under the experimenter's control. For example, if one were testing the compression strength of adobe bricks and had obtained bricks from several different suppliers, he or she would block the experiment on the basis of each supplier. Then supplier effects can be separated from the compression test measurements, as there may have been a difference in brick-forming processes between suppliers. Another way of thinking about all this is to think of a block as a series of tests run with a reasonable homogeneous collection of parts and procedures. How responses are analyzed and how blocks

are treated mathematically in the analyses will be covered in the examples that follow.

11.2.7 Degrees of Freedom

Degrees of freedom (dof) is a recurring term in experimental studies. It is a measure of the number of independent data values present in an analysis, e.g., the mean value of a population is estimated by the sample mean $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and n is the number of data values recorded for the response. The number of dof used in this estimator is n. The sample standard deviation on the other hand is given by $(1/(n-1))\sum_{i=1}^{n}(y_y - \bar{y})$, where now the degrees of freedom are reduced to $n-1$ as one dof is used in determining the mean $\bar{y}$. Both of the above estimators are unbiased Weber and Skillings (2000). Keeping track of dof is important in any analysis of data as it keeps the experimenter from typically underestimating values of the population parameters of interest. The expressions are more complicated for unbiased estimators for the skewness and kurtosis (Alderman and Mense (2007)).

Confounding is the confusing of causal effects in an experiment. If the experimental design is a full factorial then all the main effects (e.g., factors by themselves such as A, B, C) are separated from all pairs of interactions (AB, AC, BC, etc.) and other higher-order interactions. When one reduces the number of experiments for whatever reasons, then one will confuse some factors with others. The measure of this confusion, or "confounding," to use the DOE terminology, is called **resolution**. For example, a resolution IV design will allow an experimenter to separate main effects from all second-order (paired) interactions but second-order interactions are confounded with other second-order interactions and main effects are confounded with third-order interaction.

11.2.8 Example: Full 2^3 Factorial

Consider the simple screening experiment discussed in Section 11.1 in which we are interested in the response (RF gain) of an amplifier as a function of the environment in which it must operate. The test chamber will allow us to set Temperature (A), Humidity (B), and Vibration Amplitude (at a fixed frequency) (C). Three replications were performed on a single amplifier module in this chamber and the results were recorded in an Excel spreadsheet using the software package DOEPro (shown in Table 11.2–3). Y-bar is the average of the responses for the three replications and S is the standard deviation for the three replications.

The resultant Design Sheet from DOEPro is given in Table 11.2–4. This table shows the initial general model assumed by the program and includes the main effects (A, B, C), the two factor interactions (AB, AC, BC) and the only possible three-factor interaction (ABC).

Using this information one can prune the regression equation down by looking at the p-values associated with the coefficients that multiply the factors. Low p-

Table 11.2–3. Actual experiment level values for three factors.

Factor	A	B	C	RF Gain				
Row No.	A Temp	B Humid.	C Vibr. Amp	Y1	Y2	Y3	Y-bar	S
1	30	0.1	3	−60.6	−57.6354	−59.9445	−59.3933	1.557292
2	30	0.1	30	171.6	174.242	173.6223	173.1548	1.381647
3	30	0.9	3	−57.4	−53.6342	−53.0856	−54.7066	2.348618
4	30	0.9	30	174.8	179.2172	182.7192	178.9121	3.968409
5	120	0.1	3	204.9	207.5191	211.4419	207.9537	3.292519
6	120	0.1	30	−364.8	−362.882	−360.17	−362.617	2.326074
7	120	0.9	3	208.1	210.6926	207.7624	208.8516	1.603184
8	120	0.9	30	−361.6	−358.542	−357.235	−359.126	2.240299

Table 11.2–4. Design sheet (called a Regression Table) obtained from DOEPro.

Y-hat model					
		RF Gain			
Factor	Name	Coefficient	P(2 Tail)	Tol	Active
Const		−8.371	0.0000		
A	A Temperature	−67.863	0.0000	1	X
B	B Humidity	1.854	0.0021	1	X
C	C Vibration	−84.048	0.0000	1	X
	Amplitude				
AB		−0.75680	0.1553	1	X
AC		−200.59	0.0000	1	X
BC		0.45805	0.3801	1	X
ABC		0.19038	0.7125	1	X
	R^2	0.9999			
	Adj R^2	0.9999			
	Std. Error	2.4859			
	F	28799.4465			
	Sig F	0.0000			
	F_{LOF}	NA			
	Sig F_{LOF}	NA			
	Source	SS	df	MS	
	Regression	1245834.4	7	177976.3	
	Error	98.9	16	6.2	
	$Error_{Pure}$	98.9	16	6.2	
	$Error_{LOF}$	0.0	0	NA	
	Total	1245933.3	23		

Table 11.2–5. Pruned Model from Table 11.2–4.

Y-hat model					
		RF Gain			
Factor	Name	Coefficient	P(2 Tail)	Tol	Active
Const		−8.371	0.0000		
A	A Temperature	−67.863	0.0000	1	X
B	B Humidity	1.854	0.0018	1	X
C	C Vibration	−84.048	0.0000	1	X
	Amplitude				
AC		−200.59	0.0000	1	X
	R^2	0.9999			
	Adj R^2	0.9999			
	Std. Error	2.4977			
	F	49925.6186			
	Sig F	0.0000			
	F_{LOF}	1.0600			
	Sig F_{LOF}	0.3936			
	Source	SS	df	MS	
	Regression	1245814.8	4	311453.7	
	Error	118.5	19	6.2	
	$Error_{Pure}$	98.9	16	6.2	
	$Error_{LOF}$	19.7	3	6.6	
	Total	1245933.3	23		

values are of course good in the sense that the respective regressor variables contribute to the value of the response in an important way. This figure shows that the BC and ABC factors are of much less importance and the AB factor is also somewhat less important. The AC interaction term does appear to be very important to the model. By removing the X in the row corresponding to the factor, one can drop it from the model. Running the analysis again with these less important factors removed produces the resultant spreadsheet shown in Table 11.2–5. The adjusted R^2 is already very high.

The resultant regression equation can be taken from this sheet, but it requires the use of the coded variables (i.e., A, B, C values must be translated into values between −1 and +1 using the formulas shown in Section 11.1). If the uncoded or actual values are to be used — which is much more convenient to the experimenter — then DOEPro allows for this spreadsheet to be translated into uncoded variables. This is shown in Table 11.2–6.

The regression equation or metamodel we are seeking is given by the following expression, easily derived from the information in Table 11.2–5.

Table 11.2–6. Regression equation coefficients in uncoded form. Note: In uncoded form it is impossible to tell which regressors are the most important.

Y-hat model					
		RF Gain			
Factor	Name	Coefficient	P(2 Tail)	Tol	Active
Const		−203.47	0.0000		
A	A Temperature	3.940	0.0000	0.4010	X
B	B Humidity	4.636	0.0018	1	X
C	C Vibration	18.538	0.0000	0.2647	X
	Amplitude				
AC		−.33019	0.0000	0.1897	X
	R^2	0.9999			
	Adj R^2	0.9999			
	Std. Error	2.4977			
	F	49925.6186			
	Sig F	0.0000			
	F_{LOF}	1.0600			
	Sig F_{LOF}	0.3936			
	Source	SS	df	MS	
	Regression	1245814.8	4	311453.7	
	Error	118.5	19	6.2	
	$Error_{Pure}$	98.9	16	6.2	
	$Error_{LOF}$	19.7	3	6.6	
	Total	1245933.3	23		

$$\begin{aligned}\text{RF gain} = &-203.47 + 3.940 \text{ * Temperature } +4.636 \text{ * Humidity} \\ &+18.538 \text{ * Vibration Amplitude} \\ &-0.33019 \text{ * Temperature * Vibration Amplitude}\end{aligned}$$

The ANOVA chart is shown in the bottom of Table 11.2–6 and it indicates the regression is extremely accurate with regard to fitting the data (see definition in Section A.2 of the Appendix at the end of this chapter). The measure of fitness is typically the R^2 value (or adjusted R^2), although there are other metrics that should be studied, e.g., PRESS, studentized residual plots, Variance Inflation Factor (VIF), and residuals versus actual values and predicted values (of y). Some of these will be discussed later but the best references are again Montgomery (2009) and Box, Hunter, and Hunter (2005).

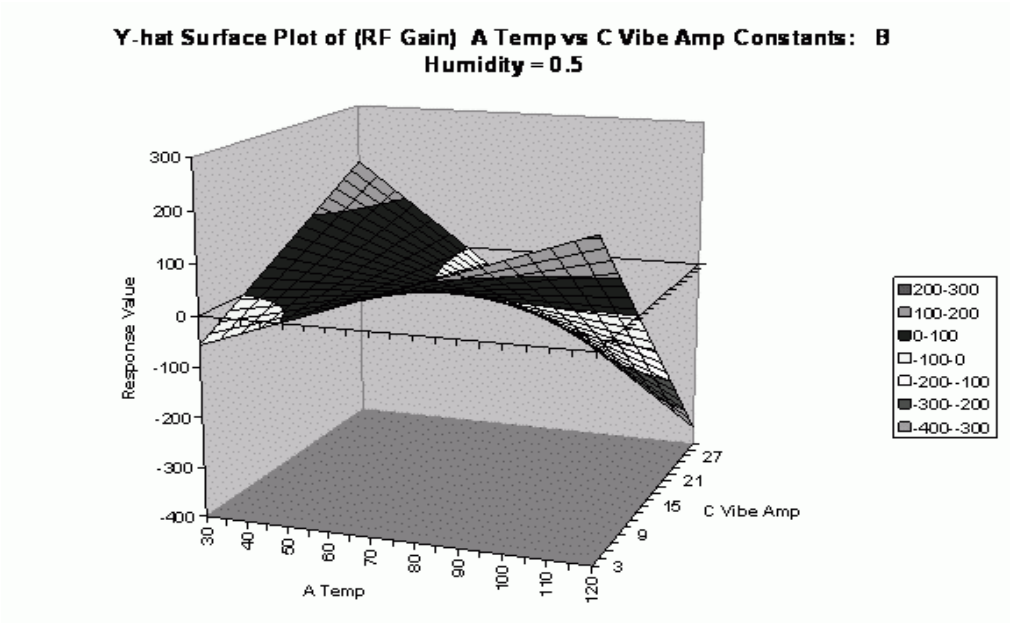


Figure 11.2–7. A surface map that shows for example loss of RF gain as vibration amplitude is increased when the temperature is high.

This software also produces plots (Figure 11.2–7) that show how the mean response varies with the IVs and Figure 11.2–8 shows such a curve with Temperature * Vibration Amplitude as the plotted axes and assumes Humidity is at median value (coded 0). A contour plot of this same figure is sometimes more useful and is shown in Figure 11.2–8.

There is clearly a saddle point around Temp = 55 and Vibe Amp = 11. The maximum RF gain occurs for minimum Vibration amplitude and maximum temperature according to the experimental data. Note that no quadratic terms are involved in these analyses due to the type of design chosen (i.e., 2^3) and thus one had only data values for the corner points. This implies the math model will only allow the response to be flat planes and twisted planes. If one wants to find a maximum or minimum, then a CCD design would be needed. CCD designs have three to five levels for each factor.

Further analyses may be performed by looking at the size of the regression coefficients in the coded space where each variable only ranges from -1 to $+1$ and thus the coefficient value tells the story of which terms are important. DOEPro again produces a chart of coefficient values and this is shown in Figure 11.2–9.

One notes that the interaction term (Temperature * Vibration Amplitude) seems to be the most important term in the regression equation. It is at least twice as important as the Temperature alone or the Vibration Amplitude alone. One could not discern this difference in looking at the regression equation in uncoded space. The coefficients in front of each term do not even have the same

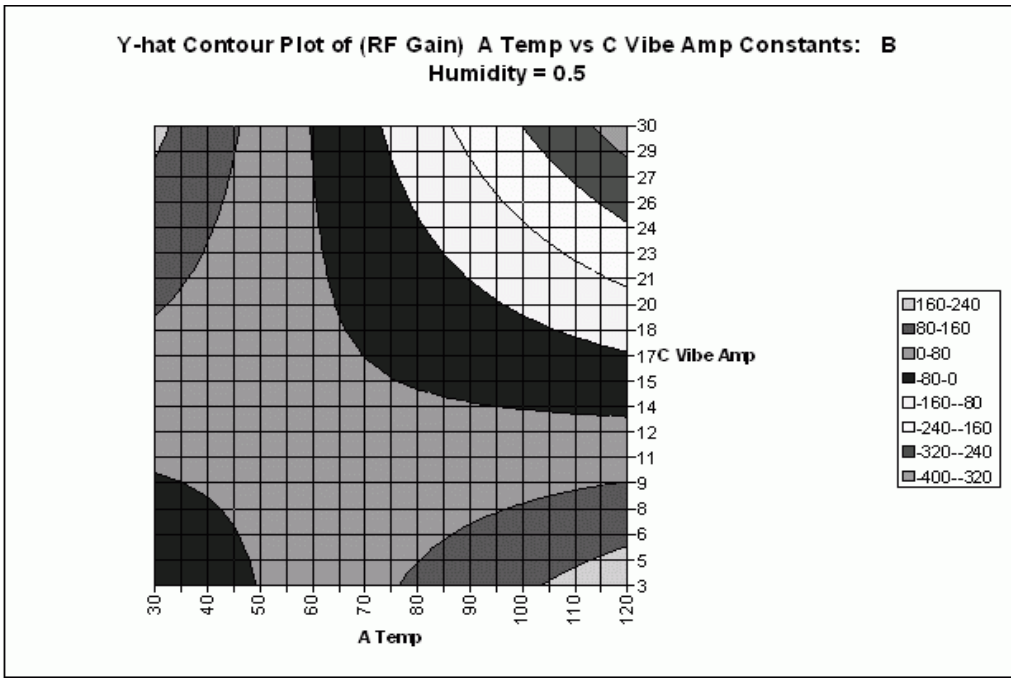


Figure 11.2–8. Contour map of RF gain versus Temperature and Vibration*Amplitude.

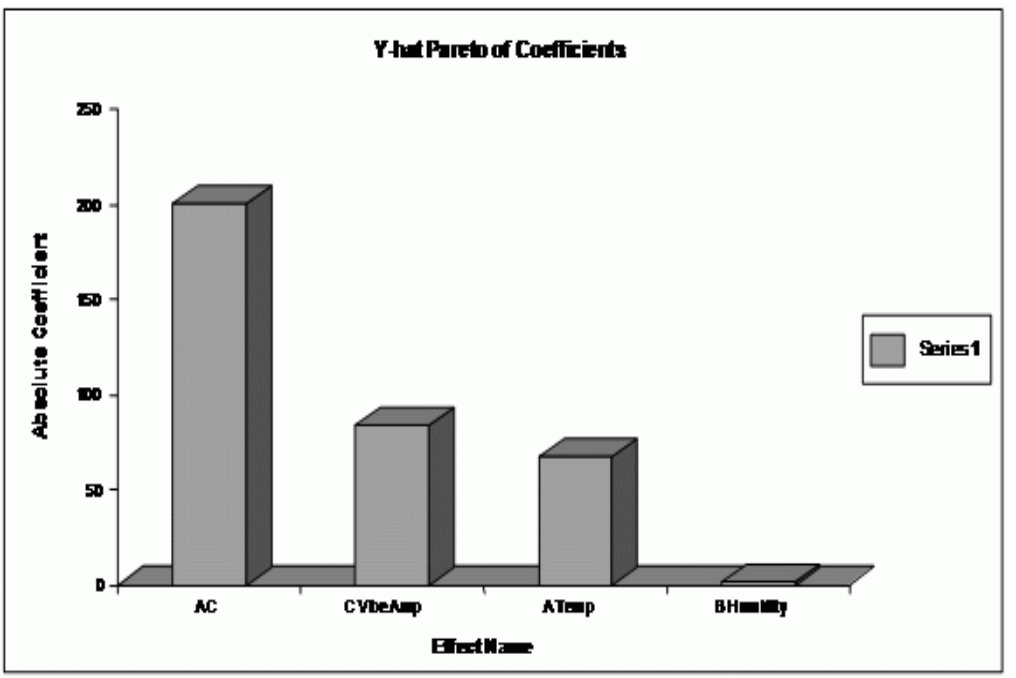


Figure 11.2–9. Coefficients that multiply their respective terms in the regression equation.

units so comparison is impossible. This is why coded variables are used in the analyses.

11.2.9 Summary

We now have a regression equation for the RF gain as a function of what we believe are the important variables, i.e., Temperature, Humidity, and Vibrational Amplitude. We have discovered that Humidity has a very small effect and that some kind of physics inside the amplifier causes the gain to increase or decrease rather dramatically with the product of Temperature * Vibration playing a critical role. This mathematical model alone does not explain why the RF gain changes so much, but it does represent the data accurately. So if this kind of effect is unacceptable, the engineers need to go back to the table and try to understand the physics to design out this effect.

A final note to the reader: We have spent some time demonstrating the full factorial 2-level design. This is done because the 2^k and 2^{k-r} experiments are probably the most useful designs available for most designed experiments during the screening phase of the design. The two-level experiments also have some very important properties as discussed in some detail by Mee (2009) and also by Montgomery (2009).

11.2.10 Example: Central Composite Designs (CCD)

A somewhat more complex experiment that requires the use of a CCD model is the following. For this example we will use Minitab 15™, which is a readily available, general-purpose statistical analysis code priced competitively with other codes on the market today. We have found it to be very good for most DOE applications.

Consider the following problem in chemical engineering process assessment: The goal is to maximize the yield of a certain chemical, and the process variables for which the experimenter has control are time and temperature in the crucible, x_1 and x_2, respectively.

The following CCD experiment was performed and the results from a single replicate recorded. The CCD design uses a 2^2 design and adds five center points and four star points, each at $\pm n^{1/4} = \pm 1.414$. The design is shown below. The time and temperature were set according to the Table 11.2–10 and the yield was recorded.

The low p-values in the output from Minitab 15™, given below, values indicate the coefficients are significant with the interaction term being right at the borderline between accept and reject. We have kept it in this analysis.

```
Response Surface Regression: y1(yield) versus x1(time), x2(temp)
The analysis was done using uncoded units.
Estimated Regression Coefficients for y1(yield)
Term Coef SE Coef T P
```

Table 11.2–10. Running the CCD design under the Response Surface option in Minitab 15 gives the following results for the yield.

x1(time)	x2(temp)	y1(yield)	x1(coded)	x2(coded)
80.00	170.00	76.5	−1.000	−1.000
80.00	180.00	77.0	−1.000	1.000
90.00	170.00	78.0	1.000	−1.000
90.00	180.00	79.5	1.000	1.000
85.00	175.00	79.5	0.000	0.000
85.00	175.00	80.3	0.000	0.000
85.00	175.00	80.0	0.000	0.000
85.00	175.00	79.7	0.000	0.000
85.00	175.00	79.8	0.000	0.000
92.07	175.00	78.4	1.414	0.000
77.93	175.00	75.6	−1.414	0.000
85.00	182.07	78.5	0.000	1.414
85.00	167.93	77.0	0.000	−1.414

```
Constant -1430.69 152.851 -9.360 0.000
x1(time) 7.81 1.158 6.744 0.000
x2(temp) 13.27 1.485 8.940 0.000
x1(time)*x1(time) -0.06 0.004 -13.630 0.000
x2(temp)*x2(temp) -0.04 0.004 -9.916 0.000
x1(time)*x2(temp) 0.01 0.005 1.878 0.103

S = 0.266290 PRESS = 2.35346
```

The regression equation is given by

$$Yield = -1430.69 + 7.80887x_1 + 13.2717x_2 - 0.05506x_1^2 - 0.04005x_2^2 + 0.1x_1x_2,$$

where x_1 is time in natural units and x_2 is temperature in natural units. Clearly, the quadratic terms are important, as is seen both from the previous output and from the ANOVA output below.

```
Analysis of Variance for y1(yield)
Source DF Seq SS Adj SS Adj MS F P
Regression 5 28.2467 28.2467 5.64934 79.67 0.000
 Linear 2 10.0430 6.8629 3.43147 48.39 0.000
 Square 2 17.9537 17.9537 8.97687 126.59 0.000
```

Interaction 1 0.2500 $b*' X_c' X_c b* /(k\sigma^2)$ 0.2500 0.25000 3.53 0.103

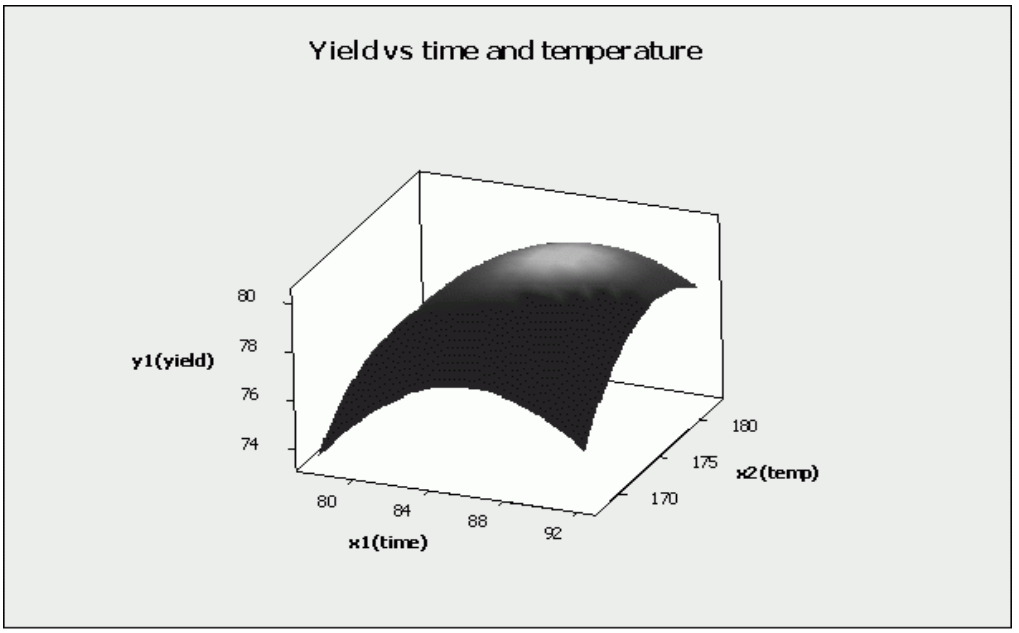


Figure 11.2–11. A contour plot of y versus x_1 and x_2.

Visualizing the response will help to understand how an optimization comes about, and this is shown in Figure 11.2–11.

This plot may help as one can clearly see the optimization region just into the right-upper quadrant of the contour plot. Most software will perform an optimization search using the resultant regression equation. A simple optimization by Minitab finds $x_1 = 86.93$, $x_2 = 176.5$ as a global optimum with $y = 80.2953$. The values $x_1 = 86.94430$, $x_2 = 176.543$ can also be found by differentiating y with respect to x_1 and x_2, respectively, and setting the resultant two equations to zero and solving. This procedure may also be carried out using software such as Maple$^{\text{TM}}$, and the script for doing this is

```
> y:=-1430.69+7.80887*x1+13.2717*x2-0.05506*x1^2
-0.04005*x2^2+0.01*x1*x2;}
```

$$y := -1430.69 + 7.80887x1 + 13.2717x2 - 0.05506x1^2 - 0.04005x2^2 + 0.01x1x2$$

```
> R0 := diff(y, x2);
```

$$R0 := 13.2717 - 0.08010x2 + 0.01x1$$

```
> R := diff(y, x1);
```

$$R := 7.80887 - 0.11012x1 + 0.01x2$$

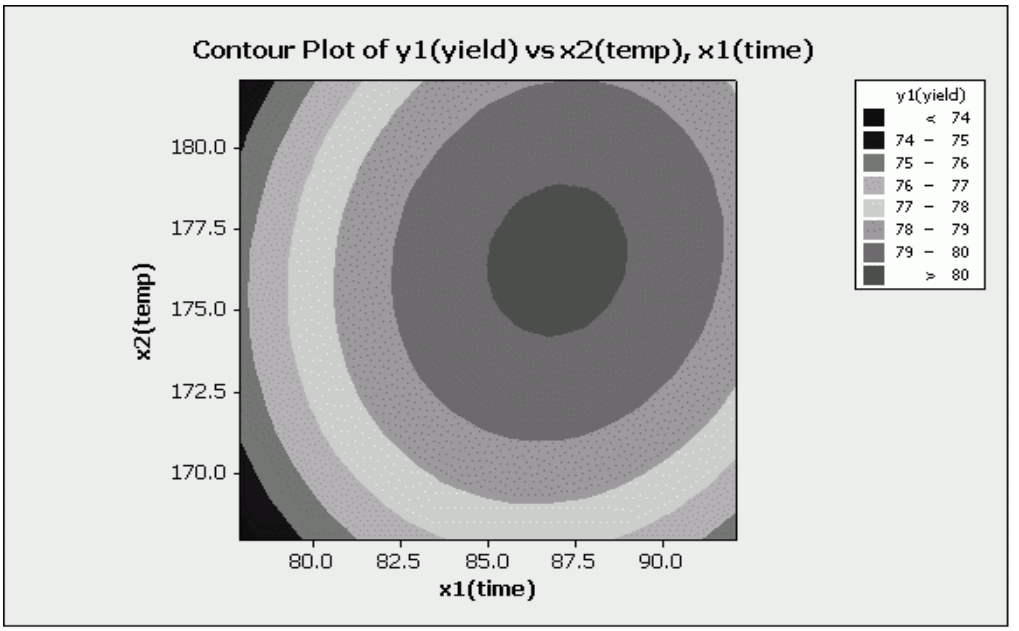


Figure 11.2–12. Contour plot of $|y|$.

```
> solve({R0,R},[x1,x2]);
```

$$[[x1 = 86.94429783,\ x2 = 176.5436077]]$$

The actual contour plot of y is shown in Figure 11.2–12, to illustrate another useful feature of the Minitab software (it is in color, of course). The region of the maximum is shown as the center darker grey (almost black).

This example shows how one can introduce higher-order polynomials into the regression equation, but to do so requires additional levels for the factors. The CCD design was chosen for its accuracy and because the design is rotatable (see Montgomery (2009)), which means the variance is the same for the response independent of the direction one moves from the center of the design space. To obtain a rotatable design one chooses the star points (in coded space) to be at $\pm(n_F)^{1/4}$ where n_F is the number of data values in just the 2^k part of the CCD design. Sometimes CCD designs are called **Box-Wilson designs**.

11.2.11 Taguchi Designs

Even though Taguchi designs (Taguchi and Konishi (1987)) may not be the most efficient (smallest number of runs with lowest variance), they are an innovative parameter design approach and deserving of attention. Taguchi designs are used often to produce so-called robust designs (Schmidt and Launsby (2003), Myers and Montgomery (1995)) to determine factor settings (levels) that achieve the

Table 11.2–13. Design Matrix for Taguchi design L8.

Factors and Interactions								Replicated			
								Responses (mileage)			average
Run	A	B	AB	C	AC	BC	D	a(rain)	b(snow)	c(dry)	mileage
1	−1	−1	1	−1	1	1	−1				
2	−1	−1	1	1	−1	−1	1				
3	−1	1	−1	−1	1	−1	1				
4	−1	1	−1	1	−1	1	−1				
5	1	−1	−1	−1	−1	1	1				
6	1	−1	−1	1	1	−1	−1				
7	1	1	1	−1	−1	−1	−1				
8	1	1	1	1	1	1	1				

desired response (e.g., attain a specific target value) while simultaneously minimizing the variability introduced by the uncontrollable or noise variables (Myers and Montgomery (1995)). Using this technique, Taguchi does not feel a need to emphasize randomization of experimental runs, but it never hurts. At each value of a control variable, Taguchi makes a separate evaluation of the effect of noise and then uses the concept of signal-to-noise ratio to find the best design.

Signal to noise is written in the traditional manner as $(S/N) = 10\log_{10}(\bar{y}^2/S^2)$ where $\bar{y}$ is the average of the response measurements and S^2 is the variance of y. Taguchi designs want to maximize (S/N). A comparison between classical and Taguchi approaches is clearly stated in Section 6.5 of Schmidt and Launsby (2003).

An example might be to find settings of factors A, B, C, and D that maximize gas mileage in your car while also making it mileage insensitive (as much as possible) to weather conditions. This might be done by running the basic DOE during, say, three different weather conditions: 1) rainy (a), 2) snowy (b), and 3) dry (c). An example of such a design matrix is shown in Table 11.2–13.

Note that one has three replicates of each run, one run under each of three environmental conditions. By minimizing the response variability over different environmental conditions, you find a process that makes the mileage less sensitive to the environment variability. The array including the factors and their products is called the "inner array" and the responses as recorded under the a, b, c columns make up the "outer array."

Taguchi based his strategy on three concepts (Schmidt and Launsby (2003)). First, they were to use orthogonal arrays based on fractional factorials (Montgomery (2009), Plackett-Burman (Plackett and Burman (1946)) or Latin Square (Schmidt and Launsby (2003)) designs; second, these designs would be modified to include information on the noise factors; and third, the designs would use the concept of signal to noise as a single measure of merit. His use of the signal-to-noise ratio as a one-parameter measure of goodness of design has come under

criticism. See Montgomery (2009) and his references for an excellent review of those concerns and how they were answered.

To produce a robust design, one in theory would find all the first derivatives of the response with respect to the values of the noise variables. These derivatives would of course be evaluated at the nominal operating values of the fixed independent variables. Now one wishes to vary those independent variables until they find the smallest first derivative with respect to the noise variables. It may be impossible to find these derivatives, so using finite differences between responses under various noise conditions is what is done.

Taguchi's philosophy uses the concept of so-called loss functions for quality improvement. Consider the following simple loss function of the form

$$L = \frac{k}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 + k\left(\bar{y} - T\right)^2 \sim \left(\sigma_y^2 + (\bar{y} - T)^2\right).$$

Traditionally, one strove only to reach the target value (T) for the response and ignored the variability. Now Taguchi uses as a measure the minimization of the sum of effects of missing the target value and the variance of the process about its mean. Even though one may not be right-on the target, i.e., $\bar{y} = T$, there may be some other value of $\bar{y}$ that results in a sufficiently lower variance that L has a minimum somewhere other than T, yet is sufficiently close to T to provide the required performance. An example of a Taguchi L8 design is shown below and hopefully will clarify the design concept. This example is presented with permission from Air Academy Associates (Schmidt and Launsby (2003), see also Myers, Montgomery, and Anderson-Cook (2009), Chapter 10 for good general discussion).

11.2.12 Example

The problem is minimizing the number of solder defects in a wave soldering process. Parts are inserted into a printed circuit board and, after some trimming of leads, the board is preheated to a prescribed temperature to prevent the solder from creating too much mechanical stress on the foils. The board is then put on a conveyor that takes the board through a solder bath that has waves of solder at a fixed wave height. After the soldering bath, the board is run through a flux cleaning process at a given temperature. These factors can all be varied; however, there is some uncontrollability. In particular, there is variability associated with the conveyor speed and the wave height. The experimental results are summarized in Table 11.2–14, where the variables are: Sol (solder pot temperature in ^{o}F), Conv (conveyor belt speed in ft./min., Flux (flux density ^{o}F), Preheat (board preheat temperature ^{o}F), Wave Ht (wave height of solder in inches). The STD column shows how a design matrix laid out the runs but the column labeled Run is the order of the actual experiments using a random number generator to decide an order of runs. Table 11.2–15 gives the results of the full Taguchi L8 model.

Table 11.2–14. Summary of experimental results.

STD	Run	Sol	Conv	Flux	Preheat	Wave Ht	Response Defects
3	1	480	10	1	200	.5	252
5	2	510	7.2	1	150	.6	244
2	3	480	7.2	.9	150	.5	145
1	4	480	7.2	1	200	.6	305
7	5	510	10	.9	200	.6	135
8	6	510	7.2	.9	200	.5	85
6	7	510	10	1	150	.5	215
4	8	480	10	.9	150	.6	195

Table 11.2–15. Full Taguchi L8 model, Note: Conveyer speed (B) and Preheat Temperature (D) do not account for many of the defects over the range of values they were varied in this set of experiments. This model is orthogonal in coded space.

Y-hat Model					
		Response No. 1			
Factor	Name	Coefficient	P(2 Tail)	Tol	Active
Const	·	197.06	0.0000	·	·
A	sold	−27.563	0.0006	1	X
B	con	2.063	0.7707	1	·
C	flux	56.813	0.0000	1	X
D	preheat	−2.563	0.7174	1	·
E	wave	22.813	0.0031	1	X
·		·			·
·	R^2	0.7800			·
·	Adj R^2	0.7377			·
·	Std Error	39.6093			·
·	F	18.4327			·
	Sig F	0.0000			·
	F_{LOF}	0.1792			
	Sig F_{LOF}	0.8370			
	Source	SS	df	MS	
	Regression	144594.6	5	28918.9	
	Error	40791.3	26	1568.9	
	ErrorPure	40191.0	24	1674.6	
	ErrorLOF	600.3	2	300.1	
	Total	185385.9	31		

Table 11.2–16. Pruning the Taguchi design reduces the number of primary factors.

Y-hat Model					
		Response No. 1			
Factor	Name	Coefficient	P(2 Tail)	Tol	Active
Const	·	197.06	0.0000	·	·
A	sold	−27.563	0.0004	1	X
C	flux	56.813	0.0000	1	X
E	wave	22.813	0.0022	1	X
·		·			·
·	R^2	0.7781			·
·	Adj R^2	0.7543			·
·	Std Error	38.3301			·
·	F	32.7273			·
·	Sig F	0.00000			·
·	F_{LOF}	0.9651			·
·	Sig F_{LOF}	0.1413			·
·		·			·
·	Source	SS	df	MS	·
·	Regression	144248.4	3	48082.8	·
·	Error	41137.5	28	1469.2	·
·	$Error_{Pure}$	40191.0	24	1674.6	·
·	$Error_{LOF}$	946.5	4	236.6	·
·	Total	185385.9	31	·	·

The pruned model appears in Table 11.2–16 along with the results of an optimization procedure (embedded in DOEPro, and shown as Table 11.2–17) that produces the minimum number of defects.

The Taguchi design, by performing four replicates of the eight different runs, should produce a more robust design with respect to the noise variables (conveyor speed tolerance and solder temperature tolerance). See Chapter 6 of Schmidt and Launsby (2003) for useful details. The method Taguchi uses forms the ratio S/N and S = mean value at given set of factor values and $N = \sum y_i^2$. When this is done, the results of Table 11.2–18 are obtained.

Table 11.2–18 indicates that factors B and D have no effect on the number of solder defects and the level settings for (A, C, E) are respectively (510, 0.9, 0.5), which is as expected since there are no quadratic terms in the model. Thus, minima and maxima occur at the boundaries of the tested region.

Finally, it is useful to look at what are called **marginal means plots**. An example is shown in Figure 11.2–19. The value of this plot to the user is its ability to show quickly the effects of the various regression variables (factors and levels) on the response. The reason for the name, marginal, is that the regression

Table 11.2–17. Optimized Values (minimized defects) occur at the factor values shown above. Note broad confidence interval due to small number of runs.

Factor	Name	Low	High	Exper
A	sold	480	510	510
B	con	7.2	10	8.806985605
C	flux	0.9	1	0.9
D	preheat	150	200	154.6028836
E	wave	0.5	0.5	0.5

Multiple Response Prediction

95	Percent	Confidence	Interval	
Y-hat	S-hat	Lower Bound	Upper Bound	
Response No. 1	89.8750	38.3765	13.122	166.628

Table 11.2–18. Reproduction of Table 6.4 of Schmidt and Launsby (2003) with permission of Air Academy Associates.

Parameters	Level	Mean # Defects	S/N
(A) Solder pot temperature	480	225	−46.87
	510	170	−44.17*
(B) Conveyor Speed	7.2	195	−45.17
	10.0	200	−45.87
(C) Solder cleaning flux density	0.9	140	−42.91*
	1.0	255	48.11
(D) Preheat temperature	150	200	−46.03
	200	194	-45.01
(E) Wave heights	0.5 inches	174	−44.50*
	0.6 inches	220	−46.54

* These are optimum factor levels of each significant factor.

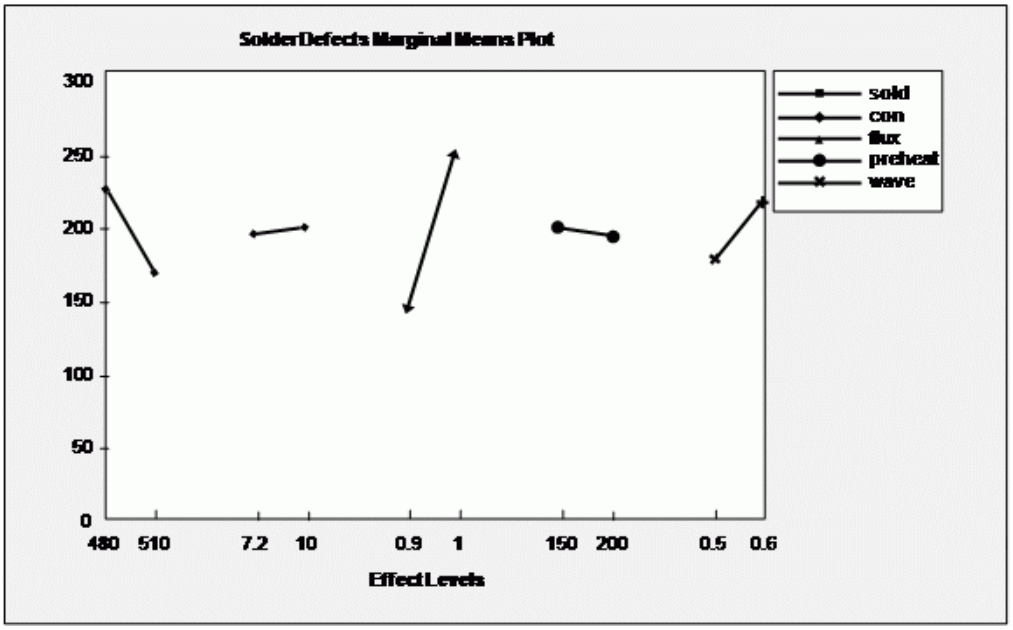


Figure 11.2–19. Marginal Means plot for Taguchi wave solder design.

variable of interest is taken at the low (−1) then high (+1) level while holding all the other regressors at their mean value (zero in coded space). This agrees, as it should, with the coefficient plot that is given in Figure 11.2–20, where one can see that the solder flux temperature has the largest effect by a factor of two.

11.2.13 Summary of Taguchi Design

This type of design is one pattern of experiments that allows for the effect of noise variables to be explicitly included in model decision-making. Taguchi designs are usually orthogonal, but do not depend much on that property because of de-emphasis of interactions (Schmidt and Launsby (2003), Section 6.5). Randomization of testing is de-emphasized. The hardest factors to change (reset) are the column vectors and the easiest to change are assigned to the outer matrix as row values. This makes these designs vulnerable to confounding of unsuspecting noise factors, but if important controllable and noise factors are included in the design, then the confounding problem is solved. Hypothesis testing is not emphasized in Taguchi analysis; instead, a graphical analysis or the S/N ratio is used. True optimal responses are not found; rather settings for the best response are based on S/N ratios.

Taguchi design method appeals to most engineers because of the tabled orthogonal arrays and marginal means analyses. It is obviously an easy way to get a design started. Eventually, however, most engineers will find the need to go to

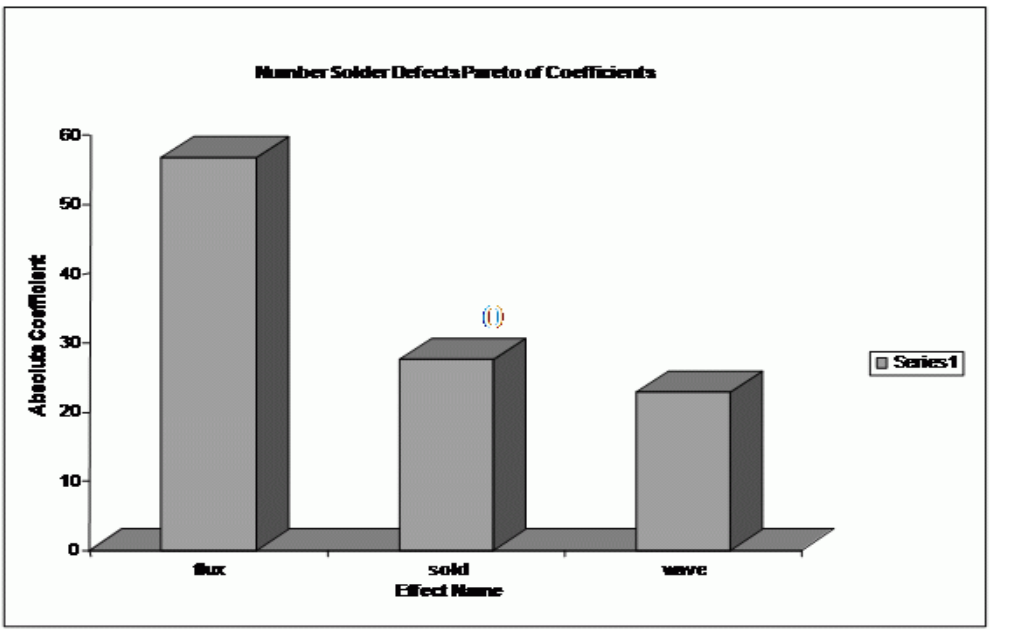


Figure 11.2–20. Regression/Factor coefficient Pareto plot.

more sophisticated designs.

11.2.14 Latin Hypercube Sampling

(Contributed by Terril Hurst, Ph.D., Raytheon Missile Systems) Latin Hypercube Sampling (LHS) is a type of experimental design that is used for statistical sampling. LHS generalizes the concept of a Latin Square (Keppel and Wickens (2004)) to an arbitrary number of dimensions. A Latin Square fills an $n \times n$ table with n symbols, each occurring exactly once in each row and exactly once in each column. The design matrix X that is produced using LHS has m rows and k columns, where m is the number of levels for each of k factors. Each factor's column in X is constrained to use each level exactly once. The size of an LHS design matrix thus grows much more slowly with k than more conventional experimental designs (e.g., factorial, fractional factorial or Central Composite Designs). LHS was created by computer analysts who were looking for efficient, high-dimensional sampling strategies (McKay, Beckman, and Conover (1979)).

LHS is sometimes confused with orthogonal Latin Hypercube Designs, such as the one published by Butler (2001). Unlike Latin Hypercube Designs, LHS produces near-orthogonal designs. The degree of LHS nonorthogonality is measured by the condition number of XTX, where X is an $m \times k$ design matrix, with all factors scaled between -1 and $+1$. Whereas XTX for an orthogonal design has a condition number of exactly 1, a design matrix that is generated using LHS

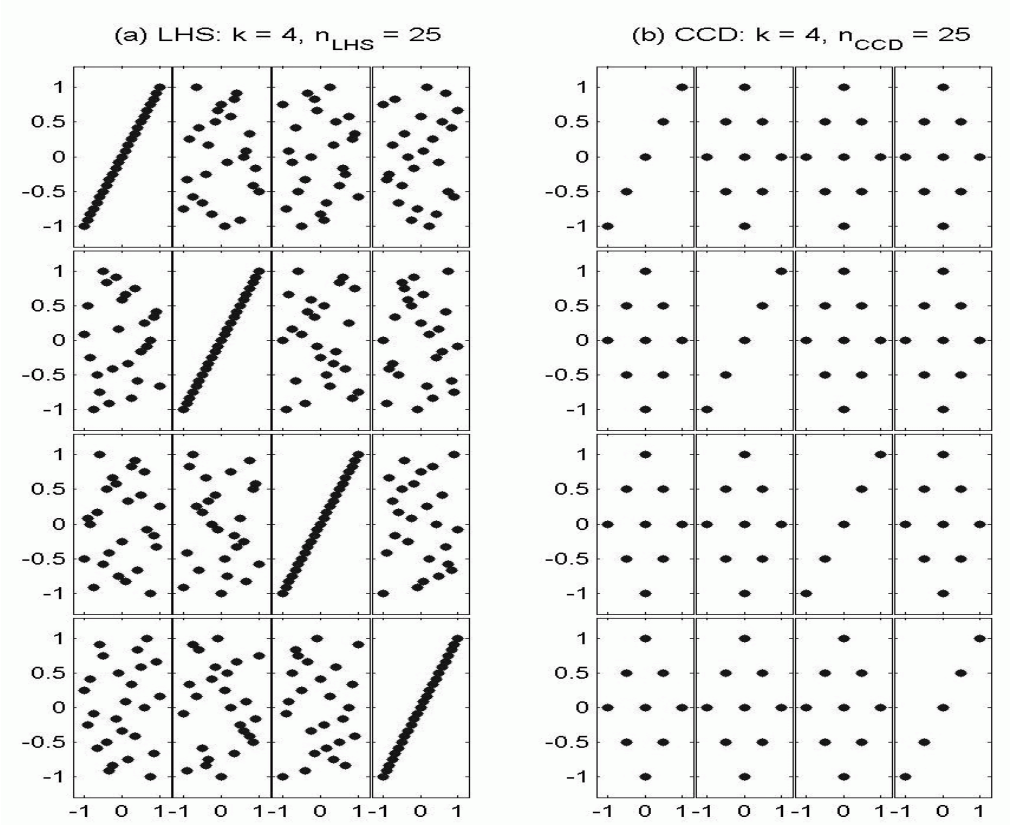


Figure 11.2–21. Comparison of four-dimensional hypercube coverage for LHS vs. CCD.

can only approach a value of 1 as the number of levels (rows) increases (Cioppa (2007)).

LHS designs can be produced by statistical software packages such as JMP and general-purpose mathematical packages such as MATLAB®. The MATLAB command and syntax for an example LHS design is

```
X = lhsdesign(m, k, criterion, correlation, iterations, 250);
```

The criterion parameter can be none if no iteration is desired beyond initial, random-level assignments; maximin, in order to achieve evenly spaced points within the hypercube (the default); or correlation, to minimize correlation of the design matrix.

LHS is characterized as a space-filling design. Figure 11.2–21 illustrates this attribute, comparing the two-dimensional subspace projections of 25 sample points in a 25-level, 4-factor LHS with a 25-point, 5-level, 4-factor Central Composite Design (MATLAB's command `plotmatrix(X, X)` was used to generate Figure 11.2–21). Even though the sample size is the same for each design, the CCD appears sparser due to sample points being regularly stacked within each 2-D subspace projection.

11.3 Analysis Procedures

In this section, some analysis procedures will be discussed that should help answer questions like: Were all potential factors and associated two-way interactions included in the design? Is there too much noise in the design? Was the appropriate number of levels selected for each factor? Was there the appropriate range for each variable? How was the number of runs (e.g., the sample size) decided upon? What is the power of the design (i.e., ability to discern Type II errors)? These and other pesky questions will be addressed below. First, there is a quote from Professor Russell V. Lenth (Lenth (2006–2009)) admonishing the user of his software that one must:

Put science before statistics. "It is easy to get caught up in statistical significance and such; but studies should be designed to meet scientific or engineering goals, and you need to keep those in sight at all times (in planning and analysis). The appropriate inputs to power/sample-size calculations are effect sizes that are deemed clinically important, based on careful considerations of the underlying scientific (not statistical) goals of the study. Statistical considerations are used to identify a plan that is effective in meeting scientific goals — not the other way around."

Conduct pilot studies. "Investigators tend to try to answer all the world's questions with one study. However, you usually cannot do a definitive study in one step. It is far better to work incrementally. A pilot study helps you establish procedures, understand and protect against things that can go wrong, and obtain variance estimates needed in determining sample size. A pilot study with 20–30 degrees of freedom for error is generally quite adequate for obtaining reasonably reliable sample-size estimates." The authors would note that very few industrial experiments allow the user to perform 20 or 30 experiments in total and certainly not just for a pilot study. Lenth goes on to say "Many funding agencies require a power/sample-size section in grant proposals. Following the above guidelines is good for improving your chances of being funded. You will have established that you have thought through the scientific issues, that your procedures are sound, and that you have a defensible sample size based on realistic variance estimates and scientifically tenable effect-size goals."

These remarks are not to be taken lightly. But the facts of life many times dictate that to be awarded the funding one has to meet certain cost goals, and unfortunately that conflicts with performing many expensive experiments. This has resulted in the modeling and simulating community working hard to develop simulations (sometimes called **6-dofs** in the space and missile community) that realistically produce solutions that closely align with the experimental results and then draw on these models to act as surrogates for experiments, thereby producing better statistics. If the researcher has neither money for experiments nor simulations that are accurate, then little is left but to sacrifice the level of confidence and the power of the experiments, e.g., one constructs confidence

intervals that are say 50 percent or 60 percent and similarly have the power to detect Type II errors that are very low, e.g., 50 percent or less. This will be demonstrated below in more detail.

11.3.1 How Many Runs?

Given the above discussion, one of the most common questions asked of a statistics consultant, particularly in an industrial setting, is how many experiments should be performed. As discussed above, experiments and even detailed simulations can be costly, so if one requires a regression equation that relates the response(s) of interest to a number of causal variables, how accurately do we need to know the regression coefficients? DOE software calculates a value for each coefficient, isn't that sufficient to use in the regression equation? The answer is yes, having a point estimate for the values of $\{b_0, b_1, \ldots, b_k\}$ is sufficient to calculate the response. However, what if one more piece of data is added — the results of one more experiment? How does this additional information affect the regression coefficient estimators? One would hope that the estimators, i.e., the b coefficients, would remain about the same as their original values calculated without the new piece of data. This is called **having stable coefficients**. It would certainly be disconcerting if the regression formula changed a lot when the results of only one more experiment were added. However, if the design experiment is not well chosen or, more typically, if no design was used for the original experiments, the probability of obtaining an unstable equation may be high. If the coefficients change wildly upon adding an additional piece of data then the regression equation is of questionable value. This then defeats the purpose of finding a regression equation.

Adding more data (hopefully from a designed set of IVs) is always useful and can stabilize the regression equation. This leads one back to the original question of how many experiments are needed.

Answering this turns out to be a rather complicated task, particularly when one considers all the details. However, there are some simple rules of thumb (Ramsey and Shafer (2002)) that one can apply before performing the experiments. The starting point for an early analysis is to consider the regression equation to be the result of performing a calculation of mean values and then applying what is called a **two-sample t-test** or **z-test** using the proposed DOE $(X'X)^{-1}$ matrix. The process for performing a two-sample t-test is described in most statistics books under the topic of hypothesis testing for two samples. See Montgomery and Runger (2006) and Ramsey and Shafer (2002) for details, and it is also discussed in the appendices in Schmidt and Launsby (2003).

The theoretical procedure is to first establish desired levels for Type I (confidence level, $1 - \alpha$) and Type II (power, $1 - \beta$) errors. Then look for sample sizes that fit those criteria. What is sought are ways to calculate the power of a given DOE given a value for α and sample size, n, before the experiments are per-

formed. There are easy ways and complex ways to determine a ballpark answer to this question. Before this is described in detail, it is important to look at the question of post-experimental power analysis. There are powerful forces on both sides of this topic. Thomas (1997) discusses the importance of post-experimental power analyses as many times experiments were not designed. They are simply performed and one needs to assess the ability of the experiment, as performed, to exclude alternate hypotheses.

However, Lenth argues that post-experimental power analysis is not of much value. Lenth discusses this point with his software, called "piface," as repeated below (Lenth (2006–2009)).

Retrospective power (a.k.a. observed power, or *post hoc* power). "You've got the data, did the analysis, and did not achieve a significant result. So you compute power retrospectively to see if the test was powerful enough. This is an empty question. Of course it wasn't powerful enough — that's why the result isn't significant. Power calculations are useful for design, not analysis."

This is a pretty strong statement. As a practical matter, the number of runs is usually determined by how much time and money management (or the customer) is willing to pay for the results. Power and significance are sacrificed all the time in the name of cost cutting.

Many times one must simply live with whatever power the experimental data provides. This is not a best practice in experimental design. In many situations project managers believe they have little choice if they are going to be competitive in bidding a project to keep costs low. Cutting the testing phase is one way to do this but it has usually been based on the past experience of the project manager when no Design Experiment Methods were employed. When designed experiments are used, the project managers need to pay attention to Lenth's admonitions given above. It is also a common belief amongst project managers with whom the authors have worked that the effects of not thoroughly exploring the causal variables do not show up until much later in the product life cycle and can be dealt with at that time, i.e., fix it when it breaks!

The good news is that power and sample size estimates can be made without performing the actual experiments/simulations, as will be seen shortly. To see how a power/sample size analysis is performed, let's consider the following problem.

We have proposed a 2^4 set of screening experiments and wish to look at confidence intervals as part of our analysis. In the initial analyses, one takes the data and calculates the estimators $b_j,\ j = 0, 1, \ldots, k$ and $s^2 = \text{MSE}$. This is standard procedure. From this information and the form of the $(X'X)^{-1}$ matrix, one can calculate the confidence intervals for all the regression coefficients, the mean of the response at any given set of x-values, the predicted value of the next y values at a given set of x-values, and the bounds on the variance, σ^2.

The first confidence interval is the one bounding the response, y. Actually we are looking for confidence bounds on the mean of y given a specific set of x-values,

Table 11.3–1. Balanced screening design.

Design Matrix for 2^4 Design				
Run #	Factor A	Factor B	Factor C	Factor D
1	1	1	1	1
2	1	1	1	−1
3	1	1	−1	1
4	1	1	−1	−1
5	1	−1	1	1
6	1	−1	1	−
7	1	−1	−1	1
8	1	−1	−1	−1
9	−1	1	1	1
10	−1	1	1	−1
11	−1	1	−1	1
12	−1	1	−1	−1
13	−1	−1	1	1
14	−1	−1	1	−1
15	−1	−1	−1	1
16	−1	−1	−1	−1

labeled say $\{\boldsymbol{x}_0\}$. This is denoted by $\mu_{y|x_0}$ and it is determined through the use of a regression equation produced from performing an ANOVA analysis on the data (all this is done for you in most computer codes). We are seeking the upper and lower bounds to a problem stated as follows: $Pr\{LSL \leq \mu_{y|x} \leq USL\} = 1 - \alpha$, where α, called the **significance level**, is the probability that a random set of data will produce a mean response that is above USL or below LSL. The quantity $1 - \alpha$, called the **confidence level**, is the probability that the quantity of interest (the mean response) is between LSL and USL. Let us observe the test matrix for a 2^4 experiment. Note that for each of the factors, we measure the response half the time with that factor at its high value and the other half of the time at its low value. Thus we have a two-sample situation for each factor, as illustrated in Table 11.3–1.

Each effect is determined by an average between eight high and eight low values so random variation effects are less likely and the mean becomes a more robust measure. For each regression coefficient β_j, one can determine the confidence interval (repeated below for easy reference) derived in all texts on linear regression such as Draper and Smith (1998), and Seber (1977). Lenth's free "piface" software can sometimes be helpful (Lenth (2006–2009)). The interval for the jth regression coefficient is given by

$$b_j - t_{\alpha/2,\,n-p}\sqrt{\hat{\sigma}^2 C_{jj}} \leq \beta_j \leq b_j + t_{\alpha/2,\,n-p}\sqrt{\hat{\sigma}^2 C_{jj}},$$

where $C_{jj} = (X'X)^{-1}$.

At any given set of causal variables, labeled by vector, $\boldsymbol{x}_0$, one can also construct the confidence interval for the expected value (i.e., mean) of y the response at $\boldsymbol{x}_0$. That interval can be shown to be

$$\hat{y}(\vec{x}_0) - t_{\alpha/2\,n-p}\sqrt{\hat{\sigma}^2\vec{x}_0'(X'X)^{-1}\vec{x}_0} \le E(y|\vec{x}_0) \le \hat{y}(\vec{x}_0) + t_{\alpha/2\,n-p}\sqrt{\hat{\sigma}^2\vec{x}_0'(X'X)^{-1}\vec{x}_0},$$

where $\hat{y}(\vec{x}_0) = \vec{x}_0'\vec{b}$.

One should specify α or the confidence level $1-\alpha$ before the experiment is performed. If one is interested in simultaneous confidence intervals, then one looks for a joint confidence region which turns out to be an ellipsoid in p-dimensional space determined by

$$\frac{(\vec{b}-\vec{\beta})'(X'X)(\vec{b}-\vec{\beta})}{pMS_E} \le F_{\alpha,\,p,\,n-p},$$

and

$$MS_E = SS_E/(n-p), \quad SS_E = \vec{y}'\vec{y} - \vec{b}'X'\vec{y}.$$

From these equations one finds critical values (LSL, USL) for all the regression coefficients $\beta_j,\ j = 0, 1, \ldots, k$. In principle, an alternative hypothesis can be formulated for each β_j and then this alternate hypothesis can be tested for a Type II error (unfortunately this probability is called β in most textbooks, but it should not be confused with the β regression coefficients). For testing the coefficients, one can use a noncentral t distribution (see Section A.4 of the Appendix at the end of this chapter) or simply a two-sample t-test for the mean of the response.

For the individual coefficients, the calculation for the probability of a Type II error is found from expressions such as those shown below (van Belle (2008)) and shown in Figure 11.3–2.

The equation to find the power $(1-\beta)$ of a design for a fixed sample size n is given by,

$$\mu_0 + Z_{1-\alpha/2}\,\sigma_0\sqrt{\frac{1}{n_0}+\frac{1}{n_1}} = \mu_1 - Z_{1-\beta}\sqrt{\frac{\sigma_0^2}{n_0}+\frac{\sigma_1^2}{n_1}}.$$

This can be put into the following more convenient form if one assumes $\sigma_0 = \sigma_1$, and $n_0 - n_1 = n/2$

$$\Delta = \frac{\delta}{\sigma} = \frac{2\left(Z_{1-\alpha/2} + Z_{1-\beta}\right)}{\sqrt{n}}, \quad \text{or} \quad n = 4\left(\frac{Z_{1-\alpha/2} + Z_{1-\beta}}{\Delta}\right)^2.$$

In some texts Δ is defined as $\delta/(2\sigma)$ so the reader should be careful.

This expression allows one to graph the required sample size for a given confidence level $1-\alpha$ and a given power $= 1-\beta$ vs. the separation ratio, Δ, between the means as stated in the null and alternate hypotheses. One seldom knows the

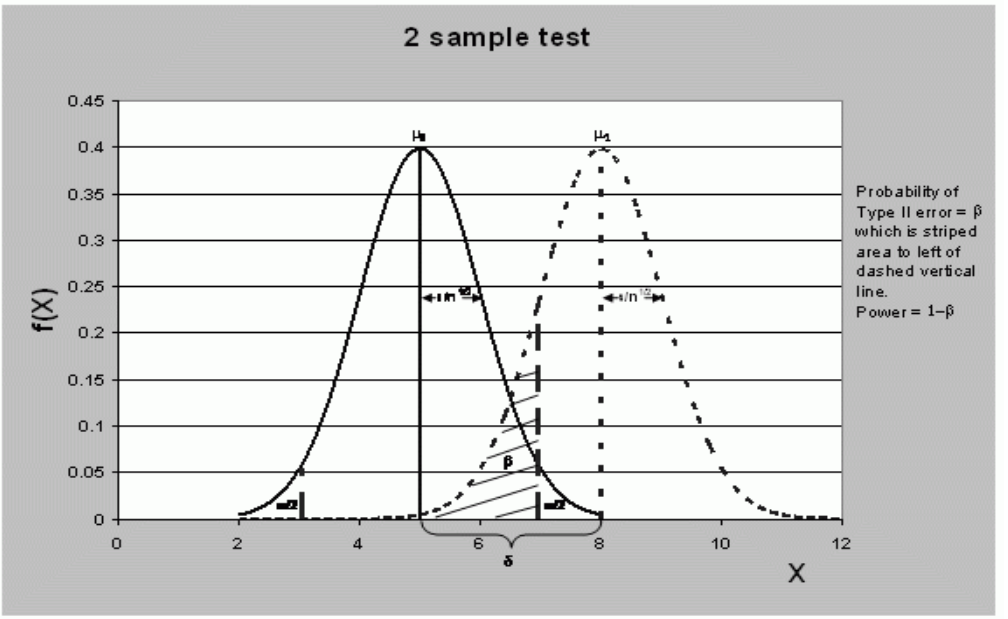


Figure 11.3–2. Graph showing the regions for Type I and Type II errors.

variance of the data before the data is actually taken, so what does one do for assessing the power of a DOE before the experiments are performed? Obviously, there is some guesswork. One can graph the power $1-\beta$ vs. Δ for several possible sample sizes since δ is independent of the standard deviation. Typically Δ ranges between 1 and 2 for most DOEs. Estimates for sample size (actually number of replications) for a given confidence level and power are given in Schmidt and Launsby (2003) in the appendices. For small sample sizes ($n < 30$), one can use t-values instead of the standard normal Z values since the standard deviation will be taken from the resultant data. However, t depends on the number of degrees of freedom, which is $n-p$ where $p = k+1$ is the number of regression coefficients in the fit to the data (which you have not taken). So you can guess p and iterate the equation for sample size to find n for fixed α and β, i.e., for $s_0 = s_1 = s$ and $n_0 = n_1 = n/2$, solve

$$\Delta = \frac{\delta}{s} = \frac{2\left(t_{\alpha/2,\,n-p} - t_{1-\beta,\,n-p}\right)}{\sqrt{n}}.$$

Or, alternatively, use the form below remembering the need to iterate a solution for sample size n

$$n = 4\left(\frac{t_{\alpha/2,\,n-p} - t_{1-\beta,\,n-p}}{\Delta}\right)^2$$

To obtain δ obviously one must estimate s based on historical data or possibly pilot studies.

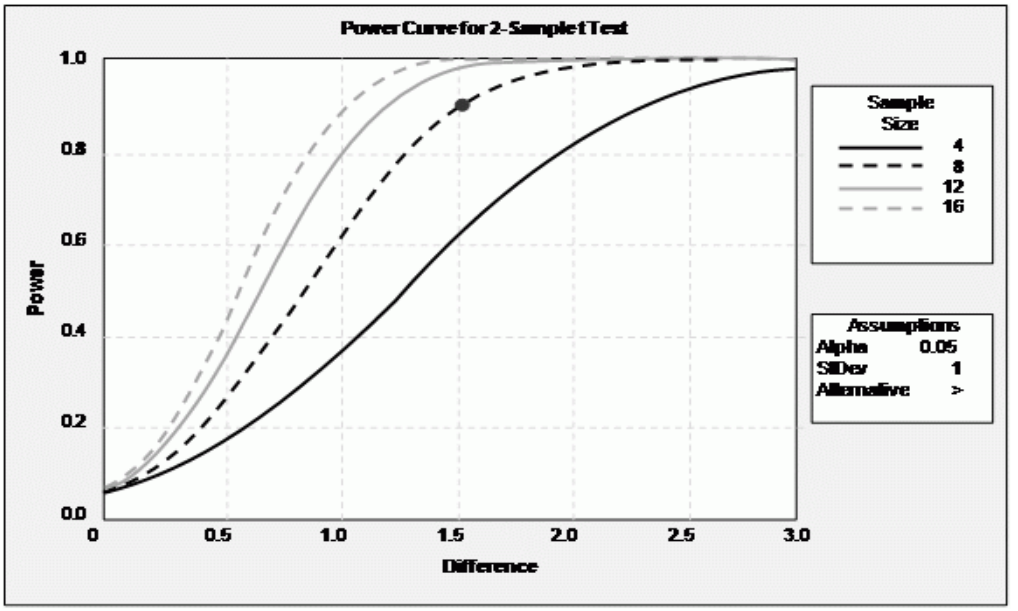


Figure 11.3–3. Plot from Minitab 15 using a standard deviation of 1 so the difference is the value shown in the previous equations.

Remember that in software such as Excel, the TINV function only produces values for a two-tailed test, so TINV($2*\gamma$, $n-p$) will give the t value needed for γ probability in one tail. In Excel, the CDF is measured from high t values to zero for the t-distribution and from lowest ($-\infty$) to the highest values for the normal distribution. This is not true in MATLAB$^{\text{TM}}$. Similarly there are problems using TINV for probabilities > 0.5. Since TINV($2*0.5$, 12) $= 0$ (or close to it) one must use the symmetry of the t-distribution to find $t_{1-b,\,n-p}$, e.g., for $1 - b = 0.95$ the value for $t_{0.95,\,12}$ to use in the above equation is $t_{0.025,\,12} = \text{TINV}(2 * .025,\ 12) = 2.179$ compared to the $Z_{0.975}$ value of 1.96. The word to the wise here is to check what each software program is actually calculating before you use it.

Software such as Minitab 15 produces graphs of power versus difference in mean values. In Figure 11.3–3, a plot shows this for four different sample sizes. Note: the sample sizes in this plot are the sizes of each group (presumed equal) so a DOE with 16 runs would be shown as a sample size of eight in Figure 11.3–3. This figure indicates that for a 16 run balanced two-level design, the sample size per level is eight. The dashed line for sample size eight shows that if we are seeking to differentiate the means from the two samples with a difference of say 1.5 standard deviations, then the probability of detecting this difference (power) is approximately 0.88.

Power of a 2^k design with center points, blocks, and replications. To execute this design one must make use of the noncentral F-distribution (see Section A.3 of the Appendix at the end of this chapter). To understand Minitab's com-

putation for the 2-level Factorial Design, we first establish the following notations and definitions.

- n_r = replicate count
- n_b = block count
- n_c = corner point count
- n_{cb} = center point count (per block)
- α = significance level
- σ = standard deviation
- δ = effect (size of effect is in the units of the variable or use $s = 1$ and then use $\Delta = \delta/\sigma$ as parameter to graph power)
- Θ_b = blocks in model (yes/no)
- ν_{OE} = degrees of freedom omitted effects (number of terms omitted)
- ν_{CP} = degrees of freedom for center points, and is determined by:

  if $n_{cb} > 0$ and Θ_b = yes then $n_b n_{cb} - 1$
  else if $n_{cb} > 0$ then $n_b n_{cb}$
  else 0

- ν_b = degrees of freedom blocks = $n_b - (2^{\text{round}[(n_b - 0.5)/(2n_r)]})$
- ν_{den} = degrees of freedom denominator = $(n_r - 1)n_c + \nu_{OE} + \nu_{CP} - \nu_b$
- fcrit(α, 1, ν_{den}) = critical value for type I error = Fcritical value of α with 1 and degree of freedom in denominator (in Excel use fcrit = FINV(α, 1, ν_{den})
- λ (non-centrality parameter) = $n_r n_c((\delta/(2\sigma))^2$
- power = $1 - \beta = 1-$ non-central F distribution CDF[fcrit, 1, $\nu_{den}|\lambda$]

Figure 11.3–4 gives a graph of the power, $1 - \beta$, based on the last item above for a specific set of parameters. Figure 11.3–5 shows a plot of an F-distribution vs. a noncentral F-distribution including the regions for Type I error (α) and for Type II error (β). This type of calculation is used in Minitab to produce graphs (as shown in Figure 11.3–6) that measure the power of an entire 2^k experiment.

Summary. Power and sample size can be determined using most DOE software programs or by using the approximations in Schmidt and Launsby (2003). It is important to have some rough idea of the power of a DOE before the experiments are undertaken so you do not undertake experiments having high expectations when such experiments cannot render the necessary statistical answers with any

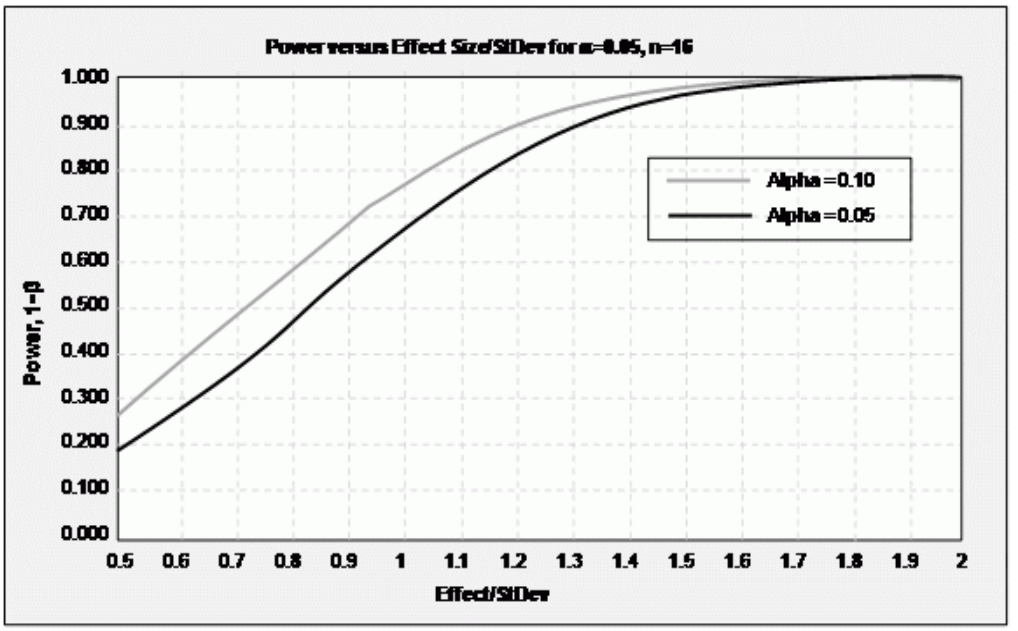


Figure 11.3–4. Graph using the formula for power CDF[fcrit, 1, $\nu_{den}|\lambda$].

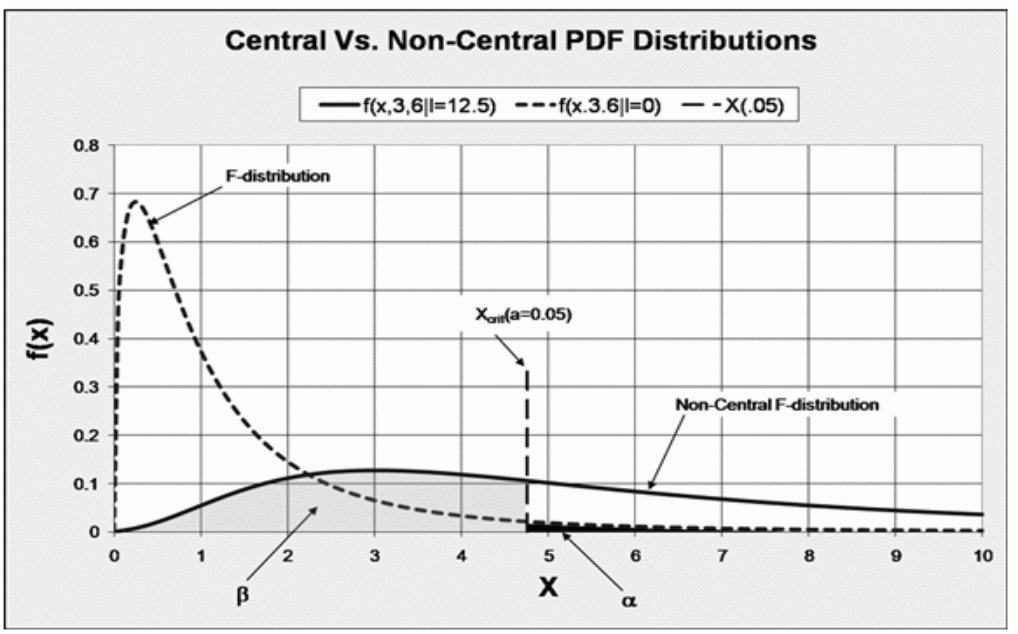


Figure 11.3–5. After determining α X_{crit} allows for the determination of β using the noncentral F-distribution. Above graph uses dof(num)=3 and dof(den)=6 for ease of plotting.

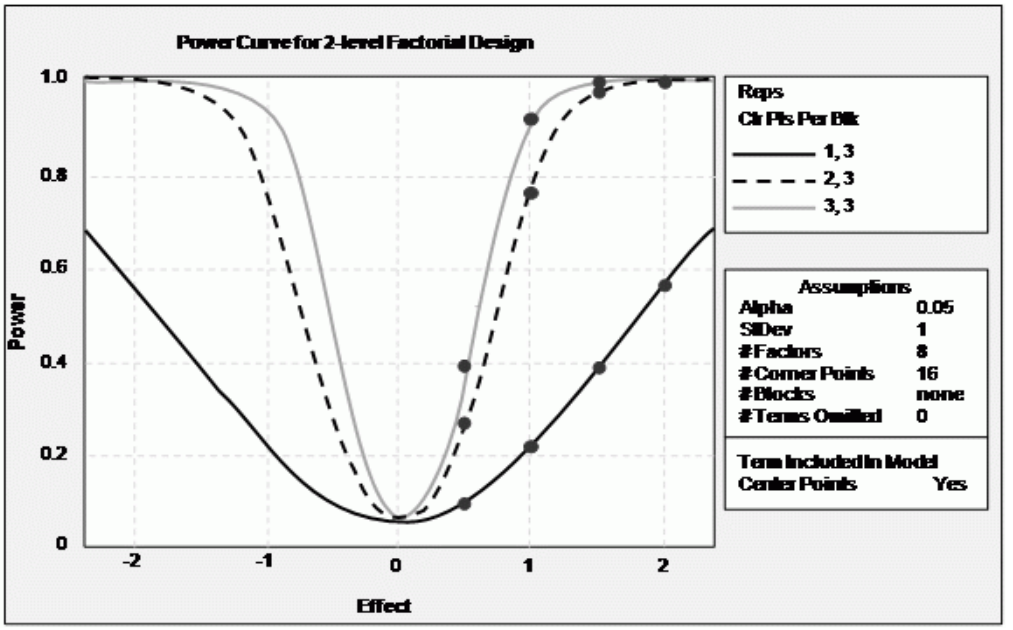


Figure 11.3–6. Curves showing Power versus the Effect (Using Mintab 15).

reasonable confidence. In fact, Keppel and Wickens (2004) warn that power analysis should precede the actual data taking. See Section 8.5 of Keppel and Wickens (2004) for further discussions and Murphy, Myors, and Wolach (2009) for a complete text on the subject.

11.3.2 Other DOE Patterns and their Usage

In this chapter, only three designs have been demonstrated: (1) the 2^k screening design, (2) the central composite design (CCD) that allows for finding minima and maxima, and (3) a Taguchi design that allows for finding the minimum signal-to-noise ratio and thus reduces extraneous noise factors. There are many other designs and in this section some of these are discussed and referenced to the literature.

In the previous material, several designs have been demonstrated including (1) the 2^k screening design, (2) the CCD response surface design, and (3) the Taguchi L8 design. In addition, it was mentioned that one can have a fractional factorial 2^{k-r} design that has the advantage of fewer runs but at the price that some effects are aliased (or confounded) with two or higher factor interactions.

It is recommended by most DOE experts, e.g., Montgomery (2009) and Mee (2009), to always include at least three center points to allow for some measure of pure error (i.e., a model independent estimate for the variance) particularly when running a single replication of a 2^{k-r} design.

There are of course 3^k and 3^{k-r} designs (Keppel and Wickens (2004), Sec-

tion 21), and higher order and Mixed Level designs (Montgomery (2009)). These designs quickly require many more experiments and need to justify their usage compared to using the 2^{k-r} or other designs. Some important designs that were not discussed are the Plackett-Burman (Montgomery (2009) and Plackett and Burman (1946)), Box-Behnken (Montgomery (2009) and Box and Behnken (1960)), and Latin Hypercube (Montgomery (2009)). Each of these designs has its own important applications and the reader is referred to the literature.

There are also sometimes fairly unique usages for DOE and one of these worth mentioning in this article is the use of DOE to determine the parameters for a distribution function when fitting data. Karian and Dudewicz (2003) used a DOE to determine a meta-model for the L^2 norm for fitting data to a Generalized Lambda Distribution (GLD). In brief, the GLD is defined through the quantile function

$$Q(y) = F^{-1}(y) = \lambda_1 + \frac{y_(^{\lambda_3} 1-y)^{\lambda_4}}{\lambda_2}.$$

For $x = Q(y)$, the pdf of the GLD is given by

$$\frac{\lambda - 2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}.$$

There are several methods for finding the four parameters of this distribution (λ_1, λ_2, λ_3, λ_4) and one method, called the **method of moments**, matches the four moments of the data (mean, variance, coefficient of skewness, coefficient of kurtosis) with the analytical expressions for the same four moments of the GLD. Suffice it to say one obtains four equations in four unknowns and the solution must be found numerically (Karian and Dudewicz (2003)). One could also use the so-called L-moments (Hosking (1990)) but they do not seem to be better (Karian and Dudewicz (2003)). Another method is to specify four percentile points and using order statistics one can again find four equations in four unknowns and solve numerically for the lambda coefficients.

To determine which method is best one needs a measure of goodness-of-fit. Karian and Dudewicz chose to use the L^2 norm. Without going into extensive detail, what they did was to construct a customized DOE for two of the four parameters (3 level for one parameter and 9 levels for the second parameter), found a regression equation (meta-model) for the value of the L^2 norm and then showed graphically that the L^2 norm that was produced using the percentile matching method produced a smaller L^2 norm than did the method of matching moments (Karian and Dudewicz (2003)). A DOE design chosen from some of the more conventional patterns may have served them better but they successfully demonstrated their point and that was using the percentile matching method gives the better fit. While there are other methods for finding these parameters (see Alderman and Mense in another chapter in this handbook) this unique use of a DOE for pdf fitting is useful and worthy of additional exploration.

Finally a note to the wise for the user. The first time or two one uses DOE, it is important to discuss the design with an experienced professional. Many users now simply plow ahead without the necessary amount of preparation and sometimes find themselves with less than expected performance from the set of experiments. Most DOE software (see references) lead the user through the steps in a clear, concise manner. Working through a tutorial such as that accompanying Design Expert™ by Stat-Ease is a good way to begin. Short courses are also available from multiple sources. Once the user builds some familiarity with the software they will find the results of a DOE are easily interpreted and more accurate than other methods when applied with some thought and planning.

References for Chapter 11

Alderman, J.L. and Mense, A.T. (2007). "Applications of Johnson PDF fitting," *American Journal of Mathematical and Management Sciences,* Vol. 27, Nos. 3 & 4, 461–478.

Anderson, M.J. and Whitcomb, P.J. (2000). *DOE Simplified,* Productivity Press.

Anthony, J. (2003). *Design of Experiments,* Butterworth-Heinemann.

Boas, M. L. (2008). *Mathematical Methods in the Physical Sciences,* 3rd Edition, John Wiley & Sons.

Box, G.E.P., Hunter, W.G., and Hunter, J.S. (2005). *Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building,* 2nd Edition, Wiley and Sons.

Box, G.E.P. and Behnken, D.W. (1960). "Some three level designs for the study of quantitative variables," *Technometrics,* Vol. 2, 455–475.

Box, G.E.P. and Wilson, K.G. (1951). "On experimental attainment of optimum conditions," *Journal of the Royal Statistical Society,* B, Vol. 13, 1–45.

Butler, N.A. (2001). "Optimal and orthogonal Latin hypercube designs for computer experiments," *Biometrika,* 88(3): 847–857.

Cioppa, T.M. (2007). "Efficient nearly orthogonal and space-filling Latin hypercubes," *Technometrics,* 49, 1 (also at http://harvest.nps.edu/papers/Cioppa.Lucas.pdf).

Draper, N.R. and Smith, H. (1998). *Applied Regression Analysis,* 3rd Edition, Wiley & Sons.

Dudewicz, E.J. and Mishra, S.N. (1988). *Modern Mathematical Statistics,* Wiley.

Golub, G.H. and Van Loan, C.F. (1996). *Matrix Computations*, 3rd ed., Johns Hopkins.

Gujarati, D.N. (2002). *Basic Econometrics,* 4th Edition, McGraw-Hill.

Hadamard, J. (1893). "Résolution d'une question relative aux déterminants," *Bulletin des Sciences Mathématiques,* 17, 240–246. (Book 1899.)

Hayter, A.J. (2002). *Probability and Statistics for Engineers and Scientists,* 2nd Edition, Duxbury Press.

Hogg, R.V. and Tanis, E.A. (1997). *Probability and Statistical Inference,* 5th Edition, Prentice Hall.

Horadam, K.J. (2007). *Hadamard Matrices and Their Applications,* Princeton University Press.

Hosking J.R.M. (1990). "*L*-Moments: Analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society, Series B*, 52, 105–124.

Householder, A. S. (1958). "Unitary triangularization of a nonsymmetric matrix," *Journal of the ACM* 5 (4): 339–342.

Johnson, N.L., Kotz, S., and Balekrishnan, N. (1995). *Continuous Univariate Distributions*, Volume 2 (Second Edition), John Wiley & Sons, Inc., New York.

Kendall, M.G. and Stuart, A. (1967). *The Advanced Theory of Statistics,* Vol. 2, 2nd Edition, Griffin & Co.

Karian, Z.A. and Dudewicz, E.J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society,* v. 2, No. 2, 171–187.

Keppel, G. and Wickens, T.D. (2004). *Design and Analysis: A Researchers Handbook,* 4th Edition, Pearson/Prentice Hall

Kirk, R.E. (2002). *Experimental Design,* Brooks/Cole Publishing Co.

Lenth, R.V. (2006–2009). "Java Applets for Power and Sample Size," (Computer software). Retrieved January 11, 2010, from http://www.stat.uiowa.edu/rlenth/Power.

Mathews, P. (2007). *Design of Experiments with MINITAB,* ASQ Press.

McKay, M.D. Beckman, R.J. and Conover, W.J. (1979). "A Comparison of three methods for selecting values of input variables in the analysis of output from a computer code," *Technometrics* 21 (2), 239–245.

Mee, R.W. (2009). *A Comprehensive Guide to Factorial Two-Level Experimentation,* Springer Science.

Montgomery, D.C. (2009). *Design and Analysis of Experiments,* 7th Edition.

Montgomery, D.C., Peck, E.A., and Vining, G.G. (2006). *Introduction to Linear Regression Analysis,* 4th Edition, Wiley & Sons.

Montgomery, D.C. and Runger, G.C. (2006). *Applied Statistics and Probability for Engineers,* 4th Edition, Wiley & Sons.

Murphy, K.R., Myors, R., and Wolach, A. (2009). *Statistical Power Analysis,* 3rd Edition, Routledge/Taylor & Francis Group.

Myers, R.H., Montgomery, D.C., and Anderson-Cook, C.M. (2009). *Response Surface Methodology: Process and Product Optimization Using Designed Experi-*

ments (Third Edition), John Wiley & Sons, Inc., Hoboken, New Jersey.

Papoulis, A. and Pillai, S.U. (2002). *Probability, Random Variables and Stochastic Processes,* 4th Edition, McGraw-Hill.

Plackett, R.L. and Burman, J.P. (1946). "The design of optimum multifactorial experiments," *Biometrika,* Vol. 33, 305–325.

Ramsey, F.L. and Shafer, D.W. (2002). *The Statistical Sleuth,* 2nd Edition, Duxbury Press.

Schmidt, S.R. and Launsby, R.G. (2003). *Understanding Industrial Designed Experiments,* 4th Edition, Air Academy Press.

Seber, G.A. (1977). *Linear Regression Analysis,* Wiley Interscience.

Stuart, A. and Ord, K. (1994). *Kendall's Advanced Theory of Statistics, Vol. I,* 6th Edition, Edward Arnold Publisher.

Tabachnick, B.G. and Fidell, L.S. (2007). *Using Multivariate Statistics,* 5th Edition, Allyn & Bacon.

Taguchi, G. and Konishi, S. (1987). *Taguchi Methods: Orthogonal Arrays and Linear Graphs,* American Supplier Institute, Inc.

Thomas, L. (1997). "Retrospective power analysis," *Conservation Biology,* Vol. 11, #1, 276–280. (See appendices for methods.)

van Belle, G. (2008). *Statistical Rules of Thumb,* 2nd Edition, Wiley & Sons.

Weber, D.C. and Skillings, J.H. (2000). *A First Course in the Design Experiments,* CRC Press.

References for DOE Software

When selecting DOE software, it *is* important to look for not only a statistical engine that's fast and accurate but also the following:

- A simple user interface that's intuitive and easy to use.
- A well-written manual with tutorials to get you off to a quick start.
- A wide selection of designs for screening and optimizing processes or product formulations.
- A spreadsheet flexible enough for data entry as well as dealing with missing data and changed factor levels.
- Software that randomizes the order of experimental runs. Randomization is crucial because it ensures that noisy factors will spread randomly across all control factors.
- Design evaluation tools that will reveal aliases and other potential pitfalls.

First-rate packages offer sharp graphics for statistical diagnoses and displaying responses. Graphics include residual plots, which confirm a model's statistical validity. Functions such as square root or log base 10 allow users to transform their responses, thus improving the statistical properties of the analysis.

- DOEPro, by Air Academy Associates, single license cost is $249, http://www.sigmazone.com/doepro.htm
- Design-Expert DX-8 by Stat-Ease, single license cost is $995, http://www.statease.com/pubs/dx7brochure.pdf
- Design-Ease DE-7.1 by Stat-Ease, single license cost is $495, http://www.statease.com/prodsoft.html#de_single

Multipurpose Software that Performs DOE Particularly Well

- Minitab 15 by Minitab Inc., single license cost is $1,195, http://www.minitab.com/en-US/products/minitab/default.aspx
- JMP 8 by SAS, single license cost is $1,495, http ://www.jmp.com/software/
- NCSS by NCSS LLC, single license for NCSS is $999.95. Note there is a power and sample size program called PASS and another program for advanced statistical analyses called GESS also sold by this company with package deals on price. http://www.ncss.com/
- Other general use programs worthy of mention that were not used or reviewed in this chapter include SPSS (http://www.spss.com/) recently acquired by IBM, Systat (http://www.systat.com/), as well as SAS (http://www.sas.com/software/), the "mother of all statistics programs," at a cost commensurate with its ability.

All of the above software programs have academic pricing lower than the commercial one-user prices shown above. Although there are freeware programs available from academic or government organizations, a thorough search was not made to find these products.

Appendix for Chapter 11

A.1 Types of DOE Design

Design Type	Comment
CCC	CCC designs provide high-quality predictions over the entire design space, but require factor settings outside the range of the factors in the factorial part. Note: When the possibility of running a CCC design is recognized before starting a factorial experiment, factor spacings can be reduced to ensure that α (star points) for each coded factor corresponds to feasible (reasonable) levels. Requires 5 levels for each factor.
CCI	CCI designs use only points within the factor ranges originally specified, but do not provide the same high-quality prediction over the entire space compared to the CCC. Requires 5 levels of each factor.
CCF	CCF designs provide relatively high-quality predictions over the entire design space and do not require using points outside the original factor range. However, they give poor precision for estimating pure quadratic coefficients. Requires 3 levels for each factor.
Box-Behnken	These designs require fewer treatment combinations than a central composite design in cases involving 3 or 4 factors. The Box-Behnken design is rotatable (or nearly so) but it contains regions of poor prediction quality like the CCI. Its "missing corners" may be useful when the experimenter should avoid combined factor extremes. This property prevents a potential loss of data in those cases. Requires 3 levels for each factor.

Comparison of Number of Runs for Central Composite Versus Box-Behnken Designs

Number of Factors	Central Composite	Box-Behnken
2	13 (5 center points)	–
3	20 (6 center point runs)	15
4	30 (6 center point runs)	27
5	33 (fractional factorial) or 52 (full factorial)	46
6	54 (fractional factorial) or 91 (full factorial)	54

A.2 Glossary of DOE Terminology

- **Alias:** When the estimate of an effect also includes the influence of one or more other effects (usually high-order interactions), the effects are said to be **aliased** (see *confounding*). For example, if the estimate of effect D in a four-factor experiment actually estimates $(D + ABC)$, then the main effect D is aliased with the three-way interaction ABC. Note: This causes no difficulty when the higher-order interaction is either nonexistent or insignificant.

- **Analysis of Variance (ANOVA):** A mathematical process for separating the variability of a group of observations into assignable causes and setting up various significance tests.

  The total sum of squares that measures the variation of the data from the grand mean can be rewritten and broken into pieces such as shown below where $SS_T = SS_R + SS_E$. These sums can be calculated from the $\boldsymbol{X}$ matrix and the $\boldsymbol{y}$ vector. The 1 vector is a column vector of all ones. Lower case $\boldsymbol{b}$ is the vector of estimates for the regression coefficients $\boldsymbol{\beta}$. See Montgomery, Peck, and Vining (2006).

$$\begin{aligned} SS_T &= y'y - N\left(\frac{\mathbf{1}'y}{N}\right)^2, \quad df = N-1, \\ SS_E &= SS_{\text{Residuals}} = (y - X'b)' = y'y - b'X'y, \quad df = N-(k+1), \\ SS_R &= SS_{\text{Regression}} = b'X'y - N\left(\frac{\mathbf{1}'y}{N}\right)^2, \end{aligned}$$

  where for SS_R, $df = k =$ # of regression variables.

  Using the above information, one can provide estimates for the variance of the response. For example, an estimator for the variance would be MSE=SSE/$(n-k-1)$ since E[MSE]= σ^2. However, if the null hypothesis is true and all the regression coefficients are zero, another estimate would be MSR = SSR/k. It can be shown that E[MSR]= $\sigma^2 + \boldsymbol{b}^{*\prime}\boldsymbol{X}_c'\boldsymbol{X}_c b^*/(k\sigma^2)$ where $\boldsymbol{X}_c$ is the X matrix minus the first column of ones and each element of the matrix has the average value of that columns values subtracted (called **mean centering**), the vector $\boldsymbol{b}^*$ is the vector of b values without b_0, i.e., there are k not $k+1$ values in the $\boldsymbol{b}^*$ vector. If the null hypothesis is true (all regression coefficients = 0), then MSR/MSE should distribute as an F distribution. If at least one coefficient $\neq 0$, then this ratio distributes as a noncentral F distribution with noncentrality parameter $\lambda = \boldsymbol{b}^{*\prime}\boldsymbol{X}_c'\boldsymbol{X}_c\boldsymbol{b}^*/\sigma^2$. See Montgomery, Peck, and Vining (2006), pages 80, 81.

- **Balanced Design:** An experimental design where all cells (i.e., treatment combinations) have the same number of observations.

- **Blocking:** A schedule for conducting treatment combinations in an experimental study such that any effects on the experimental results due to a

known change in raw materials, operators, machines, etc., become concentrated in the levels of the blocking variable. Note: The reason for blocking is to isolate a systematic effect and prevent it from obscuring the main effects. Blocking is achieved by restricting randomization.

- **Center Points:** Points at the center value of all factor ranges.

- **Central Limit Theorem (CLT):** The distribution of an average tends to be Normal, even when the distribution from which the average is computed is decidedly non-Normal. Thus, the CLT is the foundation for many statistical procedures, including quality control charts, because the distribution of the phenomenon under study does not have to be Normal because its average will be. Furthermore, this normal distribution will have the same mean as the parent distribution, and, variance equal to the variance of the parent divided by the sample size.

- **Coding Factor Levels:** Transforming the scale of measurement for a factor so that the high value becomes $+1$ and the low value becomes -1 (see *scaling*). After coding all factors in a two-level full factorial experiment, the design matrix has all orthogonal columns. Coding is a simple linear transformation of the original measurement scale. If the high value is X_h and the low value is X_L (in the original scale), then the scaling transformation takes any original X value and converts it to $(X - a)/b$, where $a = (X_h + X_L)/2$ and $b = (X_h - X_L)/2$. To go back to the original measurement scale, just take the coded value and multiply it by b and add a or, $X = b(\text{coded value}) + a$.

  As an example, if the factor is temperature and the high setting is 65°C and the low setting is 55°C, then $a = (65 + 55)/2 = 60$ and $b = (65 - 55)/2 = 5$. The center point (where the coded value is 0) has a temperature of $5(0) + 60 = 60^{o}C$.

- **Comparative Designs:** A design aimed at making conclusions about one *a priori* important factor, possibly in the presence of one or more other nuisance factors.

- **Confounding:** A confounding design is one where some treatment effects (main or interactions) are estimated by the same linear combination of the experimental observations as some blocking effects. In this case, the treatment effect and the blocking effect are said to be confounded. Confounding is also used as a general term to indicate that the value of a main effect estimate comes from both the main effect itself and also contamination or bias from higher-order interactions. Note: Confounding designs naturally arise when full factorial designs have to be run in blocks and the block size is smaller than the number of different treatment combinations. They

also occur whenever a fractional factorial design is chosen instead of a full factorial design.

- **Crossed Factors:** See *factors* below.
- **Dependent Variables:** Called DVs in this chapter. These are the response variables sometimes labeled RVs. They are random variables, i.e., for the same fixed IVs (independent variables) they take on some distribution of random values.
- **Design:** A set of experimental runs that allows you to fit a particular model and estimate your desired effects.
- **Design Matrix:** A matrix description of an experiment that is useful for constructing and analyzing experiments.
- **Effect:** How changing the settings of a factor changes the response. The effect of a single factor is also called a **main effect**. Note that for a factor A with two levels, scaled so that low $= -1$ and high $= +1$, the effect of A is estimated by subtracting the average response when A is -1 from the average response when $A = +1$ and dividing the result by 2 (division by 2 is needed because the -1 level is 2 scaled units away from the $+1$ level).
- **Error:** Unexplained variation in a collection of observations. Note: DOEs typically require understanding of both random error and lack-of-fit error.
- **Experimental Unit:** The entity to which a specific treatment combination is applied. Note that an experimental unit can be a
  - PC board;
  - silicon wafer;
  - tray of components simultaneously treated;
  - individual agricultural plants;
  - plot of land; and
  - automotive transmissions, etc.
- **Factors:** Process inputs an investigator manipulates to cause a change in the output. Some factors cannot be controlled by the experimenter but may affect the responses. If their effect is significant, these uncontrolled factors should be measured and used in the data analysis. Note that the inputs can be discrete or continuous.
  - Crossed Factors: Two factors are crossed if every level of one occurs with every level of the other in the experiment.

  - Nested Factors: A factor A is nested within another factor B if the levels or values of A are different for every level or value of B Note: Nested factors or effects have a hierarchical relationship.

- **Fixed Effect:** An effect associated with an input variable that has a limited number of levels or in which only a limited number of levels are of interest to the experimenter.

- **Hadamard Matrices:** In mathematics, a Hadamard matrix is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal. In geometric terms, this means that every two (different) rows in a Hadamard matrix represent two perpendicular vectors, while in combinatorial terms it means that every two (different) rows have matching entries in exactly half of their columns and mismatched entries in the remaining columns. It is a consequence of this definition that the corresponding properties hold for columns as well as rows. The n-dimensional parallelepiped spanned by the rows of an $n \times n$ Hadamard matrix has the maximum possible n-dimensional volume among parallelepipeds spanned by vectors whose entries are bounded in absolute value by 1. Equivalently, a Hadamard matrix has maximal determinant among matrices with entries of absolute value less than or equal to 1. Certain Hadamard matrices can almost directly be used as an error-correcting code using a Hadamard code (generalized in ReedMuller codes), and are also used in balanced repeated replication (BRR), used by statisticians to estimate the variance of a parameter estimator. Hadamard matrices are named after French mathematician Jacques Hadamard (1865–1963), whose most important result is the prime number theorem that he proved in 1896. This states that the number of primes $< n$ tends to infinity as fast as $n/(\ln n)$. Hadamard Matrices are presented in his original paper (Hadamard (1893)).

- **Independent Variables:** Called IVs in this chapter. These are the causal variables or factors or treatments. They have fixed values called **levels** in designed experiments.

- **Interactions:** Occur when the effect of one factor on a response depends on the level of another factor(s).

- **Lack-of-Fit Error:** Error that occurs when the analysis omits one or more important terms or factors from the process model. Note that including replication in a DOE allows separation of experimental error into its components: lack-of-fit and random (pure) error.

- **Model:** Mathematical relationship that relates changes in a given response to changes in one or more factors.

- **Nested Factors:** See *factors* above.

- **Orthogonality:** Two vectors of the same length are orthogonal if the sum of the products of their corresponding elements is 0. Note that an experimental design is orthogonal if the effects of any factor balance out (sum to zero) across the effects of the other factors.

- **Random Effect:** An effect associated with input variables chosen at random from a population having a large or infinite number of possible values.

- **Random Error:** Error that occurs due to natural variation in the process. Note: Random error is typically assumed to be normally distributed with zero mean and a constant variance. Note that a random error is also called an **experimental error**.

- **Randomization:** A schedule for allocating treatment material and for conducting treatment combinations in a DOE, such that the conditions in one run neither depend on the conditions of the previous run nor predict the conditions in the subsequent runs. The importance of randomization cannot be overstressed. Randomization is necessary for conclusions drawn from the experiment to be correct, unambiguous, and defensible.

- **Replication:** Performing the same treatment combination more than once. Note: Including replication allows an estimate of the random error independent of any lack-of-fit error.

- **Resolution:** A term that describes the degree to which estimated main effects are aliased (or confounded) with estimated 2-level interactions, 3-level interactions, etc. In general, the resolution of a design is one more than the smallest order interaction that some main effect is confounded (aliased) with. If some main effects are confounded with some 2-level interactions, the resolution is 3. Note that full factorial designs have no confounding and are said to have a resolution of infinity. For most practical purposes, a resolution 5 design is excellent and a resolution 4 design may be adequate. Resolution 3 designs are useful as economical screening designs.

- **Response Surface Designs:** A DOE that fully explores the process window and models the responses. These designs are most effective when there are less than five factors. Quadratic models are used for response surface designs and at least three levels of every factor are needed in the design.

- **Responses:** The output(s) of a process. Sometimes called **dependent variable(s)**.

- **Rotatability:** A design is rotatable if the variance of the predicted response at any point $\boldsymbol{x}$ depends only on the distance of $\boldsymbol{x}$ from the design center point. A design with this property can be rotated around its center point without changing the prediction variance at $\boldsymbol{x}$. Note that rotatability

is a desirable property for response surface designs (i.e., quadratic model designs).

- **Scaling Factor Levels:** Transforming factor levels so that the high value becomes +1 and the low value becomes −1.
- **Screening Designs:** A DOE that identifies which of many factors have a significant effect on the response. Typically screening designs have more than five factors.
- **SME:** Subject matter expert.
- **Treatment:** A treatment is a specific combination of factor levels whose effect is to be compared with other treatments.
- **Treatment Combination:** The combination of the settings of several factors in a given experimental trial. It is also known as a run.
- **Variance Components:** Partitioning of the overall variation into assignable components.

A.3 Useful Probability Distributions

The Student *t* Distribution

This distribution has pdf

$$t(u:n) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}}\left(1+\frac{u^2}{2}\right)^{-\frac{n+1}{2}}.$$

If $Y_1, Y_2, \ldots, Y_n$ is a random sample from a $N(y;\mu,\sigma^2)$ distribution, then U defined by

$$U = \frac{\bar{Y}-\mu}{\hat{\sigma}/\sqrt{n}}$$

is distributed as $t(u,,n-1)$ where

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2.$$

Note that

$$\int_{u=t(\alpha,n)}^{\infty}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}}\left(1+\frac{u^2}{2}\right)^{-\frac{n+1}{2}}du = \alpha.$$

This distribution has a mean of 0 when $n > 1$ (its mean does not exist when $n \leq 1$); a variance of $n/(n-2)$ when $n > 2$ (its variance does not exist when $n \leq 2$); skewness of 0 when $n > 3$ (its skewness does not exist when $n \leq 3$) and

a coefficient of kurtosis of $3 + 6/(n-4)$ when $n > 4$ (its coefficient of kurtosis does not exist when $n \leq 4$). When $n \to \infty$, the variance approaches 1 and the coefficient of kurtosis approaches 3 (the coefficient of kurtosis of the normal distribution).

The Noncentral *t* Distribution

- **Definition:** The most general representation of the noncentral t distribution is quite complicated. Johnson, Kotz, and Balakrishnan (1995) give the following formula for the probability that a noncentral t variate falls in the range $[t, t]$

$$P[-t \leq x \leq +t|\nu, \lambda] = \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda^2}{2}\right)^j}{j!} e^{\frac{\lambda^2}{2}} I\left(\frac{x^2}{\nu + x^2} \mid \frac{1}{2} + j.\frac{\nu}{2}\right),$$

  where $I(x|a, b)$ is the incomplete beta function with parameters a and b, λ is the noncentrality parameter, and ν is the number of degrees of freedom.

- **Background:** The noncentral t distribution is a generalization of Student's t distribution. Student's t distribution with $n-1$ degrees of freedom models the t statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}},$$

  where $\bar{x}$ is the sample mean and s is the sample standard deviation of a random sample of size n from a normal population with mean μ. If the population mean is actually μ_a, then the t-statistic has a noncentral t distribution with noncentrality parameter

$$\lambda = \frac{\mu_a - \mu}{\sigma/\sqrt{n}}.$$

  The noncentrality parameter is the normalized difference between μ_a and μ. The noncentral t distribution gives the probability that a t test will correctly reject a false null hypothesis of mean μ when the population mean is actually μ_a; that is, it gives the power of the t test. The power increases as the difference $\mu_a - \mu$ increases, and also as the sample size n increases. When $X \sim N(0, 1)$ and $Y \sim \chi^2(\nu)$ then

$$T = \frac{X + \lambda}{\sqrt{Y/\nu}}.$$

- **Example and a Plot of the Noncentral *t* Distribution (MATLAB 7.0):** The following commands generate a plot of the noncentral t pdf that is given in Figure 11.A–1.

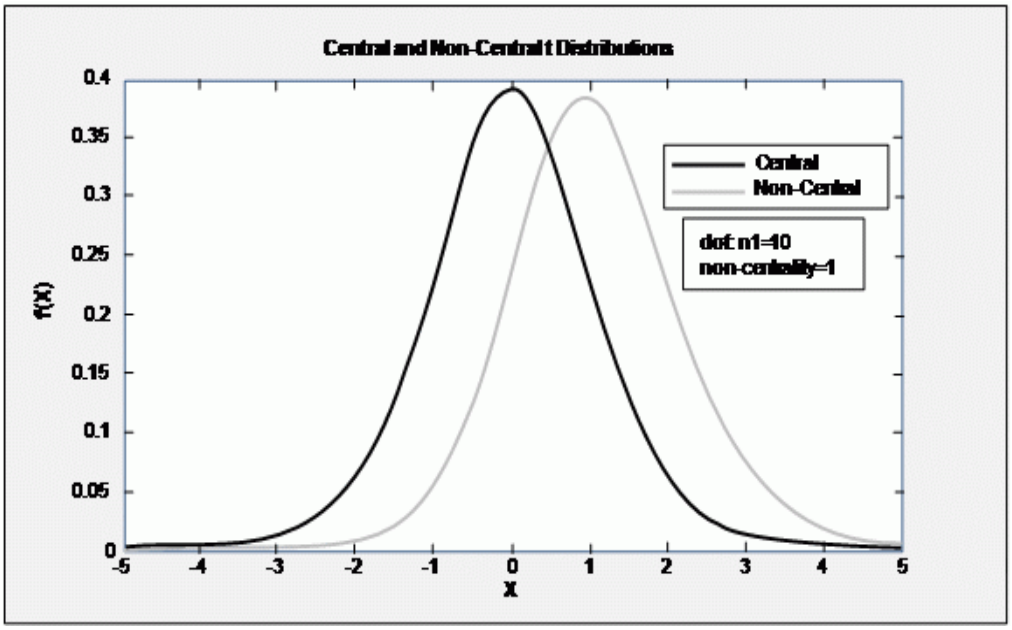


Figure 11.A–1. Central and non-central t distributions.

```
x = (-5:0.1:5)';
p1 = nctcdf(x,10,1);
p = tcdf(x,10);
plot(x,p,'-',x,p1,'--')
```

The Noncentral F Distribution

- **Definition:** Similar to the noncentral χ^2 distribution, the toolbox calculates noncentral F distribution probabilities as a weighted sum of incomplete beta functions using Poisson probabilities as the weights. For the noncentral F distribution we have

$$F(x|\nu_{num}, \nu_{den}, \lambda) = \sum_{j=1}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} e^{-\frac{\lambda}{2}} I\left(\frac{\nu_{num} x}{\nu_{den} + \nu_{num} x} \mid \frac{\nu_{num}}{2} + j, \frac{\nu_{den}}{2}\right),$$

  where $I(x|a, b)$ is the incomplete beta function with parameters a and b, and λ is the noncentrality parameter.

- **Background:** As with the χ^2 distribution, the F distribution is a special case of the noncentral F distribution. The F distribution is the result of taking the ratio of χ^2 random variables each divided by its degrees of

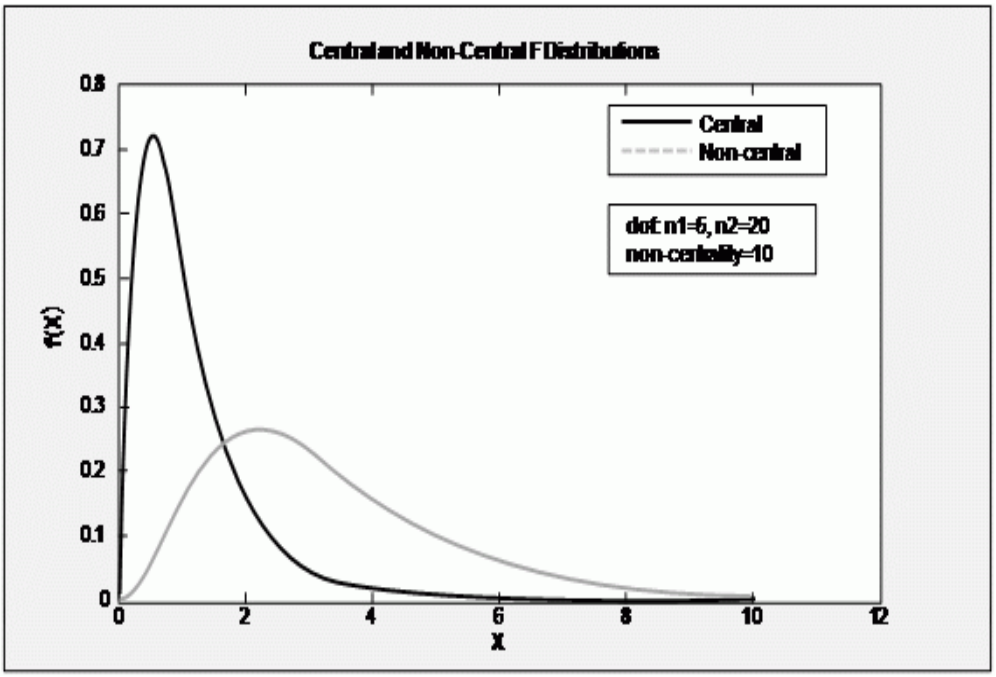


Figure 11.A–2. Central and non-central F distributions.

freedom. If the numerator of the ratio is a noncentral chi-square random variable divided by its degrees of freedom, the resulting distribution is the noncentral F distribution. The main application of the noncentral F distribution is to calculate the power of a hypothesis test relative to a particular alternative.

- **Example and a Plot of the Noncentral *F* Distribution:** The following commands generate a plot of the noncentral F pdf which is shown in Figure 11.A–2.

```
x = (0.01:0.1:10.01)';
p1 = ncfpdf(x,5,20,10);
p = fpdf(x,5,20);
plot(x,p,'-',x,p1,'-')
```

The Noncentral Chi-Square Distribution

- **Definition:** There are many equivalent formulas for the noncentral chi-square distribution function. One formulation uses a modified Bessel function of the first kind. Another uses the generalized Laguerre polynomials.

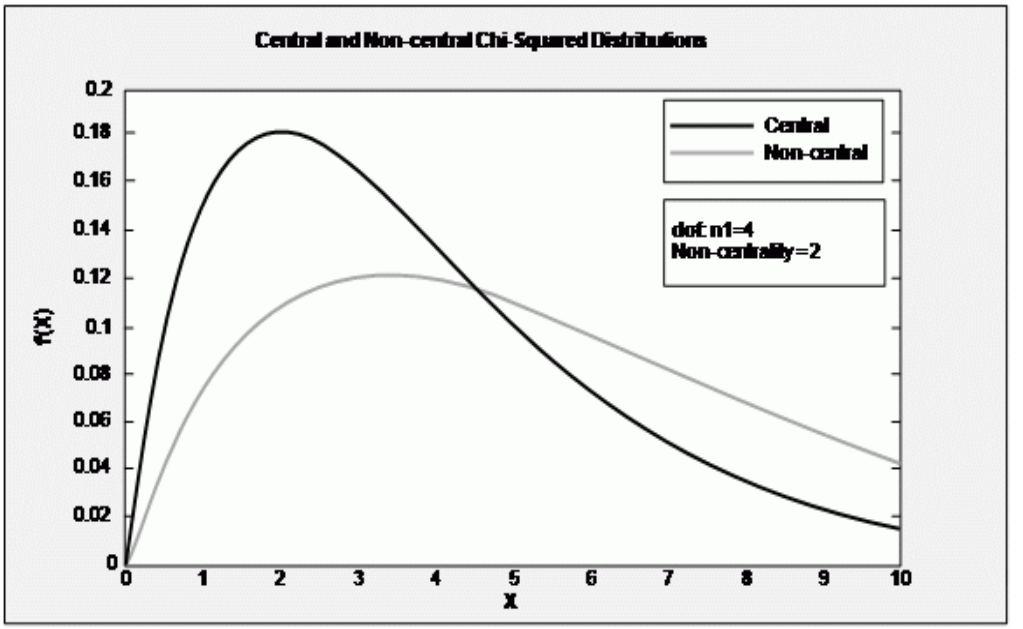


Figure 11.A–3. Central and non-central F distributions.

Statistics Toolbox computes the cumulative distribution function values using a weighted sum of χ^2 probabilities with the weights equal to the probabilities of a Poisson distribution. The Poisson parameter is one-half of the noncentrality parameter of the noncentral chi-square

$$F(x|\nu,\,\lambda) = \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} e^{-\frac{\lambda}{2}} P\left[\chi^2_{\nu+2j} \le x\right],$$

where λ is the noncentrality parameter.

- **Background:** The χ^2 distribution is actually a simple special case of the noncentral chi-square distribution. One way to generate random numbers with a χ^2 distribution (with ν degrees of freedom) is to sum the squares of ν standard normal random numbers. What if the normally distributed quantities have a mean other than zero? The sum of squares of these numbers yields the noncentral chi-square distribution. The noncentral chi-square distribution requires two parameters: the degrees of freedom and the noncentrality parameter. The noncentrality parameter is the sum of the squared means of the normally distributed quantities. The noncentral chi-square has scientific application in thermodynamics and signal processing. The literature in these areas may refer to it as the Ricean or generalized Rayleigh distribution.

- **Example of the Non-central Chi-Square Distribution:** The following commands generate a plot of the non-central chi-square distribution shown in Figure 11.A–3.

```
pdf.x = (0:0.1:10)';
p1 = ncx2pdf(x,4,2);
p = chi2pdf(x,4);
plot(x,p,'-',x,p1,'--')
```

A.4 Hypothesis Testing

Basic Issues

All hypothesis tests share the same basic terminology and structure. A null hypothesis is an assertion about a population that you would like to test. It is null in the sense that it often represents a status quo belief, such as the absence of a characteristic or the lack of an effect. It may be formalized by asserting that a population parameter, or a combination of population parameters, has a certain value. In the example given in the Introduction, the null hypothesis would be that the average price of gas across the state was \$1.15. This is written $H_0 : \mu = 1.15$. An alternative hypothesis is a contrasting assertion about the population that can be tested against the null hypothesis. In the example given in the Introduction, possible alternative hypotheses are:

- $H_1 : \mu \neq 1.15$ State average was different from \$1.15 (two-tailed test);
- $H_1 : \mu > 1.15$ State average was greater than \$1.15 (right-tail test);
- $H_1 : \mu < 1.15$ State average was less than \$1.15 (left-tail test).

To conduct a hypothesis test, a random sample from the population is collected and a relevant test statistic is computed to summarize the sample. This statistic varies with the type of test, but its distribution under the null hypothesis must be known (or assumed). The p-value of a test is the probability, under the null hypothesis, of obtaining a value of the test statistic as extreme or more extreme than the value computed from the sample. The significance level of a test is a threshold of probability α agreed to before the test is conducted. A typical value of α is 0.05. If the p-value of a test is less than α, the test rejects the null hypothesis. If the p-value is greater than α, there is insufficient evidence to reject the null hypothesis. Note that lack of evidence for rejecting the null hypothesis is not evidence for accepting the null hypothesis. Also note that the substantive significance of an alternative cannot be inferred from the statistical significance of a test. The significance level α can be interpreted as the probability of rejecting the null hypothesis when it is actually true – a Type I error. The distribution of the test statistic under the null hypothesis determines the probability α of a Type

I error. Even if the null hypothesis is not rejected, it may still be false – a Type II error. The distribution of the test statistic under the alternative hypothesis determines the probability β of a Type II error. Type II errors are often due to small sample sizes. The power of a test, $1 - \beta$, is the probability of correctly rejecting a false null hypothesis.

Results of hypothesis tests are often communicated with a confidence interval. A confidence interval is an estimated range of values with a specified probability of containing the true population value of a parameter. Upper and lower bounds for confidence intervals are computed from the sample estimate of the parameter and the known (or assumed) sampling distribution of the estimator. A typical assumption is that estimates will be normally distributed with repeated sampling (as dictated by the CLT). Wider confidence intervals correspond to poor estimates (smaller samples); narrow intervals correspond to better estimates (larger samples). If the null hypothesis asserts the value of a population parameter, the test rejects the null hypothesis when the hypothesized value lies outside the computed confidence interval for the parameter.

Hypothesis Test Assumptions

Different hypothesis tests make different assumptions about the distribution of the random variable being sampled in the data. These assumptions must be considered when choosing a test and when interpreting the results.

For example, the z-test and the t-test both assume that the data are independently sampled from a normal distribution. The MATLAB Statistics Toolbox offers a number of functions for testing this assumption, such as `chi2gof`, `jbtest`, `lillietest`, and `normplot`.

Both the z-test and the t-test are relatively robust with respect to departures from this assumption, so long as the sample size n is large enough. Both tests compute a sample mean, which, by the CLT, has an approximately normal sampling distribution with mean equal to the population mean μ, regardless of the population distribution being sampled.

The difference between the z-test and the t-test is in the assumption of the standard deviation s of the underlying normal distribution. A z-test assumes that σ is known; a t-test does not. As a result, a t-test must compute an estimate s of the standard deviation from the sample. Test statistics for the z-test and the t-test are, respectively,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \quad \text{and} \quad t = \frac{\bar{x} - \mu}{s/\sqrt{n}}.$$

Under the null hypothesis that the population is distributed with mean μ, the z-statistic has a standard normal distribution, $N(0, 1)$. Under the same null hypothesis, the t-statistic has Student's t distribution with $n - 1$ degrees of freedom. For small sample sizes, Student's t distribution is flatter and wider than

$N(0, 1)$, compensating for the decreased confidence in the estimates. As sample size increases, however, Student's t distribution approaches the standard normal distribution, and the two tests become essentially equivalent.

Knowing the distribution of the test statistic under the null hypothesis allows for accurate calculation of p-values. Interpreting p-values in the context of the test assumptions allows for critical analysis of test results. Assumptions underlying each of the hypothesis tests in Statistics Toolbox are given in the reference page for the implementing function.

PART III: Quantile Distribution Methods

Chapter 12 Statistical Modeling Based on Quantile Distribution Functions

Chapter 13 Distribution Fitting with the Quantile Function Response Modeling Methodology (RMM)

Chapter 14 Fitting GLDs and Mixture of GLDs to Data Using Quantile Matcing Method

Chapter 15 Fitting GLD to Data Using GLDEX 1.0.4 in **R**

A sketch of each of these chapters appears in Section 1.2.

Chapter 12

Statistical Modeling Based on Quantile Distribution Functions

Warren Gilchrist
Emeritus Professor,
Sheffield Hallam University, UK

The aim of this chapter is to show that the process of statistical modeling involving distributions can be approached using quantile functions instead of the classical distribution and density functions and that in many situations this has advantages. This is an extensive subject so, therefore, a "Handbook" approach is taken by presenting in the main text only the main results, outlines, and examples and leaving details to tables. See Gilchrist (1997), (2000), (2007), and (2008) for further discussion on many of the results.

12.1 Distributions Formulated as Quantile Functions

The **generalized lambda distribution** family, GLD, as discussed in Karian and Dudewicz (2000), referred to as K & D in this chapter, was one of the first distributions of general use to be defined in terms of its percentile or **quantile function**, $Q(p)$, and not to be simply expressible as a probability density, $f(x)$, or distribution function, $F(x)$. These alternative formulations of distributional models are related by

$$p = F(x), \quad x = Q(p), \tag{12.1.1}$$

X being the random variable and p the probability that $X \leq x$, so the quantile function is the inverse of the distribution function. Also,

$$f(x) = dF(x)/dx, \quad q(p) = dQ(p)/dp \quad \text{and} \quad f(Q(p))q(p) = 1, \tag{12.1.2}$$

$q(p)$ being the **quantile density function**. For example the **exponential distribution** on the positive axis is usually defined by the density function

$$f(x) = \sigma \exp(-\sigma x), \quad 0 \leq x \leq \infty. \tag{12.1.3}$$

Integrating over $(0, x)$ gives the probability as $p = F(x) = 1 - \exp(-\sigma x)$ and solving for x gives the quantile function

$$x = -(1/\sigma)\ln(1-p), \quad 0 \leq p \leq 1, \tag{12.1.4}$$

which has positive scale parameter $\eta = 1/\sigma$.

Distributions in quantile form can be expressed in the general form

$$Q(p) = \lambda + \eta S(p; \boldsymbol{\alpha}), \tag{12.1.5}$$

where λ is a position parameter, η is a scale parameter and the components of the vector $\boldsymbol{\alpha}$ are the shape parameters of the "basic" distribution $S(p)$. There are a considerable number of "basic" distributions with simple quantile function form. Table 12.1–1 lists some of these. In this table, use is made of the following quantile function properties.

1. If X has quantile distribution, $R(p)$, on the positive axis, $0 \leq x < \infty$, then the distribution $-R(1-p)$ is the quantile distribution that is its reflection in the axis at $x = 0$, called the **reflected distribution** on $-\infty < x \leq 0$.

2. Similarly the reciprocal $1/X$ has the **reciprocal distribution** $1/R(1-p)$ also on $0 \leq x < \infty$.

It should be noted from Table 12.1–1 that expressing distributions in terms of their quantile functions often gives quite simple expressions and that the relationships between distributions is particularly clear. Other distributions can be constructed from the simplest distributions of Table 12.1–1 by using these as components of more complex models. In particular quantile distributions may be added to create new distributions, $[S(p) = S_1(p) + S_2(p)]$ as shown in Table 12.1–2. For distributions on the positive axis, denoted by $R(p)$, they can also be multiplied $[R(p) = R_1(p)R_2(p)]$, as shown in the last entries of Table 12.1–2. It is possible to create new quantile distributions by using simple monotone transformations of basic distributions such as $S(p) = S_1(p^\alpha)$ and $R(p) = R_1(p)^\beta$, for $\alpha, \beta > 0$, as illustrated in Table 12.1–3. It will be noted that for $\alpha, \beta > 1$ the former transformation has the effect of moving probability to the left in the distribution, whereas the latter normally shifts it to the right, extending the right tail. The reverse behavior applies for $\alpha, \beta < 1$.

Table 12.1–1. Standard quantile form, $S(p, \alpha)$, for common basic distributions.

Distribution	$S(p;\alpha)$	Support Conditions and Alternative Names
Uniform	p	$(0, 1)$
Reciprocal Uniform	$1/(1-p)$	$(1, \infty)$
Reflected Uniform	$-(1-p)$	$(-1, 0)$
Power	p^α	$(0, 1)$, $\alpha > 0$
Reflected Power	$-(1-p)^\alpha$	$(-1, 0)$, $\alpha > 0$
Generalized Power	$(p^\alpha - 1)/\alpha$	$(-1/\alpha, 0)$, $\alpha > 0$
Limit of Generalized Power	$\ln(p)$	$(-\infty, 0)$, $\alpha \to 0$, Reflected Exponential
Pareto	$1/(1-p)^\alpha$	$(1, \infty)$, $\alpha > 0$
Reflected Pareto	$-1/p^\alpha$	$(-\infty, -1)$, $\alpha > 0$
Generalized Pareto	$[1/(1-p)^\alpha - 1]/\alpha$	$(0, \infty), \alpha > 0$
Limit of Generalized Pareto = Exponential	$-\ln(1-p)$	$(0, \infty)$, $\alpha \to 0$
Weibull	$[-\ln(1-p)]^\beta$	$(0, \infty)$, $\beta > 0$
Reciprocal Weibull	$1/[-\ln(p)]^\beta$	$(0, \infty)$, $\beta > 0$, Type 2 Extreme Value
Reflected Weibull	$-[-\ln(p)]^\beta$	$(-\infty, 0)$, $\beta > 0$, Type 3 Extreme Value
Generalized Reflected Weibull	$-[\{-\ln(p)\}^\beta - 1]/\beta$	$(-\infty, 1/\beta)$, $\beta > 0$, $(1/\beta, \infty)$, $\beta < 0$, Generalized EV
Limit of Reflected Generalized Weibull GEV	$-\ln[-\ln(p)]$	$(-\infty, \infty)$, $\beta \to 0$, Type 1 Extreme Value or Gumbel Distribution
Normal	$N(p)$	$(-\infty, \infty)$. A readily available function
Beta Distribution	$\mathrm{IIB}(p, r, n+1-r)$, Inverse of Incomplete Beta function	Distribution of r-th order statistic from n of Uniform Distribution

It can be seen from Tables 12.1–1 to 12.1–3 that there are a significant number of important distributions for which the quantile function provides a natural way to define the distribution, see Parzen (1979) and Gilchrist (2000) and (2007). A study of these tables also underlines that these models can be viewed, not as givens to be found in books, but as models that are constructed from basic components using a number of simple rules and construction techniques. Such distributions can be constructed to have desired properties, such as their skewness or the shapes of their tails.

Table 12.1–2. Quantile distributions based on addition and multiplication.

Distribution	$S(p;\, \alpha,\, \beta)$	Support and Conditions	Comments
Logistic	$\ln(p) - \ln(1-p) = \ln[p/(1-p)]$	$(-\infty,\, \infty)$	Exponential + Reflected Exponential
Skew Logistic (quantile-based form)	$(1-\delta)\ln(p) - (1+\delta)\ln(1-p)$	$(-\infty,\, \infty)$, $-1 < \delta < 1$	Weighted Exponential + Reflected Exponential. δ is a skewness parameter
Skew Normal (quantile-based form)	$N(p) + \delta[N((1+p)/2) + N((2-p)/2)]$	$(-\infty,\, \infty)$, $-1 < \delta < 1$	Normal with skewness generated by Half-Normal distributions
Symmetric Lambda or Tukey Lambda	$(1/\alpha)[p^\alpha - (1-p)^\alpha]$	$(-1/\alpha,\, 1/\alpha)$	$\alpha > 0$, Power + Reflected Power; $\alpha < 0$, Pareto + Reflected Pareto
Generalized Lambda	$p^\alpha - (1-p)^\beta$	Various support and conditions, e.g., $(-1,\, 1)$, $\alpha > 0,\, \beta > 0$; $(-\infty,\, \infty)$, $\alpha < 0, \beta < 0$; (see text)	$\alpha > 0,\, \beta > 0$, Power + Reflected Power; $\alpha < 0,\, \beta < 0$, Reflected (Pareto + Reflected Pareto), etc.
Generalized Lambda alternative	$(p^\alpha - 1)/\alpha - \{(1-p)^\beta - 1\}/\beta$	Various support	Valid for all α and β
Skew Lambda	$(1/\alpha)[((1-\delta)p^\alpha - (1+\delta)(1-p)^\alpha]$	$-1 < \delta < 1$, Various conditions and support	δ is a skewness parameter
Five Parameter Lambda	$\theta p^\alpha - \Phi(1-p)^\beta$	Various support; $\theta\alpha > 0,\, \Phi\beta > 0$	
Govindarajulu	$(\alpha+1)p^\alpha - \alpha p^{\alpha+1}$	$(0, 1);\ \alpha > 0$	
Wakeby	$[\theta(1-p)^{-\beta}/\beta - \Phi(1-p)^\alpha/\alpha]/2$	$\alpha > 0,\, \beta > 0$, (constant, ∞)	
Power-Pareto	$p^\alpha/(1-p)^\beta$	$\alpha,\, \beta > 0,\, (0, \infty)$	
Linex I	$\alpha p + \exp\{\beta(p - p_0)/p_0\} - \exp(-\beta)$	$\alpha,\, \beta > 0,\, (0,\, A)$ $A = S(1)$	
Linex II	$\alpha p + \exp\{\beta(p - p_0)/p_0(1-p)\} - \exp(-\beta)$	$\alpha,\, \beta > 0,\, (0,\, \infty)$	

Table 12.1–3. Distributions based on Transformations, such as p^α and $S(p)^\beta$.

Distribution	$S(p;\, \alpha,\, \beta)$	Support and Conditions	Comments
A	$p^\alpha/(1-p^\alpha)$	$(0,\, \infty)$, $\alpha > 0$	
Burr II Generalized Logistic	$\ln[p^\alpha/(1-p^\alpha)]$	$(-\infty,\, \infty)$, $\alpha > 0$	p^α transform of Logistic
Burr III	$[p^\alpha/(1-p^\alpha)]^\beta$	$(0,\, \infty)$, $\alpha,\, \beta > 0$	$S(p)^\beta$ transform of A, also called the Kappa Distribution
Burr XII	$[\{1-(1-p)^\alpha\}/(1-p)^\alpha]^\beta$	$(0,\, \infty)$, $\alpha,\, \beta > 0$	Reciprocal of Burr III
Exponentiated Weibull or Burr X	$[-\ln(1-p^\alpha)]^\beta$	$(0,\, \infty)$, $\alpha,\, \beta > 0$	p^α and $S(p)^\beta$ trans. of Exponential. $\beta = 0.5$ for Burr X
Log Normal	$\exp[\eta N(p)]$	$(0,\, \infty)$	
Log Logistic	$\ln(p/(1-p))$	$(0,\, \infty)$	A, exponentiated log logistic
Generalized Kappa	$[1-\{(1-p^\alpha)/\alpha\}^\beta]/\beta$	$((1-1/\alpha^\beta)/\beta,\, 1/\beta)$	
Johnson L	$\exp[\{N(p)-\gamma\}/\varepsilon]$	$(0,\, \infty)$, $\varepsilon > 0$	
Johnson B	$\exp[\{N(p)-\gamma\}/\varepsilon]/[1+\{\exp[\{N(p)-\gamma\}/\varepsilon\}]]$	$(0,\, \infty)$, $\varepsilon > 0$	
Johnson U	$\sinh[\{N(p)-\gamma\}/\varepsilon]$	$(-\infty,\, \infty)$, $\varepsilon > 0$	
Kamp	$q(p) = \eta p^\alpha (1-p)^\beta$		
Distributions of the largest and smallest observations	largest of n from $Q(p)$ is $\mathrm{Q}(p^{1/n})$;	smallest of n from $Q(p)$ is $Q(1-(1-p)^{1/n})$	
Distribution of the r-th order statistic	$Q(p) = Q(p^*(p,r))$		$p^*(p,r) =$ IIB$(p, r, n+1-r)$

It will also be noticed that the models in Tables 12.1–1 through 12.1–3 are distributions that describe raw data, they are not sampling distributions. Distributions of sampling statistics do not generally have explicit quantile functions and indeed most books of statistical tables are devoted to tables of quantile functions for these sampling distributions, since they are needed to carry out statistical tests. An outcome of the previous discussions is that when modeling in statistics one has the option of selecting a distributional model from the catalogs of distri-

Table 12.1–4. The process of modeling with quantile functions.

Stage of Modeling	Elements for Quantile Modeling
Studying Reality - Application	Study literature.
Studying Reality - Conceptual Features	Assess types of variable and support.
Studying Reality - Descriptive Features	Quantile Measures. Standard Ogive and Skewness Plot. Tail shape plots.
Develop Modeling Kit	Basic Quantile Functions appropriate for application and data. Methods for joining and developing components.
Use Modeling Experience	Playing with the Kit. Long-term experience.
Component Selection	Graphical Tail Analysis. Data Transformation.
Model Construction	Joining and modifying components. $+$, $\times$, Transformation, etc. Selecting and modifying.
Model Fitting	Method of Percentiles, Distributional Least Squares and Least Absolutes.
Model Validation	$x_{(r)}$ against $\hat{Q}(p_r)$. Distributional Residual Plots. Comparison of DLS/DLA criteria for alternate models.
Iteration	Improve features of fit and application by model modification.
Final Fit and Application	Final validation and development of control methods for application.

butions, such as Evans, Hastings, and Peacock (1993), or seeking to construct a model using a modeling kit based on basic quantile distribution functions. The construction of models is familiar in statistics in the fields of time-series and study of the deterministic component in regression, yet it has not been used in any significant fashion in distributional modeling. The stages in such quantile-based modeling, together with some of the relevant tools, are given in Table 12.1–4.

Though the list of stages is essentially as that in classical statistical modeling, there are some significant practical differences. One aspect, for example, is the frequent need to develop appropriate models for the left- and right-hand tails of a final distributional model. The quantile-based approach leads to new methods of descriptive data analysis, model construction, fitting, and validation. It is not intended to detail all the elements in Table 12.1–4. Rather, a selection of aspects are discussed and illustrated on the basis that they show particularly important features of the use of quantile-based methods.

A further practical aspect of the quantile-based approach is that by using spreadsheet formats the stages of component selection and validation, and of construction and estimation, are based on the same modeling structures and can be carried out in very flexible and iterative fashions. The methods lend themselves to the transparency and exploratory elements that spreadsheet formats provide and avoid the all too common "black box" approach to statistical computing. The layouts shown in Tables 12.5–3 and 12.6–2 are thus based on such an approach, for example, by the use of Excel. In these the steps of calculation shown in the rows of the tables would traditionally be presented as columns within the spreadsheet. Table 12.6–3 shows an Excel spreadsheet for the fitting of the Rain data from K & D, p. 103.

Where models are constructed from the combination of basic quantile functions there arise issues of model validity. **Validity** requires that quantile functions be non-decreasing functions of p in (0, 1). This depends on the values of the parameters. However a look at the literature on fitting such models reveals that there are two types of validity. A model may have **global validity**, for example the generalized lambda distribution (GLD), as discussed in K & D. Using different parameter symbols from K & D this distribution is

$$Q(p) = \lambda + \eta[p^{\alpha} - (1p)^{\beta}]. \tag{12.1.6}$$

For η, α, $\beta > 0$, this distribution is composed of power tails and for η, α, $\beta < 0$ of Pareto tails (see Table 12.1–1). There are cases, K & D p. 12, where the GLD, though being valid overall, has one component that is not itself a valid quantile function. Other examples can be found in Tarsitano (2005) where components of fitted models are sometimes invalid. In these cases the increase in one component dominates to hide the decreasing effect of another component. In these cases η may be negative. Alternatively **component validity** requires that each component quantile function in a composite model is itself a valid function. Component validity of the GLD requires η, α, $\beta > 0$, or η, α, $\beta < 0$, i.e., the two tails are both of power or of Pareto form. Note the requirement for negative η for the Pareto form, a point that is not apparent in the common use in discussing validity of only skewness and kurtosis, here depending on α, β. The parameter η will always be a positive scale factor in what follows. Note also that the distributional model where one tail is of power form and the other Pareto cannot be obtained from the GLD, but is covered by the Wakeby distribution in Table 12.1–2.

12.2 Describing and Analyzing Distributional Shape

Though it is no harder to find the moments of distributions in quantile form than in the density function form, it is more natural to use the main quantiles such as the median, quartiles and deciles as the bases for descriptive statistics. This approach has the merit that these always exist, whereas for some of the long

tailed distributions not all moments do. See Table 12.2–1 for a compendium of the measures of the shape of a distribution.

As the basic function of interest in the quantile based approach is represented by $x = Q(p)$, for $0 \le p \le 1$, much of the analysis in this chapter will be based on the use of the ordered data, denoted by $x_{(1)}, x_{(2)}, \ldots, x_{(r)}, \ldots, x_{(n)}$, the corresponding population values, $X_{(1)}, X_{(2)}, \ldots, X_{(r)}, \ldots, X_{(n)}$, being called the **order statistics**. We thus need to define in parallel a corresponding ordered sequence of probabilities, $p(r)$, $r = 1, 2, \ldots, n$. The basis for the choice of p is that any distribution $X = Q(p)$ can be generated from a uniform distribution, U, on (0, 1) by $X = Q(U)$. As $Q(p)$ is a non-decreasing function then ordering X corresponds to ordering U so $X_{(r)} = Q(U_{(r)})$. Two main forms for p will be used, dropping the subscript brackets for simplicity:

$$p'_r = r/(n+1) \quad \text{and} \quad p^*_r = \text{IIB}(0.5, r, n+1-r). \tag{12.2.1}$$

The reasoning here is that p'_r is the mean of the distribution of the r-th order statistic, $U_{(r)}$, from a uniform distribution U, and p^*_r is the median value of this statistic. This median value is obtained explicitly using the function IIB(p, a, b), which is the **inverse of the incomplete beta function**, the quantile function for the **beta distribution**. The function IIB(p, r, $n + 1 - r$) is the quantile distribution for the order statistic, $U_{(r)}$, and so can be used to derive

$$\text{Median}(X_{(r)}) = Q[\text{Median}(U_{(r)})] = Q(p^*_r). \tag{12.2.2}$$

IIB is a readily available function, for example in Excel as BETAINV(p, r, $n + 1 - r$). Using the form p^*_r has the great advantage that $Q(p^*_r)$ is the true median value for $X_{(r)}$, referred to as the **median rankit**. The **mean rankit** (or just **rankit**) is the mean of the distribution of $X_{(r)}$ and is only approximated by $Q(p'_r)$, exact values usually needing to be tabulated, see for example Balakrishnan and Chen (1997) and (1999). Often either rankit can be used, and p_r will be used to denote either. The graphical behavior of the mean rankits has no simple intuitive interpretation, as the distributions involved are variably and, in the tails, strongly skewed. However, the median rankit leads naturally to plots where there are 50:50 splits relative to some line, hence these graphics and calculations are simpler to interpret and are thus preferred.

The simplest plot using the previous quantities is the **ogive**, which **plots the ordered data, $x_{(r)}$ against p_r** and was first used by Sir Francis Galton in the middle of the 19th century, though published later, Galton (1883). In spite of its long history the ogive is a rather neglected tool of data analysis. K & D use various datasets to illustrate the fitting of GLDs, (see the later discussion of Table 12.5–4). For convenience we use the same data sets to illustrate various aspects of this chapter. Figure 12.2–2 (a) shows the ogive for the "Soil" data from which it is clear that there is a strong skew structure with a smooth pattern of points. The form suggests that standardization based on measures of position and spread

Table 13.2–1. Measures of the shapes of distributions.

Measure	Measure of	Population Form	Sample Form	Comments
r^{th} Moment		$\mu'_r = \int_0^1 Q(p)^r\,dp$ $\sigma^2 = \mu'_2 - (\mu'_1)^2$	$\bar{x}_r = (1/n)\sum x^r$ $s^2 = \bar{x}_2 - \bar{x}^2$	Standardized data $z = (x - \bar{x}_1)/s$
p-quantile		$Q(p) = \inf\{x : Pr(X \le x) \ge p\}$	$\hat{Q}(p) = (1-g)x_{[r]} + gx_{[r+1]}$	$r = np + 0.5$, $g = r - [r]$ $[r]$ = integer part of r
Probability Weighted Moments		$\omega_{r,s} = \int_0^1 Q(p)p^r(1-p)^s\,dp$	$w_{r,s} = \sum x_{(i)}(p_i^*)^r(1-p_i^*)^s$	p_i^* = BETAINV $(0.5, i, n+1-i)$
Median	Position	$M = Q(0.5)$	$m = \hat{Q}(0.5)$	$n = 2k$; $m = (x_{(k)} + x_{(k+1)})/2$, $n = 2k+1$, $m = x_{(k+1)}$
Quartiles, Q		$UQ = Q(0.75)$, $LQ = Q(0.25)$	$uq = \hat{Q}(0.75)$, $lq = \hat{Q}(0.25)$	In general use $\hat{Q}(p_r^*) = x_{(r)}$
Deciles, D		$UQ = Q(0.9)$; $LQ = Q(0.1)$	$uq = \hat{Q}(0.9)$, $lq = \hat{Q}(0.1)$	
Inter Q/D Ranges IQR/IDR	Spread	$IQR = UQ - LQ$ $IDR = UD - LD$	$iqr = uq - lq$ $idr = ud - ld$	Standardized data $z = (x - m)/iqr$
Skewness Plot	Skewness	$Q(p)$ v $Q(1-p)$	$x_{(r)}$ v $x_{(n+1-r)}$	Symmetry gives straight line. Positive skew – concave to top right
Galton Skewness Coefficient	Skewness	$G = (UQ + LQ - 2M)/IQR$, $GD = (UD + LD - 2M)/IDR$	$g = (uq + lq - 2m)/iqr$, $gd = (ud + ld - 2m)/idr$	
Left-right Tail Weight Ratio	Relative tail weight, skewness	$WQ = (M - LQ)/(UQ - M)$ $WD = (M - LD)/(UD - M)$	$wq = (m - lq)/(uq - m)$ $wd = (m - ld)/(ud - m)$	
Tail-weight Factor	Flatness, kurtosis	$T = IQR/IDR$	$t = idr/iqr$	
Moor's Kurtosis		$K = [(Q(7/8) - Q(5/8)) +$ $(Q(3/8) - Q(1/8))]/IQR$	k = sample equivalent	
Parzen Standardization	short or long tails on $(-\infty, \infty)$	$(M - Q(0.01))/2IQR$, $(Q(0.99) - M)/2IQR$ M = median or mid-IQR	sample tail standardized values	< 0.5 short tails > 1 long tails.
$R(p)$ standardization	short or long tails on $(0, \infty)$	$Q(0.99)/4M$	sample tail standardized value	< 0.5 short tail > 1 long tail.
Tail Shape Plots	Compare Exponential Tails	Ratio of above measures with those for Laplace distribution, $L(p)$	$L(p) = \ln(2p)$, $0 \le p \le 0.5$ $L(p) = -\ln(2-2p)$, $0.5 < p < 1$	First standardize both data and Laplace.

would be unhelpful for data of this form. Several studies of this data show up the largest observation as an outlier, yet it looks quite reasonable on the ogive. This suggests that a model is needed that does not show an outlier and naturally has such an extreme skewness in the right tail. A look at the ogive of the "Galaxy Velocity" data of Figure 12.2–2 (b) immediately suggests that we are looking not at one but at two clusters of stars, as referenced in K & D, p. 208. In Figure 12.2–3 the weight data on "Twins" is examined, K & D, p. 100. The ogive for the first twin with the second twin value attached suggests both have a symmetrical distribution and that there is a correlation between the weights. Ordering both data separately and plotting the ogives together shows that the two distributions are of the same shape but that the first twin to be born is in general slightly heavier than the second.

The use of empirical versions of the p-density, $f(p) = f[Q(p)]$, and ordinary density, $f(x)$, tend to be neglected in data analysis since the raw data tends to give poor and ragged plots. However, even with simple smoothing, using moving averages, enough can be gleaned to decide on the general form of the distribution and to act as a guard against the use of inappropriate models. The empirical p-density and the empirical $f(x)$, empirical and estimated values being denoted by "$\hat{}$", are directly obtained from the approximations

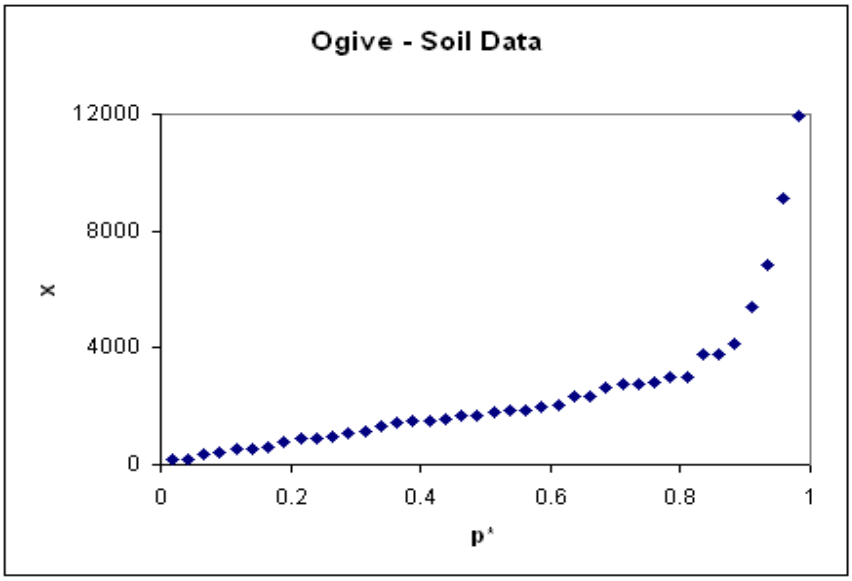
$$\hat{q}_r(\hat{p}_r) \approx (x_{(r+1)}x_{(r)})/(\hat{p}_{r+1} - \hat{p}_r), \tag{12.2.3}$$

$$\hat{f}_r(\hat{p}_r) = 1/\hat{q}_r(\hat{p}_r), \tag{12.2.4}$$

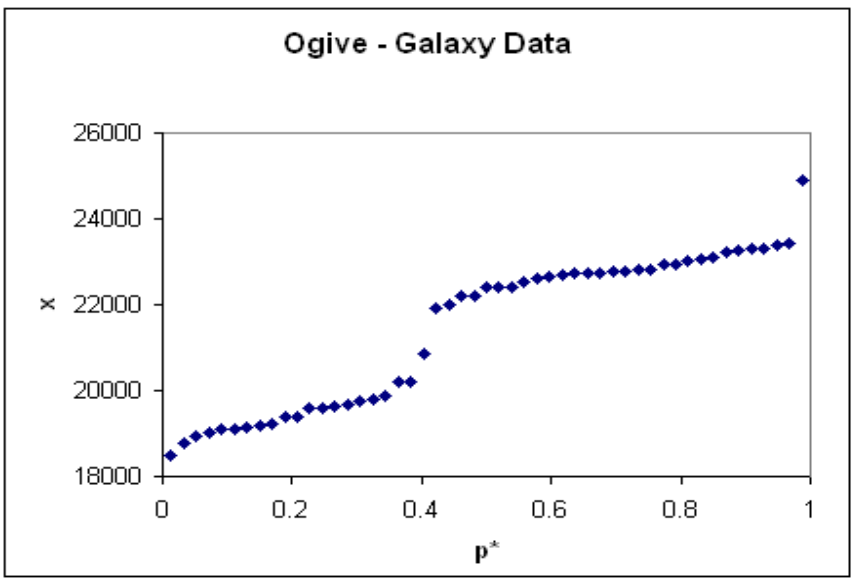
$$\hat{p}_r = (p_{r+1} + p_r)/2. \tag{12.2.5}$$

The p-density $\hat{f}_r(\hat{p}_r)$ may be plotted against $\hat{p}_r$, and also against $\hat{x}_{(r)} = (x_{(r+1)} + x_{(r)})/2$ to give $\hat{f}(x)$. With appropriate smoothing these plots give the general shape of the distribution and a clear picture of the modality of the distribution. Having looked at general shape, the functions in Table 12.2–1 provide a means for looking in more detail at specific features of the distributional shape. As an example of the use of one of these, Figures 12.2–4 (a) and (b) show the skewness plots for the "Galaxy" data separating the slower and faster sections and removing the intermediate value and the outlier. The faster galaxy shows some skewness whereas the slower is more symmetrical.

One issue of importance is the selection of components corresponding to short or long tails. A simple numerical approach, given in Table 12.2–1, is to standardize the data in appropriate ways and look at the values of the extreme observations, large values indicating long tails and small values indicating short tails. Table 12.2–1 also gives the use of the ratio of data-based measures and those of the **Laplace distribution**, which has exponential form tails in both directions. The exponential distribution is used here as a device for separating long from short tail data. Note that in terms of density functions the sequence, power–exponential-Pareto is not obvious, however in quantile function form the logarithm (power zero) is the natural boundary between the positive powers of the power distribu-

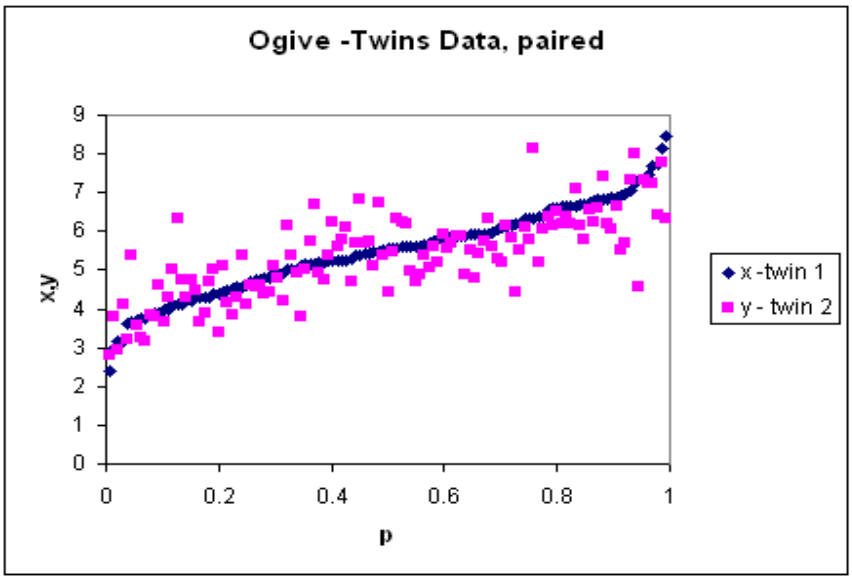


(a)



(b)

Figure 12.2–2. Ogives for soil (a) and galaxy (b) data.

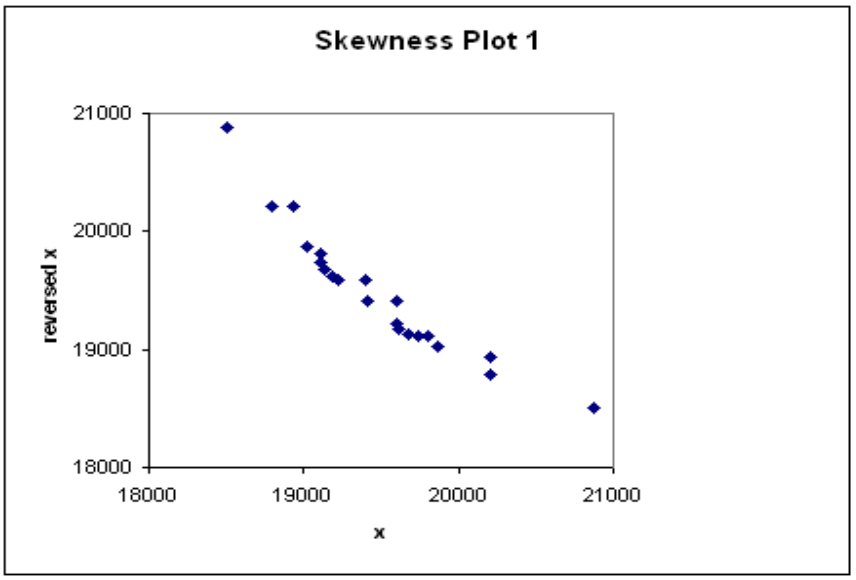


(a)

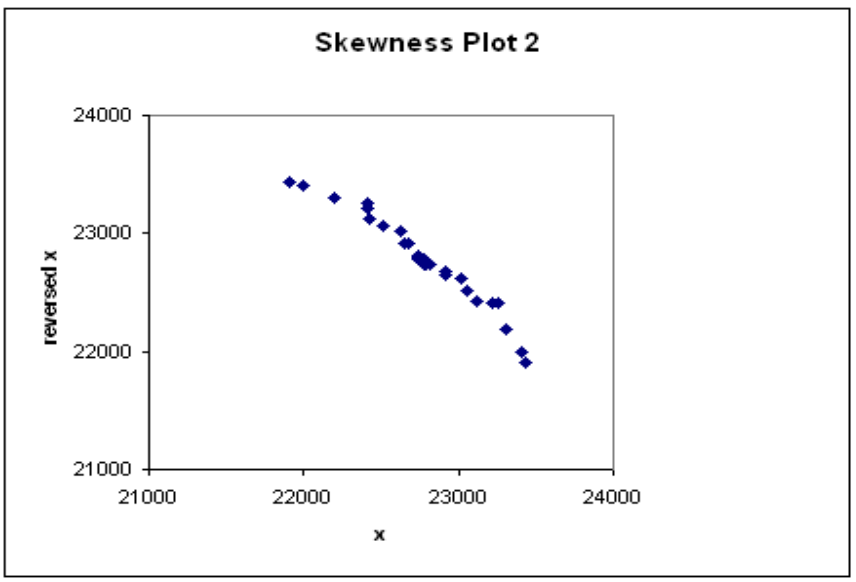


(b)

Figure 12.2–3. Ogives for paired twins data, K & D, p. 100.



(a)



(b)

Figure 12.2–4. Skewness plots for the star velocities for Galaxies.

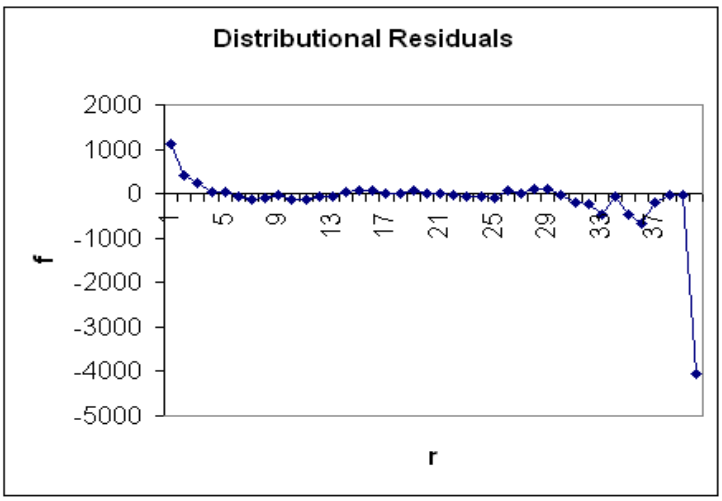


Figure 12.2–5. Distributional residuals for Soil data (K & D fit, p. 206).

tion and the negative powers of the Pareto. It is on this basis that the Laplace provides a comparator distribution. For effective use of this approach however a large data set is needed.

One practical aspect of data analysis is the identification of outliers. These show up as anomalous points in some of the types of plots discussed here. For example Figure 12.2–5 shows the distributional residuals, as defined in Table 12.5–3 for the Soil data of K & D, p. 206, with the GLD_5 model referred to there. The final observation seems to be a clear outlier. Studies with several possible models show this observation as an outlier. Having fitted a distribution, the distributions of the largest and smallest observations can be found with ease-see Table 12.1–3. Therefore, we can set 95% limits for the largest and smallest values, and outliers are more easily detected. However, care is needed, as a badly wrong model will identify reasonable observations as outliers. Outliers should only be rejected if there is some clear data-based justification, perhaps because several reasonable models all reject the observations, and also if a logic for their occurrence is available. The soil data provides an interesting example: the best fitting model found, based on a study of the ogive as discussed above, does not show an outlier for the soil data (see Table 12.5–4). Therefore, the question arises, are there conceptual and physical justifications for the form of this model.

One further aspect of data analysis is the consideration as to whether the shape properties observed might be simplified or clarified by using a simple transformation of the data, a log or power transformation being the classical approaches. Transformations within quantile analysis, to be discussed in some

detail later, need to have simple one-to-one inverses. All the methods described above may be applied to transformed data to see whether improved models are obtained. However, if an improved model is obtained, the quantile approach responds by applying the inverse transformation to the model to obtain the quantile function that can now be applied to the original data.

12.3 Model Construction

Following the data analysis there should be initial indicators of

1. the support,
2. the symmetry or otherwise,
3. data peculiarities and outliers,
4. the modality and general shape of the distribution,
5. whether one can consider the two tails separately as short or long,
6. whether some distribution involving a transformation might be relevant.

On the basis of items 1, 4, and 5, one might decide between a single basic model or a model consisting of components. If, for example, one has a distribution on the positive axis then single positive distributions such as the Weibull might be examined. For data over the whole axis or well away from the origin we begin to choose possible components on the basis of item 5 and to combine them in the light of item 2. As this is a new aspect of modeling there is little to guide one in construction, other than the information given by the analyses of Section 12.2. As with all modeling the best approach is to play with the modeling kit before attempting serious modeling. One way of doing this is to construct models that seem intuitively sensible and then explore their properties both theoretically and experimentally. This is best done on the basis of experience of the field from which the data arises. The additive form of many distributional models enables basic quantile properties to be calculated from those of the components. For experimental study three useful approaches may be adopted:

1. The generation of a **"profile"** data set from the model. This is simply either
$$y(r) = Q(p'_r) \quad \text{or} \quad Q(p^*_r),\ r = 1, \ldots, n, \tag{12.3.1}$$
in the notation of (12.2–1). The profile thus represents the "perfect" dataset and can then be used in any of the graphical analyses and their form compared with that of the data. The fitted profile is obtained by using a fitted distribution, $\hat{Q}(p)$. This is compared with the data as the basis for the estimation process in Section 12.5.

2. Simulated data sets from the fitted distribution can be obtained as $z_i = \hat{Q}(u_i)$ where u_i, for $i = 1, \ldots, n$, is a random sample from a uniform distribution on (0, 1), available in statistical and spreadsheet software.

3. The plot of $1/q(p_r)$ against $Q(p_r)$, $r = 1, \ldots, n$, which gives the plot of the density function $f(x)$.

These three approaches show the underlying form of the model and the look of random samples from it. The process of construction essentially uses the techniques and rules illustrated in Tables 12.1–2 and 12.1–3. For example, suppose the graphics suggest a distribution with a long tail, say Pareto to the left and a short decaying Power tail to the right. By using reflection of the power distribution, and shifting the origin, the right-tailed decaying power distribution has quantile function $1 - (1 - p)^{\alpha}$. The reflected Pareto to the left is $-1/p^{\beta}$. If we wish to allow for skewness we combine these with weights to give

$$Q(p) = \lambda + \eta[(1 + \delta)\{1 - (1 - p)^{\alpha}\} - (1 - \delta)/p^{\beta}], \tag{12.3.2}$$

where δ, with $-1 < \delta < 1$, controls the skewness, i.e., the balance between the two tails. Notice that in classical statistics most distributions have at most three parameters, but yet this example illustrates that there are clearly five features of a distribution that a model might seek to reflect: position (λ), spread (η), skewness (δ), right and left tail shapes α and β, respectively.

The above form for skewness suggests how any symmetric distribution can be given a skewed form. If the symmetric distribution can be expressed as the sum of two tail distributions, $R(p)$ and its reflection $-R(1 - p)$ then we have, as with the logistic in Table 12.1–2,

$$Q(p) = \lambda + \eta[(1 + \delta)R(p) - (1 - \delta)R(1 - p)], \tag{12.3.3}$$

which has symmetric form

$$Q_S(p) = \lambda + \eta[R(p) - R(1 - p)]. \tag{12.3.4}$$

Note that 12.3.3 can also be written as

$$Q(p) = Q_S(p) + \eta\delta[R(p) + R(1 - p)]. \tag{12.3.5}$$

This form suggests that for the normal distribution, $N(p)$, half-normal tails are appropriate, $R(p) = N((1 + p)/2)$. However, $N(p)$ is not constructed from these distributions, but a quantile function based **skew-normal** version that can be defined, using 12.3.5, by

$$Q(p) = \lambda + \eta N(p) + \eta\delta[N((1 + p)/2) + N((2 - p)/2)]. \tag{12.3.6}$$

Much effort in classical statistics goes into transforming data to get it to fit simple distributions. It could well be that the data analysis stage leads to the

view that a transformation of the data leads to a simple distribution. If this is the case, then the use of the quantile function approach leads to the transforming of the model and not the data. Thus, if $T(\cdot)$ is a monotone transformation then the distribution of y, where $y = T(x)$ is

$$Q_y(p) = T[Q_x(p)]. \tag{12.3.7}$$

This result is the basis for the formulae in Table 12.1–3 and implies that having found that the transform $T(x)$ gives a simple distribution identified by $Q_y(p)$ then the raw data has the distribution

$$Q_x(p) = T^{-1}[Q_y(p)]. \tag{12.3.8}$$

A classic example of this is where the logs of the data have a normal distribution and the distribution of X is

$$Q_x(p) = \exp[\lambda + \eta N(p)], \tag{12.3.9}$$

which is the quantile function for the **log-normal deistribution**. Note that the standard form of this is $\exp[\eta N(p)]$. The quantile approach is to use this quantile function with the raw data, rather than use the transformed data. This approach avoids many of the difficult, and often ignored, problems associated with transforming data and then "un-transforming" the results of analyses for the transformed data.

In the comments on the Soil data of Figure 12.2–2 (a) it was noted that the ogive plot looked reasonable but that several fitted models show the largest observation to be an outlier and give much better fits if it were dropped. However, a look at the situation and at the ogive suggest the need for a distribution that is linear for most p, but shows rapid growth for the extreme right tail. This suggests a combination of linear and exponential forms, with some value p_0 controlling the change region. The form used relates somewhat to the linex loss function, see Gilchrist (1984), and thus a suggested name is the **linex distribution**. This is defined as a distribution from λ to a constant on the positive axis,

$$Q(p) = \lambda + \eta[\alpha p + \exp\{\beta(p - p_0)/p_0\} - \exp(\beta)], \tag{12.3.10}$$

where η, α, and β are positive parameters which determine the shape and the maximum. This distribution was found to give a very good fit to the complete Soil data. A variant of this, **linex2**, on (λ, ∞) is

$$Q(p) = \lambda + \eta[\alpha p + \exp\{\beta(p - p_0)/p_0(1 - p)\} - \exp(-\beta)]. \tag{12.3.11}$$

This form has a much more extreme tail and does not model the Soil data well. It is evident from the discussion of this section, and the corresponding tables, that treating simple quantile forms as model components, and using them to build suitable models, provides a flexible and effective means of constructing distributions that mimic observed data properties. The components and the various ways of joining and developing them provide a statistician's **quantile modeling kit**.

Table 12.4–1. Estimation methods for quantile functions.

Method of	Bases	Statistics used: Population Sample	Operations: Choose parameters in $Q(p:\alpha)$ to
Moments	Population and sample moments	$\mu'_r = \int_0^1 [Q(p)]^r\, dp$ $x'_r = \sum x_i^r/n$	Equate population and sample moments for $r = 1, 2, \ldots, g$ (g = the number of parameters)
Probability weighted moments	Population and sample probability weighted moments	$\omega_{r,s} = \int_0^1 Q(p)p^r(1-p)^s\, dp$ $w_{r,s} = \sum x_{(r)}(p_r^*)^r(1-p_r^*)^s$	Equate the PWM, using $p_i^* =$ IIB$(0.5, r, n+1-r)$, or some other suitable value
Percentiles	Population and sample percentiles	$Q(p_r)$, $\hat{Q}(p_r)$ or quantile based statistics: M, m; IQR, iqr; G, g, etc.	Equate population and sample values for selected r or measures of shape.
Maximum likelihood	Log likelihood, $L(\boldsymbol{\theta})$; $\boldsymbol{\theta} =$ parameter of model	$L(\hat{\boldsymbol{\theta}}) = -\sum \ln[q(p_{(r)};\ \hat{\boldsymbol{\theta}}]$, conditional on $x_{(r)} = Q(p(r); \hat{\boldsymbol{\theta}})$	Maximize $L(\hat{\boldsymbol{\theta}})$ where, $q(p) = Q'(p)$, $f(x) = f(Q(p)) = f_p(p) = 1/q(p)$
Distributional Least Squares, DLS	Rankits $\mu_{(r)}$ $\mu_{(r)} = E[X_{(r)}; \boldsymbol{\theta}]$	$C_S = \sum((x_{(r)} - \mu_{(r)})^2 = \sum f_{(r)}^2$	Minimize C_S w.r.t. $\hat{\boldsymbol{\theta}}$; measure of fit is $DRMSE = \sqrt{C_S/n}$
Distributional Least Absolutes, DLA	Median Rankits $M_{(r)} = M[X_{(r)}; \hat{\boldsymbol{\theta}}] = Q(p_r^*)$	$C_A = \sum \lvert x_{(r)} - M_{(r)}\rvert = \sum \lvert f_{(r)}\rvert$	Minimize C_A w.r.t. $\boldsymbol{\theta}$. measure of fit is $DMAE = C_A/n$

12.4 Methods of Fitting Quantile Distributions: An Overview

Having the suitably shaped model provided by the previous methods only takes us part way. It is now necessary to scale the model and to fit the model to the data. The methods developed to fit distributions in the form of density or distribution functions can be applied to distributions in the form of quantile functions. There are, however, some methods that are designed particularly for use with quantile function models. See Gilchrist (2000) for details and Table 12.4–1 for brief mathematical summaries.

Listing the main methods briefly:

- **Maximum Likelihood**
  This standard method can be used, but requires an implicit conversion back from quantile methods to density methods and is thus not easily applied unless there is an explicit density function.

- **Method of Moments**
  As population moments can be as easily calculated from quantile functions, as from density functions this method is applied as in classical statistics. However, note that for more than three parameters the variances involved become unacceptably large and the method unreliable. Further, many long-tailed distributions do not possess higher moments.

- **Probability Weighted Moments**
  This is a modification of the method of moments suitable for long-tailed distributions. It avoids the high moments of the ordinary method. It has been widely used with distributions in quantile form.

- **The Method of Percentiles**
  This involves equating population and sample quantiles, percentiles, or suitable functions of them. Clearly this is most easily done where such percentiles can be easily calculated, as is the case for distributions defined by their quantile functions. Though in general not as efficient as the method of moments, its value increases for large numbers of parameters, for distributions with heavy tails and for situations requiring robust estimators.

- **Distributional Least Squares, DLS, and Distributional Least Absolutes, DLA**
  These methods that parallel to ordinary least squares, are based on minimizing the differences between the ordered observations and predictions of them provided by the fitted profile. They provide a natural approach to fitting quantile-based models.

12.5 Minimization Methods of Fitting Quantile Distributions

12.5.1 Rankits and Median Rankits

One feature of the use of quantile functions is the centrality of the ordered data, a feature that is ignored by most classical methods of estimation. The use of quantile plots and the methods of distributional least squares and absolutes depend on comparing the observed ordered data with their predicted positions from the fitted model. These positions, the profile, may be the expected values of the order statistics, the rankits, usually used in distributional least squares, or their

Table 12.5–1. Cases of explicit rankits.

Distribution	$S(p)$	$\mu_{(r)}$
Uniform	p	$r/(n+1)$
Power	p^{α}	$[n!/(r-1)!][\Gamma(r+\alpha)/\Gamma(n+1+\alpha)]$
Reflected Power	$-(1-p)^{\alpha}$	$[-n!/(n-r)!][\Gamma(n+1-r+\alpha)/\Gamma(n+1+\alpha)]$
Pareto	$1/(1-p)^{\beta}$	$[n!/(r-1)!][\Gamma(r-\beta)/\Gamma(n+1-\beta)]$
Reflected Pareto	$-1/p^{\beta}$	$[n!/(n-r)!][\Gamma(n+1-r-\beta)/?(n+1-\beta)]$
Power Pareto	$p^{\alpha}/(1-p)^{\beta}$	$[n!/\{(n-r)!(r-1)!\}]\times$ $[\Gamma(r-\beta)\Gamma(n+1-r+\alpha)/\Gamma(n+1+\alpha-\beta)]$

population median values, the **median rankits**, usually used in distributional least absolutes. These are considered now in more detail.

The rankits are $\mu_{(r)} = E[X_{(r)}; \boldsymbol{\theta}]$. In the discussions above, use is made of reflected distributions and the additive combinations of quantile functions. Both of these lead to simple applications of the ideas of rankits. The notation used here is to denote by $\nu_{(r)}$ the rankits of a standard distribution, $S(p)$, and by $\tau_{(r)}$ the rankits of the reflected distribution $-S(1-p)$. If we are just using the model

$$Q(p) = \lambda + \eta S(p; \boldsymbol{\alpha}), \tag{12.5.1}$$

then

$$\mu_{(r)} = \lambda + \eta \nu_{(r)}(\boldsymbol{\alpha}). \tag{12.5.2}$$

It is clear that, by the nature of reflection, $\tau_{(r)} = -\nu_{(n+1-r)}$. It follows that if a distribution is constructed so that

$$Q(p) = \lambda + \theta S(p; \boldsymbol{\alpha}) - \Phi S(1-p; \boldsymbol{\beta}), \tag{12.5.3}$$

then, in an obvious notation, allowing the two tails to have different shape parameters,

$$\mu_{(r)} = \lambda + \theta \nu_{(r)}(\boldsymbol{\alpha}) + \Phi \tau_{(r)}(\boldsymbol{\beta}). \tag{12.5.4}$$

Hence,

$$\mu_{(r)} = \lambda + \theta \nu_{(r)}(\boldsymbol{\alpha}) - \Phi \nu_{(n+1-r)}(\boldsymbol{\beta}). \tag{12.5.5}$$

Table 12.5–1 gives some cases of explicit formulae for rankits.

For many classical distributions, including the normal distribution, there are no such explicit formulae, but rather sets of published tables. Such tables are of little use in estimations based on minimization, where an explicit formula is needed, so will not be considered here. This, however, implies that the exact

Table 12.5–2. Cases of explicit rankits – recurrence relations.

Distribution	Initial Value	Recurrence Formula
Power	$\nu_{(n+1)} = 1$	$\nu_{(r)} = \nu_{(r+1)}[r/(r+\alpha)]$
Reflected power	$\tau_0 = -1$	$\tau_{(r)} = \tau_{(r-1)}[(n+1-r)/(n+1-r+\alpha)]$
Pareto	$\nu_{(0)} = 1$	$\nu_{(r)} = \nu_{(r-1)}[(n+1-r)/(n+1-r-\beta)]$
Reflected Pareto	$\tau_{(n+1)} = -1$	$\tau_{(r)} = \tau_{(r+1)}[r/(r-\beta)]$
Power Pareto B=Beta function	$\nu_{(n+1)} = nB(n+\alpha, 1-\beta)$	$\nu_{(r)} = \nu_{(r+1)}[r(n-r-\beta)/(r+\alpha)(n+1-r)]$
Reflected power Pareto	$\tau_{(0)} = -nB(n+\alpha, 1-\beta)$	$\tau_{(r)} = \tau_{(r-1)} \times [(n+1-r)(r-1-\beta)/(r-1)(n+1-r+\alpha)]$
Exponential	$\nu_{(0)} = 0$	$\nu_{(r)} = \nu_{(r-1)} + 1/(n+1-r)$
Reflected Exponential	$\tau_{(n+1)} = 0$	$\tau_{(r)} = \tau_{(r+1)} + 1/(r)$

method of DLS cannot be used for these distributions. The method can, however, be applied as an approximation using approximate formulae for the rankits. Such approximations have been explored over the years as bases for probability plotting. The approximations are usually based on the form

$$\mu_{(r)} \approx Q(p'_r), \tag{12.5.6}$$

where p'_r is a suitable function of r and n. As $x_{(r)} = Q(u_{(r)})$, where $u_{(r)}$ is an order statistic for the uniform distribution which has rankit $\nu_{(r)} = r/(n+1)$,

$$\mu_{(r)} \approx Q(r/(n+1)) \tag{12.5.7}$$

provides a natural approximation. More general approximations are of the form

$$\mu_{(r)} \approx Q[(r+a)/(n+b)]. \tag{12.5.8}$$

For example, for the normal $a = -3/8$ and $b = 1/4$ are often used. It is often the case that the best way to calculate the rankits for a distribution is by the use of recurrence formulae. Table 12.5–2 gives the recurrence formulae for some simple distributions.

The method of distributional least absolutes uses the median rankits, $Q(p^*_r)$, as defined in (12.2.1), which we have seen can be exactly evaluated and which exist in many cases where the rankits do not exist. This implies that even when distributional least squares cannot be applied distributional least absolutes can.

12.5.2 Distributional Least Squares (DLS)

As has been seen this method seeks to find the best fitting quantile function by choosing the parameters to minimize

$$C_S = \sum (x_{(r)} - \mu_{(r)})^2. \tag{12.5.9}$$

The resultant measure of best fit is the distributional root mean square error

$$DRMSE = \sqrt{(C_S/n)}. \tag{12.5.10}$$

The quantile function has both linear and non linear parameters. The optimization is simplified by using the classical least squares estimates of the linear parameters, with assumed values of the non-linear parameters, and then only optimizing by searching for the non-linear parameters. This provides starting values for a full search over all parameters. Table 12.5–3 gives the main stages in the calculation.

This method seems to have been first used to estimate the position and scale parameters of $Q(p) = \lambda + \eta S(p)$. For cases with no further parameters, see Lloyd (1952) and for fitting the generalized lambda, with its non-linear parameters, see Öztürk and Dale (1985). Both these papers refer to the method as being simply least squares. However, it is applied to a distributional model and not a deterministic one as is usual, so the terms distributional least squares and absolutes have been introduced. When we apply these methods to the regression models in Section 12.6 the terminology difference is crucial as we initially apply least squares or absolutes to the deterministic element of the model and then, as a second stage, distributional least squares or absolutes to the complete model of deterministic and stochastic elements. A problem treated by Lloyd is that the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ have differing variances and strong correlations. Thus, use was made of the generalized least squares that uses transformations to obtain constant variance and independence. This method results in estimates of the linear parameters that are best linear unbiased estimators. For a more recent treatment see David and Nagaraja (2003).

For situations with non-linear parameters, these optimum results no longer hold. DLS starts by defining a criterion based on a sensible measure of quality of fit. Applying the generalized least squares approach takes a theoretical statistical attitude that changes the problem to one of finding optimum estimators of individual parameters. Here the attitude is taken that the C_S quality of fit criterion is the one most appropriate and so C_S is used, ignoring issues of heterogeneity and correlation. The method requires knowledge of the exact rankits, which may not always be possible. Approximate rankits, such as $Q(r/(n+1))$, may be used to give a first fit of the model for use with other methods. However, Öztürk and Dale (1985) showed that for the GLD this approximation led to estimates that were little different from the correctly calculated values. For strongly skewed

Table 12.5–3. Steps in fitting quantile distributions by DLS or DLA.

Model	$Q(p) = \lambda + \eta S(p; \boldsymbol{\alpha})$	values row 1	values row n	Comments
Initial or current estimates of parameters	$\hat{\lambda}$, $\hat{\eta}$, $\hat{\boldsymbol{\alpha}}$. (A)			$\hat{\boldsymbol{\alpha}}$ are the shape parameters
Original data - z_s		z_1	z_n	
Ordered data - x_r	1, 2 for smallest	x_1	x_n	
DLS – Rankits for S(p)	$\nu_{(r)}(\hat{\boldsymbol{\alpha}})$	$\nu_{(1)}$	$\nu_{(n)}$	see Section 12.5
Rankits for DLS	$\mu_{(r)} = \lambda + \eta\nu_{(r)}(\boldsymbol{\alpha})$	$\mu_{(1)}$	$\mu_{(n)}$	
DLA – uniform median rankits	$p_r^* =$ IIB$(0.5, r, n+1-r)$	p_1^*	p_n^*	
Median rankits	$S(p_r^*; \hat{\boldsymbol{\alpha}})$	$S(p_1^*, \hat{\boldsymbol{\alpha}})$	$S(p_n^*, \hat{\boldsymbol{\alpha}})$	
DLS – Linear parameter fit, to put in (A)	$\hat{\lambda}$ =intercept of LS line through $(\mu_{(r)}, x_r)$ $\hat{\eta}$ =slope of line			initial estimates from least squres
Whole model fit DLS $\hat{x}_r = \mu_{(r)}$ DLA $\hat{x}_r = M_{(r)}$	$\mu_{(r)} = \hat{\lambda}^+\hat{\eta}\nu_{(r)}(\hat{\boldsymbol{\alpha}})$ $M_{(r)} = \hat{\lambda} + \hat{\eta}S(p_r^*; \hat{\boldsymbol{\alpha}})$	$\hat{x}_1$	$\hat{x}_n$	fitted rankits or median rankits
Distributional Residuals	$f_r = x_r - \hat{x}_r$	f_1	f_n	
Squares or $\lvert\cdot\rvert$ of distributional residuals	f_r^2 (or $\lvert f_r\rvert$)	f_1^2	f_n^2	sum $= \sum f_r^2$ (or $\sum \lvert f_r\rvert$) (B)
DLS – Choose $\hat{\boldsymbol{\alpha}}$ to minimize $\sum f_r^2$, adjusting $\hat{\lambda}$, and $\hat{\eta}$	Measure of result is Distributional Root Mean Square Error	DRMSE		$\sqrt{\sum f_r^2/n}$
DLA – Choose parameters to minimize $\sum \lvert f_r\rvert$	Measure of result is Distributional Mean Absolute Error	DMAE		$\sum \lvert f_r\rvert/n$
Validation Plotting 1	x_r against $\hat{x}_r$			fit-observation plot
Validation Plotting 2	f_r against $\hat{x}_r$			distributional residual plot

distributions there is some merit in using the p^* values in place of p', for these give slightly better approximations to the rankits.

It may be noted that the method of DLS may be used when not all the data is available, for example with censored data or with a subset of the order statistics, perhaps neglecting possible outliers. In both cases the summation is carried out over the available values of r but n is kept at its value for the whole data. A further variant on this method is available when the fitted model is to be used in frequently occuring situations where one tail is particularly important. In this case the least squares criterion is replaced by an appropriate discounted version. For example, the criterion

$$C_{DS} = \sum a^r (x_{(r)} - \mu_{(r)})^2, \quad 0 < a < 1, \tag{12.5.11}$$

will put most emphasis on the data in the left tail and so get a better fit in this region. This is parallel to the approach of discounting in time series and forecasting and is termed **distributional discounted least squares**.

12.5.3 Distributional Least Absolutes (DLA)

An alternative to DLS is the distributional least absolutes, or DLA, for which the best fitting quantile function is obtained by choosing the parameters to minimize $C_S = \sum |x_{(r)} - M_{(r)}|$, where $M_{(r)}$ is the median of the distribution of $X_{(r)}$. The resultant measure of best fit is the distributional mean absolute error, $DMAE = C_A/n$. This method has the following advantages.

1. The median rankit, $M_{(r)}$, is, as has been seen, simply and explicitly obtained from the quantile function as $Q(p_r^*)$.

2. It links naturally to the concept of median, since $\sum |Z - a|$ is minimized for $a =$ the median of Z.

3. The method gives solutions that are in general more robust than those from DLS.

4. The consequent fit-observation plot of $x_{(r)}$ against $M_{(r)}$ has the simple interpretation that half the residuals should lie above and half below the line.

5. It can be applied to distributions where the mean rankits are not known in explicit form or may not even exist.

One problem with the method is that to get optimum solutions, linear programming methods are used and the methods can require special software, though this is now readily available. However, reasonable fits can be found straightforwardly with spreadsheet software, such as by using "Solver" in Excel. The further disadvantage is that there are no explicit solutions for the linear parameters and all

parameters need to be found by searching. It is therefore useful to first get estimates from the DLS solutions. For a review of some of the literature see Dielman (2005). Table 12.5–3 gives the calculations required to implement this method as well as those needed for the DLS. The first part of Table 12.6–3 shows the spreadsheet layout for fitting a Weibull distribution to the Y rain data from K & D. The top of the table gives values of the parameters and the corresponding value of the DRMSE, calculated from the body of the spreadsheet table. "Solver," the tool for optimization in Excel, searches by sequentially altering the parameter values to minimize some selected value, here the DRMSE. Copied at the side of this are the results for minimizing the DMAE in a previous calculation. The approximation for the rankits are $Q(p')$, p' being used here for simplicity. Note also that for simplicity the methods minimize DRMSE and DMAE, rather than C_S and C_A which are equivalent.

Examples: Karian and Dudewicz (2000) give various sets of data and use them as examples of fitting the GLD, *not* of finding the best models. In Table 12.5–4 (a) this data is used to illustrate the results of exploring a variety of distributions in quantile form, seeking appropriate distributions and fitting them, using simple spreadsheet tools, by both distributional least squares and distributional least absolutes. The table shows the values of both DRMSE and DMAE for the fits of the GLD given in K & D and also the values obtained by applying DLS and DLA using both the GLD and other suitable distributions.

Table 12.5–4. (a) Distributional models for data from K & D with GLD (4 parameters).

Name given data	Page (n)	Book GLD DMAE DRMSE	DLA GLD DMAE	DLS GLD DRMSE	Alternative model	DLA fit DMAE	DLS fit DRMSE
Cadmium	96 (43)	2.66 3.32	2.56	3.23	Weibull (2)	2.48	3.20
Friction (×10000)	105 (250)	2.79 3.90	1.90	3.06	Skew Normal (3)	1.88	3.00
Brain	98 (23)	0.568 0.717	0.534	0.695	5PLD (5)	0.411	0.560
Galaxies	208 (49)	no GLD fit	334 single	data 420	$N + SkN$ (2+3)	44	56
Rainfall X (×100)	103 (47)	4.91 9.94	3.38	5.93	Weibull with p^α (3)	4.23	6.51
Rainfall Y (×100)	103 (47)	4.34 8.43	3.24 SkLD(4)	SkLD(4) 4.62	Weibull (2)	3.27	4.50
Soil	206 (39)	247 692	111	153	linex (5) 40 obs	70	113

Table 12.5–4. (b) Regression models for data from K & D with GLD (4 parameters).

Name of data	Page (n)	Criteria	Book GLD	Distribution model	Regression model	DLA/DLS
Tree Height H|D	146 (88)	DMAE DRMSE	1.706 1.400	normal 0.628 0.850	regression with SkLD	0.430 0.725
Rainfall Y|X (×100)	103 (47)	DMAE DRMSE	4.34 8.43	Weibull 3.27 4.50	Weibull with x^{Φ} regressor	2.78 3.46
Twin 2| Twin 1	100 (123)	DMAE DRMSE	0.837 1.041	GLD 0.057 0.077	regression with GLD	0.032 0.046

12.6 Fitting Parametric Regression Models Based on Quantile Functions

It is seen from the Tables 12.5–4 (a) and (b) that a wide range of distributions have been developed in the process of modeling data that arise in practice. Yet, when one turns to the literature on regression it is dominated by the assumption of normality. Even where normality does not occur, the data is often transformed to seek normality. The classic regression model takes the form

$$y = \lambda + \beta x + e, \tag{12.6.1}$$

where a dependent variable, y, relates to some regressor variables denoted by x, with independent errors e. When fitted by least squares, no assumption is made about the distribution of e other than that $E(e) = 0$ and $V(e) = \sigma$ is constant. The normality assumption appears often in the subsequent analysis and the separate estimation of σ. If the quantile function notation is used for the distribution of e, the whole model, both deterministic and stochastic elements, can be expressed as a single expression

$$Q_y(p;\, x) = \lambda + \beta x + \eta S(p), \tag{12.6.2}$$

where $S(p)$ is the quantile distribution of the error term standardized so that $E(e) = 0$, for the usual regression model. However, as an alternative one can standardize with $S(0.5) = 0$. It will be seen that then $Q_y(0.5;\, x) = \lambda + \beta x$ gives the median regression line and that in either form $Q_y(p;\, x)$ is the conditional quantile function for y given x. This model gives great flexibility and formed the original regression model introduced by Galton (1886). It can be developed in many ways, for example to many regressor variables, with $\boldsymbol{x}$ a vector of regressors, to non-constant slopes, heteroscedastic situations, where we have η as a function

Table 12.6–1. Examples of quantile form regression models.

Dependent variable $\boldsymbol{y}$	Regressor variable $\boldsymbol{x}$	Distribution form	Model $Q_y(p;\, x) =$
Bank's turnover	banks's assets	symmetric lambda	$\gamma \exp(\beta x) + \eta[1/(1-p)^{\alpha} - 1/(p)^{\alpha}]$
Stopping distance of cars	speed	skew logistic	$\beta x + \gamma x^2 + [\eta x \delta + (\eta x/2)$ $\{-(1+\delta)\ln(1-p) + (1-\delta)\ln(p)\}]$
Road distance between places	point to point distance	Weibull or Burr III	$x + [\Phi x][-\ln(1-p)/\{\ln(2)\}]^{\beta}$ $x + \Phi x[[(2^{\alpha}-1)(p^{\alpha}/(1-p^{\alpha})]^{\beta}$
Returning salmon stock	spawning stock	power-Pareto	$\alpha x \exp(-\beta x)[p^{\theta}/(1-p)^{\Phi}][2^{\theta}/2^{\Phi}]$

of x, $\eta(x)$, as is illustrated later, and to situations where the shape parameter depends on the regressors, thus:

$$Q_y(p;\, x) = \boldsymbol{x'\beta}(p) + \eta(x) S(p;\, \Phi(x)). \tag{12.6.3}$$

Table 12.6–1 gives some examples of such regression forms that have been found to give good fits to various sets of data. For a detailed discussion of these see Gilchrist (2008).

The methods available for fitting models where the distributional element is explicit are: maximum likelihood, which is the classical approach, and simple extensions of distributional least squares and least absolutes. Table 12.6–2 gives the sequence of steps required for using DLS and DLA. It should be noted that the fitting proceeds in two stages. The first fits the deterministic component by least squares or least absolutes, with assumed initial values for the other parameters. The **residuals**, the observations – the deterministic fit,

$$e_i = y_i - (\hat{\lambda} + \hat{\beta} x_i), \tag{12.6.4}$$

are calculated and then the rank order for each residual is calculated. Notice that this ordering will alter as the parameter estimates change in the optimization steps. Thus the ordering needs to make use of some ordering function in the software, such as that provided by RANK( ) in Excel. In the second stage the rankits or median rankits are obtained corresponding to the $(x_i,\, y_i)$ pairs. The process then follows that already used in Table 12.5–3. Notice that the "residuals," between the ordered data and the full quantile fit of Tables 12.5–3 and 12.6–2 are called **distributional residuals** and denoted by f.

It should be noted that the approach described here is not the same as the method referred to as **quantile regression** (see Koenker and Bassett (1978)). This, though it refers to models in quantile form, is a non-parametric approach to fitting specific quantiles that makes no use of the form of the stochastic quantile

Table 12.6–2. Fitting regression quantile models of the form $Qy(p;x) = \lambda + \boldsymbol{\beta x} + \eta S(p;\alpha)$.

Stage	Calculation	Values row 1	Values row $\boldsymbol{n}$	Comments
Conditions	$E[S(u;\alpha)] = 0$ for DLS $S(0.5,\alpha) = 0$ for DLA			u = uniform for all α
Initial or current estimates	$\hat{\lambda}$, $\hat{\boldsymbol{\beta}}$, $\hat{\eta}$, $\hat{\alpha}$			
Raw data order - i		1	n	
Regressors - x_i		x_1	x_n	
Dependent Variable - y_i	it may simplify the LS calculations to center these on their average	y_1	y_n	
Stage 1				
Regression fit (deterministic)	$\hat{y}_i = \hat{\lambda} + \hat{\boldsymbol{\beta}}\boldsymbol{x} =$ $E(y; x_i, \hat{\lambda}, \hat{\boldsymbol{\beta}}, \hat{\eta}, \hat{\alpha})$, the mean line, or $\hat{y}_i =$ $Q(1/2; \boldsymbol{x_i}, \hat{\lambda}, \boldsymbol{\beta}, \hat{\eta}, \hat{\alpha})$, the median line	$\hat{y}_1$	$\hat{y}_n$	
Regression residual	$e = y - \hat{y}$	e_1	e_n	R
Squares residuals or absolute residuals	e^2 (SR) $\lvert e\rvert$ (AR)	e_1^2 $\lvert e_1\rvert$	e_n^2 $\lvert e_n\rvert$	sum = SSR sum = SAR
Fit deterministic component by LS or LA	revise $\hat{\lambda}$, $\hat{\boldsymbol{\beta}}$ assuming $\hat{\eta}$, $\hat{\alpha}$;			for linear forms LS gives explicit solutions with LA search for minimum
r_i = rank order of regression residual e_i	1, 2 for smallest, n for largest	r_1	r_n	keep to original data order
For DLS	rankits $\mu_s = \mu_{(r)}$	μ_1	μ_n	see Tables 12.4–1, and 12.5–3
For DLA - $p^*_{r,i}$	= IIB $(0.5, r, n+1-r)$	$p^*_{r_1,1}$	$p^*_{r_n,n}$	
For DLA	median rankits $M_i = M_{(r)}$	M_1	M_n	$M_i = S(p^*_r; \hat{\alpha})$
Stage 2				
Whole model fit $Q(p; x_i, \lambda, \boldsymbol{\beta}, \eta, \alpha)$	$\hat{\hat{y}}_i = \hat{\lambda}^+ \hat{\boldsymbol{\beta}} \boldsymbol{x_i}$ $+\hat{\eta} S(p_r; \hat{\alpha})$ $S(p; \hat{\alpha}) = \mu_r$ or M_r	$\hat{\hat{y}}_1$	$\hat{\hat{y}}_n$	
Distributional residual	$f_i = y_i - \hat{\hat{y}}_i$	f_1	f_n	DR
Criteria C_S DLS or C_A DLA	f_i^2 $\lvert f_i\rvert$	f_1^2 $\lvert f_1\rvert$	f_n^2 $\lvert f_n\rvert$	sum $C_S = \sum f_i^2$ $C_A = \sum \lvert f_i\rvert$
Optimize parameters	choose all parameters to minimize C			measures of fit (Table 12.7–1) – DRMSE and DMAE
Fit-observation plot	y_i against $\hat{\hat{y}}_i$			
Distributional residual plot	f_i against $\hat{\hat{y}}_i$			

distribution. Each quantile requires a separately fitted model. The approach here defines a parametric model for the complete situation, as in Gilchrist (2000) and (2008). Note that the criteria C_S and C_A used in fitting both distributions and regression models provide simple measures, DRMSE and DMAE, see Table 12.4–1 for comparing combinations of both models and estimation methods. One can calculate these quantities simply for most situations and compare numerical values and, unlike the chi-squared statistic, use all the data in the comparison. The comparison is most natural for comparing models. For these comparison of methods, the methods of DLS and DLA will naturally give the best results by these criteria. However, if the fits of other methods by these criteria are dramatically worse, then there is useful information in the comparison. The related plots, referred to in Tables 12.5–3 and 12.6–2, are also significant tools for the comparison of models and methods.

Examples: Karian and Dudewicz (2000) do not address issues of regression. However, they do provide some sets of paired data and use them for bivariate modeling. Though there are quantile-based approaches to bivariate and multivariate distributions, see Gilchrist (2000) and Cai (2010), the data is used here to illustrate regression models of quantile form. Table 12.5–4 (b) gives examples of the results of such analysis. As a specific example consider the rainfall data given for two cities and denoted by X and Y in K & D, page 103. Fitting Y by the method of percentiles, as the method of moments is not applicable, a fit of the GLD leads to a DRMSE of 0.084. A skew lambda distribution (SkLD) gives a better fit. However, both these ignore the requirement that a reasonable model will lie on the positive axis. This suggests a Weibull distribution, for which, using DLS, the DRMSE is 0.047 or a power-Pareto distribution with DRMSE of 0.050. Using the method of DLA, the DMAE is 0.0340 for the Weibull and 0.0355 for the power-Pareto compared with 0.0434 for the GLD. However, a scatter-plot of the data suggests some relation between the variables, so X was used as a regressor influencing the heterogeneity of the Weibull distribution through the model

$$y = \eta x^{\Phi}[-\ln(1-p)]^{\beta}. \tag{12.6.5}$$

For this Weibull-based regression the method of DLS gave a DRMSE of 0.037 and the method of DLA gave 0.0283. Note that in matching the $y_{(r)}$ to the p'_r, the order r given by the Rank column in the second part of Table 12.6–3, has to be based on the ordering of y/x^{Φ}, for this correctly orders the p in $S(p)$.

This three-parameter model thus provides a good means of describing the data and illustrates how complete models are given and estimated using the quantile form. The classical separation of the analysis of the deterministic regression from that of the distributional component disappears. The search procedure in Excel will find all parameters to minimize the criterion being used. Previous comment has been made on the transparency and ease of use of spreadsheet format in analyzing data and fitting models. For illustration, the second part of Table

Table 12.6–3. Rain data fitting – spreadsheet layout, values after minimization.

Weibull Distributional Fit			current DLS	previous DLA	
		n	47	47	
		eta	0.3838	0.3884	
		beta	1.1282	1.1288	
		DRMSE	0.0450	0.0455	
		DMAE	0.0334	0.0327	
	ordered		sum f^2 -|f|	0.0951	0.0951
r	Y	p'	W(p')	f	|f|
1	0.02	0.0208	0.0049	0.0151	0.0151
2	0.04	0.0417	0.0109	0.0291	0.0291
3	0.04	0.0625	0.0174	0.0226	0.0226
4	0.04	0.0833	0.0244	0.0156	0.0156
5	0.05	0.1042	0.0318	0.0182	0.0182
6	0.05	0.1250	0.0396	0.0104	0.0104
7	0.07	0.1458	0.0477	0.0223	0.0223
8	0.08	0.1667	0.0563	0.0237	0.0237
9	0.09	0.1875	0.0652	0.0248	0.0248
10	0.09	0.2083	0.0744	0.0156	0.0156
			etc.		
43	1.05	0.8958	0.9638	0.0862	0.0862
44	1.09	0.9167	1.0717	0.0183	0.0183
45	1.25	0.9375	1.2127	0.0373	0.0373
46	1.26	0.9583	1.4146	-0.1546	0.1546
47	1.79	0.9792	1.7673	0.0227	0.0227

Regression with Multiplicative Weibull				current	previous		
					DLS	DLA	
				phi	0.0412	0.0414	
				eta	0.4003	0.4047	
				beta	1.1211	1.1077	
				DRMSE	0.0346	0.0348	
				DMAE	0.0282	0.0278	
					sum f^2 -|f|	0.0564	0.0564
X	Y	Y/X^phi	Rank	p'	fit	f	|f|
0.01	0.14	0.1692	15	0.3125	0.1102	0.0298	0.0298
0.01	0.09	0.1088	10	0.2083	0.0649	0.0251	0.0251
0.02	0.19	0.2232	23	0.4792	0.2111	-0.0211	0.0211
0.02	0.17	0.1997	19	0.3958	0.1581	0.0119	0.0119
0.03	0.05	0.0578	6	0.1250	0.0363	0.0137	0.0137
0.03	0.37	0.4274	33	0.6875	0.4105	-0.0405	0.0405
0.03	1.26	1.4556	46	0.9583	1.2668	-0.0068	0.0068
0.03	0.42	0.4852	35	0.7292	0.4676	-0.0476	0.0476
0.06	0.04	0.0449	4	0.0833	0.0231	0.0169	0.0169
0.07	0.04	0.0446	3	0.0625	0.0166	0.0234	0.0234
				etc.			
1.38	0.42	0.4145	32	0.6667	0.4508	-0.0308	0.0308
1.65	0.94	0.9208	42	0.8750	0.9286	0.0114	0.0114
1.87	0.17	0.1657	14	0.2917	0.1245	0.0455	0.0455
2.51	1.25	1.2035	45	0.9375	1.3043	-0.0543	0.0543
2.61	0.94	0.9036	41	0.8542	0.8680	0.0720	0.0720

Table 12.7–1. Validation plots.

Name of plot	y	against x	Comment
Fit-observation plot	$x_{(r)}$	$\hat{Q}(p_r)$	looking for linearity
Distributional residuals	$f_r = x_{(r)} - \hat{Q}(p_r)$	$\hat{Q}(p_r)$	looking for randomness
Control Chart based on exponential spacing	$y_{(r)} = -\ln(1 - \hat{F}(x_{(r)})$ $v(r) = (n + 1 - r)(y_{(r)} - y_{(r-1)})$	 r	P control limit at $A = -\ln(1 - P)$ looking for out of control values

12.6–3 gives sections of the spreadsheet used for this last Weibull-based regression fit. The process of optimization is as described for the distributional fitting with "Solver." Note that both mean and median regression lines are found at the final step in regression, by replacing the quantile function by its mean or median. The estimation of the parameters can therefore be carried out using either DLS or DLA and using p', p^*, or some other suitable values. It is, however, natural to use the true or approximate rankits when fitting by DLS to seek the mean regression and p^* when using DLA to find the median regression.

12.7 Validation

There are a number of measures and plots that are suitable for testing the validity of models such as those just used. Table 12.7–1 lists some of these.

The DRMSE and DMAE provide prime numerical measures, comparing the value obtained with that for rejected models. Formal tests of fit can be devised, (see Gilchrist (2000)); however, the logic of their use has several problems. Firstly, investigators rarely start with a hypothetical model as required for formal significance tests, but are searching for the best model. Secondly, by choosing models with more parameters, better fits can always be found, so a compromise is usually found between model simplicity and quality of fit. In the comparison of models in Table 12.7–1 it will be noted that in some cases an alternative model does not give a much better fit but reduces the number of parameters. The difference between the approach adopted on regression above and the "non-parametric" methods of quantile regression is that often in the latter there are more fitting parameters than observations and the object is just to look at the data present and not to build a model for any other use. Thirdly, experience shows that if we have enough data we will almost always reject a hypothesized simple model. These considerations lead normally to keeping to a graphical inspection of a suitable plot as sufficient for determining whether the model is adequate (not True). Table 12.6–3

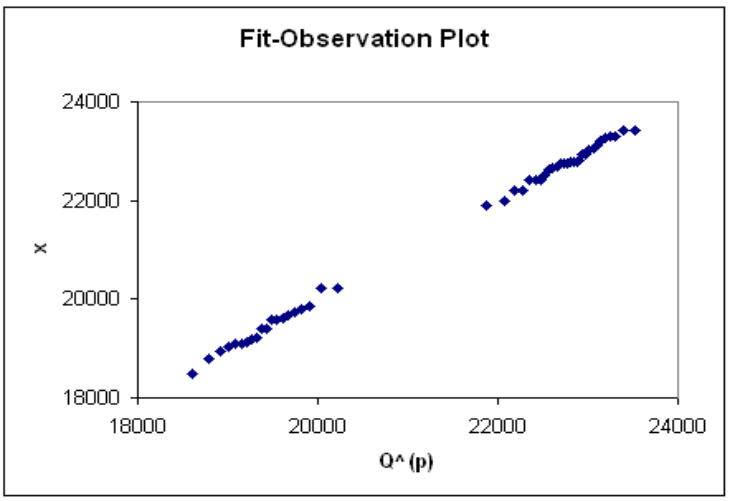


Figure 12.7–2. Fit-observation plot for the galaxy data.

gave some of the quantities used in setting up useful plots. Figure 12.7–2, as an example, shows the fit-observation plot for the "Galaxy" data based on a normal fit for the slower galaxy and a skew-normal for the faster. This plot simply plots the ordered data, $x_{(r)}$, against the fitted rankits $\hat{Q}(p_r)$. The distributional Residuals, given by $f_r = x_{(r)} - \hat{Q}(p_r)$, and illustrated for the Soil data in Figure 12.2–5, also provide a good guide as to quality of fit and particularly reveal any suspect observations.

12.8 Conclusion

The aim of this chapter has been to show an approach to statistical modeling in the distributional and regression contexts rather different from that usually presented. This leaves the question as to whether this approach has merit. Table 12.5–4 gave the distributional root mean square and mean absolute errors for the best fit given for some of the datasets in K & D, using either the method of moments or of the method of percentiles. It also gives for comparison approximate DLS and DLA fit and the criteria values for possible alternative models. It must be recognized that K & D seeks simply to illustrate the methods discussed for GLD fitting rather than seeking alternative models. However, the improved fits obtained for many of the datasets shows that the various models and methods discussed in this chapter are at least worthy of consideration. It will be seen from the tables that quantile function-based modeling provides a substantial and

flexible range of tools that focus on a careful analysis of the data. The methods avoid "black box" approaches and emphasize methods that look at the raw ordered data rather than just summary values. There are usually graphical presentations alongside the calculation elements, thus providing visual confirmation of the wisdom, or otherwise, of the methods being used.

References for Chapter 12

Balakrishnan, N. and Chen, W. W. S. (1997). *CRC Handbook of Tables for Order Statistics from Inverse Gaussian Distributions with Applications*, Boca Raton, Fl., CRC Press.

Balakrishnan, N. and Chen, W. W. S. (1999). *CRC Handbook of Tables for Order Statistics from Lognormal Distributions with Applications*, Kluwer, Dordrech, CRC Press.

Cai, Y. (2010). "Multivariate quantile function models." *Statistica Sinica*, 20, 481–496.

David, H. A. and Nagaraja, H. N. (2003). *Order Statistics (3rd Edition)*, Hoboken, New Jersey, John Wiley and Sons Inc.

Dielman, T. E. (2005). "Least absolute value regression: Recent contributions," *Journal of Statistical Computation and Simulation*, 75(4), 263–286.

Evans, M., Hastings, N., and Peacock, B. (1993). *Statistical Distributions (Second Edition)*, New York, John Wiley and Sons Inc.

Galton, F. (1883). *Enquiries into Human Faculty and its Development*, London, Macmillan.

Galton, F. (1886). "Regression towards mediocrity in hereditary stature," *Journal of the Anthropological Institute*, 15, 246–263.

Gilchrist, W. G. (1984). *Statistical Modeling*, Chichester, John Wiley and Sons Ltd.

Gilchrist, W. G. (1997). "Modeling with quantile distribution functions," *Journal of Applied Statistics*, 24, 113–122.

Gilchrist, W. G. (2000). *Statistical Modeling with Quantile Functions*, Boca Raton, Fl., Chapman and Hall/CRC Press.

Gilchrist, W. G. (2007). "Modeling and fitting quantile distributions and regressions," *American Journal of Mathematical and Management Sciences*, 27 (3 & 4), 401–439.

Gilchrist, W. G. (2008). "Regression revisited," *International Statistical Review*, 76, (3), 401–418.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*, Boca Raton, Fl., CRC Press.

Koenker, R. W. and Bassett, G. (1978). "Regression quantiles," *Econometrica*, 46 (1) 33–50.

Lloyd, E. H. (1952). "Least squares estimation of location and scale parameters using order statistics," *Biometrika*, 39, 88–95.

Öztürk, A. and Dale, R. F. (1985). "Least squares estimation of the parameters of the generalized lambda distribution," *Technometrics*, 27 (1), 81–84.

Parzen, E. (1979). "Nonparametric statistical data modeling," *Journal of the American Statistical Association*, 74, 105–121.

Tarsitano, A. (2005). "Estimation of GLD parameters for grouped data," *Communications in Statistics — Theory and Methods*, 34 (8), 1689–1710.

Chapter 13

Distribution Fitting with the Quantile Function of Response Modeling Methodology (RMM)

Haim Shore
Department of Industrial Engineering and Management,
Ben-Gurion University of the Negev,
P.O.B. 653, Beer-Sheva 84105, Israel
E-mail: Shor@bgu.ac.il

Distribution fitting aims to provide the data analyst with a general platform for modeling random variation. The need for such a "general-purpose" platform arises when either the available sample size is too small, or *a priori* knowledge about the source of the data is too scarce to identify with any acceptable degree of confidence the true underlying distribution. In such circumstances, distribution fitting proposes to substitute the unknown distribution by a member of a multi-parameter family of distributions, like the Johnson or Pearson families. It is assumed that such (known) families are flexible enough to deliver good representation to unknown distributions, no matter how diversely-shaped they might be in practice. Response Modeling Methodology (RMM), developed in the early nineties of the previous century (Shore, 2005, and references therein), provides such a platform. Although originally developed as a general methodology for empirical modeling of systematic variation (variation in a response traceable to variation of predictor variables correlated with the response), the quantile func-

tion of the RMM error distribution has been shown to deliver good representation to a wide range of variously shaped distributions. Furthermore, RMM reduces to some well-known distributions, approximations, and transformations for selected values of its parameters (Shore, (2004a), and (2005), Chapter 12).

In this chapter, we first outline the rationale for using an RMM quantile function as a general platform for distribution fitting (Section 13.1). In the following Section 13.2 we derive the quantile function of RMM's original model and some variations. Estimation via maximum likelihood, percentile matching, and moment matching are addressed.

13.1 The General Approach to Fitting by Response Modeling Methodology (RMM)

We start by demonstrating numerically a general property shared by all positively skewed unimodal distributions (distributions with one peak). This feature, once properly modeled, becomes the cornerstone for using the RMM quantile function as a general platform for distribution fitting.

Let us start by drawing a sample of some known distributions. We confine ourselves to positively skewed distributions, namely, those with elongated right tail (negatively skewed distributions may be similarly addressed). Our sample comprises the following four distributions, with different values of skewness (Sk).

1. The log normal distribution with parameters $(\alpha, \beta) = (1, 0.45)$, and $Sk = 1.5277$. The probability density function (pdf) of this distribution is

$$f(x) = \frac{0.89e^{-2.47(\ln(x)-1)^2}}{x}. \tag{13.1.1}$$

2. The exponential distribution, with $Sk = 2$ [gamma with parameters (1, 2)]. The pdf for this distribution is

$$f(x) = \frac{e^{-x/2}}{2}. \tag{13.1.2}$$

3. The Weibull distribution, with parameters (0.6, 4), and $Sk = 4.593$. The pdf of this distribution is

$$f(x) = \frac{0.26e^{-0.43x^{0.6}}}{x^{0.4}}. \tag{13.1.3}$$

4. The gamma distribution, with parameters (0.1, 2), and $Sk = 6.324$. The pdf for this distribution is

$$f(x) = \frac{0.098e^{-x/2}}{x^{0.9}}. \tag{13.1.4}$$

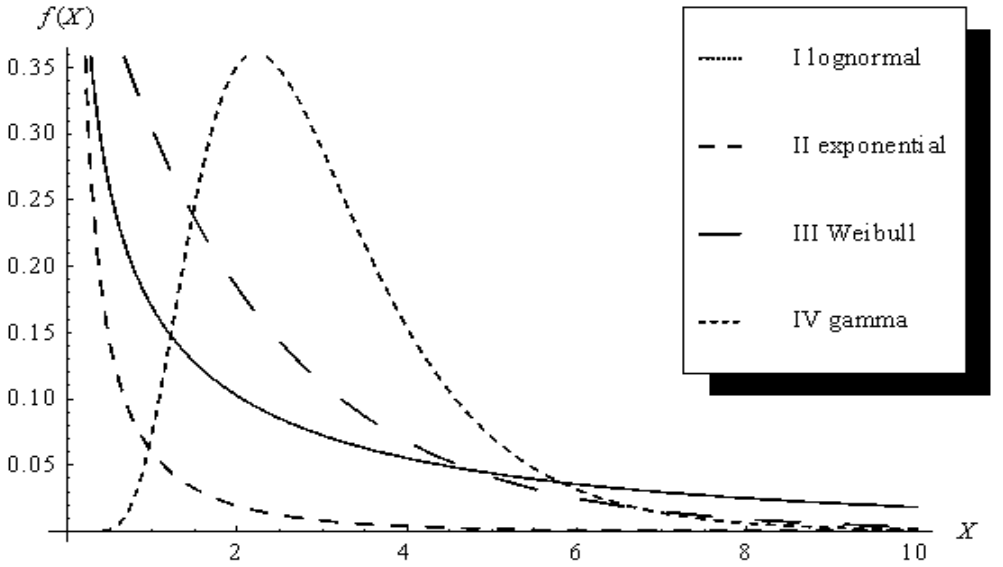


Figure 13.1–1. Probability density functions of the four distributions in the sample.

A plot of the probability density functions of these distributions is shown in Figure 13.1–1. Various levels of asymmetry, as reflected by the skewness values, are easily detected in the associated pdf plots.

Let us next select a sample of values, $\{P_i\}$, of the CDF (cumulative distribution function), and find for each P_i the corresponding quantile of the standard normal distribution, z_{p_i}. The following sample has been selected (altogether 15 pairs of values):

$\{P_i, z_{p_i}\}$: {0.001, -3.0902}, {0.01, -2.3263}, {0.05, -1.6448}, {0.10, -1.2815}, {0.2, -0.84162}, {0.3, -0.5244}, {0.4, -0.2533}, {0.5, 0}, {0.6, 0.2533}, {0.7, 0.5244}, {0.8, 0.8416}, {0.9, 1.2815}, {0.95, 1.6448}, {0.99, 2.3263}, {0.999, 3.0902}.

Next we match, for each standard normal quantile, z_p, the corresponding quantile, x_p, from each of the selected four distributions, namely,

$$F(x_p) = \Phi(z_p) = P, \tag{13.1.5}$$

where F and Φ are the CDFs of the random variable X (associated with one of the four distributions in the sample), and the standard normal variable, Z, respectively. Henceforth, the subscript p in the notation for quantiles will be omitted for the sake of simplicity.

A sample of 15 observations is obtained for each distribution, where each observation consists of two values: a standard normal quantile and the corresponding quantile of X. For each distribution, these observations may be per-

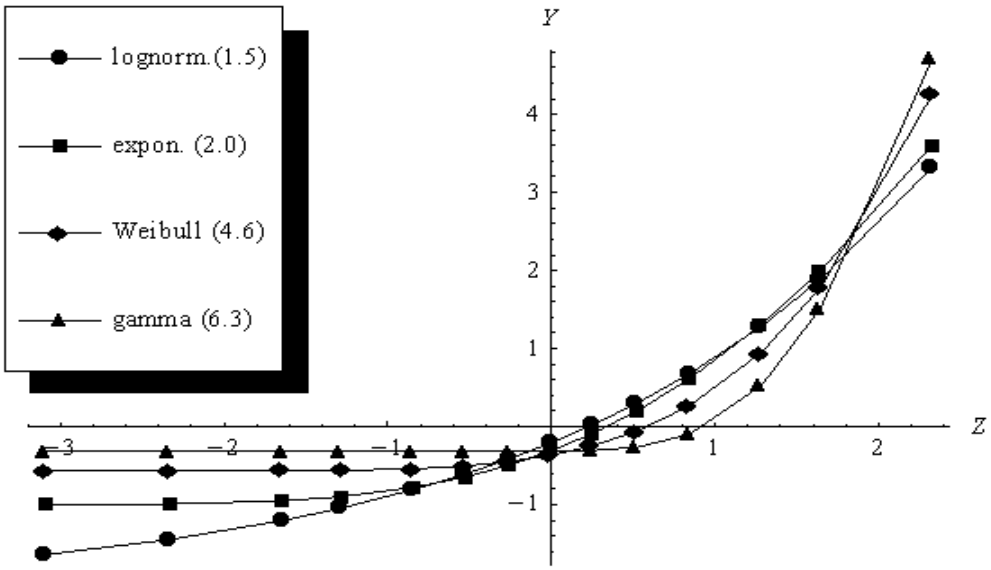


Figure 13.1–2. Standardized quantiles (Y) of lognormal ($Sk = 1.5$), exponential ($Sk = 2$), Weibull ($Sk = 4.6$) and gamma ($Sk = 6.3$) as a function of the standard normal quantile, Z.

ceived as values from an unknown quantile function that expresses quantiles of X in terms of the corresponding standard normal quantiles (generally these quantile functions are unknown and we will attempt to approximate them via RMM modeling).

Figure 13.1–2 shows plots of the resulting quantile functions for the four distributions: log-normal, exponential, Weibull, and gamma. The horizontal axis displays the standard normal quantiles and the vertical axis the corresponding standardized quantiles (denoted by Y) of the modeled distribution.

Figure 13.1–2 shows what might have been expected: all quantile functions, expressed in terms of corresponding quantiles of the symmetric standard normal distribution, are non-linear monotone-increasing **convex** functions. A similar assertion can be made regarding negatively-skewed distributions: their quantile functions will all be non-linear monotone-increasing **concave** functions. The basis for these general assertions is a theorem, proved by van Zwet (1964), that a convex (concave) transformation of a symmetrically distributed random variable (r.v.) has positive (negative) skewness. The "inverse" theorem has not been given (or proven) therein, namely, that a positively skewed r.v. is necessarily a convex transformation of a symmetrically distributed r.v. However, this is our approximating assumption here: we perceive quantile functions of positively skewed r.v.s to be monotone-increasing convex transformations of a symmetrical

r.v., in this case, the standard normal variable. This assumption implies that if a model was available, capable of representing a wide range of monotone convex functions (namely, functions widely differing in their degree of convexity), then that model may also be implemented as a general platform for modeling quantile functions. RMM provides such a model.

The underlying concept of RMM is that monotone convexity can be represented by a "ladder of monotone convex functions" (Shore, 2005), wherein certain functions are arranged in a hierarchical order according to their intensity (degree) of convexity. This hierarchy of functions consists of repeated re-appearing, as we move up the ladder, of a basic cycle of "linear, power, exponential" elements in the modeled functions. Thus, the Newton kinetic energy equation, $mV^2/2$ (m is mass and V velocity), is represented in the ladder by a power function and radioactive decay by an exponential function, both belonging to the first cycle on the ladder. The Arrhenius formula,

$$R_e(T) = A \exp[-E_\alpha/(k_B T)],$$

is an exponential-power function and R_e is the rate of a chemical reaction. The well-known Gompertz growth law, $y = e^{-\alpha e^{-bx}}$, is an exponential-exponential function. Both of these belong to the second cycle on the ladder. The types of functions represented on the ladder have been shown to be widely prevalent in the engineering and exact sciences (Shore (2004b) and (2005), Shore and Benson-Karhi (2007), and references therein). The RMM model integrates all these functions into a single model, whereof the ladder's individual functions can be derived by setting certain values to the model's parameters. This property implies that, given empirical data that exhibit monotone convexity, the final structure of the fitted model is not determined *a priori* (before parameter estimation), but rather determined by the parameters' estimates. No selection of a particular model is needed. This is a desirable property for distribution fitting, where the underlying distribution that has generated given data ("the true model") and the allied quantile function are not known.

Since the **error distribution** of RMM is associated with models of monotone convex relationships, one would expect the RMM quantile function to be also monotone convex. Given that all unimodal positively skewed distributions have monotone convex quantile functions (as discussed earlier), one would anticipate the quantile function of RMM to deliver good representation to these unknown quantile functions. Indeed, extensive research that we have conducted has shown that this conclusion is corroborated by empirical evidence (Shore (2005) and (2008), Shore and A'wad, (2010)). We will elaborate on this further in subsequent sections.

An updated list of references related to RMM may be found at the author's personal homepage, http://www.bgu.ac.il/shor/index.htm.

13.2 The Quantile Function of the RMM Model and its Estimation

13.2.1 Derivation of the RMM Quantile Function

At the center of RMM is a relational model, which describes a modeled response, Y, in terms of a linear combination of effects (the linear predictor, LP, denoted by η which transmits systematic variation to the response, two possibly correlated normal errors, ε_1 and ε_2, and a vector of parameters),

$$W = \ln Y = \frac{\alpha}{\lambda}\left[(\eta + \varepsilon_1)^{\lambda} - 1\right] + \mu_2 + \varepsilon_2, \tag{13.2.1}$$

where α, λ, μ_2 are parameters that need to be determined. This model is derived axiomatically from five basic assumptions, one of which is that the random variation component of the model (the error structure) is represented by a bivariate normal distribution (find further details in Chapter 7 of Shore, (2005)). It is easy to realize that (13.2.1) is highly flexible in modeling monotone convex relationships. Ignoring the error terms and the constant μ_2, consider the following simplified model

$$Y = \exp\left\{\frac{\alpha}{\lambda}\left[\eta^{\lambda} - 1\right]\right\}. \tag{13.2.2}$$

One can easily derive the first four "steps" (functions) of the ladder from (13.2.2):

- Step 1 - Linear (with η): $\lambda = 0$, $\alpha = 1$ (note that as $\lambda \to 0$, $1/\lambda[\eta^{\lambda} - 1] \to \ln \eta$)
- Step 2 - Power: $\lambda = 0$, $\alpha \neq 1$
- Step 3 - Exponential (also exponential-linear): $\lambda = 1$
- Step 4 - Exponential-power: $\lambda \neq 0$, $\lambda \neq 1$

How would one extract the next exponential-exponential-linear and exponential-exponential-power cases (Steps 5 and 6 on the ladder)? Insert

$$\exp\left\{(\beta/\kappa)[\eta^{\kappa} - 1]\right\} \tag{13.2.3}$$

for η in (13.2.2) (two new parameters, β and κ are introduced). One can easily verify that for $\kappa \to 0$, $\beta \to 1$,

$$\exp\left\{\frac{\beta}{\kappa}\left[\eta^{\kappa} - 1\right]\right\} \to \eta, \tag{13.2.4}$$

so that (13.2.2) is revoked (rendering it a special case of (13.2.3)). Thus, for the "price" of two additional parameters, β and κ, an additional cycle of the basic "linear, power, exponential" pattern is invoked, allowing us to climb the ladder to

its 5th and 6th "Steps." Indeed, it is a general principle that with the introduction of two additional parameters, additional repetition of the basic cycle is achieved, and we climb up the ladder to achieve functional models with ever-increasing degree of monotone convexity. As we have seen earlier (Figure 13.1–1), climbing the ladder implies, in the context of distribution fitting, moving to quantile functions associated with distributions of increasing skewness (asymmetry).

Assume that the errors in (13.2.1) derive from a bivariate normal distribution with correlation ρ. Let us express the errors as $\varepsilon_1 = \sigma_{\varepsilon_1} Z_1$ and $\varepsilon_2 = \sigma_{\varepsilon_2} Z_2$ where Z_1 and Z_2 are standard normal variables from a bivariate standard normal distribution, with correlation ρ, and σ_{ε_1} and σ_{ε_2} are the standard deviations of the errors. Expressing Z_2 in terms of Z_1 plus an additive **independent error**, we obtain

$$Z_2 = \rho Z_1 + (1 - \rho^2)^{(1/2)} U_2, \tag{13.2.5}$$

where U_2 is a standard normal variable, independent of either Z_1 or Z_2 (note that in a completely symmetrical fashion we could express Z_1 in terms of Z_2 plus an additive **independent** error). The RMM model becomes (we henceforth replace U_2 in (13.2.5) by Z_2 in order to preserve uniformity of notation)

$$W = \ln Y = \frac{\alpha}{\lambda}\left[(\eta + \sigma_{\varepsilon_1} Z_1)^{\lambda} - 1\right] + \mu_2 + \sigma_{\varepsilon_2}\left[\rho Z_1 + (1 - \rho^2)^{(1/2)} Z_2\right], \tag{13.2.6}$$

where Z_1 and Z_2 are two **independent** standard normal variates.

We now address the quantile function of this RMM model. To do that, we assume in (13.2.6) that $\eta = C$ (a constant). Model (13.2.6) becomes the quantile function of the RMM error distribution (a bivariate distribution). Without loss of generality, we can put: $\eta = 1$ (this only causes changes in the other parameters of the model), to obtain

$$W = \ln Y = \frac{\alpha}{\lambda}\left[(1 + \sigma_{\varepsilon_1} Z_1)^{\lambda} - 1\right] + \mu_2 + \sigma_{\varepsilon_2}\left[\rho Z_1 + (1 - \rho^2)^{(1/2)} Z_2\right], \tag{13.2.7}$$

which may be used as a general model for fitting to bivariate data, namely, a response with two sources of errors. Since this chapter is dedicated to distribution fitting for univariate data, that option is not explored here any further.

Let us make the simplifying assumption in (13.2.7) that $\rho = \pm 1$. Though the assumption introduces reduced precision in characterizing the error structure of the RMM model (13.2.1), it allows eliminating one parameter that needs to be estimated (namely, ρ). More importantly, an RMM quantile function is now obtained that is expressed in terms of the quantile of a single standard normal variable, Z. This will qualify (13.2.7) to be used as a platform for distribution fitting to **univariate** data. Put in a re-parameterized form, (13.2.7) becomes, with $\rho = \pm 1$,

$$W = \ln Y = \mu_2 + \frac{\alpha}{\lambda/\sigma_1}\left\{[1 \pm \sigma_1 Z]^{\lambda/\sigma_1} - 1\right\} \pm \sigma_2 Z, \tag{13.2.8}$$

where α, λ, σ_1, σ_2 are parameters that need to be determined (note that for simplicity σ_{ε_1} and σ_{ε_1} in (13.2.7) have been replaced by σ_1 and σ_2, respectively). On fitting (13.2.8) to actual data, the signs in front of either $\sigma_1\,(>0)$ or $\sigma_2\,(>0)$, but not both, may be negative. This conclusion is easily derived, regarding σ_2, from (13.2.7) on introducing therein $\rho = \pm 1$ (find further details for the validity of this assertion in Shore (2005)). Note also that the parameters in (13.2.8) are so written (relative to (13.2.1)) for computational convenience. For example, if estimates of λ or σ_1 tend to zero, limiting forms for model (13.2.8) are easily suggested.

For $Z = 0\,(P = 0.5)$, we have $\ln Y = \mu_2 = \ln M$, where M is the median of Y. Equation (13.2.8) may therefore be re-written as

$$W = \ln Y = \ln M + \frac{\alpha}{\lambda/\sigma_1}\left\{[1 \pm \sigma_1 Z]^{\lambda/\sigma_1} - 1\right\} \pm \sigma_2 Z. \tag{13.2.9}$$

Re-parameterized for simplicity, (13.2.9) becomes

$$W = \ln Y = \ln M + \frac{A}{B}\left\{[1 + CZ]^B - 1\right\} + DZ, \tag{13.2.10}$$

with M, A, B, C as parameters that need to be estimated. If M is estimated directly from the data (as the sample median), (13.2.10) has only four parameters that need to be estimated.

One may reasonably assume that in (13.2.10) the error term, CZ, is in absolute value much smaller than 1, designated by $|CZ| << 1$. An equivalent statement is that the error associated with the linear predictor is much smaller than the linear predictor itself; observe (13.2.1)). This suggests that a variation of (13.2.10), with a reduced number of parameters, may be derived as an asymptotic case. We obtain (find details in Shore (2005), p. 109):

$$W = \ln Y = \ln M + \left(\frac{A}{B}\right)\left[e^{BZ} - 1\right] + CZ, \tag{13.2.11}$$

(assuming that M is estimated directly by the sample median). While (13.2.10) delivers as special cases both the normal distribution ($A = 1$, $B = 0$, $D = 0$) and the lognormal distribution ($A = 0$, $D = 0$), (5.2.11) has only the lognormal as a special case ($A = 0$); however, (13.2.11) represents well also the normal distribution (refer to Table 19.1 in Shore, 2005). Also note that for obvious reasons, in some earlier references, (13.2.10) and (13.2.11) were denoted "Inverse Normalizing Transformations (INTs)."

The support of Y in (13.2.11) is determined by M. If M is positive the support is $[0, \infty)$. If data suggest that there is a strictly positive lower limit on Y (like with censored reliability data), or if such a limit is required by theory, an additional location parameter, L, is needed so that (13.2.11) becomes

$$W = \ln(Y - L) = \ln(M - L) + \left(\frac{A}{B}\right)\left[e^{BZ} - 1\right] + CZ. \tag{13.2.12}$$

If L is unknown, setting it equal to the lowest available observation has been found to be good practice. Alternatively, L may be an additional parameter that needs estimation. Note, however, that introducing a location parameter to an RMM model requires that parameters' estimation via non-linear regression be applied to observations in the original scale (not the log scale), namely, to the expression

$$Y = L + (M - L)e^{\frac{A(e^{BZ}-1)}{B}+CZ}. \tag{13.2.13}$$

Another case that needs attention is when sample observations have mixed signs, where some observations are positive and some are negative. Since values derived from the RMM model all share the same sign (as determined by the sign of M), this also requires re-locating the model by adding a location parameter (denoted here by δ) to obtain

$$W = \ln(Y + \delta) = \ln(M + \delta) + \left(\frac{A}{B}\right)\left[e^{BZ} - 1\right] + CZ, \tag{13.2.14}$$

(One should be aware that L and δ are the same location parameter; however two different notations are introduced here to emphasize the two different scenarios when such re-locating might be needed.)

For most applications, (13.2.11) is preferred to (13.2.10) in modeling quantile functions. This is for two reasons: first, when (13.2.10) or (13.2.11) are fitted to data, computer search for the parameters in (13.2.10) may occasionally result in imaginary values (that is, when $1 + CZ < 0$). Secondly, assuming that M is estimated directly from the sample median, (13.2.11) contains only three parameters that need estimation (versus four for (13.2.10)). This may be beneficial, for example, in implementing distribution fitting via moment-matching. Estimates of only the first three moments will then be required, allowing us to avoid the large sampling errors associated with fourth-moment estimates (estimates of kurtosis).

Both (13.2.10) and (13.2.11) have been shown to deliver good representation for the (exact) quantile functions of various distributions like the binomial, Poisson, gamma, Weibull, and the extreme-value distributions. Fitting RMM to these distributions (no data sample involved!) had been carried out either by minimizing the L_2 norm (a functional distance measure) or by quantile matching, using non-linear least-squares applied to a representative sample of (exact) quantile values. The accuracy obtained, as assessed by the achieved optimal value of the L_2 norm, or by the preservation of the first four moments of the fitted distribution, was satisfactory even for highly skewed distributions (Shore (2005), (2007), (2008) and Shore and A'wad, (2010)).

For reasons expounded earlier, we will henceforth address only quantile function (13.2.11). Estimates for the parameters of (13.2.11) may be obtained via maximum likelihood, percentile matching, and moment matching.

13.2.2 Estimating RMM Quantile Function

Estimating via maximum likelihood

Quantile function (13.2.11) is an invertible function so that Z may be explicitly expressed in terms of Y and the function $PL(t) = \text{ProductLog}(t)$. The latter gives the principal solution for w in the equation $t = w\exp(w)$. For $t > -1/e$, this solution is always real. From the quantile function of Z in terms of Y,

$$Z(y) = \frac{1}{BC}\left\{A + (B)\ln(\frac{y}{M}) - (C)PL\left(\frac{Ae^{\frac{A+B\ln(y/M)}{C}}}{C}\right)\right\}, \tag{13.2.15}$$

the pdf of Y, $f(y)$, may be easily derived (find details in Shore (2007)) from

$$f(y) = \left\{\frac{\frac{\partial Q[Z(y)]}{\partial Z(y)}}{\phi[Z(y)]}\right\}^{-1} \tag{13.2.16}$$

where

$$\frac{\partial Q}{\partial Z} = Me^{\frac{A(e^{BZ}-1)}{B}+CZ}\left(C + Ae^{BZ}\right). \tag{13.2.17}$$

One realizes that in order for $f(y)$ to be a proper pdf, it should fulfill

$$M\left(C + Ae^{BZ}\right) > 0. \tag{13.2.18}$$

Given a sample of observations of Y, maximum likelihood (ML) estimates for the parameters of (13.2.11) may be derived from equations (13.2.15) through (13.2.18). This may require use of the EM (expectation-maximization) algorithm, which has become a standard tool for ML estimation when the ML function is algebraically complex.

Currently, an **R** program is being developed to facilitate use of this option in fitting RMM models. Since this chapter deals with direct fitting of the RMM quantile function, we will not pursue this option here any further.

Estimating via percentile matching

To demonstrate application of RMM to distribution fitting using percentile matching, we take an example given in Karian and Dudewicz (KD, 2000, p. 98), originally introduced in Dudewicz, Levy, Lienhart, and Wehrli (1989). The data are 23 observations on the brain tissue MRI scan parameter, denoted AD. The following observations are associated with scans from the left thalamus mimography

108.7	107.0	110.3	110.0	113.6	99.2	109.8	104.5
108.1	107.2	112.0	115.5	108.4	107.4	113.4	101.2
98.4	100.9	100.0	107.1	108.7	102.5	103.3	

Since AD was found to be non-normal (KD, 2000), distribution fitting is called for. We first fit the data to the RMM model (13.2.11), and then compare with results obtained by fitting the generalized lambda distribution (GLD). The latter's quantile function is given, in terms of the CDF value, p, by

$$Q(p) = Q(p;\, \lambda_1,\, \lambda_2,\, \lambda_3,\, \lambda_4) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2},\ 0 \le p \le 1. \qquad (13.2.19)$$

For both families of distributions (RMM and GLD), it is assumed that the sample median is preserved by the fitted quantile function, namely, for GLD,

$$\lambda_1 = M - \frac{1}{\lambda_2}\left[\left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}\right] \qquad (13.2.20)$$

(refer to (13.2.19)), where M is replaced by the sample median (107.4). Similarly, for RMM the parameter M is replaced by the sample median in (13.2.11). This practice for estimating M allows us to reduce by one the number of parameters that need estimation in the fitting procedure. In the second example below we discard this practice.

To fit the RMM quantile function (13.2.11), we sort the observations in an ascending order, and find for the i-th order statistic (there are $i-1$ smaller observations) the estimated CDF value, p_i, which is given by ($n = 23$)

$$p_i = \frac{i - 3/8}{n + 1/4}. \qquad (13.2.21)$$

The corresponding standard normal quantile is then calculated based on this value of p_i. For example, the first number above, 108.7, is the fifteenth smallest observation (there are 14 smaller observations). Therefore, for $i = 15$, $p_{15} = 0.6290$, and the corresponding standard normal percentile is $z_{15} = 0.3292$.

Once the 23 pairs of observations, $\{z_i,\, y_i\}$, are known, non-linear regression may be applied either to $\{z_i,\, y_i\}$ or to $\{z_i,\, w_i\}$, with $\exp(W)$ and W as the fitted quantile functions, respectively, $w_i = \ln(y_i)$, and W as given by (13.2.11). However, examining the data strongly suggest that there is a strictly positive lower limit. Therefore (13.2.12), with the additional location parameter, L,

$$Y = L + (M - L)\exp\left\{\frac{A}{B}\left[e^{BZ} - 1\right] + CZ\right\}, \qquad (13.2.22)$$

is used. Non-linear regression can now be applied to the data. A good option is to use a procedure such as `NonlinearModelFit` of Mathematica[1] (version 7.0 or later). This procedure provides all necessary statistical analysis tools for non-linear regression. Procedures for implementation of non-linear regression exist in all other available mathematical and statistical packages.

[1] Mathematica™ is a registered trademark of Wolfram Research.

Similarly, GLD may be fitted to the data using p_i values obtained from (13.2.21), with the parameter λ_1 as given by (13.2.20) (in order to preserve the sample median). The resulting parameters are

- For RMM (equation (13.2.22)):

$$A = 0.26890, \quad B = -2.6615, \quad C = 0.30497, \quad L = 98.958, \quad M = 107.4$$

- For GLD (equations (13.2.19) and (13.2.20)):

$$\lambda_1 = 111.67, \quad \lambda_2 = 0.059314, \quad \lambda_3 = 0.50834, \quad \lambda_4 = 0.064046, \quad M = 107.4.$$

We realize that M, A, C are all positive so that (13.2.18) is preserved. Figures 13.2–1 and 13.2–2 display the errors obtained for the fitted quantile functions (observations' values are given on the horizontal axis). One realizes that RMM and GLD deliver about equal goodness-of-fit, although the associated error variance is slightly smaller for the former (0.3034 for RMM versus 0.5321 for GLD). Note, that without altering the resultant goodness-of-fit, one could introduce for the RMM parameter, L, the smallest order statistic (98.4), instead of the value of 98.96 obtained from the estimation routine. This would spare us a search for estimates of four parameters in the least-squares procedure, a sometimes numerically prohibitive task.

Figures 13.2–3 and 13.2–4 display the respective pdfs of RMM and GLD.

Estimating via moment matching

Denote the k-th non-central moment (moment about zero) of Y in (13.2.22) by μ'_k. From (13.2.22), with $L = 0$, we have:

$$\mu'_k = E(Y^k) = \mu(kA,\, B,\, kC\,, M^k), \tag{13.2.23}$$

where $\mu(A,\, B,\, C,\, M)$ is an expression for the mean of Y in (13.2.22) in terms of the parameters A, B, C, and M (equation (13.2.23) may be easily deduced from (13.2.22) based on the definition of the k-th non-central moment). Expanding (13.2.22), with $L = 0$, into a series in powers of Z, around zero, and taking expectation of the first six terms in the expansion, we obtain (since all odd-order moments of Z are identically zero, and $E(Z^2) = 1$, $E(Z^4) = 3$),

$$\begin{aligned} E\left(\frac{Y}{M}\right) &= \frac{\mu}{M} \cong \frac{\mu(A,B,C,M)}{M} \qquad (13.2.24)\\ &= 1 + \frac{1}{2}\left[AB + (A+C)^2\right] + \frac{1}{8}\left\{(AB^3)\left[1 + \left(\frac{A}{B}\right)^3 + 6\left(\frac{A}{B}\right)^2 + 7\left(\frac{A}{B}\right)\right]\right.\\ &+ \left. 4AB^2C\left[1 + \left(\frac{A}{B}\right)^2 + 3\left(\frac{A}{B}\right)\right] + 6ABC^2\left(1 + \frac{A}{B}\right) + 4AC^3 + C^4)\right\}. \end{aligned}$$

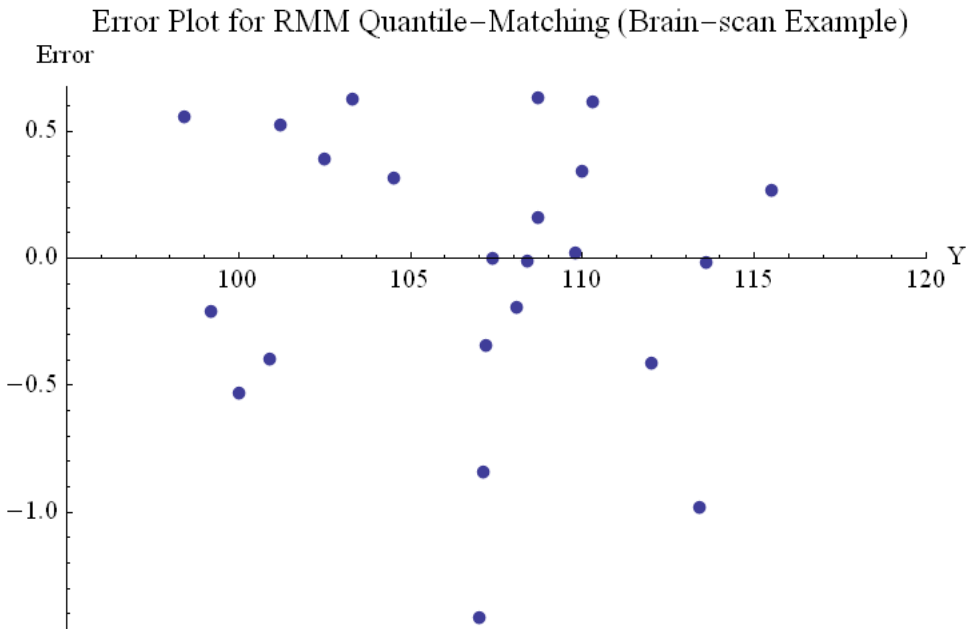


Figure 13.2–1. Error plot from fitting RMM to the brain-scan example ($\sigma_\varepsilon^2 = 0.3034$).

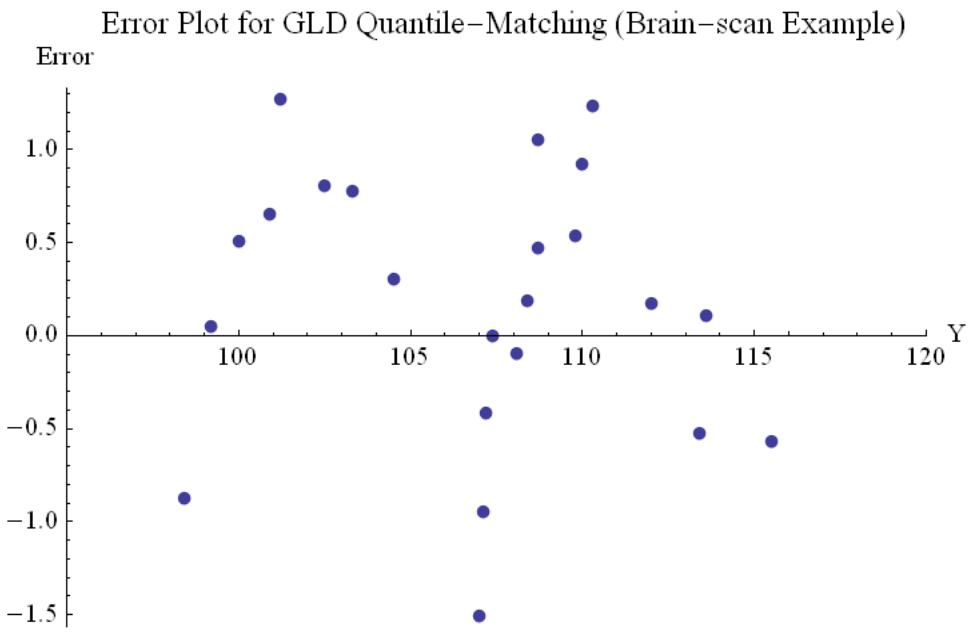


Figure 13.2–2. Error plot from fitting GLD to the brain-scan example ($\sigma_\varepsilon^2 = 0.5321$).

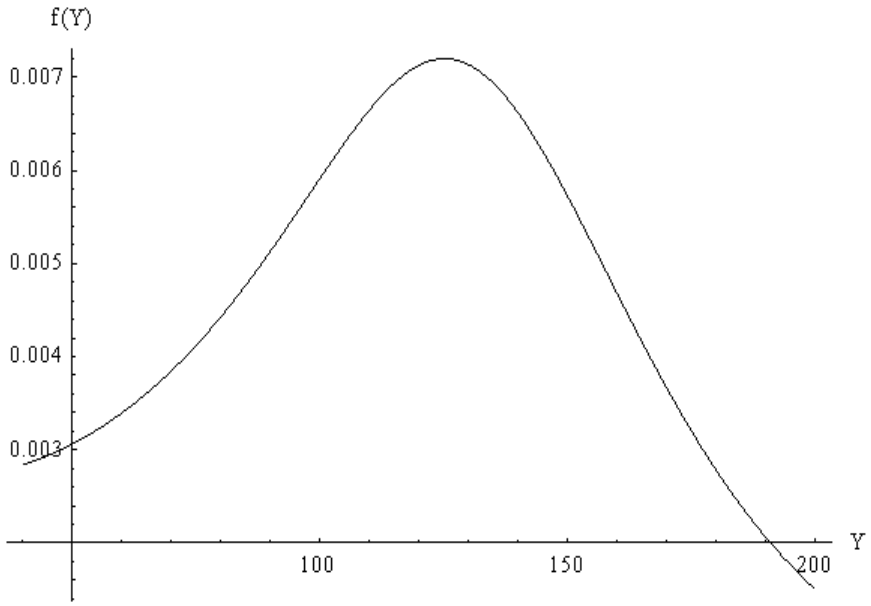


Figure 13.2–3. The RMM fitted density function for the brain-scan data.

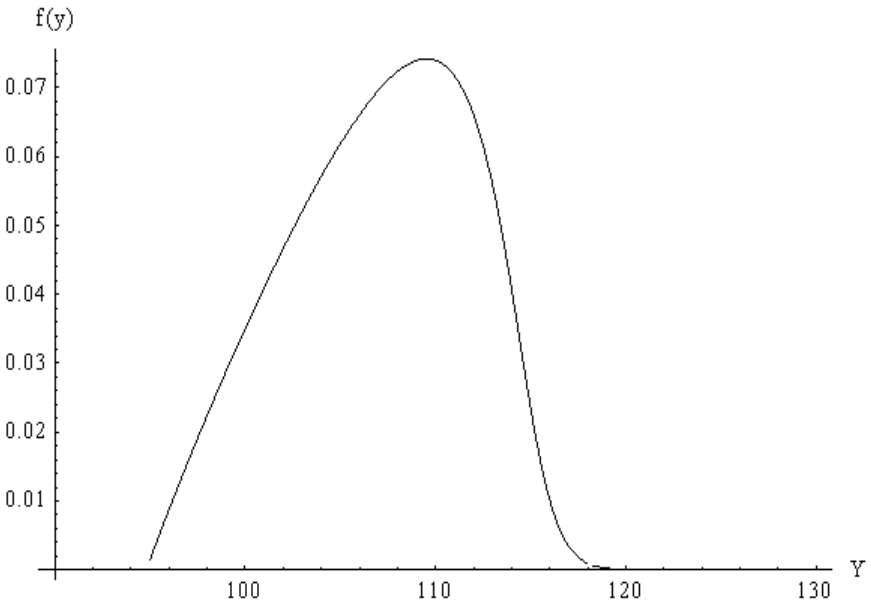


Figure 13.2–4. The GLD fitted density function for the brain-scan data.

From (13.2.23) and (13.2.24) an approximate explicit expression for the non-central k-th moment of Y in (13.2.22), with $L = 0$, is:

$$\begin{aligned}
\mu'_k &= E(Y^k) \cong \mu(kA, B, kC, M^k) \qquad (13.2.25)\\
&= M^k \left\{ 1 + \frac{1}{2}\left[kAB + (kA + kC)^2 \right] \right.\\
&+ \frac{1}{8}\left\{ (kAB^3)\left[1 + \left(\frac{kA}{B}\right)^3 + 6\left(\frac{kA}{B}\right)^2 + 7\left(\frac{kA}{B}\right)\right]\right.\\
&+ 4k^2AB^2C\left[1 + \left(\frac{kA}{B}\right)^2 + 3\left(\frac{kA}{B}\right)\right]\\
&+ \left.\left. 6k^3ABC^2\left(1 + \frac{kA}{B}\right) + 4k^4AC^3 + k^4C^4)\right\}\right\}.
\end{aligned}$$

Based on the approximate expressions for the non-central first, second and third moments (as given explicitly in (13.2.25) in terms of the unknown parameters), the mean, the variance, and the skewness measure may be obtained

$$\mathrm{Var}(Y) = \sigma^2 = \mu'_2 - \mu^2 \qquad (13.2.26)$$

and

$$Sk = [\mu'_3 - 3\mu'_2\mu + 2\mu^3]/\sigma^3. \qquad (13.2.27)$$

Three-moment matching may now be applied to estimate the parameters of (13.2.22) via moment matching. Given sample estimates of the moments, a root-finding routine (or a minimization routine) may be employed to estimate the parameters.

To demonstrate the application of this method, we refer to data about birth weights of twins, given in KD (2000). This example uses data from the Indiana Twin Study (find details in the afore-cited source). The data consist of $n = 123$ birth weights of twins, $\{Y_1, Y_2\}$, appearing in Table 13.2–5 in an ascending order of Y_1 (read lines).

KD fitted a GLD to Y_1, using moment matching, to obtain

$$\lambda_1 = 5.5872,\ \lambda_2 = 0.2266,\ \lambda_3 = 0.2089\ \lambda_4 = 0.1762.$$

We will analyze the distribution of Twin 1 (Y_1), using (13.2.22). Since there is probably a lower limit to the weights of babies, and as suggested earlier for such scenarios, we first subtract from the data the lowest observation (for the Y_1 data, this is 2.44) to obtain estimates for the first four moments of the re-located data

$$\begin{aligned}
\mu &= E(Y_1 - 2.44) = 5.49 - 2.44 = 3.05,\\
\sigma^2 &= \mathrm{Var}(Y_1) = 1.32,\\
Sk &= \sqrt{\beta_1} = E[(Y_1 - \mu)^3]/\sigma^3 = -0.0461 \text{ (Skewness)},\\
Ku &= \beta_2 - 3 = E[(Y_1 - \mu)^4]/\sigma^4 - 3 = -0.267 \text{ (Kurtosis)}.
\end{aligned}$$

Table 13.2–5. Birth Weights of Twins.

$\{Y_1$	$Y_2\}$	$\{Y_1$	$Y_2\}$	$\{Y_1$	$Y_2\}$	$\{Y_1$	$Y_2\}$
2.44	2.81	3.00	3.78	3.15	2.93	3.17	4.13
3.63	3.19	3.68	5.38	3.69	3.56	3.74	3.24
3.75	3.16	3.83	3.83	3.91	3.81	3.91	4.60
4.00	3.66	4.00	4.28	4.10	5.00	4.12	4.75
4.12	6.31	4.19	4.31	4.20	4.75	4.28	4.50
4.31	3.66	4.31	3.88	4.31	4.69	4.38	3.40
4.38	5.00	4.44	5.13	4.49	4.15	4.53	3.83
4.56	4.31	4.56	5.38	4.63	4.12	4.69	4.63
4.75	4.56	4.75	4.63	4.81	4.38	4.81	4.44
4.91	5.13	4.94	4.78	4.95	4.22	5.00	5.38
5.00	6.16	5.03	4.94	5.13	3.81	5.15	5.00
5.16	5.75	5.16	6.69	5.19	4.94	5.22	4.75
5.25	5.38	5.25	5.63	5.25	5.81	5.25	6.10
5.25	6.25	5.31	4.69	5.38	5.69	5.38	6.81
5.41	5.69	5.44	5.75	5.47	5.13	5.47	6.75
5.50	5.38	5.56	4.44	5.56	5.48	5.56	6.31
5.59	6.22	5.61	6.18	5.63	4.69	5.63	4.88
5.63	4.97	5.66	5.38	5.72	5.06	5.75	5.63
5.81	5.19	5.81	5.94	5.84	5.56	5.88	4.88
5.88	5.69	5.88	5.88	5.88	5.88	5.91	5.50
5.94	4.81	5.94	5.41	5.94	5.75	5.94	6.31
5.97	5.63	6.06	5.31	6.10	5.19	6.10	6.13
6.16	5.85	6.19	4.44	6.19	5.50	6.31	5.81
6.31	6.10	6.33	8.14	6.38	5.19	6.38	6.05
6.56	6.38	6.59	6.16	6.60	6.53	6.63	6.19
6.63	6.19	6.63	6.38	6.66	7.10	6.69	5.81
6.69	6.13	6.75	6.56	6.78	6.22	6.81	6.19
6.81	6.60	6.81	7.41	6.88	6.06	6.88	6.63
6.94	5.50	6.95	5.72	7.06	7.31	7.25	8.00
7.31	4.58	7.31	7.31	7.46	7.22	7.69	7.25
7.72	6.44	8.13	7.75	8.44	6.31		

Values of Sk and Ku close to zero suggest that the data derive from a normal distribution. This implies that on fitting (13.2.10) one can expect the parameters' estimates to be close to $A = 1$, $B = 0$, $D = 0$ (as related to earlier). Here we fit (13.2.11) only.

Using (13.2.25), (13.2.26), and (13.5.27) we wish to obtain parameters' estimates by minimizing the sum of the squared deviations of the sample moments from the corresponding moments (equations (13.2.25) (13.2.26), and (13.2.27)) of the fitted RMM quantile function. Unlike in the earlier example, we do search for M in implementing the minimization routine (instead of estimating M by the sample median). Ensuring in applying the routine that (13.2.18) is maintained (we have achieved this by constraining: $A, C > 0$, $B < 0$), we obtain,

$$A = 0.3549, \quad B = -0.4729, \quad C = 0.05355.$$

We also get $M = 3.038$ (versus the sample median for the re-located data of $5.56 - 2.44 = 3.12$).

This solution ensures that the estimated (13.2.22) has first three moments identical to those of the sample moments. It is good practice to plot the estimated function (13.2.22) in order to ensure that it is indeed a monotone-growing function of Z (maintaining (13.2.18) also ensures that). The parameters obtained indeed maintain (13.2.18). Note that instead of using a root-finding routine we have used a minimization routine. The use of a single objective function in a minimization routine (instead of the usual root-finding routine that searches for a solution to a system of three equations) allowed us the freedom to search for **four** parameters (including M) in the search routine. The final model is:

$$Y = 2.44 + 3.038 \exp\left\{\frac{0.3549(e^{-0.4729Z} - 1)}{-0.4729} + 0.05355Z\right\} \tag{13.2.28}$$

Numerical examples for values obtained from the estimated (13.2.28)

- For the 10th smallest value (sample value: **3.83**, there are nine smaller values)

$$p_{10} = \frac{10 - 3/8}{123 + 1/4} = 0.07809, \; Z = -1.41804, \; Y_{app} = 3.82 \text{ (from(13.2.7))}$$

- For the 101st smallest value (sample value: **6.63**, there are 100 smaller values):

$$p_{101} = \frac{101 - 3/8}{123 + 1/4} = 0.8164, \; Z = 0.9017, \; Y_{app} = 6.58 \text{ (from(13.2.7))}.$$

Figures 13.2–6 and 13.2–7 display the error plots obtained for the fitted RMM and GLD, respectively. Error variances (σ_ε^2) are nearly identical for the two moment-matching methods. Figures 13.2–8 and 13.2–9 display the pdfs of RMM and GLD, respectively. The reader may wish to implement the distribution fitting procedure, as expounded above, for values of Y_2 given in Table 13.2–5.

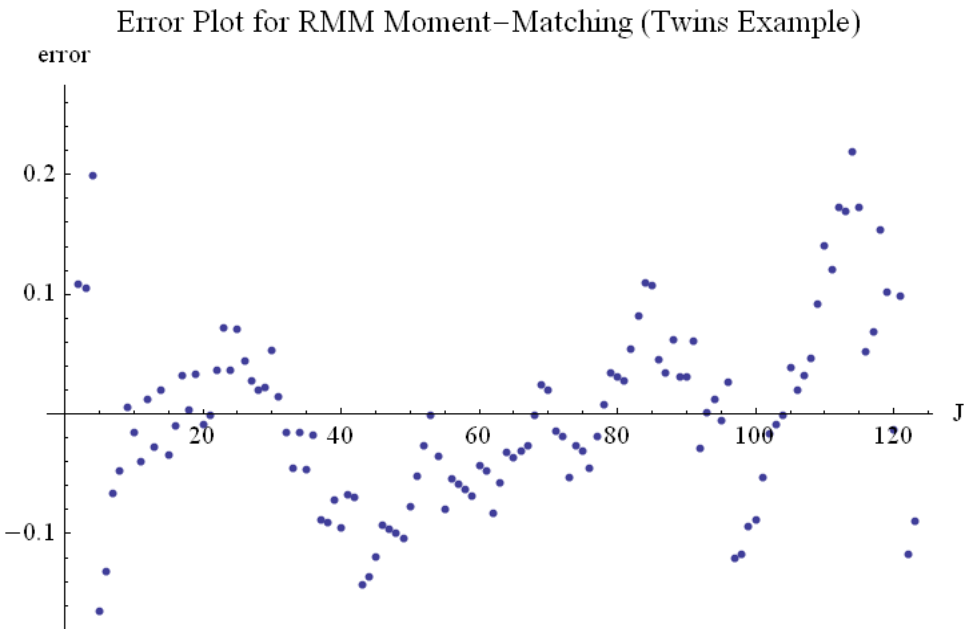


Figure 13.2–6. Error plot from fitting RMM to the twins example ($\sigma_\varepsilon^2 = 0.00729$). Horizontal axis is observation number (for the sorted Y_1 data).

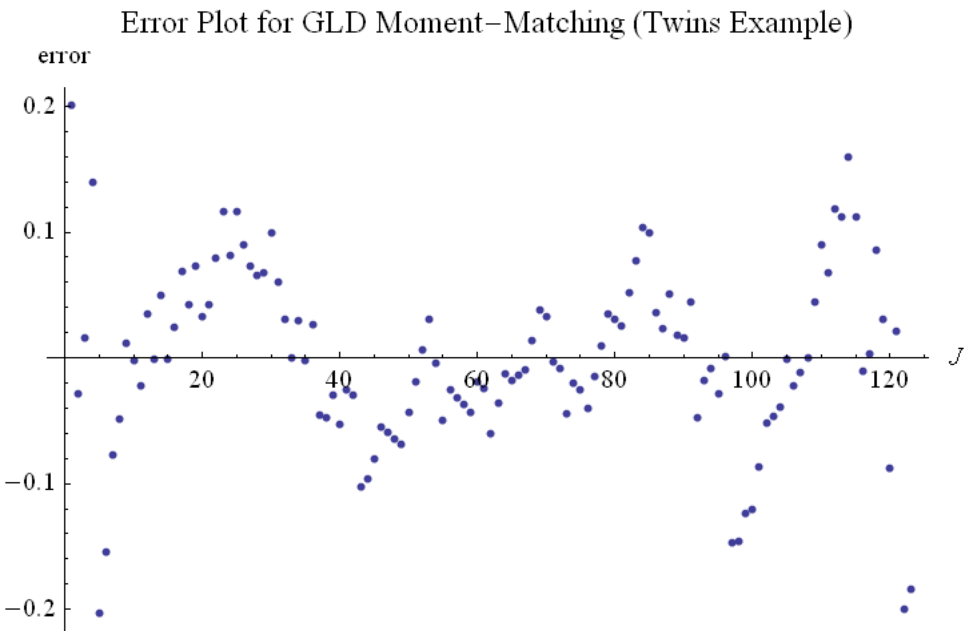


Figure 13.2–7. Error plot from fitting GLD to the twins example ($\sigma_\varepsilon^2 = 0.00499$). Horizontal axis is observation number (for the sorted Y_1 data).

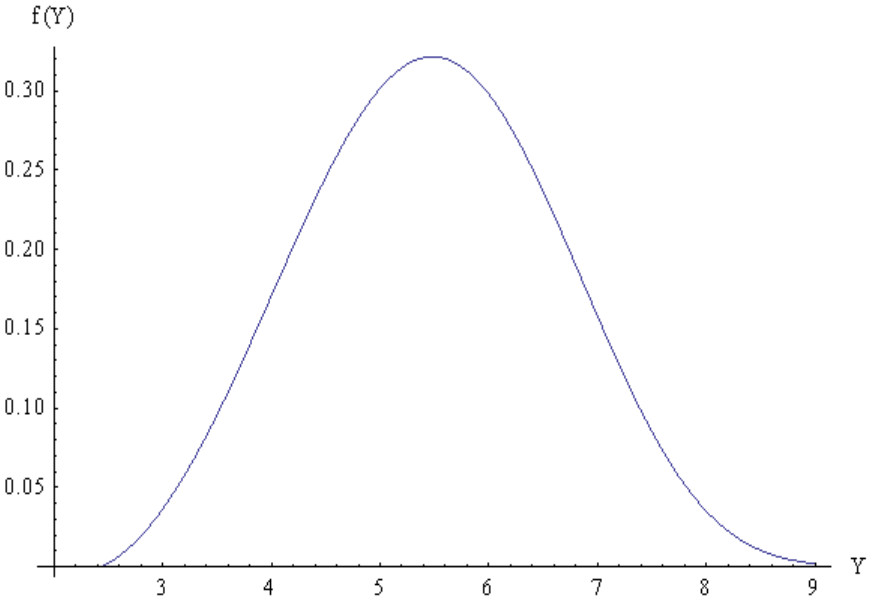


Figure 13.2–8. Error plot from fitting RMM to the twins example ($\sigma_\varepsilon^2 = 0.00729$). Horizontal axis is observation number (for the sorted Y_1 data).

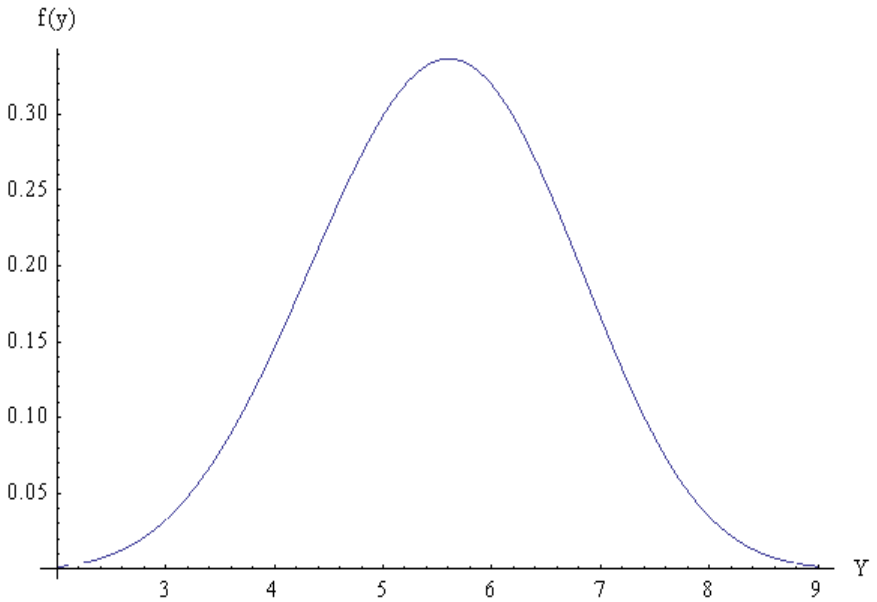


Figure 13.2–9. Error plot from fitting GLD to the twins example ($\sigma_\varepsilon^2 = 0.00499$). Horizontal axis is observation number (for the sorted Y_1 data).

References for Chapter 13

Dudewicz, E. J., Levy, G. C., Lienhart, J.L., and Wehrli, F. (1989). "Statistical analysis of magnetic resonance imaging data in the normal brain data (screening normality, discrimination, variability), and implications for expert statistical programming for ESSTM (the Expert Statistical System)," *American Journal of Mathematical and Management Sciences*, 9, 299–359.

Karian, Z. A., Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods.* CRC Press, Boca Raton, Florida, USA.

Shore, H. (2004a). "Response Modeling Methodology (RMM) – Current distributions, transformations and approximations as special cases of the RMM error distribution," *Communications in Statistics (Theory and Methods)*, 33(7), 1491–1510.

Shore, H. (2004b). "Response Modeling Methodology (RMM) – Validating evidence from engineering and the sciences," *Quality and Reliability Engineering International*, 20, 61–79.

Shore, H. (2005). *Response Modeling Methodology – Empirical Modeling for Engineering and Science*, World Scientific Publishing Co. Ltd., Singapore.

Shore, H. (2007). "Comparison of Generalized Lambda Distribution (GLD) and Response Modeling Methodology (RMM) as general platforms for distribution fitting," *Communications in Statistics (Theory and Methods),* 36(15), 2805–2819.

Shore, H. (2008). "Distribution fitting with Response Modeling Methodology (RMM) some recent results, *American Journal of Mathematical and Management Sciences*, 28(1 & 2), 3–18.

Shore, H. and A'wad, F. (2010). "Statistical comparison of the goodness-of-fit delivered by five families of distributions used in distribution fitting," *Communications in Statistics (Theory and Methods)*, 39, 1707–1728.

Shore, H., Benson-Karhi, D. (2007). "Forecasting S-shaped diffusion processes via Response Modeling Methodology (RMM)," *Journal of the Operational Research Society,* 58(6), 720–729.

van Zwet, W. R. (1964). *Convex Transformations of Random Variables,* Mathematisch Centrum, Amsterdam.

Chapter 14

Fitting GLDs and Mixture of GLDs to Data Using Quantile Matching Method

Steve Su
School of Mathematics and Statistics,
University of Western Australia,
M019, 35 Stiring Highway, Crawley, 6009, WA, Australia
E-mail: allegro.su@gmail.com

This chapter has two primary aims:

- To introduce a simple yet effective method of fitting both RS (Ramberg and Schmeiser (1974)) and FMKL[1] (Freimer, Kollia, Mudholkar, and Lin (1988)) GLDs to data using the quantile matching method.
- To compare the bias and efficiency of fitting FMKL GLD to data between the following methods: quantile matching, method of moments (Karian, Dudewicz, and McDonald (1996), Karian and Dudewicz (2000)), *L*-moments (Asquith (2007), Karvanen and Nuutinen (2008)), starship method (King and MacGillivray (1999)) and numerical maximum likelihood estimation (Su (2007c) and (2007a)).

The comparison of various methods is done using the FMKL GLD since it has the property of being a valid distribution as long as $\lambda_2 > 0$, while the RS GLD is

[1]The correct abbreviation should be FKML however it is commonly known as FMKL GLD in the statistical literature.

only valid for specified ranges of parameters. This means a fitting method using the RS GLD will require a check to ensure the parameters are in the valid range, which makes programming more involved. Additionally, to allow a fair comparison, all fitting methods use exactly the same algorithm in obtaining initial values.

14.1 Introduction

We begin by noting that the FMKL GLD has a different probability density function than the RS GLD, but it also has a rich range of shapes and its quantile function takes the form:

$$F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2}, \qquad 0 \le u \le 1. \tag{14.1.1}$$

Under (14.1.1), λ_1 and λ_2 are, respectively, the location and scale parameters of the GLD and λ_3 and λ_4 are the GLD shape parameters. The only restriction is that $\lambda_2 > 0$.

When either or both λ_3, $\lambda_4 = 0$, the distribution takes a different limiting functional form:

$$F^{-1}(u) = \lambda_1 + \frac{\ln(u) - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2} \qquad 0 \le u \le 1 \qquad \lambda_3 = 0, \tag{14.1.2}$$

$$F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \ln(1-u)}{\lambda_2} \qquad 0 \le u \le 1 \qquad \lambda_4 = 0, \tag{14.1.3}$$

$$F^{-1}(u) = \lambda_1 + \frac{\ln(u) - \ln(1-u)}{\lambda_2} \qquad 0 \le u \le 1 \qquad \lambda_3 = \lambda_4 = 0. \tag{14.1.4}$$

The cases of $\lambda_3 = \lambda_4 = 0$ for FMKL GLD rarely happen in practice and are provided mainly for completeness. Also, the FMKL GLD was not covered in L-moments and method of moment matching literature listed above, but it is relatively easy to adapt these techniques to the FMKL GLD. All of the methods mentioned above have been programmed into the GLDEX package (Su (2007b)) in **R** (**R** Development Core Team (2008)). The GLDEX package is updated regularly with new developments and applications in GLD research and it is currently distributed as freeware. Readers are welcome to support the continuing development of this package by providing feedback to the author.

Given that GLDs are defined by their inverse distribution or quantile function, it is natural to fit GLDs based on the quantiles of the data. Surprisingly, to date,

no one has examined the fitting of GLDs using the quantile matching method. The closest effort to the quantile matching method is perhaps found in Öztürk and Dale (1985), where the authors minimize the squared deviations of expected values of order statistics from RS GLDs with the order statistics from the data. Order statistics can be viewed as a subset of quantiles and a quantile may or may not be an order statistic. For example, the median (quantile = 0.5) is not an order statistic if the number of observations is even.

The quantile matching method is intuitively appealing and is akin to the concept of trying to approximate the empirical data as closely as possible over all of the important data points (quantiles) chosen by the user. The following sections discuss the basic algorithm of using the quantile matching method and illustrate the performance of quantile matching method in terms of efficiency and bias against a number of other methods. Simulation studies are given to show the performance of the quantile matching method in capturing the true probability density function and the performance of the quantile matching method in fitting a mixture of GLDs is demonstrated using four examples.

14.2 Methods

There are two main difficulties with fitting GLDs: selecting initial values to initiate the optimization process and avoiding infeasible regions for which GLDs may not exist. Reasonable solutions to these problems have been proposed in Su (2007c), (2005) (2007a). In particular, the avoidance of infeasible regions is done partially in the initial value selection process and implicitly through use of the Nelder-Mead simplex optimization scheme. The selection of initial values is firstly done by generating a series of quasi-random numbers (usually in the quantity of 10,000 or more) for λ_3, λ_4 (and subsequently λ_1, λ_2). The next step is to only retain sets of initial values for which GLDs span the entire dataset and are valid distributions. The best set of initial values is chosen on the basis of closest match (in terms of sum of squared deviations) to the third and fourth percentiles of the dataset for RS GLD and the closest match to the third and fourth moments of the dataset for FMKL GLD.

The best set of initial values is ultimately used to optimize the GLD fit to the dataset using a chosen criterion. In this chapter, the criterion is simply to minimize the sum of squared deviations between the fitted quantiles (1000 quantiles from 0 to 1 in 1000 equally spaced intervals) with the corresponding sample quantiles. Fitted quantiles resulting in infinity are removed along with sample quantiles. The number of quantiles chosen does not have to be 1000, but this is known to work well in a wide variety of situations and the user can choose more or fewer quantiles if desired. There are also many sample quantiles that can be used (Hyndman and Fan (1996)) and the default is to use the type 7 quantiles, in conformity with current settings in **R** 2.9.0 and Splus 8.0.

This chapter also examines the performance between type 4, 5, 6, 7, and 8 quantiles in terms of efficiency and bias in Figures 14.3–2 to 14.3–17. These quantiles are continuous sample quantile types and they are obtained by linear interpolation between the observed r-th order statistic and $q(r)$. The definitions of the various types of quantiles are as follows, with n being the sample size:

$$\texttt{Type 4:}q(r) = \frac{r}{n} \tag{14.2.1}$$

$$\texttt{Type 5:}q(r) = \frac{r - 0.5}{n} \tag{14.2.2}$$

$$\texttt{Type 6:}q(r) = \frac{r}{n+1} \tag{14.2.3}$$

$$\texttt{Type 7:}q(r) = \frac{r-1}{n-1} \tag{14.2.4}$$

$$\texttt{Type 8:}q(r) = \frac{r - \frac{1}{3}}{n + \frac{1}{3}}. \tag{14.2.5}$$

For example, if we have four observations, 1, 2, 3, 4, the 40% quantile point under type 4 quantile would be 1.6. Type 4 quantiles consider 2 as 50% ($\frac{2}{4}$) quantile and 3 as the 75% ($\frac{3}{4}$) quantile. The required 40% quantile is estimated via linear interpolation between these two observations. If another type of quantile is used to estimate the 40% quantile of our data, a different answer may be obtained. For example, type 5 quantiles consider 2 and 3 at 37.5% ($\frac{2-0.5}{4}$) and 62.5% ($\frac{3-0.5}{4}$) quantile points for the data, so the 40% quantile point under type 5 quantile would be 2.1.

The simplicity of the quantile matching method is well suited for modeling mixture data. Instead of using a more complex method like maximum likelihood estimation, a direct matching of fitted quantiles and their corresponding sample quantiles can be used to obtain mixture distributions. Consider a mixture distribution of the form

$$p_1 f_1 + p_2 f_2 + \ldots p_k f_k, \tag{14.2.6}$$

where $p_1, \ldots p_k$ are positive numbers with $p_1 + p_2 + \ldots p_k = 1$ and $f_1, \ldots f_k$ are probability density functions. A critical process in mixture distribution modeling is splitting the data into different groups to obtain the initial values. In this chapter, the sorted data are usually divided into different groups using the *clara* clustering scheme (Kaufman and Rousseeuw (1990)), with each group consisting of say, 10% of random data from other groups. The percentage chosen can be determined by the user, as long as the shape of each subset of the data is not distorted in such a way that will lead to inappropriate initial values. Additionally, the minimum and maximum values of the entire dataset are selected into each group. Data in each group is then fitted with a GLD using the sum of minimum

squared quantile deviations criterion, which gives an approximate set of initial values for (14.2.6). The final set of parameters for the mixture distribution is obtained by numerically minimizing the sum of squared deviations between the 1000 mixture GLDs quantiles and their corresponding sample quantiles from the entire dataset.

14.3 Results

The analysis of this section is divided into three parts. To illustrate the performance of the quantile matching method, the first part compares the sampling variance and bias of quantile matching, maximum likelihood estimation, L-moment estimation, starship, and method of moments in fitting five FMKL GLDs for a range of sample sizes at 25, 50, 100, 200, and 400. The parameters of five FMKL GLDs are chosen to represent five classes of FMKL GLD as described in Freimer, Kollia, Mudholkar, and Lin (1988) and these are shown in Table 14.3–1. The variance of the parameters and the mean of absolute bias (mean in absolute difference between the fitted parameters and actual parameters) are investigated over 500 simulations for each of the five different FMKL GLDs over sample sizes 25, 50, 100, 200, and 400. The results are given as dot plots in Figures 14.3–2 through 14.3–17.

The second part is a theoretical comparison between data fitting methods with well-known statistical distributions. The GLD approximation is obtained by fitting data on 1000 random observations from ten well-known statistical distributions. To assess the quality of the GLD approximation, a graphical analysis comparing the theoretical and estimated results is given in Table 14.3–18.

The last part is an illustration of the use of quantile matching method to fit a mixture of GLDs (Figure 14.3–19). While perhaps not as sophisticated as the maximum likelihood method, the use of quantile matching is simpler and can often provide convincing fits to the dataset. It can also be used to generate initial values to obtain mixture of GLDs.

Table 14.3–1. Five FMKL GLDs used to demonstrate the sampling variance and bias of quantile matching estimation.

λ_1	λ_2	λ_3	λ_4
0	1	0.5	0.6
0	1	2	0.5
0	1	1.5	1.5
0	1	2.5	1.5
0	1	3	3

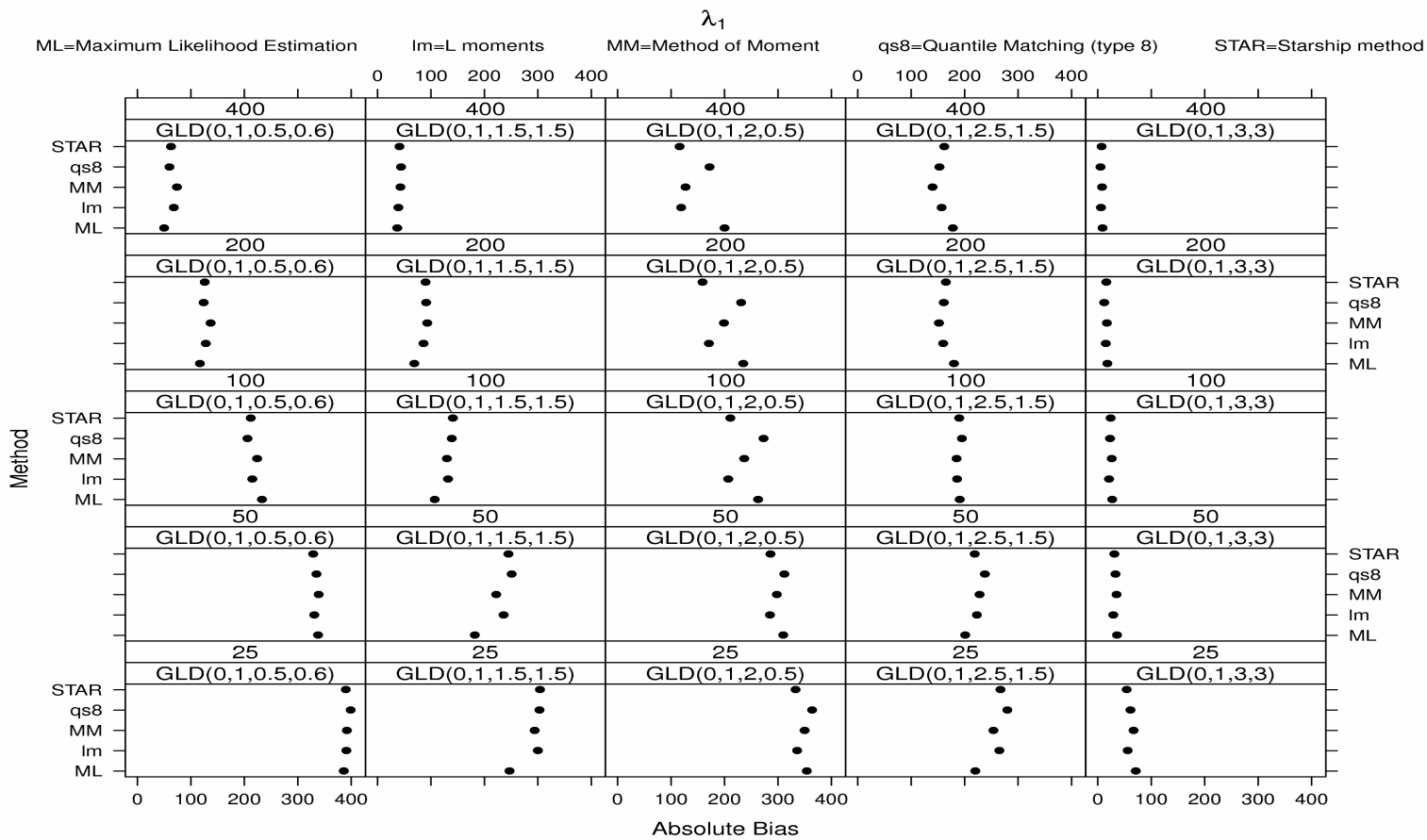


Figure 14.3–2. Absolute bias for λ_1 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

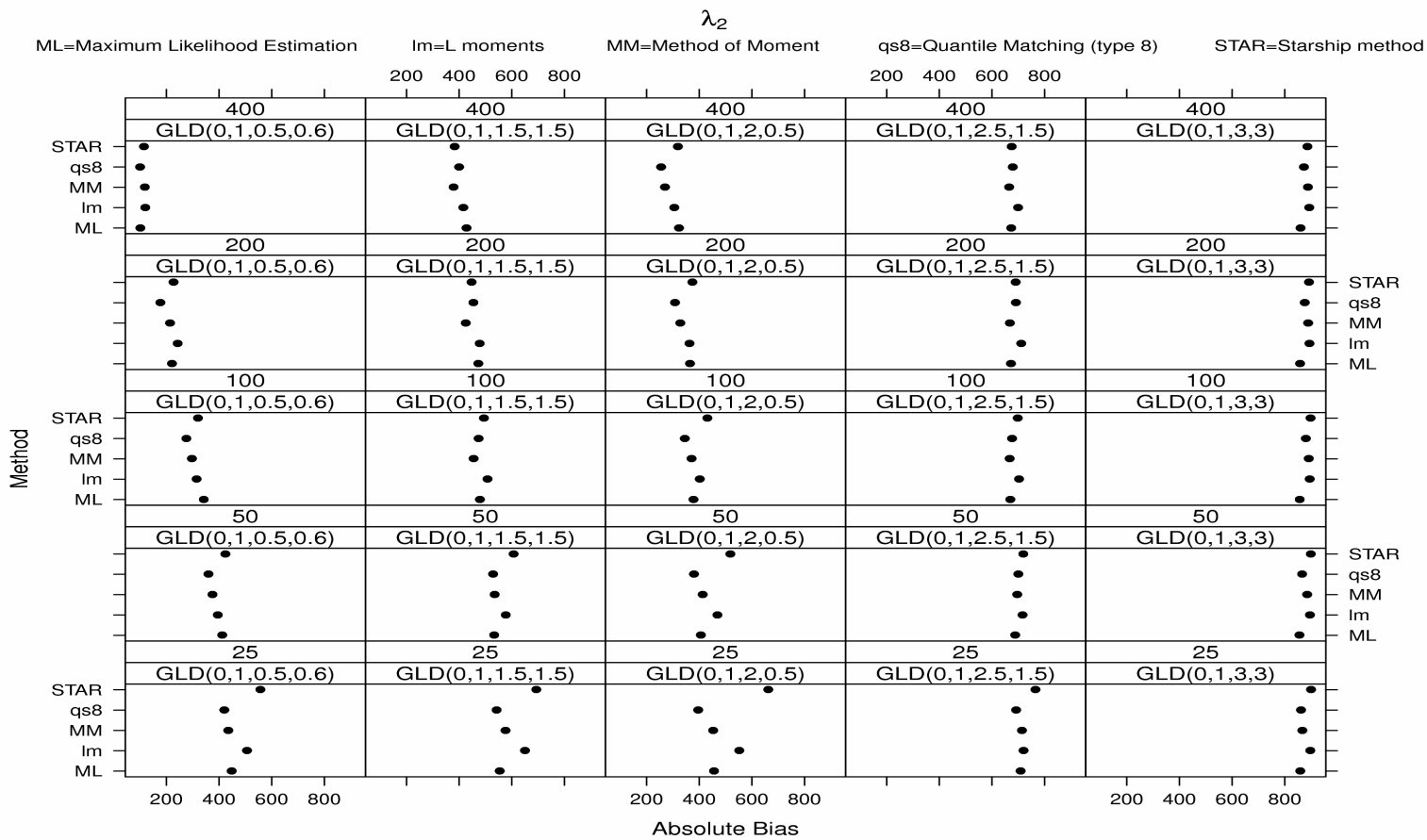


Figure 14.3–3. Absolute bias for λ_2 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

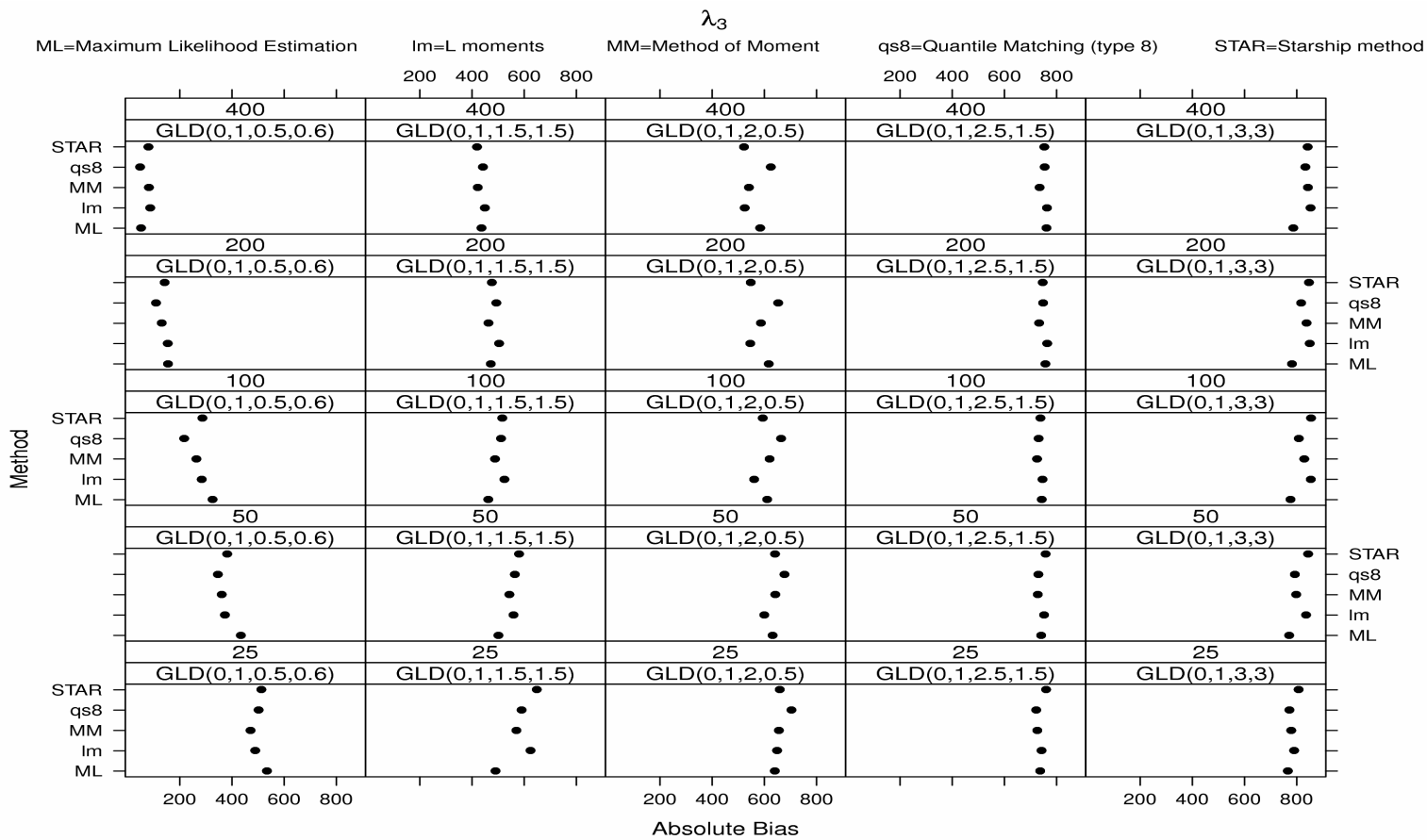


Figure 14.3–4. Absolute bias for λ_3 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

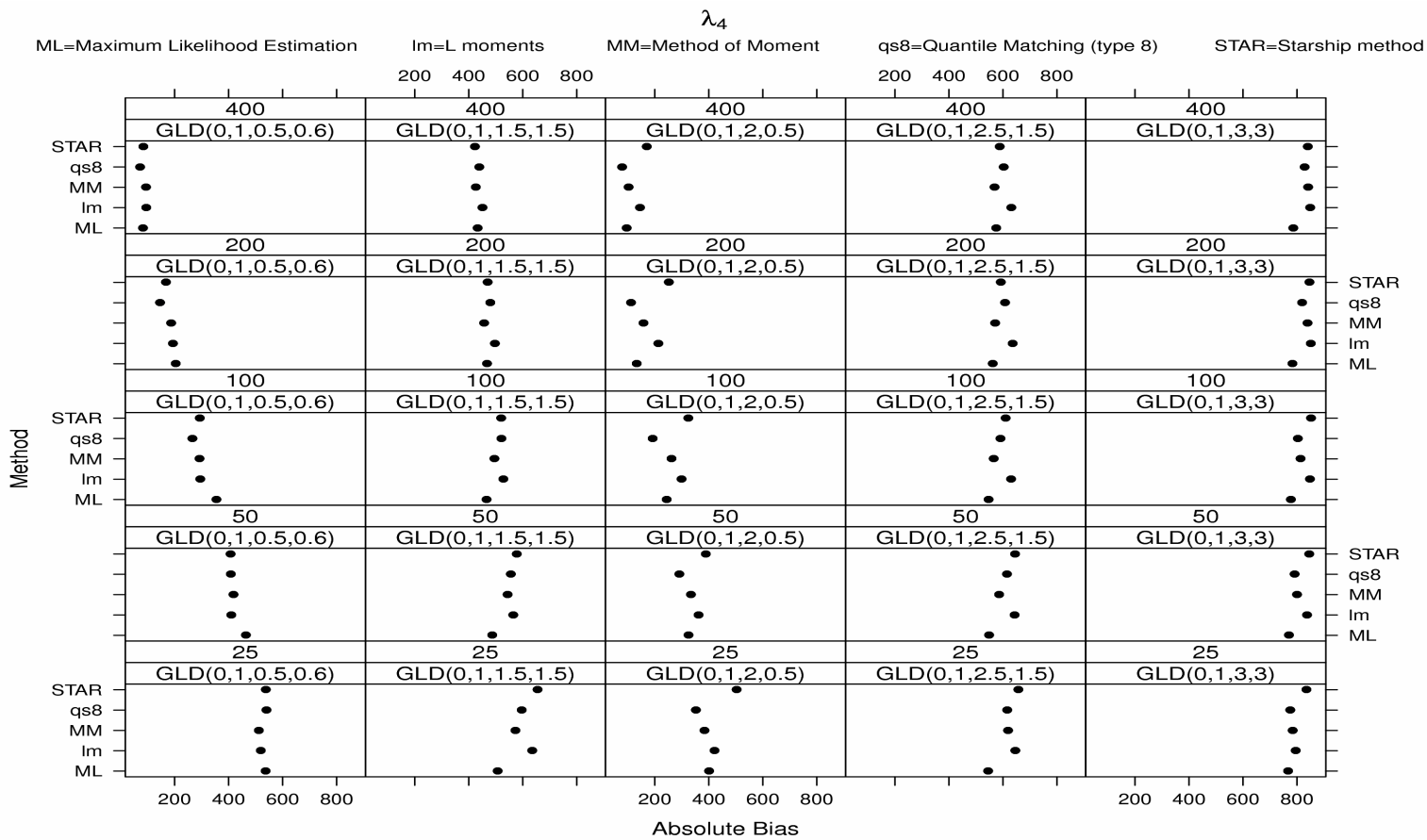


Figure 14.3–5. Absolute bias for λ_4 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

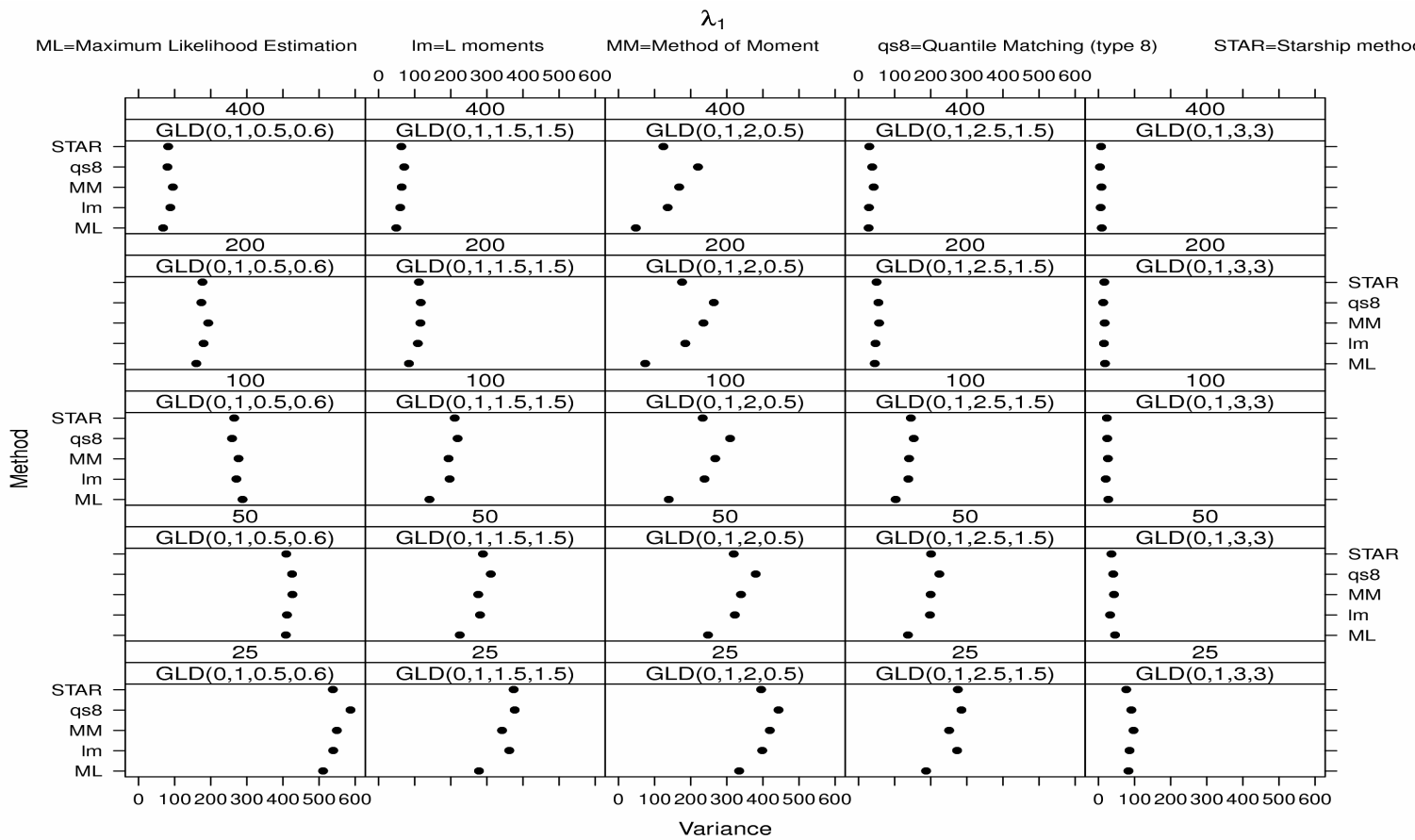


Figure 14.3–6. Variance for λ_1 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

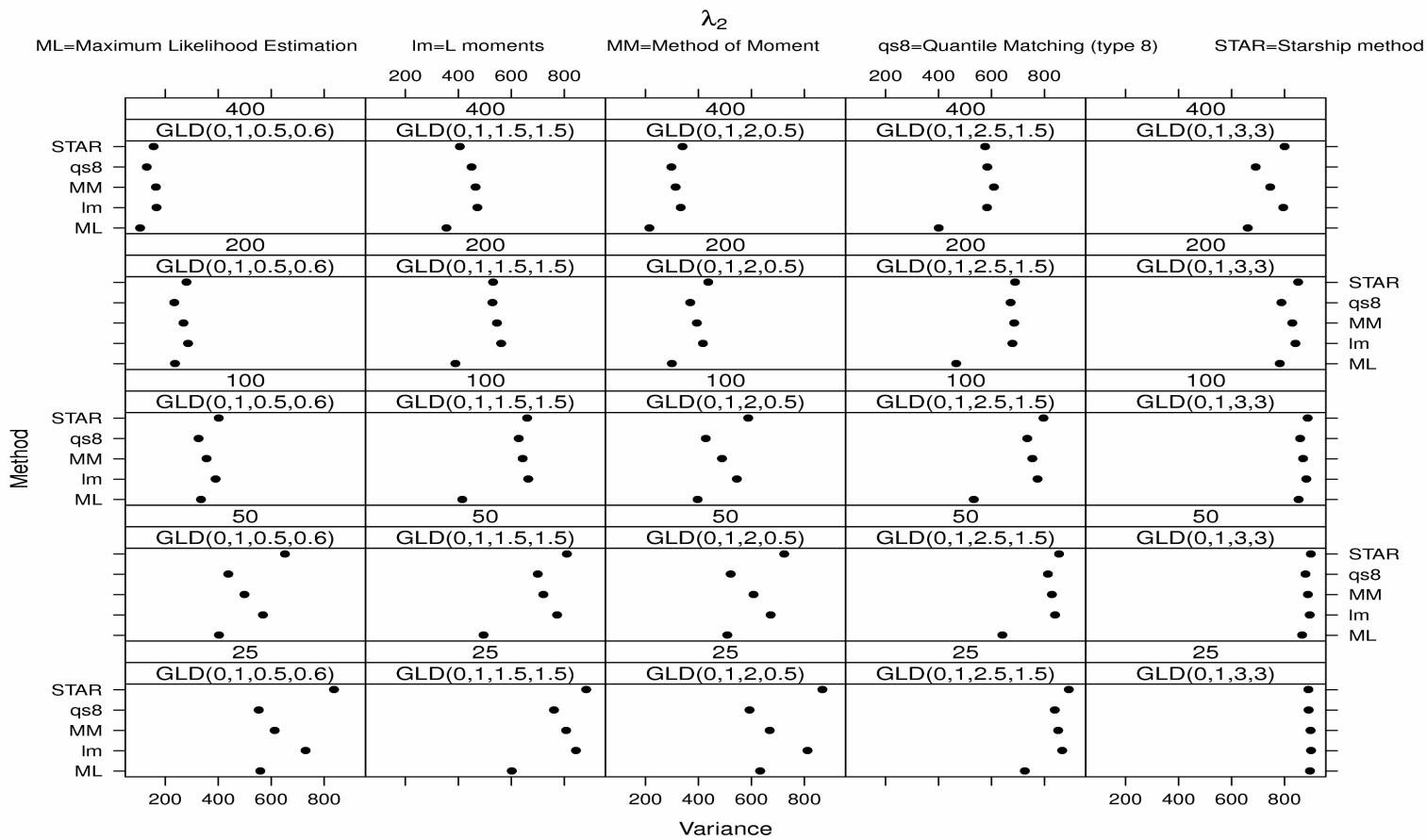


Figure 14.3–7. Variance for λ_2 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

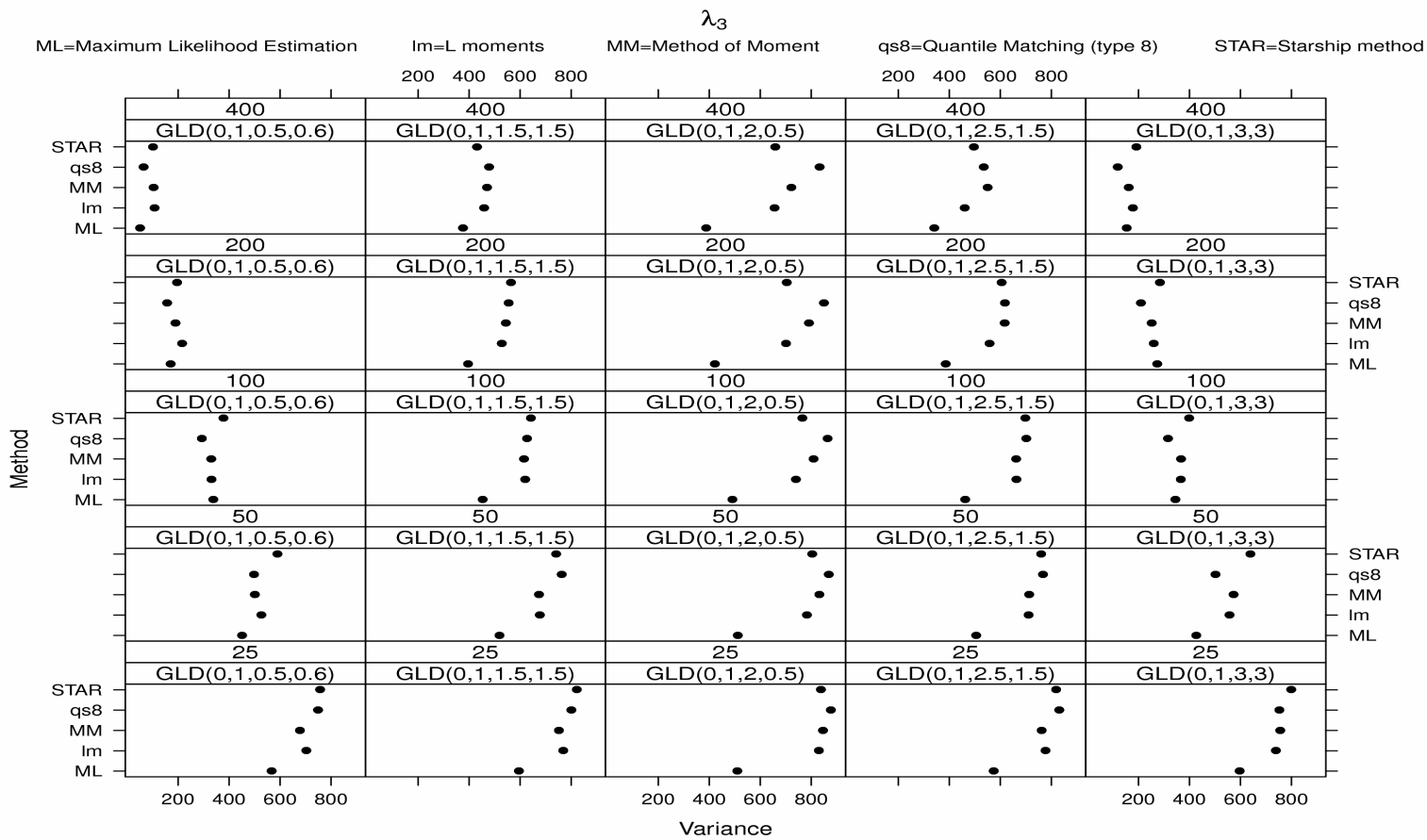


Figure 14.3–8. Variance for λ_3 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

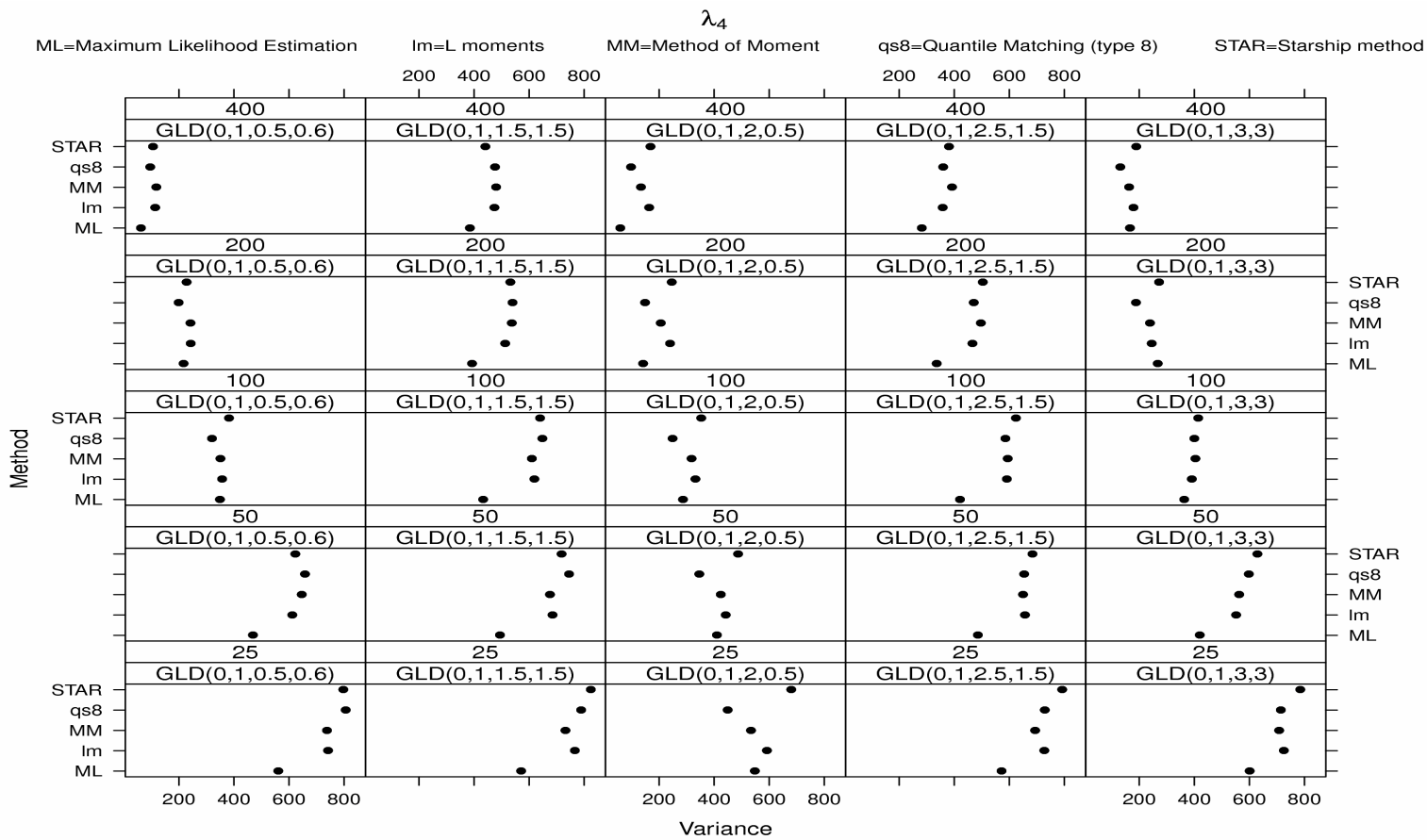


Figure 14.3–9. Variance for λ_4 among different FMKL GLD estimation methods for samples of sizes 25, 50, 100, 200, 400.

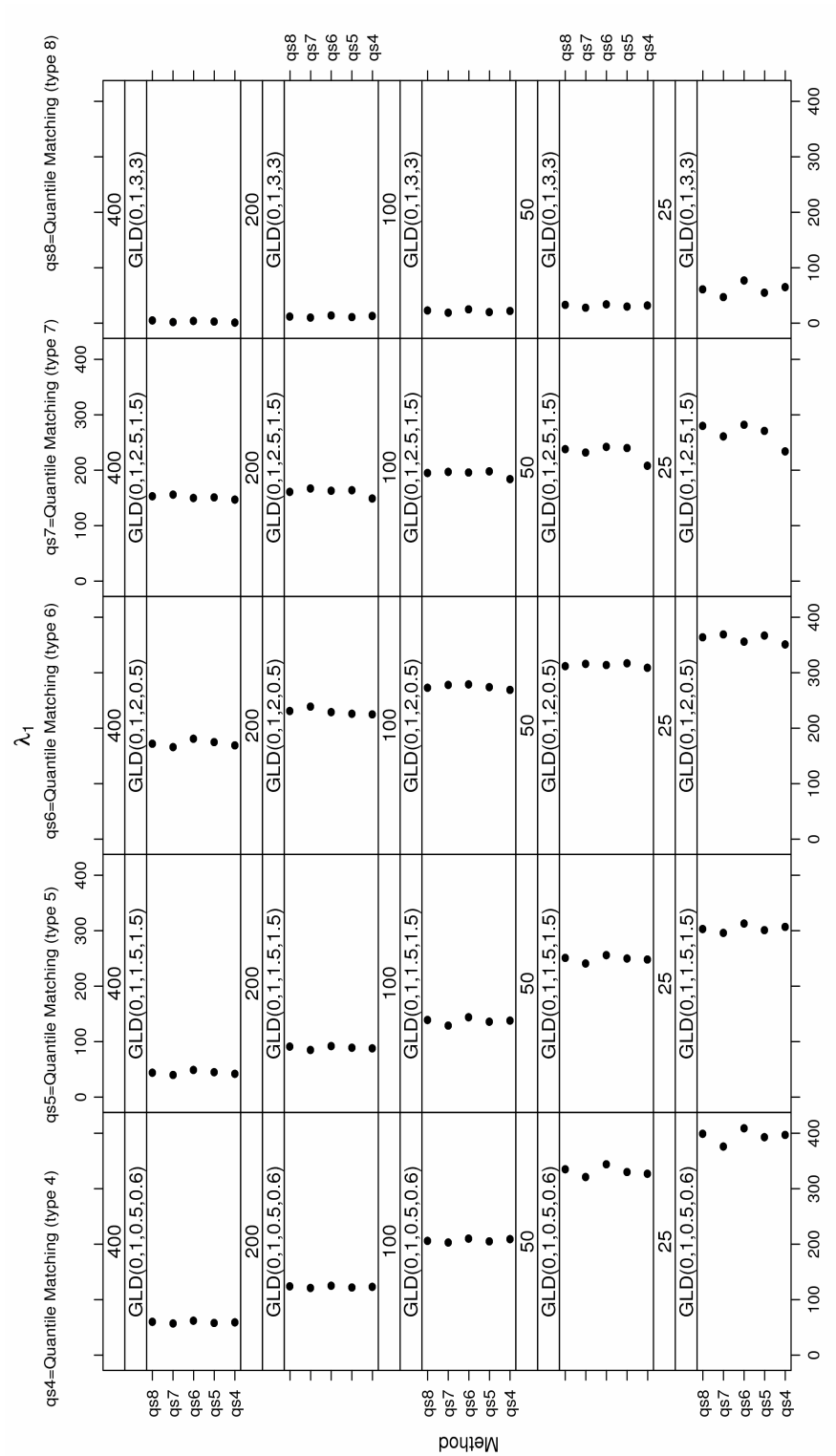


Figure 14.3–10. Absolute bias for λ_1 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

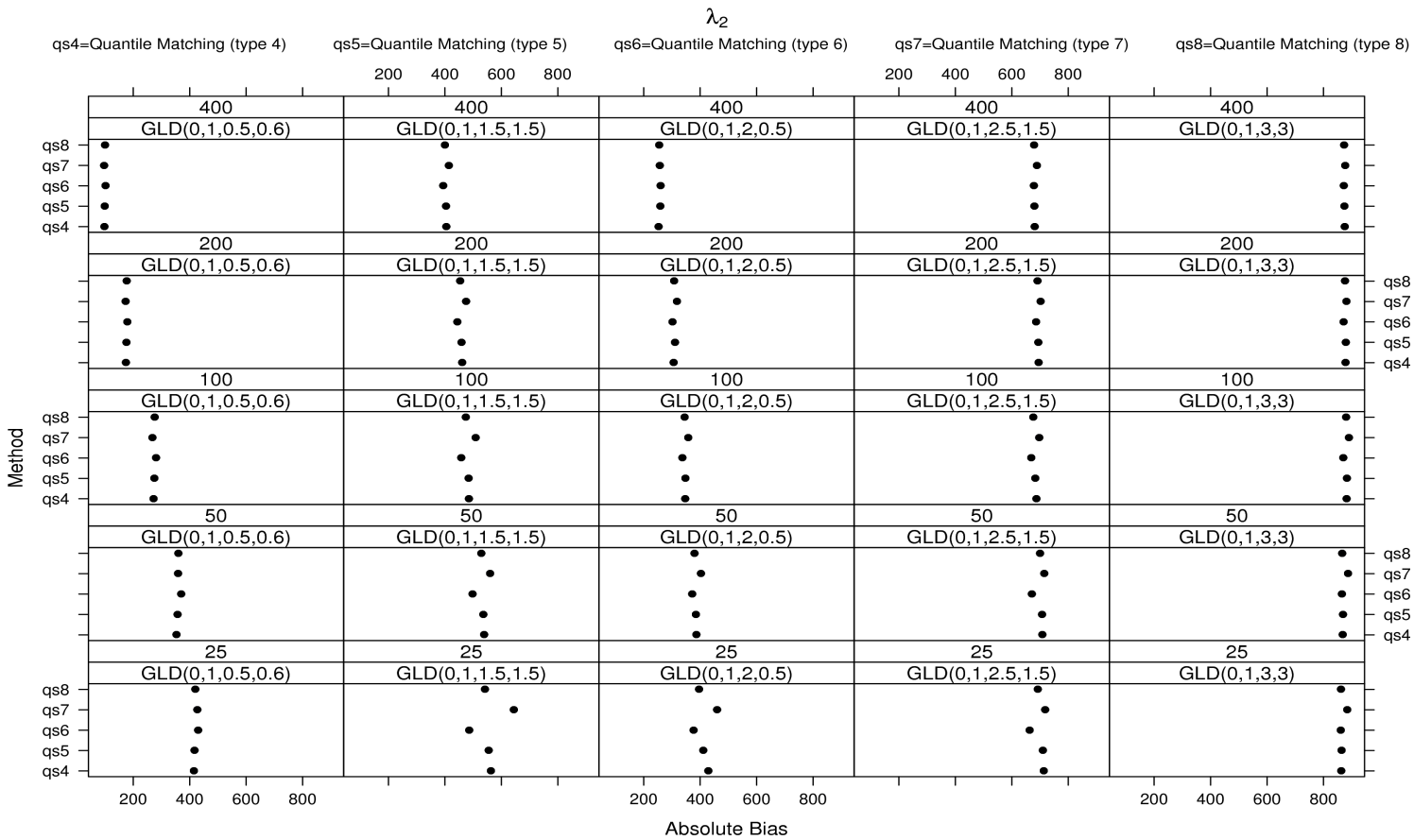


Figure 14.3–11. Absolute bias for λ_2 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

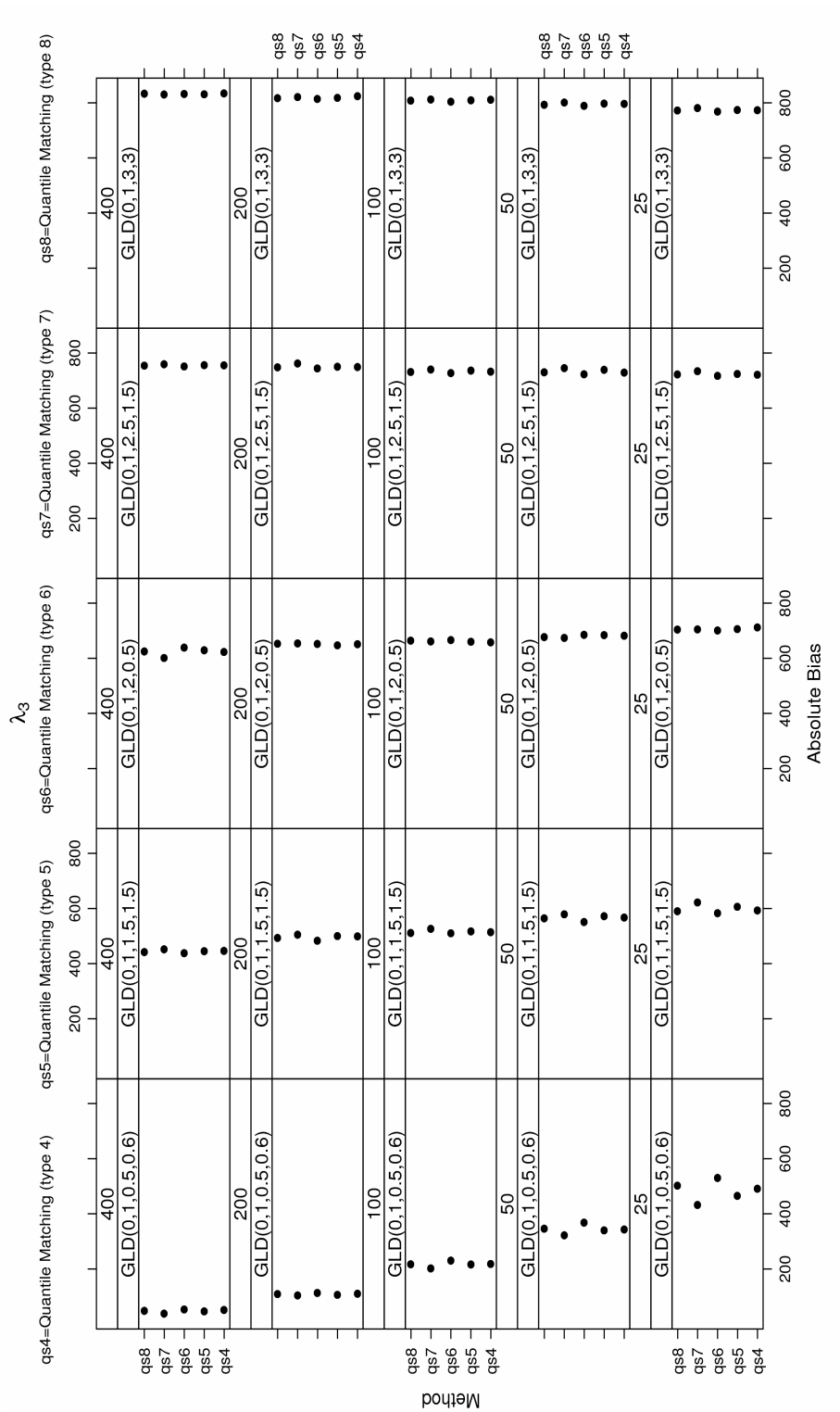


Figure 14.3–12. Absolute bias for λ_3 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

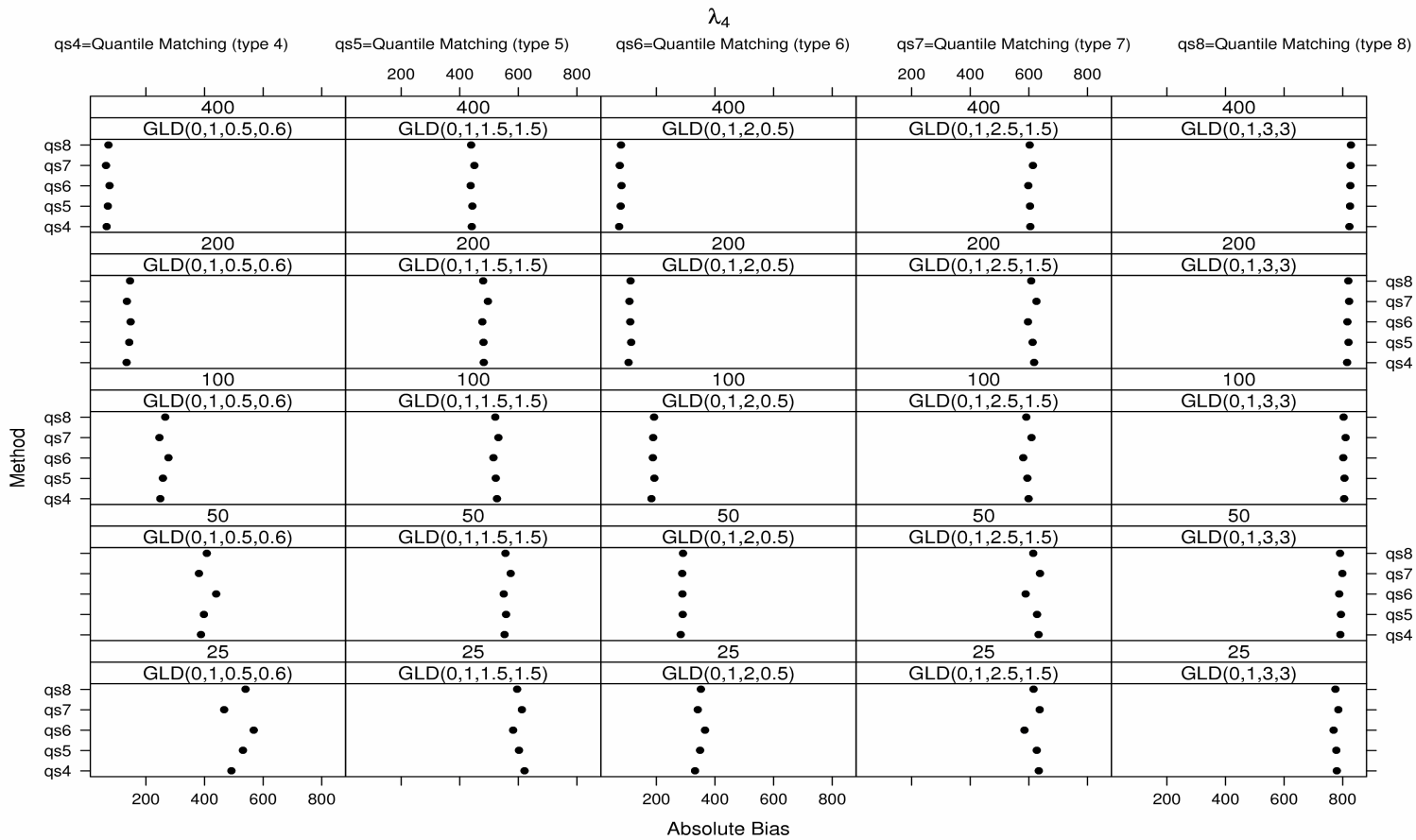


Figure 14.3–13. Absolute bias for λ_4 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

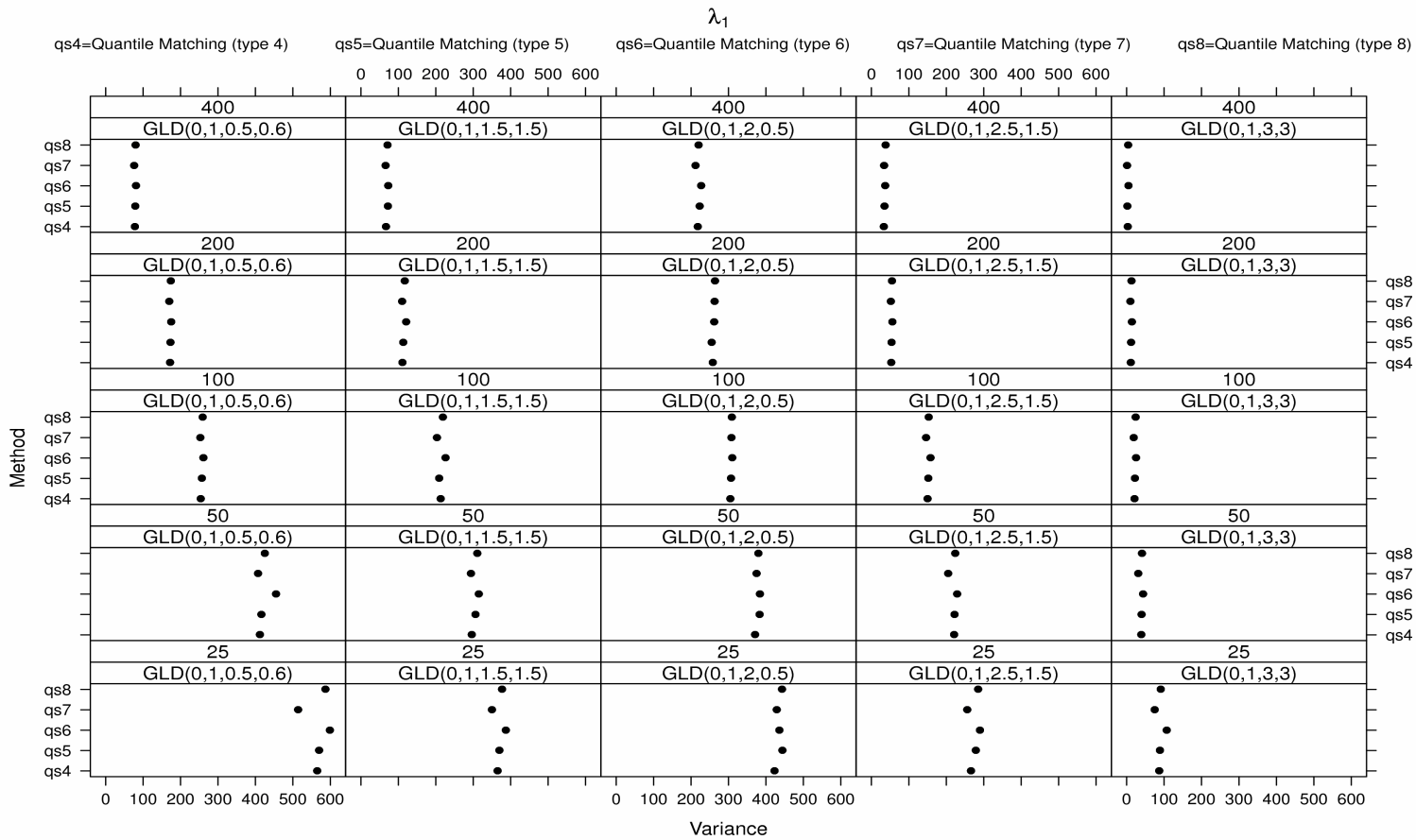


Figure 14.3–14. Variance for λ_1 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

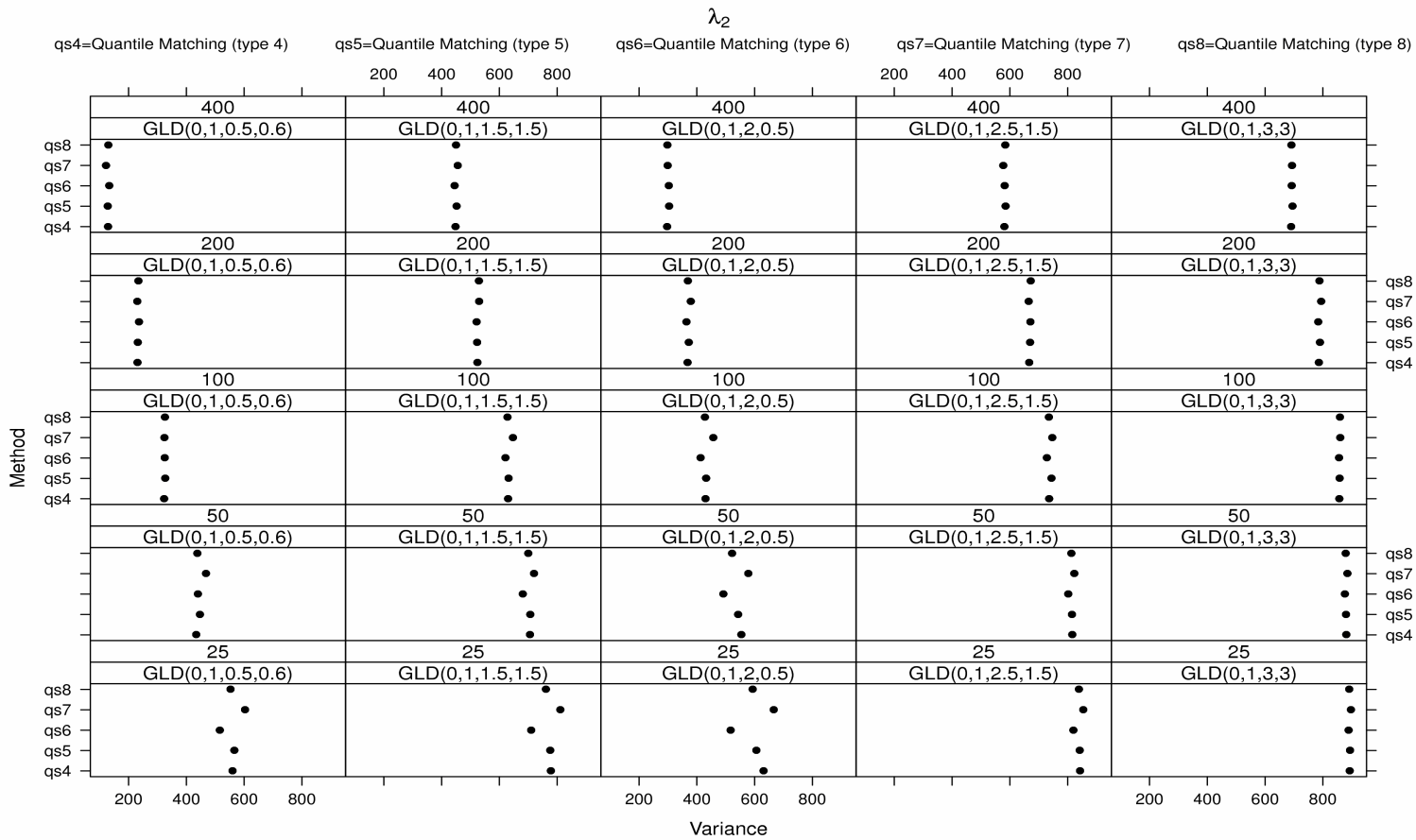


Figure 14.3–15. Variance for λ_2 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

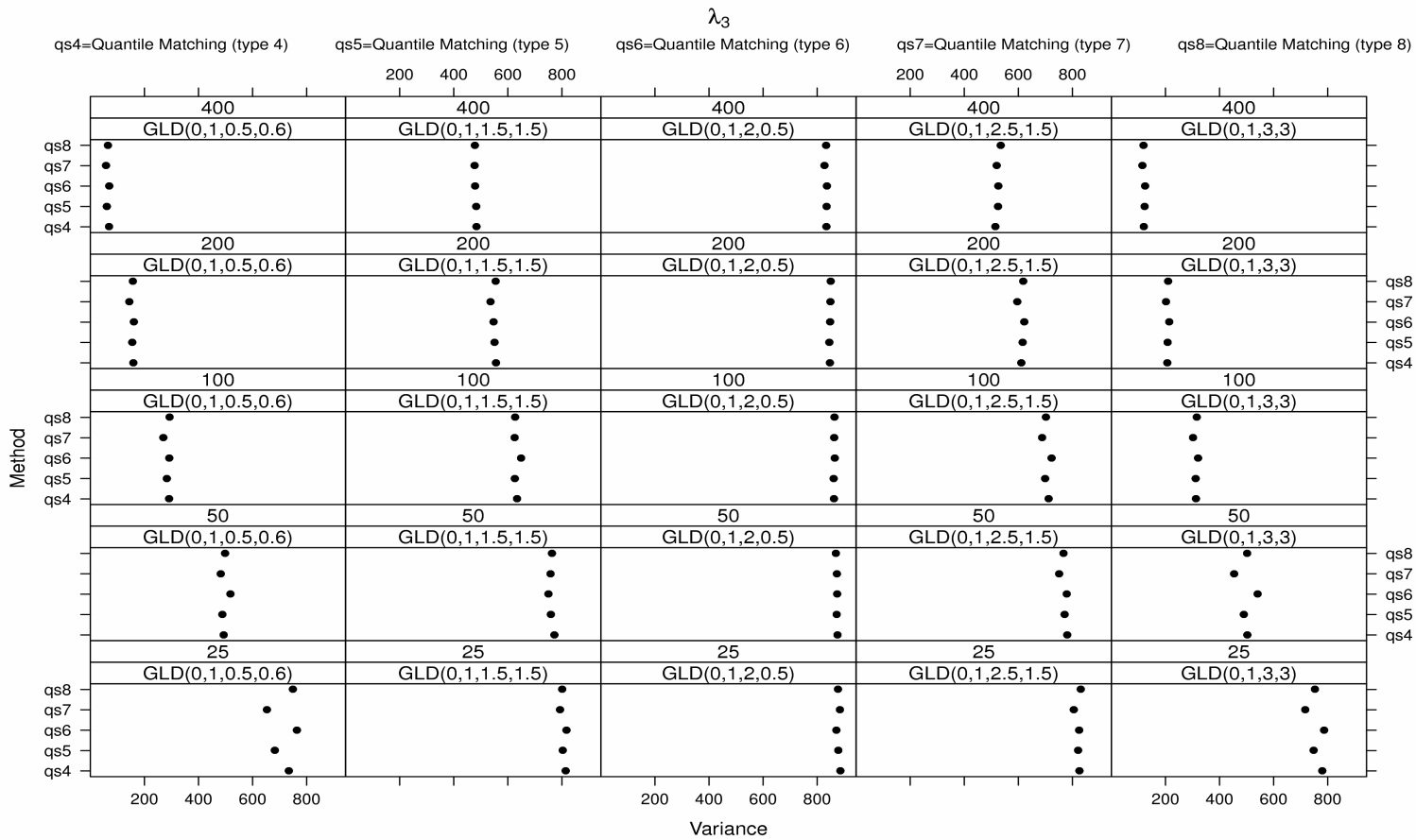


Figure 14.3–16. Variance for λ_3 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

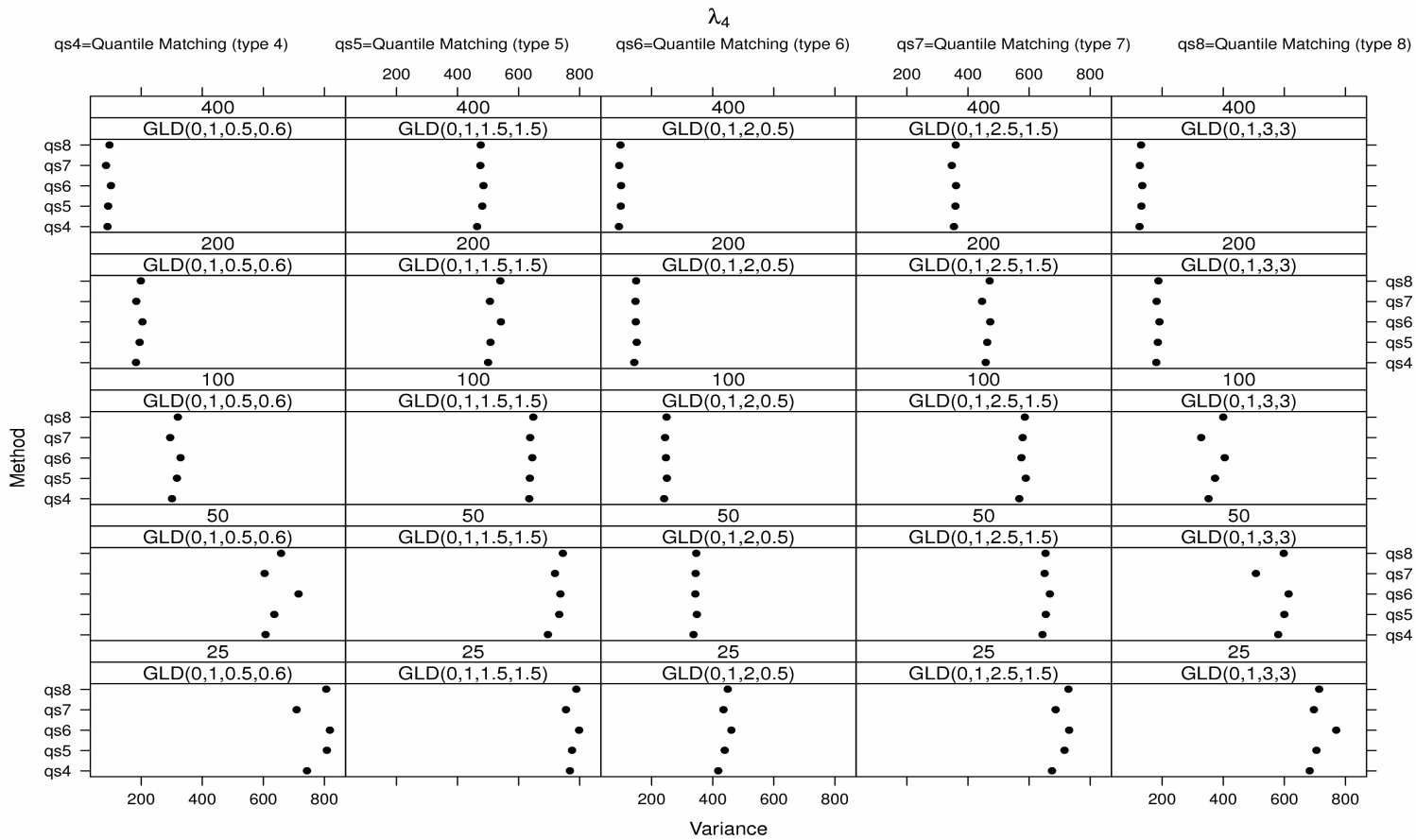


Figure 14.3–17. Variance for λ_4 among different FMKL GLD quantile methods for sizes 25, 50, 100, 200, 400.

Table 14.3–18. Mean quantile difference between approximating GLD versus true underlying distribution.

	RS	FMKL
Normal(0,1)	0.010	0.009
Student(5)	0.059	0.057
Exp(1)	-0.009	-0.012
Gamma(5,3)	0.020	0.010
Lognormal(0,1/3)	0.011	0.009
Weibull(5,2)	-0.018	-0.015
Beta(1,1)	0.007	0.007
Beta(3,3)	0.007	0.007
F(6,25)	0.035	0.020
Chisq(5)	0.089	0.028

14.3.1 Performance of Quantile Matching Estimation

In Figures 14.3–2 through 14.3–17, each column of the graphs in these figures represents each of the five different FMKL GLDs (as indicated in (14.1.1)) and each row represents sample sizes at 25, 50, 100, 200, and 400. Within the different quantile estimation methods, there are no substantial differences in terms of bias or efficiency as can be seen in Figures 14.3–10 through 14.3–17. Based on this result, quantile 8, a random selection of all quantile types considered, is used as the representative quantile matching method to compare against other estimation methods in Figures 14.3–2 through 14.3–9. Figures 14.3–6 through 14.3–9 show that the variance of the parameters under maximum likelihood estimation tend to be the lowest among all the methods, this result is consistent with Su (2007c). In terms of mean absolute bias, there appear to be no major differences between different methods of fitting as shown in Figures 14.3–2 through 14.3–5.

The use of the quantile matching method to estimate well-known statistical distributions is quite convincing as shown in Figure 14.3–19. These GLDs are attained from fitting 1000 random observations from a true underlying distribution. As there appears to be no major differences between different types of quantiles, type 7 quantile, default of **R**/Splus is used. Figure 14.3–19 shows that there are slight departures between the true distribution and fitted distribution as in the case of the F, Gamma, Chi-squared, and uniform distributions but overall the use of quantile matching does provide a reasonable fit. The mean deviations in terms of quantiles between fitted and actual distribution are also low, as can be seen in Table 14.3–18. This result gives confidence as to the capability of the quantile matching method to closely capture the true underlying distribution of the sample data.

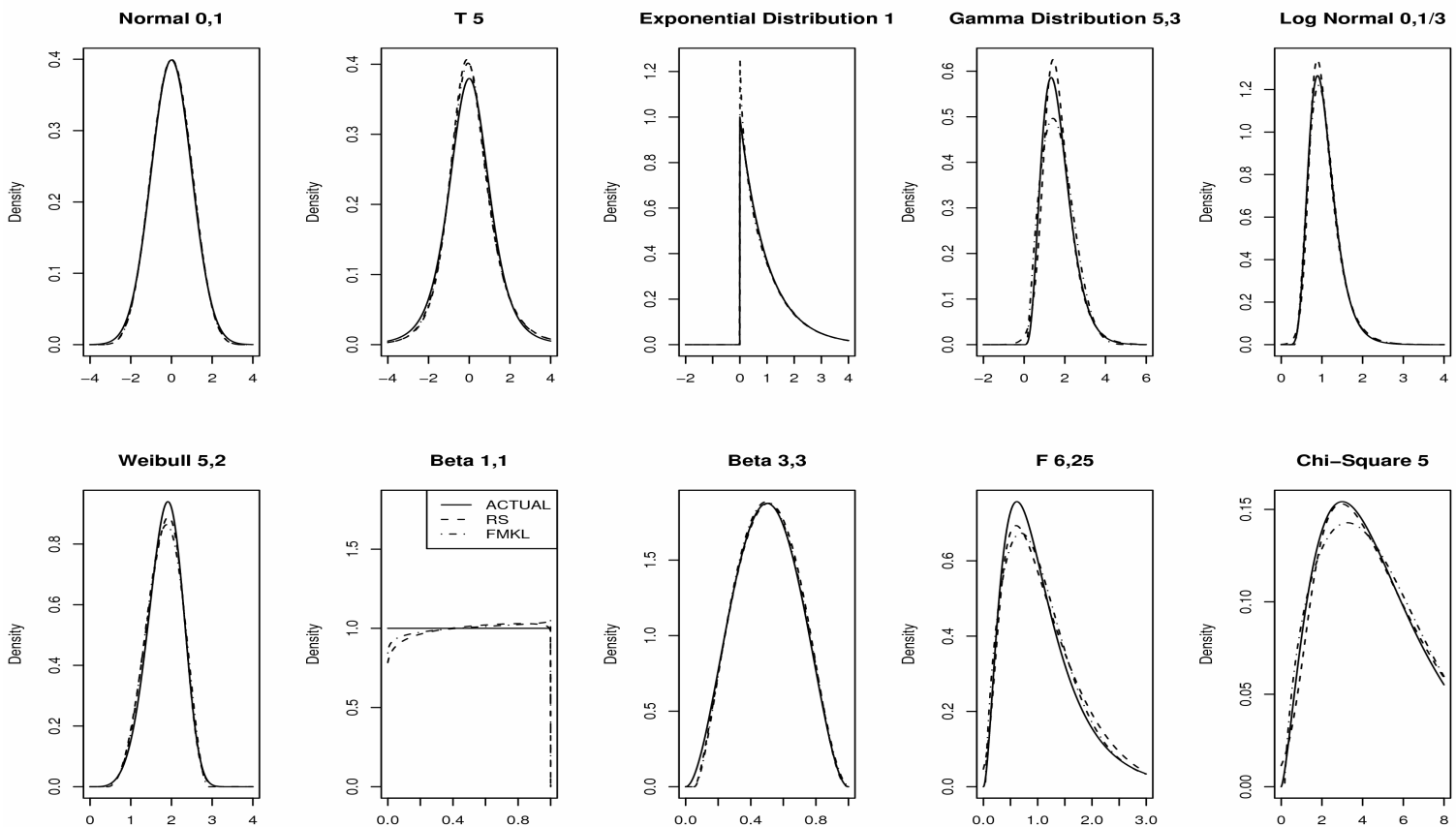


Figure 14.3–19. Quantile matching method: capturing the true probability density function of well-known distributions.

Table 14.3–20. Galaxy data.

9.172	9.35	9.483	9.558	9.775	10.227	10.406	16.084	16.17	18.419
18.552	18.6	18.927	19.052	19.07	19.33	19.343	19.349	19.44	19.473
19.529	19.541	19.547	19.663	19.846	19.856	19.863	19.914	19.918	19.973
19.989	20.166	20.175	20.179	20.196	20.215	20.221	20.415	20.629	20.795
20.821	20.846	20.875	20.986	21.137	21.492	21.701	21.814	21.921	21.96
22.185	22.209	22.242	22.249	22.314	22.374	22.495	22.746	22.747	22.888
22.914	23.206	23.241	23.263	23.484	23.538	23.542	23.666	23.706	23.711
24.129	24.285	24.289	24.366	24.717	24.99	25.633	26.96	26.995	32.065
32.789	34.279								

Table 14.3–21. Comparing the first four moments of fitted GLD with the empirical data in Figure 14.3–22.

	Data (a)	Fitted (a)	Data (b)	Fitted (b)
Mean	2.6	2.6	7.2	7.2
Variance	10	9.9	26.9	26.5
Skewness	0.4	0.4	-0.2	-0.2
Kurtosis	2.2	2.1	1.7	1.8
	Data (c)	Fitted (c)	Data (d)	Fitted (d)
Mean	201.6	201.6	187.8	187.8
Variance	29.4	29.4	4870	4866.1
Skewness	0.2	0.2	-0.2	-0.2
Kurtosis	2.4	2.4	3.9	3.9

14.3.2 Quantile Matching Method for Mixture Data

Quantile matching methods can be very effective in fitting mixture data as indicated by the graphical and numerical results in Figure 14.3–22 and Table 14.3–21. Figure 14.3–22 (a) and (b) are simulated data and Figure 14.3–22 (c) and (d) are galaxy data downloaded from the Internet by the author (see Table 14.3–20). In particular, the data in Figure 14.3–22 (a) is generated by 700 Normal(10,3) and 300 Exp(3) random numbers and data in Figure 14.3–22 (b) is generated by 500 Double Exponential(1) and 500 Normal(5,2) random numbers. In Figure 14.3–22 except (d), the default *clara* clustering scheme is used to split the initial data into subgroups. In Figure 14.3–22 (d), a manual split of data into three subgroups ([0, 100), [100, 300), [300, ∞)) was made to induce a better fitting result than the default *clara* clustering scheme.

The graphical outputs indicate a striking fit as evidenced by the QQ plots and the first four moments of the fitted distributions are quite close to the first four moments of the underlying data. These examples show the potential of the

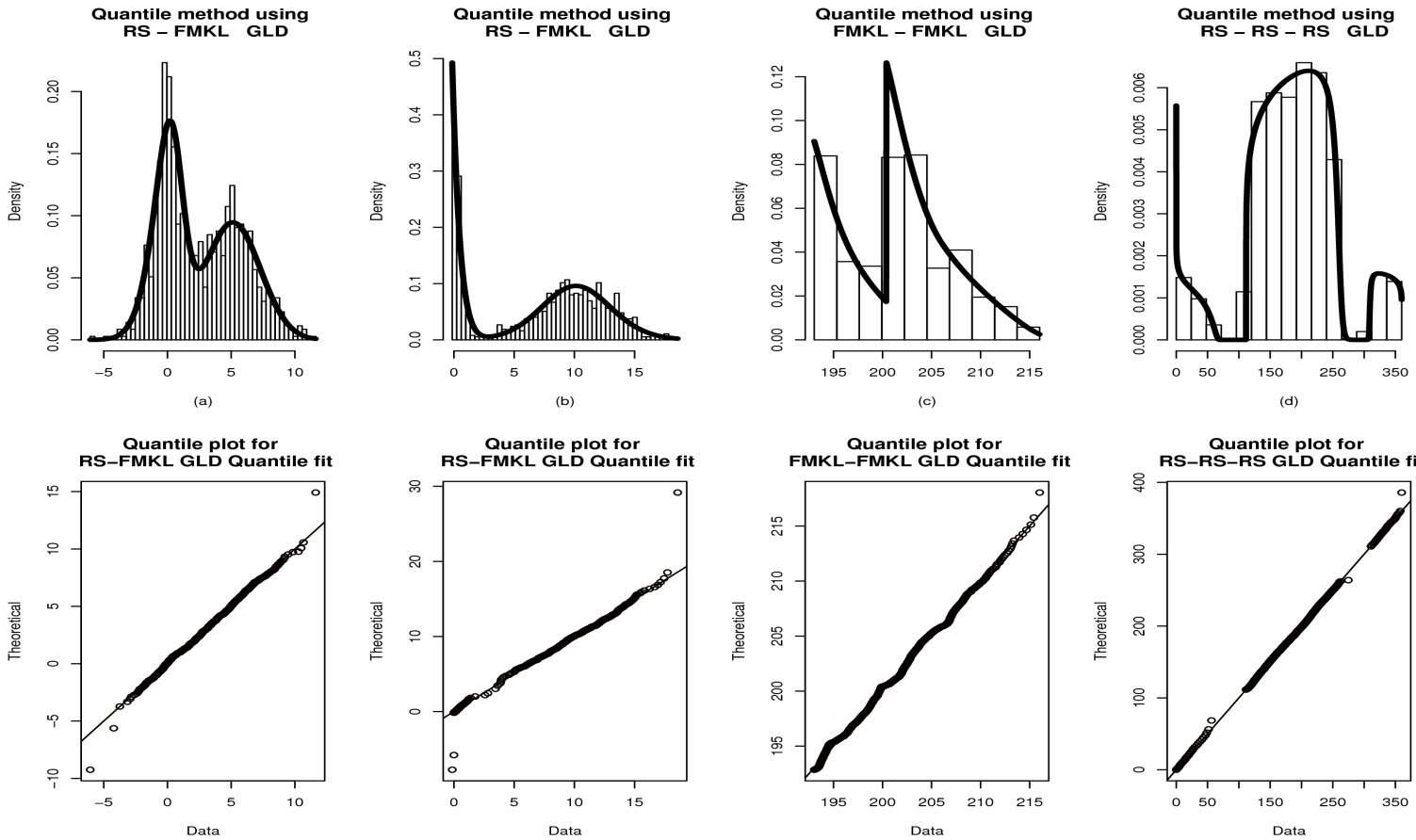


Figure 14.3–22. Using mixtures of GLDs with quantile matching estimation method.

quantile matching method as an effective tool to estimate mixture of GLDs for multi-modal data.

14.4 Conclusion

While the simple quantile matching method for fitting GLDs in this chapter is not as efficient as maximum likelihood estimation, it can often provide quite convincing alternative fits to data as demonstrated in various examples. When dealing with complex, messy real-world data, there is no one fitting method that would be best in all situations. For example, when the moments of the data are unstable, it is perhaps not advisable to use the moment matching method. Sometimes, it may be difficult or impossible to attain the maximum likelihood estimation. In these cases, the quantile matching method can be a useful alternative as the quantile of the data is always available. The main drawback, however, of the quantile matching method is that an accurate estimation of quantiles usually comes with larger samples; this can limit the use of this fitting method in some datasets.

References for Chapter 14

Asquith, W.H. (2007). "L-Moments and TL-Moments of the Generalized Lambda Distribution," *Computational Statistics and Data Analysis*, 51, 4484–4496.

Freimer, M., Kollia, G., Mudholkar, G.S., and Lin, C.T. (1988). "A Study of the Generalised Tukey Lambda Family, *Communications in Statistics–Theory and Methods*, 17, 3547–3567.

Hyndman, R.J. and Fan, Y. (1996). " Sample quantiles in statistical packages," *American Statistician*, 50, 361–365.

Karian, Z.A. and Dudewicz, E.J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalised Bootstrap Methods*, Chapman and Hall, Boca Raton.

Karian, Z.A., Dudewicz, E.J., and McDonald, P. (1996). "The extended Generalized Lambda Distribution systems for fitting distributions to data: History, completion of theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics-Computation and Simulation*, 25, 611–642.

Karvanen, J. and Nuutinen, A. (2008). "Characterizing the Generalized Lambda Distribution by L-Moments," *Computational Statistics and Data Analysis*, 52, 1971–1983.

Kaufman, L. and Rousseeuw, P.J. (1990). *Finding Groups in Data: An Introduction to Cluster Analysis*, Wiley, New York.

King, R. and MacGillivray, H.L. (1999). "A Starship estimation method for the Generalised Lambda Distributions," *Australia and New Zealand Journal of*

Statistics, 41, 353–374.

Öztürk, A and Dale, R.F. (1985). "Least squares estimation of the parameters of the Generalised Lambda Distribution," *Technometrics*, 27, 8–84.

R Development Core Team (2008). ***R*** *A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, ISBN: 3-900051-07-0.

Ramberg, J.S. and Schmeiser, B.W. (1974). "An approximate method for generating asymmetric random variables," *Communications of the Association for Computing Machinery*, 17, 78–82.

Su, S. (2005). "A discretized approach to flexibly fit Generalized Lambda Distributions to data," *Journal of Modern Applied Statistical Methods*, November, 408–424.

Su, S. (2007a). "Fitting single and mixture of Generalised Lambda Distributions to data via discretized and maximum likelihood methods: GLDEX in **R**, *Journal of Statistical Software*, 21, 1–17

Su, S. (2007b). *GLDEX: Fitting Single and Mixture of Generalized Lambda Distributions (RS and FMKL) Using Discretized and Maximum Likelihood Methods.* **R** package version 1.0.1, URL: http://cran.r-project.org/.

Su, S. (2007c). "Numerical maximum log-likelihood estimation for Generalized Lambda Distributions," *Computational Statistics and Data Analysis*, 51, 3983–3998.

Chapter 15

Fitting GLD to Data Using GLDEX 1.0.4 in R

Steve Su
School of Mathematics and Statistics,
University of Western Australia,
M019, 35 Stiring Highway, Crawley, 6009, WA, Australia
E-mail: allegro.su@gmail.com

The statistical software **R** available from http://www.r-project.org/ (**R** Development Core Team (2008)) is a freeware with many additional packages or add-ons written by users to extend the statistical capabilities of **R**. The `GLDEX` package (Su (2007b)) is one such package, primarily used to fit both RS and FMKL GLD distributions using a variety of methods. Currently, the following methods of fitting are available from `GLDEX 1.0.4`.

1. Discretized methods, weighted and unweighted (Su (2005), (2007a));
2. Method of moment matching (Karian and Dudewicz (2000));
3. *L*-moment matching (Karian and Dudewicz (2003), Asquith (2007), Karvanen and Nuutinen (2008));
4. Quantile matching (Su (2010));
5. Maximum likelihood estimation (Su (2007c), (2007a));
6. Starship method (King and MacGillivray (1999), King (2006)).

This chapter introduces the basic functionality of `GLDEX 1.0.4` in terms of generating basic probability density functions for RS and FMKL GLDs. It then gives a broad overview of various functions for each of the six methods above and illustrates the diagnostic tools available in this package to assess goodness-of-fit. The last part of the chapter illustrates the use of `GLDEX 1.0.4` through examples that fit statistical distributions and shows how better fits may be obtained using a hybrid of existing methods.

15.1 Introduction

The availability of a range of methods is important as there is no single method that would work best in every single case. The availability of several methods allows the user to choose the best method of fit for a given dataset based on a QQ plot or other goodness-of-fit measure. In cases where multiple methods provide apparently equally good fits, it may be desirable to choose a method that is most relevant to the research problem at hand. For example, it may be the case that the quantile matching method is preferred if the principal purpose is to model the quantiles of the data set using GLD.

The methods of moment matching and L-moment matching in `GLDEX` do not follow the algorithms as described by the aforementioned authors in the literature. The generation of initial values for these methods follows the strategy described in Su (2007a) but the principal goal of matching the moments or L-moments is retained in these fitting methods.

`GLDEX 1.0.4` also provides diagnostic tools by way of QQ plots and KS resample tests (repeated sampling of data against fitted distribution to assess adequacy of fit using the Kolmogorov-Smirnov or KS test). These methods have been available since the first version of `GLDEX` and have been illustrated and described in detail in Su (2007a).

A comprehensive set of functions available within `GLDEX 1.0.4` are listed in Tables 15.1–1 through 15.1–4. Table 15.1–1 gives the functions associated with fitting RS and FMKL GLDs through a variety of methods, Table 15.1–2 does the same for fitting a mixture of two GLDs, Table 15.1–3 lists the functions that are available for goodness-of-fit assessments, and Table 15.1–4 gives the functions that can be used for hybrid fitting methods. Readers should refer to these tables in conjunction with examples so they can identify clearly what the function is doing in each step. In the following demonstrations, only selected graphical outputs associated with the examples under consideration will be shown.

Users are assumed to have a basic working knowledge of **R**. The official introduction to **R** is available from the Website

http://cran.r-project.org/doc/manuals/R-intro.pdf

and there are a number of resources and textbooks available on how to use **R** (e.g., Dalgaard (2008), Rizzo (2008), Spector (2008)).

Table 15.1–1. Functions for fitting single RS and FMKL GLD.

Single GLD fit			
Method	**GLD**	**Name of function**	**Note**
Discretized method (Weighted)	FMKL RS	`fun.RMFMKL.hs` `fun.RPRS.hs`	To do both use `fun.data.fit.hs`
Discretized method (Unweighted)	FMKL RS	`fun.RMFMKL.hs.nw` `fun.RPRS.hs.nw`	To do both use `fun.data.fit.hs.nw`
Maximum Likelihood	FMKL RS	`fun.RMFMKL.ml` `fun.RPRS.ml`	To do both and starship method with FMKL GLD use `fun.data.fit.ml`
Method of moments	FMKL RS	`fun.RMFMKL.mm` `fun.RPRS.mm`	To do both use `fun.data.fit.mm`
Starship method	FMKL RS	`starship` `starship`	Declare `param="rs"` if RS GLD is desired
L-moments	FMKL RS	`fun.RMFMKL.lm` `fun.RPRS.lm`	To do both use `fun.data.fit.lm`
Quantile matching	FMKL RS	`fun.RMFMKL.qs` `fun.RPRS.qs`	To do both use `fun.data.fit.qs`

Table 15.1–2. Functions for fitting mixture of two GLDs.

Mixture of two GLDs		
Method	**Name of function**	**Note**
Partition maximum likelihood estimation	`fun.auto.bimodal.pml`	Specify the type of GLD
Maximum likelihood Likelihood estimation	`fun.auto.bimodal.ml`	required using `param1`
Quantile matching method	`fun.auto.bimodal.qs`	and `param2`.

Table 15.1–3. Functions to examine the goodness-of-fit.

Diagnostics	
Name of function	**Note**
QQ plot for single fit	`qqplot.gld`
QQ plot for mixture of two GLDs fit	`qqplot.gld.bi`
KS resample test for single fit	`fun.diag.ks.g`
KS resample test for mixture of two-GLDs fit	`fun.diag.ks.g.bimodal`
Compute theoretical first four moments-single GLD	`fun.theo.mv.gld`
Compute theoretical first four L-moments-single GLD	`fun.lm.theo.gld`
Compute theoretical first four moments-Mixture of two GLDs	`fun.theo.bi.mv.gld`
Histogram plot-single GLD	`fun.plot.fit`
Histogram plot-mixture of two GLDs	`fun.plot.fit.bm`
Compute first four L-moments of data	`Lmoments`
Compute first four moments of data	`fun.moments.r`

Table 15.1–4. Optimization functions useful for implementing hybrid fitting methods.

Optimization Functions	
Discretized-weighted	`optim.fun2`
Discretized-unweighted	`optim.fun2.nw`
Maximum likelihood estimation	`optim.fun3`
Method of moments	`optim.fun7`
Starship method	`starship`
L-moments	`optim.fun.lm`
Quantile matching	`optim.fun6`
Partition maximum likelihood estimation-Mixture of two GLDs	`optim.fun4`
Maximum likelihood estimation-Mixture of two GLDs	`optim.fun5`
Quantile matching method-Mixture of two GLDs	`optim.fun.qs`
Final maximum likelihood maximizer	`optim.fun.bi.final`

15.2 Installation and Basic GLDEX Functions

Users with an Internet connection can install the latest `GLDEX` package by using the drop-down menu within the **R** interface. Once installed the following command would list all the major functionalities of this package. This is a useful page that enables you to recall any function you may want to use in this package.

```
>?GLDEX
```

As is customary with all statistical distributions in **R**, this package contains functions to evaluate a density function (`dgl`), a distribution function (`pgl`), and a quantile function (`qgl`). Generation of random observations can be done using `rgl` (single GLD) and `fun.simu.bimodal` (mixture of two GLDs). Some examples are given below.

```
> # Density function for RS and FMKL GLD 1.5,2,3,1 with x=2
> dgl(x=2,1.5,2,3,1,param='rs')
[1] 0.5
>
> dgl(x=2,1.5,2,3,1,param='fmkl')
[1] 1
>
> # Distribution function for RS and FMKL GLD 1.5,2,3,1 with q=1.5
> pgl(q=1.5,1.5,2,3,1,param='rs')
[1] 0.6823278
>
> pgl(q=1.5,1.5,2,3,1,param='fmkl')
[1] 0.3221854
>
> # Quantile function for RS and FMKL GLD 1,2,3,1 with q=0.8
> qgl(p=0.8,1,2,3,1,param='rs')
[1] 1.156
>
> qgl(p=0.8,1,2,3,1,param='fmkl')
[1] 1.318667
>
> # simulate 5 random numbers from RS GLD 1,2,3,1 distribution
> set.seed(10)
> rgl(5,1,2,3,1,"rs")
[1] 0.8190856 0.6678188 0.7523558 1.0130309 0.5428765

> # simulate 5 random numbers from 0.6 of RS GLD 1,2,3,1 + 0.4 of
> # FMKL GLD 1,2,3,1 -- do this only once.
> set.seed(10)
```

```
> fun.simu.bimodal(c(1,2,3,1),c(1,2,3,1),prop1=0.6,prop2=0.4,
+ param1='rs', param2='fmkl',no.test=1,len=5)
$`1`
[1] 0.8190856 0.6678188 0.7523558 1.2353776 0.8760042
```

15.3 Fitting Examples

The following example illustrates the use of different fitting algorithms to fit a single GLD to data consisting of 300 random observations from a Weibull (shape = 3, scale = 2) distribution.

The discretized method (Su (2005)) is designed as a smoothing device over the number of bins of histogram and it is not designed to provide a definite fit. It can also be used as a way of generating initial values to be used in other optimization schemes.

The function `fun.data.fit.hs()` would fit a GLD to the dataset (labeled as `junk`[1] in this example) using a weighted discretized method. The random seed is set at 100, to allow reproducibility of results.

The following commands fit, using the weighted discretized method, then compute the theoretical first four moments of the resulting fit and conduct the KS (Kolmogorov-Smirnov) resample test over 1000 runs (reporting the number of times p-values exceed 0.05 in 1000 simulation runs for a sub-selection of random samples from both fitted GLD and empirical data).

```
> fun.plot.fit(obj.fit1.hs,junk,nclass=50,param=c("rs","fmkl"),
+ xlab="x")
>
> fun.theo.mv.gld(obj.fit1.hs[,1],param="rs")
      mean   variance   skewness   kurtosis
 1.7604277  0.1990901 -0.3541727  2.5790095
> fun.theo.mv.gld(obj.fit1.hs[,2],param="fmkl")
      mean   variance   skewness   kurtosis
 1.8256010  0.2014603 -0.0382985  2.6067425
>
> fun.diag.ks.g(obj.fit1.hs[,1],junk,param="rs")
[1] 236
> fun.diag.ks.g(obj.fit1.hs[,2],junk,param="fmkl")
[1] 186
>
> fun.moments.r(junk)
```

[1]The designation of the data in this chapter, as `junk` is not a reflection of the importance of the data; it is simply a convenient way of allowing the user to identify unnecessary objects from the **R** workspace.

```
     mean  variance  skewness  kurtosis
1.7653202 0.3800481 0.1408347 2.5487294
```

These results seem adequate but should be verified by further testing. It is possible to assess the goodness-of-fit through use of the resample KS test. The KS resample test demonstrates the fit is inadequate and indicates that there is no difference between the fitted and simulated distribution at 5% significance level in approximately 20% of the tests. Also there are some departures in some of the first four moments of the fitted distribution with the simulated `junk` data, particularly in variance and skewness statistics. The advantage of the discretized method is that it is possible to change the number of classes to improve the fit. Alternatively, the unweighted discretized version using `fun.data.fit.hs.nw` can be used to avoid accentuating the peak of the data and suppressing the tails of the distribution. As a further example, assume the `junk` data is refitted using the weighted discretized method with the number of classes = 15.

```
> # The second fit using discretized methods and diagnostics
>
> obj.fit2.hs<-fun.data.fit.hs(junk,rs.default="N",
+ fmkl.default="N",no.c.rs = 15, no.c.fmkl = 15)
>
> fun.plot.fit(obj.fit2.hs,junk,nclass=50,param=c("rs","fmkl"),
+ xlab="x")
>
> fun.theo.mv.gld(obj.fit2.hs[,1],param="rs")
     mean  variance  skewness  kurtosis
1.7678618 0.3858987 0.1401209 2.6825784
> fun.theo.mv.gld(obj.fit2.hs[,2],param="fmkl")
     mean  variance  skewness  kurtosis
1.7720938 0.3893295 0.2367630 2.7743679
>
> fun.diag.ks.g(obj.fit2.hs[,1],junk,param="rs")
[1] 924
> fun.diag.ks.g(obj.fit2.hs[,2],junk,param="fmkl")
[1] 923
>
```

In this example, the theoretical moments are closer to the empirical moments than before and the KS resample tests suggest that the resulting fits are better than the previous fits. More than 90% of the time, the KS tests indicate there is no difference between the fitted distribution and the empirical data[2].

[2]Alternatively a simple Kolmogorov-Smirnov test can be used directly from R: `ks.test(junk,"pgl",obj.fit2.hs[,1],param="rs")`.

It is usually unnecessary to explore different number of classes using the discretized method to get a definite fit, several methods such as the maximum likelihood estimation or the quantile matching method can give more definite fits to the data. The following example illustrates fitting the `junk` data using the maximum likelihood, quantile matching, L-moment matching, starship, and moment matching methods.

```
> # Using other methods
>
> obj.fit.ml<-fun.data.fit.ml(junk)
> obj.fit.qs<-fun.data.fit.qs(junk)
> obj.fit.lm<-fun.data.fit.lm(junk)
> obj.fit.star.rs<-starship(junk,param="rs")
> obj.fit.mm<-fun.data.fit.mm(junk)
There were 50 or more warnings (use warnings() to see the first 50)
>
> # Diagnostics
>
> fun.theo.mv.gld(obj.fit.ml[,1],param="rs")
     mean  variance  skewness  kurtosis
1.7687531 0.3819470 0.1060090 2.4598450
> fun.theo.mv.gld(obj.fit.ml[,2],param="fmkl")
     mean  variance  skewness  kurtosis
1.7660913 0.3810808 0.1100645 2.4475679
>
> fun.theo.mv.gld(obj.fit.qs[,1],param="rs")
      mean   variance   skewness   kurtosis
1.76541996 0.37063405 0.03679414 2.45763945
> fun.theo.mv.gld(obj.fit.qs[,2],param="fmkl")
     mean  variance  skewness  kurtosis
1.7649492 0.3716488 0.1150162 2.4173316
>
> fun.theo.mv.gld(obj.fit.mm[,1],param="rs")
     mean  variance  skewness  kurtosis
1.7653202 0.3800481 0.1408346 2.5487294
> fun.theo.mv.gld(obj.fit.mm[,2],param="fmkl")
     mean  variance  skewness  kurtosis
1.7653202 0.3800481 0.1408347 2.5487294
>
> fun.theo.mv.gld(obj.fit.lm[,1],param="rs")
     mean  variance  skewness  kurtosis
1.7653202 0.3833225 0.1250159 2.6357313
> fun.theo.mv.gld(obj.fit.lm[,2],param="fmkl")
```

```
     mean  variance  skewness  kurtosis
1.7653202 0.3834375 0.1529988 2.6547775
>
> Lmoments(junk)
           L1        L2          L3        L4
[1,] 1.765320 0.3519766 0.009712951 0.0356177
> fun.lm.theo.gld(obj.fit.lm[,1],param="rs")
[1] 1.765320202 0.351976646 0.009712945 0.035617710
> fun.lm.theo.gld(obj.fit.lm[,2],param="fmkl")
[1] 1.765320195 0.351976617 0.009712941 0.035617705
>
>
> fun.diag.ks.g(obj.fit.ml[,1],junk,param="rs")
[1] 922
> fun.diag.ks.g(obj.fit.ml[,2],junk,param="fmkl")
[1] 913
>
> fun.diag.ks.g(obj.fit.qs[,1],junk,param="rs")
[1] 927
> fun.diag.ks.g(obj.fit.qs[,2],junk,param="fmkl")
[1] 926
>
> fun.diag.ks.g(obj.fit.mm[,1],junk,param="rs")
[1] 931
> fun.diag.ks.g(obj.fit.mm[,2],junk,param="fmkl")
[1] 927
>
> fun.diag.ks.g(obj.fit.lm[,1],junk,param="rs")
[1] 930
> fun.diag.ks.g(obj.fit.lm[,2],junk,param="fmkl")
[1] 918
>
> fun.diag.ks.g(obj.fit.star.rs$lambda,junk,param="rs")
[1] 927
> fun.diag.ks.g(obj.fit.ml[,3],junk,param="fmkl")
[1] 931
>
> par(mfrow=c(2,3))
>
> qqplot.gld(fit=obj.fit.ml[,1],data=junk,param="rs",
+ type="str.qqplot",name="Maximum likelihood, RS GLD")
> qqplot.gld(fit=obj.fit.mm[,1],data=junk,param="rs",
+ type="str.qqplot",name="Method of Moment, RS GLD")
```

```
> qqplot.gld(fit=obj.fit.lm[,1],data=junk,param="rs",
+ type="str.qqplot",name="L moment, RS GLD")
> qqplot.gld(fit=obj.fit.qs[,1],data=junk,param="rs",
+ type="str.qqplot",name="Quantile matching, RS GLD")
> qqplot.gld(fit=obj.fit.star.rs$lambda,data=junk,param="rs",
+ type="str.qqplot",name="Starship method, RS GLD")
> qqplot.gld(fit=obj.fit2.hs[,1],data=junk,param="rs",
+ type="str.qqplot",name="Discretised weighted method, RS GLD")
>
```

The long series of commands fit the `junk` data using a number of methods, examine the first four moments for each of the methods (including first four L-moments for L-moment matching method), and compute the resample KS tests. To illustrate, a QQ plot for the RS GLD part of the fit allows users to examine graphically, the adequacy of fit. This is shown in Figure 15.3–1. As KS resample tests suggest, all fitting schemes are quite adequate in this case; this is confirmed by the QQ plots. Users can verify the FMKL GLD parts of the fit are also adequate in terms of QQ plots by modifying the codes above. As an aside, the default QQ plot shows the entire range of fitted distribution against the empirical data, so in the case where the fitted GLD have extremely large or small values at one or both of its tails, it is advisable to change the `range` argument in `qqplot.gld` or `qqplot.gld.bi` into a slightly smaller range, say `c(0.01,0.99)`.

In most situations, the functions provided would do the job. However, there could be situations where the use of the above function does not directly give desired results. For example, setting random seed to 1000, the use of `fun.RPRS.mm` did not give a good fit as shown by the QQ plot and produced strange first four moments. This is an example where a fitting function method may fail because of the choice of an inappropriate initial value. One way to resolve this is to look at the options available in `fun.RPRS.mm` and change the range of initial values or the type of quasi-number sequences. Alternatively, one can use another fitting method and use the result of that fitting method as initial values to maximize the first four moment match.

The following example uses L-moment matching to generate initial values and use the function `optim` in `R` to optimize the first four moment match. As Table 15.1–4 indicates, the function `optim.fun7` is used for the moment matching method and the resulting fit, `fit.junk1.improve`, is much better than the initial fit that was obtained directly through `fun.RPRS.mm`. This improved fit is quite adequate as can be seen in the QQ plot and has much closer first four moments to the empirical data. The user can choose any method available in the `GLDEX` package to generate initial values and then use `optim` with the required optimization function in Table 15.1–4 to get the desired final fit for a given dataset. This hybrid method approach can be very useful in cases where it may be difficult to obtain the correct initial values using the default methods and it is possible to

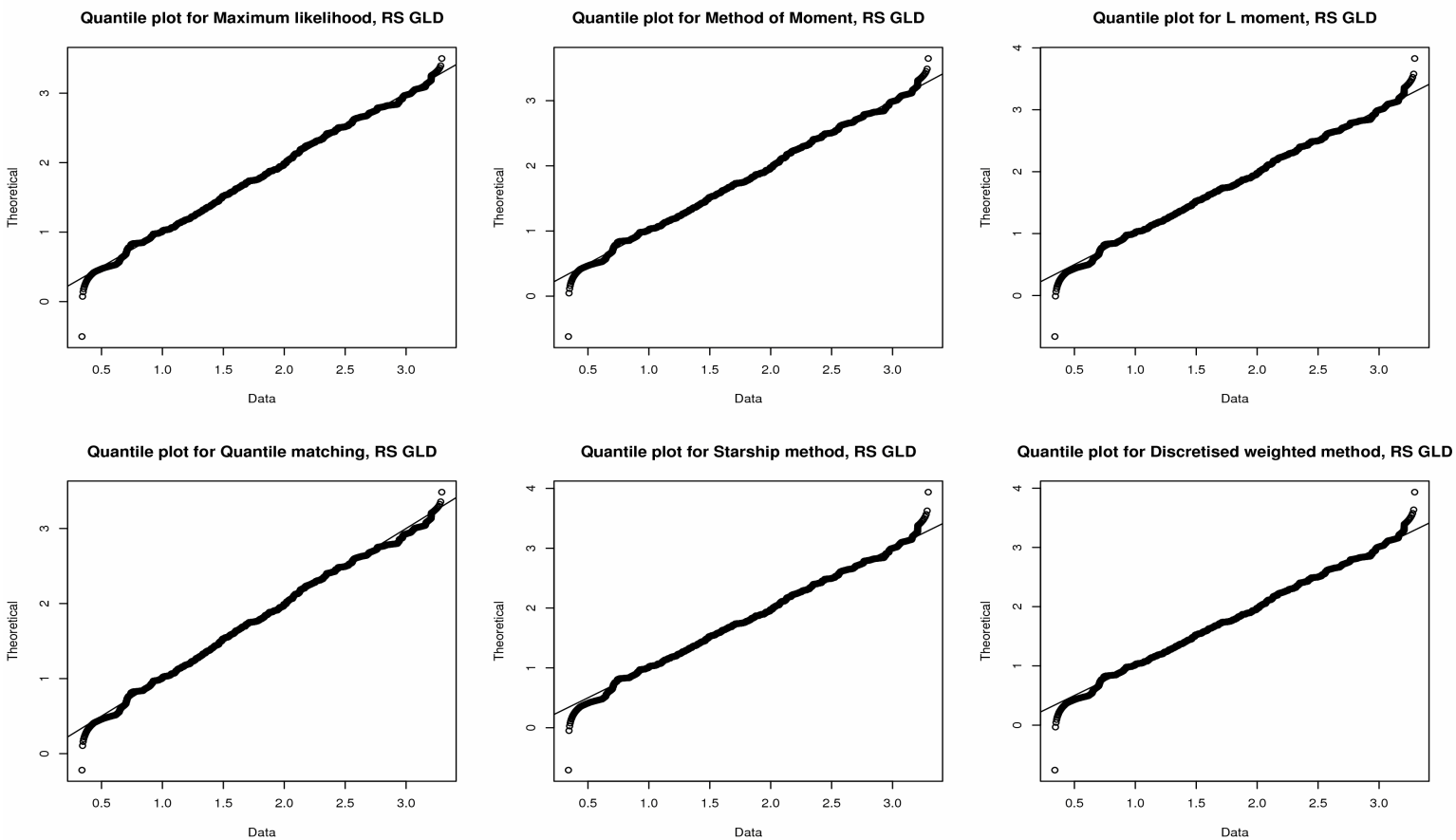


Figure 15.3–1. QQ plots for fitting RS GLD to simulated data using six methods.

improve the fit beyond the capability of using only a single fitting scheme.

```
> # Example of a bad fit, and improving the fit using
> # resources available in GLDEX
>
> set.seed(1000)
> junk1<-rweibull(300,3,2)
>
> fit.junk1<-fun.RPRS.mm(junk1)
There were 50 or more warnings (use warnings() to see the first 50)
> fit.junk1.lm<-fun.RPRS.lm(junk1)
>
> fun.theo.mv.gld(fit.junk1,param="rs")
         mean      variance      skewness      kurtosis
 1.665853e+00  1.024440e-06 -7.427503e-02  2.749049e+00
> qqplot.gld(fit=fit.junk1,data=junk1,param="rs",type="str.qqplot",
+ name="Method of Moment, RS GLD")
>
> fit.junk1.improve <- optim(fit.junk1.lm,optim.fun7,param="rs",
+ data=junk1)
>
> fun.theo.mv.gld(fit.junk1.improve$par,param="rs")
     mean  variance  skewness  kurtosis
1.7583039 0.3800870 0.2766095 2.7230469
> fun.moments.r(junk1)
     mean  variance  skewness  kurtosis
1.7583304 0.3800741 0.2766219 2.7230887
> qqplot.gld(fit=fit.junk1.improve$par,data=junk1,param="rs",
+ type="str.qqplot",name="Method of Moment, RS GLD")
>
```

GLDEX 1.0.4 also has the capability to fit a mixture of two GLDs of any combination (RS-RS, RS-FMKL, FMKL-RS, FMKL-FMKL) using maximum likelihood, partition maximum likelihood, and the quantile matching method. Details of these methods can be found in Su (2007a) and (2010).

```
> # Fitting bimodal data, faithful data set example
>
> bimodal.ml<-fun.auto.bimodal.ml(faithful[,1])
> bimodal.pml<-fun.auto.bimodal.pml(faithful[,1])
> bimodal.qs<-fun.auto.bimodal.qs(faithful[,1])
>
> par(mfrow=c(1,3))
> qqplot.gld.bi(data=faithful[,1],fit=bimodal.ml$par,param1="rs",
```

```
+ param2="rs",type="str.qqplot",range=c(0.01,0.99),
+ name="Maximum likelihood method")
> qqplot.gld.bi(data=faithful[,1],fit=bimodal.pml$par,param1="rs",
+ param2="rs",type="str.qqplot",range=c(0.01,0.99),
+ name="Partition Maximum likelihood method")
> qqplot.gld.bi(data=faithful[,1],fit=bimodal.qs$par,param1="rs",
+ param2="rs",type="str.qqplot",range=c(0.01,0.99),
+ name="Quantile matching method")
>
```

The above example shows the default use of functions `fun.auto.bimodal.ml`, `fun.auto.bimodal.pml`, and `fun.auto.bimodal.qs`. The default setting uses two RS GLDs. Details on how to specify different GLDs for the maximum likelihood estimation method, for example, can be found by typing the following command into **R**.

```
>? fun.auto.bimodal.ml
```

The above results (QQ plots in Figure 15.3–2) indicate a fairly adequate fit for both maximum likelihood estimation and the quantile matching method. The partition maximum likelihood method, however, does not seem to perform as well. It is possible to improve the fit by changing the options available in these functions. The argument `clustering.m` specifies how data should be split to generate initial values for mixture fits (see Su (2007a) for details) and instead of using the default `clara` classification scheme, it is possible to split the data based on a user defined cut-off point, say 2.75, as was done in the following example. As can be seen, this results in a much better fit as indicated by the QQ plot in Figure 15.3–3.

```
> # Improving the fit by choosing a different cut off point
>
> bimodal.pml.improve<-fun.auto.bimodal.pml(faithful[,1],
+ clustering.m=faithful[,1]>2.75)
> par(mfrow=c(1,1))
> qqplot.gld.bi(data=faithful[,1],fit=bimodal.pml.improve$par,
+ param1="rs",param2="rs",type="str.qqplot",range=c(0.01,0.99),
+ name="Partition Maximum likelihood method")
>
```

Numerical summaries of examining goodness-of-fit are shown below, while there is an improvement in the new partition maximum likelihood method, it does not result in a closer first four moment match. All methods however, show reasonable results in terms of the KS resample test, with more than 90% of the time, p-values exceed 0.05 indicating no difference between the data and the fitted distribution.

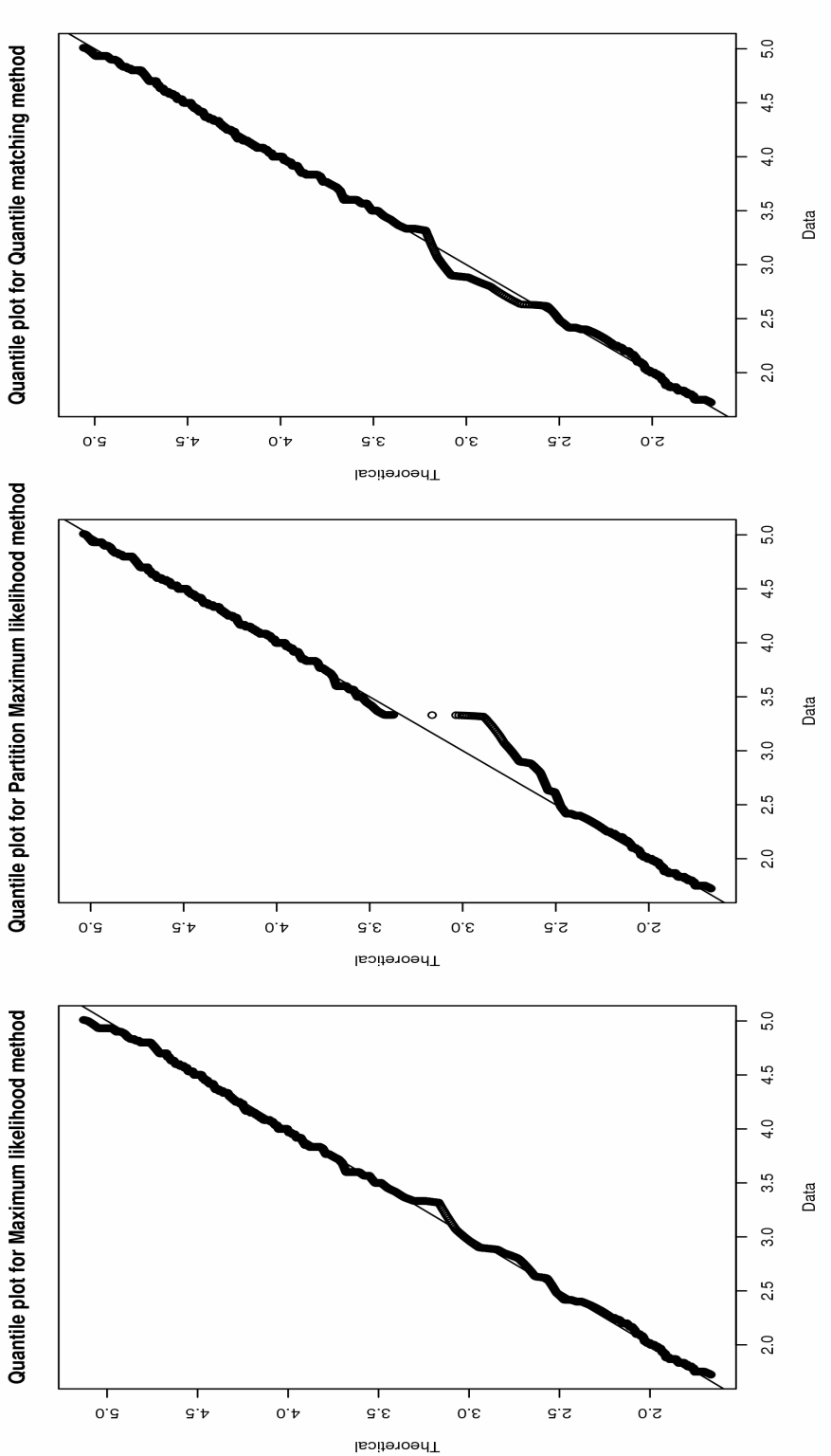


Figure 15.3–2. QQ plots for fitting Old Faithful Geyser Data using mixture of GLDs.

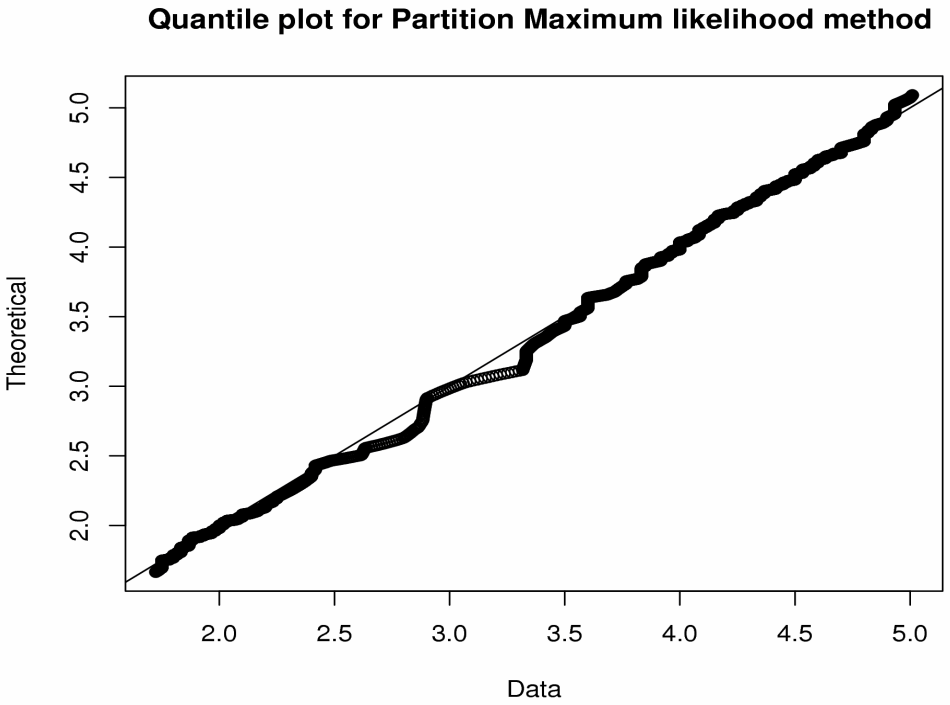


Figure 15.3–3. QQ plot for fitting Old Faithful Geyser Data using mixture of GLDs-Improved Partition Maximum Likelihood Fit.

```
> # Goodness of fit
>
> fun.moments.r(faithful[,1])
      mean   variance   skewness   kurtosis
 3.4877831  1.2979389 -0.4158410  1.4993996
> fun.theo.bi.mv.gld(bimodal.ml$par,param1="rs",param2="rs")
      mean   variance   skewness   kurtosis
 3.4914459  1.3104099 -0.4122768  1.4997445
> fun.theo.bi.mv.gld(bimodal.pml$par,param1="rs",param2="rs")
      mean   variance   skewness   kurtosis
 3.4863561  1.2927591 -0.4093697  1.4992246
> fun.theo.bi.mv.gld(bimodal.pml.improve$par,param1="rs",
+ param2="rs")
         mean      variance      skewness      kurtosis
 3.479535e+00  1.317610e+00 -2.252693e-01 -1.204597e+11
> fun.theo.bi.mv.gld(bimodal.qs$par,param1="rs",param2="rs")
        mean     variance     skewness     kurtosis
3.488557e+00 1.251372e+00 3.075544e+06 2.691940e+14
```

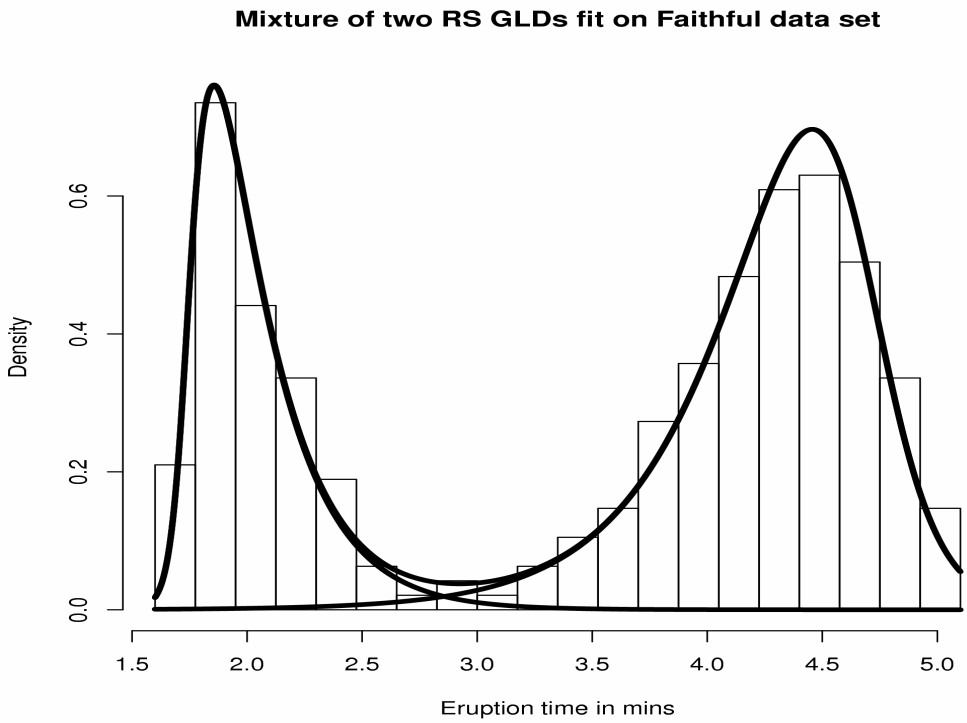


Figure 15.3–4. Old Faithful Geyser Data using mixture of GLDs-Maximum Likelihood Estimation.

```
>
> fun.diag.ks.g.bimodal(bimodal.ml$par,data=faithful[,1],
+ param1="rs",param2="rs")
[1] 943
> fun.diag.ks.g.bimodal(bimodal.pml$par,data=faithful[,1],
+ param1="rs",param2="rs")
[1] 955
> fun.diag.ks.g.bimodal(bimodal.pml.improve$par,data=faithful[,1],
+ param1="rs",param2="rs")
[1] 938
> fun.diag.ks.g.bimodal(bimodal.qs$par,data=faithful[,1],
+ param1="rs",param2="rs")
[1] 937
```

And finally, it is often preferred to examine the shape of the data and the fitted distribution, an example is given below and is shown in Figure 15.3–4. The area under the histogram in Figure 15.3–4 is 1, as is the area under the mixture probability density function (pdf). What is actually shown in Figure 15.3–4, however, is the portion between x-values of 1.6 and 5.1. The mixture pdf

is $p_1 f_1 + (1-p_1) f_2$, the weighted sum of the two density functions f_1 and f_2. The individual curves shown are $p_1 f_1$ and $(1-p_1) f_2$. Thus their areas are less than 1 due to the multipliers p_1 and $1-p_1$.

```
> # Do a plot
>
> par(mfrow=c(1,1))
>
> fun.plot.fit.bm(bimodal.ml,data=faithful[,1],
+ param.vec=c("rs","rs"),main="Mixture of two RS GLDs fit on
+ Faithful data set",nclass=20,xlab="Eruption time in mins")
```

It is also possible to improve mixture distributional fit by using a hybrid of method as was done in the case of a single distribution fit. The methods in Table 15.1–1 and/or Table 15.1–2 can be used to generate initial values. These initial values can then be used to find the optimal solution using `optim` and the required mixture distribution optimization function in Table 15.1–4.

15.4 Fitting Empirical Data

An illustration of using `GLDEX` to fit the tree diameter data (Table 3.5-6 on page 148 of Karian and Dudewicz (2000) and reproduced here as Table 15.4–1) is given in this section. The initial attempt is to fit the data using the usual maximum likelihood method and starship method.

```
> # Read in data from text file
> tree<-scan("tree diameter.txt")
>
> # Fit data using ML and starship methods
> tree.fit<-fun.data.fit.ml(tree)
```

To visually examine the goodness-of-fit, the usual strategy is to use QQ plots, quantile plots, and histograms, as given in Figure 15.4–2.

```
> par(mfrow=c(3,3))
>
> qqplot.gld(fit=tree.fit[,1],data=tree,param="rs",
+ type="str.qqplot",name="Maximum likelihood, RS GLD",
+ main="QQ plot-Maximum likelihood, RS GLD")
> qqplot.gld(fit=tree.fit[,2],data=tree,param="fmkl",
+ type="str.qqplot",name="Maximum likelihood, FMKL GLD",
+ main="QQ plot-Maximum likelihood, FMKL GLD")
> qqplot.gld(fit=tree.fit[,3],data=tree,param="fmkl",
+ type="str.qqplot",name="Starship method, FMKL GLD",
```

Table 15.4–1. Tree Diameter Data.

3.3	5.9	3.8	7	5.8
4.4	4.6	8.2	3.3	10.2
11.3	4.8	9.1	4.4	5.6
8.5	8.2	4.7	8	8.2
7.7	10	4.8	7.1	5.5
12.8	8.9	9.8	7.1	6.1
10	8.1	2.2	7.3	10.8
5.6	6.4	4.3	9.5	7.7
7.4	3.6	5.5	6.5	10.3
4.9	5.8	14.8	6.4	4.9
10	3.4	6.3	8.9	8.1
10.3	3	2.4	4.7	4.1
4	4.2	10.7	4	2.5
10.2	3.5	9.3	8.6	9.1
8.1	6.9	5.5	5.8	5.5
10.4	4.4	4.7	4.7	4.5
4.9	3	10.3	5.3	8.8
7.8	6.5	7.2	7.2	

```
+ main="QQ plo-Starship method, FMKL GLD")
>
> qqplot.gld(fit=tree.fit[,1],data=tree,param="rs",
+ name="Maximum likelihood, RS GLD",main="Quantile plot",
+ xlab="Quantile")
> qqplot.gld(fit=tree.fit[,2],data=tree,param="fmkl",
+ name="Maximum likelihood, FMKL GLD",main="Quantile plot",
+ xlab="Quantile")
> qqplot.gld(fit=tree.fit[,3],data=tree,param="fmkl",
+ name="Starship method, FMKL GLD",main="Quantile plot",
+ xlab="Quantile")
>
> fun.plot.fit(tree.fit[,1],tree,nclass=nclass.scott,
+ param=c("rs"),xlab="Tree Diameter",name="RS GLD-ML")
> fun.plot.fit(tree.fit[,2],tree,nclass=nclass.scott,
+ param=c("fmkl"),xlab="Tree Diameter",name="FMKL GLD-ML")
> fun.plot.fit(tree.fit[,3],tree,nclass=nclass.scott,
+ param=c("fmkl"),xlab="Tree Diameter",name="FMKL GLD-STARSHIP")
```

The resulting fits are reasonable except for tree diameter larger than 10.4 inches. Only five observations were greater than 10.4 inches (this is where the

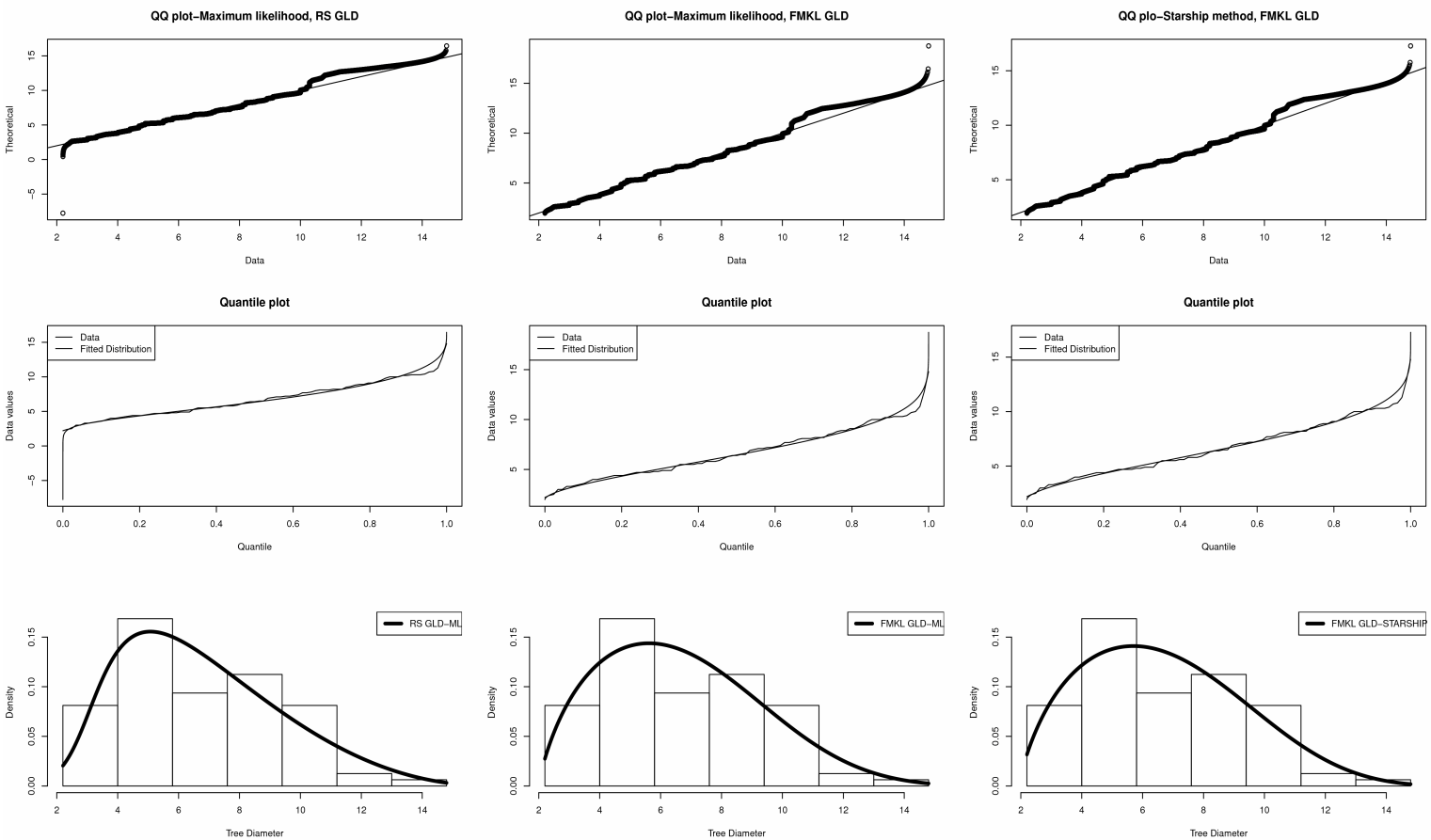


Figure 15.4–2. Tree Diameter Distributional Fit using only one GLD.

departure from the straight line begins in the QQ plots) so it is reasonable that the fit is not spectacular for these data values. It is also worthwhile to compare the moments and examine the goodness-of-fit via resample KS tests:

```
> # Check goodness of fit
>
> fun.diag.ks.g(tree.fit[,1],tree,param="rs")
[1] 936
> fun.diag.ks.g(tree.fit[,2],tree,param="fmkl")
[1] 961
> fun.diag.ks.g(tree.fit[,3],tree,param="fmkl")
[1] 941
>
> # Check moments
>
> fun.comp.moments.ml(tree.fit,tree,name="")
$r.mat
             DATA      RPRS    RMFMKL      STAR
mean     6.740449 6.7172841 6.7278053 6.7457708
variance 6.672072 6.9327013 6.7510053 6.7467638
skewness 0.454393 0.5735039 0.4971632 0.4334617
kurtosis 2.744956 2.8197232 2.7412534 2.5997545

$eval.mat
    RPRS    RMFMKL      STAR
0.4776736 0.1380501 0.2461458
>
```

The best GLD for tree diameter data is FMKL GLD using maximum likelihood estimation, it has the closest match to the first four moments of the data and tends to have the highest value of the KS resample test over a couple of simulation runs. The parameters of the resulting FMKL GLD fit can be extracted via the following command.

```
> tree.fit[,2]
[1] 6.2929550 0.4394609 0.5282997 0.1828417
>
```

It is sometimes a good idea to fit the same data using a mixture of GLDs to see if it is possible to improve the fit beyond the capability of a single GLD. An example of fitting a mixture of two RS GLDs using the maximum likelihood estimation is given below.

```
> # Fit data using mixture of two RS GLDs, using default settings
```

```
> tree.bimodal.fit<-fun.auto.bimodal.ml(tree)
>
> # Graph the resulting fit
> par(mfrow=c(2,2))
>
> qqplot.gld.bi(data=tree,fit=tree.bimodal.fit$par,param1="rs",
+ param2="rs",type="str.qqplot",range=c(0.01,0.99),
+ name="Maximum likelihood method",
+ main="QQ plot-Mixture of two RS GLDs")
> qqplot.gld.bi(data=tree,fit=tree.bimodal.fit$par,param1="rs",
+ param2="rs",range=c(0.01,0.99),name="Maximum likelihood method",
+ main="Quantile plot-Mixture of two RS GLDs",xlab="Quantile")
>
> par(fig=c(0,1,0,1/2),new=T)
>
> fun.plot.fit.bm(nclass=nclass.scott,fit.obj=tree.bimodal.fit,
+ data=tree,main="Maximum likelihood using two RS GLDs",
+ xlab="Data",param.vec=c("rs","rs"))
>
> # Check goodness of fit
> fun.diag.ks.g.bimodal(tree.bimodal.fit$par[1:4],
+ tree.bimodal.fit$par[5:8],prop1=tree.bimodal.fit$par[9],
+ data=tree,param1="rs",param2="rs")
[1] 957
>
> # Check moments
>
> fun.theo.bi.mv.gld(tree.bimodal.fit$par,param1="rs",param2="rs")
     mean  variance  skewness  kurtosis
 6.785803 12.468708        NA        NA
Warning messages:
1: In beta(a, b) : NaNs produced
2: In beta(a, b) : NaNs produced
```

Figure 15.4–3 shows quite a striking fit except for large values of tree diameter. The departure from the straight line in the QQ plot begins at around 11.2 inches and only three observations in the tree data set were larger than this value. The quantile plot is perhaps a more suitable graph to examine the goodness-of-fit in this case since the departure between the empirical data and the resulting fit is not too magnified compared to the QQ plot. The area under the histogram in Figure 15.4–3 is 1, as is the area under the mixture pdf (what is actually shown in Figure 15.4–3, however, is the portion between x-values of 2.2 and 14.8). The mixture pdf is $p_1 f_1 + (1 - p_1) f_2$, the weighted sum of the two density functions

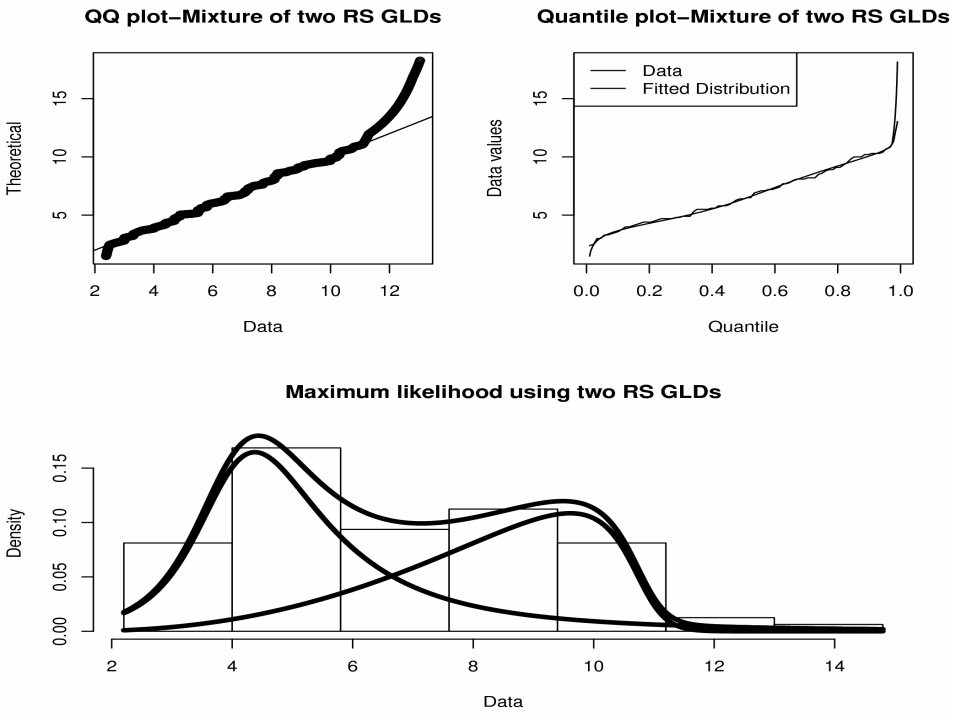


Figure 15.4–3. Tree Diameter Distributional Fit using mixture of two RS GLDs.

f_1 and f_2. The individual curves shown are $p_1 f_1$ and $(1-p_1)f_2$. Thus their areas are are less than 1 due to the multipliers p_1 and $1-p_1$.

While the resample KS test gives a reasonable goodness-of-fit, only the mean of the mixture distributions fit is close to the empirical data. It is perhaps preferable in this case to stick with the single FMKL GLD fit using maximum likelihood estimation, since improvement made by more complex mixture distributions is rather marginal and has very different higher moments compared to empirical data. In practice, the decision to choose the appropriate fitting result will depend on how the distribution is to be used. An analysis requiring more accurate estimates of the extreme values of the dataset will favor the use of mixture distributions and perhaps even more complex mixture distributions will need to be fitted. On the other hand, an analysis attempting to capture the main features of the data in terms of moments without paying much attention to extreme events would favor the use of the single GLD in this case.

15.5 Future Possible Improvements to `GLDEX 1.0.4`

`GLDEX 1.0.4` currently has most of its code written in **R** and it may be possible to increase the speed of fitting methods by rewriting some of the codes to be compiled in `Fortran` or `C`. In case of a very large dataset, however, it is usually

sufficient to just take a random sample, say 1000 observations, and fit a GLD based on this random sample. If the resulting fit from GLD gives adequate estimation to the full data, then it is not necessary to use the method on the entire data.

Also, it is possible that the methods provided in `GLDEX 1.0.4` are not yet optimal, as GLD research is still ongoing and improvements are being made as scientific discovery progresses. Quicker and more reliable initial value generation methods and customized optimization functions for GLDs can all substantially improve the fitting methods provided in this package. Other research works involving the fitting of a GLD, such as finding a confidence interval for quantiles (Su (2009)), will be progressively added to the `GLDEX` package.

15.6 Conclusion

`GLDEX 1.0.4` provides a number of fitting schemes to fit RS and FMKL GLD to data and it is updated from time to time as more methods and new techniques become available. It is hoped that this free package will encourage the use of GLD in the statistical community and users are welcome to provide feedback and suggestions to the author to improve the usability and stability of this package.

References for Chapter 15

Asquith, W. H. (2007). "L-Moments and TL-Moments of the Generalized Lambda Distribution," *Computational Statistics and Data Analysis*, 51, 4484–4496.

Dalgaard, P. (2008). *Introductory Statistics with* **R**, 2nd ed., Springer, ISBN 978-0-387-79053-4.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*, Chapman and Hall, New York.

Karian, Z. A. and Dudewicz, E. J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society*, 2, 171–187.

Karvanen, J. and Nuutinen, A. (2008). "Characterizing the Generalized Lambda Distribution by L-Moments," *Computational Statistics and Data Analysis*, 52, 1971–1983.

King, R. (2006). *GLD: Basic Functions for the Generalised (Tukey) Lambda Distribution*, **R** package version 1.8.2.

King, R. and MacGillivray, H. L. (1999). "A Starship estimation method for the Generalised Lambda Distributions," *Australia and New Zealand Journal of Statistics*, 41, 353–374.

R Development Core Team (2008). **R**: *A Language and Environment for Statistical Computing*, **R** Foundation for Statistical Computing, Vienna, Austria, ISBN 3-900051-07-0.

Rizzo, M. (2008). *Statistical Computing with* **R**, Chapman and Hall, Boca Raton, FL, ISBN 1-584-88545-9.

Spector, P. (2008). *Data Manipulation with* **R**, Springer, New York, ISBN 978-0-387-74730-9.

Su, S. (2005). "A discretized approach to flexibly fit Generalized Lambda Distributions to data," *Journal of Modern Applied Statistical Methods*, November, 408–424.

Su, S. (2007a). "Fitting single and mixture of Generalised Lambda Distributions to data via discretized and maximum likelihood methods: GLDEX in **R**," *Journal of Statistical Software*, 21, 1–17.

Su, S. (2007b). *GLDEX: Fitting Single and Mixture of Generalized Lambda Distributions (RS and FMKL) Using Discretized and Maximum Likelihood Methods.*

Su, S. (2007c). "Numerical maximum log likelihood estimation for Generalized Lambda Distributions," *Computational Statistics and Data Analysis*, 51, 3983–3998.

Su, S. (2009). "Confidence intervals for quantiles using Generalized Lambda Distributions," *Computational Statistics and Data Analysis*, 53, 3324–3333.

Su, S. (2010). "Fitting GLDs and mixture of GLDs to data via quantile matching method," in *Handbook of Distributional Fitting Methods*, Karian, Z. and Dudewicz, E., CRC Press.

PART IV: Other Families of Distributions

Chapter 16 Fitting Distributions and Data with the Johnson System via the Method of Moments

Chapter 17 Fitting Distributions and Data with the Kappa Distribution through L-Moments and Percentiles

Chapter 18 Weighted Distributional L_α Estimates

Chapter 19 A Multivariate Gamma Distribution for Linearly Related Proportional Outcomes

A sketch of each of these chapters appears in Section 1.2.

Chapter 16

Fitting Distributions and Data with the Johnson System via the Method of Moments

The Johnson system of distributions consist of families of distributions that, through specified transformations, can be reduced to the standard normal random variable. The method of using transformations in connection to the normal distribution was initiated by Edgeworth (1898) who restricted himself to polynomial transformations. Subsequent researchers studied the use of transformations that are based on transcendental functions.

The Johnson system, proposed by N. L. Johnson (1949a), consists of four families of distributions and has been widely used by a number of investigators (Johnson (1965), Leslie (1959), Mage (1980)) for fitting statistical distributions. Although most of the literature on the Johnson system is concentrated on the uses of the method of moments to obtain fits, some work also has been done with percentiles (e.g., Bukac (1972) and Mage (1980)). In this chapter we describe the Johnson system and use the method of moments to fit distributions from this system to known distributions as well as to various datasets.

16.1 Components of the Johnson System

The Johnson system consists of four components, each consisting of a transform of the normal distribution and each covering a distinct portion of (α_3, α_4)-space. It had been assumed for some time that, collectively, the components of the Johnson system covered all of (α_3, α_4)-space and did so uniquely (for example, Swain, Venkatraman, and Wilson (1988) state that "Together these families of the Johnson system can fit any distribution to its first four moments; these families prove a unique distribution for each feasible combination of the skewness α_3 and kurtosis α_4."). Even though several papers in the literature had mentioned this

claim of completeness and uniqueness for the Johnson system, this assertion was only recently proved by Dudewicz, Zhang, and Karian (2004).

Suppose Z has the standard normal distribution, i.e. Z is $N(0,1)$. Then the random variable Y defined by

$$Z = \gamma + \delta f(Y) \tag{16.1.1}$$

has a distribution depending on the two parameters γ and δ, as well as the function f. Location and scale parameters can be introduced by taking

$$Y = (X - \xi)/\lambda. \tag{16.1.2}$$

In the Johnson system then X is the general variable and has a distribution depending on the four parameters ξ, λ, γ, and δ and the function f. Four Johnson system families result from different choices of f.

1. S_N is the simplest case and it arises when $f(Y) = Y$, $\gamma = 0$, $\delta = 1$. This makes $X = \lambda Z + \xi$, a normal random variable.

2. S_L is obtained when $f(Y) = \ln Y$, where $Y > 0$. This is the *lognormal* family since $Y = (X - \xi)/\lambda$ and $\ln Y$ is a normal random variable. In this case
$$Z = \gamma + \delta \ln[(X - \xi)/\lambda], \qquad \xi < X. \tag{16.1.3}$$

3. S_U results from $f(Y) = \sinh^{-1} Y$, where $-\infty < Y < \infty$. This is the *unbounded* Johnson family since the range of Y is $(-\infty, \infty)$. Here
$$\begin{aligned} Z &= \gamma + \delta \sinh^{-1}\left(\frac{X-\xi}{\lambda}\right) \\ &= \gamma + \delta \log\left[\frac{X-\xi}{\lambda} + \sqrt{\left(\frac{X-\xi}{\lambda}\right)^2 + 1}\right], -\infty < X < \infty. \end{aligned} \tag{16.1.4}$$

4. S_B arises when $f(Y) = \ln[Y/(1-Y)]$, where $0 < Y < 1$. This is the *bounded* Johnson family since the range of Y is $(0,1)$. In this case
$$Z = \gamma + \delta \ln[(X - \xi)/(\xi + \lambda - X)], \qquad \xi < X < \xi + \lambda. \tag{16.1.5}$$

The S_N region consists of a single point $(\alpha_3, \alpha_4) = (0, 3)$ that represents the family of normal distributions. Fitting a normal distribution using its first two moments is quite simple and we will not consider this point further. In the three sections that follow, we describe how the Johnson system fits can be obtained in the other three regions through the method of moments.

16.2 The S_L Component

Since the random variables X, Y, and Z of the previous section can be obtained from one another through transformations, we start by establishing a well-known result that associates the p.d.f. and d.f. of a random variable U with those of the random variable $T(U)$ where T is a transformation.

Theorem 16.2–1. *Let X be a continuous random variable with p.d.f. $f(x)$ and d.f. $F(x)$ and let U be an increasing function with inverse V. If the random variable Y is defined by $Y = U(X)$ then*

1. *$G(y)$, the d.f. of Y, is given by $G(y) = F(V(y))$*
2. *$g(y)$, the p.d.f. of Y, is given by $g(y) = f(V(y))V'(y)$.*

Proof. By its definition,

$$G(y) = P[Y \leq y] = P[V(Y) \leq V(y)] = P[X \leq V(y)] = F(V(y)).$$

When $G(y)$ is differentiated with respect to y we get

$$g(y) = G'(y) = F'(V(y))V'(y) = f(V(y))V'(y),$$

as claimed by the Theorem. ∎

In the S_L region, $Z = V(Y) = \gamma + \delta \ln(Y)$ and the transformation that defines Y from Z is the inverse $Y = V^{-1}(Z) = U(Z) = \exp((Z - \gamma)/\delta)$. Since Z is the $N(0, 1)$ random variable, it has p.d.f.

$$p_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \text{ with } -\infty < z < \infty,$$

and by Theorem 16.2–1, $p_Y(y)$, the p.d.f. of Y must be given by

$$\begin{aligned} p_Y(y) &= p_Z(\gamma + \delta \ln(y))\, dZ/dY = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[\gamma + \delta \ln(y)]^2\right\} dZ/dY \\ &= \frac{\delta}{y\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[\gamma + \delta \ln(y)]^2\right\} \end{aligned} \tag{16.2.2}$$

where, because of the transformation, $0 < y < \infty$. When Theorem 16.2–1 is applied again to $p_Y(y)$, as it is transformed to X by (16.1.2), it yields

$$p_X(x) = \frac{\delta}{(x - \xi)\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\gamma + \delta \ln\left(\frac{x - \xi}{\lambda}\right)\right]^2\right\} \tag{16.2.3}$$

with $\xi < x < \infty$.

It is clear from the definition of each family within the Johnson system that the skewness and kurtosis of X and $Y = (X - \xi)/\lambda$ are the same and do not

depend on either the location ξ or the scale λ. This allows us to focus on the random variable Y's skewness and kurtosis. To obtain the moments of Y, we first note that

$$E(Y^m) = \frac{\delta}{\sqrt{2\pi}} \int_0^\infty y^{m-1} \exp\left\{-\frac{1}{2}[\gamma + \delta \ln(y)]^2\right\} dy = \exp\left[\frac{m^2 - 2m\gamma\delta}{2\delta^2}\right]. \tag{16.2.4}$$

To obtain expressions for α_3 and α_4 in a simplified form from (16.2.4), the substitutions

$$\omega = \exp(\delta^{-2}) \qquad \text{and} \qquad \Omega = \gamma/\delta \tag{16.2.5}$$

are used, giving,

$$\mu_{_Y} = \omega^{1/2} \exp(-\Omega), \tag{16.2.6}$$

$$\sigma^2_{_Y} = \exp(-2\Omega)(\omega^2 - \omega) = \omega(\omega - 1)\exp(-2\Omega), \tag{16.2.7}$$

$$E(Y - \mu_{_Y})^3 = \omega^{3/2}(\omega - 1)^2(\omega + 2)\exp(-3\Omega), \tag{16.2.8}$$

$$E(Y - \mu_{_Y})^4 = \omega^2(\omega - 1)^2(\omega^4 + 2\omega^3 + 3\omega^2 - 3)\exp(-4\Omega). \tag{16.2.9}$$

Equations (16.2.7), (16.2.8), and (16.2.9) produce the following expressions, in terms of ω, for α_3 and α_4 (recall that $\alpha_3 = E(Y - \mu_{_Y})^3/\sigma^3_{_Y}$ and $\alpha_4 = E(Y - \mu_{_Y})^4/\sigma^4_{_Y}$).

$$\alpha_3 = (\omega - 1)^{1/2}(\omega + 2), \tag{16.2.10}$$

$$\alpha_4 = \omega^4 + 2\omega^3 + 3\omega^2 - 3. \tag{16.2.11}$$

Since α_3 and α_4 both depend on the single parameter ω (with $\omega > 1$), the S_L region is represented by a curve in (α_3, α_4)-space. This S_L curve and the curve $\alpha_4 = 1 + \alpha_3^2$ that delineates the impossible region of (α_3, α_4)-space are shown in Figure 16.2–1. As will be shown in subsequent sections, the (α_3, α_4) points associated with the S_U family lie above the S_L curve and the points associated with the S_B family are located between the S_L curve and the impossible region that is below the $\alpha_4 = 1 + \alpha_3^2$ curve.

When fitting through the method of moments, the S_L curve is useful only on those occasions when (α_3, α_4) is located on the S_L curve. To determine if this is the case, we need to solve (16.2.10) for ω and substitute that value in (16.2.11) to see if the equality of (16.2.11) holds. The solution of (16.2.10), the only potentially complicated part in all this, can be simplified by squaring both sides of (16.2.10) to get the cubic polynomial

$$\alpha_3^2 = (\omega - 1)(\omega + 2)^2. \tag{16.2.12}$$

The only real root, ω_0, of this polynomial can be obtained by first setting

$$A = 8 + 4\alpha_3 + 4\sqrt{4\alpha_3 + \alpha_3^2}$$

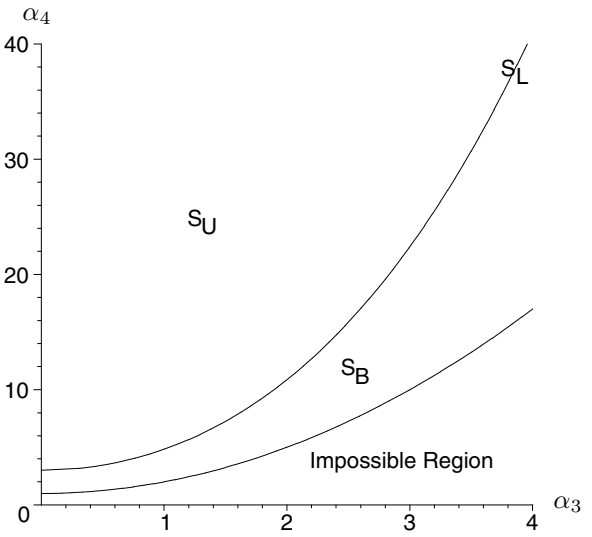


Figure 16.2–1. The S_L curve and the S_U and S_B regions of the Johnson system.

and then determining ω_0 from

$$\omega_0 = \frac{4 + A^{2/3} - 2A^{1/3}}{2A^{1/3}}. \tag{16.2.13}$$

Table 16.2–2 is provided to facilitate the determination of ω_0, which gives values for ω_0 for specified $(\hat{\alpha}_3, \hat{\alpha}_4)$. Algorithm Jnsn–$S_L$, given below, determines ξ, λ, δ, γ from $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$.

Algorithm Jnsn–S_L: Fitting a Johnson distribution from the S_L family by the method of moments.

1. Jnsn–S_L: Use $\hat{\alpha}_3$ to obtain ω_0 from (16.2.13) or from Table 16.2–2,

2. Jnsn–S_L: Assume $\sigma_Y^2 = \hat{\alpha}_2$, then substitute the value of ω_0 in (16.2.7) and get a value for Ω; equivalently, set $\Omega = \ln(\sqrt{\omega_0(\omega_0 - 1)/\hat{\alpha}_2})$,

3. Jnsn–S_L: Set $\delta = 1/\sqrt{\ln \omega_0}$ and $\gamma = \Omega\delta$ (see 16.2.5),

4. Jnsn–S_L: Set $\lambda = 1$ and $\xi = \hat{\alpha}_1 - \sqrt{\omega_0}e^{-\Omega}$.

The assumption in step 2 forces λ to be 1. This seems, and in fact is, quite arbitrary. We could have just as easily assumed $\mu_Y = \alpha_1$ and solved (16.2.6) for Ω to get $\Omega_0 = \ln(\sqrt{\omega}/\alpha_1)$. Both of these approaches lead to the same S_L region fit.

Table 16.2–2. Values of ω_0 for specified $(\hat{\alpha}_3, \hat{\alpha}_4)$ on the S_L curve.

$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0		$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0		$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0
0.01	3.0002	1.0000		0.41	3.3003	1.0184		0.81	4.1889	1.0696
0.02	3.0007	1.0000		0.42	3.3153	1.0193		0.82	4.2190	1.0713
0.03	3.0016	1.0001		0.43	3.3305	1.0203		0.83	4.2494	1.0730
0.04	3.0028	1.0002		0.44	3.3462	1.0212		0.84	4.2803	1.0746
0.05	3.0044	1.0003		0.45	3.3622	1.0222		0.85	4.3116	1.0763
0.06	3.0064	1.0004		0.46	3.3786	1.0232		0.86	4.3433	1.0781
0.07	3.0087	1.0005		0.47	3.3953	1.0242		0.87	4.3754	1.0798
0.08	3.0114	1.0007		0.48	3.4124	1.0252		0.88	4.4079	1.0816
0.09	3.0144	1.0009		0.49	3.4299	1.0262		0.89	4.4408	1.0833
0.10	3.0178	1.0011		0.50	3.4478	1.0273		0.90	4.4740	1.0851
0.11	3.0215	1.0013		0.51	3.4660	1.0284		0.91	4.5077	1.0869
0.12	3.0256	1.0016		0.52	3.4846	1.0295		0.92	4.5418	1.0887
0.13	3.0301	1.0019		0.53	3.5036	1.0306		0.93	4.5763	1.0906
0.14	3.0349	1.0022		0.54	3.5229	1.0317		0.94	4.6112	1.0924
0.15	3.0400	1.0025		0.55	3.5426	1.0329		0.95	4.6466	1.0943
0.16	3.0455	1.0028		0.56	3.5627	1.0341		0.96	4.6823	1.0961
0.17	3.0514	1.0032		0.57	3.5832	1.0353		0.97	4.7184	1.0980
0.18	3.0577	1.0036		0.58	3.6040	1.0365		0.98	4.7550	1.0999
0.19	3.0642	1.0040		0.59	3.6252	1.0377		0.99	4.7919	1.1019
0.20	3.0712	1.0044		0.60	3.6468	1.0390		1.00	4.8293	1.1038
0.21	3.0785	1.0049		0.61	3.6688	1.0403		1.01	4.8671	1.1058
0.22	3.0862	1.0054		0.62	3.6912	1.0416		1.02	4.9053	1.1077
0.23	3.0942	1.0059		0.63	3.7139	1.0429		1.03	4.9439	1.1097
0.24	3.1026	1.0064		0.64	3.7370	1.0442		1.04	4.9830	1.1117
0.25	3.1113	1.0069		0.65	3.7605	1.0456		1.05	5.0224	1.1137
0.26	3.1204	1.0075		0.66	3.7844	1.0469		1.06	5.0623	1.1157
0.27	3.1299	1.0081		0.67	3.8086	1.0483		1.07	5.1026	1.1178
0.28	3.1397	1.0087		0.68	3.8333	1.0497		1.08	5.1433	1.1198
0.29	3.1499	1.0093		0.69	3.8583	1.0511		1.09	5.1844	1.1219
0.30	3.1604	1.0099		0.70	3.8837	1.0526		1.10	5.2260	1.1240
0.31	3.1713	1.0106		0.71	3.9095	1.0540		1.11	5.2680	1.1261
0.32	3.1826	1.0113		0.72	3.9357	1.0555		1.12	5.3104	1.1282
0.33	3.1942	1.0120		0.73	3.9623	1.0570		1.13	5.3533	1.1303
0.34	3.2062	1.0127		0.74	3.9892	1.0585		1.14	5.3965	1.1324
0.35	3.2186	1.0135		0.75	4.0166	1.0601		1.15	5.4402	1.1346
0.36	3.2313	1.0143		0.76	4.0443	1.0616		1.16	5.4844	1.1368
0.37	3.2444	1.0151		0.77	4.0724	1.0632		1.17	5.5289	1.1389
0.38	3.2578	1.0159		0.78	4.1009	1.0648		1.18	5.5739	1.1411
0.39	3.2716	1.0167		0.79	4.1299	1.0664		1.19	5.6194	1.1433
0.40	3.2858	1.0176		0.80	4.1592	1.0680		1.20	5.6652	1.1455

Table 16.2–2. (Cont.) Values of ω_0 for specified $(\hat{\alpha}_3, \hat{\alpha}_4)$ on the S_L curve.

$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0	$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0	$\hat{\alpha}_3$	$\hat{\alpha}_4$	ω_0
1.21	5.7116	1.1478	1.61	7.9370	1.2460	2.01	10.9491	1.3582
1.22	5.7583	1.1500	1.62	8.0023	1.2487	2.02	11.0353	1.3612
1.23	5.8055	1.1523	1.63	8.0681	1.2513	2.03	11.1220	1.3641
1.24	5.8531	1.1545	1.64	8.1345	1.2540	2.04	11.2093	1.3671
1.25	5.9012	1.1568	1.65	8.2013	1.2567	2.05	11.2971	1.3700
1.26	5.9497	1.1591	1.66	8.2686	1.2594	2.06	11.3855	1.3730
1.27	5.9987	1.1614	1.67	8.3364	1.2621	2.07	11.4745	1.3760
1.28	6.0481	1.1637	1.68	8.4047	1.2648	2.08	11.5640	1.3789
1.29	6.0979	1.1660	1.69	8.4736	1.2675	2.09	11.6540	1.3819
1.30	6.1482	1.1684	1.70	8.5429	1.2702	2.10	11.7447	1.3849
1.31	6.1990	1.1707	1.71	8.6127	1.2730	2.11	11.8359	1.3879
1.32	6.2502	1.1731	1.72	8.6830	1.2757	2.12	11.9276	1.3909
1.33	6.3018	1.1754	1.73	8.7539	1.2785	2.13	12.0200	1.3939
1.34	6.3539	1.1778	1.74	8.8252	1.2812	2.14	12.1129	1.3969
1.35	6.4065	1.1802	1.75	8.8971	1.2840	2.15	12.2063	1.3999
1.36	6.4595	1.1826	1.76	8.9694	1.2867	2.16	12.3004	1.4029
1.37	6.5130	1.1850	1.77	9.0423	1.2895	2.17	12.3950	1.4059
1.38	6.5669	1.1874	1.78	9.1157	1.2923	2.18	12.4902	1.4090
1.39	6.6213	1.1899	1.79	9.1896	1.2951	2.19	12.5859	1.4120
1.40	6.6761	1.1923	1.80	9.2641	1.2979	2.20	12.6823	1.4150
1.41	6.7315	1.1948	1.81	9.3390	1.3007	2.21	12.7792	1.4180
1.42	6.7872	1.1973	1.82	9.4145	1.3035	2.22	12.8767	1.4211
1.43	6.8435	1.1997	1.83	9.4905	1.3063	2.23	12.9748	1.4241
1.44	6.9002	1.2022	1.84	9.5670	1.3092	2.24	13.0734	1.4272
1.45	6.9574	1.2047	1.85	9.6440	1.3120	2.25	13.1727	1.4302
1.46	7.0150	1.2072	1.86	9.7216	1.3148	2.26	13.2725	1.4333
1.47	7.0731	1.2097	1.87	9.7996	1.3177	2.27	13.3729	1.4364
1.48	7.1317	1.2123	1.88	9.8783	1.3205	2.28	13.4740	1.4394
1.49	7.1908	1.2148	1.89	9.9574	1.3234	2.29	13.5756	1.4425
1.50	7.2503	1.2174	1.90	10.0371	1.3263	2.30	13.6778	1.4456
1.51	7.3103	1.2199	1.91	10.1173	1.3292	2.31	13.7806	1.4487
1.52	7.3708	1.2225	1.92	10.1980	1.3320	2.32	13.8839	1.4517
1.53	7.4318	1.2251	1.93	10.2793	1.3349	2.33	13.9879	1.4548
1.54	7.4932	1.2276	1.94	10.3612	1.3378	2.34	14.0925	1.4579
1.55	7.5552	1.2302	1.95	10.4435	1.3407	2.35	14.1977	1.4610
1.56	7.6176	1.2328	1.96	10.5264	1.3436	2.36	14.3035	1.4641
1.57	7.6805	1.2355	1.97	10.6099	1.3465	2.37	14.4099	1.4672
1.58	7.7439	1.2381	1.98	10.6938	1.3494	2.38	14.5169	1.4703
1.59	7.8078	1.2407	1.99	10.7784	1.3524	2.39	14.6245	1.4734
1.60	7.8721	1.2434	2.00	10.8635	1.3553	2.40	14.7327	1.4766

16.3 The S_U Component

For this region, we use the transformation $f(Y) = \sinh^{-1} Y$ and Theorem 16.2–1 to obtain p_Y, the p.d.f. of Y

$$p_Y(y) = \frac{\delta}{\sqrt{2\pi(y^2+1)}} \exp\left\{-\frac{1}{2}\left[\gamma + \delta \ln\left(y + \sqrt{y^2+1}\right)\right]^2\right\}. \tag{16.3.1}$$

Applying the transformation (16.1.2) to Y, gives p_X, the p.d.f. of X

$$p_X(x) = \frac{\delta}{\sqrt{2\pi((x-\xi)^2+\lambda^2)}} \exp\left\{-\frac{1}{2}\left[\gamma + \delta \ln\left(\frac{x-\xi+\sqrt{(x-\xi)^2+\lambda^2}}{\lambda}\right)\right]^2\right\}. \tag{16.3.2}$$

As in the case of S_L, X and Y have the same α_3 and α_4. In this case,

$$E(Y^m) = \frac{1}{2^m\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{e^{(z-\gamma)/\delta} - e^{-(z-\gamma)/\delta}\right\}^m e^{-z^2/2}\, dz.$$

Using ω and Ω, as defined in (16.2.5), the following can be derived

$$\begin{aligned}
\mu_Y &= -\sqrt{\omega}\sinh\Omega, && (16.3.3)\\
\sigma_Y^2 &= \frac{1}{2}(\omega-1)(\omega\cosh 2\Omega + 1), && (16.3.4)\\
E(Y-\mu_Y)^3 &= -\frac{1}{4}\sqrt{\omega}(\omega-1)^2[\omega(\omega+2)\sinh 3\Omega + 3\sinh\Omega],\\
E(Y-\mu_Y)^4 &= \frac{1}{8}(\omega-1)^2\left[\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh 4\Omega\right.\\
&+ \left. 4\omega^2(\omega+2)\cosh 2\Omega + 3(2\omega+1)\right],
\end{aligned}$$

from which α_3 and α_4 can be obtained:

$$\alpha_3 = \frac{\sqrt{\omega(\omega-1)}\,(\omega(\omega+2)\sinh 3\Omega + 3\sinh\Omega)}{\sqrt{2}(\omega\cosh 2\Omega+1)^{3/2}}, \tag{16.3.5}$$

$$\alpha_4 = \frac{\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh 4\Omega + 4\omega^2(\omega+2)\cosh 2\Omega + 6\omega + 3}{2(\omega\cosh 2\Omega+1)^2}. \tag{16.3.6}$$

To verify that the S_U region conforms to its depiction in Figure 16.2–1, we use (16.3.5) and (16.3.6) to plot (shown in Figure 16.3–1) the (α_3, α_4) points along curves associated with $\Omega = 0.025, 0.05, 0.075, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.7, 1$ as ω ranges between 1 and 2 (the curve with $\Omega = 0.025$ is the one closest to the vertical axis). The slower rising curves of Figure 16.3–1 are associated with $\omega = 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6$, as Ω ranges between 0 and 2 (the curve with $\omega = 1.2$ is the lowest one). The curves drawn in thick lines are the S_L curve and the boundary of the impossible region.

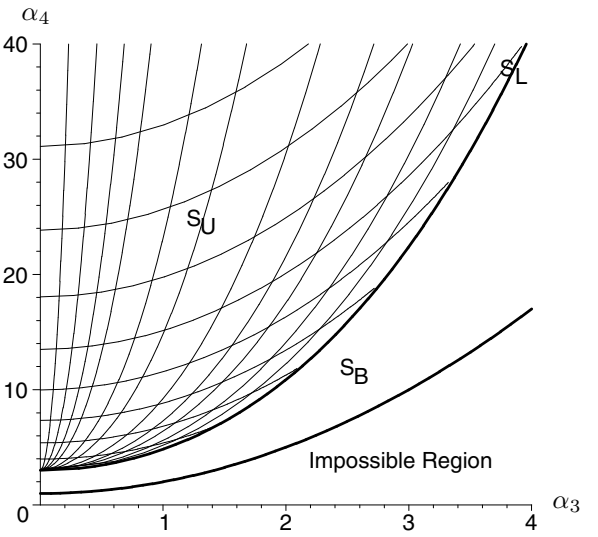


Figure 16.3–1. The S_U region of the Johnson system. The sharply rising curves have $\Omega = 0.025$, 0.05, 0.075, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.7, 1, and $1 < \omega < 2$. The slower rising curves have $\omega = 1.2$, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6, and $0 < \Omega < 2$.

To obtain a fit (when α_1, α_2, α_3, α_4 are known) we need to determine the four parameters ξ, λ, δ, and γ that appear in (16.3.2). This can be accomplished by taking the following steps (it is assumed that prior to the application of these steps, it has been established that (α_3, α_4) is in the S_U region).

1. Solve (16.3.5) and (16.3.6) for ω and Ω,

2. Use these values of ω and Ω in (16.2.5) to determine first $\delta = 1/\sqrt{\ln \omega}$ and then $\gamma = \delta\Omega$,

3. Substitute the values of ω and Ω obtained in step 1 in (16.2.6) and (16.2.7) to get $\mu_{_Y}$ and $\sigma^2_{_Y}$,

4. Set $\lambda = \sqrt{\alpha_2}/\sigma_{_Y}$ and then set $\xi = \alpha_1 - \lambda\mu_{_Y}$ (see (16.1.2)).

Steps 2 through 4 involve relatively simple substitutions and of the four steps, only the first presents a computational challenge. For this reason attempts have been made to construct tables that assist in the execution of the first step. Johnson (1965) gives a reasonably complete table for determining δ and γ when α_3 ranges between 0 and 2 (this takes care of our steps 1 and 2). A similar but slightly expanded table appears in Pearson and Hartley (1972). The table given in Appendix H continues to expand (slightly) the coverage of the (α_3, α_4) points from the S_U region and additionally gives values for $\mu_{_Y}$ and $\sigma_{_Y}$, taking care of the computation required in step 3. Note that in all cases, table entries give $-\gamma$

instead of γ. We now give an algorithm for using the tables of Appendix H to fit a data set with a Johnson system distribution from the S_U region.

Algorithm Jnsn–S_U: Fitting a Johnson distribution from the S_U family by the method of moments.

1. Jnsn–S_U: Compute the moments $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ of the data (see (3.3.1) through (3.3.4) of Chapter 3).

2. Jnsn–S_U: Determine from Table 16.2–2 if $(\hat{\alpha}_3, \hat{\alpha}_4)$ is in the S_U region. If not, this algorithm does not apply; otherwise continue.

3. Jnsn–S_U: From the closet point to $(\hat{\alpha}_3, \hat{\alpha}_4)$ in Appendix H extract values for μ_Y and σ_Y, δ, γ (remember to reverse the signs of γ).

4. Jnsn–S_U: If $\alpha_3 < 0$, replace γ by $-\gamma$ and μ_Y by $1 - \mu_Y$.

5. Jnsn–S_U: Set $\lambda = \sqrt{\hat{\alpha}_2}/\sigma_Y$.

6. Jnsn–S_U: Set $\xi = \hat{\alpha}_1 - \lambda\mu_Y$.

As an example, suppose we determine from a dataset that

$$(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4) = (1, 1, 0.758, 4.947).$$

It is clear from checking the $\hat{\alpha}_4$ entry associated with $\hat{\alpha}_3 = 0.76$ in Table 16.2–2 (this turns out to be 4.0443), that the (α_3, α_4) for this data is above the S_L line, hence it is in the S_U region. The closest point to $(\alpha_3, \alpha_4) = (0.758, 4.947)$ in Appendix H is $(0.75, 4.90)$ which gives $\mu_Y = 0.576$, $\sigma_Y = 0.586$, $\delta = 2.227$, $\gamma = -1.113$. Implementing steps 5 and 6 of Algorithm Jnsn–S_U, we obtain $\lambda = 1/0.568 = 1.7606$ and $\xi = 1 - (1.7606)(0.576) = -0.0141$ for the fit with parameters

$$(\xi, \lambda, \delta, \gamma) = (-0.0141, 1.7606, 2.227, -1.113).$$

Maple programs have been written, and and distributed with this book, to provide fits for all of the regions of the Johnson system, making it possible to get a more accurate fit through direct computation (the Maple program for fitting S_U distributions is `FitJnsnSU`). For the example under consideration, the invocation of this program and its output will be

```
> FitJnsnSU([1, 1, 0.758, 4.947]);
[-.002842114, 1.742184027, 2.206749025, -1.10016216].
```

The output consists of the values of $(\xi, \lambda, \delta, \gamma)$, in that order. There is clearly some discrepancy between the $(\xi, \lambda, \delta, \gamma)$ obtained through Appendix H and the more accurate values provided by `FitJnsnSU`. To see the consequence of this on the p.d.f.s that result from these fits we consider the superimposed plots of the two p.d.f.s (Figure 16.3–2) and see that they are visually indistinguishable.

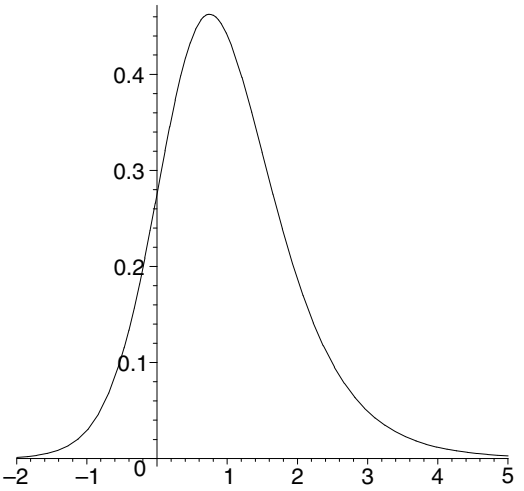


Figure 16.3–2. Two indistinguishable p.d.f.s resulting from a tabled fit and a fit through direct computation.

Solving equations (16.3.5) and (16.3.6) simultaneously for ω and Ω turns out to be a rather simple matter in Maple. Moreover, the errors (as measured by $\max(|\alpha_3 - \hat{\alpha}_3|, |\alpha_3 - \hat{\alpha}_3|)$) associated with such solutions tend to be quite small — generally, less than 10^{-8}.

16.4 The S_B Component

Using Theorem 16.2.1. with the transformation defined in (16.2.5) gives p_Y the p.d.f. of Y as

$$p_Y(y) = \frac{\delta}{\sqrt{2\pi}y(1-y)} \exp\left\{-\frac{1}{2}\left[\gamma + \delta \ln\left(\frac{y}{1-y}\right)\right]^2\right\} \tag{16.4.1}$$

from which, using (16.1.2), we get $p_X(x)$, the p.d.f. of X

$$p_X(x) = \frac{\delta\lambda}{\sqrt{2\pi}(x-\xi)(\lambda - x + \xi)} \exp\left\{-\frac{1}{2}\left[\gamma + \delta \ln\left(\frac{x-\xi}{\lambda - x + \xi}\right)\right]^2\right\}, \tag{16.4.2}$$

with $\xi < x < \xi + \lambda$. For the S_B family,

$$E(Y^m) = \frac{1}{2^m\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \left(e^{(z-\gamma)/\delta)} - e^{-(z-\gamma)/\delta}\right)^m \, dz. \tag{16.4.3}$$

The usual procedure for fitting S_B distributions would be to derive expressions for μ_Y, σ_Y, α_3, α_4, set these equal to their counterparts, and solve the resulting equations for the parameters ξ, λ, δ, and γ. Unfortunately, the desired expressions

for μ_Y, σ_Y, α_3, α_4 are not available and we must rely on numerical solutions at every step. This is particularly troublesome at the very beginning when δ and γ need to be determined from α_3 and α_4.

To facilitate the fitting of S_B distributions Johnson and Kitchen (1971a) discuss these computational difficulties and provide a very limited, in the sense of coverage of the available (α_3, α_4)-space, table. In a subsequent paper, Johnson and Kitchen (1971b) provide a considerably expanded version of their table, which is further expanded by Pearson and Hartley (1972). These tables constrain α_3 between 0 and 2 (in increments of 0.05) and do not cover the smaller values of α_4 for a specified α_3. The Table in Appendix I is the most comprehensive table available at this time; it allows α_3 to range over the interval [0, 3], in increments of 0.05, it covers α_4 from near its smallest possible value in S_B (near the curve $\alpha_4 = 1 + \alpha_3^2$) up to $\alpha_4 = 12$, and it provides values for μ_Y and σ_Y.

It should be noted that when α_3 is fixed, values of α_4 near the boundary curve $\alpha_4 = 1 + \alpha_3^2$ give rise to bimodal distributions until α_4 gets sufficiently large and S_B produces unimodal distributions (see Johnson and Kitchen (1971a) for details).

Algorithm Jnsn–S_B, given below, uses the table of Appendix I to determine values of ξ, λ, δ, γ from $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$.

Algorithm Jnsn–S_B: Fitting a Johnson distribution from the S_B family by the method of moments.

1. Jnsn–S_B: Compute the moments $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ of the data.

2. Jnsn–S_B: Determine from the Table 16.2–2 if $(\hat{\alpha}_3, \hat{\alpha}_4)$ is in the S_B region. If it is not, this algorithm does not apply; otherwise continue.

3. Jnsn–S_B: From the closet point to $(\hat{\alpha}_3, \hat{\alpha}_4)$ in Appendix I extract values for μ_Y, σ_Y, δ, γ.

4. Jnsn–S_B: If $\alpha_3 < 0$, replace γ by $-\gamma$ and μ_Y by $1 - \mu_Y$.

5. Jnsn–S_B: Set $\lambda = \sqrt{\hat{\alpha}_2}/\sigma_Y$.

6. Jnsn–S_B: Set $\xi = \hat{\alpha}_1 - \lambda\mu_Y$.

As an example, suppose we determine from a dataset that

$$(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4) = (1, 1, 0.758, 2.947).$$

It is clear from checking the $\hat{\alpha}_3$ entry associated with $\hat{\alpha}_3 = 0.76$ in Appendix I (this turns out to be 4.0443), that the (α_3, α_4) for this data is below the S_L curve, hence it is in the S_B region. The closest point to $(\alpha_3, \alpha_4) = (0.758, 2.947)$ in Appendix I is (0.75, 2.90), which gives

$$\mu_Y = 0.3011, \quad \sigma_Y = 0.1899, \quad \delta = 0.9452, \quad \gamma = 0.9735.$$

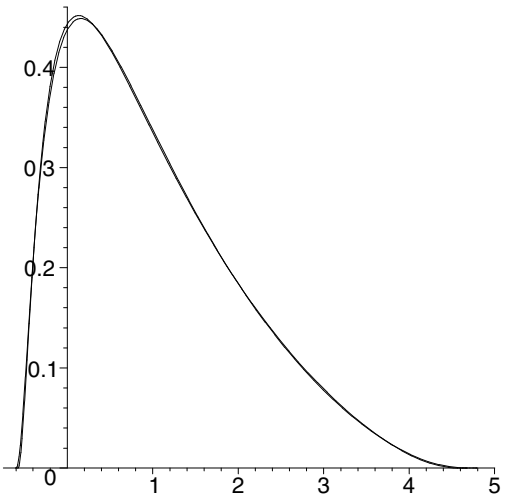


Figure 16.4–1. Two almost indistinguishable p.d.f.s resulting from a tabled fit and a fit through direct computation in the S_B region.

Implementing steps 3 and 4 of Algorithm Jnsn–S_B, we obtain

$$\lambda = 1/0.1899 = 5.2659 \quad \text{and} \quad \xi = 1 - (5.2659)(0.3011) = -0.5856$$

for the fit with parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (-0.5856,\ 5.2659,\ 0.9452,\ 0.9735).$$

From the Maple program `FitJnsnSB` (on the CD that accompanies this book) we can, through the following interaction, obtain a more accurate fit by direct computation.

```
> FitJnsnSB([1, 1, 0.758, 2.947]);
[-.609047345, 5.406077951, .9722437539, 1.011671917, 8.23967e-5].
```

The first four numbers in the output of `FitJnsnSB` are the parameters $(\xi,\ \lambda,\ \delta,\ \gamma)$ for this fit. The last number indicates that

$$\max\left(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|\right) = 8.23967 \times 10^{-5}.$$

There is some discrepancy between the $(\xi,\ \lambda,\ \delta,\ \gamma)$ obtained through Appendix I and the more accurate values provided by `FitJnsnSB`. To see the consequence of this on the p.d.f.s that result from these fits we consider the superimposed plots of the two p.d.f.s (Figure 16.4–1) and see that they are almost indistinguishable (the one that rises highest at its mode is the fit from Appendix I).

From (16.1.5), the support for a fitted distribution is $(\xi,\ \xi + \lambda)$. It is strongly recommended that the support be checked after a fit is obtained to make sure

that the fit conforms to the understanding that we have of the data. The support for the first fit is $(-0.5856, 4.6803)$ and the support for the second fit is $(-0.609047345, 4.797030607)$.

16.5 Approximations of Some Well-Known Distributions

In Sections 3.4, 4.4, 5.4, and 6.4, approximations to some well-known distributions were obtained based on GLD moment fits, GBD moment fits, GLD percentile fits, and GLD L-moment fits. In this section we consider such approximations based on Johnson system fits. For each distribution we first use Table 16.2–2 to determine which Johnson system component is appropriate and then, either through the use of the tables in Appendices H or I, or through direct computation, we obtain an approximation from the chosen component. In the case of S_L and S_U, direct computations result in very small values of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$, typically less than 10^{-9} and always less than 10^{-6}. For S_B, when direct computation is used, $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$ can be made less than 10^{-4} but this is somewhat larger when tabled values from Appendix I are used.

Direct computations for S_L and S_U approximations can be obtained through the Maple programs `FitJnsnSL([a1, a2, a3, a4])` and `FitJnsnSU([a1, a2, a3, a4])`, where `a1, a2, a3`, and `a4` are the α_1, α_2, α_3, α_4 of the distribution to be approximated. The outcome of these programs is a list with the four entries: $[\xi, \lambda, \delta, \gamma]$.

For S_B region fits, the program `FitJnsnSB` can be used with three arguments: `FitJnsnSB([a1, a2, a3, a4], Grid, Iter)` where `Grid` indicates the number of partitions to be used in both δ and γ dimensions and `Iter` specifies the number of "zoom-ins" that are repeated during the search. Setting both `Grid` and `Iter` to 9 generally gives good results within a reasonable computation time. It is also possible to use `FitJnsnSB` with two additional arguments as in

`FitJnsnSB([a1, a2, a3, a4], Grid, Iter, [dmin, dmax], [gmin, gmax])`

where `dmin, dmax, gmin, gmax` are the lower and upper boundaries for the search for δ and γ. However `FitJnsnSB` is used, it returns a list with five entries: $[\xi, \lambda, \delta, \gamma, \text{error}]$. The last entry gives the value of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$ for the approximation associated with ξ, λ, δ, γ. In a typical use of the five argument version of `FitJnsnSB`, Appendix I is used to get a first estimate for `dmin, dmax, gmin, gmax` and if the initial invocation of `FitJnsnSB` produces a fit with an error that is deemed too large, information from this fit can be used to refine `[dmin, dmax]` and `[gmin, gmax]` and reinvoke `FitJnsnSB`, if necessary. Typically, after two or three repeated uses of `FitJnsnSB`, the error can be made less than 10^{-4}.

In the following subsections, whenever we are able to get an approximation, we give quantitative assessments of the quality of the approximation by providing $\sup|\hat{f}(x) - f(x)|$, $\sup|\hat{F}(x) - F(x)|$, $||\hat{f} - f||_1$ and $||\hat{f} - f||_2$. We adhere to notation (established in previous chapters) of letting $\hat{f}$ and f denote the p.d.f.s of the approximating and the approximated distributions, respectively, and $\hat{F}$ and F denote the d.f.s of the approximating and the approximated distributions, respectively.

16.5.1 The Normal Distribution

Since the normal distribution, $N(\mu, \sigma^2)$, is one of the four components of the Johnson system, it (representing itself within the Johnson system) approximates itself perfectly. It is worth noting, however, that $N(\mu, \sigma^2)$ can also be realized as a limiting case in S_L, S_U, and S_B (see Figure 16.2–1) and, in each case, using some (α_3, α_4) close to $(0, 3)$ we can obtain approximations to $N(\mu, \sigma^2)$.

If, for an approximation in S_L, we were to take $\alpha_3 = 0.0001$, we would get (following the algorithm Jnsn–S_L of Section 16.2)

$$\begin{aligned} \omega &= 1.00001, \quad \alpha_4 = 3.00018, \quad \Omega = -5.70378, \\ \delta &= 300.002, \quad \gamma = -1711.15, \quad \xi = -300.001, \quad \lambda = 1 \end{aligned}$$

and the approximating p.d.f.

$$\hat{f}(x) = 150.0011667 \frac{\sqrt{2} e^{-0.5\,(-1711.147885 + 300.0023334 \ln(x + 300.0014998))^2}}{(x + 300.0014998)\sqrt{\pi}},$$

with support $(\xi, \infty) = (-300.001, \infty)$. Visually, $\hat{f}(x)$ is indistinguishable from $f(x)$, the $N(0, 1)$ p.d.f., and $\hat{F}(x)$ is indistinguishable from $F(x)$ and

$$\begin{aligned} \sup|\hat{f}(x) - f(x)| &= 0.0009199, \qquad \sup|\hat{F}(x) - F(x)| = 0.0006650, \\ ||\hat{f} - f||_1 &= 0.002516, \qquad ||\hat{f} - f||_2 = 0.001212, \end{aligned}$$

confirming that we have a high quality fit.

For the region S_U, if we stipulate $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 3.001)$ and invoke `FitJnsnSU([0, 1, 0, 3.001])`, we get `[0., 63.2494, 63.2573, 0.]`, indicating

$$(\xi, \lambda, \delta, \gamma) = (0., 63.2494, 63.2573, 0.)$$

and the p.d.f.

$$\hat{f}(x) = 212.133 \frac{e^{-2000.74\left(\ln\left(0.01581x + 4.74315 + 0.01581\sqrt{x^2 + 600.003x + 94001.4}\right)\right)^2}}{\sqrt{3.14159\,(x + 300.001)^2 + 12567.9}},$$

with support $(-\infty, \infty)$. This p.d.f. and its corresponding d.f. are also visually indistinguishable from their $N(0, 1)$ counterparts. For this approximation

$$\sup|\hat{f}(x) - f(x)| = 4.985 \times 10^{-5}, \qquad \sup|\hat{F}(x) - F(x)| = 2.323 \times 10^{-5},$$
$$||\hat{f} - f||_1 = 0.0001166, \qquad ||\hat{f} - f||_2 = 5.6676 \times 10^{-5},$$

indicating another good fit to $N(0, 1)$.

To obtain an approximation to $N(0, 1)$ from the S_B region, we start with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 2.99)$ and, since there is no entry for $(\alpha_3, \alpha_4) = (0, 2.99)$ in Appendix I, we invoke

```
FitJnsnSB([0, 1, 0, 2.99], 9, 9, [13.5, 14.5], [0, 0.001])
```

to obtain `[-27.7019, 55.4055, 13.8333, 0.0008148, 0.00006910]`. This gives us the approximation

$$(\xi, \lambda, \delta, \gamma) = (-27.7019, 55.4055, 13.8333, 0.0008148)$$

with $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 6.910 \times 10^{-5}$. The p.d.f. of this approximation is

$$\hat{f}(x) = \frac{305.765 \exp\left\{-0.5\left(0.0008148 + 13.8333 \ln\left(\frac{x+27.7019}{27.7036-1.0x}\right)\right)^2\right\}}{(x + 27.7091)(27.7036 - x)},$$

with support $(\xi, \xi + \lambda) = (-27.7019, 27.7036)$. As in the two previous cases, this p.d.f. and its corresponding d.f. look identical to their $N(0, 1)$ counterparts and for this approximation

$$\sup|\hat{f}(x) - f(x)| = 0.0005195 \qquad \sup|\hat{F}(x) - F(x)| = 0.0002392,$$
$$||\hat{f} - f||_1 = 0.001219, \qquad ||\hat{f} - f||_2 = 0.0005924,$$

giving a third good approximation.

16.5.2 The Uniform Distribution

Regardless of its parameters, the uniform distribution has $(\alpha_3, \alpha_4) = (0, 9/5)$ (see Section 3.4.2), placing it in the S_B region. To keep things simple, we consider the uniform distribution on the interval (0, 1) for which

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1/2, 1/12, 0, 9/5).$$

Use of

```
FitJnsnSB([1/2, 1/12, 0, 9/5], 13, 7, [0.63, 0.67], [0, 0.01])
```

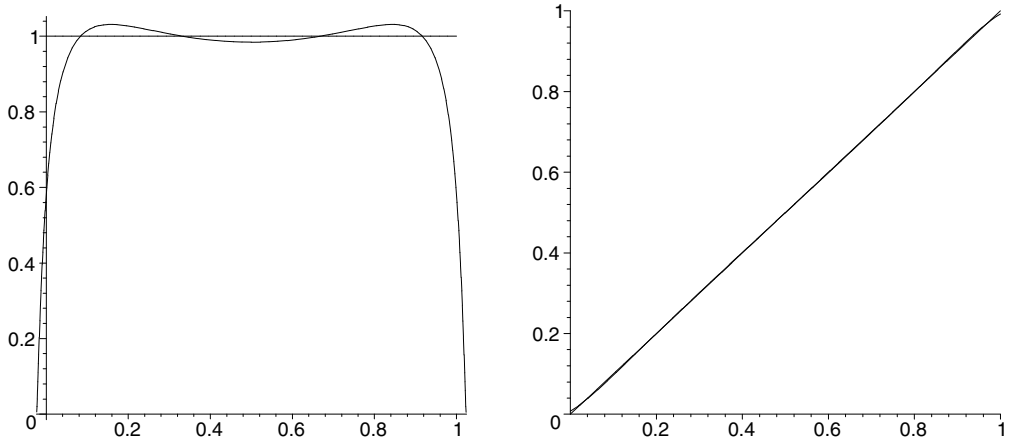


Figure 16.5–1. Uniform distribution: p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

yields the S_B parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (-0.0240,\ 1.0480,\ 0.6464,\ -0.000009103),$$

with $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|) = 4.2156 \times 10^{-5}$ and support $(-0.0240,\ 1.0240)$. As can be seen in Figure 16.5–1 (a), the p.d.f. of the approximating distribution is not particularly close to the p.d.f. of the uniform distribution on (0, 1). The discrepancy of the two d.f.s, as can be seen in Figure 16.5–1 (b), seems considerably smaller, with discrepancies visible only near (0, 0) and (0, 1). These visual indications can be made more precise through the quantitative observations

$$\sup|\hat{f}(x) - f(x)| = 0.5793 \qquad \sup|\hat{F}(x) - F(x)| = 0.007622,$$
$$||\hat{f} - f||_1 = 0.04961, \qquad ||\hat{f} - f||_2 = 0.10734.$$

This S_B approximation compares quite unfavorably with the GLD approximations (both moment-based and percentile-based, see Sections 3.4.2 and 5.4.2) as well as with the GBD approximation of Section 4.4.2. In all these cases perfect fits were available to the uniform distribution.

16.5.3 The Student's t Distribution

The α_1, α_2, α_3, α_4 of $t(\nu)$, the Student's t distribution with ν degrees of freedom, are given by

$$0, \quad \frac{\nu}{\nu - 2} \quad , 0, \quad \frac{3(\nu - 2)}{\nu - 4}$$

and for these to exist, we must have $\nu > 4$. For all ν, $\alpha_3 = 0$ and $\alpha_4 > 3$, placing $t(\nu)$ in the S_U region. Details about the t distribution are given in Section 3.4.3

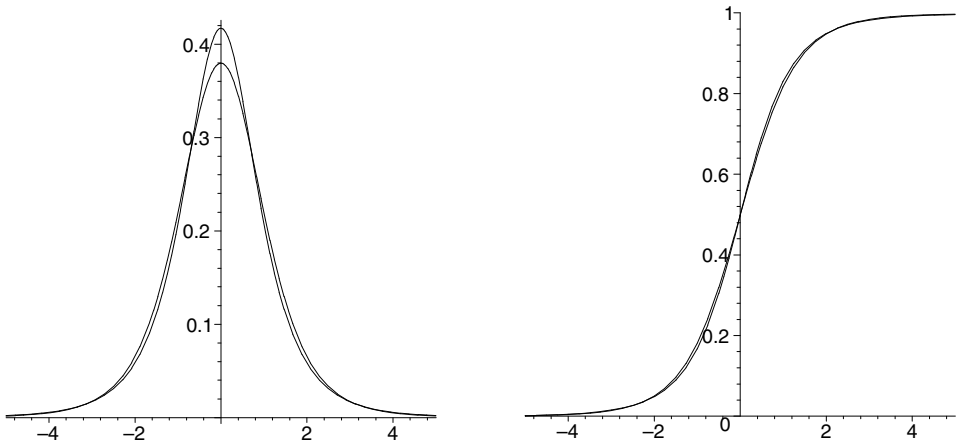


Figure 16.5–2. The $t(5)$ distribution: p.d.f. (higher at center) and its S_U; approximation (a); d.f. and its S_U approximation (b).

and Sections 3.4.3 and 5.4.3 provide GLD moment and percentile approximations to $t(\nu)$ for $\nu = 5$, 6, 10, and 30. To facilitate the comparisons of various fitting methods we consider the same choices of ν in this section.

When $\nu = 5$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 5/3, 0, 9)$, we can use the Maple command `FitJnsnSU([0, 5/3, 0, 9])` to obtain the S_U approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (0,\ 1.2910, 1.3493, 0).$$

Figures 16.5–2 (a) and 16.5–2 (b) show, respectively, the p.d.f. and d.f. of $t(5)$ with the p.d.f. and d.f. of the approximating distribution (in Figure 16.5–2 (a) the $t(5)$ p.d.f is the one that is lower at the center). For this approximation,

$$\sup|\hat{f}(x) - f(x)| = 0.03734 \qquad \sup|\hat{F}(x) - F(x)| = 0.01587,$$
$$||\hat{f} - f||_1 = 0.06817, \qquad ||\hat{f} - f||_2 = 0.3543,$$

values that are larger than their counterparts in the case of the GLD moment-based approximation of Section 3.4.3 and considerably larger than those for the percentile-based GLD approximation of Section 5.4.3.

For $t(6)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1.5, 0, 6)$ and the parameters of its S_U approximation are

$$(\xi, \lambda, \delta, \gamma) = (0, 1.6066, 1.6104, 0).$$

In this case,

$$\sup|\hat{f}(x) - f(x)| = 0.01716 \qquad \sup|\hat{F}(x) - F(x)| = 0.007253,$$
$$||\hat{f} - f||_1 = 0.03289, \qquad ||\hat{f} - f||_2 = 0.01687,$$

making this approximation comparable to its moment-based GLD counterpart, but not as good as the GLD approximation obtained through percentiles.

When $\nu = 10$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1.25, 0, 4)$, and the S_U approximation parameters are

$$(\xi, \lambda, \delta, \gamma) = (0, 2.3584, 2.3212, 0).$$

For this approximation,

$$\sup|\hat{f}(x) - f(x)| = 0.003540 \qquad \sup|\hat{F}(x) - F(x)| = 0.002450,$$
$$||\hat{f} - f||_1 = 0.007274, \qquad ||\hat{f} - f||_2 = 0.003667,$$

making this approximation superior to its moment-based GLD counterpart but not as good as the GLD approximation obtained through percentiles.

For $t(30)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1.0714, 0, 3.2308)$ and the parameters of the S_U approximation are

$$(\xi, \lambda, \delta, \gamma) = (0, 1.0714, 0, 3.2308).$$

The results

$$\sup|\hat{f}(x) - f(x)| = 0.0002628 \qquad \sup|\hat{F}(x) - F(x)| = 0.0002836,$$
$$||\hat{f} - f||_1 = 0.0005801, \qquad ||\hat{f} - f||_2 = 0.0002876,$$

indicate that this S_U approximation to $t(30)$ is superior to both of the GLD approximations.

16.5.4 The Exponential Distribution

For the positive parameter θ, the $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the exponential distribution are $(\theta, \theta^2, 2, 9)$ (see Section 3.4.4 for details). Since for $\alpha_3 = 2$ the α_4 point on the S_L curve is 10.8635 (from Table 16.2–2), the approximation of the exponential distribution should be located in the S_B region of the Johnson system. As done in other approximations of the exponential distribution in Sections 3.4.4, 4.4.4, 5.4.4, and 6.4.4, we consider the special case with $\theta = 1$.

After determining from Appendix I that δ should be close to 1.2 and γ close to 3.2 (and possibly after one or two unsuccessfull attempts because of $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) > 10^{-4}$) we use

```
FitJnsnSB([1, ,1, ,2, ,9], 9, 9, [1.192, 1.2], [3.16, 3.21])
```

to obtain

$$(\xi, \lambda, \delta, \gamma) = (-.2586, 15.0723, 1.1949, 3.1929),$$

with $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 7.8081 \times 10^{-5}$. The support of the S_B distribution with these parameters is $(\xi, \xi + \lambda) = (-.25861, 14.8137)$ and a comparison of the p.d.f and d.f. of the distribution with their counterparts of the approximating distribution (shown in Figures 16.5–3 (a) and 16.5–3 (b), respectively) makes it apparent that this is not a particularly good fit. This visual impression is

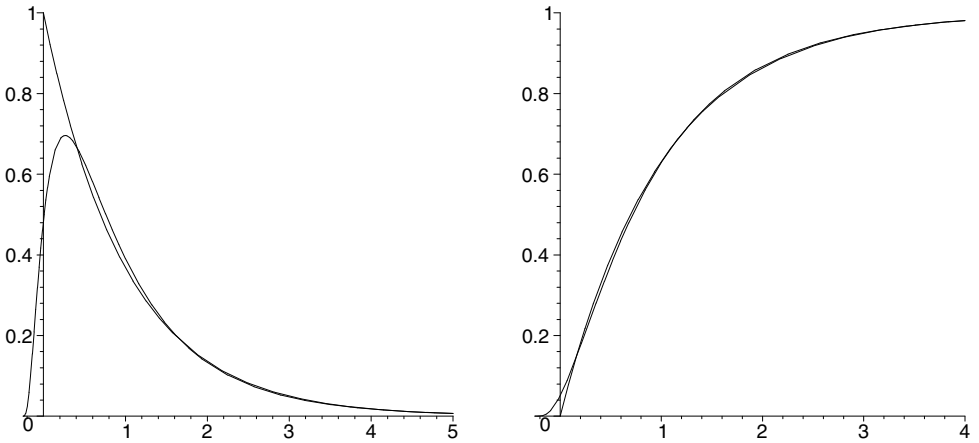


Figure 16.5–3. The exponential distribution ($\theta = 1$): p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

reinforced when

$$\sup|\hat{f}(x) - f(x)| = 0.5128 \qquad \sup|\hat{F}(x) - F(x)| = 0.04974,$$
$$||\hat{f} - f||_1 = 0.1479, \qquad ||\hat{f} - f||_2 = 0.1937$$

are computed. These values are similar to those obtained with the moment based GLD approximation (Section 3.4.4), slightly larger than those for the GLD percentile-based approximation (Section 5.4.4), and considerably larger than what was obtained for the GBD approximation in Section 4.4.4. However, the exponential distribution can be realized as a limiting case for both the GLD and GBD families of distributions; hence, arbitrarily close GLD and GBD approximations to the exponential can be realized.

16.5.5 The Chi-Square Distribution

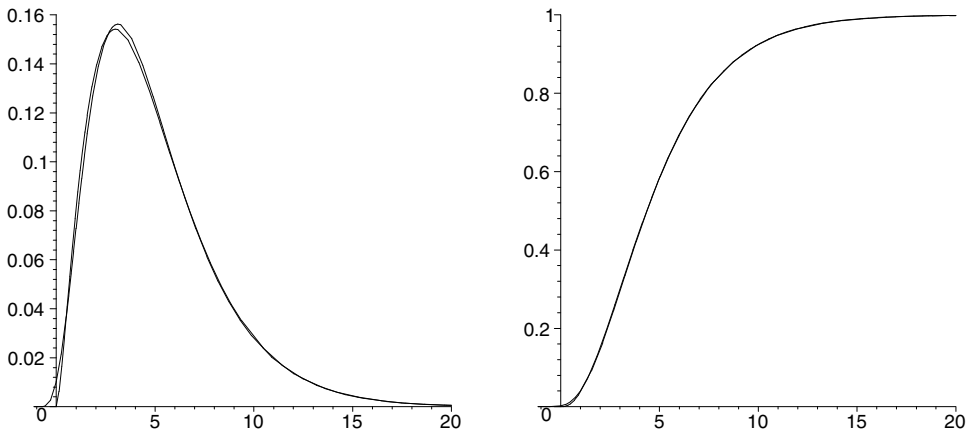
The α_1, α_2, α_3, α_4 of $\chi^2(\nu)$, the chi-square distribution with ν degrees of freedom, are

$$\nu, \quad 2\nu, \quad 2\frac{\sqrt{2}}{\sqrt{\nu}}, \quad 3 + \frac{12}{\nu}$$

(see Section 3.4.5 for details). For all $\nu > 0$, the (α_3, α_4) of the $\chi^2(\nu)$ is below the corresponding (α_3, α_4) point on the S_L curve, placing $\chi^2(\nu)$ in the S_B region of the Johnson system for all permissible ν.

Following the pattern established in Sections 3.4.5, 4.4.5, 5.4.5, and 6.4.5, we first consider $\chi^2(5)$ for which,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (5, 10, 1.2649, 5.4)$$



(a) (b)

Figure 16.5–4. The $\chi^2(5)$ distribution: p.d.f. (lower at peak) and its S_B approximation (a); d.f. and its S_B approximation (b).

and

```
FitJnsnSB([5, 10, 1.2649, 5.4], 13, 7, [1.65, 1.8], [3.62, 3.75])
```

produces the approximation with parameters

$$(\xi,\, \lambda,\, \delta,\, \gamma) = (-1.1486,\, 52.6792,\, 1.7285,\, 3.7127)$$

with $\max(|\hat{\alpha}_3 - \alpha_3|,\, |\hat{\alpha}_4 - \alpha_4|) = 7.6418 \times 10^{-5}$.

The support of this approximation of $\chi^2(5)$ is the interval $(-1.1486,\, 51.5306)$ and when we look at the $\chi^2(5)$ p.d.f. superimposed on the p.d.f. of this distribution (shown in Figures 16.5–4 (a)), we see some discrepancies near the origin and at the peak of the curves (the $\chi^2(5)$ p.d.f. is the lower one near its peak). A comparison of the d.f.s (Figure 16.5–4 (b)), shows close agreement between the two curves except near the origin.

For this approximation we have

$$\sup|\hat{f}(x) - f(x)| = 0.01068 \qquad \sup|\hat{F}(x) - F(x)| = 0.005320,$$
$$||\hat{f} - f||_1 = 0.02648, \qquad ||\hat{f} - f||_2 = 0.01129,$$

which are better than what was obtained for the GLD moment-based fit (Section 3.4.5) and the GLD percentile-based fit (Section 5.4.5), but not as good as what the GBD approximation produced in Section 4.4.5 (recall that the chi-square distribution is a limiting case of the GBD).

For $\chi^2(1)$, $(\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4) = (1,\, 2,\, 2.8284,\, 15)$ and the parameters of the approximating S_B distribution are

$$(\xi,\, \lambda,\, \delta,\, \gamma) = (-0.2938,\, 22.2203,\, 0.9383,\, 3.0522),$$

giving us a distribution with support $(-0.2938, 21.9266)$. For this approximation

$$\sup|\hat{f}(x) - f(x)| = 14.7196 \qquad \sup|\hat{F}(x) - F(x)| = 0.1561,$$
$$||\hat{f} - f||_1 = 0.3831, \qquad ||\hat{f} - f||_2 = 0.6680,$$

GLD and GBD approximations to $\chi^2(1)$ were not possible.

When $\nu = 3$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 6, 1.6330, 7)$ and the S_B approximating distribution of $\chi^2(3)$ has

$$(\xi, \lambda, \delta, \gamma) = (-0.7312, 37.7701, 1.3983, 3.3656)$$

with support $(-.7312, 37.0388)$. For this approximation,

$$\sup|\hat{f}(x) - f(x)| = 0.08167 \qquad \sup|\hat{F}(x) - F(x)| = 0.1837,$$
$$||\hat{f} - f||_1 = 0.06667, \qquad ||\hat{f} - f||_2 = 0.04378,$$

which are larger than their counterparts obtained in connection with GLD moment and percentile fits (Sections 3.4.5, 4.4.5, 5.4.5, and 5.4.6 and note that the GBD can represent the chi-square distribution as a limiting case).

In the case of $\chi^2(10)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (10, 20, 0.8944, 4.2)$ and we obtain an S_B approximation

$$(\xi, \lambda, \delta, \gamma) = (-2.1612, 89.0740, 2.3483, 4.4829).$$

The support of this fit is $(-2.1612, 86.9128)$ and from this approximation we get

$$\sup|\hat{f}(x) - f(x)| = 0.001250 \qquad \sup|\hat{F}(x) - F(x)| = 0.001782,$$
$$||\hat{f} - f||_1 = 0.008466, \qquad ||\hat{f} - f||_2 = 0.002364.$$

These figures are about the same as those obtained for the GLD moment-based approximation in Section 3.4.5 and slightly smaller than the corresponding figures for the GLD percentile-based fit of Section 5.4.5.

For $\nu = 30$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (30, 60, 0.5164, 3.4)$ and an S_B approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (-6.1829, 233.6754, 3.9437, 6.7797),$$

is obtained. The support of this distribution is $(-6.1829, 227.4925)$ and

$$\sup|\hat{f}(x) - f(x)| = 9.3010 \times 10^{-5} \qquad \sup|\hat{F}(x) - F(x)| = 0.0002944,$$
$$||\hat{f} - f||_1 = 0.001546, \qquad ||\hat{f} - f||_2 = 0.0002868,$$

making this approximation superior to the GLD fits obtained through either the moment (Section 3.5.4) or the percentile (Section 5.5.4) methods.

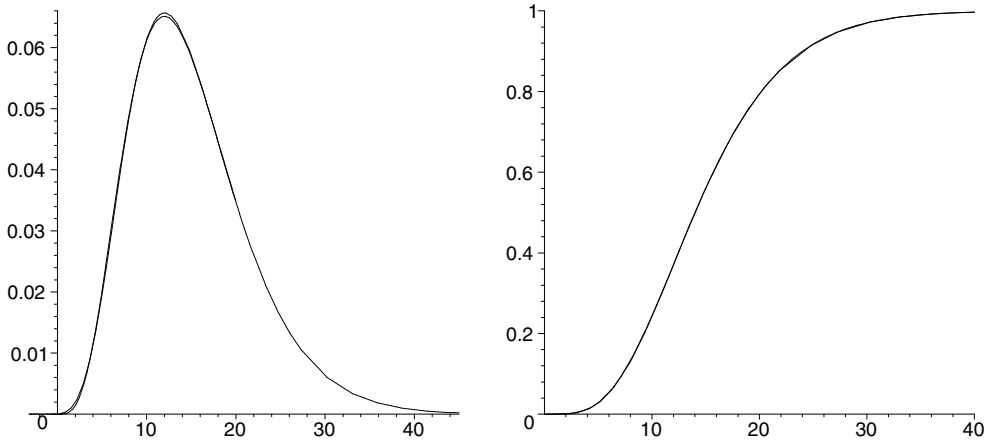


Figure 16.5–5. The gamma ($\alpha = 5$, $\theta = 3$) distribution: p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

16.5.6 The Gamma Distribution

The gamma distribution with parameters α and θ, described in Section 3.4.6, has

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\alpha\theta, \quad \alpha\theta^2, \quad \frac{2}{\sqrt{\alpha}}, \quad 3 + \frac{6}{\alpha}\right).$$

As was done in Sections 3.4.6, 4.4.6, 5.4.6, and 6.4.6, we consider the specific case where $\alpha = 5$ and $\theta = 3$, making

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (15, 45, \frac{2\sqrt{5}}{5}, \frac{21}{5}) = (15, \ 45, \ 0.8944, \ 4.2).$$

We first note, from Table 16.2–2, that (α_3, α_4) is located in the S_B region, and then use the Maple command

```
FitJnsnSB([15, 45, 0.8944, 4.2], 13, 8),
```

which produces the S_B approximation

$$(\xi, \lambda, \delta, \gamma) = (-3.2502, 133.7929, 2.3498, 4.4880)$$

with support $(-3.2502, 130.5427)$.

A graphic comparison of the gamma distribution p.d.f. and d.f. with the p.d.f. and d.f. of the approximating distribution, shown in Figures 16.5.–5 (a) and 16.5–5 (b), respectively, give us a sense that we have a reasonably good fit. This observation is reinforced by

$$\sup|\hat{f}(x) - f(x)| = 0.0008361 \qquad \sup|\hat{F}(x) - F(x)| = 0.001785,$$
$$||\hat{f} - f||_1 = 0.008463, \qquad ||\hat{f} - f||_2 = 0.001933.$$

These figures indicate that this approximation is decidedly better than the GLD approximations (Sections 3.4.6 and 5.4.6).

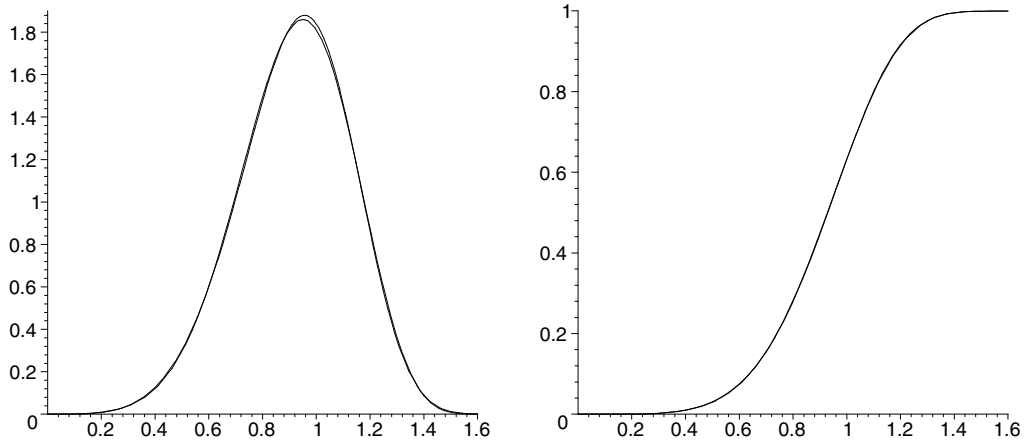


Figure 16.5–6. The Weibull distribution ($\alpha = 1$, $\beta = 5$): p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

16.5.7 The Weibull Distribution

Using the results of Section 3.4.7 regarding the $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the Weibull distribution and the specifications $\alpha = 1,\ \beta = 5$, we have

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.9182, 0.04423, -0.2541, 2.8803).$$

Since it is clear that the Weibull distribution with $\alpha = 1,\ \beta = 5$ is in the S_B region of the Johnson system, we can use

$$\texttt{FitJnsnSB([0.9182, 0.04423, -0.2541, 2.8803], 13, 7)},$$

to get the approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (-0.6120,\ 2.4458,\ 2.6438,\ -1.4037)$$

which has support $(-0.6120,\ 1.8338)$ and $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 6.5166 \times 10^{-5}$. The p.d.f. of the Weibull distribution with $\alpha = 1$ and $\beta = 5$ and its approximation are shown in Figure 16.5–6 (a) and the d.f.s of these distributions are given in Figure 16.5–6 (b). It appears that the approximation is a good one and for this approximation

$$\sup|\hat{f}(x) - f(x)| = 0.02804 \qquad \sup|\hat{F}(x) - F(x)| = 0.002564,$$
$$||\hat{f} - f||_1 = 0.01414, \qquad ||\hat{f} - f||_2 = 0.01498,$$

which is a slight improvement over the moment-based GLD fit of Section 3.4.7 but is not as good as the GBD or GLD percentile fits obtained in Sections 4.4.7 and 5.4.7.

16.5.8 The Lognormal Distribution

The lognormal distribution, with its two parameters μ and $\sigma > 0$, is defined in Section 3.4.8. Exact fits will be available in this situation since the lognormal distribution is one of the components of the Johnson system. Section 3.4.8 also gives formulas for $(\alpha_1, \alpha_2, \alpha_3, \alpha_3)$ and a GLD approximation for the lognormal with $\mu = 0$ and $\sigma = 1/3$. For these μ and σ, we have, as shown in Section 3.4.8,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_3) = (1.0571, 0.1313, 1.0687, 5.0974).$$

If we were to try to fit an S_L distribution by the Jnsn–S_L algorithm of Section 16.2, we would obtain

$$\begin{aligned} \omega &= 1.1175, \quad \alpha_4 = 5.0973, \quad \Omega = 0.00009577, \\ \delta &= 3.0000, \quad \gamma = 0.0002873, \quad \xi = 0.00007527, \quad \lambda = 1. \end{aligned}$$

The roundoff errors in the computation of $(\alpha_1, \alpha_2, \alpha_3, \alpha_3)$ produce some minor discrepancies since an exact fit would require $\delta = 3$, $\gamma = 0$, $\xi = 0$, $\lambda = 1$.

From Figure 16.2–1 we can see that every member of the lognormal family can also be realized as a limiting case from the S_B and S_U regions. If we were to use a slightly smaller α_4, say $\alpha_4 = 5.095$, we would obtain the S_B approximation specified by

$$(\xi, \lambda, \delta, \gamma) = (0.006590,\ 278.7265,\ 2.9692,\ 16.7270)$$

with support $(0.006590, 278.7331)$ and

$$\begin{aligned} \sup|\hat{f}(x) - f(x)| &= 0.001427 \qquad \sup|\hat{F}(x) - F(x)| = 0.0002000, \\ ||\hat{f} - f||_1 &= 0.0007895, \qquad ||\hat{f} - f||_2 = 0.0007861. \end{aligned}$$

With a slightly larger α_4 such as $\alpha_4 = 5.1$, we would look for a fit in the S_U region and obtain the distribution specified by

$$(\xi, \lambda, \delta, \gamma) = (0.004877, 0.0710, 2.9939, -9.9849)$$

with support $(-\infty, \infty)$ and

$$\begin{aligned} \sup|\hat{f}(x) - f(x)| &= 0.0005742 \qquad \sup|\hat{F}(x) - F(x)| = 6.861 \times 10^{-5}, \\ ||\hat{f} - f||_1 &= 0.0003460, \qquad ||\hat{f} - f||_2 = 0.0003174. \end{aligned}$$

16.5.9 The Beta Distribution

Section 3.4.9 gives the definition and the $(\alpha_1, \alpha_2, \alpha_3, \alpha_3)$ of the beta distribution. In that section an approximation is obtained for the beta distribution with $\beta_3 =$

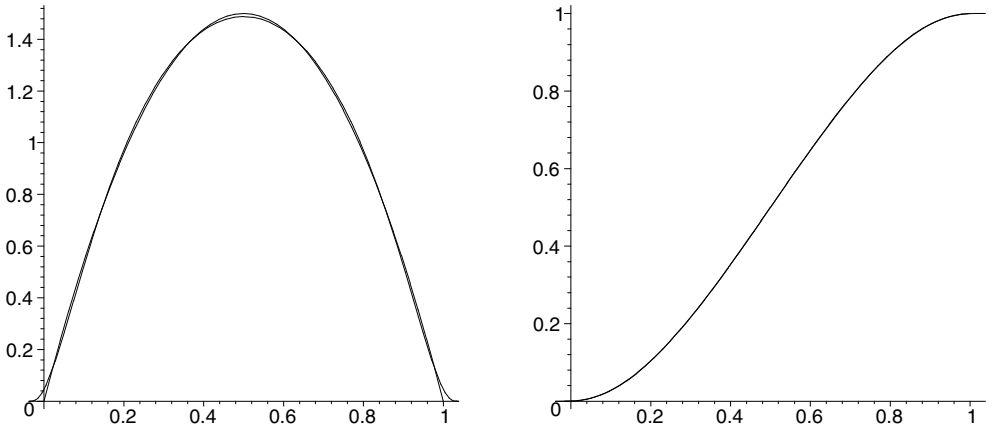


Figure 16.5–7. The beta distribution ($\beta_3 = \beta_4 = 1$): p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

$\beta_4 = 1$ and it is determined that GLD fits are not available for many members of this family of distributions. In the case $\beta_3 = \beta_4 = 1$, we have

$$(\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_3) = \left(\frac{1}{2},\ \frac{1}{20},\ 0,\ \frac{15}{7}\right)$$

which places this distribution in the S_B region. The Maple command

```
FitJnsnSB([0.5, 0.05, 0, 2.1429], 13, 7, [.98, 1.05], [0., 0.01]),
```

produces the S_B distribution specified by

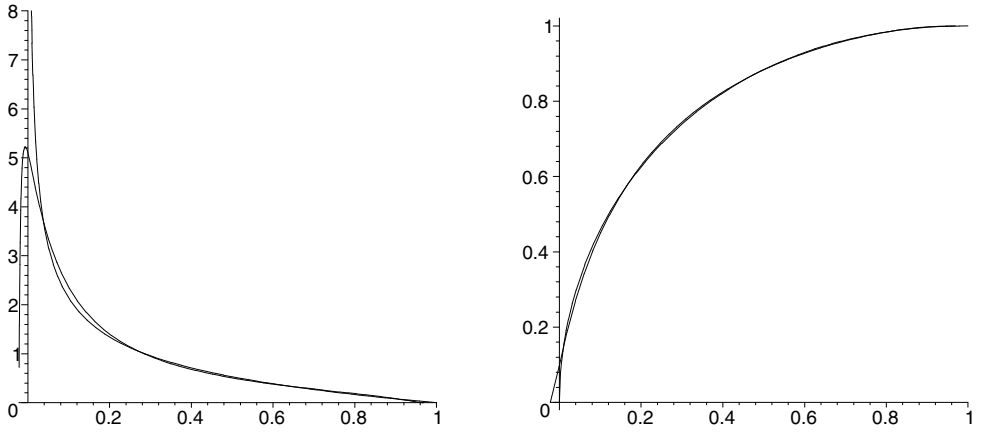
$$(\xi,\, \lambda,\, \delta,\, \gamma) = (-0.03851,\ 1.0770,\ 1.0044,\ 3.111 \times 10^{-5}),$$

with $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|) = 8.4197 \times 10^{-5}$ and support $(-0.03851, 1.0385)$ (it should be noted that the support of any member of the beta distribution family is $(0,\ 1)$). We can see from the comparison of the distribution and approximating p.d.f.s (Figure 16.5–7 (a)) and d.f.s (Figure 16.5–7(b)) that we have a reasonably good fit, except perhaps near the support endpoints. For this approximation we have

$$\sup|\hat{f}(x) - f(x)| = 0.04502 \qquad \sup|\hat{F}(x) - F(x)| = 0.001049,$$
$$||\hat{f} - f||_1 = 0.01171, \qquad ||\hat{f} - f||_2 = 0.01370.$$

These figures are slightly lower than those obtained for a moment-based GLD fit (Section 3.4.9) but somewhat higher than those for percentile-based fits (Section 5.4.9). The GBD, of course, provides perfect fits to the beta distribution.

It was shown in Section 3.4.9 that moment-based GLD approximations were not available for the beta distribution with $\beta_3 = -1/2,\ \beta_4 = 1$. This, however, is



(a) (b)

Figure 16.5–8. The beta distribution ($\beta_3 = 1/2$, $\beta_4 = 1$): p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

not the case for the Johnson system. For this distribution,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_3) = \left(\frac{1}{5}, \frac{8}{175}, \frac{\sqrt{14}}{3}, \frac{42}{11}\right) = (0.2,\ 0.04571,\ 1.2472,\ 3.8182)$$

and

$$\texttt{FitJnsnSB}(\texttt{[0.2, 0.04571, 1.2472, 3.8182]}, \texttt{13}, \texttt{7})$$

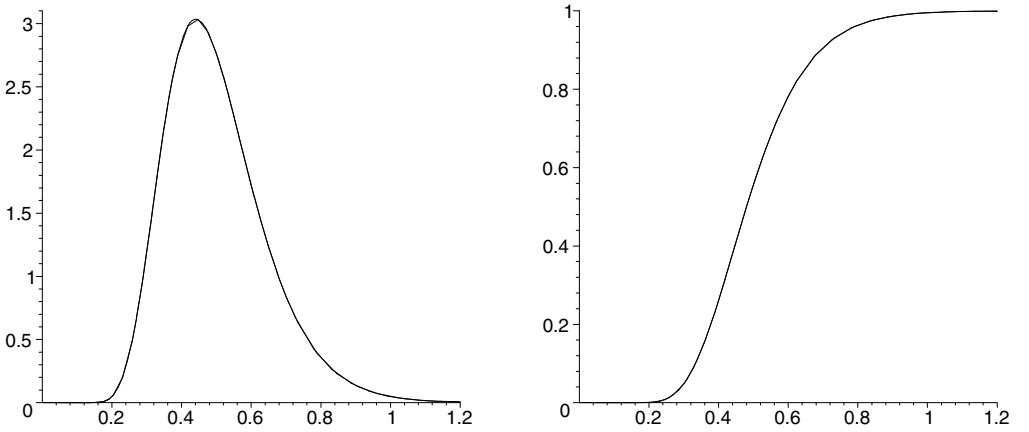
produces the S_B approximation given by

$$(\xi, \lambda, \delta, \gamma) = (-0.02228,\ 0.9915,\ 0.6395,\ 1.1239)$$

with $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|) = 6.6075 \times 10^{-5}$ and support $(-0.02228, 0.9692)$. The first difficulty with this approximation is that its support does not cover the entire region over which the original beta distribution is defined. More difficulties are observed when the p.d.f.s and d.f.s of this beta and its approximating distribution are compared (Figures 16.5–8 (a) and 16.5–8 (b), respectively). As $x \to 0^+$, the p.d.f. of the beta distribution tends to ∞ but the approximating distribution attains a value a little higher than 5 for its maximum. This problem is further clarified by the computation

$$\sup|\hat{f}(x) - f(x)| = 744.8935 \qquad \sup|\hat{F}(x) - F(x)| = 0.09738,$$
$$||\hat{f} - f||_1 = 0.2684, \qquad ||\hat{f} - f||_2 = 2.2448.$$

Note that in the computation of $\sup|\hat{f}(x) - f(x)|$, a finite set of discrete values of x are used and the largest of these is taken for $\sup|\hat{f}(x) - f(x)|$.



(a) (b)

Figure 16.5–9. The inverse Gaussian distribution ($\mu = 1/2$, $\lambda = 6$): p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

16.5.10 The Inverse Gaussian Distribution

For parameters $\mu > 0$ and $\lambda > 0$ the inverse Gaussian distribution (defined in Section 3.4.10), has

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_3) = \left(\mu, \ \frac{\mu^3}{\lambda}, \ 3\sqrt{\frac{\mu}{\lambda}}, \ 3 + \frac{15\mu}{\lambda}\right).$$

It is easy to see that for all inverse Gaussian distributions $\alpha_4 = 3 + (5/3)\alpha_3^2$, placing these distributions in the S_B region of the Johnson system.

As was done in Section 3.4.10, we again consider the case $\mu = 1/2, \ \lambda = 6$, for which

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_3) = \left(\frac{1}{2}, \ \frac{1}{48}, \ \frac{\sqrt{3}}{2}, \ \frac{17}{4}\right) = (0.5, \ 0.02083, \ 0.8660, \ 4.25).$$

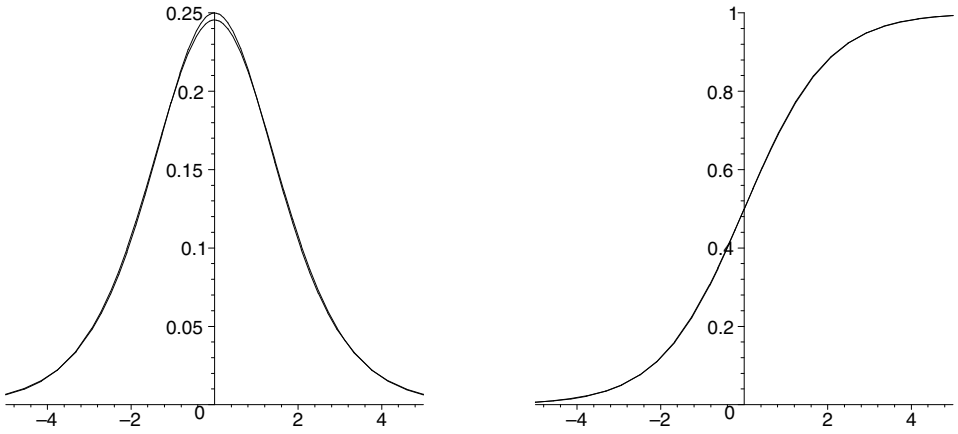
When

```
FitJnsnSB([0.5, 0.02083, 0.8660, 4.25], 11, 11, [2.9, 2.94], [7.1, 7.22])
```

is used, the S_B fit with parameters

$$(\xi, \lambda, \delta, \gamma) = (0.04611, \ 5.5084, \ 2.9162, \ 7.1704)$$

that has $\max(|\hat{\alpha}_3 - \alpha_3|, \ |\hat{\alpha}_4 - \alpha_4|) = 8.1040 \times 10^{-5}$ and support $(0.04611, 5.5545)$ is obtained. Figures 16.5–9 (a) and 16.5–9 (b) compare the inverse Gaussian p.d.f. and d.f. with those of the fitted distribution. These p.d.f.s and d.f.s are



(a) (b)

Figure 16.5–10. The logistic distribution ($\mu = 0$, $\sigma = 1$): p.d.f. and its S_U approximation (a); d.f. and its S_U approximation (b).

virtually indistinguishable, indicating that we have a good approximation; this is borne out by

$$\sup|\hat{f}(x) - f(x)| = 0.001725 \qquad \sup|\hat{F}(x) - F(x)| = 9.2842 \times 10^{-5},$$
$$||\hat{f} - f||_1 = 0.0004567, \qquad ||\hat{f} - f||_2 = 0.0006560.$$

These figures are considerably smaller than those obtained for the GLD moment and percentile approximations (Sections 3.4.11 and 5.4.11, respectively). The GBD is not able to provide an approximation in this case.

16.5.11 The Logistic Distribution

The logistic distribution, with parameters μ and $\sigma > 0$, is defined in Section 3.4.11 where its α_1, α_2, α_3, α_4 are given by

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\mu, \frac{\pi^2\sigma^2}{3},\ 0,\ \frac{21}{5}\right).$$

Since $\alpha_4 > 3$, this distribution is always in the S_U region of the Johnson system.

As was done in Section 3.4.11, we consider the case $\mu = 0$ and $\sigma = 1$ for which

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(0, \frac{\pi^2}{3},\ 0,\ \frac{21}{5}\right) = (0,\ 3.2899,\ 0,\ 4.2).$$

From the Maple command `FitJnsnSU([0, 3.2899, 0, 4.2])`, we get the S_U approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (0,\ 3.5240,\ 2.1689,\ 0).$$

Figure 16.5–10 (a) shows the p.d.f. of the logistic distribution with its approximating p.d.f. (the curve that rises at the center is the logistic p.d.f.) and Figure 16.5–10 (b) show the d.f.s of these distributions. In this case we have

$$\sup|\hat{f}(x) - f(x)| = 0.004465 \qquad \sup|\hat{F}(x) - F(x)| = 0.002843,$$
$$||\hat{f} - f||_1 = 0.01414, \qquad ||\hat{f} - f||_2 = 0.005693,$$

making this approximation quite good but still inferior to both the moment and the percentile GLD fits (the GBD does not provide an approximation in this situation).

16.5.12 The Largest Extreme Value Distribution

The largest extreme value distribution has parameters μ and $\sigma > 0$ (see Section 3.4.12 for the definition) and has

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\mu + \gamma\sigma,\ \frac{\pi^2\sigma^2}{6},\ \sqrt{1.29857},\ \frac{27}{5}\right),$$

where $\gamma \approx 0.57722$ is Euler's constant. For this distribution, regardless of the parameter values, $\alpha_3 = \sqrt{1.2987} = 1.1395$ and $\alpha_4 = 5.4$. The point on the S_L curve (Figure 16.2–1) at $\alpha_4 = 5.4$ has $\alpha_3 = 1.1408$. Therefore, this family of distributions is represented by a point in the S_U region that is very close to the S_L curve.

We specify, as in Section 3.4.12, $\mu = 0$ and $\sigma = 1$ and establish

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\gamma, \frac{\pi^2}{6},\ \sqrt{1.29857},\ \frac{27}{5}\right) = (0.5772,\ 1.6449,\ 1.1395,\ 5.4).$$

The Maple command

```
FitJnsnSU([0.5772, 1.6449, 1.1395, 5.4])
```

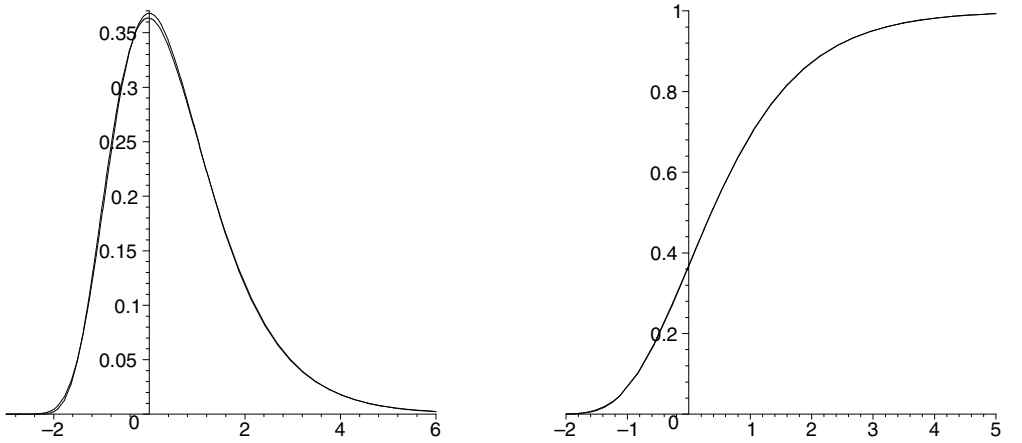
yields the S_U approximation with parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (-2.9202,\ 0.3121,\ 2.8265,\ -8.6183).$$

Figure 16.5–11 (a) gives the p.d.f. of this extreme value distribution with the approximating p.d.f. (the extreme value p.d.f. is slightly lower at the center) and Figure 16.5–11 (b) shows the corresponding d.f.s, which look identical. As a quantitative measure of the quality of the approximation,

$$\sup|\hat{f}(x) - f(x)| = 0.007395 \qquad \sup|\hat{F}(x) - F(x)| = 0.003234,$$
$$||\hat{f} - f||_1 = 0.01420, \qquad ||\hat{f} - f||_2 = 0.007452$$

are computed. These figures indicate a somewhat higher quality approximation than the GLD approximations obtained through moments and percentiles (Sections 3.4.12 and 5.4.12). The extreme value distributions are out of the range of the GBD.



(a) (b)

Figure 16.5–11. The largest extreme value distribution ($\mu = 0$, $\sigma = 1$): p.d.f. and its S_U approximation (a); d.f. and its S_U approximation (b).

16.5.13 The Extreme Value Distribution

This distribution, discussed in some detail in Section 3.4.13, has $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-a_1, a_2, -a_3, a_4)$ where (a_1, a_2, a_3, a_4) is its counterpart for the largest extreme value distribution with the same μ and σ. Therefore, fitting a member of the Johnson system to this distribution can be accomplished by a minor adjustment to the fitting described in the previous section.

16.5.14 The Double Exponential Distribution

The double exponential distribution, defined in Section 3.4.14, has a single positive parameter λ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 2\lambda^2, 0, 6)$. Thus, distributions in this family are represented by a single point in the S_U region of the Johnson system. We consider, as was done in Section 3.4.13, the specific double exponential distribution with $\lambda = 1$. For this distribution, `FitJnsnSU([0, 2, 0, 6])` provides the fit with parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (0,\ 1.8551,\ 1.6104,\ 0).$$

We can see from Figures 16.5–12 (a) and 16.5–12 (b), that give, respectively, the double exponential p.d.f. and d.f. with $\lambda = 1$ with its approximating distribution, that this S_U fit is not a particularly good one. For this fit we have

$$\sup|\hat{f}(x) - f(x)| = 0.1530, \qquad \sup|\hat{F}(x) - F(x)| = 0.03077,$$
$$||\hat{f} - f||_1 = 0.1385, \qquad ||\hat{f} - f||_2 = 0.08613,$$

which are slightly larger than the corresponding figures obtained for the GLD moment and percentile approximations (Sections 3.4.14 and 5.4.14). Being outside of the GBD (α_3, α_4)-space, the GBD does not provide a fit in this case.

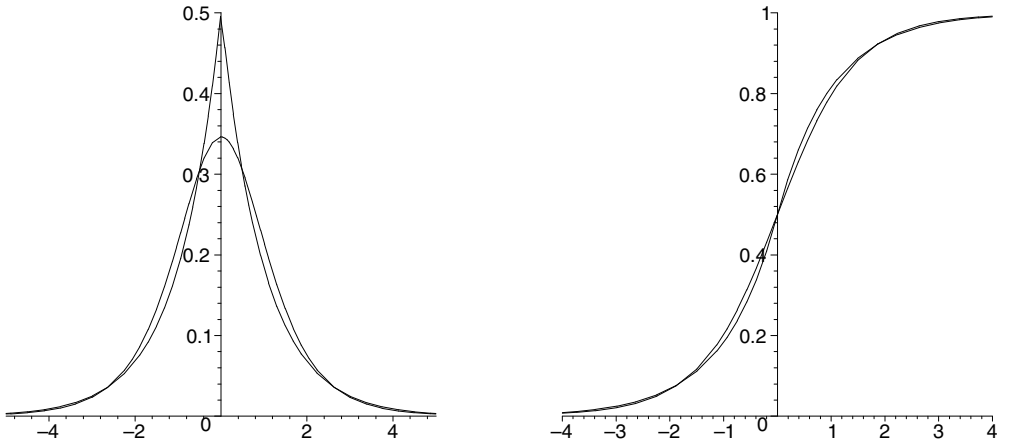


Figure 16.5–12. The double exponential distribution ($\lambda = 1$): p.d.f. and its S_U approximation (a); d.f. and its S_U approximation (b).

16.5.15 The F–Distribution

The F-distribution, $F(\nu_1,\ \nu_2)$, and its $(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)$ are discussed in Section 3.4.15. This distribution has two positive parameters ν_1 and ν_2 and, as shown in Section 3.4.15, we must have $\nu_2 > 8$ for the first four moments to exist. We first consider $F(6,\ 25)$, the F-distribution with $\nu_1 = 6$, $\nu_2 = 25$, as is done in Sections 3.4.15 and 5.4.15, where it is shown that

$$(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4) = (1.0869,\ 0.5439,\ 1.8101,\ 9.1986).$$

It is clear from Appendix I that this places us in the S_B region and from

`FitJnsnSB([1.0869, 0.5439, 1.8101, 9.1986], 13, 7)`

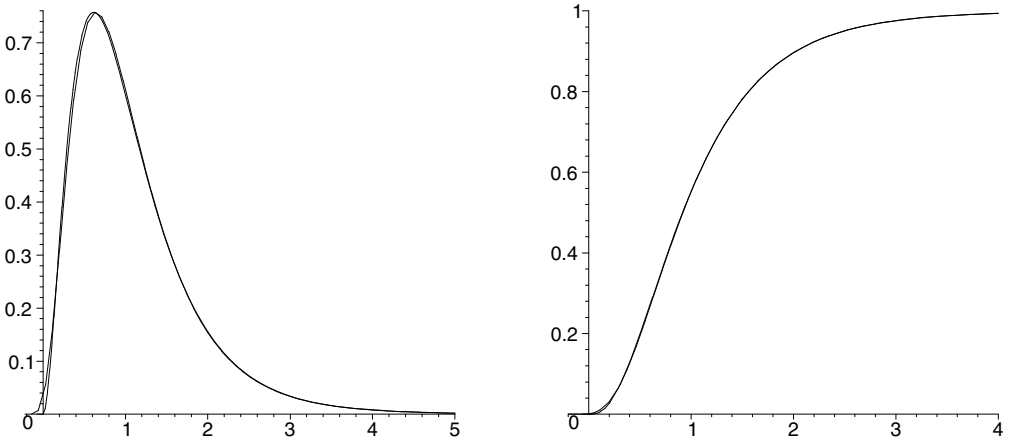
we get the S_B approximation with parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (-0.2264,\ 125.7050,\ 1.8800,\ 8.8152)$$

which has support $(-0.2264, 125.4786)$ and $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|) = 9.6447 \times 10^{-5}$. Figures 16.5–13 (a), showing the p.d.f. of this F-distribution with its S_B approximation, and 16.5–13 (b), showing the corresponding d.f.s, indicate a reasonably good fit. This is borne out by

$$\sup|\hat{f}(x) - f(x)| = 0.04034 \qquad \sup|\hat{F}(x) - F(x)| = 0.005037,$$
$$||\hat{f} - f||_1 = 0.02149, \qquad ||\hat{f} - f||_2 = 0.02078.$$

These figures compare favorably with the ones obtained for moment and percentile based GLD fits. The $(\alpha_3,\ \alpha_4)$ of the $F(6,\ 25)$ distribution is not in the $(\alpha_3,\ \alpha_4)$-space of the GBD.



(a) (b)

Figure 16.5–13. The $F(6, 25)$ distribution: p.d.f. and its S_B approximation (a); d.f. and its S_B approximation (b).

Section 3.4.15 also indicates that the (α_3, α_4) of $F(6, 16)$ is outside of the region covered by the GLD moments. For $F(6, 16)$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{8}{7}, \frac{320}{441}, \frac{26\sqrt{5}}{25}, \frac{366}{25}\right) = (1.1429, 0.7256, 2.3255, 14.64),$$

and $(\alpha_3, \alpha_4) = (2.3255, 14.64)$ is located in the S_U region of the Johnson system (on the S_L curve when $\alpha_3 = 2.3255$, $\alpha_4 = 13.9411$). The F-distribution is our first encounter with a family whose (α_3, α_4) points extend across the S_B, S_L and S_U regions. Using `FitJnsnSU([1.1429, 0.7256, 2.3255, 14.64])`, we get the S_U approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (0.06896, 0.4443, 1.5483, -2.2086).$$

Figure 16.5–14 (a) shows the $F(6, 16)$ p.d.f. with this S_U approximation and Figure 16.5–14 (b) gives the corresponding distribution functions. In this case we have

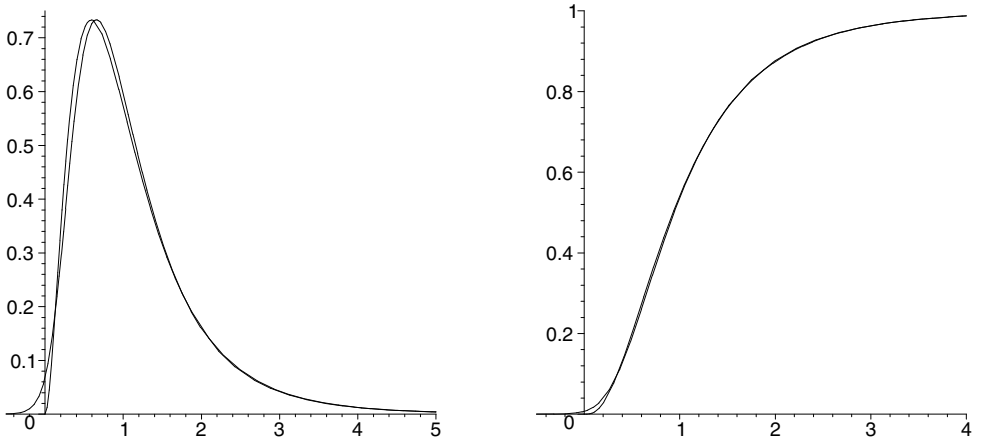
$$\sup|\hat{f}(x) - f(x)| = 0.08900 \qquad \sup|\hat{F}(x) - F(x)| = 0.01352,$$
$$||\hat{f} - f||_1 = 0.05797, \qquad ||\hat{f} - f||_2 = 0.05218.$$

In a similar fashion, the $F(6, 12)$ distribution can be approximated with a Johnson S_U distribution with parameters

$$(\xi, \lambda, \delta, \gamma) = (0.6456, 0.5703, 1.1370, -0.7048),$$

with

$$\sup|\hat{f}(x) - f(x)| = 0.2899 \qquad \sup|\hat{F}(x) - F(x)| = 0.06964,$$
$$||\hat{f} - f||_1 = 0.2481, \qquad ||\hat{f} - f||_2 = 0.1877.$$



(a) (b)

Figure 16.5–14. The $F(6, 16)$ distribution: p.d.f. and its S_U approximation (a); d.f. and its S_U approximation (b).

16.5.16 The Pareto Distribution

The Pareto distribution, with the positive parameters β and λ, is defined in Section 3.4.16. From the expressions for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ given in Section 3.4.16, we see that α_1 and α_2 depend on both β and λ but α_3 and α_4 depend only on β. Moreover, for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to exist, we must have $\beta > 4$. In the specific case of $\beta = 5$ and $\lambda = 1$ that is considered in Section 5.4.16, the Pareto p.d.f. $f(x) = 5/x^6$ for $1 < x < \infty$ and

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{5}{4}, \frac{5}{48}, \frac{6\sqrt{3}}{\sqrt{5}}, \frac{369}{5}\right) = (1.25,\ 0.1042,\ 4.6476,\ 73.8).$$

On the S_L curve $\alpha_3 = 4.647$ results in $\alpha_4 = 57.5675$; hence, the approximation of this Pareto distribution must be located in the S_U region.

From `FitJnsnSU([1.25, 0.1042, 4.6476, 73.8])` we obtain the S_U approximation with parameters

$$(\xi, \lambda, \delta, \gamma) = (1.0656,\ 0.1289,\ 1.0179,\ -0.8105).$$

Figure 16.5–15 (a) shows the p.d.f. of this approximation with $f(x)$ and Figure 16.5–15 (b) shows their corresponding d.f.s. It is clear that this is a very poor fit, an observation that is verified by

$$\sup|\hat{f}(x) - f(x)| = 3.7626 \qquad \sup|\hat{F}(x) - F(x)| = 0.09154,$$
$$||\hat{f} - f||_1 = 0.3813, \qquad ||\hat{f} - f||_2 = 0.7336.$$

The reason that this Pareto distribution presents a challenge to the Johnson system is its large α_4 relative to its α_3, forcing the approximation to the S_U region

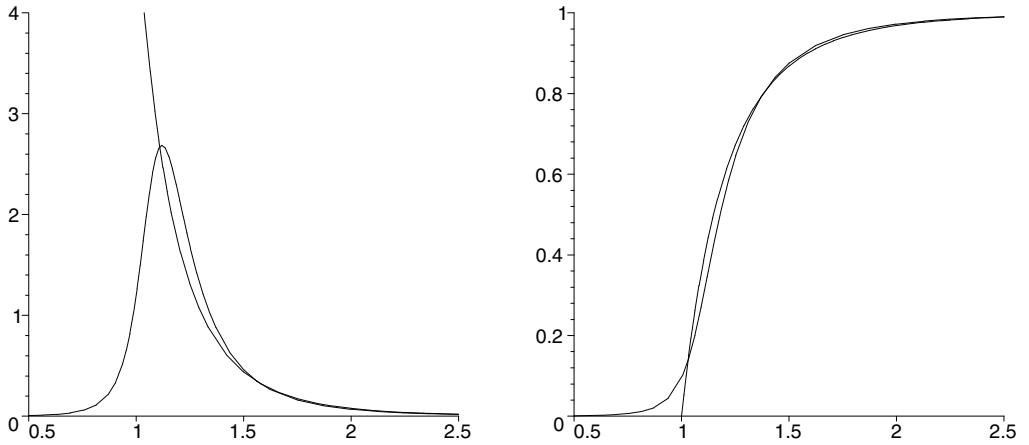


Figure 16.5–15. The Pareto distribution ($\beta = 5$, $\lambda = 1$): p.d.f. and its S_U approximation (a); d.f. and its S_U approximation (b).

where all distributions are defined over $(-\infty,\ \infty)$. Similar poor approximations are obtained for other values of the Pareto parameters (e.g., $\lambda = 1$ and $\beta = 10$ or 20). This combination of α_3 and α_4 also makes it impossible to obtain a GLD moment-based fit or a GBD fit. However, a good quality GLD fit is obtained through percentiles in Section 5.4.16.

16.6 Examples of Johnson System Fits to Data

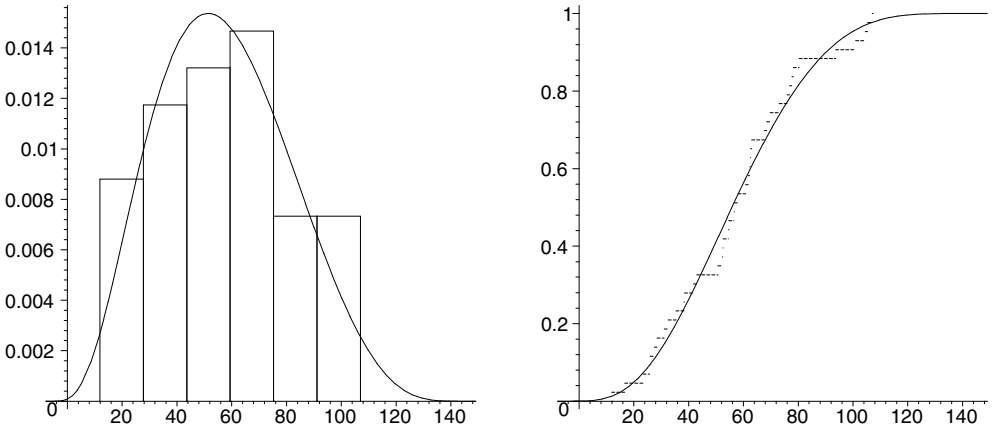
We now apply the methods of this chapter to fit distributions from the Johnson system to data that have been introduced in other chapters. When we tried to fit a GBD or a GLD (moment or percentile) distribution to data, we found, on some occasions, that a fit was not possible. The principle advantage of the Johnson system is that this cannot happen — we may have difficulty with computations, but there is a theoretical guarantee that a unique member of the Johnson system will match the first four moments of the data. This allows us, as will be detailed in the following sections, to fit a Johnson system distribution to every dataset considered in this book.

16.6.1 Example: Cadmium Concentration in Horse Kidneys

The data for this example was introduced in Section 3.5.2 and it was shown there that

$$\hat{\alpha}_1 = 57.2442, \quad \hat{\alpha}_2 = 576.0741, \quad \hat{\alpha}_3 = 0.2546, \quad \hat{\alpha}_4 = 2.5257.$$

After observing that the fit for this data must come from the S_B region of the Johnson system, we appeal to the Maple program `FitJnsnSB` to obtain the fit



(a) (b)

Figure 16.6–1. Histogram of cadmium concentration and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

with parameters

$$(\xi,\ \lambda,\ \delta,\ \gamma) = (-7.9073,\ 156.9935,\ 1.4421,\ 0.5487),$$

which we abbreviate by

$$\text{JnsnSB}(-7.9073,\ 156.9935,\ 1.4421,\ 0.5487).$$

The support of this fit is $(-7.9073,\ 149.0862)$, which covers the range of the data. Figure 16.6–1 (a) shows the p.d.f. of the fitted distribution with a superimposed histogram of the data and Figure 16.6–1 (b) shows the d.f. of the fitted distribution along with the e.d.f. of the data. These figures indicate a reasonably good fit, which we now try to quantify through the chi-square and Kolmogorov-Smirnov tests.

Using the classes

$$(-\infty,\ 30),\quad [30,\ 50),\quad [50,\ 60),\quad [60,\ 70),\quad [70,\ 85),\quad [85,\ \infty)$$

whose frequencies are

$$7,\quad 7,\quad 9,\quad 9,\quad 6,\quad 5,$$

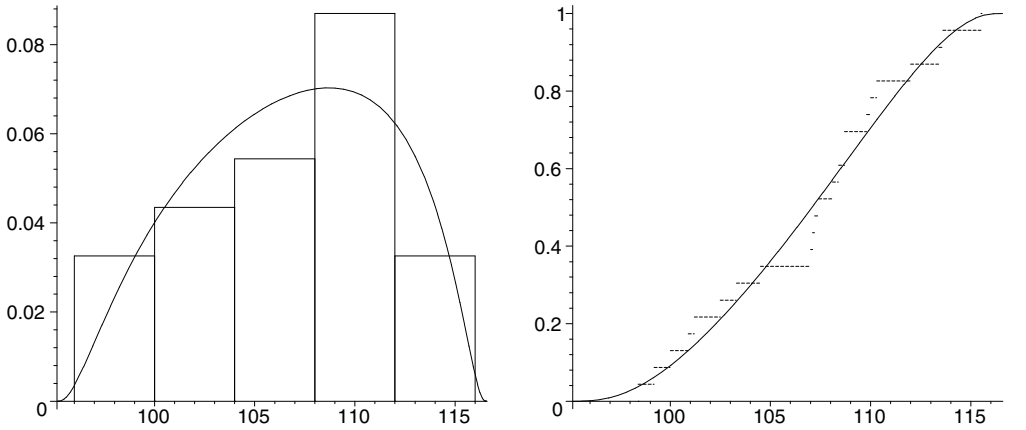
we calculate the expected frequencies

$$5.8142,\ \ 11.8431,\ \ 6.5290,\ \ 5.9534,\ \ 6.8781,\ \ 5.9822,$$

obtaining the following chi-square statistic and associated p-value, respectively,

$$4.9900 \quad \text{and} \quad 0.02549.$$

The Kolmogorov-Smirnov statistic, KS, in this case is 0.09713 with a sample size of $n = 43$. This gives $\sqrt{n}KS = 0.637$ and (from Appendix J) a p-value of 0.81 for the Kolmogorov-Smirnov test.



(a) (b)

Figure 16.6–2. Histogram of MRI scans and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

16.6.2 Example: Brain MRI Scan

This data, given in Section 3.5.3, has

$$\hat{\alpha}_1 = 106.8348, \quad \hat{\alpha}_2 = 22.2988, \quad \hat{\alpha}_3 = -0.1615, \quad \hat{\alpha}_4 = 2.1061.$$

Since this directs us to the S_B region of the Johnson system, we use `FitJnsnSB` to obtain the fit

$$\text{JnsnSB}(95.1280,\ 21.4607,\ 0.9167, -0.2081),$$

with support $(95.1280,\ 116.5886)$ that covers the range of the data. A histogram of the data with the fitted p.d.f. and the e.d.f. of the data with the d.f. of the fitted distribution are shown in Figures 16.6–2 (a) and 16.6–2 (b), respectively. These indicate a fit of questionable quality.

To obtain a chi-square statistic, the class intervals

$$(-\infty,\ 103), \qquad [103,\ 107), \qquad [107,\ 111), \qquad [111,\ \infty)$$

are used, for which the frequencies are

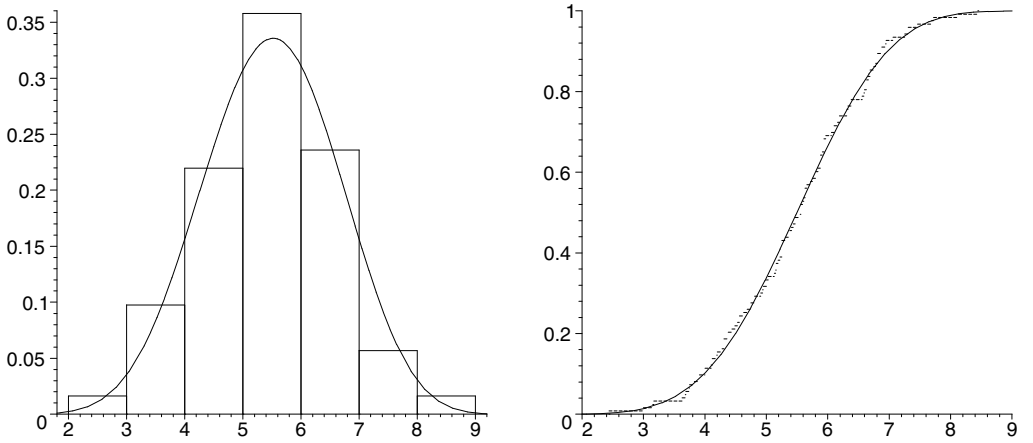
$$6, \qquad 2, \qquad 11, \qquad 4$$

with expected frequencies:

$$5.5042, \quad 5.8835, \quad 6.3919, \quad 5.2204,$$

yielding a chi-square value of 6.2155.

In this case we have a sample size of $n = 23$ and Kolmogorov-Smirnov statistic of $KS = 0.1473$. This makes $\sqrt{n}KS = 0.706$ and from Appendix J we get a p-value of 0.70 for the Kolmogorov-Smirnov test.



(a) (b)

Figure 16.6–3. Histogram of birth weights X and the fitted p.d.f (a); the e.d.f. of X with the d.f. of the fitted distribution (b).

16.6.3 Example: Human Twin Data

This data, discussed in some detail in Section 3.5.4, consists of birth weights $(X,\ Y)$ of twins, where X is the weight of the twin who is born first. For the weights X,

$$\hat{\alpha}_1 = 5.4863, \quad \hat{\alpha}_2 = 1.3083, \quad \hat{\alpha}_3 = -0.04715, \quad \hat{\alpha}_4 = 2.7332,$$

leading to the S_B fit

$$\text{JnsnSB}(-0.5810,\ 11.6203,\ 2.4378,\ -0.2246).$$

The support of this fit, $(-0.5810,\ 11.0394)$, covers the data range. Figures 16.6–3 (a) and 16.6–3 (b), that show the density of the fitted distribution with a histogram of X and the d.f. of the fit with the e.d.f. of X, indicate a reasonably good fit.

To verify the visual indication of a good fit, we take the class intervals

$$(-\infty,\ 4),\ [4,\ 4.5),\ [4.5,\ 5),\ [5,\ 5.4),\ [5.4,\ 5.8),\ [5.8,\ 6.2),\ [6.2,\ 6.8),\ [6.8,\ \infty),$$

to obtain class frequencies

$$12, \quad 15, \quad 12, \quad 17, \quad 16, \quad 19, \quad 16, \quad 16$$

and expected frequencies of

$$12.5103,\ 12.1550,\ 16.9499,\ 15.9040,\ 16.4045,\ 15.2517,\ 17.8543,\ 15.9704.$$

This leads to the chi-square statistic and associated p-value, respectively, of

$$3.3316 \qquad \text{and} \qquad 0.3433.$$

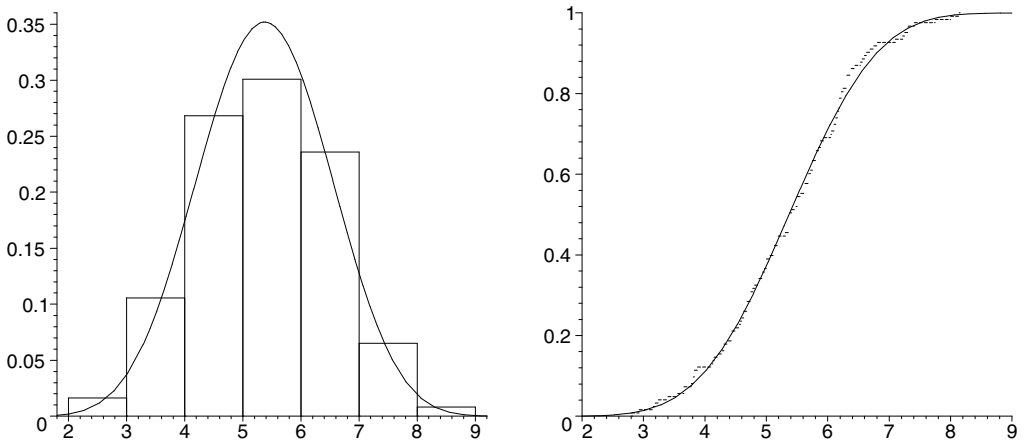


Figure 16.6–4. Histogram of birth weights Y and the fitted p.d.f (a); the e.d.f. of Y with the d.f. of the fitted distribution (b).

The Kolmogorov-Smirnov statistics for this fit is $KS = 0.03871$ and, since in this case $n = 123$, we get $\sqrt{n}KS = 0.429$ and a p-value larger than 0.99. The chi-square and Kolmogorov-Smirnov p-values certainly reinforce our earlier observation of a good fit.

For data Y, the birth weights of the second-born twins,

$$\hat{\alpha}_1 = 5.3666, \quad \hat{\alpha}_2 = 1.2033, \quad \hat{\alpha}_3 = -0.01219, \quad \hat{\alpha}_4 = 2.7665,$$

that produces the S_B fit

$$\text{JnsnSB}(-0.7539,\ 12.0931,\ 2.6659,\ -0.06736).$$

The support of this distribution, $(-0.7539, 11.3393)$, covers the range of Y and Figures 16.6–4 (a) and 16.6–4 (b), that compare, respectively, the p.d.f of the fitted distribution with a histogram of Y and the d.f of the fitted distribution with the e.d.f. of Y, indicate that this is a reasonable fit.

When a chi-square test is applied, using the same classes that were used for X, we obtain the frequencies

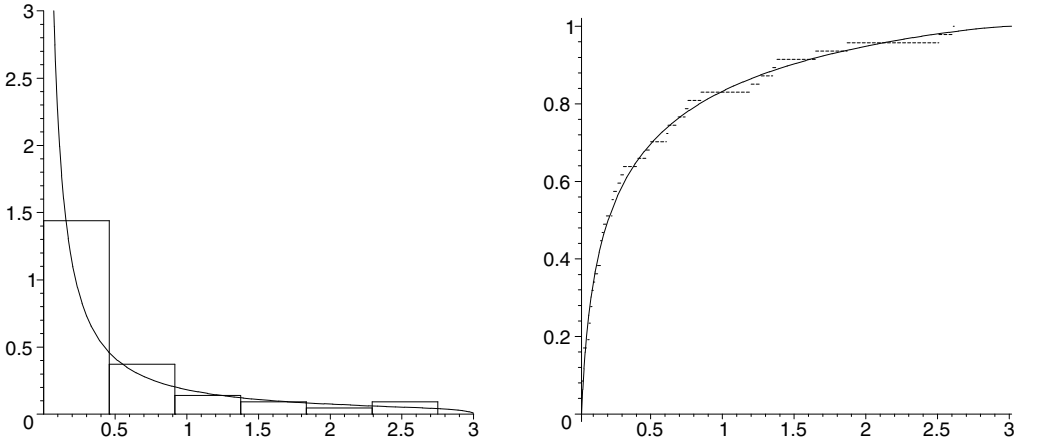
$$15, \quad 11, \quad 19, \quad 17, \quad 16, \quad 19, \quad 16, \quad 10$$

and expected frequencies

$$13.5622,\ 13.5623,\ 18.6726,\ 17.0401,\ 16.9167,\ 14.9533,\ 16.1821,\ 12.1107.$$

This gives a chi-square statistic and associated p-value, respectively, of

$$2.1570 \qquad \text{and} \qquad 0.5405.$$



(a) (b)

Figure 16.6–5. Histogram of rainfall X and its fitted p.d.f (a); the e.d.f. of X with the d.f. of the fitted distribution (b).

For Y and its fitted distribution we have $n = 123$ and $KS = 0.04855$, from which we obtain $\sqrt{n}KS = 0.538$ and a p-value of 0.93. As was the case for X, the chi-square and Kolmogorov-Smirnov p-values reinforce the graphic observation of a good fit.

16.6.4 Example: Rainfall Distribution

This example was initially considered in Section 3.5.5 and consists of X, rainfall data (in inches) in Rochester, N.Y., and Y, similar data for Syracuse, N.Y. For X,

$$\hat{\alpha}_1 = 0.4913, \quad \hat{\alpha}_2 = 0.4074, \quad \hat{\alpha}_3 = 1.8321, \quad \hat{\alpha}_4 = 5.7347,$$

indicating a fit from the S_B region. The fit that is obtained is

$$\text{JnsnSB}(0.01805,\ 2.9987,\ 0.4810,\ 1.3106)$$

which has support, $(0.01805,\ 3.0167)$ that **does not covers the range of** X on the left ($\min(X) = 0.01$), an indication that the fit is likely to be a poor one. This view is reinforced by the graphic comparison of a histogram of X with the p.d.f. of the fitted distribution (Figure 16.6–5 (a)) and the e.d.f. of X with the d.f. of X (Figure 16.6–5 (b)).

To perform a chi-square test, we use the intervals

$$(-\infty,\ 0.07), \quad [0.07,\ 0.1), \quad [0.1,\ 0.2), \quad [0.2,\ 0.45), \quad [0.45,\ 1.0), \quad [1.0,\ \infty)$$

and obtain observed frequencies of

$$9, \quad 6, \quad 9, \quad 7, \quad 8, \quad 8$$

and the expected frequencies

$$12.3993,\ \ 3.6670,\ \ 7.3017,\ 8.3525,\ \ 7.4113,\ \ 7.8681.$$

The chi-square statistic and its p-value are, respectively,

$$3.0791 \quad \text{and} \quad 0.07930.$$

For this fit the Kolmogorov-Smirnov statistic, $KS = 0.08091$ and we have a sample size of $n = 47$, giving us $\sqrt{n}KS = 0.555$ and (from Appendix J) a p-value of 0.92. There are inconsistencies throughout the fitting process for X: First, the support of the fit does not cover the data range, potentially a sufficient reason for rejecting this fit; second, the chi-square test indicates an acceptable but not a particularly good fit; and third, the Kolmogorov-Smirnov test suggests a very good fit.

For Y, the rainfall in Syracuse, N.Y.,

$$\hat{\alpha}_1 = 0.3906, \quad \hat{\alpha}_2 = 0.1533, \quad \hat{\alpha}_3 = 1.6164, \quad \hat{\alpha}_4 = 5.2245,$$

requiring a fit from the S_B region, which produces

$$\text{JnsnSB}(0.03005,\ 2.0740,\ 0.6380,\ 1.3953).$$

As was the case for X, the support of this distribution, $(0.03005, 2.1041)$, **does not cover the range of** Y on the left ($\min(Y) = 0.02$), leading us to expect a poor fit. Figure 16.6–6 (a) gives a comparison of a histogram of Y with the p.d.f. of its fit and Figure 16.6–6 (b) compares the e.d.f of Y with the d.f. of the fitted distribution (these reinforce our initial concern about the quality of the fit).

We now perform a chi-square test using the intervals

$$(-\infty,\ 0.08),\quad [0.08,\ 0.15),\quad [0.15,\ 0.22),\quad [0.22,\ 0.38),\quad [0.38,\ 0.73),\quad [0.73,\ \infty)$$

and obtain observed frequencies of

$$7,\quad 7,\quad 9,\quad 8,\quad 8,\quad 8$$

and the expected frequencies

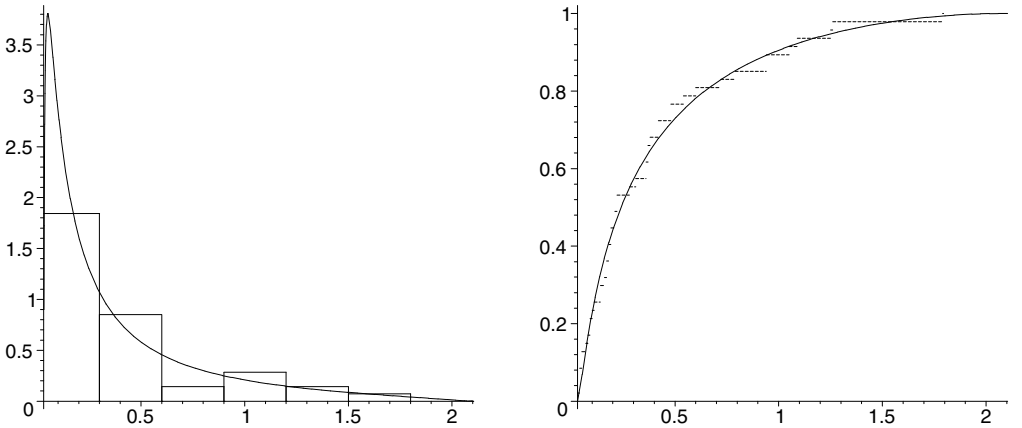
$$7.8459,\ \ 8.6092,\ \ 5.7616\ \ ,8.2045,\ \ 8.71637\ \ ,7.8626.$$

The chi-square statistic and its p-value are

$$2.2786 \quad \text{and} \quad 0.1312,$$

respectively.

For this fit the Kolmogorov-Smirnov statistic is given by $KS = 0.07327$ with $n = 47$. This makes $\sqrt{n}KS = 0.502$ and Appendix J yields a p-value of 0.96. With slight variations, the inconsistencies mentioned in connection with the fit for X are also applicable here.



(a) (b)

Figure 16.6–6. Histogram of rainfall Y and its fitted p.d.f (a); the e.d.f. of Y with the d.f. of the fitted distribution (b).

16.6.5 Example: Data Simulated from GBD(3, 5, 0, -0.5)

The data introduced in Section 4.5.1 was obtained through simulation form the GBD(3, 5, 0, −0.5) distribution, the generalized beta distribution with parameters 3, 5, 0, and −0.5. For this data

$$\hat{\alpha}_1 = 6.4975, \quad \hat{\alpha}_2 = 2.0426, \quad \hat{\alpha}_3 = -0.7560, \quad \hat{\alpha}_4 = 2.3536,$$

indicating a fit from the S_B region, which gives

$$\text{JnsnSB}(3.0394,\ 5.0756,\ 0.5190,\ -0.6316).$$

The support of this fit, which is (3.0394, 8.1149), covers the range of the data. Figure 16.6–7 (a) shows a histogram of the data with the p.d.f. of the fitted distribution and Figure 16.6–7 (b) gives the e.d.f. of the data with the d.f. of the Johnson system fit.

As done in Section 4.5.1, the classes

$$(-\infty,\ 4.5),\ \ [4.5,\ 5.5),\ \ [5.5,\ 6.5),\ \ [6.5,\ 7.0),\ \ [7.0,\ 7.5),\ \ [7.5,\ 7.9),\ \ [7.9,\ \infty)$$

are used to obtain the observed frequencies

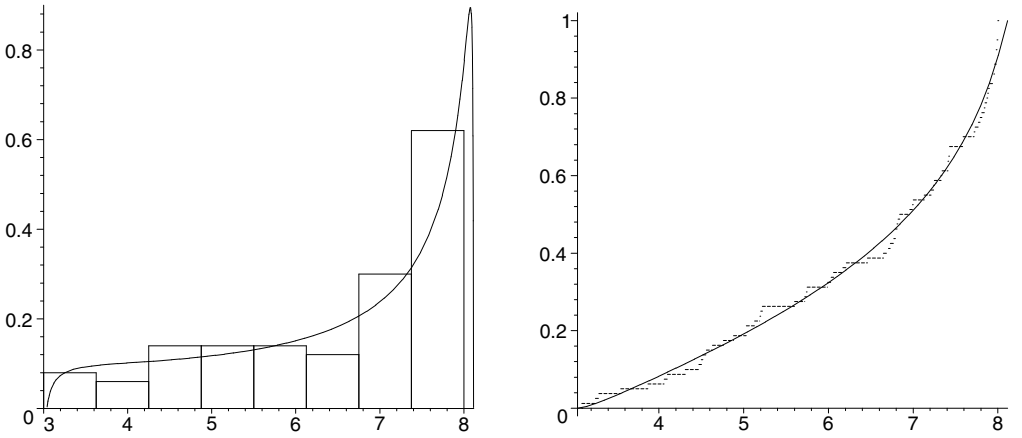
$$9, \quad 12, \quad 10, \quad 11, \quad 12, \quad 12, \quad 14.$$

For this fit, the expected frequencies are

$$10.8201,\ \ 9.4696,\ \ 12.2476,\ \ 8.3031,\ \ 11.5011,\ \ 14.7130,\ \ 12.9455,$$

yielding the chi-square statistic and its p-value of

$$2.8786 \quad \text{and} \quad 0.2371,$$



(a) (b)

Figure 16.6–7. Histogram of GBD$(3, 5, 0, -0.5)$ data and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

respectively.

For this data we have a sample size of $n = 80$ and for this fit we get a Kolmogorov-Smirnov statistic of $KS = 0.09301$. From $\sqrt{n}KS = 0.832$, using Appendix J, we get a p-value of 0.49 for the Kolmogorov-Smirnov test. The graphic depictions, and the chi-square and Kolmogorov-Smirnov p-values are consistent, indicating that we have a reasonably good fit.

16.6.6 Example: Data Simulated from GBD(2, 7, 1, 4)

The data for this example, introduced in Section 4.5.2, is simulated from GBD(2, 7, 1, 4) and has

$$\hat{\alpha}_1 = 4.1053, \quad \hat{\alpha}_2 = 1.2495, \quad \hat{\alpha}_3 = 0.6828, \quad \hat{\alpha}_4 = 3.3000.$$

From the S_B region of the Johnson system we obtain the fit

$$\text{JnsnSB}(1.4435,\ 9.9261,\ 1.6558,\ 1.7937)$$

whose support, $(1.4435,\ 11.3696)$ covers the range of the data. Figure 16.6–8 (a) gives a histogram of the data with the p.d.f. of the fitted S_B distribution and Figure 16.6–8 (b) shows the e.d.f. of the data with the d.f. of the S_B fit. These figures indicate that this is a reasonably good fit.

Using, as in Section 4.5.2, the classes

$$(-\infty,\ 3),\ \ [3,\ 3.5),\ \ [3.5,\ 4),\ \ [4,\ 4.5),\ \ [4.5,\ 5),\ \ [5,\ \infty)$$

we obtain the observed frequencies

$$6, \quad 14, \quad 11, \quad 11, \quad 7, \quad 11$$

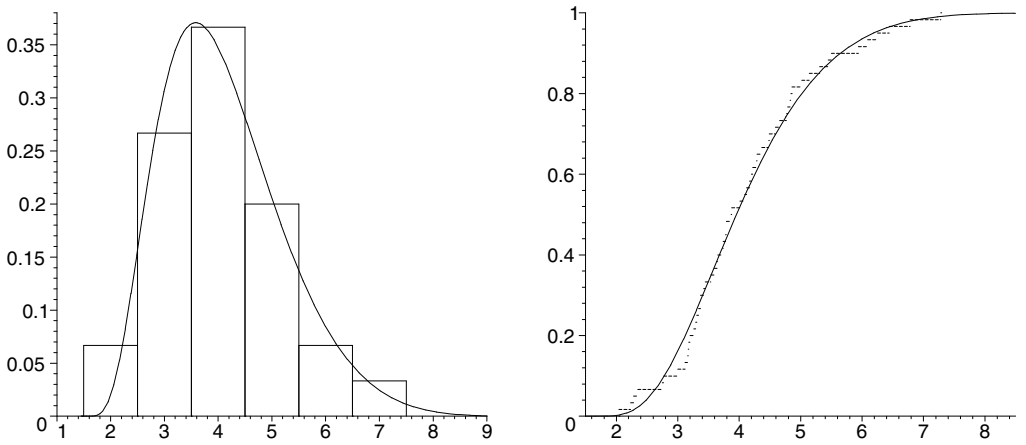


Figure 16.6–8. Histogram of GBD$(2, 7, 1, 4)$ data and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

and the expected frequencies

$$9.6417,\ 10.4098,\ 10.9219,\ 9.5054,\ 7.3041,\ 12.2169.$$

These, in turn produce the chi-square statistic and its p-value

$$2.9831 \quad \text{and} \quad 0.08414,$$

respectively.

For this fit the Kolmogorov-Smirnov statistic is $KS = 0.08247$ and the sample size is $n = 60$, producing $\sqrt{n}KS = 0.639$ and a p-value for the Kolmogorov-Smirnov test of 0.81. The chi-square p-value indicates a fit of modest quality, contrary to our earlier observation of a good fit from Figures 16.6–8 (a) and 16.6–8 (b) and from the results of the Kolmogorov-Smirnov test.

16.6.7 Example: Tree Stand Heights and Diameters

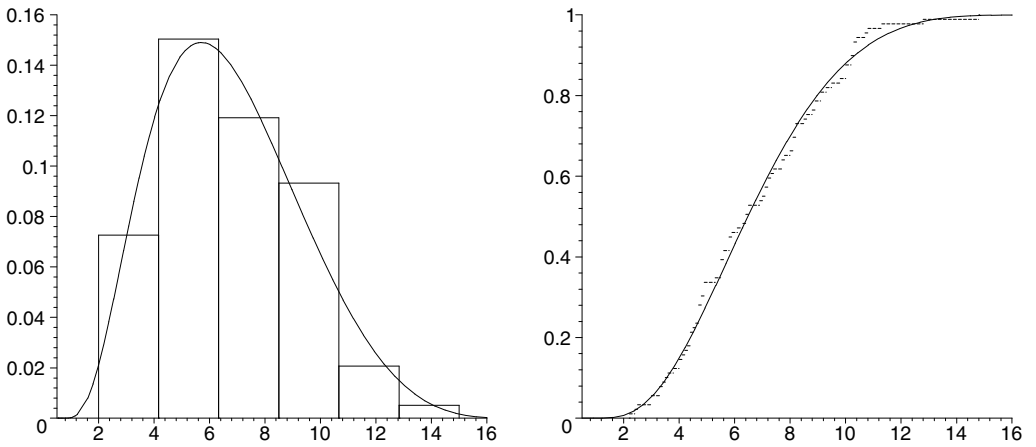
Data from a forestry study involving the height of trees in feet, H, and the diameters in inches at breast height, DBH, was introduced in Section 4.5.5 (Table 4.5–6). For the DBH data it was shown that

$$\hat{\alpha}_1 = 6.7404, \quad \hat{\alpha}_2 = 1.2495, \quad \hat{\alpha}_3 = 0.6828, \quad \hat{\alpha}_4 = 3.3000.$$

From the S_B region of the Johnson system we obtain the fit

$$\text{JnsnSB}(0.5195,\ 17.6684,\ 1.4276,\ 0.9651)$$

whose support, $(0.5195,\ 18.1879)$, covers the range of the data. Figure 16.6–9 (a) shows a histogram of DBH with the p.d.f. of the fitted distribution and Figure



(a) (b)

Figure 16.6–9. Histogram of DBH and its fitted p.d.f (a); the e.d.f. of DBH with the d.f. of the fitted distribution (b).

16.6–9 (b) gives the e.d.f. of DBH with the d.f. of the fit. The fit seems to be a reasonable one.

To perform a chi-square test, we use the same intervals

$$(-\infty,\ 3.75),\ \ [3.75,\ 4.5),\ \ [4.5,\ 5),\ \ [5,\ 6),\ \ [6,\ 7),$$
$$[7,\ 8),\ \ [8,\ 9),\ \ [9,\ 10.25),\ \ [10.25,\ \infty)$$

that were used in Section 4.5.5 to obtain the observed frequencies

$$10,\quad 9,\quad 11,\quad 11,\quad 7,\quad 10,\quad 12,\quad 10,\quad 9$$

and the expected frequencies

$$10.7251,\ 8.1753,\ 6.2193,\ 13.1637,\ 12.7656,\ 11.2402,\ 9.1355,\ 8.2656,\ 9.30976.$$

These lead to the chi-square statistic and its p-value

$$8.1760 \quad \text{and} \quad 0.08534,$$

respectively.

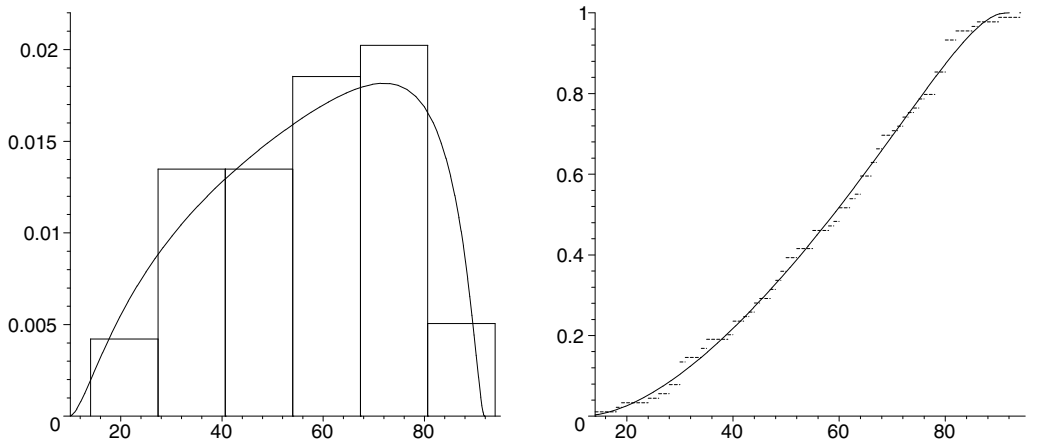
For this fit the Kolmogorov-Smirnov statistic is $KS = 0.06919$ and the sample size is $n = 89$. From $\sqrt{n}KS = 0.653$, we conclude (using Appendix J) that the p-value for this test is 0.79.

For the H (tree height) data, we have

$$\hat{\alpha}_1 = 57.1348, \quad \hat{\alpha}_2 = 365.3751, \quad \hat{\alpha}_3 = -0.2825, \quad \hat{\alpha}_4 = 2.1046.$$

Again, from the S_B region of the Johnson system we obtain the fit

$$\text{JnsnSB}(9.4989, 82.5291, 0.8350, -0.3339)$$



(a) (b)

Figure 16.6–10. Histogram of H and its fitted p.d.f (a); the e.d.f. of H with the d.f. of the fitted distribution (b).

that has support $(9.4989, 92.0280)$ that **does not cover the data range** $(\max(\mathrm{H}) = 94)$. Figure 16.6–10 (a) compares a histogram of H with the fitted p.d.f. and Figure 16.6–10 (b) compares the e.d.f of H with the d.f of this fitted S_B distribution.

For the chi-square test, the classes we choose are the same ones chosen in Section 4.5.5:

$$(-\infty,\ 30),\ \ [30,\ 37.5),\ \ [37.5,\ 47.5),\ \ [47.5,\ 55),\ \ [55,\ 62.5),$$
$$[62.5,\ 67.5),\ \ [67.5,\ 75),\ \ [75\ 80),\ \ [80,\ \infty).$$

These lead to the observed frequencies

$$7,\quad 10,\quad 11,\quad 9,\quad 11,\quad 11,\quad 9,\quad 8,\quad 13$$

and the expected frequencies

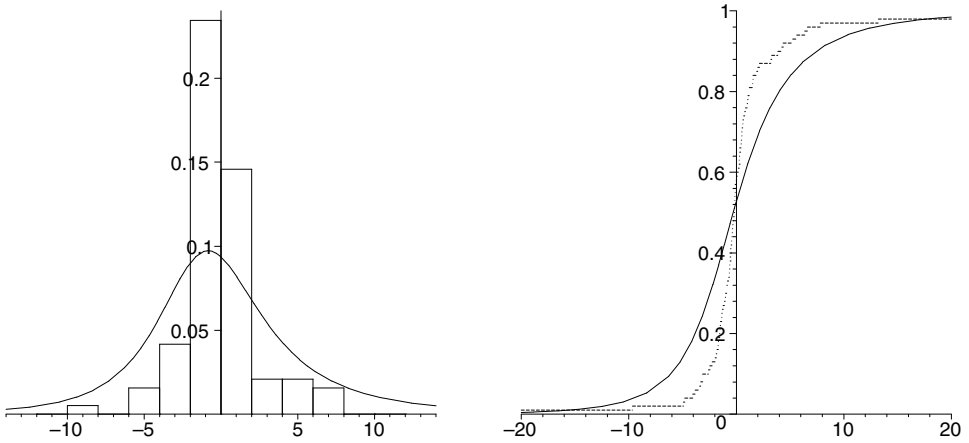
$$9.2677,\ 7.3416,\ 11.9167,\ 10.2529,\ 11.1872,\ 7.8652,\ 12.0817,\ 7.8218,\ 11.2650.$$

From these we get the chi-square statistic and its p-value

$$4.051 \quad \text{and} \quad 0.3991,$$

respectively.

For this fit we have a Kolmogorov-Smirnov statistic of $KS = 0.05916$, giving us $\sqrt{n}KS = 0.05916\sqrt{89} = 0.558$. From Appendix J we get a p-value of 0.915 for this test. Figures 16.6–10 (a) and 16.6–10 (b) and the two p-values associated with the chi-square and Kolmogorov-Smirnov tests all indicate a good fit.



(a) (b)

Figure 16.6–11. Histogram of Cauchy distribution data and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

16.6.8 Example: Data from the Cauchy Distribution

Section 5.6.1 used data simulated from the Cauchy distribution in order to get a percentile-based GLD fit. For that data,

$$\hat{\alpha}_1 = 0.3464, \quad \hat{\alpha}_2 = 49.4909, \quad \hat{\alpha}_3 = 1.8671, \quad \hat{\alpha}_4 = 31.3916,$$

which takes us to the S_U region of the Johnson system. (We note here that this is the first time, and as will become apparent the only time, that we encounter data with its (α_3, α_4) outside of the S_B region). From the S_U region we get the fit

$$\text{JnsnSU}(-1.4475,\ 4.2592,\ 1.0592,\ -0.2823)$$

with support $(-\infty, \infty)$. Figure 16.6–11 (a) gives a histogram of this data and the p.d.f. of the fitted distribution. Although the histogram covers values from -12 to 12, the data ranges from -37.9 to 48.4 and four data points are excluded from the histogram. Figure 16.6–11 (b) shows the e.d.f of the data with the d.f of this fitted S_U distribution, both figures indicate a poor fit.

For the chi-square test we need to make sure that all expected frequencies are at least 5. For this reason, we choose intervals that are different from those used in Section 5.6.1 and use the intervals

$$(-\infty,\ -6),\ [-6,\ -3.5),\ [-3.5,\ -2),\ [-2,\ -1),\ [-1,\ 0),$$
$$[0,\ 1),\ [1,\ 2),\ [2\ 3.5),\ [3.5,\ 6),\ [6,\ \infty)$$

to obtain the observed frequencies

$$2,\quad 4,\quad 7,\quad 16,\quad 29,\quad 18,\quad 10,\quad 3,\quad 5,\quad 6,$$

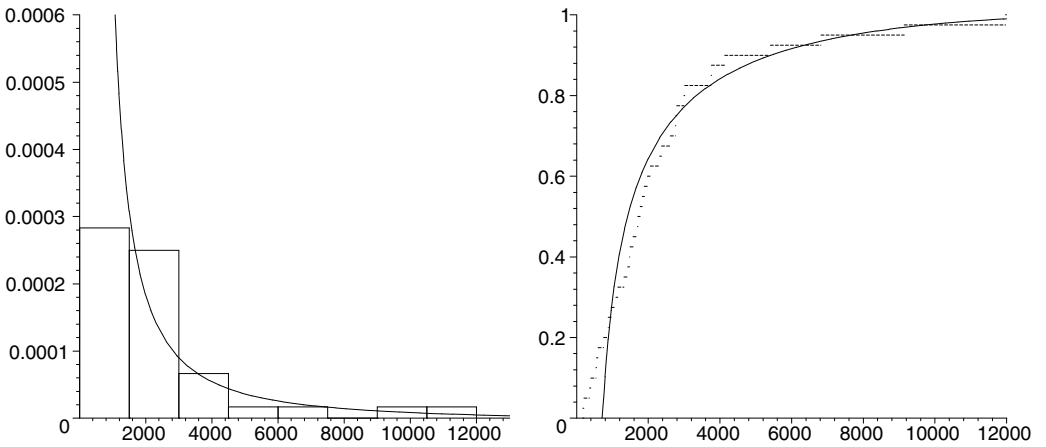


Figure 16.6–12. Histogram of radiation data and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

expected frequencies

10.2655, 11.6570, 11.8256, 9.4532, 9.6305, 8.8447, 7.5217, 8.6623, 9.0313, 13.1082,

and the chi-square statistic and its p-value

$$76.7939 \quad \text{and} \quad 0.0000,$$

respectively.

For this fit the Kolmogorov-Smirnov statistic is $KS = 0.2093$ and $n = 100$, making $\sqrt{n}KS = 2.093$ and giving us, through Appendix J, a p-value of 0.00 for this test. This gives us yet another reason to conclude that the fit is an extremely poor one.

16.6.9 Example: Radiation in Soil Samples

Radiation data in soil samples, discussed more fully in Section 5.6.2, has

$$\hat{\alpha}_1 = 2384.8422, \quad \hat{\alpha}_2 = 5.5198 \times 10^6, \quad \hat{\alpha}_3 = 2.3811, \quad \hat{\alpha}_4 = 9.1440,$$

with its ($\hat{\alpha}_3$, $\hat{\alpha}_4$) in the S_B region of the Johnson system. This leads us to the fit

$$\text{JnsnSB}(710.5480,\ 15357.2104,\ 0.5798,\ 1.7554)$$

with support (710.5480, 16067.7584) that covers the range of the data. Figure 16.6–12 (a) shows a histogram of the data with a superimposed p.d.f. of its fit and Figure 16.6–12 (b) gives the corresponding data e.d.f and the d.f. of the fitted distribution. Both figures indicate that the S_B fit obtained is not a good one.

To make sure that, for a chi-square test, all expected frequencies are at least 5, we choose intervals that are different from those used in Section 5.6.2 and use the intervals

$$(-\infty,\ 850),\ \ [850\ 1050),\ \ [1050,\ 1400),\ \ [1400,\ 2100),\ \ [2100,\ 3300),\ \ [3300,\ \infty)$$

to obtain the observed frequencies

$$8,\quad 3,\quad 3,\quad 11,\quad 8,\quad 7.$$

The expected frequencies from these intervals and the S_B fit are

$$6.6888,\ 6.4846,\ 6.5511,\ 6.74754, 5.4017,\ 8.1262$$

leading to the chi-square statistic and its p-value

$$8.1403 \quad \text{and} \quad 0.004329,$$

respectively.

For this fit the Kolmogorov-Smirnov statistic is $KS = 0.1750$ and the sample size is $n = 40$. This makes $\sqrt{n}KS = 0.1750\sqrt{40} = 1.107$ and gives us a p-value of 0.17. This is somewhat of an endorsement, albeit a weak one, of this fit; the p-value of the chi-square statistic and the visual impression created by Figures 16.6–12 (a) and 16.6–12 (b) clearly indicate that the fit is of poor quality.

16.6.10 Example: Velocities within Galaxies

The main reason for the consideration of the data on velocities within galaxies in Section 5.6.3, was to illustrate that it is difficult to find a good fit for certain datasets. Here we attempt to find a Johnson system fit to the data of Section 5.6.3. The $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ in this case are given by

$$\hat{\alpha}_1 = 21456.5882,\ \ \hat{\alpha}_2 = 3.0384 \times 10^6,\ \ \hat{\alpha}_3 = -0.2639,\ \ \hat{\alpha}_4 = 1.5473.$$

The $(\hat{\alpha}_3, \hat{\alpha}_4)$ of this data is in the S_B region of the Johnson system, leading us to the fit

$$\text{JnsnSB}(18636.9305,\ 4981.5805,\ 0.3738, -0.1984)$$

with support $(18636.9305, 23618.5110)$ that **does not cover the data range** (the minimum of the data is 18499). Figure 16.6–13 (a) depicts a histogram of the data with the superimposed p.d.f. of the fitted distribution and Figure 16.6–13 (b) shows the e.d.f of the data with the d.f. of the S_B fit. It is clear from these figures that this is a rather poor fit.

To perform a chi-square test, the classes

$$(-\infty,\ 19000),\ \ [19000\ 20000),\ \ [20000,\ 21000),\ \ [21000,\ 22000),$$

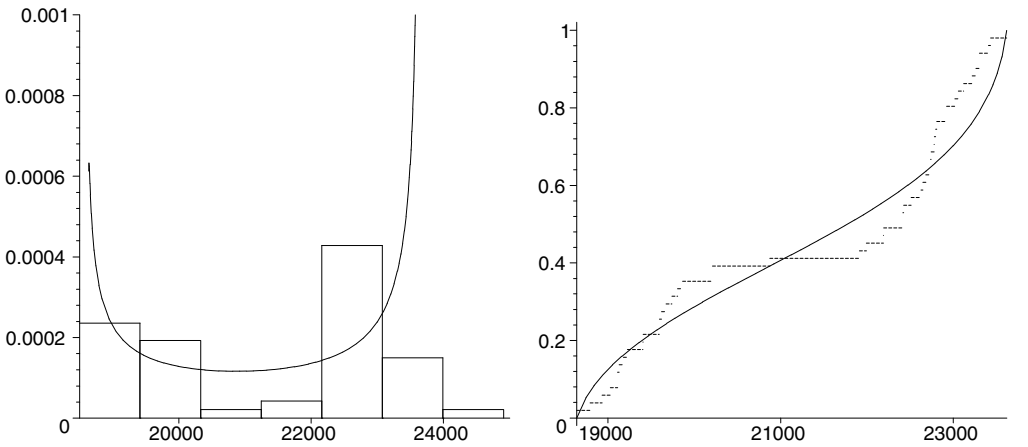


Figure 16.6–13. Histogram of velocities within galaxies and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

$$[22000,\ 23000),\quad [23000,\ \infty)$$

are used and the observed frequencies

$$3,\quad 15,\quad 3,\quad 2,\quad 18,\quad 10,$$

and expected frequencies

$$6.3899,\ 8.2276,\ 6.1101,\ 6.2959,\ 8.8070,\ 15.1695$$

are obtained. These lead to the chi-square statistic and its p-value of

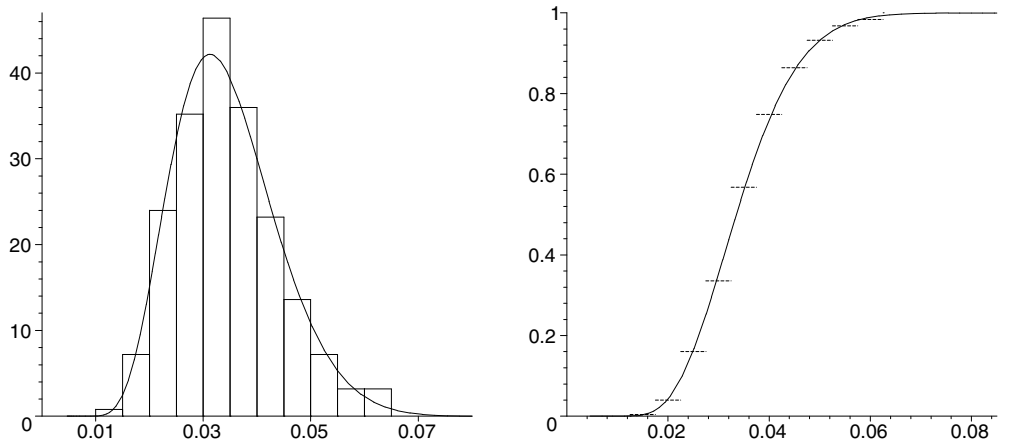
$$23.2450 \quad \text{and} \quad 1.4262 \times 10^{-6},$$

respectively.

For this data, the sample size is $n = 51$ and for this fit the Kolmogorov-Smirnov statistic is $KS = 0.1506$. From $\sqrt{n}KS = 0.1506\sqrt{51} = 1.075$, we get a p-value of 0.20 for this test. Although this lends some support for this fit, the p-value of the chi-square test, the visual impression of Figures 16.6–13 (a) and 16.6–13 (b), and the fact that the support of the fit does not cover the data range, indicate that the fit is a poor one.

16.7 Fitting Data Given by a Histogram

Data on coefficient of friction frequencies, taken from a histogram, was presented in tabular form (Table 3.6–1) in Section 3.6. For purposes of fitting a GLD distribution to this data, it was assumed there that all members within a class



(a) (b)

Figure 16.7–1. Histogram of coefficients of friction (assuming data is located at interval centers) and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

interval were located at the center of the interval. With this same assumption, we have

$$\hat{\alpha}_1 = 0.03448, \quad \hat{\alpha}_2 = 9.2380 \times 10^{-5}, \quad \hat{\alpha}_3 = 0.5475, \quad \hat{\alpha}_4 = 3.2130$$

and the Johnson fit to this data, from the S_B region, is

$$\text{JnsnSB}(0.004617,\ 0.1095,\ 2.1926,\ 2.2512).$$

The data range is covered by the support of this fit, (0.004617, 0.1141). A graphic appraisal of the fit is given in Figures 16.7–1 (a), comparing a histogram of the data with the p.d.f. of the fitted distribution, and 16.7–1 (b), showing the e.d.f. of the data and the d.f. of the fit. Figure 16.7–1 (a) indicates a reasonably good fit, but the concentration of data at interval midpoints creates a visible distortion in Figure 16.7–1 (b).

To perform a chi-square test, we choose the intervals

$$(-\infty,\ 0.02),\ \ [0.02,\ 0.025),\ \ [0.025,\ 0.03),\ \ [0.03,\ 0.035),\ \ [0.035,\ 0.04),$$
$$[0.04,\ 0.045),\ \ [0.045,\ 0.05),\ \ [0.05,\ 0.055),\ \ [0.055,\ \infty)$$

which have observed frequencies

$$10,\quad 30,\quad 44,\quad 58,\quad 45,\quad 29,\quad 17,\quad 9,\quad 8$$

and expected frequencies

10.6607, 29.9725, 47.6803, 51.7870, 43.7390, 30.6943, 18.5195, 9.7604, 7.1863.

For this test, the chi-square statistic and its p-value are, respectively,

$$1.4764 \quad \text{and} \quad 0.8308.$$

The Kolmogorov-Smirnov statistic for this fit is $KS = 0.1224$ with $n = 250$, leading to $KS\sqrt{n} = 1.94$ and the p-value of 0.00. It is clear that this glaring disparity between the two p-values is due to the initial assumption that every data point is located at the center of its interval.

In Section 5.7, when a percentile-based GLD fit was sought to this data a different assumption, one more appropriate for percentile computations, was made: the data in an interval was assumed to be uniformly distributed throughout the interval. From Table 3.6–1, using this principle, we would set the first data point (the only one in its interval) to 0.0125; for the 9 points in the next interval (0.015, 0.020), we obtain 11 points by dividing the interval into 10 equal parts (each of length 0.0005) and assign the second through the 10th points as values of the next 9 observations (these will be $0.0155, 0.0160, \ldots, 0.0195$). This process is then repeated for each interval. It is not surprising that when the data is recast in this way, the recomputed $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ will be close to what was obtained earlier. The new $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$,

$$\hat{\alpha}_1 = 0.03448, \quad \hat{\alpha}_2 = 9.4301 \times 10^{-5}, \quad \hat{\alpha}_3 = 0.5289, \quad \hat{\alpha}_4 = 3.1886,$$

give the slightly different S_B fit

$$\text{JnsnSB}(0.003550,\ 0.1118,\ 2.2366,\ 2.2462)$$

with support (0.003550, 0.1154) that covers the range of the data.

Figure 16.7–2 (a) shows a histogram of the recast data with the p.d.f. of its new S_B fit and Figure 16.7–2 (b) gives the e.d.f. of the recast data with the d.f. of its fit. We do not expect (and we do not see) a significant difference between Figures 16.7–1 (a) and 16.7–2 (a). But, as expected there is a dramatic difference between Figures 16.7–1 (b) and 16.7–2 (b).

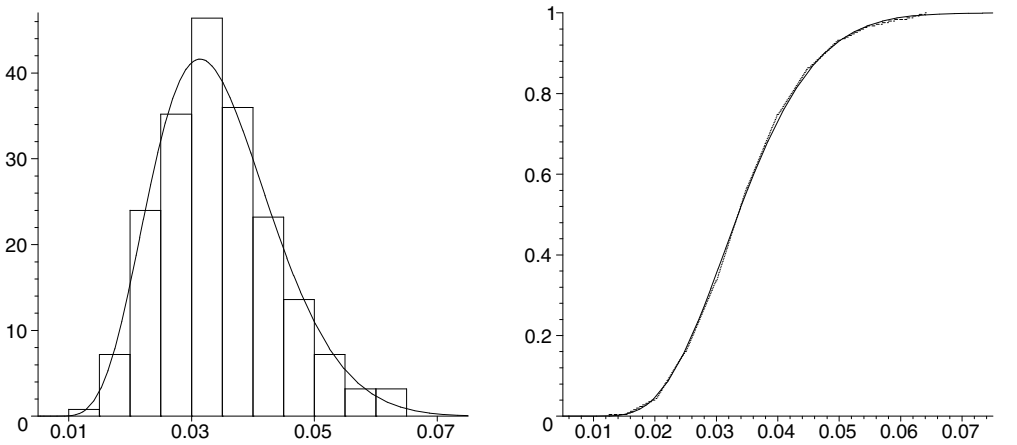
With the class intervals used above, we get the same observed frequencies but the new expected frequencies

11.5153, 29.9740, 47.0317, 51.1792, 43.5237, 30.7829, 18.7081, 9.92141, 7.36379,

that lead to the chi-square statistic and its p-value, respectively, of

$$1.7537 \quad \text{and} \quad 0.7809.$$

These indicate that, as measured by the chi-square p-value, this fit is almost as good as the previous one. The dramatic change comes in the recalculated Kolmogorov-Smirnov statistic because we now get $KS = 0.02158$ and $KS\sqrt{n} = 0.02158\sqrt{250} = 0.341$, with a p-value in excess of 0.99.



(a) (b)

Figure 16.7–2. Histogram of coefficients of friction (assuming data is uniform over intervals) and its fitted p.d.f (a); the e.d.f. of the data with the d.f. of the fitted distribution (b).

References for Chapter 16

Bukac, J. (1972). "Fitting S_B curves using symmetrical percentile points," *Biometrika*, 59, 688–690.

Dudewicz, E.J., Zhang, C.X., and Karian, Z.A. (2004). "The completeness and uniqueness of johnson's system in skewness-kurtosis space," *Communications in Statistics: Theory and Methods*, 33, 2097–2116.

Edgeworth, F.Y. (1898). "On the representation of statistics by mathematical formulae," *J. Royal Statist. Soc.*, 61, 670–700.

Johnson, N.L. (1949a). "Systems of frequency curves generated by methods of translation," *Biometrika*, 36, 149–176.

Johnson, N.L. (1949b). "Bivariate distributions based on simple translation systems," *Biometrika*, 36, 297–304.

Johnson, N. L. (1965). "Tables to facilitate fitting S_U frequency curves," *Biometrika*, 52, 547–571.

Johnson, N.L. and Kitchen, J.O. (1971a). "Some notes on tables to facilitate fitting S_B curves," *Biometrika*, 58, 223–226.

Johnson, N.L. and Kitchen, J.O. (1971b). "Tables to facilitate fitting S_B curves II: Both terminals known," *Biometrika*, 58, 657–668.

Leslie, D.C.M. (1959). "Determination of parameters in the Johnson system of probability distributions," *Biometrika*, 46, 229–243.

Mage, David T. (1980). "An explicit solution for S_B parameters using four percentile points," *Technometrics*, 22, 247–251.

Pearson, E.S. and Hartley, H.O. (Eds.) (1972). *Biometrika Tables for Statisticians*, vol. 2, 288–303, Cambridge University Press, London.

Swain, J.J., Venkatraman, S., and Wilson, J.R. (1988). "Least-squares estimation of distribution functions in Johnson's translation system," *Statist. Comput. Simul.*, 29, 271–297.

Chapter 17

Fitting Distributions and Data with the Kappa Distribution through L-Moments and Percentiles

The kappa distribution, with three positive parameters a, b, and θ, was defined initially by Mielke (1973), through its distribution function

$$G(x) = \left(\frac{x}{b}\right)^{\theta} \left(a + \left(\frac{x}{b}\right)^{a\theta}\right)^{-1/a}, \quad x \geq 0 \tag{17.0.1}$$

Hosking (1994) generalized this formulation to a four-parameter family of distributions and described how this family could be fitted to data through the use of L-moments.

In this chapter we consider Hosking's use of L-moments as well as an approach based on the use of percentiles. Section 17.1 describes the four-parameter kappa distribution family and some of its attributes. Following the development in Hosking (1994), Section 17.2 gives a method based on L-moments for fitting kappa distributions to data. The definitions, properties, as well as the advantages of using L-moments, are described in Chapter 6 and Section 17.2 of this Chapter will rely on some of the developments in Sections 6.1 and 6.2 of Chapter 6. Section 17.3 devises a method for fitting kappa distributions through percentiles. This method parallels the percentile approach that was used in Karian and Dudewicz (2000), also described in Chapter 5 of this *Handbook*. Section 17.3 will use the notation and many of the definitions of Section 5.1 of Chapter 5. Sections 17.4 uses both fitting methods to approximate some well-known distributions with kappa family members and Section 17.5 applies the L-moment and percentile-based approaches to fitting kappa distributions to various datasets.

17.1 The Kappa Distribution

The distribution function of the four-parameter generalization of the kappa distribution is given by

$$F(x) = \begin{cases} \left(1 - h\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k}\right)^{1/h}, & \text{if } k \neq 0, h \neq 0, \\ \exp\left(-\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k}\right), & \text{if } k \neq 0, h = 0, \\ \left(1 - h\exp\left[-\frac{x-\xi}{\alpha}\right]\right)^{1/h}, & \text{if } k = 0, h \neq 0, \\ \exp\left(-\exp\left[-\frac{x-\xi}{\alpha}\right]\right)^{1/h}, & \text{if } k = 0, h = 0. \end{cases} \tag{17.1.1}$$

The shape parameters, h and k, can range over all real values, as can the location parameter, ξ. But the scale parameter, α, must be restricted to positive values for $F(x)$ to define a valid distribution function.

Setting

$$\xi = b, \quad \alpha = \frac{b}{a\theta}, \quad k = -\frac{1}{a\theta}, \quad h = -a \tag{17.1.2}$$

in (17.1.1) converts it to (17.0.1), verifying that, indeed, (17.1.1) is a generalization of (17.0.1).

When $h \neq 0$ and $k \neq 0$,

$$F(x) = \left(1 - h\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k}\right)^{1/h} \tag{17.1.3}$$

and the other three cases where $h = 0$, or $k = 0$, or $h = 0$ and $k = 0$, can be obtained from 17.1.3 by taking the appropriate limits as $h \to 0$, $k \to 0$ or both h and k tend to 0. This makes it convenient to use 17.1.3 as the kappa distribution function in all cases, keeping in mind that the proper limits need to be applied for the situations when h, k, or both are 0.

To obtain the kappa density function, $f(x)$, we differentiate $F(x)$ and get

$$f(x) = \frac{1}{h}\left\{1 - h\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k}\right\}^{1/h-1}\left\{-\frac{h}{k}\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k-1}\left(\frac{-k}{\alpha}\right)\right\}. \tag{17.1.4}$$

Simplifying and noting that the expression enclosed in the first set of braces is $F(x)^{1-h}$, $f(x)$ can be written as

$$f(x) = \frac{1}{\alpha}\left[1 - k\frac{x-\xi}{\alpha}\right]^{1/k-1} F(x)^{1-h}. \tag{17.1.5}$$

Setting $y = F(x)$ and solving for x produces $Q(y)$, the kappa quantile function

$$Q(y) = \xi + \frac{\alpha}{k}\left(1 - \left(\frac{1 - y^h}{h}\right)^k\right). \tag{17.1.6}$$

The left and right endpoints of the support for the kappa distribution can be derived by considering

$$\lim_{y \to 0^+} Q(y) \quad \text{and} \quad \lim_{y \to 1^-} Q(y),$$

respectively. It is easy to see that for the case $h > 0,\ k > 0$, these limits yield $\xi + \alpha(1 - h^{-k})/k$ and $\xi + \alpha/k$, respectively. When $h > 0,\ k = 0$, we see from (17.1.1) that

$$F(x) = \left(1 - h \exp\left[-\frac{x - \xi}{\alpha}\right]\right)^{1/h}.$$

This makes $Q(y) = \xi - \alpha \ln((1 - y^h)/h)$, whose limits as $y \to 0^+$ and $y \to 1^-$ are $\xi + \alpha \ln h$ and ∞, respectively. Using similar arguments in the remaining cases, we get the following support intervals for the various h and k combinations summarized in Table 17.1–1 below.

Table 17.1–1. Kappa distribution support intervals.

$[\ \xi + \alpha(1 - h^{-k})/k,\ \ \xi + \alpha/k\ ]$	if $h > 0,\ k > 0$
$[\ \xi + \alpha \ln h,\ \ \infty\ )$	if $h > 0,\ k = 0$
$[\ \xi + \alpha(1 - h^{-k})/k,\ \ \infty\ )$	if $h > 0,\ k < 0$
$(\ -\infty,\ \ \xi + \alpha/k\ ]$	if $h \le 0,\ k > 0$
$(\ -\infty,\ \ \infty\ )$	if $h \le 0,\ k = 0$
$[\ \xi + \alpha/k,\ \ \infty\ )$	if $h \le 0,\ k < 0.$

We can also verify that $f(x)$ is the density of a valid distribution by first noting that $f(x) \ge 0$ and then observing that

$$\lim_{x \to a^+} F(x) = 0 \quad \text{and} \quad \lim_{x \to b^-} F(x) = 1,$$

where a and b are the left and right endpoints of the support of the distribution.

Although not quite as flexible as the GLD, the kappa density functions can assume a variety of shapes. The number and nature of the extremas of the kappa distribution are described in Figure 17.1–2. A kappa density

- has no minima and a unique maximum if $h < 0$ and $1/h < k < 1$ or if $0 \le h < 1$ and $k < 1$ (region bounded by heavy lines in Figure 17.1–2),
- has no maxima and a unique minimum if $h > 1$ and $k > 1$ (region bounded by dashed lines in Figure 17.1–2),

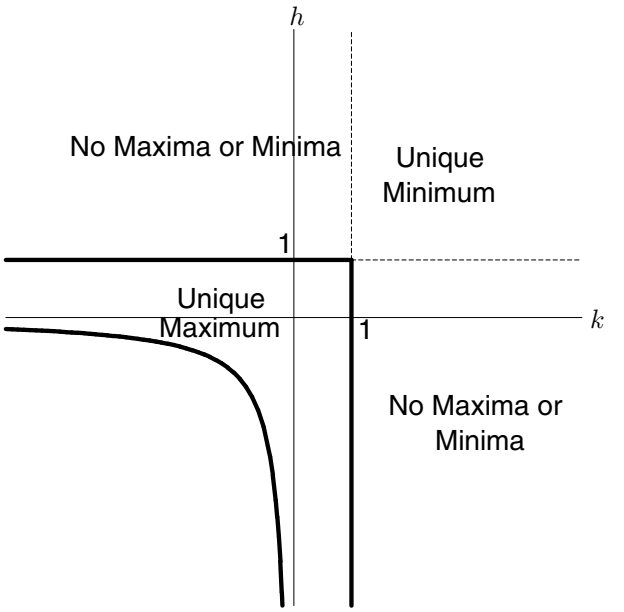


Figure 17.1–2. Shapes of the kappa density for various $(h,\ k)$ combinations

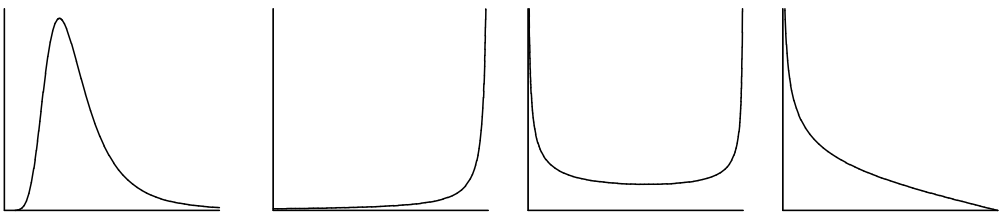


Figure 17.1–3. Shapes of kappa density functions. (a) $(h, k) = (-.15, -.23)$; (b) $(h, k) = (-1, 2)$; (c) $(h, k) = (2, 2)$; (d) $(h, k) = (1.5, .5)$.

- either decreases or increases for all admissible x in the regions $h > 1,\ k < 1$ and $h < 1,\ k > 1$.

The four possible shapes of the kappa density function are illustrated in Figure 17.1–3 (a) through (d) for $(h, k) = (-.15, -.23)$, $(h, k) = (-1, 2)$, $(h, k) = (2, 2)$, $(h, k) = (1.5, .5)$, respectively.

One of the objectives of Hosking (1994) was to devise a distribution that could include a number of well-known distributions as special cases. Table 17.1–4 summarizes the distributions that the kappa family can represent as special cases.

Table 17.1–4. Special cases of the kappa distribution

Distribution	h	k
Generalized Pareto	1	*
Generalized extreme value	0	*
Generalized logistic	−1	0
Exponential	1	0
Largest extreme value	0	0
Logistic	−1	0
Uniform	1	1

* The value of k in these cases depends on the particular member of the family (see Sections 17.4.16 and 17.4.13, respectively.

17.2 Estimation of Kappa Parameters via L-Moments

L-moments were introduced in Chapter 6, where their use in fitting GLD distributions was considered. In this section we establish a method for using L-moments to fit kappa distributions. Much of the content of this section relies on development of Hosking (1994). With the definitions and notation that were used in Section 6.1, we know from (17.1.1) that for $j = 0, 1, \ldots$,

$$\begin{aligned}
\beta_j &= E\left[X(F(X))^j\right] = \int_0^1 Q(y)y^j\,dy \\
&= \int_0^1 \left(\xi + \frac{\alpha}{k} - \frac{\alpha}{k}\left(\frac{1-y^h}{h}\right)^k\right) y^j\,dy \\
&= \int_0^1 \left(\xi + \frac{\alpha}{k}\right) y^j\,dy - \frac{\alpha}{k}\int_0^1 \left(\frac{1-y^h}{h}\right)^k y^j\,dy \\
&= \left(\xi + \frac{\alpha}{k}\right)\left(\frac{1}{j+1}\right) - \frac{\alpha}{k}\int_0^1 \left(\frac{1-y^h}{h}\right)^k y^j\,dy. \qquad (17.2.1)
\end{aligned}$$

When $h > 0$, the remaining integral in (17.2.1) can be evaluated through the substitution $u = y^h$ to yield

$$\begin{aligned}
\int_0^1 \left(\frac{1-y^h}{h}\right)^k y^j\,dy &= \int_0^1 \left(\frac{1-u}{h}\right)^k u^{j/h}\left(\frac{1}{h}\right)u^{1/h-1}\,du \\
&= \left(\frac{1}{h^{k+1}}\right)\int_0^1 (1-u)^k u^{(j+1)/h-1}\,du \\
&= \left(\frac{1}{h^{k+1}}\right)\beta(k+1,\,(j+1)/h) \\
&= \left(\frac{1}{h^{k+1}}\right)\frac{\Gamma(k+1)\Gamma\left((j+1)/h\right)}{\Gamma\left((j+1)/h+k+1\right)}, \qquad (17.2.2)
\end{aligned}$$

where β and Γ represent, respectively, the beta and gamma functions (see (3.1.6), (4.1.2), and Theorem 4.1.5). From (17.2.1) and (17.2.2) we now obtain

$$(j+1)\beta_j = \xi + \frac{\alpha}{k}\left(1 - \frac{(j+1)\Gamma(k+1)\Gamma\left((j+1)/h\right)}{h^{k+1}\Gamma\left((j+1)/h+k+1\right)}\right), \quad j = 0, 1, \ldots . \tag{17.2.3}$$

The case of $h = 0$ is a good bit simpler and the one for $h < 0$ is slightly more complicated, requiring the substitution $u = y^{-h} - 1$ for its integration. The following gives $(j+1)\beta_j$ for all cases.

$$(j+1)\beta_j = \begin{cases} \xi + \frac{\alpha}{k}\left(1 - \frac{(j+1)\Gamma(k+1)\Gamma((j+1)/h)}{h^{k+1}\Gamma((j+1)/h+k+1)}\right), & \text{if } h > 0,\ k > -1 \\ \xi + \frac{\alpha}{k}\left(1 - \frac{\Gamma(k+1)}{(j+1)^k}\right), & \text{if } h = 0,\ k > -1 \\ \xi + \frac{\alpha}{k}\left(1 - \frac{(j+1)\Gamma(k+1)\Gamma(-k-(j+1)/h)}{(-h)^{k+1}\Gamma(1-(j+1)/h)}\right), & \text{if } h < 0,\ -1 < k < -1/h. \end{cases} \tag{17.2.4}$$

Using (6.1.5) through (6.1.8) it is now possible to get expressions for the first four L-moments, $\Lambda_1, \ldots, \Lambda_4$ of the kappa distribution and subsequently for $\tau_3 = \Lambda_3/\Lambda_2$ and $\tau_4 = \Lambda_4/\Lambda_2$. As was done in Chapter 6, we use the notation ℓ_i, $\hat{\tau}_3$, $\hat{\tau}_4$ to designate the empirical counterparts of Λ_i, τ_3, and τ_4, respectively (for definitions of ℓ_i, $\hat{\tau}_3$, $\hat{\tau}_4$ see (6.3.1) through (6.3.3)). It is clear that τ_3 and τ_4 are independent of the location and scale parameters ξ and α. This provides an opportunity to solve the system of equations $\{\tau_3 = \hat{\tau}_3,\ \tau_4 = \hat{\tau}_4\}$ for h and k and eventually obtain α and ξ from the equations $\Lambda_2 = \ell_2$ and $\Lambda_1 = \ell_1$, respectively.

Hosking (1994) indicates that constraining h and k by

$$\{h < 0 \text{ and } hk > -1\}, \quad h > -1, \quad k > -1, \quad k + 0.725h > -1 \tag{17.2.5}$$

guarantees the existence of the first four kappa L-moments and the uniqueness of h and k for these L-moments. We also know, from Theorem 6.2.5, that the τ_3 and τ_4 of any nondegenerate random variable with finite mean must have

$$|\tau_3| < 1, \quad |\tau_4| < 1, \quad \frac{5}{4}\tau_3^2 - \frac{1}{4} \le \tau_4. \tag{17.2.6}$$

Figure 17.2–1 shows the portion of the admissible (τ_3, τ_4)-space that is covered when h and k range over the region restricted by (17.2.5). The thick curves indicate the boundary of the (τ_3, τ_4)-space given by (17.2.6). The curves within the boundary are associated (from top to bottom) with

$$h = -0.99, -0.7, -0.4, -0.2, -0.03, 0.05, 0.3, 0.7, 1.0, 1.5, 2.2, 3.0, 4.0, 5.5, 7.0$$

where k is restricted by (17.2.5) for the given h.

A comparison of Figure 17.2–1 with Figure 17.2–2 shows that the L-moments of the GLD and kappa complement each other very well in the sense that the

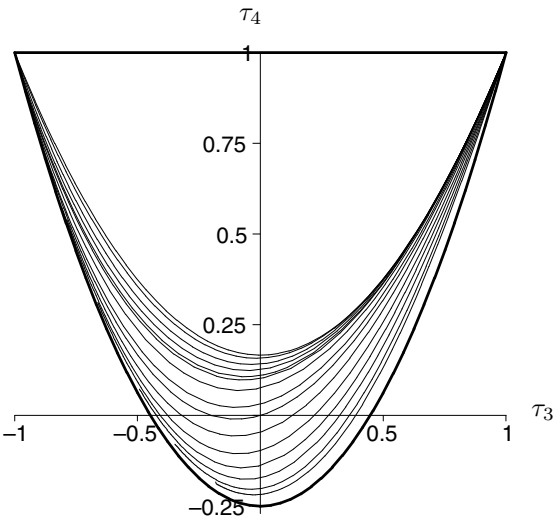


Figure 17.2–1. Portion of the (τ_3, τ_4)-space covered by the kappa distributions.

portion of (τ_3, τ_4)-space that is not covered by one is covered by the other. As shown in Figure 17.2–2, the upper portion of (τ_3, τ_4)-space is covered by the GLD and the kappa distribution does not extend to this area, there is a middle region where the (τ_3, τ_4) of the GLD and kappa distributions overlap, and a lower region which is covered only by the kappa distribution. It will be seen in Sections 17.4 and 17.5 that in many, but not all, situations (τ_3, τ_4) is in the middle region of Figure 17.2–2 where both the GLD and kappa distributions can be used.

The restrictions on h and k given in (17.2.5) are necessary to make sure that the arguments of the gamma function are positive. Beyond these restrictions, however, there are pragmatic computational problems when the argument of a gamma function gets too large or too small, since $\Gamma(x) \to \infty$ when either $x \to \infty$ or $x \to 0$. Most programming environments that use 64 or fewer bits for floating point computations are not able to evaluate $\Gamma(x)$ for $x > 170$. This indicates that for computational purposes we cannot allow h to get too small (in this case, smaller than $1/171$). Similar difficulties arise when k is close to -1 and we need to evaluate $\Gamma(k+1)$. Furthermore, more stringent restrictions may be required when both conditions prevail, i.e., we have a small h and k is close to -1.

To get a fit for a specified set of $\ell_1, \ell_2, \ell_3, \ell_4$, suppose we have already solved $\{\tau_3 = \hat{\tau}_3, \tau_4 = \hat{\tau}_4\}$ for h and k. Now to obtain α, we use $\Lambda_2 = 2\beta_1 - \beta_0$ (see (6.1.6)) and determine that $\Lambda_2 = A\alpha$ where

$$A = \begin{cases} \frac{\Gamma(k+1)}{kh^{k+1}} \left(\frac{\Gamma(1/h)}{\Gamma(1+k+1/h)} - \frac{2\Gamma(2/h)}{\Gamma(1+k+2/h)} \right), & \text{if } h > 0,\ k > -1 \\ \left(\frac{1}{k}\right) \Gamma(k+1) \left(1 - \frac{1}{2^k}\right), & \text{if } h = 0,\ k > -1 \\ \frac{\Gamma(k+1)}{k(-h)^{k+1}} \left(\frac{\Gamma(k-1/h)}{\Gamma(1-1/h)} - \frac{2\Gamma(k-2/h)}{\Gamma(1-2/h)} \right) & \text{if } h < 0,\ -1 < k < -1/h. \end{cases} \tag{17.2.7}$$

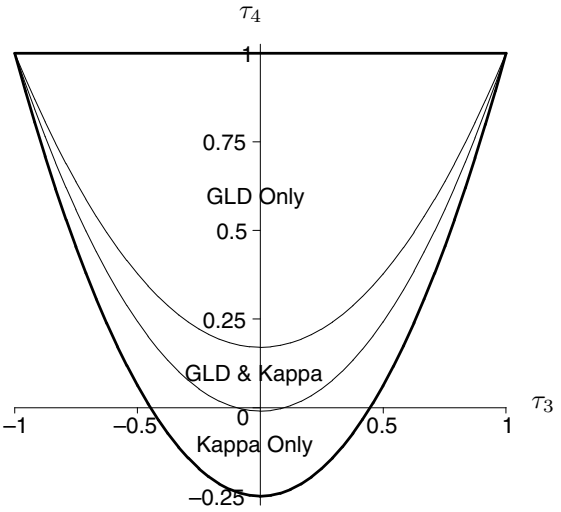


Figure 17.2–2. Portions of the (τ_3, τ_4)-space covered by the GLD and kappa distributions.

Thus, setting $\ell_2 = A\alpha$, gives us α. Next, we consider $\Lambda_1 = \beta_0$ (see (6.1.5)) and obtain $\xi = \hat{\Lambda}_1 - B\alpha$ where

$$B = \begin{cases} \frac{1}{k} - \frac{\Gamma(k+1)\Gamma(1/h)}{kh^{k+1}\Gamma(k+1+1/h)}, & \text{if } h > 0,\ k > -1 \\ \left(\frac{1}{k}\right)(1 - \Gamma(k+1)), & \text{if } h = 0,\ k > -1 \\ \frac{1}{k} - \frac{\Gamma(k+1)\Gamma(-k-1/h)}{k(-h)^{k+1}\Gamma(1-1/h)} & \text{if } h < 0,\ -1 < k < -1/h. \end{cases} \tag{17.2.8}$$

By far the most difficult part of determining ξ, α, h, k from ℓ_1, ℓ_2, ℓ_3, ℓ_4, is the determination of h and k. The software that accompanies this book has three programs (written in **R**) that provide values for ξ, α, h, and k. All programs attempt to minimize

$$\max\left(|\tau_3 - \hat{\tau}_3|,\ |\tau_4 - \hat{\tau}_4|\right).$$

The first of these is `FindKappaL(LM)`, that takes the single argument `LM` (a numeric array of Λ_1, Λ_2, Λ_3, Λ_4) and returns an array consisting of the values of ξ, α, h, and k.

The second program, `FitKappaLA(Data)`, takes the numeric array `Data` for its argument, computes ℓ_1, ℓ_2, ℓ_3, ℓ_4 from `Data` and searches for a solution for h and k to the equations $\{\tau_3 = \hat{\tau}_3,\ \tau_4 = \hat{\tau}_4\}$ in the region $0.04 \le h \le 8$ and $-0.95 \le k \le 8$. The first (h, k) pair encountered for which $\max\left(|\tau_3 - \hat{\tau}_3|,\ |\tau_4 - \hat{\tau}_4|\right) < 10^{-4}$ is taken as the h and k portion of the solution and α and ξ are determined next from $\Lambda_2 = \ell_2$ and $\Lambda_1 = \ell_1$, respectively. If there is no solution in this region, `FitKappaLA(Data)` returns an error; if there is one, it prints information related to the fit and returns values for ξ, α, h, and k. In a similar fashion,

`FitKappaLB(Data)` searches and, if successful, returns the solution from the region $h \le -0.9999 \le h \le -0.04$ and $-1 - 0.725h < k < -1/h$ (see 17.2.5).

`RefinesearchKappaL(LM, Hrange, Krange, Grid, Iterations)`, the last of these programs, accepts as arguments, ℓ_1, ℓ_2, ℓ_3, ℓ_4 as the array `LM`, minimum and maximum values for the search range of h as the array `Hrange`, minimum and maximum values for the search range of k as the array `Krange`, the refinement for the search to be conducted as `Grid` and the number of repeated applications of the search with the specified `Grid` as `Iterations`. `RefinesearchKappaL` returns ξ, α, h, and k and the value of $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|)$, which need not be less than 10^{-4}, associated with the search for h and k.

Another, slightly less precise, way of fitting a kappa distribution to a dataset is to use the table of Appendix F to estimate the kappa parameters. The columns of this table, headed by A, B, h, and k, provide estimates of the two parameters h and k and the A and B values of the expressions defined in (17.2.7) and (17.2.8). To obtain a fit through the use of the table of Appendix F the following algorithms can be used.

Algorithm Kappa–L: Fitting kappa distributions to data via L-moments.

1. Kappa–L. Use (6.1.4), (6.3.1), and (6.3.2) to compute ℓ_1, ℓ_2, ℓ_3, ℓ_4;
2. Kappa–L. Let $\hat{\tau}_3 = \ell_3/\ell_2$ and $\hat{\tau}_4 = \ell_4/\ell_2$;
3. Kappa–L. Find the closest entry to $(\hat{\tau}_3, \hat{\tau}_4)$ in the table of Appendix F and extract the values of A, B, h, and k from the table;
4. Kappa–L. Set $\alpha = \ell_2/A$;
5. Kappa–L. Set $\xi = \ell_1 - \alpha B$.

To illustrate the use of Algorithm Kappa–L, suppose we have a dataset for which

$$\ell_1 = 0, \quad \ell_2 = 1, \quad \ell_3 = 0.11, \quad \ell_4 = 0.058.$$

Step 2. Kappa–L sets $\tau_3 = 0.11$ and $\tau_4 = 0.058$. Following Step 3. Kappa–L, we find the closest entry in Appendix F to $(\hat{\tau}_3, \hat{\tau}_4) = (0.11, 0.058)$ to be $(\hat{\tau}_3, \hat{\tau}_4) = (0.100, 0.06)$ with entries of

$$A = 0.3393529, \quad B = 0.5943, \quad h = 0.6864, \quad k = 0.4322.$$

Executing Step 4. Kappa–L we find $\alpha = 1/0.3393529 = 2.9468$ and from Step 5. Kappa–L, we get $\xi = 0 - 2.9468 \times 0.5943 = -1.7513$. For convenience we designate the kappa distribution with parameters ξ, α, h, k by $\kappa(\xi, \alpha, h, k)$ so that in this case the fitted distribution is $\kappa(-1.7513, 2.9468, 0.6864, 0.4322)$.

If, instead of Appendix F, the `RefineSearchKappaL` program is invoked, we would obtain the fit $\kappa(-1.8747, 3.0652, 0.7442, 0.4397)$, with an indication that $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|) = 6.5 \times 10^{-7}$. Figure 17.2–3 provides a visual comparison of the two density functions associated with these fits (the density of the distribution obtained from Appendix F is the one that rises to a higher peak).

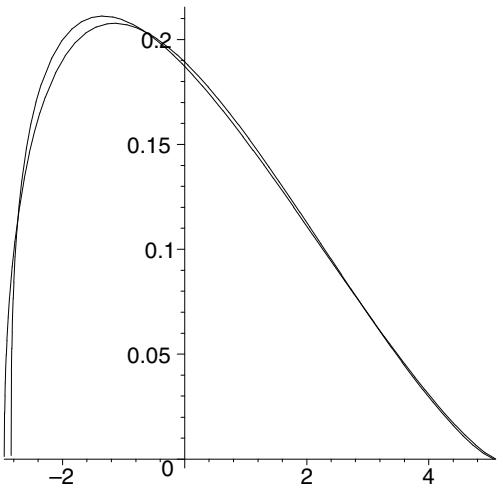


Figure 17.2–3. $\kappa(-1.7513,\ 2.9468,\ 0.6864,\ 0.4322)$ (rising to a higher peak) and $\kappa(-1.8747,\ 3.0652,\ 0.7442,\ 0.4397)$ density functions.

17.3 Estimation of Kappa Parameters via Percentiles

To develop a percentile-based approach to fitting kappa distributions to data, we start with the definitions of ρ_1, ρ_2, ρ_3, ρ_4 given in Chapter 5 (see (5.1.6) through (5.1.9)). For the kappa distribution,

$$\rho_1 = \xi + \alpha \left(1 - \left(\frac{1-(1/2)^h}{h}\right)^k\right) k^{-1} \tag{17.3.1}$$

$$\rho_2 = \frac{\alpha}{k}\left(-\left(\frac{1-9^h 10^{-h}}{h}\right)^k + \left(\frac{1-10^{-h}}{h}\right)^k\right) \tag{17.3.2}$$

$$\rho_3 = \left(\left(\frac{1-2^{-h}}{h}\right)^k - \left(\frac{1-10^{-h}}{h}\right)^k\right) \times \left(\left(\frac{1-9^h 10^{-h}}{h}\right)^k - \left(\frac{1-2^{-h}}{h}\right)^k\right)^{-1} \tag{17.3.3}$$

$$\rho_4 = \left(\left(\frac{1-3^h 4^{-h}}{h}\right)^k - \left(\frac{1-4^{-h}}{h}\right)^k\right) \times \left(\left(\frac{1-9^h 10^{-h}}{h}\right)^k - \left(\frac{1-10^{-h}}{h}\right)^k\right)^{-1}. \tag{17.3.4}$$

From the discussion of Section 5.1, we know that ρ_1 can assume any real value,

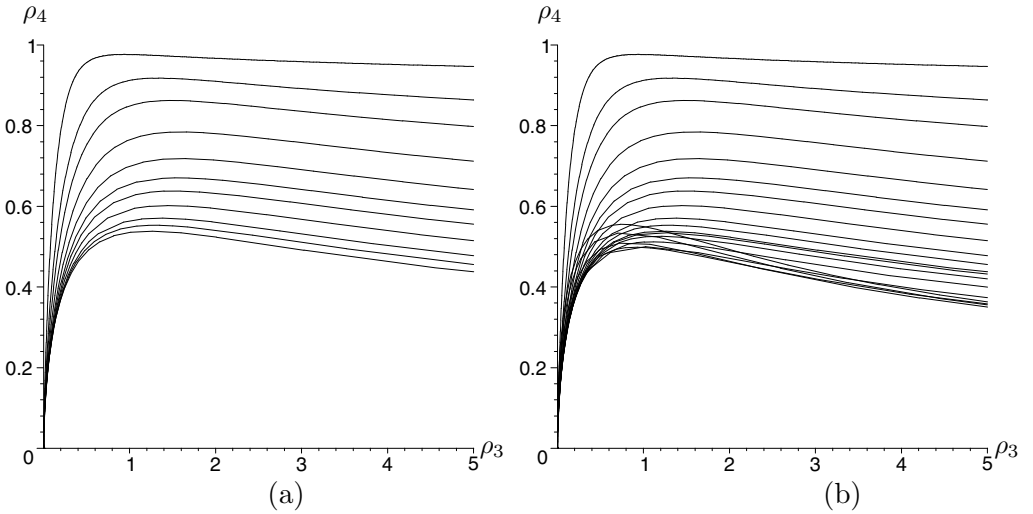


Figure 17.3–1. (ρ_3, ρ_4) points generated by $h = .05$ (lowest curve), .2, .4, .7, 1, 1.25, 1.6, 2.8, 3.5, 5.0 and $-20 \le k \le 70$ (a); additional (ρ_3, ρ_4) points with $h = -.05$ (lowest curve), $-.2$, $-.5$, -1, -1.6, -2.5, -3.8, -5 (b).

ρ_2 and ρ_3 must be non-negative, and $0 \le \rho_4 \le 1$. To get a sense of the portion of the (ρ_3, ρ_4)-space that can be covered by the kappa distribution, (ρ_3, ρ_4) points can be plotted for various (h, k) combinations. Figure 17.3–1 (a) shows such points along curves specified (lowest to highest) by

$$h = 0.0.5,\ \ 0.2,\ \ 0.4,\ \ 0.7,\ \ 1.0,\ \ 1.25,\ \ 1.6,\ \ 2.1,\ \ 2.8,\ \ 3.5,\ \ 5.0$$

and for each curve k ranging from -20 to 70. At first glance it looks as if a given (ρ_3, ρ_4) is associated with a unique (h, k). Although this is true for most of the region shown in Figure 17.3–1 (a), that is not the case for (ρ_3, ρ_4) points near the origin where the curves are "crowded."

Additional coverage of the (ρ_3, ρ_4)-space is provided by negative values of h. Curves associated with

$$h = -0.05,\quad -0.2,\quad -0.5,\quad -1.0,\quad -1.6,\quad -2.5,\quad -3.8,\quad -5.0$$

are included, along with the curves of Figure 17.3–1 (a), in Figure 17.3–1 (b). These additional curves appear with $h = -0.05$ representing the lowest curve and $h = -5$ the highest one.

It is clear from Figure 17.3–1 (b) that the (h, k) values associated with a given (ρ_3, ρ_4) will not be unique in the lower portion of the (ρ_3, ρ_4)-space covered by the kappa distribution. For example, $\rho_3 = \rho_4 = 0.5$ leads to the (h, k) pairs $(0.2923, -0.0629)$ and $(-2.7985, -0.3309)$. However, the shapes of the distributions $\kappa_1(0, 1, 0.2923, -0.0629)$ and $\kappa_2(0, 1, -2.7985, -0.3309)$ are quite different from one another as seen in Figure 17.3–2 ($\kappa_1(0, 1, 0.2923, -0.0629)$ rises higher).

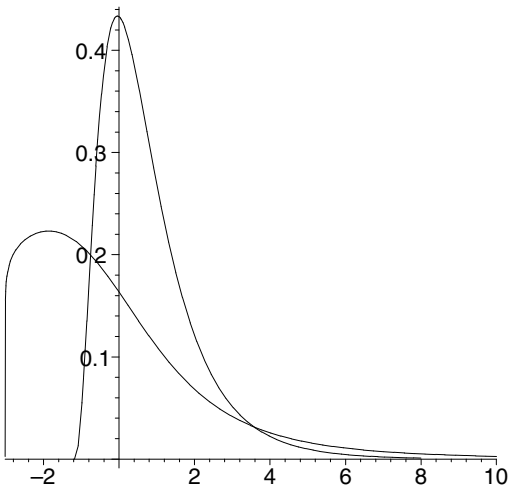


Figure 17.3–2. $\kappa_1(0,\ 1,\ 0.2923,\ -0.0629)$ (rising to a higher peak) and $\kappa_2(0,\ 1,\ -2.7985,\ -0.3309)$ density functions.

As was the case with L-moment fits, the coverage of $(\rho_3,\ \rho_4)$-space by the kappa distribution and the GLD complement each other very well (see Figure 5.3–1 of Chapter 5). Figure 17.3–3 shows the region covered by each family. The kappa distribution covers the upper portion of this space and the GLD the lower portion. The middle region is covered by both distribution families and fits from either of these systems will be available when $(\rho_3,\ \rho_4)$ is in this middle area. It will become evident in Sections 17.4 and 17.5 that in many applications $(\rho_3,\ \rho_4)$ ends up in the region covered by both distributions.

It should be noted that all three boundaries of Figure 17.3–3 are approximations. When compared to Figure 5.3–1 of Chapter 5, the middle curve of Figure 17.3–3 seems to be wrong because first it peaks and then it gets lower whereas the upper GLD boundary of Figure 5.3–1 does not peak but keeps getting higher. This seeming anomaly is due primarily to the fact that in the case of the GLD we imposed the restriction $\rho_3 \leq 1$ because for the GLD, $(\rho_3,\ \rho_4)$ is an admissible point if and only if $(\rho_3,\ 1/\rho_4)$ is. Thus, if Figure 5.3–1 were to be extended to the right of $\rho_3 = 1$, the boundary curve would get lower.

To fit a kappa distribution to data using percentiles, we need to first compute $\hat{\rho}_1,\ \hat{\rho}_2,\ \hat{\rho}_3,\ \hat{\rho}_4$ from the data (see (5.1.2) through (5.1.5)), then solve $\{\rho_3 = \hat{\rho}_3,\ \rho_4 = \hat{\rho}_4\}$ for h and k, next substitute the h and k values just obtained into the equation $\rho_2 = \hat{\rho}_2$ and solve it for α. This is equivalent to solving $\alpha A = \hat{\rho}_2$ where

$$A = \frac{1}{k}\left[\left(\frac{1-10^{-h}}{h}\right)^k - \left(\frac{1-9^h 10^{-h}}{h}\right)^k\right].$$

Finally, we substitute h, k, α into the equation $\rho_1 = \hat{\rho}_1$ to obtain ξ. This can be

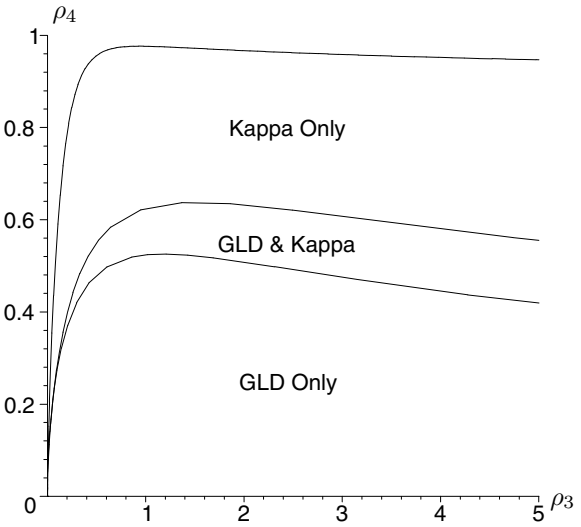


Figure 17.3–3. Portions of (ρ_3, ρ_4)-space covered by the kappa and GLD distributions.

done by solving $\xi + \alpha B = \hat{\rho}_1$, where

$$B = \frac{1}{k}\left[1 - \left(\frac{1 - 2^{-h}}{h}\right)^k\right].$$

The **R** program `FindKappaP(Rhos)`, with `Rhos` representing the numeric array of ρ_1, ρ_2, ρ_3, ρ_4, returns the vector of values of ξ, α, h, and k. The program `FitKappaP(Data)` takes the argument `Data` as a numeric array and produces one or more approximate solutions to the system of equations $\rho_i = \hat{\rho}_i$, $i = 1, \ldots, 4$. `FitKappaP` determines its solutions by first finding, if possible, h and k that make $\max(|\rho_3 - \hat{\rho}_3|, \max(|\rho_4 - \hat{\rho}_4|) < 10^{-4}$. Following this, α and ξ are determined from $\rho_2 = \hat{\rho}_2$ and $\rho_1 = \hat{\rho}_1$, respectively.

It is also possible to search for solutions in specific regions through the invocation of `RefineSearchKappaP(Data, Hrange, Krange, Grid, Interations)`. By contrast to `FitKappaP`, this program returns that "solution" for which a minimum value of $\max(|\rho_3 - \hat{\rho}_3|, \max(|\rho_4 - \hat{\rho}_4|)$ is attained in the specified region, without the restriction that $\max(|\rho_3 - \hat{\rho}_3|, \max(|\rho_4 - \hat{\rho}_4|)$ be less than 10^{-4}. As a third alternative, solutions, when available, can be obtained through the table of Appendix G. For specified $\hat{\rho}_3$ and $\hat{\rho}_4$, this table provides values of h, k, A, and B. An algorithm for using Appendix G is given below.

Algorithm Kappa–P: Fitting kappa distributions to data via percentiles.

1. Kappa–P. Use (5.1.1) through (5.1.5) to compute $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$;

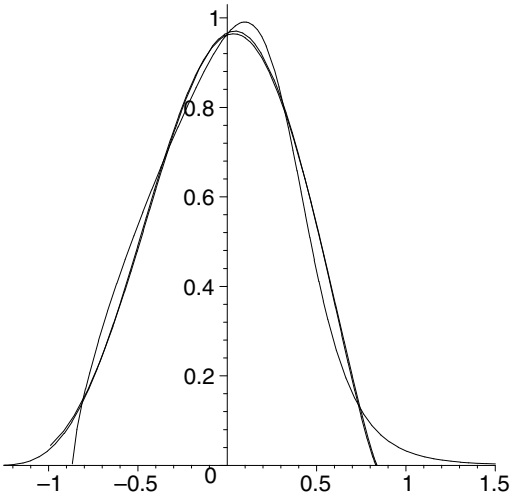


Figure 17.3–4. $\kappa_3(-0.1827, 0.4681, 0.1848, 0.4584)$, $\kappa_4(-0.1747, 0.4458, 0.1535, 0.4539)$ and $\kappa_5(0.2097, 0.1195, -5.2446, -0.1110)$ p.d.f.s.

2. Kappa–P. Find the closest entry to $(\hat{\rho}_3, \hat{\rho}_4)$ in the table of Appendix G and extract the values of A, B, h, and k from the table;

3. Kappa–P. Set $\alpha = \hat{\rho}_2/A$;

4. Kappa–P. Set $\xi = \hat{\rho}_1 - \alpha B$.

As an example of the use of Algorithm Kappa–P, suppose that we have

$$\hat{\rho}_1 = 0, \quad \hat{\rho}_2 = 1, \quad \hat{\rho}_3 = 1.12, \quad \hat{\rho}_4 = 0.548.$$

From Step 2. Kappa–P, the closest point to $(\hat{\rho}_3, \hat{\rho}_4)$ is (1.10, 0.55), yielding

$$h = 0.1848, \quad k = 0.4584, \quad A = 2.1362303, \quad B = 0.3902.$$

From Step 3. Kappa–P we obtain $\alpha = 1/2.1362303 = 0.4681$ and from Step 4. Kappa–P we get $\xi = -0.1827$ for the fit $\kappa_3(-0.1827, 0.4681, 0.1848, 0.4584)$, with support $[-1.38, 0.84]$. If instead we were to use direct computation by using

```
RefineSearchKappaP(Rhos, c(0.001,0.5), c(0.1,0.6), 100, 3)
```

we would get the fit $\kappa_4(-0.1747, 0.4458, 0.1535, 0.4539)$ with $[-0.99, 2.34]$ for its support, along with an indication that $\max(|\rho_3 - \hat{\rho}_3|, |\rho_4 - \hat{\rho}_4|) = 2.3 \times 10^{-6}$.

Locating $(\hat{\rho}, \hat{\rho}_4) = (1.12, 0.548)$ in Figure 17.3–1 (b) gives us the indication that there may well be another fit when h is negative and, indeed, a special search produces the fit $\kappa_5(0.2097, 0.1195, -5.2446, -0.1110)$ with support $[-0.87, \infty)$. The density functions of κ_3, κ_4, and κ_5 are shown in Figure 17.3–4 where the density of κ_5 is the one that rises to the highest peak and stretches farthest to the right. The κ_3 and κ_4 densities are almost indistinguishable but a careful look can locate κ_4 since its support does not extend to the left of -1.

17.4 Approximations of Some Well-Known Distributions

In previous chapters when new methods for fitting distributions were devised, these methods were put to the test by applying them to approximate various known distributions. That pattern is continued here in connection with the kappa distribution and the L-moment and percentile fitting approaches discussed in the previous two sections. When L-moments are used, the L-moments themselves as well as the (τ_3, τ_4) that will be needed were computed, either in exact form or through numeric approximations, in the various subsections of Section 6.4. Similarly, in the case of percentile-based approximations, the ρ_1, ρ_2, ρ_3, ρ_4 that will be needed were computed in the subsections of Section 5.4.

In the subsections that follow, for each distribution that we consider we attempt to obtain an approximation, κ_L, through L-moments and another one, κ_P, through percentiles. Whenever such approximations are available we provide values for

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)|, &\quad \sup|\hat{F}_L(x) - F(x)| \\ ||\hat{f}_L - f||_1, &\quad ||\hat{f}_L - f||_2, \\ \sup|\hat{f}_P(x) - f(x)|, &\quad \sup|\hat{F}_P(x) - F(x)|, \\ ||\hat{f}_P - f||_1, &\quad ||\hat{f}_P - f||_2, \end{aligned}$$

where $f(x)$ and $F(x)$ are the density and distribution function, respectively, of the distribution being approximated, $\hat{f}_L(x)$ and $\hat{F}_L(x)$ are the density and distribution function, respectively, of the L-moment kappa approximation and $\hat{f}_P(x)$ and $\hat{F}_P(x)$ are the density and distribution function, respectively, of the kappa approximation obtained through the use of percentiles.

17.4.1 The Normal Distribution

For $N(\mu, \sigma^2)$, the normal distribution with mean μ and variance σ^2, we have (from Section 6.4.1)

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.5642\sigma, 0, 0.06917\sigma) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.1226).$$

Since τ_3 and τ_4 are independent of μ and σ, we consider only the special case $\mu = 0$ and $\sigma = 1$. Through `FindKappaL(c(0, 0.5642, 0, 0.06917))` we obtain the L-moment approximation of $N(0, 1)$ given by

$$\kappa_L(-0.2675, 0.8764, -0.1613, 0.2138)$$

with support $(-\infty, 3.83]$.

To obtain a percentile fit to $N(0, 1)$, we note (see section 5.4.1) that

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 2.5631, 1, 0.52631)$$

and from `FindKappaP(c(0, 2.5631, 1, 0.52631))` we get the fit

$$\kappa_P(-0.2677,\, 0.8862,\, -0.1524,\, 0.2321)$$

that has support $(-\infty,\, 3.55]$.

Although for both κ_L and κ_P, the supports are not ideal, the probability under $N(0,\, 1)$ that is not covered by κ_L is about 0.00006 and it is 0.00011 for κ_P. The density and distribution functions of $N(0,\, 1)$ and the two approximations κ_L and κ_P cannot be distinguished visually and

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.005190, & \sup|\hat{F}_L(x) - F(x)| &= 0.002072, \\ ||\hat{f}_L - f||_1 &= 0.01455, & ||\hat{f}_L - f||_2 &= 0.006695, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.003854, & \sup|\hat{F}_P(x) - F(x)| &= 0.002618, \\ ||\hat{f}_P - f||_1 &= 0.01070, & ||\hat{f}_P - f||_2 &= 0.004673. \end{aligned}$$

Although both these approximations are reasonably good, they are not as good as the GLD approximations (via moments, percentiles, or L-moments).

17.4.2 The Uniform Distribution

From 6.4.2, the L-moments and $(\tau_3,\, \tau_4)$ for the uniform distribution on the interval $[a, b]$ are given by

$$(\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, \Lambda_4) = \left(\frac{a+b}{2},\, \frac{1}{6},\, 0,\, 0\right) \quad \text{and} \quad (\tau_3,\, \tau_4) = (0,\, 0).$$

Since $\tau_3 = \tau_4 = 0$ are independent of a and b, we only consider the uniform distribution on the interval $[0, 1]$. If the **R** command `FindKappaL(c(1/2, 1/6, 0, 0))` is executed, we would get the L-moment approximation given by

$$\kappa_L(4.1 \times 10^{-6},\, 0.9999,\, 0.9999,\, 1)$$

with support [0.0000, 1.0000]. What we actually should have is $\kappa_L(0,\, 1,\, 1,\, 1)$ because, as indicated in Table 17.1–4, the uniform distribution can be realized as a special case of the kappa distribution. (This can easily be verified by substituting $h = 1$ and $k = 1$ in the (17.1.3) and obtaining the distribution function $F(x) = (x - \xi)/\alpha$.)

A similar situation is encountered when a percentile fit is sought. Starting with

$$(\rho_1,\, \rho_2,\, \rho_3,\, \rho_4) = (1/2,\, 4/5,\, 1,\, 5/8),$$

we get, through `FindKappaP(c(1/2, 4/5, 1, 5/8)`, the approximation

$$\kappa_P(-1.5 \times 10^{-6},\, 1.0000,\, 1.0000,\, 1.0000).$$

Since both $\kappa_L = \kappa_P$ approximations are perfect, it is unnecessary to consider errors associated with the fit.

17.4.3 The Student's t Distribution

In previous chapters we have considered $t(\nu)$, the Student's t distribution with ν degrees of freedom, for $\nu = 1, 5, 6, 10$, and 30. The case $\nu = 1$ does not lend itself to L-moment fits because $t(1)$ does not have a finite mean and hence its L-moments do not exist. To find a percentile-based approximation, we note (see Section 5.4.3) that

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 6.1554, 1, 0.3249).$$

Positioning $(\rho_3, \rho_4) = (1, 0.3249)$ on Figure 17.3–1 (b), makes us realize that percentile-based approximations are not available either.

For $t(5)$, from Section 6.4.3, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.6839, 0, 0.1259) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.1841).$$

We can see from Figure 17.3–3 (this is also evident from Appendix F) that this (τ_3, τ_4) is located near the border of the region covered by the kappa distribution. Because of this, `FindKappaL` fails to provide an approximation to $t(5)$. However, a search, conducted with

```
LM <- c(0, 0.6839, 0, 0.1259)
RefineSearchKappaL(LM, c(-1.5, -.9), c(-0.1, -0.01), 100, 3),
```

leads to the L-moment approximation

$$\kappa_L(0.1027, 0.6134, -1.3405, -0.02775)$$

with support $[-21.99, \infty)$. Because of the location of (τ_3, τ_4), this κ_L fit has an unusually large $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|)$ of 0.014.

To obtain a percentile fit to $t(5)$, we note (see section 5.4.3) that

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 2.9518, 1, 0.4924)$$

and we can see from Figure 17.3–1 (b) that (ρ_3, ρ_4) is close to the boundary of the region covered by the kappa distribution. As was the case with L-moments, `FindKappaP` is not able to return an approximation. But using

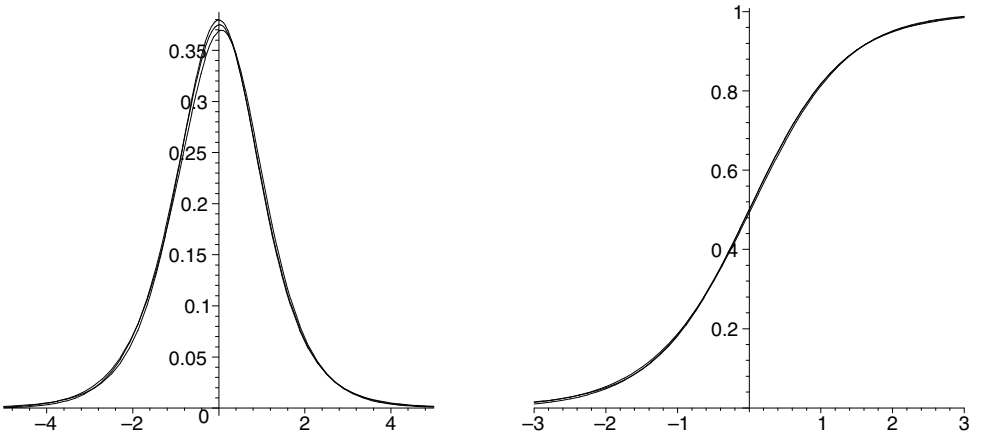
```
Rhos <- c(0, 2.9518, 1, 0.4924)
RefineSearchKappaP(Rhos, c(-3, -0.05), c(-3, -0.001), 100, 3)
```

we get the fit

$$\kappa_P(0.09645, 0.5947, -1.4132, -0.05601)$$

that has support $[-10.52, \infty)$ and the somewhat large $\max(|\rho_3 - \hat{\rho}_3|, |\rho_4 - \hat{\rho}_4|)$ of 0.005.

Figure 17.4–1 (a) and (b), illustrate, respectively, the density and distribution functions of $t(5)$ with those of κ_L and κ_P. In Figure 17.4–1 (a), the $t(5)$ density function rises highest near the center, followed by the density function of κ_P and then that of κ_L. Regardless of the large errors encountered for $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 -$



(a) (b)

Figure 17.4–1. The $t(5)$, κ_P, and κ_L p.d.f.s ($t(5)$ is highest at the center and κ_L is lowest) (a); the c.d.f.s of $t(5)$, κ_P, and κ_L (b).

$\hat{\tau}_4|$) and $\max(|\rho_3 - \hat{\rho}_3|, |\rho_4 - \hat{\rho}_4|)$, both κ_L and κ_P approximations seem to be reasonably good, an observation supported by

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.01598, & \sup|\hat{F}_L(x) - F(x)| &= 0.009624, \\ ||\hat{f}_L - f||_1 &= 0.04202, & ||\hat{f}_L - f||_2 &= 0.01825, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.006755, & \sup|\hat{F}_P(x) - F(x)| &= 0.005040, \\ ||\hat{f}_P - f||_1 &= 0.02270, & ||\hat{f}_P - f||_2 &= 0.008625. \end{aligned}$$

On the basis of these figures it seems that κ_L and κ_P provide better approximations to $t(5)$ than the GLD through moments but not as good as the GLD through percentiles or L-moments.

When $\nu = 6$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.6555, 0, 0.1093) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.1667).$$

Again, (τ_3, τ_4) is close to the boundary of the points covered by the kappa distribution and the approximation we obtain through

```
RefineSearchKappaL(LM, c(-2,-0.05), c(-2,-0.05), 100, 3)
```

where `LM <- c(0, 0.6555, 0, 0.1093)`, is

$$\kappa_L(0.1020, 0.6139, -1.3377, -0.02747)$$

with support $[-22.24, \infty)$. Again, we have a rather large $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|)$ of 0.014.

To obtain a percentile fit to $t(6)$, we first compute

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0., 2.8795, 1, 0.4984)$$

and, although (ρ_3, ρ_4) is close to the boundary of points covered by the kappa distribution, through

```
Rhos <- c(0.  2.8795, 1, 0.4984)
RefineSearchKappaP(Rhos, c(-1.2,-0.9), c(-0.1,-0.0001), 100, 3)
```

we get the approximation

$$\kappa_P(0.04081, 0.6207, -1.1689, -0.02689)$$

that has support $[-23.04, \infty)$ and the small $\max(|\rho_3 - \hat{\rho}_3|, |\rho_4 - \hat{\rho}_4|)$ of 2.60×10^{-6}.

For the κ_L and κ_P approximations to $t(6)$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.01961, & \sup|\hat{F}_L(x) - F(x)| &= 0.01104 \\ ||\hat{f}_L - f||_1 &= 0.04558, & ||\hat{f}_L - f||_2 &= 0.02105, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.002333, & \sup|\hat{F}_P(x) - F(x)| &= 0.002266, \\ ||\hat{f}_P - f||_1 &= 0.007935, & ||\hat{f}_P - f||_2 &= 0.002753. \end{aligned}$$

Comparison of these figures with their counterparts when $t(6)$ was fitted by other means, indicates that κ_L and κ_P provide approximations that are at about the same level as the GLD approximation via moments, but not as good as the GLD fits through percentiles or L-moments.

For $\nu = 10$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.6183, 0, 0.09394) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.1519).$$

Now (τ_3, τ_4) is within the set of points covered by the kappa distribution and we obtain the approximation

$$\kappa_L(-0.1243, 0.7439, -0.5700, 0.08281)$$

with support $[-\infty, 8.86)$. By contrast with $t(5)$ and $t(6)$, we have a much smaller $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|)$ of 7.11×10^{-8}.

For a percentile fit to $t(10)$, we first compute

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 2.7444, 1, , 0.5100)$$

and note that this is within the (ρ_3, ρ_4)-space covered by the kappa distribution. The approximation that results is

$$\kappa_P(-0.1300, 0.7557, -0.5400, 0.1022),$$

with support $(-\infty, 7.26]$ and $\max(|\rho_3 - \hat{\rho}_3|, |\rho_4 - \hat{\rho}_4|) = 3.28 \times 10^{-6}$.

For the κ_L and κ_P approximations to $t(10)$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.004491, & \sup|\hat{F}_L(x) - F(x)| &= 0.001837 \\ ||\hat{f}_L - f||_1 &= 0.01177, & ||\hat{f}_L - f||_2 &= 0.005366, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.002836, & \sup|\hat{F}_P(x) - F(x)| &= 0.002315, \\ ||\hat{f}_P - f||_1 &= 0.008360, & ||\hat{f}_P - f||_2 &= 0.003316. \end{aligned}$$

These indicate that κ_L and κ_P provide approximations that are better than the one provided by the GLD through moments, but not as good as the GLD fits through percentiles or L-moments.

When $\nu = 30$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 0.5812, 0, 0.07681) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.1322).$$

This (τ_3, τ_4) is well within the set of points covered by the kappa distribution, leading to the approximation

$$\kappa_L(-0.2251, 0.8339, -0.2734, 0.1716),$$

with support $[-\infty, 4.63)$.

For a percentile fit to $t(30)$, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 2.6208, 1, 0.5210).$$

Similar to the situation with the L-moment fit, this (ρ_3, ρ_4) is well within the space covered by the kappa distribution. The approximation that results is

$$\kappa_P(-0.2267, 0.8450, -0.2588, 0.1916),$$

with support $(-\infty, 4.18]$.

For the κ_L and κ_P approximations to $t(30)$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.005327, & \sup|\hat{F}_L(x) - F(x)| &= 0.002157 \\ ||\hat{f}_L - f||_1 &= 0.01473, & ||\hat{f}_L - f||_2 &= 0.006706, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.003613, & \sup|\hat{F}_P(x) - F(x)| &= 0.002595, \\ ||\hat{f}_P - f||_1 &= 0.01051, & ||\hat{f}_P - f||_2 &= 0.004370. \end{aligned}$$

The figures for κ_P are comparable with those in connection with the GLD approximation through moments but not as good as the ones for GLD fits via percentiles or L-moments. The figures for κ_L do not compare favorably with those for the GLD fits through moments, percentiles, or L-moments.

17.4.4 The Exponential Distribution

The exponential distribution, defined in Section 3.4.4, has a single positive parameter, θ, and it can be realized as a special case of the kappa distribution with $h = 1$ and $k = 0$ (see Table 17.1–4). In previous chapters we have considered the exponential distribution with $\theta = 1$ and the kappa distribution that represents this specific exponential is $\kappa(0, 1, 1, 0)$. In general, the exponential distribution with parameter θ is represented by $\kappa(0, \theta, 1, 0)$.

17.4.5 The Chi-Square Distribution

The chi-square distribution with parameter ν, and designated by $\chi^2(\nu)$, is defined in Section 3.4.5. Following the pattern set in previous chapters we begin by attempting to approximate $\chi^2(5)$, for which

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (5, 1.6977, 0.3553, 0.2330) \quad \text{and} \quad (\tau_3, \tau_4) = (0.2093, 0.1372).$$

This leads to the L-moment approximation

$$\kappa_L(2.9122, 2.9490, 0.3726, 0.04874)$$

with support $[-0.070, 63.42]$.

To obtain a percentile fit to $\chi^2(5)$, we note (see Section 5.4.5) that

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (4.3515, 7.6260, 0.5611, 0.5181)$$

and in this case the fit

$$\kappa_P(2.8961, 2.9767, 0.3816, 0.05684)$$

with support $[-0.052, 55.27]$ is obtained.

It is clear from Figures 17.4–2 (a) and (b) which show, respectively, the density and distribution functions of $\chi^2(5)$, κ_L, and κ_P, that these three p.d.f.s and c.d.f.s cannot be distinguished visually. Some measure of the differences between the $\chi^2(5)$ distribution and its κ_L and κ_P approximations are given by

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.002000, & \sup|\hat{F}_L(x) - F(x)| &= 0.0006041 \\ ||\hat{f}_L - f||_1 &= 0.004669, & ||\hat{f}_L - f||_2 &= 0.001690, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.001642, & \sup|\hat{F}_P(x) - F(x)| &= 0.0005795, \\ ||\hat{f}_P - f||_1 &= 0.003023, & ||\hat{f}_P - f||_2 &= 0.001129. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are superior to those provided by the GLD through moments, percentiles, or L-moments.

When $\nu = 3$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (3, 1.2732, 0.3459, 0.1902) \quad \text{and} \quad (\tau_3, \tau_4) = (0.2717, 0.1494).$$

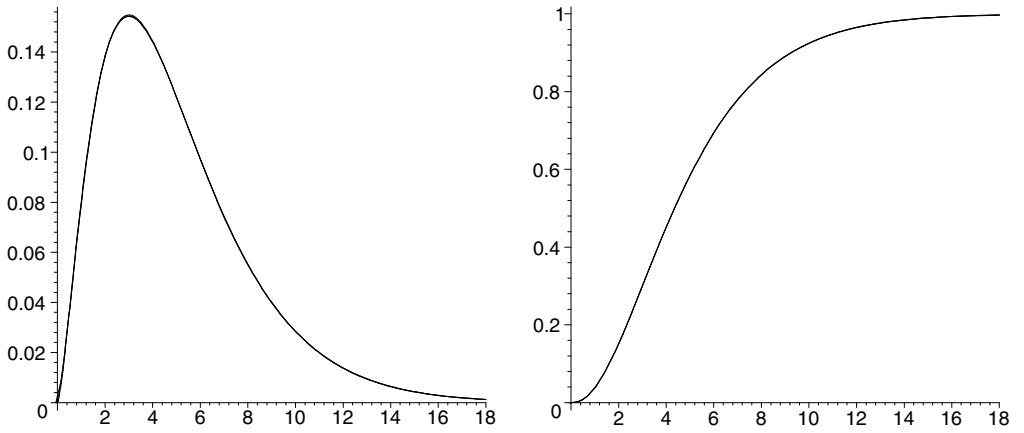
This leads to the approximation

$$\kappa_L(0.9927, 2.3598, 0.6557, 0.02127)$$

with support $[-0.0076, 111.96]$.

For a percentile fit to $\chi^2(3)$, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (2.3660, 5.6670, 0.4585, 0.5110)$$



(a) (b)

Figure 17.4–2. The $\chi^2(5)$, κ_P, and κ_L p.d.f.s (a); the c.d.f.s of $\chi^2(5)$, κ_P, and κ_L (b).

and the approximation that results is

$$\kappa_P(0.9841,\, 2.3728,\, 0.6606,\, 0.02541),$$

with support $[-0.0049,\, 94.37]$.

For these κ_L and κ_P approximations to $\chi^2(3)$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.01248, & \sup|\hat{F}_L(x) - F(x)| &= 0.0004487, \\ ||\hat{f}_L - f||_1 &= 0.002311, & ||\hat{f}_L - f||_2 &= 0.001847, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.009192, & \sup|\hat{F}_P(x) - F(x)| &= 0.0003190, \\ ||\hat{f}_P - f||_1 &= 0.001350, & ||\hat{f}_P - f||_2 &= 0.001233. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles, or L-moments.

For the $\chi^2(10)$ distribution,

$$(\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, \Lambda_4) = (10,\, 2.4609,\, 0.3617,\, 0.3184) \quad \text{and} \quad (\tau_3,\, \tau_4) = (0.1470,\, 0.1294).$$

This leads to the L-moment fit

$$\kappa_L(7.6710,\, 4.0562,\, 0.1471,\, 0.08373),$$

with support $[-0.76,\, 56.11]$.

To get a percentile fit to $\chi^2(10)$, we first compute

$$(\rho_1,\, \rho_2,\, \rho_3,\, \rho_4) = (9.3418,\, 11.1220,\, 0.6736,\, 0.5225)$$

and then obtain the approximation

$$\kappa_P(7.6461,\, 4.1071,\, 0.1595,\, 0.09588),$$

that has support $[-0.60,\, 50.48]$.

For these κ_L and κ_P approximations to $\chi^2(10)$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.0007896, & \sup|\hat{F}_L(x) - F(x)| &= 0.0009740, \\ ||\hat{f}_L - f||_1 &= 0.007298, & ||\hat{f}_L - f||_2 &= 0.001715, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.0005229, & \sup|\hat{F}_P(x) - F(x)| &= 0.001027, \\ ||\hat{f}_P - f||_1 &= 0.004964, & ||\hat{f}_P - f||_2 &= 0.001090. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles, or L-moments.

When $\nu = 30$,

$$(\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, \Lambda_4) = (30,\, 4.3338,\, 0.3655,\, 0.5405) \quad \text{and} \quad (\tau_3,\, \tau_4) = (0.08434,\, 0.1247).$$

From this we get the L-moment fit

$$\kappa_L(26.9114,\, 6.8710,\, -0.02501,\, 0.1290)$$

with support $(-\infty,\, 80.17]$ and $\max(|\tau_3 - \hat{\tau}_3|,\, |\tau_4 - \hat{\tau}_4|) = 0.0001445$, which is slightly larger than our threshold of 10^{-4}.

For a percentile fit to $\chi^2(30)$, we first compute

$$(\rho_1,\, \rho_2,\, \rho_3,\, \rho_4) = (29.3360,\, 19.6568,\, 0.8001,\, 0.5251)$$

and then obtain the approximation

$$\kappa_P(26.8696,\, 6.9727,\, -.009797,\, 0.1450)$$

that has support $(-\infty,\, 74.97]$.

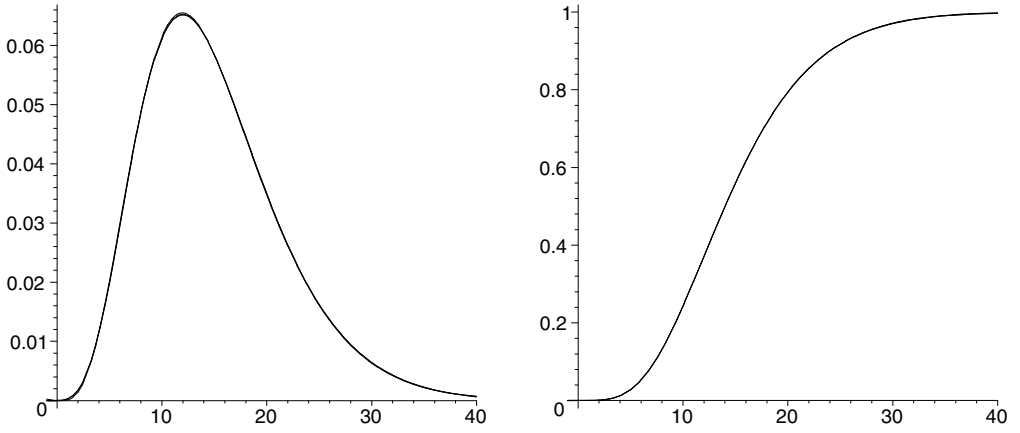
The support of the $\chi^2(30)$ distribution is $[0,\, \infty)$ but the supports for both κ_L and κ_P extend to $-\infty$, indicating the possibility of a poor fit. However, this is not the case because

$$f_L(0) \approx 9 \times 10^{-9}, \quad F_L(0) \approx 6 \times 10^{-9}, \quad f_P(0) \approx 6 \times 10^{-9}, \quad F_P(0) \approx 4 \times 10^{-9}$$

and

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.0004922, & \sup|\hat{F}_L(x) - F(x)| &= 0.001354 \\ ||\hat{f}_L - f||_1 &= 0.01023, & ||\hat{f}_L - f||_2 &= 0.001686, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.0003137, & \sup|\hat{F}_P(x) - F(x)| &= 0.001601, \\ ||\hat{f}_P - f||_1 &= 0.007216, & ||\hat{f}_P - f||_2 &= 0.001079. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles, or L-moments.



(a) (b)

Figure 17.4–3. The p.d.f.s of the gamma distribution with $\alpha = 5$ and $\theta = 3$ and the κ_L, κ_P c.d.f.s (a); the p.d.f.s of the gamma distribution with $\alpha = 5$ and $\theta = 3$ and the κ_L, κ_P c.d.f.s (b).

17.4.6 The Gamma Distribution

As done in previous considerations of the gamma distribution (see Sections 3.4.6, 5.4.6, and 6.4.6), we set the parameters α and θ to 5 and 3, respectively. In this situation,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (15, 3.6914, 0.5415, 0.4765) \quad \text{and} \quad (\tau_3, \tau_4) = (0.1467, 0.1291).$$

From this we get the L-moment fit

$$\kappa_L(11.4991, 6.0967, 0.1493, 0.08498)$$

with support $[-1.09, 83.24]$.

For a percentile fit to the gamma distribution with $\alpha = 5$ and $\theta = 3$, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (14.0129, 16.683, 0.6736, 0.5225)$$

which leads to the fit

$$\kappa_P(11.4714, 6.1579, 0.1589, 0.09560),$$

that has support $[-0.91, 75.89]$.

Figure 17.4–3 (a) shows the p.d.f.s of κ_L, κ_P, and the gamma distribution with $\alpha = 5$ and $\beta = 3$; Figure 17.4–3 (b) shows the c.d.f.s of these distributions. A very slight disparity is visible near the peak of Figure 17.4–3 (a).

Since the supports of κ_L and κ_P both extend to negative values that are outside of the support for the gamma distribution, we note that

$$f_L(0) \approx 1.1\times10^{-5}, \quad F_L(0) \approx 2.0\times10^{-6}, \quad f_P(0) \approx 9.1\times10^{-6}, \quad F_P(0) \approx 1.4\times10^{-6}$$

and for these approximations of the gamma distribution with $\alpha = 5$ and $\theta = 3$,

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.0004882, & \sup|\hat{F}_L(x) - F(x)| &= 0.0008703 \\ ||\hat{f}_L - f||_1 &= 0.006919, & ||\hat{f}_L - f||_2 &= 0.001302, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.0003596, & \sup|\hat{F}_P(x) - F(x)| &= 0.001011, \\ ||\hat{f}_P - f||_1 &= 0.004975, & ||\hat{f}_P - f||_2 &= 0.0009035. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles, or L-moments.

17.4.7 The Weibull Distribution

Following the pattern set in previous approximations of the Weibull distribution (see Sections 3.4.7, 5.4.7, and 6.4.7) we consider the special case of the Weibull distribution with $\alpha = 1$ and $\beta = 5$. For this distribution,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.9182, 0.1189, -0.005664, 0.01361)$$

and $(\tau_3, \tau_4) = (-0.04764, 0.1145)$. This produces the L-moment fit

$$\kappa_L(0.8649, 0.19978, -0.1037, 0.3184)$$

with support $(-\infty, 1.49]$.

For a percentile fit to this Weibull distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0.9293, 0.5439, 1.1567, 0.52959)$$

which leads to the fit

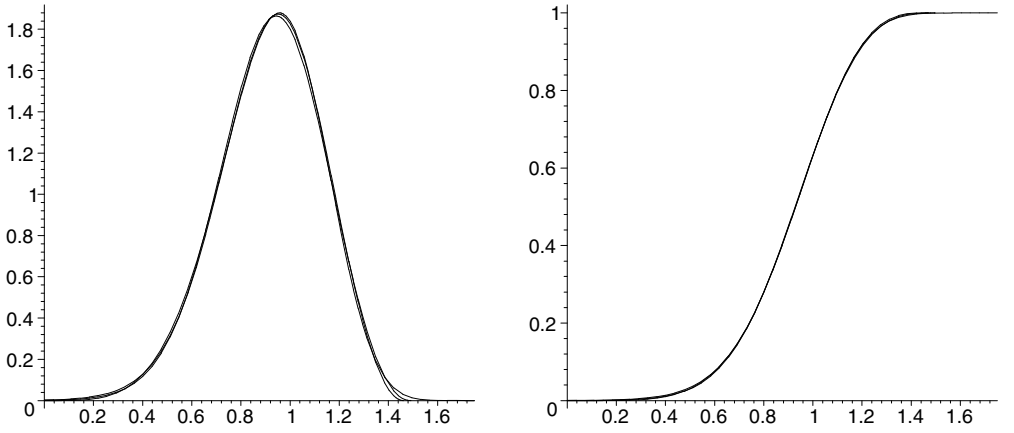
$$\kappa_P(0.8690, 0.1980, -0.1291, 0.3343),$$

that has support $(-\infty, 1.46]$.

Figure 17.4–4 (a) shows the plots of the density function of the Weibull distribution with $\alpha = 1$ and $\beta = 5$, along with the κ_L and of κ_P density functions and Figure 17.4–4 (b) shows the c.d.f.s of these distributions. We can see some separation of the curves near their peaks as well as at the right tail in Figure 17.4–4 (a). The Weibull distribution rises to the highest point at the peak and extends farthest to the right. The lowest curve at the peak is the κ_L density. Near the right tail the Weibull p.d.f. is again the highest one but the density of κ_P is the lowest.

The support of the Weibull distribution is $[0, \infty)$ but the supports of κ_L and κ_P extend to $-\infty$ and are truncated on the right. This prompts us to check the p.d.f. and c.d.f. values at 0 and at the right endpoints. Computations at 0 yield

$$f_L(0) \approx 0.001348, \quad F_L(0) \approx 0.00010853, \quad f_P(0) \approx 0.002570, \quad F_P(0) \approx 0.0002463$$



(a) (b)

Figure 17.4–4. The p.d.f.s of the Weibull distribution with $\alpha = 1$ and $\beta = 5$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of the Weibull distribution with $\alpha = 1$ and $\beta = 5$ and the κ_L, κ_P c.d.f.s (b).

and at the right endpoints we have

$$f(1.46) = 0.02988, \quad F(1.46) = 0.9987, \quad f(1.49) = 0.01593, \quad F(1.49) = 0.9994.$$

For the κ_L and κ_P approximations of the Weibull distribution with $\alpha = 1$ and $\beta = 5$, we also have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.04669, & \sup|\hat{F}_L(x) - F(x)| &= 0.003844, \\ ||\hat{f}_L - f||_1 &= 0.02675, & ||\hat{f}_L - f||_2 &= 0.02764, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.04019, & \sup|\hat{F}_P(x) - F(x)| &= 0.004635, \\ ||\hat{f}_P - f||_1 &= 0.02044, & ||\hat{f}_P - f||_2 &= 0.02094. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are not as good as those provided by the GLD through moments, percentiles, or L-moments.

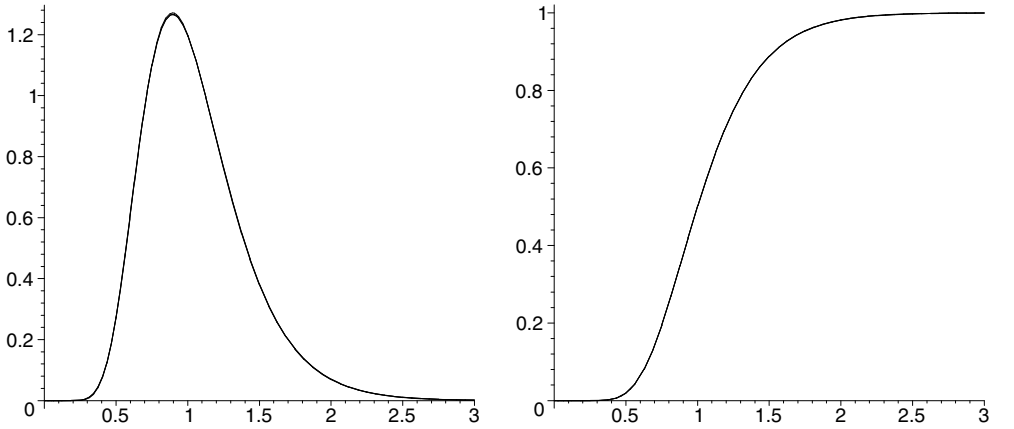
17.4.8 The Lognormal Distribution

In previous considerations of the lognormal distribution, defined in Section 3.4.8, (see Sections 3.48, 5.4.8, and 6.48) we restricted the parameters μ and σ to $\mu = 0$ and $\sigma = 1/3$. With this specification, we have

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.0571, 0.1970, 0.03188, 0.02821)$$

and $(\tau_3, \tau_4) = (0.1618, 0.1432)$. This leads us to the L-moment fit

$$\kappa_L(0.8860, 0.2971, 0.05075, 0.02769)$$



(a) (b)

Figure 17.4–5. The p.d.f.s of the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$ and the κ_L, κ_P c.d.f.s (b).

with support $[-0.037, 11.61]$.

For a percentile fit to this lognormal distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (1, 0.8806, 0.6523, 0.5149)$$

which leads to the fit

$$\kappa_P(0.8839, 0.3006, 0.06581, 0.03817),$$

that has support $[0.022, 8.76]$.

Figure 17.4–5 (a) shows the plots of the density function of the lognormal distribution with $\mu = 0$ and $\sigma = 1/3$, along with density functions of its κ_L and κ_P approximations and Figure 17.4–5 (b) shows the c.d.f.s of these distributions. We can see only a slight separation of the curves near their peaks in Figure 17.4–5 (a).

The support of the lognormal distribution is $[0, \infty)$ but the support of κ_L extends to negative values. The truncation of the supports of κ_L and κ_P on the right should not be a problem since $f(8.76) \approx 8.7 \times 10^{-11}$ and $1 - F(8.76) < 10^{-9}$. For the κ_L and κ_P approximations of this lognormal distribution with $\mu = 0$ and $\sigma = 1/3$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.006375, & \sup|\hat{F}_L(x) - F(x)| &= 0.0006929, \\ ||\hat{f}_L - f||_1 &= 0.005285, & ||\hat{f}_L - f||_2 &= 0.004286, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.003786, & \sup|\hat{F}_P(x) - F(x)| &= 0.0007492, \\ ||\hat{f}_P - f||_1 &= 0.003216, & ||\hat{f}_P - f||_2 &= 0.002258. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles, or L-moments.

17.4.9 The Beta Distribution

Section 3.4.9 defines the beta distribution with its parameters $\beta_3 > -1$ and $\beta_4 > -1$. As done in Sections 3.4.9, 5.4.9, and 6.4.9, we consider here the two cases where $\beta_3 = \beta_4 = 1$ and $\beta_3 = -1/2$, $\beta_4 = 1$.

When $\beta_3 = \beta_4 = 1$, we have from Section 6.4.9,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.5, 0.1286, 0, 0.007193) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.05593),$$

which leads us to the L-moment fit

$$\kappa_L(0.3323, 0.3669, 0.4745, 0.5584)$$

with support $[-0.007, 0.895]$.

For a percentile fit to this beta distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0.5, 0.6084, 1, 0.5708)$$

which leads to the fit

$$\kappa_P(0.3324, 0.3679, 0.4743, 0.5634),$$

that has support $[-0.0087, 0.985]$.

Figure 17.4–6 (a) shows the plots of the density function of the beta distribution with $\beta_3 = \beta_4 = 1$, along with density functions of its κ_L and κ_P approximations and Figure 17.4–6 (b) shows the c.d.f.s of these distributions. We can see some separation of the curves near both endpoints in Figure 17.4–6 (a).

For the κ_L and κ_P approximations of the beta distribution with $\beta_3 = \beta_4 = 1$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.03590, & \sup|\hat{F}_L(x) - F(x)| &= 0.0008426 \\ ||\hat{f}_L - f||_1 &= 0.006256, & ||\hat{f}_L - f||_2 &= 0.007187, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.07787, & \sup|\hat{F}_P(x) - F(x)| &= 0.001592, \\ ||\hat{f}_P - f||_1 &= 0.006230, & ||\hat{f}_P - f||_2 &= 0.009211. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through percentiles or L-moments but not as good as the GLD approximation based on moments.

When $\beta_3 = -1/2$, $\beta_4 = 1$, we have from Section 6.4.9,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.2, 0.1125, 0.03653, 0.01023)$$

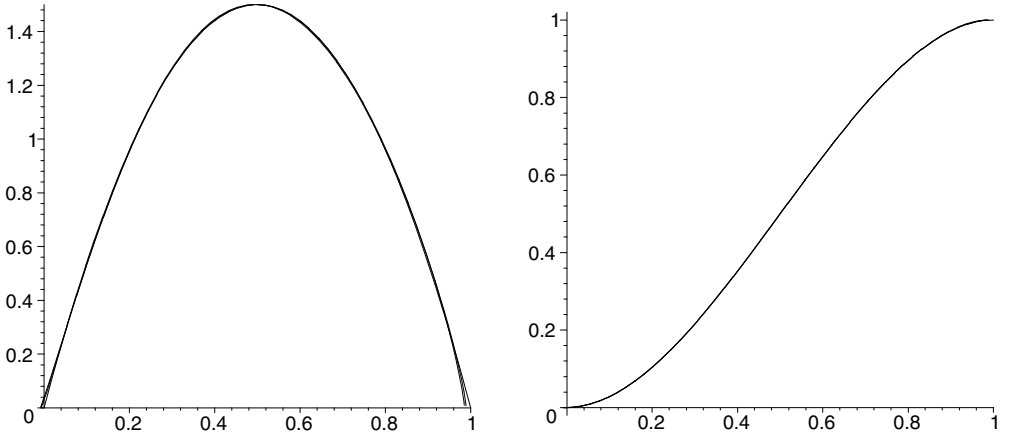


Figure 17.4–6. The p.d.f.s of the beta distribution with $\beta_3 = \beta_4 = 1$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of these distributions (b).

and $(\tau_3, \tau_4) = (0.3247, 0.09093)$, which leads us to the L-moment fit

$$\kappa_L(-0.3724, 0.6166, 2.0188, 0.4407)$$

with support $[0.00015, 1.03]$.

For a percentile fit to this beta distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0.1206, 0.5274, 0.2824, 0.5364)$$

which leads to the fit

$$\kappa_P(-0.3647, 0.6049, 2.0089, 0.4290),$$

with support $[5 \times 10^{-5}, 1.05]$.

Figure 17.4–7 (a) shows the plots of the density function of the beta distribution with $\beta_3 = -1/2$, $\beta_4 = 1$, along with density functions of its κ_L and κ_P approximations and Figure 17.4–7 (b) shows the c.d.f.s of these distributions. In both Figures 17.4–7 (a) and (b) it seems that all three curves are coincident. However, a large gap develops as the curves approach the vertical axis. This becomes evident when we consider

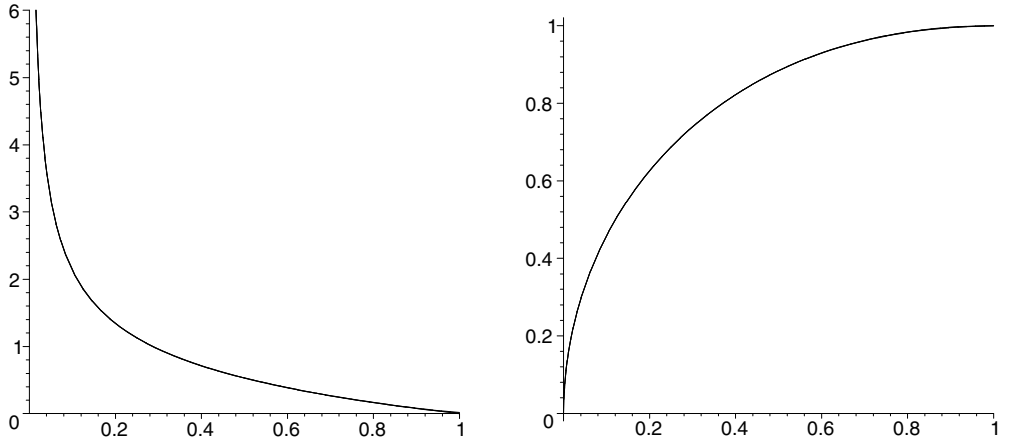
$$f(0.001) = 23.69 \quad \hat{f}_L(0.001) = 25.98 \quad \hat{f}_P(0.001) = 24.44,$$

and when we take $x = 0.00015$, close to the left support endpoint of κ_L, we have

$$f(0.00015) = 61.23 \quad \hat{f}_L(0.00015) = 885.92 \quad \hat{f}_P(0.00015) = 76.19.$$

At $x = 5.1 \times 10^{-5}$, near the left support endpoint of κ_P, we get

$$f(5.1 \times 10^{-5}) = 104.80 \quad \text{and} \quad \hat{f}_P(5.1 \times 10^{-5}) > 10^5.$$



(a) (b)

Figure 17.4–7. The p.d.f.s of the beta distribution with $\beta_3 = -1/2$, $\beta_4 = 1$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of these distributions (b).

For the κ_L and κ_P approximations of the beta distribution with $\beta_3 = -1/2\, \beta_4 = 1$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &> 824.689, & \sup|\hat{F}_L(x) - F(x)| &= 0.01651, \\ ||\hat{f}_L - f||_1 &= 0.03211, & ||\hat{f}_L - f||_2 &= 0.3254, \\ \sup|\hat{f}_P(x) - f(x)| &> 10^5, & \sup|\hat{F}_P(x) - F(x)| &= 0.01836, \\ ||\hat{f}_P - f||_1 &= 0.01651, & ||\hat{f}_P - f||_2 &= 0.08406. \end{aligned}$$

17.4.10 The Inverse Gaussian Distribution

As we have done on previous occasions (see Sections 3.4.10, 4.4.10, 5.4.10, and 6.4.10), we consider the inverse Gaussian distribution with its parameters $\mu = 0.5$ and $\lambda = 6$. In this situation we have, from Section 6.4.10,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0.5, 0.07944, 0.01097, 0.01066)$$

and $(\tau_3, \tau_4) = (0.1381, 0.1342)$, which leads us to the L-moment fit

$$\kappa_L(0.4327, 0.1242, 0.05403, 0.06744)$$

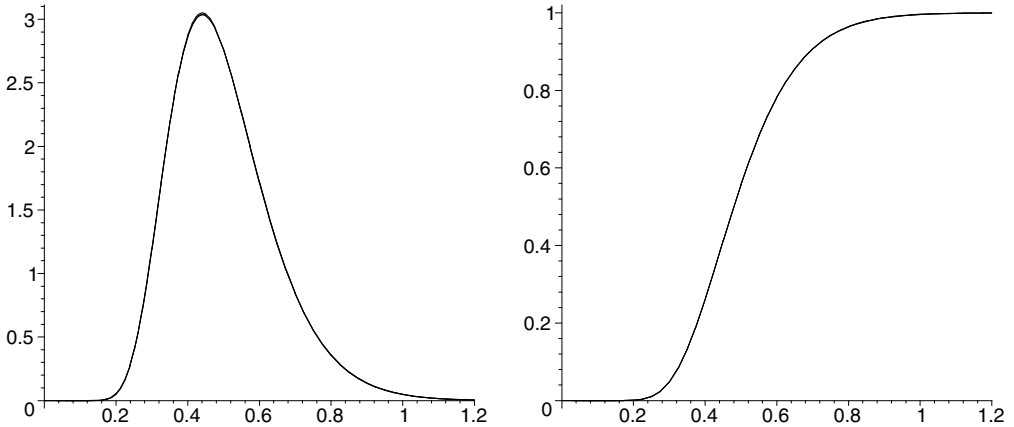
with support [0.032, 2.28].

For a percentile fit to this distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0.4801, 0.3578, 0.6915, 0.5197)$$

and the fit

$$\kappa_P(0.4317, 0.1258, 0.06968, 0.07864),$$



(a) (b)

Figure 17.4–8. The p.d.f.s of the inverse Gaussian distribution with $\mu = 0.5$, $\lambda = 6$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of these distributions (b).

with support [0.058, 2.03].

Figure 17.4–8 (a) shows the plots of the density function of the inverse Gaussian distribution with $\mu = 0.5$, $\lambda = 6$, along with density functions of its κ_L and κ_P approximations and Figure 17.4–8 (b) shows the c.d.f.s of these distributions. The sets of three curves in each of these figures seem to be coincident and for the κ_L and κ_P approximations of the inverse Gaussian distribution with $\mu = 0.5\,\lambda = 6$, we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.03590, & \sup|\hat{F}_L(x) - F(x)| &= 0.0008426 \\ ||\hat{f}_L - f||_1 &= 0.005661, & ||\hat{f}_L - f||_2 &= 0.007187, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.07787, & \sup|\hat{F}_P(x) - F(x)| &= 0.001592, \\ ||\hat{f}_L - f||_1 &= 0.005449, & ||\hat{f}_L - f||_2 &= 0.009211. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P provide approximations that are better than those provided by the GLD through moments, percentiles or L-moments.

17.4.11 The Logistic Distribution

It was pointed out in Section 17.1 (see Table 17.1–4) that the logistic distribution is a special case of the kappa distribution with $h = -1$ and $k = 0$. When we compare the distribution function of the kappa with $h = -1$ and $k = 1$ to that of the logistic (given in Section 3.4.11), we see that the two are identical when μ is set to ξ and α is set to σ. Therefore, the exact kappa fit to the logistic distribution with parameters μ and σ is $\kappa(\mu, \sigma, -1, 0)$.

17.4.12 The Largest Extreme Value Distribution

It can easily be seen that the largest extreme value distribution, defined through its distribution function in Section 3.4.12, is the special case of the kappa distribution with $\xi = \mu$, $\alpha = \sigma$, $h = 0$, $k = 0$. This gives us the exact kappa fit of $\kappa(\mu, \sigma, 0, 0)$ to the extreme value distribution with parameters μ and σ.

17.4.13 The Extreme Value Distribution

Like the largest extreme value distribution, this too is a special case of the kappa distribution.

17.4.14 The Double Exponential Distribution

The double exponential distribution with parameter λ, defined in Section 3.4.14, has

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (0, 3\lambda/4, 0, 17\lambda/96) \quad \text{and} \quad (\tau_3, \tau_4) = (0, 0.2361).$$

It is clear from Figure 17.2–2 that this (τ_3, τ_4) is not covered by the kappa distribution and a kappa fit that is based on L moments cannot be attained for any double exponential distribution.

For a percentile fit, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 2\lambda \ln 5, 1, \ln 2/\ln 5) = (0, 3.2189\lambda, 1, 0.4307)$$

and (ρ_3, ρ_4) is not in the region covered by the kappa distribution (see Figure 17.3–1 (b)). Thus, percentile-based approximations to any double exponential distribution cannot be attained from the kappa distribution.

17.4.15 The F-Distribution

The F-distribution with parameters ν_1 and ν_2, designated by $F(\nu_1, \nu_2)$, was defined in Section 3.4.15 and fits to this distribution have been attempted for several (ν_1, ν_2) combinations. For $F(6, 25)$,

$$(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.0870, 0.3806, 0.09666, 0.06319)$$

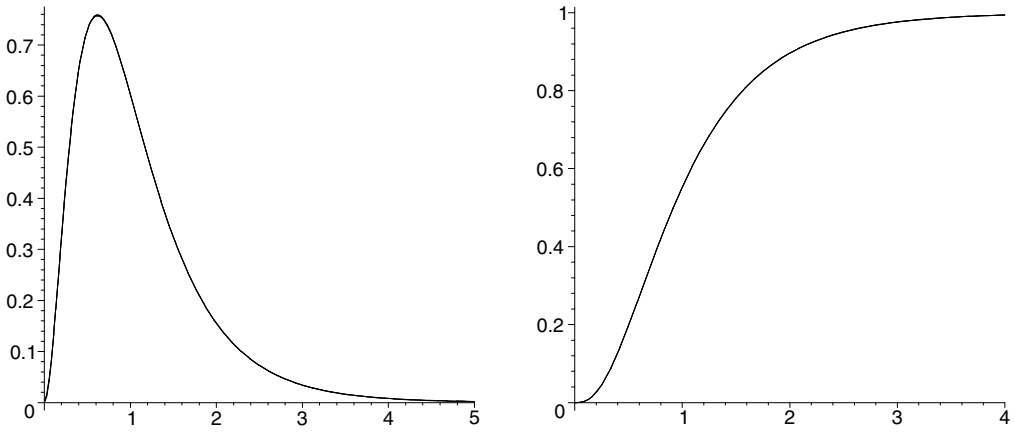
and $(\tau_3, \tau_4) = (0.2540, 0.1660)$, leading us to the L-moment fit

$$\kappa_L(0.6376, 0.5794, 0.3159, -0.04872),$$

with support $[-0.012, \infty)$.

For a percentile fit to this F-distribution, we have

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0.9158, 1.6688, 0.5058, 0.5031)$$



(a) (b)

Figure 17.4–9. The p.d.f.s of the $F(6, 25)$ and the κ_L, κ_P p.d.f.s (a); the c.d.f.s of these distributions (b).

that leads to the fit

$$\kappa_P(0.6343, 0.5841, 0.3254, -0.04222),$$

with support $[-6.06, \infty)$.

Figure 17.4–9 (a) shows the plots of the density function of $F(6, 25)$, along with density functions of its κ_L and κ_P approximations and Figure 17.4–9 (b) shows the c.d.f.s of these distributions. In both Figures 17.4–9 (a) and (b), the sets of three curves that are being depicted seem coincident.

Since the support of $F(6, 25)$ is $[0, \infty)$ and the supports of κ_L and κ_P extend to negative values, we verify that

$$\hat{f}_L(0) = 0.001328, \quad \hat{F}_L(0) = 5.0 \times 10^{-6}, \quad \hat{f}_P(0) = 0.0004596, \quad \hat{F}_P(0) = 9.1 \times 10^{-7}$$

are not too large. For the κ_L and κ_P approximations of the $F(6, 25)$ distribution we have

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.005023, & \sup|\hat{F}_L(x) - F(x)| &= 0.0004273, \\ ||\hat{f}_L - f||_1 &= 0.003253, & ||\hat{f}_L - f||_2 &= 0.002476, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.003113, & \sup|\hat{F}_P(x) - F(x)| &= 0.0003652, \\ ||\hat{f}_P - f||_1 &= 0.001726, & ||\hat{f}_P - f||_2 &= 0.01219. \end{aligned}$$

The indication from these figures is that both κ_L and κ_P are better approximations to $F(6, 25)$ than the GLD fits obtained through moments, L-moments, and percentiles.

Since, in previous chapters we have also considered approximations to the F-distribution with $(\nu_1, \nu_2) = (2, 4), (4, 6), (6, 12), (6, 16)$ we summarize below the kappa distribution approximations to these F-distributions.

- $(\nu_1, \nu_2) = (2, 4)$:
  $F(2, 4)$ is the same distribution as $\kappa(0, 1, 1, -1/2)$ and has a perfect kappa distribution fit.

- $(\nu_1, \nu_2) = (4, 6)$:
  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.5, 0.7711, 0.3503, 0.2317)$;
  $(\tau_3, \tau_4) = (0.4543, 0.3005)$;
  $\kappa_L(0.4721, 0.7935, 0.5195, -0.3337)$;
  $(\rho_1, \rho_2, \rho_3, \rho_4) = (0.9419, 2.9314, 0.3093, 0.4454)$;
  $\kappa_P(0.4727, 0.7914, 0.5178, -0.3361)$;

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.02767, & \sup|\hat{F}_L(x) - F(x)| &= 0.0004860, \\ ||\hat{f}_L - f||_1 &= 0.002072, & ||\hat{f}_L - f||_2 &= 0.002873, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.02806, & \sup|\hat{F}_P(x) - F(x)| &= 0.0005549, \\ ||\hat{f}_P - f||_1 &= 0.001560, & ||\hat{f}_P - f||_2 &= 0.002978. \end{aligned}$$

- $(\nu_1, \nu_2) = (6, 12)$:
  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.2, 0.4699, 0.13, 0.09643)$;
  $(\tau_3, \tau_4) = (0.2767, 0.2052)$;
  $\kappa_L(0.7843, 0.5543, -0.06331, -0.1717)$;
  $(\rho_1, \rho_2, \rho_3, \rho_4) = (0.9434, 1.9868, 0.4318, 0.4845)$;
  $\kappa_P(0.6276, 0.6306, 0.3429, -0.1498)$;

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.1229, & \sup|\hat{F}_L(x) - F(x)| &= 0.02536, \\ ||\hat{f}_L - f||_1 &= 0.10316, & ||\hat{f}_L - f||_2 &= 0.08418, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.001792, & \sup|\hat{F}_P(x) - F(x)| &= 0.0001204, \\ ||\hat{f}_P - f||_1 &= 0.0005518, & ||\hat{f}_P - f||_2 &= 0.0004995. \end{aligned}$$

- $(\nu_1, \nu_2) = (6, 16)$:
  $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (1.1429, 0.4246, 0.1223, 0.07883)$;
  $(\tau_3, \tau_4) = (0.2880, 0.1857)$;
  $\kappa_L(0.6340, 0.6046, 0.3258, -0.1040)$;
  $(\rho_1, \rho_2, \rho_3, \rho_4) = (0.9300, 1.8290, 0.4652, 0.4935)$;
  $\kappa_P(0.6307, 0.6085, 0.3346, -0.09922)$;

$$\begin{aligned} \sup|\hat{f}_L(x) - f(x)| &= 0.003303, & \sup|\hat{F}_L(x) - F(x)| &= 0.0002951, \\ ||\hat{f}_L - f||_1 &= 0.002093, & ||\hat{f}_L - f||_2 &= 0.001591, \\ \sup|\hat{f}_P(x) - f(x)| &= 0.0007446, & \sup|\hat{F}_P(x) - F(x)| &= 0.0002260, \\ ||\hat{f}_P - f||_1 &= 0.0008201, & ||\hat{f}_P - f||_2 &= 0.0004125. \end{aligned}$$

17.4.16 The Pareto Distribution

The Pareto distribution, with positive parameters β and λ, was defined in Section 3.4.16. In Chapter 3 we determined that the Pareto distribution is a special case

of the GLD and Table 17.1–4 indicates that the Pareto is also a special case of the kappa distribution. The particular Pareto distribution that has been considered previously is the one with $\lambda = 1$ and $\beta = 5$ and this distribution can be realized as $\kappa(1, 1/5, 1, -1/5)$.

17.5 Examples of Kappa Distribution Fits to Data

This section uses the L-moment and percentile methods developed in this chapter to fit kappa distributions to the datasets that have been fitted, not always successfully, in previous chapters. The results described in the following subsections are based on the output of the programs `FitKappaLA(Data)`, `FitKappaLB(Data)`, or `FitKappaP(Data)`. For all three programs `Data` is a numeric vector that represents the items in the dataset under consideration. Each of the first two of the programs, if successful, produce an L-moment kappa distribution fit based on solution searches in different parts of (h, k)-space. The third program, `FitKappaP` produces one or more kappa fits based on percentiles. Additional information related to a fit is also printed as these programs execute. In the case of an L-moment fit, this information consists of the sample L-moments, ℓ_1, ℓ_2, ℓ_3, ℓ_4; the $\hat{\tau}_3$ and $\hat{\tau}_4$ values obtained from ℓ_3/ℓ_2 and ℓ_4/ℓ_2, respectively; the value of $\max(|\tau_3 - \hat{\tau}_3|, |\tau_4 - \hat{\tau}_4|)$ for the fit; the support of the fitted distribution; the minimum and maximum data entries; and the Kolmogorov-Smirnov statistics associated with the fit. Similar information is provided for percentile fits.

17.5.1 Example: Cadmium Concentration in Horse Kidneys

This data was introduced in Section 3.5.2 and when we apply the **R** command `FitKappaLB(Data)`, where `Data` has been assigned the values of this dataset, we get

$$\ell_1 = 57.2441, \quad \ell_2 = 13.8771, \quad \ell_3 = 0.6352, \quad \ell_4 = 1.6543$$

and the kappa fit with $\xi = 48.4477$, $\alpha = 22.4275$, $h = -0.05538$, $k = 0.18114$. Following the notation established in the previous section we will denote this fit by

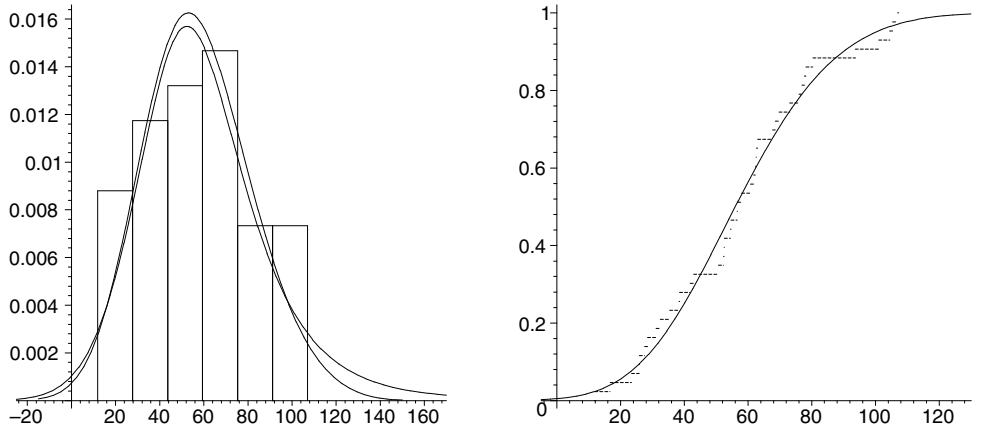
$$\kappa_L(48.4477, 22.4275, -0.05538, 0.18114).$$

The support of κ_L is $(-\infty, 172.2606]$ and it covers all values of the data which has a minimum of 11.9 and a maximum of 107.

In a similar way, we attempt to find a percentile fit by first noting that

$$\hat{\rho}_1 = 56.7, \quad \hat{\rho}_2 = 72.24, \quad \hat{\rho}_3 = 0.7458, \quad \hat{\rho}_4 = 0.4817.$$

However, `FitKappaP` is not able to return a solution. It is clear from Figures 17.3–1 (b) that this $(\hat{\rho}_3, \hat{\rho}_4)$ is at the very edge of the region where percentile fits are available. Also, the table of Appendix G indicates that a solution is not available for this $(\hat{\rho}_3, \hat{\rho}_4)$. If we use



(a) (b)

Figure 17.5–1. Histogram of cadmium concentration and the κ_L (higher at its peak), κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

```
> R <- FindRhos(Data)
> RefineSearchKappaP(R, c(-2, -0.0001), c(-2, -0.0001), 50, 4),
```

we get the fit

$$\kappa_P(57.8190,\ 15.4468,\ -1.18688,\ -0.1619)$$

which has support $[-37.55, \infty)$ and a rather large $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|) = 0.01182$.

Figure 17.5–1 (a) shows a histogram of the data with the p.d.f.s of κ_L and κ_P (the κ_L p.d.f. rises to a higher peak) and Figure 17.5–1 (b) shows the empirical distribution function of the data with the c.d.f.s of κ_L and κ_P. Although Figure 17.5–1 (b) shows no visible distinctions between the κ_L and κ_P distribution functions, there is a clear separation of the p.d.f. curves.

As we have done in previous chapters, we use a chi-square and Kolmogorov-Smirnov test to get some sense of the quality of the κ_L and κ_P fits. For a chi-square test, when the intervals

$$(-\infty,\ 30),\quad [30,\ 50),\quad [50,\ 60),\quad [60,\ 70),\quad [70,\ 85),\quad [85,\ \infty)$$

are used, we get observed frequencies of

$$7,\quad 7,\quad 9,\quad 9,\quad 6,\quad 5$$

and expected frequencies of

$$5.6260,\ \ 11.6924,\ \ 6.9232,\ \ 6.2238,\ \ 6.7546,\ \ 5.7800$$

and

$$5.7107,\ \ 11.2688,\ \ 6.6789,\ \ 5.8241,\ \ 6.0911,\ \ 7.4264$$

for κ_L and κ_P, respectively. The chi-square statistics and p-values for these fits are

$$\kappa_L \text{ fit: chi-Square statistic} = 4.2696, \ \ p\text{-value} = 0.03880$$
$$\kappa_P \text{ fit: chi-Square statistic} = 5.2047, \ \ p\text{-value} = 0.02206.$$

The use of `FitKappaLB` and `FitKappaP` also gives us the Kolmogorov-Smirnov statistics for these two fits, which lead to

$$\kappa_L \text{ fit: K-S statistic} = 0.09118, \ \ p\text{-value} = 0.87$$
$$\kappa_P \text{ fit: K-S statistic} = 0.09294, \ \ p\text{-value} = 0.85.$$

Note that the p-values are obtained by first multiplying the K-S statistics with $\sqrt{n}$, the square root of the number of data entries, and then using Appendix J.

17.5.2 Example: Brain MRI Scan

Following the pattern of the previous example, we find that for this data, introduced in Section 3.5.3,

$$\ell_1 = 106.8348, \ \ \ell_2 = 2.7988, \ \ \ell_3 = -0.1542, \ \ \ell_4 = 0.2401,$$

which leads to the fit

$$\kappa_L(104.8023, 6.0646, 0.1598, 0.4796),$$

with support [86.98, 117.45]. For a percentile fit we get

$$\hat{\rho}_1 = 107.4, \ \ \hat{\rho}_2 = 14, \ \ \hat{\rho}_3 = 1.2876, \ \ \hat{\rho}_4 = 0.5357$$

and the fit

$$\kappa_P(105.5416, 5.6532, -0.03054, 0.4524)$$

that has support $(-\infty, 118.04]$. Both the supports of κ_L and κ_P cover the span of the data which extends from 98.4 to 115.5.

Figure 17.5–2 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P fits (the κ_L p.d.f. rises to a higher peak) and Figure 17.5–2 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. There is a clear separation of the κ_L and κ_P density functions as well as the distribution functions.

For a chi-square test, when the intervals

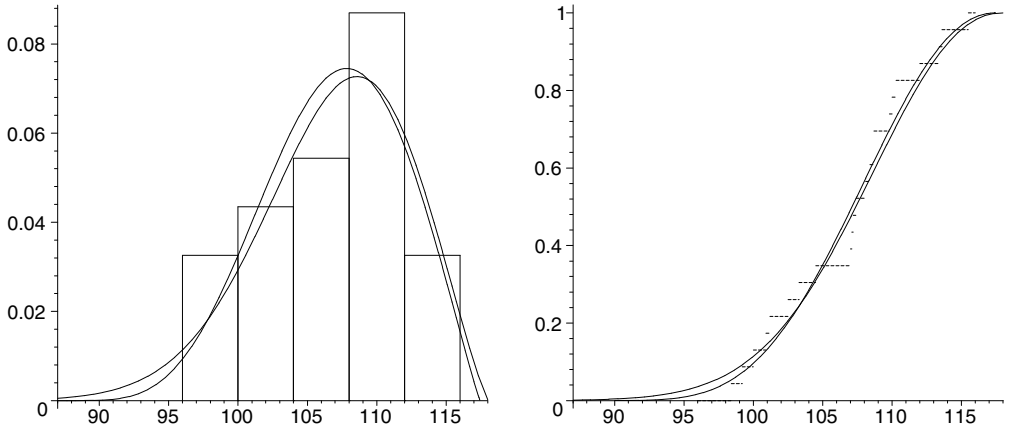
$$(-\infty, 103), \qquad [103, 107), \qquad [107, 111), \qquad [111, \infty)$$

are used, we get observed frequencies of

$$6, \qquad 2, \qquad 11, \qquad 4$$

and expected frequencies of

$$5.2208, \qquad 6.0819, \qquad 6.6021, \qquad 5.0952$$



(a) (b)

Figure 17.5–2. Histogram of MRI scans and κ_L(higher at its peak) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

and

$$5.2772, \quad 5.5720, \quad 6.5356, \quad 5.6152$$

for κ_L and κ_P, respectively. The chi-square statistics for κ_L and κ_P are, respectively, 6.0210 and 5.9031. For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic$= 0.1436$, p-value$= 0.73$
κ_P fit: K-S statistic$= 0.1239$, p-value$= 0.88$.

17.5.3 Example: Human Twin Data

This data, introduced in Section 3.5.4, gives the birth weights of twins. We let X designate the birth weight of the first-born and Y that of the second-born. For X,

$$\ell_1 = 5.4858, \quad \ell_2 = 0.6535, \quad \ell_3 = -0.009286, \quad \ell_4 = 0.07139$$

and the fit

$$\kappa_L(5.1408, 1.1055, -0.06919, 0.2823),$$

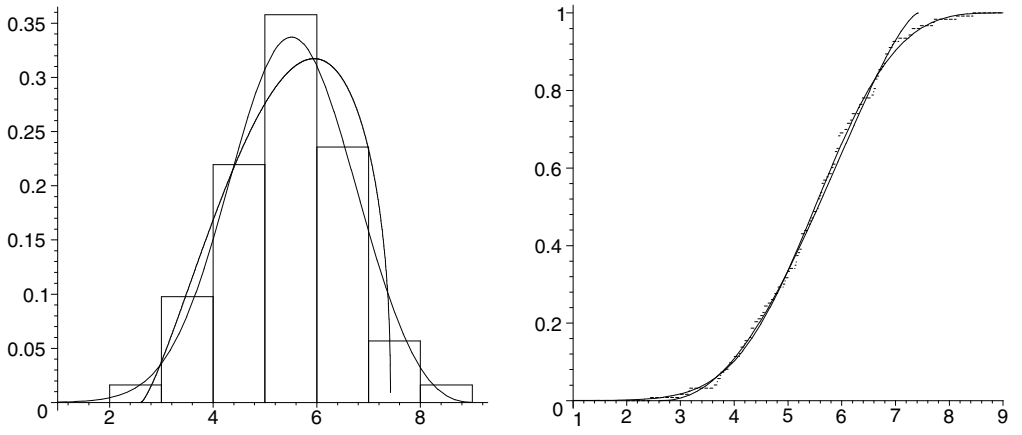
with support $(-\infty, 9.06]$, is obtained. For a percentile fit we have

$$\hat{\rho}_1 = 5.5600, \quad \hat{\rho}_2 = 2.9340, \quad \hat{\rho}_3 = 1.22276, \quad \hat{\rho}_4 = 0.5726$$

and the fit

$$\kappa_P(4.7585, 1.8640, 0.4290, 0.6993)$$

that has support $[2.61, 7.42]$. The support of κ_L covers the span of the data, which extends from 98.4 to 115.5, but the support of κ_P does not.



(a) (b)

Figure 17.5–3. Histogram of X and κ_L(higher at its peak) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

Figure 17.5–3 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P fits (the κ_L p.d.f. rises to a higher peak) and Figure 17.5–3 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. There is a clear separation of the κ_L and κ_P density functions as well as the distribution functions. From these figures it seems that κ_L is the better of the two fits. This observation is reinforced when we perform a chi-square test by using the intervals

$$(-\infty, 4),\ [4, 4.5),\ [4.5, 5),\ [5, 5.4),\ [5.4, 5.8),\ [5.8, 6.2),\ [6.2, 6.8),\ [6.8, \infty),$$

to get observed frequencies of

$$12,\quad 15,\quad 12,\quad 17,\quad 16,\quad 19,\quad 16,\quad 16$$

and expected frequencies of

$$12.5414,\ \ 11.8823,\ \ 16.8887,\ \ 15.9795,\ \ 16.4691,\ \ 15.2374,\ \ 17.7797,\ \ 16.2039$$

and

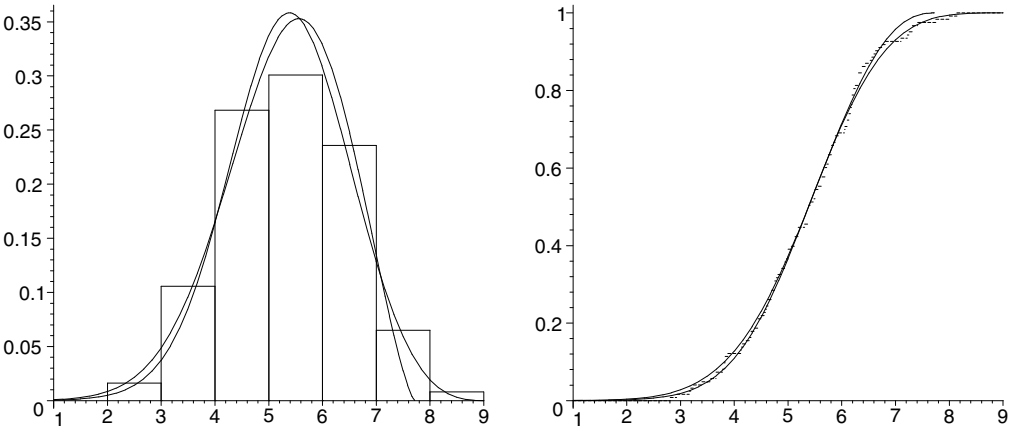
$$13.3959,\ \ 12.2395,\ \ 15.5644,\ \ 14.2733,\ \ 15.2771,\ \ 15.5669,\ \ 21.8130,\ \ 14.8698$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic= 3.4483, p-value= 0.3275
κ_P fit: chi-Square statistic= 4.5316, p-value= 0.2095.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic = 0.03809, p-value = 0.99



(a) (b)

Figure 17.5–4. Histogram of Y and κ_L(lower at its peak) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

$$\kappa_P \text{ fit: K-S statistic} = 0.06336, \;\; p\text{-value} = 0.71.$$

For the birth weights of the second twins, Y,

$$\ell_1 = 5.3666, \;\; \ell_2 = 0.6251, \;\; \ell_3 = -0.01047, \;\; \ell_4 = 0.07481$$

and we have the fit

$$\kappa_L(5.0769, 0.9965, -0.1429, 0.2484),$$

with support $(-\infty, 9.09]$.

For percentile fits we have

$$\hat{\rho}_1 = 5.3800, \;\; \hat{\rho}_2 = 2.8480, \;\; \hat{\rho}_3 = 1.2146, \;\; \hat{\rho}_4 = 0.5372,$$

which leads to

$$\kappa_P(4.9917, 1.1541, -0.01023, 0.4233)$$

that has support $(-\infty, 7.72]$. The support of κ_L covers the data range but that of κ_P does not.

Figure 17.5–4 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P fits (the κ_L p.d.f. rises to a higher peak) and Figure 17.5–4 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. There is a clear separation of the κ_L and κ_P density functions as well as the distribution functions. From these figures it seems that both of these fits are good ones. This observation

is reinforced when we perform a chi-square test by using the same intervals as we did for X to get observed frequencies of

$$15, \quad 11, \quad 19, \quad 17, \quad 16, \quad 19, \quad 16, \quad 10$$

and expected frequencies of

$$13.4353, \quad 13.0393, \quad 18.5987, \quad 17.2937, \quad 17.2102, \quad 15.1100, \quad 16.2036, \quad 12.1092$$

and

$$15.6872, \quad 12.6366, \quad 17.4789, \quad 16.5567, \quad 17.2746, \quad 16.1191, \quad 18.2031, \quad 9.0436$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic = 1.9713, p-value = 0.5784
κ_P fit: chi-Square statistic = 1.1363, p-value = 0.7142.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic = 0.04650, p-value = 0.95
κ_P fit: K-S statistic = 0.04720, p-value = 0.95.

17.5.4 Example: Rainfall Distribution

This data, introduced in Section 3.5.5, consists of rainfall measurements in Rochester NY, designated by X, and in Syracuse NY, designated by Y. For X,

$$\ell_1 = 0.4913, \quad \ell_2 = 0.3105 \quad \ell_3 = 0.1495, \quad \ell_4 = 0.06617$$

and its l-moment fit is

$$\kappa_L(-1.1237, 1.2170, 2.7395, 0.1271),$$

with support [0.027, 8.45]. For a percentile fit we get

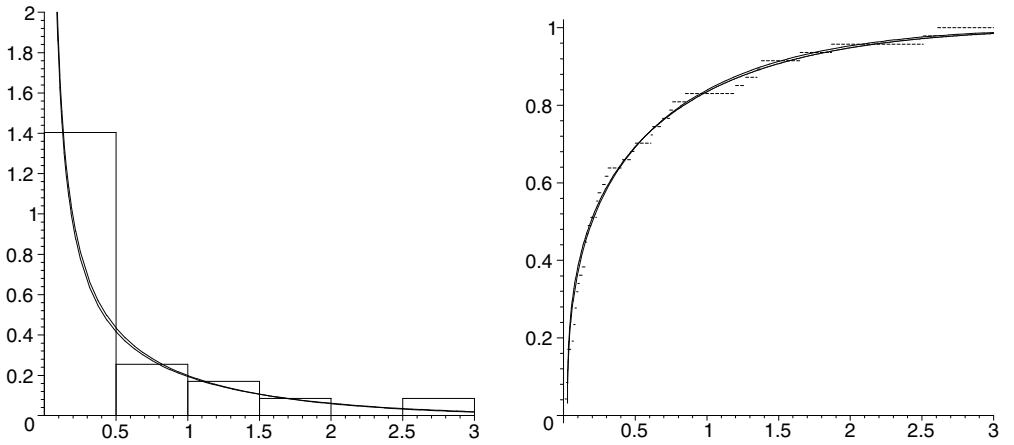
$$\hat{\rho}_1 = 0.1900, \quad \hat{\rho}_2 = 1.4060, \quad \hat{\rho}_3 = 0.1302, \quad \hat{\rho}_4 = 0.4339$$

and the fit

$$\kappa_P(-1.2959, 1.3297, 2.9069, 0.1336)$$

that has support [0.027, 8.66]. The support of neither of these fits covers the data, which ranges from 0.01 to 2.61.

Figure 17.5–5 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P fits and Figure 17.5–5 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. There is a small but definite separation of the κ_L and κ_P density



(a) (b)

Figure 17.5–5. Histogram of X and κ_L(higher at its peak) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

functions as well as the distribution functions. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 0.06), \quad [0.06, 0.1), \quad [0.1, 0.2), \quad [0.2, 0.45), \quad [0.45, 1), \quad [1, \infty),$$

and obtain observed frequencies of

$$8, \quad 7, \quad 9, \quad 7, \quad 8, \quad 8$$

and expected frequencies of

$$13.0585, \quad 4.3416, \quad 6.1272, \quad 7.9309, \quad 7.9945, \quad 7.5472$$

and

$$13.8433, \quad 4.2102, \quad 5.9008, \quad 7.5906, \quad 7.6924, \quad 7.76266$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

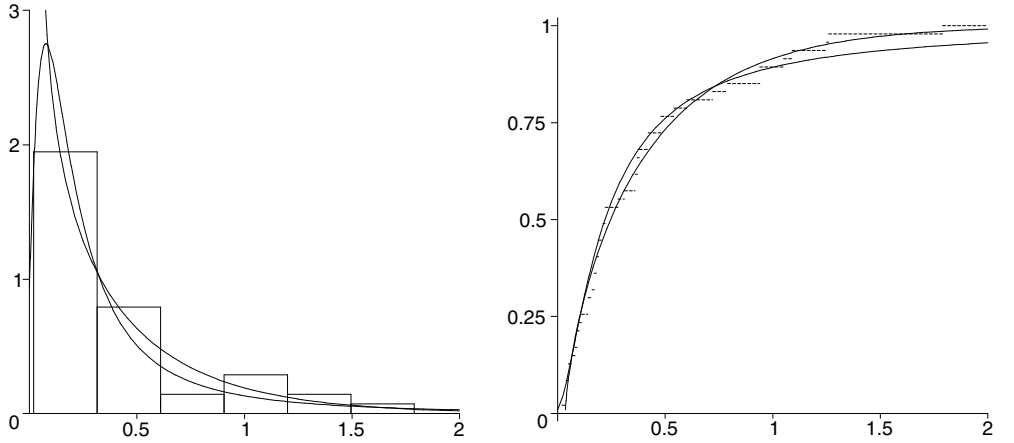
κ_L fit: chi-Square statistic = 5.0707, p-value = 0.02433
κ_P fit: chi-Square statistic = 6.0083, p-value = 0.01424.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic = 0.1145, p-value = 0.57
κ_P fit: K-S statistic = 0.1304, p-value = 0.40.

For the rainfall in Syracuse, NY, Y,

$$\ell_1 = 0.3906, \quad \ell_2 = 2008, \quad \ell_3 = -0.07777, \quad \ell_4 = 0.03654$$



(a) (b)

Figure 17.5–6. Histogram of Y and κ_L (asymptotic to the vertical axis) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

and we have the fit

$$\kappa_L(-0.1756,\, 0.4952,\, 1.5538,\, 0.03472).$$

The support of κ_L, $[0.041, 14.08)$ does not cover the data span.

For a percentile fit we get

$$\hat{\rho}_1 = 0.22, \quad \hat{\rho}_2 = 1.0100, \quad \hat{\rho}_3 = 0.2052, \quad \hat{\rho}_4 = 0.3664$$

and `FitKappaP` does not provide any fits. However, through a search with the use of `RefineSearchKappaP`, we locate the fit

$$\kappa_P(0.1913,\, 0.1559,\, -0.5427,\, -0.7294),$$

whose support, $[-0.022, \infty)$, covers the data span.

Figure 17.5–6 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P fits (the κ_L p.d.f. is asymptotic to the vertical axis) and Figure 17.5–6 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. There is a clear separation of the κ_L and κ_P density functions as well as the distribution functions. From these figures it seems that both fits are reasonable. To perform a chi-square test, we choose the intervals

$$(-\infty, 0.08), \quad [0.08, 0.15), \quad [0.15, 0.23), \quad [0.23, 0.38), \quad [0.38, 0.73), \quad [0.73, \infty)$$

and obtain observed frequencies of

$$7, \quad 7, \quad 11, \quad 6, \quad 8, \quad 8$$

and expected frequencies of

$$9.0326, \quad 7.7093, \quad 5.9768, \quad 7.5729, \quad 9.2713, \quad 7.4378$$

and

$$8.7588, \quad 8.3829, \quad 7.1084, \quad 7.9446, \quad 7.4583, \quad 7.3471$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic = 5.2878, p-value = 0.02147
κ_P fit: chi-Square statistic = 3.2852, p-value = 0.06991.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic = 0.08511, p-value = 0.89
κ_P fit: K-S statistic = 0.09388, p-value = 0.80.

17.5.5 Example: Data Simulated from GBD(3, 5, 0, −0.5)

In this example, the data (listed in Section 4.5.1) is simulated from the generalized beta distribution with parameters 3, 5, 0, and −0.5. For this data,

$$\ell_1 = 6.4975, \quad \ell_2 = 0.7984 \quad \ell_3 = -0.1903, \quad \ell_4 = 0.01103.$$

which leads to the fit

$$\kappa_L(3.4742, 9.8075, 0.9583, 2.1605),$$

with support [3.04, 8.01]. For a percentile fit we get

$$\hat{\rho}_1 = 6.9, \quad \hat{\rho}_2 = 3.654, \quad \hat{\rho}_3 = 2.3833, \quad \hat{\rho}_4 = 0.7115$$

and the fit

$$\kappa_P(-12.3093, 67.4601, 1.6352, 3.3235)$$

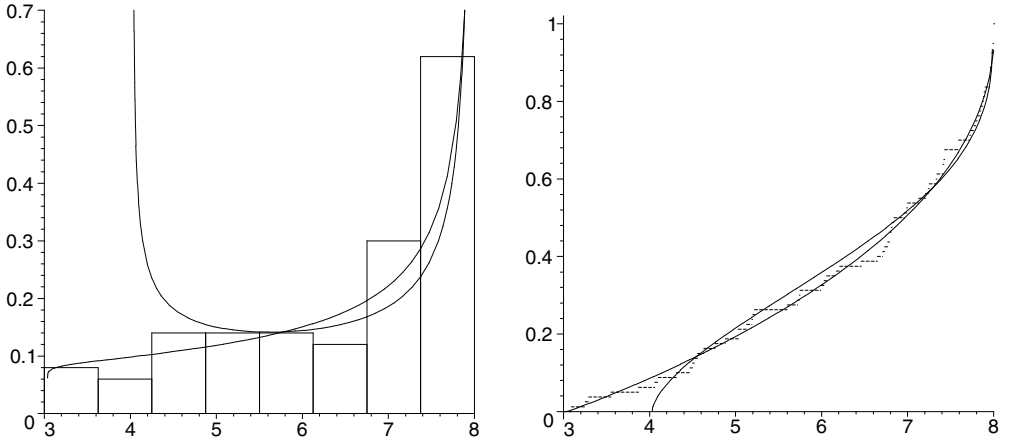
that has support [4.03, 7.99]. The support of κ_L covers the data which ranges from 3.09 to 8 but κ_P leaves portions of the data span, on both extremes — but particularly on the left — uncovered.

Figure 17.5–7 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P ("U" shaped) fits and Figure 17.5–7 (b) shows the empirical c.d.f. of the data with the c.d.f.s of κ_L and κ_P. The fit κ_L seem like a good one and it is clear that κ_P is not a good fit. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 4.5), \quad [4.5, 5.5), \quad [5.5, 6.5), \quad [6.5, 7.0), \quad [7.0, 7.5), \quad [7.5, 7.9), \quad [7.9, \infty),$$

and obtain observed frequencies of

$$9, \quad 12, \quad 10, \quad 11, \quad 12, \quad 12, \quad 14$$



(a) (b)

Figure 17.5–7. Histogram of GBD(3,5,0,-0.5) data and κ_L and κ_P ("U" shaped) p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

and expected frequencies of

$$10.9579,\ \ 9.5285,\ \ 12.1566,\ \ 7.8886,\ \ 10.5402,\ \ 14.4681,\ \ 14.4601$$

and

$$10.7174,\ \ 12.2988,\ \ 11.6684,\ \ 6.7709,\ \ 8.7730,\ \ 13.0306,\ \ 16.7407$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic = 3.2386, p-value = 0.1980
κ_P fit: chi-Square statistic = 4.8796, p-value = 0.08718.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic = 0.06801, p-value = 0.85
κ_P fit: K-S statistic = 0.07500, p-value = 0.76.

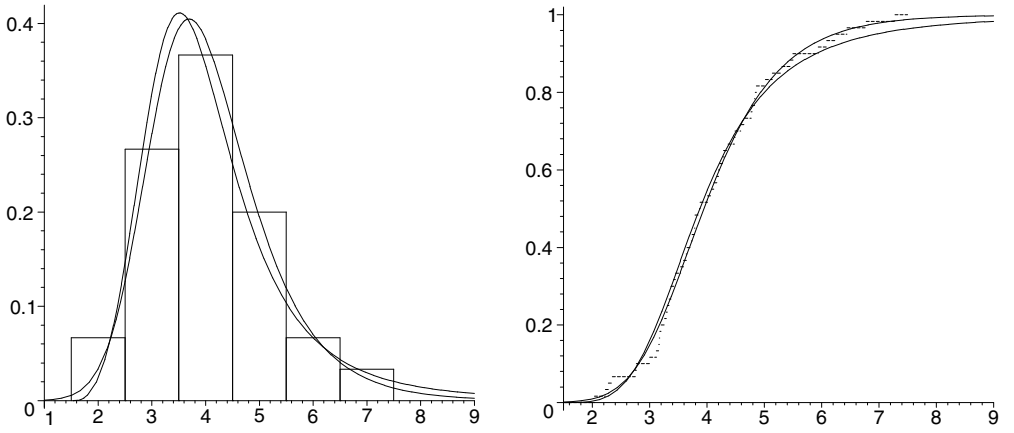
17.5.6 Example: Data Simulated from GBD(2, 7, 1, 4)

This data, simulated from the generalized beta distribution with parameters 2, 7, 1, and 4, is listed in Section 4.5.2. For this data,

$$\ell_1 = 4.1053,\ \ \ell_2 = 0.6276\ \ \ell_3 = 0.08993,\ \ \ell_4 = 0.09960$$

and we get the fit

$$\kappa_L(3.7207,\ 0.8096,\ -0.2514,\ -0.02644),$$



(a) (b)

Figure 17.5–8. Histogram of GBD(2, 7, 1, 4) data and κ_L and κ_P ("U" shaped) p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

with support $[-26.91, \infty)$. For a percentile fit we get

$$\hat{\rho}_1 = 3.875, \quad \hat{\rho}_2 = 3.103, \quad \hat{\rho}_3 = 0.5354, \quad \hat{\rho}_4 = 0.4729$$

and note that $(\hat{\rho}_3, \hat{\rho}_4)$ is close to the boundary of the (ρ_3, ρ_4) region where percentile-based fits are available (see Figure 17.3–1 (b)) and for this reason, the fit that we get,

$$\kappa_P(3.8678, 0.6650, -0.9720, -0.2847),$$

has $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|) \approx 0.003$. The support of κ_L is $[1.53, \infty)$ and the supports of both κ_L and κ_P fits cover the data which ranges from 2.04 to 7.28.

Figure 17.5–8 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P (rises higher at its peak) fits and Figure 17.5–8 (b) shows the empirical d.f. of the data with the d.f.s of κ_L (higher on the right) and κ_P. Both κ_L and κ_P seem to be good fits. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 3), \quad [3, 3.5), \quad [3.5, 4), \quad [4, 4.5), \quad [4.5, 5), \quad [5, \infty),$$

and obtain observed frequencies of

$$6, \quad 14, \quad 11, \quad 11, \quad 7, \quad 11$$

and expected frequencies of

$$8.8303, \quad 10.4376, \quad 11.9687, \quad 10.2526, \quad 7.2827, \quad 11.2282$$

and

$$9.5490, \quad 11.4512, \quad 11.7528, \quad 9.1043, \quad 6.1296, \quad 12.0130$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

$$\kappa_L \text{ fit: chi-Square statistic} = 2.2715, \ p\text{-value} = 0.1318$$
$$\kappa_P \text{ fit: chi-Square statistic} = 2.5383, \ p\text{-value} = 0.1111.$$

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

$$\kappa_L \text{ fit: K-S statistic} = 0.06667, \ p\text{-value} = 0.95$$
$$\kappa_P \text{ fit: K-S statistic} = 0.08376, \ p\text{-value} = 0.79.$$

17.5.7 Example: Tree Stand Heights and Diameters

This data, introduced in Section 4.5.5, consists of pairs (DBH, H), where DBH stands for the diameters (in inches) of trees at breast height and H stands for the heights (in feet) of the trees. For DBH,

$$\ell_1 = 6.740, \quad \ell_2 = 1.4870, \quad \ell_3 = 0.1298, \quad \ell_4 = 0.08624,$$

which leads to the fit

$$\kappa_L(4.2263, 4.3568, 0.6604, 0.4489),$$

with support [2.24, 13.93]. For a percentile fit we get

$$\hat{\rho}_1 = 6.4, \quad \hat{\rho}_2 = 6.8, \quad \hat{\rho}_3 = 0.7436, \quad \hat{\rho}_4 = 0.5882$$

and the fit

$$\kappa_P(3.4113, 5.4917, 0.8834, 0.5929)$$

that has support [2.70, 12.67]. Neither of the supports of κ_L and κ_P covers the data which ranges from 2.2 to 14.8.

Figure 17.5–9 (a) shows a histogram of the data with the p.d.f.s of the κ_L and κ_P (rises higher at its peak) fits and Figure 17.5–9 (b) shows the empirical d.f. of the data with the c.d.f.s of κ_L and κ_P. Neither κ_L or κ_P, particularly the latter, seem to be good fits. To evaluate these fits through a chi-square test, we choose the intervals

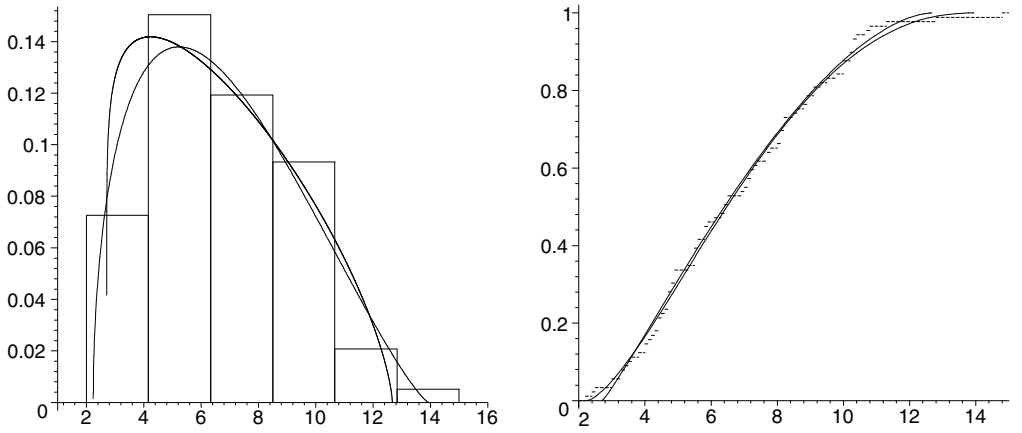
$$(-\infty, 3.75), \quad [3.75, 4.5), \quad [4.5, 5), \quad [5, 6), \quad [6, 7),$$
$$[7, 8), \quad [8, 9), \quad [9, 10.5), \quad [10.5, \infty),$$

and obtain observed frequencies of

$$10, \quad 9, \quad 11, \quad 11, \quad 7, \quad 10, \quad 12, \quad 10, \quad 9$$

and expected frequencies of

$$11.9280, \ 8.6648, \ 6.0746, \ 12.2123, \ 11.6377, \ 10.5138, \ 9.0294, \ 8.8716, \ 10.0677$$



(a) (b)

Figure 17.5–9. Histogram of DBH and κ_L and κ_P (rises higher at its peak) p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

and

$$12.0047,\ 9.4497,\ 6.26346,\ 12.1359,\ 11.3381,\ 10.2959,\ 9.0489,\ 9.2467,\ 9.2165$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic $= 7.5459$, p-value $= 0.1097$
κ_P fit: chi-Square statistic $= 6.7416$, p-value $= 0.1502$.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic $= 0.05120$, p-value $= 0.97$
κ_P fit: K-S statistic $= 0.04712$, p-value $= 0.99$.

For the H portion of the data of this section, we have,

$$\ell_1 = 57.1348,\ \ \ell_2 = 11.0570\ \ \ell_3 = -0.8635,\ \ \ell_4 = 0.3934$$

and we get the fit

$$\kappa_L(40.2236,\, 42.2105,\, 0.5852,\, 0.8748),$$

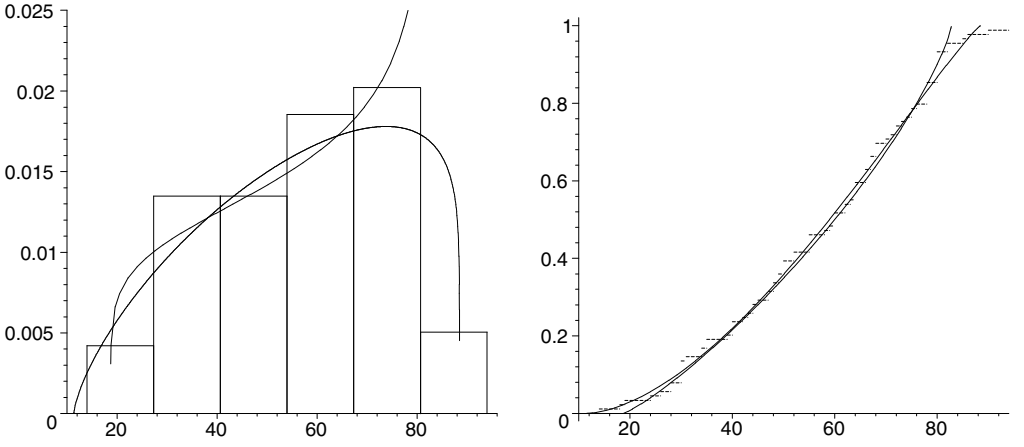
with support [11.37, 88.48]. For a percentile fit we get

$$\hat{\rho}_1 = 60,\ \ \hat{\rho}_2 = 50,\ \ \hat{\rho}_3 = 1.5,\ \ \hat{\rho}_4 = 0.62$$

and the fit

$$\kappa_P(30.4330,\, 67.2230,\, 0.8548,\, 1.2852)$$

that has support [18.75, 82.74]. Neither the support of κ_L nor that of κ_P covers the data which ranges from 14 to 94.



(a) (b)

Figure 17.5–10. Histogram of H and the κ_L (with a peak) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

Figure 17.5–10 (a) shows a histogram of H with the p.d.f.s of the κ_L (with a peak) and κ_P fits and Figure 17.5–10 (b) shows the empirical d.f. of the data with the c.d.f.s of κ_L and κ_P. Both κ_L and κ_P seem to be poor fits. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 30),\quad [30, 37.5),\quad [37.5, 47.5),\quad [47.5, 55),\quad [55, 62.5),$$
$$[62.5, 67.5),\quad [67.5, 75),\quad [75, 80),\quad [80, \infty),$$

and obtain observed frequencies of

$$7,\quad 10,\quad 11,\quad 9,\quad 11,\quad 11,\quad 9,\quad 8,\quad 13$$

and expected frequencies of

$$9.5624,\ 7.2349,\ 11.7960,\ 10.1560,\ 11.0219,\ 7.7009,\ 11.8318,\ 7.8551,\ 11.8409$$

and

$$8.9000,\ 7.5755,\ 11.4573,\ 9.6115,\ 10.6236.\ 7.7906,\ 13.2693,\ 10.8722,\ 8.9000$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic $= 4.1360$, p-value $= 0.3879$
κ_P fit: chi-Square statistic $= 6.5953$, p-value $= 0.1589$.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic $= 0.06563$, p-value $= 0.84$
κ_P fit: K-S statistic $= 0.05870$, p-value $= 0.92$.

17.5.8 Data from the Cauchy Distribution

This data, given in Section 5.6.1, is generated from the Cauchy distribution (or Student's t-distribution with 1 degree of freedom). Since the distribution from which the data is generated does not have a mean (and hence L-moments), it is unlikely that we could obtain a kappa distribution fit through L-moments. For this data,

$$\ell_1 = 0.3464, \quad \ell_2 = 2.3658, \quad \ell_3 = 0.4343, \quad \ell_4 = 1.3295$$

and our expectation is confirmed when `FitKappaL` indicates that a fit is not available for this data.

In this case we also have

$$\hat{\rho}_1 = -0.182, \quad \hat{\rho}_2 = 7.26, \quad \hat{\rho}_3 = 0.6632, \quad \hat{\rho}_4 = 0.2981$$

and, when we use `FitKappaP`, we get an indication that a percentile-based fit is not available.

17.5.9 Example: Radiation in Soil Samples

This data, introduced in Section 5.6.2, has

$$\ell_1 = 2384.842, \quad \ell_2 = 1112.377, \quad \ell_3 = 450.6757, \quad \ell_4 = 343.1844,$$

with $(\hat{\tau}_3, \hat{\tau}_4) = (0.4051, 0.3085)$ located just outside of the region where L-moment-based fits are available. An attempt to get a percentile-based fit yields

$$\hat{\rho}_1 = 1742.28, \quad \hat{\rho}_2 = 4867.311, \quad \hat{\rho}_3 = 0.3776, \quad \hat{\rho}_4 = 0.3881,$$

which places $(\hat{\rho}_3, \hat{\rho}_4)$ outside of the region where kappa distribution fits are available.

17.5.10 Velocities within Galaxies

This data, described in Section 5.6.3, has

$$\ell_1 = 21456.5882, \quad \ell_2 = 981.1686, \quad \ell_3 = -108.7784, \quad \ell_4 = -68.8118,$$

which leads to the fit

$$\kappa_L(3015.542, 56685.91, 1.6732, 2.7647),$$

with support [18578.46, 23519.]. For a percentile fit we get

$$\hat{\rho}_1 = 22417, \quad \hat{\rho}_2 = 4183.6, \quad \hat{\rho}_3 = 3.7671, \quad \hat{\rho}_4 = 0.7682$$

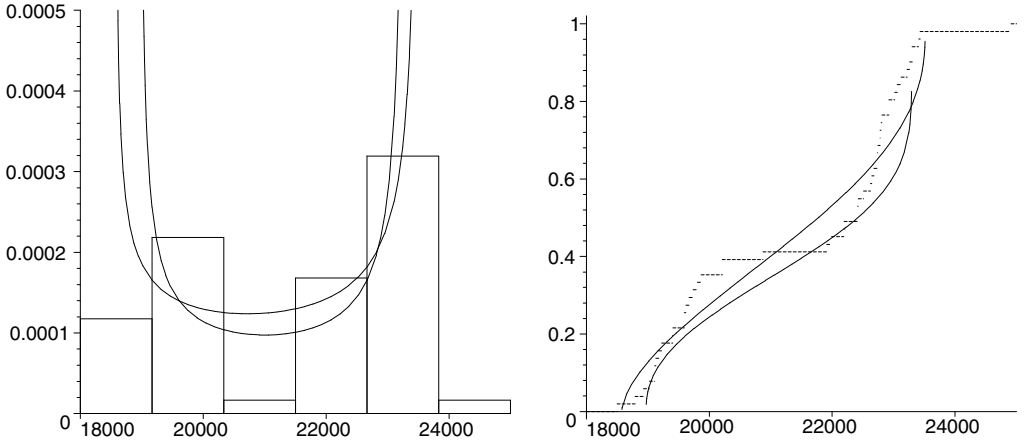


Figure 17.5–11. Histogram of velocities within galaxies and κ_L (lower at center) and κ_P p.d.f.s (a); the data e.d.f. with the c.d.f.s of κ_L and κ_P (b).

and the fit

$$\kappa_P(-1944366., 14203845., 2.3345, 7.2186)$$

that has support [18968.50, 23294.67]. Neither the support of κ_L nor that of κ_P cover the data which ranges from 18499 to 24909.

Figure 17.5–11 (a) shows a histogram of the data with the p.d.f.s of the κ_L (lower at center) and κ_P fits and Figure 17.5–11 (b) shows the empirical e.d.f. of the data with the d.f.s of κ_L and κ_P. Neither κ_L nor κ_P seem to be good fits. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 19125), \quad [19125, 20250), \quad [20250, 21500), \quad [21500, 22750),$$
$$[22750, 23250), \quad [23250, \infty),$$

and obtain observed frequencies of

$$6, \quad 14, \quad 1, \quad 14, \quad 10, \quad 6$$

and expected frequencies of

$$7.6165, \quad 8.0488, \quad 7.9892, \quad 9.6324, \quad 6.1738, \quad 11.5393$$

and

$$5.3023, \quad 8.5210, \quad 6.3482, \quad 7.9626, \quad 8.7112, \quad 14.1546$$

for κ_L and κ_P, respectively. The chi-square statistics and their associated p-values for these fits are

κ_L fit: chi-Square statistic $= 17.8683$, p-value $\approx 2.4 \times 10^{-5}$
κ_P fit: chi-Square statistic $= 17.5870$, p-value $\approx 2.7 \times 10^{-5}$.

For these two fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_L fit: K-S statistic $= 0.1487$, p-value $= 0.21$
κ_P fit: K-S statistic $= 0.2200$, p-value < 0.005.

17.6 Fitting Data Given by a Histogram

Section 3.6 (see Table 3.6–1) gives measurements of coefficients of friction in 11 non-overlapping intervals. The two avenues that we have explored in obtaining fits to this data have made one of two assumptions regarding the data: the data within an interval is located at the center of the interval or the data within an interval is spread uniformly throughout the interval. For this data, under the first of these assumptions, we get

$$\ell_1 = 0.03448, \quad \ell_2 = 0.005328, \quad \ell_3 = 0.0005443, \quad \ell_4 = 0.0007216,$$

which leads to the fit

$$\kappa_{L1}(0.03089, 0.007876, -0.09764, 0.07617),$$

with support $(-\infty, 0.13]$. For a percentile fit we get

$$\hat{\rho}_1 = 0.0325, \quad \hat{\rho}_2 = 0.025, \quad \hat{\rho}_3 = 0.6667, \quad \hat{\rho}_4 = 0.6$$

and the fit

$$\kappa_{P1}(0.01801, 0.02514, 1.1170, 0.6718)$$

that has support $[0.021, 0.055]$. The support of κ_{L1} does cover the data, which ranges from 0.0125 to 0.0625, but the support of κ_{P1} falls short at both endpoints.

Under our second assumption that the data is spread uniformly throughout each interval, we get

$$\ell_1 = 0.03448, \quad \ell_2 = 0.005453, \quad \ell_3 = 0.0005196, \quad \ell_4 = 0.0007312,$$

which leads to the fit

$$\kappa_{L2}(0.03088, 0.008093, -0.1030, 0.08502),$$

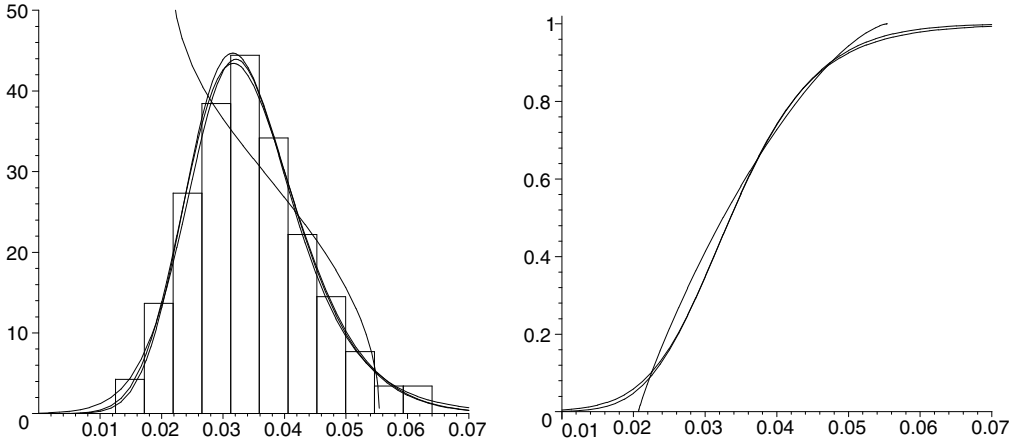
with support $(-\infty, 0.13]$. For a percentile fit we get

$$\hat{\rho}_1 = 0.03352, \quad \hat{\rho}_2 = 0.02531, \quad \hat{\rho}_3 = 0.7786, \quad \hat{\rho}_4 = 0.5009$$

and the fit

$$\kappa_{P2}(0.03264, 0.006416, -0.6427, -0.05397)$$

that has support $[-0.086, \infty)$. The supports of both κ_{L2} and κ_{P2} cover the data, which ranges from 0.0125 to 0.064.



(a) (b)

Figure 17.6–1. Histogram of coefficients of friction and κ_{L1} (highest at its maximum) κ_{P1} (no maximum), κ_{L2} (lowest at its maximum) and κ_{P2} p.d.f.s (a); the c.d.f.s of κ_{L1}, κ_{P1}, κ_{L2} and κ_{P2} (b).

Figure 17.6–1 (a) shows a histogram of the data with the p.d.f.s of the κ_{L1} (with highest maximum), κ_{P2} (next highest maximum), κ_{L2} (with lowest maximum) and κ_{P1} (no maximum) fits. Figure 17.6–1 (b) shows the c.d.f.s of κ_{L1}, κ_{L2}, and κ_{P2}. The empirical d.f. is not included in this figure for two reasons: it is different for each of the two assumptions regarding the location of points within their intervals, and in both cases, it stays close to the c.d.f.s of κ_{L1}, κ_{L2}, and κ_{P2}.

The fits κ_{L1}, κ_{L2}, and κ_{P2} seem to be good ones but κ_{P1} seems to be a very poor fit. To evaluate these fits through a chi-square test, we choose the intervals

$$(-\infty, 0.02),\ \ [0.02, 0.025),\ \ [0.025, 0.03),\ \ [0.03, 0.035),\ \ [0.035, 0.04),$$
$$[0.04, 0.045),\ \ [0.045, 0.05),\ \ [0.05, 0.055),\ \ [0.055, \infty)$$

and obtain observed frequencies of

$$10,\quad 30,\quad 44,\quad 58,\quad 45,\quad 29,\quad 17,\quad 9,\quad 8$$

and expected frequencies of

10.4257, 27.4983, 48.4575, 54.8161, 45.1382, 29.9634, 17.2163, 8.9449, 7.5395;

0, 53.5293, 49.4500, 42.4829, 36.4541, 30.3796, 23.5078, 13.8654, 0.3309;

11.9311, 27.8123, 47.2783, 53.3692, 44.5771, 30.1676, 17.6543, 9.30677, 7.9032;

14.7599, 26.1226, 45.8236, 54.0133, 44.8053, 29.1429, 16.5560, 8.8470, 9.9295

for κ_{L1}, κ_{P1}, κ_{L2}, and κ_{P2}, respectively.

It is clear from Figure 17.6–1 (a) that κ_{P1} is a poor fit and the support of κ_{P1} fails to cover both extremes of the data range. The judgment of the poor quality of the κ_{P1} fit also becomes apparent when we note that the expected frequencies in the first and last intervals of the partition we are using have values of 0 and 0.3309. This last concern makes it impossible to apply a chi-square test for the goodness of κ_{P1} and we drop further considerations of this fit. For the other three fits, the chi-square statistics and their associated p-values are

κ_{L1} fit: chi-Square statistic $= 0.9025$, p-value $= 0.9242$
κ_{L2} fit: chi-Square statistic $= 1.1985$, p-value $= 0.8783$
κ_{P2} fit: chi-Square statistic $= 2.8684$, p-value $= 0.5801$.

For these three fits, the Kolmogorov-Smirnov statistics and associated p-values are

κ_{L1} fit: K-S statistic $= 0.12077$, p-value < 0.005
κ_{L2} fit: K-S statistic $= 0.01570$, p-value > 0.999
κ_{P2} fit: K-S statistic $= 0.02124$, p-value > 0.999.

References for Chapter 17

Hosking, J. R. M. (1994). "The four-parameter kappa distribution," *IBM J. Res, Development*, 38, 251–258.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Method,* Chapman & Hall/CRC, Boca Raton

Mielke, P. W. (1973). "Another family of distributions for describing and analyzing percipitation data," *J. Appl. Meteorol.*, 12, 275–280.

Chapter 18

Weighted Distributional L_α Estimates

Agostino Tarsitano
Dipartimento di Economia e Statistica - Università della Calabria
Via Pietro Bucci, cubo 1C, Rende (CS) - Italy
agotar@unical.it

The fortune of ordinary least squares is built on the myth that disturbances are normally distributed. In fact, this method is the best unbiased estimator under normality while its relative efficiency is reduced when the error distribution is asymmetric, or has a finite lower and/or upper bound, or has heavier tails then the Gaussian. Nevertheless, natural phenomena often produce departures from normality (outliers that appear on one side of the data provide a clear indication that errors are not normally distributed) and many recent findings suggest that the most commonly used estimation methods exhibit varying degrees of nonrobustness to certain violations of the assumption of normality.

In current literature, there is a tacit assumption that the Euclidean distance is the suitable measure for gauging the efficiency of the fit of a regression model. This is not necessarily the case if the distribution of residuals diverges from the normal and other distance functions such as Minkowski metrics become eligible.

Another unrealistic hypothesis underlying standard use of the regression method is that each point on one side of the regression hyperplane provides equally precise information about the deterministic part of the response variable. Elementary accounts of statistical methods commonly give little attention to the possibility that the experimental values analyzed may not all be equally reliable. This is a simplification that we cannot afford to make for a host of practical applications. In situations where it may not be reasonable to assume that every

observation should be treated equally, a weighting system can often be used to maximize the efficiency of parameter estimation.

In practice, many professionals believe that the Gaussian distribution is to be expected to hold since it is called the normal distribution. Therefore, in this chapter we will call the distribution with probability density function (pdf)

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

the (standard univariate) Gaussian distribution, rather than the (standard univariate) normal distribution. A solution to all these problems is to develop efficient estimators of coefficients in a multiple linear regression where the underlying distribution is not-necessarily Gaussian. Since some of the distributional families we will use, such as the EGLD and the FPLD, include distributions which are close to the Gaussian, the Gaussian case is included. In accordance with the line of thought indicated by Parzen, Karian and Dudewicz and Gilchrist, we believe that such a result can be more usefully obtained in the quantile domain $[p, Q(p)]$ than in the probability domain $[x, F(x)]$ or, at least, that the two approaches should be considered on the same footing. In particular, we intend to implement a least squares regression procedure based on a distributional approach in which the stochastic component is parametrically defined either in the Q domain or in the F domain.

18.1 Introduction

The common task of predicting a single interval or metric variable is carried out using the ordinary regression model $y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$ where the errors $\{u_i\}$ are independent and identically distributed. Usually the estimates of the parameters $\boldsymbol{\beta}$ are obtained with no specific reference to the distribution $f(\cdot)$ of the disturbance; only the deterministic component is parametrically defined. If information is needed about the stochastic element, the residuals are used to provide it in a totally separate specification and, in most cases, the information contained in the data is not used sufficiently. A standard assumption in this context is that the error term is distributed as a multivariate Gaussian (alias normal) distribution $N(\mathbf{0}, \sigma\mathbf{I})$ or that gaussianity can be induced by an easily applicable transformation. The Gaussian distribution is evoked in statistical inference procedures concerning the regression parameters, the interval prediction of the response variable as well the model fitting; moreover, the gaussianity is a suitable hypothesis when minimum variance unbiased estimates are required.

There are two good reasons for the prevalence of the Gaussian among real-world distributions. First, the properties of the Central limit theorem. Second, the connected technique of least squares is the best unbiased estimator for the unknown parameters under $N(\mathbf{0}, \sigma\mathbf{I})$, while its relative efficiency is reduced in situations where the error distribution is asymmetric or has heavy tails.

18.1.1 Is the Normal Distribution Normal?

Zeckhauser and Thompson (1970) observe that various conditions may render the Central limit theorem inoperative and, thus, vitiate the *a priori* hypothesis of a Gaussian error distribution. Data involved in life and reliability analysis, process capability, and earth science are rarely Gaussian. (See, for example, Phadnis (2002) and Pyzdek (1999)). In practice, it is not normal to get normal data. Banks have millions of savings accounts, each of which is like a random variable, but unless they are independent and they have finite standard deviations, the central limit theorem is inappropriate. Stochastic volatility models, for instance, do not converge as observations lengthen or convergence is not at a reasonable speed. Casti (2009) notes:

> The key reason fat tails exist in financial market returns is that investors decisions are not fully independent (a key assumption underlying the normal distribution). At extreme lows, investors are gripped with fear and they become more risk-averse, while at extreme market highs investors become irrationally exuberant. This type of interdependence then leads to herding behavior, which, in turn, causes investors to buy at ridiculous highs and sell at illogical lows. This behavior, coupled with random events from the outside world, push market averages to extremes much more frequently than models based on the normal distribution would have one believe.

In many real-world applications, researchers are uncertain about the true distribution of errors and, often, the shape of the distribution is unknown because too few measurements will have been taken to draw a reliably smooth curve. Cooke (1984) noticed that there might, then, be a temptation to assume nevertheless that a Gaussian distribution is applicable and to make an unjustified link between the actual estimates and large sample properties. To increase awareness of the limitations of extreme and dogmatic thinking regarding the Gaussian distribution, Bower (2003) points out that

> ... Some Six Sigma practitioners are encouraged to discover why the data are nonnormal and to continue to look for explanations until normality is obtained. This may be poor advice and frustrate the investigator because, despite best efforts, the assumption of normality frequently cannot (reasonably) be obtained.

A naive application of the Gaussian model can give the user the wrong impression that one can obtain useful inferential results "on demand." Anscombe (1967, p. 16) noted that:

> The disposition of present-day statistical theorists to suppose that all "error" distributions are exactly normal can be ascribed to their ontological perception that gaussianity is too good not to be true.

Several analysts (e.g., Zeckhauser and Thompson (1970), Goldfeld and Quandt (1981), McDonald and Newey (1998), McDonald (1991), and McDonald and White (1993)) have argued against complacency with regards to Gaussian distributed errors.

The researcher often comes up against random errors whose distribution function is either unknown or known not to be Gaussian. In this case, if one is not inclined to make unwarranted assumptions about the errors, then certain practical difficulties arise. For instance, how should we choose $f(\cdot)$ when we do not know the true distribution or we have an insufficient amount of data for a residual analysis (or time to spend on it)? Many not-necessarily Gaussian distributions can be used to model a response, and a reasonable procedure can be determined for each of them; but, if an alternative to the Gaussian distribution is going to be viable, then simple models such as Laplace, or Johnson's SU or simple transformations, such as asymmetric logistic, log-gamma, lognormal, or, log-Weibull distributions usually suffice. If our experiences, feelings, and circumstances point to the errors having a certain type of probability distribution, then we want that distribution to be included as a special case of a more general family of distributions.

One practical approach to dealing with non-Gaussian residuals is the partially adaptive estimation, which fits a model selected from within a general parametric family of distributions to the error distribution of the data being analyzed. The model has some unknown parameters which permit enough flexibility to represent the characteristics of the observed data and to accommodate most commonly encountered distributions without any requirement of symmetry, finiteness of the variance, or support of $(-\infty, +\infty)$. Of course, there must be a good reason for introducing a complex distribution, particularly if it requires more degrees of freedom than many distributions currently in use. Overly sophisticated models are difficult to utilize and identify and they often add more noise than signal to estimation procedures. The instability of higher moments is a case in point.

If the selected family of distribution includes the true error distribution as a special case then the corresponding estimator should perform similarly to the maximum likelihood estimator, except for some possible efficiency loss due to over-parameterization. See Hansen, McDonald, and Turley (2006).

18.1.2 Weighted L_α Regression

The method of ordinary least squares (OLS) has dominated the applications of regression models for decades largely due to its computational simplicity and theoretical elegance. OLS is an optimal procedure when the error distribution is Gaussian or when linear estimates are required (Gauss-Markov Theorem). The tacit assertion is that the Euclidean distance between observed and estimated values is a valuable measure of the efficiency of the fit of a regression hyperplane. It is evident that the validity of this metric diminishes as the underlying error distribution differs from the Gaussian and a few observations can account for most

of the variance. For example, in accounting research, it is almost a commonplace that the underlying distributions could have infinite variances, thus requiring the use of a Minkowski metric L_α with an exponent α which is lower than the Euclidean.

While data may be containing outliers, the method of least squares has a clear disadvantage as it may be pulled by extremely large errors. A valid alternative is the least absolute deviations (LAD) criterion obtained for $\alpha = 1$. The L_1 metric is more robust than L_2 in the sense that the estimate of the regression parameters are less sensitive to outliers in the response variable, but we cannot *a priori* exclude the possibility that some other metric may be able to provide more efficient estimates. If the selection of the flexible density function is based on the Minkowski metric L_α of the residual vector, then not only will the estimation process be based on a weaker assumption on the error term, but the estimator will also exhibit a certain degree of robustness to outliers or to the fatness of the tails. Particularly, if the power of the metric α is data-driven.

Unfortunately, the robustness of partially adaptive estimation (PAE) cannot be extended to the effect of observations that are distant in the space of regressors, where even a single point dragged far enough towards infinity, may cause the regression plane to go through it. One assumption underlying the regression model is that each point on one side of the regression plane provides equally precise information about the deterministic part of the response variable. This is almost never the case in real applications.

In situations when it may not be reasonable to assume that every observation should be treated equally, a weighting system may often be used to maximize the efficiency of parameter estimation. The theory behind this method is based on the assumption that the weights are fixed and known in advance. This assumption rarely holds, so estimated weights must be used instead. In the OLS framework an iterative estimation procedure is invoked and a similar approach could be considered for the partially adaptive estimation. In this chapter we consider two system of weights: one related to the response variable that is considered given; the other is related to the disturbances of the model and is derived from the variances of the order statistics involved in the estimation process.

Iteratively reweighted least squares is part of robust statistics aiming to reduce the influence of single measurements with a very large error on the parameter estimation. A similar approach could also be considered for the estimation based on a general Minkoswki α-metric. In fact, recursive least squares reduces the L_α problem to the solution of a sequence of easily solvable weighted least squares problem.

Partially adaptive estimation has been popularized by McDonald and Newey (1998) in the F world (probability-based approach) and by Gilchrist (2000) in the Q world (quantile-based approach). In both cases the method of maximum likelihood (ML) has been suggested for estimating the unknown parameters. Although ML has several advantages, at least, from a theoretical point of view, its use in

the context of PAE is vulnerable to outliers and nonlinearity of the distributional parameters. For the two approaches we have developed a unifying framework to represent adaptation and, although discussed in two different sections with separate algorithms, the computer program implementing them is the same. The two algorithms are based on a two-stage inner/outer optimization process. The first stage is the inner step in which the linear parameters are estimated for fixed values of the nonlinear parameters. The second stage is the outer step in which a controlled random search (CRS) technique is applied to determine the best companion value for the nonlinear parameters. The two stages are carried out consecutively, and the process is stopped when the difference between the values of the minima of two successive steps becomes smaller than a prefixed tolerance threshold.

The content of this chapter is organized as follows. In Section 18.2, we describe the estimation algorithm for the distributional regression in the F domain. In Section 18.3, we discuss a quantile-based partially adaptive estimation. Section 18.4 is devoted to controlled random search algorithms, especially developed for dealing with global optimization problems which other optimization techniques have difficulties with due to the existence of multiple extremes and/or difficulties in defining functions or their gradient analytically. Section 18.5 explores the evaluation of the goodness-of-fit, which is always problematic when the ordinary least squares are not involved. In Section 18.6 the effectiveness of the proposed methods is explored via application to real data. Conclusions and future research directions are the subject of the final section.

18.2 Probability-Based Partially Adaptive Estimation

In recent years, there has been increasing awareness that departure from gaussianity occurs and that the Gaussian distribution should be considered an exception rather than the rule in applied modeling work. In the meantime, there has been a growing interest in the study of a flexible class of very rich distributional models that cover the Gaussian and other common distributions.

In the (PAE) method the distribution of errors in the linear regression model belongs to a parametric family of distributions which is adaptable enough to capture a wide variety of probability densities of interest in statistics, economics, physical sciences (e.g., agronomy, ecology, climate science, energy systems), health sciences, and general management. The primary objective of PAE is to extract from observed data hidden or implied relationships which were missed or neglected by traditional regression analysis; therefore a common effective framework to obtain full error distribution handling capabilities is established and kept operational for a vast range of applications.

We assume that the data are generated in the following scenario

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i \quad \text{for } i = 1, 2, \cdots, n, \tag{18.2.1}$$

where y_i denotes the response variable of the i-th observation, $\mathbf{x}_i$ is the $m \times 1$ i-th vector of observations of the exogenous variables including, if needed, the intercept term, and $n > m + 1$; the symbol $\boldsymbol{\beta}$ denotes a conformable vector of unknown regression coefficients or regression parameters. Finally, u_i is the error or residual term corresponding to the i-th observation. In this chapter we adopt the standard assumption that the $u_i,\ i = 1, 2, \ldots, n$ are unobservable independent and identically distributed random variables. We also assume that errors are independent of the regressors. Equation (18.2.1) tells us that u_i is distributed according to the same model regardless of the value assumed by $\mathbf{x}_i$.

Suppose we know that the residuals in (18.2.1) are distributed according to the probability density function $f(\mathbf{u}, \boldsymbol{\lambda})$ which, in turn, depends on a vector $\boldsymbol{\lambda}$ of k parameters called **distributional parameters**. In this setting, a random sample $\{y_i, \mathbf{x}_i,\ i = 1, 2, \ldots, n\}$ yields indirect observations on the residuals $\mathbf{u}$ from $f(\mathbf{u}, \boldsymbol{\lambda})$ obtained as $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$ where $\mathbf{X}$ is the design matrix of order $n \times (m+1)$; $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ are the true but unknown values of the parameters. The vector $\boldsymbol{\lambda}$ makes it possible to acquire original and reliable models of the error term, which may be of use in the analysis of the data at hand; it also allows a correct evaluation of the shape of the error distribution: for instance, very diverse tail behavior can be described. If the regression hyperplane has an intercept and $f(\mathbf{u}, \boldsymbol{\lambda})$ is asymmetric, then the estimate of the intercept and the mean of the estimated errors are indistinguishable unless we specify that $E(u_i) = 0,\ i = 1, 2, \ldots, n$.

In the standard scheme of partially adaptive estimation, the error distribution is known up to $\boldsymbol{\lambda}$ so we can obtain efficient estimates using the maximum likelihood (ML) estimation method. An assumption which is implicit in the above procedure is that at least one of the many subordinate models is able to provide a suitable approximation to the true distribution.

Given the observations and the model, we want to minimize

$$S(\boldsymbol{\beta}, \boldsymbol{\lambda}) = -\sum_{i=1}^{n} \ln \left[f\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}; \boldsymbol{\beta}, \boldsymbol{\lambda}\right) \right] \tag{18.2.2}$$

over $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$. A recurrent hypothesis is that the log-likelihood function in (18.2.2) is differentiable; consequently, if ML estimators exist they must satisfy the following partial differential equations

$$\frac{\partial S(\boldsymbol{\beta})}{\partial \beta_j} = \frac{1}{f\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}; \boldsymbol{\beta}, \boldsymbol{\lambda}\right)} \frac{\partial \left[f\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}; \boldsymbol{\beta}, \boldsymbol{\lambda}\right) \right] x_{ij}}{\partial \beta_j} = 0 \quad j = 1, 2, \ldots, m+1 \tag{18.2.3}$$

and

$$\frac{\partial S(\boldsymbol{\beta})}{\partial \lambda_r} = -\frac{1}{\left[f\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}; \boldsymbol{\lambda}\right) \right]} \frac{\partial \left[f\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}; \boldsymbol{\beta}, \boldsymbol{\lambda}\right) \right]}{\partial \lambda_r} = 0 \quad r = 1, 2, \ldots, k \tag{18.2.4}$$

Statistical theory shows that, under standard regularity conditions, ML estimators are invariant to the parameterization, asymptotically unbiased, consistent

and asymptotically efficient irrespective of the sample size and the complexity of the model (this last property means that, in the limit, there is no other unbiased estimator that produces more accurate parameter estimates). Furthermore, the maximum-likelihood method generates, along with the estimates themselves, useful information about the accuracy of the parameter estimates. In fact, likelihood inference offers a convenient apparatus to establish the large-sample properties of partially adaptive estimators. For instance, suppose that each $\mathbf{x}_i$ is redefined as the deviation from its own mean

$$x_{i0} = 1 \quad \text{for } i = 1, 2, \cdots, n, \qquad \sum_{i=1}^{n} x_{ij} = 0 \quad \text{for } j = 1, 2, \cdots, m. \tag{18.2.5}$$

The ML estimate for the regression parameters $\boldsymbol{\beta}$ will be asymptotically independent of the ML estimate of the distributional parameters $\boldsymbol{\lambda}$ included in the error distribution (Cox and Hinkley (1968)). However, when the error distribution is asymmetric none of these estimates gives a consistent estimate of the intercept and therefore, the corresponding prediction of a conditional mean, given the regressors, is also inconsistent. The estimates of the other regression parameters are consistent, but they may lose their high efficiency. If the design matrix $\mathbf{X}$ has its columns centered as in (18.2.5), then the intercept is absorbed in the function. The estimate of the intercept needs a bias-correction when $E(u)$ is not equal to zero.

Since the true distribution function of the errors does not necessarily belong to the hypothesized family $f(\mathbf{u}, \boldsymbol{\lambda})$, the minimization of (18.2.2) should more precisely be called the **pseudo** or **quasi-maximum likelihood method** (Gourieroux, Monfort, and Trognon (1984)). However, if the estimated density approximates the underlying distribution well, the efficiency is expected to be close to that of the ML estimation based on knowledge of the actual distribution of the errors.

18.2.1 Not-Necessarily Gaussian Error Distributions

The crucial point of partially adaptive estimation is the distributional family used for modeling the residuals of the linear regression model. The higher the degree of flexibility of the function the better tends to be the approximation to the true model. Moreover, the models of the error distribution must be sufficiently different to allow their respective parameters to be reliably estimated from the residuals. This is the issue of parameter identifiability discussed, for example in Dee and Da Silva (1999): no method can produce meaningful estimates of poorly identifiable parameters.

18.2.1.1 Skewed Generalized t Distribution

There are several proposals in the literature that can be considered valid candidates and are, at the same time, analytically tractable. One example is the

skewed generalized t (SGT) discussed, among others, in Hansen, McDonald, and Turley (2006). The probability density function of this distribution is

$$f(u;\boldsymbol{\lambda}) = \frac{C_1}{\left\{1+\lambda_3^{-1}\left[\frac{|u|}{\lambda_1\left(1+\lambda_4 \text{sign}(u)\right)}\right]^{\lambda_2}\right\}^{\lambda_3+\frac{1}{\lambda_2}}}, \tag{18.2.6}$$

where

$$C_1 = \frac{\lambda_2}{2\lambda_1\lambda_3^{\frac{1}{\lambda_2}} B\left(\lambda_2^{-1},\lambda_3\right)},$$

with $\lambda_2, \lambda_3 > 0$ and where $\lambda_1 > 0$ is a scale parameter. Larger values of λ_2 and λ_3 yield thinner tails of the density, while smaller values of λ_2 and λ_3 are associated with thicker tails. The parameter $-1 \leq \lambda_4 \leq 1$ introduces the possibility of skewness; a symmetric distribution is obtained for $\lambda_4 = 0$. Setting $\lambda_2 = 2$, formula (18.2.6) yields the skewed t distribution which includes the Student t distribution when $\lambda_4 = 0$.

18.2.1.2 Generalized Beta Distribution

Dudewicz and Karian (1996) and Karian, Dudewicz, and McDonald (1996) use a hybrid distribution, the extended generalized lambda distribution (EGLD), to fit the distribution of observed data. The components are derived from two different domains: the first part is the generalized lambda distribution (GLD) expressed from the Q perspective; the other part is the generalized beta distribution GBD expressed from the F perspective. Since the two distributions cover different areas of the skewness/kurtosis plane, the switch from the two kinds of component depends on the third and fourth sample moments. In the area where GLD and GBD overlap either model may be used.

The GLD will be discussed (although in a different parameterization) in section 18.3.1. The other component is characterized by a linear transformation of the standard form of the beta distribution

$$f(u;\boldsymbol{\lambda}) = \frac{(\lambda_1-\lambda_0)^{-(\lambda_2+\lambda_3-1)}(u-\lambda_0)^{\lambda_2-1}(\lambda_1-u)^{\lambda_3-1}}{B(\lambda_2,\lambda_3)}, \tag{18.2.7}$$

for $\lambda_0 \leq u \leq \lambda_1$. The shape parameters λ_2, λ_3 do not possess a clear-cut meaning. The GBD is used as a general and flexible model to represent distribution of a variable that takes values in any finite range. The moments of GBD exist provided that $\lambda_2, \lambda_3 > 0$

$$E(u) = \lambda_0 + \frac{(\lambda_1-\lambda_0)\lambda_2}{\lambda_2+\lambda_3}, \qquad \sigma^2(u) = \frac{(\lambda_1-\lambda_0)^2\lambda_2\lambda_3}{(\lambda_2+\lambda_3)^2(\lambda_2+\lambda_3+1)}. \tag{18.2.8}$$

If the parameters λ_0, λ_1 are known, the ML estimators of λ_2, λ_3 can be obtained by operating on a standard beta distribution (the shape factors remain unchanged). When both λ_0 and λ_1 are unknown, the support of the GBD depends on the unknown parameters λ_0 and λ_1, which must be estimated from the data: this is a form of non-regularity for the maximum likelihood estimators. Wang (2005) found that if an ML solution exists, then $\hat{\lambda_2}$ and $\hat{\lambda_3}$ must be both greater than one. For the GBD distribution, λ_0 and λ_1 represent the minimal and maximal values, respectively, that an observation may take. If a distribution is skewed to the right (to the left), the minimum (maximum) allowable value λ_0 should be near the first (n-th) order statistics, but nothing can be said about the other. Also, if the shape of the density is symmetrical then there is no reason to assume that any observation is near the allowable minima or maxima. See McGarvey, Del Castillo, Cavalier, and Lehtihet (2002). Only if $\lambda_2 = \lambda_3 = 1$ (uniform distribution) are $\hat{\lambda_0} = u_{(1)}$ and $\hat{\lambda_1} = u_{(n)}$, where $u_{(i)}, i = 1, 2, \ldots, n$, are the order statistics of a random sample of size n drawn from (18.2.7). The many problems with ML tend to support the claim that the use of the likelihood approach is not advisable in this situation. See Carnahan (1989) and Wang (2005).

To use GBD as a model for the residuals of the regression analysis, we have considered a heuristic based upon a mix of the method of moments and the method of maximum likelihood. First, the errors are reduced to the interval $[0, 1]$ by subtracting the observed minimum and dividing by the observed range.

$$u_i^* = \frac{u_i - u_{(1)}}{u_{(n)} - u_{(1)}} \qquad i = 1, 2, \ldots, n \tag{18.2.9}$$

then the ML estimates $\hat{\lambda_2}$ and $\hat{\lambda_3}$ are obtained for the transformed sample (18.2.9). Finally, the end points are estimated solving the equations (18.2.8) for given λ_2 and λ_3

$$\hat{\lambda_1} = \hat{\lambda_3}\hat{\sigma}(u)\sqrt{\frac{\hat{\lambda_2} + \hat{\lambda_3} + 1}{\hat{\lambda_2} + \hat{\lambda_3}}} \quad \text{and} \quad \hat{\lambda_0} = -\hat{\lambda_1}\left(\frac{\hat{\lambda_2}}{\hat{\lambda_3}}\right) \tag{18.2.10}$$

where $\hat{\sigma}(u)$ is the standard error of the residuals and $E(u) = 0$. The solution is feasible if $\hat{\lambda_0} < u_{(1)}$ and $\hat{\lambda_1} > u_{(n)}$.

18.2.1.3 Skewed Generalized Error Distribution

The assumption that the residuals follow an exponential power distribution was one of the first distributional alternatives considered in regression analysis (Zeckhauser and Thompson (1970)). Applications of this model were limited to symmetric residuals. A more flexible family of distributions, which can model either asymmetric and/or thick-tailed empirical distributions, is the skewed generalized

error distribution (SGED) suggested by Bali and Theodossiou (2008):

$$f(u;\boldsymbol{\lambda}) = \frac{\lambda_3}{2C_1\lambda_1\Gamma\left(\lambda_3^{-1}\right)}\exp\left\{-\left[\frac{|u|}{\lambda_1\left(1+\operatorname{sign}(u)\,\lambda_2\right)}\right]^{\lambda_3}\right\}, \tag{18.2.11}$$

where

$$C_1 = \frac{\Gamma\left(\lambda_3^{-1}\right)}{\Gamma\left(3\lambda_3^{-1}\right)\left\{1+\lambda_2^2\left[3-\frac{4\Gamma\left(2\lambda_3^{-1}\right)^2}{\Gamma\left(\lambda_3^{-1}\right)\Gamma\left(3\lambda_3^{-1}\right)}\right]\right\}}.$$

The parameter λ_1 is the standard deviation of the disturbances. The shape parameters obey the following constraints: $\lambda_3 > 0, -1 < \lambda_2 < 1$. The SGED densities are positively (negatively) skewed if $\lambda_2 > 0$ ($\lambda_2 < 0$). The SGED becomes the symmetric generalized error distribution for $\lambda_2 = 0$. In particular, (18.2.11) yields the Laplace distribution for $(\lambda_2 = 0, \lambda_3 = 1)$ and the Gaussian distribution for $(\lambda_2 = 0, \lambda_3 = 2)$. If $\lambda_3 \to \infty$, the SGED tends to the uniform distribution. Lower values of λ_3 generate higher values of the kurtosis. If $\lambda_3 \to \infty$, then the kurtosis decreases and achieves its global lower bound 1.8 corresponding to the uniform distribution. For the SGED, interchanging λ_2 and $-\lambda_2$ changes the skewness from positive to negative, but the kurtosis remains the same.

18.2.1.4 Generalized Beta Distribution of the Second Kind

Another class of analytical models that have a certain importance, is the exponential generalized beta distribution of the second kind (EGB2) introduced by McDonald and Xu (1995) and (1998). The probability density function is

$$f(u;\boldsymbol{\lambda}) = \frac{e^{\frac{\lambda_2 u}{\lambda_1}}}{\lambda_1 B(\lambda_2,\lambda_3)\left(1+e^{\frac{u}{\lambda_1}}\right)^{\lambda_2+\lambda_3}}. \tag{18.2.12}$$

where $\lambda_1 > 0$ is a scale parameter. The density (18.2.12) can also be interpreted as the logarithm of an F statistic with $(2\lambda_2, 2\lambda_3)$ degrees of freedom. The EGB2 densities are positively or negatively skewed according to whether $\lambda_2 > \lambda_3$ or $\lambda_2 < \lambda_3$. The distribution at (λ_2, λ_3) is a reflection around zero of that at (λ_3, λ_2). The density (18.2.12) is symmetric if $\lambda_2 = \lambda_3$. The limiting case in which both λ_2 and λ_3 tend to infinity produces a Gaussian distribution although the approximation is satisfactory for $(\lambda_2 \geq 16, \lambda_3 \geq 16)$. Values of $(1,1), (1,\infty), (\infty,1)$ for (λ_2, λ_3) correspond, respectively, to logistic, extreme value for minima, and extreme value for maxima. For $\lambda_2 = 1$ ($\lambda_3 = 1$) too, the EGB2 gives the logarithm of the Burr/12 (Burr/3) distribution. The lognormal distribution is a well-known special case of EGB2.

18.2.1.5 Johnson's SU

Johnson's SU distribution is an additional family of distributions which is worthy of note in the context of partially adaptive regression. The density function is

$$f(u;\boldsymbol{\lambda}) = \frac{\lambda_2}{\lambda_1\sqrt{1+\left(\frac{u}{\lambda_1}\right)^2}}\phi\left[\lambda_3 + \lambda_2 \sinh^{-1}\left(\frac{u}{\lambda_1}\right)\right], \tag{18.2.13}$$

where $\phi(\cdot)$ is the density function of the standardized normal random variable and $\sinh^{-1}$ is the inverse hyperbolic sine function. The parameter $\lambda_1 > 0$ is a scale parameter; λ_3 is prevalently linked to the skewness of the distribution and λ_2 can be interpreted as a kurtosis parameter. The distribution is positively or negatively skewed according to whether $\lambda_3 < 0$ or $\lambda_3 > 0$; holding the value of λ_3 and increasing that of λ_2 reduces the kurtosis.

The use of the families described above (and many others not mentioned here for reason of brevity: e.g., Lye and Martin (1993), Philips (1994), Tiku, Islam, and Selcuk (2001)) allows exploration, identification, and comparison of data without imposing over-restrictive models. It may be that a dataset could be fitted reasonably well by a subordinate model of the larger distribution, but generalized distributions include this information without presupposing it. See King and MacGillivray (1999).

18.2.2 Estimation of the Parameters

When the density function is known, the ML estimators are solutions of (18.2.3) with respect to the parameters $\beta_j, i = 1, 2, \ldots, m$ and $\lambda_j, j = 1, 2, \ldots, k$. Likelihood functions are rarely sufficiently regular, e.g., convex, so that it is not usually possible to obtain a closed-form solution of the likelihood equations and computationally intensive procedures are required. This is particularly true in the area of partially adaptive estimation because the "flexible" functional forms employed to model the error distribution are highly non-linear and include a large number of parameters. Perhaps, it is useful to recall that these parameters do not end in themselves, but are necessary tools for acknowledging and capturing characteristics associated with many phenomena of statistical interest. However, the widespread availability of versatile and powerful software packages and the improved performance and reduced costs of home computing platforms on which to run them, encourage the regular use of nonlinear parameter estimation when necessary.

A number of multivariable optimization methods are available for computing the ML estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$. The most common techniques are:

(a) To search negative gradient directions successively for the minimum value of the objective function.

(b) To solve iteratively equations (18.2.3)–(18.2.4).

The technique used most often is the direct minimization of $S(\boldsymbol{\beta}, \boldsymbol{\lambda})$ reported in (18.2.2), in which regression and distributional parameters are estimated simultaneously. Most iterative algorithms for numerical iterative optimization of an objective function use the Gauss-Newton method, steepest descent method, or a combination of these two methods. These procedures frequently incorporate a one-dimensional search algorithm and an option for generalized inverses. The usual process starts from an initial estimate of the entire set of parameters; with each iteration the estimates are refined by computing a correction factor for each parameter by using the information in the gradient and in the Hessian (analytically or numerically determined); iteration ceases when the gradient is sufficiently close to zero or the correction factors become sufficiently small. A Newton-Raphson or a quasi-Newton method works well and generates asymptotic standard errors as a by-product of the estimation procedure.

Such algorithms are not immune from common weaknesses: local optima, inappropriate starting values, divergence, slow convergence, and solutions outside the feasible range of the parameters; in some cases the calculation of derivatives is completely impractical except by finite difference approximation. These difficulties are essentially due to the use of a large number of unknowns and to the effect of nonlinearity. A high degree of non-linearity, in fact, can generate very high variability of the estimates, intense correlation between these estimates, and numerical singularities due to heavy cancellation in the density function of the errors. Furthermore, the basic model often nests more simpler models as a limiting case for some parameters that may be difficult to handle numerically.

To circumvent all these difficulties it is often possible to take advantage of special structures that exist in certain types of optimization problems. For instance, if the distribution of the residuals is normal, fitting by maximum likelihood is equivalent to fitting by least squares, but the latter is much simpler. Also, the parameters of the likelihood function need not all be treated as nonlinear; in fact, the replacement of linear parameters by their linear least squares estimates, given the values of the nonlinear parameters, leads to a reduced model involving only nonlinear parameters. (See Lawton and Sylvestre (1971), Oberhofer and Kmenta (1974), Gallant and Goebel (1976)). This can be helpful when the model of the error distribution provides box constraints for the distributional parameters which can be exploited by the optimization algorithm, whereas no significant bounds can be given for the regression parameters. Smyth (1996) discusses some methods, which are aimed at making iterative estimation algorithms more manageable by breaking the iterations into a greater number of more simple or faster steps.

In this section we propose a new algorithm for PAE, which deals at each iteration with a proper subset of the parameters. The method computes the unknown parameters for different groups separately: first, regression parameters $\boldsymbol{\beta}$ are estimated, while considering the distributional parameters $\boldsymbol{\lambda}$ as given and then distributional parameters are computed on the basis of a temporary estimate of the regression parameters. Our objective is to gain numerical efficiency by

using ordinary or weighted estimates of the regression parameters leaving only the distributional parameter subject to a nonlinear optimization routine.

18.2.3 Probability-Based Distributional Regression

Gilchrist (2000) suggested using a quantile function to model the errors in (18.2.1). An observation on the response variable y_i, taken under condition $\mathbf{x}_i$, has p-th quantile

$$y_{p_{r_i}} = \mathbf{x}_i'\boldsymbol{\beta} \; + \; Q\left(p_{r_i}, \boldsymbol{\lambda}\right) \quad \text{for } 0 < p_{r_i} < 1, \tag{18.2.14}$$

where $Q\left(p, \boldsymbol{\lambda}\right)$ is the quantile function of the residuals and p is a uniform random variable in the interval $(0, 1)$. The symbol r_i denotes the anti-rank of $u_i = y_i - \mathbf{x}_i'\boldsymbol{\beta}$, that is $r_i = j$ if and only if u_i is the j-th smallest in a set of the n residuals. The essential requirement of the partially adaptive estimation approach is that the model that describes the residuals, however expressed (in the F or in the Q domain), belongs to a parametric family of distributions spanning a large collection of possible distributions. The merit of this framework over standard regression schemes is that all the regression and distributional parameters arising in the estimation procedure are explicit in the one equation that represents the model. Gilchrist (2000) defined this approach as distributional regression.

An estimator that weights each observation equally can be seriously affected by a few points that deviate from the majority of the sample in the regressor space. In many cases, we may not want to treat all of the errors in the same way so we resort to methods that enable the observations to be weighted unequally in fitting the model. Essentially observations that produce large residuals are downweighted and if some observations are assumed to have greater precision than others, we may want to apply a greater weight. Let $w_i, i = 1, 2, \ldots, n$ be a system of non-negative weights. To fit (18.2.14) we consider the criterion

$$C_\alpha\left(\boldsymbol{\beta}, \boldsymbol{\lambda}\right) = \left[\sum_{i=1}^{n} \left| w_i^{1/\alpha} \left[\left(y_i - \mathbf{x}_i'\boldsymbol{\beta}\right) - Q\left(p_{r_i}, \boldsymbol{\lambda}\right)\right]\right|^\alpha\right]^{1/\alpha} = \text{minimum}, \tag{18.2.15}$$

with $\alpha \geq 1$. For $0 < \alpha < 1$, C_α is no longer a metric because the triangle inequality does not hold. The first addend of (18.2.15) constitutes the deterministic part of the distributional regression criterion and the second addend represents the stochastic or erratic part. The Minkowski metrics are strictly convex, implying that a single minimum always exists (but this is not true for $\alpha = 1$ or $\alpha \to \inf$). We assume that the weights are known. In this case, to solve the weighted problem, it is sufficient to multiply the rows of the design matrix $\mathbf{x}_i$, the component of $\tilde{y}_i^{(t)}$, and the estimated residuals by $w_i^{1/\alpha}$.

To estimate the unknown parameters we used an algorithm which operates on separate parts swinging repeatedly from one group of parameters to another. We conjecture that the value of $\hat{\boldsymbol{\beta}}$ that minimizes (18.2.15) for fixed $\boldsymbol{\lambda}$ and the value

of $\boldsymbol{\lambda}$ that minimizes (18.2.15) for fixed $\boldsymbol{\beta}$ in subsequent iterations, also minimizes the criterion (18.2.15) as a whole.

Let $\tilde{u}^{(0)} = Q\left(p_{r_i}, \boldsymbol{\lambda}\right) = 0,\ i = 1, 2, \ldots, n$ be a first approximation to the regression residuals and define $\tilde{y}_i^{(0)} = y_i - \tilde{u}_{r_i}^{(0)}$. Let $t = 0, 1, 2, \ldots$, be a counter iteration. The first step of the procedure is the calculation of the L_α estimator $\hat{\boldsymbol{\beta}}^{(t)}$ which minimizes

$$S_\alpha\left(\boldsymbol{\beta}\right) = \left[\sum_{i=1}^{n} \left|\tilde{y}_i^{(t)} - \mathbf{x}_i'\boldsymbol{\beta}\right|^\alpha\right] \tag{18.2.16}$$

with respect to $\boldsymbol{\beta}$. The outer exponent has been removed because the value of $\boldsymbol{\beta}$ which minimizes $\left[S_\alpha\left(\boldsymbol{\beta}\right)\right]$ also minimizes $S_\alpha\left(\boldsymbol{\beta}\right)$. (Fletcher, Grant, and Hebden (1971)). Späth (1987) and Gentle (2004), p. 107 applied iteratively reweighed least squares to find the L_α estimate of the regression parameters for given α. The recursion is

$$S_\alpha\left[\boldsymbol{\beta}^{(h+1)}\right] = \min_{\boldsymbol{\beta}} \sum_{i=1}^{n} \left|d_i^{(h)}\left[\tilde{y}_i^{(t)} - \mathbf{x}_i'\boldsymbol{\beta}\right]\right|^2 \qquad h = 1, 2, \ldots. \tag{18.2.17}$$

Expression (18.2.17) implements the procedure of weighted least squares with weights

$$d_i^{(h)} = \left[\frac{1}{\left|\tilde{y}_i^{(t)} - \mathbf{x}_i'\boldsymbol{\beta}^{(h)}\right|}\right]^{\frac{\alpha-2}{2}}. \tag{18.2.18}$$

During the iterative procedure, certain of the $d_i^{(h)}$ will come close to zero causing some problems with the use of the reciprocals of powers as weights. We handled this problem by setting the weight for the i-th residual in the h-th iteration to zero and reinstating it when it became important again. This was suggested in Schlossmacher (1973) (see also Adcock and Meade (1995)). Using the recursion (18.2.17)–(18.2.18) does not always lead to the right answer and has numerical problems. To operate at its best, it needs a good starting point (which is available if the initial least squares estimator is near the final solution) and the data are not too affected by outliers. For the applications discussed in this paper, we have employed the adaptation of the Newton-Raphson's method described in Späth (1987).

In our procedure, the exponent α is not fixed but must be optimized to fit the underlying data. Gonin and Money (1985) and Mineo and Ruggeri (2005) devised a procedure that takes into account the relationship between the index of the metric α and a measure of kurtosis of the residuals. We preferred using a combination of golden section search and successive parabolic interpolation executing the recursion (18.2.17) for each required value of α.

The second step of the probability-based PAE algorithm is the computation of the residuals from (18.2.16),

$$\hat{u}_i^{(t)} = \tilde{y}_i^{(t)} - \mathbf{x}_i'\hat{\boldsymbol{\beta}}^{(t)} \qquad i = 1, 2, \ldots, n, \tag{18.2.19}$$

and determining their anti-ranks, i.e., $\hat{u}_{r_j}^{(t)}, j = 1, 2, \ldots, n$. Of course, a sample $\{u_1, u_2, \ldots, u_n\}$ from the density $f(\mathbf{u}, \boldsymbol{\lambda})$ referred to the linear regression model (18.2.14) cannot be obtained; however, we can use $\hat{u}_i^{(t)}, i = 1, 2, \ldots, n$ as a reasonable proxy for it. In practice, we consider the estimated residuals $\hat{\mathbf{u}}^{(t)}$ a sample of observations from $f(\mathbf{u}, \boldsymbol{\lambda})$ for a given $\boldsymbol{\lambda}$. McDonald and Newey (1998) pointed out:

> The motivation for this approach is the usual one, that the properties of the residuals should be descriptive of the properties of the disturbance distribution, so that it should be possible to adjust the distributional parameters to fit properties of the disturbance distribution by using residuals.

Given the current parameter estimates $\hat{\boldsymbol{\beta}}$ for the deterministic component and $\hat{\alpha}$ for the index of the metric, a complete solution to (18.2.15) requires the minimization of

$$\left[\sum_{i=1}^{n} \left|\hat{u}_{r_i}^{(t)} - Q(p_{r_i}, \boldsymbol{\lambda})\right|^{\alpha}\right]^{1/\alpha} \tag{18.2.20}$$

with respect to $p_{r_i}, i = 1, 2, \ldots, n$ (the cumulated probabilities associated with the theoretical errors) and to the distributional parameters $\boldsymbol{\lambda}$. We have reintroduced the outer exponent $1/\alpha$ because the minimum of (18.2.20) changes depending on the value of α.

If the stochastic component in (18.2.15) is described by one of the models discussed in Section 18.2.1, then virtually all the k parameters in $\boldsymbol{\lambda}$ are nonlinear; moreover, a complex interaction exists between the cumulated percentages $p_{r_i}, i = 1, 2, \ldots, n$ and the regression parameters $\boldsymbol{\beta}$. Consequently, (18.2.20) is very difficult to minimize. A part of the problem is that, in the probability-based version of the distributional regression, we know the density of the residuals whereas (18.2.20) requires the quantile function.

To overcome this obstacle, we must move around the quantile function. Wilson (1983) proposed a regression-based method for fitting cumulative distribution functions from the Johnson distribution system based on sample data. Swain, Venkatraman, and Wilson (1988) extended this estimation method and implemented it as part of an interactive software package for fitting Johnson distributions. Other authors have experimented by using the Wilson method with other models such as the beta, the Weibull or Pareto, and Rayleigh. In this paper, we reconsider this approach not only for errors due to lack of fit of a density, but also for errors of the regression.

Let $F(u, \boldsymbol{\lambda})$ the cumulative distribution function of the residuals. Then, for a fixed value of $\boldsymbol{\lambda}$, we have $P[u \leq u_{r_i}] = F(u_{r_i}, \boldsymbol{\lambda}) \quad i = 1, 2, \ldots, n$. It is well known that if $F(u_{r_i}, \boldsymbol{\lambda})$ is continuous, in view of the probability integral transformation, then $F(u_{r_i}, \boldsymbol{\lambda})$ has uniform distribution in $(0, 1)$ and that, consequently

$$E\left[F\left(u_{r_i},\boldsymbol{\lambda}\right)\right] = \frac{r_i}{n+1} = p_{r_i}, \tag{18.2.21}$$

$$\mathrm{Var}\left[F\left(u_{r_i},\boldsymbol{\lambda}\right)\right] = \frac{r_i\left(n-r_i+1\right)}{\left(n+1\right)^2\left(n+2\right)} = v_{r_i} \quad i=1,2,\ldots,n. \tag{18.2.22}$$

Since actual errors only approximate to true errors, we set

$$F\left(\hat{u}_{r_i}^{(t)},\boldsymbol{\lambda}\right) = \frac{r_i}{n+1} + e_{r_i}, \quad i=1,2,\ldots,n, \tag{18.2.23}$$

where each error e_i has mean $E\left(e_i\right)=0$ and $\mathrm{Var}\left(e_i\right)=v_i$. Abourizk, Halpin, and Wilson (1994) noticed that the basic idea is to view (18.2.23) as a pseudo non-linear regression model even though the unknown parameters appear on the left-hand side as part of the dependent variable rather than on the right-hand side with the regressor variable.

Although this setup appears to be unconventional, it allows for a legitimate application of the principle of least-squares estimation. An L_α estimator for the distributional parameters can now be obtained by minimizing the following expression with respect to $\boldsymbol{\lambda}$

$$T_\alpha\left(\boldsymbol{\lambda}\right) = \left[\sum_{i=1}^{n}\left|F\left(\hat{u}_{r_i},\boldsymbol{\lambda}\right) - \frac{r_i}{n+1}\right|^\alpha\right]^{1/\alpha}. \tag{18.2.24}$$

The quantity (18.2.24) is centered on $F\left(u,\boldsymbol{\lambda}\right)=Pr\left(U\leq u\right)$ and, in this sense, we have defined a probability-based approach. In the absence of a simple formula for $F\left(\hat{u}_{r_i}^{(t)},\boldsymbol{\lambda}\right)$, we must apply a numerical technique for its evaluation. The obvious criticism is that, as it stands, (18.2.24) ignores the high correlations between order statistics and the heterogeneity of their variances. A paper by Lloyd (1952) applies generalized least squares to deal with this in the case where there are only position and scale parameters and there are precise formulas for the variances and covariances. However, a weighted version of (18.2.24) is also available as

$$T_{\alpha w}\left(\boldsymbol{\lambda}\right) = \left[\sum_{i=1}^{n}\left(\frac{1/v_{r_i}}{\sum_{i=1}^{n}\left(1/v_i\right)}\right)\left|F\left(\hat{u}_{r_i},\boldsymbol{\lambda}\right) - \frac{r_i}{n+1}\right|^\alpha\right]^{1/\alpha}. \tag{18.2.25}$$

where $v_i, i=1,2,\ldots,n$ is the u-shaped system of weights (18.2.21). The weights are inversely proportional to the corresponding variances of the i-th order statistics of the current estimated residuals (these weights are specific to the stochastic component of the model and should not be confused with the weights for the expression (18.2.15)). In either case, the non-linear minimization procedure that forms the third step of the algorithm, has to only be applied to the group of parameters associated with the error distribution because the anti-ranks $r_i, i=1,2,\ldots,n$ and α are given. The procedure that we have generally used is

of the direct search type (so that it is not necessary to program the derivative calculations) and this has been demonstrated to be efficient for obtaining global minima in applications with noisy objective function. More specifically, we have determined $\tilde{\boldsymbol{\lambda}}$ by performing a controlled random search (discussed in Section 18.4) either for (18.2.24) or (18.2.25). We observe that now the outer exponent $1/\alpha$ cannot be ignored.

Let $\hat{\boldsymbol{\lambda}}^{(t)}$ be the estimate of the distributional parameters at the t-th iteration obtained in the preceding step and let $\hat{p}_{r_i} = F\left(p_{r_i}, \hat{\boldsymbol{\lambda}}\right), i = 1, 2, \ldots, n$ be the cumulative percentages corresponding to the estimated distributional parameters $\hat{\boldsymbol{\lambda}}$.

The fourth step of our procedure consists of computing the new estimated errors

$$\hat{u}_{r_i}^{(t+1)} = Q\left(\hat{p}_{r_i}, \hat{\boldsymbol{\lambda}}\right) \quad i = 1, 2, \ldots, n \tag{18.2.26}$$

by using the quantile determined by $\hat{p}_{r_i}$ and $\hat{\boldsymbol{\lambda}}$. It should be noted that only rarely can the cumulative distribution function be analytically inverted and it is necessary to compute the quantiles numerically. Note also that residuals here have to be consistent with the zero mean constraint. To this end, we correct (18.2.26) as

$$\tilde{u}_{r_i}^{(t+1)} = \hat{u}_{r_i}^{(t+1)} - n^{-1} \sum_{i=1}^{n} \hat{u}_i^{(t+1)} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{18.2.27}$$

According to Gilchrist (2000), we expect that the criterion $C_\alpha\left(\boldsymbol{\beta}, \boldsymbol{\lambda}\right)$ is decreasing as the number of iterations increases because the two criteria take into consideration the shape of the estimated errors and the procedure goes through successive refinements. However, in several applications, all the criteria proposed to orient the probability-based algorithm were found to exhibit a zigzag behavior: both the Minkowski metric of the regression residuals (either weighted or unweighted) and the negative log-likelihood function, showed ups and downs during the iterations so that a sub-counter of iterations had to be introduced to control the inversion of tendency.

Given the new approximation (18.2.26), a better value for the regression parameters can be established by defining $\tilde{y}_i^{(t+1)} = y_i - \tilde{u}_{r_i}^{(t+1)} \quad i = 1, 2, \ldots, n$. The algorithm is iterated by returning to the first step in (18.2.16) until the correction for the selected criterion becomes sufficiently small.

18.3 Quantile-Based Partially Adaptive Estimation

Emanuel Parzen has encouraged quantile and conditional quantile statistical thinking for the first time in 1976, arguing that it would be immensely valuable in opening up a whole new approach to statistical methods of data modeling. Other

authors have worked along this line: e.g., Gilchrist (2000), Karian and Dudewicz (2000).

The probability law of a continuous univariate random variable U can be specified in two ways

(a) The distribution function $F(u)$ giving the probability that U is no larger than u.

(b) The quantile function $Q(p)$ that, for each $0 < p < 1$, determines the values of u such that $F(u) = p$.

There is an inverse relation of sorts between the quantiles and the cumulative distribution values, but the relation is more complicated than that of a function and its ordinary inverse function, because the distribution function is not necessarily one-to-one. If F strictly increases from zero to one on a given interval so that the underlying distribution is continuous and is supported on that interval, then Q is the ordinary inverse of F.

In general, the inverse of a function is obtained essentially by reversing the roles of independent and dependent variables. A simple exchange of axis does not add new information to that already contained in the conventional coordinate system. If, by the term additional information, one means a mere increase in the number of observations or new relationships and constraints on the parameters, a rotation (either clockwise or counter-clockwise) is neutral. If, on the other hand, the notion of information is properly extended by adding the new perspectives to the same set of data, then even a 90-degree rotation of axis becomes informative. When we look at an unknown object for the first time, we try to decipher its nature and content by turning it around and looking at every part in the hope of seeing what no one else has seen before. In the same way, revisiting the linear regression model from a quantile point of view can be highly relevant and give us insights we wouldn't have had otherwise.

18.3.1 Quantile Models

Statistical literature is not very helpful in providing models to fit empirical distributions naturally arising from a quantile framework (a few examples will be given here taking Gilchrist (2000) as a guideline). It is regrettable that this topic has been neglected in the statistical literature (interesting exceptions are Jones (2007) and Lampasi (2008)) because the explicit form of the quantile function makes it straightforward to fit the empirical distribution of the regression residuals; in fact, the new perspective avoids the intermediate step of a preliminary fitting of an F model to the current residuals. Historically, it was the modeling in the Q domain that was first introduced by Galton in the 1880s but was then largely forgotten (Gilchrist (2008)).

18.3.1.1 The Wakeby Distribution

Quantile functions are often used for modeling extreme floods. To be a reasonable candidate, a distribution has to be able to accommodate positively skewed histograms. Moreover, since the form of empirical distributions encountered in hydrology is found to vary considerably, it seems likely that a model with four or five parameters would be needed to describe the data. The Wakeby distribution (WAD), discussed in Houghton (1977), has gained widespread use in this field due to its reliability in curve fitting, although the estimation of its parameters can be very difficult. The Wakeby quantile function is

$$Q(p,\lambda) = \lambda_0 - \left(\lambda_1 q^{\lambda_3} + \lambda_2 q^{\lambda_4}\right), \qquad q = 1-p \quad 0 \leq \ p \ \leq \ 1, \tag{18.3.1}$$

where λ_0 is a location parameter, λ_1, λ_2 with $\lambda_1 + \lambda_2 > 0$ are linear parameters prevalently related to the scale of the variable and λ_3, λ_4 are exponential parameters determining the shape of the quantile function. The Wakeby distribution has a finite lower bound $\lambda_0 - (\lambda_1 + \lambda_2)$. The upper bound is λ_0 if $\lambda_3 > 0, \lambda_4 > 0$ and infinite if λ_3 or λ_4 or both are negative. The probability density function is defined implicitly by the density quantile function, that is, the density expressed in terms of the cumulative probability p.

$$\frac{1}{\frac{dQ(p,\lambda)}{dp}} = h\left[Q(p;\lambda)\right] = \left[\lambda_3\lambda_1 q^{\lambda_3 - 1} + \lambda_4\lambda_2 q^{\lambda_4 - 1}\right]^{-1} = h(p,\lambda) \tag{18.3.2}$$

The regions in which (18.3.2) is a valid density function are

$$\begin{aligned}
R_1 &: \ \lambda_3\lambda_1 > 0, \lambda_4\lambda_2 \geq 0 \\
R_2 &: \ \lambda_3\lambda_1 \geq 0, \ \lambda_4\lambda_2 > 0 \\
R_3 &: \ \lambda_3\lambda_1 < 0, \ \ \lambda_4\lambda_4 > |\lambda_3\lambda_1|, \lambda_3 \geq \lambda_4 \\
R_4 &: \ \lambda_4\lambda_2 < 0, \ \ \lambda_3\lambda_1 > |\lambda_4\lambda_2|, \lambda_3 \leq \lambda_4
\end{aligned} \tag{18.3.3}$$

The parameters λ_3 and λ_4 determine the type of tails. For example, if $\lambda_3, \lambda_4 > 0$ then (18.3.2) has increasing peakedness and short tails; if $\lambda_3, \lambda_4 < 0$ the tails have increasing heaviness. The Wakeby distribution includes models with a variety of short as well as medium and very long upper tails and generates many standard models as particular cases: uniform, exponential, Pareto. It can give rise to occasional high outliers, a phenomenon often observed in hydrology and finance. Until now no applications of the Wakeby distribution to the regression area have been reported although applications in censored regression models could benefit from it.

18.3.1.2 Class I Quantile Model

Deng and Jiang (2002) proposed a class of stochastic volatility models for electricity prices using quantile functions that exhibit heavy, flexible, and asymmetric

tail behaviors. Specifically, they fit marginal distributions of power prices to two special classes of distributions by matching the quantile of an empirical distribution to that of a theoretical distribution. We only consider distributions from the first class that has closed-form formulae for probability densities, probability distribution functions, and quantile functions

$$Q\left(p,\boldsymbol{\lambda}\right)=\lambda_0+\lambda_1^{\frac{1}{\lambda_3}}\left[\log\left(\frac{p^{\lambda_2}}{1-p^{\lambda_2}}\right)\right]^{\left(\frac{1}{\lambda_3}\right)}, \tag{18.3.4}$$

where $\lambda_1,\lambda_2,\lambda_3>0$. The superscript (a) for $a>0$ represents the following operations

$$x^{(a)}=\begin{cases} x^{\alpha} & \text{if} \quad x>0 \\ 0 & \text{if} \quad x=0 \\ -\left(-x\right)^{a} & \text{if} \quad x<0. \end{cases} \tag{18.3.5}$$

All parameters in (18.3.4) have some intuitive interpretation: λ_0 is a location parameter, λ_1 works as a scale parameter, λ_2 acts as a tail balance adjuster. In particular $\lambda_2=1$ means a balanced tail, and $\lambda_2<(>)\,1$ means that the left (right) tail is fatter than the right (left) tail. Finally, λ_3 indicates the tail order in the sense that the smaller λ_3, the fatter the tail of a distribution belonging to (18.3.4).

The corresponding probability density function can be expressed in either the F domain or the Q domain. The density in the F domain is given by

$$f\left(u,\boldsymbol{\lambda}\right)=\frac{\lambda_1^{-1}u^{(\lambda_3)-1}\exp\left[-\frac{(\lambda_3)}{\lambda_2}\right]}{\left[1+\exp\left[-\frac{(\lambda_3)}{\lambda_2}\right]\right]^{1+1/\lambda_2}} \tag{18.3.6}$$

The appealing features of the quantile approach are that it can effectively model the heavy tail behavior of electricity prices caused by jumps and stochastic volatility and that the resulting distributions are easy to simulate. This latter feature enables users to perform both parameter estimation and derivative pricing tasks based on price data gathered directly from real markets. The application of (18.3.4) developed in Jiang, Chen, and Wu (2008) is in line with the partial adaptive estimation method discussed in this chapter.

18.3.1.3 Davies Distribution

An interesting proposal to fit censored data is the quantile model described in Hankin and Lee (2006), known as the Davies distribution (Gilchrist (2000) defines it as the Power-Pareto distribution), with

$$Q\left(p,\boldsymbol{\lambda}\right)=\lambda_0+\lambda_1\left[\frac{p^{\lambda_2}}{(1-p)^{\lambda_3}}\right], \tag{18.3.7}$$

where, as usual, λ_0 is a location parameter, $\lambda_1 > 0$ is a scale parameter, and $\lambda_2, \lambda_3 > 0$ are shape parameters. It is also possible that for λ_2 or λ_3 (but not both) to be zero. The logistic distribution is obtained as the case $\lambda_2 = \lambda_3 = 1$; the distribution associated with $\lambda_2 = \lambda_3 = 1$ is the Student t distribution with two degrees of freedom. In the general case of $\lambda_2 = \lambda_3$, one can readily obtain the distribution function

$$F(u, \boldsymbol{\lambda}) = \left[1 + \left(\frac{u - \lambda_0}{\lambda_1}\right)^{1/\lambda_2}\right]^{-1} \quad u > \lambda_0. \tag{18.3.8}$$

Except for some special cases, neither the density nor the distribution function is available in closed form, but can be calculated by numerical inversion of the quantile function.

18.3.1.4 Minimax Distributions

Jones (2007) explored a family of distributions on bounded support that has many similarities to the beta distribution. The minimax random variables, so-called because of their genesis in terms of uniform order statistics, have quantile functions that do not depend on special functions (and hence afford very easy random variate generation). Two forms of the minimax distribution are

$$Q_1(p, \boldsymbol{\lambda}) = \lambda_0 + \lambda_1 \left[1 - (1-p)^{\lambda_2}\right]^{\lambda_3} \tag{18.3.9}$$

$$Q_2(p, \boldsymbol{\lambda}) = \lambda_0 + \lambda_1 \left[1 - \left(1 - p^{\lambda_2}\right)^{\lambda_3}\right], \tag{18.3.10}$$

where λ_0 is a location parameter, $\lambda_1 > 0$ is a scale parameter and λ_2, λ_3 are shape parameters such that $\lambda_2 > 0, \lambda_3 > 0$. The standardized versions of (18.3.9) are inverses of one another with the parameters of one equal to the reciprocal of the other. It can be shown that the minimax distributions have the same basic shape properties as the beta distribution. Minimax distributions might have a significant role when a quantile-based approach to statistical modeling is adopted.

18.3.1.5 Five-Parameter Generalized Lambda Distribution

To describe the error distribution of the linear regression model we adopt the five-parameter version of the generalized lambda distribution (FPLD) suggested by Gilchrist (2000) (pp. 163–164), with

$$Q(p, \boldsymbol{\lambda}) = \lambda_0 + \frac{\lambda_1}{2}\left\{(1 - \lambda_2)\left(\frac{p^{\lambda_3} - 1}{\lambda_3}\right) - (1 + \lambda_2)\left[\frac{q^{\lambda_4} - 1}{\lambda_4}\right]\right\} \tag{18.3.11}$$

which is a flexible and manageable tool for modeling a broad class of empirical and theoretical distributions. The references Ramberg and Schmeiser (1974), Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979), Freimer, Mudholkar,

Kollia, and Lin (1988), Karian, Dudewicz, and McDonald (1996), Karian and Dudewicz (2000), Gilchrist (2000), and Tarsitano (2010) are illustrative of the history of the various parameterizations of the distribution. See also King (2009).

If $\lambda_1 \geq 0$ and $\lambda_2 \in [-1, 1]$ then (18.3.11) is a continuous and increasing function of p. Here, λ_0 controls, albeit not exclusively, the location of $Q(p, \boldsymbol{\lambda})$; λ_1 is a scale parameter, while $\lambda_2, \lambda_3, \lambda_4$ influence the shape of $Q(p, \boldsymbol{\lambda})$. We observe that parameters $\lambda_0, \lambda_1, \lambda_2$ appear in linear form, whereas λ_3 and λ_4 are in non-linear form.

The quantile-density function of a FPLD random variable is

$$h(p, \boldsymbol{\lambda}) = \lambda_1 \left[\phi_1 p^{\lambda_3 - 1} + \phi_2 q^{\lambda_4 - 1}\right], \quad \phi_1 = (1 - \lambda_2)/2, \phi_2 = (1 + \lambda_2)/2. \tag{18.3.12}$$

This function appears, for example, in the log-likelihood of a FPLD for a sample of size n,

$$L(\boldsymbol{\lambda}) = -\sum_{i=}^{n} \mathrm{Ln}\left[h(p_i, \boldsymbol{\lambda})\right]. \tag{18.3.13}$$

The density-quantile function is the reciprocal of the quantile-density function

$$f\left[Q(p, \boldsymbol{\lambda})\right] = \frac{1}{h(p, \boldsymbol{\lambda})} = \frac{\lambda_1^{-1}}{\phi_1 p^{\lambda_3 - 1} + \phi_2 q^{\lambda_4 - 1}}. \tag{18.3.14}$$

The function (18.3.14) quantifies the density of a FPLD at the value expressed by the cumulative probability at p.

The range of a FPLD is given by $[Q(0, \boldsymbol{\lambda}), Q(1, \boldsymbol{\lambda})]$ and it is different for different values of $\boldsymbol{\lambda}$; in particular, the extremes are $[\lambda_0 - \lambda_1\phi_1, \lambda_0 + \lambda_1\phi_2]$ if $\lambda_3, \lambda_4 > 0$; $[\lambda_0 - \lambda_1\phi_1, \infty)$ if $\lambda_3 > 0$, $\lambda_4 \leq 0$; and $(-\infty, \lambda_0 + \lambda_1\phi_2]$ if $\lambda_3 \leq 0$, $\lambda_4 > 0$. Hence FPLD can model both infinite distributions and random variables which have a physically limited range.

The FPLD has great sensitivity to its parameter values; in fact, the moments of $Q(p, \boldsymbol{\lambda})$ contains all the elements of the vector $\boldsymbol{\lambda}$. If $\lambda_3 = \lambda_4$ and $\lambda_2 = 0$ then the FPLD is symmetric about λ_0 because in this case (18.3.11) satisfies the condition $Q(p, \boldsymbol{\lambda}) = -Q(q, \boldsymbol{\lambda})$. When $\lambda_3 = \lambda_4$ then (18.3.14) is skewed to the left (right) if $\lambda_2 < 0$ ($\lambda_2 > 0$), which suggests a natural interpretation of λ_2 as a parameter that is prevalently related to the asymmetry of $Q(p, \boldsymbol{\lambda})$. Hence, by interchanging λ_3 and λ_4 and simultaneously changing the sign of λ_2 we obtain a density that is the mirror image of the original density. If $\lambda_2 = -1$ ($\lambda_2 = 1$), then λ_3 (λ_4) indicates the left (right) tail flatness in the sense that the smaller the λ_3 (λ_4) the flatter the left (right) tail of the FPLD density. In practice, λ_3 and λ_4 capture the tail order on the two sides of the support of $Q(p, \boldsymbol{\lambda})$.

The densities (18.3.14) are:

zeromodal if	$\{\max(\lambda_3, \lambda_4) > 1\} \wedge \{\min(\lambda_3, \lambda_4) < 1\}$;
unimodal with continuous tails if	$\{\max(\lambda_3, \lambda_4) < 1\}$;
U-shaped if	$1 < \lambda_3, \lambda_4 < 2$;
unimodal with truncated tails if	$\{\min(\lambda_3, \lambda_4) > 2\}$;
S-shaped if	$\{\max(\lambda_3, \lambda_4) > 2\} \wedge \{\min(\lambda_3, \lambda_4) > 1\}$.

Curves corresponding to large positive values of λ_3 and λ_4 have extreme peakedness and short high tails. For $\lambda_1 = 0$ the FPLD degenerates to a one-point distribution $Q(p, \boldsymbol{\lambda}) = \lambda_0$.

The FPLD family is a good model for use in Monte Carlo simulation and in robustness studies because it contains densities that range over a wide spectrum of arbitrarily shaped densities. For example, if $\lambda_3 \to 0, \lambda_4 \to 0$ then $Q(p, \boldsymbol{\lambda})$ converges to a potentially asymmetric form of the logistic distribution; if $\lambda_3 \to \infty$ and $\lambda_4 \to 0$, then $Q(p, \boldsymbol{\lambda})$ is an exponential distribution whereas for $\lambda_3 \to 0, \lambda_4 \to \infty$, $Q(p, \boldsymbol{\lambda})$ becomes a reflected exponential distribution. Furthermore, the FPLD fits data containing extreme values well; in fact, for $\lambda_3 \to \infty, |\lambda_4| < \infty$ the FPLD corresponds to the generalized Pareto distribution. For $\lambda_4 \to \infty, |\lambda_3| < \infty$, the FPLD generates the power-function distribution. The rectangular random variable is present in four versions:

$$(\lambda_2 = -1, \lambda_3 = 1), (\lambda_2 = 1, \lambda_4 = 1), (\lambda_3 = 1, \lambda_4 = 1), (\lambda_2 = 0, \lambda_3 = 2, \lambda_4 = 2).$$

When the actual distribution is not known the FPLD could have an important role as an alternative if it is able to produce good approximations to many of the densities commonly encountered in applications. Table 18.3.1 shows the accuracy of its fit to some familiar standard random variables. The closeness between the theoretical quantile function $Q(p, \boldsymbol{\lambda})$ and the FPLD is quantified by the maximum absolute difference (Maxd) between observed and fitted quantiles:

$$\max_{1 \le i \le 500} \left| X_{(i)} - Q(p_i, \boldsymbol{\lambda}) \right|, \quad \text{where} \quad p_i = \frac{i}{501}, \; i = 1, 2, \ldots, 500. \tag{18.3.15}$$

The five parameters of $Q(p, \boldsymbol{\lambda})$ have been estimated using the controlled random search method described in Section 18.4.

Our findings show that the FPLD provides reasonably good fits for all the models included in the Table 18.3–1, with the only possible exception being the Cauchy random variable. In this sense, the FPLD is a valid candidate for being fitted to data when the experimenter does not want to be committed to the use of a particular distribution.

18.3.2 Quantile-Based Distributional Regression

The regression procedure based on a distributional regression approach in the quantile domain has been discussed in Perri and Tarsitano (2007a), (2007b),

Table 18.3–1. Comparison of FPLD approximations.

Model	λ_0	λ_1	λ_2	λ_3	λ_4	Maxd
Normal	0.00000	1.35921	0.00000	0.13312	0.13312	0.0065
Laplace	0.00000	1.32397	0.00000	-0.09556	-0.09556	0.0957
Student(2)	0.00000	0.91014	0.00000	-0.45155	-0.45155	0.0973
Cauchy	0.00000	0.66267	0.00000	-0.99250	-0.99250	0.1673
Chi3	1.81280	2.83162	0.62823	0.47342	0.01430	0.0059
LN(4,2)	3.42830	1.50529	0.18708	0.19707	0.05117	0.0062
Weibull(3)	0.81158	0.72618	-0.01090	0.44992	0.09235	0.0031
Gumbel	0.01416	1.50529	0.31747	0.19723	-0.00211	0.0020

(2008) as a direct outgrowth of a seminal idea from Chapter 12 of Gilchrist (2000).

The p-th percentile of the distribution of response y for a given $\mathbf{x}_i$ is

$$y_p - r_i = \mathbf{x}_i'\boldsymbol{\beta} \; + \; Q\left(p_{r_i}, \boldsymbol{\lambda}\right) \quad \text{for } 0 < p_{r_i} < 1. \tag{18.3.16}$$

To gain additional flexibility in modeling the error distribution, we use a five-parameter version of the generalized lambda distribution since it contains (or almost contains) many common distributions and enables us to obtain reasonable estimates in the presence of a broad range of deviations from the Gaussian model. For the FPLD,

$$Q\left(p, \boldsymbol{\lambda}\right) = \lambda_0 + \lambda_1 \left[\phi_1 \left(\frac{p^{\lambda_3} - 1}{\lambda_3}\right) - \phi_2 \left(\frac{q^{\lambda_4} - 1}{\lambda_4}\right)\right], \tag{18.3.17}$$

with $\phi_1 = (1 - \lambda_2)/2$ and $\phi_2 = (1 + \lambda_2)/2$. The model (18.3.17) has zero mean and

$$\int_0^1 Q\left(p, \boldsymbol{\lambda}\right) dp = \lambda_0 + \lambda_1 \left[\frac{\phi_2}{1 + \lambda_4} - \frac{\phi_1}{1 + \lambda_3}\right] = 0, \tag{18.3.18}$$

which implies the exclusion of the parameter λ_0 from the estimation procedure.

To fit (18.3.16) we continue to use the criterion (18.2.15) starting the iterations from the same initial configuration with $\tilde{u}^{(0)} = Q\left(p_{r_i}, \boldsymbol{\lambda}\right) = 0$, $\tilde{y}_i^{(0)} = y_i - \tilde{u}_{r_i}^{(0)}$ $i = 1, 2, \ldots, n$. This leads to computing of the ordinary (or weighted) L_α estimator $\hat{\boldsymbol{\beta}}^{(t)}$ which minimizes (18.2.16), and to the estimation of the power α of the Minkowski metric.

The second step of the new algorithm is the same as that of the probability-based algorithm because the residuals derive from the estimated regression model

$$\hat{u}_i^{(t)} = \tilde{y}_i^{(t)} - \mathbf{x}_i'\hat{\boldsymbol{\beta}}^{(t)} \quad \text{for} \;\; i = 1, 2, \ldots, n. \tag{18.3.19}$$

As we have already seen, a solution to (18.2.15) requires the minimization of

$$\left[\sum_{i=1}^{n} \left|\hat{u}_{r_i} - Q\left(p_{r_i}, \boldsymbol{\lambda}\right)\right|^{\hat{\alpha}}\right]^{1/\hat{\alpha}} \tag{18.3.20}$$

with respect to $p_{r_i}, i = 1, 2, \ldots, n$ and of the distributional parameters $\boldsymbol{\lambda}$. The term **quantile-based** for this algorithm derives from the centrality of the error distribution quantiles. The major difference between this approach and the probability-based approach is that the quantile function is now put in explicit terms. The use of a quantile function for modeling the error distribution simplifies the procedure of the distributional regression because some of the steps are now combined in just one step. For fixed α, the objective function (18.3.20) can be converted into

$$H_\alpha(\boldsymbol{\lambda}) = \left[\sum_{i=1}^{n} \left|\hat{u}_{r_i}^{(t)} - \gamma_0 - \gamma_1 V_{1,r_i}(\boldsymbol{\lambda}) - \gamma_2 V_{2,r_i}(\boldsymbol{\lambda})\right|^{\hat{\alpha}}\right], \tag{18.3.21}$$

where $\gamma_0 = \lambda_0$, $\gamma_1 = \phi_1\lambda_1, \gamma_2 = \phi_2\lambda_1$ and

$$V_{1,r_i}(\boldsymbol{\lambda}) = \lambda_3^{-1}\left[(p_{r_i})^{\lambda_3} - 1\right], \quad V_{2,r_i}(\boldsymbol{\lambda}) = \lambda_4^{-1}\left[1 - (p_{n+1-r_i})^{\lambda_4}\right]. \tag{18.3.22}$$

The following expression is a weighted variant using the weights given in (18.2.25)

$$H_{\alpha w}(\boldsymbol{\lambda}) = \left[\sum_{i=1}^{n} \left(\frac{1/v_{r_i}}{\sum_{i=1}^{n}(1/v_i)}\right)\left|\hat{u}_{r_i}^{(t)} - \gamma_0 - \gamma_1 V_{1,r_i}(\boldsymbol{\lambda}) - \gamma_2 V_{2,r_i}(\boldsymbol{\lambda})\right|^{\hat{\alpha}}\right]^{1/\hat{\alpha}}. \tag{18.3.23}$$

Each weight is an approximation to the reciprocal of the variance of the order statistics of the residuals. The minimization of (18.3.21) or (18.3.23) is accomplished by first making an initial guess at the value of (λ_3, λ_4) and then applying the iteratively reweighted least squares to solve the linear problem for $\gamma_0, \gamma_1, \gamma_2$. The outer exponent $1/\hat{\alpha}$ has been temporarily set aside. The recursion is

$$\hat{\boldsymbol{\gamma}}^{(h+1)} = \left(\mathbf{V}_{\boldsymbol{\lambda}}^t \boldsymbol{\Delta}^{(h)} \mathbf{V}_{\boldsymbol{\lambda}}\right)^{-1} \mathbf{V}_{\boldsymbol{\lambda}}^t \boldsymbol{\Delta}^{(u)} \mathbf{u}^{(t)}, \tag{18.3.24}$$

where $\mathbf{V}_{\boldsymbol{\lambda}}$ denotes the $(n \times 2)$ matrix with pseudo regressors given by (18.3.22) and $\mathbf{u}^{(t)}$ is the vector of the residuals contained in (18.3.20). The vector $\mathbf{u}^{(t)}$ is fixed at this stage of the algorithm. The matrix $\boldsymbol{\Delta}^{(r)}$ is the diagonal matrix of weights with elements

$$\delta_i^{(h)} = \left|\hat{u}_{r_i}^{(t)} - \mathbf{v}_{\boldsymbol{\lambda},i}^t \hat{\boldsymbol{\gamma}}^{(r)}\right|^{(\hat{\alpha}-2)/2} \quad i = 1, 2, \ldots, n. \tag{18.3.25}$$

The process terminates when the absolute change in the value of the objective function $H_\alpha(\boldsymbol{\lambda})$ is less than a given tolerance or the absolute change in each estimated parameter value is less than a set threshold. The conversion of γ_1, γ_2 into λ_1, λ_2 is straightforward

$$\hat{\lambda}_2 = \frac{1 - \hat{\gamma}_1/\hat{\gamma}_2}{1 + \hat{\gamma}_1/\hat{\gamma}_2}, \qquad \hat{\lambda}_1 = \frac{\hat{\gamma}_1}{\left(1 - \hat{\lambda}_3\right)} + \frac{\hat{\gamma}_2}{\left(1 + \hat{\lambda}_3\right)}, \qquad \hat{\lambda}_0 = \hat{\gamma}_0. \tag{18.3.26}$$

The value of $\hat{\lambda}_2$ is set at equal to the arithmetic mean of the two equivalent estimates deriving from (18.3.17). We can now define the reduced form of criterion (18.3.21) in which $\hat{\boldsymbol{\gamma}}$ is substituted to $\boldsymbol{\gamma}$ and the outer exponent reinstated in it. Clearly, the reduced criterion depends only on the unknown pair (λ_3, λ_4). We might try to find a better estimate of (λ_3, λ_4) by using one of the function minimization techniques and, then, repeating linear and non-linear steps until the correction for (18.3.20) is sufficiently small. This procedure corresponds to the third and fourth step of the probability-based algorithm. In fact, at the end of the iterations we obtain an estimation of the distributional parameters $\boldsymbol{\lambda}$ and a new set of quantiles $\tilde{u}_{r_i}^{(t+1)}$.

Given the new estimates of the distributional parameters, we reset the values of $\hat{\beta}^{(t)}$ with $\hat{\beta}^{(t+1)}$ and go through exactly the same procedure described above restarting from (18.3.19) with $\tilde{y_i}^{(t+1)} = y_i - \tilde{u}_{r_i}^{(t+1)}$ for $i = 1, 2, \ldots, n$. The process is repeated until the correction for the general criterion $C(\boldsymbol{\beta}, \boldsymbol{\lambda})$ becomes negligible.

18.4 Controlled Random Search

The stochastic component in (18.2.15) contains two or more non-linear parameters and it is, almost certainly, multimodal. Consequently, $C(\boldsymbol{\beta}, \boldsymbol{\lambda})$ is a highly non-linear optimization problem. The inner/outer scheme outlined in previous sections has the merit of reducing the dimension of the non-linear problem but the objective function is still difficult to solve using traditional optimization approaches.

Instead of attempting to solve the minimization problem with methods that use derivatives, we considered a systematic inspection of the $C(\boldsymbol{\beta}, \boldsymbol{\lambda})$. More specifically, we resorted to a direct search optimization technique called **controlled random search (CRS)** proposed by Price (1977), suitable for searching for global minima in continuous functions that are not differentiable or whose derivatives are difficult to compute or to approximate (see Ali, Törn, and Viitanen (1997) and Ali, Storey, and Törn (1996)). In addition, CRS allows box constraints i.e., each unknown parameter can be given lower and/or upper restrictions.

Some well-known direct search procedures such as the downhill simplex method of Nelder and Mead (1965) or the pattern-search method of Hooke and Jeeves (1961) are familiar in statistics. However, these techniques are really local optimization methods; in practice, they are designed so as to converge towards a single optimum point and so they, unavoidably, discard the information relating to all the other possible optima. On the other hand, a wise application of the CRS allows these limits to be overcome since this method is especially suited for multimodal functions. Of course, global optimality cannot be guaranteed in finite time.

A CRS procedure employs a storage A of points, the number of which is deter-

mined by the user for the particular problem to be solved. Repeated evaluation of the function to be optimized $f(\mathbf{x})$ is performed at points randomly chosen from the storage of points and the set A is progressively contracted by substituting the worst point with a better one. The search continues until an iteration limit is reached, or until a desired tolerance between minimum and maximum values in the $f(\mathbf{x})$ value storage is achieved. These methods are less known and practiced in statistics (two works have appeared in literature up to now: Shelton, Khuri, and Cornell (1983) and Krivy and Tvrdik (1995)).

For practical purposes, it is necessary to confine the search within a prescribed bounded domain. Let $M \subset \mathcal{R}^k$ be the set of admissible values of the k non-linear parameters. The most promising region that we have found in our applications is the rectangle

$$\mathbf{M} = \{\boldsymbol{\lambda} \mid l_i \leq \lambda_i \ \leq s_i \ ; \ i = 1, 2, \ldots, k\}. \tag{18.4.1}$$

Other types of constraints could be imposed by means of a suitable penalization scheme applied to the objective function.

The theory and practice of global optimization has progressed rapidly during the last few years, and a wide variety of modifications of the basic CRS algorithm are now available. In particular, we refer to the scheme discussed in Brachetti, Ciccoli, Pillo, and Lucidi (1997).

Any CRS implies many subjective choices of constants. **The CRS version used here is as follows**

1. Generate a set A of $m_0 > (k+1)$ random admissible points in $\mathbf{M}$ for the objective function $C(\boldsymbol{\beta}, \boldsymbol{\lambda}) = f(\boldsymbol{\lambda})$. Find the minimum and the maximum in A and their function values: $(\mathbf{x}_L, f_L), (\mathbf{x}_H, f_H)$.

2. If the convergence criterion

$$\frac{|f_H - f_L|}{10^{-7} + |f_H| + |f_L|} < \epsilon \tag{18.4.2}$$

   is satisfied then stop. Also stop if the limit to the number of function evaluations has been achieved. The point with the lowest objective function value found is returned. If required, return all the storage points contained in A.

3. Choose at random, without duplication, a set B of k points in $(A - \mathbf{x}_L)$, include $\mathbf{x}_L$ in B and compute the centroid $\bar{\mathbf{x}} = (k+1)^{-1} \sum_{\mathbf{x} \in B} \mathbf{x}$. Define $A_1 = A - B$

4. Select randomly another point $\mathbf{x}_r \in A_1$.

5. Generate uniform random numbers $d_1, d_2, \cdots, d_k$ in $[0, 2]$ and compute the trial point $x_{i,T} = (1 + d_i)\, \bar{\mathbf{x}}_i - d_i \mathbf{x}_{r,i}$, for $i = 1, 2, \cdots, k$; thus, the exploration of the space around the centroid is based on a randomized reflection in the simplex originally used by Price (1977).

6. If $\mathbf{x}_T \in \mathbf{M}$ then compute $f_T = f(\mathbf{x}_T)$ and go to step 7. If $\mathbf{x}_T \notin \mathbf{M}$, then repeat step 5 a fixed number m_1 of times. If $\mathbf{x}_T$ is still unfeasible, then exclude $\mathbf{x}_r$ from A_1, redefine the set A_1, if it is not empty, and repeat step 4 a fixed number m_2 of times; otherwise go to step 9.

7. If $f_L < f_T < f_H$, then replace f_H with f_T, $\mathbf{x}_H$ with $\mathbf{x}_T$ and then redetermine the maximum $\mathbf{x}_H$ and f_H.

8. If $f_T < f_L$, then replace f_H with f_L, $\mathbf{x}_H$ with $\mathbf{x}_L$, f_L with f_T, $\mathbf{x}_L$ with $\mathbf{x}_T$ and then $\mathbf{x}_H$ and f_H.

9. This is an alternative heuristics. Choose randomly two distinct points $\mathbf{x}_2, \mathbf{x}_3$ in $(A - \mathbf{x}_L)$ and compute

$$d_{Q,i} = (x_{2,i} - x_{3,i})\, f_L + (x_{3,i} - x_{L,i})\, f_2 + (x_{L,i} - x_{2,i})\, f_3 \quad i = 1, 2, \ldots, k. \tag{18.4.3}$$

10. If $|d_{Q,i}| > 10^{-7}$ for each i then build the following quadratic interpolation of the objective function

$$x_{Q,i} = \left[\frac{(x_{2,i} - x_{3,i})^2 f_L + (x_{3,i} - x_{L,i})^2 f_2 + (x_{L,i} - x_{2,i})^2 f_3}{2 d_{Q,i}} \right]. \tag{18.4.4}$$

If some of the $|d_{Q,i}| \leq 10^{-7}$, then generate k uniform random numbers $d_i, i = 1, 2, \ldots, k$ in $[0, 1]$ and compute

$$x_{i,Q} = d_i x_{i,L} + (1 - d_i)\, x_{i,H}, \quad \text{for } i = 1, 2, \ldots, k. \tag{18.4.5}$$

11. If $\mathbf{x}_Q \notin \mathbf{M}$, then repeat steps 9–10 a fixed number m_3 of times. If $\mathbf{x}_Q$ is still unfeasible then return to step 2.

12. Execute steps 7–9 with x_Q in place of x_T, f_Q in place of f_T, and then return to step 2.

We experimented with the CRS algorithm using $\epsilon = 10^{-8}$, $m_0 = 25\,(k+1)$, $m_1 = 3\,(k+1)$, $m_2 = 6\,(k+1)$, $m_3 = m_0/4$.

The CRS procedure has several desirable features. Boundaries and objective function can be arbitrarily complex because the procedure does not require the calculation of derivatives. Shelton, Khuri, and Cornell (1983) wisely observed that, if more then one optimal point is found, the procedure will make this clear to the user.

A limitation that controlled random search techniques have is the lack of guarantee that the global optimum will be achieved, though it works very well on a range of practical problems. Another major concern is that controlled random search has a numerically intensive nature and, hence, is time consuming although not to a degree that is of likely to be practical importance in an age of increasing availability of computing power.

18.5 Goodness-of-Fit Assessment

The measurement of the goodness-of-fit for the distributional regression has two aspects: the degree of proximity between the model adopted to fit the observed residuals to the true distribution that generates data and the agreement between observed and estimated responses. The true distribution of the errors is, obviously, unknown but we are attempting to find a model that can assume a wide variety of curve shapes (Gaussian included) and uses only one general formula over the entire range of data. It should be noted that, when the distribution family includes the Gaussian, using these methods instead of Gaussian-assumption methods does not require a stringent testing of goodness-of-fit, since the distribution can/will approach the Gaussian if that is the true underlying distribution.

In any case, one can test this aspect of the fit as follows: first, use a general family, such as one of those discussed in this chapter. Second, use the Gaussian family. Compare the results to see if they differ enough to matter in practice. If not, then either assumption (non-Gaussian or Gaussian) will yield essentially the same results. On the other hand, if they do differ enough to matter, then (if the family includes the Gaussian) we know that Gaussian-based results are not appropriate for use in the problem. The adequacy of the Gaussian distribution can be assessed in several ways, all well known. For instance, Islam and Tiku (2004) use the *q-q* plot of the observed and estimated responses to ascertain the goodness-of-fit for various possible models of the error distribution. We will not discuss such methods here, but we observe that a Minkowski metric around $\alpha = 2$ is an indirect signal of the gaussianity of the residuals.

To examine the standardized distance from the observed data to the modeled values for estimation methods which are alternative to the ordinary least squares is a difficult task. There are a variety of procedures that one may wish to consider to assess the fitting of a model to experimental data. In this section we explore the predictive capability of the proposed estimation method by measuring the agreement between observed and predicted data. Hagquist and Stenbeck (1998) summarized some of the main arguments of the debate concerning the importance of goodness-of-fit in regression analysis. Following the line of work of Dehon and Croux (2002), the reduction in variation due to fitting a distributional regression model can be measured by

$$r_2 = 1 - \sqrt{\frac{\sum_{i=1}^{n} [y_i - \hat{y}_i]^2}{\sum_{i=1}^{n} [y_i - \bar{y}]^2}} \qquad \bar{y} = n^{-1} \sum_{i=1}^{n} y_i. \tag{18.5.1}$$

The numerator in the ratio (18.5.1) is the variance of the residuals in the full model, while the denominator is the variance of the residuals in the reduced model when the regression is a fixed constant. The r_2 statistic measures, in the unit scale of the data, the proportionate reduction in the deviation due to fitting the complete model with the least squares criterion. See Cade and Richards

(1996). We have $r_2 = 0$ when $\hat{y}_i = \bar{y}, i = 1, 2, \ldots, n$. This means that regression is a line parallel to the x-axis; r_2 increases when moving from a reduced model to a more informative model. Finally, $r_2 = 1$ when the model perfectly fits the data.

The coefficient r_2 is misleading when used to compare the fit of regressions using different numbers of predictors; in fact, the regression sum of squares cannot decrease when new predictors are added to the regression, so that increasing the number of predictors will usually increase r_2 even when the true values of the new coefficients are zero (see Healy (1984)). A general dilemma in multiple regression analysis is the essential compromise between an optimal goodness-of-fit statistic and the number of coefficients in the regression equation. Sonnergaard (2006) discusses a variant of (18.5.1) which takes into account the total number of unknown parameters involved in the estimation process

$$\bar{r}_2 = 1 - \sqrt{\left(\frac{n-1}{n-m-k}\right) \frac{\sum_{i=1}^{n} [y_i - \hat{y}_i]^2}{\sum_{i=1}^{n} [y_i - \bar{y}]^2}}. \tag{18.5.2}$$

The extremes are the same as those of r_2, but the intermediate values are much better covered. Another common measure of agreement is

$$r_c = \frac{\sum_{i=1}^{n} (y_i - \bar{y}) (\hat{y}_i - \bar{\hat{y}})}{\sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2 \sum_{i=1} (\hat{y}_i - \bar{\hat{y}})^2}}. \tag{18.5.3}$$

The index r_c is the cosine of the angle between the original data $\mathbf{y}$ and the predicted vector $\hat{\mathbf{y}}$ centered about their respective means. The model fits the observed data well for values of r_c near one, while the fit is poor when r_c is near zero. From another point of view, (18.5.3) can be considered to be a measure of the linearity of the probability plot reporting the ordered value of the observed and estimated responses. The correlation coefficient (18.5.3) is often used and more often misused as an index expressing the quality in linear regression analysis. Models yielding a good fit are systematically associated with high values of r_c, the converse is not true. In addition, r_c tends to show a relatively high value even though no relationship could be shown between response and regressors.

A robust competitor of the correlation coefficient is the correlation median proposed by Falk (1997)

$$\delta = \frac{\text{med}\,\{[y_i - \text{med}\,(y)]\,[\hat{y}_i - \text{med}\,(\hat{y})]\}}{\text{med}\,[|y_i - \text{med}\,(y)|]\,\text{med}\,[|\hat{y}_i - \text{med}\,(\hat{y})|]}, \tag{18.5.4}$$

where the operator mean in the popular correlation coefficient has been substituted by the operator median. The index δ is symmetrical and invariant under affine transformations; it is null in the case of independence between observed and calculated values, although it is not always guaranteed to be in the range $[0, 1]$.

Many of the problems with the measurement of fit arise from the failure of the classical variability decomposition

$$\sum_{i=1}^{n}(y_i-\bar{y})^2=\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2+\sum_{i=1}^{n}(y_i-\hat{y}_i)^2\,, \tag{18.5.5}$$

which does not generally hold for a metric other than the Euclidean metric. The application of the coefficient of determination or other measures based on (18.5.5) in settings different from the traditional linear regression model, generally leads to a value which may lie outside the $[0,1]$ range and decreases as regressors are added. See Kvålset (1985). For example, since the term $\mathbf{x}_i'\boldsymbol{\beta}$ is not necessarily a minimum for the LAD criterion in the distributional regression method, the value of R^2 may be greater than one and r_2 less than zero. Cameron and Windmeijer (1997) described alternative R^2-type goodness-of-fit summary statistics for the class of exponential family regression models. Koenker and Machado (1999) and Royston and Wright (1998) discussed the model adequacy problem for the quantile regression. Nevertheless, the measures proposed in these contexts are not really suitable for the distributional regression fitting, since definitions and estimation procedures are different from those of the classical regression model.

There are other measures for goodness-of-fit based on the absolute deviations of the observed responses from their median. Using the triangle inequality, we have

$$\sum_{i=1}^{n}|y_i-\hat{y}_i|\le\sum_{i=1}^{n}|y_i-\operatorname{med}(y)|+\sum_{i=1}^{n}|\hat{y}_i-\operatorname{med}(y)|, \tag{18.5.6}$$

with $\operatorname{med}(y)=\operatorname{median}\{y_1,y_2,\cdots,y_n\}$. The first term on the right-hand side of (18.5.6) is proportional to the mean of the absolute deviation of the observed responses from the median; the second term is proportional to the mean of the absolute deviations of the hypothetical responses from the median $\operatorname{med}(y)$ of the observed responses. Thus, a normalized index able to measure the agreement between the observed and the predicted data, may be expressed as:

$$\tau=1-\frac{\sum_{i=1}^{n}|y_i-\hat{y}_i|}{\sum_{i=1}^{n}|y_i-\operatorname{med}(y)|+\sum_{i=1}^{n}|\hat{y}_i-\operatorname{med}(y)|}. \tag{18.5.7}$$

The index (18.5.7) takes values in the $[0,1]$ interval. In particular it turns out that $\tau=1$ if $y_i=\hat{y}_i, i=1,2,\ldots,n$, i.e., when the model perfectly fits the observed data. On the other hand, the value of τ is zero if $\hat{y}_i=\operatorname{med}(y)\,, i=1,2,\ldots,n$ implying that $y_i=\beta_0+u_i, i=1,2,\ldots,n$ provides an extremely succinct, but reasonable explanation of the responses.

In order to judge the overall goodness-of-fit of estimated models, Bonferroni (1940–41, p. 223) proposed a descriptive index of concordance between observed and modeled values based on the median of the modeled values given by

$$b_o=\frac{\sum_{i=1}^{n}|\hat{y}_i-\operatorname{med}(\hat{y})|}{\sum_{i=1}^{m}|\hat{y}_i-\operatorname{med}(\hat{y})|+\sum_{i=1}^{n}|y_i-\hat{y}_i|} \tag{18.5.8}$$

where med $(\hat{y})$ is the median of the modeled values. The index b_o varies in the $[0, 1]$ interval. More specifically, $b_o = 0$ denotes a non-informative model $\hat{y}_i = \text{constant}, i = 1, 2, \ldots, n$. The value of b_o increases when passing to a more informative model. Finally, $b_o = 1$ means that $y_i = \hat{y}_i, i = 1, 2, \ldots, n$, i.e., the model fits the observed data perfectly. In practice, the Bonferroni index of adequacy measures, in the unit scale of the data, the proportionate reduction in the absolute deviation due to fitting the complete model.

An effective measure of goodness-of-fit should be directly linked to the fitting criterion and the algorithm used to implement it. Therefore, with the aim of evaluating different estimation methods, an omnibus and compatible measure should be used to make the comparisons realistic and reliable. With this intent, we consider two coefficients. The first index is

$$R_\alpha = 1 - \frac{\sum_{i=1}^{n} |y_i - \mathbf{x}_i'\boldsymbol{\beta}|^\alpha}{\sum_{i=1}^{n} |y_i - \theta_\alpha|^\alpha}, \tag{18.5.9}$$

where θ_α minimizes $\sum_{i=1}^{n} |y_i - \theta|^\alpha$ with respect of θ (the value of θ_α can be determined by applying the iteratively reweighted least squares algorithm already used in the previous sections). For $\alpha = 1$ the index (18.5.9) becomes the coefficient proposed by Seppo and Salmi (1994); for $\alpha = 2$ it yields the coefficient analyzed by Kvålset (1985). In general, R_α compares a model with only the constant term, to a model with all regressors. In the former, each y_i value is predicted to be the θ_α of the dependent variables; note that θ_α would be our best guess without having any additional information. The ratio in (18.5.9) is indicative of the degree to which the model parameters improve upon the prediction of the null or non-informative model. The smaller this ratio, the greater the improvement and the higher the R_α. Typically $0 \leq R_\alpha \leq 1$, but this is not guaranteed in the area of distributional regression.

The second index is

$$\psi_\alpha = 1 - \frac{\sum_{i=1}^{n} |(y_i - \mathbf{x}_i'\boldsymbol{\beta}) - Q\,(p_{r_i}, \boldsymbol{\lambda})|^\alpha}{\sum_{i=1}^{n} |y_i - \theta_\alpha|^\alpha}, \tag{18.5.10}$$

where the numerator is the criterion used for distributional regression and the denominator is the same as the denominator in (18.5.9). The index ψ_α is attractive for a number of reasons: it is dimensionless and varies in the $[0, 1]$ interval; it has an intuitively reasonable interpretation. Before accepting the more complicated models, we need to ask whether the improvement in goodness-of-fit is more than we had expected by chance. If the distributional regression model were really correct then one should find values of (18.5.10) close to one; conversely, if ψ_α is near to zero, the regression plan determined by minimizing the Minkowski metric under the given distribution of errors is most likely to be wrong.

18.6 Empirical Examples

The introduction of a not necessarily Gaussian probability density function or density-quantile function $f(u;\boldsymbol{\beta},\boldsymbol{\lambda})$ to model the errors of the linear regression raises a number of questions. Following Goldfeld and Quandt (1981) we mention some of the most relevant.

1. Are estimates with the selected family of distributions routinely computable?
2. What practical difference does it make whether the error distribution is assumed to be Gaussian or to belong to the family $f(u;\boldsymbol{\beta},\boldsymbol{\lambda})$?
3. Does the new error model yield an advantage from the point of view of both fitting and efficiency?
4. Is the $f(u;\boldsymbol{\beta},\boldsymbol{\lambda})$ flexible enough to change with data and to transform naturally, i.e., without interaction with the user, into a Gaussian when the sample is drawn from a population in which the true distribution is Gaussian or when the conditions of the Central limit theorem are met?
5. Which family of distributions will be most useful overall?

In order to provide tentative answers to the above points, we have applied the techniques discussed in the previous sections to some illustrative examples taken from the literature on this topic.

The regression parameters under the L_α metric were estimated by using iteratively reweighted least squares with $0.01 \leq \alpha \leq 5$ allowing a maximum of 40 iterations and a convergence test given by $\left|\boldsymbol{\beta}^{(h+1)} - \boldsymbol{\beta}^{(h)}\right| < 0.0000001\left|\boldsymbol{\beta}^{(h+1)}\right|$.

The non-linear distributional parameters were computed using the controlled random search (CRS) technique discussed in Section 18.4 with the constraints:

EGB2	$L_b : [0.1\sigma(y), 0.01, 0.01,]$	$U_b : [5\sigma(y), 10, 10]$
SGED	$L_b : [0.1\sigma(y), -0.99, 0.01,]$	$U_b : [5\sigma(y), 0.99, 10]$
SGT	$L_b : [0.1\sigma(y), 1, 1, -0.99,]$	$U_b : [5\sigma(y), 10, 10^6, 0.99]$
JSU	$L_b : [0.1\sigma(y), 0.01, 0.01,]$	$U_b : [5\sigma(y), 10, 10]$
GBD	$L_b : (1, 1)$	$U_b : (15, 15)$
FPLD	$L_b : (-0.99, -0.99)$	$U_b : (4, 4)$

For each dataset, for each error distribution, and for each iteration, the CRS algorithm had a maximum of 2500 function evaluations possible. We have employed the special procedures outlined in Sections 18.2.1 and 18.3.2 for the linear parameters of the GBD and the FPLD, respectively. The two stages: L_α optimization and CRS, which form the distributional regression algorithm, have been applied for 40 iterations and a maximum of 15 subiterations in case of inversion of the decreasing tendency of the objective function $C(\boldsymbol{\beta},\boldsymbol{\lambda})$.

We compare the estimates resulting from the above computations on the basis of the criteria discussed in Section 18.5. The comparison is also made on the

basis of the negative of the loglikelihood $-L(\boldsymbol{\beta}, \boldsymbol{\lambda})$ and the Schwarz information criterion:

$$Sic = -L(\boldsymbol{\beta}, \boldsymbol{\lambda}) + (m'/2)\log(n)$$

where m' is the total number of estimated parameters and n is the number of observations. Sic is the sum of two addends: the first quantifies the goodness of the fit of maximum likelihood estimates and tends to decrease as more parameters are added to the model; the second is a function of m' and n which penalizes for the loss of degrees of freedom when additional parameters are introduced. There is an abundance of this type of indices and there is no general agreement on which is most useful for practical applications. However, models with smaller Sic values are to be preferred to those with larger values.

As a general result we note that the power of the Minkowski metric assumes values in the range $[1, 2]$ in all datasets; this is not surprising, at least not for those that support the idea that an estimator between LAD and OLS (extremes included) might be the right procedure to use in most situations. Another unexpected finding is the fact that the estimated power $\hat{\alpha}$ appears to be insensitive to the family of distributions chosen for modeling the residuals. Thus, the decisions on α and on $f(u; \boldsymbol{\beta}, \boldsymbol{\lambda})$ could be considered orthogonal.

Partially adaptive estimation of linear regression models is based on distributions versatile enough to adapt to many and different datasets. One important finding that emerges from our investigation, is that the shape parameters of the error distributions used and the related measures of fitting, are extremely sensitive to any change in the data and therefore not applicable when a certain amount of noise tolerance is required.

Two other remarks are in order. The first concerns the better quality of the results obtained by using the U-shaped system of weights for the estimation of the distributional parameters, thus confirming the validity and the importance of this option over unweighted estimation. Second, we emphasize that the calculations of the distributional regression estimates for the quantile-based algorithm are much faster than those needed for the probability-based algorithm. This, unfortunately, does not imply that distributional regression is, at the moment, routinely computable. A possible and desirable development is the implementation of the software used to perform the analysis described in this chapter in the form of an **R** package, (**R** Development Core Team (2009)), made freely available.

The final general point to note is that the agreement between modeled and experimental data is generally higher for the FPLD and for the GBD than their other counterparts. Such consideration leads to a postulation that partially adaptive estimation based on a hybrid of the two distributions, i.e., the EGLD (Dudewicz and Karian (1996) and Karian, Dudewicz, and McDonald (1996)) offers a good point of departure. The appendix at the end of this chapter lists the data associated with each of the five applications that follow, along with the **R** code that was used in the analyses of these examples.

18.6.1 Mayer's Data

This example comes from Dodge and Jurecková (2000, Section 3.4). Johann Tobias Mayer used measurements of the location of the crater Manilius on the surface of the moon (a point always observable from earth) to locate the moon's equator and its angle of inclination to the earth. The dataset comprises $n = 27$ observations where the response is the observed latitude of Manilius in relation to the moon's apparent equator and the two regressors are the sine and the cosine of the same observed angle (of course the sum of the square of the two angles equals one). The real values of the parameters are $\beta_0 \in [14.41667, 14.58333]$, $\beta_1 \in [1.54047, 1.54436]$, $\beta_2 = 0.03$.

According to Dodge and Jurecková, the OLS seems to be the best possible solution for all three parameters. The results summarized in the Table 18.6–1 show that this is true with respect to the slopes β_1 and β_2, but not in terms of goodness-of-fit for which the not necessarily Gaussian disturbances ($\gamma_1 = -0.53$, $\gamma_2 = 4.31$) are accurately represented by a GBD distribution.

The most obvious point in Table 18.6–1 is the scarce or null difference between the estimated slope coefficients. The similarity between these estimates derives from a substantial Gaussianity of the regression residuals which can be easily verified with the q-q plot, and from the fact that the Gaussian distribution is a special case common to all the parametric families studied in this chapter. Hansen, McDonald, and Turley (2006) noted that, the approximate equality of parameter estimates in partially adaptive estimation, illustrates the limiting relationship between the various distributions.

18.6.2 Martin Marietta Data

This example consists of $n = 60$ measurements from January 1982 to December 1986 for the Martin Marietta Corporation (NYSE) given in Butler, McDonald, Nelson, and White (1990). Both the rate of return on equity for the firm and the rate of return on the market portfolio were considered. The dataset includes an evident outlier and two more points are suspected outliers. The analysis of the OLS residuals conducted by Lye and Martin (1993) gives a coefficient of skewness $\gamma_1 = 2.25$ and a coefficient of kurtosis $\gamma_2 = 12.21$. These statistics suggest that a not necessarily Gaussian distribution which takes into account asymmetry, leptokurtosis, and heaviness of one of the tails should give better results than a Gaussian distribution.

A comparison of the OLS and distributional regression shows that, by incorporating the shape information contained in the data, the regression parameter estimates have been improved. It should be noted that, the estimation of the beta coefficient (β_1) is important for the risk classification. A value of 1.80 means that the Martin Marietta company rates as very aggressive, whereas a value of 1.4 rates the company as moderately aggressive. It is apparent from Table 18.6–2 that each of the not-necessarily Gaussian distributions causes the β_1 estimate

Table 18.6–1. DR estimation for Mayer's data.

Model Weighting	OLS	FPLD F	FPLD T	GBD F	GBD T	JSU F	JSU T	SGT F	SGT T	SGED F	SGED T	EGB2 F	EGB2 T
β_0	14.558	14.553	14.551	14.549	14.554	14.554	14.554	14.554	14.554	14.554	14.554	14.554	14.554
β_1	1.506	1.507	1.504	1.507	1.507	1.507	1.507	1.507	1.507	1.507	1.507	1.507	1.507
β_2	-0.072	-0.106	-0.109	-0.107	-0.106	-0.106	-0.106	-0.106	-0.107	-0.106	-0.107	-0.106	-0.106
α	2.000	1.414	1.424	1.375	1.397	1.397	1.397	1.397	1.393	1.397	1.393	1.397	1.397
λ_1		0.158	0.167	-1.212	-1.186	0.293	0.183	0.210	0.176	0.157	0.168	0.207	0.170
λ_2		-0.909	-0.756	1.000	1.000	2.022	1.344	1.858	1.345	0.028	0.007	4.096	2.893
λ_3		0.082	0.167	5.385	5.254	0.010	0.010	70.99	360.5	1.761	1.331	3.978	2.852
λ_4		-0.990	-0.472	4.442	4.430			0.029	0.008				
r^2	0.867	0.877	0.887	0.953	0.866	0.866	0.866	0.866	0.866	0.866	0.866	0.866	0.866
$\bar{r}^2$	0.859	0.864	0.874	0.948	0.850	0.854	0.854	0.850	0.850	0.854	0.854	0.854	0.854
$cor\,(y, \hat{y})$	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991	0.991
$com\,(y, \hat{y})$	1.032	1.086	1.097	1.091	1.088	1.088	1.088	1.088	1.089	1.088	1.089	1.088	1.088
τ	0.938	0.945	0.947	0.979	0.939	0.939	0.939	0.939	0.939	0.939	0.939	0.939	0.939
b_o	0.889	0.901	0.905	0.960	0.893	0.893	0.893	0.893	0.893	0.893	0.893	0.893	0.893
R_α		0.952	0.957	0.986	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944
ψ_α		0.998	0.997	0.968	0.969	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944
S_α	0.754	0.118	0.165	0.913	0.894	1.362	1.362	1.362	1.371	1.362	1.371	1.362	1.362
- L	27.90	-15.62	-17.60	40.52	39.85	-13.35	-12.94	-13.40	-13.05	-13.38	-13.04	-13.47	-13.51
Sic	34.50	-5.730	-7.715	50.40	49.74	-5.113	-4.696	-3.507	-3.162	-5.136	-4.795	-5.229	-5.272

Table 18.6–2. DR estimation for Martin Marietta data.

Model	OLS	FPLD	FPLD	GBD	GBD	JSU	JSU	SGT	SGT	SGED	SGED	EGB2	EGB2
Weight		F	T	F	T	F	T	F	T	F	T	F	T
β_0	0.001	-0.003	-0.002	0.001	0.001	-0.003	-0.003	-0.001	-0.001	-0.001	-0.001	-0.003	-0.003
β_1	1.803	1.392	1.379	1.396	1.396	1.396	1.396	1.431	1.434	1.395	1.396	1.396	1.396
α	2.000	1.335	1.331	1.341	1.341	1.340	1.341	1.311	1.307	1.303	1.324	1.341	1.341
λ_1		0.069	0.044	-0.620	-0.622	0.517	0.525	0.096	0.088	0.079	0.089	0.129	0.106
λ_2		-0.771	-0.659	2.113	2.134	8.085	8.324	2.743	2.128	0.047	0.057	7.859	5.799
λ_3		0.244	0.035	4.319	4.329	0.161	0.156	12241	729547	1.987	1.453	8.476	6.332
λ_4		-0.990	-0.990	14.707	14.853			-0.069	-0.095				
r^2	0.225	0.339	0.285	0.067	0.068	0.372	0.384	0.274	0.265	0.261	0.256	0.384	0.379
$\bar{r}^2$	0.205	0.380	0.317	0.074	0.075	0.409	0.422	0.305	0.294	0.284	0.278	0.422	0.416
$cor\,(y, \hat{y})$	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632
$com\,(y, \hat{y})$	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426	0.426
τ	0.557	0.544	0.553	0.782	0.781	0.524	0.522	0.551	0.554	0.552	0.548	0.522	0.523
b_o	0.491	0.446	0.450	0.627	0.626	0.436	0.435	0.454	0.456	0.450	0.448	0.435	0.436
R_α		0.336	0.368	0.751	0.749	0.297	0.292	0.354	0.360	0.364	0.364	0.292	0.294
ψ_α		0.989	0.970	0.689	0.694	0.297	0.292	0.421	0.406	0.375	0.364	0.292	0.294
S_α	0.719	0.067	0.140	0.798	0.789	1.470	1.476	1.333	1.367	1.433	1.400	1.476	1.473
Lik	-53.97	-75.68	-85.68	655.21	675.61	-52.07	-50.22	-62.84	-65.32	-68.32	-69.31	-55.17	-55.95
Sic	-45.78	-63.40	-73.40	667.49	687.90	-41.84	-39.98	-50.56	-53.03	-58.09	-59.07	-44.94	-45.72

to decrease from 1.8 to a value in the interval $[1.379 - 1.434]$ thus changing the classification of the company.

Both Butler, McDonald, Nelson, and White (1990) and Lye and Martin (1993) considered the possibility of deleting one or more extreme observations. A very interesting result has been obtained by Lye and Martin (1993). The β_1 estimate, as determined by assuming that the error term has a GET (generalized student t-distribution, which is a skew form of the t distribution), was similar to the OLS estimate when three outliers in the data are deleted: $\beta_1 = 1.24$. On the other hand, Butler, McDonald, Nelson, and White (1990) observe that a regression which excludes only the very extreme data point, gives an almost identical β_1 estimate to the one which excludes all three. We have reanalyzed the Martin Marietta data after deleting record 8 obtaining $\beta_1 \in [1.14, 1.21]$ and the records $8, 15$, and 34 obtaining $\beta_1 \in [1.077, 1.091]$. This result lends support to the argument that distributional regression algorithms recognize as significant every single data point.

18.6.3 Prostate Cancer Data

This application is reported in Hansen, McDonald, and Turley (2006) and comprises $n = 97$ patients' data. The objective of this study is to predict the magnitude of prostate cancer as a function of several independent variables. The dependent variable is the log of prostate specific antigen (lpsa), and the independent variables are: log of cancer volume (lcavol), log of weight (lweight), log of benign prostatic hyperplasica amount (lbph), seminal vesicle invasion (svi), log of capsular penetration (lcp), Gleason score (G), and an indication of which observations were used as the "training set" and which 30 as the "test set" (Tr). In the data set used in Hansen, McDonald, and Turley (2006) there was an error. Subject 32 had a value of 6.1 for log of weight, which translates to a 449 gm prostate! The correct value is 44.9 gm. However, to keep the results comparable, we used the data reported in the first edition of Hastie, Tibshirani, and Friedman (2009).

Restricting attention to the SGT distribution, we can compare the results in Table 18.6–3 with those obtained in Hansen, McDonald, and Turley (2006). The two sets of values are different and not only for the intercept (this is attributable to a different evaluation of the constraint of zero-mean residuals), but also for the distributional parameters, particularly for λ_4, the skewness parameter. Such a finding is in line with the suggestion that distributional regression estimators are very near, but not coincident with ML or quasi-ML estimators.

Table 18.6–4 reports the computation for the corrected dataset thus helping us to evaluate the influence of a grossly anomalous value in one of the regressors. The changes are significative in the intercept and in the slope coefficient connected to the logarithm of weight (β_2) where the wrong value has been incorrectly entered. The agreement figures are only slightly improved by the correction. This

Table 18.6–3. DR results for prostate cancer data (with a wrong value).

Model	OLS	FPLD	FPLD	GBD	GBD	JSU	JSU	SGT	SGT	SGED	SGED	EGB2	EGB2
Weighting		F	T	F	T	F	T	F	T	F	T	F	T
β_0	0.146	0.037	0.038	0.046	0.046	0.027	0.027	0.027	0.035	0.025	0.035	0.035	0.035
β_1	0.550	0.510	0.510	0.511	0.511	0.510	0.510	0.510	0.510	0.510	0.510	0.510	0.510
β_2	0.391	0.437	0.436	0.434	0.434	0.440	0.440	0.440	0.437	0.440	0.437	0.437	0.437
β_3	0.090	0.101	0.101	0.102	0.102	0.101	0.101	0.101	0.101	0.101	0.101	0.101	0.101
β_4	0.712	0.642	0.641	0.644	0.644	0.643	0.643	0.641	0.642	0.639	0.642	0.644	0.644
α	2.000	1.033	1.075	1.062	1.062	1.064	1.064	1.080	1.041	1.080	1.041	1.147	1.147
λ_1		0.942	11.940	-1.089	-1.077	1.171	2.542	0.803	0.942	0.795	0.729	1.039	0.864
λ_2		-0.511	0.039	1.000	1.000	1.764	3.609	1.297	1.721	0.018	-0.001	4.747	3.285
λ_3		0.222	4.000	5.015	4.959	0.010	0.010	532296	735754	1.296	1.722	4.691	3.281
λ_4		-0.138	4.000	4.604	4.603			0.018	-0.001				
r^2	0.397	0.404	0.419	0.480	0.480	0.394	0.394	0.394	0.394	0.394	0.394	0.394	0.394
$\bar{r}^2$	0.388	0.387	0.403	0.466	0.466	0.381	0.381	0.378	0.378	0.381	0.381	0.381	0.381
$cor\,(y, \hat{y})$	0.798	0.797	0.797	0.797	0.797	0.796	0.796	0.796	0.797	0.796	0.797	0.797	0.797
$com\,(y, \hat{y})$	0.784	0.923	0.922	0.927	0.927	0.922	0.922	0.921	0.921	0.919	0.921	0.920	0.920
τ	0.665	0.668	0.673	0.710	0.711	0.663	0.663	0.662	0.662	0.662	0.662	0.663	0.663
b_o	0.576	0.571	0.575	0.605	0.605	0.568	0.568	0.568	0.567	0.568	0.567	0.568	0.568
R_α		0.407	0.429	0.494	0.495	0.406	0.406	0.411	0.400	0.411	0.400	0.430	0.430
ψ_α		0.994	0.989	0.925	0.925	0.406	0.406	0.411	0.400	0.411	0.400	0.430	0.430
S_α	6.818	0.555	0.963	5.812	5.845	40.772	40.770	38.338	44.392	38.372	44.392	30.597	30.597
-Log L	120.50	101.59	95.77	130.58	129.28	104.14	102.89	103.80	102.60	103.81	102.60	102.69	102.84
Sic	129.65	115.31	109.49	144.30	143.00	115.58	114.33	117.52	116.32	115.24	114.04	114.12	114.28

Table 18.6–4. DR results for prostate cancer data (corrected).

Model	OLS	FPLD	FPLD	GBD	GBD	JSU	JSU	SGT	SGT	SGED	SGED	EGB2	EGB2
Weighting		F	T	F	T	F	T	F	T	F	T	F	T
β_0	-0.341	-0.366	-0.363	-0.363	-0.363	-0.363	-0.363	-0.363	-0.364	-0.363	-0.364	-0.363	-0.363
β_1	0.528	0.487	0.487	0.487	0.487	0.487	0.487	0.487	0.487	0.487	0.487	0.487	0.487
β_2	0.536	0.550	0.549	0.549	0.549	0.549	0.549	0.549	0.550	0.549	0.550	0.549	0.549
β_3	0.079	0.095	0.096	0.096	0.096	0.096	0.096	0.096	0.096	0.096	0.096	0.096	0.096
β_4	0.705	0.699	0.697	0.697	0.697	0.697	0.697	0.697	0.697	0.697	0.697	0.697	0.697
α	2.000	1.148	1.128	1.128	1.128	1.128	1.128	1.128	1.129	1.128	1.129	1.128	1.128
λ_1		0.833	0.960	-1.104	-1.098	0.725	1.426	0.743	0.825	0.776	0.730	0.907	0.710
λ_2		-0.043	0.415	1.000	1.000	1.218	2.181	1.239	1.451	0.011	0.005	3.948	2.398
λ_3		0.031	-0.092	5.255	5.226	0.010	0.010	214631	686632	1.237	1.451	3.902	2.374
λ_4		0.041	0.243	4.761	4.760			0.011	0.005				
r^2	0.403	0.411	0.401	0.401	0.401	0.401	0.401	0.401	0.401	0.401	0.401	0.401	0.401
$\bar{r}^2$	0.393	0.395	0.385	0.385	0.385	0.388	0.388	0.385	0.385	0.388	0.388	0.388	0.388
$cor\,(y, \hat{y})$	0.802	0.801	0.801	0.801	0.801	0.801	0.801	0.801	0.801	0.801	0.801	0.801	0.801
$com\,(y, \hat{y})$	0.783	0.826	0.827	0.827	0.827	0.827	0.827	0.827	0.827	0.827	0.827	0.827	0.827
τ	0.675	0.677	0.672	0.672	0.672	0.672	0.672	0.672	0.672	0.672	0.672	0.672	0.672
b_o	0.584	0.580	0.576	0.576	0.576	0.576	0.576	0.576	0.576	0.576	0.576	0.576	0.576
R_α		0.452	0.435	0.435	0.435	0.435	0.435	0.435	0.435	0.435	0.435	0.435	0.435
ψ_α		0.994	0.988	0.940	0.939	0.435	0.435	0.435	0.435	0.435	0.435	0.435	0.435
S_α	6.752	0.555	1.035	4.406	4.419	31.991	31.998	31.986	31.918	31.986	31.915	31.998	31.991
-Log L	120.17	99.56	101.77	130.38	129.72	104.13	101.94	102.16	101.40	102.16	101.40	101.45	101.67
Sic	129.32	113.29	115.49	144.10	143.44	115.57	113.38	115.88	115.12	113.60	112.84	112.89	113.11

supports the hypothesis that part of the negative impact of the outlier on the fitting has been absorbed by the reduction in the power of the Minkowki metric: from $\alpha = 1.2$ for the corrected dataset to $\alpha = 1.06$ for the wrong dataset. As the dataset transitions from a wrong to a correct content (even if for a single datum), the distributional parameters of all the models have undergone a dramatic change in their values. This has a double meaning: first, it can be referred to the sensitivity to outliers and instability of distributional regressions estimates which, in turn, might lead one to suspect a nonexistent structural modification in the phenomenon underlying the data; second, to the flexibility of the family of distributions which is able to capture any contamination of the data. It is difficult to establish what would be the prevalent attitude towards high sensitivity parameters in any given situation.

Since SGED is a special case of the SGT, it is not unexpected that there is similarity between the regression and distributional parameters of these two distributions.

18.6.4 Salinity Data

The data are based on $n = 28$ measurements of water salinity taken in North Carolina's Pamlico Sound. The salinity is regressed against the salinity lagged by two weeks, the trend and the volume of river discharge into the sound. According to Ruppert and Carroll (1980), this dataset has outliers: observations 1, 13, 15, and 17. (see also Dodge and Jurecková (2000, Section 3.4)). The true parameters of this dataset are unknown. The corruption of the data appears evident from the values of the skewness parameters that, in all the distributions, are pushed to the limits. Table 18.6–5 summarizes the results for this data.

A comparison with the findings reported both in Ruppert and Carroll (1980) and in Dodge and Jurecková (2000) shows that the distributional regression estimates are near the M estimates using the Huber's proposal with tuning constant equal to 1.2.

18.6.5 Gaussian Data

The artificial dataset discussed by Narula (1987) contains $n = 31$ observations generated from the model $y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u$ with parameters $\beta_0 = 0, \beta_1 = \beta_2 = 1$. The values of (X_1, X_2, u) were obtained from independent standard normal distributions. Results for OLS estimates are shown in Table 18.6–6 together with our findings related to the not-necessarily Gaussian models. The major bias appears to be in the estimate of the intercept term obtained by OLS. The slope coefficients are close to the real ones, while the measures of goodness-of-fit do not reveal any significant change in the fitting data performance with respect to OLS. This result is perhaps less obvious than it appears at a first glance since the OLS are performed under the assumption that, for this dataset, is the true

Table 18.6–5. DR results for salinity data.

Model Weight	OLS	FPLD F	FPLD T	GBD F	GBD T	JSU F	JSU T	SGT F	SGT T	SGED F	SGED T	EGB2 F	EGB2 T
β_0	9.590	13.865	13.396	13.944	13.914	13.981	13.981	13.984	13.984	13.981	13.984	13.962	13.981
β_1	0.777	0.740	0.740	0.741	0.741	0.741	0.741	0.741	0.741	0.741	0.741	0.741	0.741
β_2	-0.026	-0.099	-0.075	-0.103	-0.102	-0.106	-0.106	-0.106	-0.106	-0.106	-0.106	-0.106	-0.106
β_3	-0.295	-0.445	-0.428	-0.448	-0.447	-0.449	-0.449	-0.449	-0.449	-0.449	-0.449	-0.448	-0.449
α	2.000	1.083	1.179	1.051	1.035	1.072	1.072	1.071	1.071	1.072	1.071	1.073	1.072
λ_1		0.701	1.230	-1.094	-0.614	0.551	0.530	0.919	0.945	1.298	1.337	0.904	1.004
λ_2		-0.632	-0.828	1.732	1.000	0.721	0.712	1.000	1.000	0.006	-0.002	2.249	1.589
λ_3		-0.280	0.179	5.346	2.919	0.010	0.020	35.31	656494	1.000	1.000	2.192	1.660
λ_4		-0.990	-0.989	8.460	4.756			0.007	-0.002				
r^2	0.583	0.585	0.616	0.589	0.606	0.565	0.565	0.565	0.565	0.565	0.565	0.566	0.565
$\bar{r}^2$	0.558	0.541	0.575	0.544	0.564	0.529	0.529	0.519	0.519	0.529	0.529	0.530	0.529
$cor\,(y, \hat{y})$	0.909	0.902	0.903	0.902	0.902	0.902	0.902	0.902	0.902	0.902	0.902	0.902	0.902
$com\,(y, \hat{y})$	0.831	1.062	1.032	1.067	1.065	1.068	1.068	1.069	1.069	1.068	1.069	1.067	1.068
τ	0.797	0.822	0.828	0.824	0.831	0.814	0.814	0.814	0.814	0.814	0.814	0.814	0.814
b_o	0.698	0.724	0.731	0.726	0.735	0.715	0.715	0.715	0.715	0.715	0.715	0.715	0.715
R_α		0.680	0.721	0.675	0.685	0.661	0.661	0.661	0.661	0.661	0.661	0.662	0.661
ψ_α		0.989	0.981	0.984	0.970	0.661	0.661	0.661	0.661	0.661	0.661	0.662	0.661
S_α	6.517	0.874	1.517	1.167	2.131	20.698	20.698	20.710	20.710	20.691	20.710	20.630	20.698
-Log L	58.93	43.15	42.91	160.38	19.19	44.76	44.78	44.35	44.36	44.32	44.36	47.17	45.75
Sic	65.59	53.15	52.91	170.38	29.18	53.10	53.11	54.35	54.36	52.65	52.69	55.50	54.08

Table 18.6–6. DR results for Narula Gaussian data.

Model Weight	OLS	FPLD F	FPLD T	GBD F	GBD T	JSU F	JSU T	SGT F	SGT T	SGED F	SGED T	EGB2 F	EGB2 T
β_0	-0.161	-0.041	-0.042	-0.040	-0.040	-0.034	-0.034	-0.037	-0.034	-0.037	-0.034	-0.034	-0.034
β_1	0.924	1.083	1.086	1.083	1.083	1.083	1.083	1.083	1.083	1.083	1.083	1.083	1.083
β_2	1.128	1.046	1.046	1.045	1.045	1.045	1.045	1.046	1.045	1.046	1.045	1.045	1.045
α	2.000	1.208	1.205	1.209	1.209	1.209	1.209	1.212	1.209	1.212	1.209	1.209	1.209
λ_1		0.922	1.506	-1.545	-1.524	0.953	1.066	1.233	1.246	1.743	1.645	1.396	0.597
λ_2		0.291	0.840	1.000	1.000	0.868	0.983	1.001	1.044	0.056	0.035	2.737	0.635
λ_3		-0.687	-0.990	7.906	7.794	0.010	0.010	129.83	380379	1.000	1.043	2.587	0.598
λ_4		-0.234	0.214	5.116	5.115			0.056	0.035				
r^2	0.363	0.390	0.426	0.385	0.385	0.354	0.354	0.355	0.354	0.355	0.354	0.354	0.354
$\bar{r}^2$	0.328	0.331	0.371	0.326	0.327	0.307	0.307	0.293	0.293	0.307	0.307	0.307	0.307
$cor\,(y, \hat{y})$	0.771	0.766	0.766	0.766	0.766	0.766	0.766	0.766	0.766	0.766	0.766	0.766	0.766
$com\,(y, \hat{y})$	0.569	0.773	0.775	0.774	0.774	0.774	0.774	0.773	0.774	0.773	0.774	0.774	0.774
τ	0.680	0.707	0.717	0.705	0.705	0.690	0.690	0.690	0.690	0.690	0.690	0.690	0.690
b_o	0.568	0.597	0.606	0.595	0.596	0.584	0.584	0.584	0.584	0.584	0.584	0.584	0.584
R_α		0.524	0.545	0.519	0.520	0.490	0.490	0.491	0.490	0.491	0.490	0.490	0.490
ψ_α		0.985	0.972	0.987	0.987	0.490	0.490	0.491	0.490	0.491	0.490	0.490	0.490
S_α	8.452	1.120	1.859	0.971	0.980	20.52	20.52	20.37	20.52	20.37	20.52	20.52	20.52
-Log L	62.00	53.72	52.66	54.25	53.66	56.35	55.97	55.81	55.65	55.79	55.65	56.97	55.91
Sic	68.86	64.02	62.96	64.55	63.96	64.94	64.55	66.11	65.95	64.37	64.23	65.55	64.50

population is Gaussian. Partial adaptive estimation has correctly reproduced the real values of the regression parameters and provided a slightly better fit to the observed data than did the Gaussian distribution.

18.7 Conclusions and Future Research

Linear regression is a powerful statistical tool for quantitative investigations in many research studies. An essential task of partially adaptive estimation in linear regression analysis is to screen a large number of potential error distributions to select models which fit the information contained in the response variable both efficiently and concisely. Since estimates that are optimal for one residuals behavior may be quite inadequate for another, owing to differences in the tails and asymmetries, it is desirable to have a procedure that is sufficiently tractable over a vast range of different distributions.

Distributional regression (DR) is potentially useful in this connection. The specific peculiarity of this method is that, for a fixed class of not necessarily Gaussian distributions for the error term, the estimation procedure selects the subordinate model most compatible with the deterministic component of the regression. The benefits derived from a distributional approach mainly rely on the possibility of estimating, by a procedure, both the coefficients of the regression hyperplane and the parameters of the error distribution: the information derived from the deterministic part is treated in such a way as to increase efficiency of the estimation for the stochastic part and vice-versa. Consequently, the DR estimates may substantially improve upon OLS estimates in finite sample applications, particularly when the data errors are unevenly distributed around the mean, or show peakedness and/or fat tails. On the other hand, the DR method gives insignificantly different results from OLS for Gaussian distributed errors.

In this chapter we have outlined two approaches to adaptive estimation in linear regression: one probability-based and the other quantile-based. The two views are complementary, while quantiles and probabilities are in strict relationship both approaches should lead to qualitatively similar results. At the same time, the quantile domain is fundamentally different from the probability domain, since it focuses on tangible objects: the values rather than on abstract probabilities. The use of quantile functions such as FPLD is a new proposal in this context and it has revealed itself to be adaptable to many critical behaviors. Of course, our algorithm of DR for partially adaptive estimation is equally applicable to all the more complicated modeling situations such as time-series modeling.

The properties of the estimator proposed in this chapter have not been fully established, but they promise to hold interesting implications for the future. However, distributional regression estimation competes with ML estimation. In this sense, it is necessary to conduct a series of simulation studies assessing the performance of the DR method under various scenarios in comparison to the ML

method.

A possible drawback of the PAE approach is the selection procedure for choosing the parametric family of distributions for the residuals. This point deserves a specific investigation which must take into account the great amount of work done in the F domain and the opportunity offered by the Q domain.

References for Chapter 18

Abourizk, S. M., Halpin, D. W., and Wilson, J. R. (1994). "Fitting beta distribution based on sample data," *Journal of Construction Engineering and Management*, 120, 288–305.

Adcock, C. J. and Meade, N. (1995). "A simple algorithm to incorporate transactions costs in quadratic optimisation," *European Journal of Operational Research*, 79, 85–94.

Ali, M. M., Storey, A., and Törn, A. (1996). "Application of some stochastic global optimization algorithms to practical problems," *Journal of Optimization Theory and Applications*, 10, 545–563.

Ali, M. M., Törn, A., and Viitanen, S. (1997). "A numerical comparison of some modified controlled random search algorithms," *Journal of Optimization Theory and Applications*, 11, 377–385.

André, C. D. S., Elian, S. N., Narula, S. C., and Tavares R. A. (2000). "Coefficient of determination for variable selection in the MSAE regression," *Communications in Statistics — Theory and Methods*, 29, 623–642.

Anscombe, F. J. (1967). "Topics in the investigation of linear relations fitted by the method of least squares," *Journal of the Royal Statistical Society. Series B*, 29, 1–52.

Bali, T. G. and Theodossiou, P. (2008). "Risk measurement performance of alternative distribution functions," *Journal of Risk & Insurance*, 75, 411–437.

Bonferroni, C. E. (1940–41). *Elementi di statistica generale*, Litografia Gili, Torino.

Bower, K. M. (2003). "Some misconceptions about the normal distribution," *American Society for Quality, Six Sigma Forum*,
http://www.minitab.com/uploadedFiles/Shared_Resources/Documents_Articles/normal_distribution_misconceptions.pdf.

Brachetti, P., Ciccoli, M.D.F., Pillo, G. D., and Lucidi, S. (1997). "A new version of the Price's algorithm for global optimization," *Journal of Optimization Theory and Applications*, 10, 165–184.

Burnham, K. P. and Anderson, D. R. (2002). *Model Selection and Multimodal Inference*, Springer-Verlag, New York.

Butler, R.J., McDonald, J.B., Nelson, R.D., and White, S.B. (1990). "Robust and partially adaptive estimation of regression models," *The Reviews of Economics and Statistics* 72, 321–327.

Cade, B. S. and Richards, J. D. (1996). "Permutation tests for LAD regression," *Biometrics*, 52, 886–902.

Cameron, A. C. and Windmeijer, F. A. G. (1997). "An R^2 measure of goodness of fit for some common nonlinear regression models," *Journal of Econometrics*, 77, 329–342.

Carnahan, J. V. (1989). "Maximum likelihood estimation for the 4-parameter beta distribution," *Communications in Statistics: Simulation and Computation*, 18, 513–536.

Casti, J. (2009). "How normal is normal?," accessed date = 11/2009, ttp://kn.theiet.org/magazine/issues/0907/how-normal-0907.cfm.

Charnes, A., Frome E. L., and Yu, P. L. (1976). "The equivalence of generalized least squares and maximum likelihood estimates in the exponential family," *Journal of the American Statistical Association*, 71, 214–222.

Cooke, C. (1984). "Is the normal distribution normal?," *Physics Education*, 19, 305–307.

Cox, D. R. and Hinkley, D. V. (1968). "A note on the efficiency of least-squares estimates," *Journal of the Royal Statistical Society. Series B*, 30, 284–289.

Deng, S. J. and Jiang, W. (2002). "An inverse-quantile function approach for modeling electricity price," *Hawaii International Conference on System Sciences*, 2, 55b, IEEE Computer Society, Los Alamitos, CA.

Dee, D. P. and da Silva, A. M. (1999). "Maximum-likelihood estimation of forecast and observation error covariance parameters. Part I: Methodology," *Monthly Weather Review,* 127, 1822–1834.

Dodge, Y. and Jurecková, J. (2000). *Adaptive Regression*, Springer-Verlag, New York.

Dehon, C. and Croux, C. (2002). "Statistical inference for a robust measure of multiple correlation," in *Compstat 2002: Proceedings in Computational Statistics*, Berlin, Germany, eds. W. Härdle and B. R"onz, Physika-Verlag/Heidelberg, 557–562.

Dudewicz, E. J. and Karian, Z. A. (1996). "The extended generalized lambda distribution (EGLD) system for fitting distributions to data with moments, II: Tables," *American Journal of Mathematical and Management Sciences*, 16, 271–332.

Falk, M. (1997). "On mad and comedians," *Annals of the Institute of Statistical Mathematics*, 49, 615–644.

Filliben, J. J. (1975). "The probability plot correlation coefficient test for nor-

mality," *Technometrics*, 17, 111–117.

Fletcher, F., Grant, J. A., and Hebden, M. D. (1971). "The calculation of linear best L_p approximations," *Computer Journal*, 14, 276–279.

Freimer, M., Mudholkar, S., Kollia, G., and Lin, T. C. (1988). "A study of the generalized Tukey lambda family," *Communications in Statistics — Theory and Methods*, 17, 3547–3567.

Gallant, A. R. and Goebel, J. J. (1976). "Nonlinear regression with auto correlated errors," *Journal of the American Statistical Association*, 71, 961–967.

Gentle, J. E. (2004). "Optimization Methods for Applications in Statistics," George Mason University, Madison (VA),
http://aleria.info/temp/87ffc9f7f9_Gentle-OptimiStat.pdf.

Gilchrist, W. G. (2000). *Statistical Modelling with Quantile Functions*, Chapman & Hall, CRC, Boca Raton (FL).

Gilchrist, W. (2008). "Regression revisited," *International Statistical Review*, 76, 401–418.

Goldfeld, S. M. and Quandt, R. E. (1981). "Econometric modelling with nonnormal disturbances," *Journal of Econometrics*, 17, 141–155.

Gonin, R. and Money, A. H. (1985). "Nonlinear L_p-norm estimation: Part I — On the choice of the exponent, p, where the errors are adaptive," *Communications in Statistics — Theory and Methods*, 14, 827–840.

Gourieroux, C., Monfort, A., and Trognon, A. (1984). "Pseudo maximum likelihood method: Theory," *Econometrica*, 52, 681–700.

Hagquist, C. and Stenbeck, M. (1998). "Goodness of fit regressione analysis — R^2 and G^2 reconsidered," *Quality & Quantity*, 32, 229–245.

Hankin, R. K. S. and Lee, A. (2006). "A new family of non-negative distributions," *Australian & New Zealand Journal of Statistics*, 48, 67–78.

Hansen, J. V., McDonald, J. B., and Turley, R. S. (2006). "Partially adaptive robust estimation of regression models and applications," *European Journal of Operational Research*, 170, 132–143.

Harrison, D. and Rubinfeld, D. (1978). "Hedonic prices and the demand for clean air," *Journal of Environmental Economics and Management*, 5, 81–102.

Hastie, T., Tibshirani, R. and Friedman, J. (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, Springer-Verlag, New York.

Healy, M. J. R. (1984). "The use of R^2 as a measure of goodness of fit," *Journal of the Royal Statistical Society. Series A*, 147, 608–609.

Hooke, R. and Jeeves, T. A. (1961). "Direct search solution of numerical and statistical problems," *Journal of Assoc. Computing Machinery*, 8, 212–229.

Houghton, J. C. (1977). "Birth of a parent: The Wakeby distribution for model-

ing flood flows," Technical Report, MIT, MIT-EL77-033WP.

Islam, M. Q. and Tiku, M. L. (2004). "Multiple linear regression models under non-normality," *Communications in Statistics — Theory and Methods*, 33, 2443–2467.

Jiang, W., Chen, G., and Wu, Z. (2008). "A new quantile function based model for modeling price behaviors in financial markets," *Statistics and Its Interface*, 1, 327–332.

Jones, M. C. (2007). "The minimax distribution: A beta-type distribution with some tractability advantages," The Open University - Statistics Group, *Technical Report.*

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*, CRC Press, Boca Raton (FL).

Karian, Z. A., Dudewicz, E. J. and McDonald, P. (1996). "The extended generalized lambda distribution (EGLD) system for fitting distributions to data: Completation of the theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics — Simulation and Computation*, 25, 611–642.

King, R. (2009) "GLD: Estimation and use of the generalised (Tukey) lambda distribution," **R** package version 1.8.4,
http://tolstoy.newcastle.edu.au/ rking/publ/rprogs/information.html.

King, A. A. R. and MacGillivray, H. L. (1999). "A starship estimation method for the generalized λ distributions," *Australian and New Zealand Journal of Statistics*, 41, 353–374.

Koenker, R. and D'Orey, V. (1987). "Computing regression quantiles," *Applied Statistics*, 36, 383–393.

Koenker, R. and Machado, J. (1999). "Goodness of fit and related inference process for quantile regression," *Journal of the American Statistical Association*, 94, 1296–1309.

Krivy, I. and Tvrdik, J. (1995). "The controlled random search algorithm in optimizing regression models," *Computational Statistics & Data Analysis*, 20, 229–234.

Kvålset, T. O. (1985). "Cautionary note about R^2," *The American Statistician*, 39, 279–285.

Lampasi, D. A. (2008). "An alternative approach to measurement based on quantile functions," *Measurement*, 41, 994–1013.

Lawton, W. H. and Sylvestre, E. A. (1971). "Elimination of linear parameters in nonlinear regression," *Technometrics*, 13, 461–467.

Lloyd, E. H. (1952). "Least squares estimation of location and scale parameters using order statistics," *Biometrika*, 39, 88–95.

Lye, J.N. and Martin, V.L. (1993). "Robust estimation, nonnormalities, and generalized exponential distributions," *Journal of the American Statistical Society*, 88, 261–267.

McGarvey, R. G., Del Castillo, E., Cavalier, T. M., and Lehtihet, E. A. (2002). "Four-parameter beta distribution estimation and skewness test," *Quality and Reliability Engineering International*, 18, 395–402.

McDonald, J. B. (1991). "Parametric models for partially adaptive estimation with skewed and leptokurtic residuals," *Economic Letters*, 12, 103–104.

McDonald, J. B. and Newey, W. K. (1998). "Partially adaptive estimation of regression models via the generalized t distribution," *Econometric Theory*, 4, 428–457.

McDonald, J. B. and White, S. B. (1993). "Comparison of robust, adaptive, and partially adaptive estimators of regression models," *Econometric Reviews*, 37, 273–278.

McDonald, J. B. and Xu, Y. J. (1995). "A generalization of the beta distribution with applications," *Journal of Econometrics*, 66, 133–152.

McDonald, J. B. and Xu, Y. J. (1998). "A generalization of the beta distribution with applications. *Erratum*," *Journal of Econometrics*, 69, 427–428.

McKean, J. W. and Sievers, G. L. (1987). "Coefficients of determination for least absolute deviation analysis," *Statistics & Probability Letters*, 5, 49–54.

Milner, S. As yet untitled paper, In preparation, for submission to the A5 symposium on Control of Railways – the human challenge. PUMA working paper WP34. See the World Wide Web pages at
http://www.cs.mdx.ac.uk/puma/.

Mineo, A. and Ruggeri, M. (2005). "A software tool for the exponential power distribution: The NORMALP package," *Journal of Statistical Software*, 12, 1–24.

Narula, S. C. (1987). "The minimum sum of absolute errors regression," *Journal of Quality Technology*, 19, 37–45.

Nelder, J. A. and Mead, R. (1965). "A simplex method for function minimization," *Computer Journal*, 7, 308–313.

Oberhofer, W. and Kmenta, J. (1974). "A general procedure for obtaining maximum likelihood estimates in generalized regression models," *Econometrica*, 42, 579–590.

Öztürk, A. and Dale, R. F. (1985). "Least squares estimation of the parameters of the generalized lambda distributions," *Technometrics*, 19, 81–84.

Parzen, E. (1979). "Nonparametric statistical data modelling," *Journal of the American Statistical Association*, 74, 105–131.

Parzen, E. (2004). "Quantile probability and statistical data modeling," *Statistical Science*, 19, 652–662.

Perri, P. F. and Tarsitano, A. (2007a). "A quantile function-based approach for least absolute deviations," *Fifth Conference on Complex Models and Computational Intensive Methods for Estimation and Prediction*, Cleup, 386–391, Padova.

Perri, P. F. and Tarsitano, A. (2007b). "Partially adaptive estimation via quantile functions," *Communications in Statistics — Simulation and Computation*, 36, 277-296.

Perri, P. F. and Tarsitano, A. (2008). "Distributional Least Squares Based on the Generalized Lambda Distribution," *COMPSTAT 2008, Proceedings in Computational Statistics, 18th Symposium*, ed. Brito, P., Springer, Berlin, 341–348.

Phadnis, S. (2002). "Handling non-normal data," accessed-date: 11/2009, http://www.isixsigma.com/library/content/c020121a.asp.

Phillips, R.F. (1994). "Partially adaptive estimation via a normal mixture," *Journal of Econometrics*, 64, 123–144.

Price, W. L. (1977). "Global optimization by controlled random search," *Computer Journal*, 20, 367–370.

Pyzdek, T. (1999). "Non-normal distributions in the real word: What to do when non-normal is the norm," *Quality Digest*, http://www.qualitydigest.com/dec99/html/nonnormal.html.

R Development Core Team. (2009). "**R**: A Language and Environment for Statistical Computing," **R** *Foundation for Statistical Computing*, Vienna, Austria, http://www.R-project.org.

Ramberg, J. S., Dudewicz, E. J., Tadikamalla, P. R., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics*, 21, 201–214.

Ramberg, J. S. and Schmeiser, B. W. (1974). "An approximate method for generating asymmetric random variables," *Communications of the ACM*, 17, 78–82.

Rao, R. (1988). "Methodology based on the L_1 norm in statistical inference," *Sankhya*, 50, 289–313.

Royston, P. and Wright, E. M. (1998). "Goodness-of-fit statistics for age specific reference intervals," *Statistics in Medicine*, 19, 2943–2962.

Ruppert, D. and Carroll, R. J. (1980). "Trimmed least squares estimation in the linear model," *Journal of the American Statistical Society*, 75, 828–838.

Schlossmacher, E. J. (1973). "An iterative technique for absolute deviations curve fitting," *Journal of the American Statistical Society*, 68, 857–859.

Seppo, P. and Salmi, T. (1994). "A report on least absolute deviation regression with ordinary linear programming," *The Finnish Journal of Business Economics*, 43, 33–49.

Shelton, J. T., Khuri, A. I., and Cornell, J. A. (1983). "Selecting check points

for testing lack of fit in response surface models," *Technometrics*, 25, 357–365.

Smyth, G. K. (1996). "Partitioned algorithms for maximum likelihood and other non-linear estimation," *Statistics and Computing*, 16, 201–216.

Sonnergaard, J. M.(2006). "On the misinterpretation of the correlation coefficient in pharmaceutical sciences," *International Journal of Pharmaceutics*, 321, 12–17.

Späth, H. (1987). *Mathematical Algorithms for Linear Regression*, Academic Press, San Diego (CA).

Swain, J., Venkatraman, S., and Wilson, J. (1988). "Least squares estimation of distribution functions in Johnson's translation system," *Journal of Statistical Computation and Simulation*, 29, 271–297.

Tarsitano, A. (2010). "Comparing estimation methods for the FPLD," *Journal of Probability and Statistics*. To appear.

Tiku, M.L., Islam, M.Q., and Selcuk, A.S. (2001). "Non-normal regression, II: Symmetric distributions," *Communications in Statistics — Theory and Methods*, 30, 1021–1045.

Wang, J. Z. (2005). "A note on estimation in the four-parameter beta distribution," *Communications in Statistics: Simulation and Computation*, 34, 495–501

Wilson, J. (1983). "Modeling multivariate populations with Johnson translation systems," Technical Report, Mechanical Engineering Department, University of Texas at Austin.

Zeckhauser, R. and Thompson, M. (1970). "Linear regression with non-normal error terms," *The Reviews of Economics and Statistics*, 52, 280–286.

Appendix for Chapter 18: Data Listings and Code

A.1 Data Listing

Mayer's Data
(See Section 18.6.1 for a description of the data.)

Latitude	Sine	Cosine	Latitude	Sine	Cosine
13.1667	−0.8836	0.4682	13.9667	−0.3608	−0.9326
13.1333	−0.9996	0.0282	14.2333	−0.1302	−0.9915
13.2	−0.9899	−0.1421	14.9333	0.1068	−0.9943
14.25	−0.2221	−0.975	14.7833	0.3363	−0.9418
14.7	−0.0006	−1	15.9333	0.856	−0.517
13.0167	−0.9308	0.3654	13.4833	−0.8002	−0.5997
14.5167	−0.0602	−0.9982	15.9167	0.9952	0.0982
14.95	0.157	−0.9876	15.65	0.8409	−0.5412
13.0833	−0.9097	0.4152	16.15	0.9429	−0.333
13.0333	−1	−0.0055	16.3667	0.9768	−0.2141
13.2	−0.9689	−0.2476	15.6333	0.6262	0.7797
13.1833	−0.8878	−0.4602	14.9	0.4091	0.9125
13.5667	−0.7549	−0.6558	13.1167	−0.9284	0.3716
13.8833	−0.5755	−0.8178			

Martin Marietta Data
(See Section 18.6.2 for a description of the data.)

Rate of Return			Rate of Return	
On Equity	Market Portfolio		On Equity	Market Portfolio
−0.1365	−0.03		0.0876	−0.0238
−0.0769	−0.0584		0.1025	0.1031
−0.0575	−0.0181		−0.0499	−0.0065
0.0526	0.0306		0.1953	−0.0067
−0.0449	−0.0397		−0.0714	−0.0167
−0.0859	−0.0295		0.0469	0.0188
−0.0742	−0.0316		0.1311	0.0733
0.6879	0.1176		0.0461	0.0105
−0.077	0.0075		−0.0328	−0.007
0.085	0.1098		−0.0096	−0.0099
0.003	0.0408		0.1272	0.0521
0.0754	0.0095		−0.0077	0.0117
−0.0412	0.0301		0.0165	−0.0099
−0.089	0.0221		−0.015	−0.0102
0.2319	0.0269		−0.1479	−0.0428
0.1087	0.0655		−0.0065	0.0376
0.0375	−0.003		0.039	0.0628
0.0958	0.0325		0.0223	0.0391
0.0174	−0.0374		−0.069	0.0002
−0.0724	0.0049		0.1338	0.0688
0.075	0.0105		0.1458	0.0486
−0.0588	−0.0257		0.0063	−0.0174
−0.062	0.0186		0.0692	0.046
−0.0378	−0.0155		−0.0239	0.01
0.0169	−0.0165		−0.0568	−0.0594
−0.0799	−0.044		0.0814	0.068
−0.0147	0.0094		−0.0889	−0.0839
0.0106	−0.0028		−0.0887	0.0481
−0.0421	−0.0591		0.1037	0.0136
−0.0036	0.0158		−0.1163	−0.0322

Prostate Data – with wrong value.
(See Section 18.6.3 for a description of the data.)

lpsa	lcavol	lweight	age	lbph	svi	lcp	G	P	Tr
−0.4308	−0.5798	2.7695	50	−1.3863	0	−1.3863	6	0	T
−0.1625	−0.9943	3.3196	58	−1.3863	0	−1.3863	6	0	T
−0.1625	0.5108	2.6912	74	−1.3863	0	−1.3863	7	20	T
−0.1625	−1.204	3.2828	58	−1.3863	0	−1.3863	6	0	T
0.3716	0.7514	3.4324	62	−1.3863	0	−1.3863	6	0	T
0.7655	−1.0498	3.2288	50	−1.3863	0	−1.3863	6	0	T
0.7655	0.7372	3.4735	64	0.6152	0	−1.3863	6	0	F
0.8544	0.6931	3.5395	58	1.5369	0	−1.3863	6	0	T
1.0473	−0.7765	3.5395	47	−1.3863	0	−1.3863	6	0	F
1.0473	0.2231	3.2445	63	−1.3863	0	−1.3863	6	0	F
1.2669	0.2546	3.6041	65	−1.3863	0	−1.3863	6	0	T
1.2669	−1.3471	3.5987	63	1.2669	0	−1.3863	6	0	T
1.2669	1.6134	3.0229	63	−1.3863	0	−0.5978	7	30	T
1.3481	1.477	2.9982	67	−1.3863	0	−1.3863	7	5	T
1.3987	1.206	3.442	57	−1.3863	0	−0.4308	7	5	F
1.4469	1.5412	3.0611	66	−1.3863	0	−1.3863	6	0	T
1.4702	−0.4155	3.516	70	1.2442	0	−0.5978	7	30	T
1.4929	2.2885	3.6494	66	−1.3863	0	0.3716	6	0	T
1.5581	−0.5621	3.2677	41	−1.3863	0	−1.3863	6	0	T
1.5994	0.1823	3.8254	70	1.6582	0	−1.3863	6	0	T
1.639	1.1474	3.4194	59	−1.3863	0	−1.3863	6	0	T
1.6582	2.0592	3.501	60	1.4748	0	1.3481	7	20	F
1.6956	−0.5447	3.3759	59	−0.7985	0	−1.3863	6	0	T
1.7138	1.7817	3.4516	63	0.4383	0	1.1787	7	60	T
1.7317	0.3853	3.6674	69	1.5994	0	−1.3863	6	0	F
1.7664	1.4469	3.1246	68	0.3001	0	−1.3863	6	0	F
1.8001	0.5128	3.7197	65	−1.3863	0	−0.7985	7	70	T
1.8165	−0.4005	3.866	67	1.8165	0	−1.3863	7	20	F
1.8485	1.0403	3.129	67	0.2231	0	0.0488	7	80	T
1.8946	2.4096	3.3759	65	−1.3863	0	1.6194	6	0	T
1.9242	0.2852	4.0902	65	1.9629	0	−0.7985	6	0	T
2.0082	0.1823	3.8044	65	1.7047	0	−1.3863	6	0	F
2.0082	1.2754	3.0374	71	1.2669	0	−1.3863	6	0	T
2.0215	0.01	3.2677	54	−1.3863	0	−1.3863	6	0	F
2.0477	−0.0101	3.2169	63	−1.3863	0	−0.7985	6	0	T
2.0857	1.3083	4.1199	64	2.1713	0	−1.3863	7	5	F
2.1576	1.4231	3.6571	73	−0.5798	0	1.6582	8	15	T
2.1917	0.4574	2.3749	64	−1.3863	0	−1.3863	7	15	T
2.2138	2.661	4.0851	68	1.3737	1	1.8326	7	35	T
2.2773	0.7975	3.0131	56	0.9361	0	−0.1625	7	5	T
2.2976	0.6206	3.142	60	−1.3863	0	−1.3863	9	80	T
2.3076	1.4422	3.6826	68	−1.3863	0	−1.3863	7	10	F
2.3273	0.5822	3.866	62	1.7138	0	−0.4308	6	0	T
2.3749	1.7716	3.8969	61	−1.3863	0	0.8109	7	6	F
2.5217	1.4861	3.4095	66	1.7492	0	−0.4308	7	20	T
2.5533	1.6639	3.3928	61	0.6152	0	−1.3863	7	15	T
2.5688	2.7279	3.9954	79	1.8795	1	2.6568	9	100	T
2.5688	1.1632	4.0351	68	1.7138	0	−0.4308	7	40	F

Prostate Data – with wrong value (continued).

lpsa	lcavol	lweight	age	lbph	svi	lcp	G	P	Tr
2.5915	1.7457	3.498	43	−1.3863	0	−1.3863	6	0	F
2.5915	1.2208	3.5681	70	1.3737	0	−0.7985	6	0	F
2.6568	1.0919	3.9936	68	−1.3863	0	−1.3863	7	50	T
2.6776	1.6601	4.2348	64	2.0732	0	−1.3863	6	0	T
2.6844	0.5128	3.6336	64	1.4929	0	0.0488	7	70	F
2.6912	2.127	4.1215	68	1.7664	0	1.4469	7	40	F
2.7047	3.1536	3.516	59	−1.3863	0	−1.3863	7	5	F
2.718	1.2669	4.2801	66	2.1223	0	−1.3863	7	15	T
2.7881	0.9746	2.8651	47	−1.3863	0	0.5008	7	4	F
2.7942	0.4637	3.7647	49	1.4231	0	−1.3863	6	0	T
2.8064	0.5423	4.1782	70	0.4383	0	−1.3863	7	20	T
2.8124	1.0613	3.8512	61	1.2947	0	−1.3863	7	40	T
2.842	0.4574	4.5245	73	2.3263	0	−1.3863	6	0	T
2.8536	1.9974	3.7197	63	1.6194	1	1.9095	7	40	F
2.8536	2.7757	3.5249	72	−1.3863	0	1.5581	9	95	T
2.882	2.0347	3.917	66	2.0082	1	2.1102	7	60	F
2.882	2.0732	3.623	64	−1.3863	0	−1.3863	6	0	F
2.8876	1.4586	3.8362	61	1.3218	0	−0.4308	7	20	F
2.9205	2.0229	3.8785	68	1.7834	0	1.3218	7	70	T
2.9627	2.1983	4.0509	72	2.3076	0	−0.4308	7	10	T
2.9627	−0.4463	4.4085	69	−1.3863	0	−1.3863	6	0	T
2.973	1.1939	4.7804	72	2.3263	0	−0.7985	7	5	T
3.0131	1.8641	3.5932	60	−1.3863	1	1.3218	7	60	T
3.0374	1.16	3.3411	77	1.7492	0	−1.3863	7	25	T
3.0564	1.2149	3.8254	69	−1.3863	1	0.2231	7	20	F
3.075	1.839	3.2367	60	0.4383	1	1.1787	9	90	F
3.2753	2.9992	3.8491	69	−1.3863	1	1.9095	7	20	T
3.3375	3.1411	3.2638	68	−0.0513	1	2.4204	7	50	T
3.3928	2.0109	4.4338	72	2.1223	0	0.5008	7	60	T
3.4356	2.5377	4.3548	78	2.3263	0	−1.3863	7	10	T
3.4579	2.6483	3.5821	69	−1.3863	1	2.584	7	70	T
3.513	2.7794	3.8232	63	−1.3863	0	0.3716	7	50	F
3.516	1.4679	3.0704	66	0.5596	0	0.2231	7	40	T
3.5308	2.5137	3.4735	57	0.4383	0	2.3273	7	60	T
3.5653	2.613	3.8888	77	−0.5276	1	0.5596	7	30	T
3.5709	2.6776	3.8384	65	1.1151	0	1.7492	9	70	F
3.5877	1.5623	3.7099	60	1.6956	0	0.8109	7	30	T
3.631	3.3028	3.519	64	−1.3863	1	2.3273	7	60	T
3.6801	2.0242	3.7317	58	1.639	0	−1.3863	6	0	T
3.7124	1.7317	3.369	62	−1.3863	1	0.3001	7	30	T
3.9843	2.8076	4.7181	65	−1.3863	1	2.4639	7	60	T
3.9936	1.5623	3.6951	76	0.9361	1	0.8109	7	75	T
4.0298	3.2465	4.1018	68	−1.3863	0	−1.3863	6	0	T
4.1296	2.5329	3.6776	61	1.3481	1	−1.3863	7	15	T
4.3851	2.8303	3.8764	68	−1.3863	1	1.3218	7	60	T
4.6844	3.821	3.8969	44	−1.3863	1	2.1691	7	40	T
5.1431	2.9074	3.3962	52	−1.3863	1	2.4639	7	10	F
5.4775	2.8826	3.7739	68	1.5581	1	1.5581	7	80	T
5.5829	3.472	3.975	68	0.4383	1	2.9042	7	20	F

Prostate Data – corrected

(See Section 18.6.3 for a description of the data.)

lpsa	lcavol	lweight	age	lbph	svi	lcp	G	P	Tr
−0.4308	−0.5798	2.7695	50	−1.3863	0	−1.3863	6	0	T
−0.1625	−0.9943	3.3196	58	−1.3863	0	−1.3863	6	0	T
−0.1625	−0.5108	2.6912	74	−1.3863	0	−1.3863	7	20	T
−0.1625	−1.204	3.2828	58	−1.3863	0	−1.3863	6	0	T
0.3716	0.7514	3.4324	62	−1.3863	0	−1.3863	6	0	T
0.7655	−1.0498	3.2288	50	−1.3863	0	−1.3863	6	0	T
0.7655	0.7372	3.4735	64	0.6152	0	−1.3863	6	0	F
0.8544	0.6931	3.5395	58	1.5369	0	−1.3863	6	0	T
1.0473	−0.7765	3.5395	47	−1.3863	0	−1.3863	6	0	F
1.0473	0.2231	3.2445	63	−1.3863	0	−1.3863	6	0	F
1.2669	0.2546	3.6041	65	−1.3863	0	−1.3863	6	0	T
1.2669	−1.3471	3.5987	63	1.2669	0	−1.3863	6	0	T
1.2669	1.6134	3.0229	63	−1.3863	0	−0.5978	7	30	T
1.3481	1.477	2.9982	67	−1.3863	0	−1.3863	7	5	T
1.3987	1.206	3.442	57	−1.3863	0	−0.4308	7	5	F
1.4469	1.5412	3.0611	66	−1.3863	0	−1.3863	6	0	T
1.4702	−0.4155	3.516	70	1.2442	0	−0.5978	7	30	T
1.4929	2.2885	3.6494	66	−1.3863	0	0.3716	6	0	T
1.5581	−0.5621	3.2677	41	−1.3863	0	−1.3863	6	0	T
1.5994	0.1823	3.8254	70	1.6582	0	−1.3863	6	0	T
1.639	1.1474	3.4194	59	−1.3863	0	−1.3863	6	0	T
1.6582	2.0592	3.501	60	1.4748	0	1.3481	7	20	F
1.6956	−0.5447	3.3759	59	−0.7985	0	−1.3863	6	0	T
1.7138	1.7817	3.4516	63	0.4383	0	1.1787	7	60	T
1.7317	0.3853	3.6674	69	1.5994	0	−1.3863	6	0	F
1.7664	1.4469	3.1246	68	0.3001	0	−1.3863	6	0	F
1.8001	0.5128	3.7197	65	−1.3863	0	−0.7985	7	70	T
1.8165	−0.4005	3.866	67	1.8165	0	−1.3863	7	20	F
1.8485	1.0403	3.129	67	0.2231	0	0.0488	7	80	T
1.8946	2.4096	3.3759	65	−1.3863	0	1.6194	6	0	T
1.9242	0.2852	4.0902	65	1.9629	0	−0.7985	6	0	T
2.0082	0.1823	6.1076	65	1.7047	0	−1.3863	6	0	F
2.0082	1.2754	3.0374	71	1.2669	0	−1.3863	6	0	T
2.0215	0.01	3.2677	54	−1.3863	0	−1.3863	6	0	F
2.0477	−0.0101	3.2169	63	−1.3863	0	−0.7985	6	0	T
2.0857	1.3083	4.1199	64	2.1713	0	−1.3863	7	5	F
2.1576	1.4231	3.6571	73	−0.5798	0	1.6582	8	15	T
2.1917	0.4574	2.3749	64	−1.3863	0	−1.3863	7	15	T
2.2138	2.661	4.0851	68	1.3737	1	1.8326	7	35	T
2.2773	0.7975	3.0131	56	0.9361	0	−0.1625	7	5	T
2.2976	0.6206	3.142	60	−1.3863	0	−1.3863	9	80	T
2.3076	1.4422	3.6826	68	−1.3863	0	−1.3863	7	10	F
2.3273	0.5822	3.866	62	1.7138	0	−0.4308	6	0	T
2.3749	1.7716	3.8969	61	−1.3863	0	0.8109	7	6	F
2.5217	1.4861	3.4095	66	1.7492	0	−0.4308	7	20	T
2.5533	1.6639	3.3928	61	0.6152	0	−1.3863	7	15	T
2.5688	2.7279	3.9954	79	1.8795	1	2.6568	9	100	T
2.5688	1.1632	4.0351	68	1.7138	0	−0.4308	7	40	F

Prostate Data – corrected (continued).

lpsa	lcavol	lweight	age	lbph	svi	lcp	G	P	Tr
2.5915	1.7457	3.498	43	−1.3863	0	−1.3863	6	0	F
2.5915	1.2208	3.5681	70	1.3737	0	−0.7985	6	0	F
2.6568	1.0919	3.9936	68	−1.3863	0	−1.3863	7	50	T
2.6776	1.6601	4.2348	64	2.0732	0	−1.3863	6	0	T
2.6844	0.5128	3.6336	64	1.4929	0	0.0488	7	70	F
2.6912	2.127	4.1215	68	1.7664	0	1.4469	7	40	F
2.7047	3.1536	3.516	59	−1.3863	0	−1.3863	7	5	F
2.718	1.2669	4.2801	66	2.1223	0	−1.3863	7	15	T
2.7881	0.9746	2.8651	47	−1.3863	0	0.5008	7	4	F
2.7942	0.4637	3.7647	49	1.4231	0	−1.3863	6	0	T
2.8064	0.5423	4.1782	70	0.4383	0	−1.3863	7	20	T
2.8124	1.0613	3.8512	61	1.2947	0	−1.3863	7	40	T
2.842	0.4574	4.5245	73	2.3263	0	−1.3863	6	0	T
2.8536	1.9974	3.7197	63	1.6194	1	1.9095	7	40	F
2.8536	2.7757	3.5249	72	−1.3863	0	1.5581	9	95	T
2.882	2.0347	3.917	66	2.0082	1	2.1102	7	60	F
2.882	2.0732	3.623	64	−1.3863	0	−1.3863	6	0	F
2.8876	1.4586	3.8362	61	1.3218	0	−0.4308	7	20	F
2.9205	2.0229	3.8785	68	1.7834	0	1.3218	7	70	T
2.9627	2.1983	4.0509	72	2.3076	0	−0.4308	7	10	T
2.9627	−0.4463	4.4085	69	−1.3863	0	−1.3863	6	0	T
2.973	1.1939	4.7804	72	2.3263	0	−0.7985	7	5	T
3.0131	1.8641	3.5932	60	−1.3863	1	1.3218	7	60	T
3.0374	1.16	3.3411	77	1.7492	0	−1.3863	7	25	T
3.0564	1.2149	3.8254	69	−1.3863	1	0.2231	7	20	F
3.075	1.839	3.2367	60	0.4383	1	1.1787	9	90	F
3.2753	2.9992	3.8491	69	−1.3863	1	1.9095	7	20	T
3.3375	3.1411	3.2638	68	−0.0513	1	2.4204	7	50	T
3.3928	2.0109	4.4338	72	2.1223	0	0.5008	7	60	T
3.4356	2.5377	4.3548	78	2.3263	0	−1.3863	7	10	T
3.4579	2.6483	3.5821	69	−1.3863	1	2.584	7	70	T
3.513	2.7794	3.8232	63	−1.3863	0	0.3716	7	50	F
3.516	1.4679	3.0704	66	0.5596	0	0.2231	7	40	T
3.5308	2.5137	3.4735	57	0.4383	0	2.3273	7	60	T
3.5653	2.613	3.8888	77	−0.5276	1	0.5596	7	30	T
3.5709	2.6776	3.8384	65	1.1151	0	1.7492	9	70	F
3.5877	1.5623	3.7099	60	1.6956	0	0.8109	7	30	T
3.631	3.3028	3.519	64	−1.3863	1	2.3273	7	60	T
3.6801	2.0242	3.7317	58	1.639	0	−1.3863	6	0	T
3.7124	1.7317	3.369	62	−1.3863	1	0.3001	7	30	T
3.9843	2.8076	4.7181	65	−1.3863	1	2.4639	7	60	T
3.9936	1.5623	3.6951	76	0.9361	1	0.8109	7	75	T
4.0298	3.2465	4.1018	68	−1.3863	0	−1.3863	6	0	T
4.1296	2.5329	3.6776	61	1.3481	1	−1.3863	7	15	T
4.3851	2.8303	3.8764	68	−1.3863	1	1.3218	7	60	T
4.6844	3.821	3.8969	44	−1.3863	1	2.1691	7	40	T
5.1431	2.9074	3.3962	52	−1.3863	1	2.4639	7	10	F
5.4775	2.8826	3.7739	68	1.5581	1	1.5581	7	80	T
5.5829	3.472	3.975	68	0.4383	1	2.9042	7	20	F

Salinity Data

(See Section 18.6.4 for a description of the data.)

Salinity	Lagsal	Trend	Waterd
7.6	8.2	4	23.005
7.7	7.6	5	23.873
4.3	4.6	0	26.417
5.9	4.3	1	24.868
5	5.9	2	29.895
6.5	5	3	24.2
8.3	6.5	4	23.215
8.2	8.3	5	21.862
13.2	10.1	0	22.274
12.6	13.2	1	23.83
10.4	12.6	2	25.144
10.8	10.4	3	22.43
13.1	10.8	4	21.785
12.3	13.1	5	22.38
10.4	13.3	0	23.927
10.5	10.4	1	33.443
7.7	10.5	2	24.859
9.5	7.7	3	22.686
12	10	0	21.789
12.6	12	1	22.041
13.6	12.1	4	21.033
14.1	13.6	5	21.005
13.5	15	0	25.865
11.5	13.5	1	26.29
12	11.5	2	22.932
13	12	3	21.313
14.1	13	4	20.769
15.1	14.1	5	21.393

Gaussian Data

(See Section 18.6.5 for a description of the data.)

Ydep	X1	X2
0.9797	0.5172	0.2667
-2.9153	-1.8203	-0.9067
1.9162	2.2095	0.1622
3.3493	-0.7685	2.0982
1.259	-1.1293	2.5264
-6.4798	-2.248	-1.9282
-0.992	3.5005	0.3594
2.5974	1.7413	0.0142
-1.8367	-0.9008	0.6612
-1.4193	0.1993	-0.0746
1.7248	-0.3334	1.4317
0.4704	1.126	-1.1948
1.6926	0.2027	0.671
-0.7337	0.5215	-0.3461
2.0666	0.7076	0.379
-1.0316	-0.6142	-0.5287
0.8469	0.1812	2.5557
-2.4232	-1.2812	-0.7426
3.0334	1.6236	-0.7988
2.0915	1.5206	0.0123
-3.3738	-2.9653	1.3155
3.5124	1.7911	1.7359
-1.3748	0.4146	-1.1015
-0.1605	-0.4697	0.4283
-0.9695	-1.2368	-0.0497
4.0729	-0.2617	2.0894
-2.158	-1.9194	-0.5292
2.2126	-0.5849	-0.2807
3.0598	2.3668	0.46
-0.7415	0.5606	-0.6821
-1.3162	0.6216	-0.082

A.2 R Code of the Procedures

```
Density<-function(y,x,dmodel)
{# Johnson's SU
if (dmodel=="JSU")
 {C1<-x[2]/x[1];arg1<-sqrt(1+(y/x[1])^2);arg1<-C1/arg1
     arg2<-dnorm(x[3]+x[2]*asinh(y/x[1]))
     arg<-arg1*arg2
     return(arg)}
# Skewed generalized t-Student
 if (dmodel=="SGT")
    {Rl2<-1/x[2]
     C1<-x[2]/(2*x[1]*(x[3]^Rl2)*beta(Rl2,x[3]))
     arg<-(abs(y)/(x[1]*(1+x[4]*sign(y))))^x[2]
     arg<-(1+arg/x[3])^(x[3]+Rl2)
     arg<-C1/arg
     return(arg)}
# Skewed generalized error distribution
if (dmodel=="SGED")
    {Rl2<-1/x[3];G1<-gamma(Rl2);G2<-gamma(2*Rl2);G3<-gamma(3*Rl2)
     Theta<-G1/(G3*(1+(3-4*G2^2/(G3*G1))*x[2]^2));Theta<-sqrt(Theta)
     C1<-x[3]/(2*x[1]*G1*Theta)
     arg<-abs(y)/((1+sign(y)*x[2])*x[1]*Theta)
     arg<-C1*exp(-(arg^x[3]))
     return(arg)}
# Exponential generalized beta of the second type
if (dmodel=="EGB2")
   {arg1<-(1+exp(y/x[1]))^(x[2]+x[3])
    C1<-x[1]*beta(x[2],x[3])
    arg2<-exp(x[2]*y/x[1])
    arg<-arg2/(C1*arg1)
    return(arg)}
}
########################################
Likelihood<-function(err,x,Ppos,dmodel)
{n<-length(err)
 if (dmodel=="JSU")
 {Term1<-n*log(x[2]/x[1])
     C1<-sum(log(dnorm(x[3]+x[2]*asinh(err/x[1]))))
     C2<-0.5*sum(log(1+(err/x[1])^2))
     arg<--Term1+C2-C1
     return(arg)}
# Skewed generalized t-Student
 if (dmodel=="SGT")
    {Rl2<-1/x[2]
     Term1<-n*(log(x[2])-log(2*x[1])-Rl2*log(x[3])-lbeta(Rl2,x[3]))
     arg<-(abs(err)/(x[1]*(1+x[4]*sign(err))))^x[2]
     arg<-log(1+arg/x[3])
     arg<-(x[3]+Rl2)*sum(arg)-Term1
     return(arg)}
# Skewed generalized error distribution
if (dmodel=="SGED")
   {Rl2<-1/x[3];G1<-gamma(Rl2);G2<-gamma(2*Rl2);G3<-gamma(3*Rl2)
    Theta<-G1/(G3*(1+(3-4*G2^2/(G3*G1))*x[2]^2));Theta<-sqrt(Theta)
    C1<-x[3]/(2*x[1]*G1*Theta);Term1<-n*log(C1)
    arg<-abs(err)/((1+sign(err)*x[2])*x[1]*Theta)
    arg<-sum(arg^x[3])-Term1
    return(arg)}
# Exponential generalized beta of the second type
if (dmodel=="EGB2")
   {C2<-(x[2]/x[1]);C3<-x[2]+x[3]
```

```
    Term1<-n*(log(x[1])+lbeta(x[2],x[3]))
    arg<-C2*err-C3*log(1+exp(err/x[1]))
    arg<-Term1-sum(arg)
    return(arg)}
if (dmodel=="GBD")
   {Term1<-n*(lbeta(x[3],x[4])+(x[3]+x[4]+1)*log(x[2]-x[1]))
    arg1<-(x[3]-1)*log(err-x[1])+(x[2]-1)*log(x[2]-err)
    arg<-Term1-sum(arg1)
    return(arg)}
if (dmodel=="FPLD")
   {Lah<<-matrix(0,5)
    Lah[3]<<-(1-(gam[2]/gam[3]))/(1+(gam[2]/gam[3]))
    ax<-2*gam[2]/(1-Lah[3]);ay<-2*gam[3]/(1+Lah[3])
    Lah[2]<<-(ax+ay)/2;Lah[4]<<-x[1];Lah[5]<<-x[2]
    G1<-(1-Lah[3])/2;G2<-(1+Lah[3])/2
    Lah[1]<<-Lah[2]*(G1/(1+Lah[4])-G2/(1+Lah[5]))
    arg1<-Lah[2]*(G1*Ppos^(Lah[4]-1)+G2*(1-Ppos)^(Lah[5]-1))
    arg<-sum(log(arg1))
    return(arg)}
}
########
Dfun<-function(y,tlef,para,imodel,term)
{z<-integrate(Density,lower=tlef,upper=y,subdivisions=100,x=para,dmodel=imodel)$value
 fz<-abs(z-term);return(fz)}
Quafun<-function(low,upp,para,imodel,term)
{Val<-optimize(Dfun,c(low,upp),tol=0.000001,tlef=low,
+ para=para,imodel=imodel,term=term)$min}
########
Lmpnew<-function(p,Ydep,Datix)
{cost1<-1/(p-1);cost2<-(p-2);cost3<-(p-2)/2;Maxpas<-30
 ipas<-0;Nors<-TRUE;bold<-rep(0,(ncol(Datix)+1))
 W<-as.matrix(diag(rep(1,nrow(Datix))))
 while(Nors)
      {ipas<-ipas+1
       Zdep<-W %*% Ydep; Datiz<-W %*% as.matrix(cbind(1,Datix))
       Ols<-lm(Zdep~Datiz-1)
       bnew<-Ols$coeff;Lx<-which(is.na(bnew))
       if(length(Lx)>0)
         {Fit<-as.matrix(cbind(1,Datix)) %*% bold;
          Resx<-Ydep-Fit;Resx<-Resx-mean(Resx)
          Res<-abs(Resx);Trasm<<-bold
          Crit3<-sum(abs(Resx));return(Crit3)}
       if (p>2) {bnew<-as.matrix(cost1*(cost2*bold+bnew))}
 S<-sum(abs(bnew));Test<-sum(abs(bold-bnew))
       Fit<-as.matrix(cbind(1,Datix)) %*% bnew
       Resx<-Ydep-Fit;Res<-abs(Resx)
       if (Test<0.0000001*S) Nors<-FALSE
       if (ipas>Maxpas) Nors<-FALSE
 lz<-which(Res<=10^(-3))
 if (length(lz)>0) {Res[-lz]<-Res[-lz]^cost3;Res[lz]<-0} else
                   {Res<-Res^cost3}
       diag(W)<-Res;
       bold<-as.matrix(bnew)}
       Fit<-as.matrix(cbind(1,Datix)) %*% bold;Resx<-Ydep-Fit;
       Resx<-Resx-mean(Resx)
       Res<-abs(Resx);Trasm<<-bold
       Crit3<-sum(abs(Resx))
  return(Crit3)}
##############################
Lp_Reg<-function(Ydep,Datix)
{Ydep<-as.matrix(Ydep);Datix<-as.matrix(Datix)
```

```
 A<-optimize(Lmpnew,c(1.01,5),Ydep,Datix)
 Fit<-as.matrix(cbind(1,Datix)) %*% Trasm;Resx<-Ydep-Fit;
 Resx<-Resx-mean(Resx)
 Rev<-list(coef=Trasm,alpha=A$minimum,residuals=Resx,fitted=Fit)
 return(Rev)}
 ##############################
Lmpni<-function(p,Ydep,Datix)
{cost1<-1/(p-1);cost2<-(p-2);cost3<-(p-2)/2;Maxpas<-60
 ipas<-0;Nors<-TRUE;bold<-rep(0,3)
 W<-as.matrix(diag(rep(1,nrow(Datix))))
 while(Nors)
      {ipas<-ipas+1;Zdep<-W %*% Ydep; Datiz<-W %*% Datix
       Ols<-lm(Zdep~Datiz-1)
bnew<-Ols$coef;Res<-Ols$residuals;
if(p==2){return(bold)}
       if (p>2) {bnew<-as.matrix(cost1*(cost2*bold+bnew))}
S<-sum(abs(bnew));Test<-sum(abs(bold-bnew));
       if (Test<0.000001*S) Nors<-FALSE
       if (ipas>Maxpas) Nors<-FALSE
       eto<<-Datix %*% bnew;eto<<-eto-mean(eto);
       Resx<-Ydep-eto;Res<-abs(Resx)
lz<-which(Res<=10^(-9))
 if (length(lz)>0) {Res[-lz]<-Res[-lz]^cost3;Res[lz]<-0} else
                   {Res<-Res^cost3}
       diag(W)<-Res;bold<-as.matrix(bnew)}
       #stop()
   return(bold)}
############################
Lmean<-function(p,Ydep)
{Maxpas<-30;ipas<-0;Nors<-TRUE
 cost1<-1/(p-1);cost2<-(p-2);cost3<-(p-2)/2
 nobs<-length(Ydep);Datix<-matrix(1,nobs,1)
W<-as.matrix(diag(rep(1,nobs)));bold<-rep(0,1)
 while(Nors)
      {ipas<-ipas+1;Zdep<-W %*% Ydep; Datiz<-W %*% Datix
       Ols<-lm(Zdep~Datiz-1)
bnew<-Ols$coef;Res<-Ols$residuals;
if(p==2){return(bold)}
       if (p>2) {bnew<-as.matrix(cost1*(cost2*bold+bnew))}
S<-sum(abs(bnew));Test<-sum(abs(bold-bnew))
Crit3<-sum(abs(Ydep-bnew[1])^p)
       if (Test<0.000001*S) Nors<-FALSE
       if (ipas>Maxpas) Nors<-FALSE
       fit<-Datix %*% bnew;Resx<-Ydep-fit;Res<-abs(Resx)
lz<-which(Res<=10^(-14))
 if (length(lz)>0) {Res[-lz]<-Res[-lz]^cost3;
                    Res[lz]<-0} else {Res<-Res^cost3}
       diag(W)<-Res;bold<-as.matrix(bnew)}
    Crit3<-sum(abs(Ydep-bold[1])^p)
return(Crit3)}
###############################################
ObjFun<-function(pa,Appr,ebar,Ppos,Weig,alpha,dmodel)
{n<-length(ebar);pa<-as.matrix(pa)
 if (Appr)
    {alim1<-7*ebar[1]*(-sign(ebar[1]))
     if (dmodel=="SGED") Olk<-0.001 else Olk<-0
     eto[1]<<-integrate(Density,lower=alim1,upper=ebar[1],x=pa,dmodel=dmodel)$value
     for (i in 2:n){eto[i]<<-integrate(Density,lower=ebar[i-1]-Olk,
+         upper=ebar[i],x=pa,dmodel=dmodel)$value}
     eto<<-cumsum(eto)
     Suer<-sum((Weig*abs(eto-Ppos))^alpha)^(1/alpha)
```

```
     return(Suer)}
 if (!Appr & dmodel=="FPLD")
    {Z1<-(Ppos^pa[1]-1)/(pa[1]);Z2<-(1-(1-Ppos)^pa[2])/(pa[2])
     Wr<-as.matrix(diag(Weig));Zdep<-Wr %*% ebar
     Datiz<-as.matrix(cbind(1,Z1,Z2));Datiz<-Wr %*% Datiz
     gam<<-Lmpni(alpha,Zdep,Datiz);gam<<-as.matrix(gam)
     Laz<-(1-(gam[2]/gam[3]))/(1+(gam[2]/gam[3]))
     if(abs(Laz)>1){Suer<-10^64;return(Suer)}
     Suer<-sum(abs(eto-Zdep)^alpha)^(1/alpha)
     return(Suer)}
 if (!Appr & dmodel=="GBD")
    {xz<-matrix(0,5);xz[3]<-pa[1];xz[4]<-pa[2]
     xz[2]<-sd(ebar)*xz[4]*sqrt((xz[3]+xz[4]+1)/(xz[3]+xz[4]))
     xz[1]<--xz[2]*xz[3]/xz[4]
     if(xz[1]>ebar[1]|xz[2]<ebar[n]){Suer<-10^64;return(Suer)}
      eto<<-pbeta(ebar,pa[1],pa[2])
     Suer<-sum((Weig*abs(eto-Ppos))^alpha)^(1/alpha)
     return(Suer)}
}
#####################################################################
Adaptive<-function(Resp,Regp,Pwei,DataMat,dmodel,Iwgt,Idata,alpha,Neval=500,
                   Niter=30,Subiter=15,Start=178107418,Iwrite=3000)

{set.seed(Start)
 Ydep<-as.matrix(DataMat[,Resp]);Datix<-as.matrix(DataMat[,Regp]);
 W<-as.matrix(DataMat[,Pwei])
 Appr<-TRUE;n<-nrow(Datix);nm1<-n-1;np1<-n+1;s1n<-1:n;Ppos<-s1n/np1
 ehat<-matrix(0,n);Sub<-matrix(0,9);eto<<-matrix(0,n)
 sedv<-sd(Ydep);La1m<-0.1*sedv;La1M<-5*sedv
 rownames(Sub)<-c("R2","aR2","Cos","Cmd","Tau","Bo","R_a","Psi_a","S_a")
 Npar<-0
 if (dmodel=="EGB2"){Npar<-3;Lbound<-c(La1m,0.01,0.01);
    Ubound<-c(La1M,10,10)}
 if (dmodel=="SGED") {Npar<-3;Lbound<-c(La1m,-0.99,1);
    Ubound<-c(La1M,0.99,10)}
 if (dmodel=="SGT") {Npar<-4;Lbound<-Lbound<-c(La1m,1,1,-0.99);
    Ubound<-c(La1M,10,1000000,0.99)}
 if (dmodel=="JSU") {Npar<-3;Lbound<-c(La1m,0.01,0.01);
    Ubound<-c(La1M,10,10)}
 if (dmodel=="FPLD"){Npar<-2;Lbound<-c(-0.99,-0.99);
    Ubound<-c(4,4);Appr<-FALSE}
 if (dmodel=="GBD") {Npar<-2;Lbound<-c(1,1);
    Ubound<-c(15,15);Appr<-FALSE}
 if (Npar==0) stop("Wrong model")
 Ntot<-length(Resp)+Npar
 if (!Appr) Ntot<-Ntot+2
 Mudep<-mean(Ydep);Vardep<-var(Ydep)
 Mediandep<-median(Ydep);Tep<-Ydep-Mediandep;tdep<-sum(abs(Tep))
 Qdep<-median(abs(Ydep-Mediandep))
 Region<-matrix(0,Npar,3);La<-matrix(0,Npar)
 Region[,1]<-Lbound;Region[,2]<-Ubound
 Region[,3]<-Region[,2]-Region[,1]
 if (Iwgt){Weig<-((n+2)*(n+1)^2)/(s1n*(n+1-s1n));Weig<-Weig/sum(Weig)}
 else {Weig<-rep(1/n,n)}
 Ols<-Lp_Reg(Ydep,Datix)
 R<-rank(Ols$residuals);betaold<-Ols$Trasm
 ebar<-sort(Ols$residuals);alpha<-Ols$alpha;aold<-alpha
 T2old<-sum(Weig*abs(ebar)^alpha)^(1/alpha)
 Laold<-rep(NA,Npar);betaold<-Ols$coef;Likold<-NA;Sicold<-NA
 count<-0;Iters<-0;Pursue<-TRUE
 while (Pursue)
```

```
   {count<-count+1;Weigp<-Weig^(1/alpha)
     if (count>Niter) Pursue<-FALSE
     if (!Appr & dmodel=="GBD")
      {ebas<-(ebar-ebar[1])/(ebar[n]-ebar[1])
       Lax<-ConRandSer(Npar,Neval,Region,Iwrite,Appr,ebas,Ppos,Weigp,alpha,dmodel)
         ObjFun(Lax$Lam,Appr,ebas,Ppos,Weigp,alpha,dmodel)}
     else
      {Lax<-ConRandSer(Npar,Neval,Region,Iwrite,Appr,ebar,Ppos,Weigp,alpha,dmodel)
         ObjFun(Lax$Lam,Appr,ebar,Ppos,Weigp,alpha,dmodel)}
     La<-as.matrix(Lax$Lam);T2new<-Lax$Crit
     if (Appr)
        {alim1<-5*ebar[1]*(-sign(ebar[1]))
        for (i in s1n){ehat[i]<-Quafun(alim1,ebar[i],La,dmodel,eto[i])}
        ehat<-ehat-mean(ehat)}
     if (!Appr & dmodel=="FPLD") {ehat<-eto}
     if (!Appr & dmodel=="GBD")
        {ehat<-qbeta(eto,La[1],La[2])
         xz<-matrix(0,4);xz[3]<-La[1];xz[4]<-La[2]
         xz[2]<-sd(ehat)*xz[4]*sqrt((xz[3]+xz[4]+1)/(xz[3]+xz[4]))
         xz[1]<--xz[2]*xz[3]/xz[4];ehat<-(ehat-xz[1])/(xz[2]-xz[1])
         ehat<-ehat-mean(ehat);Lak<<-xz}
     Test<-(T2old-T2new)/(1+T2old+T2new)
     if (Test>0)
      {betaold<-Ols$coef;Yest<-Ols$fitted;ehat<-as.numeric(ehat)
       if (!Appr & dmodel=="GBD") {lik<-Likelihood(ehat,xz,Ppos,dmodel)}
       else {lik<-Likelihood(ehat,La,Ppos,dmodel)}
       Sic<-lik+(Ntot+1)*log(n)/2
       cat("Test of conv.",Test,"\n")
         cat("Data set:",Idata,"Model:",dmodel,"Weighting:",Iwgt,"Iteration:",count,"\n")
         cat("Criterion:",T2new,"Log-Likh:",lik,"Schwarz I.C.=",Sic,"\n")
     #if (dmodel=="FPLD") betaold[1]<-betaold[1]+Lah[1]
         cat("Reg.par: ",betaold,"alpha.r: ",alpha,"\n");
         if (dmodel=="FPLD") cat("Dis.par: ",Lah,"\n") else
            cat("Dis.par: ",La,"\n")
         Ras<-var(ebar)/Vardep;Sub[1]<-1-sqrt(Ras)
         Sub[2]<-1-sqrt(Ras*nm1/(nm1-Ntot))
         sbar<-sum(abs(ebar));ptau<-sum(abs(Yest-Mediandep))
         Sub[5]<- 1-sbar/(tdep+ptau)
         Ndep<-median(Yest)
         qtau<-sum(abs(Yest-Ndep))
         Sub[6]<-qtau/(qtau+sbar)
         Edep<-Lmean(alpha,Ydep)
         Sub[3]<-cor(Ydep,Yest)
         Falk1<-median((Yest-median(Yest))*Tep)
         Falk2<-Qdep*median(abs(Yest-median(Yest)))
         Sub[4]<-Falk1/Falk2
         Sub[7]<-1-(sum(abs(ebar)^alpha)/Edep)
         Sub[8]<-1-(sum(abs(ehat)^alpha)/Edep)
         Sub[9]<-(sum(abs(ehat)^alpha))^(1/alpha)
         T2old<-T2new;Likold<-lik;Sicold<-Sic;Iters<-1;Laold<-La;aold<-alpha
         print(trunc(Sub*100000+0.5)/100000)
         cat("================================================================","\n")
         if (Test<0.0000000001) Pursue<-FALSE
         }
     else
       {cat("Inversion - Iteration:",count,"sub counter:",Iters,"Crit.value:",
+       T2new,Test, "\n")
         Iters<-Iters+1;count<-count-1
         if (Iters>Subiter) {Pursue<-FALSE}
         }
     if (Pursue)
```

```
       {ytilde<-Ydep-ehat[R]
        Ols<-Lp_Reg(ytilde,Datix);alpha<-Ols$alpha
        R<-rank(Ols$residuals);ebar<-sort(Ols$residuals)}
 }
 FinRes<-list(RegPar=betaold,alpha.r=aold,DisPar=Laold,Crit=T2old,
+  Likh=Likold,SicIC=Sicold,Fitm=round(Sub,5))
 return(FinRes)
}
####################
ConRandSer<-function(N3,Maxev,Region,Iwrite,Appr,ebar,Ppos,Weig,alpha,dmodel)
{Nver<-25*N3;Count<-0;N4<-N3+1;Isim<-1:Nver;M1<-N3*3;M2<-N3*2;M3<-floor(Nver/4)
 Por<-matrix(0,Nver,N3);Pory<-matrix(0,Nver);dq<-matrix(0,3)
 for (i in 1:Nver)
  {Por[i,]<-Region[,1]+crossprod(diag(runif(N3)),Region[,3])
   X<-ObjFun(Por[i,],Appr,ebar,Ppos,Weig,alpha,dmodel);Pory[i]<-X}
 Count<-Nver;Itera<-Nver
 Fmax<-max(Pory);Imax<-which.max(Pory)
 Fmin<-min(Pory);Imin<-which.min(Pory)
 Done2<-FALSE;Itera<-1
 while (!Done2)
{Test<-(Fmax-Fmin)/(0.0000001+abs(Fmin)+abs(Fmax))
 if (Test<0.0000001)
  {La<-Por[Imin,];Fmin<-ObjFun(La,Appr,ebar,Ppos,Weig,alpha,dmodel)
             Ext<-list(Crit=Fmin,Lam=La);Done2<-TRUE;return(Ext)}
 if (Count>Maxev)
 {La<-Por[Imin,];Fmin<-ObjFun(La,Appr,ebar,Ppos,Weig,alpha,dmodel)
  Ext<-list(Crit=Fmin,Lam=La);Done2<-TRUE;return(Ext)}
 if (Itera>=Iwrite){cat("Evaluations:",Count,"Test=",Test,"Criterion: ",Fmin,"\n")
               cat("Distr.Par.=",round(Por[Imin,],4),"\n")
            cat("..........................................................","\n")
  Itera<-0}
# Choose at random without replacement a set B of k points; include Xmin into B
 Isim1<-Isim[-Imin]
 Icen<-sample(Isim1,N3,replace=FALSE)
 Icen1<-c(Imin,Icen);Bmean<-apply(Por[Icen1,],2,mean)
 M2c<-0;Isim2<-Isim[-Icen1]
 Done3<-FALSE;Targ<-FALSE
 while (!Done3)
{#Select randomly another point not in B
 M2c<-M2c+1; Done4<-FALSE;M1c<-0
 Ide<-sample(Isim2,1,replace=FALSE)
 while (!Done4)
       {M1c<-M1c+1;v<-runif(N3,0,2)
        Betp<-(1+v)*Bmean-v*Por[Ide,]
        ff<-which((Betp>=Region[,1]) & (Betp<=Region[,2]))
        if (length(ff)==N3) {Done4<-TRUE;Succ<-TRUE}
        if (length(ff)<N3 & M1c>M1){Done4<-TRUE;Succ<-FALSE}
        }
  if (!Succ) {Isim2<-Isim2[-Ide]}
  if (Succ)
  {Fbar<-ObjFun(Betp,Appr,ebar,Ppos,Weig,alpha,dmodel)
   Count<-Count+1;Itera<-Itera+1
   if (Fbar<Fmax & Fbar>Fmin)
      {Por[Imax,]<-Betp;Pory[Imax]<-Fbar;Targ<-TRUE
       Fmax<-max(Pory);Imax<-which.max(Pory)}
   if (Fbar<Fmin)
     {Por[Imax,]<-Por[Imin,];Pory[Imax]<-Fmin
    Por[Imin,]<-Betp;Pory[Imin]<-Fbar;Fmin<-Fbar;Targ<-TRUE
    Fmax<-max(Pory);Imax<-which.max(Pory)}
       }
       if (Targ | M2c>M2) Done3<-TRUE
```

```
      }
   if (Targ) Done5<-TRUE else Done5<-FALSE
   M3c<-0
   while (!Done5)
      {M3c<-M3c+1
       Ide<-sample(Isim1,2,replace=FALSE);Ord<-c(Imin,Ide)
       Ws1<-(Por[Ord[2],]^2-Por[Ord[3],]^2)*Pory[Ord[1]]
       Ws2<-(Por[Ord[3],]^2-Por[Ord[1],]^2)*Pory[Ord[2]]
       Ws3<-(Por[Ord[1],]^2-Por[Ord[2],]^2)*Pory[Ord[3]]
       Ws4<-(Por[Ord[2],]-Por[Ord[3],])*Pory[Ord[1]]
       Ws5<-(Por[Ord[3],]-Por[Ord[1],])*Pory[Ord[2]]
       Ws6<-(Por[Ord[1],]-Por[Ord[2],])*Pory[Ord[3]]
       temp<-Ws4+Ws5+Ws6
       gg<-which(abs(temp)>0.00000001)
       if (length(gg)==N3){Done1<-TRUE;Betp<-0.5*(Ws1+Ws2+Ws3)/(Ws4+Ws5+Ws6)} else
          {Done1<-FALSE;uni<-runif(N3);Betp<-uni*Por[Imin,]+(1-uni)*Por[Imax,]}
       ff<-which((Betp>=Region[,1]) & (Betp<=Region[,2]))
       if (length(ff)>=N3) Done6<-TRUE else Done6<-FALSE
       if (Done6)
       {Fbar<-ObjFun(Betp,Appr,ebar,Ppos,Weig,alpha,dmodel)
        Count<-Count+1;Itera<-Itera+1
     if (Fbar<Fmax & Fbar>Fmin)
         {Por[Imax,]<-Betp;Pory[Imax]<-Fbar
          Fmax<-max(Pory);Imax<-which.max(Pory);Done5<-TRUE}
     if (Fbar<Fmin)
        {Por[Imax,]<-Por[Imin,];Pory[Imax]<-Pory[Imin]
          Por[Imin,]<-Betp;Pory[Imin]<-Fbar;Fmin<-Fbar
          Fmax<-max(Pory);Imax<-which.max(Pory);Done5<-TRUE}
     }
if(M3c>=M3) Done5<-TRUE
}
}
}
```

A.3 R Interaction for Using the Functions of A.2

```
source("Routines_PAEDR.R")
rowlab<-c("beta_0","beta_1","beta_2","beta_3","beta_4","alpha","lambda_1",
+ "lambda_2","lambda_3","lambda_4","R^2","Ad.R^2","Cos","Comd","tau","Bonf",
+ "R_alpha","psi_alpha","S_alpha","-L","Sic")
#Data sets  c("Mayers.csv","Prostate.csv","Prostate1.csv","Salinity.csv","Narula.csv")
Models<-c("FPLD","GBD","JSU","SGT","SGED","EGB2")
#############################################################################
DataMat<-read.table("Mayers.csv",sep=",",dec=".",header=TRUE)
PosDip<-1        # Position of the response variable
PosReg<-c(2,3)   # Position of the regressor variables
PosWei<-4        # Position of the weights for the deterministic component
Active<-matrix(0,21);rownames(Active)<-rowlab
dmodel<-"FPLD"
Iwgt<-FALSE
Result<-Adaptive(PosDip,PosReg,PosWei,DataMat,dmodel,Iwgt,"Mayers.csv")
Npar<-length(Result$RegPar);Npar1<-Npar+1
Active[1:Npar]<-Result$RegPar
if (Npar<5) Active[Npar1:5]<-NA
Active[6,1]<-Result$alpha.r
kmodel<-match(dmodel,Models)
if (kmodel==1) Result$DisPar<-Lah[2:5]
if (kmodel==2) Result$DisPar<-Lak
Npad<-length(Result$DisPar);Npad1<-Npad+6
```

```
Active[7:Npad1]<-Result$DisPar
if (Npad<4) Active[(Npad1+1):10]<-0
Active[11:19]<-Result$Fitm
Active[20]<-Result$Likh
Active[21]<-Result$SicIC
print(Active)
write(t(Active),ncolumns=12,sep=",")
```

Chapter 19

A Multivariate Gamma Distribution for Linearly Related Proportional Outcomes[1]

Norou Diawara
Old Dominion University,
Norfolk, Virginia 23529, USA
ndiawara@odu.edu

Mark Carpenter
Auburn University,
Auburn, Alabama 36849, USA
carpedm@auburn.edu

In many applied problems, heavy tailed distributions are used to model unbalanced phenomena and to find estimates of parameters. Among the heavy tailed distributions, the gamma distribution is used often for probability modeling. Its multivariate form has been shown to be valuable in image registration, computer communications, and change detection. Moreover, its multivariate form has generated a lot of interest since its characterization is not unique and its distribution form is not computationally efficient. To alleviate such concerns, we use the Laplace transform to characterize a multivariate gamma distribution for linearly related proportional events.

[1] The authors are grateful to the referees for their valuable suggestions that helped in the presentation and improvement of the contents.

We motivate the multivariate gamma distribution model on positively correlated gamma marginal distributions capturing the idea from Marshall and Olkin (1967) of non-zero probability of simultaneous occurrence. Although the likelihood function is potentially unbounded, the characterization of the multivariate gamma distribution has several forms that are tractable, interpretable, and computationally efficient. We describe each of those forms and discuss a simulated example.

19.1 Introduction

The multivariate gamma distribution belongs to the class of multivariate lifetime distributions, where the variables are associated. Hougaard (2000) gives many references and covers study for the multivariate analysis. As he mentions, the field has gaps, and is still growing both in aspects of theory and application.

Many other authors have proposed versions of multivariate gamma distributions from a linear combination of independent gamma random variables. Mathai and Moschopoulos (1991) define a form of multivariate gamma where the coefficient of relationship is specifically chosen. To advance these concepts further, we develop a form of a multivariate gamma distribution based on a general coefficient and where there is a random component in the model, which assumes that the observations are independent conditional to an unobserved covariate. The model we propose extends that of Mathai and Moschopoulos (1991) as the coefficient of linear relationship is more general. Moreover, the proposed multivariate distribution function is attractive and is completely characterized when its density function captures the dependence structure within the random covariate vector. We discuss the properties of the proposed multivariate gamma distribution, which incorporates covariance and extends probability properties. We propose a characterization of the unobserved covariates.

The one-dimensional case model with gamma marginal is given special attention as it plays a fundamental role in survival analysis and in reliability. It is simple and adapts to many situations. Several examples are used to motivate such a model structure. Even with its simpler form of the exponential, it is, however, not easy to work with. With the development of proportional hazard models or frailty models by Cox (1972), Ghosh and Gelfand (1998), and Hougaard (2000), some progress has been made. However, not much has been done without the assumption of the proportional hazard rate, and that is why we propose our alternative to describe the multivariate gamma.

In Section 19.2, we define the multivariate gamma and review some basic definitions. Section 19.3 presents some known results about infinite divisibility and complete monotonicity of distributions. Section 19.4 presents the form of the unknown random variable based on several versions of our proposed multivariate gamma. The various scenarios will be useful to practitioners in many different

areas, and will also extend the scope of applicabilities. In Section 19.5, a simulated example is discussed where the gamma distribution is fitted, and estimators are developed. We also discuss cases where such a model is an alternative to be used in modeling and data fitting. We end this chapter with our conclusion and an appendix that lists the SAS and **R** code that is used in the example of Section 19.5.

Throughout this chapter we use the symbol $\mathcal{R}$ to denote the real numbers and $\mathcal{N}$ to denote the natural numbers.

19.2 Definitions

Positively skewed distributions of random variables (r.v.'s) with positive support occur quite often in practical applications, such as in reliability and survival analysis. Lawless (2003) gives some examples of the most common models, and their applications. The gamma family of distributions is right-tailed, and seems a natural choice for positive valued distributions with heavy tails. It is widely used in reliability and survival analysis. For example, the gamma distribution is used in statistical studies of times between earthquakes. So estimation of its parameters can be quite important. We consider the univariate three parameter gamma distribution X, denoted here as $Ga(\mu, \lambda, \alpha)$.

Definition 19.2.1. *The three parameter gamma distribution, see Cohen and Whitten (1986) and Billingsley (1986) for examples, is defined by the density function:*

$$f_X(.; \mu, \lambda, \alpha) : t \in (\mu, \infty) \mapsto f_X(t; \mu, \lambda, \alpha) \in \mathcal{R}^+$$

of the three parameters $\mu \in \mathcal{R}, \lambda > 0$ and $\alpha > 0$, given as:

$$f_X(t; \mu, \lambda, \alpha) = \frac{\lambda^\alpha}{\Gamma(\alpha)}(t-\mu)^{\alpha-1}e^{-\lambda(t-\mu)}I_{[\mu,\infty)}(t) \tag{19.2.2}$$

where:

- *$\mu \in \mathcal{R}$, $\lambda > 0$, and $\alpha > 0$ are the location, scale and shape parameters, respectively.*
- *$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function.*

The three parameter gamma distribution is also known as a Type III of distributions in the Pearson's system of distribution. When the location $\mu = 0$, we denote X as $Ga(\lambda, \alpha)$. Rather than starting the distribution at the origin, it is sometimes more convenient to shift it on the real axis, allowing the domain to be $[\mu, \infty)$, as it is done in (19.2.2). The scale parameter λ is also called the **rate** or **inverse scale under inverse parametrization**. The larger the value of λ, the

wider the density, and vice-versa. Whereas for the normal distribution, the scale parameter is the standard deviation. When $0 < \alpha < 1$, this distribution has a shape with a peak. When $\alpha > 1$, the distribution has a mode at $(\alpha - 1)/\lambda + \mu$ and heavy tails.

The gamma function $\Gamma(\alpha)$, with $\Gamma(1/2) = \sqrt{\pi}$, is the normalizing constant of the distribution, and satisfies the relation $\alpha\Gamma(\alpha) = \Gamma(\alpha + 1)$. The incomplete gamma function is defined as

$$I(\alpha, t) = \frac{1}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-x}\, dx,$$

as in Lawless (2003), and

$$\Gamma(\alpha, t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty x^{\alpha-1} e^{-x}\, dx$$

is called the **complementary incomplete gamma function**.

Several special cases of the gamma, such as the exponential or the Erlang, exist and are widely used. The gamma distribution of (19.2.2) with location at the origin and scale 1 is known as the standard gamma distribution. A gamma distribution with unit shape at the origin is known as the exponential distribution. Most statistical packages generate random numbers from exponential distributions. Its density is

$$f(t) \quad = \quad \lambda e^{-\lambda t}, \quad \text{where } \lambda > 0, \text{ and } t \geq 0. \tag{19.2.3}$$

The displaced exponential is the exponential shifted by some value μ, denoted $exp(\mu, \lambda)$ and defined by

$$f(t) \quad = \quad \lambda e^{-\lambda(t-\mu)}, \quad \text{where } \lambda > 0, \text{ and } t \geq \mu. \tag{19.2.4}$$

The Erlang distribution is also a particular case of the gamma distribution where the shape parameter α in (19.2.2) is a positive integer value, denoted here as n. Its pdf is

$$f(t) = \frac{\lambda^n}{\Gamma(n)} (t - \mu)^{n-1} e^{-\lambda(t-\mu)}, \quad \text{where } \lambda > 0,\ n \in \mathcal{N}, \text{ and } t \geq \mu. \tag{19.2.5}$$

The Erlang distribution predicts the waiting time in a queuing system, particularly in the case of telephone traffic engineering. It is the probability distribution of the waiting time until the n^{th} arrival in a one-dimensional Poisson process with intensity λ. The case $n = 1$ reduces to the exponential distribution, and is the difference between successive occurrences of events given by a Poisson process.

Another interpretation, due to Barlow and Proschan (1965), explains the Erlang distribution as the waiting time until failure by a random shock in a device, and the multivariate format is described in Hougaard (1986). The device fails

when there are exactly α shocks that occur randomly over time, $\alpha \in \mathcal{N}$, at a Poisson rate with parameter λ. That is the convolution of α exponential distributions.

The class of gamma distributions which includes the exponential, the Erlang and the Chi-Square distributions, is widely used in applied and theoretical statistics. Unlike the normal distribution, the gamma distribution is asymmetric, and cannot be specified by its mean and variance alone, known as first and second moments, respectively. In addition to the mean and variance, this distribution is characterized by its third- and fourth-order moments known as skewness and kurtosis, and by higher moments. See Grove and Coddington (2005), for example.

The skewness describes the degree of symmetry of the distribution. A positively skewed (or right-skewed) distribution rises rapidly, reaches its maximum, and falls slowly with a heavy right tail. A negatively skewed (or left-skewed) distribution rises slowly, reaches its maximum through a pronounced left tail, and falls rapidly. The kurtosis describes the peakedness of the distribution near its mode.

The expected value of the gamma distribution of (19.2.2) and its variance are

$$E(X) = \alpha/\lambda + \mu \quad \text{and} \quad \text{Var}(X) = \alpha/\lambda^2, \tag{19.2.6}$$

respectively. The probability of an event of interest being alive up to time x, the survival function, is given by

$$S_X(x) = 1 - F_X(x) = 1 - I\Big(\alpha; \lambda(x-\mu)\Big). \tag{19.2.7}$$

In engineering, this is known as **reliability function**, and the data may represent the times to failures in a system.

The instantaneous rate of occurrence of an event, also called **hazard function**, or **failure rate function**, is defined as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{(x-\mu)^{\alpha-1}}{(1 - I(\alpha; \lambda(x-\mu)))} e^{-\lambda(x-\mu)}, \tag{19.2.8}$$

and the cumulative hazard rate is

$$H(x) = \int_\mu^x h(u)du = -\log S(x). \tag{19.2.9}$$

The hazard function of (19.2.2) is increasing when $\alpha > 1$ from 0 to λ, and decreasing when $\alpha < 1$ from ∞ to λ.

Cox (1972) used the hazard function to get estimators of the effects of different covariates on the times to failures of a system.

Balakrishnan and Wang (2000) point out that the estimation problem for the three parameter gamma has always been a challenging and interesting one, in particular when the shape parameter α is small (less than 2.5). As the shape

$\alpha \to \infty$, the gamma distribution in (19.2.2) approaches normality. However, the normality assumption cannot always be made as described in Hougaard (2001). Hence the need to characterize the distributional form of the variables under a gamma probability model.

Another property related with the gamma density is that it is closed under scalar products. More precisely, if $X \sim Ga(\mu, \lambda, \alpha)$, and if

$$Y = cX \sim Ga(c\mu, \frac{\lambda}{c}, \alpha) \tag{19.2.10}$$

for all $c > 0$, then

$$f_Y(y) = f_X(\frac{y}{c}) \times \frac{1}{c} = \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)}(y - c\mu)^{\alpha-1}e^{-\frac{\lambda}{c}(y-c\mu)}.$$

Some advantages of the gamma distribution are in the simple forms of its density (19.2.2) and its Laplace transform.

Another important property is that in the Erlang type distribution, $Ga(\mu, \lambda, m)$ can be thought of as the sum of m independent distributions that are each displaced exponential as in (19.2.4) with parameters μ_i and θ such that $\sum_{i=1}^m \mu_i = \mu$. More precisely, if $X_i \sim \exp(\mu_i, \lambda),\ i = 1, 2, \ldots, m$, then

$$X = \sum_{i=1}^m X_i \sim Ga\left(\sum_{i=1}^m \mu_i, \lambda, m\right).$$

Dudewicz and Mishra (1988) and Billingsley (1986) showed that a sum of independent gamma r.v.'s with the same location and shape parameters μ and λ, is again a gamma r.v. with an added scale parameters. The class of distributions $\{Ga(\mu, \lambda, m);\ m \in \mathcal{N}\}$ is then additive as it has a well-defined sum operation. This is analogous to the informally defined *multiplicative distribution* for distributions that have a well-defined multiplicative operation.

We now define the multivariate gamma model as follows:

Definition 19.2.11. *Let $X_0, X_1, \ldots, X_p$ be gamma random variables with shape parameters α_i, scale parameters λ_i and location parameters μ_i (that is, $X_i \sim Ga(\alpha_i, \lambda_i, \mu_i)$ for $i = 0, 1, \ldots, p$). Let Z_i be unknown independent random variables satisfying*

$$X_i = a_i X_0 + Z_i \tag{19.2.12}$$

for $a_i > 0$ and Z_i independent with respect to X_0 for $i = 1, 2, \ldots, p$. We define the joint distribution of the random vector $\boldsymbol{X} = (X_1, X_2, ..., X_p)'$ as the multivariate gamma distribution for some fixed integer p, or simply multivariate gamma.

This model offers the option of having marginal distributions that are of the gamma types. In Carpenter and Diawara (2007), some of the properties of the model are discussed. One interesting question is to characterize the distribution of Z_i for $i = 1, \ldots, p$. We will present results in that regard using the linear relationship given in (19.2.12), and the fact that $Z_i,\ i = 1, .., p$, are independent

of each other and of X_0. As we will explain later, depending on the choice of the constants a_i, the distributions of Z_i take on many forms. In fact, in some cases, Z_i are gamma, and in other cases, Z_i are a mixture of distributions.

The correlation between X_0 and X_i is expressed as:

$$\rho(X_0, X_i) = \frac{\text{Cov}(X_0, X_i)}{\sqrt{\text{Var}(X_0)}\sqrt{\text{Var}(X_i)}},$$

where $\text{Cov}(X_0, X_i) = E(X_0X_i) - E(X_0)E(X_i)$ is the covariance between X_0 and X_i, and $\text{Var}(X_0)$ and $\text{Var}(X_i)$ denote the variances of X_0 and X_i, $i = 1, \ldots, p$, respectively.

Linear correlation is a measure of linear dependence. In the case where $a \in \mathcal{R}$, and a perfect linear dependence is established with $Z_i \equiv b_i$ degenerate, i.e. $X_i = a_iX_0 + b_i$, $a_i, b_i \in \mathcal{R}$, we have that $\rho(X_0, X_i) = \pm 1$ and $\rho(X_i, X_j) = sgn(a_ia_j)$ with $X_j = a_jX_0 + b_j, i, j = 1, \ldots, p$. This piecewise linear relation seems to be natural as pointed out in Iyer, Srilanth, Manjunath, and Manivasakan (2002) and Iyer, Srilanth, and Manjunath (2004), for examples. From the structural probability distribution, Carpenter and Diawara (2007) derive the corresponding form of the density, values of the means, variances, and covariances. It is clear from (19.2.12) that X_i and X_j, $i \neq j$, $i, j = 1, \ldots, p$, are positively correlated. This is a restriction in our model also, but it is not unusual. He and Lawless (2005) give more examples of positively correlated lifetime data.

We use the linear relationship between X_0 and X_i through latent unobserved random variables Z_i for $i = 1, \ldots, p$, Z_i independent between them and with respect to X_0, such as in (19.2.12).

19.3 Basic Concepts

The Laplace transform (the equivalent concept of moment generating function) is used in many areas of statistics, probability theory, and risk analysis. We are interested in the Laplace transform associated with real arguments. The characterization by the means of the real Laplace transform does not presume the existence of moments.

Definition 19.3.1. *If X is a r.v. defined on $\mathcal{R}_+$ with cdf F_X, satisfying $P(X = 0) < 1$, then its Laplace-Stieltjes transform* (LST) *is the function valued in $\mathcal{R}$ defined in Abramowitz and Stegun (1972) by*

$$L_X(s) = Ee^{-sX} = \int_0^\infty e^{-sx}\, dF_X(x), \quad s > 0. \tag{19.3.2}$$

Here are some properties associated with the LST:

- Existence: The integral in (19.3.2) exists with respect to Lebesgue-Stieltjes integration in discrete and continuous cases. In our cases of positive support distributions, (19.3.2) always exists. In fact, $0 < L_X(s) \leq 1$.

- $L_X(s)$ is infinitely differentiable, and $\dfrac{d^n L_X}{ds^n}(s)$ exist for all n.

- For $m \in \mathcal{N}$, the m^{th} moment of X is given by $E\left(X^m\right) = (-1)^m L_X^{(m)}(0)$.

- Additivity: The LST of the sum of independent r.v.'s is obtained by taking the product of the LST of the individual r.v.'s. If $X_1, \ldots, X_n$ are independent r.v.'s, then $X = \sum_{i=1}^n X_i$ has LST:
$$L_X(s) = Ee^{-sX} = E\prod_{i=1}^n e^{sX_i} = \prod_{i=1}^n Ee^{sX_i} = \prod_{i=1}^n L_{X_i}(s).$$

- The LST can be defined for a $p-$variate r.v., say $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ as:
$L_{\mathbf{X}}(\mathbf{s}) = Ee^{-(s_1 X_1 + \cdots + s_p X_p)}$, for $\mathbf{s} = (s_1, s_2, \ldots, s_p)' \in \mathcal{R}_+^p$.

- Uniqueness: If X_1 and X_2 are two r.v.'s such that $L_{X_1}(s) = L_{X_2}(s)$ then $f_{X_1}(x) = f_{X_2}(x)$, for all x except on a set of measure 0.

- The LST completely characterizes the distribution.

These transforms help in the computations and in the linear combinations of r.v.'s associated with some distributions.

Example 19.3.3. *If $X \sim Ga(\mu, \lambda, \alpha)$ with pdf as given in (19.2.2), then its* LST *is given by*

$$L_X(s) \;=\; e^{-s\mu}\left(\frac{\lambda}{\lambda + s}\right)^{\alpha}.$$

The gamma distribution, shifted at the origin, with unit mean has LST

$$L_X(t) = \left(\frac{1}{1 + t/\alpha}\right)^{\alpha} \longrightarrow e^{-t} \quad \text{as } \alpha \to \infty.$$

Example 19.3.4. *Let $X_1, \ldots, X_n$, $n \geq 1$, be independent r.v.'s distributed as $Ga(\mu_i, \lambda, \alpha_i)$ for $1 \leq i \leq n$. Then, using* LST*, it is easy to show that $X = X_1 + \ldots + X_n$ has density of*

$$Ga\left(\sum_{i=1}^n \mu_i, \lambda, \sum_{i=1}^n \alpha_i\right).$$

In the remaining cases, we consider situations where the λ's are different. When two or more of the distributions have the same parameter λ, we can add them first to obtain another gamma distribution with the same parameter λ.

Example 19.3.5. *Suppose X and Y are independent discrete and positive support continuous r.v.'s with pmf and pdf $p(x)$ and $f(y)$, respectively. Then*

$$\begin{aligned} L_{XY}(s) &= Ee^{-sXY} = \int_0^\infty \sum_x e^{-sxy} p(x) f(y) dy \\ &= \sum_x \Big(\int_0^\infty e^{-sxy} f(y) dy \Big) p(x) \\ &= \sum_x L_Y(sx) p(x). \end{aligned}$$

We later take up, in Example 19.3.5, the case when X is a Bernoulli random variable with probability p, and Y is a positive support distribution. In that case, we have that:

$$L_{XY}(s) = p + (1-p) L_Y(s).$$

We are interested in the sum of r.v.'s. For two r.v.'s $X_1 \sim f_1$ and $X_2 \sim f_2$, their sum, $X = X_1 + X_2$, has density obtained from the joint pdf as:

$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x-x_1} f(x_1, x_2) dx_1 dx_2.$$

The convolution of two functions $f, g : \mathcal{R} \to \mathcal{R}$ is the function

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt.$$

If the r.v.'s are independent, the density of their sum is the convolution of their densities, and can be represented as below.

Theorem 19.3.6. *Assume that X_1 and X_2 are independent r.v.'s defined on $\mathcal{R}_+$ with pmf/pdf f_1 and f_2, respectively. Then $X = X_1 + X_2$ has density*

$$f_X(x) = (f_1 \star f_2)(x) = \int_0^x f_1(t) f_2(x-t) dt.$$

Proof: Proved in Hunter and Nachtergaele (2001). ∎

However, a lot of work can be alleviated as the LST of X in Theorem 19.3.6 is given by $L_X(t) = L_{X_1}(t) L_{X_2}(t)$, and is recognized in some distributional form.

Example 19.3.7. *Consider X to be the sum of two iid exponential type r.v.'s with parameter θ as defined in (19.2.3). Then*

$$f_X(x) = \int_0^x f_1(t) f_2(x-t) dt = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x},$$

which is a $Ga(0, \lambda, 2)$.

Theorem 19.3.8. *For $X_1 \sim Ga(\mu_1, \lambda_1, \alpha_1)$ and $X_2 \sim Ga(\mu_2, \lambda_2, \alpha_2)$, the r.v. $X = X_1 + X_2$ has a gamma distribution $Ga(\mu, \lambda, \alpha)$ iff $\lambda_1 = \lambda_2 = \lambda$, with $\alpha = \alpha_1 + \alpha_2$ and $\mu = \mu_1 + \mu_2$.*

Proof: The LST of X is given by

$$L_X(s) = e^{-(\mu_1+\mu_2)s} \frac{\lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}{(\lambda_1 + s)^{\alpha_1} (\lambda_2 + s)^{\alpha_2}}.$$

If $\lambda_1 \neq \lambda_2$, then the LST is not representative of a gamma distribution. Considering the case where $\lambda_1 = \lambda_2 = \lambda$, the LST and the density of the sum $X = X_1 + X_2$ become, respectively,

$$L_X(s) = e^{-s\mu} \left(\frac{\lambda}{\lambda + s} \right)^{\alpha}$$

and

$$f_X(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} (x - \mu)^{\alpha-1} e^{-\lambda(x-\mu)},$$

where $\alpha = \alpha_1 + \alpha_2$ and $\mu = \mu_1 + \mu_2$.

The sum of independent gamma distribution with the same scale parameter gives a gamma distribution with the same scale parameter with the location and scale updated. In that sense, the gamma distribution follows an additive property. ∎

For $X_i \sim Ga(\mu_i, \lambda_i, \alpha_i)$ for $i = 0, 1, \ldots, p$, based on the LST given in (19.3.2) and using the binomial theorem, we have the following

$$\frac{d^m L_{X_i}(s)}{ds^m} = \lambda^{\alpha} \sum_{k=0}^{m} \binom{m}{k} \left\{ \frac{d^k (\lambda_i + s)^{-\alpha}}{ds^k} \right\} \left\{ \frac{d^{m-k} e^{-\mu_i s}}{ds^{m-k}} \right\}, \quad \text{for } m \in \mathcal{N}.$$

Setting $s = 0$, we have

$$E(X_i^m) = \sum_{k=0}^{m} \binom{m}{k} \lambda_i^{\alpha+k} \mu_i^{m-k} (\alpha_i)_k,$$

where $(\alpha)_k = \alpha(\alpha + 1) \ldots (\alpha + k - 1)$.

The moments and cross-moments are given for m and $l \in \mathcal{N}$, by:

$$\begin{aligned} E(X_i^m) &= E(a_i X_0 + Z_i)^m = \sum_{r=0}^{m} \binom{m}{r} a_i^r E X_0^r E Z_i^{m-r} \\ &= \sum_{r=0}^{m} \binom{m}{r} E Z_i^{m-r} a_i^r \sum_{k=0}^{r} \binom{r}{k} \lambda_0^{\alpha+k} \mu_0^{r-k} (\alpha_0)_k \end{aligned}$$

and

$$\begin{aligned} E(X_i^m X_j^l) &= E(a_i X_0 + Z_i)^m (a_j X_0 + Z_j)^l \\ &= E \sum_{r=0}^{m} \binom{m}{r} a_i^r X_0^r Z_i^{m-r} \sum_{s=0}^{l} \binom{l}{s} a_j^s X_0^s Z_j^{l-s} \\ &= \sum_{r=0}^{m} \sum_{s=0}^{l} \binom{m}{r} \binom{l}{s} a_i^r a_j^s E X_0^{r+s} E Z_i^{m-r} E Z_j^{l-s}, \end{aligned}$$

for $i \neq j,\ i, j = 1, 2, \ldots, p$.

The LST has many helpful properties as discussed earlier. In particular, we use it for many of the derivations of the distributions of the latent variables. The multivariate LST for any p-variate random vector $\mathbf{X}$ defined in (19.2.12) is

$$\begin{aligned} L_{\mathbf{X}}(\mathbf{s}) &= E e^{-\mathbf{s'X}} = E e^{-\sum_{i=1}^{p} s_i X_i} \\ &= E e^{-\sum_{i=1}^{p} s_i (a_i X_0 + Z_i)} = E \prod_{i=1}^{p} e^{-a_i s_i X_0} e^{-s_i Z_i} \\ &= E e^{-X_0 \sum_{i=1}^{p} a_i s_i} \prod_{i=1}^{p} E e^{-s_i Z_i} = L_{X_0}(\sum_{i=1}^{p} a_i s_i) \prod_{i=1}^{p} L_{Z_i}(s_i) \end{aligned}$$

for $\mathbf{s} = (s_1, s_2, \ldots, s_p)'$. So

$$L_{\mathbf{X}}(\mathbf{s}) = e^{-\mu_0 \sum_{i=1}^{p} a_i s_i} \left(\frac{\lambda_0}{\lambda_0 + \sum_{i=1}^{p} a_i s_i} \right)^{\alpha_0} \prod_{i=1}^{p} \frac{L_{X_i}(s_i)}{L_{X_o}(a_i s_i)}. \quad (19.3.9)$$

Note that from the above, the LST of Z_i is

$$L_{Z_i}(s) = \frac{L_{X_i}(s)}{L_{X_o}(a_i s)}, \quad \text{for } i = 1, 2, .., p\,. \quad (19.3.10)$$

There are important results obtained from the concept of infinitely divisibility and with many applications in the theory of limit distributions for the sum of independent r.v.'s. In general it is difficult to determine whether a given distribution is infinite divisible or not. We would like to consider what conditions are required for the pdf of the gamma distributions, and its mixture to be infinitely divisible. We first give a notation and the definition. Let the symbol $\stackrel{d}{=}$ denote equality in distribution.

Definition 19.3.11. *Consider a random vector X. Its distribution is said to be infinitely divisible if for every $n \in \mathcal{N}$ there exist iid random vectors $X_{n1}, X_{n2}, \ldots, X_{nn}$ with $\sum_k X_{nk} \stackrel{d}{=} X$. In other words, an infinitely divisible r.v. X has pdf $f(x)$ that can be represented as the sum of an arbitrary number of iid r.v.'s $X_1, X_2, \ldots, X_n$, with close cdf F_n, that is,*

$$X \stackrel{d}{=} X_1 + X_2 + \ldots + X_n.$$

Hence, the term infinitely divisible. Borrowing from Billingsley (1986), the distribution F of X is the n-fold convolution $F_n * F_n * \ldots * F_n$ where F_i is the distribution function of X_i, $1 \leq i \leq n$.

Two simple examples of infinitely divisible distributions are the Poisson distribution and the negative Binomial. We use that property of infinite divisibility to characterize the distribution of the unknown r.v. from a linear relationship as given in Definition 19.2.11.

Definition 19.3.12. *A function ϕ on the interval $I = [0, \infty)$ is completely monotone if it possesses derivatives $\phi^{(n)}$ at all orders which alternate in sign, i.e. if*

$(-1)^n \phi^{(n)}(s) \geq 0$, *for all s in the interior of I, and $n = 0, 1, 2, \ldots$*

Theorem 19.3.13. *ϕ is completely monotone iff ϕ is the Laplace transform of some measure.*

Proof: See Feller (1971). ∎

For two real-valued functions ϕ_1 and ϕ_2 that are completely monotone, so is their product and their compositions, when appropriately chosen. It is important to note that any completely monotone probability density function is infinitely divisible. See Feller (1971). Moreover, if ϕ is completely monotone on $[0, \infty)$ and $\phi(c) = 0$ for some $c > 0$, then ϕ must be identically zero on $[0, \infty)$.

All Erlang type distributions are infinitely divisible. The Erlang distribution defined in (19.2.5), is infinitely divisible in the sense that it can be represented as the sum of n independent r.v.'s with a common distribution (see Feller (1971)). In fact, n need not be an integer as the gamma can be expressed in a canonical form of an infinitely divisible distribution.

Notice that the distribution of the sum of mutually independent gamma r.v.'s with different scale parameters, is not gamma, even if they are mutually independent. Rather, it is described as a mixed gamma with mixing shape parameter. Using ideas from Moschopoulos (1985), Mathai and Moschopoulos (1991), and Mathai and Saxena (1978), we characterize the distribution of the sums.

So, as long as L_{Z_i} in equation (19.3.10) is completely monotone as in Definition 19.3.12, then conditions of Definition 19.2.11 are satisfied, and the marginals are gamma. Equation (19.3.10) will be used extensively throughout, but for now we present an important theorem which gives some properties on the scale and shift transformations of the multivariate gamma distribution given in Definition 19.2.11.

Theorem 19.3.14. *The class of multivariate gamma distributions based on Definition 19.2.11 is closed under scale transformation in the sense of products with diagonal matrices with positive entries, shift transformations, and under finite independent convolutions.*

Proof: Let $\mathbf{X} = (X_1, \ldots, X_p)'$, be a multivariate gamma and let $C = (c_i)_{i=1,..,p}$

be a diagonal matrix with positive entries. Set $\mathbf{Y} = C\mathbf{X}$. Then we have that

$$\begin{aligned} L_{\mathbf{Y}}(\mathbf{s}) &= Ee^{-\mathbf{s}'\mathbf{Y}} = Ee^{-\sum_{i=1}^{p} s_i Y_i} \\ &= Ee^{-\sum_{i=1}^{p} c_i s_i (a_i X_0 + Z_i)} = E\prod_{i=1}^{p} e^{-a_i c_i s_i X_0} e^{-c_i s_i Z_i} \\ &= Ee^{-X_0 \sum_{i=1}^{p} a_i c_i s_i} \prod_{i=1}^{p} Ee^{-c_i s_i Z_i} = L_{X_0}(\sum_{i=1}^{p} a_i c_i s_i) \prod_{i=1}^{p} L_{Z_i}(c_i s_i) \\ &= e^{-\mu_0 \sum_{i=1}^{p} a_i c_i s_i} \left(\frac{\lambda_0}{\lambda_0 + \sum_{i=1}^{p} a_i c_i s_i}\right)^{\alpha_0} \prod_{i=1}^{p} \frac{L_{X_i}(c_i s_i)}{L_{X_0}(a_i c_i s_i)}, \end{aligned}$$

for $\mathbf{s} = (s_1, s_2, \ldots, s_p)'$, which is of the same form as in (19.3.9). The mean is $E\mathbf{Y} = CE\mathbf{X}$ and the variance is is $\mathrm{Var}(\mathbf{Y}) = C\mathrm{Var}(\mathbf{X})C'$.

Let $\mathbf{d} = (d_1, d_2, \ldots, d_p)'$ and $\mathbf{W}$ defined as: $\mathbf{W} = (W_1, W_2, \ldots, W_p)' = \mathbf{X} + \mathbf{d} = (X_1 + d_1, X_2 + d_2, \ldots, X_p + d_p)'$. Then, we have the following:

$$\begin{aligned} L_{\mathbf{W}}(\mathbf{s}) &= Ee^{-\mathbf{s'W}} = Ee^{-\sum_{i=1}^{p} s_i W_i} = Ee^{-\sum_{i=1}^{p} s_i (X_i + d_i)} \\ &= Ee^{-\sum_{i=1}^{p} s_i d_i} e^{-\sum_{i=1}^{p} s_i (a_i X_0 + Z_i)} = Ee^{-\sum_{i=1}^{p} s_i d_i} \prod_{i=1}^{p} e^{-a_i s_i X_0} e^{-s_i Z_i} \\ &= e^{-\sum_{i=1}^{p} s_i d_i} Ee^{-X_0 \sum_{i=1}^{p} a_i s_i} \prod_{i=1}^{p} Ee^{-s_i Z_i} \\ &= e^{-\sum_{i=1}^{p} s_i d_i} L_{X_0}(\sum_{i=1}^{p} a_i s_i) \prod_{i=1}^{p} L_{Z_i}(s_i) \\ &= e^{-\sum_{i=1}^{p} s_i d_i} e^{-\mu_0 \sum_{i=1}^{p} a_i s_i} \left(\frac{\lambda_0}{\lambda_0 + \sum_{i=1}^{p} a_i s_i}\right)^{\alpha_0} \prod_{i=1}^{p} \frac{L_{X_i}(s_i)}{L_{X_0}(a_i s_i)} \\ &= e^{-\sum_{i=1}^{p} (d_i + \mu_0 a_i) s_i} \left(\frac{\lambda_0}{\lambda_0 + \sum_{i=1}^{p} a_i s_i}\right)^{\alpha_0} \prod_{i=1}^{p} \frac{L_{X_i}(s_i)}{L_{X_0}(a_i s_i)}, \end{aligned}$$

which is of the same form as in (19.3.9). So the multivariate gamma is closed under shift transformation of the form $\mathbf{W} = \mathbf{X} + \mathbf{d}$.

Consider now $\mathbf{X_i}$=$(X_{i1}, X_{i2}, \ldots, X_{ip})'$, n independent p-variate gamma distributions for $i = 1, \ldots, n$, and set

$$\mathbf{X} = \mathbf{X_1} + \ldots + \mathbf{X_n} = (X_{11} + \ldots + X_{n1}, X_{12} + \ldots + X_{n2}, \ldots, X_{1p} + \ldots + X_{np})'.$$

Then,

$$\begin{aligned} L_{\mathbf{X}}(\mathbf{s}) &= Ee^{-\mathbf{s}'(\mathbf{X_1} + \ldots + \mathbf{X_n})} = Ee^{-\sum_{i=1}^{p} s_i (X_{1i} + \ldots + X_{ni})} \\ &= \prod_{i=1}^{p} Ee^{-s_i (X_{1i} + \ldots + X_{ni})} = L_{\mathbf{X_1}}(s) \ldots L_{\mathbf{X_n}}(s), \end{aligned}$$

which has the form of n product of (19.3.9). ∎

Recall that the distribution of Z_i has not been specified, but its LST is given in (19.3.10). In the next theorem, we state a very interesting property, as it gives a necessary and sufficient condition for all the latent variables Z_i to be of the gamma type distributions (19.2.2).

Theorem 19.3.15. *Suppose that for some* $a_i > 0$, $i = 1, \ldots, p$, $X_i \sim Ga(\lambda_i, \alpha_i)$,
$i = 0, 1, \ldots, p$, *are related as in Definition 19.2.11 with* $Z_i, i = 1, \ldots, p$. *Then for* $i = 1, \ldots, p$,

$$Z_i \sim Ga(\lambda_i, \alpha_i - \alpha_0) \Longleftrightarrow a_i = \frac{\lambda_0}{\lambda_i} \quad \textit{and} \quad p_i = P(Z_i = 0) = 0.$$

Proof: From the expression of $a_i = \lambda_0/\lambda_i$ and the result in (19.2.10), we have that $a_i X_0 \sim Ga(\lambda_i, \alpha_0)$. Since $X_i \sim Ga(\lambda_i, \alpha_i)$, it follows, from Theorem 19.3.8 (the additive property of the gamma distribution), that Z must be distributed as $Z_i \sim Ga(\lambda_i, \alpha_i - \alpha_0)$.

Conversely, assume that $Z_i \sim Ga(\lambda_i, \alpha_i - \alpha_0)$. Then from Example 19.3.3 and the independence between X_0 and Z_i, we have $L_{X_i}(t) = L_{X_0}(a_i t) L_{Z_i}(t)$, which implies

$$L_{X_0}(a_i t) = \left(\frac{\lambda_0}{\lambda_0 + a_i t}\right)^{\alpha_0} = \left(\frac{\lambda_i}{\lambda_i + t}\right)^{\alpha_i} \left(\frac{\lambda_i}{\lambda_i + t}\right)^{\alpha_o - \alpha_i}.$$

Hence,

$$\frac{\lambda_0}{\lambda_0 + a_i t} = \frac{\lambda_i}{\lambda_i + t},$$

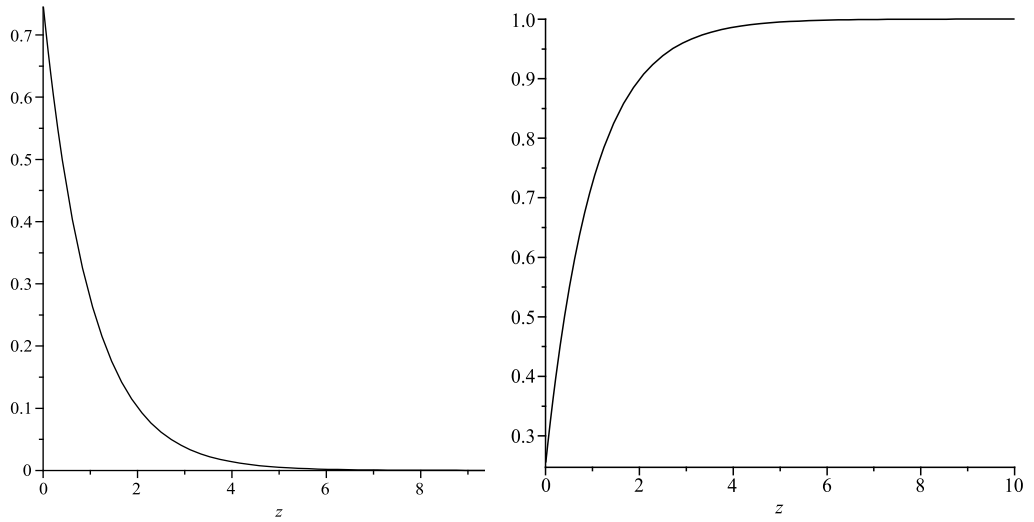
and solving for a_i gives

$$\frac{\lambda_0 + a_i t}{\lambda_0} = \frac{\lambda_i + t}{\lambda_i},$$

which implies

$$1 + \frac{a_i t}{\lambda_0} = 1 + \frac{t}{\lambda_i} \quad \text{or} \quad a_i = \frac{\lambda_0}{\lambda_i}. \quad ∎$$

Mathai and Moschopoulos (1991), among other authors, made use of this particular coefficient of linear relationship, a, between two gamma distributions, and developed properties, approximations, and inequalities based on that particular model. We have asked and answered the question of what happens in the general setting where a is not necessarily this fixed particular constant. We further examined the Mathai and Moschopoulos (1991) and (1992) specific form, and call it the continuous class of distributions. In all other cases where the conditions of Theorem 19.3.15 are not satisfied, we call that form the "fatal shock" class,



(a) The probability density function (b) The cumulative distribution function

Figure 19.3–1. The pdf and cdf of the latent randon variable Z with $\lambda_1 = 4$, $\lambda_2 = 1$ and $a = 1$.

because there is a positive probability of simultaneous or proportional occurrences. In the next section, we focus on the characterization of the distribution of $Z_i,\ i = 1, 2, .., p$ in the fatal shock class. All the Z_i's take on similar structures. To simplify the notations, without lost of generality, we drop the indices, and we rewrite (19.2.12) as

$$Y = aX + Z, \qquad \text{for some } a > 0. \tag{19.3.16}$$

So Y, X, a, and Z in (19.3.16) play the role of X_i, X_0, a_i, and Z_i in (19.2.12), respectively.

A simple computation shows that the cdf of Z is

$$\begin{aligned} P(Z \le z) &= P(Y - aX \le z) \\ &= P(Y = aX) + (1-p)P(X \ge \frac{Y-z}{a}) \\ &= P(Y = aX) + (1-p)\Big[1 - P(X \le \frac{Y-z}{a})\Big] \\ &= P(Y = aX) + (1-p)\Big[1 - \int_z^\infty F_X(\frac{y-z}{a}) f_Y(y)dy\Big]. \end{aligned}$$

Since

$$\begin{aligned} \int_z^\infty F_X(\frac{y-z}{a}) f_Y(y)dy &= F_X(\frac{y-z}{a})F_Y(y)\Big]_z^\infty - \int_z^\infty \frac{1}{a} f_X(\frac{y-z}{a})F_Y(y)dy \\ &= 1 - \int_z^\infty \frac{1}{a} f_X(\frac{y-z}{a})F_Y(y)dy, \end{aligned}$$

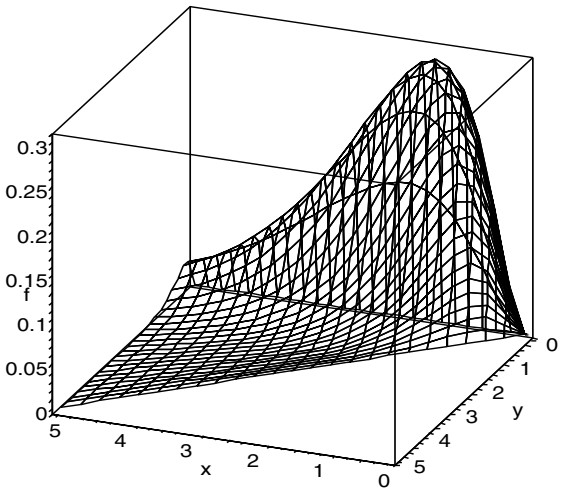


Figure 19.4–1. Joint density of $X \sim Ga(0, 4, 1)$ and $Y \sim Ga(0, 1, 4)$ with $a = 1$.

the cdf can be further simplified as

$$P(Z \leq z) \quad = \quad P(Y = aX) + \frac{1-p}{a} \int_z^\infty f_X(\frac{y-z}{a}) F_Y(y) dy,$$

where $P(Y = aX) = P(Z = 0)$ is the probability of proportional occurrence. The graph of the pdf and cdf of Z is given in Figure 19.3–1. Note that if the conditions of Theorem 19.3.15 are not satisfied, then $P(Y = aX) = P(Z = 0) > 0$. In the next section, we further characterize the distribution of Z. In Carpenter, Diawara, and Han (2006), an example based on such a model has been suggested in the simple exponential case. Our goal is to extend that theory to include the Erlang and Gamma distributions.

19.4 The Fatal Shock Model

In this section, we study the form of the distribution of Z obtained in (19.3.16) for a general $a > 0$ where $a \neq \lambda_0/\lambda_i$. Suppose that X and Y are distributed as $Ga(\mu, \lambda, \alpha)$ and $Ga(\mu', \lambda', \alpha')$, respectively. Assuming a linear relationship as in (19.3.16), Figure 19.4–1 gives the density of the joint distribution.

Using equation (19.3.10) , where a is some positive constant, and the fact that Z is independent of X, the LST of Z becomes

$$L_Z(s) = \frac{L_Y(s)}{L_X(as)} = e^{(\mu-\mu')s} \left(\frac{\lambda'}{\lambda'+s}\right)^{\alpha'} \left(\frac{\lambda+as}{\lambda}\right)^{\alpha}.$$

All distributions Z satisfying this relation (19.3.16) for all $a \in (0, 1)$, are called **members of the self decomposable r.v.'s**. See also Gaver and Lewis

(1980). We have studied properties of the r.v. Z and derived its LST in (19.3.10). However, the form of the distribution of Z has not been given. In the theorems that follow, we give the forms that the distribution of Z takes on to be able to obtain the multivariate gamma marginals. We state the main theorems, and start with the Erlang distributions described earlier in equation (19.2.5) subjected to the model in (19.3.16).

Theorem 19.4.1: Erlang-type distributions.
Let $X \sim Ga(\mu, \lambda, \alpha)$ and $Y \sim Ga(\mu', \lambda', \alpha')$ such that $\alpha, \alpha' \in \mathcal{N}, \alpha \leq \alpha'$ and as in (19.3.16), Z is independent of X. Then Z is the shifted sum of independent

- *α r.v.'s, each being product of Bernoulli and exponential distribution with parameter λ',*
- *and of $\alpha' - \alpha$ gamma r.v.'s with scale λ' and shape $k = 1, .., \alpha' - \alpha$.*

Proof: Using the Laplace transform technique, we have

$$\begin{aligned} L_Z(s) &= \frac{L_Y(s)}{L_X(as)} = e^{(\mu-\mu')s}\frac{\lambda'^{\alpha'}}{\lambda^{\alpha}}\frac{(\lambda+as)^{\alpha}}{(\lambda'+s)^{\alpha'}} \\ &= e^{(\mu-\mu')s}\frac{\lambda'^{\alpha'}}{\lambda^{\alpha}}\frac{(\lambda+as)^{\alpha}}{(\lambda'+s)^{\alpha}}\left[\frac{1}{(\lambda'+s)^{\alpha'-\alpha}}\right]. \end{aligned}$$

By the theorem of expansion in partial fractions, with α,

$$\frac{1}{(\lambda'+s)^{\alpha}} = \sum_{k=1}^{\alpha}\frac{C_k}{(\lambda'+s)^k}$$

where C_k are real numbers. Hence,

$$L_Z(s) = e^{-(\mu-\mu')s}\frac{\lambda'^{\alpha'}}{\lambda^{\alpha}}\Big[p + (1-p)L_Y(s)\Big]^{\alpha}\sum_{k=1}^{\alpha'-\alpha}\frac{C_k}{\lambda'^k}\left(\frac{\lambda'}{\lambda'+s}\right)^k,$$

where $C_k,\ k = 1, 2, \ldots, \alpha' - \alpha$, are constant and $p = a\lambda/\lambda'$. Based on the properties of the LST from Example 19.3.5, we have that $[p + (1-p)L_Y(s)]$ is the product of Bernoulli and exponential r.v.'s and in Carpenter, Diawara, and Han (2006). So

$$[p + (1-p)L_Y(s)]^{\alpha}$$

is the product of α independent of those terms.
Also, since the LST of a sum of functions is the sum of their LST,

$$\sum_{k=1}^{\alpha'-\alpha}\frac{C_k}{\lambda'^k}\left(\frac{\lambda'}{\lambda'+s}\right)^k$$

is the LST of the sum of $\alpha' - \alpha$ gamma distributions. ∎

The special case of the exponential, where $\alpha = \alpha' = 1$, has been studied in Carpenter, Diawara, and Han (2006), and by Iyer, Srilanth, Manjunath, and Manivasakan (2002). A similar result in the gamma context has been given in Carpenter and Diawara (2007).

Theorem 19.4.2. *Suppose $X \sim Ga(\mu, \lambda, \alpha)$, with $\alpha \in \mathcal{N}$, and $Y \sim Ga(\mu', \lambda', \alpha')$ as in (19.3.16), and $\alpha \leq \alpha'$. Then Z is the sum of two r.v.'s:*

- *a $Ga(\mu' - \mu, \lambda', \alpha' - \alpha)$ and*
- *another distribution which is the sum of α independent r.v.'s each of which is a product of a Bernoulli and an exponential with parameter λ'.*

Proof: From the expression of L_Z,

$$L_Z(s) = \left[\frac{\lambda'}{\lambda}\frac{(\lambda + as)}{(\lambda' + s)}\right]^{\alpha} e^{(\mu-\mu')s}\left[\frac{\lambda'}{\lambda' + s}\right]^{\alpha'-\alpha},$$

and the result follows. ∎

Assuming $Z = X_2 - aX_1$, and using the Taylor approximation of

$$\log(1 + s) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} s^{n+1},$$

the cumulants of Z are obtained as

$$\begin{aligned}
\log L_Z(s) &= \log L_{X_2}(s) - \log L_{X_1}(as) \\
&= \alpha_1 \log\left(1 - \frac{as}{\lambda_1}\right) - \alpha_2 \log\left(1 - \frac{s}{\lambda_2}\right) \\
&= \alpha_1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{as}{\lambda_1}\right)^{n+1} - \alpha_2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{s}{\lambda_2}\right)^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} s^{n+1}\left[\alpha_1\left(\frac{a}{\lambda_1}\right)^{n+1} - \alpha_2\left(\frac{1}{\lambda_2}\right)^{n+1}\right].
\end{aligned}$$

Another characterization of Z is obtained by considering several cases for the $Ga(\mu, \lambda, \alpha)$ and $Ga(\mu', \lambda', \alpha')$.

Corollary 19.4.3. *Let $\alpha = \alpha' \in \mathcal{N}$, then Z is a shifted sum of α independent r.v.'s, each being a product of a Bernoulli with mean $(1 - p)$, $p = a \times \lambda'/\lambda$, and an exponential with parameter λ', where the value of the shift is $\mu - \mu'$.*

Proof: $\alpha \in \mathcal{N}$. Then the LST of Z can be expressed as:

$$\begin{aligned}
L_Z(s) &= e^{(\mu-\mu')s}\left[\frac{\lambda'}{\lambda}\frac{(\lambda + as)}{(\lambda' + s)}\right]^{\alpha} \\
&= e^{(\mu-\mu')s}\left[(1 - p)\left(\frac{\lambda'}{\lambda' + s}\right) + p\right]^{\alpha}. \quad ∎
\end{aligned}$$

Theorem 19.4.4. *Suppose $X \sim Ga(\mu, \lambda, \alpha)$, and $Y \sim Ga(\mu', \lambda', \alpha')$ as in (19.3.16), with $\alpha < \alpha'$, and assume that $[\alpha + 1] \leq \alpha'$, where $[x]$ is the greatest integer less than or equal to x, also called the integer part of x. Then Z is the sum of*

- $[\alpha + 1]$ *independent r.v.'s, each being a product of a Bernoulli with mean* $(1 - p)$ *and an exponential with parameter* λ'
- $Ga(\mu' - \mu, \lambda', \alpha' - [\alpha + 1])$
- $Ga(0, \lambda/a, [\alpha + 1] - \alpha)$

Proof: The result is obtained as follows:

$$L_Z(s) = \left[\frac{\lambda'}{\lambda}\frac{\lambda + as}{\lambda' + s}\right]^{[\alpha+1]} e^{(\mu-\mu')s}\left[\frac{\lambda'}{\lambda' + s}\right]^{\alpha'-[\alpha+1]}\left[\frac{\lambda + as}{\lambda}\right]^{\alpha-[\alpha+1]}$$

and the following completes the proof:

$$\left[\frac{\lambda + as}{\lambda}\right]^{\alpha-[\alpha+1]} = \left[\frac{\lambda}{\lambda + as}\right]^{[\alpha+1]-\alpha}. \quad \blacksquare$$

In fact, instead of assuming $[\alpha + 1] \leq \alpha'$, it suffices to assume that

$$[\alpha + \epsilon] \leq \alpha' \text{ for some } \epsilon \in [0, 1].$$

Theorem 19.4.5. *Suppose $X \sim Ga(\mu, \lambda, \alpha)$, and $Y \sim Ga(\mu', \lambda', \alpha')$ as in (19.3.16), with $\alpha = \alpha' \notin \mathcal{N}$, then Z does not have the* LST *of some measure.*

Proof: $\alpha \notin \mathcal{N}$, The explicit formula in terms of the LST is

$$L_Z(s) = \left[\frac{\lambda'}{\lambda}\frac{(\lambda + as)}{(\lambda' + s)}\right]^{[\alpha]}\left[\frac{\lambda'}{\lambda' + s}\right]^{\alpha-[\alpha]} e^{(\mu-\mu')s}\left[\frac{\lambda + as}{\lambda}\right]^{\alpha-[\alpha]}$$

The question we have is how to interpret

$$\left[\frac{\lambda + as}{\lambda}\right]^{\alpha-[\alpha]}.$$

However, if we set $\phi(s) = \left[\frac{\lambda + as}{\lambda}\right]^{\beta}$, where $\beta = \alpha - [\alpha] \in (0, 1)$, we have

$$\phi^{(1)}(s) = \beta\frac{a}{\lambda}\left[\frac{\lambda + as}{\lambda}\right]^{\beta-1},$$
$$\phi^{(2)}(s) = \beta(\beta - 1)\left(\frac{a}{\lambda}\right)^2\left[\frac{\lambda + as}{\lambda}\right]^{\beta-2},$$
$$\vdots$$
$$\phi^{(n)}(s) = \beta(\beta - 1)\ldots(\beta - n + 1)\left(\frac{a}{\lambda}\right)^n\left[\frac{\lambda + as}{\lambda}\right]^{\beta-n}.$$

So $\phi(0) = 1$, and $\phi(s)$ has derivatives that alternate in sign. However, by Definition 19.3.12 and Theorem 19.3.7, from Feller (1971), $(-1)^n\phi(s) \leq 0$, for all $n = 0, 1, 2, \ldots$, ϕ is not completely monotone. We have that ϕ is not the Laplace transform of some probability distribution. ∎

Theorem 19.4.6. *Let $X \sim Ga(\mu, \lambda, \alpha)$ and $Y \sim Ga(\mu', \lambda', \alpha')$ such that $\alpha \in \mathcal{N}$, $\alpha' \in \mathcal{R}_+$, $\alpha' \leq \alpha$ and as in (19.3.16). Then Z has* LST *of some measure iff $\alpha = \alpha'$.*

Proof:

$$L_Z(s) = \frac{L_Y(s)}{L_X(as)} = e^{(\mu-\mu')s}\left(\frac{\lambda'}{\lambda'+s}\right)^{\alpha'}\left(\frac{\lambda+as}{\lambda}\right)^{\alpha}$$

$$= e^{(\mu-\mu')s}\left[\frac{\lambda'}{\lambda}\frac{(\lambda+as)}{(\lambda'+s)}\right]^{\alpha'}\left[\frac{\lambda+as}{\lambda}\right]^{\alpha-\alpha'}.$$

The expression

$$\left[\frac{\lambda+as}{\lambda}\right]^{\alpha-\alpha'}$$

is completely monotone iff $\alpha \leq \alpha'$. Since $\alpha' \leq \alpha$ must also be satisfied, $\alpha = \alpha'$. ∎

We have seen that the distribution of Z takes on many different forms depending on the parameters that are given. In Theorem 19.3.9, we found the only case where the latent variable is of continuous gamma type as the marginals. The results from Theorem 19.3.9 up to 19.4.6 give the various forms that the distribution of Z can take. Theorem 19.3.9 and Theorem 19.4.1 are similar, as they express the distribution of Z as a sum of products of Bernoulli and exponential, and that of gamma r.v.'s. The special case, given in Corollary 19.4.3, states that the Erlang case where $\alpha = \alpha' \in \mathcal{N}$, leads to just the product of α Bernoulli and exponential. These results give alternative ways of characterizing the distribution of the latent variables, an opportunity to replace the unconditional expectation, density by the unconditional expectation, density. That allows us to increase the efficiency in the variance covariance matrix of parameters. In all those cases, $\alpha \leq \alpha'$ is a required condition for the model validation. Theorem 19.4.4 shows that there must be an integer value between α and α' to justify the gamma marginals for Y and X in the equation (19.3.16). If there is no such integer value between α and α', then Z does not have the LST of some measure. It cannot be represented by a distributional form. Theorem 19.4.6 confirms that finding in the sense that it gives back the result obtained in Corollary 19.4.3.

We can see that it is possible to exactly represent gamma long-tailed probability distributions by convenient short-tailed probability distributions. Another advantage of the proposed form is that the statistical estimation will be exact, not approximate, all based on the explicit Laplace transform.

19.5 Example

To illustrate our result, we consider a simulation experiment in which we fit Erlang probability distributions to a sample of 50 data points drawn from two related distributions. Suppose the Erlang random variables X and Y are distributed as $Ga(0, 4, 3)$ and $Ga(0, 1, 3)$, respectively, and related by $Y = X + Z$, with Z unknown. Here $a = 1$ in (19.2.12), meaning that there is a non-zero probability of simultaneous occurrence between the two random variables. As described above in Corollary 19.4.3, Z is a mixture of three independent r.v.'s, each being a product of a Bernoulli with mean $1 - p = 3/4$, and an exponential with unit parameter. We simulate data from such a distribution using SAS, and we propose to get estimates for the parameters using **R**. The SAS data code and the **R** code are gived in the appendix at the end of this chapter.

The estimates are $\widehat{\alpha} = 3$, $\widehat{\lambda_1} = 4.19$, and $\widehat{\lambda_2} = 0.98$. The estimated values produced are quite close to the original parameters and show that our proposed method and algorithm produce excellent approximations, even with a reasonable sample size. This is not surprising since our fitting method is designed to treat data, and we have model simplicity and computational efficiency using the **R** tools of constraint optimization.

Many important probability distributions, such as those associated with network traffic, have long tails. We propose these techniques to model highly variable datasets. They capture the dynamical relation between events. Consider two hydrologic systems related linearly through a random component. It is quite common to use the gamma distribution to characterize such systems. The approximation to the normal distribution should not be considered if we have underlying lifetime distribution functions. As such, we propose techniques to parametrically link the data from these distributions. These techniques will allow the scientist to efficiently estimate the parameters of the models without the use of infinite functions. The estimates are then exact, and not approximate. In computer communications, Feldmann and Whitt (1998) proposed models for large scale network intrusion detection systems related to traffic volume in which the literature omits practical difficulties. Other authors such as Harris, Brill, and Fischer (2000) have explained that the suggested solutions are typically complicated and never analytically tractable, especially with the presence of degenerate state system probability distributions. This is to emphasize that solutions to such problems are needed. The techniques are based on the mixtures of gamma, exponential, and Bernoulli distributions. In that sense, this is an expansion of the ideas proposed by Feldmann and Whitt (1998), who fitted mixtures of exponentials to long tail distributions models.

In health care services, Venturini, Dominici, and Parmigiani (2008) explain that medical expenditures, under a simple form, are highly skewed heavy-tailed distributions, and they cannot be understood by a simple variable. We consider the gamma distributions under the linear model and show that its joint distri-

bution is based on a mixture of gammas, with mixing weights obtained from the shape parameter. We propose an extension in the sense that the idea of simultaneous or proportional occurrence from Marshall and Olkin (1967) is included in our formulation. The parametric approach is an efficient solution to their estimation problem.

19.6 Conclusion

We have described, in a computational form, the density and possible structures of the distribution of Z, based on equation (19.3.16). Our proposed method appears as the multivariate gamma form of $\mathbf{X}$ as in Definition 19.2.11. That means we need to study the distribution of $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_p)$ based on equation (19.2.12). Also, we study the case where the latent variables Z_i are all of gamma types, as in the result obtained from Theorem 19.3.9. In that case, the probability of proportional occurrence is $p_i = 0$, for $i = 1, 2, .., p$. Mathai and Moschopoulos (1991) used that form to give a version of a multivariate gamma. We have proposed a more general model, as in Definition 19.2.11, of a multivariate gamma, that can be simplified to yield other models proposed by Mathai and Moschopoulos (1991), and Mathai and Moschopoulos (1992). Nadarajah and Kotz (2005) derive a linear combination of exponential and gamma r.v.'s. They assume that there is no probability of simultaneous occurrence between X_i and X_0. However, as given in the famous paper by Marshall and Olkin (1967), "shock model" conditions are quite meaningful and require attention. Our model captures the shock model that Marshall and Olkin (1967) studied in the simple case of the multivariate exponential distribution and provides a more applicable solution. The various scenarios will be useful to practitioners in many different areas.

References for Chapter 19

Abramowitz, M. and Stegun, I.A. (1972). *Handbook of Mathematical Functions*, Selected Government Publications, Chapter 6, New York, Dover.

Balakrishnan, N. and Wang, J. (2000). "Simple efficient estimation of the three-parameter gamma distribution," *Journal of Statistical Planning and Inference,* 85, pp. 115–126.

Barlow, R.E. and Proschan, F. (1965). *Mathematical Theory of Reliability*, John Wiley & Sons, Library of Congress.

Billingsley, P. (1986). *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics, 2nd edition.

Carpenter, M. and Diawara, N. (2007). "A multivariate gamma distribution, and its characterizations," *American Journal of Mathematical and Management*

Sciences, Vol. 27 (3 & 4), pp. 499–507.

Carpenter, M., Diawara, N., and Han, Y. (2006). "A new class of bivariate Weibull survival distributions," *American Journal of Mathematical and Management Sciences,* Vol. 26, pp. 164–184.

Cohen, A.C. and Whitten, B.J. (1986). "Modified moment estimation for the three-parameter gamma distribution," *Journal of Quality Technology,* 18, pp. 53–62.

Cox, D.R. (1972). "Regresssion models and life tables," *Journal of the Royal Statististical Society: Series B,* Vol. 34, pp. 187–220.

Dudewicz, E.J. and Mishra, S. (1988). *Modern Mathematical Statistics*, Wiley Series in Probability and Mathematical Statistics.

Feldmann, A. and Whitt, W. (1998). "Fitting mixtures of exponentials to long-tail distributions to analyze network performance models," *Performance Evaluation,* Vol. 31 (3–4), pp. 245–279.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, 2nd Edition, Wiley Series in Probability and Mathematical Statistics.

Gaver, D.P. and Lewis, P.A.W. (1980). "First-order autoregressive gamma sequences and point processes," *Advances in Applied Probability,* Vol. 12, pp. 727–745.

Ghosh, S.K. and Gelfand, A.E. (1998). "Latent waiting time models for bivariate event times with censoring," *Sankhyā,* Vol. 60, Series A, pp. 31–47.

Grove, D.A. and Coddington, P.D. (2005). "Analytic models of probability distributions for MPI point-to-point communication times on distributed memory parallel computers," *Proc. 6th International Conference on Algorithms and Architectures for Parallel Processing,* Melbourne, Australia.

Harris, C.M., Brill P.H., and Fischer, M.J. (2000). "Internet-type queues with power-tailed interarrival times and computational methods for their analysis," *Informs Journal on Computing,* Vol. 12, 4, pp. 261–271.

He, W. and Lawless, J.F. (2005). "Bivariate location-scale models for regression analysis, with applications to lifetime data," *J. Royal Statist. Soc. B*, Vol. 67, part 1, pp. 63–78.

Hougaard, P. (1986). "A class of multivariate failure time distributions," *Biometrika,* 73, 3, pp. 671–678.

Hougaard, P. (2000). *Analysis of Multivariate Survival Data,* Springer.

Hougaard, P. (2001). "The normality assumption: Pro and Contra," *Biometrika,* 73, 3, pp. 671–678.

Hunter, J.K., and Nachtergaele, B. (2001). *Applied Analysis*, River Edge, NJ, World Scientific.

Iyer, Srilanth K., Manjunath, D., and Manivasakan, R. (2002). "Bivariate exponential distributions using linear structures," *Sankhyā,* V. 64, Series A, pp. 156–166.

Iyer, Srilanth K., and Manjunath, D. (2004). "Correlated bivariate sequence for queueing and reliability applications," *Communications in Statistics,* Vol. 33, pp. 331–350.

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*, 2nd edition, Wiley Series in Probability and Statistics.

Marshall, A.W. and Olkin, I. (1967). "A multivariate exponential distribution," *Journal of the American Statistical Association,* 63, pp. 30–44.

Mathai, A.M. and Moschopoulos, P.G. (1991). "On a multivariate gamma," *Journal of Multivariate Analysis,* 39, pp. 135–153.

Mathai, A.M. and Moschopoulos, P.G. (1992). "A form of multivariate gamma distribution," *Annals of the Institute of Statistical Mathematics,* 44, pp. 97–106.

Mathai, A.M. and Saxena, R.K. (1978). *The H-function with Applications in Statistics and Other Disciplines*, Wiley New York.

Moschopoulos, P.G. (1985). "The distribution of the sum of independent gamma random variables," *Annals of the Institute of Statistical Mathematics,*" V. 37, pp. 541–544.

Nadarajah, S. and Kotz, S. (2005). "On the linear combinations of exponential and gamma random variables," *Entropy,* V. 7(2), pp. 161–171.

Venturini, S., Dominici, F., and Parmigiani, G. (2008). "Gamma shape mixtures for heavy-tailed distributions," *The Annals of Applied Statistics,* Vol. 2, Number 2 (2008), pp. 756–776.

Appendix for Chapter 19

A.1 SAS Code for Data Generation

```
%macro erlangrv(alpha,lambda1,lambda2,a,n);

data sim1;
%do i=1 %to &n;
  *generating the random variable Z;
  p=(&a*&lambda2)/&lambda1;
  z=0;

  %do j=1 %to \&alpha;
    seed1=12356;seed2=345678;seed3=9898;
    E=ranexp(seed1)/&lambda2;
    B=ranbin(seed2,1,1-p);
    y=B*E;
    z=z+y;
  %end;

 *generating the random variable X1;
  x1=\&lambda1*rangam(seed3,&alpha);
 *obtaining the random variable X2;
  x2=&a*x1+z;
  output;
%end;

run;

proc print data=sim1; var Z B x1 x2;run;
data sim2;set sim1; if (z=0) then count=1; else count=0; keep count; run;
proc freq data=sim2; tables count;run;
```

A.2 R Code for Estimation

```
log.likelihood <- function(params,data,a=1){
    x1 <- data[,1]
    x2 <- data[,2]
    density <- density.expmix(x1,x2,lambda1=params[1],
+       lambda2=params[2], alpha=params[3],a=a)
    sum(log(density))
}

log.likelihood.fix.alpha <- function(params,data,alpha,a=1){
    if (params[2] > params[1]){return(NA)}
    x1 <- data[,1]
    x2 <- data[,2]
```

```
    density <- dens2(x1,x2,lambda1=params[1],lambda2=params[2],
+       alpha=alpha,a=a)
    sum(log(density))
}

combined.estimator <- function(theta,f,data,alpha.range=2:10){
    big.list <- list()
    for (alpha in alpha.range){
        estimate <- constrOptim(theta=theta,f=f,grad=NULL,
+           ui=rbind(c(1,-1),c(0,1)),ci=c(0,0), method="Nelder-Mead",
+           data=data,alpha=alpha,control=list(fnscale=-1))
            if(!is.null(estimate$message)) {
                warning(c("alpha=",alpha,estimate\$message))
            }
        big.list[[alpha]] <- estimate
        }
    big.unlist <- unlist(big.list)
    par <- cbind(big.unlist[names(big.unlist)=="par1"],
+       big.unlist[names(big.unlist)=="par2"])
    dimnames(par) <- list(paste("alpha.eq.",alpha.range,sep=""),
+       c("lambda1","lambda2"))
    value <- big.unlist[names(big.unlist)=="value"]
    counts <-  big.unlist[names(big.unlist)=="counts.function"]
    df.version <-  cbind(alpha.range,par,value,counts)
    dimnames(df.version) <- list(NULL,c("alpha","lambda1","lambda2",
+       "log.likelihood","Fn.evaluations"))
    return.list <- list(raw.constrOptim=big.list,par=par,value=value,
    Function.evaluations=counts,df.version=df.version)
    class(return.list) <- "comb.est"
    return.list
    }

print.comb.est <- function(x){ cat("Parameter estimates\n")
print(x$df.version) }
combined.estimator(theta=c(4.7,3.6),f=log.likelihood.fix.
+   alpha,data,alpha=2:10)
results
combined.estimator(theta=c(6,3.6),f=log.likelihood.fix.
+    alpha,data,alpha=2:3)
results
results$par
results$raw.constrOptim
```

PART V: The Generalized Bootstrap and Monte Carlo Methods

Chapter 20 The Generalized Bootstrap (GB) and Monte Carlo (MC) Methods

Chapter 21 The Generalized Bootstrap: A New Fitting Strategy and Simulation Study Showing Advantage over Bootstrap Percentile Methods

Chapter 22 Generalized Bootstrap Confidence Intervals for High Quantiles

A sketch of each of these chapters appears in Section 1.2.

Chapter 20

The Generalized Bootstrap (GB) and Monte Carlo (MC) Methods

In Chapter 1 we noted the key role of statistical models, in particular of probability distributions, in modern human endeavor, and the recent explosive developments in **fitting distributions to data**. In Chapters 2 through 10 (through the use of the EGLD) and in Chapters 16 and 17 we have provided modern, cutting-edge methods for this area.

The fitted distributions are often used in a **Monte Carlo Simulation (MC) problem solution**, where one samples from the fitted distribution using random numbers — recall that we saw how to accomplish such sampling in Section 2.5 (GLD Random Variate Generation), Section 4.6 (EGLD Random Variate Generation), and Section 9.6 (GLD–2 Random Variate Generation). The resulting samples from the fitted distribution can then be used to assess the performance of the simulated system, making it possible to optimize the system before it is built or modified (we saw in Section 3.7 how design of experiments could be used in this process with a fitted GLD). For some examples of MC in optimization, vehicle routing, and inventory management, see, e.g., Golden, Assad, and Zanakis (1984), Golden and Eiselt (1992), and Dudewicz (1997), respectively. For uses in quality control, see Dudewicz (1999, pp. 44–84).

An alternative approach in use as early as 1967, that did not gain wide use until it was given the name **bootstrap** by B. Efron in 1979, after which interest in it increased markedly, does not fit a distribution and then sample from it. Rather, **it draws its samples at random with replacement from the data at hand.**

The Bootstrap Method has attracted much interest since it is simple and seems to yield solutions to problems. However, the method can behave badly if the sample size is not large, and most of the theory supporting it is developed

as the sample size becomes infinite. For example, if we have 50 years of data on rainfall in an area and we are designing a levee, the bootstrap method will never yield observations more extreme than the most extreme in the dataset of size 50. This can easily lead to design of a levee that will not withstand the 100-year flood (a typical design criterion in the United States) or the 1000-year flood (a design criterion in Europe) with high probability. For more details, see Dudewicz (1992) and Dudewicz and Mishra (1988, Section 5.6). For these reasons, **we will not detail the Bootstrap Method further here; it is a method fraught with danger of seriously inadequate results.**

We do detail the **Generalized Bootstrap**, which was introduced formally in Dudewicz (1992) (and discussed earlier in Karian and Dudewicz (1991), Section 6.6)).

20.1 The Generalized Bootstrap (GB) Method

The essence of the **Generalized Bootstrap (GB) Method** is to fit an EGLD to the available data, and then take samples from the fitted distribution (and work with them as the BM does with its samples from the data itself). This method has been shown to do better than the BM when the number of data points is not very large, and do as well as the BM when the number of data points is large. Sun and Müller–Schwarze (1996) have an excellent exposition with real-data examples. More recent examples in **data mining**, including interactive computer systems, consumer purchases, crop damage, and pathogens, are considered by Dudewicz and Karian (1999).

Suppose that we are interested in

$$\theta = \theta(F), \tag{20.1.1}$$

some function of the distribution $F(\cdot)$, but $F(\cdot)$ is unknown (if it were known we could simply calculate $\theta(F)$). However, we have a random sample $X_1, X_2, \ldots, X_n$ from $F(\cdot)$. How can we use this sample to estimate $\theta(F)$?

The Generalized Bootstrap (GB) approaches the problem as follows. Suppose that one would typically estimate θ by $\hat{\theta} = \hat{\theta}(X_1, X_2, \ldots, X_n)$. Then, instead, proceed as follows:

GB–1. Estimate F by $\hat{F}$. (The estimating $\hat{F}$ should always be one that has the properties known for the true F. For example, $\hat{F}$ should be continuous if it is known that F is continuous. Fitting with the Generalized Lambda Distribution or with the empiric p.d.f. will often be appropriate.)

GB–2. Independently generate N random samples of size n from $\hat{F}$.

From

$Y_1, Y_2, \ldots, Y_n$ estimate $\theta(F)$, calling the estimate $\hat{\theta}_1$.

From

$Y_{n+1}, Y_{n+2}, \ldots, Y_{2n}$ estimate $\theta(F)$, calling the estimate $\hat{\theta}_2$.

$\vdots$

From

$Y_{(N-1)n+1}, \ldots, Y_{Nn}$ estimate $\theta(F)$, calling the estimate $\hat{\theta}_N$.

GB–3. Use the sample $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$ to estimate $\hat{\theta}$. For example, one may calculate

$$\overline{\theta} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \hat{\theta}_i \tag{20.1.2}$$

and

$$\hat{\sigma}^2 = \left(\frac{1}{N-1}\right) \sum_{i=1}^{N} (\hat{\theta}_i - \overline{\theta})^2, \tag{20.1.3}$$

which give the sample mean and sample variance, respectively, of the GB estimators found in GB–2. Then, assuming approximate normality, we have

$$\mathrm{Var}(\overline{\theta}) = \mathrm{Var}(\hat{\theta}_i)/N \approx \hat{\sigma}^2/N, \tag{20.1.4}$$

and **an approximate** $100(1-\alpha)\%$ **confidence interval for** θ **is**

$$\hat{\theta} \pm \Phi^{-1}(1-\alpha/2)\hat{\sigma}\,. \tag{20.1.5}$$

The GB method uses the original estimator based on the original sample. However, the GB method uses the samples generated in Step GB–2 to arrive at an estimate of the variability of the original estimator. The method of (20.1.5) is called the **"standard method"** (e.g., see Chernick (1999, p. 54)), and is appropriate if $\hat{\theta}$ has an approximate normal distribution.

A widely used alternative to the standard method is the **"percentile method,"** which uses the upper and lower $\alpha/2$ percentiles of the GB sample estimators as the confidence interval. Specifically, **the percentile method proceeds as follows:**

PM–1. Use the Generalized Bootstrap GB (Steps GB–1 and GB–2) to obtain N estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$.

PM–2. Place the N estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$ from Step PM–1 in increasing numerical order, obtaining

$$\hat{\theta}_{(1)} \le \hat{\theta}_{(2)} \le \ldots \le \hat{\theta}_{(N)}, \tag{20.1.6}$$

the order statistics, where $\hat{\theta}_{(1)}$ is the smallest of $\hat{\theta}_{(1)} \ldots \hat{\theta}_{(N)}$, $\hat{\theta}_{(2)}$ is the second smallest of $\hat{\theta}_{(1)} \ldots \hat{\theta}_{(N)}$, etc., and $\hat{\theta}_{(N)}$ is the largest of $\hat{\theta}_{(1)} \ldots \hat{\theta}_{(N)}$.

PM–3. The percentile method's (approximate) $100(1-\alpha)\%$ confidence interval for θ is

$$(\hat{\theta}_{(a)}, \hat{\theta}_{(b)}) \tag{20.1.7}$$

where

$$a = \lfloor N\frac{\alpha}{2} \rfloor, \quad b = \lceil N(1-\frac{\alpha}{2}) \rceil. \tag{20.1.8}$$

(That is, the lower $100\alpha/2\%$, and upper $100\alpha/2\%$, of the samples are deleted. Note that $\lfloor \cdot \rfloor$ rounds down and $\lceil \cdot \rceil$ rounds up; e.g., $\lfloor 24.8 \rfloor = 24$, $\lceil 76.3 \rceil = 77$.)

Example: We have the $n = 60$ observations $X_1, X_2, \ldots, X_{60}$ given in Table 20.1–1. We wish to obtain a 95% (approximate) confidence interval for p, the probability that a value at least 7.30 will be found. (This data was considered, in another context, in Dudewicz and Karian (1999).)

Table 20.1–1. Data ($n = 60$ data points).

3.88	5.31	3.26	3.65	6.78	3.31	6.09	3.42	3.45	4.83
4.01	4.37	3.34	4.20	2.23	3.17	3.20	3.12	5.50	5.45
4.28	4.77	4.49	3.74	4.59	3.64	3.16	4.66	6.44	2.04
3.87	3.37	2.35	4.29	6.25	2.77	3.73	7.28	4.22	4.48
3.54	3.79	3.78	2.28	4.09	5.02	3.37	3.00	2.75	4.12
4.86	4.17	5.94	5.14	3.29	3.79	3.59	4.79	3.18	4.84

Using the Generalized Bootstrap, in Step GB–1 we fit a GLD using the methods of Chapter 2. From the data we find

$$\hat{\alpha}_1 = 4.1053, \quad \hat{\alpha}_2 = 1.2495, \quad \hat{\alpha}_3 = 0.6828, \quad \hat{\alpha}_4 = 3.2998$$

and the fit is

$$\lambda_1 = 3.2530, \quad \lambda_2 = 0.1549, \quad \lambda_3 = 0.04033, \quad \lambda_4 = 0.2059$$

or the fit

$$\text{GLD}(3.2530, 0.1549, 0.04033, 0.2059).$$

This fit, with a histogram of the $n = 60$ data points, is shown in Figure 20.1–2.

Following Step GB–2, we generate $N = 500$ random samples of size $n = 60$ from the fitted GLD $(3.2530, 0.1549, 0.04033, 0.2059)$. (For details of the random number generator used, see Karian and Dudewicz (1999, p. 133), where it is called

URN41. It is a generator with high-quality test results and an astronomic period 2^{62}. Also see Dudewicz and Karian (1999, p. 262).) Samples $1, 2, 3, 4, 5, 6,$

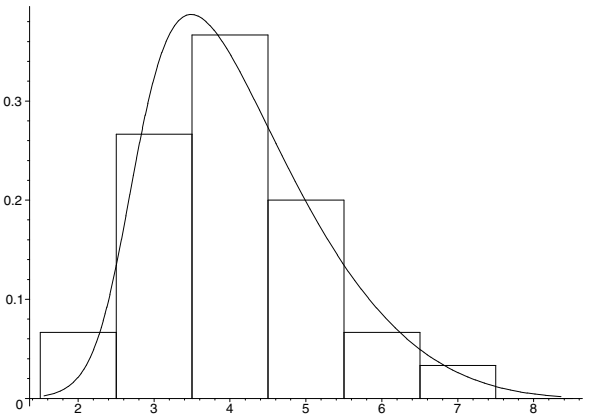


Figure 20.1–2. Data in the form of a histogram, and fitted GLD(3.2530, 0.1549, 0.04033, 0.2059).

and 500 are shown in Table 20.1–3. From these we calculate the estimates of p, probability of values at least 7.30, as

$$\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \hat{p}_4 = 0, \quad \hat{p}_5 = 0.0333, \quad \hat{p}_6 = 0, \ldots, \hat{p}_{500} = 0.$$

Note that in Table 20.1–3 values at or above 7.30 are highlighted for the reader's ease. The only such values, 8.15 and 7.76, are in the fifth sample, hence $\hat{p}_5 = 2/60 = .0333$.

The resulting $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{500}$ are shown in Table 20.1–4 where the entry in the i-th row and j-th column is $\hat{p}_{10(i-1)+j}$. The order statistics (20.1.6) of the entries of Table 20.1–4 are shown in Table 20.1–5; the latter are needed for the percentile method of obtaining an approximate confidence interval (Step PM–3).

For an approximate 95% confidence interval, we have

$$N\alpha/2 = 12.5 \quad \text{and} \quad N(1 - \alpha/2) = 487.5;$$

hence, our interval for p is

$$\left(\hat{p}_{(12)}, \hat{p}_{(488)}\right) = (0, 0.0333).$$

We know (see Dudewicz and Karian (1999, p. 262)) that the fitted GLD actually has $p = 0.0056$, so this interval covers the true p of the fitted GLD. (As the distribution from which the original data was drawn is not known, we are not in a position to conclude with certainty whether its p is also covered by the interval. This is the case with all "bootstrap" methods, which allow us to make inferences from data but cannot guarantee correctness of those inferences.)

Table 20.1–3. Samples 1 through 6 and 500, each of size 60, from the fitted GLD. Highlighted observations are at least 7.30.

$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$\cdots$	$i = 500$
3.30	3.02	6.03	5.14	2.73	3.67	$\cdots$	3.25
5.53	4.01	2.87	5.84	3.36	4.25	$\cdots$	4.25
2.90	3.73	5.13	5.56	6.52	3.74	$\cdots$	4.66
5.01	3.45	3.54	5.70	2.48	2.87	$\cdots$	6.08
5.78	2.73	2.84	3.49	3.91	5.91	$\cdots$	4.81
4.94	5.35	3.49	4.33	4.12	3.60	$\cdots$	5.20
3.56	4.29	4.42	4.16	3.99	2.47	$\cdots$	5.73
5.64	3.82	4.75	3.85	3.06	4.77	$\cdots$	4.52
5.15	2.66	2.65	3.62	3.78	5.43	$\cdots$	4.00
3.02	3.81	4.72	5.09	2.96	3.59	$\cdots$	4.91
3.48	4.34	6.29	3.71	3.17	4.31	$\cdots$	3.18
5.10	3.73	4.17	6.35	5.51	5.85	$\cdots$	2.51
4.33	3.21	1.99	4.66	6.94	4.51	$\cdots$	4.37
2.81	3.26	3.68	3.02	5.69	4.18	$\cdots$	3.28
2.36	2.21	3.01	5.26	4.37	4.42	$\cdots$	3.22
5.10	5.49	2.54	6.35	3.71	5.14	$\cdots$	5.65
4.81	4.13	4.33	3.08	3.78	2.82	$\cdots$	3.49
2.46	4.21	2.67	5.50	3.56	2.49	$\cdots$	4.21
5.11	4.77	3.20	2.44	2.84	4.95	$\cdots$	4.03
2.94	4.36	2.42	4.05	4.43	4.29	$\cdots$	3.48
5.29	4.75	3.95	2.30	4.49	3.52	$\cdots$	4.01
4.18	7.21	6.23	3.39	3.83	4.43	$\cdots$	3.78
2.65	4.13	3.56	3.86	3.59	2.83	$\cdots$	3.39
4.44	3.03	2.87	3.79	3.21	4.43	$\cdots$	4.31
5.44	3.64	4.09	3.42	5.13	1.84	$\cdots$	2.76
3.81	5.60	3.12	3.15	3.56	4.03	$\cdots$	4.81
2.77	6.59	4.49	6.19	5.21	3.46	$\cdots$	4.82
3.76	3.71	4.85	3.41	3.39	3.48	$\cdots$	6.15
3.76	4.11	3.17	2.57	3.05	3.03	$\cdots$	3.08
5.55	3.52	3.25	4.07	4.48	5.71	$\cdots$	4.30
3.12	4.35	2.98	3.52	3.78	3.75	$\cdots$	2.89
3.83	4.97	3.39	3.36	4.28	6.03	$\cdots$	5.79
4.99	4.64	4.47	3.72	4.16	3.16	$\cdots$	3.48
3.18	4.75	4.83	4.97	4.36	4.20	$\cdots$	3.47
2.70	4.46	4.11	5.73	**8.15**	3.42	$\cdots$	3.48
3.55	3.66	2.87	5.56	3.25	3.12	$\cdots$	4.94
2.70	5.07	4.95	4.33	6.48	5.27	$\cdots$	4.04
1.95	3.50	4.06	4.25	4.83	3.12	$\cdots$	3.15
2.99	2.93	4.98	3.67	3.22	4.77	$\cdots$	3.27
3.44	3.64	4.51	3.08	3.75	5.03	$\cdots$	4.19
2.98	6.56	2.97	6.17	3.68	5.20	$\cdots$	2.94
3.12	6.32	6.68	3.52	3.57	4.32	$\cdots$	4.83
3.78	3.88	6.24	4.27	3.27	4.05	$\cdots$	4.75
5.71	4.58	5.11	2.67	3.30	4.20	$\cdots$	5.25
4.84	3.26	4.58	3.60	3.28	5.91	$\cdots$	4.64
5.74	3.63	3.78	4.77	4.36	4.02	$\cdots$	4.02
3.45	3.37	6.48	5.02	4.79	5.73	$\cdots$	2.94
3.05	4.69	4.41	2.67	3.73	4.81	$\cdots$	3.25
3.55	6.11	4.30	3.86	5.44	3.97	$\cdots$	4.79
3.32	2.44	3.49	3.72	4.23	6.26	$\cdots$	3.30
4.47	4.93	2.23	3.99	4.31	6.34	$\cdots$	4.11
4.16	4.02	6.29	3.04	4.91	4.52	$\cdots$	5.64
2.34	2.51	4.64	5.02	3.72	3.86	$\cdots$	3.52
5.08	3.17	6.22	4.36	3.95	5.89	$\cdots$	5.13
2.95	4.91	3.52	3.49	3.43	4.16	$\cdots$	4.83
3.30	5.94	2.87	2.93	**7.76**	5.29	$\cdots$	4.15
3.22	3.12	2.42	4.07	3.22	4.81	$\cdots$	4.24
6.48	4.67	2.36	5.19	3.20	3.45	$\cdots$	3.67
4.50	3.62	2.27	6.10	4.07	3.39	$\cdots$	4.59
2.94	2.77	3.89	5.36	2.30	3.37	$\cdots$	3.85

Table 20.1–4. The 500 GB estimates $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{500}$.

0	0	0	0	.0333	0	.0333	0	0	0
0	0	0	.0500	0	.0167	0	0	.0500	0
.0167	.0167	0	0	.0167	0	0	0	0	.0167
0	0	0	0	.0167	0	0	.0333	0	.0167
0	.0167	0	.0333	.0167	0	.0167	0	.0167	.0167
.0167	.0167	0	0	.0167	.0167	.0333	0	0	0
0	0	.0167	.0667	0	.0167	.0167	0	.0167	.0167
.0333	.0333	.0167	.0167	0	.0167	0	.0167	0	0
0	.0167	0	0	0	0	0	0	0	0
0	.0333	.0167	0	.0167	.0167	0	0	0	0
0	0	0	.0667	.0500	.0167	0	0	0	.0500
.0333	0	0	0	0	.0167	0	.0167	.0500	0
0	.0167	0	0	.0500	0	0	.0167	0	0
.0333	0	0	0	0	0	0	.0167	0	.0167
0	0	0	0	.0167	0	.0167	0	0	0
0	0	0	0	0	0	.0167	.0500	.0167	0
.0167	.0167	0	0	0	.0333	0	.0167	0	0
0	0	0	0	0	0	.0167	0	0	0
.0167	0	0	0	.0333	.0333	0	0	0	.0167
.0167	0	0	0	0	0	0	.0167	0	0
0	.0333	.0333	0	.0167	.0500	0	0	.0167	.0167
.0167	0	0	0	0	.0333	0	.0333	0	0
.0333	.0333	.0167	0	.0167	0	0	0	0	0
.0167	0	.0333	.0167	.0167	.0333	.0333	.0333	0	0
0	0	.0167	.0167	0	.0333	.0333	.0167	0	.0333
0	.0333	0	0	0	.0167	0	.0167	0	0
0	.0167	0	.0333	0	0	0	0	0	.0167
0	.0167	0	.0167	0	.0167	0	0	.0333	0
.0167	0	0	0	.0167	0	0	.0167	.0333	0
0	0	0	0	.0167	.0167	0	0	.0167	0
.0167	.0333	0	.0333	.0167	0	0	0	0	0
0	0	0	.0167	.0167	0	.0167	.0333	.0167	0
0	0	.0333	0	.0167	.0167	.0333	.0167	.0167	0
.0167	.0167	.0167	.0167	.0167	0	.0167	0	0	0
0	0	0	0	.0167	.0167	0	.0167	0	0
0	0	0	0	.0333	0	0	.0167	0	.0167
0	0	0	.0333	0	.0333	.0167	.0167	0	.0333
.0333	0	.0167	0	.0167	0	.0167	0	.0333	0
0	.0167	.0167	.0167	0	.0333	0	0	0	.0167
.0167	.0167	0	.0333	0	0	0	.0167	0	0
0	0	.0167	.0167	.0333	.0167	.0333	0	0	0
0	.0167	.0167	0	0	0	0	.0333	0	0
0	.0167	0	.0167	0	.0167	0	.0167	.0167	0
.0167	0	0	.0333	.0167	.0167	0	0	0	0
0	.0167	.0167	0	0	0	0	0	0	.0167
0	0	.0167	.0167	0	.0167	0	.0500	0	0
0	.0167	0	0	.0167	0	0	.0167	0	0
.0167	0	.0167	.0167	0	.0167	0	.0167	0	0
.0167	0	0	.0167	.0167	0	0	.0167	0	0
0	0	0	.0167	0	0	.0167	.0167	0	0

Table 20.1–5. The order statistics $\hat{p}_{(1)} \le \hat{p}_{(2)} \le \cdots \le \hat{p}_{(500)}$.

0	0	0	0	0	0	0	0	0	0

The following theorem, discussed in Dudewicz (1992), implies that by proper use of the GB one will obtain better results than with the BM.

Theorem 20.1.9. *The Bootstrap Method is a special case of the Generalized Bootstrap (namely, the BM takes the empiric d.f. in Step GB–1).*

Sun and Müller-Schwarze (1996) compared the performances of the Bootstrap Method and the Generalized Bootstrap of this section (as well as another method — the jackknife). Focusing on a particular applied problem and dataset, they provided the necessary computer code in the C programming language and drew general implications. They reported two important conclusions.

1. The Generalized Bootstrap is more consistent in parameter estimation than is the Bootstrap Method.

2. The number of bootstrap samples N should be at least 500.

Asymptotic properties of GB have been shown by Lin (1997), who notes (p. 302) that "... some advantages of using [GB] ... over the ordinary bootstrap [BM] have been found ... ." For further discussion and examples, see Dudewicz (1992).

20.2 Comparisons of the GB and BM Methods

Since the BM has defects that are not generally well-known, in addition to the comments already made in Section 20.1, we detail here some of the recent results relevant to the comparison of GB and BM. These comparisons, in our opinion, make GB the method of choice when one intends to use MC methods.

In one study of distributional inference, Kroese (1994, pp. 41–42) stated, "A disadvantage of the Bootstrap procedure is that discrete distributions are generated." This is because the data is resampled by the BM, so one never sees any but the same data points one started with when the BM is used. Indeed, criticism of the BM has been growing in recent years (Young (1994)).

Meanwhile, recent theoretical work (e.g., Lin (1997)) has shown that methods like the GB, which beat the BM for small to moderate sample sizes, also give good asymptotic performance. In this light, it is not surprising that the most recent comprehensive book on bootstrap methods states, ". . . the generalized bootstrap. . . [is] a promising alternative to the. . . bootstrap since it has the advantage of taking account of the fact that the data are continuous but it does not seem to suffer the drawbacks. . . "(Chernick (1999, p. 109)).

Problems for Chapter 20

20.1 Suppose we are faced with the question: What is the probability that a value of a random variable X greater than 6.30 will be encountered? We have available a random sample of data of size 20, namely

3.88	4.01	4.28	3.87	3.54	4.86	5.31	4.37	4.77	3.37
3.79	4.17	3.26	3.34	4.49	2.35	3.78	5.94	3.65	4.20

(this is part of the data considered in Dudewicz and Karian (1999, p. 260) in another context). Use the GB method to answer this question. [Hint: Fit an EGLD to the data. Then either (a) find the tail probability above 6.30 in the fitted distribution, or (b) take a sample of size 10,000 from the fitted EGLD and estimate the desired probability by $Y/10{,}000$ where Y is the number of the 10,000 values that are 6.30 or larger.]

20.2 Use the BM to attempt to answer the question of Problem 20.1. [Hint: Sample 10,000 from the data at random with replacement, and estimate by $Z/10{,}000$ where Z is the number of the 10,000 values which are 6.30 or larger.]

20.3 In Problem 20.1, find an approximate 95% confidence interval for the desired probability p as follows. Do the sampling of 10,000 values 1,000 times. This yields 1,000 estimates of p, say $p_1, \ldots, p_{1000}$. Discard the 25 smallest of the p_is and also discard the 25 largest, leaving the middle 95% of the estimates. Claim this is a 95% confidence interval for p.

20.4 As in Problem 20.3, but using the BM.

20.5 Comparing the results of Problem 20.1 with those of Problem 20.2, and of Problem 20.3 with those of Problem 20.4, given an assessment of the GB in comparison with the BM.

References for Chapter 20

Chernick, M. R. (1999). *Bootstrap Methods, A Practitioner's Guide,* John Wiley & Sons, Inc., New York.

Dudewicz, E. J. (1992). "The Generalized Bootstrap," *Bootstrapping and Related Techniques* (K.-H. Jöckel, G. Rothe, and W. Sendler, eds.), V. 376 of Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, pp. 31–37.

Dudewicz, E. J. (Editor) (1997). *Modern Digital Simulation Methodology, III: Advances in Theory, Application, and Design–Electric Power Systems, Spare Parts Inventory, Purchase Interval and Incidence Modeling, Automobile Insurance Bonus-Malus Systems, Genetic Algorithms–DNA Sequence Assembly, Education, & Water Resources Case Studies,* American Sciences Press, Inc., Columbus, Ohio.

Dudewicz, E. J. (1999). "Basic statistical methods," Chapter 44, *Juran's Quality Handbook, Fifth Edition* (edited by J. M. Juran, A. Blanton Godfrey, R. E. Hoogstoel, and E. G. Schilling), McGraw-Hill, New York.

Dudewicz, E. J. and Karian, Z. A. (1999). "The role of statistics in IS/IT: Practical gains from mined data," *Information Systems Frontiers*, 1, 259–266.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, John Wiley & Sons, New York.

Golden, B. L., Assad, A. A., and Zanakis, S. H. (eds.) (1984). *Statistics and Optimization: The Interface*, American Sciences Press, Inc., Columbus, Ohio.

Golden, B. L. and Eiselt, H. A. (eds.) (1992). *Location Modeling in Practice: Applications (Site Location, Oil Field Generators, Emergency Facilities, Postal Boxes), Theory, and History*, American Sciences Press, Inc., Columbus, Ohio.

Karian, Z. A. and Dudewicz, E. J. (1991). *Modern Statistical, Systems, and GPSS Simulation* (First Edition), Computer Sciences Press/W. H. Freeman and Company, Publishers, New York. (Now available from American Sciences Press, Inc., Columbus, Ohio. For the Second Edition, see Karian and Dudewicz (1999).)

Karian, Z. A. and Dudewicz, E. J. (1999). *Modern Statistical, Systems, and GPSS Simulation*, Second Edition, CRC Press, Boca Raton, Florida.

Kroese, A. H. (1994). *Distributional Inference: A Loss Function Approach*, Ph.D. Thesis, University of Groningen, The Netherlands.

Lin, Y. (1997). "Asymptotics of bootstrapping mean on some smoothed empirical distribution," *Statistics & Decisions*, 15, 301–306.

Sun, L. and Müller-Schwarze, D. (1996). "Statistical resampling methods in biology: A case study in beaver dispersal patterns," *American Journal of Mathematical and Management Sciences*, 16, 463–502.

Young, G. A. (1994). "Bootstrap: More than a stab in the dark?," *Statistical Science*, 9, 382–415.

Chapter 21

The Generalized Bootstrap: A New Fitting Strategy and Simulation Study Showing Advantage over Bootstrap Percentile Methods[1]

Weixing Cai
Department of Mathematics
Syracuse University
Syracuse, New York 13244 USA

Edward J. Dudewicz
Department of Mathematics
Syracuse University
Syracuse, New York 13244 USA
dudewicz@syr.edu

Chernick (1999, pp. 54–55) notes that Efron and Tibshirani (1993) describe four methods for constructing approximate confidence intervals for a parameter. One of these methods, in fact perhaps the most popular method and the method covered in such excellent introductory works as that of Moore and McCabe (2003),

[1]This chapter appeared, under the same title, in the *American Journal of Mathematical and Management Sciences*, Vol. 27, pp. 275–318. Copyright © 2007 by the American Sciences Press, Inc., Syracuse, New York 13224-2104. Reprinted by permission.

is the **bootstrap percentile method**. The generalized bootstrap (GB) was introduced formally in Dudewicz (1992). It has as its essence fitting a suitable distribution to the available data, and then taking samples from the fitted distribution. The fitted GLD implementation of the GB chooses to fit a distribution F with an $\hat{F}$ from the Generalized Lambda Distribution family. Karian and Dudewicz (2000) introduce three methods of fitting as the GB. We combine these methods, which we describe with corresponding algorithms, into a fitting strategy. This strategy is then used in a simulation study. Through the simulation study, **we show that the GB has an advantage over the bootstrap** for the goal of covering the true value of the upper limit of a uniform distribution with empirical confidence intervals.

21.1 Introduction

In this chapter, we give up-to-date algorithms for generalized bootstrap (GB) methods, and for bootstrap methods (BM). The bootstrap resampling technique, which was first introduced by Efron (1979), gained widespread use because of its capability of investigating uncertainty of statistical quantities. When Moore and McCabe (2003, p. 428) discuss the bootstrap and use a percentile method, they say, "If we want a 95% confidence interval, for example, we use the middle 95% of this distribution."

The following is an **algorithm for the BM.**

BM–1. Suppose F is a d.f., and $\theta((F)$ is a function of F of interest to us.

BM–2. Form the empirical d.f. F_n of the data $X_1, X_2, \ldots, X_n$. Then from N successive random samples of size n, each taken with replacement from F_n, find estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$ of $\theta(F)$.

BM–3. A statistic of interest calculated from the distribution of $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$ is called a **bootstrap estimator** of $\theta(F)$.

For example, consider the problem of estimating θ when F is uniform on $(0, \theta)$, where $\theta = 8$. Consider $\{X_1, X_2. \ldots, X_n\} =$

$$\{0.776, 4.705, 3.333, 5.144, 6.311\, 0.443, 7.337, 4.726, 7.034, 7.119\},$$

a randomly generated sample of size $n = 10$. If we know F is uniform on $[0, \theta]$, we may wish to obtain a point estimate of θ by using its MLE, $\hat{\theta} = \max(X_1, X_2. \ldots, X_n) = 7.337$, or by an interval estimate by using the exact confidence interval which is available.

If we only know the point estimate and **we desire a confidence interval**, then the BM is often used. In fact, one of its attractions is that we need not know much statistics. Also, if one does not wish to rely on F being uniform on

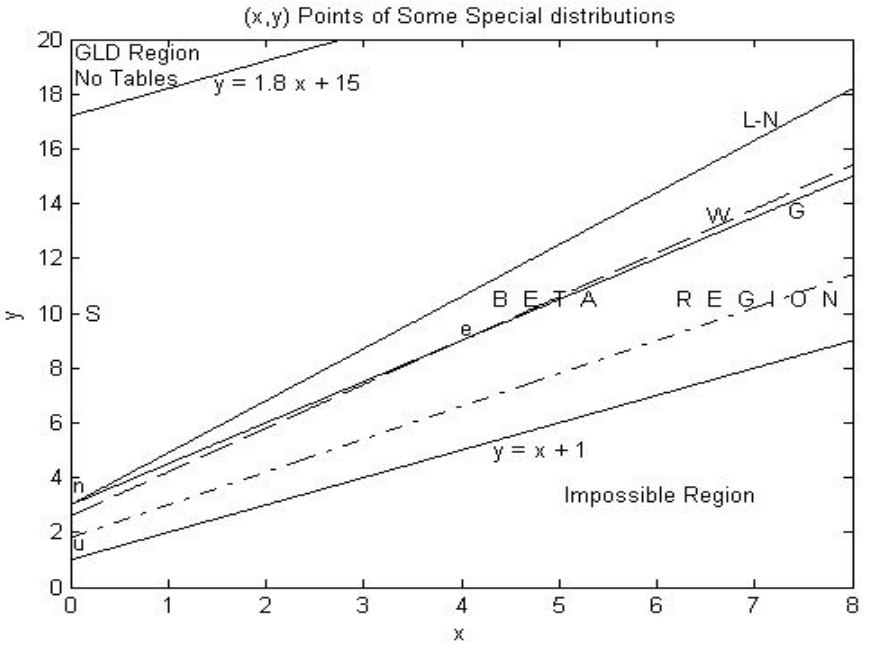


Figure 21.1–1 (Skewness squared (x), kurtosis (y)) of some distributions.

$(0, \theta)$, but simply any distribution with support $(0, \theta)$, we can use the BM, one then proceeds as follows:

BM–1: Generate $N = 40$ samples of size $n = 10$, and rearrange the MLEs in ascending order.

$$\bar{\theta}_1 = 7.0340, \qquad \bar{\theta}_2 = 7.1190 \qquad \bar{\theta}_{39} = 7.3370. \qquad \bar{\theta}_{40} = 7.3370.$$

BM–2: An empirical 95% CI for θ is then $[7.1190, 7.3370]$.

However, the true value of θ used to generate the data was $\theta = 8$. Hence, the BM produced a "CI," which does not cover the true value of θ.

The **generalized bootstrap** (GB) was introduced formally in Dudewicz (1992) and discussed earlier in Karian and Dudewicz (1991), Section 6.6. It has as its essence fitting a suitable distribution to the available data, and then taking samples from the fitted distribution. The fitted GLD implementation of the GB chooses to fit F with an $\hat{F}$ from the Generalized Lambda Distribution family, a broad family which can match any mean, any variance, and skewness and kurtosis in a very broad range (virtually any values if we use the Extended GLD form of the family). This is shown in Figure 21.1–1 (Figure 2.2–5 of Karian and Dudewicz (2000, p. 53)).

In Figure 21.1–1, $x = \alpha_3^2$ and $y = \alpha_4$, where α_3 and α_4 are the skewness and kurtosis of a distribution, respectively. The lines that are designated by "W," "L-N," "G," and "S" ($\alpha_3 = 0$) show the pairs for the Weibull, lognormal, gamma, and Students t distributions, respectively. The area designated by "BETA REGION"

shows the points that can be produced by the Beta distribution. "u" located at (0, 1.8) represents the uniform distribution; "n" located at (0, 3) means the normal distribution; and "e" located at (4, 9) stands for the exponential distribution. The region above the line $y = 1.8x + 15$ is able to be fitted but tables are not available. The region for the points of the GLD is the region above the line (the dot-and-dash line) that starts at the point labeled "u" on the vertical axis, and heads to the upper right.

Theorem 21.1.1. (Theorem 1.5.1 in Karian and Dudewicz (2000)): *If $Q(\cdot)$ is the percentile function of a random variable X, and U is a uniform random variable on $(0, 1)$, then $Q_X(U)$ has the same d.f. as does X.*

The GLD family is specified by its percentile function. If X is a GLD random variable with parameters λ_1, λ_2, λ_3, λ_4 (designated by GLD(λ_1, λ_2, λ_3, λ_4)), then its percentile function is

$$Q_X(y) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}.$$

For example, we use GLD$(0, 0.1975, 0.1349, 0.1349)$ to fit the standard normal distribution $N(0, 1)$ and use GLD$(0.5, 2, 1, 1)$ to fit the uniform distribution $U(0, 1)$.

An **algorithm for the Generalized Bootstrap (GB)** is given as follows:

GB–1. Suppose F is a d.f., and $\theta(F)$ is a function of F of interest to us.

GB–2. A random sample $X_1, \ldots, X_n$ is available from F and $\hat{\theta}(X_1, \ldots, X_n)$ is used to estimate $\theta(F)$.

GB–3. Estimate $F \in F_c$, where F_c is a class of d.f.s, by $\hat{F}$.

GB–4. Independently generate N random samples of size n from $\hat{F}$ and obtain $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$.

GB–5. Use $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ to estimate $\hat{\theta}$. For example, upper and lower $\alpha/2$ percentiles can be used as a $(1-\alpha)\%$ confidence interval (CI).

The following sections are organized as follows: In Section 21.2, we introduce three methods of fitting distributions for the generalized bootstrap (GB), with corresponding algorithms; in Section 21.3, through a simulation study, we show that the GB has an advantage over the BM in constructing a confidence interval (CI) by using bootstrap percentile methods to cover the upper limit parameter of a uniform distribution; in Section 21.4, six special examples are given to illustrate the process of applying the GB (with the three different methods given in Section 21.2) to construct empirical CIs to cover the true parameter; a summary is given in Section 21.5.

21.2 Algorithms for Three Methods of Distribution Fitting

21.2.1 Fitting the GLD through the Method of Moments (MOM)

$$\alpha_1 = \bar{X} = \sum_{i=1}^{n} x_i/n, \tag{21.2.1}$$

$$\alpha_2 = \hat{\sigma}^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n, \tag{21.2.2}$$

$$\alpha_3 = \sum_{i=1}^{n} (X_i - \bar{X})^3/(n\hat{\sigma}^3), \tag{21.2.3}$$

$$\alpha_4 = \sum_{i=1}^{n} (X_i - \bar{X})^4/(n\hat{\sigma}^4). \tag{21.2.4}$$

Algorithm GLD–M: Fitting a GLD distribution to data by the method of moments.

GLD–M–1. Use formulas (21.2.1) through (21.2.4) to compute $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, and $\hat{\alpha}_4$;

GLD–M–2. Find the entry point in the table of Appendix B of Karian and Dudewicz (2000) closest to $(|\hat{\alpha}_3|, \hat{\alpha}_4)$;

GLD–M–3. Using $(|\hat{\alpha}_3|, \hat{\alpha}_4)$ from Step GLD–M–2 extract $\lambda_1(0,1)$, $\lambda_2(0,1)$, λ_3, λ_4 from the table;

GLD–M–4. If $\hat{\alpha}_3 < 0$, interchange λ_3 and λ_4 and change the sign of $\lambda_1(0,1)$;

GLD–M–5. Compute $\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1$ and $\lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2}$.

21.2.2 EGLD: Method of Generalized Beta Distribution (MGBD)

Definition 21.2.5. If X is a beta random variable with parameters β_3 and β_4, for $\beta_2 > 0$, and any β_1, $Y = \beta_1 + \beta_2 X$ is said to have a Generalized Beta Distribution, GBD(β_1, β_2, β_3, β_4).

Theorem 21.2.6. The p.d.f. of the GBD(β_1, β_2, β_3, β_4) random variable is

$$f(x) = \begin{cases} \dfrac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2)^{\beta_4}}{\beta(\beta_3+1,\,\beta_4+1)\beta_2^{(\beta_4+\beta_4+1)}} & \text{for } \beta_1 \le x \le \beta_1+\beta_2 \\ 0, & \text{otherwise.} \end{cases} \tag{21.2.7}$$

Algorithm GBD–M: Fitting a GBD distribution to data by the method of moments.

GBD–M–1. Using (21.2.1) through (21.2.4) compute $\hat{\alpha}_1$, ; $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$;

GBD–M–2. Find the entry point in the table of Appendix C of Karian and Dudewicz (2000) closest to $(|\alpha_3|, \alpha_4)$;

GBD–M–3. Using $(|\alpha_3|, \alpha_4)$ from Step GBD–M–2 extract β_3 and β_4 from the table;

GBD–M–4. If $\hat{\alpha}_3 < 0$, interchange β_3 and β_4;

GBD–M–5. Substitute β_3 and β_4 (obtained in GBD–M–3 or GBD–M–4) and $\hat{\alpha}_2$, for β_3, β_4, and α_2 to compute β_2 from

$$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{\frac{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2}{(\beta_3 + 1)(\beta_4 + 1)}}; \tag{21.2.8}$$

GBD–M–6. Substitute β_2, β_3, and β_4 (obtained in GBD–M–3 or GBD–M–4 and GBD–M–5) and $\hat{\alpha}_1$, respectively, to compute β_1 from

$$\beta_1 = \hat{\alpha}_1 - \frac{\beta_2(\beta_3 + 1)}{\beta_3 + \beta_4 + 2}. \tag{21.2.9}$$

21.2.3 Method of Percentiles (MOP)

Denote by $\hat{\pi}_p$ the $(100p)$-th percentile of the data. We write $(n + 1)p$ as $r + a/b$, where r is a positive integer and a/b is a proper fraction, possibly zero. If $Y_1, Y_2, \ldots, Y_n$ are the order statistics of a given dataset $X_1, X_2, \ldots, X_n$, then $\hat{\pi}_p$ can be obtained from

$$\hat{\pi}_p = Y_r + \frac{a}{b}\left(Y_{r+1} - Y_r\right). \tag{21.2.10}$$

Define

- $\hat{\rho}_1$ as the sample median, i.e.,

$$\hat{\rho}_1 = \hat{\pi}_{0.5}; \tag{21.2.11}$$

- $\hat{\rho}_2$ as the inter-decile range, the range between the 10th percentile and 90th percentile, i.e.,

$$\hat{\rho}_2 = \hat{\pi}_{0.9} - \hat{\pi}_{0.1}; \tag{21.2.12}$$

- $\hat{\rho}_3$ as the left-right tail-weight ratio, i.e.,

$$\hat{\rho}_3 = \frac{\hat{\pi}_{0.5} - \hat{\pi}_{0.1}}{\hat{\pi}_{0.9} - \hat{\pi}_{0.5}}; \tag{21.2.13}$$

- $\hat{\rho}_4$ as the tail-weight factor or the ratio of the inter-quartile range to the inter-decile range, i.e.,

$$\rho_4 = \frac{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}{\hat{\rho}_2}. \tag{21.2.14}$$

Algorithm GLD–P: Fitting a GLD distribution to data, percentile method.

GLD–P–1. Use (21.2.11) through (21.2.14) to compute $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$;

GLD–P–2. Find the entry point in one or more of the tables of Appendix D of Karian and Dudewicz (2000) closest to $(\hat{\rho}_3, \hat{\rho}_3)$; if $\hat{\rho}_3 > 1$, use $(1/\hat{\rho}_3, \hat{\rho}_4)$ instead of $\hat{\rho}_3$, $\hat{\rho}_4$;

GLD–P–3. Using the entry point from Step GLD–P–2, extract $\hat{\lambda}_3$ and $\hat{\lambda}_4$; if $\hat{\rho}_3 > 1$, interchange $\hat{\lambda}_3$ and $\hat{\lambda}_4$;

GLD–P–4. Use $\hat{\lambda}_3$ for λ_3 and $\hat{\lambda}_4$ for λ_4 to compute $\hat{\lambda}_2$ from

$$\hat{\lambda}_2 = \frac{0.9^{\lambda_3} - 0.1^{\lambda_4} + 0.9^{\lambda_4} - 0.1^{\lambda_3}}{\hat{\rho}_2}. \tag{21.2.15}$$

GLD–P–5. Use $\hat{\lambda}_2$ for λ_2, $\hat{\lambda}_3$ for λ_3 and $\hat{\lambda}_4$ for λ_4, to compute $\hat{\lambda}_1$ from

$$\hat{\lambda}_1 = \hat{\rho}_1 = \frac{(1/2)^{\lambda_3} - (1/2)^{\lambda_4}}{\lambda_2}. \tag{21.2.16}$$

Note: The heading of Table D–2 (pp. 382, Karian and Dudewicz (2000)) says $\lambda_3 < 0$, $\lambda_4 < 0$. If e.g., we have (our sample no. 8 on page 860, Second Option) $\rho_4 = 0.598$ and $\rho_3 = 0.68$, then the tabled values are $\lambda_3 = 3.06435$, $\lambda_4 = 1.54108$. The λ_3 and λ_4 should not be < 0 (negative), but positive. This also would be in agreement with Example 1 on pp. 166–167 of Karian and Dudewicz (2000), where there are no minus signs.

21.3 Fitting Strategy and Simulation Study

We generate 40 samples with sample size 10 from the uniform distribution $X \sim \text{Uniform}(0, b)$, where $b = 8$. The details of the results of the 40 original samples are listed in Appendix A.1. From each 40 original samples, we use resampling techniques to generate 40 new samples, and get 40 MLEs for the parameter b from the 40 new samples. We list 40 MLEs based on new samples in increasing order, so that we can construct the empirical confidence interval and see whether $b = 8$, is covered by it. We will compare the BM and GB by using these procedures.

The random number generator we use to generate original samples and parts of resamples is called URN41 in Karian and Dudewicz ((1999), p. 133). It is

a linear congruential generator $x_{i+1} = ax_i + c \pmod m$ with $m = 2^{64}$, $a = 39468974943$, $x_0 = 123456789$, $c = 0$.

The probability of getting the upper limit $b = 8$ from a sample of size 10 is 0. As one may expect, no observation is equal to 8 in 40 original samples. Because the observations in the resample randomly come from the original sample with replacement, the estimate of the parameter upper limit will not be equal to (in fact it will be less than) 8 in each resample. Based on each of the 40 original samples, the 95% empirical confidence interval obtained by trimming the smallest and the largest estimates in 40 ordered estimates, will NOT cover $b = 8$ either.

Fitting Strategy for GB: The fitting for the GB proceeds in the following order:

1. MOP (Method of Percentiles). (If there is more than one choice for the parameters, we follow in this order T1, T2, T3, T4, and T5 (as defined on page 357 in Karian and Dudewicz (2000)) to check the corresponding parameter values.
2. MOM (Method of moments).
3. MGBD (Method of Generalized Beta Distribution, also called EGLD in Karian and Dudewicz (2000)).

The reason to consider the MOP as the first choice is that the MOP has been shown to be superior to the MOM or the MGBD in Karian and Dudewicz (2003). The reason to consider MGBD as the third choice is that MOM was proposed earliest and proved to be a good fitting method, and that MGBD can be applied to a wider region (see Figure 21.1–1), i.e., $\hat{\alpha}_3^2 < \hat{\alpha}_4 < 1.8(\hat{\alpha}_3^2 + 1)$, where we can not use MOM.

The MOP is chosen when $0 < \hat{\rho}_4 < 0.625$. If $\hat{\rho}_4 > 0.625$, then we consider the other two methods. If $1.8(\hat{\alpha}_3^2+1) \le \hat{\alpha}_4 \le 1.8\hat{\alpha}_3^2+15$, we use the MOM; otherwise if $\hat{\alpha}_3^2 < \hat{\alpha}_4 < 1.8(\hat{\alpha}_3^2 + 1)$, we use the MGBD. If $\hat{\alpha}_4 > 1.8\hat{\alpha}_3^2 + 15$, we are unable to fit the data. Recall that $\{(\hat{\alpha}_3^2, \hat{\alpha}_4) : \hat{\alpha}_4 \le 1 + \hat{\alpha}_3^2\}$ is the Impossible Region, as shown in Figure 21.1–1.

The third and fourth moments of the uniform distribution $U(0, 8)$ are $\alpha_3 = 0$ and $\alpha_4 = 9/5$. The values of ρ_3 and ρ_4 for the uniform distribution $U(0, 8)$ are $\rho_3 = 1$ and $\rho_4 = 5/8$. In our simulation results, there are no samples that we are unable to fit among the 40 original samples. Figure 21.3–1 shows 40 sample points $(\hat{\rho}_3, \hat{\rho}_4)$, Figure 21.3–2 shows 40 sample $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ points based on the 40 original samples and Figure 21.3–3 shows the $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ sample points with sample 5 excluded.

Below we give the sample numbers for each type of fit.

MOP:	5 8 10 13 15 20 25 28 30 31 32 37 38 39 40
MGBD:	1 2 3 4 6 7 9 11 14 16 17 18 21 22 23 24 26 27 29 33 34 35 36
MOM:	12 19

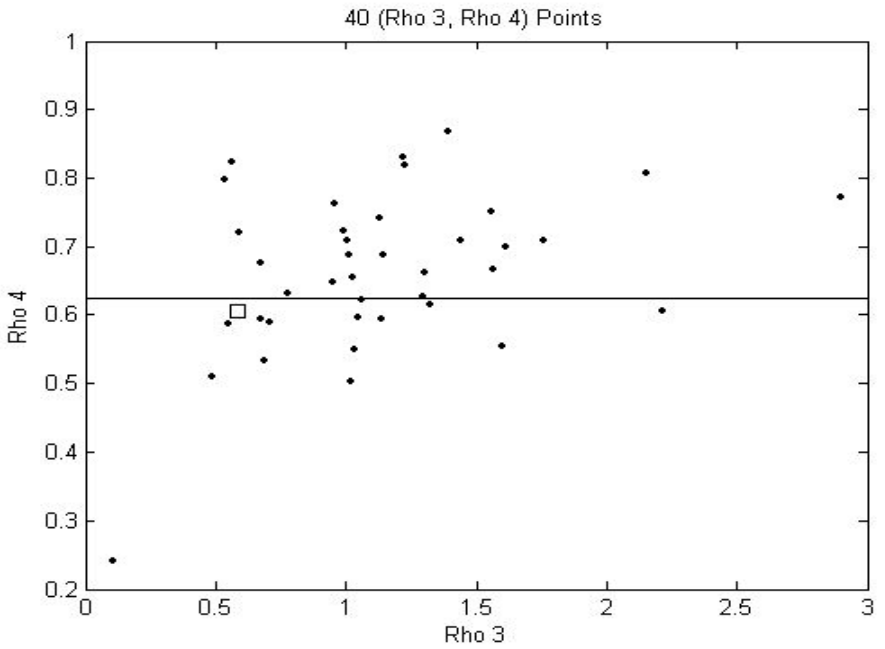


Figure 21.3–1. The $(\hat{\rho}_3, \hat{\rho}_4)$ points. Note the point designated by a square is based on sample 5.

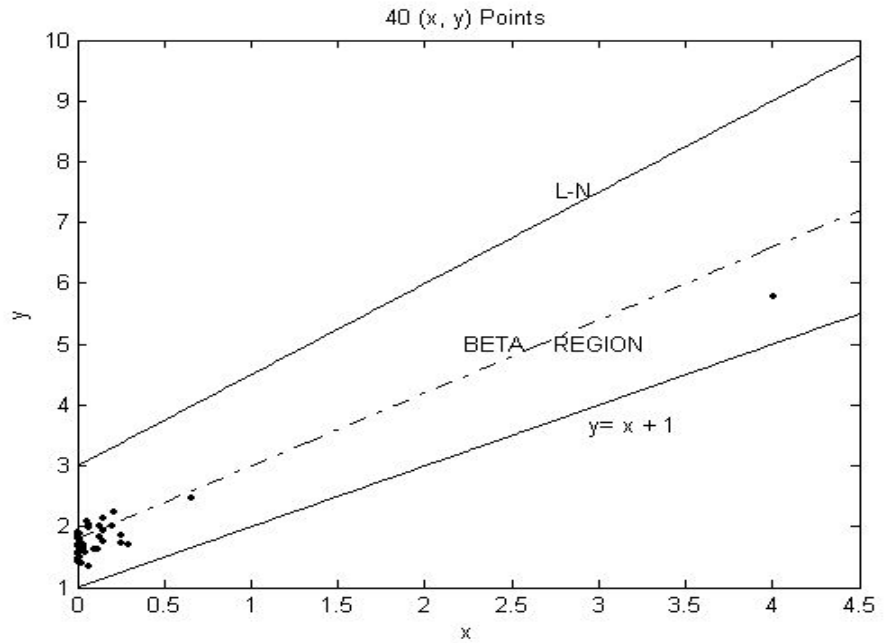


Figure 21.3–2. The $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ points; L-N designates the lognormal.

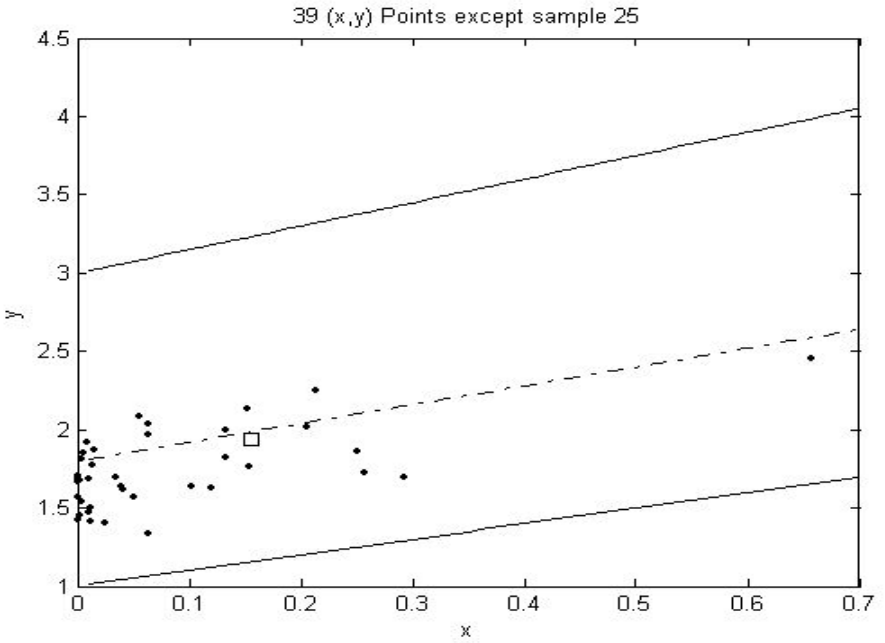


Figure 21.3–3. The $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ points, excluding the point of the fifth sample.

In the following summary we indicate whether the confidence intervals associated with a fit cover the parameter (a "Y" indicating that it does and an "N" indicating that it does not).

MOP:	5Y 8Y 10Y 13Y 15Y 20Y 25Y 28N 30Y 31N 32Y 37Y 38Y 39Y 40Y
MGBD:	1N 2N 3N 4N 6N 7N 9N 11N 14N 16N 17N 18N 21Y 22N 23N 24Y 26N 27N 29N 33N 34N 35N 36N
MOM:	12N 19Y

We can also summarize the results as follows:

1, 2, 3, 4	GBD	N	
5	MOP	Y	
6	GBD	N	
7	GBD	N	
8	MOP	Y	(The first choice is No, the second is Yes)
9	GBD	N	
10	MOP	Y	
11	GBD	N	
12	MOM	N	
13	MOP	Y	

14	GBD	N	
15	MOP	Y	
16,17,18	GBD	N	
19	MOM	Y	
20	MOP	Y	
21	GBD	Y	
22, 23	GBD	N	
24	GBD	Y	
25	MOP	Y	
26, 27	GBD	N	
28	MOP	N	(All three choices are No)
29	GBD	N	
30	MOP	Y	(The first choice: No; the second and third choices are both Yes)
31	MOP	N	(All four choices are No)
32	MOP	Y	
33,34,35,36	GBD	N	
37,38,39,40	MOP	Y	

We conclude that when GB is used, $b = 8$ is covered in 16 cases in the total 40 cases. However, when BM is used, $b = 8$ is not covered in any of the 40 cases.

21.4 Specific Examples

We will give examples in detail for the following 6 settings:

Sample	Method of Fit	Covered Yes/No
1	GB	No
5	MOP	Yes
12	MOM	No
19	MOM	Yes
21	GBD	Yes
28	MOP	No

This will give details for the six situations which can occur from the three methods and two possible results. In each situation, we point out which method (MOP, MOM, or MGBD) to use and whether the empirical confidence interval covers $b = 8$ or not. The details for the fitting process are given for each situation. Details for all samples are given in Appendix A.3 at the end of this chapter.

21.4.1 Sample 1: GBD, Not Covered

From the original sample of

2.1132, 1.2999, 7.8368, 5.1515, 3.7352, 1.5031, 7.1337, 6.3546, 2.0321, 3.7062

we have

$$\hat{\alpha}_1 = 4.0866\, \hat{\alpha}_2 = 5.2450\, \hat{\alpha}_3 = 0.3210\, \hat{\alpha}_4 = 1.6344.$$

To use the tables, we set $\hat{\alpha}_3 = 0.325$, $\hat{\alpha}_4 = 1.63 and get \beta_3 = -0.5469$, $\beta_4 = -0.3767$. Thus,

$$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1)}$$

and

$$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_4 + 2)$$

yield

$$\beta_2 = 6.6843 \quad \text{and} \quad \beta_1 = 1.2729.$$

The `RSGBD` MATLAB® program listed in Appendix A.2 of this chapter. When `RSGBD` is invoked through the command,

```
RSGBD(1.2729, 6.6843, -0.3767, -0.5469)
```

it provides the following ordered estimates, each based on a sample of size 10.

7.3955	7.6244	7.6931	7.7409	7.7773	7.7996	7.8337	7.8448
7.8731	7.8755	7.9035	7.9122	7.9182	7.9195	7.9233	7.9239
7.9303	7.9308	7.9316	7.9363	7.9415	7.9449	7.9457	7.9480
7.9493	7.9529	7.9538	7.9541	7.9553	7.9557	7.9560	7.9562
7.9563	7.9566	7.9567	7.9568	7.9570	7.9570	7.9571	7.9572

The 95% empirical confidence interval of [7.6244, 7.9572] does not contain the value of the parameter, $b = 8$. Table 21.4–1 gives a listing of each of the 40 samples of size 10.

21.4.2 Sample 5: MOP, Covered

From the original sample of

4.4071 5.3312 6.8055 3.1827 0.7040 1.5343 2.1024 0.3859 3.0009 7.7374

we have

$$\hat{\rho}_1 = 3.0918\, \hat{\rho}_2 = 7.2266\, \hat{\rho}_3 = 0.5874\, \hat{\rho}_4 = 0.6051.$$

We use the tables, with $\hat{\rho}_3 = 0.58$, $\hat{\alpha}_4 = 0.610$ to obtain $\lambda_3 = 1.65593$, $\lambda_4 = -0.47138$. Thus,

$$\lambda_2 = \left[(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}\right]/\hat{\rho}_2 \quad \text{with} \quad u = 0.1$$

and

$$\lambda_1 = \hat{\rho}_1 - \left[(0.5)^{\lambda_3} - (0.5)^{\lambda_4}\right]/\lambda_2$$

Table 21.4–1. Details of 40 samples: each row is a sample of size 10.

7.6836	4.4695	3.9207	7.9566	5.0676	7.7703	3.3424	1.4886	5.5637	2.6305
2.0636	2.6077	6.3431	2.9812	6.1406	2.9876	7.9427	3.2615	6.9997	7.9529
4.2932	6.3170	4.3210	5.3851	6.9938	7.9233	4.6102	4.4488	1.6148	4.1859
5.7448	2.9252	4.7268	6.5913	4.8541	5.4578	2.5513	7.9570	6.2373	2.2033
7.9553	4.5331	7.9034	7.9185	6.7317	1.7607	5.1149	3.7672	2.1561	5.0386
4.7685	4.6919	3.9460	6.6688	7.2377	1.5596	5.4595	4.4816	7.7065	7.7996
1.3040	7.5594	6.0309	5.0906	7.9563	7.6346	6.8103	7.1481	1.4693	7.8398
6.0742	7.9303	6.8971	6.9638	7.5794	7.8759	3.2613	6.9807	1.3006	2.7656
3.0655	1.6881	2.7010	7.0315	7.8731	2.5019	3.2924	1.6774	5.7830	1.6502
1.3357	7.8927	5.7557	7.3716	4.8175	7.4829	6.8144	7.1104	2.4295	7.9449
7.0362	5.3274	5.6757	7.9035	1.4539	1.3105	4.8460	7.2521	1.3564	5.0375
6.1751	1.3040	2.7945	7.9053	3.9893	1.3154	1.8773	6.9925	7.2394	7.9239
7.9570	4.8515	7.9127	2.4497	1.5032	3.6450	7.2238	1.286	3.3219	1.2971
5.1606	6.7007	6.6324	5.7979	5.4631	5.0447	2.3891	7.9572	6.8899	4.6550
4.7904	7.9323	6.1230	1.3719	6.5171	7.9457	7.1303	1.8540	6.2218	6.1162
7.8552	6.9311	7.9567	2.3058	5.6012	7.7712	4.4894	7.6434	6.2131	4.9189
7.9560	5.7029	2.0256	7.4656	1.5901	7.9378	3.6786	1.4655	5.5278	7.5905
5.0944	7.0400	7.9182	6.1647	7.4307	6.1955	6.6692	2.6655	7.1811	1.9591
3.7080	5.8502	3.9942	1.2884	2.9910	5.5828	1.7858	1.5016	2.1795	7.8755
2.5695	6.1142	3.5802	1.4618	7.5738	7.6931	3.2454	1.2787	6.8749	4.5341
6.8668	6.4903	3.2822	6.7479	7.0474	3.3367	1.3368	1.9176	4.9360	7.6244
7.7525	5.4451	1.2794	7.9541	7.9259	1.4945	7.1728	7.7375	7.5771	1.6987
7.9415	1.5783	7.5407	3.8057	7.9369	7.4043	6.5515	6.8371	5.9437	2.1551
6.9412	7.3955	2.1702	3.7895	5.2208	2.0346	4.5010	6.4777	2.0505	3.8239
3.5207	7.5872	7.6568	1.3090	2.5680	2.8118	3.6096	4.2678	7.9308	1.2843
4.1570	1.3220	7.7475	5.9051	4.7870	1.2805	1.4642	7.9122	7.3518	7.4957
7.8337	4.6579	4.5631	3.4276	4.2437	5.0948	2.7972	4.7461	3.1080	3.2057
5.8730	4.8081	7.7857	5.3602	3.9050	6.3936	3.8941	7.8448	7.1612	5.3974
4.2839	4.2306	7.6864	7.9363	7.7338	1.3011	7.1600	7.6476	7.6397	5.1345
7.9121	2.3941	3.8618	3.3585	7.3942	7.9568	1.4074	7.5361	4.7145	4.6390
7.9480	5.9714	7.3178	4.0415	7.2407	5.8313	1.3972	1.7637	4.7834	6.9831
7.9571	6.0915	6.7212	6.5187	4.8272	7.8474	1.3562	7.9346	7.3183	7.8578
7.6701	6.5855	7.9195	1.2793	2.0127	7.4854	7.8718	1.6972	7.3344	5.9806
7.5598	7.1445	1.8917	1.3683	1.8892	4.5594	7.7734	1.7289	6.2150	7.7773
7.9562	6.1276	1.4942	7.6186	7.3374	1.3530	5.7516	3.0952	1.3086	7.6484
7.9043	1.3430	4.2992	7.8243	7.8451	3.2682	7.9316	1.4384	2.3742	4.3102
5.1228	1.3726	5.9611	7.7409	1.9223	1.7275	6.8015	7.3886	6.8244	7.5511
7.3455	7.9538	7.7729	2.4962	1.4900	3.3163	6.0683	7.8073	7.4150	6.3317
4.6282	1.8162	7.2092	7.9493	5.7974	7.7546	2.7451	7.2257	7.9244	7.5407
1.5408	6.2091	6.1343	5.0984	7.9557	5.3560	2.6885	1.2962	1.9075	2.4791

yield

$$\lambda_2 = 0.1981 \quad \text{and} \quad \lambda_1 = 5.1309.$$

The `RSGLD` MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(5.1309, 0.1981, 1.65593, 0.47138),
```

provides the following ordered estimates, each based on a sample of size 10.

3.2767	5.3111	5.5372	6.1709	6.3080	6.3170	6.4732	7.0756
7.2027	7.5058	7.5553	7.5966	7.6779	7.7046	8.0603	8.0941
8.1158	8.1452	8.2496	8.5057	8.5633	8.7910	8.8336	8.8990
8.9374	8.9600	9.0843	9.1591	9.1775	9.2035	9.2109	9.2300
9.2923	9.4734	9.5417	9.6053	9.7972	9.8553	9.9250	9.9894

The 95% empirical confidence interval of [5.3111, 9.9250] covers the value of the parameter, $b = 8$. Table 21.4–2 gives a listing of each of the 40 samples of size 10. Based on these 40 resamples, we calculate $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ and $(\hat{\rho}_3, \hat{\rho}_4)$ of each resample and obtain the plots that are shown in Figure 21.4–3 and Figure 21.4–4.

21.4.3 Sample 12: MOM, Not Covered

From the original sample of

6.8524 3.6359 7.7509 2.9874 0.1493 2.8390 0.0477 3.1906 7.4462 4.0421

we have

$$\hat{\alpha}_1 = 3.8941\ \hat{\alpha}_2 = 6.7580\ \hat{\alpha}_3 = 0.09316\ \hat{\alpha}_4 = 1.9153$$

We use the tables, with $\hat{\alpha}_3 = 0.1$, $\hat{\alpha}_4 = 1.9$ to get $\lambda_1(0,1) = -1.5049$, $\lambda_2(0,1) = 0.2911$, $\lambda_3 = 0.01562$, $\lambda_4 = 0.8296$. Thus,

$$\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1$$

and

$$\lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2},$$

yield

$$\lambda_1 = -0.181 \quad \text{and} \quad \lambda_2 = 0.112.$$

The `RSGLD` Matlab program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(-1.5049, 0.112, 0.01562, 0.8296),
```

provides the following ordered estimates, each based on a sample of size 10.

2.4840	4.2892	4.4739	4.9776	5.0841	5.0911	5.2111	5.6629
5.7559	5.9741	6.0093	6.0385	6.0957	6.1144	6.3592	6.3820
6.3966	6.4163	6.4858	6.6522	6.6887	6.8298	6.8555	6.8945
6.9171	6.9303	7.0015	7.0430	7.0531	7.0672	7.0712	7.0815
7.1144	7.2049	7.2367	7.2649	7.3408	7.3604	7.3813	7.3977

Table 21.4–2. Details for 40 resamples: each row contains a sample of size 10.

1.3194	0.7368	9.2035	4.4638	2.8084	0.8714	7.5359	6.1832	1.2561	2.7780
3.5776	0.9089	6.1427	9.4734	8.8708	6.8555	0.3473	5.8082	0.5665	5.5864
0.5075	6.1216	0.2041	7.6009	7.7046	0.2943	5.0906	1.5269	2.5829	3.9243
0.5030	8.4732	5.3670	2.0115	1.3977	6.8826	5.8049	2.2014	0.8752	9.8553
3.5503	4.6999	6.9334	2.2536	0.3876	0.8927	1.3109	0.2325	2.0817	8.8990
2.4652	8.5110	2.3650	9.6162	1.8056	0.4347	0.9345	1.3434	9.9894	3.0866
5.4801	6.1709	2.3574	0.6724	0.1495	3.7179	3.7617	0.2396	0.2784	3.9624
2.9659	0.9415	6.8924	7.2027	0.6613	4.0294	2.1549	2.2312	3.6324	1.2398
4.7596	0.7774	5.3111	0.6821	5.1672	0.9143	4.3603	1.5659	3.0587	2.8436
1.5251	5.0937	0.1810	9.7972	5.1562	1.9191	1.6900	5.9034	7.2130	1.2518
1.9524	6.2205	6.0326	2.2988	0.6467	0.3249	2.6226	6.3080	4.8650	4.0006
7.0160	2.7050	8.9374	2.0692	0.1345	1.9332	0.0982	2.2611	8.1797	3.1380
6.1674	4.7836	3.9501	7.7560	1.7577	0.9360	0.0956	3.5984	4.1919	9.0843
0.7607	2.7211	4.5730	0.0839	0.3513	4.0874	2.5987	1.3932	5.5372	0.1016
9.1775	6.2704	0.6906	4.6067	1.3293	3.7923	1.6565	3.6167	1.5706	4.0391
1.0969	5.1593	0.3959	7.3179	7.5058	0.4778	4.2905	2.2912	1.5641	5.7493
7.8952	0.8187	2.7839	4.6274	8.5633	0.9579	2.0890	6.3838	5.5029	2.9098
0.1974	0.9966	1.3914	0.0938	2.6705	7.0756	3.6599	5.9996	4.1430	5.7728
4.0171	6.3206	3.3093	7.8514	2.1696	0.3564	0.8646	1.6318	8.0603	4.0777
4.0075	8.7910	1.0006	2.1276	5.7695	7.1036	1.2201	2.3215	4.7701	0.1180
0.1875	4.8680	1.7919	2.3748	3.7524	0.9800	5.6062	0.2906	7.0577	9.2923
7.8149	8.8336	7.6119	0.1043	6.5341	0.5043	4.8863	1.4436	2.9491	3.2244
1.0569	6.3711	7.5966	0.7519	3.1614	3.8675	0.2673	0.2981	3.4961	4.1165
8.9600	1.0302	1.7713	7.4354	4.3403	4.1848	7.1987	2.3519	0.2177	1.3372
0.7054	0.8608	0.9729	0.7621	0.9020	1.0022	0.5209	1.6717	9.2300	2.9766
6.0083	5.2697	3.2480	0.1152	0.8966	2.0331	6.3170	6.0970	1.9312	1.3496
6.8112	6.4220	1.3355	2.4209	4.3104	0.3888	8.1452	6.5119	0.7556	4.1555
1.8333	2.8353	2.6947	2.1865	3.1184	2.0475	3.0079	2.3825	2.3880	3.2767
1.4039	4.9533	0.3434	8.0148	6.4440	1.0246	2.9995	3.7457	0.4463	9.5417
4.5423	2.8518	0.6180	0.1266	3.6691	4.2438	8.0941	1.4947	1.0855	0.1473
2.7757	6.4732	4.4004	4.7731	5.6410	3.9433	6.2543	3.7643	6.0936	4.1946
5.2002	5.3366	3.7710	7.5553	2.0744	0.5494	0.4898	2.4132	6.0814	5.9393
2.2590	0.9035	8.5057	4.7941	2.5513	1.0472	7.0717	6.5898	1.0683	3.0441
3.2917	1.0871	5.7577	0.1383	8.2496	7.3034	0.2281	6.1956	0.4251	5.9634
0.3714	6.5248	0.1062	8.1158	7.2273	0.4220	4.7492	1.7392	2.3345	4.2340
0.3673	9.1591	5.0148	2.2461	1.2022	7.3325	5.4347	2.4441	0.7104	0.1659
3.2655	5.0394	6.5082	2.4986	0.2637	1.0699	1.1201	0.3524	1.8541	9.9250
2.2216	9.2109	2.1255	0.1503	1.5903	0.5771	0.7657	1.5464	8.9150	3.3647
5.1234	6.5767	2.1182	0.8344	9.6053	4.0198	3.4691	0.3604	0.1681	4.2735
2.7027	1.1217	6.4696	7.6779	0.5120	4.3431	1.9241	2.4752	3.3445	1.4372

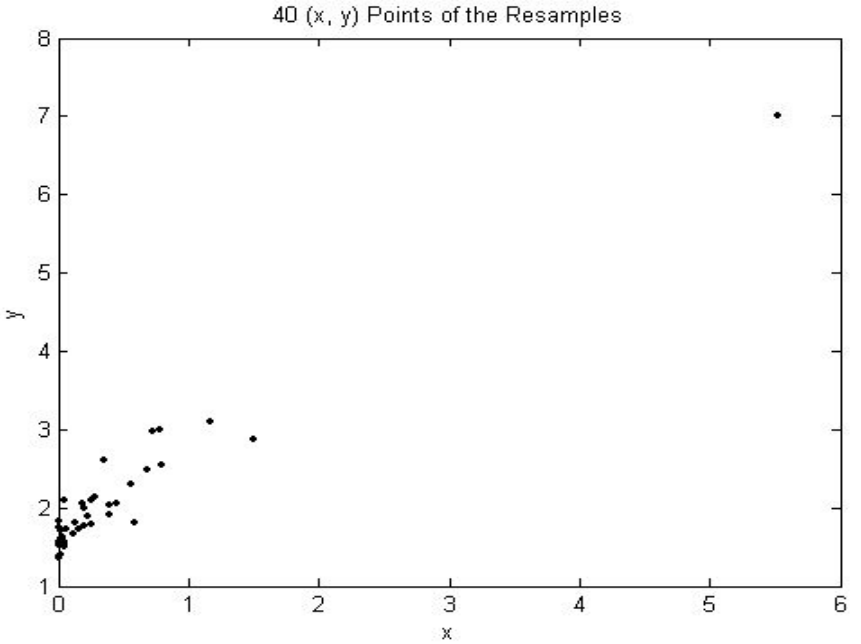


Figure 21.4–3. The $(\hat{\alpha}_3^2, \hat{\alpha}_4)$ points of the 40 samples.

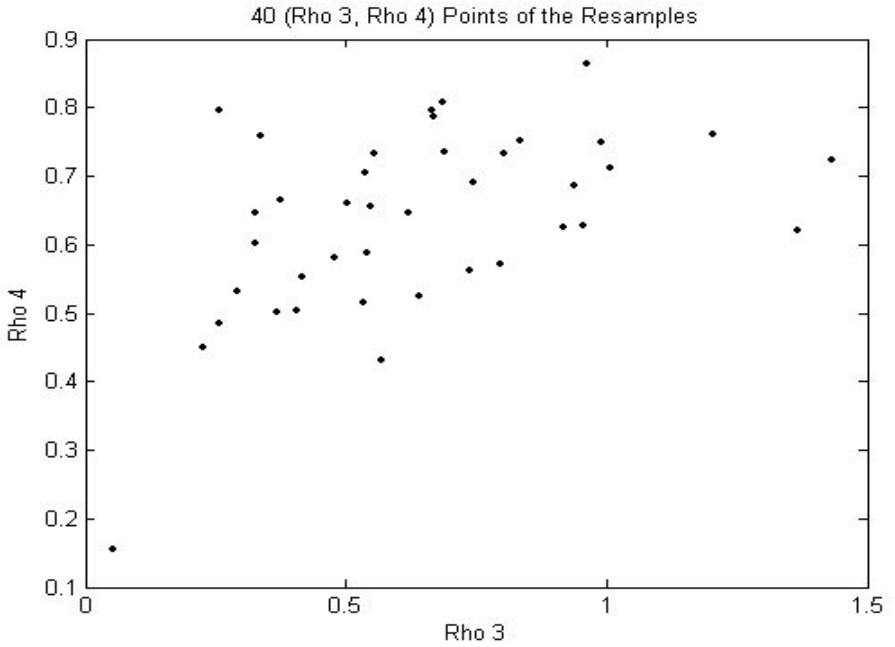


Figure 21.4–4. The $(\hat{\rho}_3, \hat{\rho}_4)$ points of the 40 samples.

The 95% empirical confidence interval of [4.2892, 7.3813] does not cover the value of the parameter, $b = 8$. Table 21.4–5 gives a listing of each of the 40 samples of size 10.

21.4.4 Sample 19: MOM, Covered

From the original sample of

4.7977 6.4406 4.1962 7.2921 3.0946 0.6437 1.4930 2.4952 7.3915 4.8468

we have

$$\hat{\alpha}_1 = 4.2691\ \hat{\alpha}_2 = 4.9598\ \hat{\alpha}_3 = -0.0689\ \hat{\alpha}_4 = 1.8087.$$

We use the tables, with $\hat{\alpha}_3 = 0.05$, $\hat{\alpha}_4 = 1.8$ to get $\lambda_1(0,1) = 1.4668$, $\lambda_2(0,1) = 0.2996$, $\lambda_3 = 0.8558$, $\lambda_4 = 0.02295$. (Note that since $\hat{\alpha}_3 < 0$, we need to interchange λ_3 and λ_4 and change the sign of $\lambda_1(0,1)$.) Thus,

$$\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1$$

and

$$\lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2},$$

yield

$$\lambda_1 = 7.5358 \quad \text{and} \quad \lambda_2 = 0.1343.$$

The `RSGLD` MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(7.5358, 0.1343, 0.8558, 0.02295),
```

provides the following ordered estimates, each based on a sample of size 10.

4.4755	5.9384	6.0806	6.4622	6.5417	6.5469	6.6361	6.9683
7.0361	7.1949	7.2204	7.2417	7.2833	7.2969	7.4758	7.4926
7.5034	7.5180	7.5695	7.6950	7.7231	7.8345	7.8554	7.8877
7.9067	7.9180	7.9805	8.0188	8.0283	8.0419	8.0458	8.0558
8.0890	8.1910	8.2325	8.2734	8.4186	8.4733	8.5511	8.6422

The 95% empirical confidence interval of [5.9384, 8.5511] covers the value of the parameter, $b = 8$. Table 21.4–6 gives a listing of each of the 40 samples of size 10.

21.4.5 Sample 21: GBD, Covered

From the original sample of

0.2821 5.4562 2.6805 3.3077 4.5790 1.6593 5.9774 0.5109 6.8759 7.8624

we have

$$\hat{\alpha}_1 = 3.9191\ \hat{\alpha}_2 = 6.3185\ \hat{\alpha}_3 = -0.0099\ \hat{\alpha}_4 = 1.7043.$$

We use the tables, with $|\hat{\alpha}_3| = 0$, $\hat{\alpha}_4 = 1.71$ to get $\beta_3 = -0.1744$, $\beta_4 = -0.1744$, which in turn give us $\beta_2 = 8.1857$ and $\beta_1 = -0.1738$.

Table 21.4–5. Details for 40 resamples: each row contains a sample of size 10.

0.3173	-0.5333	7.0672	3.5717	2.0197	-0.3193	5.9955	4.9873	0.2329	1.9887
2.7706	-0.2619	4.9557	7.2049	6.8777	5.4999	-1.2698	4.6918	-0.8277	4.5137
-0.9380	4.9392	-1.6438	6.0416	6.1144	-1.3955	4.1065	0.5847	1.7869	3.0910
-0.9465	6.6314	4.3352	1.1636	0.4199	5.5201	4.6891	1.3767	-0.3134	7.3604
2.7450	3.7759	5.5578	1.4343	-1.1806	-0.2865	0.3061	-1.5591	1.2432	6.8945
1.6628	6.6556	1.5555	7.2697	0.9244	-1.0818	-0.2233	0.3490	7.3977	2.2984
4.4275	4.9776	1.5472	-0.6412	-1.8380	2.9014	2.9420	-1.5392	-1.4357	3.1257
2.1786	-0.2128	5.5274	5.7559	-0.6602	3.1864	1.3252	1.4096	2.8219	0.2108
3.8270	-0.4673	4.2892	-0.6245	4.1702	-0.2538	3.4811	0.6335	2.2708	2.0555
0.5824	4.1091	-1.7195	7.3408	4.1611	1.0573	0.7858	4.7674	5.7634	0.2271
1.0958	5.0162	4.8694	1.4836	-0.6852	-1.3217	1.8284	5.0841	3.9167	3.1603
5.6190	1.9138	6.9171	1.2291	-1.9048	1.0736	-2.1471	1.4425	6.4394	2.3489
4.9749	3.8475	3.1146	6.1503	0.8673	-0.2210	-2.1774	2.7901	3.3320	7.0015
-0.4944	1.9305	3.6666	-2.5351	-1.2608	3.2385	1.8035	0.4140	4.4739	-2.1133
7.0531	5.0550	-0.6103	3.6958	0.3305	2.9702	0.7451	2.8072	0.6394	3.1951
0.0133	4.1637	-1.1630	5.8395	5.9741	-0.9954	3.4196	1.4754	0.6312	4.6447
6.2467	-0.4014	1.9947	3.7136	6.6887	-0.1883	1.2514	5.1425	4.4461	2.1223
-1.6650	-0.1311	0.4117	-2.2020	1.8782	5.6629	2.8476	4.8435	3.2884	4.6635
3.1752	5.0938	2.5154	6.2165	1.3416	-1.2493	-0.3298	0.7149	6.3592	3.2298
3.1666	6.8298	-0.1253	1.2947	4.6609	5.6835	0.1842	1.5083	3.8360	-1.9927
-1.6974	3.9193	0.9081	1.5660	2.9334	-0.1556	4.5297	-1.4048	5.6497	7.1144
6.1912	6.8555	6.0493	-2.0897	5.2576	-0.9441	3.9347	0.4790	2.1618	2.4332
-0.0437	5.1328	6.0385	-0.5086	2.3718	3.0392	-1.4642	-1.3861	2.6938	3.2647
6.9303	-0.0822	0.8836	5.9239	3.4635	3.3257	5.7530	1.5413	-1.6024	0.3409
-0.5854	-0.3357	-0.1661	-0.4920	-0.2724	-0.1230	-0.9124	0.7637	7.0815	2.1893
4.8503	4.2551	2.4562	-2.0101	-0.2806	1.1881	5.0911	4.9200	1.0713	0.3571
5.4668	5.1719	0.3386	1.6155	3.4371	-1.1781	6.4163	5.2406	-0.5026	3.2995
0.9570	2.0471	1.9032	1.3603	2.3297	1.2045	2.2205	1.5743	1.5802	2.4840
0.4280	3.9913	-1.2788	6.3284	5.1887	-0.0904	2.2121	2.9272	-1.0582	7.2367
3.6400	2.0638	-0.7353	-1.9446	2.8561	3.3782	6.3820	0.5441	-0.0030	-1.8472
1.9864	5.2111	3.5163	3.8385	4.5577	3.1083	5.0425	2.9444	4.9173	3.3345
4.1976	4.3101	2.9506	6.0093	1.2350	-0.8591	-0.9721	1.6073	4.9077	4.7959
1.4402	-0.2701	6.6522	3.8564	1.7538	-0.0577	5.6600	5.2999	-0.0274	2.2564
2.4985	-0.0007	4.6515	-1.8870	6.4858	5.8290	-1.5719	4.9969	-1.1017	4.8149
-1.2161	5.2505	-2.0738	6.3966	5.7738	-1.1079	3.8181	0.8451	1.5225	3.3695
-1.2250	7.0430	4.0430	1.4260	0.1598	5.8500	4.3905	1.6403	-0.5771	-1.7735
2.4731	4.0637	5.2379	1.6982	-1.4738	-0.0251	0.0459	-1.2583	0.9815	7.3813
1.3991	7.0712	1.2923	-1.8344	0.6638	-0.8085	-0.4861	0.6091	6.9039	2.5687
4.1338	5.2900	1.2841	-0.3768	7.2649	3.1777	2.6682	-1.2404	-1.7651	3.4045
1.9115	0.0482	5.2083	6.0957	-0.9294	3.4659	1.0631	1.6734	2.5493	0.4709

Table 21.4–6. Details for 40 resamples: each row contains a sample of size 10.

2.5251	1.6924	8.0419	5.3731	4.0767	1.9056	7.2104	6.4694	2.4443	4.0497
4.7170	1.9624	6.4458	8.1910	7.8738	6.8490	0.9449	6.2468	1.3954	6.1111
1.2834	6.4334	0.5720	7.2439	7.2969	0.8175	5.7964	2.7787	3.8729	4.9827
1.2747	7.6791	5.9739	3.3146	2.6228	6.8639	6.2448	3.5077	1.9115	8.4733
4.6955	5.5361	6.8915	3.5594	1.0358	1.9381	2.5144	0.6543	3.3870	7.8877
3.7632	7.6976	3.6678	8.2807	3.0953	1.1366	2.0005	2.5554	8.6422	4.3173
6.0450	6.4622	3.6605	1.5841	0.3949	4.8260	4.8596	0.6739	0.7770	5.0111
4.2142	2.0108	6.8692	7.0361	1.5649	5.0609	3.4612	3.5372	4.7598	2.4231
5.5766	1.7585	5.9384	1.6008	5.8461	1.9705	5.3002	2.8245	4.2936	4.1077
2.7765	5.7984	0.5007	8.4186	5.8390	3.2175	2.9669	6.3041	7.0416	2.4387
3.2528	6.4911	6.3810	3.6036	1.5397	0.8922	3.9094	6.5417	5.6475	5.0396
6.9362	3.9843	7.9067	3.3742	0.3397	3.2325	0.1838	3.5668	7.5350	4.3605
6.4602	5.5928	5.0020	7.3231	3.0425	2.0027	0.1704	4.7332	5.1796	7.9805
1.7314	3.9988	5.4491	0.0987	0.9540	5.1035	3.8875	2.6173	6.0806	0.2004
8.0283	6.5200	1.6152	5.4723	2.5377	4.8830	2.9290	4.7475	2.8300	5.0680
2.2321	5.8410	1.0538	7.0970	7.1949	1.2248	5.2506	3.5963	2.8224	6.2110
7.3933	1.8241	4.0549	5.4865	7.7231	2.0349	3.3944	6.5852	6.0593	4.1656
0.5518	2.0911	2.6151	0.1606	3.9531	6.9683	4.7812	6.3615	5.1441	6.2253
5.0518	6.5490	4.5022	7.3713	3.4760	0.9657	1.8952	2.9007	7.4758	5.0964
5.0447	7.8345	2.0968	3.4336	6.2233	6.9833	2.3974	3.6257	5.5837	0.2740
0.5212	5.6495	3.0802	3.6772	4.8525	2.0670	6.1234	0.8081	6.9587	8.0890
7.3529	7.8554	7.2496	0.2132	6.6705	1.2771	5.6617	2.6789	4.1998	4.4324
2.1766	6.5780	7.2417	1.7172	4.3800	4.9400	0.7484	0.8269	4.6526	5.1248
7.9180	2.1390	3.0576	7.1584	5.2860	5.1745	7.0340	3.6552	0.6120	2.5476
1.6402	1.8894	2.0568	1.7337	1.9520	2.0991	1.3094	2.9463	8.0558	4.2235
6.3666	5.9119	4.4519	0.2621	1.9439	3.3370	6.5469	6.4190	3.2304	2.5631
6.8248	6.6070	2.5455	3.7213	5.2648	1.0383	7.5180	6.6580	1.7231	5.1532
3.1254	4.1005	3.9750	3.4929	4.3441	3.3519	4.2503	3.6846	3.6899	4.4755
2.6305	5.7062	0.9357	7.4532	6.6195	2.1310	4.2431	4.8474	1.1607	8.2325
5.4278	4.1149	1.4891	0.3089	4.7883	5.2171	7.4926	2.7404	2.2162	0.3870
4.0476	6.6361	5.3286	5.5857	6.1448	4.9969	6.5107	4.8616	6.4170	5.1816
5.8674	5.9546	4.8668	7.2204	3.3795	1.3635	1.2485	3.7140	6.4098	6.3256
3.5647	1.9543	7.6950	5.5999	3.8437	2.1629	6.9662	6.7018	2.1924	4.2812
4.4878	2.2185	6.2162	0.3540	7.5695	7.0893	0.6417	6.4767	1.1164	6.3399
0.9996	6.6653	0.2222	7.5034	7.0491	1.1100	5.5696	3.0220	3.6384	5.2100
0.9905	8.0188	5.7467	3.5520	2.3739	7.1046	6.0166	3.7433	1.6485	0.4514
4.4663	5.7629	6.6559	3.7946	0.7388	2.1947	2.2637	0.9566	3.1479	8.5511
3.5278	8.0458	3.4315	0.3979	2.8529	1.4149	1.7397	2.8016	7.8956	4.5472
5.8177	6.6945	3.4241	1.8486	8.2734	5.0538	4.6312	0.9749	0.4589	5.2384
3.9822	2.2660	6.6341	7.2833	1.2920	5.2880	3.2229	3.7726	4.5309	2.6712

The RSGBD MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGBD(-0.1739, 8.1857, -0.1744, -0.1744),
```

provides the following ordered estimates, each based on a sample of size 10.

5.4721	5.8123	6.0720	6.1309	6.4231	6.6631	6.7638	6.7802
7.0105	7.0994	7.2186	7.2586	7.3463	7.4096	7.4943	7.5574
7.6123	7.6875	7.7228	7.7590	7.7877	7.8082	7.8209	7.8485
7.8655	7.8740	7.8831	7.8914	7.9111	7.9162	7.9305	7.9331
7.9355	7.9657	7.9692	7.9785	7.9834	8.0050	8.0110	8.0116

The 95% empirical confidence interval of [5.8123, 8.0110] covers the value of the parameter, $b = 8$. Table 21.4–7 gives a listing of each of the 40 samples of size 10.

21.4.6 Sample 28: MOP, Not Covered

From the original sample of

2.7274 3.7609 3.6259 3.1124 4.0243 2.9640 3.9227 3.3156 3.3212 4.1671

we have

$$\hat{\rho}_1 = 3.4735\, \hat{\rho}_2 = 1.4018\, \hat{\rho}_3 = 1.0635\, \hat{\rho}_4 = 0.6226.$$

Since $\hat{\rho}_1 > 1$, we use $1/\hat{\rho}_1 = 0.9403$ and obtain, from the tables, $\lambda_3 = 43.3598$ and $\lambda_4 = 1.0967$. These, in turn, give $\lambda_2 = 0.5858$ and $\lambda_1 = 4.2717$.

The RSGLD MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(4.2717, 0.5858, 43.3598, 1.0967),
```

provides the following ordered estimates, each based on a sample of size 10.

3.5100	3.8520	3.8842	3.9689	3.9862	3.9873	4.0065	4.0778
4.0928	4.1304	4.1371	4.1428	4.1546	4.1587	4.2258	4.2339
4.2394	4.2470	4.2773	4.3767	4.4053	4.5473	4.5796	4.6331
4.6667	4.6873	4.8111	4.8940	4.9154	4.9462	4.9551	4.9783
5.0565	5.3004	5.3957	5.4841	5.7329	5.7975	5.8648	5.9148

The 95% empirical confidence interval of [3.8520, 5.5648] does not cover the value of the parameter, $b = 8$. Table 21.4–8 gives a listing of each of the 40 samples of size 10.

There is a second percentile-based fit in this case (obtained form Table D–2, p. 383 of Karian and Dudewicz (2000)). For this fit, $\lambda_3 = 1.94929$ and $\lambda_4 = 1.16511$, which lead to $\lambda_2 = 1.1551$ and $\lambda_1 = 3.6354$.

The RSGLD MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(3.6354, 1.1551, 1.94929, 1.16511),
```

Table 21.4–7. Details for 40 resamples: each row contains a sample of size 10.

4.0001	2.2148	7.6875	1.0360	6.2221	7.1313	3.1373	2.7403	1.2466	6.5724
5.5689	1.0818	7.9305	2.4593	4.8064	0.4234	6.3975	3.2186	7.2716	0.2133
7.5130	1.3605	5.6899	2.5156	7.7590	0.1219	0.3911	6.4946	4.8899	2.3496
8.0116	4.9387	3.9121	2.1655	-0.165	2.8077	5.2234	3.2729	6.0366	1.6517
6.0896	0.0904	8.0110	2.4180	2.8633	3.3789	1.1445	4.7492	4.1340	1.2565
2.7152	0.4865	4.8621	4.5925	4.8683	1.2852	0.6318	5.8123	4.1400	3.4863
4.5405	2.9321	7.2761	6.3443	0.0853	0.8818	0.8812	1.2224	4.4986	7.8655
3.1547	3.8974	2.2334	0.1492	3.4202	5.3778	4.3364	-0.1524	7.0213	7.2186
1.2303	7.2664	0.2111	3.4135	4.9961	5.1914	7.9331	7.2620	1.6898	7.1584
5.8808	5.2194	7.9785	5.2603	6.7904	-0.0373	7.2349	1.0406	5.1174	3.6948
1.3554	0.1921	6.4315	4.5647	3.7618	6.7802	3.4658	6.1274	5.6213	2.0751
4.9180	6.0623	0.9371	4.3281	0.1716	7.0105	3.7204	5.3005	0.5616	3.3028
5.2883	3.2020	6.6631	2.7220	1.4647	4.2491	0.6626	4.9309	5.2551	0.9656
3.5689	0.1035	5.8774	1.1900	0.8577	5.5470	5.8196	7.9834	3.7384	4.5908
6.0720	4.3534	1.9615	-0.1572	0.1699	2.7925	-0.0853	1.4860	0.0186	0.5416
6.7782	1.4568	3.0151	0.1240	1.6786	2.4082	0.6204	3.8744	2.9885	7.7228
7.1413	3.1543	0.2363	7.5140	8.0050	7.3770	0.9460	4.2629	7.2691	3.9331
-0.0457	1.8578	0.6880	5.5968	2.0485	1.1939	3.5654	0.3329	2.0267	6.4231
0.0783	2.0076	5.1396	5.8970	4.9872	2.0651	4.7747	0.9004	7.9111	-0.0114
4.5169	4.5445	6.4074	6.9294	-0.0652	5.7629	7.1712	2.8111	7.5574	3.2699
-0.1722	0.2622	1.4590	3.0327	5.4721	3.9830	2.4724	2.8851	1.0106	3.0841
2.8154	1.4628	0.9556	1.1931	6.2413	7.2373	7.4943	5.2824	0.4621	1.9246
2.5739	4.3457	7.1850	6.0617	7.0011	-0.1119	3.5541	6.7598	7.3463	4.9336
5.6338	7.9355	5.3340	7.1921	0.2132	5.5932	3.1653	7.7470	4.4701	6.5833
7.8485	7.5424	1.2001	5.5929	3.2107	1.4363	7.0743	4.3373	3.3883	1.1327
0.1584	7.5888	4.4356	7.8082	5.3989	6.2729	4.3979	7.7003	4.5820	6.2314
7.7401	0.5634	5.5906	7.8831	1.0022	0.6549	2.1418	2.9505	5.0748	6.9969
3.3188	3.4527	7.2586	4.8887	6.1182	6.7217	1.1028	1.5935	5.3505	5.8344
1.2101	2.7035	7.2122	3.5396	7.9692	7.1685	7.4236	1.5173	2.0412	5.9911
6.1309	0.7973	1.0905	5.3310	-0.0893	3.6538	4.7014	3.6995	0.5641	4.2243
2.7052	3.4902	0.6230	3.1409	1.3042	2.7581	6.3141	7.8914	7.3380	4.9345
7.4096	6.8167	3.9408	5.4097	5.5446	0.5337	5.2097	3.6121	0.5110	-0.0387
-0.1714	7.4097	1.3228	7.1922	-0.1081	3.9114	2.6991	7.1567	7.9657	-0.0827
7.0994	0.3552	3.4437	6.8222	0.2863	0.8568	6.9993	0.1152	4.5855	6.877
3.9278	2.2802	7.8209	2.9933	4.4471	-0.0339	6.2291	0.0844	6.4966	2.7031
5.6484	3.2477	7.9132	6.0935	-0.1222	6.3546	7.9162	6.8738	7.5687	7.4942
6.0961	4.5677	1.2655	5.5623	7.1179	0.8351	7.8740	6.5789	0.0742	4.2583
2.5888	3.9742	0.3546	6.7638	0.4818	6.5342	0.0272	4.8393	4.1954	4.2360
2.0490	2.3907	4.1747	6.8142	2.5432	2.6735	0.3585	7.7877	3.2663	7.3389
1.9198	7.6123	7.5006	0.7832	0.9365	5.4864	6.9193	5.7176	2.8705	4.8202

Table 21.4–8. Details for 40 resamples: each row contains a sample of size 10.

3.0523	2.8663	4.9462	3.7216	3.4154	2.9130	4.1345	3.9705	3.0338	3.4090
3.5672	2.9255	3.9653	5.3004	4.6094	4.0521	2.7105	3.9214	2.8028	3.8911
2.7793	3.9626	2.6396	4.1434	4.1587	2.6856	3.8196	3.1105	3.3671	3.6300
2.7775	4.3617	3.8601	3.2354	3.0746	4.0553	3.9210	3.2808	2.9143	5.7975
3.5621	3.7596	4.0612	3.2930	2.7287	2.9201	3.0498	2.6547	3.2524	4.6331
3.3411	4.3793	3.3186	5.4992	3.1840	2.7491	2.9340	3.0592	5.9148	3.4725
3.8762	3.9689	3.3168	2.8429	2.6087	3.5930	3.6010	2.6584	2.6778	3.6367
3.4480	2.9363	4.0564	4.0928	2.8388	3.6484	3.2698	3.2878	3.5774	3.0290
3.7690	2.8807	3.8520	2.8465	3.8310	2.9273	3.7046	3.1211	3.4668	3.4227
3.1100	3.8201	2.6269	5.7329	3.8294	3.2126	3.1541	3.9341	4.0940	3.0326
3.2209	3.9752	3.9511	3.3034	2.8334	2.7001	3.3757	3.9862	3.7854	3.6434
4.0708	3.3935	4.6667	3.2494	2.5995	3.2161	2.5758	3.2947	4.2565	3.4827
3.9685	3.7727	3.6346	4.1668	3.1717	2.9345	2.5740	3.5711	3.6763	4.8111
2.8748	3.3969	3.7393	2.5653	2.7123	3.6585	3.3705	3.0734	3.8842	2.5781
4.9154	3.9815	2.8496	3.7448	3.0551	3.6065	3.1453	3.5745	3.1224	3.6501
2.9858	3.8298	2.7323	4.1066	4.1304	2.7672	3.6930	3.3017	3.1206	3.9135
4.1910	2.8951	3.4102	3.7481	4.4053	2.9416	3.2541	3.9956	3.8794	3.4365
2.6360	2.9541	3.0729	2.5727	3.3861	4.0778	3.5824	3.9468	3.6680	3.9167
3.6463	3.9877	3.5163	4.1830	3.2733	2.7147	2.9107	3.1388	4.2258	3.6568
3.6446	4.5473	2.9554	3.2633	3.9162	4.0811	3.0232	3.3086	3.7706	2.5891
2.6305	3.7858	3.1805	3.3208	3.5993	2.9488	3.8939	2.6838	4.0757	5.0565
4.1766	4.5796	4.1450	2.5800	4.0139	2.7780	3.7886	3.0875	3.4446	3.4998
2.9733	3.9940	4.1428	2.8717	3.4874	3.6199	2.6724	2.6874	3.5520	3.6635
4.6873	2.9649	3.1753	4.1213	3.7013	3.6751	4.0923	3.3156	2.6469	3.0574
2.8550	2.9094	2.9465	2.8753	2.9232	2.9559	2.7847	3.1494	4.9783	3.4502
3.9479	3.8460	3.5044	2.5873	2.9214	3.2406	3.9873	3.9594	3.2156	3.0610
4.0469	4.0003	3.0569	3.3312	3.6963	2.7292	4.2470	4.0112	2.8730	3.6701
3.1911	3.4210	3.3913	3.2773	3.4788	3.2441	3.4566	3.3225	3.3238	3.5100
3.0764	3.7989	2.7087	4.2154	4.0030	2.9631	3.4549	3.5981	2.7540	5.3957
3.7344	3.4245	2.8226	2.5946	3.5841	3.6851	4.2339	3.1017	2.9822	2.6073
3.4085	4.0065	3.7112	3.7711	3.8987	3.6334	3.9795	3.6014	3.9590	3.6768
3.8359	3.8557	3.6026	4.1371	3.2506	2.7961	2.7721	3.3295	3.9574	3.9389
3.2942	2.9237	4.3767	3.7744	3.3602	2.9702	4.0773	4.0206	2.9768	3.4639
3.5129	2.9827	3.9146	2.6019	4.2773	4.1048	2.6524	3.9720	2.7450	3.9421
2.7214	4.0128	2.5813	4.2394	4.0957	2.7437	3.7673	3.1670	3.3116	3.6835
2.7196	4.8940	3.8082	3.2912	3.0178	4.1084	3.8698	3.3364	2.8568	2.6183
3.5078	3.8119	4.0108	3.3485	2.6706	2.9773	2.9929	2.7129	3.1963	5.8648
3.2855	4.9551	3.2629	2.6092	3.1277	2.8069	2.8766	3.1158	4.6468	3.5270
3.8245	4.0191	3.2611	2.9004	5.4841	3.6468	3.5469	2.7165	2.6196	3.6901
3.3930	2.9934	4.0061	4.1546	2.7811	3.7018	3.2139	3.3433	3.5231	3.0858

provides the following ordered estimates, each based on a sample of size 10.

3.5109	3.8989	3.9393	4.0488	4.0718	4.0733	4.0991	4.1946
4.2139	4.2585	4.2656	4.2715	4.2829	4.2866	4.3344	4.3387
4.3415	4.3452	4.3583	4.3887	4.3952	4.4196	4.4240	4.4304
4.4341	4.4363	4.4476	4.4541	4.4556	4.4577	4.4583	4.4599
4.4647	4.4774	4.4816	4.4852	4.4940	4.4960	4.4980	4.4994

The 95% empirical confidence interval of [3.8989, 4.4980] does not cover the value of the parameter, $b = 8$. Table 21.4–9 gives a listing of each of the 40 samples of size 10.

There is yet a third percentile-based fit in this case (obtained from Table D–3, p. 390 of Karian and Dudewicz (2000)). For this fit, $\lambda_3 = 0.88746$ and $\lambda_4 = 1.02218$, which lead to $\lambda_2 = 1.1300$ and $\lambda_1 = 3.4309$.

The `RSGLD` MATLAB program listed in Appendix A.2 of this chapter, when invoked through the command,

```
RSGLD(3.4309, 1.1300, 0.88746, 1.02218),
```

provides the following ordered estimates, each based on a sample of size 10.

3.5098	3.8542	3.8873	3.9749	3.9930	3.9941	4.0142	4.0876
4.1022	4.1358	4.1411	4.1455	4.1540	4.1568	4.1923	4.1955
4.1976	4.2003	4.2099	4.2324	4.2372	4.2552	4.2583	4.2631
4.2658	4.2674	4.2758	4.2805	4.2817	4.2833	4.2837	4.2848
4.2884	4.2979	4.3010	4.3037	4.3103	4.3119	4.3134	4.3145

The 95% empirical confidence interval of [3.8542, 4.3134] does not cover the value of the parameter, $b = 8$. Table 21.4–10 gives a listing of each of the 40 samples of size 10.

Table 21.4–9. Details for 40 resamples: each row contains a sample of size 10.

3.0944	2.9563	4.4577	3.7425	3.4156	2.9893	4.2628	4.0509	3.0800	3.4093
3.5709	2.9984	4.0441	4.4774	4.4277	4.1605	2.8541	3.9868	2.9132	3.9480
2.8977	4.0405	2.8117	4.2721	4.2866	2.8389	3.8590	3.1413	3.3688	3.6390
2.8965	4.3850	3.9090	3.2477	3.1122	4.1647	3.9862	3.2884	2.9903	4.4960
3.5655	3.7869	4.1726	3.2996	2.8654	2.9945	3.0925	2.8205	3.2628	4.4304
3.3442	4.3893	3.3232	4.4858	3.2030	2.8782	3.0045	3.0999	4.4994	3.4725
3.9292	4.0488	3.3215	2.9402	2.7940	3.5986	3.6072	2.8226	2.8342	3.6464
3.4479	3.0062	4.1662	4.2139	2.9374	3.6594	3.2785	3.2948	3.5818	3.0762
3.7981	2.9664	3.8989	2.9427	3.8730	2.9997	3.7229	3.1501	3.4668	3.4228
3.1409	3.8596	2.8043	4.4940	3.8710	3.2277	3.1776	4.0033	4.2154	3.0790
3.2350	4.0572	4.0254	3.3091	2.9337	2.8477	3.3771	4.0718	3.8176	3.6538
4.1855	3.3942	4.4341	3.2602	2.7888	3.2308	2.7757	3.3012	4.3496	3.4829
4.0482	3.8025	3.6440	4.2937	3.1925	3.0049	2.7747	3.5750	3.6907	4.4476
2.9622	3.3976	3.7631	2.7701	2.8552	3.6706	3.3721	3.1112	3.9393	2.7770
4.4556	4.0655	2.9448	3.7695	3.0967	3.6132	3.1702	3.5787	3.1511	3.6613
3.0430	3.8715	2.8676	4.2311	4.2585	2.8898	3.7096	3.3075	3.1496	3.9766
4.3127	2.9765	3.4105	3.7733	4.3952	3.0101	3.2644	4.0844	3.9332	3.4364
2.8096	3.0194	3.1108	2.7740	3.3871	4.1946	3.5872	4.0198	3.6813	3.9807
3.6570	4.0739	3.5174	4.3068	3.2817	2.8566	2.9877	3.1647	4.3344	3.6687
3.6552	4.4196	3.0203	3.2727	3.9801	4.1989	3.0717	3.3140	3.8000	2.7830
2.8064	3.8182	3.2000	3.3252	3.6054	3.0154	3.9515	2.8378	4.1919	4.4647
4.3018	4.4240	4.2736	2.7780	4.1090	2.8969	3.8215	3.1226	3.4445	3.5003
3.0336	4.0823	4.2715	2.9601	3.4876	3.6279	2.8309	2.8400	3.5548	3.6762
4.4363	3.0273	3.1955	4.2483	3.7191	3.6894	4.2133	3.3204	2.8159	3.0985
2.9485	2.9868	3.0137	2.9626	2.9967	3.0207	2.9013	3.1736	4.4599	3.4501
4.0213	3.8915	3.5051	2.7820	2.9954	3.2524	4.0733	4.0364	3.2304	3.1013
4.1535	4.0907	3.0981	3.3349	3.7134	2.8657	4.3452	4.1054	2.9610	3.6837
3.2090	3.4211	3.3921	3.2853	3.4789	3.2555	3.4565	3.3268	3.3280	3.5109
3.1137	3.8339	2.8530	4.3285	4.0943	3.0260	3.4547	3.6041	2.8814	4.4816
3.7574	3.4245	2.9264	2.7861	3.5890	3.7007	4.3387	3.1341	3.0403	2.7932
3.4088	4.0991	3.7305	3.8005	3.9576	3.6427	4.0628	3.6077	4.0358	3.6913
3.8789	3.9035	3.6091	4.2656	3.2613	2.9087	2.8930	3.3333	4.0337	4.0094
3.3007	2.9971	4.3887	3.8045	3.3622	3.0313	4.1940	4.1180	3.0363	3.4638
3.5139	3.0407	3.9780	2.7901	4.3583	4.2289	2.8191	4.0530	2.8756	4.0136
2.8608	4.1075	2.7787	4.3415	4.2176	2.8748	3.7961	3.1884	3.3167	3.6988
2.8597	4.4541	3.8452	3.2980	3.0675	4.2333	3.9211	3.3398	2.9497	2.7994
3.5086	3.8497	4.1048	3.3512	2.8299	3.0367	3.0484	2.8555	3.2136	4.4980
3.2928	4.4583	3.2722	2.7942	3.1555	2.9159	2.9635	3.1457	4.4320	3.5285
3.8650	4.1159	3.2707	2.9804	4.4852	3.6576	3.5494	2.8578	2.8002	3.7064
3.3937	3.0488	4.0985	4.2829	2.8989	3.7196	3.2288	3.3463	3.5245	3.1212

Table 21.4–10. Details for 40 resamples: each row contains a sample of size 10.

3.0557	2.8691	4.2833	3.7217	3.4158	2.9162	4.1390	3.9766	3.0373	3.4095
3.5669	2.9288	3.9712	4.2979	4.2611	4.0616	2.7089	3.9256	2.8044	3.8943
2.7803	3.9684	2.6333	4.1460	4.1568	2.6826	3.8211	3.1136	3.3680	3.6296
2.7784	4.2296	3.8625	3.2375	3.0779	4.0648	3.9252	3.2825	2.9175	4.3119
3.5618	3.7601	4.0709	3.2945	2.7278	2.9234	3.0532	2.6496	3.2543	4.2631
3.3422	4.2328	3.3199	4.3041	3.1866	2.7491	2.9373	3.0626	4.3145	3.4725
3.8790	3.9749	3.3182	2.8453	2.5990	3.5926	3.6006	2.6535	2.6744	3.6363
3.4482	2.9396	4.0660	4.1022	2.8412	3.6481	3.2716	3.2894	3.5770	3.0325
3.7696	2.8836	3.8542	2.8490	3.8327	2.9306	3.7046	3.1242	3.4669	3.4231
3.1131	3.8215	2.6193	4.3103	3.8310	3.2149	3.1569	3.9388	4.1034	3.0360
3.2231	3.9815	3.9564	3.3049	2.8357	2.6980	3.3765	3.9930	3.7862	3.6431
4.0806	3.3941	4.2658	3.2513	2.5887	3.2184	2.5607	3.2963	4.2035	3.4827
3.9745	3.7734	3.6342	4.1621	3.1744	2.9378	2.5585	3.5707	3.6761	4.2758
2.8777	3.3975	3.7396	2.5471	2.7108	3.6582	3.3714	3.0767	3.8873	2.5636
4.2817	3.9880	2.8521	3.7451	3.0585	3.6061	3.1482	3.5741	3.1254	3.6498
2.9893	3.8315	2.7316	4.1152	4.1358	2.7678	3.6929	3.3032	3.1237	3.9174
4.1762	2.8981	3.4107	3.7484	4.2372	2.9450	3.2560	4.0028	3.8823	3.4368
2.6293	2.9576	3.0762	2.5568	3.3868	4.0876	3.5820	3.9519	3.6677	3.9207
3.6459	3.9946	3.5161	4.1718	3.2751	2.7132	2.9139	3.1417	4.1923	3.6565
3.6443	4.2552	2.9589	3.2652	3.9202	4.0909	3.0266	3.3100	3.7713	2.5766
2.6233	3.7867	3.1831	3.3221	3.5989	2.9522	3.8972	2.6807	4.0855	4.2884
4.1681	4.2583	4.1471	2.5658	4.0219	2.7790	3.7895	3.0908	3.4448	3.4997
2.9768	4.0011	4.1455	2.8745	3.4873	3.6195	2.6685	2.6845	3.5517	3.6632
4.2674	2.9683	3.1779	4.1282	3.7012	3.6749	4.1018	3.3169	2.6412	3.0608
2.8576	2.9126	2.9499	2.8782	2.9265	2.9594	2.7859	3.1522	4.2848	3.4504
3.9531	3.8480	3.5043	2.5745	2.9247	3.2427	3.9941	3.9651	3.2179	3.0643
4.0562	4.0077	3.0603	3.3324	3.6962	2.7284	4.2003	4.0191	2.8758	3.6699
3.1936	3.4214	3.3919	3.2790	3.4788	3.2461	3.4567	3.3238	3.3250	3.5098
3.0797	3.8000	2.7070	4.1879	4.0105	2.9665	3.4550	3.5977	2.7542	4.3010
3.7346	3.4248	2.8247	2.5830	3.5837	3.6850	4.1955	3.1049	2.9857	2.5975
3.4090	4.0142	3.7112	3.7717	3.9021	3.6330	3.9859	3.6010	3.9646	3.6766
3.8377	3.8580	3.6022	4.1411	3.2526	2.7975	2.7729	3.3307	3.9630	3.9437
3.2958	2.9270	4.2324	3.7750	3.3611	2.9737	4.0871	4.0289	2.9803	3.4640
3.5128	2.9862	3.9186	2.5913	4.2099	4.1136	2.6471	3.9782	2.7448	3.9470
2.7203	4.0208	2.5674	4.1976	4.1050	2.7434	3.7679	3.1697	3.3130	3.6833
2.7184	4.2805	3.8095	3.2928	3.0213	4.1168	3.8724	3.3376	2.8595	2.6098
3.5077	3.8133	4.0187	3.3496	2.6666	2.9808	2.9964	2.7113	3.1988	4.3134
3.2871	4.2837	3.2647	2.5996	3.1307	2.8086	2.8795	3.1189	4.2642	3.5268
3.8261	4.0273	3.2630	2.9036	4.3037	3.6464	3.5466	2.7151	2.6112	3.6900
3.3936	2.9969	4.0138	4.1540	2.7822	3.7017	3.2162	3.3444	3.5229	3.0890

21.5 Conclusions

The original GB algorithm did not point out when to use which fitting method; in this paper, we give up-to-date algorithms for GB, and for BM. One pilot study shows that the putative 95% CI from BM can and does in fact have confidence 0%, i.e. it always fails to cover the true value of the parameter $b = 8$ because of an **inherent difficulty in the bootstrap method: the inability to draw samples outside the range of the original dataset**. We also show that in the same difficult case the GB CI has an observed confidence level of 40%, i.e. it covers the true value in 16 cases out of 40. This indicates that the GB in this case has a true confidence level between 25% and 55% (40% plus or minus twice the square root of (.40)(.60)/40 expressed as a %). **This parametric bootstrap method based on the GLD family for fitting the original data offers an alternative having major practical advantages**. Moreover, it remains for a future research topic as to how to improve the GB to achieve a desired 95% confidence interval since 40% on average and 25%–55% still leaves room for improvement.

Acknowledgments

The authors wish to thank the referees for their patient reading and pointing out several typographical errors. Their comments and suggestions have improved the presentation.

References for Chapter 21

Chernick, M.R. (1999). *Bootstrap Methods, A Practitioners Guide*, John Wiley & Sons, Inc., New York.

Dudewicz, E.J. (1992). "The Generalized Bootstrap," in *Bootstrapping and Related Techniques* (K.-H. Jckel, G. Rothe, and W. Sendler, eds.), V. 376/ Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, pp. 3137.

Efron, B. (1979). "Bootstrap methods: Another look at the jackknife," *Ann. Statist.*, 7, pp. 1–26

Efron, B., and Tibshirani, R.J. (1993). *An Introduction to the Bootstrap*, Chapman & Hall, New York.

Karian, Z.A. and Dudewicz, E.J. (1991). *Modern Statistical Systems and GPSS Simulation* 1st Ed., Computer Sciences Press/W.H. Freeman and Co., New York. (Now available from American Sciences Pres Inc., Columbus, Ohio.

Karian, Z.A. and Dudewicz, E.J. (1999). *Modern Statistical, Systems, and GPSS Simulation,* 2nd Ed., CRC Press, Boca Raton, FL.

Karian, Z.A. and Dudewicz, E.J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* CRC Press, Boca Raton, FL.

Karian, Z.A. and Dudewicz, E.J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm, *Journal of the Iranian Statistical Society*, 2, 171–187.

Moore, D. S. and McCabe, G. P. (2003). *Introduction to the Practice of Statistics,* 4th Ed. W. H. Freeman and Company, New York.

Appendix for Chapter 21

A.1 MATLAB Code and Results for the Four Original Samples

A.1.1 The MATLAB Code for Obtaining the Samples

```
function  [x]=originalsamples
a=0; %lower limit
b=8; %upper limit
m1=(a+b)./2; m2=2./(b-a); m3=2;m4=2;
    %% assign values for four lambda parameters
x=[];z=[];s=123456789; d=39468974943;m=2^64;
for i=1:40
    for j=1:10
     s=mod(d*s,m); %% random number generator
     y=s./m;  %% generating a random number between 0 and 1;
     z(j)=Q(y,m1,m2,m3,m4);
         %% uniformly generating a random number between a and b;
     x(i,j)=z(j);
    end;
end;
```

A.1.2 The 40 Original Samples

Sample 1:	2.1132	1.2999	7.8368	5.1515	3.7352
	1.5031	7.1337	6.3546	2.0321	3.7062
Sample 2:	4.4305	1.5577	6.3290	7.9093	7.7274
	6.7609	0.6259	6.1124	1.0243	5.9640
Sample 3:	0.9227	6.3156	0.3212	7.1671	7.2196
	0.5187	5.6180	2.3702	3.5165	4.7218
Sample 4:	0.9149	7.5734	5.8134	2.9249	2.2118
	6.7765	6.1102	3.1281	1.5087	7.9796
Sample 5:	4.4071	5.3312	6.8055	3.1827	0.7040
	1.5343	2.1024	0.3859	3.0009	7.7374
Sample 6:	3.3993	7.5890	3.2977	7.9406	2.6962
	0.7921	1.5946	2.1437	7.9930	3.9952
Sample 7:	5.8915	6.3468	3.2899	1.1984	0.1884
	4.5499	4.5868	0.4015	0.4852	4.7531
Sample 8:	3.8837	1.6046	6.7821	6.9562	1.1805
	4.8077	3.0790	3.1593	4.4774	2.0109
Sample 9:	5.3759	1.3624	5.7743	1.2140	5.6727
	1.5656	5.0712	2.4171	3.9696	3.7687
Sample 10:	2.3680	5.6202	0.2665	7.9718	5.6649

	2.8234	2.5634	6.1749	6.9618	2.0265
Sample 11:	2.8602	6.3781	6.2587	3.2296	1.1571
	0.5812	3.5555	6.4327	5.4540	4.7843
Sample 12:	6.8524	3.6359	7.7509	2.9874	0.1493
	2.8390	0.0477	3.1906	7.4462	4.0421
Sample 13:	6.3446	5.3937	4.7430	7.2452	2.6415
	1.5968	0.0399	4.4484	4.9384	7.7999
Sample 14:	1.3368	3.6515	5.2352	0.0031	0.6337
	4.8546	3.5321	2.2062	5.9305	0.0577
Sample 15:	7.8290	6.4093	1.2274	5.2609	2.1259
	4.6124	2.5243	4.4640	2.4227	4.8156
Sample 16:	1.8212	5.6671	0.7196	7.0187	7.1180
	0.8702	5.0165	3.2218	2.4149	6.0734
Sample 17:	7.3134	1.4249	3.7118	5.2765	7.6102
	1.6281	3.0087	6.4796	5.9071	3.8312
Sample 18:	0.3055	1.6827	2.2040	0.0343	3.6024
	6.8859	4.5009	6.2374	4.8993	6.0890
Sample 19:	4.7977	6.4406	4.1962	7.2921	3.0946
	0.6437	1.4930	2.4952	7.3915	4.8468
Sample 20:	4.7899	7.6984	1.6884	3.0499	6.0868
	6.9015	1.9852	3.2531	5.3837	0.1046
Sample 21:	0.2821	5.4562	2.6805	3.3077	4.5790
	1.6593	5.9774	0.5109	6.8759	7.8624
Sample 22:	7.2743	7.7140	7.1727	0.0656	6.5712
	0.9171	5.4696	2.2687	3.8680	4.1202
Sample 23:	1.7665	6.4718	7.1649	1.3234	4.0634
	4.6749	0.4618	0.5265	4.3602	4.8781
Sample 24:	7.7587	1.7296	2.6571	7.0812	5.0555
	4.9327	6.9540	3.2843	0.3524	2.1359
Sample 25:	1.2509	1.4874	1.6493	1.3390	1.5477
	1.6906	0.9462	2.5421	7.8446	3.8937
Sample 26:	6.2430	5.7452	4.1415	0.0968	1.5399
	2.9484	6.4384	6.2999	2.8368	2.1515
Sample 27:	6.7352	6.5031	2.1337	3.3546	5.0321
	0.7062	7.4305	6.5577	1.3290	4.9093
Sample 28:	2.7274	3.7609	3.6259	3.1124	4.0243
	2.9640	3.9227	3.3156	3.3212	4.1671
Sample 29:	2.2196	5.5187	0.6180	7.3702	6.5165
	1.7218	3.9149	4.5734	0.8134	7.9249
Sample 30:	5.2118	3.7765	1.1102	0.1281	4.5087
	4.9796	7.4071	2.3312	1.8055	0.1827
Sample 31:	3.7040	6.5343	5.1024	5.3859	6.0009
	4.7374	6.3993	4.5890	6.2977	4.9406

```
Sample 32:   5.6962    5.7921    4.5946    7.1437    2.9930
             0.9952    0.8915    3.3468    6.2899    6.1984
Sample 33:   3.1884    1.5499    7.5868    5.4015    3.4852
             1.7531    6.8837    6.6046    1.7821    3.9562
Sample 34:   4.1805    1.8077    6.0790    0.1593    7.4774
             7.0109    0.3759    6.3624    0.7743    6.2140
Sample 35:   0.6727    6.5656    0.0712    7.4171    6.9696
             0.7687    5.3680    2.6202    3.2665    4.9718
Sample 36:   0.6649    7.8234    5.5634    3.1749    1.9618
             7.0265    5.8602    3.3781    1.2587    0.2296
Sample 37:   4.1571    5.5812    6.5555    3.4327    0.4540
             1.7843    1.8524    0.6359    2.7509    7.9874
Sample 38:   3.1493    7.8390    3.0477    0.1906    2.4462
             1.0421    1.3446    2.3937    7.7430    4.2452
Sample 39:   5.6415    6.5968    3.0399    1.4484    7.9384
             4.7999    4.3368    0.6515    0.2352    5.0031
Sample 40:   3.6337    1.8546    6.5321    7.2062    0.9305
             5.0577    2.8290    3.4093    4.2274    2.2609
```

A.2 MATLAB Codes for Resampling Functions

```
function [N,Y]=RSGBD(beta1,beta2,beta3,beta4)
Y=[];M=[];
for i=1:40
   R=betarnd(beta3+1,beta4+1,1,10);
   Y(i,:)=beta1+beta2*R;
   z=sort(Y(i,:));
   M(i)=z(10);
end;
N=sort(M);
```

Note: Since we use the beta random generator from the toolbox in MATLAB, the samples are non-reproducible, but the empirical confidence intervals are close. The Beta random number generator in the toolbox of MATLAB is based on the methods provided by L. Devroye, *Non-Uniform Random Variate Generation*, Springer-Verlag, 1986, page 430. They use derive beta random numbers as a ratio of gamma random numbers: $r = g_1/(g_1+g_2)$, where g_1 and g_2 are generated from the gamma random numbers, which is also based on the same reference (pp. 401–402).

```
function [N,x]=RSGLD(lamb1,lamb2,lamb3,lamb4)
    %% The results and details for resamples are reproducible.
x=[]; M=[];
```

```
s=123456789; d=39468974943;m=2^64;
for i=1:40
   for j=1:10
     s=mod(d*s,m); %% random number generator
     y=s./m;  %% generating a random number between 0 and 1;
         %%y=unifrnd(0,1);
     x(i,j)=Q(y,lamb1,lamb2,lamb3,lamb4);
         %% uniformly generating a random number between a and b;
   end;
  z=sort(x(i,:));
   M(i)=z(10);
end;
N=sort(M);

function r=Q(y,m1,m2,m3,m4)
r=m1+(y^m3-(1-y)^m4)./m2
```

The empirical confidence intervals are constructed using the empirical estimators based on the 40 resamples of size 10 each, deleting the smallest and largest estimators (so 95% of the 40 estimates are constructed).

A.3 Details for Obtaining Empirical Confidence Intervals Based on the 40 Original Samples

SAMPLE 1: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.0866, \quad \hat{\alpha}_2 = 5.2450, \quad \hat{\alpha}_3 = 0.3210, \quad \hat{\alpha}_4 = 1.6344.$
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.325, 1.63)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.5469, -0.3767)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.6843$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 1.2729$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.2729, 6.6843, –0.3767, –0.5469))

7.3955	7.6244	7.6931	7.7409	7.7773	7.7996	7.8337	7.8448
7.8731	7.8755	7.9035	7.9122	7.9182	7.9195	7.9233	7.9239
7.9303	7.9308	7.9316	7.9363	7.9415	7.9449	7.9457	7.9480
7.9493	7.9529	7.9538	7.9541	7.9553	7.9557	7.9560	7.9562
7.9563	7.9566	7.9567	7.9568	7.9570	7.9570	7.9571	7.9572

Empirical 95% Confidence Interval: [7.6244, 7.9571] – does not cover $b = 8$.

SAMPLE 2: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.8441, \quad \hat{\alpha}_2 = 6.9806, \quad \hat{\alpha}_3 = -0.5431, \quad \hat{\alpha}_4 = 1.6953.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.55, 1.71)$.

Extracted Values from Table: (after β_3 and β_4 interchange) $(\beta_3, \beta_4) = (-0.5937, -0.7678)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.0303$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.4105$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.4105, 7.0303, –0.5937, –0.7678))

5.9343	7.4188	7.4223	7.4252	7.4282	7.4301	7.4332	7.4343
7.4352	7.4359	7.4383	7.4393	7.4400	7.4402	7.4404	7.4405
7.4407	7.4407	7.4407	7.4407	7.4407	7.4408	7.4408	7.4408
7.4408	7.4408	7.4408	7.4408	7.4408	7.4408	7.4408	7.4408
7.4408	7.4408	7.4408	7.4408	7.4408	7.4408	7.4408	7.4408

Empirical 95% Confidence Interval: [7.4188, 7.4408] – does not cover $b = 8$.

SAMPLE 3: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.8691$, $\hat{\alpha}_2 = 6.6739$, $\hat{\alpha}_3 = -0.0996$, $\hat{\alpha}_4 = 1.4704$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.1, 1.47)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.5300, -0.5768)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.1189$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.1231$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.1231, 7.1189, –0.5768, –0.53))

6.4982	6.5396	6.8613	6.8737	6.9193	6.9999	7.0074	7.0366
7.0396	7.0432	7.0484	7.0644	7.0993	7.1173	7.1247	7.1413
7.1424	7.1474	7.1619	7.1674	7.1737	7.1885	7.2073	7.2150
7.2200	7.2203	7.2211	7.2235	7.2287	7.2335	7.2353	7.2374
7.2378	7.2387	7.2404	7.2408	7.2418	7.2419	7.2420	7.2420

Empirical 95% Confidence Interval: [6.5396, 7.2420] – does not cover $b = 8$.

SAMPLE 4: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.4941$, $\hat{\alpha}_2 = 6.2453$, $\hat{\alpha}_3 = -0.0053$, $\hat{\alpha}_4 = 1.4222$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0, 1.41)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.6132, -0.6132)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.6563$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 1.1660$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.166, 6.6563, –0.6132, –0.6132))

5.6159	5.8914	6.3166	6.5726	6.8809	7.0586	7.0999	7.3764
7.5159	7.5490	7.5976	7.6260	7.6525	7.7114	7.7702	7.7778
7.7786	7.7794	7.7822	7.7838	7.7861	7.7873	7.7978	7.8047
7.8094	7.8130	7.8135	7.8190	7.8191	7.8200	7.8201	7.8209
7.8211	7.8212	7.8216	7.8217	7.8220	7.8220	7.8223	7.8223

Empirical 95% Confidence Interval: [5.8914, 7.8223] – does not cover $b = 8$.

SAMPLE 5: (MOP)
Rhos: $\hat{\rho}_1 = 3.0918, \quad \hat{\rho}_2 = 7.2266, \quad \hat{\rho}_3 = 0.5874, \quad \hat{\rho}_4 = 0.6051.$
Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.58, 0.610)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.65593, 0.47138)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.1981$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 5.1309$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through Q(y, 5.1309, 0.1981, 1.65593, 0.47138))

3.2767	5.3111	5.5372	6.1709	6.3080	6.3170	6.4732	7.0756
7.2027	7.5058	7.5553	7.5966	7.6779	7.7046	8.0603	8.0941
8.1158	8.1452	8.2496	8.5057	8.5633	8.7910	8.8336	8.8990
8.9374	8.9600	9.0843	9.1591	9.1775	9.2035	9.2109	9.2300
9.2923	9.4734	9.5417	9.6053	9.7972	9.8553	9.9250	9.9894

Empirical 95% Confidence Interval: [5.3111, 9.9250] – covers $b = 8$.

SAMPLE 6: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.1441, \quad \hat{\alpha}_2 = 6.6222, \quad \hat{\alpha}_3 = 0.5065, \quad \hat{\alpha}_4 = 1.7279.$
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.50, 1.74)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.6648, -0.4361)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.3338$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 1.4099$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.4099, 7.3338, –0.4361, –0.6648))

8.4122	8.5902	8.6531	8.6549	8.6602	8.6630	8.6681	8.6860
8.7095	8.7193	8.7211	8.7308	8.7317	8.7329	8.7347	8.7353
8.7369	8.7376	8.7378	8.7387	8.7396	8.7405	8.7409	8.7421
8.7421	8.7423	8.7431	8.7431	8.7432	8.7433	8.7435	8.7435
8.7436	8.7436	8.7436	8.7436	8.7437	8.7437	8.7437	8.7437

Empirical 95% Confidence Interval: [8.5902, 8.7437] – does not cover $b = 8$.

SAMPLE 7: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.169, \quad \hat{\alpha}_2 = 5.1581, \quad \hat{\alpha}_3 = -0.1106, \quad \hat{\alpha}_4 = 1.4112.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.1, 1.41)$.

Extracted Values from Table: $(\beta_3, \beta_4) = (-0.6068, -0.6456)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.0129$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.0066$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.0066 6.0129, –0.6068, –0.6456))

5.3489	5.6310	5.7384	5.7850	5.8488	5.8505	5.8629	5.9104
5.9228	5.9462	5.9760	5.9776	5.9805	5.9824	5.9861	5.9934
6.0022	6.0043	6.0065	6.0075	6.0096	6.0108	6.0129	6.0135
6.0138	6.0140	6.0142	6.0166	6.0174	6.0175	6.0181	6.0184
6.0185	6.0185	6.0187	6.0191	6.0192	6.0194	6.0195	6.0195

Empirical 95% Confidence Interval: [5.6310, 6.0195] – does not cover $b = 8$.

SAMPLE 8: (MOP)
Rhos: $\hat{\rho}_1 = 3.5215, \quad \hat{\rho}_2 = 5.7158, \quad \hat{\rho}_3 = 0.6726, \quad \hat{\rho}_4 = 0.5934.$

- First Option (Table D–1)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.68, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.6306, 27.7884)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.1526$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 1.4052$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(1.4052, 0.1526, 1.6306, 27.7884))

3.6676	5.2561	5.4275	5.8980	5.9976	6.0041	6.1164	6.5367
6.6224	6.8218	6.8536	6.8801	6.9316	6.9484	7.1652	7.1850
7.1978	7.2149	7.2748	7.4154	7.4457	7.5602	7.5806	7.6112
7.6288	7.6390	7.6932	7.7240	7.7314	7.7417	7.7446	7.7520
7.7755	7.8376	7.8583	7.8761	7.9207	7.9311	7.9415	7.9490

  Empirical 95% Confidence Interval: $[5.2561, 7.9415]$ – does not cover $b = 8$.

- Second Option (Table D–2)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.68, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (3.06435, 1.54108)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.2702$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 4.3508$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(4.3508, 0.2702, 3.06435, 1.54108))

3.6616	5.1983	5.3692	5.8456	5.9480	5.9547	6.0707	6.5108
6.6017	6.8144	6.8486	6.8770	6.9324	6.9505	7.1852	7.2068
7.2206	7.2392	7.3045	7.4582	7.4914	7.6169	7.6393	7.6729
7.6922	7.7034	7.7628	7.7966	7.8047	7.8160	7.8192	7.8273
7.8531	7.9210	7.9436	7.9630	8.0113	8.0226	8.0338	8.0418

Empirical 95% Confidence Interval: [5.1983, 8.0338] – does not cover $b = 8$.

SAMPLE 9: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.6191, \quad \hat{\alpha}_2 = 3.0741, \quad \hat{\alpha}_3 = -0.1596, \quad \hat{\alpha}_4 = 1.4033.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.15, 1.41)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.6143, -0.6698)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 4.6073$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 1.1369$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.1369, 4.6073, –0.6143, –0.6698))

5.5549	5.6282	5.6525	5.6535	5.6754	5.6914	5.7033	5.7054
5.7124	5.7191	5.7215	5.7217	5.7248	5.7249	5.7299	5.7318
5.7367	5.7411	5.7417	5.7418	5.7425	5.7428	5.7429	5.7432
5.7433	5.7434	5.7439	5.7440	5.7440	5.7441	5.7442	5.7442
5.7442	5.7442	5.7442	5.7442	5.7442	5.7442	5.7442	5.7442

Empirical 95% Confidence Interval: [5.6282, 5.7442] – does not cover $b = 8$.

SAMPLE 10: (MOP)
Rhos: $\hat{\rho}_1 = 4.2218, \quad \hat{\rho}_2 = 7.4283, \quad \hat{\rho}_3 = 1.0357, \quad \hat{\rho}_4 = 0.5505.$ (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)

- First Option (Table D–1)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.94, 0.55)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (18.6537, 1.4743)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1296$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 6.9988$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(6.9988, 0.1296, 18.6537, 1.4743))

4.3911	5.8463	5.9767	6.3468	6.4332	6.4391	6.5437	7.0473
7.1841	7.5731	7.6463	7.7099	7.8414	7.8865	8.5898	8.6671
8.7181	8.7881	9.0490	9.7703	9.9483	10.7047	10.8545	11.0891
11.2289	11.3122	11.7760	2.0586	12.1284	12.2270	12.2551	12.3273
12.5625	13.2261	13.4625	13.6721	14.2211	14.3551	14.4912	14.5904

Empirical 95% Confidence Interval: [5.8463, 14.4912] – covers $b = 8$.

- Second Option (Table D–2)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.94, 0.55)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (2.83899, 3.36762)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.194$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 4.1310$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(4.131, 0.194, 2.83899, 3.36762))

4.5076	6.1044	6.2804	6.7772	6.8854	6.8925	7.0159	7.4916
7.5915	7.8277	7.8660	7.8978	7.9602	7.9806	8.2478	8.2727
8.2886	8.3101	8.3855	8.5649	8.6040	8.7520	8.7794	8.8196
8.8428	8.8563	8.9280	8.9690	8.9789	8.9926	8.9965	9.0064
9.0379	9.1215	9.1495	9.1737	9.2342	9.2484	9.2627	9.2729

  Empirical 95% Confidence Interval: [6.1044, 9.2627] – covers $b = 8$.

- Third Option (Table D–3) Not needed.

SAMPLE 11: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.0691$, $\hat{\alpha}_2 = 4.1217$, $\hat{\alpha}_3 = -0.3656$, $\hat{\alpha}_4 = 1.8199$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.375, 1.81)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.1385, -0.4281)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.4673$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.1821$.
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.1821, 6.4673, –0.1385, –0.4281))

6.0202	6.1009	6.1773	6.2582	6.3191	6.3358	6.4215	6.4283
6.4840	6.4897	6.4906	6.5024	6.5088	6.5130	6.5334	6.5360
6.5382	6.5388	6.5491	6.5597	6.5651	6.5788	6.5832	6.5906
6.5956	6.5966	6.6097	6.6179	6.6200	6.6325	6.6348	6.6369
6.6411	6.6451	6.6460	6.6465	6.6481	6.6485	6.6486	6.6489

Empirical 95% Confidence Interval: [6.1009, 6.8486] – does not cover $b = 8$.

SAMPLE 12: (MOM)
Alphas: $\hat{\alpha}_1 = 3.8941$, $\hat{\alpha}_2 = 6.7580$, $\hat{\alpha}_3 = 0.09316$, $\hat{\alpha}_4 = 1.9153$.
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.1, 1.9)$.
Extracted Values from Table: $\lambda_1(0,1) = -1.5049$, $\lambda_2(0,1) = 0.2911$, $\lambda_3 = 0.01562$, $\lambda_4 = 0.8296$
$\lambda_1 = \lambda_1(0,1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1 = -0.181$, $\lambda_2 = \lambda_2(0,1)/\sqrt{\hat{\alpha}_2} = 0.112$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(–1.5049, 0.112, 0.01562, 0.8296))

2.4840	4.2892	4.4739	4.9776	5.0841	5.0911	5.2111	5.6629
5.7559	5.9741	6.0093	6.0385	6.0957	6.1144	6.3592	6.3820
6.3966	6.4163	6.4858	6.6522	6.6887	6.8298	6.8555	6.8945
6.9171	6.9303	7.0015	7.0430	7.0531	7.0672	7.0712	7.0815
7.1144	7.2049	7.2367	7.2649	7.3408	7.3604	7.3813	7.3977

Empirical 95% Confidence Interval: [4.2892, 7.3813] – does not cover $b = 8$.

SAMPLE 13: (MOP)
Rhos: $\hat{\rho}_1 = 4.8407$, $\hat{\rho}_2 = 7.5489$, $\hat{\rho}_3 = 1.5997$, $\hat{\rho}_4 = 0.5505$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)
Table Entry Point : (Table D–2) $(\hat{\rho}_3, \hat{\rho}_4) = (0.62, 0.55)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (20.7842, 2.0233)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1206$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 6.8805$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(6.8805, 0.1206, 20.7842, 2.0233))

5.0095	6.2670	6.3593	6.6066	6.6634	6.6672	6.7365	7.0941
7.1995	7.5164	7.5787	7.6333	7.7480	7.7879	8.4369	8.5110
8.5601	8.6280	8.8838	9.6138	9.7984	10.5998	10.7615	11.0162
11.1691	11.2605	11.7740	12.0902	12.1688	12.2799	12.3116	12.3933
12.6603	13.4219	13.6961	13.9404	14.5855	14.7441	14.9057	15.0238

Empirical 95% Confidence Interval: [6.267, 14.9057] – covers $b = 8$.

SAMPLE 14: (MGBD)
Alphas: $\hat{\alpha}_1 = 2.7441$, $\hat{\alpha}_2 = 4.3711$, $\hat{\alpha}_3 = 0.0670$, $\hat{\alpha}_4 = 1.5433$.
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.075, 1.53)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.4892, -0.4467)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.0123$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.1420$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD((–0.142, 6.0123, –0.4892, –0.4467))

4.2836	4.6603	5.0457	5.1210	5.1422	5.3324	5.3910	5.4044
5.4817	5.4834	5.4862	5.5341	5.6121	5.6178	5.6428	5.6697
5.6800	5.6906	5.7282	5.7357	5.7779	5.7892	5.7962	5.7975
5.8195	5.8502	5.8503	5.8526	5.8574	5.8587	5.8626	5.8637
5.8638	5.8667	5.8674	5.8680	5.8681	5.8684	5.8687	5.8702

Empirical 95% Confidence Interval: [4.6603, 5.8687] – does not cover $b = 8$.

SAMPLE 15: (MOP)
Rhos: $\hat{\rho}_1 = 4.5382$, $\hat{\rho}_2 = 6.3698$, $\hat{\rho}_3 = 1.0229$, $\hat{\rho}_4 = 0.5023$. (We use $1/\hat{\rho}_3$,

since $\hat{\rho}_3 > 1$.)
Table Entry Point : (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.98, 0.526)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (12.7274, 1.8123)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1684$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 6.2282$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(6.2282, 0.1684, 12.7274, 1.8123))

4.6647	5.7375	5.8476	6.1992	6.2889	6.2951	6.4060	6.9419
7.0820	7.4618	7.5303	7.5891	7.7084	7.7488	8.3420	8.4037
8.4439	8.4988	8.6993	9.2242	9.3480	9.8536	9.9502	10.0993
10.1870	10.2389	10.5227	10.6917	10.7330	10.7911	10.8075	10.8498
10.9863	11.3625	11.4937	11.6088	11.9053	11.9766	12.0487	12.1010

Empirical 95% Confidence Interval: [5.7375, 12.0487] – covers $b = 8$.

SAMPLE 16: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.1691, \quad \hat{\alpha}_2 = 5.5369, \quad \hat{\alpha}_3 = -0.0565, \quad \hat{\alpha}_4 = 1.4536.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.05, 1.44)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.6065, -0.6264)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.2580$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.9589$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.9589, 6.2580, –0.6065, –0.6264))

5.5687	6.3125	6.5392	6.7479	6.7779	6.7863	6.8106	6.8652
7.0490	7.0636	7.0739	7.0833	7.0913	7.1075	7.1111	7.1765
7.1784	7.1888	7.1947	7.1949	7.1999	7.2039	7.2065	7.2071
7.2096	7.2100	7.2133	7.2140	7.2159	7.2159	7.2161	7.2162
7.2165	7.2166	7.2167	7.2167	7.2168	7.2168	7.2169	7.2169

Empirical 95% Confidence Interval: [6.3125, 7.2169] – does not cover $b = 8$.

SAMPLE 17: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.6191, \quad \hat{\alpha}_2 = 4.4945, \quad \hat{\alpha}_3 = -0.1006, \quad \hat{\alpha}_4 = 1.6811.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.1, 1.68)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.2056, -0.2888)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.7219$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 1.0724$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.0724, 6.7219, –0.2025, –0.2888))

5.9180	6.3579	6.3768	6.4355	6.4459	6.5804	7.0172	7.1342
7.1813	7.2530	7.3463	7.3953	7.4115	7.4233	7.4286	7.4364
7.4704	7.4925	7.5559	7.5630	7.5821	7.5964	7.6361	7.6377
7.6605	7.6869	7.6914	7.6988	7.7010	7.7207	7.7294	7.7321
7.7522	7.7559	7.7666	7.7781	7.7793	7.7802	7.7846	7.7918

Empirical 95% Confidence Interval: [6.3579, 7.846] – does not cover $b = 8$.

SAMPLE 18: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.6441$, $\hat{\alpha}_2 = 5.5621$, $\hat{\alpha}_3 = -0.1987$, $\hat{\alpha}_4 = 1.6406$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.2, 1.64)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.2978, -0.4346)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.1446$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.3137$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD((–0.3137, 7.1446, –0.2978, –0.4346))

5.7292	6.1380	6.1385	6.2150	6.2845	6.4038	6.4463	6.4628
6.5015	6.5070	6.5519	6.6049	6.6138	6.6331	6.7255	6.7300
6.7384	6.7398	6.7413	6.7431	6.7572	6.7673	6.7699	6.7755
6.7763	6.7794	6.7835	6.7859	6.7993	6.8027	6.8043	6.8146
6.8213	6.8232	6.8239	6.8258	6.8269	6.8275	6.8281	6.8309

Empirical 95% Confidence Interval: [6.1380, 6.8281] – does not cover $b = 8$.

SAMPLE 19: (MOM)
Alphas: $\hat{\alpha}_1 = 4.2691$, $\hat{\alpha}_2 = 4.9598$, $\hat{\alpha}_3 = -0.0689$, $\hat{\alpha}_4 = 1.8087$.
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.05, 1.8)$.
Extracted Values from Table: $\lambda_1(0, 1) = 1.4668$, $\lambda_2(0, 1) = 0.2996$, $\lambda_3 = 0.8558$, $\lambda_4 = 0.02295$
$\lambda_1 = \lambda_1(0, 1)\sqrt{\hat{\alpha}_2} + \hat{\alpha}_1 = 7.5358$, $\lambda_2 = \lambda_2(0, 1)/\sqrt{\hat{\alpha}_2} = 0.134$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(7.5358, 0.1343, 0.8558, 0.02295))

4.4755	5.9384	6.0806	6.4622	6.5417	6.5469	6.6361	6.9683
7.0361	7.1949	7.2204	7.2417	7.2833	7.2969	7.4758	7.4926
7.5034	7.5180	7.5695	7.6950	7.7231	7.8345	7.8554	7.8877
7.9067	7.9180	7.9805	8.0188	8.0283	8.0419	8.0458	8.0558
8.0890	8.1910	8.2325	8.2734	8.4186	8.4733	8.5511	8.6422

Empirical 95% Confidence Interval: [5.9384, 8.5511] – does not cover $b = 8$.

SAMPLE 20: (MOP)
Rhos: $\hat{\rho}_1 = 4.0215$, $\hat{\rho}_2 = 7.3557$, $\hat{\rho}_3 = 1.0448$, $\hat{\rho}_4 = 0.5954$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)

Table Entry Point : (Table D–2) $(\hat{\rho}_3, \hat{\rho}_4) = (0.96, 0.598)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (2.3686, 2.56204)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.2088$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 3.9061$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(3.9061, 0.2088, 2.3686, 2.56024))

4.1999	5.9376	6.1129	6.5894	6.6898	6.6964	6.8094	7.2327
7.3192	7.5209	7.5532	7.5800	7.6323	7.6494	7.8704	7.8908
7.9038	7.9214	7.9828	8.1276	8.1589	8.2776	8.2987	8.3306
8.3489	8.3596	8.4161	8.4484	8.4562	8.4669	8.4700	8.4777
8.5024	8.5677	8.5896	8.6084	8.6555	8.6665	8.6776	8.6855

Empirical 95% Confidence Interval: [5.9376, 8.6776] – does not cover $b = 8$.

SAMPLE 21: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.9191, \quad \hat{\alpha}_2 = 6.3185, \quad \hat{\alpha}_3 = -0.0099, \quad \hat{\alpha}_4 = 1.7043.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0, 1.71)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.1744, -0.1744)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 8.1857$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.1738$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD((–0.1739, 8.1857, –0.1744, –0.1744))

3.0399	5.3281	5.9204	6.1393	6.6460	6.6749	6.9518	6.9730
7.0230	7.0801	7.0925	7.0999	7.1042	7.2233	7.2590	7.2677
7.2989	7.4420	7.4435	7.4820	7.5170	7.6798	7.6990	7.7698
7.7740	7.7991	7.8054	7.8354	7.8502	7.8689	7.9124	7.9186
7.9313	7.9362	7.9395	7.9569	7.9975	8.0002	8.0067	8.0105

Empirical 95% Confidence Interval: [5.3281, 8.0067] – does not cover $b = 8$.

SAMPLE 22: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.5441, \quad \hat{\alpha}_2 = 6.8404, \quad \hat{\alpha}_3 = -0.3925, \quad \hat{\alpha}_4 = 1.7617.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.4, 1.76)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.2635, -0.5203)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.9667$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.2803$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD((–0.2803, 7.9667, –0.2635, –0.5203))

6.7371	6.8436	6.8443	6.8840	7.2633	7.3566	7.4385	7.5057
7.5084	7.5296	7.5354	7.5546	7.5632	7.5689	7.5718	7.6035
7.6234	7.6299	7.6320	7.6381	7.6426	7.6510	7.6552	7.6573
7.6623	7.6720	7.6750	7.6767	7.6769	7.6773	7.6782	7.6783
7.6799	7.6843	7.6846	7.6851	7.6859	7.6862	7.6863	7.6864

Empirical 95% Confidence Interval: [6.8436, 7.6863] – does not cover $b = 8$.

SAMPLE 23: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.5691, \quad \hat{\alpha}_2 = 5.2368, \quad \hat{\alpha}_3 = -0.0016, \quad \hat{\alpha}_4 = 1.67.$
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0, 1.68)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.2273, -0.2273)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.3020$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.0819$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(–0.0819, 7.3020, –0.2273, –0.2273))

5.3559	5.9481	6.0734	6.0838	6.1857	6.2367	6.2879	6.3028
6.4160	6.4724	6.4917	6.5643	6.6336	6.6637	6.6679	6.7131
6.7176	6.7590	6.9324	6.9472	6.9489	6.9506	6.9795	6.9907
6.9985	7.0186	7.0521	7.0758	7.0905	7.1221	7.1389	7.1437
7.1519	7.1540	7.1672	7.1759	7.1832	7.1940	7.2074	7.2170

Empirical 95% Confidence Interval: [5.9481, 7.2074] – does not cover $b = 8$.

SAMPLE 24: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.1941, \quad \hat{\alpha}_2 = 5.8205, \quad \hat{\alpha}_3 = 0.0473, \quad \hat{\alpha}_4 = 1.6787.$
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.05, 1.68)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.2536, -0.2114)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.686$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.4577$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.4577, 7.686, –0.2536, –0.211))

6.3495	6.6575	6.9097	7.1309	7.1351	7.1572	7.1946	7.2104
7.2858	7.3116	7.3798	7.3965	7.4303	7.4968	7.5084	7.5395
7.5481	7.6221	7.6401	7.6683	7.7692	7.7875	7.7968	7.8594
7.9042	7.9444	7.9813	7.9969	8.0188	8.0375	8.0475	8.0778
8.0895	8.0898	8.1124	8.1143	8.1206	8.1278	8.1310	8.1355

Empirical 95% Confidence Interval: [6.6575, 8.1310] – does not cover $b = 8$.

SAMPLE 25: (MOP)
Rhos: $\hat{\rho}_1 = 1.5985, \quad \hat{\rho}_2 = 6.4729, \quad \hat{\rho}_3 = 1.1063, \quad \hat{\rho}_4 = 0.2415.$ (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)
Table Entry Point : (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.90, 0.241)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (8.4448, 7.2918)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1351$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 6.4989$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(6.4989, 0.1351, 8.4448, 7.2918))

6.4943	6.9699	7.0895	7.5468	7.6729	7.6815	7.8389	8.5849
8.7718	9.2591	9.3442	9.4164	9.5614	9.6099	10.2937	10.3620
10.4063	10.4665	10.6829	11.2287	11.3536	11.8493	11.9417	12.0830
12.1653	12.2137	12.4755	12.6289	12.6662	12.7183	12.7331	12.7709
12.8922	13.2213	13.3343	13.4327	13.6834	13.7430	13.8032	13.8466

Empirical 95% Confidence Interval: [6.9699, 13.8032] – covers $b = 8$.

SAMPLE 26: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.8441$, $\hat{\alpha}_2 = 4.6252$, $\hat{\alpha}_3 = -0.1857$, $\hat{\alpha}_4 = 1.6974$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.175, 1.71)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.1505, -0.2403)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.9587$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.1706$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.1706, 6.9587, –0.1505, –0.2403))

4.9969	5.6162	6.0231	6.0373	6.1821	6.2487	6.2727	6.3637
6.3929	6.4231	6.6123	6.6273	6.6676	6.8350	6.8566	6.8667
6.9030	6.9196	6.9197	6.9684	6.9772	6.9822	7.0029	7.0141
7.0147	7.0215	7.0359	7.0625	7.0663	7.0727	7.0768	7.0773
7.0829	7.0994	7.1007	7.1037	7.1076	7.1121	7.1269	7.1272

Empirical 95% Confidence Interval: [5.6162, 7.1269] – does not cover $b = 8$.

SAMPLE 27: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.4691$, $\hat{\alpha}_2 = 5.3632$, $\hat{\alpha}_3 = -0.3460$, $\hat{\alpha}_4 = 1.6291$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.35, 1.63)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.3978, -0.573)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.6956$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.5514$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.5514, 6.6956, –0.3978, –0.573))

6.7767	6.8430	6.8641	6.9462	6.9933	7.0520	7.0588	7.1856
7.1890	7.1919	7.2015	7.2032	7.2063	7.2086	7.2115	7.2153
7.2179	7.2236	7.2282	7.2284	7.2285	7.2288	7.2302	7.2327
7.2327	7.2338	7.2377	7.2378	7.2392	7.2415	7.2430	7.2435
7.2457	7.2465	7.2467	7.2467	7.2468	7.2468	7.2468	7.2470

Empirical 95% Confidence Interval: [6.8430, 7.2468] – does not cover $b = 8$.

SAMPLE 28: (MOP)
Rhos: $\hat{\rho}_1 = 3.4735$, $\hat{\rho}_2 = 1.4018$, $\hat{\rho}_3 = 1.0635$, $\hat{\rho}_4 = 0.6226$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)

- First Option (Table D–1)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.94, 0.622)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (43.3598, 1.0967)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.5858$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 4.2717$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(4.2717, 0.5858, 43.3598, 1.0967))

3.5100	3.8520	3.8842	3.9689	3.9862	3.9873	4.0065	4.0778
4.0928	4.1304	4.1371	4.1428	4.1546	4.1587	4.2258	4.2339
4.2394	4.2470	4.2773	4.3767	4.4053	4.5473	4.5796	4.6331
4.6667	4.6873	4.8111	4.8940	4.9154	4.9462	4.9551	4.9783
5.0565	5.3004	5.3957	5.4841	5.7329	5.7975	5.8648	5.9148

  Empirical 95% Confidence Interval: [3.8520, 5.5648] – does not cover $b = 8$.

- Second Option (Table D–2)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.44, 0.622)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.94929, 1.16511)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 1.1551$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 3.6354$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(3.6354, 1.1551, 1.94929, 1.16511))

3.5109	3.8989	3.9393	4.0488	4.0718	4.0733	4.0991	4.1946
4.2139	4.2585	4.2656	4.2715	4.2829	4.2866	4.3344	4.3387
4.3415	4.3452	4.3583	4.3887	4.3952	4.4196	4.4240	4.4304
4.4341	4.4363	4.4476	4.4541	4.4556	4.4577	4.4583	4.4599
4.4647	4.4774	4.4816	4.4852	4.4940	4.4960	4.4980	4.4994

  Empirical 95% Confidence Interval: [3.8989, 4.4980] – does not cover $b = 8$.

- Third Option (Table D–3)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.44\,0.622)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (0.88746, 1.02218)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 1.1300$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 3.4309$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(3.4309, 1.1300, 0.88746, 1.02218))

3.5098	3.8542	3.8873	3.9749	3.9930	3.9941	4.0142	4.0876
4.1022	4.1358	4.1411	4.1455	4.1540	4.1568	4.1923	4.1955
4.1976	4.2003	4.2099	4.2324	4.2372	4.2552	4.2583	4.2631
4.2658	4.2674	4.2758	4.2805	4.2817	4.2833	4.2837	4.2848
4.2884	4.2979	4.3010	4.3037	4.3103	4.3119	4.3134	4.3145

Empirical 95% Confidence Interval: [3.8542, 4.3134] – does not cover $b = 8$.

SAMPLE 29: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.1191, \quad \hat{\alpha}_2 = 6.5549, \quad \hat{\alpha}_3 = 0.0382, \quad \hat{\alpha}_4 = 1.5690.$
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.05, 1.56)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.4363, -0.4052)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.5257$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.4573$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.4573, 7.5257, –0.4363, –0.4052))

6.1907	6.4197	6.4583	6.8022	7.1752	7.4635	7.4674	7.4748
7.5271	7.7729	7.7816	7.8240	7.8359	7.8407	7.8454	7.8465
7.8526	7.8559	7.8572	7.8643	7.8702	7.8778	7.8877	7.9031
7.9081	7.9187	7.9190	7.9239	7.9410	7.9415	7.9418	7.9546
7.9583	7.9632	7.9695	7.9707	7.9736	7.9778	7.9804	7.9812

Empirical 95% Confidence Interval: [6.4197, 7.9804] – does not cover $b = 8$.

SAMPLE 30: (MOP)
Rhos: $\hat{\rho}_1 = 3.0538, \quad \hat{\rho}_2 = 7.0541, \quad \hat{\rho}_3 = 0.7065, \quad \hat{\rho}_4 = 0.5896.$

- First Option (Table D–1)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.70, 0.586)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.6587, 25.0667)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.1260$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 5.5675$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(5.5675, 0.126, 1.6587, 25.0667))

8.2157	10.1167	10.3227	10.8890	11.0090	11.0169	11.1523	11.6595
11.7630	12.0039	12.0424	12.0744	12.1367	12.1570	12.4192	12.4433
12.4587	12.4794	12.5519	12.7221	12.7588	12.8975	12.9222	12.9593
12.9806	12.9930	13.0586	13.0960	13.1049	13.1174	13.1209	13.1299
13.1584	13.2336	13.2588	13.2804	13.3344	13.3470	13.3596	13.3687

  Empirical 95% Confidence Interval: [10.1167, 13.3596] – covers $b = 8$.

- Second Option (Table D–2)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.70, 0.586)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (3.36261, 1.69831)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.2151$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 4.0344$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(4.0344, 0.2151, 3.36261, 1.69831))

3.2206	5.0583	5.2658	5.8495	5.9759	5.9842	6.1281	6.6786
6.7933	7.0634	7.1069	7.1432	7.2140	7.2372	7.5390	7.5669
7.5848	7.6089	7.6935	7.8937	7.9372	8.1020	8.1315	8.1759
8.2014	8.2162	8.2950	8.3400	8.3508	8.3658	8.3701	8.3809
8.4153	8.5062	8.5366	8.5628	8.6282	8.6435	8.6588	8.6698

  Empirical 95% Confidence Interval: [5.0583, 8.6588] – does not cover $b = 8$.

- Third Option (Table D–3)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.70, 0.586)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (0.01009, 0.58203)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.0993$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = -0.2192$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD((–0.2192, 0.0993, 0.01009, 0.58203))

3.2230	5.0356	5.2367	5.8053	5.9297	5.9379	6.0804	6.6391
6.7591	7.0490	7.0968	7.1370	7.2161	7.2422	7.5949	7.6289
7.6509	7.6806	7.7867	8.0511	8.1114	8.3530	8.3987	8.4694
8.5110	8.5357	8.6719	8.7546	8.7750	8.8040	8.8122	8.8335
8.9032	9.1076	9.1852	9.2574	9.4747	9.5396	9.6160	9.6846

  Empirical 95% Confidence Interval: [5.0356, 9.6160] – covers $b = 8$.

SAMPLE 31: (MOP)
Rhos: $\hat{\rho}_1 = 5.2441$, $\hat{\rho}_2 = 2.7283$, $\hat{\rho}_3 = 1.1371$, $\hat{\rho}_4 = 0.5948$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)

- First Option (Table D–1)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.88, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (27.8295, 1.2979)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.3208$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 3.9961$

The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(3.9961, 0.3208, 27.8295, 1.2979))

2.7966	3.4040	3.4578	3.5983	3.6276	3.6295	3.6628	3.8028
3.8380	3.9387	3.9581	3.9751	4.0108	4.0232	4.2308	4.2554
4.2717	4.2944	4.3814	4.6420	4.7108	5.0215	5.0865	5.1906
5.2539	5.2920	5.5106	5.6486	5.6833	5.7326	5.7467	5.7833
5.9037	6.2564	6.3866	6.5039	6.8195	6.8983	6.9791	7.0384

Empirical 95% Confidence Interval: [3.4040, 6.9791] – does not cover $b = 8$.

- Second Option (Table D–2)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.88, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (2.15460, 2.74901)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.5632$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 5.1094$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(.1094, 0.5632, 2.15460, 2.74901))

5.3101	5.9362	5.9979	6.1644	6.1993	6.2016	6.2408	6.3868
6.4166	6.4858	6.4968	6.5060	6.5239	6.5297	6.6052	6.6121
6.6166	6.6226	6.6435	6.6927	6.7034	6.7436	6.7508	6.7616
6.7678	6.7715	6.7906	6.8015	6.8041	6.8078	6.8088	6.8115
6.8198	6.8419	6.8493	6.8556	6.8715	6.8752	6.8790	6.8816

  Empirical 95% Confidence Interval: [5.9362, 6.8790] – does not cover $b = 8$.

- Third Option (Table D–3)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.88, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (0.50286, 0.83472)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.5145$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 3.9961$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(4.9622, 0.5145, 0.50286, 0.83472))

5.3107	5.9439	6.0055	6.1710	6.2055	6.2077	6.2464	6.3897
6.4188	6.4865	6.4973	6.5063	6.5238	6.5295	6.6037	6.6106
6.6150	6.6209	6.6416	6.6909	6.7016	6.7427	6.7501	6.7613
6.7678	6.7715	6.7918	6.8035	6.8064	6.8103	6.8115	6.8143
6.8235	6.8485	6.8572	6.8648	6.8850	6.8901	6.8954	6.8996

  Empirical 95% Confidence Interval: [5.9439, 6.8954] – does not cover $b = 8$.

- Fourth Option (Table D–4)
  Table Entry Point: $(\hat{\rho}_3, \hat{\rho}_4) = (0.88, 0.598)$.
  Extracted Values from Table: $(\lambda_3, \lambda_4) = (0.75741, 0.03444)$.
  $\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.3010$
  $\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 6.5227$
  The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(6.5227, 0.3010, 0.75741, 0.03444))

5.3108	5.9390	5.9999	6.1640	6.1983	6.2006	6.2392	6.3840
6.4139	6.4844	6.4958	6.5053	6.5240	6.5301	6.6115	6.6193
6.6243	6.6310	6.6550	6.7142	6.7277	6.7820	6.7924	6.8086
6.8182	6.8239	6.8559	6.8758	6.8809	6.8880	6.8901	6.8954
6.9132	6.9691	6.9925	7.0159	7.1011	7.1339	7.1810	7.2365

  Empirical 95% Confidence Interval: [5.9390, 7.1810] – does not cover $b = 8$.

SAMPLE 32: (MOP)
Rhos: $\hat{\rho}_1 = 5.1454$, $\hat{\rho}_2 = 6.1564$, $\hat{\rho}_3 = 2.2184$, $\hat{\rho}_4 = 0.6055$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)
Table Entry Point: (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.46, 0.610)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (37.5208, 2.0083)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.133$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 7.0143$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(7.0143, 0.133, 37.5208, 2.0083))

5.2989	6.4385	6.5169	6.6987	6.7317	6.7338	6.7692	6.8979
6.9281	7.0185	7.0372	7.0540	7.0907	7.1039	7.3581	7.3921
7.4152	7.4479	7.5791	8.0227	8.1510	8.7816	8.9232	9.1560
9.3012	9.3900	9.9167	10.2641	10.3531	10.4808	10.5176	10.6133
10.9340	11.9151	12.2922	12.6385	13.6000	13.8466	14.1022	14.2916

Empirical 95% Confidence Interval: [6.4385, 14.1022] – covers $b = 8$.

SAMPLE 33: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.2191$, $\hat{\alpha}_2 = 6.6345$, $\hat{\alpha}_3 = 0.2239$, $\hat{\alpha}_4 = 1.5728$.
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.225, 1.58)$.
Extracted Values from Table: $(\beta_3, \beta_4) = ()$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = -0.5324$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.4060$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(1.4766, 6.2264, –0.5324, –0.406))

6.0649	6.0798	6.1324	6.3342	6.3690	6.4096	6.6911	6.7110
6.7311	6.7598	6.9083	6.9110	7.2876	7.2910	7.3539	7.3639
7.4015	7.4267	7.4658	7.4759	7.4818	7.4870	7.4909	7.5012
7.5073	7.5457	7.5798	7.5870	7.5901	7.5991	7.6470	7.6528
7.6540	7.6687	7.6785	7.6836	7.6903	7.6909	7.6950	7.6995

Empirical 95% Confidence Interval: [6.0798, 7.6950] – does not cover $b = 8$.

SAMPLE 34: (MGBD)
Alphas: $\hat{\alpha}_1 = 4.0441$, $\hat{\alpha}_2 = 7.9072$, $\hat{\alpha}_3 = -0.2539$, $\hat{\alpha}_4 = 1.3319$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.25, 1.34)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.7327, -0.7926)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 6.8846$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.1674$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.1674, 6.8846, –0.7327, –0.7926))

6.1752	6.6854	6.9672	6.9867	7.0171	7.0418	7.0423	7.0445
7.0445	7.0485	7.0495	7.0501	7.0511	7.0512	7.0512	7.0513
7.0516	7.0516	7.0518	7.0519	7.0519	7.0519	7.0519	7.0520
7.0520	7.0520	7.0520	7.0520	7.0520	7.0520	7.0520	7.0520
7.0520	7.0520	7.0520	7.0520	7.0520	7.0520	7.0520	7.0520

Empirical 95% Confidence Interval: [6.6854, 7.0520] – does not cover $b = 8$.

SAMPLE 35: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.8691$, $\hat{\alpha}_2 = 6.9112$, $\hat{\alpha}_3 = -0.1136$, $\hat{\alpha}_4 = 1.5026$.
Table Entry Point: $(|\hat{\alpha}_3|, \hat{\alpha}_4) = (0.125, 1.5)$.
Extracted Values from Table: $(\beta_3, \beta_4) = (-0.4917, -0.5545)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.3653$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = -0.056$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(–0.056, 7.3653, –0.4917, –0.5545))

4.9585	6.1720	6.6667	7.0409	7.0422	7.0605	7.0936	7.1471
7.1474	7.1882	7.2106	7.2172	7.2175	7.2263	7.2536	7.2676
7.2689	7.2691	7.2836	7.2848	7.2879	7.2888	7.2952	7.2959
7.2962	7.2981	7.3006	7.3046	7.3058	7.3069	7.3075	7.3077
7.3077	7.3079	7.3080	7.3086	7.3088	7.3090	7.3092	7.3093

Empirical 95% Confidence Interval: [6.1720, 7.3092] – does not cover $b = 8$.

SAMPLE 36: (MGBD)
Alphas: $\hat{\alpha}_1 = 3.6941$, $\hat{\alpha}_2 = 6.6822$, $\hat{\alpha}_3 = 0.2021$, $\hat{\alpha}_4 = 1.6143$.
Table Entry Point: $(\hat{\alpha}_3, \hat{\alpha}_4) = (0.2, 1.61)$.

Extracted Values from Table: $(\beta_3, \beta_4) = (-0.4735, -0.3475)$.
$\beta_2 = (\beta_3 + \beta_4 + 2)\sqrt{(\beta_3 + \beta_4 + 3)\hat{\alpha}_2/((\beta_3 + 1)(\beta_4 + 1))} = 7.6763$
$\beta_1 = \hat{\alpha}_1 - \beta_2(\beta_3 + 1)/(\beta_3 + \beta_2 + 2) = 0.267$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGBD(0.267, 7.6763, –0.4735, –0.3472))

5.5203	5.9860	6.2683	6.4682	6.5066	6.7023	6.8371	6.9284
7.0061	7.1491	7.2442	7.2848	7.3610	7.4071	7.4591	7.4952
7.5402	7.5848	7.6514	7.7154	7.7412	7.7591	7.7599	7.7737
7.7970	7.8083	7.8180	7.8274	7.8574	7.8687	7.8689	7.8810
7.9051	7.9112	7.9197	7.9250	7.9262	7.9342	7.9411	7.9432

Empirical 95% Confidence Interval: [5.9860, 7.9411] – does not cover $b = 8$.

SAMPLE 37: (MOP)
Rhos: $\hat{\rho}_1 = 3.0918$, $\hat{\rho}_2 = 7.3721$, $\hat{\rho}_3 = 0.5512$, $\hat{\rho}_4 = 0.5870$.
Table Entry Point: (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.56, 0.586)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.9265, 26.9273)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1171$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 0.8453$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(0.8453, 0.1171, 1.9265, 26.9273))

3.2761	5.4022	5.6427	6.3125	6.4560	6.4654	6.6279	7.2422
7.3686	7.6642	7.7116	7.7509	7.8278	7.8529	8.1778	8.2077
8.2269	8.2526	8.3429	8.5556	8.6016	8.7755	8.8066	8.8533
8.8800	8.8957	8.9783	9.0255	9.0368	9.0526	9.0570	9.0683
9.1043	9.1995	9.2313	9.2587	9.3271	9.3431	9.3592	9.3707

Empirical 95% Confidence Interval: [5.4022, 9.3592] – covers $b = 8$.

SAMPLE 38: (MOP)
Rhos: $\hat{\rho}_1 = 2.7469$, $\hat{\rho}_2 = 7.5537$, $\hat{\rho}_3 = 0.4862$, $\hat{\rho}_4 = 0.5098$.
Table Entry Point: (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.48, 0.514)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (2.5694, 20.1614)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2($ with $u = 0.1) = 0.1165$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 1.30083$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(1.3008, 0.1165, 2.5694, 20.1614))

2.9073	5.0150	5.2788	6.0362	6.2027	6.2137	6.4041	7.1396
7.2940	7.6589	7.7179	7.7671	7.8632	7.8946	8.3056	8.3437
8.3681	8.4011	8.5167	8.7910	8.8506	9.0773	9.1180	9.1791
9.2143	9.2348	9.3437	9.4059	9.4209	9.4417	9.4476	9.4626
9.5103	9.6367	9.6790	9.7156	9.8070	9.8284	9.8499	9.8653

Empirical 95% Confidence Interval: [5.015, 9.8499] – covers $b = 8$.

SAMPLE 39: (MOP)
Rhos: $\hat{\rho}_1 = 4.5684$, $\hat{\rho}_2 = 7.5273$, $\hat{\rho}_3 = 1.3262$, $\hat{\rho}_4 = 0.6152$. (We use $1/\hat{\rho}_3$, since $\hat{\rho}_3 > 1$.)
Table Entry Point: (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.76, 0.61)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (31.0673, 1.4307)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.1144$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 7.811$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(7.811, 0.1144, 31.0673, 1.4307))

4.7605	6.4097	6.5488	6.9016	6.9724	6.9770	7.0562	7.3730
7.4501	7.6702	7.7131	7.7507	7.8304	7.8584	8.3402	8.3990
8.4383	8.4932	8.7065	9.3678	9.5472	10.3793	10.5574	10.8447
11.0210	11.1277	11.7461	12.1421	12.2423	12.3852	12.4261	12.5323
12.8844	13.9305	14.3217	14.6766	15.6414	15.8844	16.1345	16.3188

Empirical 95% Confidence Interval: [6.4097, 16.1345] – covers $b = 8$.

SAMPLE 40: (MOP)
Rhos: $\hat{\rho}_1 = 3.5215$, $\hat{\rho}_2 = 6.1158$, $\hat{\rho}_3 = 0.6907$, $\hat{\rho}_4 = 0.5342$.
Table Entry Point: (Table D–1) $(\hat{\rho}_3, \hat{\rho}_4) = (0.70, 0.538)$.
Extracted Values from Table: $(\lambda_3, \lambda_4) = (1.9530, 18.5336)$.
$\lambda_2 = [(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}]/\hat{\rho}_2(\text{ with } u = 0.1) = 0.1545$
$\lambda_1 = \hat{\rho}_1 - [(0.5)^{\lambda_3} - (0.5)^{\lambda_4}]/\lambda_2 = 1.8498$
The 40 Ordered Estimates: (each based on a resample of size 10, obtained through RSGLD(1.8498, 0.1545, 1.953, 18.5336))

3.6606	5.2739	5.4572	5.9682	6.0778	6.0850	6.2091	6.6789
6.7757	7.0020	7.0384	7.0685	7.1274	7.1466	7.3957	7.4186
7.4333	7.4530	7.5223	7.6855	7.7207	7.8542	7.8781	7.9139
7.9345	7.9464	8.0099	8.0461	8.0548	8.0669	8.0703	8.0790
8.1067	8.1798	8.2042	8.2252	8.2778	8.2901	8.3024	8.3113

Empirical 95% Confidence Interval: [5.2739, 8.3024] – covers $b = 8$.

Chapter 22

Generalized Bootstrap Confidence Intervals for High Quantiles

Bin Wang
bwang@jaguar1.usouthal.edu

Satya N. Mishra
mishra@jaguar1.usouthal.edu

Madhuri S. Mulekar
mmulekar@jaguar1.usouthal.edu

Nutan S. Mishra
nmishra@usouthal.edu

Kun Huang
kunkunh@gmail.com

Department of Mathematics and Statistics,
University of South Alabama
ILB 325, University Blvd
Mobile, AL 36688-0002 USA

22.1 Introduction

22.1.1 High Quantile Estimation

Let $X = \{X_1, X_2, \ldots, X_n\}$ be a random sample from a probability distribution $F(x)$. For any $0 < p < 1$, let ξ_p be the p-th *quantile of level* p of $F(x)$, which is any root of the equation $F(\xi_p) = p$. When the quantile level p is high, ξ_p is called a **high quantile**. High quantile estimation has many practical applications. In real life, it is often necessary to evaluate the risk of large but possibly rare losses such as earthquakes and hurricanes and the safety of nuclear power stations. In foodborne disease control, risk assessors are usually more interested in high quantiles of contaminant exposure distributions rather than the averages of the distributions (Albert and Gauchi (2002)); accurate estimation of the far right tail of the future lifetime distribution of the annuitants is crucial in setting adequate insurance premiums in life annuities (Einmahl, Fils-Villetard, and Guillou (2008)). In a value-at-risk context, precisely predicting the probability of an extreme movement in the value of a portfolio is essential for both risk management and regulatory purpose (Gençay, Selçuk, and Ulugülyağci (2003)). Applications of high quantile estimation include customer wallet estimation (Perlich, Rosset, Lawrence, and Zadrozny (2007)), network performance evaluation (Choi, Moon, Cruz, Zhang, and Diot (2006)), and many others. For patients with life-threatening diseases, an estimate of the mean or median lifetime of patients with the disease might be too optimistic for patients with very bad conditions and too discouraging for otherwise very healthy patients. People may be more interested in the answer to a question such as, "How long can the top 10% of very healthy men/women survive AIDS?"

Usually, high quantiles are located at the boundary of an observed sample or beyond the range of the sample. The accuracy of high quantile estimation relies on the quality of modeling the distribution F, especially the right-tail behavior of F. It is well known that the tail behavior of F is closely related to the asymptotic distribution of the sample extreme values. Fitting extreme value distributions is a well recognized, specialized, and highly technical area of statistical theory (Mudholkar and Kollia (1994)). It is easy to see that for random samples of small or moderate sizes, the probability that ξ_p falls beyond the range of the sample could be large when p is close to 1. The traditional approach of an order statistic or a weighted average of two order statistics is not suitable. One needs to infer beyond these observations via extrapolating from intermediate quantiles when the underlying distribution has a regularly varying tail. In practice, the functional form of F, or equivalently the probability density function f, is seldom known to the researchers. Due to the fact that the empirical distribution function (e.d.f.) $F_n(x) = 1$ for any $x \geq \max(X)$, many techniques (such as the non-parametric bootstrapping) won't work to make inferences of ξ_p. The main idea behind most, if not all, estimators for high quantiles is first to select some auxiliary pilot estimates

of a quantile inside the range of the sample, then to move this pilot estimate to the right (Markovich (2007)). If f is heavy-tailed (that is, its tail decays to 0 slower than an exponential tail), with the violation of Cramér's condition, many results of large deviation theory can not be used. This makes inferences on high quantiles even harder.

Most of the existing methods for high quantile estimation address cases when p is close to 1 and the estimators are derived based on asymptotic tail models such as the extreme value distribution, which is based on the distribution of the largest order statistic. Based on the extreme value distribution, three typical high quantile estimators have been proposed in literature: (1) the peaks over thresholds high quantile estimator by (McNeil and Saladin (1997) and McNeil, Frey, and Embrechts (2005)); (2) the Pareto tail high quantile estimator by (Weissman (1978)); (3) the combined parametric-nonparametric method by (Markovich and Krieger (2002)). In this study, we will focus on quantiles with level around 0.90.

22.1.2 The Bootstrap Method

The bootstrapping is a form of sampling procedure for statistical inference. It was first proposed in Efron (1979). Since then, many theoretical contributions have been made to the development of the bootstrap. The key idea behind the bootstrap is that if a sample is representative of the underlying population, then one can make inferences about the population characteristics by resampling from the current sample.

Algorithm 22.1.1 (Bootstrapping Procedure) *Let $\theta = T(F)$ be a feature of F of interest. The basic steps in the bootstrap procedure are*

Step 1 Estimate the distribution function $F(x)$ based on the observed sample X, and denote by $\widehat{F}$.

Step 2 Draw a bootstrap sample, $X^* = \{X_1^*, X_2^*, \ldots, X_n^*\}$ from $\widehat{F}$.

Step 3 Compute θ^*, which is the $\hat{\theta}$ value based on the above bootstrap sample X^*.

Step 4 Repeat Steps 2 and 3 for B number of times.

Step 5 Using B values of θ^* obtained in Steps 2 through 4, prepare the sampling distribution of θ^*.

Step 6 Make inferences about $\hat{\theta}$ based on this sampling distribution.

Depending on whether $\widehat{F}$ is estimated parametrically or non-parametrically, the bootstrap method can be classified as parametric bootstrap method and non-parametric bootstrap method.

- Non-parametric bootstrap (NPB)

In nonparametric bootstrap, no assumption is made about the functional form of F or its probability density function f; and F is estimated by the empirical distribution function F_n defined as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x),$$

where $I(A)$ is an indicator function of event A. By the *strong law of large numbers*, if $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid), $F_n(x)$ converges to $F(x)$ almost surely for any x. Glivenko (1933) and Cantelli (1933) strengthened this result by proving uniform convergence of F_n to F. The accuracy of an estimator θ is assessed in terms of $F_n(x)$ and the inferences about θ are made based on the sampling distribution of $\hat{\theta}$, which is obtained by sampling independently with replacement from F_n and computing the $\hat{\theta}$ values from the bootstrap samples.

- Parametric bootstrap (PB)

  In parametric bootstrap, the functional form of F is assumed to be known with a vector of unknown parameters λ. F can be the distribution function from a well-known family of distributions. The parameters λ can be estimated by the maximum likelihood method or the moment method, and the estimator is denoted by $\hat{\lambda}$. Then B bootstrap samples are drawn repeatedly from the distribution $\hat{F}(x; \hat{\lambda})$ and $\hat{\theta}$ is computed from each sample and inferences about $\hat{\theta}$ are made based on the sampling distribution of $\hat{\theta}$ generated from B values.

The only difference between the nonparametric bootstrap and parametric bootstrap is how F is estimated. Nonparametric bootstrap is distribution free and does not depends on a particular class of distributions. Thus nonparametric bootstrap is more robust and can reduce the risk of model mis-specification. However, it is poor in estimating extremes or higher/lower quantiles due to the fact $F_n(x) = 1$ for any $x \geq \max(X_1, X_2, \cdots, X_n)$. Parametric bootstrap is capable of estimating beyond the range of the sample, however, its application in practice is limited due to the fact that $F(x; \lambda)$ is seldom known to the researchers.

In this chapter, estimation of high quantiles using a more general parametric bootstrap procedure known as the *generalized bootstrap* (GB) is studied. The rest of this chapter is arranged as follows. In Section 22.2, the generalized lambda distribution (GLD), GB, and the methods to fit a GLD to data are discussed. The performance of GB is compared with that of PB and NPB through a simulation study involving 95% confidence intervals (CIs). The settings and algorithm of the simulation study are presented in Section 22.3. In Section 22.4 the simulation results are discussed in detail. Summary results along with suggestions for further research are presented in Section 22.5. The **R** and C source codes are available in the Appendix at the end of this chapter.

22.2 Generalized Lambda Distribution and Generalized Bootstrap

The GB method was first discussed by Karian and Dudewicz (1991) and formally introduced by Dudewicz (1992). It is a type of parametric bootstrap where F is estimated by fitting a GLD to the sample data. In GB, instead of assuming that the observed data are from a classical family of distributions such as the normal, beta, or gamma, the underlying distribution is assumed to follow a GLD. The GLD was developed from the Tukey-Lambda family of distributions originally proposed by Tukey (1960). It was generalized to multi-parameter extensions by Hogben (1963), Shapiro and Wilk (1965), and Joiner and Rosenblatt (1971) among others. The GLD specifies a distribution by a *quantile function* as follows:

$$Q(y) = F^{-1}(y) = \inf\{x : F(x) \geq y\} \quad \text{for } 0 \leq y \leq 1.$$

Ramberg and Schmeiser (1972) extended the Tukey-lambda family of distributions to a four-parameter GLD, which is also known as Ramberg-Schmeiser-Tukey (RST) lambda distribution,

$$Q(y; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left[y^{\lambda_3} - (1-y)^{\lambda_4}\right] \quad \text{for } 0 \leq y \leq 1, \qquad (22.2.1)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a vector of parameters, which includes a shift, a spread, a skewness, and a kurtosis parameter, respectively. The GLD in (22.2.1) is very rich in shapes and tails. It can provide good approximations to most well-known distributions such as the normal, uniform, student's t, exponential, chi-square, gamma, Weibull, lognormal, beta, F, extreme value distribution, Pareto, the largest extreme value distribution, and many other distributions. However, fitting a GLD to a given data set is not an easy task. Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979) first used the method of moments to estimate the GLD parameters. Mykytka and Ramberg (1979) modified this method to allow the use of measures of location, scale, symmetry, and tail weight as functions of order statistics. Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979) also suggested a *least squares method* to estimate the GLD parameters by minimizing the squared distance between the percentile function of a GLD and the empirical percentile function. Öztürk and Dale (1985) showed by simulation that no appreciable differences can be found between the least squares method and method of moments in terms of accuracy and precision of the fits. King and MacGillivray (1999) applied a starship method (Owen (1988)) on GLD fitting, which is built on the fact that the inverse transformation of the GLD, $Q^{-1}(x; \lambda)$, is a uniform distribution. The λ value that makes $Q^{-1}(x; \lambda)$ the closest to a uniform distribution or passes the Kolmogorov-Smirnov goodness-of-fit test is regarded as the starship estimate of λ.

It is worth noting that not all combinations of λ_1, λ_2, λ_3, and λ_4 will lead to valid distributions. The GLD in (22.2.1) specifies a valid distribution if and only

if

$$g(y, \lambda_3, \lambda_4) \equiv \lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1 - y)^{\lambda_4 - 1}$$

has the same sign for all y in $[0, 1]$, as long as λ_2 takes that sign also (Corollary 1.3.9 of Karian and Dudewicz (2000), page 12). The GLD fitting was further extended to cover the distributions that are not attainable by the method of moments in Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979) by fitting an *extended* GLD (Karian, Dudewicz, and McDonald (1996), Karian and Dudewicz (1999) and (2000)). Instead of fitting a GLD by matching the first four moments, Karian and Dudewicz (2003) introduced an L-moment method by matching the first four L-moments (Greenwood, Landwehr, Matalas, and Wallis (1979) and Hosking (1990)). Karvanen and Nuutinen (2008) showed that the L-moment method can attain a wider subset of GLD than the moment-matching method.

The methods of moments, percentiles, and L-moments to fit a GLD or extended GLD work well in most situations. Each of these GLD fitting methods has advantages over the others for distributions in different regions of the parameter space of GLD. The method of L-moments could be extremely useful for biased samples. In this study, we simply choose a maximum likelihood-based method to fit a GLD due to the availability of an **R** package (Su (2007a) and (2007b)), which uses the moment estimates as the initial values in its algorithm.

Theoretically speaking, for a GLD $Q(y; \lambda)$, the density at $X = x$ can be computed by

$$f(x; \lambda) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 y^{\lambda_4 - 1}} \quad \text{at } Q(y; \lambda) = x.$$

However, this density function does not have a general closed form in terms of x and λ. Since the quantile y can be evaluated numerically for any given x using the quantile function in (22.2.1), the following likelihood function,

$$L(\lambda | x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \frac{\lambda_2}{\lambda_3 y_i^{\lambda_3 - 1} + \lambda_4 y_i^{\lambda_4 - 1}}$$

can also be numerically evaluated. By maximizing $L(\lambda | x_1, \cdots, x_n)$, a maximum likelihood estimate (MLE) of λ can be obtained numerically.

Let $\hat{\lambda}$ be a point estimate of λ. Upon the success of a GLD fitting, bootstrap samples can be drawn independently from the fitted GLD, $\hat{Q}(y; \hat{\lambda})$; and inferences about $\hat{\theta}$ can be made using the sampling distribution of θ^* obtained from B bootstrap samples from $\hat{Q}(y; \hat{\lambda})$.

22.3 Comparisons of Bootstrap Confidence Intervals for High Quantiles

In this chapter, we compare the bootstrap confidence intervals for high quantiles by three different bootstrap methods, namely (a) *NPB*, where the empirical distribution function $F_n(x)$ is used to estimate $F(x)$; (b) *PB*, where the functional

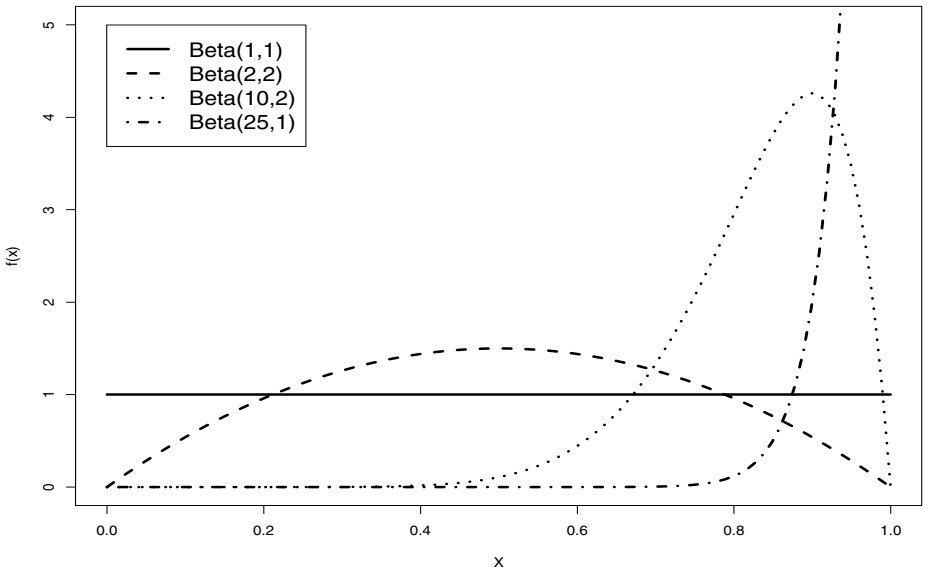


Figure 22.3–1. The beta distributions used in simulation.

form of $F(x)$ is assumed to be known with unknown parameter(s); (c) *GB*, where $F(x)$ is assumed to have functional form as in (22.2.1) and λ is estimated by an MLE.

22.3.1 Simulated Distributions

To compare the performances of the three types of bootstrap confidence intervals, we considered simulated populations with different right-tail behaviors. Distributions with limited supports were considered by choosing the beta distribution with different shape and scale parameters. The probability density function of the beta distribution is:

$$f(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad \text{for } x \in (0,1) \text{ and } a > 0, b > 0.$$

Figure 22.3–1 shows the density curves of four beta distributions used to simulate the high quantile estimation. They are

- $beta(1,1)$ is the *uniform(0,1)* as well.
- $beta(2,2)$ is a symmetric inverted bowl-shaped distribution over $[0,1]$ with decreasing densities toward the lower and upper boundaries.
- $beta(10,2)$ is a left-skewed distribution with mode at 0.9, which is a quantile with level $p = 0.697$.
- $beta(25,1)$ is a more skewed distribution than is $beta(10,2)$, and its density is monotone increasing over $[0,1]$.

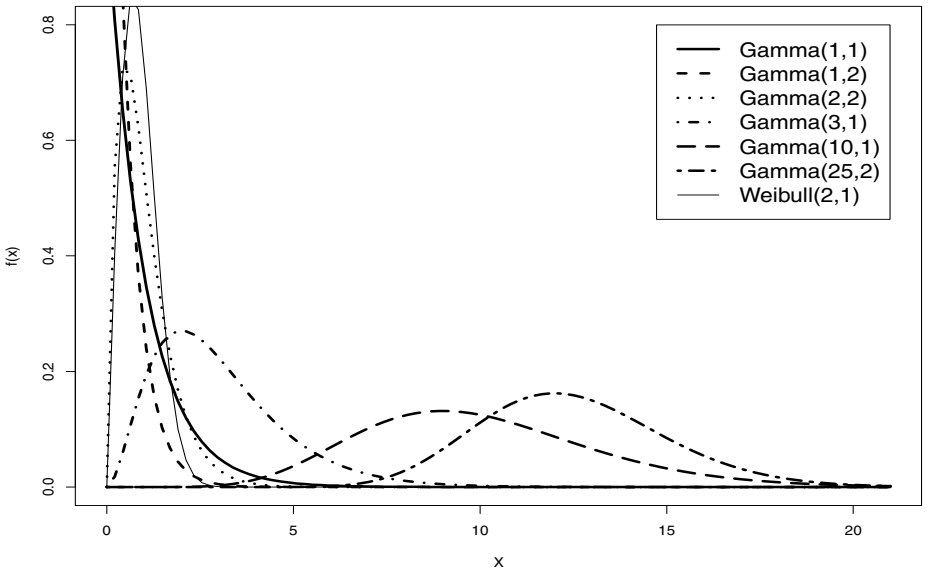


Figure 22.3–2. The gamma and Weibull distributions used in simulation.

Six gamma distributions with different shape and scale parameters were also used to simulate the high quantile estimation for distributions with infinite upper boundaries. The probability density function of the gamma distribution is:

$$f(x;a,b) = x^{a-2}\frac{e^{-x/b}}{b^a\Gamma(a)} \quad \text{for } x > 0 \text{ and } a > 0, b > 0.$$

From Figure 22.3–2, we see that the heaviness of the right tails of these gamma distributions varies with different shape parameters. When the shape parameter is as large as 25 and scale parameter is 2, the density curve is less skewed. Two special cases, $Weibull(2,1)$ and $N(3,1)$ distributions are also used in the simulation study. The probability density function of the Weibull distribution is:

$$f(x;a,b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-(x/b)^a} \quad \text{for } x \geq 0 \text{ and } a > 0, b > 0.$$

In Figure 22.3–2, the $Weibull(2,1)$ distribution was plotted as the thin solid curve, while the normal curve was not plotted.

22.3.2 Choices of Quantile Levels and Sample Sizes

According to Theorem 7.4.21 in Dudewicz and Mishra (1988, page 374),

$$\left(\frac{n}{p(1-p)}\right)^{1/2} f(\xi_p)(\hat{\xi}_p - \xi_p)$$

is asymptotically $N(0,1)$, where $f(x)$ is the probability density function. The standard deviation of $\hat{\xi}_p$ is proportional to

$$\beta = \left(\frac{p(1-p)}{nf^2(\xi_p)}\right)^{1/2}. \tag{22.3.1}$$

From (22.3.1) we see that when n is large, the standard deviation of $\hat{\xi}_p$ tends to be small. Which is reasonable because we will have more information when a larger sample is available, and further have less uncertainty on the estimates. In this simulation, we choose the following sample sizes $n = 10$, 30, 50, 75, and 100.

When p is very large, say close to 1, $p(1-p)$ tends to be small. For distributions whose right tail decay is slower than $p(1-p)$, the standard deviation of $\hat{\xi}_p$ decreases when p increases. However, if the right tail of a distribution decays faster than $p(1-p)$, this standard deviation might increase when p increase.

Lemma 22.3.1 *For a distribution satisfying*

$$\lim_{p\to 1} f(\xi_p) \to 0,$$

when $n \to \infty$, the standard deviation of $\hat{\xi}_p$ does not change with p when $p \to 1$, if the right tail of the distribution decays at a rate satisfying

$$\lim_{p\to 1} f'(\xi_p) = -c, \quad c > 0. \tag{22.3.2}$$

The standard deviation increases when p increases if

$$\lim_{p\to 1} f'(\xi_p) = 0. \tag{22.3.3}$$

Otherwise, the standard deviation decreases when p increases if

$$\lim_{p\to 1} f'(\xi_p) \to -\infty. \tag{22.3.4}$$

Proof By L'Hôpital's rule, we have

$$\lim_{p\to 1} \frac{p(1-p)}{f^2(\xi_p)} = \lim_{p\to 1} \frac{1-F(\xi_p)}{f^2(\xi_p)} = \lim_{p\to 1} -\frac{1}{2f'(\xi_p)}.$$

When $f'(\xi_p)$ is a negative constant, $f^2(\xi_p)$ and $p(1-p)$ decay at the same rate and p will not affect the asymptotic standard deviation of $\hat{\xi}_p$. While $f'(\xi_p) \to 0$ when $p \to 1$, $f^2(\xi_p)$ decays faster than $p(1-p)$ and β increases with p. Otherwise, if $f'(\xi_p) \to -\infty$ when $p \to 1$, β decreases with p.

For example,

- for $beta(1,1)$, $f(x) = 1$ for any $x \in [0,1]$, and $p(1-p)/f^2(\xi_p)$ goes to 0 at the same rate as p when p goes to 1.

- for $beta(2,2)$, $\lim_{p\to 1} f'(\xi_p) = -6$.
- for $beta(10,2)$, $\lim_{p\to 1} f'(\xi_p) = -66$.
- for $beta(25,1)$, $f(x)$ is an increasing function, so $p(1-p)/f^2(\xi_p)$ goes to 0 at the same rate as p when p goes to 1.
- for $gamma(a,b)$, $\lim_{p\to 1} f'(\xi_p) = 0$.
- for $N(3,1)$, $\lim_{p\to 1} f'(\xi_p) = 0$.
- for all heavy-tailed distributions, whose tails decay slower than the exponential distribution, also have $\lim_{p\to 1} f'(\xi_p) = 0$.

In this simulation study, we will focus on quantiles with intermediate levels. The following quantile levels were selected: $p = 0.90, 0.95, 0.975, 0.99$, and 0.995. The median ($p = 0.5$) is included in the simulation as well. The lower quantiles were not studied here because of the following two reasons. First, for symmetric distributions, bootstrap methods GB, PB, and NPB are expected to have similar performance in estimating $\hat{\xi}_p$ and $\hat{\xi}_{1-p}$. Second, the performance in estimating ξ_p for left-skewed distributions is expected to be similar to that in estimating ξ_{1-p} for right-skewed distributions. To estimate a low quantile ξ_{1-p} based on X, one can estimate the high quantile ξ_p based on $-X$ instead.

When p is close to 1.0, estimators should be derived based on asymptotic tail models such as the extreme value distribution. When the quantile level is not too high or too low, say close to the median with $p \to 0.5$, the lack of capability to infer beyond the observed data of NPB won't be a big problem, and NPB is supposed to have good performance in terms of bias and variance. However, the problem of lack of capability of inferring beyond the scope of the observations by NPB will become more severe when the sample size is small and/or the quantile level is high.

22.3.3 Criteria for Performance Evaluation

The performances of the PB, GB, and NPB were compared by comparing the 95% bootstrap confidence intervals for ξ_p. Three criteria used are

1. *Cover Rate:* The cover rate measures the likelihood that the 95% CI covers the true values of ξ_p. In the simulation study, we approximate the cover rate by
$$\bar{R} = \frac{b}{B},$$
where B is the total number of bootstrap samples generated, and b is the number of bootstrap samples based on which the bootstrap confidence intervals cover the true quantile ξ_p.

2. *Average Width:* The average width measures the precision of the 95% bootstrap confidence interval. At a cost of increasing the width of the confidence interval, the coverage rate can be increased. However, if a bootstrap confidence interval is too wide, it is less useful in practice. The average width is computed by taking the average of the width of the bootstrap confidence intervals based on B bootstrap samples. We denote the average width by $\bar{W}$.

3. *Standard Deviation* (of the width of the bootstrap confidence interval): The dispersion of the width of the bootstrap confidence intervals, denoted by s_W, were also computed and compared.

A better method is expected to have a higher $\bar{R}$ value and smaller values of $\bar{W}$ and s_W.

22.3.4 Simulation Algorithms

For each target population $F(x)$, the following algorithm was used in the simulation:

Algorithm 22.3.1 (Simulation Algorithm) *The distribution function of the underlying distribution $F(x, \theta)$ is known. We denote the p-th quantile of $F(x, \theta)$ by ξ_p. For fixed sample size n,*

Step 1: Generate a random sample $X = \{X_1, X_2, \ldots, X_n\}$ from $F(x, \theta)$.

Step 2: Based on X, estimate F by $F_n(x)$, $\hat{F}(x; \hat{\theta})$ and $\hat{Q}(y, \hat{\lambda})$ for NPB, PB, and GB, respectively.

Step 3: Draw one random sample $X^* = \{X_1^*, X_2^*, \ldots, X_n^*\}$ of size n with replacement from each of the $F_n(x)$, $\hat{F}(x; \hat{\theta})$ and $\hat{Q}(y, \hat{\lambda})$, respectively. Based on X^*, compute $\hat{\xi}_p$ for $p = 0.50, 0.90, 0.95, 0.975, 0.99$, and 0.995.

Step 4: Repeat Step 3 for B times. For each of the three bootstrap methods and each p value, find the 95% confidence interval of $\hat{\xi}_p$ by the percentile method, where the lower- and upper-limit of the 95% confidence interval were computed by taking the 2.5th and 97.5th order statistics of $\hat{\xi}_p$ based on B values of $\hat{\xi}_p$.

Step 5: For each of the three bootstrap methods and each p value, compute W for each confidence interval in Step 4. Also let $R = 1$ if the confidence interval covers ξ_p; and $R = 0$ otherwise.

Step 6: Repeat Steps 1–5 for M times. For each of the three bootstrap methods and each p value, compute $\bar{R}$, $\bar{W}$, and s_W, respectively.

In this simulation study, the default CRAN R random number generator (RNG), *Mersenne-Twister*, was used. It is a twisted GFSR with period $2^{19937} - 1$

and equidistribution in 623 consecutive dimensions (over the whole period). The "seed" is a 624-dimensional set of 32-bit integers plus a current position in that set (Matsumoto and Nishimura, (1998)). In Step 1 of the simulation algorithm, we used the built-in **R** functions `rnorm, rweibull, rbeta`, and `rgamma` to generate random samples from the normal, Weibull, beta, and gamma distributions, respectively.

In this simulation study, we used $B = 2,000$ and $M = 1,000$, respectively. In Step 3, the following estimator is used to compute ξ_p based on a random sample X of size n,

$$\hat{\xi}_p = X_{(r)} + q \cdot (X_{(r+1)} - X_{(r)}), \tag{22.3.5}$$

where r is the integer part of $1 + (n-1)p$, $q = (n-1)p - (r-1)$ is the proper fraction, and $X_{(r)}$ is the r^{th} order statistic of X. This algorithm is adopted by statistical software **R** and is the default method to compute quantiles of the **R** built-in `quantile` function (Hyndman and Fan (1996)).

22.4 Simulation Results and Discussion

The detailed simulation results for $N(3,1)$ are listed in Table 22.4–1. All the simulation results are graphically shown in Figures 22.4–2 through 22.4–13. In each figure, the solid lines with circles demonstrate the results by GB, the dotted lines with upside-down triangles show the results by PB (SPB in the figures), and the dashed lines with crosses present the results by NPB. In columns from left to right, results are presented for $p = 0.50, 0.90, 0.95, 0.975, 0.99$, and 0.995, respectively. In each column, the three plots from top to bottom show the cover rates, average widths of bootstrap confidence interval, and standard deviations of the widths of the bootstrap confidence interval for PB, GB, and NPB for different sample sizes. The sample sizes are plotted on the x-axis.

Some of the simulation results have been discussed in detail in Wang, Mishra, Mulekar, Mishra, and Huang (2010). Here we restate some of the conclusions as follows:

1. For all simulated populations and at different quantile levels, all three methods have better performance when the sample size increased in terms of the cover rate, average CI width and standard deviation of CI width, which is consistent with the conclusion in (22.3.1): with a larger sample, we have more information and less uncertainty in estimating ξ_p.

2. For fixed sample sizes, all three methods have worse performances as the quantile level increased, especially when the sample sizes are small.

3. Due to the fact that PB uses extra information, i.e., the known functional form of the distribution (density) function, it showed overall better performance as expected. Throughout this study, the performance of PB will be used as a reference in the comparisons.

Table 22.4–1. Simulation results for $N(3, 1)$.

		GB			PB			NPB		
n	Level	$\bar{R}$	$\bar{W}$	s_W	$\bar{R}$	$\bar{W}$	s_W	$\bar{R}$	$\bar{W}$	s_W
10	0.50	0.923	1.377	0.327	0.952	1.408	0.267	0.938	1.447	0.353
10	0.90	0.686	1.51	0.736	0.884	1.727	0.327	0.653	1.343	0.446
10	0.95	0.535	1.6	1.124	0.826	1.825	0.344	0.403	1.158	0.431
10	0.975	0.432	1.73	1.397	0.774	1.994	0.377	0.227	1.102	0.439
10	0.99	0.356	1.828	1.573	0.647	2.126	0.401	0.09	1.073	0.446
10	0.995	0.316	1.865	1.634	0.533	2.174	0.41	0.044	1.064	0.449
30	0.50	0.963	0.934	0.171	0.973	0.86	0.091	0.933	0.86	0.175
30	0.90	0.889	1.007	0.249	0.95	1.123	0.118	0.858	1.001	0.272
30	0.95	0.773	1.061	0.392	0.936	1.27	0.136	0.792	1.135	0.361
30	0.975	0.636	1.154	0.55	0.922	1.431	0.154	0.518	0.97	0.34
30	0.99	0.503	1.298	0.761	0.88	1.649	0.176	0.254	0.873	0.339
30	0.995	0.436	1.38	0.85	0.803	1.77	0.189	0.13	0.849	0.34
50	0.50	0.967	0.707	0.102	0.983	0.678	0.056	0.949	0.681	0.128
50	0.90	0.926	0.842	0.156	0.967	0.901	0.075	0.904	0.853	0.214
50	0.95	0.858	0.94	0.254	0.96	1.045	0.087	0.874	1.024	0.29
50	0.975	0.77	1.088	0.386	0.955	1.239	0.106	0.718	1.018	0.35
50	0.99	0.64	1.239	0.558	0.938	1.427	0.119	0.398	0.841	0.327
50	0.995	0.56	1.37	0.672	0.912	1.59	0.134	0.228	0.796	0.328
75	0.50	0.976	0.573	0.065	0.982	0.561	0.039	0.946	0.556	0.099
75	0.90	0.952	0.71	0.102	0.975	0.741	0.051	0.935	0.738	0.161
75	0.95	0.916	0.828	0.164	0.975	0.884	0.061	0.894	0.878	0.234
75	0.975	0.854	0.977	0.252	0.979	1.058	0.073	0.856	1.035	0.322
75	0.99	0.742	1.15	0.382	0.97	1.263	0.088	0.53	0.85	0.307
75	0.995	0.654	1.288	0.493	0.956	1.426	0.1	0.312	0.768	0.306
100	0.50	0.974	0.491	0.049	0.984	0.484	0.03	0.948	0.484	0.079
100	0.90	0.957	0.636	0.079	0.978	0.651	0.04	0.932	0.654	0.141
100	0.95	0.94	0.757	0.131	0.982	0.789	0.049	0.888	0.743	0.189
100	0.975	0.886	0.879	0.201	0.983	0.931	0.058	0.876	0.904	0.252
100	0.99	0.803	1.117	0.326	0.986	1.202	0.075	0.636	0.888	0.311
100	0.995	0.712	1.198	0.414	0.974	1.308	0.081	0.398	0.76	0.294

4. When $p = 0.5$, ξ_p is the median, and all three methods perform well with similar performances.

22.4.1 Performances for the Beta Distributions

Figures 22.4–2 through 22.4–5 show the simulation results for the four beta distributions. From Figure 22.4–2 we find that the cover rates of GB and NPB drop faster than that of PB. When $p = 0.99$ and 0.995, GB has higher cover rates than NPB at different sample sizes. When the sample size reaches 100, GB and NPB have similar cover rates. GB has a similar cover rate as PB when the sample size is small; and the performance of GB does not improve much as n increases.

For $beta(2, 2)$, as shown in Figure 22.4–3, GB has obviously better performance than NPB and are inferior to PB as expected. The $beta(10, 2)$ distribution is left skewed and data are relatively more dense at the right tail. For $beta(10, 2)$, the differences in cover rates for GB and NPB become more noticeable than those for $beta(2, 2)$. For $beta(25, 1)$, the density reaches its maximum at $x = 1$ and data are expected to be dense near $x = 1$. As a result, $\hat{\xi}_p$ is over-estimated and the cover rates of GB and NPB are low (see Figure 22.4–5). The performance of GB is very similar to that of NPB for $beta(25, 1)$, which is poor. For all four beta distributions, the standard deviations of the CI width show a decreasing trend as n increases. For $beta(1, 1)$ and $beta(25, 1)$, the standard deviations of the CI widths show a decreasing trend as p increases, which is not very obvious for $beta(2, 2)$, and $beta(10, 2)$ when p is large.

22.4.2 Performances for the Gamma, Weibull, and Normal Distributions

All the gamma, Weibull, and normal distributions studied have long right tails. In Figure 22.4–6, we see that GB attains a cover rate as high as PB for $gamma(1, 1)$. However, the average CI widths of GB are larger with large variances, especially for large quantile levels and small sample sizes. Figure 22.4–7 through 22.4–11 show the results for the other gamma distributions. We find that for $gamma(1, 1)$ and $gamma(1, 2)$, GB has a much higher cover rate than NPB, and is close to that of PB at a cost of larger average CI width.

For $gamma(2, 2)$, $gamma(3, 1)$, $gamma(10, 1)$, and $gamma(25, 2)$, in terms of cover rate GB outperforms NPB in estimating quantiles with high levels. Also, the average CI widths of GB are not significantly different from those of PB, and they are even smaller than those of PB for high quantile levels.

GB also performed well for $Weibull(2, 1)$ (Figure 24.4–12), and $N(3, 1)$ (Figure 22.4–13). For $p = 0.99$ and 0.995, GB has significantly higher cover rates than NPB, and smaller average CI widths than those of PB.

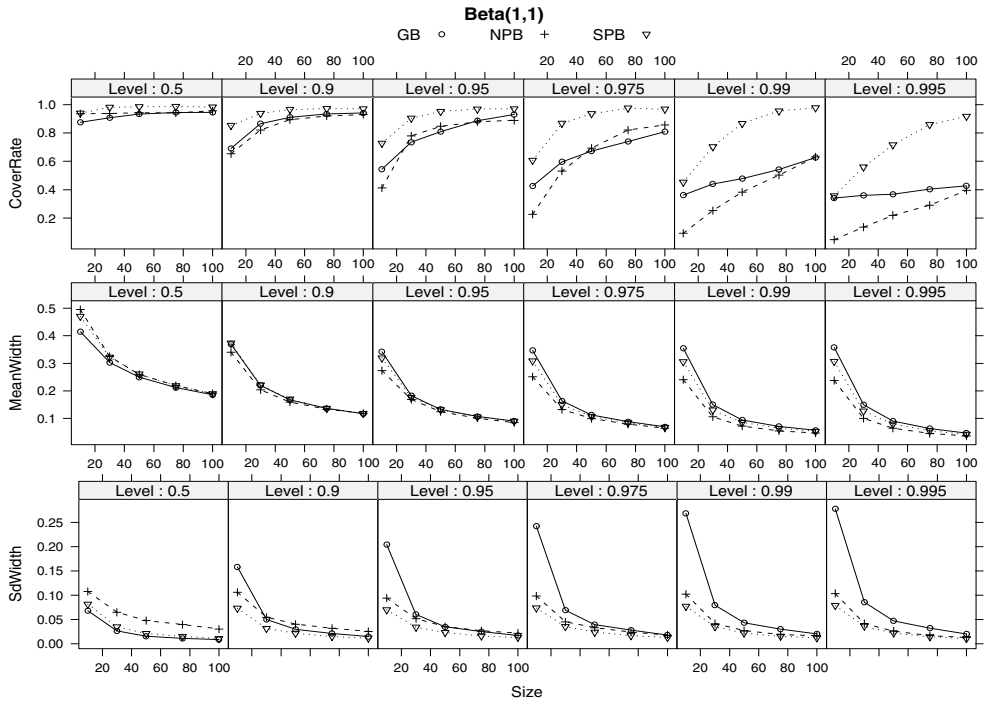


Figure 22.4–2. Simulation results for *beta*(1,1).

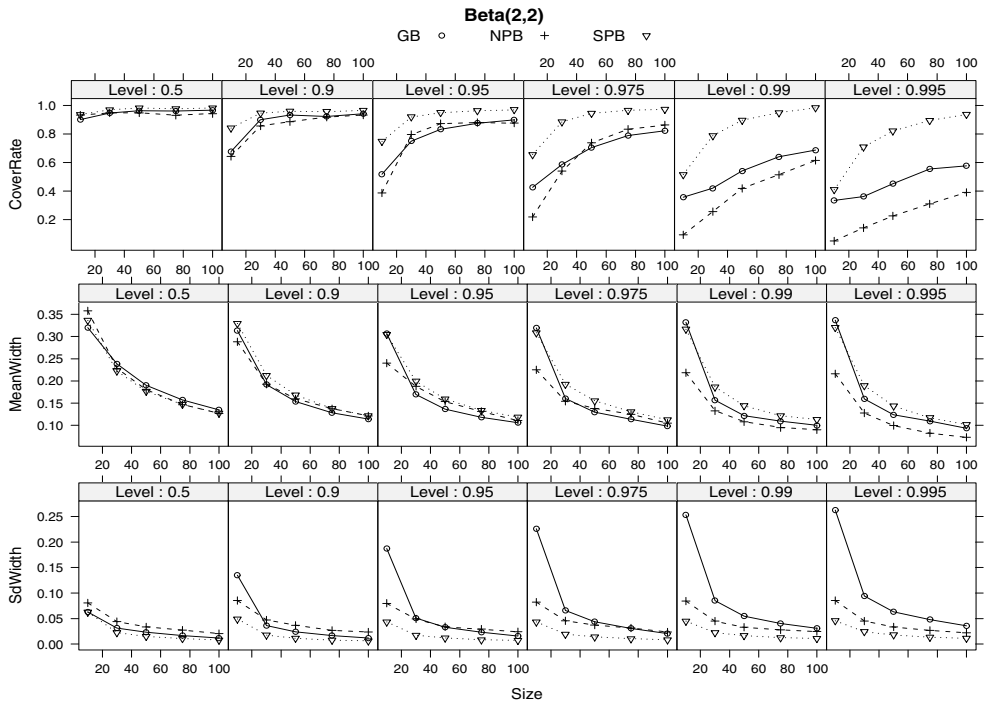


Figure 22.4–3. Simulation results for *beta*(2,2).

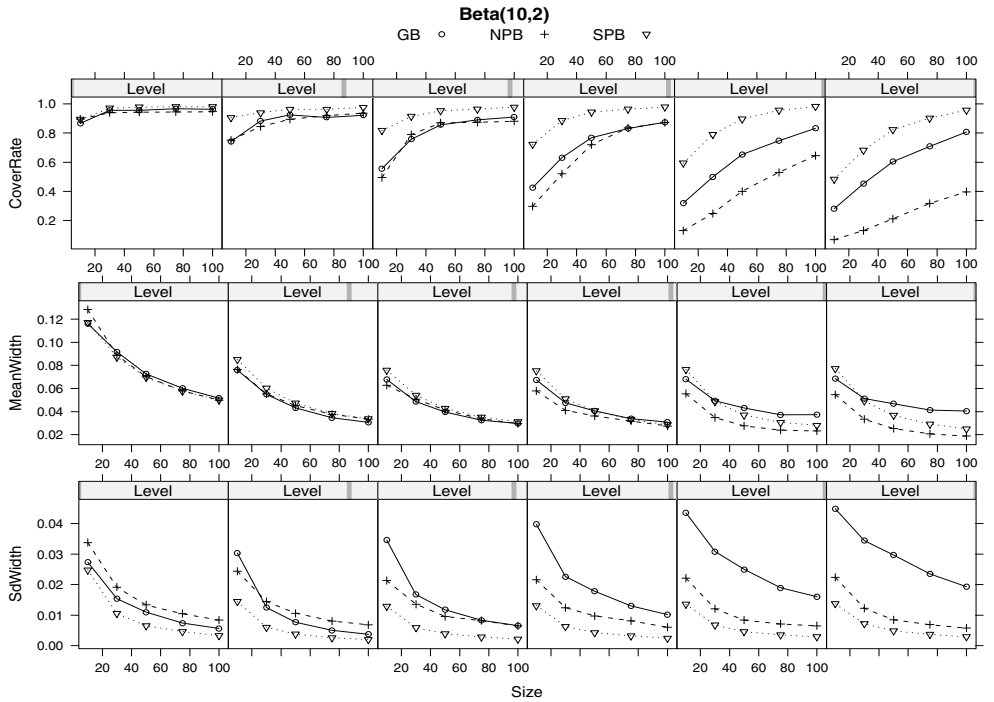


Figure 22.4–4. Simulation results for *beta*(10,2).

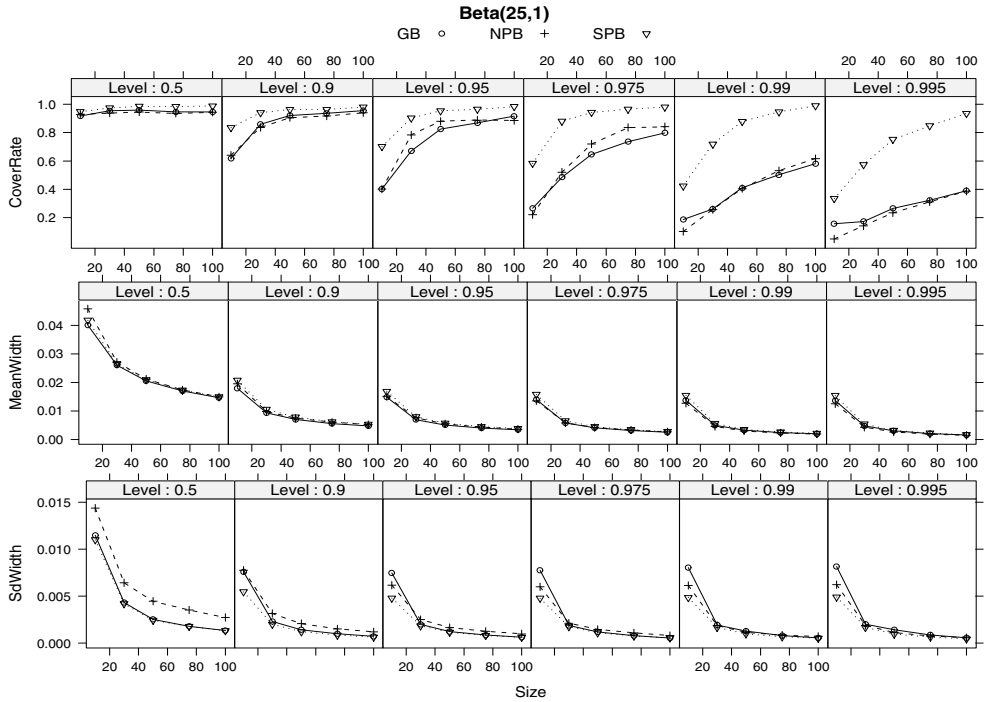


Figure 22.4–5. Simulation results for *beta*(25,1).

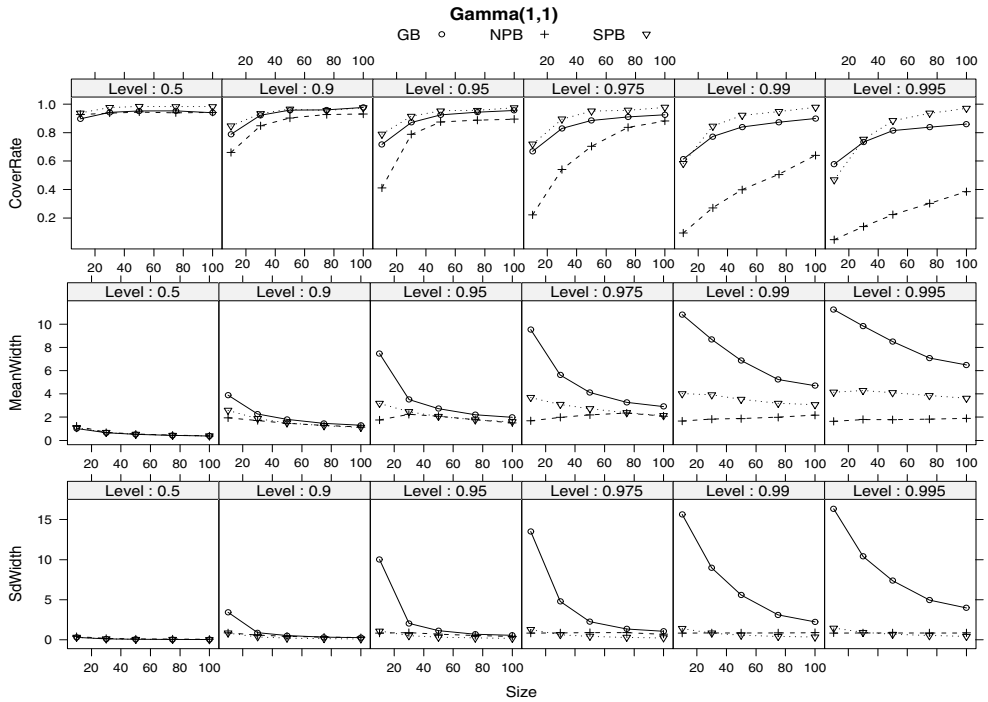


Figure 22.4–6. Simulation results for *gamma*(1,1).

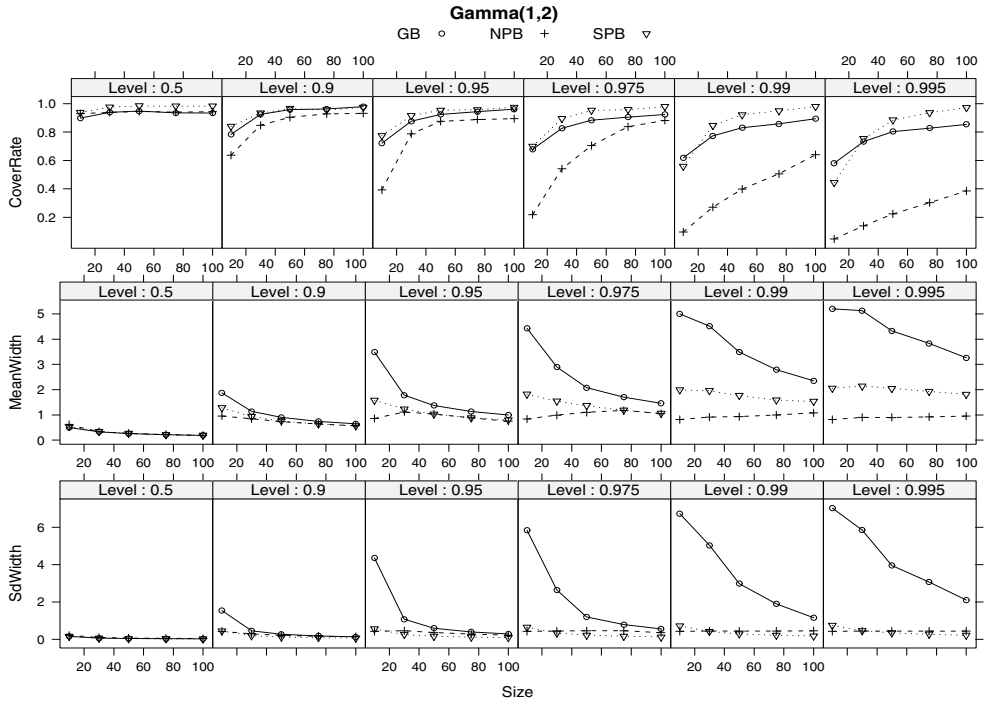


Figure 22.4–7. Simulation results for *gamma*(1,2).

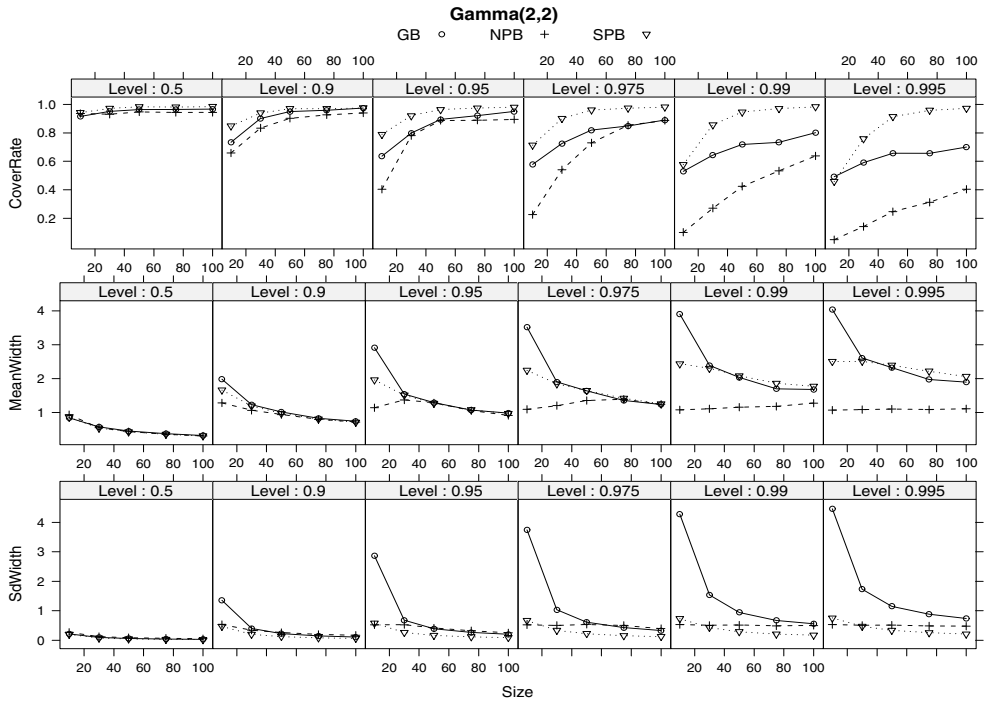


Figure 22.4–8. Simulation results for *gamma*(2,2).

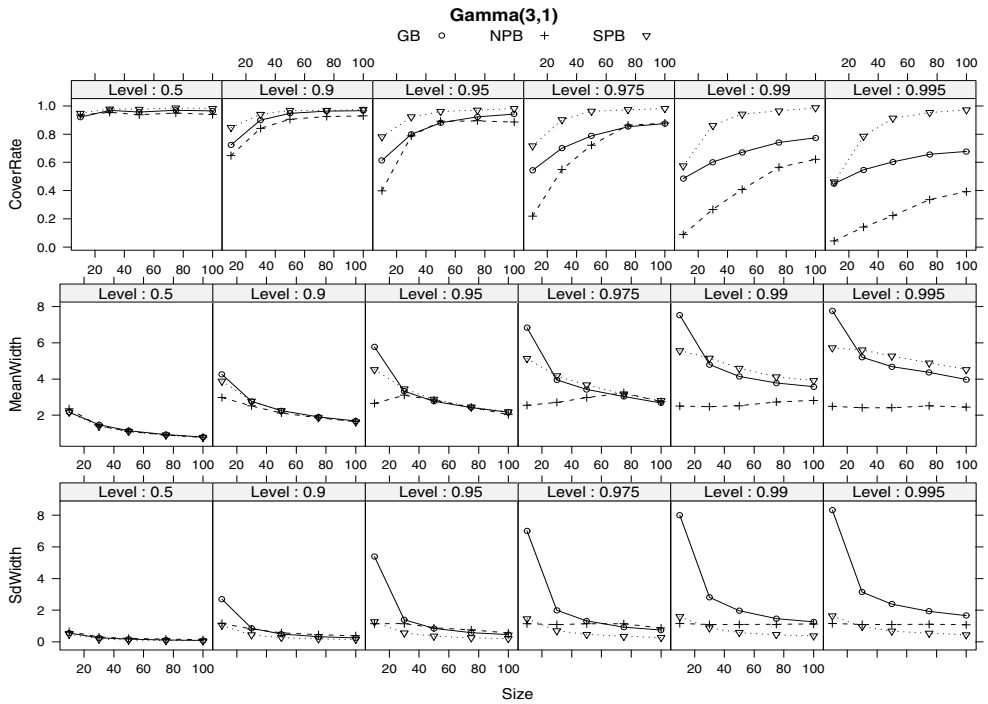


Figure 22.4–9. Simulation results for *gamma*(3,1).

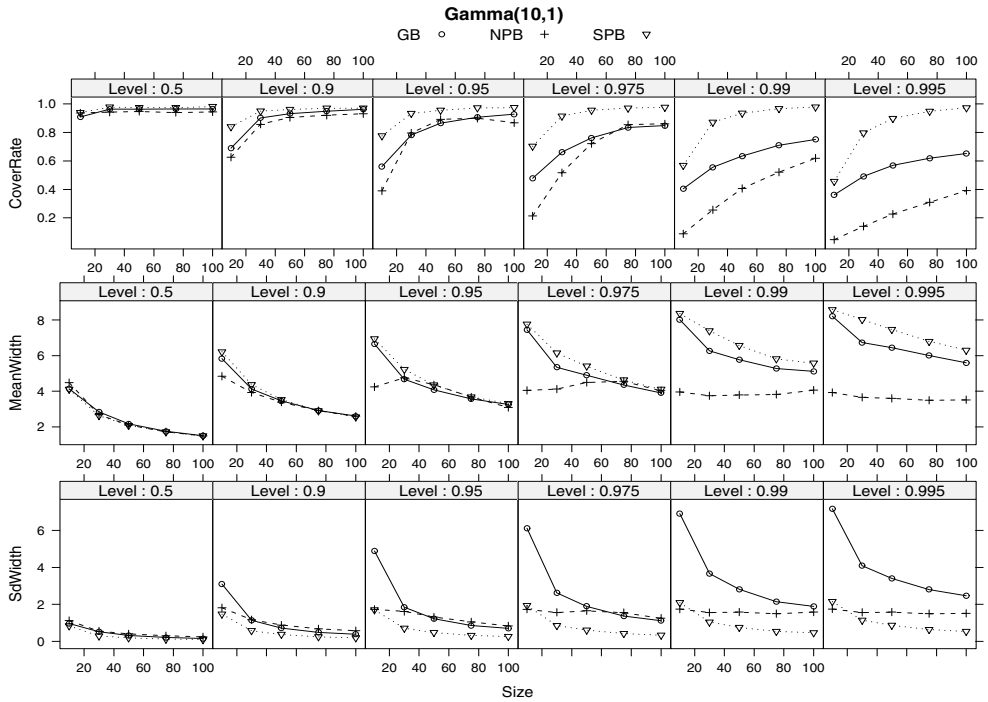


Figure 22.4–10. Simulation results for *gamma*(10,1).

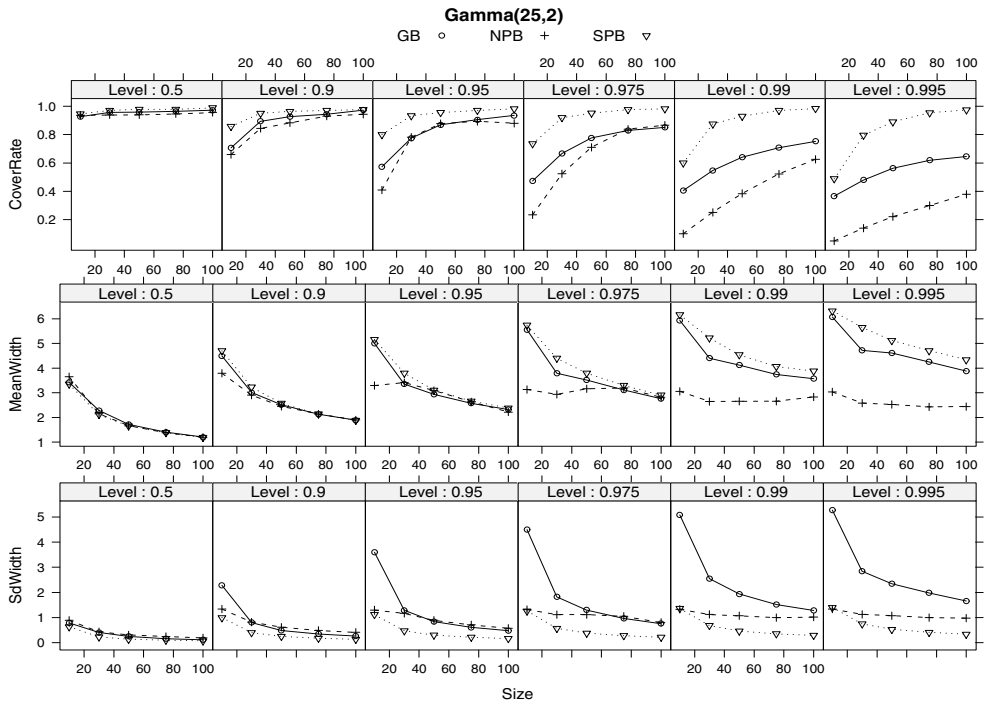


Figure 22.4–11. Simulation results for *gamma*(25,2).

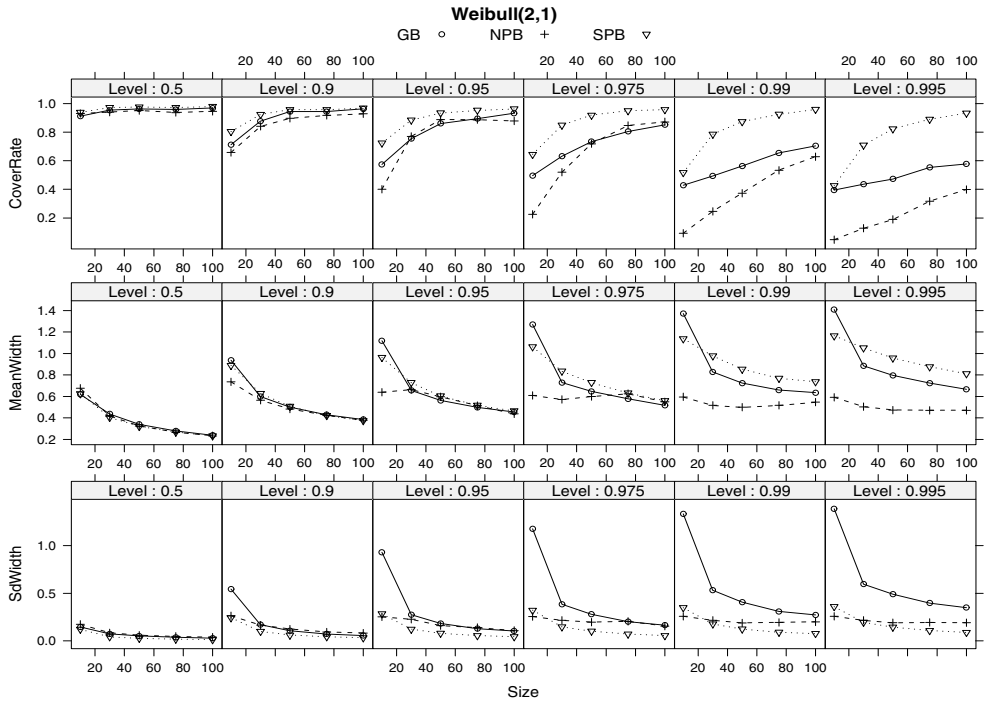


Figure 22.4–12. Simulation results for *Weibull*(2,1).

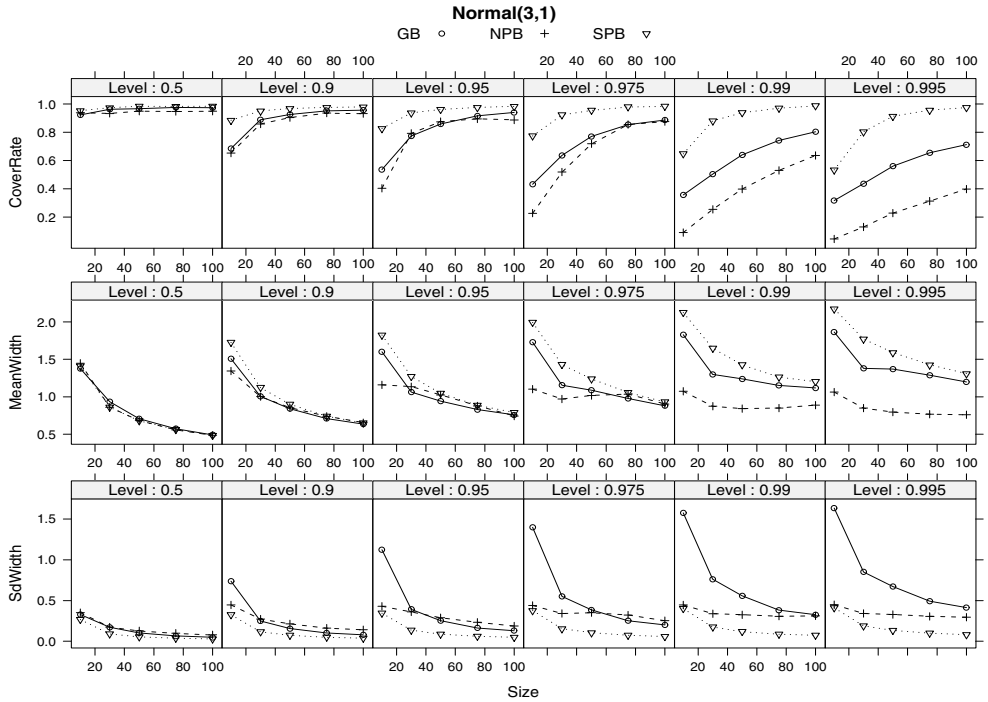


Figure 22.4–13. Simulation results for $N(3,1)$.

22.5 Conclusions

In summary, GB performed well in estimating high quantiles for right skewed distributions. For distributions with light right tails, NPB might totally fail to estimate high quantiles. However, the performance of GB heavily depends on the goodness of the GLD fitting. GLD is rich in shapes and tails, which provides the flexibility in model fitting. On the other hand, as a trade-off, the GLD fitting could be sensitive to sampling, which will result in large variability in the estimates based on the fitted GLD. This has also been verified by the simulation results. The use of GB in high quantile estimation is limited because the GLD fitting method might approximate a distribution with unlimited support by a distribution with limited support. GLD sometimes provides good approximation for most well-known distributions, but sometimes it might fail to fit a distribution to specific data. In practice, if prior information about the underlying distribution is available, one can use the extra information to enhance the model fitting. For example, for data comeing from a beta family of distributions, one could consider fitting an extended GLD; in lifetime data analysis, the generalized Weibull family (GWD) and the exponentiated Weibull family (EWD) are more specific than the GLDs (Mudholkar and Srivastava (1993), Mudholkar and Kollia (1994), and Ashour and Afify (2007)). Freimer, Mudholkar, Kollia, and Lin (1989) showed that the expansions of quantile functions of GWD and a two-parameter generalization of the Tukey family of distribution can provide a simple approach to obtain extreme value distributions.

An alternative solution is to consider a hybrid GLD fitting strategy. Different GLD model-fitting methods outperform the others to fit distribution from different regions in the moment spaces. We can first fit GLDs via various methods including the moment method, L-moment method, and percentile method among many others; second, we choose the best fitting from the fitted models according to some criteria. A potential barrier is the computing speed. All existing GLD fitting algorithms are computationally intensive and slow. The computational burden might prevent the adoption of such a hybrid GLD fitting strategy. Efficient computing algorithms need to be developed to facilitate the use of GB.

Simulation results show that GB outperforms NPB and still has a smaller average bootstrap CI width than PB. There is still room to improve the performance of GB to improve its cover rates.

References for Chapter 22

Albert, I. and Gauchi, J.P. (2002). "Sensitivity analysis for high quantiles of ochratoxin A exposure distribution," *International Journal of Food Microbiology*, 75, 143–155.

Ashour, S.K. and Afify, W.M. (2007). "Statistical analysis of exponentiated Weibull family under type I progressive interval censoring with random removals,"

Journal of Applied Sciences Research, 3(12) 1851–1863.

Cantelli, F.P. (1933). "Sulla determinazione empirica di une legge di probabilita," *Giorna 1st. Ital. Attuari*, 4, 421–424.

Choi, B-Y., Moon, S., Cruz, R., Zhang, Z-L., and Diot, C. (2006). "Quantile sampling for practical delay monitoring in internet backbone networks," *Computer Networks*, 51 2701–2716.

Dudewicz, E.J., (1992). "The Generalized Bootstrap," Invited Paper in Bootstrapping and Related Techniques, Proceedings of an International Conference held in Trier, West Germany, June 4–8, 1990, edited by K.H. Jöckel, G. Rothe, and W. Sendler, V. 376 of *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, 31–37.

Dudewicz, E.J. and Mishra, S. (1988). *Modern Mathematical Statistics*, John Wiley and Sons, New York.

Efron, B.F., (1979). "Bootstrap methods: Another look at the jackknife," *Ann. Statist.* 7, 1–26.

Einmahl, J.H.J., Fils-Villetard, A., and Guillou, A. (2008). "Statistics of extremes under random censoring," *Bernoulli*, 14(1), 207–227.

Freimer, M., Mudholkar, G.S., Kollia, G., and Lin, C.T. (1989). "Extremes, extreme spacings and outliers in the tukey and Weibull families." *Comm. in Statist. — Theory and Methods*, 18(11), 4261–4274.

Galambos, J., (1978). *The Asymptotic Theory of Extreme Order Statistics*, John Wiley and Sons, New York.

Gençay, R., Selçuk, F., and Ulugülyağci, A. (2003). "High volatility, thick tails and extreme value theory in value-at-risk estimation," *Insurance: Mathematics and Economics*, 33 337–356.

Greenwood, J.A., Landwehr, J.M., Matalas, M.C., and Wallis, J.R. (1979). "Probability weighted moments: Definition and relation to parameters of several distributions expressible in inverse Form," *Water Resources Res.*, 15, 1049–1054.

Glivenko, V. (1933). "Sulla determinazione empirica delle leggi di probabilita," *Giorna 1st. Ital. Attuari*, 4, 92–99.

Hogben, D. (1963). *Some Properties of Tukey's Test for Non-Additivity*, Unpublished Ph.D. Thesis, Rutgers-The State University of New Jersey.

Hosking, J.R.M. (1990). "L-Moments: Analysis and estimation of distributions using linear combinations of order statistics," *J. Royal Stat. Soc., Ser. B*, 52, 105–124.

Hyndman, R. J. and Fan, Y. (1996). "Sample quantiles in statistical packages," *American Statistician*, 50, 361–365.

Joiner, B.L. and Rosenblatt, J.R. (1971). "Some properties of the range in samples from Tukey's symmetric lambda distributions," *J. Amer. Statist. Assoc.* 66, 394–399.

Karian, Z.A. and Dudewicz, E.J. (1991). *Modern Statistical, Systems, and GPSS Simulation*, Computer Sciences Press/W.H. Freeman Publishers, New York.

Karian, Z.A. and Dudewicz, E.J. (1999). "Fitting the Generalized Lambda distribution to Data: A method based on percentiles," *Comm. in Statist.- Simul. & Comput.* 28(3), 793–819.

Karian, Z.A. and Dudewicz, E.J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*, Chapman and Hall/CRC.

Karian, Z.A. and Dudewicz, E.J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *J. Iranian Statist. Soc.* 2, 171–187.

Karian, Z.A. and Dudewicz, E.J., and McDonald, P. (1996). "The extended Generalized Lambda Distribution system for fitting distributions to data: history,completion of theory, tables, applications, the 'final word' on moment fits," *Comm. in Statist. — Simul. & Comput.* 25(3), 611–642.

Karvanen, J. and Nuutinen, A. (2008). "Characterizing the generalized lambda distribution by L-moments," *Computational Statistics and Data Analysis*, 52, 1971–1983.

King, A.R. and MacGillivray, H.L. (1999). "A starship method for fitting the generalised λ distribution," *Australian and New Zealand Journal of Statistics*, 41, 353–374.

Markovich, N. (2007). *Nonparametric Analysis of Univariate Heavy-Tails Data: Research and Practice*, John Wiley & Sons, England.

Markovich, N.M. and Krieger, U.R. (2002). "The estimation of heavy-tailed probability density functions, their mixtures and quantiles," *Computer Networks*, 40(3) 459–474.

Matsumoto, M. and Nishimura, T. (1998). "Mersenne Twister: A 623-dimensionally equidistributed uniform pseudo-random number generator," *ACM Transactions on Modeling and Computer Simulation*, 8, 3–30.

McNeil, A.J. and Saladin, T. (1997). "The Peaks over Threshold Method for Estimating High Quantiles of Loss Distributions," *Proceedings of 28th International ASTIN Colloquium.*

McNeil, A.J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, Tool*, Princeton University Press, Princeton, NJ.

Mudholkar, G.S. and Kollia G.D. (1994). "Generalized Weibull family: A structural analysis," *Comm. in Statist. - Theory and Methods*, 23(4) 1149–1171.

Mudholkar, G.S. and Srivastava, D.K. (1993). "Exponentiated Weibull family for analyzing bathtub failure rate data," *IEEE Trans. Reliability*, 42 299–302.

Mykytka, E. and Ramberg, J. (1979). "Fitting Distributions to Data Using an Alternative to Moments," *IEEE Proceedngs of the 1979 Winter Simulation Conference,* 361–374.

Owen, D.B. (1988). "The Starship," *Comm. in Statist. — Simul. & Comput.* 17, 315–323.

Öztürk, A. and Dale R.F. (1985). "Least squares estimation of the parameters of the Generalized Lambda Distribution," *Technometrics.* 27(1), 81–84.

Perlich, C., Rosset, S., Lawrence, R., and Zadrozny, B. (2007). "High-quantile modeling for customer wallet estimation and other applications," *Industrial and Government Track Paper.* 977–985.

Ramberg, J.S. and Schmeiser, B.W. (1972). "An approximate method for generating symmetric random variables," *Communications, Association for Computing Machinery*, 15, 987–990.

Ramberg, J.S., Dudewicz, E., Tadikamalla, P., and Mykytka, E. (1979). "A probability distribution and its uses in fitting data," *Technometrics.* 21(2), 201–214.

Shapiro, S.S. and Wilk, M.B. (1965). "An analysis of variance test for normality (complete samples)," *Biometrika* 52, 591–611.

Su, S. (2007a). "Numerical maximum log likelihood estimation for generalized lambda distributions," *Computational Statistics and Data Analysis*, 51(8), 3983–3998.

Su, S. (2007b). "Fitting single and mixture of Generalized Lambda Distributions to data via discretized and maximum likelihood methods: GLDEX in R," *Journal of Statistical Software*, 21(9), 1–7.

Tukey, J.W. (1960). *The Practical Relationship between the Common Transformations of Percentages of Counts and of Amounts*, Technical Report 36, Statistical Techniques Research Group, Princeton University.

Wang, B., Mishra, S.N., Mulekar, M.S., Mishra, N.S., and Huang, K. (2010). "Comparison of bootstrap and generalized bootstrap methods for estimating high quantiles," *Journal of Statistical Planning and Inference*, 140, 2926–2435.

Weissman, I. (1978). "Estimation of parameters and large quantiles based on the k largest observations," *Journal of American Statistical Association.* 73, 812–815.

Appendix for Chapter 22

A.1 C source code

The following subroutines are called by the **R** programs.

```
#include <R.h>
#include <Rmath.h>
#define INLINE

void GammaMle(double *x, int *n, double *para){
  double ex1=0.,ex2=0., elx1=0., lex1=0., c=0.;
  int i;
  for(i=0;i<n[0];i++){
    ex1 += x[i];
    elx1 += log(x[i]);
  }
  ex1 = ex1/n[0];
  elx1 = elx1/n[0];
  lex1 = log(ex1);
  for(i=0;i<n[0];i++){
    ex2 += pow(x[i]-ex1,2.0);
  }
  ex2 = ex2/(n[0]-1.);
  c = elx1 - lex1;

  double x0=0.,x1, eps=0.00000001;
  int Iter=20;
  i=0;
  x1 = pow(ex1,2.0)/ex2;
  while(i<Iter && fabs(x1-x0)>eps){
    x0 = x1;
    x1 = x1 - g0(x1,c)/g1(x1);
    i=i+1;
  }
  x0 = x1/ex1;

  para[0] = x1;
  para[1] = x0;
  if(ISNAN(x0)||ISNAN(x1)||!R_finite(x0)
     ||!R_finite(x1)||(i>=Iter)||x0<=0.||x1<=0.){
    c=0.;
  }else{
    c=1.;}
  para[2] = c;
}

void BetaMle(double *x, int *n, double *para){
```

```
  double ex1=0.,ex2=0., elx1=0., elx2=0.;
  int i;
  for(i=0;i<n[0];i++){
    ex1 += x[i];
    elx1 += log(x[i]);
    elx2 += log(1.-x[i]);
  }
  ex1 = ex1/n[0];
  elx1 = elx1/n[0];
  elx2 = elx2/n[0];
  for(i=0;i<n[0];i++){
    ex2 += pow(x[i]-ex1,2.0);
  }
  ex2 = ex2/(n[0]-1.);

  double a0=0.,a1=0., b0=0., b1=0., eps=0.00000001;
  int Iter=20;
  double F1, F2, a,b,c;
  i=0;
  a1 = ex1*(ex1*(1.-ex1)/ex2-1.);
  b1 = (1-ex1)*(ex1*(1.-ex1)/ex2-1.);
  while(i<Iter && fabs(a1-a0)>eps && fabs(b1-b0)>eps){
    a0 = a1; b0=b1;
    b = -trigamma(a1+b1);
    a = trigamma(a1) +b;
    c = trigamma(b1)+b;
    F1 = digamma(a1)-digamma(a1+b1)-elx1;
    F2 = digamma(b1)-digamma(a1+b1)-elx2;
    a1 = a1 - (c*F1-b*F2)/(a*c-pow(b,2.));
    b1 = b1 - (-b*F1+a*F2)/(a*c-pow(b,2.));
    i=i+1;
  }
  if(ISNAN(a1)||ISNAN(b1)||!R_finite(a1)
     ||!R_finite(b1)||(i>=Iter)||a1<=0.||b1<=0.){
    c=0.;
  }else{
    c=1.;}
  para[0] = a1;
  para[1] = b1;
  para[2] = c;
}

void WeibullMle(double *x, int *n, double *para){
  double mulx=0.;
  int i,j;
  double x0=1.,x1=1.5, eps=0.00000001;
  double g0x, g1x, a1,a2,a3,a4,c1,c2;
  int Iter=20;
```

```
  i=0;
  while(i<Iter && fabs(x1-x0)>eps){
    x0 = x1;
    a1=0.;a2=0.;a3=0.;a4=0.;
    c1=0.;c2=0.;
    for(j=0;j<n[0];j++){
      c1 = pow(x[j],x1);
      c2 = log(x[j]);
      a1 += c2*c1;
      a2 += c1;
      a3 += c2;
      a4 += pow(c2,2.)*c1;
    }
    g0x= 1./x1-a1/a2+a3/n[0];
    g1x = -1./pow(x1,2.) - (a4*a2 - pow(a1,2.))/pow(a2,2.);
    x1 = x1 - g0x/g1x;
    i=i+1;
  }

  x0 = pow(a2/n[0],1./x1);

  para[0] = x1;
  para[1] = x0;
  if(ISNAN(x0)||ISNAN(x1)||!R_finite(x0)
     ||!R_finite(x1)||(i>=Iter)||x0<=0.||x1<=0.){
    c1=0.;
  }else{
    c1=1.;}
  para[2] = c1;
}
```

A.1 R source code

```
#########################################################################
##        Generalized Bootstrapping
#########################################################################

GB.gamma.mle <- function(x){
  return(.C("GammaMle", as.double(x), as.integer(length(x)),
  as.double(rep(0,3)))[[3]])}
GB.beta.mle <- function(x){
  return(.C("BetaMle", as.double(x), as.integer(length(x)),
  as.double(rep(0,3)))[[3]])}
GB.weibull.mle <- function(x){
  return(.C("WeibullMle", as.double(x),as.integer(length(x)),
  as.double(rep(0,3)))[[3]])}
GB.norm.mle <- function(x){  return(c(mean(x), sd(x)));}
GB.gld.mle <- function(x){  return(fun.data.fit.ml(x)[,1]);}
```

```
GB.ci <- function(x, dist='norm',slevel=.95){
  ## define subroutines to compute the quantiles
  GB.pctl <- function(data){return(quantile(data,c(.5,.95,.99)))}
  ## define subroutines to generate bootstrapping random samples.
  rsgld <- function(data,mle){rGld(length(data),mle)}
  rsbeta <- function(data,mle)
+           {return(rbeta(length(data), mle[1],mle[2]))}
  rsgamma <- function(data,mle)
+           {return(rgamma(length(data), mle[1],mle[2]))}
  rsweibull <- function(data,mle)
+           {return(rweibull(length(data), mle[1],mle[2]))}
  rsnorm <- function(data,mle)
+           {return(rnorm(length(data), mle[1],mle[2]))}
  rsnp <-  function(data,mle){return(sample(data, replace=TRUE));}

  ltail=(1-slevel)/2;
  rtail=1-ltail;
  X = sort(x)
  F.edf = Edf(X)

  if(dist =='beta'){
    out = GB.beta.mle(x)
    al = out[1]
    bl = out[2]
    if(out[3]==1){
      F.mle = pbeta(X,al,bl);
      ksp.mle = KSPvalue(max(abs(F.mle-F.edf)),n)[[1]]
    }else{
      ksp.mle=0.;
    }
    if(ksp.mle>.05){
      bsample = boot(x, GB.pctl, R=999, sim="parametric",
        ran.gen=rsbeta,mle=c(al,bl))
      mle.ci.50 = c(quantile(bsample[[2]][,1], c(ltail,rtail)))
      mle.ci.95 = c(quantile(bsample[[2]][,2], c(ltail,rtail)))
      mle.ci.99 = c(quantile(bsample[[2]][,3], c(ltail,rtail)))
    }else{
      mle.ci.50 = NA;
      mle.ci.95 = NA;
      mle.ci.99 = NA;
    }
  }
  else if(dist=='gamma'){
    out = GB.gamma.mle(x)
    al = out[1]
    bl = out[2]
    if(out[3]==1){
      F.mle = pgamma(X,al,bl);
```

```
    ksp.mle = KSPvalue(max(abs(F.mle-F.edf)),n)[[1]]
  }else{
    ksp.mle=0.;
  }
  if(ksp.mle>.05){
    bsample = boot(x, GB.pctl, R=999, sim="parametric",
      ran.gen=rsgamma,mle=c(al,bl))
    mle.ci.50 = c(quantile(bsample[[2]][,1], c(ltail,rtail)))
    mle.ci.95 = c(quantile(bsample[[2]][,2], c(ltail,rtail)))
    mle.ci.99 = c(quantile(bsample[[2]][,3], c(ltail,rtail)))
  }else{
    mle.ci.50 = NA;
    mle.ci.95 = NA;
    mle.ci.99 = NA;
  }
 }
else if(dist=='weibull'){
  out = GB.weibull.mle(x)
  al = out[1]
  bl = out[2]
  if(out[3]==1){
    F.mle = pweibull(X,al,bl);
    ksp.mle = KSPvalue(max(abs(F.mle-F.edf)),n)[[1]]
  }else{
    ksp.mle=0.;
  }
  if(ksp.mle>.05){
    bsample = boot(x, GB.pctl, R=999, sim="parametric",
      ran.gen=rsweibull,mle=c(al,bl))
    mle.ci.50 = c(quantile(bsample[[2]][,1], c(ltail,rtail)))
    mle.ci.95 = c(quantile(bsample[[2]][,2], c(ltail,rtail)))
    mle.ci.99 = c(quantile(bsample[[2]][,3], c(ltail,rtail)))
  }else{
    mle.ci.50 = NA;
    mle.ci.95 = NA;
    mle.ci.99 = NA;
  }
}else{
  al = mean(X)
  bl = sd(X)
  F.mle = pnorm(X,al,bl);
  ksp.mle = KSPvalue(max(abs(F.mle-F.edf)),n)[[1]]
  bsample = boot(x, GB.pctl, R=999, sim="parametric",
    ran.gen=rsnorm,mle=c(al,bl))
  mle.ci.50 = c(quantile(bsample[[2]][,1], c(ltail,rtail)))
  mle.ci.95 = c(quantile(bsample[[2]][,2], c(ltail,rtail)))
  mle.ci.99 = c(quantile(bsample[[2]][,3], c(ltail,rtail)))
}
```

```
mle = list(mle.ci.50,mle.ci.95,mle.ci.99);

fit1 = fun.data.fit.hs(x)  # discretized weighted approach
(lmd.da = fit1[,1])
fit2 = fun.data.fit.ml(x)  # ml and starship estimates
(lmd.ml = fit2[,1])
(lmd.star= fit2[,3])
(lmd = FitGld(x))

(k = lmd$GLD);lmd.hat=lmd$Lambda;
if(k==2){
  F1 = RGbdFx(X,lmd.hat)
  f1 = RGbdfx(X,lmd.hat)
}else{
  F1 = RGldFx(X,lmd.hat)
  f1 = RGldfx(X,lmd.hat)
}

F.da = RGldFx(X,lmd.da)
F.ml = RGldFx(X,lmd.ml)
F.star = RGldFx(X,lmd.star)

ksp1 = KSPvalue(max(abs(F1-F.edf)),n)[[1]]
ksp.ml = KSPvalue(max(abs(F.ml-F.edf)),n)[[1]]
ksp.da = KSPvalue(max(abs(F.da-F.edf)),n)[[1]]
ksp.star = KSPvalue(max(abs(F.star-F.edf)),n)[[1]]
KSP = c(ksp1,ksp.ml, ksp.da, ksp.star,ksp.mle)
(b1 = boot(x, GB.pctl, R=999, sim="parametric",
   ran.gen=rsnp,mle=0.))

(np.ci.50 = c(quantile(b1[[2]][,1], c(ltail,rtail))))
(np.ci.95 = c(quantile(b1[[2]][,2], c(ltail,rtail))))
(np.ci.99 = c(quantile(b1[[2]][,3], c(ltail,rtail))))
np = list(np.ci.50,np.ci.95,np.ci.99);

if(ksp1>.05){
  b1 = boot(x, GB.pctl, R=999, sim="parametric",
     ran.gen=rsgld, mle=c(lmd.hat))
  (mmp.ci.50 = c(quantile(b1[[2]][,1], c(ltail,rtail))))
  (mmp.ci.95 = c(quantile(b1[[2]][,2], c(ltail,rtail))))
  (mmp.ci.99 = c(quantile(b1[[2]][,3], c(ltail,rtail))))
}else{
  mmp.ci.50 = NA;
  mmp.ci.95 = NA;
  mmp.ci.99 = NA;
 }
mmp = list(mmp.ci.50,mmp.ci.95,mmp.ci.99);
```

```
  if(ksp.ml>.05){
    b1 = boot(x, GB.pctl, R=999, sim="parametric",
       ran.gen=rsgld, mle=c(lmd.ml))
    (ml.ci.50 = c(quantile(b1[[2]][,1], c(ltail,rtail))))
    (ml.ci.95 = c(quantile(b1[[2]][,2], c(ltail,rtail))))
    (ml.ci.99 = c(quantile(b1[[2]][,3], c(ltail,rtail))))
  }else{
    ml.ci.50 = NA;
    ml.ci.95 = NA;
    ml.ci.99 = NA;
   }
  ml = list(ml.ci.50,ml.ci.95,ml.ci.99);

  if(ksp.da>.05){
    b1 = boot(x, GB.pctl, R=999, sim="parametric",
       ran.gen=rsgld, mle=c(lmd.da))
    (da.ci.50 = c(quantile(b1[[2]][,1], c(ltail,rtail))))
    (da.ci.95 = c(quantile(b1[[2]][,2], c(ltail,rtail))))
    (da.ci.99 = c(quantile(b1[[2]][,3], c(ltail,rtail))))
  }else{
    da.ci.50 = NA;
    da.ci.95 = NA;
    da.ci.99 = NA;
   }
  da = list(da.ci.50,da.ci.95,da.ci.99);

  if(ksp.star>.05){
    b1 = boot(x, GB.pctl, R=999, sim="parametric",
       ran.gen=rsgld, mle=c(lmd.star))
    (star.ci.50 = c(quantile(b1[[2]][,1], c(ltail,rtail))))
    (star.ci.95 = c(quantile(b1[[2]][,2], c(ltail,rtail))))
    (star.ci.99 = c(quantile(b1[[2]][,3], c(ltail,rtail))))
  }else{
    star.ci.50 = NA;
    star.ci.95 = NA;
    star.ci.99 = NA;
   }
  star = list(star.ci.50,star.ci.95,star.ci.99);
  list(ks.pvalue=KSP, np=np, mle=mle,mmp=mmp,ml=ml,da=da,star=star);
}

GB.ci.compare <- function(n,para1,para2,dist='norm'){
  cover=rep(0,18);
  size = rep(0,18);
  if(dist =='beta'){
    x = sort(rbeta(n,para1,para2))
    Pctl = qbeta(c(.5,.95,.99), para1,para2)
```

```
}
else if(dist=='gamma'){
  x = sort(rgamma(n,para1,para2))
  Pctl = qgamma(c(.5,.95,.99), para1,para2)
}
else if(dist=='weibull'){
  x = sort(rweibull(n,para1,para2))
  Pctl = qweibull(c(.5,.95,.99), para1,para2)
}
else{# if(dist=='norm'){  # if not specified, use normal
  x = sort(rnorm(n,para1,para2))
  Pctl = qnorm(c(.5,.95,.99), para1,para2)
}

ci.hat = GB.ci(x, dist=dist);
(ksp = ci.hat$ks.pvalue);

i=1;
for(l in 1:3){
  ll = ci.hat$np[[l]][1]; ul = ci.hat$np[[l]][2]
  if(is.na(ll)){
    cover[i]=0;
    size[i]=NA;
  }else{
    if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
    size[i]=ul-ll;
  }
  i=i+1;
}
for(l in 1:3){
  ll = ci.hat$mle[[l]][1]; ul = ci.hat$mle[[l]][2]
  if(is.na(ll)){
    cover[i]=0;
    size[i]=NA;
  }else{
    if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
    size[i]=ul-ll;
  }
  i=i+1;
}
for(l in 1:3){
  ll = ci.hat$mmp[[l]][1]; ul = ci.hat$mmp[[l]][2]
  if(is.na(ll)){
    cover[i]=0;
    size[i]=NA;
  }else{
    if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
    size[i]=ul-ll;
```

```
    }
    i=i+1;
  }

  for(l in 1:3){
    ll = ci.hat$ml[[l]][1]; ul = ci.hat$ml[[l]][2]
    if(is.na(ll)){
      cover[i]=0;
      size[i]=NA;
    }else{
      if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
      size[i]=ul-ll;
    }
    i=i+1;
  }

   for(l in 1:3){
    ll = ci.hat$da[[l]][1]; ul = ci.hat$da[[l]][2]
    if(is.na(ll)){
      cover[i]=0;
      size[i]=NA;
    }else{
      if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
      size[i]=ul-ll;
    }
    i=i+1;
  }

   for(l in 1:3){
    ll = ci.hat$star[[l]][1]; ul = ci.hat$star[[l]][2]
    if(is.na(ll)){
      cover[i]=0;
      size[i]=NA;
    }else{
      if(Pctl[l]>ll && Pctl[l]<ul){cover[i]=1;}
      size[i]=ul-ll;
    }
    i=i+1;
  }
  c(ksp,size,cover)
 }


rsnorm=function(size, mle){return(rnorm(size, mle[1], mle[2]))}
rsgld=function(size, mle){return(rGld(size, mle))}
rsbeta=function(size, mle){return(rbeta(size, mle[1], mle[2]))}
rsgamma = function(size, mle){return(rgamma(size, mle[1], mle[2]))}
rsweibull=function(size, mle){return(rweibull(size, mle[1], mle[2]))}
```

```
npb=function(result, x, qlevel){
  bx = sample(x, replace=T);
  return(quantile(bx,  probs=qlevel,na.rm=TRUE))
}

spb=function(size, mlehat, qlevel){
  bx = rsample(size, mlehat);
  out=quantile(bx, probs=qlevel,na.rm=TRUE);
  return(out);
}

gb=function(size, mlehat, qlevel){
  bx = rsgld(size, mlehat);
  out=quantile(bx, probs=qlevel,na.rm=TRUE);
  return(out);
}

crsize = function(x, cilevel){
  cill = (1-cilevel)/2; ciul=1-cill;
  out = quantile(x, probs=c(cill,ciul), na.rm=TRUE);
  return(out);
}

cicompare = function(size,para1,para2,qlevel=0.50,cilevel=.95, B=10000){
  mle=c(para1,para2);
  X0 = rsample(size, mle);
#Non-parametric bootstrapping
  tmp = matrix(0, nrow=B);
  thetahats=apply(tmp, 1, npb, x=X0, qlevel=qlevel);
  results = apply(thetahats,1,crsize, cilevel=cilevel);
  output = c(results);
#parametric bootstrapping
  tmp = matrix(size, nrow=B);
  mlehat1 = get.mle(X0);
  if(mlehat1[3]==1){
    thetahats=apply(tmp, 1, spb, mlehat=mlehat1, qlevel=qlevel);
    results = apply(thetahats,1,crsize, cilevel=cilevel);
  }else{
    results = rep(NA, 2*length(qlevel));
  }
  output=c(output,results);
#generalized bootstrapping
  tmp = matrix(size, nrow=B);
  mlehat2 = GB.gld.mle(X0);
  thetahats=apply(tmp, 1, gb, mlehat=mlehat2, qlevel=qlevel);
  results = apply(thetahats,1,crsize, cilevel=cilevel);
  output=c(output,results);
  return(c(output)); # (ll50,ul50,...ll995,ul995)
```

```
}

cisummary <- function(x,thetas, n){
  Result=x;
  Thetas=thetas;
  MyRes = NULL;
  for(i in 1:3){
    for(j in 1:6){
      BM = i;
      QL = QLevel[j];
      CovRate = ifelse(
         Result[j*2+(i-1)*12-1,]<Thetas[j])&
+           (Result[j*2+(i-1)*12,]>Thetas[j],1,0);
      CovRate=mean(CovRate,na.rm=TRUE);
      cisize = apply(Result[(j*2+(i-1)*12-1):(j*2+(i-1)*12),],2,diff);
      cisize=abs(sort(cisize));
      MuSize = mean(cisize,na.rm=TRUE);
      SdSize = sd(cisize,na.rm=TRUE);
      MyRes=rbind(MyRes, c(n, BM, QL, CovRate,MuSize,SdSize));
      }
  }
  return(MyRes)
}


trimsum0 <- function(x,thetas,n){
  Result=x;
  Thetas=thetas;
  MyRes = NULL;
  for(i in 1:3){
    for(j in 1:6){
      BM = i;
      QL = QLevel[j];
      CovRate = ifelse(
         Result[j*2+(i-1)*12-1,]<Thetas[j])&
+           (Result[j*2+(i-1)*12,]>Thetas[j],1,0);
      CovRate=mean(CovRate,na.rm=TRUE);
      cisize = apply(Result[(j*2+(i-1)*12-1):(j*2+(i-1)*12),],2,diff);
      cisize=abs(sort(cisize));
      tsize = sort(cisize[!(is.na(cisize))]);
      sele = c((floor(length(tsize)*0.05)):(floor(length(tsize)*0.95)))
      tsize = tsize[sele];
      MuSize = mean(tsize,na.rm=TRUE);
      SdSize = sd(tsize,na.rm=TRUE);
      MyRes=rbind(MyRes, c(n, BM, QL, CovRate,MuSize,SdSize));
      }
  }
  return(MyRes)
```

```
}

trimsum <- function(x,thetas,n){
  Result=x;
  Thetas=thetas;
  MyRes = NULL;
  for(i in 1:3){
    for(j in 1:6){
      BM = i;
      QL = QLevel[j];
      CovRate = ifelse(
         Result[j*2+(i-1)*12-1,]<Thetas[j])&
+           (Result[j*2+(i-1)*12,]>Thetas[j],1,0);
      CovRate=mean(CovRate,na.rm=TRUE);
      cisize = apply(Result[(j*2+(i-1)*12-1):(j*2+(i-1)*12),],2,diff);
      cisize=abs(sort(cisize));
      tsize = sort(cisize[!(is.na(cisize))]);
      sele = c((floor(length(tsize)*0.10)):(floor(length(tsize)*0.90)))
      tsize = tsize[sele];
      MuSize = mean(tsize,na.rm=TRUE)/Thetas[j];
      SdSize = sd(tsize,na.rm=TRUE)/Thetas[j];
      MyRes=rbind(MyRes, c(n, BM, QL, CovRate,MuSize,SdSize));
      }
  }
  return(MyRes)
}

trimplot <- function(mtitle){
  postscript('tfig.ps',paper='letter')
  newdat = read.table('trimall.txt')
  names(newdat) = c("Size","Method","Level","CoverRate","MeanWidth",
+                 "SdWidth")
  newdat$Method =
   ifelse(newdat$Method==1, "NPB", ifelse(newdat$Method==2, "SPB", "GB"));
  newdat$Level = factor(newdat$Level);
  superpose.line <- trellis.par.get("superpose.line")
  plotSize <- xyplot(MeanWidth~Size|Level, data=newdat,
allow.multiple=T||outer, outer=T,
        groups=Method, layout=c(6,1), #ylim=c(0,2),
strip = function(...)
strip.default(..., strip.names = c(T,T)),
type='b', lty=1:3, xlab=NULL)
  plotSd <- xyplot(SdWidth~Size|Level, data=newdat,
allow.multiple=T||outer, outer=T,
        groups=Method, layout=c(6,1), #ylim=c(0,1),
strip = function(...)
strip.default(..., strip.names = c(T,T)),
        type='b', lty=1:3)
```

```
  print(plotSd, position=c(0,0,1,.404), more=T)
  print(plotSize, position=c(0,.298,1,.68), more=T)
  plotCR <- xyplot(CoverRate~Size|Level, data=newdat,
allow.multiple=T||outer, outer=T,
        groups=Method, layout=c(6,1),
strip = function(...)
strip.default(..., strip.names = c(T,T)),
type='b', auto.key=list(space='top', columns=3),
                  main=mtitle, lty=1:3, xlab=NULL)
  print(plotCR, position=c(0,.57,1,1))
  dev.off()
}

trimsum2 <- function(x,thetas,fs,n){
  Result=x;
  Thetas=thetas;
  Fs = fs;
  MyRes = NULL;
  for(i in 1:3){
    for(j in 1:6){
      BM = i;
      QL = QLevel[j];
      CovRate = ifelse(
         Result[j*2+(i-1)*12-1,]<Thetas[j])&
+           (Result[j*2+(i-1)*12,]>Thetas[j],1,0);
      CovRate=mean(CovRate,na.rm=TRUE);
      cisize = apply(Result[(j*2+(i-1)*12-1):(j*2+(i-1)*12),],2,diff);
      cisize=abs(sort(cisize));
      tsize = sort(cisize[!(is.na(cisize))]);
      sele = c((floor(length(tsize)*0.01)):(floor(length(tsize)*0.99)))
      tsize = tsize[sele];
      MuSize = mean(tsize,na.rm=TRUE) - 2*qnorm(.975)*Fs[j]/sqrt(n);
      SdSize = sd(tsize,na.rm=TRUE)/Fs[j]*sqrt(n);
      MyRes=rbind(MyRes, c(n, BM, QL, CovRate,MuSize,SdSize));
      }
  }
  return(MyRes)
}
```

PART VI: Assessment of the Quality of Fits

Chapter 23 Goodness-of-Fit Criteria Based on Observations Quantized by Hypothetical and Empirical Percentiles

Chapter 24 Evidential Support Continuum (ESC): A New Approach to Goodness-of-Fit Assessment, which Addresses Conceptual and Practical Challenges

Chapter 25 Estimation of Sampling Distributions of the Overlapping Coefficient and Other Similarity Measures

A sketch of each of these chapters appears in Section 1.2.

Chapter 23

Goodness-of-Fit Criteria Based on Observations Quantized by Hypothetical and Empirical Percentiles

I. Vajda
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4
182 08 Prague, Czech Republic

E. C. van der Meulen
Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Heverlee, Belgium
E-mail: edward.vandermeulen@wis.kuleuven.be

Goodness-of-fit disparity statistics are defined as appropriately scaled ϕ-disparities or ϕ-divergences of quantized hypothetical and empirical distributions. It is shown that the classical Pearson-type statistics are obtained if we quantize by means of hypothetical percentiles, and that new spacings-based disparity statistics are obtained if we quantize by means of empirical percentiles. The main attention is paid to the asymptotic properties of the new disparity statistics and their comparisons with the spacings-based statistics known from the literature. First the asymptotic equivalence between them is proved, and then for the new

statistics a general law of large numbers is proved, as well as an asymptotic normality theorem both under local and fixed alternatives. Special attention is devoted to the limit laws for the power divergence statistics of orders $\alpha \in I\!R$. Parameters of these laws are evaluated for $\alpha \in (-1, \infty)$ in a closed-form and their continuity in α on the subinterval $(-1/2, \infty)$ is proved. These closed form expressions are used to compare local asymptotic powers of the tests based on these statistics, which allows us to extend previous asymptotic optimality results to the class of power divergence statistics. Tables of values of the asymptotic parameters are presented for selected representative orders of $\alpha > -1/2$. Programs are provided for the evaluation of three families of spacings-based statistics, including two families of true divergence statistics. These programs are applied to compare the three families for a specific hypothetical distribution and two examples of datasets.

23.1 Data and their Statistical Models

In this chapter we consider the explanation of *observed data* $x_1, x_2, ..., x_n$ statistically as a sequence of independent outcomes from a statistical model. Our aim is to review and extend the criteria of goodness-of-fit of the model and data, and to study their properties and applications in decisions about the acceptability of concrete models for concrete data.

Let us start with the example of data first studied by Pearson (1894), which represent measurements of the ratio of the forehead to body of $n = 1000$ crabs. Pearson partitioned the original domain of measurements $(a, b) = (0.5795, 0.6995)$ into intervals of equidistant size 0.004 and counted the frequency for each interval. Table 23.1–1 gives the measurement values v_j represented by midpoints of the intervals and the corresponding frequencies φ_j for $1 \le j \le 30$.

As a second example we use the data studied recently by Ning, Gao, and Dudewicz (2008), which are results of measurements of cadmium concentrations in the kidney cortex of $n = 43$ horses. These are presented in Table 23.1–2 below.

Statistical models are probability distributions on data spaces $\mathcal{X}$. A general probability distribution is specified by a **probability measure** (briefly, **p.m.**) P defined on appropriate subsets of $\mathcal{X}$. The data spaces of the examples in Tables

Table 23.1–1 Ratio of the forehead to body of 1000 crabs.

Value	Freq.	Value	Freq.	Value	Freq.	Value	Freq.	Value	Freq.
0.5815	1	0.5855	3	0.5895	5	0.5935	2	0.5975	7
0.6015	10	0.6055	13	0.6095	19	0.6135	20	0.6175	25
0.6215	40	0.6255	31	0.6295	60	0.6335	62	0.6375	54
0.6415	74	0.6455	84	0.6495	86	0.6535	96	0.6575	85
0.6615	75	0.6655	47	0.6695	43	0.6735	24	0.6775	19
0.6815	9	0.6855	5	0.6895	0	0.6935	1	0.6975	0

Table 23.1–2 Amounts of cadmium in horse kidneys.

11.9	16.7	23.4	25.8	25.9	27.5	28.5	31.1	32.5	35.4	38.3
38.5	41.8	42.9	50.7	52.3	52.5	52.6	54.5	54.7	56.6	56.7
58.0	60.8	61.8	62.3	62.5	62.6	63.0	67.7	68.5	69.7	73.1
76.0	76.9	77.7	78.2	80.3	93.7	101.0	104.5	105.4	107.0	-

23.1–1 and 23.1–2 are the halflines $(0, \infty)$. Throughout this chapter we restrict ourselves to real valued data with the data space being an interval $\mathcal{X} = (a, b) \subseteq \mathbb{R}$. Then each probability measure P is uniquely specified by the **distribution function** (briefly, **d.f.**)

$$F(x) = P((-\infty, x]), \quad x \in \mathbb{R}$$

We restrict ourselves to statistical models with increasing and continuously differentiable d.f.'s $F(x)$ on the data space. These are uniquely specified by the positive continuous **probability density functions** (briefly, **p.d.f.**'s)

$$f(x) = \frac{dF(x)}{dx} \quad x \in (a, b)$$

as well as by the increasing **percentile functions** (briefly, **p.f.**'s)

$$Q(y) = F^{-1}(y) \quad \text{(i.e. } x \in [a, b] \text{ such that } F(x) = y), \quad y \in [0, 1],$$

where $Q(0) = a$ and $Q(1) = b$, due to the assumption that $F(x)$ is increasing on (a, b). Therefore the observed data $x_1, x_2, \ldots, x_n$ are interpreted as realizations of independent copies $X_1, X_2, \ldots, X_n$ of a random variable X which can be specified equivalently by a p.m. P on $\mathcal{X}$ or a d.f. $F(x)$ on $\mathbb{R}$ or a p.d.f. $f(x)$ on the data space (a, b), or by a p.f. $Q(y)$ on the percentile space $[0, 1]$.

The data $x_1, x_2, \ldots, x_n$ themselves can be represented by the so-called **empirical probability measure** (briefly, **e.p.m.**)

$$P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \tag{23.1.1}$$

on the data space $\mathcal{X} = (a, b)$ where δ denotes the Dirac probability measure, or by the related **empirical distribution function** (briefly, **e.d.f.**)

$$F_n(x) = P_n((-\infty, x)) = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{I}(x \geq x_i) \tag{23.1.2}$$

on $\mathbb{R}$ where $\boldsymbol{I}$ denotes the indicator function. Both of these representations are unique up to the ordering of the data. In other words, P_n as well as F_n are one-to-one related to the order statistics $(x_{n:1}, x_{n:2}, \ldots, x_{n:n})$ of the data vector

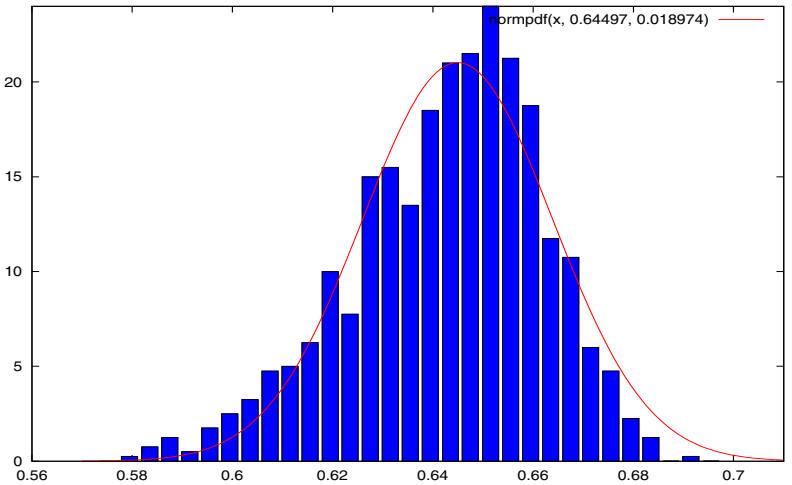


Figure 23.1–3. Normal approximation of the crab data from Table 23.1–1.

$(x_1, x_2, \ldots, x_n)$. The loss of order means no loss of statistical information, because the order statistics are known to be statistically sufficient for the statistical models independent identically distributed (i.i.d.) observations (see Lehmann and Romano (2005)).

An alternative form of presentation of the data $x_1, x_2, ..., x_n$ is by the histogram $f_n(x)$ on the data space (a, b). This is the density of the restriction of the e.p.m. P_n on the algebra generated by the $k+1$ intervals obtained by slicing (a, b) at the cutpoints

$$a = c_0 < c_1 < \cdots < c_k < c_{k+1} = b.$$

Then for every $x \in [c_j, c_{j+1})$

$$f_n(x) = \#x_i \in [c_j, c_{j+1}).$$

For example, the frequencies φ_j of Table 23.1–1 define the histogram

$$f_n(x) = \varphi_j \quad \text{for} \quad x \in [c_j, c_{j+1}) \tag{23.1.3}$$

where

$$c_j = 0.5795 + 0.004 \cdot j \quad \text{for} \quad 0 \le j \le 30 \tag{23.1.4}$$

are uniform cutpoints of the interval $(a, b) = (0.5795, 0.6995)$. The histogram form of presentation of data is statistically sufficient in exceptional situations only.

Figure 23.1–3 compares the histogram (23.1.3) presenting the data from Table 23.1–1 with the p.d.f. $f(x)$ of the maximum likely normal model $\mathrm{N}(\mu,\sigma^2)$ of these data where

$$\mu = \frac{1}{1000}\sum_{i=1}^{1000} x_i = 0.6447 \quad \text{and} \quad \sigma^2 = \frac{1}{999}\sum_{i=1}^{1000}(x_i - \mu)^2 = 0.00036. \tag{23.1.5}$$

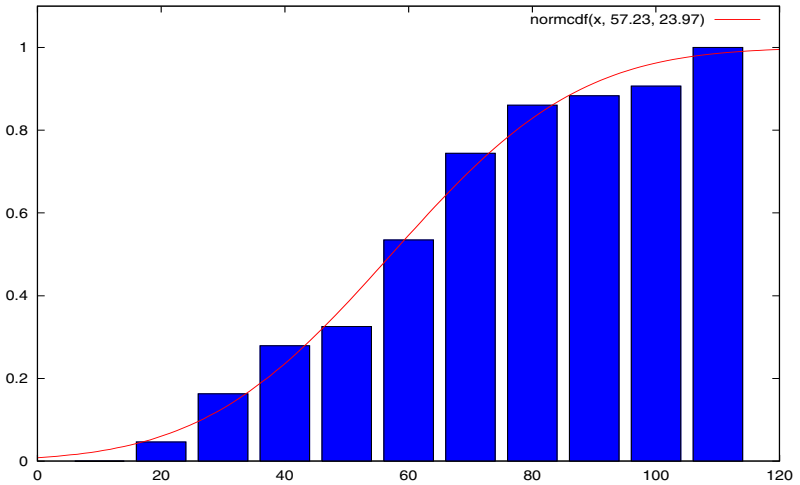


Figure 23.1–4. Normal approximation of the horse data from Table 23.1–2.

Figure 23.1–4 shows the e.d.f. $F_n(x)$ of the data from Table 23.1–2 together with the d.f. $F(x)$ of the normal model $\mathrm{N}(\mu,\sigma^2)$ with sample mean and variance

$$\mu = \frac{1}{43}\sum_{i=1}^{43} x_i = 57.2326 \quad \text{and} \quad \sigma^2 = \frac{1}{42}\sum_{i=1}^{43}(x_i-\mu)^2 = 574.98. \qquad (23.1.6)$$

In this chapter we deal with quality criteria for fitting various possible statistical models $F(x)$ to the data represented by e.d.f.'s $F_n(x)$, including the asymptotic properties of these criteria for $n \to \infty$. The basic concepts and notations introduced in this section are used throughout all what follows below.

23.2 Assessment of Goodness-of-Fit

Intuitively one can expect that all numerical goodness-of-fit criteria will be measures of distance, divergence, or disparity between on the one hand the empirical reality represented by a data-based p.m. P or d.f. F, e.g. the e.p.m. P_n or e.d.f. F_n, and on the other the hypothetical statistical model given by a p.m. P_0 or a d.f. F_0. In this chapter the terms distance, divergence and disparity have a specific mathematical meaning which is specified in the definitions below, where we deal primarily with p.m.'s P, P_0 rather than with d.f.'s F, F_0.

Definition 23.2.1 (**i**) By a *distance* $D(P, P_0)$ of p.m.'s P, P_0 we mean a standard mathematical metric distance, taking values in the interval $[0, \infty)$, which is reflexive (i.e. $D(P, P_0) = 0$ iff $P = P_0$), symmetric (i.e. $D(P, P_0) = D(P_0, P)$) and satisfies the triangle inequality (i.e. $D(P, P_0) \le D(P, P_1) + D(P_1, P_0)$ for all p.m.'s P, P_0, P_1).

(**ii**) A **divergence** (more precisely, an **information-theoretic divergence**) $D(P, P_0)$ is a reflexive functional taking values in the interval $[0, \infty)$ and satisfying the information processing law. To formulate this law, let $T : \mathcal{X} \to \mathcal{Y}$ be a mapping describing the results of processing the data x from the observation space $\mathcal{X}$ towards another space $\mathcal{Y}$, with $y = T(x)$. The information processing law says that no processing rule T can increase the divergence, in symbols

$$D(PT^{-1}, P_0T^{-1}) \leq D(P, P_0), \tag{23.2.2}$$

with the equality being valid if and only if T is statistically sufficient for $\{P, P_0\}$.
(**iii**) A **disparity** of p.m.'s P, P_0 is a nonnegative functional $D(P, P_0)$ which is reflexive in the above sense.

Convention 23.2.3 Throughout the chapter we denote by P, P_0 an arbitrary pair of probability measures on a general observation space $\mathcal{X}$. They will be represented by means of their p.d.f.'s

$$p = \frac{dP}{d\lambda} \quad \text{and} \quad p_0 = \frac{dP_0}{d\lambda} \tag{23.2.4}$$

with respect to (w.r.t.) a dominating measure λ on $\mathcal{X}$ (in symbols, $P \sim p, P_0 \sim p_0$). The only restriction imposed on the hypothetical model P_0 is the positivity of p_0 almost everywhere w.r.t. λ.

Continuous case. If $\mathcal{X} = (a, b) \subseteq \mathbb{R}$ then the p.m.'s (P, P_0) are in a one-to-one manner represented by the d.f.'s (F, F_0), and for absolutely continuous F, F_0 and Lebesgue measure λ it holds that

$$p = f \quad \text{and} \quad p_0 = f_0 \tag{23.2.5}$$

where (f, f_0) are the usual p.d.f.'s of (F, F_0) (in symbols, $F \sim f, F_0 \sim f_0$). As stated above, we assume $f_0 > 0$ on (a, b).

Discrete case. If $\mathcal{X} = (1, 2, ..., k)$, then the densities (p, p_0) of (P, P_0) w.r.t. the counting measure $\lambda(1) = \lambda(2) = ... = \lambda(k) = 1$ reduce to the stochastic vectors

$$\boldsymbol{p} = (p_j := P(j) : 1 \leq j \leq k), \quad \boldsymbol{p}_0 = (p_{0j} := P_0(j) : 1 \leq j \leq k), \tag{23.2.6}$$

(in symbols, $P \sim \boldsymbol{p}$, $P_0 \sim \boldsymbol{p_0}$). As stated above, we assume $p_{0j} > 0$ for all $1 \leq j \leq k$.

23.2.1 Special Distances, Divergences, and Disparities

(a) The L_1**-distance**

$$L_1(P, P_0) = \int |p - p_0| d\lambda \quad \text{for} \quad P \sim p, P_0 \sim p_0 \tag{23.2.7}$$

is an example of a distance on the class of all p.m.'s, which satisfies only partly the information processing law. Namely it satisfies inequality (23.2.2) but fails to satisfy the necessary condition for equality. To see this it suffices to consider the discrete p.m.'s $P \sim \boldsymbol{p}$ and $P_0 \sim \boldsymbol{p_0}$ given by (23.2.6), where

$$L_1(P, P_0) = L_1(\boldsymbol{p}, \boldsymbol{p}_0) = ||\boldsymbol{p} - \boldsymbol{p}_0||_1 = \sum_{j=1}^{k} |p_j - p_{0j}|. \tag{23.2.8}$$

Example 23.2.9 The binary coding $T : \mathcal{X} \to \mathcal{Y} = \{1, 0\}$ of the ternary observations $x \in \mathcal{X} = \{1, 2, 3\}$ defined by

$$T(1) = 1 \quad \text{and} \quad T(2) = T(3) = 0 \tag{23.2.10}$$

significantly reduces the information for discrimination between the discrete models

$$P = (1/10,\, 5/10,\, 4/10) \quad \text{and} \quad P_0 = (9/10, 1/10, 0).$$

Indeed, the discrimination rule

$$\delta(1) = P_0 \quad \text{and} \quad \delta(2) = \delta(3) = P$$

based on the original uncoded observations from $\mathcal{X} = \{1, 2, 3\}$ is errorless if $x = 3$ and the discrimination errors for the remaining observations x take place with the probability $1/10$. On the other hand, arbitrary discrimination rule $\delta : \mathcal{Y} \to \{P, P_0\}$ based on the encoded data from $\mathcal{Y} = \{1, 0\}$ admits discrimination errors with the probability $9/10$. This reduction of discernibility is caused by the loss of information due to the coding which is evidently not a statistically sufficient transformation. However, this evidence is not reflected by the L_1- distance which remains unaffected by the coding, namely

$$L_1((1/10,\, 5/10,\, 4/10), (9/10, 1/10, 0)) = L_1((1/10,\, 9/10),\, (9/10, 1/10)) = 8/5.$$

Note that, nevertheless, the L_1-distance $L_1(f_n, f)$ is widely used as a goodness-of-fit criterion between the model p.d.f.'s f and histogram-like representations f_n of the observed data, since being introduced to the nonparametric statistics by Devroye and Györfi (1985).

(b) The L_2-**distance**

$$L_2(P, P_0) = \left(\int (p - p_0)^2 d\lambda \right)^{1/2} \quad \text{for} \quad P \sim p, P_0 \sim p_0$$

does not satisfy the information processing law in the sense that processing of the observations can increase the L_2-distance between the models P, P_0. To this end

it suffices to consider the discrete p.m.'s $P \sim \boldsymbol{p}$ and $P_0 \sim \boldsymbol{p}_0$ given by (23.2.6) where

$$L_2(P, P_0) = L_2(\boldsymbol{p}, \boldsymbol{p}_0) = ||\boldsymbol{p} - \boldsymbol{p}_0||_2 = \left(\sum_{j=1}^{k} (p_j - p_{0j})^2 \right)^{1/2}. \tag{23.2.11}$$

The mentioned violation of the information processing law can be verified by taking $k = 3$ and the models $P \sim \boldsymbol{p} = (0,\ 1/2,\ 1/2)$ and $P_0 \sim \boldsymbol{p}_0 = (1/2,\ 1/4,\ 1/4)$ on the observation space $\mathcal{X} = \{1, 2, 3\}$. Applying the coding (23.2.10) to the observations $x \in \mathcal{X}$ we obtain

$$L_2(PT^{-1}, P_0T^{-1}) = \sqrt{1/2} > L_2(P, P_0) = \sqrt{3/8}.$$

This flaw of the L_2-distance justifies the preference of the above-mentioned L_1-distance method over the L_2-method and underlines the importance of the L_1-method in statistical research.

(c) The **Kolmogorov distance**

$$K(F, F_0) = \sup_{x \in \mathbb{R}} |F(x) - F_0(x)| \tag{23.2.12}$$

introduced by Kolmogorov (1933) is the distance in the above-stated metric sense and defines the well-known **Kolmogorov-Smirnov goodness-of-fit statistic** $T_n = \sqrt{n} K(F_n, F_0)$ (Smirnov (1948); for more see Lehmann and Romano (2005)). Applicability of the Kolmogorov distance is restricted by the fact that it contradicts the information processing law in a similar way as the L_2-distance.

A simple example where

$$K(FT^{-1}, F_0T^{-1}) = 1 > K(F, F_0) = 1/2$$

is obtained by using the p.d.f.'s

$$f(x) = \boldsymbol{I}(-1 < x < -1/2) + \boldsymbol{I}(0 < x < 1/2), \tag{23.2.13}$$

$$f_0(x) = \boldsymbol{I}(-1/2 < x < 0) + \boldsymbol{I}(1/2 < x < 1) \tag{23.2.14}$$

and the data processing formula

$$T(x) = x + \frac{1}{2}\left[\boldsymbol{I}(-1/2 < x < 0) - \boldsymbol{I}(0 < x < 1/2)\right] \tag{23.2.15}$$

which transforms the interval $(-1/2, 0)$ on $(0, 1/2)$ and vice versa. This formula is skew symmetric about $x = 0$ and thus the data processing is reversible in the sense $T^{-1} = T$. The d.f.'s F, F_0 for this example are shown in Figure 23.2–1 and their modifications resulting from the data processing (23.2.15) in Figure 23.2–2 below.

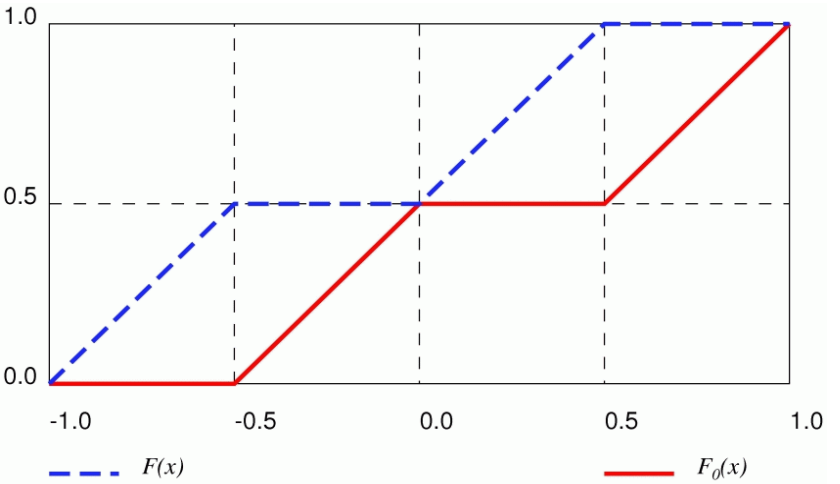


Figure 23.2–1. Distribution functions F and F_0.

For the special discrete binary p.m.'s

$$P_x \sim \boldsymbol{p}_x = (F(x), 1 - F(x)) \quad \text{and} \quad P_{0x} \sim \boldsymbol{p}_{0x} = (F_0(x), 1 - F_0(x)) \tag{23.2.16}$$

(23.2.12) and (23.2.8) imply

$$K(F, F_0) = \frac{1}{2} \sup_{x \in I\!R} L_1(P_x, P_{0x}). \tag{23.2.17}$$

Similar relations as (23.2.17) take place for other measures of goodness-of-fit between F, F_0 and their one-point approximations $F(x), F_0(x)$ formally represented by the binary p.m.'s (23.2.16). Next we present two well-known measures and the alternatives to the relation (23.2.17) are in (**f**) and (**g**) below.

(d) The **Pearson divergence** is given by the formula

$$\chi^2(P, P_0) = \int \frac{(p - p_0)^2}{p_0} d\lambda \quad \text{for} \quad P \sim p, P_0 \sim p_0 \tag{23.2.18}$$

which in the discrete case considered in (23.2.6) reduces to

$$\chi^2(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(p_j - p_{0j})^2}{p_{0j}}. \tag{23.2.19}$$

It defines the well-known **Pearson χ^2 goodness-of-fit test statistic**

$$T_n = n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(np_{nj} - np_{0j})^2}{np_{0j}} = \sum_{j=1}^{k} \frac{(\varphi_{nj} - np_{0j})^2}{np_{0j}} \tag{23.2.20}$$

for testing hypotheses $\mathcal{H}_0 : P_0$ on the basis of observations represented by the e.p.m. P_n where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are the restrictions of the hypothetical and empirical

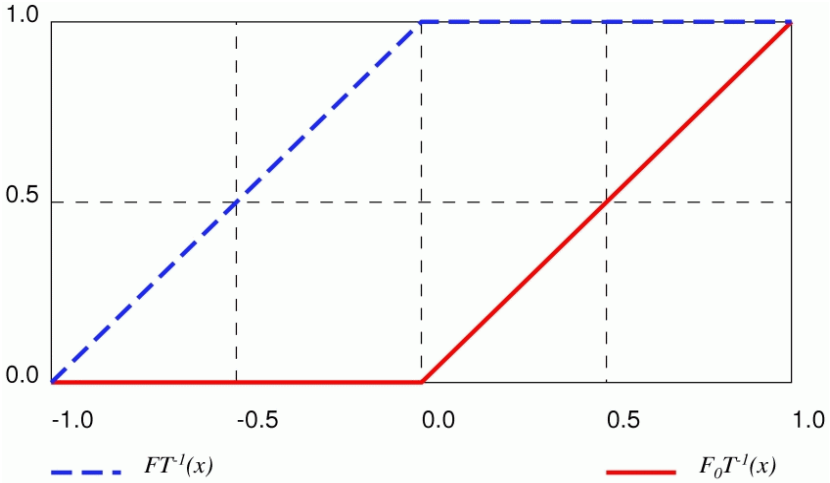


Figure 23.2–2. Distribution functions F and F_0.

p.m.'s P_0 and P_n on the test cells $C_j = [c_j, c_{j+1})$ and $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \le j \le k$. The Pearson divergence is a divergence in the rigorous sense stated above, but it is not a distance since it is neither symmetric nor satisfies the triangle inequality. Of course, the same holds for the **reversed Pearson divergence**

$$R\chi^2(P, P_0) := \chi^2(P_0, P). \tag{23.2.21}$$

(e) Another example of a divergence is the **LeCam divergence**

$$LC(P, P_0) = \int \frac{(p - p_0)^2}{p + p_0} d\lambda \quad \text{for} \quad P \sim p, P_0 \sim p_0 \tag{23.2.22}$$

which in the discrete case considered in (23.2.6) reduces to

$$LC(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(p_j - p_{0j})^2}{p_j + p_{0j}}. \tag{23.2.23}$$

Since it is a divergence in the sense of Definition 23.2.1, all its roots are divergences in the sense of the same definition too but, as proved by Kafka, Österreicher, and Vincze (1991), the square root $\sqrt{LC(P, P_0)}$ is distinguished by being a metric distance in the space of p.m.'s P, P_0.

(f) In the discrete case considered in (23.2.16) and (23.2.17) the relations (23.2.19) and (23.2.16) imply

$$\chi^2(P_x, P_{0x}) = \frac{(F(x) - F_0(x))^2}{F_0(x)} + \frac{(F(x) - F_0(x))^2}{1 - F_0(x)} = \frac{(F(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))}.$$

Since $\chi^2(P_x, P_{0x})$ is a disparity of binary distributions P_x, P_{0x} for all $x \in I\!R$, the integral over $I\!R$,

$$AD(F, F_0) = \int \chi^2(P_x, P_{0x})dx = \int \frac{(F(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))}dx, \tag{23.2.24}$$

is a disparity of d.f.'s F, F_0 in the above-stated sense. We call it the **Anderson-Darling disparity** because its scaled version

$$T_n = n \cdot AD(F_n, F_0) = n \int \frac{(F_n(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))}dx \tag{23.2.25}$$

is the well known **Anderson-Darling goodness-of-fit statistic** for testing the hypothesis that the data represented by the e.d.f. F_n were generated by the d.f. F_0 (hypothesis $\mathcal{H}_0$).

(g) Similarly, we call

$$CM(F, F_0) = \int \chi^2(P_x, P_{0x})F_0(x)(1-F_0(x))dx = \int (F(x)-F_0(x))^2 dx \tag{23.2.26}$$

the **Cramér-von Mises disparity** because it is a disparity of p.d.f.'s F, F_0 in the above-defined sense and

$$T_n = n \cdot CM(F_n, F_0) = n \int_{\mathcal{X}} (F_n(x) - F_0(x))^2 dx \tag{23.2.27}$$

is the **Cramér-von Mises goodness-of-fit statistic** for testing the hypothesis $\mathcal{H}_0 = F_0$ under the empirical evidence F_n.

The goodness-of-fit statistics mentioned in **(f)** and **(g)** were introduced by von Mises (1947) and Anderson and Darling (1954) (see also Darling (1957)). We refer in this respect to pp. 58–64 in Serfling (1980) or to D'Agostino and Stephens (1986).

23.2.2 Examples

The divergences and disparities will be more systematically studied in the next section. In the remainder of this section we apply the goodness-of-fit criteria introduced in this section to the data from Tables 23.1–1 and 23.1–2.

Example 23.2.28 Consider the discrete e.p.m. $\boldsymbol{p}_n = (p_{n1}, \ldots, p_{n,30})$ representing the data given by the frequencies φ_j of Table 23.1–1 and defined by the formula

$$p_{nj} = \varphi_j 10^{-3}, \quad 1 \le j \le 30. \tag{23.2.29}$$

In addition to the normal model $\mathrm{N}(\mu,\sigma^2)$ for these data given by (23.1.5), we consider the mixed normal model

$$\mathrm{MixN}(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \pi \mathrm{N}(\mu_1, \sigma_1^2) + (1 - \pi)\mathrm{N}(\mu_2, \sigma_2^2)$$

for the parameters

$$\pi = 0.5, (\mu_1, \sigma_1^2) = (0.6343, 0.000361), (\mu_2, \sigma_2^2) = (.6551, 0.00014641)$$

used by Pearson (1894). The third model considered by us is the mixed generalized lambda model

$$\text{MixGLD}(\pi, \theta, \vartheta) = \pi\text{GLD}(\theta) + (1-\pi)\text{GLD}(\vartheta)$$

from p. 88 of Ning, Gao, and Dudewicz (2008) where the generalized lambda component models are given by the percentile functions

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1-y)^{\theta_4}}{\theta_2} \quad \text{and} \quad Q_\vartheta(y) = \vartheta_1 + \frac{y^{\vartheta_3} - (1-y)^{\vartheta_4}}{\vartheta_2}.$$

for the parameters

$$\begin{aligned} \pi &= 0.802 \\ \theta &= (0.6415, 13.218, 0.135, 0.205) \\ \vartheta &= (0.6564, 11.328, 0.55, 0.15). \end{aligned} \tag{23.2.30}$$

Each d.f. $F_0 \in \{\text{N}(\mu,\sigma^2), \text{MixN}(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2), \text{MixGLD}(\pi, \theta, \vartheta)\}$ represents a different hypothesis about the stochastic source of the data from Table 23.1–1. Each one defines a hypothetical p.d.f. f_0 and a hypothetical discrete p.m. $\boldsymbol{p}_0 = (p_{01}, \ldots, p_{0k})$ obtained by quantization of the observation space $\mathcal{X} = (a, b) \subseteq I\!R$ into k cells. In the present example $(a, b) = (0.5795, 0.6995)$ and we consider quantization into $k = 30$ partition intervals symmetric about the centers v_j given in Table 23.1–1 leading to $\boldsymbol{p}_0 = (p_{01}, \ldots, p_{0,30})$ with the components

$$p_{0j} = \int_{v_j - 0.002}^{v_j + 0.002} f_0(x)dx = F_0(v_j + 0.002) - F_0(v_j - 0.002). \tag{23.2.31}$$

Table 23.2–1 presents the values of the distance or divergence criteria $L_1(\boldsymbol{p}, \boldsymbol{p}_0)$, $L_2(\boldsymbol{p}, \boldsymbol{p}_0)$, $\chi^2(\boldsymbol{p}, \boldsymbol{p}_0)$ and $LC(\boldsymbol{p}, \boldsymbol{p}_0)$ given by (23.2.8), (23.2.11), (23.2.19), and (23.2.23) for the e.p.m. $\boldsymbol{p} = \boldsymbol{p}_n$ given by (23.2.29) and hypothetical p.m.'s $\boldsymbol{p}_0$ given by (23.2.31). The estimate MixGLD was evaluated by the package GLDEX of the program R (project CRAN) by Steve Su. For description of the method and further citations we refer to

http://cran.r-project.org/web/packages/GLDEX/GLDEX.pdf

We see from this table that all four criteria are isotone in the sense that the MixN model fits best the data and the MixGLD fits worst. This can be explained by the fact that the criterion for selection of the mixed models was the best possible approximation of the first 4 moments of the e.d.f. and the MixGLD model is in this sense better than the MixN, as observed in Ning, Gao, and Dudewicz (2008). The normal model approximates the first 2 moments only

Table 23.2–1. Values of several criteria for the models N, MixN and MixGLD of the data from Table 23.1–1.

Model	L_1	L_2	*Pearson*	*LeCam*
Normal	0.17476	0.047971	0.075765	0.034877
Mix of Normals	0.0958	0.025351	0.020596	0.011084
Mix of Lambdas	0.24636	0.063542	0.14132	0.054918

and this allows it to approximate the data source better in the sense of all four selected criteria. However, the relative differences between the accuracies of the models are evaluated by each criterion in a slightly different way. The relative difference between N and MixGLD is 20% through the prism of the L_2, while the Pearson criterion evaluates the same difference as 100% larger. No simple recipe can be given for the choice of the most adequate criterion. It usually requires a deep insight into the properties of the criteria and into the details of the studied application.

Example 23.2.32 Consider the e.d.f. F_n defined by the data of Table 23.1–2. In addition to the normal model $\mathrm{N}(\mu,\sigma^2)$ for these data given by (23.1.6), we consider the generalized lambda models $\mathrm{GLD}(\theta)$ given by the percentile function

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1-y)^{\theta_4}}{\theta_2}$$

for the parameters

$$\theta = (41.7897, 0.01134, 0.09853, 0.3606) \tag{23.2.33}$$

obtained on page 97 of Karian and Dudewicz (2000). We use also the mixed generalized lambda model

$$\mathrm{MixGLD}(\pi, \theta, \vartheta) = \pi\mathrm{GLD}(\theta) + (1-\pi)\mathrm{GLD}(\vartheta)$$

with the component generalized lambda models given by the percentile functions

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1-y)^{\theta_4}}{\theta_2} \quad \text{and} \quad Q_\vartheta(y) = \vartheta_1 + \frac{y^{\vartheta_3} - (1-y)^{\vartheta_4}}{\vartheta_2}.$$

for the parameters

$$\begin{aligned} \pi &= 0.4 \\ \theta &= (57.8233, 0.0076, 0.1432, 0.1356) \\ \vartheta &= (56.2136, 0.0193, 0.4601, 0.4838) \end{aligned} \tag{23.2.34}$$

Table 23.2–2. Values of several criteria for the models N, GLD and MixGLD of the data from Table 23.1–2.

Model	$K(F_n, F_0)$	$AD(F_n, F_0)$	$CM(F_n, F_0)$
Normal	0.45177	0.38039	0.05513
Lambda	0.07917	0.39040	0.06034
Mix of Lambdas	0.21747	78.4530	0.78635

introduced on page 91 of Ning, Gao, and Dudewicz (2008). The present models $F_0 \in \{\mathrm{N}(\mu,\sigma^2),\ \mathrm{GLD}(\theta),\ \mathrm{MixGLD}(\pi,\theta,\vartheta)\}$ differ from those considered in the previous example but, similarly as above, they represent three different hypotheses about the sources of data from Table 23.1–2. Table 23.2–2 presents values of the distance or disparity $K(F, F_0)$, $AD(F, F_0)$, and $CM(F, F_0)$ given by (23.2.12), (23.2.24), and (23.2.26) for the d.f. $F(x) = F_n(x)$ from Figure 23.1–4 and the present models $F_0(x)$. Distribution functions of the models GLD and MixGLD were evaluated by the package GLDEX of the program R mentioned in Example 23.2.28.

The preferences attributed to the models by the criteria are given in Table 23.2–3 below. We see from there that an isotony similar to that observed in Table 23.2–1 takes place only for the Anderson-Darling and Cramér-von Mises criteria and that their preferences contradict those resulting from the Kolmogorov criterion. The extremely large value of the Anderson-Darling's $AD(F_n, F_0)$ in Table 23.2–2 underlines the importance of analysis of the criterion formulas and algorithms from the point of view of numerical stability, i.e. resistence (robustness) with respect to small computational inaccuracies. In the $AD(F_n, F_0)$ case, for $x \approx 1$ the empirical values $F_n(x)$ are close to 1 while the program computes the hypothetical values $F_0(x)$ by two orders closer to 1 without possibility to control the accuracy of computation. This obviously results into the uncontrolled increase of values during numerical integration in the formula

$$AD(F_n, F_0) = \int \frac{(F_n(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))} dx \quad \text{(see (23.2.25))}$$

for x such that the numerator is of the order of 1 while the denominator is of the order $< 10^{-2}$.

Table 23.2–3. The orders of preferences in Table 23.2–2.

Criterion	1	2	3
$K(F_n, F_0)$	Lambda	Mix of Lambdas	Normal
$AD(F_n, F_0)$	Normal	Lambda	Mix of Lambdas
$CM(F_n, F_0)$	Normal	Lambda	Mix of Lambdas

Better performance of the simple lambda than of the mix of lambdas observed in Table 23.2–3 can be explained by the fact that the mix of lambdas is optimal in the sense of best approximation of the first 4 moments of the e.d.f. (c.f. Ning, Gao, and Dudewicz (2008)). Similarly as mentioned above in the comments on Table 23.2–1, the best fit of the several first moments does not mean the best fit of d.f.'s which is measured by the goodness-of-fit criteria appearing in Tables 23.2–2 and 23.2–3. This is reflected also in the results of Section 23.8 where the fit of the models of this section and the data of Tables 23.1–1 and 23.1–2 is treated by the new criteria introduced in the next section.

23.3 Criteria of Goodness-of-Fit

This section studies more systematically those criteria of goodness-of-fit between the empirical evidence represented by the e.d.f. F_n on the one hand and the hypothetical model specified by its d.f. F_0 on the other, that are measures of dissimilarity $D(F_n, F_0)$ between the distribution functions F_n and F_0. Since $F_n \to F_0$ a.s. for $n \to \infty$ and, consequently, $D(F_n, F_0) \to 0$ stochastically as $n \to \infty$ for reasonable dissimilarity measures D, the statistics T_n for testing the hypothesis $\mathcal{H}_0 : F_0$ on the basis of empirical evidence represented by F_n, are considered in the form $T_n = m_n D(F_n, F_0)$. Here $m_n \to \infty$ is an appropriate scaling sequence for which T_n tends to a limit distribution. The value $Q(1-\alpha)$ of the p.f. of this distribution is then used as a critical value of T_n for the asymptotically α-level test of hypothesis $\mathcal{H}_0$. Examples were given in Section 23.2, e.g. the Kolmogorov distance $K(F_n, F_0)$ and corresponding Kolmogorov-Smirnov statistic $T_n = \sqrt{n}K(F_n, F_0)$.

Similarly, if the empirical evidence is represented by a discrete distribution $\boldsymbol{p}_n$ and the hypothetical model is specified by the discrete distribution $\boldsymbol{p}_0$ then the dissimilarity between the former and the sequel is $D(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and the statistic for testing the hypothesis $\mathcal{H}_0 : \boldsymbol{p}_0$ is $T_n = m_n D(\boldsymbol{p}_n, \boldsymbol{p}_0)$. Examples were given in Section 23.2, e.g. the Pearson distance $\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and the corresponding Pearson statistic $T_n = n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0)$.

Goodness-of-fit criteria are not only studied for e.d.f.'s F_n or related e.p.m.'s $\boldsymbol{p}_n$ simply representing the datasets $\{x_1, x_2, ..., x_n\}$ in a straightforward manner (23.1.2), but also for more sophisticated models F and $\boldsymbol{p}$ obtained by data-based statistical inference like e.g. those obtained by maximum likelihood estimation. Therefore we deal in the rest of this section with arbitrary d.f.'s F and p.m.'s $\boldsymbol{p}$.

23.3.1 Disparities, Divergences, and Metric Distances

To unify the treatment of the situation when goodness-of-fit is considered for d.f.'s (F, F_0) or p.m.'s $(\boldsymbol{p}, \boldsymbol{p}_0)$, it is convenient to represent simultaneously both hypothetical models $F_0, \boldsymbol{p}_0$ by the corresponding general p.m.'s P_0, and alternative

models $F, \boldsymbol{p}$ by the corresponding general p.m.'s P. Thus we deal in this subsection primarily with dissimilarity measures $D(P, P_0)$. We respect Convention 23.2.3 and use the notations (23.2.4)–(23.2.6) introduced there.

We define the class of **dissimilarities** of probability measures P, P_0 on $\mathcal{X}$ by

$$D_\phi(P, P_0) = \int_{\mathcal{X}} p_0 \phi\left(\frac{p}{p_0}\right) d\lambda \quad \text{for} \quad P \sim p, P_0 \sim p_0 \tag{23.3.1}$$

generated by continuous functions $\phi : (0, \infty) \mapsto [0, \infty)$ with continuous extension $0 \le \phi(0) \le \infty$, such that the integral (23.3.1) exists. In particular, for arbitrary d.f. F and hypothetical d.f. F_0 on an interval observation space $\mathcal{X} = (a, b) \subseteq I\!R$,

$$D_\phi(F, F_0) = \int_a^b f_0(x) \phi\left(\frac{f(x)}{f_0(x)}\right) dx \quad \text{for} \quad F \sim f, F_0 \sim f_0, \tag{23.3.2}$$

and, in the discrete case, for

$$P \sim \boldsymbol{p} = (p_1, p_2, ..., p_k) \quad \text{and} \quad P_0 \sim \boldsymbol{p}_0 = (p_{01}, p_{02}, ..., p_{0k}) \tag{23.3.3}$$

see (23.2.6),

$$D_\phi(P, P_0) = D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} p_{0j} \phi\left(\frac{p_j}{p_{0j}}\right). \tag{23.3.4}$$

Let us clarify for which functions ϕ the dissimilarities (23.3.1)–(23.3.4) are disparities, divergences, or distances in the sense of the previous section. Denote by $\boldsymbol{\Phi}$ the class of all differentiable functions $\phi : (0, \infty) \mapsto I\!R$ with continuous extension $\phi(0)$ and the property

$$(\phi'(t) - \phi'(1))\text{sign}(t - 1) > 0 \quad \text{for all } t \in (0, \infty). \tag{23.3.5}$$

Then the standardized version

$$\tilde{\phi}(t) = \phi(t) - \phi(1) - \phi'(1)(t - 1) \tag{23.3.6}$$

of $\phi(t)$ is increasing in the domain $t \ge 0$ and decreasing in the domain $t \le 0$, i.e. is positive for $t \in (0, \infty)$ except for $t = 1$ where $\tilde{\phi}(1) = 0$. Therefore the integral

$$D_{\tilde{\phi}}(P, P_0) = \int_{\mathcal{X}} p_0 \tilde{\phi}\left(\frac{p}{p_0}\right) d\lambda = D_\phi(P, P_0) - \phi(1) \tag{23.3.7}$$

exists, takes on values in the closed interval $[0, \infty]$, and the dissimilarity $D_{\tilde{\phi}}(P, P_0)$ is reflexive in the sense that equality holds if and only if $P = P_0$. Hence the expressions $D_{\tilde{\phi}}(P, P_0)$ are well-defined disparities, and also the expressions $D_\phi(P, P_0)$ are disparities up to shifts $\phi(1)$. This justifies speaking about all dissimilarities

$D_\phi(P, P_0)$, $D_\phi(F, F_0)$, and $D_\phi(\boldsymbol{p}, \boldsymbol{p}_0)$ given by (23.3.1) - (23.3.4) for $\phi \in \boldsymbol{\Phi}$ as **disparities in the wide sense**, transformed by constant shifts $\phi(1)$ to disparities defined in the precise sense of the previous section. Of course, the subset

$$\boldsymbol{\Phi}_{\text{disp}} = \{\phi \in \boldsymbol{\Phi} : \phi(1) = 0\}, \quad \boldsymbol{\Phi}_{\text{disp}} \subset \boldsymbol{\Phi}$$

defines proper disparities by equations (23.3.1) through (23.3.4).

Let $\boldsymbol{\Phi}_{\text{div}}$ be the class of differentiable convex functions $\phi : (0, \infty) \mapsto I\!R$ with continuous extension $\phi(0)$ and strict convexity at $t = 1$. Then $\phi(1) + \phi'(1)(t-1)$ is the support straight line of the function $\phi(t)$, which is strictly smaller than $\phi(t)$ at all $t \neq 1$ due to the strict convexity of $\phi(t)$ at $t = 1$. However, the assumed convexity of $\phi(t)$ on the whole domain $(0, \infty)$ means that the function (23.3.6) is increasing on $(1, \infty)$ and decreasing on $(0, 1)$ so that (23.3.5) holds. Consequently,

$$\boldsymbol{\Phi}_{\text{div}} \subset \boldsymbol{\Phi}_{\text{disp}},$$

i.e. expression (23.3.7) is reflexive. As proved in Csiszár (1967) or Liese and Vajda (1987), the disparities (23.3.7) with $\phi \in \boldsymbol{\Phi}_{\text{div}}$ satisfy the information processing law, i.e. they are *divergences* in the sense defined in the previous section.

Example 23.3.8 The functions defined on $(0, \infty)$ by

$$\phi_0(t) = -\ln t \quad \text{and} \quad \phi_1(t) = t \ln t \tag{23.3.9}$$

with extensions $\phi_0(0) = \infty$ and $\phi_1(0) = 0$ satisfy condition (23.3.5). Thus they belong to $\boldsymbol{\Phi}$ and define the wide-sense disparities

$$D_1(P, P_0) := D_{\phi_1}(P, P_0) = \int_{\mathcal{X}} p \ln\left(\frac{p}{p_0}\right) d\lambda \tag{23.3.10}$$

and

$$D_0(P, P_0) := D_{\phi_0}(P, P_0) = \int_{\mathcal{X}} p_0 \ln\left(\frac{p_0}{p}\right) d\lambda = D_1(P_0, P). \tag{23.3.11}$$

Since $\phi_i(1) = 0$, both these functions belong to the subset $\boldsymbol{\Phi}_{\text{disp}} \subset \boldsymbol{\Phi}$, so that (23.3.10) and (23.3.11) are proper disparities. Further, both functions $\phi_i(t)$ are differentiable and strictly convex on the domain $(0, \infty)$. Therefore they belong to $\boldsymbol{\Phi}_{\text{div}}$ and (23.3.10) and (23.3.11) are divergences. In fact, $D_1(P, P_0)$ is known as the **information divergence** or **Kullback divergence**, and $D_0(P, P_0)$ is usually called the **reversed information divergence** or **reversed Kullback divergence**. Note that

$$T_n = nD_1(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} np_{nj} \ln \frac{np_{nj}}{np_{0j}} = \sum_{j=1}^{k} \varphi_{nj} \ln \frac{\varphi_{nj}}{np_{0j}} \tag{23.3.12}$$

is the well-known likelihood ratio test statistic for testing the hypothesis $\mathcal{H}_0 : P_0$ on the basis of observations represented by the e.p.m. P_n, where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are restrictions of the hypothetical and empirical p.m.'s P_0 and P_n to the test cells $C_j = [c_j, c_{j+1})$, and $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \le j \le k$.

Example 23.3.13 The functions defined on $(0, \infty)$ by

$$\bar{\phi}_{-1}(t) = \frac{1}{t} \quad \text{and} \quad \bar{\phi}_2(t) = t^2$$

with extensions $\bar{\phi}_{-1}(0) = \infty$ and $\bar{\phi}_2(t) = 0$ satisfy condition (23.3.5). Thus they belong to $\boldsymbol{\Phi}$ and define the wide-sense disparities

$$\bar{D}_{-1}(P, P_0) := D_{\bar{\phi}_{-1}}(P, P_0) = \int_{\mathcal{X}} \frac{p_0^2}{p}\, d\lambda$$

and

$$\bar{D}_2(P, P_0) := D_{\bar{\phi}_2}(P, P_0) = \int_{\mathcal{X}} \frac{p^2}{p_0}\, d\lambda = \bar{D}_{-1}(P_0, P).$$

Since $\bar{\phi}_{-1}(1) = \bar{\phi}_2(1) = 1$, the functions $\delta_{-1}(t) := \bar{\phi}_{-1}(t) - 1$ and $\delta_2(t) = \bar{\phi}_2(t) - 1$ define the proper disparities

$$\Delta_{-1}(P, P_0) := D_{\delta_{-1}}(P, P_0) = \int_{\mathcal{X}} \frac{(p - p_0)^2}{p}\, d\lambda \tag{23.3.14}$$

and

$$\Delta_2(P, P_0) := D_{\delta_2}(P, P_0) = \int_{\mathcal{X}} \frac{(p - p_0)^2}{p_0}\, d\lambda. \tag{23.3.15}$$

The same proper disparities are defined by the standardized versions

$$\delta_{-1}^+(t) = \delta_{-1}(t) + t - 1 = \frac{(t-1)^2}{t} \quad \text{and} \quad \delta_2^+(t) = \delta_2(t) + 2(1-t) = (t-1)^2$$

belonging to $\boldsymbol{\Phi}_{\text{disp}} \subset \boldsymbol{\Phi}$. Of course, the functions $\delta_{-1}(t)$ and $\delta_2(t)$ as well as $\delta_{-1}^+(t)$ and $\delta_2^-(t)$ belong to $\boldsymbol{\Phi}_{\text{div}}$ and define the Pearson divergences of Section 23.2.1, part **d** (briefly 23.2.1.(**d**)),

$$\Delta_2(P, P_0) = \chi^2(P, P_0) \text{ (see (23.2.18))}$$

and

$$\Delta_{-1}(P, P_0) = R\chi^2(P, P_0) \text{ (see (23.2.21))}.$$

Example 23.3.16 Power divergences The functions defined on $(0, \infty)$ by

$$\phi_\alpha(t) = \frac{t^\alpha - 1}{\alpha(\alpha - 1)} \quad \text{for the powers } \alpha \in I\!R - \{0, 1\} \tag{23.3.17}$$

and for the remaining powers by

$$\phi_1(t) = t \ln t \quad \text{and} \quad \phi_0(t) = -\ln t \text{ (see Example 23.3.8)} \tag{23.3.18}$$

belong to $\mathbf{\Phi}$ defined in Example 23.3.8. They satisfy the condition $\phi_\alpha(1) = 0$ so that they belong also to the subset $\mathbf{\Phi}_{\text{disp}} \subset \mathbf{\Phi}$. Since they are differentiable and strictly convex on $(0, \infty)$, they belong in fact also to $\mathbf{\Phi}_{\text{div}}$ and define the divergences

$$D_\alpha(P, P_0) := D_{\phi_\alpha}(P, P_0) = \frac{\int_{\mathcal{X}} p^\alpha p_0^{1-\alpha}\, d\lambda - 1}{\alpha(\alpha - 1)} \quad \text{for} \quad \alpha \in I\!R - \{0, 1\} \tag{23.3.19}$$

and

$$D_1(P, P_0) := D_{\phi_1}(P, P_0) = \int_{\mathcal{X}} p \ln\left(\frac{p}{p_0}\right) d\lambda = D_0(P_0, P) := D_{\phi_0}(P_0, P), \tag{23.3.20}$$

for $\alpha \in \{0, 1\}$. The classes of modified power divergences $\alpha D_\alpha(P, P_0), \alpha > 0$ are one-to-one related to the logarithmic power divergences

$$R_\alpha(P, P_0) = \frac{\ln[\alpha(\alpha - 1)D_\alpha(P, P_0) + 1]}{\alpha - 1} = \frac{\ln \int_{\mathcal{X}} p^\alpha p_0^{1-\alpha}\, d\lambda}{\alpha - 1}, \quad \alpha > 0 \tag{23.3.21}$$

of Rényi (1961), where $R_1(P, P_0) = \lim_{\alpha \uparrow 1} R_\alpha(P, P_0) = D_1(P, P_0)$. The slightly restricted versions $\alpha D_\alpha(P, P_0)$, $\alpha > 0$, were used in the 1960s and 1970s by several authors, e.g. by Perez (1967). The essential parts of the power divergences

$$H_\alpha(P, P_0) = \alpha(\alpha - 1)D_\alpha(P, P_0) + 1 = \int_{\mathcal{X}} p^\alpha p_0^{1-\alpha}\, d\lambda, \quad \alpha \in I\!R \tag{23.3.22}$$

called **Hellinger integrals** were used e.g. by Chernoff (1952), Vajda (1971) and Liese (1982). The Hellinger integrals are skew symmetric about $\alpha = 1/2$ in the sense that

$$H_\alpha(P, P_0) = H_{1-\alpha}(P_0, P), \quad \alpha \in I\!R. \tag{23.3.23}$$

The power divergences (23.3.19), (23.3.20) are skew symmetric in the same sense, i.e.

$$D_\alpha(P, P_0) = D_{1-\alpha}(P_0, P), \quad \alpha \in I\!R. \tag{23.3.24}$$

The skew-symmetrization of the formerly used divergences $\alpha D_\alpha(P, P_0)$, $\alpha > 0$, resulting into the class of power divergences (23.3.19), (23.3.20) was introduced by Cressie and Read (1984), and the power divergences are in the last two decades used as standard representatives of the whole class of divergences $D_\phi(P, P_0), \phi \in$

$\mathbf{\Phi}_{\rm div}$. A similar skew-symmetrization of the Rényi divergences (23.3.21) resulting into the formula

$$R_\alpha(P,P_0)=\frac{\ln\int_{\mathcal{X}}p^\alpha p_0^{1-\alpha}\,d\lambda}{\alpha(\alpha-1)},\quad \alpha\in I\!R \tag{23.3.25}$$

was introduced by Liese and Vajda (1987) who used them among other as auxiliary tools for rigorous proofs of a number of general properties of the power divergences (23.3.24).

Remarks (i) Since the functions (23.3.18) are the same as in Example 23.3.8, the members $D_1(P,P_0)$ and $D_0(P,P_0)$ of the power divergence family are the **Kullback** and **reversed Kullback divergences** introduced in Example 23.3.8.

(ii) Further,

$$D_\alpha(P,P_0)=\Delta_\alpha(P,P_0)/2\quad\text{for}\quad\alpha=-1\text{ and }\alpha=2$$

where on the right are the disparities introduced in Example 23.3.13. Therefore $2D_2(P,P_0)$ and $2D_{-1}(P,P_0)$ are the Pearson and reversed Pearson divergences of 23.2.1.**(d)**.

(iii) Another well-known member of the family of power divergences not mentioned before is the **Hellinger divergence**

$$D_{1/2}(P,P_0)=4H(P,P_0)=4\int_{\mathcal{X}}\left(\sqrt{p}-\sqrt{p_0}\right)^2\,d\lambda, \tag{23.3.26}$$

which in the discrete case (23.2.6) reduces to

$$D_{1/2}(\boldsymbol{p},\boldsymbol{p}_0)=4\sum_{j=1}^k\left(\sqrt{p_j}-\sqrt{p_{0j}}\right)^2. \tag{23.3.27}$$

The divergence $H(\boldsymbol{p},\boldsymbol{p}_0)$ was introduced by Matusita (1956) but it is better known as **squared Hellinger distance**. Indeed,

$$\sqrt{H(P,P_0)}=||\sqrt{p}-\sqrt{p_0}||_2$$

is the L_2-distance of the square roots of p.d.f.'s and, as such, is a metric distance. Of course, $\sqrt{D_{1/2}(\boldsymbol{p},\boldsymbol{p}_0)}$ is a metric distance also.

(iv) The expression

$$T_n=nD_{1/2}(\boldsymbol{p}_n,\boldsymbol{p}_0)/2=2n\sum_{j=1}^k\left(\sqrt{p_j}-\sqrt{p_{0j}}\right)^2 \tag{23.3.28}$$

is known as the **Freeman-Tukey statistic** for testing the hypothesis $\mathcal{H}_0:P_0$ on the basis of observations represented by the e.p.m. P_n, where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are the restrictions of the hypothetical and empirical p.m.'s P_0 and P_n on the test cells $C_j=[c_j,c_{j+1})$ for $1\le j\le k$.

(v) The square root $\sqrt{D_{1/2}(\boldsymbol{p},\boldsymbol{p}_0)}$ is the only member among all powers of all power divergences which is a distance in the sense introduced above (see Definition 23.2.1). The unicity follows already from the fact that $D_{1/2}(P,P_0)$ is the only power divergence which is symmetric in the variables P, P_0.
(vi) The nonnegative standardized versions of the functions (23.3.17) are

$$\tilde{\phi}_\alpha(t) = \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(\alpha-1)} \quad \text{for} \quad \alpha \in I\!R - \{0,1\} \tag{23.3.29}$$

and those corresponding to (23.3.18) are the limits

$$\tilde{\phi}_1(t) = t\ln t - t + 1 \quad \text{and} \quad \tilde{\phi}_0(t) = -\ln t + t - 1 \tag{23.3.30}$$

of $\tilde{\phi}_\alpha(t)$ for $\alpha \to 0$ and $\alpha \to 1$. Obviously

$$D_{\tilde{\phi}_\alpha}(P,P_0) = D_{\phi_\alpha}(P,P_0) = D_\alpha(P,P_0) \quad \text{for all } \alpha \in I\!R. \tag{23.3.31}$$

23.3.2 Metricity and Robustness

The power divergences $D_\alpha(P,P_0), \alpha \in I\!R$, do not represent all aspects of the class of all divergences $D_\phi(P,P_0), \phi \in \boldsymbol{\Phi}_{\rm div}$. For example, $\sqrt{D_{1/2}(P,P_0)}$ is the only power divergence which is a metric distance. This might suggest that the metricity of a divergence is a rare property. But, in fact, the class of all divergences contains uncountably many of them with this property. For example, all functions

$$\varphi_\beta(t) = \frac{\text{sign}\beta}{1-\beta}\left[(t^{1/\beta}+1)^\beta - 2^{\beta-1}(t+1)\right] \quad \text{for} \quad -\infty < \beta \le 2 \tag{23.3.32}$$

with the terms for $\beta = 0$ and $\beta = 1$ obtained by the continuous extension rule as

$$\varphi_0(t) = |t-1|/2 \quad \text{and} \quad \varphi_1(t) = t\ln t + (t+1)\ln\frac{2}{t+1} \tag{23.3.33}$$

belong to $\boldsymbol{\Phi}_{\rm div}$. The square roots $\sqrt{D_{\phi_\beta}(P,P_0)}$ of the corresponding divergences are metric distances (see Vajda (2009)). LeCam's divergence of 23.2.1.**(e)** is among them, since

$$D_{\varphi_{-1}}(P,P_0) = LC(P,P_0).$$

The total variation function $\varphi_0(t)$ formally does not belong to $\boldsymbol{\Phi}_{\rm div}$ because it is not differentiable at $t = 1$, but this may be cured by defining this derivative as the mean of the left- and right-hand derivatives,

$$\varphi_0'(1) = \frac{\varphi_0'(1+) + \varphi_0'(1-)}{2} = \frac{1}{2} - \frac{1}{2} = 0. \tag{23.3.34}$$

Let us now look at the robustness of the testing or estimation based on the disparity statistics

$$T_{\phi,n} = nD_\phi(P_n, P_0), \quad \phi \in \mathbf{\Phi}_{\rm disp}$$

reflecting the proximity of the hypothetical p.m. P_0 and e.p.m. P_n, or their special forms

$$T_{\phi,n} = nD_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} np_{0j}\phi\left(\frac{np_{nj}}{np_{0j}}\right) = \sum_{j=1}^{k} np_{0j}\phi\left(\frac{\varphi_{nj}}{np_{0j}}\right), \quad \phi \in \mathbf{\Phi}_{\rm disp}$$

using restrictions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ of P_0 and P_n on the test cells $C_j = [c_j, c_{j+1})$ where $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \le j \le k$.

If $\phi \in \mathbf{\Phi}_{\rm div}$, i.e. if $D_\phi(P, P_0)$ is a divergence then $\phi(t)$ is always unbounded and its derivative $\phi'(t)$ is usually unbounded on the domain $(0, \infty)$. For example, all power divergence functions $\phi_\alpha(t)$ as well their derivatives $\phi'_\alpha(t)$ are unbounded. The statistical disparity measures were introduced by Lindsay (1994) and more systematically investigated by Menéndez, Morales, Pardo and Vajda (1998). In these papers it is argued that the robustness of statistical inference based on minimization of disparities between models requires bounded generating functions $\phi \in \mathbf{\Phi}_{\rm disp}$, or at least bounded derivatives ϕ' because these two functions represent the criterion function and the influence function of robust statistics. Thus, from the point of robustness of statistical decisions based on disparity statistics, the attention is concentrated on the functions $\rho(t-1) \in \mathbf{\Phi}_{\rm disp}$ for the classical criterial ρ-functions of robust statistics leading to bounded influence functions $\psi(t)$ proportional to the derivatives $\rho'(t)$.

Example 23.3.35 A classical example is the family of **Huber ρ-functions**

$$\rho_k(t) = \boldsymbol{I}(|t| \le k)t^2, \quad k > 0$$

smoothly extended as linear functions in the domain $\{t \in I\!R : |t| > k\}$ with the constantly bounded influences $\psi_k(t) = \rho'_k(t) = 2k$ of the observations t from this domain (see Hampel, Ronchetti, Rousseeuw, and Stahel (1986) or Jurečková and Sen (1996)). The Huber functions define the family

$$\phi_k(t) = \rho_k(t-1), \quad k > 0 \tag{23.3.36}$$

of functions from $\mathbf{\Phi}_{\rm div}$ with the bounded derivatives

$$\psi_k(t) = \phi'_k(t) = 2[(t-1)\boldsymbol{I}(|t-1| \le k) + k\boldsymbol{I}(|t-1| > k)], \quad k > 0. \tag{23.3.37}$$

These functions generate the family of robust divergences $D_{\phi_k}(P, P_0), k > 0$.

Example 23.3.38 Another classical example is the family of ρ-functions

$$\rho_\alpha(t) = (\alpha - 1)t\boldsymbol{I}(t < 0) + \alpha t\boldsymbol{I}(t > 0), \quad 0 < \alpha < 1$$

leading to the robust statistical inference based on the so-called **regression quantiles** (see Jurečková and Sen (1996)). The total variation generating function is the special case

$$\varphi_0(t) = \rho_{1/2}(t - 1).$$

We put

$$\phi_\alpha(t) = \rho_\alpha(t - 1), \quad 0 < \alpha < 1 \tag{23.3.39}$$

and similarly as in (23.3.34), we use the generalized form

$$\phi'_\alpha(1) = \frac{\phi'_\alpha(1+) + \phi'_\alpha(1-)}{2} = \alpha - \frac{1}{2} \tag{23.3.40}$$

to extend the derivative $\phi'_\alpha(t)$ of the family of robust divergence generating functions to $t = 1$ in order to achieve the formal validity for including this family in $\boldsymbol{\Phi}_{\rm div}$. The derivatives

$$\psi_\alpha(t) = \phi'_\alpha(t) = (\alpha - 1)\boldsymbol{I}(t < 1) + \alpha\boldsymbol{I}(t > 1), \quad 0 < \alpha < 1 \tag{23.3.41}$$

of these functions are bounded on the domain $(0, \infty)$ so that the functions (23.3.39) generate the family of robust divergences $D_{\phi_\alpha}(P, P_0)$, $0 < \alpha < 1$.

23.4 Disparities Based on Partitions

In the previous section we assessed goodness-of-fit between two statistical models given by p.m.'s P and P_0 by the disparity, divergence, or distance

$$D_\phi(P, P_0) = \int_{\mathcal{X}} p_0 \phi\left(\frac{p}{p_0}\right) d\lambda \tag{23.4.1}$$

where the concrete type depends on the extended real valued function $\phi \in \boldsymbol{\Phi}$ and, according to Convention 23.2.3, p, p_0 are densities of P, P_0 given by (23.2.4) and p_0 is positive on $\mathcal{X}$. The specification (23.2.4) of the densities p, p_0 means that for every $A \subset \mathcal{X}$

$$P(A) = \int_A p d\lambda \quad \text{and} \quad P_0(A) = \int_A p_0 \, d\lambda \quad \text{for} \quad \lambda = P + P_0. \tag{23.4.2}$$

In some situations it is necessary to restrict the p.m.'s P and P_0 on a partition $\mathcal{P} = \{C_1, ..., C_k\}$ of $\mathcal{X}$ into disjoint cells C_j, resulting into the quantizations

$$\boldsymbol{p} = (p_j := P(C_j) : 1 \le j \le k) \quad \text{and} \quad \boldsymbol{p}_0 = (p_{0j} := P_0(C_j) : 1 \le j \le k) \tag{23.4.3}$$

of these p.m.'s and to the reduced disparity, divergence, or distance

$$D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} p_{0j} \phi\left(\frac{p_j}{p_{0j}}\right). \tag{23.4.4}$$

Partitioning of the observation space means that observations $x \in \mathcal{X}$ are replaced by the indices of the partition sets containing these observations, i.e. by

$$T(x) \in \{1, 2, ..., k\} \text{ where } T^{-1}(j) = C_j \text{ for } 1 \le j \le k. \tag{23.4.5}$$

Among others this means that if ϕ generates the divergence $D_\phi(P, P_0)$, then the information processing law implies that

$$D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) \le D_\phi(P, P_0) \tag{23.4.6}$$

where the equality holds if and only if the partition $\mathcal{P}$ is statistically sufficient for (P, P_0).

The situation described above takes place when the first of the p.m.'s is an e.p.m., i.e. when it is uniform on the observation support set $S_n = \{x_1, x_2, ..., x_n\}$ according to (23.1.1). Then (23.4.2) holds for

$$p(x) = p_n(x) := \boldsymbol{I}(x \in S_n) \quad \text{and} \quad p_0(x) := \boldsymbol{I}(x \notin S_n)$$

because

$$(P_n + P_0)(S_n \cap A) = P_n(A) \text{ and } (P_n + P_0)((\mathcal{X} - S_n) \cap A) = P_n(A),$$

for every $A \subset \mathcal{X}$. This, together with the fact that P_n is supported by S_n and P_0 is supported by $\mathcal{X} - S_n$ implies that

$$\begin{aligned} D_\phi(P_n, P_0) &= \int_{\mathcal{X}} p_0 \, \phi\left(\frac{p_n}{p_0}\right) d(P_n + P_0) \\ &= \int_{S_n} \boldsymbol{I}(x \notin S_n) \, \phi\left(\frac{\boldsymbol{I}(x \in S_n)}{\boldsymbol{I}(x \notin S_n)}\right) dP_n \\ &+ \int_{\mathcal{X} - S_n} \boldsymbol{I}(x \notin S_n) \, \phi\left(\frac{\boldsymbol{I}(x \in S_n)}{\boldsymbol{I}(x \notin S_n)}\right) dP_0 \\ &= \int_{S_n} 0 \, \phi\left(\frac{1}{0}\right) dP_n + \int_{\mathcal{X} - S_n} 1 \phi\left(\frac{0}{1}\right) dP_0 \\ &= \lim_{t \to \infty} \frac{\phi(t)}{t} + \phi(0), \end{aligned} \tag{23.4.7}$$

where in the last line we replaced the undefined expression by the limit on the basis of the principle of continuous extension. The existence of the limit is guaranteed only for convex ϕ, i.e. for divergences, but even then the value (23.4.7) is constant, often infinite (for the power divergences it is finite only for the powers $0 < \alpha < 1$). Thus, without any further specification or restriction, the disparities, divergences or distances $D_\phi(P_n, P_0)$ are meaningless for statistical inference. Hence for the rest of the chapter we adopt the following convention.

Convention 23.4.8 The disparities, divergences or distances $D_\phi(P_n, P_0)$ are replaced by their discrete versions (23.4.4) resulting from finite partitions $\mathcal{P} = \{C_1, ..., C_k\}$ of the observation space $\mathcal{X}$ by the quantization rule (23.4.3). Further, we restrict ourselves to **real valued observations** and assume interval observation spaces $\mathcal{X} = (a, b) \subseteq I\!R$ on which statistical models P_n, P, P_0 are given by d.f.'s F_n, F, F_0 where both F and F_0 are assumed to have positive densities $f > 0$ and $f_0 > 0$. Moreover, we restrict ourselves to the interval partitions

$$\mathcal{P} = \{C_j := (c_{j-1}, c_j] : 1 \le j \le k\} \quad \text{for} \quad a = c_0 < c_1 < \cdots < c_{k-1} < c_k = b \tag{23.4.9}$$

where the quantization rule (23.4.3) takes on the form

$$\boldsymbol{p} = (p_j := F(c_j) - F(c_{j-1}) : 1 \le j \le k)\,, \tag{23.4.10}$$

$$\boldsymbol{p}_n = (p_{nj} := F_n(c_j) - F_n(c_{j-1}) : 1 \le j \le k) \tag{23.4.11}$$

$$\boldsymbol{p}_0 = (p_{0j} := F_0(c_j) - F_0(c_{j-1}) : 1 \le j \le k)\,. \tag{23.4.12}$$

By (23.4.4), the **disparities, divergences,** or **distances** $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ depend on the frequencies of the observations $x_1, \ldots, x_n$ in the intervals of the partition $\mathcal{P}$ but not on the order of these observations. Therefore the vector $(x_1, \ldots, x_n)$ of observations can be replaced by the order statistics $(x_{n:1}, x_{n:2}, \ldots, x_{n:n})$. Further, the **hypothetical model** F_0, which is compared with the **alternative model** F or the **empirical model** F_n, is known and by the assumptions the function $F_0(y)$ is strictly increasing on the observation space $\mathcal{X} = (a, b)$. Hence it can be used to transform in a one-to-one manner this observation space into the simple standardized observation space $\mathcal{Y} = (F_0(a), F_0(b)) = (0, 1)$ commonly used in the literature dealing with testing hypotheses $\mathcal{H}_0 : F_0$ against alternatives $\mathcal{A} : F$. The d.f. F governs in an i.i.d. manner the random observations $X_1, X_2, \ldots, X_n$ generating the realizations $x_1, x_2, \ldots, x_n$. In this new observation space we deal with the ordered observations

$$F_0(a) := 0 = Y_0 < Y_1 := F_0(X_{n:1}) \le \cdots \le Y_n := F_0(X_{n:n}) < 1 = Y_{n+1} := F_0(b) \tag{23.4.13}$$

and with the hypotheses $\mathcal{H}_0 : F_0(Q_0)$, alternatives $\mathcal{A} : F(Q_0)$ and e.d.f.'s $F_n(Q_0)$, all defined on $[0, 1]$ by means of the increasing **hypothetical percentile function** (briefly, **h.p.f.**)

$$Q_0(y) = F_0^{-1}(y) \quad \text{on} \quad [0, 1]. \tag{23.4.14}$$

This motivates the next convention, which will also hold for the remainder of this chapter.

Convention 23.4.15 We assume without loss of generality that the hypothesis of uniformity $\mathcal{H}_0 : F_0$ has constant p.d.f. $f_0(y) = 1$ (i.e. that the hypothetical d.f. is linear on $(0, 1)$, $F_0(y) = y$, which defines the same linear h.p.f. $Q_0(y) = y$).

This hypothesis is tested against the alternative $\mathcal{A} : F$ with a p.d.f. $f(y)$ positive on $(0, 1)$ (i.e. with the **alternative d.f.** $F(y)$ increasing on $(0, 1)$, which defines **alternative p.f.** $Q(y) = F^{-1}(y)$ increasing on $[0, 1]$). The testing is based on the e.d.f. $F_n(y)$ on $[0, 1]$ with jumps $1/n$ at the points

$$0 < Y_1 \leq \cdots \leq Y_n < 1 \tag{23.4.16}$$

obtained by ordering the random observations

$$X_1, X_2, \ldots, X_n \text{ i.i.d. by the p.d.f. } f(y) \text{ on the observation space } (0, 1). \tag{23.4.17}$$

Alternatively, the testing can be based on the one-to-one related **empirical percentile function** (briefly, **e.p.f.**)

$$Q_n(y) = F_n^{-1}(y) = \inf\{z \in (0, 1] : F_n(z) \geq y\} \text{ on } [0, 1]. \tag{23.4.18}$$

Consequently

$$Q_n\left(\frac{j}{n}\right) = Y_j, \quad 0 \leq j \leq n \tag{23.4.19}$$

and the disparities, divergences, or distances $D_\phi(P, P_0)$ and $D_\phi(P_0, P)$ are given by the formulas

$$D_\phi(P, P_0) = \int_0^1 \phi\,(p)\; dy \quad \text{and} \quad D_\phi(P_0, P) = \int_0^1 p\phi\left(\frac{1}{p}\right)\; dy. \tag{23.4.20}$$

This is the the basic conceptual framework for the rest of the chapter.

The information available to the statistician when he faces the problem of testing hypothesis $\mathcal{H}_0 : F_0$ is represented by the d.f.'s F_n and F_0. It is used by him to calculate the decision tool $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ or $T_n = m_n D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ using the rules (23.4.4), (23.4.11), and (23.4.12). Naturally, this procedure is simplified if either the distribution $\boldsymbol{p}_0$ or $\boldsymbol{p}_n$ is uniform,

$$\boldsymbol{p}_0 = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right) \quad \text{or} \quad \boldsymbol{p}_n = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right) \quad \text{for some } k > 1 \tag{23.4.21}$$

where k may increase with the sample size n, i.e. the dependence

$$k = k_n \tag{23.4.22}$$

is admitted but the subscript n is suppressed unless it plays an explicit role. These two possibilities are mutually exclusive for large k and are studied separately in the next two subsections.

23.4.1 Partitioning by Hypothetical Percentiles

Let us start with the first case considered in (23.4.21). It takes place if the partition (23.4.9) is defined by the $k+1$ cutpoints

$$c_j = Q_0\left(\frac{j}{k}\right) = \frac{j}{k}, \quad 0 \le j \le k \quad \text{for the h.p.f. } Q_0 \quad \text{(see (23.4.14))}, \tag{23.4.23}$$

which are the **hypothetical percentiles** of the uniformly distributed orders $0, 1/k, \ldots, (k-1)/k, 1$. Thus, the partitioning by uniformly distributed hypothetical percentiles leads to the discrete alternative, empirical, and hypothetical distributions

$$\boldsymbol{p} = \left(p_j := F\left(\frac{j}{k}\right) - F\left(\frac{j-1}{k}\right) : 1 \le j \le k\right), \tag{23.4.24}$$

$$\boldsymbol{p}_n = \left(p_{nj} := F_n\left(\frac{j}{k}\right) - F_n\left(\frac{j-1}{k}\right) : 1 \le j \le k\right), \tag{23.4.25}$$

$$\boldsymbol{p}_0 = \left(p_{0j} := F_0\left(\frac{j}{k}\right) - F_0\left(\frac{j-1}{k}\right) = \frac{1}{k} : 1 \le j \le k\right), \tag{23.4.26}$$

respectively. Based on formula (23.4.4), the disparities, divergences, or distances $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ are given by the formulas

$$D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = k \sum_{j=1}^{k} \phi\left(k p_{nj}\right) \quad \text{(see (23.4.25), (23.4.26))} \tag{23.4.27}$$

and

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = \sum_{j=1}^{k} p_{nj} \phi\left(\frac{1}{k p_{nj}}\right) \quad \text{(see (23.4.25), (23.4.26))} \tag{23.4.28}$$

where the latter is meaningful only if all p_{nj} are positive.

The probability that all p_{nj} are positive decreases when k increases and vanishes for $k \ge n+1$, since there are only n observations, i.e. F_n has at most n jumps. Therefore the version (23.4.28) is not used in the sequel and attention is restricted to the goodness-of-fit criteria (23.4.27) or their scaled versions $T_{\phi,n} = nD_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$. The above-introduced concrete statistics (23.2.20), (23.2.25), (23.2.27), (23.3.12), and (23.3.28) are examples of such scaled versions if their cells C_j are specified by the cutpoints (23.4.23).

23.4.2 Partitioning by Empirical Percentiles

In this subsection we study the second possibility considered in (23.4.21). It takes place if the partition (23.4.9) is defined by the same number of $k+1$ cutpoints

$$0 = c_0 < c_1 < \cdots < c_{k-1} < c_k = 1 \tag{23.4.29}$$

as in Subsection 23.4.1, but the first k of them are the **empirical percentiles** of the uniformly distributed orders $0, 1/k, \ldots, (k-1)/k$, i.e.,

$$c_j = \begin{cases} Q_n\left(\frac{j}{k}\right), & 0 \le j \le k-1 \ \text{ for the e.p.f. } Q_n \ \text{ (see (23.4.18))} \\ 1, & j = k. \end{cases} \tag{23.4.30}$$

This formula is not a complete parallel to (23.4.23) of the previous subsection, because here the last cutpoint is $c_k = 1$ and not $Q_n(k/k) = Q_n(1) = Y_n < 1$ (see in this respect Remark 23.4.39 below).

The cutpoints take on a more obvious form when k divides n. Therefore, unless otherwise explicitly stated, we assume that

$$k = k_n = \frac{n}{m} \ \text{ for } \ m = m_n = 1, 2, \ldots \quad \text{(see (23.4.22))}. \tag{23.4.31}$$

In accordance with the agreement above, the subscript n is suppressed in k_n, m_n and all expressions involving k_n, m_n unless it is explicitly needed to display.

By the definitions of F_n, F_0, under the assumption (23.4.31) the cutpoints formula (23.4.30) implies

$$c_j = Q_n\left(\frac{jm}{n}\right) = Y_{jm}, \quad 0 \le j \le k-1 \tag{23.4.32}$$

so that

$$F_0(c_j) = c_j = Y_{jm}, \quad 0 \le j \le k-1. \tag{23.4.33}$$

Since $c_k = 1$ so that $F_n(c_k) = 1 = km/n$, we get from (23.4.11) and (23.4.32)

$$\boldsymbol{p}_n = \left(p_{nj} := \frac{jm}{n} - \frac{(j-1)m}{n} = \frac{m}{n} = \frac{1}{k} : 1 \le j \le k\right). \tag{23.4.34}$$

Similarly, from (23.4.12) and (23.4.33) we get

$$\boldsymbol{p}_0 = \left(p_{0j} := Y_{jm} - Y_{(j-1)m} : 1 \le j \le k-1 \ \text{ and } \ p_{0k} := 1 - Y_{(k-1)m}\right). \tag{23.4.35}$$

In this formula $p_{0k} := 1 - Y_{(k-1)m}$ cannot be replaced by $p_{0k} := Y_{km} - Y_{(k-1)m}$ because $c_k = 1 = F_0(c_k)$ is almost surely greater than $Y_{km} = Y_n$. Notice that under the assumption (23.4.31) the cutpoints formula (23.4.30) implies that each cell of the partition of the interval contains exactly m observations.

By (23.4.4), the disparities, divergences, or distances $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ of the distributions (23.4.34) and (23.4.35) are given by the formulas

$$D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = D_{\phi^*}(\boldsymbol{p}_n, \boldsymbol{p}_0) \ \text{ for } \ \phi^*(t) = t\phi\left(\frac{1}{t}\right) \ \text{ on } (0, \infty) \tag{23.4.36}$$

where

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = \sum_{j=1}^{k-1} \frac{1}{k}\phi\left(k(Y_{jm} - Y_{(j-1)m})\right) + \frac{1}{k}\phi\left(k(1 - Y_{(k-1)m})\right)$$
$$= \frac{m}{n}\sum_{j=1}^{k-1} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + \frac{m}{n}\phi\left(\frac{n}{m}(1 - Y_{(j-1)m})\right). \quad (23.4.37)$$

The version (23.4.36) leads to a too complicated formula in terms of the original function ϕ. Thus for theoretical analysis as well as for practical applications it is more convenient to work with version (23.4.37). The stochastic differences $Y_{jm} - Y_{(j-1)m}$ are generally referred to as the m-spacings.

The disparities, divergences, or distances (23.4.37) define the m-spacings based goodness-of-fit statistics

$$T_\phi^{(m)} = T_{\phi,n}^{(m)} = nD_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$$
$$= m\sum_{j=1}^{k-1} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + m\phi\left(\frac{n}{m}(1 - Y_{(k-1)m})\right). \quad (23.4.38)$$

Remark 23.4.39 If in full analogy with (23.4.23) the cutpoint scheme (23.4.29), (23.4.30) is replaced by

$$c_j = Q_n\left(\frac{j}{k}\right), \quad 0 \le j \le k \quad \text{for the e.p.f. } Q_n \quad \text{(see (23.4.18))}$$

then $c_k = Y_n < 1$, so that the components $p_{0j} = F_0(c_j) - F_0(c_{j-1}) = c_j - c_{j-1}$ of the hypothetical distribution $\boldsymbol{p}_0$ satisfy the strict inequality

$$\sum_{j=1}^{k} p_{0j} = c_k = Y_n < 1$$

i.e., they cannot be normalized to 1. This can be solved by adding the cutpoint $c_{k+1} = 1$ when the collection of cutpoints

$$0 = c_0 < c_1 < \cdots < c_{k-1} < c_k < c_{k+1} = 1$$

generates hypothetical and empirical distributions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ with $k+1$ components, both normalized to 1. However, then the last component of the empirical distribution is

$$p_{n,k+1} = F_n(c_{k+1}) - F_n(c_k) = F_n(1) - F_n(Y_n) = 0$$

so that the empirical distribution is not uniform and its components are not strictly positive. Nevertheless, for $\phi \in \boldsymbol{\Phi}_{\text{div}}$ the ϕ-divergences $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ remain

to be well defined by the formula

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = D_{\phi^*}(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{1}{k} \phi\left(k(Y_{jm} - Y_{(j-1)m})\right) + (1 - Y_n)\phi^*(0) \tag{23.4.40}$$

for the function $\phi^*(t)$ defined in (23.4.36) (for details see e.g. Liese and Vajda (2006)). Thus the ϕ-divergences with finite limit $\phi^*(0) = \lim_{t\to\infty} \phi(t)/t$ define meaningful spacings-based divergence statistics

$$T_\phi^{*(m)} = T_{\phi,n}^{*(m)} = m \sum_{j=1}^{k} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + n(1 - Y_n)\phi^*(0). \tag{23.4.41}$$

In what follows we would like to deal with statistics $T_\phi^{(m)}$ for more general functions ϕ than just $\phi \in \boldsymbol{\Phi}_{\rm div}$. Therefore we prefer the more universal statistics (23.4.38).

In the remaining sections we study the properties and applications of the simple spacings-based variants

$$T_\phi = T_{\phi,n} = \sum_{j=1}^{n-1} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(1 - Y_{n-1})\right) \tag{23.4.42}$$

of the statistics $T_\phi^{(1)}$ in (23.4.38), i.e. $T_\phi^{(m)}$ when $m = 1$. We decompose them into **representative parts** R_ϕ and **asymptotically vanishing parts** V_ϕ as follows:

$$T_\phi = R_\phi + V_\phi \quad \text{with} \quad R_\phi = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \boldsymbol{\Phi} \tag{23.4.43}$$

and

$$V_\phi = \phi\left(n(1 - Y_{n-1})\right) - \phi\left(n(Y_n - Y_{n-1})\right) - \phi\left(n(1 - Y_n)\right) \tag{23.4.44}$$

where we put as before $Y_{n+1} = 1$.

Example 23.4.45 The power divergences $D_\alpha(\boldsymbol{p}_0, \boldsymbol{p}_n)$ of orders $\alpha \in I\!R$ from Example 23.3.16 define spacings-based statistics $T_\alpha = T_{\alpha,n}$ obtained by inserting in (23.4.42) the generating power functions ϕ_α or $\tilde{\phi}_\alpha$. For example, the power divergence of order 2 with the generating function $\phi_2(t) = (t-1)^2/2$ given in (23.3.17) defines the spacings-based statistic $T_2 = T_{2,n}$ with the representative part

$$R_2 = R_{2,n} = \frac{1}{2} \sum_{j=1}^{n+1} \left(n(Y_j - Y_{j-1}) - 1\right)^2 = \frac{n^2}{2} \sum_{j=1}^{n+1} \left(Y_j - Y_{j-1} - \frac{1}{n}\right)^2. \tag{23.4.46}$$

Obviously, this is a spacings-based version of the classical Pearson goodness-of-fit statistic

$$T_n = nD_2(\boldsymbol{p}_n, \boldsymbol{p}_0) = \frac{1}{2} n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0) \quad \text{(see (23.2.20))}.$$

Similarly, the power divergence of order 0 with the generating function $\phi_0(t) = -\ln t$ from (23.3.18) defines the spacings-based statistic $T_0 = T_{0,n}$ with the representative part

$$R_0 = R_{0,n} = -\sum_{j=1}^{n+1} \ln\left(n\left(Y_j - Y_{j-1}\right)\right), \tag{23.4.47}$$

which is nothing but a spacings-based version of the classical likelihood ratio goodness-of-fit statistic

$$T_n = nD_0(\boldsymbol{p}_0, \boldsymbol{p}_n) = nD_1(\boldsymbol{p}_n, \boldsymbol{p}_0) \quad \text{(see (23.3.12))}.$$

23.5 Goodness-of-Fit Statistics Based on Spacings

23.5.1 Objectives of the Following Sections

This chapter is devoted to the systematic analysis of the disparity and divergence spacings statistics and to their comparisons with the spacings statistics studied in the previous literature. The primary aim is to show that while the motivation of the latter is not based on the concept of similarity between empirical and hypothetical distributions, this idea is in fact hidden somewhere behind because they asymptotically coincide with the former. Therefore the *first objective* is to prove the mutual asymptotic equivalence of the disparity and divergence spacings statistics introduced in this chapter and the spacings statistics known from the literature. This equivalence helps to understand why many ad hoc defined spacings-based statistics exhibit desirable asymptotic properties. The secondary aim is to present in a relatively simple unified manner the asymptotic properties of the many various types of spacings statistics studied in the previous literature. Thus the *second objective* of this chapter is to prove the consistency and asymptotic normality under fixed and local alternatives for a sufficiently wide variety of our spacings-type disparity or divergence statistics. These results are important for applications of the spacings statistics in the testing of goodness-of-fit, and they may also be useful in the estimation of functionals of the type of ϕ-disparity or ϕ-divergence. The last aim is to apply this asymptotic theory to the spacings-based power divergence statistics and compare their asymptotic parameters and properties for various divergence orders $\alpha \in \mathbb{R}$. Therefore the *third objective* is the explicit evaluation of the asymptotic parameters of spacings-based power divergence statistics and an analysis of their properties including their continuity in the parameter $\alpha \in \mathbb{R}$. To achieve all these objectives within a reasonably

limited space, we pay the main attention to the simple spacings with $m = 1$ and, starting with Section 23.6, we deal exclusively with simple spacings.

It seems that the spacings-based goodness-of-fit test statistics given in the literature lacked so far the motivation of taking into account the notion of disparity between hypothetical and empirical distributions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$. This contrasts with the goodness-of-fit statistics based on deterministic partitions specified by the uniformly distributed constant cutpoints c_j given in (23.4.23) and by the related random frequency counts (23.4.25), where the typical statistics, including the most classical Pearson statistic T_1 and likelihood ratio statistic T_0, can easily be recognized as appropriately scaled power divergences between $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ for the powers $\alpha = -1$ and $\alpha = 0$, respectively.

The classical spacings-based statistics, however, appear to have been motivated rather by other considerations such as the analytic simplicity of formulas and the possibility to achieve desired asymptotic properties. In fact, as pointed out by Pyke (1965) in his landmark paper, most of the classical spacings-based statistics were proposed within the context of testing the randomness of events in time, in which differences between successive order statistics (spacings) were considered to play an important role. Also, in the period 1946–1953, when most of the classical tests based on spacings were proposed, research focused mostly on studying the behavior of these tests under the null-hypothesis, rather than under an alternative, making it unnecessary to motivate the test statistic from the point of view of divergence or disparity. Although the concept of dispersion of spacings around the uniform distribution has been mentioned as a motivation for a test statistic by some authors, all known spacings-based statistics are close to the divergence statistic $T_\phi^{(m)}$ of (23.4.38) or T_ϕ of (23.4.42) for some ϕ in $\boldsymbol{\Phi}_{\mathrm{div}}$, but none of them happens to be precisely equal to this divergence statistic. This situation is illustrated in the next examples for the simple-spacings statistics $T_\phi = T_{\phi,n}$ with $\phi \in \boldsymbol{\Phi}$ given by (23.4.43) as the sum $R_\phi + V_\phi$ where the representative terms

$$R_\phi = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \boldsymbol{\Phi} \tag{23.5.1}$$

slightly differ from the majority of the statistics known from the literature, which are of the form

$$S_\phi = \sum_{j=1}^{n+1} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \boldsymbol{\Phi} \tag{23.5.2}$$

where ϕ is often from the divergence subclass $\boldsymbol{\Phi}_{\mathrm{div}} \subset \boldsymbol{\Phi}$. Hence the departure from the divergence statistics is mainly the scaling of the spacings by $n+1$ instead of n. A possible explanation for this is suggested in the next example. We prove in the next section that the departure from the divergence statistics as such is asymptotically negligible in the sense that the so-called asymptotically vanishing term

$$V_\phi = \phi\left(n(1 - Y_{n-1})\right) - \phi\left(n(Y_n - Y_{n-1})\right) - \phi\left(n(1 - Y_n)\right) \tag{23.5.3}$$

really vanishes asymptotically and the modification of the scaling factor by $n/(n+1)$ is asymptotically negligible.

Example 23.5.4 The first known statistic of the type (23.5.2) is

$$\mathcal{G} = \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 \tag{23.5.5}$$

of Greenwood (1946) who devised it for testing the hypothesis that the intervals between successive events in epidemiology are exponentially distributed. Obviously,

$$(n+1)^2 \mathcal{G} = S_\phi = \sum_{j=1}^{n+1} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right) \tag{23.5.6}$$

for $\phi(t) = t^2$ from $\mathbf{\Phi}$. Therefore the Greenwood proposal was neither the divergence nor the disparity spacing statistic. However, Irwin in the discussion of Greenwood (1946), and Kimball (1947) suggested to replace $\mathcal{G}$ by the modification of the power divergence spacing statistic (23.4.46) defined by

$$\mathcal{K} = \sum_{j=1}^{n+1} \left(Y_j - Y_{j-1} - \frac{1}{n+1}\right)^2 = \frac{2}{(n+1)^2} \sum_{j=1}^{n+1} \phi_2\left((n+1)\left(Y_j - Y_{j-1}\right)\right) \tag{23.5.7}$$

for $\phi_2(t) = (t-1)^2/2$ from $\mathbf{\Phi}_{\text{div}}$, which is the same function that generated the quadratic spacing statistics R_2 in (23.4.46). The motivation of the Irwin and Kimball statistic $\mathcal{K}$ may be deduced from the inequality

$$E \sum_{j=1}^{n+1} \left(Y_j - Y_{j-1} - \frac{1}{n+1}\right)^2 \geq E \sum_{j=1}^{n+1} \left(Y_j - Y_{j-1} - E(Y_j - Y_{j-1})\right)^2$$

where the inequality is strict unless

$$E\left(Y_j - Y_{j-1}\right) = \frac{1}{n+1} \quad \text{for} \quad 1 \leq j \leq n+1$$

which in turn takes place if and only if $F = F_0$. Therefore the minimal expected values of the Kimball criterion $\mathcal{K}$ characterize the hypothesis F_0 and the larger expected values are reserved for the alternatives $F \neq F_0$.

Example 23.5.8 Another classical spacing statistic which is a slight modification of the power divergence spacings statistic (23.4.47) was defined by Moran (1951) as

$$\mathcal{M} = S_{\phi_0} = -\sum_{j=1}^{n+1} \ln\left((n+1)\left(Y_j - Y_{j-1}\right)\right) = \sum_{j=1}^{n+1} \phi_0\left((n+1)\left(Y_j - Y_{j-1}\right)\right) \tag{23.5.9}$$

where $\phi_0(t) = -\ln t$ from $\mathbf{\Phi}_{\text{div}}$ generated the logarithmic spacing statistics R_0 in (23.4.47). Van Es (1992) considered a statistic $\mathcal{VE}$ which for $m = 1$ takes on the form

$$\mathcal{VE} = \frac{1}{n-1} \sum_{j=2}^{n} \ln\left((n+1)\,(Y_j - Y_{j-1})\right). \tag{23.5.10}$$

Hence $(1-n)\mathcal{VE}$ is a modification of (23.5.9).

23.5.2 Types of Statistics Studied

As stated above, the analysis of the spacings-based disparity or divergence statistics generated by functions $\phi \in \mathbf{\Phi}$ is, in the rest of this chapter, restricted to the case of the simple spacings with $m = 1$. In the previous subsection we defined for these spacings three different statistics, viz. T_ϕ of (23.4.42), R_ϕ of (23.5.1), and S_ϕ of (23.5.2). The first of these was obtained by application of the partition of the observation space by n empirical percentiles of equidistant orders to the disparity or divergence $D_\phi(F_0, F_n)$ of the hypothetical and empirical distribution. The remaining two were modifications of T_ϕ representing the spacing statistics from the pioneering work of Greenwood, Kimball, and Moran. In the present subsection some other modifications are introduced, which represent the spacing statistics known from the literature subsequent to the mentioned pioneering work. Since those statistics generally used the m-spacings, we return temporarily in this subsection to our disparity or divergence statistics $T_\phi^{(m)}$ from (23.4.38) based on m-spacings, in order to make the comparisons more realistic and credible.

Let us start with Del Pino's (1979) class of statistics of the form

$$S_\phi^{(m)} = m \sum_{j=1}^{k} \phi\left(\frac{n+1}{m}(Y_{mj} - Y_{m(j-1)})\right) \tag{23.5.11}$$

where it is assumed that $n+1$ is divisible by k and that $m = (n+1)/k \geq 1$. Here the notation in our chapter is consistent in the sense that (23.5.11) reduces for $m = 1$ to the formula for S_ϕ in (23.5.2). Del Pino found $\phi(t) = t^2$ to be optimal among the functions ϕ considered by him. The class (23.5.11) was later investigated by Jammalamadaka, Zhou, and Tiwari (1989) and many others. The mentioned authors studied the asymptotics of $S_\phi^{(m)}$ for m tending slowly to infinity as $n \to \infty$. In such case these asymptotics depend only on the local properties of $\phi(t)$ in the neighborhood of $t = 1$, and in this regard a wide class of functions ϕ can be admitted, including those with $\phi''(1) = 0$, so that they can be used for functions which generate disparities or divergences. However, as we have seen in Examples 23.5.4 and 23.5.8 for some $\phi \in \mathbf{\Phi}_{\text{div}}$, the statistics (23.5.11) differ from those in (23.4.38). Other examples of well-known spacings-based statistics which differ from our spacings-type ϕ-disparity statistics (23.4.38) will be given below. Therefore it is important to look at the problem whether

the classical spacings-based statistics and our spacings-type disparity statistics are asymptotically equivalent for $n \to \infty$, and, if yes, then in what precise sense.

Let us return to our spacings-type ϕ-disparity statistic $T_\phi^{(m)}$ introduced in (23.4.38). Notice that $T_\phi^{(m)}$ cannot be efficient when $m > 1$, because then it ignores the observations Y_{mj+r} for $1 \le j \le k-1$ and $1 \le r \le m-1$. Shifting the orders j/k of the percentiles in (23.4.32) by a quantity depending on r, we obtain the shifted empirical percentiles

$$c_j^{(r)} = Q_n\left(\frac{mj+r}{n}\right) = Y_{mj+r}, \quad 1 \le j \le k-1, \quad 1 \le r \le m-1 \tag{23.5.12}$$

as cutpoints and, instead of $p_{0j} = Y_{mj} - Y_{m(j-1)} = p_{0j}^{(0)}$, the shifted hypothetical probabilities $p_{0j}^{(r)} = Y_{mj+r} - Y_{m(j-1)+r}$, while still preserving the uniform shifted empirical probabilities $p_{nj}^{(r)} = 1/k = m/n$ on the cells $(c_{j-1}^{(r)}, c_j^{(r)}]$, $1 \le r \le m-1$. Replacing each term $\phi(\frac{n}{m}(Y_{mj} - Y_{m(j-1)}))$ in (23.4.38) by the average

$$\frac{1}{m}\sum_{r=0}^{m-1} \phi\left(\frac{n}{m}(Y_{mj+r} - Y_{m(j-1)+r})\right) \tag{23.5.13}$$

of all $\phi(np_{0j}^{(r)}/m)$ for $0 \le r \le m-1$, we get a potentially more efficient version of $T_\phi^{(m)}$, namely

$$\hat{T}_\phi^{(m)} = \sum_{j=m}^{n-1} \phi\left(\frac{n}{m}(Y_j - Y_{j-m})\right) + m\phi\left(\frac{n}{m}(1 - Y_{n-m})\right) \tag{23.5.14}$$

which for $m = 1$ reduces to T_ϕ of (23.4.42), so that the notation of our chapter is again consistent.

A similar procedure can be carried out for $S_\phi^{(m)}$ of (23.5.11), which involves the observations Y_{mj} for $1 \le j \le k$, but ignores the observations Y_{mj+r} for $0 \le j \le k-1$ and $1 \le r \le m-1$. Replacing each term $\phi(\frac{n+1}{m}(Y_{mj} - Y_{m(j-1)}))$ in (23.5.11) by the average of the m terms $\phi(\frac{n+1}{m}(Y_{mj+r} - Y_{m(j-1)+r}))$, $0 \le r \le m-1$, and excluding the terms containing undefined expressions (that is, the terms $Y_{mk+r} - Y_{m(k-1)+r}, 1 \le r \le m-1$, where $mk + r > n+1$), we get a similar possibly more efficient version

$$\hat{S}_\phi^{(m)} = \sum_{j=m}^{n+1} \phi\left(\frac{n+1}{m}(Y_j - Y_{j-m})\right) \tag{23.5.15}$$

of Del Pino's statistic $S_\phi^{(m)}$ of (23.5.11). Notice that if $m = 1$, then $\hat{S}_\phi^{(m)}$ of (23.5.15) reduces to S_ϕ of (23.5.2) above, so that our notation is in this sense still consistent. The statistics (23.5.15) are formally well defined for all $1 \le m \le n$, and not only for $m = (n+1)/k \ge 1$ corresponding to the integers $1 < k \le n+1$.

Both Del Pino's statistic $S_\phi^{(m)}$ of (23.5.11) and $\hat{S}_\phi^{(m)}$ of (23.5.15) can be viewed as generalizations of S_ϕ of (23.5.2) from the case of simple spacings to that of m-spacings, but whereas (23.5.11) involves *non-overlapping* m-spacings, (23.5.15) uses m-th order *overlapping* spacings. Cressie (1976, 1979) and Ekström (1999) are among the authors dealing with the statistics (23.5.15) for fixed $m \geq 1$ and eventually also for m slowly tending to ∞ when $n \to \infty$.

In spite of the fact that, when carrying out our analysis above, we went through several important papers (and many other ones listed in these as references), covering altogether four decades of research on spacings-based statistics, we did not in this literature come across the framework of the statistics T_ϕ and its modification R_ϕ when restricting ourselves to the simple spacings with $m = 1$. To make this connection, take into account that if $m > 1$, and in particular if $m \to \infty$, then the statistics (23.5.15) assign more weight to the central spacings than to those in the tails. To avoid this, Hall (1986) proposed to wrap the observations $Y_1, Y_2, ..., Y_n$ around the circle of unit circumference and to define the m-spacings $Y_{m+j} - Y_j$ for arbitrary $1 \leq m \leq n$ and j as the distance between observations Y_j and Y_{j+m} on this circle. This leads to the following two possible extensions of the ordered observations $Y_1, \ldots, Y_n$.

(i) By the formula

$$Y_{n+j} = 1 + Y_j \quad \text{for} \quad j = 1, 2, \ldots, n \tag{23.5.16}$$

where the previous dummy observation $Y_0 = 0$ is suppressed and the other dummy observation $Y_{n+1} = 1$ is redefined in accordance with (23.5.16) by $Y_{n+1} = 1 + Y_1$, leading to the m-spacing $Y_{j+m} - Y_j$ to be equal to $1 + Y_{m+j-n} - Y_j$ if $n + 1 - m \leq j \leq n$, and remaining unaltered, i.e. equal to $Y_{j+m} - Y_j$, if $1 \leq j \leq n - m$.

(ii) By the alternative formula

$$Y_{n+j} = 1 + Y_{j-1} \quad \text{for} \quad j = 0, 1, \ldots, n \tag{23.5.17}$$

where the dummy observations $Y_0 = 0$ and $Y_{n+1} = 1$ are placed on the circle as well, resulting in the m-spacing $Y_{j+m} - Y_j$ to be defined as $1 + Y_{m+j-n-1} - Y_j$ if $n + 2 - m \leq j \leq n$ and remaining equal to $Y_{j+m} - Y_j$ if $0 \leq j \leq n + 1 - m$.

Both these extensions of the ordered observations Y_j beyond $j > n$ allow to add in (23.5.15) the tail evidence missing there by adding to the substituted averages (23.5.13) also the previously excluded terms. Depending on whether we use (23.5.16) or the alternative extension (23.5.17), we get in this manner two different extensions of (23.5.15), namely

$$\tilde{S}_\phi^{(m)} = \sum_{j=1}^{n} \phi\left((n+1)\,\frac{Y_{j+m} - Y_j}{m}\right) \quad \text{where} \quad Y_{j+m} = 1 + Y_{j+m-n} \tag{23.5.18}$$

if $j = n+1-m, \ldots, n$ (see (23.5.16)), or

$$\tilde{\tilde{S}}_{\phi}^{(m)} = \sum_{j=0}^{n} \phi\left((n+1)\,\frac{Y_{j+m} - Y_j}{m}\right) \quad \text{where} \quad Y_{j+m} = 1 + Y_{j+m-n-1} \tag{23.5.19}$$

if $j = n+2-m, \ldots, n$, $Y_0 = 0$ and $Y_{n+1} = 1$ (see (23.5.17)).
The statistics from the class (23.5.18) were studied for example by Hall (1986) and Morales, Pardo, and Vajda (2003), while those from the class (23.5.19) were investigated among others by Cressie (1978), Rao and Kuo (1984), Ekström (1999) and Misra and van der Meulen (2001) and others cited there.

Recently Jiménez and Shao (2009) studied for convex functions ϕ the statistics

$$JS_{\phi}^{(m)} = m \sum_{j=1}^{k} \phi\left(\frac{n+1}{m}\left[F(Y_{mj}) - F(Y_{m(j-1)})\right]\right)$$

for m, n such that $n+1$ is divisible by k and $m = (n+1)/k \geq 1$. Under the hypothetical d.f.'s $F(y) = y$ considered in this chapter these statistics reduce to the statistic $S_{\phi}^{(m)}$ of Del Pino's (23.5.11).

As said above, this chapter deals only with the ordinary spacings where $m = 1$. We have seen that then the statistic T_{ϕ} takes on the form presented in (23.4.42) and both $S_{\phi}^{(m)}$ of (23.5.11) and $\hat{S}_{\phi}^{(m)}$ of (23.5.15) reduce to the statistic

$$S_{\phi} = \sum_{j=1}^{n+1} \phi\left((n+1)\,(Y_j - Y_{j-1})\right), \quad \text{where} \quad Y_{n+1} = 1 \text{ and } Y_0 = 0 \tag{23.5.20}$$

introduced in (23.5.2). Consequently, $JS_{\phi}^{(m)}$ reduces for $m = 1$ to S_{ϕ} too. It is easy to see that in this case also $\tilde{\tilde{S}}_{\phi}^{(m)}$ of (23.5.19) reduces to S_{ϕ}. However, $\tilde{S}_{\phi}^{(m)}$ of (23.5.18) does not do so unless ϕ is linear. Indeed, if $m = 1$, $\tilde{S}_{\phi}^{(m)}$ reduces to

$$\tilde{S}_{\phi} = \sum_{j=1}^{n-1} \phi\left((n+1)\,(Y_{j+1} - Y_j)\right) + \phi\left((n+1)\,(Y_1 + 1 - Y_n)\right) \tag{23.5.21}$$

which coincides with

$$S_{\phi} = \sum_{j=1}^{n-1} \phi\left((n+1)\,(Y_{j+1} - Y_j)\right) + \phi\left((n+1)\,Y_1\right) + \phi\left((n+1)\,(1 - Y_n)\right) \tag{23.5.22}$$

only if

$$\phi\left((n+1)\,Y_1\right) + \phi\left((n+1)\,(1 - Y_n)\right) = \phi\left((n+1)\,(Y_1 + 1 - Y_n)\right)$$

which takes place with a positive probability only for linear ϕ.

In addition to the statistics T_ϕ, S_ϕ, R_ϕ and $\tilde{S}_\phi$ introduced above, we will study two more single-spacings statistics. First, we introduce another new type of spacings-type disparity statistic denoted by $\tilde{T}_\phi$. To obtain it, we redefine the partition $\mathcal{Q} = \{(0, Y_1], \ldots, (Y_{n-2}, Y_{n-1}], (Y_{n-1}, 1)\}$ of $(0, 1)$ considered in (23.4.35). The new partition $\tilde{\mathcal{Q}}$ of $(0, 1)$ is obtained by rearranging the n intervals of the partition $\mathcal{Q}$ into n new intervals by the rule

$$(0, Y_1] \mapsto (0, Y_1] \cup (Y_n, 1) \quad \text{and} \quad (Y_{n-1}, 1) \mapsto (Y_{n-1}, Y_n] \tag{23.5.23}$$

while keeping the remaining intervals $(Y_{j-1}, Y_j]$, $\quad 2 \le j \le n-1$ unaltered. This new partition $\tilde{\mathcal{Q}}$ leads to the **modified hypothetical distribution**

$$\tilde{\boldsymbol{p}}_0 = (\tilde{p}_{01} = Y_1 + 1 - Y_n,\, \tilde{p}_{02} = Y_2 - Y_1, \ldots, \tilde{p}_{0n} = Y_n - Y_{n-1})$$

which similar to (23.4.34) preserves the original uniform empirical distribution $\boldsymbol{p}_n$ on the cells, as each of the new n intervals still contains exactly one of the observations $Y_1, \ldots, Y_n$. Therefore the new partition leads to the new spacings-type disparity statistics

$$\begin{aligned} \tilde{T}_\phi &= n\, D_\phi\,(\tilde{\boldsymbol{p}}_0, \boldsymbol{p}_n) = \sum_{j=1}^{n} \phi(n\tilde{p}_{0j}) \\ &= \sum_{j=2}^{n} \phi\,(n(Y_j - Y_{j-1})) + \phi\,(n(Y_1 + 1 - Y_n)) \end{aligned} \tag{23.5.24}$$

which differ from T_ϕ of (23.4.42). Applying the functions

$$\phi^{(n)}(t) = \phi\left(\frac{n+1}{n}\, t\right) \tag{23.5.25}$$

we obtain the useful relations

$$\tilde{S}_\phi = \tilde{T}_{\phi^{(n)}} \quad \text{and} \quad S_\phi = R_{\phi^{(n)}}. \tag{23.5.26}$$

In addition to the statistics $R_\phi, S_\phi, \tilde{S}_\phi$, $T_\phi, \tilde{T}_\phi$, defined above in (23.5.1), (23.5.2), (23.5.21), (23.4.42), and (23.5.24), respectively, we use in this chapter also the auxiliary spacings-based statistics

$$\tilde{R}_\phi = \sum_{j=1}^{n-1} \phi\,(n(Y_{j+1} - Y_j)) = R_\phi - \phi(nY_1) - \phi\,(n(1 - Y_n)) \tag{23.5.27}$$

investigated previously by authors neglecting the tail probabilities such as for example Hall (1984). Thus we can conclude this subsection by introducing the sets

$$\mathcal{U}_\phi = \left\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\right\}, \quad \phi \in \boldsymbol{\Phi} \tag{23.5.28}$$

of the spacings-based statistics of the special types introduced here and studied in the following sections. The statistics from $\mathcal{U}_\phi$ are representative in the sense that they cover almost all spacings based statistics studied in the previous literature as special cases when the attention is restricted to the simple spacings. The only exception known to us are the statistics

$$\bar{R}_\phi = \tilde{R}_{\phi^{(n)}} = \tilde{S}_\phi - \phi\left((n+1)\left(Y_1 + 1 - Y_n\right)\right) \tag{23.5.29}$$

studied by van Es (1992), see e.g. (23.5.10). However, it is seen from the proof of Assertion 23.6.3 below that the asymptotic properties of $\bar{R}_\phi$ are the same as the asymptotic properties of the statistics from the class $\mathcal{U}_\phi$ studied in the rest of this chapter.

The set $\mathbf{\Phi}$ of differentiable functions $\phi : (0, \infty) \to I\!R$ was introduced in the Subsection 23.3.1 by mild additional conditions guaranteeing the existence of the integrals

$$D_\phi(P, P_0) = \int p_0 \phi\left(\frac{p}{p_0}\right) d\lambda \quad \text{(see (23.3.7))}$$

called **disparities** in the wide sense, which were justified as measures of disparity, divergence, or distance only for ϕ from the subclasses $\mathbf{\Phi}_{\text{disp}} \supset \mathbf{\Phi}_{\text{div}}$ of $\mathbf{\Phi}$. On the other hand, the functions ϕ which defined the statistics $U_\phi \in \mathcal{U}_\phi$ considered in the cited literature imposed on the functions $\phi : (0, \infty) \to I\!R$ usually somewhat different additional conditions, namely the continuity and the continuous second-order differentiability in the neighborhood of 1 with $\phi''(1) \neq 0$ and $\phi(1) = 0$. Therefore in the sequel we study the class of statistics

$$\mathcal{U}_\phi = \left\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\right\}, \quad \phi \in \mathbf{\Phi}_0 \tag{23.5.30}$$

where $\mathbf{\Phi}_0$ is the set of all continuous functions $\phi : (0, \infty) \mapsto I\!R$ which are twice continuously differentiable in a neighborhood of 1 with $\phi''(1) > 0$ and $\phi(1) = 0$. The replacement of $\mathbf{\Phi}$ by $\mathbf{\Phi}_0$ means no loss of generality and guarantees that the class (23.5.30) contains all disparity and divergence statistics R_ϕ, $\tilde{R}_\phi$, S_ϕ, $\tilde{S}_\phi$, T_ϕ, $\tilde{T}_\phi$, defined above or, more generally, that

$$\mathbf{\Phi}_{\text{div}} \subset \mathbf{\Phi}_{\text{disp}} \subset \mathbf{\Phi}_0. \tag{23.5.31}$$

For references later, we summarize the definition formulas

$$R_\phi = R_{\phi,n} = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right) \tag{23.5.32}$$

$$S_\phi = S_{\phi,n} = \sum_{j=1}^{n+1} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right) \tag{23.5.33}$$

(e.g., Jammalamadaka, Zhou, and Tiwari (1989), Jiménez and Shao (2009))

$$T_\phi = T_{\phi,n} = \sum_{j=1}^{n-1} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(1 - Y_{n-1})\right) = nD_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) \quad (23.5.34)$$

(see (23.4.38), (23.4.42))

$$\tilde{R}_\phi = \tilde{R}_{\phi,n} = \sum_{j=2}^{n} \phi\left(n(Y_j - Y_{j-1})\right) \quad (23.5.35)$$

(e.g., Hall (1984), Cressie (1976), (1978), and (1979))

$$\tilde{S}_\phi = \tilde{S}_{\phi,n} = \sum_{j=2}^{n} \phi\left((n+1)(Y_j - Y_{j-1})\right) + \phi\left((n+1)(Y_1 + 1 - Y_n)\right) \quad (23.5.36)$$

(e.g., Hall (1986), Morales, Pardo, and Vajda (2003))

$$\tilde{T}_\phi = \tilde{T}_{\phi,n} = \sum_{j=2}^{n} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(Y_1 + 1 - Y_n)\right) = nD_\phi(\tilde{\boldsymbol{p}}_0, \boldsymbol{p}_n) \quad (23.5.37)$$

(see (23.5.24)), where in all these formulas $Y_0 = 0, Y_{n+1} = 1$ and

$$Y_j = F_0(X_{n:j}) \text{ for } X_{n:j} \sim F, \quad 1 \le j \le n \text{ and } \mathcal{H} : F_0. \quad (23.5.38)$$

In the next subsection and in the rest of the chapter we present a series of assertions about the properties of the statistics (23.5.32) through (23.5.37). Some of these assertions follow from the papers dealing with general m-spacing statistics cited in this and the following sections. Our simple proofs are to some extent based on the arguments established there.

23.5.3 Structural Spacings Statistics

In this subsection and in the rest of this chapter we study the subclasses

$$\mathcal{U}_\phi = \left\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\right\} \text{ for } \phi \in \boldsymbol{\Phi}_2 \text{ or } \phi \in \boldsymbol{\Phi}_1, \quad (23.5.39)$$

where $\boldsymbol{\Phi}_2 \subset \boldsymbol{\Phi}_1 \subset \boldsymbol{\Phi}_0$ of the class (23.5.30) of statistics defined by (23.5.32) through (23.5.38). Here $\boldsymbol{\Phi}_1 \subset \boldsymbol{\Phi}_0$ is the subset of functions $\phi \in \boldsymbol{\Phi}_0$ for which there exist functions $\xi, \eta, \zeta : (0, \infty) \mapsto I\!R$ satisfying the functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \zeta(t)\,\phi(s) + \eta(s)\,(t-1) \quad \text{for all } s, t \in (0, \infty). \quad (23.5.40)$$

The narrower class $\boldsymbol{\Phi}_2$ consists of all $\phi \in \boldsymbol{\Phi}_1$ which admit functions $\xi, \eta, \zeta : (0, \infty) \mapsto I\!R$ satisfying the stronger functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \phi(s) + \eta(s)\,(t-1). \quad (23.5.41)$$

Assertion 23.5.42 The functions ξ, ζ, and η are continuous on $(0, \infty)$ and satisfy the relations

$$\xi(1) = \zeta(1) = 1 \quad \text{and} \quad \eta(1) = 0. \tag{23.5.43}$$

Proof The continuity of ξ and η can be obtained by putting $t = 2$ and $t = 3$, and that of ζ by putting $s = 2$ in (23.5.40). If we put $s = 1$ in (23.5.40) or (23.5.41) and use the assumption $\phi(1) = 0$, then we obtain that for all $t \in (0, \infty)$

$$(\xi(1) - 1)\,\phi(t) + \eta(1)\,(t - 1) = 0.$$

This contradicts the assumption $\phi''(1) > 0$, unless $\xi(1) = 1$, which implies also $\eta(1) = 0$. By putting $t = 1$ in (23.5.40) we find that $\zeta(1) = 1$. □

Assertion 23.5.44 Every $\phi \in \mathbf{\Phi}_1$ is differentiable on $(0, \infty)$, the corresponding functions ξ and η are differentiable at 1, and for every $t > 0$

$$\phi'(t) = \xi'(1)\,\frac{\phi(t)}{t} + \phi'(1)\,\frac{\zeta(t)}{t} + \eta'(1)\,\frac{t-1}{t}. \tag{23.5.45}$$

Proof Putting $s = 1 + \varepsilon$ and

$$\xi^*(\varepsilon) = \frac{\xi(1+\varepsilon) - \xi(1)}{\varepsilon}, \quad \eta^*(\varepsilon) = \frac{\eta(1+\varepsilon) - \eta(1)}{\varepsilon}$$

we obtain from (23.5.40) for every $t > 0$ and ε close to 0

$$t\,\frac{\phi(t+\varepsilon t) - \phi(t)}{\varepsilon t} = \xi^*(\varepsilon)\,\phi(t) + \frac{\phi(1+\varepsilon) - \phi(1)}{\varepsilon}\,\zeta(t) + \eta^*(\varepsilon)\,(t-1). \tag{23.5.46}$$

Since ϕ is differentiable in a neighborhood of 1, we have for t close to 1

$$\xi^*(\varepsilon)\,\phi(t) + \eta^*(\varepsilon)\,(t-1) = t\,\phi'(t) - \phi'(1)\,\zeta(t) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

By the assumptions concerning $\mathbf{\Phi}$, $\phi(t)$ is not linear in a neighborhood of $t = 1$. Therefore the last relation implies that the limits of $\xi^*(\varepsilon)$ and $\eta^*(\varepsilon)$ for $\varepsilon \to 0$ exist, that is,

$$\xi^*(\varepsilon) = \xi'(1) + o(\varepsilon) \quad \text{and} \quad \eta^*(\varepsilon) = \eta'(1) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Now (23.5.45) follows from (23.5.46) for all $t > 0$. □

Example 23.5.47 The function $\phi(t) = (1-t)/t$, $t > 0$, belongs to $\mathbf{\Phi}$ and satisfies (23.5.41) for $\xi(t) = 1/t$ and $\eta(t) = 0$. Therefore it belongs to $\mathbf{\Phi}_2 \subset \mathbf{\Phi}$. The function $\phi(t) = (1-t)^2/t$, $t > 0$, belongs to $\mathbf{\Phi}$ too and satisfies (23.5.41) for the same $\xi(t)$ as above and $\eta(t) = t - 1/t$. Therefore it belongs to $\mathbf{\Phi}_2$. The functions defined on $(0, \infty)$ by

$$\phi_\alpha(t) = \frac{t^\alpha \ln t}{(2\alpha - 1)}, \quad \alpha \in I\!R - \{\tfrac{1}{2}\}$$

belong to $\mathbf{\Phi}$ and satisfy (23.5.40) for $\xi(t) = \zeta(t) = t^\alpha$ and $\eta(t) = 0$. Therefore

$$\{\phi_\alpha : \alpha \in I\!R - \{\tfrac{1}{2}\}\} \subset \mathbf{\Phi}_1$$

and $\phi_0 \in \mathbf{\Phi}_2$. But ϕ_1 satisfies also (23.5.41) for $\xi(t) = t$ and $\eta(t) = t \ln t$ and therefore ϕ_1 belongs to $\mathbf{\Phi}_2$.

23.5.4 Organization of the Following Sections

The rest of the chapter deals with the asymptotic properties and applications of the classes of statistics considered in (23.5.39). Let us mention briefly how the following sections are organized. Section 23.6 establishes the asymptotic equivalence of the statistics from the class $\mathcal{U}_\phi = \left\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\right\}$, $\phi \in \mathbf{\Phi}_1$, and presents the general asymptotic theory of the structural statistics from the class $\mathcal{U}_\phi = \left\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\right\}$, $\phi \in \mathbf{\Phi}_2$. Section 23.7 applies this theory to and makes it precise for the power divergence statistics of Example 23.3.16 and comments on comparable results in the previous literature. Section 23.8 presents a universal program for evaluation of power divergence spacings statistics and their applications in testing the goodness-of-fit. Finally, Section 23.9 contains proofs of the assertions of Sections 23.6 and 23.7.

23.6 Asymptotic Properties of Structural Statistics

In the remainder of this chapter the observations are assumed to be distributed on $[0, 1]$ in two possible ways:

(i) *under a fixed alternative*,

(ii) *under local alternatives.*

Case (i) means that the observations are distributed by a fixed distribution function $F \sim f$ with f positive and continuous on $[0, 1]$. Case (ii) means that the observations from samples of sizes $n = 1, 2, \ldots$ are distributed by distribution functions

$$F^{(n)}(x) = F_0(x) + \frac{L_n(x)}{\sqrt[4]{n}} = x + \frac{L_n(x)}{\sqrt[4]{n}} \tag{23.6.1}$$

on $[0,1]$, where the functions $L_n : I\!R \mapsto I\!R$ are continuously differentiable, with $L_n(0) = L_n(1) = 0$, and with derivatives $\ell_n(x) = L_n'(x)$ tending on $[0,1]$ to a continuously differentiable function $\ell : I\!R \mapsto I\!R$ uniformly in the sense that

$$\sup_{0\le x\le 1} |\ell_n(x) - \ell(x)| = o(1) \quad \text{as } n \to \infty. \tag{23.6.2}$$

The two possibilities (i) and (ii) are not mutually exclusive: their conjunction is "under the hypothesis $\mathcal{H}_0$ " where $F(x) = F_0(x)$, $f(x) = f_0(x) = \boldsymbol{I}_{[0,1]}(x)$ and $L_n(x) = 0$ on $I\!R$ for all n. This means that the asymptotic results obtained under local alternatives for $\ell(x)$ of (23.6.2) being identically equal to 0 must coincide with the results obtained under the fixed alternative for $F(x) = F_0(x)$.

23.6.1 Asymptotic Equivalence

The theorems below demonstrate that if $\phi \in \boldsymbol{\Phi}_2$ defines a ϕ-divergence or ϕ-disparity, then the statistics S_ϕ, $\tilde{S}_\phi$, R_ϕ, and $\tilde{R}_\phi$, which are formally not scaled ϕ-divergences or ϕ-disparities of the hypothetical and empirical distributions F_0 and F_n, share the most important statistical properties with the statistics T_ϕ and $\tilde{T}_\phi$, which are scaled ϕ-divergences or ϕ-disparities of this type. Therefore they provide a key argument for the thesis of the present chapter formulated at the beginning of Subsection 23.5.1, namely that the spacings-based goodness-of-fit statistics considered in the previous literature actually measure a disparity between the hypothetical and empirical distributions F_0 and F_n, although this was possibly not so intended by the various authors. But the main purpose of the following theorems is to present a systematic asymptotic theory for the whole set of statistics (23.5.39) and to demonstrate that the small modifications distinguishing these statistics from one another are asymptotically negligible. The restriction to the functions from $\boldsymbol{\Phi}_2$ or even $\boldsymbol{\Phi}_1$ is not essential – it only simplifies the proof of the next theorem.

Assertion 23.6.3 Consider the observations under fixed or local alternatives, and the set of statistics $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$ defined in (23.5.1), (23.5.27), (23.5.2), (23.5.21), (23.4.42), and (23.5.24). If $\phi \in \boldsymbol{\Phi}_1$, then for any statistic $U_\phi \in \{R_\phi, T_\phi, \tilde{T}_\phi\}$

$$U_\phi - \tilde{R}_\phi = O_p(1) \quad \text{as } n \to \infty, \tag{23.6.4}$$

and, if $\phi \in \boldsymbol{\Phi}_2$, then

$$S_\phi - R_\phi = \varepsilon_n R_\phi + \delta_n \quad \text{and} \quad \tilde{S}_\phi - \tilde{T}_\phi = \varepsilon_n \tilde{T}_\phi + \delta_n \tag{23.6.5}$$

where $\varepsilon_n = o(1)$ and $\delta_n = \phi'(1) + o(1)$ as $n \to \infty$. □

The proofs of this assertion and of the remaining ones of the chapter are deferred to Section 23.9.

23.6.2 Assumptions and Notations

In this subsection we study spacings-type ϕ-disparity statistics R_ϕ, $\tilde{R}_\phi$, S_ϕ, $\tilde{S}_\phi$, T_ϕ, and $\tilde{T}_\phi$, defined by (23.5.1), (23.5.27), (23.5.2), (23.5.21), (23.4.42), and (23.5.24), for ϕ from $\mathbf{\Phi}_2$ or $\mathbf{\Phi}_1$ as in the previous subsection. Unless otherwise explicitly stated, these statistics are assumed to be distributed under the fixed or local alternatives introduced as case (i) and case (ii) in the beginning of this section.

For every continuous function $\psi : (0, \infty) \mapsto I\!R$ we define the condition

$$\lim_{t\to\infty} t^{-\alpha}|\psi(t)| = \lim_{t\downarrow 0} t^{\beta}|\psi(t)| = 0 \quad \text{for some } \alpha \geq 0 \text{ and } \beta < 1 \tag{23.6.6}$$

and the integral

$$\langle\psi\rangle = \langle\psi(t)\rangle = \int_0^\infty \psi(t)\, e^{-t}\, dt. \tag{23.6.7}$$

Obviously, if (23.6.6) holds then $\langle\psi\rangle$ exists and is finite.

Let $\phi \in \mathbf{\Phi}_1$ satisfy (23.6.6) and let

$$\xi = \xi_\phi, \quad \zeta = \zeta_\phi \quad \text{and} \quad \eta = \eta_\phi \tag{23.6.8}$$

be the corresponding functions satisfying the functional equation (23.5.40). Then all functions

$$\psi(t) = \phi(ts) - \phi(t)\,\zeta(s), \quad s > 0,$$

satisfy (23.6.6) too, and by (23.5.40) the linear combinations

$$\psi(t) = \xi(t)\,\phi(s) + \eta(t)\,(s-1), \quad s > 0,$$

of functions $\xi(t)$ and $\eta(t)$ also satisfy (23.6.6). Since $\phi(s)$ is not linear in the neighborhood of $s = 1$, it follows from here that $\xi(t)$ and $\eta(t)$ themselves satisfy (23.6.6). Therefore the integrals $\langle\xi\rangle$ and $\langle\eta\rangle$ exist and are finite.

For the fixed alternatives $F \sim f$ we shall consider the linear combinations

$$\mu_\phi(f) = \langle\xi\rangle\, D_\phi(F_0, F) + \langle\phi\rangle\, D_\zeta(F_0, F) \tag{23.6.9}$$

where

$$D_\phi(F_0, F) = \int_0^1 f(x)\phi\left(\frac{1}{f(x)}\right)\, dx \tag{23.6.10}$$

and

$$D_\zeta(F_0, F) = \int_0^1 f(x)\zeta\left(\frac{1}{f(x)}\right)\, dx \tag{23.6.11}$$

are disparities of the distributions F_0 and F, well defined by (23.4.1) under the present assumptions about the densities f_0 and f and finite. If $\phi(t)$ is convex on $(0, \infty)$, or $\phi(t) - \phi'(1)\,(t-1)$ is monotone on $(0, 1)$ and $(1, \infty)$, then $D_\phi(F_0, F)$ is a nonnegative ϕ-divergence or ϕ-disparity of F_0 and F. Similarly, if $\zeta(t)$ is convex

on $(0,\infty)$, or $\zeta(t)-\zeta(1)-\zeta'(1)\,(t-1)$ is monotone on $(0,1)$ and $(1,\infty)$, then the ϕ^*-divergence or ϕ^*-disparity of F_0 and F for

$$\phi^*(t)=\zeta(t)-\zeta(1)=\zeta(t)-1 \quad \text{(see (23.5.43))}$$

satisfies the relation $D_{\phi^*}(F_0,F)=D_\zeta(F_0,F)-1$. Hence the formula for $\mu_\phi(f)$ can be written for every $\phi\in\mathbf{\Phi}_1$ in the more intuitive form

$$\mu_\phi(f)=\langle\xi\rangle\, D_\phi(F_0,F)+\langle\phi\rangle\,[D_{\phi^*}(F_0,F)+1] \tag{23.6.12}$$

where ξ and ϕ^* depend on ϕ as specified above, and $D_\phi(F_0,F)$, $D_{\phi^*}(F_0,F)$ are divergences or disparities between the hypothesis F_0 and the alternative F for typical $\phi\in\mathbf{\Phi}_1$. For $\phi\in\mathbf{\Phi}_2\subset\mathbf{\Phi}_1$ it holds that $\zeta\equiv 1$,so that (23.6.12) then simplifies to

$$\mu_\phi(f)=\langle\xi\rangle\, D_\phi(F_0,F)+\langle\phi\rangle. \tag{23.6.13}$$

In particular for $\phi\in\mathbf{\Phi}_2$

$$\mu_\phi(f_0)=\langle\phi\rangle. \tag{23.6.14}$$

23.6.3 Consistency under Hypothesis and Fixed Alternatives

The consistency results of this section are in fact laws of large numbers for the spacings based disparity and divergence statistics. They are used in the following sections dealing with asymptotic normality under local and fixed alternatives. Note that under fixed alternatives they provide tools for nonparametric estimates of disparities or divergences. Such tools were used e.g. for nonparametric estimation of entropy by Vasicek (1976) or by Dudewicz and van der Meulen (1987). Our main result is formulated for fixed alternatives.

Assertion 23.6.15 Consider the observations under a fixed alternative $F\sim f$ with f positive and continuous on $[0,1]$, and denote by U_ϕ any statistic from the class $\{R_\phi,\tilde{R}_\phi,T_\phi,\tilde{T}_\phi\}$. If $\phi\in\mathbf{\Phi}_1$ satisfies (23.6.6), then

$$\frac{U_\phi}{n}\xrightarrow{\;p\;}\mu_\phi(f) \quad \text{for } n\to\infty \tag{23.6.16}$$

where $\mu_\phi(f)$ is given by (23.6.12). If $\phi\in\mathbf{\Phi}_2$ satisfies (23.6.6), then the asymptotic relation (23.6.16) remains valid also for $U_\phi=\tilde{S}_\phi$ and $U_\phi=S_\phi$, and $\mu_\phi(f)$ is given by the simpler formula (23.6.13). □

Corollary 23.6.17 Under $\mathcal{H}_0$ (23.6.16) reduces to

$$\frac{U_\phi}{n}\xrightarrow{\;p\;}\mu_\phi(f_0)=\langle\phi\rangle=\int_0^\infty\phi(t)\,e^{-t}\,dt \qquad \text{for } n\to\infty. \tag{23.6.18}$$

In the sequel we use the L_2-norm

$$\|\ell\| = \left(\int_0^1 \ell^2(x)\,dx\right)^{1/2}$$

and we denote the integral (23.6.7) usually by $\langle\psi(t)\rangle$ rather than $\langle\psi\rangle$.

23.6.4 Asymptotic Normality under Local Alternatives

Assertion 23.6.19 Consider the observations under the local alternatives given in (23.6.1) with a limit function $\ell(x)$ introduced in (23.6.2), and denote by U_ϕ any statistic from the set $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$. If $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (23.6.6) with $\beta < 1/2$ then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi) \xrightarrow{\mathcal{D}} N(m_\phi(\ell), \sigma_\phi^2) \quad \text{as } n \to \infty \tag{23.6.20}$$

where

$$\mu_\phi = \langle\phi(t)\rangle, \quad \sigma_\phi^2 = \langle\phi^2(t)\rangle - \langle\phi(t)\rangle^2 - (\langle t\phi(t)\rangle - \langle\phi(t)\rangle)^2 \tag{23.6.21}$$

and

$$m_\phi(\ell) = \frac{\|\ell\|^2}{2}\left(\langle t^2\phi(t)\rangle - 4\langle t\phi(t)\rangle + 2\langle\phi(t)\rangle\right). \tag{23.6.22}$$

□

23.6.5 Asymptotic Normality under Fixed Alternatives

Let us now consider the fixed alternative $F \sim f$ defined at the beginning of this section under (i), and $\phi \in \mathbf{\Phi}_2$ with $\xi = \xi_\phi$, $\eta = \eta_\phi$, satisfying the functional equation (23.5.41), and denote by ϕ', ξ', η' the derivatives of ϕ, ξ, η as in Assertion 23.5.44. To express the asymptotic normality under this alternative, we need auxiliary functions $\Psi_i = \Psi_{i,\phi}$ of the variable $x \in (0,1)$:

$$\begin{aligned}\Psi_1(x) &= \xi'(1)\,\langle\phi(t)\rangle\, f(x)\,\xi\left(\frac{1}{f(x)}\right) + \xi'(1)\,f(x)\,\phi\left(\frac{1}{f(x)}\right) \\ &\quad + [\phi'(1) - \eta'(1)]\,f(x) + \eta'(1)\end{aligned} \tag{23.6.23}$$

$$\begin{aligned}\Psi_2(x) &= \left(\langle\phi^2(t)\rangle - \langle\phi(t)\rangle^2\right) f(x)\,\xi^2\left(\frac{1}{f(x)}\right) + f(x)\,\eta^2\left(\frac{1}{f(x)}\right) \\ &\quad + 2(\langle t\phi(t)\rangle - \langle\phi(t)\rangle) f(x)\,\xi\left(\frac{1}{f(x)}\right)\eta\left(\frac{1}{f(x)}\right),\end{aligned} \tag{23.6.24}$$

$$\Psi_3(x) = (\langle t\phi(t)\rangle - \langle\phi(t)\rangle)\sqrt{f(x)}\,\xi\left(\frac{1}{f(x)}\right) + \sqrt{f(x)}\,\eta\left(\frac{1}{f(x)}\right), \tag{23.6.25}$$

and also

$$\Psi_4(x) = \frac{\sqrt{f(x)}}{F(x)} \int_0^x \left(1 - \frac{F(y)\, f'(y)}{f^2(y)}\right) \Psi_1(y)\, dy \tag{23.6.26}$$

when the alternative density has a continuous derivative $f'(x)$ on $(0,1)$.

Assertion 23.6.27 Consider the observations under the fixed alternative $F \sim f$ with f positive and continuous on $[0,1]$ and continuously differentiable on $(0,1)$ with the derivative f' bounded. If U_ϕ is a statistic from the set $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$, and $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (23.6.6) with $\beta < 1/2$, then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \sigma_\phi^2(f)) \quad \text{as} \quad n \to \infty \tag{23.6.28}$$

where $\mu_\phi(f)$ is given by (23.6.13) and

$$\sigma_\phi^2(f) = \int_0^1 \Psi_2(x)\, dx - 2\int_0^1 \Psi_3(x)\, \Psi_4(x)\, dx + \int_0^1 \Psi_4^2(x)\, dx \tag{23.6.29}$$

for $\Psi_2(x)$, $\Psi_3(x)$, and $\Psi_4(x)$ defined by (23.6.24) through (23.6.26). □

Remark 23.6.30 Under the hypothesis $F_0 \sim f_0 = 1$ both Assertions 23.6.19 and 23.6.27 deal with the same statistical model. Therefore, if $f = f_0$, the asymptotic parameters $(\mu_\phi, \sigma_\phi^2)$ from (23.6.21) and $(\mu_\phi(f_0), \sigma_\phi^2(f_0))$ from (23.6.13) and (23.6.29) must be the same, that is, the equalities

$$\mu_\phi(f_0) = \langle \phi \rangle \quad \text{and} \quad \sigma_\phi^2(f_0) = \langle \phi^2 \rangle - \langle \phi \rangle^2 - (\langle t\phi(t) \rangle - \langle \phi \rangle)^2$$

must hold. The first equality is clear from (23.6.14). For $f = f_0$ we get from (23.9.14) by partial integration

$$\Psi_1(y) = \langle t\phi'(t) \rangle = \langle t\phi(t) \rangle - \langle \phi \rangle \quad \text{for all } y \in (0,1).$$

Thus, by (23.6.26), $\Psi_4(x)$ is under the hypothesis constant, equal to $\langle t\phi(t) \rangle - \langle \phi \rangle$. Similarly, by (23.6.24), (23.6.25) and Assertion 23.5.42, $\Psi_2(x) = \langle \phi^2 \rangle - \langle \phi \rangle^2$ and $\Psi_3(x) = \Psi_4(x)$. Hence (23.6.29) implies the desired result

$$\sigma_\phi^2(f_0) = \Psi_2(x) - 2\Psi_4^2(x) + \Psi_4^2(x) = \sigma_\phi^2.$$

Remark 23.6.31 The expressions μ_ϕ, σ_ϕ^2 are well defined by (23.6.21) for every continuous function $\phi : (0,\infty) \mapsto I\!R$ satisfying the condition (23.6.6) with $\beta < 1/2$. If this condition holds for some function $\psi : (0,\infty) \mapsto I\!R$, then it holds also for all linear transformations $\phi(t) = a\psi(t) + b(t-1) + c$ and

$$\mu_\phi = a\mu_\psi + c, \quad \sigma_\phi^2 = a^2\sigma_\psi^2. \tag{23.6.32}$$

Let us now consider a fixed alternative $F \sim f$ with the density continuously differentiable on $(0,1)$. Then, using expression (23.9.3) for $\mu_\phi(f)$, and (23.9.6) through (23.9.8) for $s_i^2(f)$, the formulas

$$\mu_\phi(f) = \int_0^1 f(x) \left\langle \phi\left(\frac{t}{f(x)}\right)\right\rangle dx \text{ and } \sigma_\phi^2(f) = s_1^2(f) + s_2^2(f) + s_3^2(f) \quad (23.6.33)$$

define $\mu_\phi(f)$ and $\sigma_\phi^2(f)$ for all continuously differentiable functions $\phi : (0,\infty) \mapsto I\!R$ such that both $\phi(t)$ and $\tilde{\phi}(t) = t\phi'(t)$ satisfy (23.6.6) with $\beta < 1/2$. If ψ is one of the functions satisfying all these conditions then all linear transformations $\phi(t) = a\psi(t) + b(t-1) + c$ satisfy these conditions too and

$$\mu_\phi(f) = a\mu_\psi(f) + c, \quad \sigma_\phi^2(f) = a^2\sigma_\psi^2(f). \quad (23.6.34)$$

Formulas (23.6.32) and (23.6.34) are verifiable from the definitions mentioned in this remark and are useful for the evaluation of asymptotic means and variances.

Remark 23.6.35 We observe that the asymptotic results of Assertions 23.6.15, 23.6.19, and 23.6.27 are in each case for a fixed ϕ the same for *any* statistic U_ϕ from the class of statistics considered, confirming the asymptotic equivalence of these statistics.

23.7 Asymptotic Properties of Power Spacings Statistics

23.7.1 Power Spacing Statistics

In the rest of this chapter we deal with and frequently refer to the statistics generated by the power functions $\phi = \tilde{\phi}_\alpha$ introduced in (23.3.29), (23.3.30) as nonnegative linear transforms of the simpler functions $\phi = \phi_\alpha$ defined by (23.3.17), (23.3.18). In order to simplify the notations, we interchange the symbols $\phi_\alpha \longleftrightarrow \tilde{\phi}_\alpha$, i.e. we use the functions defined by

$$\phi_\alpha(t) = \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(\alpha-1)} \text{ if } \alpha \notin \{0,1\}, \quad (23.7.1)$$

and otherwise by the corresponding limits

$$\phi_1(t) = t\ln t - t + 1 \quad \text{and} \quad \phi_0(t) = -\ln t + t - 1, \quad (23.7.2)$$

and their simpler alternatives

$$\tilde{\phi}_\alpha(t) = \frac{t^\alpha - 1}{\alpha(\alpha-1)} \quad \text{for} \quad \alpha \notin \{0,1\}, \quad \tilde{\phi}_1(t) = t\ln t, \quad \tilde{\phi}_0(t) = -\ln t. \quad (23.7.3)$$

The rest of this chapter pays special attention to the subclass

$$\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}, \quad \alpha \in I\!R \tag{23.7.4}$$

of the spacings-based structural disparity statistics studied in the previous section, which are generated by the power functions $\phi = \phi_\alpha : (0,\infty) \mapsto I\!R$ defined for all powers $\alpha \in I\!R$ in (23.7.1) and (23.7.2). It is easy to verify that these functions belong to the subset $\mathbf{\Phi}_2$, that is, they satisfy the functional equation (23.5.41) with

$$\xi(t) = \xi_\alpha(t) = t^\alpha \text{ and } \eta(t) = \eta_\alpha(t) = \begin{cases} \frac{t^\alpha - t}{\alpha - 1} & \text{if} \quad \alpha \neq 1 \\ \lim\limits_{\alpha \to 1} \frac{t^\alpha - t}{\alpha - 1} = t \ln t & \text{if} \quad \alpha = 1 \end{cases} \tag{23.7.5}$$

In other words, if $\alpha \in I\!R$ then

$$\phi_\alpha(st) = s^\alpha \phi_\alpha(t) + \phi_\alpha(s) + (t-1) \cdot \begin{cases} \frac{s^\alpha - s}{\alpha - 1} & \text{if} \quad \alpha \neq 1 \\ s \ln s & \text{if} \quad \alpha = 1 \end{cases} \tag{23.7.6}$$

for all $s,\, t > 0$.

It is also easy to verify that the functions ϕ_α, $\alpha \in I\!R$ are convex, belong to $\mathbf{\Phi}_{\rm div}$ and define ϕ_α-divergences (or briefly, α-divergences). Referring to definitions (23.3.1) and (23.3.2), we introduce the following simplified notation for these ϕ_α-divergences:

$$D_\alpha(\boldsymbol{p}, \boldsymbol{p}_0) = D_{\phi_\alpha}(\boldsymbol{p}, \boldsymbol{p}_0) = \frac{1}{\alpha(\alpha-1)} \left(\sum_{j=1}^k p_j^\alpha p_{0j}^{1-\alpha} - 1 \right) = D_{1-\alpha}(\boldsymbol{p}_0, \boldsymbol{p})$$

if $\alpha \notin \{0, 1\}$ (see (23.3.4)), and

$$D_1(\boldsymbol{p}, \boldsymbol{p}_0) = D_{\phi_1}(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^k p_j \ln \frac{p_j}{p_{0j}} = D_0(\boldsymbol{p}_0, \boldsymbol{p})$$

otherwise. Similarly (see (23.3.2)),

$$D_0(F_0, F) = D_{\phi_0}(F_0, F) = \int_0^1 f \ln \frac{f}{f_0}\, dx = \int_0^1 f(x) \ln f(x)\, dx, \tag{23.7.7}$$

$$D_1(F_0, F) = D_{\phi_1}(F_0, F) = \int_0^1 f_0 \ln \frac{f_0}{f}\, dx = -\int_0^1 \ln f(x)\, dx, \tag{23.7.8}$$

$$\begin{aligned} D_\alpha(F_0, F) &= D_{\phi_\alpha}(F_0, F) = \frac{1}{\alpha(\alpha-1)} \left(\int_0^1 f \left(\frac{f_0}{f} \right)^\alpha dx - 1 \right) \\ &= \frac{1}{\alpha(\alpha-1)} \left(\int_0^1 f(x)^{1-\alpha} dx - 1 \right) \quad \text{if} \quad \alpha \notin \{0, 1\}. \end{aligned} \tag{23.7.9}$$

Similar to the corresponding ϕ_α-divergences themselves, the ϕ_α-divergence statistics T_{ϕ_α}, $\tilde{T}_{\phi_\alpha}$, and S_{ϕ_α} are not altered if the nonnegative convex functions $\phi_\alpha \in \mathbf{\Phi}_2$ are replaced by the convex functions $\tilde{\phi}_\alpha(t)$ from $\mathbf{\Phi}_2$ given by (23.7.3).

For illustration and references later we present formulas for selected statistics from the class (23.7.4). In the first set are our true divergence statistics

$$T_{\phi_\alpha} = T_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)}\left[n^\alpha\left(\sum_{j=1}^{n-1}(Y_j - Y_{j-1})^\alpha + (1-Y_{n-1})^\alpha\right) - n\right] \quad (23.7.10)$$

if $\alpha \notin \{0,1\}$, and otherwise

$$T_{\phi_1} = T_{\phi_1,n} = \sum_{j=1}^{n-1} n(Y_j - Y_{j-1})\ln[n(Y_j - Y_{j-1})] + n(1-Y_{n-1})\ln[n(1-Y_{n-1})] \quad (23.7.11)$$

$$T_{\phi_0} = T_{\phi_0,n} = -\sum_{j=1}^{n-1}\ln[n(Y_j - Y_{j-1})] - \ln[n(1-Y_{n-1})] \quad (23.7.12)$$

(see (23.5.34)). In the second set are the modified divergence statistics

$$S_{\phi_\alpha} = S_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)}\left[(n+1)^\alpha\sum_{j=1}^{n+1}(Y_j - Y_{j-1})^\alpha - n - 1\right] \quad (23.7.13)$$

if $\alpha \notin \{0,1\}$, and otherwise

$$S_{\phi_1} = S_{\phi_1,n} = \sum_{j=1}^{n+1}(n+1)(Y_j - Y_{j-1})\ln[(n+1)(Y_j - Y_{j-1})] \quad (23.7.14)$$

$$S_{\phi_0} = S_{\phi_0,n} = -\sum_{j=1}^{n+1}\ln[(n+1)(Y_j - Y_{j-1})] \quad (23.7.15)$$

(see (23.5.33)) extensively used in the literature (see Jammalamadaka and Tiwary (1987) and Jammalamadaka, Zhou, and Tiwary (1989), Misra and van der Meulen (2001), Jiménez and Shao (2009) and others cited there). In the third set are again the true divergence statistics

$$\tilde{T}_{\phi_\alpha} = \tilde{T}_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)}\left[n^\alpha\left(\sum_{j=2}^{n}(Y_j - Y_{j-1})^\alpha + (Y_1 + 1 - Y_n)^\alpha\right) - n\right] \quad (23.7.16)$$

if $\alpha \notin \{0,1\}$, and otherwise

$$\tilde{T}_{\phi_1} = \tilde{T}_{\phi_1,n} = \sum_{j=2}^{n} n(Y_j - Y_{j-1})\ln[n(Y_j - Y_{j-1})] + n(Y_1 + 1 - Y_n)\ln[n(Y_1 + 1 - Y_n)] \quad (23.7.17)$$

$$\tilde{T}_{\phi_0} = \tilde{T}_{\phi_0,n} = -\sum_{j=2}^{n} \ln\left[n(Y_j - Y_{j-1})\right] - \ln\left[n(Y_1 + 1 - Y_n)\right] \tag{23.7.18}$$

(see (23.5.37)) slightly different from those used by Hall (1986), Morales, Pardo, and Vajda (2003), Vajda and van der Meulen (2006), Vajda (2007), and others cited there (they differ by the normalizing constant n instead of $n+1$).

Example 23.7.19 For $\alpha = 2$ we obtain the statistic

$$S_{\phi_2} = \frac{1}{2}\left[(n+1)^2 \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 - (n+1)\right] = \frac{n+1}{2}\left[(n+1)\mathcal{G} - 1\right] \tag{23.7.20}$$

where $\mathcal{G}$ is the Greenwood statistic of Example 23.5.4.

Since the general asymptotic theory of the statistics $U_\alpha \in \mathcal{U}_\alpha$ specified by (23.7.4) is covered by Assertion 23.5.44 and Assertions 23.6.15, 23.6.19, and 23.6.27, the assertions that follow in this section are basically their corollaries. However, they bring explicit formulas and additional important new results, the proofs of which are not trivial. These proofs are partly based on a continuity theory for the asymptotic parameters

$$\begin{aligned} \mu_\alpha(f) &= \mu_{\phi_\alpha}(f), \quad \sigma^2_\alpha(f) = \sigma^2_{\phi_\alpha}(f), \\ \mu_\alpha &= \mu_{\phi_\alpha}, \qquad\quad \sigma^2_\alpha = \sigma^2_{\phi_\alpha}, \\ m_\alpha(\ell) &= m_{\phi_\alpha}(\ell), \end{aligned} \tag{23.7.21}$$

defined by (23.6.33), (23.6.21), and (23.6.22), as functions of the structural parameter $\alpha \in I\!R$. Such a theory enables us to avoid a direct calculation of the asymptotic parameters at some $\alpha_0 \in I\!R$, if these calculations are tedious and the asymptotic parameters are known at the neighboring parameters α. This theory is summarized in Assertion 23.7.24 below using Assertion 23.7.22. In Assertion 23.7.24 we take the representations (23.6.13) and (23.6.29) for $\mu_{\phi_\alpha}(f)$ and $\sigma^2_{\phi_\alpha}(f)$ rather than (23.6.33).

Assertion 23.7.22 Let $g(y)$ be a continuous positive function on a compact interval $[a,b] \subset I\!R$ and $\Phi(u,v)$ a continuous function of variables $u, v \in I\!R$. Furthermore let, for all α from an interval $(c,d) \subset I\!R$, $\psi_\alpha : (0,\infty) \mapsto I\!R$ be convex or concave functions differentiable at some point $t_* \in (0,\infty)$. If the values $\psi_\alpha(t)$, $t \in (0,\infty)$, and the derivatives $\psi'_\alpha(t_*)$ depend continuously on $\alpha \in (c,d)$, then for every $\alpha_0 \in (c,d)$

$$\lim_{\alpha \to \alpha_0} \int_a^b \Phi(g, \psi_\alpha(g))\, dy = \int_a^b \Phi(g, \psi_{\alpha_0}(g))\, dy. \tag{23.7.23}$$

□

Assertion 23.7.24 The asymptotic parameters μ_α, σ_α^2 and $m_\alpha(\ell)$, specified by (23.7.21), (23.6.21), and (23.6.22), are continuous in the variable $\alpha \in (-1/2, \infty)$. If the density f satisfies the assumptions of Assertion 23.6.15, then the asymptotic mean $\mu_\alpha(f)$ specified by (23.7.21) and (23.6.13) is continuous in the variable $\alpha \in (-1, \infty)$. If f satisfies the stronger assumptions of Assertion 23.6.27, then the asymptotic variance $\sigma_\alpha^2(f)$ specified by (23.7.21) and (23.6.29) is continuous in the variable $\alpha \in (-1/2, \infty)$. □

23.7.2 Consistency

In the assertion below and in the rest of the chapter, we use the gamma function of the variable $\alpha \in \mathbb{R}$ and the Euler constant,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\, e^{-t}\, dt \quad \text{and} \quad \gamma = 0.577\ldots. \tag{23.7.25}$$

Assertion 23.7.26 Consider the observations under the fixed alternative $F \sim f$ assumed in Assertion 23.6.15 and denote by U_α any statistic from the class $\mathcal{U}_\alpha$ of (23.7.4). If $\alpha > -1$, then

$$\frac{U_\alpha}{n} \xrightarrow{p} \mu_\alpha(f) \quad \text{as } n \to \infty \tag{23.7.27}$$

for

$$\mu_\alpha(f) = D_\alpha(F_0, F)\, \Gamma(\alpha+1) + \mu_\alpha, \tag{23.7.28}$$

where

$$\mu_0 = \gamma, \quad \mu_1 = 1 - \gamma, \quad \text{and} \quad \mu_\alpha = \frac{\Gamma(\alpha+1) - \Gamma(1)}{\alpha(\alpha-1)} \quad \text{for } \alpha \notin \{0, 1\}, \tag{23.7.29}$$

and $D_\alpha(F_0, F)$ are the ϕ_α-divergences (23.7.7) through (23.7.9). The ϕ_α-divergences are zero if and only if $F = F_0$, so that under the hypothesis $\mathcal{H}_0 : F = F_0$

$$\mu_\alpha(f_0) = \mu_\alpha, \quad \alpha \in \mathbb{R}. \tag{23.7.30}$$

Both parameters μ_α and $\mu_\alpha(f)$ are continuous in the variable $\alpha \in (-1, \infty)$ and satisfy the inequality $\mu_\alpha(f) \geq \mu_\alpha$, which is strict unless $F = F_0$. □

Since $\Gamma(\alpha+1) = \alpha(\alpha-1)\,\Gamma(\alpha-1)$, (23.7.29) and (23.7.28) can be replaced for $\alpha \notin \{0, 1\}$ by

$$\mu_\alpha = \Gamma(\alpha-1) - \frac{1}{\alpha(\alpha-1)} \quad \text{and} \quad \mu_\alpha(f) = \Gamma(\alpha-1) \int_0^1 f^{1-\alpha}\, dx - \frac{1}{\alpha(\alpha-1)}. \tag{23.7.31}$$

Assertion 23.7.26 can be illustrated by Table 23.7–1 , in which actual values of the parameters μ_α and $\mu_\alpha(f)$ are presented for selected parameters α. In this table, f denotes any density considered in Assertion 23.6.15, and the expressions for $D_\alpha(F_0, F)$, $H(F_0, F)$, and $\chi^2(F_0, F)$ can be easily discerned from those used in Examples 23.3.8 and 23.3.16, thereby replacing P and P_0 by F_0 and F and sums by integrals.

Table 23.7–1. Values of μ_α and $\mu_\alpha(f)$ for selected $\alpha > -1$.

α	μ_α	$\mu_\alpha(f)$
$-\frac{1}{2}$	$\frac{4}{3}(\sqrt{\pi}-1) \doteq 1.030$	$\sqrt{\pi}\, D_{-1/2}(F_0,F) + \mu_{-1/2} = \frac{4\sqrt{\pi}}{3}\int_0^1 f^{3/2}\,dx - \frac{4}{3}$
0	$\gamma \doteq 0.577$	$D_0(F_0,F) + \mu_0 = \int_0^1 f \ln f\, dx + \gamma$
$\frac{1}{2}$	$4 - 2\sqrt{\pi} \doteq 0.455$	$2\sqrt{\pi}\, H(F_0,F) + \mu_{1/2} = 4 - 2\sqrt{\pi}\int_0^1 \sqrt{f} dx$
1	$1 - \gamma \doteq 0.423$	$D_1(F_0,F) + \mu_1 = 1 - \gamma - \int_0^1 \ln f\, dx$
$\frac{3}{2}$	$\sqrt{\pi} - \frac{4}{3} \doteq 0.439$	$\frac{3\sqrt{\pi}}{4} D_{3/2}(F_0,F) + \mu_{3/2} = \sqrt{\pi}\int_0^1 \frac{dx}{\sqrt{f}} - \frac{4}{3}$
2	$\frac{1}{2} = 0.500$	$\chi^2(F_0,F) + \mu_2 = \int_0^1 \frac{dx}{f} - \frac{1}{2}$
$\frac{5}{2}$	$\frac{\sqrt{\pi}}{2} - \frac{4}{15} \doteq 0.620$	$\frac{15\sqrt{\pi}}{8} D_{5/2}(F_0,F) + \mu_{5/2} = \frac{\sqrt{\pi}}{2}\int_0^1 \frac{dx}{f^{3/2}} - \frac{4}{15}$
3	$\frac{5}{6} \doteq 0.833$	$6D_3(F_0,F) + \mu_3 = \int_0^1 \frac{dx}{f^2} - \frac{1}{6}$

23.7.3 Asymptotic Normality under Local Alternatives

Assertion 23.7.32 Consider the observations under local alternatives (23.6.1) with the limit function $\ell(x)$ introduced in (23.6.2), and denote by U_α any statistic from the class $\mathcal{U}_\alpha$ of (23.7.4). If $\alpha > -1/2$, then

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha) \xrightarrow{\mathcal{D}} N(m_\alpha(\ell), \sigma_\alpha^2) \qquad \text{as } n \to \infty \tag{23.7.33}$$

where the parameters μ_α, $m_\alpha(\ell)$, and σ_α^2 are continuous in the variable $\alpha \in (-1/2, \infty)$, and are given by (23.7.29) and the formulas

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\,\Gamma(\alpha+1) \tag{23.7.34}$$

$$\sigma_\alpha^2 = \frac{\Gamma(2\alpha+1) - (\alpha^2+1)\,\Gamma^2(\alpha+1)}{\alpha^2(\alpha-1)^2} \qquad \text{for } \alpha \notin \{0,1\} \tag{23.7.35}$$

and

$$\sigma_0^2 = \frac{\pi^2}{6} - 1, \quad \sigma_1^2 = \frac{\pi^2}{3} - 3. \tag{23.7.36}$$

□

Assertion 23.7.32 provides the possibility to compute and compare asymptotic relative efficiencies of tests of the hypothesis $\mathcal{H}_0 : F_0 \sim f_0$ based on the statistics $U_\alpha \in \mathcal{U}_\alpha$, $\alpha > -1/2$, for various values of α. The Pitman asymptotic relative efficiency (ARE) of one test relative to another is defined as the limit of the inverse ratio of sample sizes required to obtain the same limiting power at the sequence

Table 23.7–2. The asymptotic parameters $m_\alpha(\ell)$, σ_α^2, and $\mathrm{eff}(U_\alpha)$ for selected statistics U_α of Assertion 23.7.32.

α	$m_\alpha(\ell)$	σ_α^2	$\mathrm{eff}(U_\alpha)$
0	$\frac{\|\ell\|^2}{2}$	$\frac{\pi^2}{6} - 1 \doteq 0.645$	1.550
$\frac{1}{2}$	$\|\ell\|^2 \frac{\sqrt{\pi}}{4} \doteq \frac{\|\ell\|^2}{2} \times 0.886$	$16 - 5\pi \doteq 0.292$	2.690
1	$\frac{\|\ell\|^2}{2}$	$\frac{\pi^2}{3} - 3 \doteq 0.290$	3.448
$\frac{3}{2}$	$\|\ell\|^2 \frac{3\sqrt{\pi}}{8} \doteq 1.329$	$\frac{32}{3} - \frac{13\pi}{4} \doteq 0.457$	3.871
2	$\|\ell\|^2 = \frac{\|\ell\|^2}{2} \times 2$	1	4.000
$\frac{5}{2}$	$\|\ell\|^2 \frac{15\sqrt{\pi}}{16} \doteq \frac{\|\ell\|^2}{2} \times 3.323$	$\frac{128}{15} - \frac{29\pi}{16} \doteq 2.839$	3.890
3	$\|\ell\|^2 3 = \frac{\|\ell\|^2}{2} \times 6$	10	3.600

of alternatives converging to the null hypothesis. If we define the "efficacies" of the statistics $U_\alpha \in \mathcal{U}_\alpha$ of Assertion 23.7.32 by

$$\mathrm{eff}(U_\alpha) = \frac{\Gamma^2(\alpha+1)}{\sigma_\alpha^2} = \frac{(m_\alpha(\ell))^2}{\sigma_\alpha^2} \left(\frac{2}{\|\ell\|^2}\right)^2 \quad \text{for } \|\ell\|^2 \neq 0$$

then under the assumptions of Assertion 23.7.32 we get in accordance with Section 23.4 in Del Pino (1979)

$$\mathrm{ARE}(U_{\alpha_1}, U_{\alpha_2}) = \frac{\mathrm{eff}(U_{\alpha_1})}{\mathrm{eff}(U_{\alpha_2})}$$

where U_{α_1} and U_{α_2} are arbitrary statistics from $\mathcal{U}_{\alpha_1}$ and $\mathcal{U}_{\alpha_2}$. Notice that arbitrary statistics U_α from the set $\mathcal{U}_\alpha$, α fixed, all have the same efficacy (see also Remark 23.6.35). In Table 23.7–2 we present the parameters $m_\alpha(\ell)$, σ_α^2, and $\Gamma^2(\alpha+1)/\sigma_\alpha^2$ for selected values of $\alpha > -1/2$. This table indicates that the statistics $U_2 \in \{R_{\phi_2}, \tilde{R}_{\phi_2}, S_{\phi_2}, \tilde{S}_{\phi_2}, T_{\phi_2}, \tilde{T}_{\phi_2}\}$ are most asymptotically efficient in the Pitman sense among all statistics U_α, $\alpha > -1/2$. This extends the result on p. 1457 in Rao and Kuo (1984) about the asymptotic efficiency of the Greenwood statistic $\mathcal{G} = (2S_{\phi_2} + n + 1)/(n+1)^2$ (see Example 23.5.4 (formula (23.5.5)), Example 23.7.19 (formula (23.7.20)), and formula (23.7.53) below).

The general form of the asymptotic normality (23.7.33), as well as the continuity of the parameters μ_α, $m_\alpha(\ell)$ and σ_α^2 in $\alpha \in (-1/2, \infty)$ established in Assertion 23.7.32 appear to be new results. The special result for $\alpha = 0$ also seems to be new. The particular result for $\alpha \in (-1/2, \infty) - \{0, 1\}$ and $U_\alpha = S_{\phi_\alpha}$ follows from the asymptotic normality obtained for the statistics

$$\sum_{j=1}^{n+1} \left((n+1)\,(Y_j - Y_{j-1})\right)^\alpha = \alpha(\alpha-1)\,S_{\phi_\alpha} + n + 1 \tag{23.7.37}$$

(see (23.7.54) below) by Del Pino, see p. 1062 in Del Pino (1979). The particular result for $\alpha = 1$ and the statistic $U_1 = S_{\phi_1}$ with μ_1 and σ_1^2 given in Tables 23.7–1 and 23.7–2 was obtained previously by Misra and van der Meulen (2001), who however considered m-spacings for arbitrary $m \geq 1$. They compared also the efficiency of the test statistics for $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$ with a similar conclusion as in Table 23.7–2.

23.7.4 Asymptotic Normality Under Fixed Alternatives

In this subsection we study the asymptotic distributions of the spacings-type power divergence statistics U_α from the sets $\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ for $\alpha > -1/2$ under the assumption that the observations are distributed by a fixed alternative $F \sim f$ satisfying the assumptions of Assertion 23.6.27. If $\alpha > -1/2$ then ϕ_α satisfies the assumption of Assertion 23.6.27 too. Therefore this theorem implies that

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha(f)) \xrightarrow{\mathcal{D}} N(0, \sigma_\alpha^2(f)) \quad \text{for } n \to \infty \tag{23.7.38}$$

where the asymptotic parameters $\mu_\alpha(f)$, $\sigma_\alpha^2(f)$ are given by (23.7.21). Similarly as in the previous section, we are interested in explicit formulas for these parameters. By Assertion 23.6.27, the asymptotic mean is for all $\alpha \in \mathbb{R}$ given by the explicit formula (23.7.28) presented in Assertion 23.7.26. The only problem which remains is the formula for $\sigma_\alpha^2(f)$, $\alpha \in \mathbb{R}$.

The functions $\psi_\alpha(t) = t^\alpha$ with $\alpha > -1/2$ satisfy all assumptions of Remark 23.6.31 so that we can consider the quantities

$$\tau_\alpha^2(f) := \sigma_{\psi_\alpha}^2(f), \quad \alpha \in (-1/2, \infty)$$

obtained from the formula introduced there. By (23.6.34),

$$\sigma_\alpha^2(f) = \frac{\tau_\alpha^2(f)}{\alpha^2(\alpha-1)^2} \quad \text{for } \alpha \in (-1/2, \infty) - \{0, 1\}. \tag{23.7.39}$$

One can find on p. 521 of Hall (1984) an expression for $\tau_\alpha^2(f)$ for all $\alpha \in (-1/2, \infty) - \{0, 1\}$, which for the case $m = 1$ can be given the form

$$\tau_\alpha^2(f) = \alpha^2(\alpha-1)^2 \left(\sigma_\alpha^2 \int_0^1 f^{1-2\alpha}\, dx + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0, F) \right) \tag{23.7.40}$$

where σ_α^2 is defined by formula (23.7.35) and

$$\Delta_\alpha(F_0, F) = \frac{1}{\alpha^2} \int_0^1 \left(\frac{1}{(f(x))^\alpha} - \frac{1}{F(x)} \int_0^x (f(y))^{1-\alpha}\, dy \right)^2 f(x)\, dx \quad \text{for } \alpha \in \mathbb{R} - \{0\}. \tag{23.7.41}$$

Since Hall (1984) gave no hint about the derivation of his formula, let us mention that (23.7.40) is obtained if one substitutes ψ_α for ϕ in the expressions (23.9.6) through (23.9.8) below for $s_j^2(f)$, $j \in \{1,2,3\}$, given in the proof of Assertion 23.6.27 (thereby employing the expression

$$\begin{aligned} G(x) &= \alpha E(Z^\alpha) \int_0^x \left(1 - \frac{F f'}{f^2}\right) \frac{1}{f^{\alpha-1}}\, dy \\ &= \Gamma(\alpha+1) \left((\alpha-1) \int_0^x (f(y))^{1-\alpha}\, dy + (f(x))^{-\alpha} F(x)\right) \end{aligned}$$

for $G(x)$ of (23.9.5) when ϕ is replaced by ψ_α), and then forms the sum $s_1^2(f) + s_2^2(f) + s_3^2(f)$. By (23.7.39) and (23.7.40),

$$\sigma_\alpha^2(f) = \sigma_\alpha^2 \int_0^1 f^{1-2\alpha}\, dx + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0, F), \quad \alpha \in (-1/2, \infty) - \{0,1\}. \tag{23.7.42}$$

The final, intuitively appealing, form of the asymptotic variance

$$\sigma_\alpha^2(f) = (1 + 2\alpha(2\alpha-1)\, D_{2\alpha}(F_0, F))\, \sigma_\alpha^2 + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0, F) \tag{23.7.43}$$

(with $\sigma_\alpha^2(f_0) = \sigma_\alpha^2$ given in (23.7.35)), follows for $\alpha \in (-1/2, \infty) - \{0,1\}$ by taking into account the formula for $D_{2\alpha}(F_0, F)$ obtained from (23.7.9). The peculiar expression $\Delta_\alpha(F_0, F)$ figuring in (23.7.41) and (23.7.43) can be better understood if we take into account the following assertion, after which we extend (23.7.43) to include also the values $\alpha \in \{0,1\}$.

Assertion 23.7.44 If the fixed alternative $F \sim f$ satisfies the assumptions of Assertion 23.6.27, then the class $\{\Delta_\alpha(F_0, F) : \alpha \in \mathbb{R} - \{0\}\}$ consists of the variances

$$\begin{aligned} \Delta_\alpha(F_0, F) &= \int_0^1 \left(\frac{f^{-\alpha}}{\alpha} - \int_0^1 \frac{f^{-\alpha}}{\alpha} f\, dy\right)^2 f\, dx \\ &= \int_0^1 \left(\frac{f^{-\alpha}}{\alpha}\right)^2 f\, dx - \left(\int_0^1 \frac{f^{-\alpha}}{\alpha} f\, dx\right)^2 \end{aligned} \tag{23.7.45}$$

of the functions $f^{-\alpha}(X)/\alpha$ of the random argument X distributed by F. This class is continuously extended to all $\alpha \in \mathbb{R}$ by introducing the variance

$$\begin{aligned} \Delta_0(F_0, F) &= \int_0^1 \left(\ln f - \int_0^1 (\ln f)\, f\, dy\right)^2 f\, dx \\ &= \int_0^1 f \ln^2 f\, dx - \left(\int_0^1 f \ln f\, dx\right)^2 \end{aligned} \tag{23.7.46}$$

of the function $\ln f(X)$ of the random argument X that was introduced above. All $\Delta_\alpha(F_0, F)$, $\alpha \in \mathbb{R}$, are nonnegative measures of divergence of F_0 and F, reflexive in the sense that $\Delta_\alpha(F_0, F) = 0$ if and only if $F = F_0$. □

We are now in a position to formulate the general results obtained in this chapter regarding the asymptotic normality of spacings-type power divergence statistics U_α from the sets $\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ for $\alpha > -1/2$ under the assumption of the fixed alternative, thereby specifying the parameters $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ in (23.7.38) for all $\alpha > -1/2$. Inspecting once more formula (23.7.43), we observe that if $\alpha > -1/2$ differs from 0 and 1, then the asymptotic variance $\sigma_\alpha^2(f)$ under the alternative f exceeds the asymptotic variance $\sigma_\alpha^2 = \sigma_\alpha^2(f_0)$ achieved under the hypothesis $F_0 \sim f_0$ by a linear function of σ_α^2 given by

$$2\alpha(2\alpha - 1)\, D_{2\alpha}(F_0, F)\sigma_\alpha^2 + \Gamma^2(\alpha + 1)\, \Delta_\alpha(F_0, F) \tag{23.7.47}$$

with the coefficients $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ positive unless $F = F_0$. By using Assertion 23.7.24, we can now find the formulas for $\sigma_0^2(f)$ and $\sigma_1^2(f)$ which are missing in (23.7.42) by taking limits in (23.7.43) for $\alpha \to 0$ and $\alpha \to 1$. Since the limits σ_0^2 and σ_1^2 were already calculated in Assertion 23.7.26, and the limit $\Delta_0(F_0, F)$ is clear from Assertion 23.7.44, we obtain

$$\sigma_0^2(f) = \lim_{\alpha \to 0} \sigma_\alpha^2(f) = \sigma_0^2 + \Delta_0(F_0, F) \tag{23.7.48}$$

and

$$\sigma_1^2(f) = \lim_{\alpha \to 1} \sigma_\alpha^2(f) = (1 + 2D_2(F_0, F))\, \sigma_1^2 + \Delta_1(F_0, F) \tag{23.7.49}$$

where (see (23.7.45))

$$\Delta_1(F_0, F) = \int_0^1 \frac{1}{f}\, dx - 1. \tag{23.7.50}$$

Together with (23.7.42), (23.7.48), and (23.7.49) provide formulas for $\sigma_\alpha^2(f)$ for all $\alpha > -1/2$. It is clear that $\sigma_0^2(f)$ and $\sigma_1^2(f)$ are of the form (23.7.43), so that the representation (23.7.43) holds for all $\alpha > -1/2$. We summarize our results as follows.

Assertion 23.7.51 If the alternative $F \sim f$ satisfies the assumptions of Assertion 23.6.27, then the asymptotic formula of (23.7.38) is valid for all $\alpha > -1/2$. The asymptotic means $\mu_\alpha(f)$ are given by the explicit formulas (23.7.28) and (23.7.29), thereby taking into account expressions (23.7.7) through (23.7.9). The asymptotic variances $\sigma_\alpha^2(f)$ are given by (23.7.43), where the explicit formulas for $D_{2\alpha}(F_0, F)$ can be found in (23.7.7) through (23.7.9), those for σ_α^2 in (23.7.35) and (23.7.36), and the formulas for $\Delta_\alpha(F_0, F)$ in (23.7.45) and (23.7.46). The asymptotic means and variances are continuous in the variable $\alpha \in (-1/2, \infty)$. The asymptotic means satisfy the inequality $\mu_\alpha(f) \geq \mu_\alpha$ mentioned in Assertion 23.7.26. The asymptotic variances satisfy the inequality $\sigma_\alpha^2(f) \geq \sigma_\alpha^2$. Both inequalities become equalities if and only if $F = F_0$. □

Concrete forms of $\mu_\alpha(f)$ and $\sigma_\alpha^2(f_0) = \sigma_\alpha^2$ were illustrated in Tables 23.7–1 and 23.7–2. The next table illustrates $\sigma_\alpha^2(f)$ given by (23.7.43) for arbitrary f

Table 23.7–3. Asymptotic variances $\sigma_\alpha^2(f)$ for selected $\alpha > -1/2$.

α	$\sigma_\alpha^2(f)$		
0	$\sigma_0^2 + \Delta_0(F_0, F)$	$=$	$\frac{\pi^2}{6} - 1 + \int_0^1 f \ln^2 f \, dx$ $- \left(\int_0^1 f \ln f \, dx\right)^2$
$\frac{1}{2}$	$\sigma_{\frac{1}{2}}^2 + \frac{\pi}{4}\Delta_{\frac{1}{2}}(F_0, F)$	$=$	$16 - 4\pi - \pi \left(\int_0^1 \sqrt{f} \, dx\right)^2$
1	$[1 + \chi^2(F_0, F)]\, \sigma_1^2 + \Delta_1(F_0, F)$	$=$	$\int_0^1 \frac{dx}{f} \left(\frac{\pi^2}{3} - 2\right) - 1$
2	$[1 + 12D_4(F_0, F)\, \sigma_2^2 + 4\Delta_2(F_0, F)]$	$=$	$2 \int_0^1 \frac{dx}{f^3} - \left(\int_0^1 \frac{dx}{f}\right)^2$
3	$[1 + 30D_6(F_0, F)]\, \sigma_3^2 + 36\Delta_3(F_0, F)$	$=$	$14 \int_0^1 \frac{dx}{f^5} - 4 \left(\int_0^1 \frac{dx}{f^2}\right)^2$

satisfying the assumptions of Assertion 23.6.27 and selected values of α. In each line of Table 23.7–3 two expressions for $\sigma_\alpha^2(f)$ are given: the first one is obtained by substituting α in (23.7.43), the second one by actually calculating $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ in each case and putting the results in a closed form. As presumed, for $f = 1$ the illustrated values reduce to σ_α^2 from Table 23.7–2.

23.7.5 Discussion

The general form of the asymptotic normality (23.7.38) established by Assertion 23.7.51, as well as the continuity of the asymptotic means and variances $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ in the parameter $\alpha > -1/2$ proved in Assertion 23.7.24, and the explicit formulas (23.7.28) and (23.7.43) for these parameters for general α seem to be new results. However, in the references cited in Subsections 23.5.1 and 23.5.2 one can find particular versions of these results for some of the statistics U_α from the set $\{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ or their linear functions, and for some $\alpha > -1/2$ and some distributions $F \sim f$.

Let us start with the statistic S_{ϕ_0} proposed by Moran (1951), and denoted by $\mathcal{M}$ in Example 23.5.8 (equation(23.5.9)). The asymptotic normality (23.7.38) for $\alpha = 0$, $U_0 = S_{\phi_0}$, and $f = f_0 = 1$, with the parameters $\mu_0(f_0) = \mu_0$ and $\sigma_0^2(f_0) = \sigma_0^2$ given in Tables 23.7–1 and 23.7–2, was proved by Darling (1953), yielding specifically that under $\mathcal{H}_0$

$$\frac{1}{\sqrt{n}} (\mathcal{M} - n\gamma) \xrightarrow{\mathcal{D}} N\left(0, \frac{\pi^2}{6} - 1\right) \quad \text{as } n \to \infty. \tag{23.7.52}$$

The result of Darling was extended to all positively valued step functions f on $[0, 1]$ by Cressie (1976), who also obtained $\mu_0(f)$ and $\sigma_0^2(f)$ given in Tables 23.7–2 and 23.7–3. The result of Cressie was extended by van Es (1992) to the alternatives f considered in the present chapter which satisfy a Lipschitz condition on $[0, 1]$, and to all f considered in this chapter by Shao and Hahn (1995). Cressie

(1976) and van Es (1992) studied S_{ϕ_0} as the special case obtained for $m = 1$ from a more general statistic based on m-spacings with $m \geq 1$. Van Es extended ideas and methods developed for $m > 1$ by Vasicek (1976) and Dudewicz and van der Meulen (1981) for proving the consistency and asymptotic normality of a spacings-based estimator of entropy. The latter authors considered only $\phi(t) = -\ln t$.

Greenwood (1946) introduced the statistic

$$\mathcal{G} = \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 = \frac{2S_{\phi_2} + n + 1}{(n+1)^2}, \tag{23.7.53}$$

discussed in Examples 23.5.4 and 23.7.19. Kimball (1950) proposed the generalization

$$\sum_{j=1}^{n+1} (Y_j - Y_{j-1})^\alpha = \frac{\alpha(\alpha-1)S_{\phi_\alpha} + n + 1}{(n+1)^\alpha}, \quad \alpha > 0, \tag{23.7.54}$$

and Darling (1953) proved an asymptotic normality theorem equivalent to (23.7.38) for $U_\alpha = S_{\phi_\alpha}$, $\alpha \in (0, \infty) - \{1\}$, and $f = f_0 = 1$. Weiss (1957) extended this result of Darling to positive piecewise constant densities f. Hall (1984) obtained the asymptotic normality

$$\frac{1}{\sqrt{n}} \left(\tilde{U}_\alpha - n\alpha(\alpha-1)\, \mu_\alpha(f) - n \right) \xrightarrow{\mathcal{D}} N(0, \alpha^2(\alpha-1)^2 \sigma_\alpha^2(f)), \tag{23.7.55}$$

as $n \to \infty$, for all statistics

$$\begin{aligned} \tilde{U}_\alpha &= \sum_{j=2}^{n} (n(Y_j - Y_{j-1}))^\alpha \\ &= \alpha(\alpha-1)\, \tilde{R}_{\phi_\alpha} - \alpha n(1 - Y_n + Y_1) + n + \alpha - 1 \\ &= \alpha(\alpha-1)\, \tilde{R}_{\phi_\alpha} + n + O_p(1), \end{aligned}$$

with $\alpha \in (-1/2, \infty) - \{0, 1\}$ for any f considered in Assertion 23.7.51. Here $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ are the same as in Assertion 23.7.51, with $\mu_\alpha(f)$ given by the right-hand side of (23.7.31) and $\sigma_\alpha^2(f)$ by (23.7.43), $\tilde{R}_{\phi_\alpha}$ is defined as in (23.5.27) with $\phi = \phi_\alpha$, and the $O_p(1)$ statement follows from the proof of Assertion 23.6.3. In fact, this result of Hall (1984) was one of the arguments used in the proof of Assertion 23.7.51.

The statistic S_{ϕ_1} was proposed by Misra and van der Meulen (2001), who proved the asymptotic normality (23.7.38) for $U_1 = S_{\phi_1}$ and any f considered in Assertion 23.7.51, with the parameters $\mu_1(f)$ and $\sigma_1^2(f)$ given in Tables 23.7–1 and 23.7–2, yielding the result

$$\frac{1}{\sqrt{n}} \left(S_{\phi_1} - n \left(1 - \gamma - \int_0^1 \ln f \, dx \right) \right) \xrightarrow{\mathcal{D}} N \left(0, \int_0^1 \left(\frac{\pi^2}{3} - 2 \right) \frac{dx}{f} - 1 \right) \tag{23.7.56}$$

as $n \to \infty$. We see that the present Assertion 23.7.51 unifies and extends the results proved separately in the literature in three different situations for two particular statistics from the set (23.7.4). The formulas for all asymptotic parameters $\mu_\alpha(f)$ and $\sigma^2_\alpha(f)$ of the statistics U_α are shown to follow via the asymptotic equivalence of these statistics (see Assertion 23.6.3) and the continuity of these parameters in α (see Assertion 23.7.24) from just Hall's formula (see (23.7.55)) for the asymptotic parameters of $\tilde{U}_\alpha$ with $\alpha \in (-1/2, \infty)$ different from 0 and 1.

23.8 The PODISTAT Program Package

This section presents a computer program package PODISTAT for evaluation of power divergence statistics. It consists of three MATLAB® based functions for evaluation of three selected families of spacings-based statistics parametrized by $\alpha \in I\!R$ from the class (23.7.4) of Section 23.7. Two of them are the families of the true divergence statistics $T_{\phi_\alpha,n}$ and $\tilde{T}_{\phi_\alpha,n}$ proposed in this chapter, and the third one is one of the families of classical spacings-based statistics studied in this chapter, namely $S_{\phi_\alpha,n}$. The statistics $T_{\phi_\alpha,n}$, $S_{\phi_\alpha,n}$, and $\tilde{T}_{\phi_\alpha,n}$ were explicitly defined for all $\alpha \in I\!R$ by the formulas (23.7.10) through (23.7.18). The programs are illustrated by applications to the models introduced in Section 23.2 for the data of Tables 23.1–1 and 23.1–2. More precisely, the hypothesis that these data were generated by their MixGLD models is tested by means of these statistics at the 95% confidence level.

Input parameters of the mentioned MATLAB functions are the row vector of observed data **X**, the string definition of the hypothetical distribution function **F0**, and the order **alpha** of the statistic. The program uses **X** and **F0** to generate by the formula (23.4.13) the ordered observations $0 < Y_1 \le ... \le Y_n < 1$ introduced in Convention 23.4.15 and appearing in the formulas (23.7.10) through (23.7.18) for the statistics $T_{\phi_\alpha,n}$, $S_{\phi_\alpha,n}$ and $\tilde{T}_{\phi_\alpha,n}$. The alternative model $F(y) = F(Q_0(y))$ considered in the same convention, and obtained from the original alternative model $F(x)$ by means of the h.p.f. Q_0 given by (23.4.14), is a theoretical concept needed neither for the evaluation of these statistics nor for the related testing of hypotheses.

1. Evaluation of the statistic $T_{\phi_\alpha,n}$ given by (23.7.10) through (23.7.12) and denoted as function T = Tpdt.

```
function T = Tpdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: Tpdt(X, F0, {alpha})
%    x       row vector; observed data
%    F0      string: hypothetical cumulative distribution
%                    distribution of vector x
```

```
%    {alpha} number: from interval (-1/2, inf) order of
%                    power-divergence (optional, default = 2)
%
% Example: Tpdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: Tpdt(x, F0, {alpha})'); end
Xsize = size(X);
if (Xsize(1)~=1), error('The first parameter is not a row vector');
   end
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end Y = [0 sort(Y) 1]; dY = diff(Y(1:n)); if (abs(alpha)<eps)
% equation 7.12
   T = -sum(log(n*dY))-log(n*(1-Y(n)));
elseif (abs(1-alpha)<eps)
% equation 7.11
   T = log(n*dY)*(n*dY)'+log(n*(1-Y(n)))*n*(1-Y(n));
else
% equation 7.10
   T = (sum((n*dY).^alpha)+(n*(1-Y(n)))^alpha-n)/alpha/(alpha-1);
end return
```

2. Evaluation of the statistic $S_{\phi_\alpha,n}$ given by (23.7.13) through (23.7.15) and denoted as function S = Spdt.

```
function S = Spdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: Spdt(X, F0, {alpha})
%    x       row vector; observed data
%    F0      string: hypothetical cumulative
%                    distribution distribution of vector x
%    {alpha} number: from interval (-1/2, inf) order of
%                    power-divergence (optional, default = 2)
%
% Example: Spdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: Spdt(X, F0, {alpha})'); end
```

```
Xsize = size(X);
if (Xsize(1)~=1), error('The first parameter is not a row vector');
   end
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end
for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end
Y = [0 sort(Y) 1];
dY = diff(Y);
if (abs(alpha)<eps)
% equation 7.15
   S = -sum(log((n+1)*dY));
elseif (abs(1-alpha)<eps)
% equation 7.14
   S = log((n+1)*dY)*((n+1)*dY)';
else
% equation 7.13
   S = (sum(((n+1)*dY).^alpha)-n-1)/alpha/(alpha-1);
end
return
```

3. Evaluation of the statistic $\tilde{T}_{\phi_\alpha,n}$ given by (23.7.16) through (23.7.18) and denoted as function TW = TWpdt.

```
function TW = TWpdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: TWpdt(X, F0, {alpha})
%    x       row vector; observed data
%    F0      string: hypothetical cumulative distribution
%                    distribution of vector x
%    {alpha} number: from interval (-1/2, inf) order of
%                    power-divergence (optional, default = 2)
%
% Example: TWpdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: TWpdt(X, F0, {alpha})');
   end
```

```
Xsize = size(X);
if (Xsize(1)~=1), error('The first parameter is not a row vector'); end
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end
for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end
Y = [0 sort(Y) 1];
dY = diff(Y(2:n+1));
if (abs(alpha)<eps)
% equation 7.18
   TW = -sum(log(n*dY))-log(n*(Y(2)+1-Y(n+1)));
elseif (abs(1-alpha)<eps)
% equation 7.17
   TW = log(n*dY)*(n*dY)'+log(n*(Y(2)+1-Y(n+1)))*n*(Y(2)+1-Y(n+1));
else
% equation 7.16
TW=(sum((n*dY).^alpha)+(n*(Y(2)+1-Y(n+1)))^alpha-n)/alpha/(alpha-1);
end
return
```

Example 23.8.1 Applicability of the PODISTAT package was verified by evaluating all three statistics in the domain $-1 < \alpha \leq 3$ for the empirical distributions resulting from the Pearson crab measurement data of Table 23.1–1 and for the hypothetical distributions given by the mixed generalized lambda models MixGLD of Example 23.2.28. The problem faced here is that the data in this table are in fact discrete, obtained by quantization of the original continuous measurement outcomes. Due to this, the majority of the formally evaluated spacings is zero. These data can be brought into conformity with the assumptions of the present section by reconstructing the original nonquantized Pearson's data by a suitable random smoothing. We used the method of spreading the data from each quantization interval over all this interval uniformly randomly. The results for various realizations of this smoothing procedure were practically the same. They are presented in graphical form in Figure 23.8–1. Moreover, the asymptotic theory of Section 23.7 for the statistics from the class (23.7.4) was applied to evaluate the critical values as functions of $-1 < \alpha \leq 3$ for rejection of the MixGLD models at the 95% significance level. The model (i.e. the corresponding hypothetical d.f. $F_0 \equiv \mathbf{F0}$) is rejected by a statistic if its value exceeds the critical value. More precisely, we applied Assertion 23.7.32 which in accordance with Remark 23.6.30

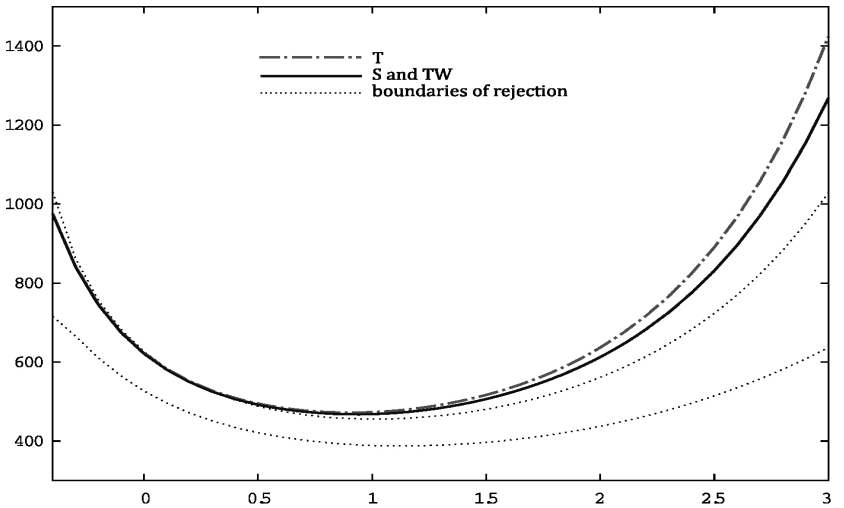


Figure 23.8–1. Statistics T= $T_{\phi_\alpha,n}$, S= $S_{\phi_\alpha,n}$ and TW= $\tilde{T}_{\phi_\alpha,n}$ for $-1/2 < \alpha \leq 3$ and the data and MixGLD model of Example 23.2.28.

implies under the hypothesis (i.e. for $f = f_0$) the asymptotic normality

$$\frac{1}{\sqrt{n}}(U_{\alpha,n} - n\mu_\alpha) \xrightarrow{\mathcal{D}} N(0, \sigma_\alpha^2) \qquad \text{as } n \to \infty \tag{23.8.2}$$

where $U_{\alpha,n}$ stands for any of the statistics $T_{\phi_\alpha,n}$, $S_{\phi_\alpha,n}$, and $\tilde{T}_{\phi_\alpha,n}$. Hence the percentage points $U^*_{\alpha,\frac{\gamma}{2}}$ and $U^*_{\alpha,1-\frac{\gamma}{2}}$ defined by $P(U^*_{\alpha,\frac{\gamma}{2}} < U_\alpha < U^*_{\alpha,1-\frac{\gamma}{2}}) = 1 - \gamma$ may, for reasonably large n, under $\mathcal{H}_0 : F_0$ be approximated by

$$U^*_{\alpha,\frac{\gamma}{2}} = n\mu_\alpha - z_{1-\frac{\gamma}{2}}\sigma_\alpha\sqrt{n} \tag{23.8.3}$$

and

$$U^*_{\alpha,1-\frac{\gamma}{2}} = n\mu_\alpha + z_{1-\frac{\gamma}{2}}\sigma_\alpha\sqrt{n} \tag{23.8.4}$$

respectively, where $z_{1-\frac{\gamma}{2}}$ is the $100(1-\frac{\gamma}{2})$ percentage point of the standard normal distribution. The 95% critical values are presented by the finely dotted lines in Figure 23.8–1. They determine boundaries of the 95% non-rejection area between them and the 5% rejection area outside of them.

We see from Figure 23.8–1 that the differences between the statistics T = $T_{\phi_\alpha,n}$, S = $S_{\phi_\alpha,n}$ and TW = $\tilde{T}_{\phi_\alpha,n}$ are in the whole domain of powers $-1/2 < \alpha \leq 3$ small. This indicates that the asymptotic equivalence of Subsection 23.6.1 works well for the concrete sample size $n = 1000$ in the case of Table 23.1–1. On the other hand, we also see from this figure that all three continua of statistics presented there are in their common rejection domain for the orders $0 < \alpha \leq 3$ and slightly inside the non-rejection domain for the orders $-1/2 < \alpha < 0$. The 5% rejection domain is conservative in the sense that it guarantees at most 0.05 probability of the wrong rejection decision while the alternative

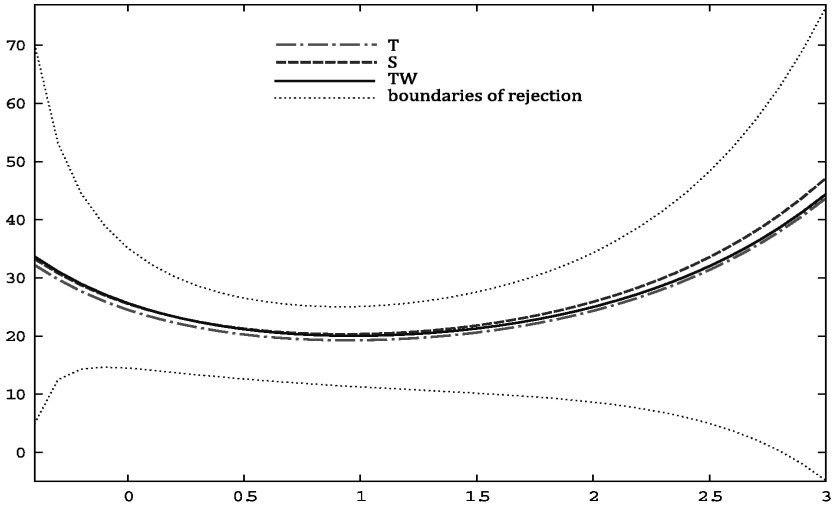


Figure 23.8–2. Statistics T= $T_{\phi_\alpha,n}$, S= $S_{\phi_\alpha,n}$, and TW= $\tilde{T}_{\phi_\alpha,n}$ for $-1/2 < \alpha \leq 3$ and the data and MixGLD model of Example 23.2.32.

non-rejection decision has no formal guarantee. This means that the statistics of the positive orders find a sufficient stochastic argument to reject the hypothesis that the crab data under consideration were generated by the MixGLD model of Example 23.2.28 while the statistics of the negative orders do not go so far and admit (without a formal stochastic guarantee) this hypothesis. Similarly as in the comments to the tables of Section 23.2, one has take into account that the rejection decisions are based here on disparity between the hypothetical d.f. F and the empirical d.f. F_n. This does not exclude the possibility that the fit between model and data is decided differently by the criteria taking into account some local properties of F and F_n.

We have seen in the last example that the infinite families of statistics introduced in this chapter provide a broader picture of agreement or disagreement between the statistical model and data than just one such statistic, thus creating a safer basis for the final statistician's decision. At the same time we have seen that the continuum of statistics may impose a new problem for the statistician, namely the need to weight eventual differences between recommendations resulting from different subclasses of statistics. However, sometimes it is not so, as is illustrated by the next example.

Example 23.8.5 The PODISTAT package was verified also by evaluating its three families of statistics in the domain $-1/2 < \alpha \leq 3$ for the empirical distributions resulting from the horse kidney measurement data of Table 23.1–2 and for the hypothetical distributions given by the mixed generalized lambda models MixGLD of Example 23.2.32. The results are presented in graphical form in Figure 23.8–2. Similarly as in the previous example, the asymptotic normality

result of Assertion 23.7.32 was applied to evaluate the critical values as functions of $-1/2 < \alpha \leq 3$ for rejection of the MixGLD models at the 5% significance level. These values are printed in the finely dotted curves. We see from Figure 23.8–2 that, similarly as in the previous example, the differences between the statistics $\mathrm{T} = T_{\phi_\alpha,n}$, $\mathrm{S} = S_{\phi_\alpha,n}$, and $\mathrm{TW} = \tilde{T}_{\phi_\alpha,n}$ are negligible for all powers $-1/2 < \alpha \leq 3$ which indicates that the asymptotic equivalence of Subsection 23.6.1 works already for moderate sample sizes like $n = 43$ in the case of Table 23.1–2. Further we see from Figure 23.8–2 that the MixGLD of Example 23.2.32 is not rejected at the 5% significance level. The conservative character of the hypothesis rejection rule mentioned in Example 23.8.1 is manifested here by a much wider non-rejection domain corresponding to the small sample size $n = 43$ than the domain corresponding to the large sample size $n = 1000$ in Example 23.8.1. All statistics under consideration indicate that the MixGLD of Example 23.2.32 is well fitted to the horse kidney data in the sense that all of them are localized deeply in the non-rejection domain, in its central area.

23.9 Proofs of Assertions

Proofs of the assertions stated in this chapter above can be found in Vajda and van der Meulen (2010). They are presented here for the sake of completeness.

23.9.1 Proofs for Structural Spacings Statistics

Proof of Assertion 23.6.3 We shall consider the fixed alternative $F(x)$ with a continuous density $f(x) > 0$ for $0 \leq x \leq 1$. For the local alternatives the argument is similar. By inspecting the definitions of T_ϕ, $\tilde{T}_\phi$, and R_ϕ we see that for (23.6.4) it suffices to prove that as $n \to \infty$

$$\phi(np_{01}) = O_p(1) \quad \text{and} \quad \phi(n(p_{01} + p_{02})) = O_p(1) \tag{23.9.1}$$

where the p_{0j} 's are given by (23.4.35) with $m = 1$.

It is known (see for example page 208 in Hall (1986)) that $p_{01} = F^{-1}(Z_1/W_{n+1})$ and $p_{01} + p_{02} = F^{-1}((Z_1 + Z_2)/W_{n+1})$, where $Z_1, \ldots, Z_{n+1}$ are independent standard exponential variables and $W_{n+1} = Z_1 + \cdots + Z_{n+1}$, so that, for $n \to \infty$,

$$\frac{W_{n+1}}{n} \xrightarrow{p} 1 \quad \text{and} \quad V_n = \frac{Z_1}{W_{n+1}} \xrightarrow{p} 0.$$

Setting

$$\Upsilon_n = \frac{F^{-1}(V_n)}{V_n} = \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n}$$

and using the mean value theorem and the assumed continuity of f in the neighborhood of 0, we find that

$$\Upsilon_n \xrightarrow{p} \frac{1}{f(0)} \quad \text{as } n \to \infty$$

where, by assumptions about f, $0 < f(0) < \infty$. Thus

$$np_{01} = \frac{n}{W_{n+1}}\, Z_1\, \Upsilon_n$$

and, by applying (23.5.40),

$$\phi(np_{01}) = \xi\left(\frac{n}{W_{n+1}}\right)\, \phi(Z_1\, \Upsilon_n) + \zeta(Z_1\, \Upsilon_n)\, \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right)\, (Z_1\, \Upsilon_n - 1).$$

Since $Z_1\, \Upsilon_n = O_p(1)$ as $n \to \infty$, we obtain from Assertion 23.5.42

$$\begin{aligned} \phi(np_{01}) &= \left[\xi\left(\frac{n}{W_{n+1}}\right) + \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right)\right] O_p(1) \\ &= [\xi(1) + \phi(1) + \eta(1) + o_p(1)]\, O_p(1) \\ &= O_p(1) \quad \text{(see (23.5.43))}, \end{aligned}$$

thus proving the first relation of (23.9.1). Replacing $V_n = Z_1/W_{n+1}$ by $V_n = (Z_1 + Z_2)/W_{n+1}$, and using the fact that now

$$(Z_1 + Z_2)\, \Upsilon_n = (Z_1 + Z_2)\, \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n} = O_p(1)$$

we obtain the second relation of (23.9.1). Next we prove (23.6.5). From (23.5.41) we get for any $p > 0$

$$\phi((n+1)\, p) = \xi\left(\frac{n+1}{n}\right)\, \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right)\, (np - 1)$$

so that

$$\phi((n+1)\, p) - \phi(np) = \varepsilon_n \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right)\, (np - 1) \qquad (23.9.2)$$

where $\varepsilon_n = \xi((n+1)/n) - 1 = o(1)$ as $n \to \infty$ by Assertion 23.5.42. Replacing p by the probabilities $p_{0j} = Y_j - Y_{j-1}$ figuring in the definitions of S_ϕ and R_ϕ (see (23.5.2) and (23.5.1)), and summing over $1 \le j \le n+1$, we get the equality

$$S_\phi - R_\phi = \varepsilon_n\, R_\phi + \delta_n$$

for

$$\begin{aligned} \delta_n &= (n+1)\, \phi\left(\frac{n+1}{n}\right) - \eta\left(\frac{n+1}{n}\right) \\ &= \frac{n+1}{n}\, \frac{\phi\left(1 + \frac{1}{n}\right) - \phi(1)}{\frac{1}{n}} - \eta\left(\frac{n+1}{n}\right). \end{aligned}$$

By Assertion 23.5.42,

$$\delta_n = \phi'(1) + o(1) \quad \text{as } n \to \infty.$$

This completes the proof of the first relation in (23.6.5). The proof of the second relation is the same: we just replace p in (23.9.2) by the probabilities $\tilde{p}_{0j}$ figuring in the definition (23.5.24) of $\tilde{T}_\phi$. □

Proof of Assertion 23.6.15 By Theorem 1 of Hall (1984), the statistic $\tilde{R}_\phi$ defined by (23.5.27) satisfies under a fixed alternative $F \sim f$ the relation

$$\frac{\tilde{R}_\phi}{n} \xrightarrow{p} \tilde{\mu}_\phi(f) = \int_0^1 f^2(x) \left(\int_0^\infty \phi(t)\, e^{-tf(x)} dt \right) dx \quad \text{as } n \to \infty$$

provided $\phi : (0, \infty) \mapsto \mathbb{R}$ is continuous and exponentially bounded in the sense that $|\phi(t)| \leq K(t^\alpha + t^{-\beta})$ for some $K > 0$, $\alpha \geq 0$, $\beta < 1$, and f is bounded, piecewise continuous, and bounded away from 0 (see also part (i) of Theorem 3.1 in Misra and van der Meulen (2001)). Thus (23.6.16) is proved for $U_\phi = \tilde{R}_\phi$ as soon as it is shown that for $\phi \in \mathbf{\Phi}_1$ the limit $\tilde{\mu}_\phi(f)$ coincides with $\mu_\phi(f)$. By substituting s for $tf(x)$ in the last integral, and using the assumption $0 < f(x) < \infty$ and the functional equation (23.5.40),

$$\begin{aligned}
\tilde{\mu}_\phi(f) &= \int_0^1 f(x) \left(\int_0^\infty \phi\left(\frac{s}{f(x)}\right) e^{-s} ds \right) dx && (23.9.3) \\
&= \int_0^1 f(x) \left(\int_0^\infty \left[\xi(s)\, \phi\left(\frac{1}{f(x)}\right) + \zeta\left(\frac{1}{f(x)}\right) \phi(s) \right. \right. && (23.9.4) \\
&\quad \left. \left. + \eta(s) \left(\frac{1}{f(x)} - 1 \right) \right] e^{-s} ds \right) dx \\
&= \mu_\phi(f) + \int_0^\infty \eta(s)\, e^{-s}\, ds \int_0^1 (1 - f(x))\, dx = \mu_\phi(f).
\end{aligned}$$

The extension of (23.6.16) to $U_\phi \in \{T_\phi, \tilde{T}_\phi, R_\phi\}$ follows from Assertion 23.6.3. For $\phi \in \mathbf{\Phi}_2$ the extension of (23.6.16) to $U_\phi \in \{S_\phi, \tilde{S}_\phi\}$ follows from Assertion 23.6.3 too. □

Proof of Assertion 23.6.19 For $U_\phi = S_\phi$ the relations (23.6.20) through (23.6.22) follow from the result of Kuo and Rao (1981), see also Del Pino (1979) and Theorem 3.2 in Misra and van der Meulen (2001). The extension to the remaining statistics U_ϕ follows from Assertion 23.6.3. □

Proof of Assertion 23.6.27 Consider $U_\phi = \tilde{R}_\phi$ for $\phi \in \mathbf{\Phi}_2$. By Assertion 23.5.44, $\phi(t)$ has a continuous derivative $\phi'(t)$ on $(0, \infty)$. By (23.5.45), for every $c \in \mathbb{R}$

$$t^c |\phi'(t)| \leq |\xi'(1)|\, t^{c-1} |\phi(t)| + |\phi'(1)|\, t^c + |\eta'(1)|\, t^{c-1} |t - 1|.$$

Thus if ϕ satisfies (23.6.6) with $\beta < 1/2$ then there exists $\alpha \geq 0$ such that

$$\lim_{t \to \infty} t^{-\alpha} |\phi'(t)| = \lim_{t \downarrow 0} t^{1+\beta} |\phi'(t)| = 0.$$

This means that under the assumptions of Assertion 23.6.19 there exist $c > 0$, $K > 0$ and $b < 1/2$ such that for every $t \in (0, \infty)$

$$|\phi(t)| \leq K(t^a + t^{-b}) \quad \text{and} \quad |\phi'(t)| \leq K(t^a + t^{-b-1}).$$

For continuously differentiable functions ϕ satisfying these assumptions, and fixed alternatives with densities f continuously differentiable on $(0,1)$, it follows from Theorem 2 in Hall (1984) (see also part (ii) of Theorem 3.1 in Misra and van der Meulen (2001)) that $U_\phi = \tilde{R}_\phi$ satisfies the relation

$$\frac{1}{\sqrt{n}}(U_\phi - n\tilde{\mu}_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \tilde{\sigma}^2_\phi(f)) \quad \text{for } n \to \infty$$

where: **(1)** the asymptotic mean $\tilde{\mu}_\phi(f)$ was presented and proved to be equal to $\mu_\phi(f)$ in the proof of Assertion 23.6.15 under assumptions weaker than here and, **(2)** the asymptotic variance $\tilde{\sigma}^2_\phi(f)$ can be specified by means of the standard exponential variable Z and the auxiliary function

$$G(x) = \int_0^x \left(1 - \frac{F(y)\, f'(y)}{f^2(y)}\right) E\left[Z\, \phi'\left(\frac{Z}{f(y)}\right)\right] dy, \quad 0 < x < 1, \tag{23.9.5}$$

as the sum of

$$s^2_1(f) = \int_0^1 \left(E\phi^2\left(\frac{Z}{f(x)}\right) - \left[E\phi\left(\frac{Z}{f(x)}\right)\right]^2\right) f(x)\, dx \tag{23.9.6}$$

$$s^2_2(f) = -2\int_0^1 E\left[(Z-1)\,\phi\left(\frac{Z}{f(x)}\right)\right] \frac{G(x)}{F(x)}\, f(x)\, dx \tag{23.9.7}$$

and

$$s^2_3(f) = \int_0^1 \left(\frac{G(x)}{F(x)}\right)^2 f(x)\, dx. \tag{23.9.8}$$

It remains to be proved that for every $x \in (0,1)$

$$\left(E\,\phi^2\left(\frac{Z}{f(x)}\right) - \left[E\,\phi\left(\frac{Z}{f(x)}\right)\right]^2\right) f(x) = \Psi_2(x), \tag{23.9.9}$$

$$E\left[(Z-1)\,\phi\left(\frac{Z}{f(x)}\right)\right] \sqrt{f(x)} = \Psi_3(x) \tag{23.9.10}$$

and

$$\frac{G(x)\,\sqrt{f(x)}}{F(x)} = \Psi_4(x). \tag{23.9.11}$$

Indeed, then $\tilde{\sigma}^2_\phi(t) = \sigma^2_\phi(f)$ so that (23.6.28) is proved for $U_\phi = \tilde{R}_\phi$, and the extension of (23.6.28) to the remaining statistics $U_\phi \in \{R_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$ follows from Assertion 23.6.3. We shall prove (23.9.9) through (23.9.11) in the reversed order. By substituting $t = Z/f(y)$ in (23.5.45) and taking into account that $\zeta(t) = 1$ we obtain

$$\begin{aligned} E\left[Z\phi'\left(\frac{Z}{f(y)}\right)\right] &= f(y)\, E\left[\xi'(1)\,\phi\left(\frac{Z}{f(y)}\right) + \phi'(1) + \eta'(1)\left(\frac{Z}{f(y)} - 1\right)\right] \\ &= f(y)\left[\xi'(1)\, E\,\phi\left(\frac{Z}{f(y)}\right) + \phi'(1) + \eta'(1)\left(\frac{1}{f(y)} - 1\right)\right] \end{aligned}$$

and, by putting $s = 1/f(x)$ and $t = Z$ in (23.5.41), we get

$$\phi\left(\frac{Z}{f(x)}\right) = \phi(Z)\,\xi\left(\frac{1}{f(x)}\right) + \phi\left(\frac{1}{f(x)}\right) + \eta\left(\frac{1}{f(x)}\right)(Z-1). \tag{23.9.12}$$

Therefore

$$E\,\phi\left(\frac{Z}{f(x)}\right) = \langle\phi\rangle\xi\left(\frac{1}{f(x)}\right) + \phi\left(\frac{1}{f(x)}\right) \tag{23.9.13}$$

and, consequently,

$$E\left[Z\,\phi'\left(\frac{Z}{f(y)}\right)\right] = \Psi_1(y). \tag{23.9.14}$$

This, together with the definitions of $\Psi_4(x)$ and $G(x)$ in (23.6.26) and (23.9.5), implies (23.9.11). Further, from (23.9.12) and the definition of $\Psi_3(x)$ in (23.6.25) we get (23.9.10). Finally, from (23.9.12), (23.9.13) and the definition of $\Psi_2(x)$ in (23.6.24) we obtain (23.9.9), which completes the proof. □

23.9.2 Proofs for Power Spacings Statistics

Proof of Assertion 23.7.22 By the assumptions about g,

$$t_0 = \min_{y\in[a,b]} g(y) > 0 \quad \text{and} \quad t_1 = \max_{y\in[a,b]} g(y) < \infty.$$

If $\psi_\alpha(t)$ is convex, then for every $t \in [t_0, t_1]$ and $\alpha \in (c, d)$

$$\psi'_\alpha(t_*)\,(t - t_*) \le \psi_\alpha(t) \le \psi_\alpha(t_0) + \psi_\alpha(t_1).$$

If $\psi_\alpha(t)$ is concave, then, similarly,

$$\psi_\alpha(t_0) + \psi_\alpha(t_1) \le \psi_\alpha(t) \le \psi'_\alpha(t_*)\,(t - t_*).$$

Therefore in both cases

$$\max_{t_0\le t\le t_1} |\psi_\alpha(t)| \le \max\left\{|\psi_\alpha(t_0) + \psi_\alpha(t_1)|,\, |\psi'_\alpha(t_*)| \cdot |t_1 - t_0|\right\}.$$

The assumed continuity of $\psi'_\alpha(t_*)$ and $\psi_\alpha(t_0) + \psi_\alpha(t_1)$ in the variable $\alpha \in (c, d)$ implies that for all compact neighborhoods $N \subset (c, d)$ of α_0 the constant

$$k = \sup_{\alpha\in N} \max_{t_0\le t\le t_1} |\psi_\alpha(t)| = \sup_{\alpha\in N} \max_{y\in[a,b]} |\psi_\alpha(g(y))|$$

is finite. Put

$$K = \max_{[t_0,t_1]\times[-k,k]} \Phi(u, v).$$

The function $|\Phi(g, \psi_\alpha(g))|$ of variables $(y, \alpha) \in [a, b]\times(c, d)$ is bounded on $[a, b]\times N$ by $K < \infty$. Since for every $y \in [a, b]$

$$\lim_{\alpha\to\alpha_0} \Phi(g, \psi_\alpha(g)) = \Phi(g, \psi_{\alpha_0}(g)),$$

the Lebesgue dominated convergence theorem for integrals implies (23.7.23). □

Proof of Assertion 23.7.24 Since $\mu_\alpha = \mu_\alpha(f_0)$ and $\sigma_\alpha^2 = \sigma_\alpha^2(f_0)$, where the hypothetical density f_0 satisfies the assumptions of Assertions 23.6.15 and 23.6.27, the continuity of μ_α and σ_α^2 follows from the continuity of $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ proved below. By (23.7.21) and (23.6.22),

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\Big(\langle t^2\phi_\alpha(t)\rangle - 4\langle t\phi_\alpha(t)\rangle + 2\langle\phi_\alpha(t)\rangle\Big)$$

where ϕ_α is given by (23.7.1), (23.7.2), and, by (23.6.7),

$$\langle t^j\phi_\alpha(t)\rangle = \int_0^\infty t^j\phi_\alpha(t)\,dH(t), \quad j \in \{0,1,2\} \tag{23.9.15}$$

for $H(t) = 1 - e^{-t}$. All integrals (23.9.15) are finite if and only if $\alpha \in (-1,\infty)$. Further, for every fixed $t > 0$

$$\frac{d}{d\alpha}\,\alpha\phi_\alpha(t) \geq 0 \quad \text{at any } \alpha \in I\!R. \tag{23.9.16}$$

Hence the continuity of the products $\alpha\langle t^j\phi_\alpha(t)\rangle$ in the variable $\alpha \in I\!R$ follows from the monotone convergence theorem for integrals, and this implies also the desired continuity of the integrals (23.9.15) at any $\alpha \in (-1,\infty) - \{0\}$. Further, for every fixed $t > 0$

$$\frac{d}{d\alpha}\,(\alpha-1)\,\phi_\alpha(t) \geq 0 \quad \text{for any } \alpha \in I\!R. \tag{23.9.17}$$

Hence the continuity of the products $(\alpha-1)\,\langle t^j\phi_\alpha(t)\rangle$ in the variable $\alpha \in I\!R$ follows as well from the monotone convergence theorem for integrals. Similarly as above, this implies the continuity of the integrals (23.9.15) at the remaining point $\alpha = 0$. Further, by (23.7.21) and (23.6.13),

$$\mu_\alpha(f) = \langle\xi_\alpha\rangle\, D_\alpha(F_0,F) + \langle\phi_\alpha\rangle$$

where, by (23.6.7) and (23.7.5)

$$\langle\xi_\alpha\rangle = \int_0^\infty t^\alpha dH(t) \quad \text{and} \quad \langle\phi_\alpha\rangle = \int_0^\infty \phi_\alpha(t)\,dH(t).$$

These integrals are finite if and only if $\alpha \in (-1,\infty)$. The continuity of $\langle\phi_\alpha\rangle$ at $\alpha \in (-1,\infty)$ was proved above, the continuity of $D_\alpha(F_0,F)$ at $\alpha \in I\!R$ follows from the assumptions about the densities f_0 and f and from Proposition 2.14 in Liese and Vajda (1987). The continuity of $\langle\xi_\alpha\rangle$ at $\alpha \in (-1,\infty)$ follows from the monotone convergence theorem for integrals applied separately to the integration domains $(0,1)$ and $(1,\infty)$. Finally, let us consider $\sigma_\alpha^2(f)$ defined by (23.6.29) together with (23.6.23) through (23.6.26) for $\phi = \phi_\alpha$, $\xi = \xi_\alpha$, and $\eta = \eta_\alpha$ given by (23.7.1), (23.7.2) and (23.7.5). The integrals $\langle t\phi_\alpha(t)\rangle$, $\langle\phi_\alpha(t)\rangle$ and $\langle\phi_\alpha^2(t)\rangle$ are finite if and only if $\alpha \in (-1/2,\infty)$, and their continuity at $\alpha \in (-1/2,\infty)$ was

either proved above or can be proved similarly as above. The continuity of the integral

$$\int_0^1 \left[f\xi_\alpha^2\left(\frac{1}{f}\right) + f\eta_\alpha^2\left(\frac{1}{f}\right)\right] dx$$

at $\alpha \in (-1/2, \infty)$ follows from Assertion 23.7.22, which establishes the continuity of the component $\int \Psi_2(x)\,dx$ of $\sigma_\alpha^2(f)$ in (23.6.29). For the continuity of the remaining two components, we take into account that $F(x) > c_1 x$ for some $c_1 > 0$ on $[0,1]$, because f is bounded away from zero on $[0,1]$. Furthermore, both $f(x)$ and $f'(x)$ are bounded on $[0,1]$, so that there exists a constant c_2 such that in (23.6.26)

$$\frac{\sqrt{f(x)}}{F(x)} \int_0^x \left| 1 - \frac{F(y)\, f'(y)}{f^2(y)} \right| dy < c_2 \quad \text{for all } x \in [0,1]. \tag{23.9.18}$$

Using the function $\varphi_\alpha(t) = \alpha\phi_\alpha(t)$, which is for every $t > 0$ continuous and monotone in $\alpha \in I\!R$ (see (23.9.16)), we obtain from (23.6.23)

$$\Psi_1(x) = \alpha\langle\phi_\alpha\rangle\, f(x)^{1-\alpha} + f(x)\,\varphi_\alpha\left(\frac{1}{f(x)}\right) + 1 - f(x)$$

where the right-hand side is bounded on $[0,1]$, locally uniformly in α, and continuous at any $\alpha \in I\!R$. By (23.6.26) and (23.9.18), this implies that also $\Psi_4(x)$ is bounded on $[0,1]$, locally uniformly in α, and continuous at any $\alpha \in I\!R$. Since the integrands in

$$\int_0^1 \left[\sqrt{f}\xi_\alpha\left(\frac{1}{f}\right) + \sqrt{f}\eta_\alpha\left(\frac{1}{f}\right)\right] \Psi_4\, dx \quad \text{and} \quad \int_0^1 \Psi_4^2\, dx$$

are continuous on $[0,1]$ and locally bounded in the variable $\alpha \in I\!R$, the continuity of both these integrals in the variable $\alpha \in I\!R$ follows from the Lebesgue dominated convergence theorem for integrals. This clarifies the continuity of the second and third component of $\sigma_\alpha^2(f)$ in (23.6.29) and thus completes the proof. □

Proof of Assertion 23.7.26 The functions from the class $\{\phi_\alpha : \alpha \in (-1,\infty)\} \subset \mathbf{\Phi}_2$ satisfy all assumptions of Assertion 23.6.15. Hence (23.7.27) holds for all $\alpha > -1$ and the limit $\mu_\alpha(f)$ is given in accordance with (23.6.13) and (23.7.5) by the formula

$$\mu_\alpha(f) = \langle\xi_\alpha(t)\rangle\, D_\alpha(F_0, F) + \langle\phi_\alpha(t)\rangle = \langle t^\alpha\rangle\, D_\alpha(F_0, F) + \langle\tilde{\phi}_\alpha(t)\rangle$$

where $\langle t^\alpha\rangle = \Gamma(\alpha+1)$ for all $\alpha \in I\!R$. If $\alpha \notin \{0,1\}$ then

$$\langle\tilde{\phi}_\alpha(t)\rangle = \frac{1}{\alpha(\alpha-1)}\langle t^\alpha - 1\rangle = \frac{\Gamma(\alpha+1) - \Gamma(1)}{\alpha(\alpha-1)}$$

but the expressions

$$\langle\tilde{\phi}_0(t)\rangle = \langle -\ln t\rangle \quad \text{and} \quad \langle\tilde{\phi}_1(t)\rangle = \langle t\ln t\rangle$$

lead to the evaluation of unpleasant integrals. This evaluation can be avoided by employing Assertion 23.7.24. From the continuity of $\mu_\alpha = \langle\tilde{\phi}_\alpha(t)\rangle$, it follows that

$$\mu_j = \langle\tilde{\phi}_j(t)\rangle = \lim_{\alpha\to j}\frac{\Gamma(\alpha+1)-\Gamma(1)}{\alpha(\alpha-1)} \quad \text{for } j\in\{0,1\},$$

where the limit on the right can be easily evaluated by using L'Hospital's rule and the known formulas $\Gamma'(1) = -\gamma$, $\Gamma'(2) = 1-\gamma$, thus leading to the values μ_j, $j\in\{0,1\}$, given in (23.7.29). The continuity and the inequality $\mu_\alpha(f)\geq\mu_\alpha$ for $\alpha\in(-1,\infty)$ follow from (23.7.28) and (23.7.29) because $D_\alpha(F_0,F)$ is nonnegative and continuous in $\alpha\in\mathbb{R}$ and $\Gamma(\alpha+1)$ is positive and continuous in $\alpha\in(-1,\infty)$. The condition for equality follows from the fact that $D_\alpha(F_0,F)$ is positive unless $F=F_0$. □

Proof of Assertion 23.7.32 Similarly as we applied Assertion 23.6.15 in the proof of Assertion 23.7.26, (23.7.33) follows for all $\alpha > -1/2$ from Assertion 23.6.19. If $\alpha\notin\{0,1\}$, then the expressions for $m_\alpha(\ell)$ and σ_α^2 given in (23.7.34) and (23.7.35) follow easily from the formulas given for $m_{\phi_\alpha}(\ell)$ and $\sigma^2_{\phi_\alpha}$ in Assertion 23.6.19, but the direct evaluation of $m_j(\ell)$ and σ_j^2 from these formulas for $j\in\{0,1\}$ is a somewhat tedious task. However, by using the continuity of $m_\alpha(\ell)$ and σ_α^2 established in Assertion 23.7.24, we obtain $m_j(\ell)$ and σ_j^2 given in (23.7.34) and (23.7.36) as the limits

$$m_j(\ell) = \lim_{\alpha\to j} m_\alpha(\ell) \quad \text{and} \quad \sigma_j^2 = \lim_{\alpha\to j}\sigma_\alpha^2 \quad \text{for } j\in\{0,1\},$$

which expressions can be easily evaluated by using the continuity of the right-hand side of (23.7.34) and L'Hospital's rule, thereby employing the formulas

$$\begin{aligned}\Gamma(\alpha+k+1) &= (\alpha+k)\,(\alpha+k-1)\cdots(\alpha+1)\,\Gamma(\alpha+1),\\ \Gamma''(\alpha+1) &= 2\Gamma'(\alpha)+\alpha\Gamma''(\alpha)\end{aligned}$$

and

$$\Gamma''(1) = \frac{\pi^2}{6}+\gamma^2,\quad \Gamma''(2) = \frac{\pi^2}{6}-2\gamma+\gamma^2\quad \Gamma''(3) = \frac{\pi^2}{3}+2-6\gamma+2\gamma^2$$

in addition to the previously used $\Gamma'(1) = -\gamma$ and $\Gamma'(2) = 1-\gamma$. □

Proof of Assertion 23.7.44 If $\psi : [0,1] \mapsto I\!R$ is continuous then by the assumptions about f

$$\inf_{x\in[0,1]} f(x) > 0 \quad \text{and} \quad \sup_{x\in[0,1]} |\psi(x)\, f(x)| < \infty$$

and, consequently, the function

$$\Psi(x) = \int_0^x \psi(y)\, f(y)\, dy, \quad x \in (0,1)$$

is well defined. Since

$$\frac{d}{dx}\,\frac{\Psi^2}{F} = -\left(\frac{\Psi}{F}\right)^2 f + \frac{2\Psi\psi f}{F}$$

and

$$|\Psi(y)| \le y \sup_{x\in[0,1]} |\psi(x)\, f(x)| \ \text{ as well as } \ F(y) \ge y \inf_{x\in[0,1]} f(x),$$

the function Ψ satisfies the relation

$$\int_0^1 (\psi - \Psi/F)^2 f\, dx = \int_0^1 \psi^2 f\, dx - \left(\int_0^1 \psi\, f\, dx\right)^2. \tag{23.9.19}$$

To this end take into account the relations

$$\begin{aligned}
\int_0^1 (\psi - \Psi/F)^2 f\, dx &= \int_0^1 \psi^2 f\, dx - \int_0^1 \frac{2\Psi\psi f}{F}\, dx + \int_0^1 \left(\frac{\Psi}{F}\right)^2 f\, dx \\
&= \int_0^1 \psi^2 f\, dx - \left(\frac{\Psi^2(1)}{F(1)} - \lim_{y\downarrow 0} \frac{\Psi^2(y)}{F(y)}\right) \\
&= \int_0^1 \psi^2 f\, dx - \frac{\Psi^2(1)}{F(1)}.
\end{aligned}$$

Now, using (23.9.19) we obtain (23.7.45) from the definition (23.7.41). Since f is assumed to be bounded and bounded away from 0,

$$\begin{aligned}
\lim_{\alpha\to 0} \Delta_\alpha(F_0, F) &= \int_0^1 \left(\lim_{\alpha\to 0} \frac{f^{-\alpha} - 1}{\alpha} - \int_0^1 \lim_{\alpha\to 0} \frac{f^{-\alpha} - 1}{\alpha}\, f\, dy\right)^2 f\, dx \\
&= \int_0^1 \left(\ln f - \int_0^1 (\ln f)\, f\, dy\right)^2 f\, dx \\
&= \Delta_0(F_0, F)
\end{aligned}$$

which proves the continuity at $\alpha = 0$. The reflexivity is clear from (23.7.45) and (23.7.46). □

Proof of Assertion 23.7.51 The proof should be clear from what was said above. The inequality $\sigma_\alpha^2(f) \geq \sigma_\alpha^2$ and the condition for equality follow from (23.7.43), because $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ are nonnegative measures of divergence of F_0 and F, which are equal to zero if and only if $F = F_0$, in which case the excess function (23.7.47) is 0. □

Acknowledgements

The research for this chapter was supported by GAČR grants 102/07/1131, 202/10/0618 and K.U. Leuven project GOA/98/06. The authors are deeply indebted to their colleague Pavel Boček for the figures, programs, and computations presented in the chapter. The authors would also like to thank Ed Dudewicz and Zaven Karian for their editorial suggestions and assistance, which led to several improvements in the presentation of this chapter.

References for Chapter 23

Anderson, T. W. and Darling, D. A. (1954). "A test of goodness-of-fit," *J. Amer. Statist. Assoc.* **49**, 765–769.

Chernoff, H. (1952). "A measure of asymptotic efficiency of tests of a hypothesis based on a sum of observations," *Ann. Math. Statist.*, **23**, 493–507.

Cressie, N. (1976). "On the logarithms of high-order spacings," *Biometrika*, **63**, 345–355.

Cressie, N. (1978). "Power results for tests based on high-order gaps," *Biometrika*, **65**, 214–218.

Cressie, N. (1979). "An optimal statistic based on higher order gaps," *Biometrika*, **66**, 619–627.

Cressie, N. and Read, T. R. C. (1984). "Multinomial goodness-of-fit tests," *Journal of the Royal Statistical Society Series B*, **46**, 440–464.

Csiszár, I. (1963). "Informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Kette," *Publ. Math. Inst. Hungarian Acad. Sci., Ser. A*, **8**, 85–108.

Csiszár, I. (1967). "On topological properties of f-divergence," *Studia Scientiarum Math. Hungarica*, **2**, 329–339.

D'Agostino, R. B. and Stephens, M. A. (1986). *Goodness-of-fit Techniques*. M. Dekker, New York.

Darling, D. A. (1953). "On a class of problems related to the random division of an interval," *Ann. Math. Statist.*, **24**, 239–253.

Darling, D. A. (1957). "The Kolmogorov-Smirnov, Cramer-von Mises tests," *Ann. Math. Statist.*, **28**, 823–838.

Del Pino, G. E. (1979). "On the asymptotic distribution of k-spacings with applications to goodness of fit tests," *Ann. Statist.*, **7**, 1058–1065.

Devroye, L. and Györfi, L. (1985). *Nonparametric Density Estimation: The L1 View*. J. Wiley, New York.

Dudewicz, E. J. and van der Meulen, E. C. (1981). "Entropy-based tests of uniformity," *J. Amer. Statist. Assoc.*, **76**, 967–974.

Dudewicz, E. J. and van der Meulen, E. C. (1987). "The empiric entropy, a new approach to nonparametric entropy estimation," *New Perspectives in Theoretical and Applied Statistics* (eds. M.L. Puri, J.P. Vilaplana, and W. Wertz), John Wiley & Sons, New York, 207–227.

Durbin, J. (1973). *Distribution Theory for Tests Based on the Sample Distribution Function*, Regional Conference Series in Applied Mathematics, Vol. 9. Philadelphia: SIAM.

Ekström, M. (1999). "Strong limit theorems for sums of logarithms of high order spacings," *Statistics*, **33**, 153–169.

Greenwood, M. (1946). "The statistical study of infectious diseases," *J. Roy. Statist. Soc., Ser. A*, **109**, 85–110.

Hall, P. (1984). "Limit theorems for sums of general functions of m-spacings," *Math. Proc. Cambridge Philos. Soc.*, **96**, 517–532.

Hall, P. (1986). "On powerful distributional tests based on sample spacings," *J. Multivariate Anal.*, **19**, 201–224.

Hampel, F.R., Ronchetti, E.M., Rousseeuw, P.J. and Stahel, W.A. (1986). *Robust Statistics, The Approach Based on Influence Functions.* Wiley: New York.

Jammalamadaka, S. R. and Tiwari, R. C. (1987). "Efficiencies of some disjoint spacings tests relative to a χ^2 test," *New Perspectives in Theoretical and Applied Statistics*, (eds. M.L. Puri, J.P. Vilaplana, and W. Wertz), John Wiley & Sons, New York, 311–317.

Jammalamadaka, S. R., Zhou, X., and Tiwari, R. C. (1989). "Asymptotic efficiencies of spacings tests for goodness of fit," *Metrika*, **36**, 355–377.

Jiménez, R. and Shao, Y. (2009). "Limit theorems for ϕ-divergences based on k-spacings," *Comm. Statist. Theory Methods*, **38**, 695–710.

Jurečková, J. and Sen, P.K. (1996). *Robust Statistical Procedures.* New York: Wiley.

Kafka, P., Österreicher, F., and Vincze, I. (1991). "On powers of Csiszár f-divergences defining a distance," *Studia Sci. Math. Hungar.* **26**, 415–422.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions. The Generalized Lambda Distribution and Generalized Bootstrap Methods.* Boca Raton, FL: CRC Press.

Kimball, B. F. (1947). "Some basic theorems for developing tests of fit for the case of nonparametric probability distribution functions," *Ann. Math. Statist.*, **18**, 540–548.

Kimball, B. F. (1950). "On the asymptotic distribution of the sum of powers of unit frequency differences," *Ann. Math. Statist.*, **21**, 263–271.

Kolmogorov A. N. (1933). "Sulla determinazione empirica di una legge di distibuzione," *Giorn. Istit. Attuari.* **4**, 83–91.

Kuo, M. and Rao, J. S. (1981). "Limit Theory and Efficiencies for Tests Based on Higher Order Spacings," in *Statistics – Applications and New Directions*, Proc. Golden Jubilee Conference of the Indian Statistical Institute, 333–352, Calcutta: Statistical Publishing Society.

Lehmann, E. L. and Romano, J. P. (2005). *Testing Statistical Hypotheses*, Third Edition. Berlin: Springer.

Liese, F. (1982). "Hellinger integrals of Gaussian processes with independent increments," *Stochastics* **6**, 81–96.

Liese, F. and Vajda, I. (1987). *Convex Statistical Distances*. Leipzig: Teubner.

Liese, F. and Vajda, I. (2006). "On divergences and informations in statistics and information theory," *IEEE Trans. Inform. Theory*, **52,** 4394–4412.

Lindsay, G. G. (1994). "Efficiency versus robustness. The case of minimum Hellinger distance and other methods," *Ann. Statist.*, **22**, 1081–1114.

Matusita, K. (1956). "Distances and decision rules," *Ann. Inst. Statist. Math.*, **16**, 305–320.

Menéndez, M., Morales, D. Pardo, L., and Vajda, I. (1998). "Two approaches to grouping of data and related disparity statistics," *Comm. Statist. Theory Methods*, **27**, 609–633.

Misra, N. and van der Meulen, E. C. (2001). "A new test of uniformity based on overlapping sample spacings", *Comm. Statist. Theory Methods*, **30**, 1435–1470.

Morales, D., Pardo, M. C., and Vajda, I. (2003). "Limit laws for disparities of spacings," *J. Nonparametr. Statist.*, **15**, 325–342.

Moran, P. A. P. (1951). "The random division of an interval–Part II," *J. Roy. Statist. Soc. Ser. B*, **13**, 147–150.

Ning, W., Gao, Y., and Dudewicz, E.J. (2008). "Fitting mixture distributions using generalized lambda distributions and comparison with normal mixtures," *American Journal of Mathematical and Management Sciences*, **28**, Nos. 1 & 2, 81–99.

Pearson, K. (1894). "Contributions to the mathematical theory of evolution," *Phil. Trans. of the Roy. Soc. of London A.* **185**, 71–110.

Perez, A. (1967). "Information-theoretic risk estimates in statistical decision,"

Kybernetika, **3**, 1–21.

Pyke, R. (1965). "Spacings," *J. Roy. Statist. Soc. Ser. B*, **27**, 395–449.

Rao, J. S. and Kuo, M. (1984). "Asymptotic results on Greenwood statistic and some of its generalizations," *J. Roy. Statist. Soc. Ser. B*, **46**, 228–237.

Read, T. R. C. and Cressie, N. (1988). *Goodness of Fit Statistics for Discrete Multivariate Data.* Berlin: Springer.

Rényi, A. (1961). "On measures of Entropy and Information," *Proc. 4th Berkeley Symp. on Probab. Th. Math. Statist.*, Vol. 1, 547–561.

Serfling, R. H. (1980). *Approximation Theorems of Mathematical Statistics.* Wiley, New York.

Shao, Y. and Hahn, M. G. (1995). "Limit theorems for the logarithm of sample spacings," *Statist. Probab. Lett.*, **24**, 121–132.

Smirnov, N. V. (1948). "Tables for estimating the goodness of fit of empirical distributions," *Ann. Math. Statist.*, **19**, 279–281.

Vajda, I. (1971). "Limit theorems for total variation of cartesian product measures," *Studia Sci. Math. Hungar.* **6**, 317–333.

Vajda, I. (2007). "Asymptotic comparisons of divergence-based goodness-of-fit statistics," *Publ. Statist. Inst. Univ. Paris* , **51**, 49–66.

Vajda, I. (2009). "On metric divergences of probability measures," *Kybernetika,* **45**, 885–900.

Vajda, I. and van der Meulen, E. C. (2006). "On Estimation and Testing by Means of ϕ-Disparities Based on m-Spacings," in *Proceedings Prague Stochastics 2006*, (Eds Hušková M., Janžura M.), 701–708, Matfyzpress, Prague.

Vajda, I. and van der Meulen, E. C. (2010). *Divergences Between Models and Data under Hypothetical and Empirical Quantiles.* Res. Report No. 2275, Institute of Information Theory and Automation, Prague (available online at http://simu0292.utia.cas.cz/vajda/RR2274).

van Es, B. (1992). "Estimating functionals related to a density by a class of statistics based on spacings," *Scand. J. Statist.*, **19**, 61–72.

Vasicek, O. (1976). "A test for normality based on sample entropy," *J. Roy. Statist. Soc. Ser. B*, **38**, 54–59.

von Mises, R. (1947). "On the asymptotic distribution of differential statistical functions," *Ann. Math. Statist.*, **18**, 309–348.

Weiss, L. (1957). "The asymptotic power of certain tests of fit based on sample spacings," *Ann. Math. Statist.*, **28**, 783–786.

Chapter 24

Evidential Support Continuum (ESC): A New Approach to Goodness-of-Fit Assessment, which Addresses Conceptual and Practical Challenges

Dwight J. Thé
College of Technology, Montana Tech
25 Basin Creek Road
Butte, MT 59701 USA
E-mail: dthe@mtech.edu

Edward J. Dudewicz
Department of Mathematics,
Syracuse University
Syracuse, NY 13244 USA
E-mail: dudewicz@syr.edu

Zaven A. Karian
Department of Mathematics and Computer Science
Denison University
Granville, OH USA
E-mail: karian@denison.edu

Lori Ploutz-Snyder
Exercise Science Department
Syracuse University
Syracuse, NY 13244 USA
E-mail: llploutz@syr.edu

Assessing the fit of various statistical distributions to sample data from an unknown distribution remains a problem in many areas of statistical application. Selecting appropriate methodologies presents numerous challenges as both the concept and practice of goodness-of-fit (G-O-F) has been heavily debated since its inception around 1900. Much of the conceptual debate has focused on whether distributional approximations of data (e.g., probability density functions) should be used only for description or also for inference (e.g., hypothesis testing). In addition to these conceptual challenges, the practice of model selection, fitting distributions to data, and assessing G-O-F is becoming increasingly confusing due to the growing number of recognized statistical distributions as well as more recent developments in building application-specific models. Furthermore, the inherent ambiguity in the results of G-O-F tests is sometimes viewed as problematic or disconcerting. To partially alleviate such concerns, it is important to use the most direct and accurate methods of G-O-F testing available. For example, when performing the χ^2 G-O-F test, we suggest simulating the χ^2 test statistics' true null distribution as opposed to relying on the χ^2 distributional approximation even though the latter method is generally more common. Despite having the best intentions and using the most accurate methods available, results obtained from a single G-O-F test are typically inconclusive and results obtained from multiple G-O-F tests are often contradictory. We propose a strategy, termed the **evidential support continuum (ESC)**, that permits: (1) combination of quantitative hypothesis tests with graphical evaluations; (2) simultaneous evaluation of multiple distributions; and (3) construction of an overall G-O-F assessment criterion based on eight different pieces of evidential support.

24.1 Introduction

The overall purpose of a goodness-of-fit (G-O-F) test can be described succinctly as "a test of the fit of some model to a set of data" (Upton and Cook (2002), p. 150). However, since its inception around 1900, numerous conceptual and practical issues related to the development, implementation, and interpretation of G-O-F testing have challenged would-be users. For example, consider the case of the relatively straightforward uniform distribution where a recent text cites no fewer than eight different G-O-F tests: Kolmogorov-Smirnov D, Cramér-von Mises W^2, Kuiper V, Watson U^2, Anderson-Darling A^2, log-statistic Q, χ^2, and entropy (Karian and Dudewicz (2000)).

Karian and Dudewicz (2000) have pointed out several key features of G-O-F assessment: distributions may be chosen for many different reasons including their theoretical appropriateness as well as their ability to produce good approximations; the type of sample information available may vary considerably (e.g., raw observations, histogram of the data, sample moments based on the data, or some other type of complete or incomplete information); and both practical as well as

theoretical issues should always be considered (e.g., Is the distribution support adequate for the spread of the data? If it is adequate, is it also appropriate?). Finally, these authors offer a word of caution,

> These are not simple questions for which there is a simple formula that can provide a correct answer—they are some of the most difficult questions that arise in modeling, and usually require serious interaction between the statistician and the subject matter specialists—but it is essential that they be carefully and fully considered if we are to have confidence in the model. (Karian and Dudewicz (2000), p. 94)

The work presented in this chapter uses real data from an unknown distribution of origin in an attempt to comprehensively address some of the aforementioned G-O-F issues cited in Karian and Dudewicz (2000). Several conceptual/theoretical as well as practical/applied challenges to the G-O-F assessment process are discussed and a new approach, termed the evidential support continuum (ESC), is proposed for maximizing available information and extending the results of formal hypothesis tests.

24.2 Challenges to Effective G-O-F Testing

Many issues contribute to the difficulties surrounding effective G-O-F assessment. One issue is that although the performance of G-O-F assessment is generally regarded as important, the reasons for why this is the case are neither agreed upon nor fully understood. For example, many theoreticians and practitioners alike do not possess a well-developed sense of what constitutes reasonable outcomes or of how to interpret the results. Some common questions include: Will G-O-F testing provide only basic information related to distributional shape or form? Or will it also permit the evaluation of specific hypotheses, aid in decision making, etc.?

24.2.1 Conceptual and Theoretical Challenges

Current conceptual challenges stem from early disagreements in ideology over what constituted appropriate uses of G-O-F testing. Historically, there were two dominant viewpoints: (1) G-O-F testing is **only** appropriate for description; and (2) G-O-F testing is appropriate for **both** description and inference. Viewpoint (1) addresses how well certain theoretical distributions describe data. For example, is the model a good approximation of the data? If so, how good? In contrast, viewpoint (2) asserts that results can (and perhaps should) be used to perform formal hypothesis tests as well as assist in various types of decision making.

Despite the fact that these key conceptual disputes have not been resolved, recent investigations into G-O-F testing have tended to ignore theoretical issues and focus instead on specific practices or applications. Appreciating the present

emphasis in the literature while also acknowledging that "theory informs practice," we offer a brief overview of some relevant theoretical arguments as presented by the primary developers of each theory.

24.2.1.1 Goodness-of-Fit for Description Only

Karl Pearson (1900) originated and published the first G-O-F test (then called the "P, χ^2 test" and now known as the "χ^2 test," "chi-square test," or "chi-squared test"). He was also responsible for introducing the phrase "goodness-of-fit." The following passage from Pearson (1932) explains his rationale for the test's development as well as the concept of G-O-F,

> I introduced the P, χ^2 test to enable a scientific worker to ascertain whether or not a curve by which he was graduating observations was a reasonable "fit." On this account, and as a measure of success in graduation, I termed it a "goodness of fit" test. It had no special relation to the normal curve or any other curve. The scientific worker in the past had chosen any curve he pleased to graduate his observations, but rarely applied any measure of aptness, beyond looking at a graph to "see" whether it was a "good fit." (Inman (1994), p. 4)

Pearson stated on numerous occasions that scientific formulas should be used for **describing** and not for **explaining** data (Inman (1994)). He also claimed that the role of scientists was to seek verbal and mathematical summaries of relevant data and not to search for **truth** (Inman (1994)). To Pearson, any type of modeling situation, including χ^2 testing, is for measuring the reasonableness or aptness of fit of a mathematical curve to actual data. Specifically, the χ^2 test is not to be used for hypothesis testing and furthermore cannot be used to show hypotheses to be true or false. The following remark succinctly expresses Pearson's views:

> The difference between Prof. Fisher and myself lies in the use (and abuse) of the acceptance and rejection of "hypotheses." There is only one case in which an hypothesis can be definitely rejected, namely when its probability is zero. (Inman (1994), p. 6).

Of course, there are those (of earlier and current generations) who oppose Pearson's views on G-O-F testing. Perhaps the most outspoken critic and challenger of the Pearsonian viewpoint as well as the developer of an alternative theory on G-O-F was Ronald Fisher.

24.2.1.2 Goodness-of-Fit for Description and Inference

Early in his career, Ronald Fisher collaborated with Pearson on several projects including those involving extensions of the χ^2 test. However, papers published by

Fisher between 1922 and 1924 illustrate a divergence from the ideas of Pearson on the appropriate use of G-O-F tests. Ultimately, Fisher would advocate the use of G-O-F testing for inferential as well as descriptive procedures. Pearson and Fisher also had different perceptions of the relative importance of conceptual versus practical issues of G-O-F and a different understanding of what each of these entailed.

Fisher would openly disregard Pearson's assertion that distribution functions were only "... mathematical constructs to describe more or less accurately what we have observed" (Inman (1994), page 4). Specifically, with regard to the use of G-O-F tests for evaluating hypotheses, Fisher had this to say "... it is certain that the interest of statistical tests for scientific workers depends entirely from their use in rejecting hypotheses which are thereby judged to be incompatible with the observations." (Inman (1994), page 5). This remark illustrates Fisher's perception of inference as not only appropriate, but also more important to scientific researchers than description.

Furthermore, there is reason to believe that Fisher viewed these forms of G-O-F and hypothesis testing as theoretically justifiable. For example, Fisher often included tables of critical values for test statistics at specific predefined levels of significance (e.g., $p = 0.01$ and $p = 0.05$) in both his journal articles and textbooks (i.e., rather than featuring more complete tables with values representing numerous points in the distribution). There can be little doubt that such tables of critical values were provided (at least in part) for the purpose of aiding practitioners in performing G-O-F related, hypothesis tests. It is difficult to imagine why such tables would be included if Fisher believed their use to be inappropriate or otherwise invalid.

Finally, it is important to point out that although Fisher focused on what he perceived to be practically relevant issues, his explanations for **how** to correctly use the results of G-O-F tests for evaluating hypotheses are rather confusing to many would-be practitioners. His basic idea is that results obtained from statistical hypothesis tests may be used to reject scientific hypotheses, but never to accept scientific hypotheses. This notion has been described by Fisher in different ways: "... a test of significance contains no criterion for 'accepting' a hypothesis;" the significance level of a statistical test serves as a "well-defined measure of reluctance to acceptance;" and a "measure of the rational grounds for the disbelief" (Inman (1994), page 8).

Overall, it is clear that the Pearsonian and Fisherian viewpoints on G-O-F are rather divergent. What remains ambiguous is whether such ideologies can coexist and equally important contributions to our collective understanding of the conceptual aspects of G-O-F or whether (as eventual users of G-O-F methods) we shall have to choose between the two philosophies. Additionally, it seems that there is a need for some sort of bridge between the conceptual and practical aspects of G-O-F.

24.2.1.3 Limitations of Formal Hypothesis Tests

In addition to theoretical disputes over the appropriateness of hypothesis testing in G-O-F, there are also certain inherent limitations in the process of hypothesis testing itself that bear mentioning. First, consider the case in which a single hypothesis test was performed to evaluate the G-O-F of one particular distribution to a set of data. For this situation, failure to reject the null hypothesis does not necessarily indicate that the null hypothesis is true; the finding could be reflective of a sample size that is too small to detect differences (Karian and Dudewicz (2000)). Conversely, rejection of the null hypothesis does not necessarily indicate that the hypothesized distribution is generally inappropriate or theoretically unsound; the result could indicate a large sample size that is able to detect small differences from the hypothesized distribution.

Next, consider the situation in which multiple hypothesis tests were performed to evaluate the G-O-F of one particular distribution to a set of data. If two such tests are performed then three possibilities emerge: (1) both tests reject; (2) one test rejects and one test fails to reject; or (3) both tests fail to reject. If both tests reject this may be indicative of a poor fit, however the aforementioned limitation that the quality of fit relates to the specific testing situation still applies. If one test rejects while the other test fails to reject, this may indicate that the fit is inadequate on what the test that rejects evaluates, but that it is not inadequate on what the test that fails to reject evaluates. Finally, if both tests fail to reject this may indeed be indicative of an adequate fit, or (as was the case with both tests rejecting) it may just be a symptom of the particular testing situation (e.g., a small sample size).

Finally, evaluation of more than one theoretical distribution with one or more hypothesis tests can result in numerous situations. Consider the case where the same test fails to reject for two different distributions. Assuming that the data could not have originated from two different distributions, a reasonable conclusion (in such a case) is that there is insufficient information to distinguish between them (Gibbons (1992)), i.e., both distributions are consistent with the data on the aspect examined by the particular test used.

24.2.2 Practical Application Challenges

Practical challenges to effective G-O-F assessment are as numerous and diverse as those related to conceptual issues. For example, assessment of the ability of theoretical distributions to reasonably approximate sample data is at least a three-part process involving: (1) selection of one or more candidate distributions to evaluate from among the many theoretical distributions available; (2) selection or development of a methodological approach for fitting these distributions to the data; and (3) selection and application of the form of the G-O-F assessment criteria.

With regards to (1), the statistically literate practitioner will recognize the

need to choose from among a vast array of options. For example, a text on the subject addresses "40 major distributions" (Evans, Hastings, and Peacock (2000)) while another text advocates building application-specific models (see Gilchrist (2000)). It is easy to see how those less initiated can become confused over which distribution(s) to select for G-O-F testing. With regards to (2), many different methods are available for fitting distributions to data. For example, a published paper on fitting via the GLD offers a comparison between three such methods: the method of moments, a method based on percentiles, and a method based on L-moments (Karian and Dudewicz (2003)). With regards to (3), appropriate forms for the G-O-F assessment criteria depend largely on which G-O-F test(s) are selected and the manner in which they are applied to the particular situation at hand (e.g., type of variable, nature of the dataset, relevant context, etc.).

24.2.3 Addressing Goodness-of-Fit Challenges: A New Approach

In this chapter, the approach to dealing with the challenges of G-O-F assessment is to combine the results of multiple evaluations, including quantitative as well as graphical results. This approach produces a more complete understanding of the appropriateness of different theoretical distributions for a particular dataset (as opposed to relying on a single statistical test). This approach does not appear to have been even mentioned in the previous literature. This effort is called the evidential support continuum (ESC), and it seeks to include the maximum amount of available information related to the G-O-F process. Next, we describe the context used to test this new ESC approach, and then go into the ESC's full development, implementation, and interpretation.

24.3 Context

24.3.1 Data Description

The field of exercise science (like many other areas of statistical application) has exploratory variables in need of distributional classification. The data used to evaluate the ESC approach consists of exercise test scores from an estimate of leg muscular power developed for this project. The test may be formally described as an isotonic knee extension mechanical power test (I-KEPT). The test is performed in a seated position on a knee extension machine with participants kicking out "as hard and as fast as possible" against 70% of their previously determined maximum resistance. I-KEPT scores were obtained from 24 participants ($n = 20$ to 40 I-KEPT scores per participant). I-KEPT scores are ratio level measurements (units are in watts [$\mathrm{KgNm/s^2}$]) that are continuously distributed, finite, and non-negative. I-KEPT is considered an exploratory variable because both the experimental test protocol and the computer program that generated the data have not been previously investigated. Given the exploratory status of the

variable, it is important to first determine if the data follow a known theoretical distribution before analyzing the data to ascertain meaning in various content areas (e.g., point estimation, prediction, or cross-validation). That is, are there any theoretical distributions which provide a "reasonable approximation" of the data at hand (Inman (1994))?

24.3.2 Distributions Considered

To limit the distribution scope of this project, only five distributional approximations are considered: the uniform distribution, $U(\min, \max)$, chosen to match the minimum and maximum of the data; the normal distribution, $N(\mu, \sigma)$, chosen to match the mean and standard deviation of the data; the Generalized Lambda Distribution (GLD) chosen to match the first four sample moments of the data, $\text{GLD}_M(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$; the GLD chosen to match certain percentiles of the data, $\text{GLD}_P(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$; and the GLD chosen to match the first four L-moments of the data, $\text{GLD}_L(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

The U and N distributions were selected primarily because their properties are (at least somewhat) familiar to most users of statistical methods; and at the outset, it was reasonable to suppose that at least some of the datasets may have come from either a U or N distribution. These assumptions were not based on prior findings in the literature (as both the experimental test protocol and the computer program that generated the data have not been previously investigated). Instead, the expectation regarding the N was based on the popular viewpoint that the normal distribution provides an adequate approximation of many different types of variables and as such the same may hold for this particular exploratory variable. The expectation regarding the U was based on the high degree of internal consistency within the test protocol (e.g., same resistance used, same verbal instructions given, same warm-up procedures followed, and the same rest periods allowed).

The GLD is not as universally well known as the U and N, but many statistical workers (in a variety of research settings) have noted its useful features for fitting distributions to data. The definition, properties, and fitting of a GLD through the use of moments and percentiles are described in Chapters 3 and 5, respectively. Key points we need follow. The GLD is a member of a family of statistical distributions known as the Extended Generalized Lambda Distribution (EGLD) which also includes the Generalized Beta Distribution (GBD). The GLD has parameters λ_1, λ_2, λ_3, and λ_4 controlling location, scale, skewness, and kurtosis, respectively. Like any continuous probability distribution, the GLD may be specified equally well in terms of distribution functions (d.f.), probability density functions (p.d.f.), and percentile functions (p.f.). The $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is most easily specified in terms of its p.f.:

$$Q(y) = Q(y;\, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \tag{24.3.1}$$

where $0 \leq y \leq 1$. However, it is relatively easy to find the p.d.f. of the GLD via:

$$f(x) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} - (1-y)^{\lambda_4 - 1}}, \quad \text{at } x = Q(y). \tag{24.3.2}$$

The GLD is extremely flexible and therefore capable of approximating many well-known distributions. The GLD has been described as "... very rich in variety of density and tail shapes. It contains unimodal, U-shaped, J-shaped, and monotone p.d.f.s. These can be symmetric or asymmetric and their tails can be smooth, abrupt, or truncated ..." (Freimer, Kollia, Mudholkar, and Lin (1988)). We postulate that the GLD's inherent versatility makes it an efficient alternative to the direct evaluation of a large number of distributions.

24.4 The Method

24.4.1 Eight Pieces of Evidence

The results of a battery of statistical evaluations involving both quantitative and graphical methods were used to determine the G-O-F of the five different distributional approximations to the data. Specifically, this combined evidential support was composed of eight different pieces of evidence: (1) rejection or failure to reject via the χ^2 test; (2) rejection or failure to reject via the Kolmogorov-Smirnov (K-S) test; (3) adequacy of distribution support for the spread of the data; (4) appropriateness of distribution support for the data; (5) visual match of p.d.f. to the body of the histogram; (6) visual match of p.d.f. to the tails of the histogram; (7) visual match of d.f. to the body of the e.d.f.; and (8) visual match of d.f. to the tails of the e.d.f.

24.4.2 The Evidential Support Continuum (ESC) Method

The basic idea of the Evidential Support Continuum or ESC is that rather than presenting the results of individual quantitative and graphical evaluations as separate entities, all the available evidence related to each distribution under consideration is compiled and collectively expressed as a sum. For example, if the χ^2 test fails to reject, this constitutes one piece of evidence in support of that distribution. Similarly, if the K-S test also fails to reject, this constitutes a second piece of supportive evidence and so forth. This summative value of total evidential support (associated with each distribution under consideration) ranges from 0 to 8 and is represented as a single point on the ESC. There is one ESC for every participant dataset permitting a simultaneous G-O-F assessment of the U, N, GLD_M, GLD_P, and GLD_L distributional approximations. A brief description of each of the eight pieces of evidence, two of which are quantitative and six of which are graphical, follows.

24.4.3 Quantitative Evaluations

24.4.3.1 χ^2 Test with Accurate p-Values via Simulation

Overview. The χ^2 test, as developed by Pearson in 1900, can be defined briefly. Let f denote the observed frequency of values from a particular range of the data and let e denote the expected frequency of occurrence from some theoretical distribution (in this case the U, N, GLD_M, GLD_P, or GLD_L). The value of the test statistic is based on deviations between f and e and is defined as:

$$Q = \sum_{i=1}^{r} \frac{(f_i - e_i)^2}{e_i}, \tag{24.4.1}$$

where r is the number of ranges (i.e., classes) and f_i and e_i are, respectively, the observed and expected frequencies for each of the r classes. From (24.4.1), it is apparent that the test only allows for one-sided hypothesis testing as all deviations are squared. It also requires grouping the data into classes (unless groups are pre-formed) prior to the G-O-F assessment. The following guidelines were followed for grouping the data into classes:

1. If expected frequencies ($f_i = np_i$) ≥ 5.0 for all r classes except one, then that class has $np_i \geq 0.05$. If $np_i \geq 5.0$ for all r classes except two, then those classes for which the condition does not hold have $np_i \geq 1.0$ (Cochran (1952)).

2. $n/r \geq 5.0$ (Haber (1980)).

3. If $r \geq 3$ and k = the number of classes for which $np_i < 5.0$ then the minimum np_i can be as small as $5k/r$ (Williams (1950)).

Testing via χ^2 distributional approximation. After the data have been grouped into classes, we can start the process of G-O-F assessment via the χ^2 test. Perhaps the most common method of performing the χ^2 test is to begin with the assumption that the distribution of the test statistic Q can be approximated by the χ^2 distribution with a single parameter known as **degrees of freedom (df)**. Typically, $df = r - p - 1$, where r is the number of classes formed and p is the number of parameters in the distribution evaluated (e.g., $p = 2$ for U, $p = 2$ for N, and $p = 4$ for GLD_M, GLD_P, and GLD_L) (Gibbons (1997)). The problem with performing the χ^2 test in the manner just described is twofold. First, the sampling distribution of the test statistic Q is at best only approximately χ^2 distributed with $r - p - 1$ df. In other words, this is a best-case scenario when testing conditions are ideal – e.g., large samples; $df > 0$; and criteria 1), 2), and 3) are all satisfied. Second, when we use the χ^2 distribution to approximate the distribution of the test statistic Q, we have no way of knowing the accuracy of the p-values associated with this approximation.

Let us now consider an example of testing via χ^2 distributional approximation. For Participant #3, we have a sample size of $n = 29$. To satisfy our criterion of $n/r \geq 5.0$, we must have (at most) five classes (i.e., $r \leq 5$). If $r = 5$ then $n/r \geq 5.0$ because $29/5 = 5.8$. However, at $r = 5$, we cannot satisfy our other test criterion of $df > 0$ because $df = 5-4-1 = 0$ (recall that $df = r-p-1$). This is the exact type of situation in which χ^2 testing via simulation becomes important — i.e., where some or all of our testing criteria cannot be met. For the sake of comparison (to results obtained via simulation), what follows are the results of what would happen if we proceeded with testing via χ^2 distributional approximation. Our observed value for the test statistic Q is $Q_{obs} = 10.3921$. As critical values for Q (i.e., Q_{crt}) at $df = 0$ do not exist, we use $df = 1$ instead. Using the χ^2 distribution and taking $df = 1$, it would seem that our result of $Q_{obs} = 10.3921$ is rather extreme or unusual. If the test results were valid (which they are not), they would imply significance beyond the 0.05 level as $Q_{crt}(0.95, 1) = 3.84$. A result of $Q_{obs} = 10.3921$ would correspond to a p-value between 0.01 and 0.001 — i.e., $p < 0.01$ ($Q_{crt}(0.99, 1) = 6.64$) and $p > 0.001$ ($Q_{crt}(0.999, 1) = 10.83$).

Testing via simulation. A much better alternative, and the one that we propose in this chapter, is to use simulation to provide the distribution of Q. In this case, we neither make assumptions about the distribution of Q nor do we need to use an approximation of Q (which may be a good, fair, or poor approximation). Rather, the true null distribution of Q is revealed through the process of simulation. Another advantage of the simulation method (i.e., over approximating the distribution of Q via the χ^2 distribution) is that the error associated with the process may be calculated quite easily. Specifically, we can say with 95% confidence that the error in the simulation p-value is equal to $\pm 1/\sqrt{n}$, where n is the number of simulations performed (Dudewicz and van der Meulen (1984)). For these reasons, the simulation method is appropriate to use at any time (i.e., under all different sorts of testing conditions) and it should generally be regarded as the preferred (or de facto) method of χ^2 testing.

For simulating the true null distribution of Q, the following procedures were used (Note: for ease of illustration, we have used only $n = 100$ simulations. In practice, we recommend $n = 10,000$ simulations for the reasons already discussed.).

1. Begin by selecting a good Random Number Generator (RNG) and then generate as many independent uniform random variables on (0,1) as needed. For this chapter, we used the RNG named URN37 (proposed by Dudewicz, Karian, and Marshall (1985)). Details of the quality testing results for URN37 can be found in Karian and Dudewicz (1999). The Maple program used to generate independent uniform random variables on (0,1) via URN37 can be found in Section A.8 of the appendix at the end of this chapter. For example, to conduct χ^2 testing for Participant #3 ($n = 29$), we simulated 100 samples each of size $n = 29$; and so, we used URN37 to generate a total

of 2,900 numbers. All 2,900 numbers have values ranging from 0 to 1 as these numbers represent our probabilities or y_js.

2. Fit a GLD to the original dataset (e.g., Participant #3) using one of several methods described in this chapter and also elsewhere in the Handbook. From the fitted GLD, generate the first simulated dataset $x_1, x_2, \ldots, x_{29}$ by substituting the relevant GLD parameters into the formula for the GLD percentile function (p.f.). For example, with Participant #3 we have the following fit: GLD(122.4359, −0.0017, −0.0063, −0.0112). Using our y_js from step 1 and then substituting these lambda values into (24.3.1) gives us
$$122.4359 + \frac{(y^{-0.0063}(1-y)^{-0.0112}}{-0.0017},$$
which produces a sequence of values for $x_1, x_2, \ldots, x_{29}$. At this point, we have our first simulated dataset; and we only need to repeat these same procedures to obtain all 100 simulated datasets.

3. Once all of the simulated datasets have been obtained, use the formula (24.4.1) to calculate a test statistic Q for each of the simulated datasets. For example, Participant #3's test statistic values for $Q_i, Q_2, \ldots, Q_{100}$ are given in Section A.7 of the appendix at the end of this chapter. To estimate the p value for our Q_{crt} that was obtained from the original dataset, we needed to count the number of $Q_i, Q_2, \ldots, Q_{100}$ that have a value larger than Q_{crt} and then divide this number by 100. For $Q_{crt} = 10.39$, we had 7 cases of Q_i values larger than 10.39 (see Section A.7 of the appendix at the end of this chapter) and so we can estimate $P(Q > 10.39) = 7/100 = 0.07$.

4. To calculate the error associated with our simulation results of $P(Q > 10.39) = 0.07$, we recall the following formula: $\pm 1/\sqrt{n}$, where n is the number of simulations performed. As noted previously, we used only $n = 100$ simulations (i.e., for ease of illustration). With only $n = 100$ simulations, the error is quite large at ± 0.10 because $\pm 1/\sqrt{100} = \pm 0.1$. In other words, we can say with 95% confidence that the true p-value associated with our test statistic Q is between −0.03 and 0.17. Obviously, this level of precision (i.e., knowing the p-value within 0.10) is insufficient in most cases. At $n = 1,000$ simulations the error is considerably smaller at ± 0.03 and in this case we obtain $P(Q > 10.39) = 0.047$ after $n = 1,000$ simulations, allowing us to say with 95% confidence that the true p-value associated with our test statistic Q is between 0.017 and 0.077. In practice, we recommend using $n = 10,000$ simulations where the error is only 0.01 ($\pm 1/\sqrt{10000} = 0.01$). For example, $P(Q > 10.39) = 0.0390$ after $n = 10,000$ simulations, allowing us to say with 95% confidence that the true p-value associated with our test statistic Q is between 0.049 and 0.059. At $n = 10,000$ simulations, the level of accuracy associated with our test statistic p-value (i.e., ± 0.01)

exceeds that of many comparable test procedures and is certainly better than approximating the distribution of Q via the χ^2 distribution in which no accuracy results have been previously reported.

Overall, some advantages of the χ^2 test over other G-O-F tests (e.g., the K-S test) are that it is intuitive, easy to use, and generalizable though subject to the arbitrariness of the classes selected. For a good comparison of the advantages and disadvantages of the χ^2 test vs. the K-S test see Gibbons (1997).

24.4.3.2 The K-S Test

The K-S test was introduced in 1933 by Kolmogorov using earlier results obtained by Smirnov (Upton and Cook (2002)). The test statistic, D, is obtained from taking the absolute value of the largest difference between the empirical distribution function or e.d.f. (i.e., based on observations from the actual data) and the distribution function or d.f. (i.e., based on some theoretical or hypothesized distribution). The test statistic D can be defined by

$$D = \sup_x |S_N(x) - F_0(x)|, \tag{24.4.2}$$

where $S_N(x)$ represents the e.d.f., $F_0(x)$ is the d.f., and the supremum yields the largest $|S_N(x) - F_0(x)|$ for all possible x. As $F_0(x)$ is a continuous function, it is not sufficient to look at $|S_N(x) - F_0(x)|$ for only the observed values of x, but rather in the neighborhood (i.e., at jump points in $S_N(x)$ and other points near by). Of importance is that the test assesses G-O-F of observations in the dataset via each individual data point (not via grouped classes as in the χ^2 test). However, it may be lacking in power (e.g., fail to reject if consistent deviations abound but none are unduly large) (Karian and Dudewicz (2000)).

Unlike the χ^2 statistic, the exact sampling distribution of the K-S statistic is known when the null hypothesis is true. However, in many testing situations (as was the case with this particular project) the parameters of the theoretical distributions to be evaluated are not specified and are therefore estimated from the data. Ideally, the test should only be performed with fully specified distributions and any results obtained with unspecified distributions should be interpreted with caution or not taken on full merit. An extension of the test using modified critical values for D has been developed for some specific distributions (Upton and Cook (2002)), but the only applicable test for this investigation is Lilliefors' test (i.e., N with unknown parameters) (Lilliefors (1967)). The K-S test was applied in the evaluation of all five distributional approximations (even though the parameters of the U, GLD_M, GLD_P, and GLD_L are unspecified), but as Lilliefors' test was used for the evaluation of the N this particular distributional approximation was more heavily penalized than the others. That is, compared to the other K-S tests, there was more power to reject the hypothesis involving the N.

24.4.4 Graphical Evaluations

Prior to the graphical evaluations, it was necessary to first construct theoretical distribution approximations of the datasets in the form of d.f.s and p.d.f.s. The following general properties hold when the method is appropriate: for the d.f., the curve starts at a height near 0 at the left (for very small values of x) and increases continuously toward a value of 1 (as x gets large); and for the p.d.f., the curve is non-negative, the area under the whole curve is 1, and areas under the curve between values give the probability of that range of values (Karian and Dudewicz (2000)). In all cases, the theoretical distribution parameters are unspecified and their values were estimated from the data. However, the specific process of constructing d.f.s and p.d.f.s via estimated parameters differs with the distribution considered.

24.4.4.1 The Uniform and Normal Distributions

For the uniform, dataset minimum and maximum values were the parameter estimates used to construct the d.f. and p.d.f. approximating functions; data mean and standard deviation values were used for the normal.

24.4.4.2 GLD_M and GLD_P

For the GLD, three different estimation methods were used to construct the d.f. and p.d.f. approximating functions: Method of Moments (MOM) approximation designated by GLD_M; Method of Percentiles (MOP) approximation designated by GLD_P; and fitting via L-moments designated by GLD_L. The details of how to obtain the GLD_M, GLD_P, and GLD_L approximations are given in Chapters 3, 5, and 6, but are summarized here for the Handbook user's ease of use. For the MOM, the dataset's first four moments, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$, (mean, variance, skewness, and kurtosis) were defined by

$$\hat{\alpha}_1 = \overline{X} = \sum_{i=1}^{n} X_i/n, \tag{24.4.3}$$

$$\hat{\alpha}_2 = \hat{\sigma}^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2/n, \tag{24.4.4}$$

$$\hat{\alpha}_3 = \sum_{i=1}^{n} (X_i - \overline{X})^3/(n\hat{\sigma}^3), \tag{24.4.5}$$

$$\hat{\alpha}_4 = \sum_{i=1}^{n} (X_i - \overline{X})^4/(n\hat{\sigma}^4). \tag{24.4.6}$$

Next, the population moments, α_1, α_2, α_3, α_4, were set equal to the dataset moments, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$, (also known as method-of-moments estimators) and the resulting equations were solved for the GLD_M parameters λ_1, λ_2, λ_3, λ_4. The

expressions for $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$ are (see Theorem 2.1.11 of Karian and Dudewicz (2000))

$$\alpha_1 = \mu = E(X) = \lambda_1 + \frac{A}{\lambda_2}, \tag{24.4.7}$$

$$\alpha_2 = \sigma^2 = E\left[(X-\mu)^2\right] = \frac{B - A^2}{\lambda_2^2}, \tag{24.4.8}$$

$$\alpha_3 = E(X - E(X))^3/\sigma^3 = \frac{C - 3AB + 2A^3}{\lambda_2^3 \sigma^3}, \tag{24.4.9}$$

$$\alpha_4 = E(X - E(X))^4/\sigma^4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \sigma^4}, \tag{24.4.10}$$

where

$$A = \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4}, \tag{24.4.11}$$

$$B = \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3,\, 1+\lambda_4), \tag{24.4.12}$$

$$C = \frac{1}{1+3\lambda_3} - \frac{1}{1+3\lambda_4} - 3\beta(1+2\lambda_3,\, 1+\lambda_4) + 3\beta(1+\lambda_3,\, 1+2\lambda_4), \tag{24.4.13}$$

$$D = \frac{1}{1+4\lambda_3} + \frac{1}{1+4\lambda_4} - 4\beta(1+3\lambda_3,\, 1+\lambda_4) + 6\beta(1+2\lambda_3,\, 1+2\lambda_4) - 4\beta(1+\lambda_3,\, 1+3\lambda_4). \tag{24.4.14}$$

Solving the set of equations $\{\alpha_i = \hat{\alpha}_4\}$ for $i = 1, 2, 3, 4$, is sufficiently complex to prevent analytical solutions, therefore numerical methods involving computer search algorithms were used to obtain numerical solutions ("Algorithm GLD-M" and the Maple procedure "FindLambdasM" (Karian and Dudewicz (2000), pp. 63, 292–293)).

For the MOP, the dataset's rho sample statistics, (median, inter-decile range, left-right tail-weight ratio, and tail-weight factor), were calculated:

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{24.4.15}$$

$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{24.4.16}$$

$$\hat{\rho}_3 = \frac{\hat{\pi}_{0.5} - \hat{\pi}_u}{\hat{\pi}_{1-u} - \hat{\pi}_{0.5}}, \tag{24.4.17}$$

$$\hat{\rho}_4 = \frac{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}{\hat{\rho}_2}, \tag{24.4.18}$$

where $\hat{\pi}_p$ is the $(100p)$th percentile of the data and $u = 0.10$. And, as with MOM,

population counterparts of $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_4$, are determined:

$$\rho_1 = Q(0.5) = \lambda_1 + \frac{0.5^{\lambda_3} - 0.5^{\lambda_4}}{\lambda_2}, \tag{24.4.19}$$

$$\rho_2 = Q(0.9) - Q(0.1) = \frac{0.9^{\lambda_3} - 0.1^{\lambda_4} + 0.9^{\lambda_4} - 0.1^{\lambda_3}}{\lambda_2}, \tag{24.4.20}$$

$$\rho_3 = \frac{Q(0.5) - Q(0.1)}{Q(0.9) - Q(0.5)} = \frac{0.9^{\lambda_4} - 0.1^{\lambda_3} + 0.5^{\lambda_3} - 0.5^{\lambda_4}}{0.9^{\lambda_3} - 0.1^{\lambda_4} + 0.5^{\lambda_4} - 0.5^{\lambda_3}}, \tag{24.4.21}$$

$$\rho_4 = \frac{Q(0.75) - Q(0.25)}{\rho_2} = \frac{0.75^{\lambda_3} - 0.25^{\lambda_4} + 0.75^{\lambda_4} - 0.25^{\lambda_3}}{0.9^{\lambda_3} - 0.1^{\lambda_4} + 0.9^{\lambda_4} - 0.1^{\lambda_3}}. \tag{24.4.22}$$

The GLD parameters λ_1, λ_2, λ_3, λ_4 are determined by solving the equations $\{\rho_i = \hat{\rho}_4\}$ for $i = 1, 2, 3, 4$. Solving these equations is sufficiently complex to prevent analytical solutions, therefore numerical methods involving computer search algorithms were used to obtain numerical solutions ("Algorithm GLD-P" and the Maple procedure "FindLambdasP" (Karian and Dudewicz (2000), pp. 166, 296–298).

24.4.4.3 The GLD$_L$

L-moments are defined as linear combinations of expectations of order statistics. They are useful in fitting distributions because they specify location, scale, and shape (symmetry and kurtosis) attributes like ordinary moments do. Their use is detailed in Chapter 6.

24.4.4.4 Spread of Data vs. Distribution Support

The spread of data is the minimum and maximum values in the dataset, whereas the support comprises the minimum and maximum values of a particular theoretical distribution function. Two important aspects of the spread of data versus support were considered: (1) adequacy; and (2) appropriateness. "Adequacy" is concerned with whether or not the distribution's support is adequate to cover or "explain" the spread of the data. For example, support is adequate if and only if the support covers the data spread. A support is considered "appropriate" if and only if reasonable values are included and unreasonable ones are excluded. Support is not appropriate if it includes very high positive or any negative values when such values are not theoretically possible for the type of data under consideration.

24.4.4.5 Histogram of Data vs. Distribution p.d.f.s

Dataset histograms were constructed using the same class boundaries as those used for the χ^2 test. In evaluating this graphical evidence, the following subjec-

tive considerations were made: (1) Is the p.d.f. well-matched visually with the main body of the data histogram? and (2) Are the tails of the p.d.f. well-matched visually with the tails of the data histogram? These graphical evaluations were conducted systematically; the following quotation addresses this issue of subjectivity:

> The eyeball test is not a license to follow a personal whim, and it is important to guard against any lack of rigor in so important a decision as choice of distributional model for the phenomenon under study ... we should always be aware that there is subjectivity to the analysis. (Karian and Dudewicz (2000), p. 96)

24.4.4.6 Data e.d.f. vs. Distribution d.f.s

A similar process to that used for dataset histograms vs. p.d.f.s was followed. Specifically, two considerations were made: (1) Is the d.f. well-matched visually for the main body of the e.d.f.? and (2) Are the tails of the d.f. well-matched visually for the tails of the e.d.f.?

24.5 Results

Recall from the previous section that the basic idea of the Evidential Support Continuum or ESC is that rather than presenting the results of individual quantitative and graphical evaluations as separate entities, all the available evidence related to each distribution under consideration is compiled and collectively expressed as a sum. For example, if the χ^2 test fails to reject, this constitutes one piece of evidence in support of that distribution. Similarly, if the K-S test also fails to reject, this constitutes a second piece of supportive evidence and so forth. This summative value of total evidential support (associated with each distribution under consideration) ranges from 0 to 8 and is represented as a single point on the ESC.

For brevity, only the results of one example dataset (i.e., Participant #25) are presented in detail. However, the following material is presented in the Appendix at the end of this chapter for all 24 datasets: ESC diagrams (Section A.1 through A.3); raw data (Section A.4); first four sample moments αs (Section A.5); and the ρ sample statistics (Section A.6). For this dataset, we will consider eight different fits: the uniform, normal, two GLD fits via M.O.M. (GLD_{M1} and GLD_{M2}), two GLD fits via M.O.P. (GLD_{P1} and GLD_{P2}), and two GLD fits via L-moments (GLD_{L1} and GLD_{L2}). Before considering the total evidential support (for each of these eight different fits) via an ESC diagram, let us take at look at the results of the eight pieces of individual evidence and see how they are combined to produce an ESC diagram.

24.5.1 The χ^2 Test

In Section 24.4.3.1, we have demonstrated for a different participant dataset (i.e., Participant #3) how one can use simulation to find p-values even when the criteria for "approximately chi-square" do not hold. We also noted that there is no criterion for accuracy of the χ^2 approximation even when it is customarily used, while the simulation p-value estimate (with $n = 10,000$ trials) gives a value within 0.01 of the true p-value. For details of the simulation results please see Section A.7 of the appendix at the end of this chapter.

Here we do not present the simulation case. Rather, we show the results of χ^2 testing via the χ^2 distributional approximation. For Participant #25, it is reasonable to assume that our test statistic Q is "approximately chi-square" distributed because all of the necessary testing criteria have been met. Full details on the testing criteria are explained in Section 24.4.3.1. Essentially, we wanted to choose our class boundaries so that these two situations were maintained: (1) $n/r \geq 5.0$; and (2) $df \geq 1$. (Note: we could have used the simulation method for Participant #25 as well. It is never incorrect to do so; we simply elected to show χ^2 testing via the χ^2 distributional approximation method out of completeness).

The following class boundaries were used in the χ^2 testing procedures for all eight different fits: 160, 180, 200, 220, and 240. Five class boundaries produces six different classes and so $r = 6$. The df and the associated Q_{crt}, depend on the distributional approximation being considered. For all of the GLD fits, $df = 1$ and $Q_{crt}(0.95, 1) = 3.841$ because the GLD has four parameters ($df = r - p - 1 = 6 - 4 - 1 = 1$). For the uniform and normal fits, $df = 3$ and $Qcrt(0.95, 3) = 7.815$ because both the uniform and the normal have only two parameters.

Overall results of the χ^2 testing are presented in the second column of Table 24.5–1. Of the eight different fits, only GLD_{P1} failed to reject (or "accepted") the null hypothesis ($Q_{obs} = 2.888 < Q_{crt}(0.95, 1) = 3.841$). Remember, in the case of the χ^2 test, failure to reject the null hypothesis (H_0) constitutes evidence in favor of a fit being reasonable, acceptable, etc. The seven other fits all rejected H_0 ($Qobs > Q_{crt}$) implying that these seven fits are unreasonable/unacceptable.

Table 24.5–1. Summative table of evidential support for Participant 25 ($n = 40$).

Theoret. distrib.	Quantitative evaluations		Distribution support to spread of data		Fit of p.d.f. to histogram		Fit of d.f. to e.d.f.		Tally of evidence	
	χ^2	K-S	adq.	app.	body	tails	body	tailes	for	against
Uniform	reject	reject	yes	yes	no	no	no	no	2	6
Normal	accept	accept	yes	no	no	no	yes	no	4	4
GLD_{M1}	reject	reject	yes	no	no	no	no	no	1	7
GLD_{M2}	reject	accept	yes	no	yes	no	yes	no	4	4
GLD_{P1}	accept	accept	no	no	no	no	yes	no	3	5
GLD_{P2}	reject	accept	yes	yes	yes	yes	yes	yes	7	1
GLD_{L1}	reject	accept	no	no	no	no	no	no	1	7
GLD_{L2}	reject	accept	yes	no	yes	no	yes	no	4	4

24.5.2 The K-S Test

Full details on the K-S testing criteria are explained in Section 24.4.3.2. Essentially, the test statistic, D, is obtained from taking the absolute value of the largest difference between the empirical distribution function or e.d.f. (i.e., based on observations from the actual data) and the distribution function or d.f. (i.e., based on some theoretical or hypothesized distribution).

For Participant #25, the K-S testing results run almost completely contrary to the χ^2 testing results. K-S test results are presented in the third column of Table 24.5–1. Of the eight different fits, only the uniform and GLD_{M1} approximations rejected H_0 ($D = 2.52$; $p < 0.05$). The six other fits all failed to reject (or "accepted.") H_0. And so, unlike the χ^2 test (which suggested a distinctive "lack of fit" for 6 of the 8 fits), the K-S test results suggest that 6 of the 8 fits may indeed be reasonable/acceptable.

This is the type of situation so often encountered in research involving statistical methods (generally) and G-O-F (specifically) — i.e., we are faced with the need to decide what to do when we encounter results from one quantitative test that contradict the results from another quantitative test. This is also precisely the type of situation that the ESC was designed for. Let us now consider some additional information that will hopefully help us in our task of deciding what to do.

24.5.3 Distribution Support to Spread of Data (Adequacy and Appropriateness

More detailed considerations of distribution support to the spread of data are given in Section 24.4.4.4. Basically, the spread of data is the minimum and maximum values in the dataset, whereas the support comprises the minimum and maximum values of a particular theoretical distribution function. Two important aspects of the spread of data versus support (that are presented in Table 24.5–1 and included in ESC diagrams) are **adequacy** and **appropriateness**[1]. The spread of data versus support may be evaluated numerically (e.g., by having a computer program display the minimum and maximum values of a particular function and then comparing those numbers to the minimum and maximum values of a dataset) or visually (e.g., by inspecting a graph such as a histogram of the data vs. a probability density function (p.d.f.) of the distribution). Figure 24.5–2

[1]For some, the ultimate determination on appropriateness of distribution support is a matter of considering theoretical and/or practical issues related to fitting (Gilchrist (2000); Karian and Dudewicz (2000)). For example, as support for N is always $-\infty$ to $+\infty$, it may be stated formally that the support for N is always adequate as support $\geq$ spread, but never appropriate since $P(X < 0) > 0$ when we know that X cannot be negative (Karian and Dudewicz (2000)). There is also the practical matter of support not being appropriate when it includes either very high positive values or negative values, neither of which are physically possible for this type of data (i.e., power of the leg musculature measured in KgNm/s^2).

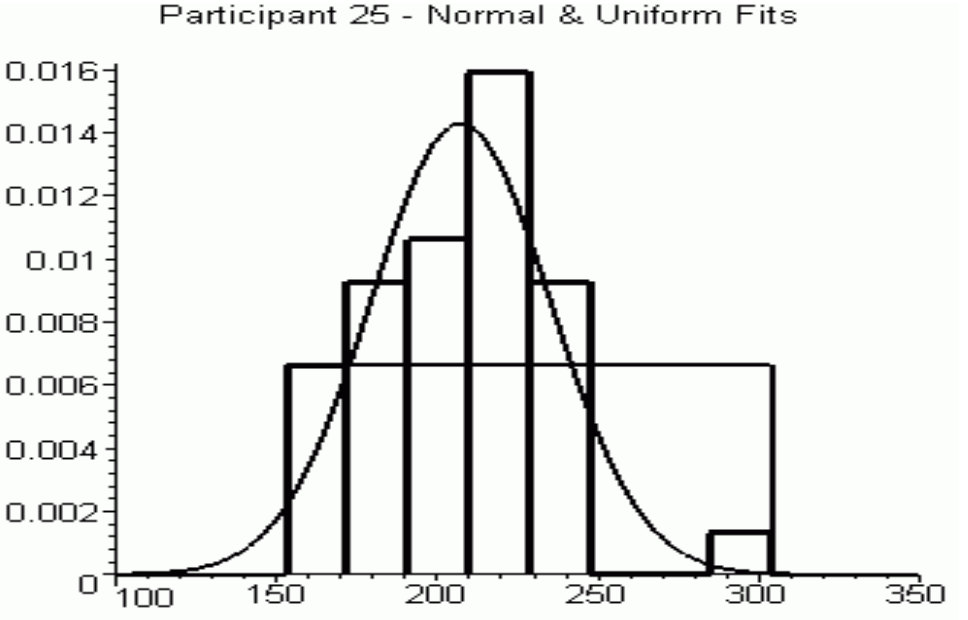


Figure 24.5–2. Dataset histogram along with p.d.f.s for $U(153.13, 303.90)$ and $N(207.2350, 27.9198)$.

shows a histogram of the data for Participant #25 and the p.d.f.s for the uniform and normal distribution approximations.

For the uniform, the support is always both adequate and appropriate as the values for the dataset minimum and maximum are the same as the two distributional parameters (i.e., a = dataset minimum; and b = dataset maximum) used to develop the function. This can be confirmed visually by inspecting the rectangular-shaped function in Figure 24.5–2 and noticing how it matches the end points of the histogram exactly. A synopsis is presented in the fourth and fifth column of Table 24.5–1.

For the various GLD fits, the results are presented in Figures 24.5–3 through 24.5–5 as well as summarized in the fourth and fifth column of Table 24.5–1. According to this piece of evidence, the best GLD fit is $\text{GLD}_{P2}(229.5922, 0.0131, 24.1675, 2.0130)$ because (like the uniform approximation) its distributional support to spread of data is not only adequate but also appropriate. The two moment-based GLD fits (i.e., $\text{GLD}_{M1}(214.0803, -0.009, -0.0110, -0.0174)$ and $\text{GLD}_{M2}(210.0893, -0.0009, -0.0161, -0.0133)$) resemble the normal distribution approximation in that the distribution support is adequate but not appropriate for the data. And the two worst fits overall (at least according to this piece of evidence) are $\text{GLD}_{P1}(195.6542, 0.0222, 1.3107, 3.8676)$ and $\text{GLD}_{L1}(206.4146, 0.0157, 4.5491, 4.9785)$ where the distribution support is both inadequate and inappropriate for the spread of the data.

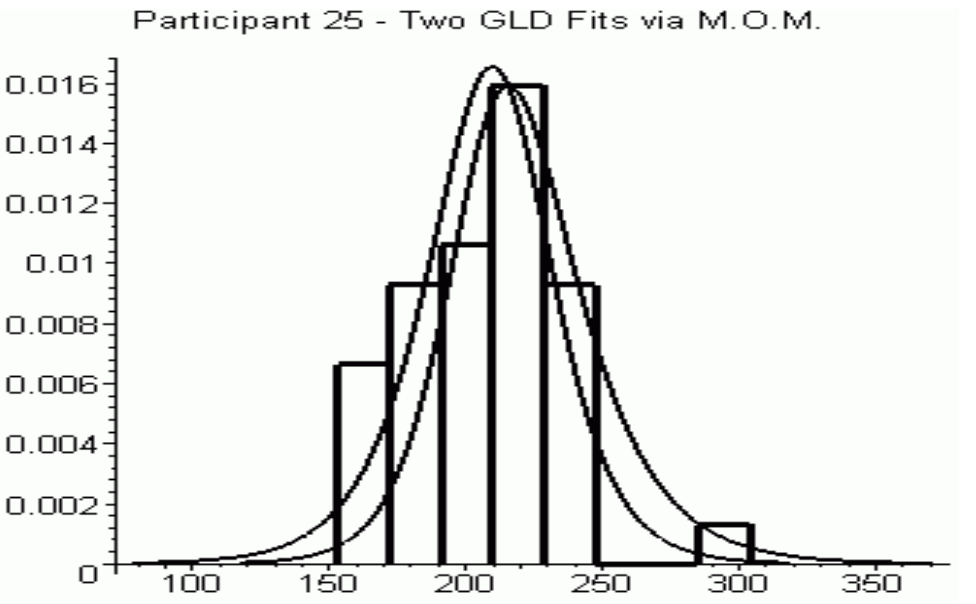


Figure 24.5–3. Dataset histogram along with p.d.f.s for two GLD fits: $\text{GLD}_{M1}(214.0803, -0.0009, -0.0110, -0.0174)$ and $\text{GLD}_{M2}(210.0893, -0.001, -0.0161, -0.0133)$.

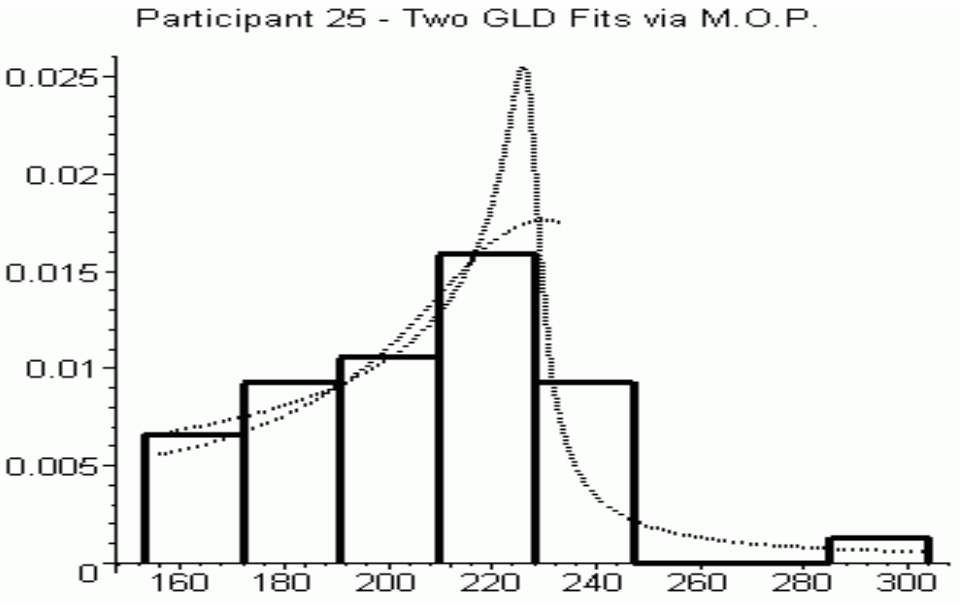


Figure 24.5–4. Dataset histogram along with p.d.f.s for two GLD fits: $\text{GLD}_{P1}(195.6542, 0.0222, 1.3107, 3.8676)$ and $\text{GLD}_{P2}(229.5922, 0.0131, 24.1675, 2.0130)$.

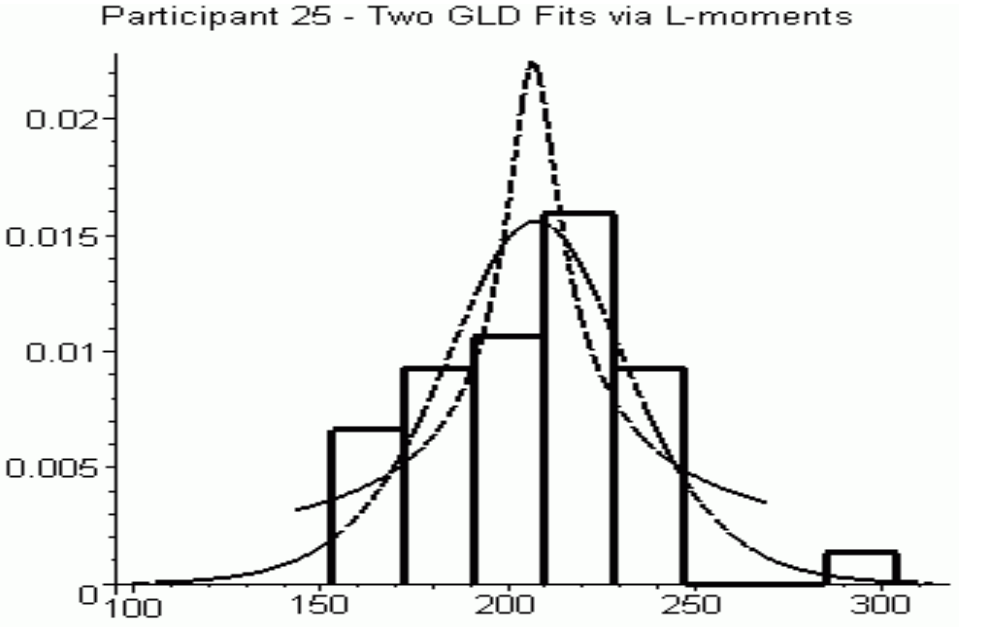


Figure 24.5–5. Dataset histogram along with p.d.f.s for two GLD fits: $\text{GLD}_{L1}(206.4146, 0.0157, 4.5491, 4.9785)$ and $\text{GLD}_{L2}(207.5276, 0.0025, 0.0425, 0.0417)$.

24.5.4 Fit of the Distribution p.d.f. to the Dataset Histogram (Main Body and Tails)

Dataset histograms were constructed using the same class boundaries as those used for the χ^2 test (Figures 24.5–2 through 24.5–5). In evaluating this graphical evidence, the following subjective considerations were made: (1) Is the p.d.f. well-matched visually with the main body of the data histogram? and (2) Are the tails of the p.d.f. well-matched visually with the tails of the data histogram? The findings are summarized in the sixth and seventh columns of Table 24.5–1.

The easiest fits to evaluate are the uniform and normal distributional approximations in Figure 24.5–2. Even with only a quick glance, it is apparent that neither the uniform nor the normal are well-matched visually with either the main body or tails of the data histogram.

Next, we consider the two fits obtained through the method of moments in Figure 24.5–3. $\text{GLD}_{M2}(210.0893, -0.0009, -0.0161, -0.0133)$ rises slightly higher and covers more of the left side of the distribution than $\text{GLD}_{M1}(214.0803, -0.0009, -0.0110, -0.0174)$. However, GLD_{M2} under-fits on the right tail. That is why Table 24.5–1 shows that "yes" GLD_{M2} fits in the body of the histogram, but "no" it does not fit very well in the tails of the histogram. GLD_{M1} is not well-matched

visually with either the body or tails of the histogram.

Figure 24.5–4 illustrates the two GLD fits via M.O.P. $\text{GLD}_{P1}(195.6542, 0.0222, 1.3107, 3.8676)$ is a very strange fit indeed. It seems to stop in midstream and therefore presents no coverage whatsoever for the entire right half of the dataset histogram. Clearly, GLD_{P1} is a case of two "no"s for our fifth and sixth pieces of evidence (i.e., not well-matched visually with either the body or tails of the histogram). In sharp contrast to GLD_{P1} we have the fit of $\text{GLD}_{P2}(229.5922, 0.0131, 24.1675, 2.0130)$. This fit seems to be the "best of the bunch." Other than a bit of under-fitting on the right side for one histogram interval, this curve is very well-matched visually for both the body and tails of the histogram (and remember from the last section that, other than the uniform, this fit was the only fit with support that was both adequate and appropriate).

Lastly, Figure 24.5–5 displays the two GLD fits obtained from L-moments. It is reminiscent of Figure 24.5–3 in that one fit (GLD_{L2}) is a much better fit than the other (GLD_{L1}). $\text{GLD}_{L1}(206.4146, 0.0157, 4.5491, 4.9785)$ rises up entirely too high at the center and also stops short of fitting all the data on the right side of the histogram. Therefore, GLD_{L1} is another case of two "no"s (i.e., not well-matched visually with either the body or tails of the histogram). Conversely, $\text{GLD}_{L2}(207.5276, 0.0025, 0.0425, 0.0417)$ is like GLD_{M2} in that it fits pretty well in the main body of the histogram and slightly under-fits on the right tail. And so, GLD_{L2} receives a "yes" for fit to the histogram body and a "no" for fit to the histogram tails (Table 24.5–1).

24.5.5 Fit of the Distribution c.d.f. to the Dataset e.d.f. (Main Body and Tails)

For our seventh and eighth piece of evidence related to the construction of ESC diagrams, we consider the following two questions: (1) Is the d.f. well-matched visually for the main body of the e.d.f.? and (2) Are the tails of the d.f. well-matched visually for the tails of the e.d.f.? Figures 24.5–6 through 24.5–9 contain graphs of the d.f.s and e.d.f.s.

Figure 24.5–6 compares the dataset e.d.f. for Participant #25 to both the uniform and normal d.f.s. This is a good example of how a different graph can present different information than another style of graph. Recall from the previous Section that when we examined the fit of the normal distribution p.d.f. to a histogram of the data, our impression was that it did not fit in either the body or the tails. Here, it is a different story. Figure 24.5–6 suggests that the normal d.f. fits the body of the e.d.f., but it does not fit the tails of the e.d.f. And merely a quick glance reveals that the uniform d.f. fits very poorly to both the body and tails of the e.d.f.

Figure 24.5–7 compares the two M.O.M.-based GLD fits. $\text{GLD}_{M2}(210.0893, -0.0009, -0.0161, -0.0133)$ is a much better fit than $\text{GLD}_{M1}(214.0803, -0.009, -0.0110, -0.0174)$. GLD_{M2} sticks pretty close to the main body of the e.d.f.

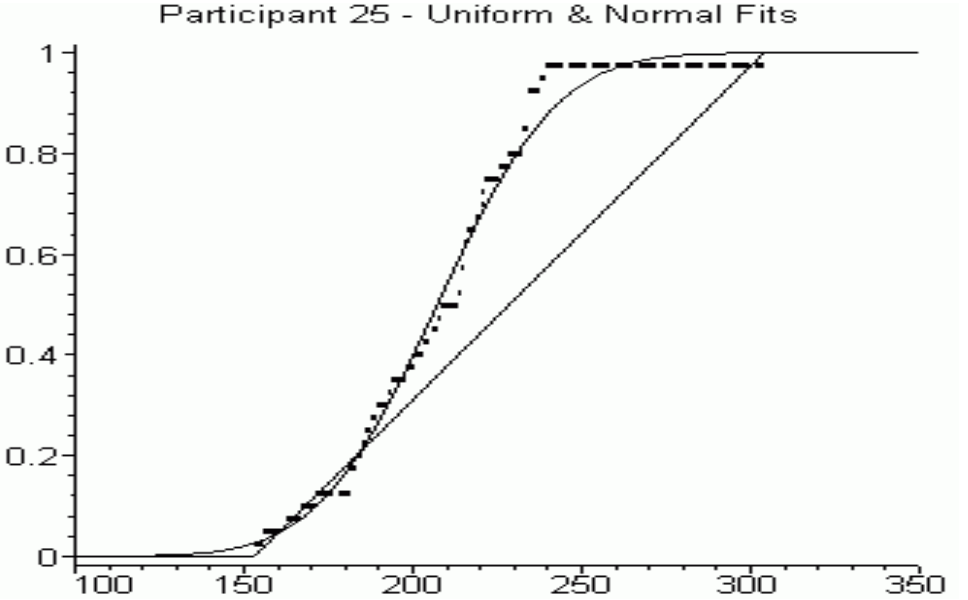


Figure 24.5–6. Dataset e.d.f. along with distribution d.f.s: $N(207.2350, 27.9198)$ and $U(153.13, 303.90)$.

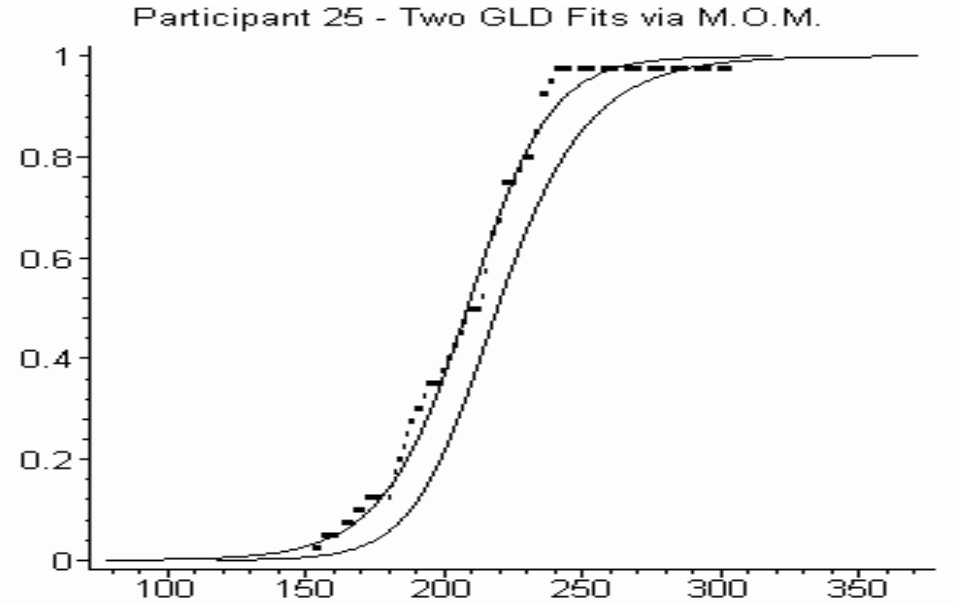


Figure 24.5–7. Dataset e.d.f. along with distribution d.f.s: $\text{GLD}_{M1}(214.0803, -0.009, -0.0110, -0.0174)$ and $\text{GLD})M2(210.0893, -0.0009, -0.0161, -0.0133)$.

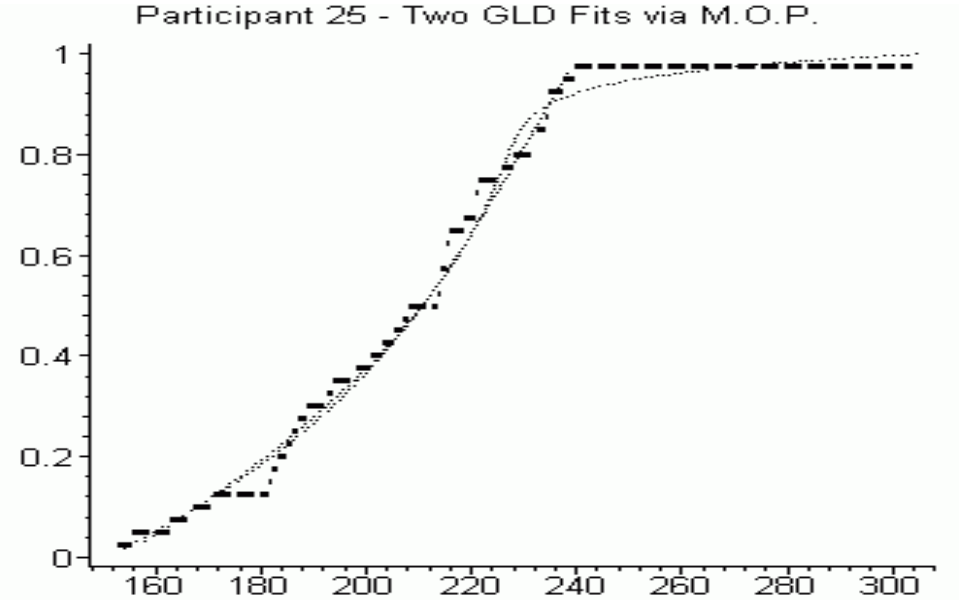


Figure 24.5–8. Dataset e.d.f. along with distribution d.f.s: $\mathrm{GLD}_{P1}(195.6542, 0.0222, 1.3107, 3.8676)$ and $\mathrm{GLD}_{P2}(229.5922, 0.0131, 24.1675, 2.0130)$.

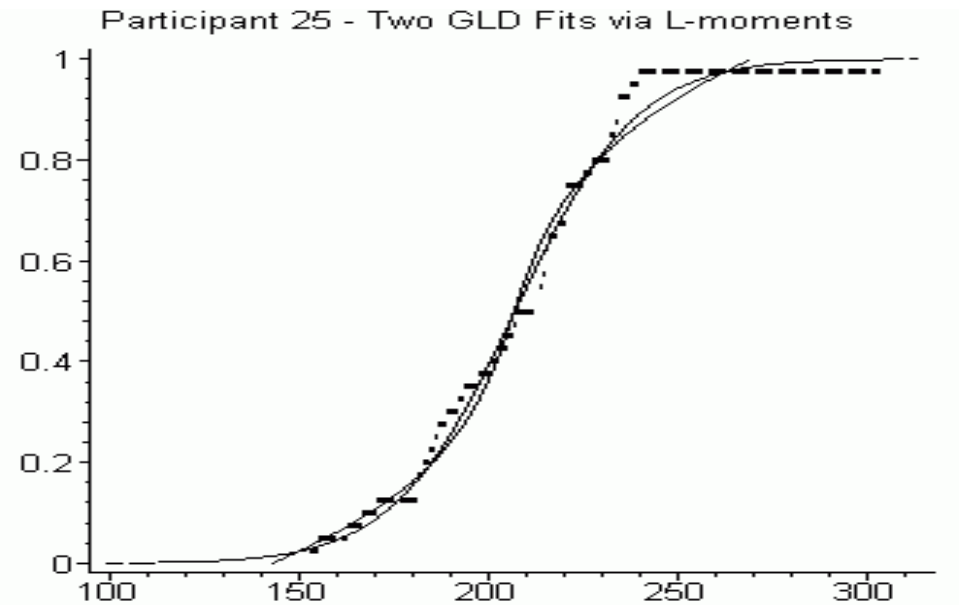


Figure 24.5–9. Dataset e.d.f. along with distribution d.f.s: $\mathrm{GLD}_{L1}(206.4146, 0.0157, 4.5491, 4.9785)$ and $\mathrm{GLD}_{L2}(207.5276, 0.0025, 0.0425, 0.0417)$.

throughout its fit; and it does a decent job of fitting at the upper tail. The main problem with the fit of GLD_{M2} is that it continues indefinitely at the lower tail where the data e.d.f. stops. In contrast, GLD_{M1} is shifted very far to the right of the e.d.f. throughout its fit in the main body; and it suffers from the same problem as GLD_{M2} in fitting of the lower tail.

The two d.f.s in Figure 24.5–8, representing the two M.O.P.-based fits, are very close together (and in some cases nearly indistinguishable), i.e., at least until the upper tail. At this point, the fit of $\text{GLD}_{P2}(229.5922, 0.0131, 24.1675, 2.0130)$ distinguishes itself from GLD_{P1}. GLD_{P2} continues to follow the dataset e.d.f. at the upper tail region whereas GLD_{P1} seems to simply stop. For this reason, Table 24.5–1 includes two "yes"s for GLD_{P2} (i.e., d.f. is well-matched visually for both the body and tails); and GLD_{P1} receives a "yes" for matching the body but a "no" for matching the tail.

Finally, Figure 25.4–9 compares the two GLD fits obtained via L-moments. Although particular aspects of the fit quality are quite different, Figure 24.5–9 is similar to Figure 24.5–7 in that one fit (GLD_{L2}) is a much better fit than the other (GLD_{L1}). $\text{GLD}_{L2}(207.5276, 0.0025, 0.0425, 0.0417)$ clings much closer to the main body of the e.d.f. than does $\text{GLD}_{L1}(206.4146, 0.0157, 4.5491, 4.9785)$. Also, GLD_{L1} offers a much poorer fit of the upper tail than GLD_{L2}.

24.5.6 Constructing the ESC Diagram

For many statistical practitioners, building a table such as Table 24.5–1 is a necessary first step in constructing ESC diagrams. Once the table has been completed, it is relatively easy to see how the ESC diagram takes shape. All that is necessary is to locate the numerical tally of evidence "for" (i.e., column 10); this value between 0 and 8 is what becomes the ESC score.

Based on the information contained in Table 24.5–1 (generally) and the ESC diagram (specifically) (Figure 24.5–10), three different categories of fit emerged for Participant #25: (a) a single very good fit (ESC = 7 for GLD_{P2}); (b) three average fits (ESC = 4 for the normal, GLD_{M2}, and GLD_{L2}); and (c) four poor fits (ESCs = 1 to 3 for the uniform, GLD_{M1}, GLD_{P1}, and GLD_{L1}).

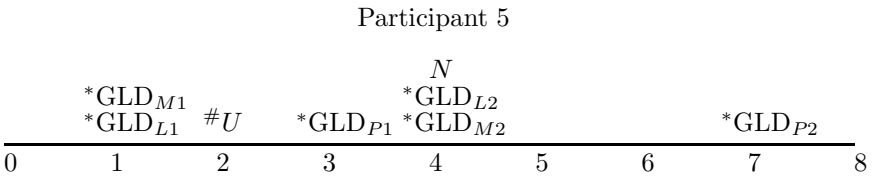


Figure 24.5–10. ESC diagram for Participant #25 ($^* = p \leq 0.05$ for χ^2 test; $^\# = p \leq 0.05$ for K-S test).

24.6 Conclusions

Many diverse statistical methods are used for fitting theoretical distributions to data (Lilliefors (1967); Gilchrist (2000); Karian and Dudewicz (2000)). The growing number of recognized theoretical distributions (Evans, Hastings, and Peacock (2000)), as well as more recent developments in building application-specific models (Gilchrist (2000)), makes the process of choosing, fitting, and determining G-O-F for different statistical models a rather complicated task (Karian and Dudewicz (2000)). Further complicating the methodological landscape is that the results obtained from different tests sometimes appear divergent or contradictory. One explanation for this phenomenon is that the statistical methods themselves (inherent in the testing procedures) differ among the various formal/quantitative tests as well as among different graphical/visual fitting methods. For example, the χ^2 statistic is based on differences in expected versus observed values for class-grouped data, whereas the K-S statistic is based on the absolute value of the single greatest difference in d.f. versus e.d.f. Similarly, although GLD fits via MOM and MOP have been shown to closely approximate many known theoretical distributions (e.g., for (0, 1) the maximal difference in support over 249 points was only 0.002 and 0.001, respectively for p.d.f. and d.f. approximations (Karian and Dudewicz (2000), pp. 67–68)), the fit for a particular dataset may be considerably less appropriate or even not at all acceptable.

It seems that what is sorely needed is a way to both organize and streamline this rather complicated process of G-O-F assessment. The ESC approach provides a way to combine diverse methodological techniques into a cohesive form thereby rendering a straightforward yet comprehensive judgment of G-O-F for several different theoretical distributions simultaneously. The following considerations represent some advantages to using ESCs for various G-O-F needs.

24.6.1 ESCs Address Conceptual and Theoretical Challenges to Effective G-O-F

In the introduction (Section 24.1) we mentioned the two dominant viewpoints which illustrate some of the presently unresolved conceptual/theoretical challenges to effective G-O-F assessment: (1) G-O-F testing is only appropriate for description, and (2) G-O-F testing is appropriate for both description and inference. We shall now discuss how the ESC handles these conceptual/theoretical challenges by offering a compromise between the Pearsonian and Fisherian perspectives. Specifically, the ESC approach enriches Pearson's view of G-O-F for description by allowing for the determination of lesser to greater evidential support. Moreover, it strengthens Fisher's argument of G-O-F for inference by extending the capabilities of the traditional hypothesis test.

24.6.1.1 ESCs Evaluate G-O-F Based on Lesser to Greater Evidential Support

It is important to remember that in most cases there is no way of knowing (or we will not know in practice) if our data originated from a particular theoretical distribution (e.g., the normal) or from some other distribution. This situation is not due to an inadequacy in the G-O-F evaluation *per se*, but rather because we lack certain specific information related to the origins of our sample dataset. For example, knowing that a G-O-F evaluation suggests that a distribution is a good fit, it is not appropriate to make statements such as "... the data are normal ..." or "... the data originated from a normal distribution ..." without knowing for certain that this is the case.

One strength of the ESC approach is that it does not restrict the examination of G-O-F to a potentially invalid yes/no interpretation (e.g., yes, the data are normal), but rather it allows for an overall G-O-F assessment based on "lesser" to "greater" evidential support. For many types of research/testing situations, this approach may be preferable to limiting discussion to the test results themselves (i.e., rejection or failure to reject a single null hypothesis).

Let us consider two examples from the results section of this chapter with Participant #25. For the first example, it was noted that the GLD_{P2} distributional approximation was a very good fit for the data because it achieved an ESC score of 7, whereas other distributional approximations only achieved ESC scores of at most 4. Although we cannot say for certain that the data from Participant #25 comes from a distribution approximated by GLD_{P2}, the ESC results do support statements such as "the GLD_{P2} is a better fit for the data than the uniform, normal, or other GLD approximations" and "there is greater evidential support for the GLD_{P2} fit than for the other distributional approximations." In the second example, because several distributional approximations received very low ESC scores of 1 to 3, we might choose to disregard these distributional approximations altogether. In other words, we might decide to do no further testing with them, i.e., at least with regards to the dataset from Participant #25.

Ultimately, the decision governing which distributional approximations to use may reside with various non-statistical factors. For example, which distribution has the most intuitive properties so that our audience of clinicians has the best chance to understand the key issues involved? Or perhaps, which distribution has the type of properties that our computer programmers will find easiest to write software for?

24.6.1.2 ESCs Extend the Results of Formal Hypothesis Tests

Another advantage to using ESCs for G-O-F assessment is that **ESCs extend or enhance the results of formal hypothesis tests**. This is important because of the sheer number of limitations associated with the use of traditional hypothesis testing in G-O-F evaluations. Some of these limitations were addressed in the

Sections 24.1 and 24.2.2. For example, the use of a single hypothesis test for G-O-F assessment almost always results in incomplete information; and the use of multiple hypothesis tests frequently ends in ambiguity (i.e., often unclear how to prioritize multiple test results or how to reconcile contradictory test results).

A willingness to adopt (or experiment with) the ESC approach does not in any way suggest a disregard for the traditional hypothesis test or doubt about its role in G-O-F assessment. Rather, using ESCs represents a commitment to the survival of the hypothesis test in G-O-F assessment by updating its profile, enhancing its capabilities, and otherwise extending its usefulness for contemporary statistical practice. ESCs are not a replacement for hypothesis testing, but rather a theoretical framework by which the results of formal hypothesis testing can be put to their fullest use. Essentially, the ESC makes a very strong showing in what all good statistical methods purport to do — i.e., it maximizes available statistical information (in this case, through the use of 8 different pieces of evidence [both quantitative and graphical]) and minimizes the loss of information contained in the data. Put differently, using ESCs provides much needed additional information (i.e., beyond hypothesis testing) which assists in determining a theoretical distributions overall fit to a set of data.

Results from the present investigation demonstrate **how** the ESC is capable of extending the results of formal hypothesis testing. When one or both quantitative tests rejected the null hypothesis, the ESC corroborated these results by illustrating lesser evidential support overall. However, when the results of quantitative testing were ambiguous because one or both tests failed to reject, the ESC helped to provide some degree of clarification by indicating either lesser or greater evidential support overall.

Finally, as noted in Section 2.1, the issue of whether distributional approximations of data (e.g., probability density functions) should be used only for description or also for inference (e.g., hypothesis testing) has not yet been resolved. This is perhaps not surprising when we consider that recent investigations into G-O-F testing have tended to ignore theoretical issues and focus instead on specific practices or applications. One strength of the ESC approach is that it actively addresses several issues of theoretical merit. Furthermore, the type of fit quality judgment rendered by an ESC diagram represents a sort of integrated philosophy on G-O-F (i.e., a compromise between the Pearsonian and Fisherian view). It remains true to Pearson's original ideas of G-O-F for description while allowing Fisher's ideas on inference to be better justified.

24.6.2 ESCs Address Practical Application Challenges to G-O-F

In Section 2.2, it was noted that the practical challenges to effective G-O-F assessment are as numerous and diverse as those related to the conceptual issues. In practical terms, G-O-F was described as a three-part process involving: (1)

selection of one or more candidate distributions to evaluate from among the many theoretical distributions available; (2) selection or development of a methodological approach for fitting these distributions to the data; and (3) selection and application of the form of the G-O-F assessment criteria.

A big advantage of the ESC approach is that it helps to both organize and streamline a process which seems rather complicated and therefore confusing to many practitioners of G-O-F evaluation. Simply put, steps (2) and (3) are already "taken care of" or "spoken for" and so one needs only to follow the recipe/algorithm of the ESC for these two parts of the process. As for step (1), decisions can be made based on those issues that are determined to be relevant by subject matter specialists/experts. Additionally, steps (2) and (3) do not have to be fixed per se; and a brief discussion of the ESCs flexibility/extensibility in accommodating users of the methodology is addressed in Section 6.2.2.2.

Finally, we have already seen how use of the ESC methodology produces a more complete understanding of the appropriateness of different theoretical distributions for particular datasets (i.e., rather than relying on a single statistical test to provide sufficient information). This increase in available information is not only theoretically sound, but is potentially quite useful in assisting G-O-F practitioners with understanding individual objects/elements of study (e.g., particular scientific variables). The additional information may also be rather beneficial for developing theories on the more complex aspects of the inner workings of these variables (e.g., proposing scientific mechanisms). Let us now discuss some of the more specific examples of how the ESC practically improves G-O-F assessment.

24.6.2.1 ESCs Provide Both Relevant and Practical Information

The interpretation of a distribution's fit to data as "lesser to greater" rather than an overly restrictive and difficult to substantiate "... yes it does fit ..." or "... no it does not fit ..." has many practical advantages. The information generated by the ESC approach seems useful and therefore relevant. For example, the following questions can be addressed and subsequently answered:

1. Does a particular theoretical distribution provide a good, fair, or poor approximation of the data?

2. If more than one distribution provides a good fit to the data, are there some that provide a better fit?

3. Is there one distribution in particular that provides a best fit?

4. If there is a best fitting distribution, are there other distributions whose fits are less appropriate but nevertheless adequate depending upon the specific application?

Questions 1 through 4 are representative of the type of questions that many users of G-O-F methods (e.g., scientific researchers) would like to have answered. Perhaps equally important is that these questions are also the kinds of questions that supervisors (e.g., laboratory directors and business managers) want answered. In this way, the ESC approach is not only theoretically sound but also practically meaningful.

24.6.2.2 ESCs Permit an Adaptable and Extensible Approach to G-O-F Assessment

Ultimately, the responsible use of ESC diagrams is not limited to particular theoretical distributions or to specific data fitting methods. This inherent flexibility in the ESC approach permits users (e.g., researchers and/or teachers of G-O-F methods) to incorporate a variety of models (if they choose) and to select these models based on things such as a research unit's familiarity with a distribution's properties or a distribution's popularity within a given discipline. Similarly, the methods used for fitting distributions to data can reflect the specific needs of the user rather than the specific directives of the test *per se*.

Finally, the authors of this chapter consider both the present form of the ESC diagram and the manner in which the evidential support totals are computed to be merely a prototype for future study of G-O-F assessment using ESC-type methods. We assume that both the physical appearance and substantive content of ESC diagrams will change over time. For example, further study may reveal a need to differently weight some aspects of evidential support (i.e., rather than assigning all eight pieces of evidence a weight of 1). Or perhaps it will become evident that more than eight pieces of evidence are required to effectively convey evidential support for most research projects. Also the look/composition of the graphic itself may need to be modified in order to appeal to the widest possible cross-section of ESC users.

24.6.2.3 ESCs Help Disseminate Information Within Relevant Communities

There are two ways in which ESCs help disseminate information within relevant, scientific communities: (1) ESCs are easy to use and interpret; and (2) use of ESCs puts more control in the hands of the practitioner.

Even though the ESC diagram may eventually need to be modified, the present form seems serviceable (i.e., at least for the moment). Regarding (1), the idea of representing collective evidence on a continuum seems intuitive and it is certainly easy to understand. Especially when compared to concepts such as characteristics of statistical distributions that are notoriously difficult for practitioners to understand. Indeed, the familiarity (or lack thereof) with particular theoretical distributions may affect the manner and extent to which investigational findings are propagated within scientific communities as well as the ease of application

for practitioners (e.g., many computer programs and other types of algorithmic procedures are pre-existing or easier to write for certain theoretical distributions).

As for (2), the use of ESCs allows for the consideration of both theoretical and practical issues surrounding G-O-F assessment; and as such, the results from an ESC analysis could provide justification for the use of certain distributions over others. Following an ESC analysis, overall G-O-F considerations can be contextualized with other relevant elements permitting subject matter specialists to make more informed decisions with regards to which distribution(s) to employ. As an example, if both the GLDM and N distributions provide adequate approximations of the data and the difference in evidential support is only minor (e.g., 8 vs. 7 pieces of evidential support) then perhaps the decision of which distributions to use will come down to other non-statistical issues such as the relative ease of writing the necessary computer programs, the overall financial cost to implement, or the average statistical literacy of the eventual users.

24.6.2.4 ESCs Display Maximum Information in Minimum Space

Finally, the ESC approach is a very efficient method for displaying large amounts of information related to G-O-F assessment. ESC diagrams require much less space than separate tables and figures of test results. They can also be drawn rather small in scale while remaining legible as there are relatively few items included in each diagram.

In this chapter, we have seen how ESCs address many of the theoretical challenges (Section 24.6.1) as well as numerous practical application challenges (Section 24.6.2) to G-O-F assessment. Based on our findings, we enthusiastically recommend use of the ESC approach with different types of research projects. Additionally, we welcome any type of correspondence/feedback from other investigators regarding how the ESC methodology might be improved to make the process even more useful.

References for Chapter 24

Cochran, W. G. (1952). "The χ^2 test of goodness of fit," *Annals of Mathematical Statistics,* 23, 315–342.

Dahiya, R.C., and Gurland, J. (1973). "How many classes in the Pearson chi-square test?" *Journal of the American Statistical Association,* 68, 707–712.

Dudewicz, E. J., Karian, Z. A., and Marshal, R. J. (1985). "Random number generation on microcomputers," *Modeling and Simulation on Microcomputers: 1985* (Edited by R. G. Lavery). The Society for Computer Simulation, La Jolla, California, 9–14.

Dudewicz, E.J., and van der Meulen, E.C. (1984). "On assessing the precision of simulation estimates of percentile points," *American Journal of Mathematical*

and Management Sciences, 4, 335–343.

Evans, M., Hastings, N., and Peacock, B. (2000). *Statistical Distributions* (3rd Ed.), John Wiley & Sons, Inc.

Freimer, M., Kollia, G., Mudholkar, G. S., and Lin, C. T. (1988). "A study of the Generalized Tukey Lambda family," *Communications in Statistics—Theory and Methods,* 17, 3547–3567.

Gan, F. F., Koehler, K. J., and Thompson, J. C. (1991). "Probability plots and distribution curves for assessing the fit of probability models," *The American Statistician,* 45, 14–21.

Gibbons, J. D. (1997). *Nonparametric Methods for Quantitative Analysis* (3rd ed.), American Sciences Press, Inc.

Gibbons, J. D. (1992). *Nonparametric Statistical Inference* (3rd ed.), Marcel Dekker, Inc.

Gilchrist, W. G. (2000). *Statistical Modeling with Quantile Functions,* Chapman & Hall/CRC.

Greenwood, P. E., and Nikulin, M. S. (1996). *A Guide to Chi-Squared Testing,* John Wiley & Sons, Inc.

Haber, M. (1980). "A comparison of some continuity corrections for the chi-square test on 2 X 2 tables," *Journal of the American Statistical Association,* 75, 510–515.

Inman, H. F. (1994). "Karl Pearson and R.A. Fisher on statistical tests: A 1935 exchange, from *Nature*," *The American Statistician,* 48, 2–11.

Karian, Z. A., and Dudewicz, E. J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L2 norm," *Journal of the Iranian Statistical Society,* 2, 171–187.

Karian, Z. A., and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* Chapman & Hall/CRC.

Karian, Z. A., and Dudewicz, E. J. (1999). *Modern Statistical, Systems, and GPSS Simulation,* 2nd Ed, Chapman & Hall/CRC.

Lilliefors, H. W. (1967). "On the Kolmogorov-Smirnov test for normality with mean and variance unknown," *Journal of the American Statistical Association,* 62, 399–402.

Moore, D. S. (1977). "Generalized inverses, Wald's Method, and the construction of chi-squared tests of fit," *Journal of the American Statistical Association,* 72, 131–137.

Munk, A., and Czado, C. (1998). "Nonparametric validation of similar distributions and assessment of goodness of fit," *Journal of the Royal Statistical Society, Series B, Statistical Methodology,* 60, 223–241.

Pearson, K. (1900). "On the theory of contingency and its relation to association and normal correlation," *Philosophy Magazine*, 50, 157–175.

Pearson, K. (1932). "Experimental discussion of the (χ^2, P) test for goodness of fit," *Biometrika,* 24, 351–381.

Rafter, J. A., Abell, M. L., and Braselton, J. P. (2003). Statistics with Maple, Elsevier Science.

Thé, D.J. (2003). "Fitting the Extended Generalized Lambda Distribution (EGLD) to Exercise Science Data," *2003 Proceedings of the American Statistical Association, Section on Statistical Graphics [CD-ROM]*, San Francisco, CA: American Statistical Association, 4218–4225.

Upton, G., and Cook, I. (2002). *Oxford Dictionary of Statistics,* Oxford University Press, Inc.

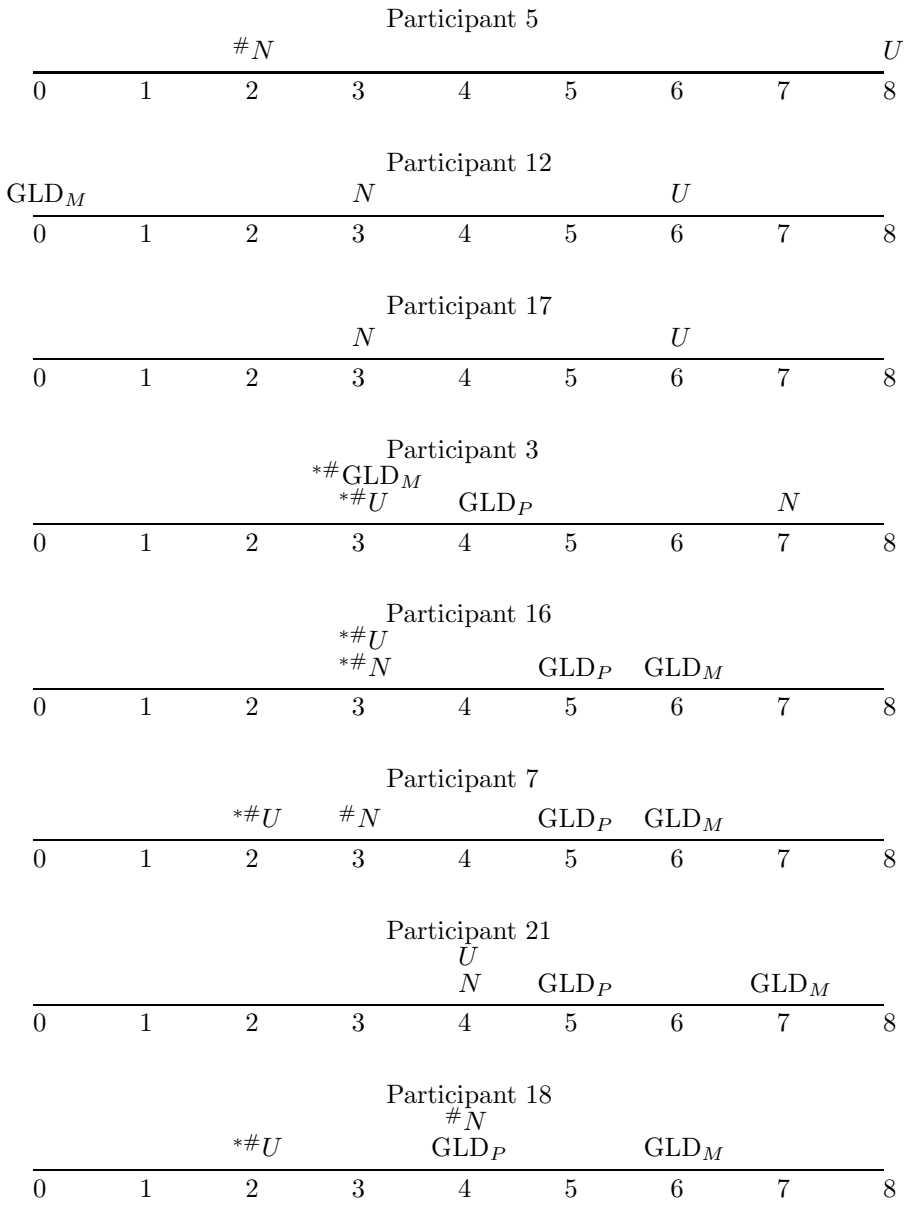
Williams, C. A., Jr. (1950). "On the choice of the number and width of classes for the chi-square test of goodness of fit," *Journal of the American Statistical Association,* 45, 77–86.

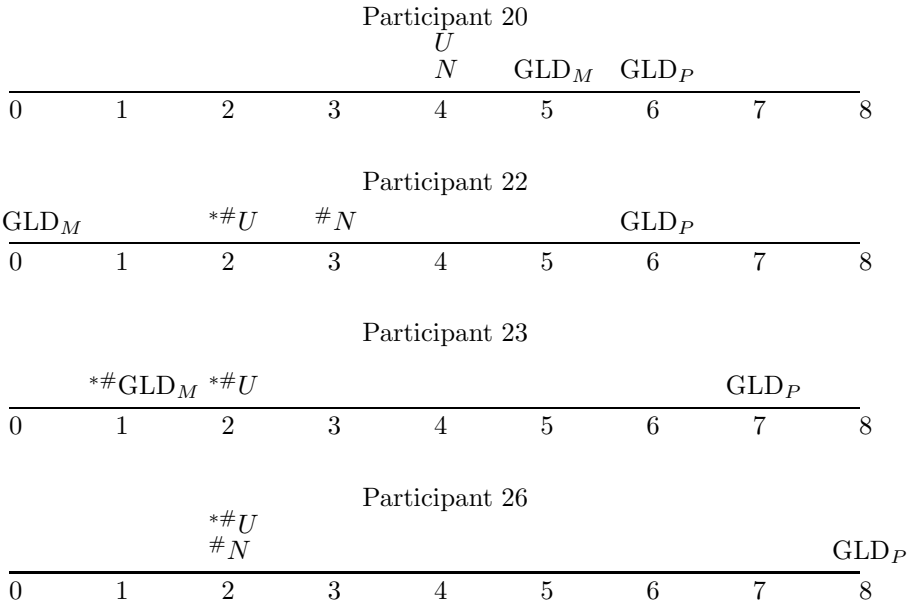
Yarnold, J. K. (1970). "The minimum expectation in χ^2 goodness of fit tests and the accuracy of approximations for the null distribution," *Journal of the American Statistical Association,* 65, 864–888.

Appendix for Chapter 24

A.1 ESC Diagrams where One Distributional Approximation is a Much Better Fit than the Others

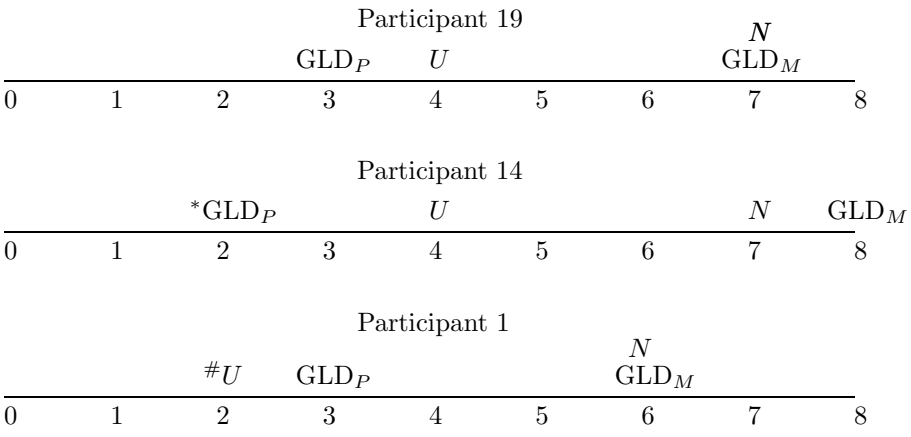
All of these datasets are more suggestive of one particular distribution. Participants 5, 12, and 17 are more suggestive of the U. Participant 3 is more suggestive of the N. Participants 16, 7, 21, and 18 are more suggestive of the GLD_M. In the following diagrams, * designates that $p \leq 0.05$ for the χ^2 test and $^\#$ indicates that $p \leq 0.05$ for the K-S test.

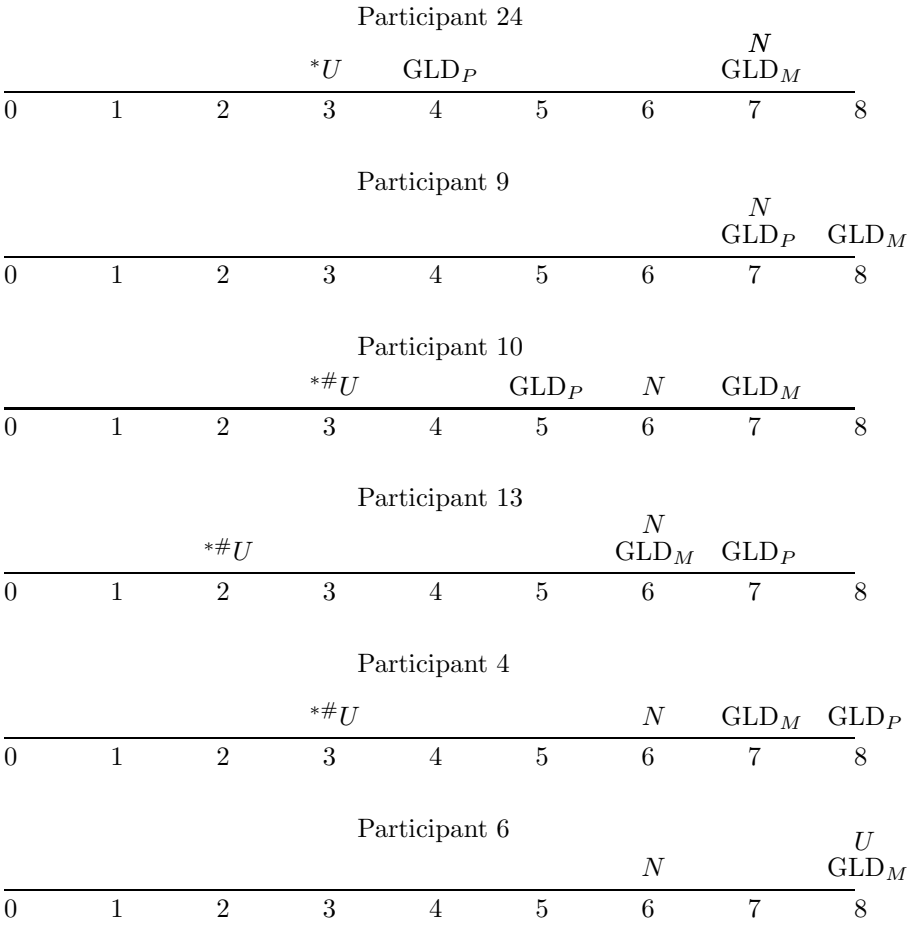




A.2 ESC Diagrams where More than One Distributional Approximation Provides a Reasonably Good Fit

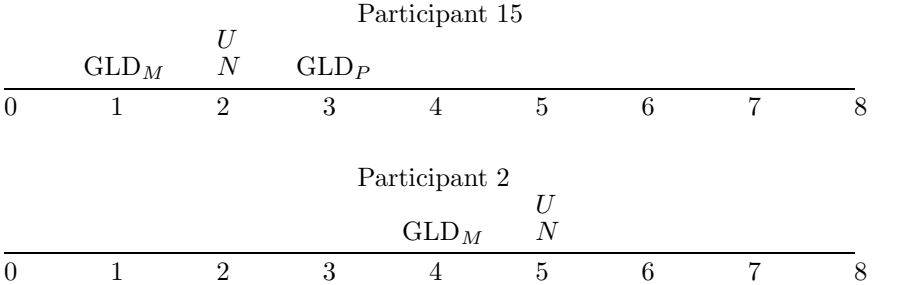
All of these datasets are suggestive of more than one distribution. Participants 19, 14, 1, and 24 are suggestive of the N and GLDM. Participants 9, 10, 13, and 4 are suggestive of the N, GLDM, and GLDP. Participant 6 is suggestive of the U and GLDM. In the following diagrams, * designates that $p \leq 0.05$ for the χ^2 test and # indicates that $p \leq 0.05$ for the K-S test.





A.3 ESC Diagrams where None of the Distributional Approximations Provides an Acceptable Fit

There were only two datasets with insufficient evidence for any of the three distributions: Participant 15 and Participant 2. In the following diagrams, * designates that $p \leq 0.05$ for the χ^2 test and # indicates that $p \leq 0.05$ for the K-S test.



A.4 Raw Data for All 24 Participant Datasets ($n = 20$ to 40)

p_1	266.70, 383.32, 358.55, 354.24, 316.96, 222.46, 409.56, 356.78, 363.89, 374.23, 347.29, 465.74, 337.52, 310.02, 319.96, 376.88, 337.93, 399.06, 305.80, 348.44, 371.23, 376.88, 345.23, 318.33, 390.18, 377.86, 354.23, 391.38, 392.47, 486.35
p_2	184.66, 138.38, 168.78, 153.84, 178.84, 205.52, 196.61, 177.28, 190.41, 164.84, 208.82, 187.56, 209.23, 220.75, 169.76, 201.53, 226.08, 193.00, 206.60, 184.98, 204.39, 201.67, 206.51, 223.85, 213.72, 196.08, 147.11, 206.09, 229.69, 208.15
p_3	101.43, 121.92, 115.82, 121.39, 124.06, 114.73, 112.64, 101.05, 120.77, 119.40, 108.64, 111.07, 123.06, 111.74, 118.93, 149.79, 133.14, 130.46, 111.86, 114.26, 124.61, 129.37, 124.94, 120.19, 125.23, 125.69, 114.55, 118.69, 117.54
p_4	241.71, 224.44, 264.90, 231.45, 276.10, 261.97, 268.89, 270.95, 315.84, 272.82, 265.89, 304.04, 275.60, 295.92, 309.08, 292.56, 307.33, 274.30, 295.14, 319.26, 279.53, 283.54, 261.28, 262.44, 276.16, 285.32, 295.68, 293.52, 300.63, 313.42
p_5	198.08, 196.01, 206.47, 202.52, 195.43, 205.01, 181.85, 206.75, 183.86, 191.34, 188.43, 178.99, 175.88, 191.25, 176.60, 169.28, 172.17, 174.66, 171.78, 189.88
p_6	172.21, 178.61, 192.51, 190.08, 172.82, 188.40, 177.44, 180.17, 186.67, 198.61, 170.36, 192.83, 188.00, 195.76, 189.59, 187.53, 189.15, 181.62, 189.73, 182.86
p_7	158.25, 190.77, 158.45, 167.13, 171.94, 185.62, 170.13, 159.47, 168.45, 157.35, 232.02, 263.03, 191.99, 196.72, 252.16, 296.51, 231.01, 219.94, 194.03, 270.53, 200.29, 188.83, 200.11, 211.10, 200.29, 190.71, 208.38, 197.60, 185.00, 203.25
p_9	211.31, 196.20, 196.72, 214.64, 208.60, 233.12, 198.43, 272.32, 187.85, 177.34, 174.23, 218.28, 204.18, 241.61, 242.21, 217.52, 223.20, 219.53, 224.95, 229.46
p_{10}	309.36, 310.90, 319.15, 326.47, 367.69, 330.94, 388.36, 350.23, 325.87, 345.00, 309.05, 341.27, 350.68, 341.27, 327.74, 355.25, 328.07, 347.59, 341.13, 353.68
p_{12}	409.24, 402.62, 396.54, 429.43, 379.53, 399.10, 390.11, 386.25, 394.49, 415.92, 480.68, 475.92, 457.64, 455.63, 473.00, 471.09, 480.28, 494.35, 483.37, 504.50
p_{13}	89.19, 88.07, 104.40, 110.49, 117.73, 160.29, 108.89, 126.40, 122.99, 117.34, + 125.56, 119.59, 124.10, 131.42, 119.42, 132.64, 117.80, 131.57, 127.44, 121.06
p_{14}	166.05, 144.59, 162.23, 152.05, 157.90, 160.67, 171.04, 169.55, 162.51, 167.52, 169.88, 177.20, 176.33, 163.48, 183.32, 188.94, 177.53, 173.43, 189.70, 170.77, 197.14, 183.42, 166.61, 185.40, 173.10, 186.38, 180.72, 199.93, 185.06, 184.93
p_{15}	170.65, 172.64, 172.93, 173.31, 174.07, 215.84, 184.93, 177.59, 176.86, 171.88, 188.93, 177.40, 171.48, 173.79, 173.68, 198.10, 182.15, 180.97, 170.33, 170.59
p_{16}	146.05, 149.43, 163.81, 169.89, 142.41, 155.56, 168.25, 144.28, 172.68, 111.46, 190.07, 196.43, 218.10, 229.34, 195.19, 211.60, 257.21, 184.95, 233.63, 203.88, 239.57, 258.43, 274.59, 243.70, 362.46, 228.23, 220.36, 347.31, 219.18, 225.63, 227.19, 191.20, 333.59, 231.72, 323.24, 265.31, 256.21, 226.89, 379.21
p_{17}	136.19, 192.35, 156.50, 193.04, 179.01, 182.81, 193.44, 203.14, 156.08, 169.10, 226.24, 274.33, 297.29, 327.46, 294.51, 294.68, 343.55, 378.50, 325.60, 276.74, 265.87, 330.26, 309.54, 281.22, 329.07, 230.79, 313.87, 349.38, 385.42, 315.02
p_{18}	94.84, 82.02, 99.08, 93.37, 96.12, 81.43, 101.31, 98.44, 77.59, 73.23, 101.51, 184.79, 113.42, 179.02, 119.88, 117.15, 130.82, 161.45, 119.17, 139.97, 117.43, 117.84, 116.67, 141.47, 134.33, 110.84, 115.89, 101.43, 97.29, 138.20
p_{19}	175.09, 175.05, 229.58, 128.29, 172.53, 206.88, 151.67, 155.12, 166.74, 192.93, 268.12, 260.42, 226.05, 298.84, 220.66, 202.04, 262.04, 193.09, 233.40, 260.71, 190.06, 210.85, 213.62, 313.89, 222.89, 246.40, 192.72, 242.43, 185.77, 256.77

p_{20}	144.67, 152.26, 140.72, 183.83, 128.39, 174.69, 129.92, 183.14, 106.60, 128.36, 190.86, 129.93, 136.33, 153.80, 128.69, 133.92, 141.56, 109.78, 182.35, 144.87
p_{21}	185.62, 236.12, 178.85, 271.26, 319.51, 281.13, 252.56, 347.80, 187.42, 240.50, 160.87, 245.23, 223.85, 148.99, 173.55, 192.47, 175.59, 209.92, 254.09, 289.25
p_{22}	626.17, 541.87, 562.86, 520.34, 579.17, 751.86, 443.36, 684.26, 600.22, 516.38, 449.67, 459.85, 457.04, 426.72, 420.17, 474.91, 470.75, 480.00, 469.96, 451.33
p_{23}	296.77, 260.30, 187.88, 228.68, 291.63, 221.53, 233.05, 371.52, 311.17, 208.66, 173.43, 201.13, 190.76, 175.90, 170.98, 209.48, 182.12, 194.74, 229.63, 187.52
p_{24}	318.59, 313.23, 250.70, 358.96, 466.36, 380.05, 287.08, 314.33, 408.95, 193.57, 357.96, 457.55, 454.44, 398.10, 510.71, 474.91, 327.27, 341.37, 355.28, 586.79, 307.83, 382.51, 345.92, 415.63, 385.89, 343.60, 333.02, 368.63, 332.71, 359.62
p_{25}	182.01, 155.81, 193.70, 186.29, 171.41, 153.13, 220.77, 181.48, 189.06, 203.20, 187.34, 163.08, 225.78, 215.59, 192.72, 205.20, 200.82, 214.34, 184.86, 207.05, 232.61, 198.10, 232.74, 213.52, 234.83, 239.92, 207.89, 303.90, 167.54, 218.63, 213.95, 183.39, 215.23, 220.97, 234.80, 237.86, 221.33, 228.09, 216.14, 234.32
p_{26}	105.73, 100.94, 113.90, 115.56, 126.36, 119.75, 124.99, 128.23, 132.05, 140.36, 152.21, 134.19, 148.31, 147.25, 136.39, 140.86, 149.31, 148.38, 138.01, 151.75, 150.89, 160.35, 149.61, 147.01, 152.42, 160.77, 154.43, 160.78, 149.63, 153.62

A.5 First Four Sample Moments α's for Each Participant ($n = 20$ to 40)

Note: α_1 = mean, α_2 = variance, α_3 = skewness, α_4 = kurtosis

Participant	n	α_1	α_2	α_3	α_4
1	30	358.64	2608.04	−0.03	4.95
2	30	193.49	533.91	−0.69	3.01
3	29	119.55	93.12	0.74	5.54
4	30	280.65	561.92	−0.48	3.07
5	30	184.83	138.23	0.43	2.00
6	20	185.24	63.01	−0.42	2.37
7	30	200.70	1209.89	1.07	3.98
9	20	214.58	548.60	0.41	3.73
10	20	338.48	406.35	0.54	3.55
12	20	438.98	1781.87	0.02	1.29
13	20	119.81	244.44	0.09	5.17
14	30	174.24	166.60	−0.11	2.84
15	20	178.90	125.83	2.28	8.74
16	40	225.69	4104.27	0.68	3.10
17	30	263.70	5254.06	−0.18	1.77
18	30	115.20	747.57	0.90	3.79
19	30	215.15	1926.70	0.26	2.77
20	20	146.23	613.59	0.47	2.31
21	20	228.72	2994.20	0.51	2.58
22	20	519.34	8137.69	1.22	3.98
23	20	226.34	2925.32	1.32	4.31
24	30	371.05	6175.47	0.54	4.29
25	40	207.23	799.50	0.61	5.26
26	30	139.80	268.82	−0.84	2.87

A.6 The ρ Sample Statistics for Each Participant (n = 20 to 40)

Note: ρ_1 = median (50th percentile), ρ_2 = inter-decile range (90th percentile − 10th percentile), ρ_3 = left-right tail-weight ratio (50th percentile − 10th percentile)/(90th percentile − 50th percentile), ρ_4 = tail-weight factor (75th percentile − 25th percentile)/(90th percentile − 10th percentile).

Participant	n	ρ_1	ρ_2	ρ_3	ρ_4
1	30	357.66	102.28	1.01	0.50
2	30	199.07	68.60	1.80	0.43
3	29	119.40	21.82	0.97	0.51
4	30	277.84	69.31	0.97	0.45
5	30	189.15	34.50	1.00	0.62
6	20	187.76	23.19	2.01	0.47
7	30	195.37	103.39	0.55	0.40
9	20	216.08	63.75	1.44	0.48
10	20	341.20	56.93	1.25	0.43
12	20	442.53	106.61	1.10	0.76
13	20	120.32	41.82	2.42	0.35
14	30	173.26	31.44	0.92	0.62
15	20	173.93	26.58	0.14	0.36
16	40	220.36	187.54	0.65	0.44
17	30	278.98	191.03	1.73	0.69
18	30	114.65	77.96	0.74	0.44
19	30	212.23	111.23	1.01	0.59
20	20	141.14	72.12	0.69	0.56
21	20	229.98	154.34	0.78	0.55
22	20	477.45	250.06	0.24	0.48
23	20	209.07	136.05	0.35	0.48
24	30	358.46	184.90	0.59	0.46
25	40	210.70	66.90	1.77	0.56

A.7 χ^2 Test Statistics Based on 100 Samples

The following table displays the χ^2 G-O-F test statistics, $Q_1, Q_2, \ldots, Q_{100}$, obtained after simulating 100 samples from GLD(122.4359, −0.0017, −0.0063, −0.0112). $Q_{crt} = 10.39$ was obtained from the original dataset (of participant #3) and $P(Q > 10.39)$ can be estimated from the simulation by dividing the number of $Q_i > 10.39$ by 100. In this case, $P(Q > 10.39) = 7/100 = 0.07$.

4.84	2.32	1.88	4.44	3.65	**22.87**	3.46	1.40	6.38	2.05
6.56	6.17	1.27	2.41	1.07	2.26	8.92	6.46	0.90	4.00
3.48	1.96	1.09	**12.51**	4.58	1.72	2.62	6.14	3.20	2.63
2.66	5.58	5.41	1.57	1.80	0.04	7.91	2.15	3.28	**11.95**
1.45	5.62	3.55	2.40	2.13	5.36	3.55	7.62	3.11	**11.60**
0.99	0.90	1.70	3.13	3.08	6.27	2.37	1.92	5.61	3.60
1.23	6.02	2.44	2.86	1.60	1.58	1.54	1.65	2.32	1.85
3.19	1.09	1.15	2.19	5.12	4.24	4.66	5.79	3.91	2.13
2.31	0.66	1.54	2.05	4.73	1.45	1.37	2.42	9.77	**10.94**
1.76	9.37	0.20	2.57	**13.32**	3.36	2.57	2.08	3.76	**14.93**

A.8 χ^2 Maple Programs Used for the χ^2 Testing Simulation

The program named `Sample` takes two arguments: `N` (the sample size) and `L` (a list of lambdas values for a GLD). `Sample` returns a random sample (in the form of a list) of size `N` from the specified GLD distribution. It uses a Random Number Generator named `URN37`[2]. The program called `ChiSqGOF` performs a χ^2 G-O-F test. `ChiSqGOF` takes three arguments: `Data` (the set of raw data), `Classes` (a list of the class boundaries for the test), and `L` (a list of lambdas values for a GLD). The last program, `NLarger` takes two arguments: a list `X` and a number (in this cse `Qcrt`) and returns the number of items in the list `X` that exceed `Qcrt`.

```
Sample:=proc(N::posint,L::list)
    local i, x, a, m, y, Q:
    global Seed:
    a:=5^13: m:=2^31:
    Q:=array(1..N):
    for i from 1 to N do
        x := a*Seed mod m:
        Seed := x:
```

[2]For details about `URN37`, including the results of various quality testing procedures, see *Modern Statistical, Systems, and GPSS Simulation* by Karian and Dudewicz (1999). Note: `Sample` has a global variable called `Seed`. This is the initial value that gets `URN37` started; you may want to take it to be 32767 as indicated in Karian and Dudewicz (1999) (page 492). Once set, the Seed variable should not be redefined.

```
        y:=1.0*x/m:
        Q[i]:=evalf(L[1]+(y^L[3]-(1-y)^L[4])/L[2]):
    od:
    convert(Q,list)
end:



ChiSqGOF := proc(Data::list, Classes::list, L::list)
  local n, cl,C, i, j, k, OF, EF, Q, Pr, ChiSq, u, v;
  n:=nops(Data): cl:=nops(Classes)+1:
  C:=array(1..cl):
  for i from 1 to cl do C[i]:=0 od:
  for j from 1 to n do
    if Data[j] < Classes[1] then C[1]:=C[1]+1 fi:
    if Data[j] >= Classes[cl-1] then C[cl]:=C[cl]+1 fi:
    for k from 1 to cl-2 do
      if Data[j] >= Classes[k] and Data[j] < Classes[k+1]
          then C[k+1]:=C[k+1]+1 fi:
    od:
  od:
  OF:=convert(C,list): lprint(OF):
  Q:=L[1]+(y^L[3] - (1-y)^L[4])/L[2]:
  Pr:=[0, seq(fsolve(Q=Classes[i],y,y=0..1), i=1..cl-1), 1]:
  EF:=seq(n*(Pr[u+1]-Pr[u]), u=1..cl): lprint(EF):
  ChiSq:=sum(((EF[v]-OF[v])^2)/EF[v], v=1..cl)
end;



NLarger:= proc(X::list,Qcrt::numeric)
    local Count, i, N; Count:=0: N:=nops(X):
    for i from 1 to N do
        if X[i]>Qcrt then Count:=Count+1 fi od:
    Count
end:
```

Then Maple interaction to produce the results of 10,000 simulations has two parts. First, the computation of the χ^2 values, which can be done as follows.

```
CSQ := array(1..10000):
Case10000:= [seq(Sample(N,L), i=1..10000)]:
```

```
for i from 1 to 10000 do
    CSQ:=ChiSqGOF(Case10000[i], B, L);
od:

X := convert(CSQ, list):
NLarger(X, 10.39)
```

Chapter 25

Estimation of Sampling Distributions of the Overlapping Coefficient and Other Similarity Measures

Madhuri S. Mulekar
Department of Mathematics and Statistics
University of South Alabama
Mobile, Alabama 36688 USA
mmulekar@jaguar1.usouthal.edu

Jeffrey M. Boone
Department of Mathematics and Statistics
University of South Alabama
Mobile, Alabama 36688 USA
jmboone1@crimson.ua.edu

Subhash Aryal
Department of Mathematics and Statistics
University of South Alabama
Mobile, Alabama 36688 USA
saryal@hsc.unt.edu

Measures of similarity are useful to quantify the extent of similarity between two populations. Their complements, known as measures of dissimilarity, are also used commonly by researchers. These measures are often used to make compar-

ative inferences about two groups, such as describing the degree of inter-specific encounter or crowdedness of two species in their resource utilization, estimating the proportion of genetic deviates in segregating populations, determining the similarity of two different radiologists' readings of the same x-rays, measuring racial segregation, or making inference about the comparison of survival of two groups after cancer treatment. One such similarity measure, also known as the overlap coefficient, has been used for comparing fits of statistical distributions by different methods. Due to the unknown nature of sampling distributions of these measures, however, decisions are often made using only point estimates. A study was designed using the Alabama Supercomputer to simulate distributions of estimates of four different similarity measures for several different parametric configurations for commonly used normal and exponential distributions. With the help of a distribution fitting technique, the sampling distributions were estimated to be from the family of beta distributions whose parameters were estimated using multiple regression techniques with transformations.

25.1 Introduction

The similarity between two densities can be considered as the commonality shared by both populations. Generally it is measured on the scale of 0 to 1. Values of measure close to 0 indicate extreme inequality of two densities and the value of 1 indicates exact equality. Scientists from different disciplines propose different measures of similarity serving different purposes. One measure of similarity, commonly known as the **overlap coefficient**, has been recently used to study goodness of fitting mixture distributions with generalized lambda distribution (Ning, Gao, and Dudewicz (2008)). The area between the norms was used to compare fits done by the percentile and moment methods (Karian and Dudewicz (2003)). In this chapter, we emphasize estimation of sampling distributions of overlap coefficient, δ, for exponential and normal densities. However, the overlap coefficient is not the only similarity measure used by researchers, hence sampling distributions of three other commonly used similarity measures are also included.

Because of the complex mathematical structure of many similarity measures, virtually no results are available on their exact sampling distributions. In most studies, the point estimates of these measures are used for decision-making. No confidence interval estimates or hypothesis tests are available for most measures of similarity. Approximate results are available for some measures. Smith (1982) derived approximate formulas using the delta method for estimating the mean and variance of the discrete version of one such measure known as Weitzman's measure (Weitzman (1970)) (also known as the overlap coefficient). Mishra, Shah, and Lefante (1986) gave the small and large sample properties of the sampling distributions for a function of this overlap measure under the assumption of homogeneity of variances. Mulekar and Mishra (1994) simulated the sampling distri-

butions of Matusita's, Morisita's, and Weitzman's measures for normal densities with equal means and obtained the approximate expressions for the bias and variance of the estimates. They also proposed confidence interval estimation of these measures using a transformation technique. The estimation of overlap measure has been studied by Inman (1981), Inman and Bradley (1989, 1994), Rom and Hwang (1996), and Reiser and Faraggi (1999). Ricklefs and Lau (1980) studied estimation of bias and variance of some of the measures.

Mueller and Altenberg (1985) studied estimation of four measures including discrete versions of overlap (also known as **coefficient of community**) and Morisita's measures using delta, bootstrap, and jackknife methods. They also studied bias and variance of these estimators. Mulekar and Mishra (2000) studied two resampling techniques, namely, jackknife and bootstrap, along with the Taylor series approximation and transformation method for the construction of confidence intervals. This study included three similarity measures frequently used by ecologists: Matusita's measure, Morisita's measure, and Weitzman's measure. In general, the transformation technique and Taylor-series approximation provided sensible and reasonably reliable confidence intervals. The bootstrap worked better for large samples and populations with similar variances, and Jackknife was the most inferior in performance compared to the others. Dixon (1993) described the use of bootstrap and jackknife techniques for the Gini coefficient of size hierarchy, a commonly used measure of similarity between income distributions of two ethnic, gender, or geographical groups, and the Jaccard index of community similarity. Mulekar (2006) described a procedure for selecting a population with the largest Gini coefficient under the assumption of approximate normality for the sampling distribution of Gini coefficient estimates. A result showing that the estimates of the Gini Coefficient are approximately normally distributed was also provided.

The measures of similarity have been used in a wide variety of applications. Here are some examples of the fields where the similarity measures are used.

- Anthropology: Czekanowski (1909, 1932, and 1962)
- Ecology: Bray and Curtis (1957), Field and McFarlane (1968), Kistler, Stephenson, and Benninghoff (1979), Feinsinger, Spears, and Poole (1981), Wolda (1981), Sale and Guy (1992), Kokko, Mackenzie, Reynolds, Lindstrom, and Sutherland (1999), and Horn (1966)
- Econometrics: Anderson (2004), Anderson, Ge, and Leo (2005), Braun (1988), Moore (1996), and Karoly (1990)
- Geology: Sneath (1977)
- Reliability analysis: Ichikawa (1993)
- Health sciences: Mizuno, Yamaguchi, Fukushima, Matsuyama, and Ohashi (2005)

Federer, Powers, and Payne (1963) used the overlap measure, described in Section 25.2.1, to estimate the proportion of genetic deviates in segregating populations. Ichikawa (1993) showed use of the overlap coefficient to estimate the lowest upper bound for the probability of failure in the stress-strength models of reliability analysis. Cleveland and Lachenbruch (1974) applied the overlap coefficient to estimate pairwise divergence for nonwhite birth weight distributions. Hanula and Engstrom (2000) used Morisita's index to study similarities in the nesting diet of the red-cocked woodpecker (*Picoides borealis*) in different locations, and Chung, Epperson, and Chung (2002) used Morisita's index of dispersion to study spatial patterns of individual Camellia japonica in age classes.

The history of development of some similarity measures is given by Inman and Bradley (1989) and some references to applications of such measures are given by Mulekar and Mishra (1994). Gastwirth (1975) discussed the use of overlap measures for studying income differentials and described how the crossing point of densities affects the outcome. He proposed using nonparametric alternatives more effectively. All the measures studied here are known to have biased estimators. Mimoto and Zitikis (2009) proposed an L_p-extension of the overlap measure (which is an L_1-type quantity) and showed that for $p > 4/3$, the L_p-extension is nearly unbiased under an appropriate choice of the bandwidth of the kernel-density estimator.

Lu, Smith, and Good (1989) studied the sampling variability of three similarity measures, namely the overlap coefficient, Morisita's measure, and MacArthur-Levins' measure, for multivariate normal densities using simulation techniques and investigated bias as affected by differences in sample sizes and dimensionality. Harner and Whitmore (1977) studied asymmetric measure α_{12} and the overlap measure for multivariate normal densities and showed that by using a discriminant variable, the multivariate problem is reduced to a univariate problem while preserving relevant information. Minami and Shimizu (1999) considered the problem of estimating Matusita's measure when samples are from multivariate normal distributions with unknown mean vectors and covariance matrices. They also derived asymptotic variances and biases of estimates and proposed bias reduction methods. Clemons and Bradley (2000) also studied the estimation of δ for multivariate normal densities.

Recently, Ning, Gao, and Dudewicz (2008) used the overlapping coefficient as a measure of goodness-of-fit. They studied fitting of mixture distributions with the generalized lambda distribution. Along with Kullback-Leibler distance, they used the overlap coefficient to measure the closeness of the target or fitted probability density function to the true probability density function. The value of the overlap coefficient was used to assess the level of goodness of the approximation of the fitted distribution. The larger value of the overlap coefficient or the smaller Kullback-Leibler distance indicated a better approximation. The overlap coefficient was also used to compare fits by two different methods, percentile and moments, and to show the superiority of percentile fits to moments in L^2 norm

(Karian and Dudewicz (2003)).

Four commonly used similarity measures, namely the overlap coefficient, Matusita's measure, Morisita's measure, and MacArthur-Levins' measure (defined in Sections 25.2.1 through 25.2.4) and the estimation of sampling distributions of their estimators are discussed in this chapter.

25.2 Measures of Similarity

The similarity indices are functions of data from two different populations that produce a single numerical value that is used for the purpose of measuring the amount of similarity among two different sets of data. What constitutes a "similarity" between two populations? The idea of "similarity" is somewhat dependent on the discipline and the situation that a researcher is interested in investigating. As a result there are many different types of similarity measures based on different ideas of "similarity." The most commonly used similarity measure is the Pearson's correlation coefficient (or its estimate, r) which is based on the idea of changing size of one type of measurements in relation to the other. It involves two different types of measurements taken on the same unit of population. By definition, the correlation coefficient is independent of units of these measurements. However, most similarity measures do not measure statistical correlation. Different measures are based on different ideas of similarity, such as distance between two populations, or the amount of overlap between two populations. Measures based on the parametric assumptions of underlying populations, as well as nonparametric measures, are available in the literature. Here we restrict our study to measuring the similarity between two exponential or two normal populations that are commonly encountered in practice.

Similarity indices are used in almost every field, from linguistics to genetics. Whenever measurements of some sort are compared on different levels, the question of similarity or dissimilarity among them arises. For example, x-rays read by two different radiologists, and how much consensus there is between their readings? When one develops a function to quantify this comparison, then it becomes a similarity measure. If researchers are interested in its complement, i.e., the amount of differences in the readings by two radiologists, then the measure is known as a measure or index of dissimilarity. Another example would be that of comparison of feeding patterns of two different species sharing the same resources. Ecologists are interested in knowing how different species share the same resources. Do they use the same resources in a given environment or different parts of the same environment?

Generally two different approaches are used to quantify the amount of similarity between two populations. One approach is to derive measures using the frequency distribution of the resource use by two different categories (May (1975), Slobodchikoff and Schulz (1980), and Hurlbert (1978)). The second approach is

to estimate measures, assuming samples of observations are drawn from continuous distributions (Slobodchikoff and Schulz (1980), Harner and Whitmore (1977), and MacArthur (1972)).

Here we study four different similarity measures: the overlap coefficient (δ), Matusita's measure (ρ), Morisita's measure (λ), and MacArthur-Levins' measure (α^*). Of these four measures, δ is the easiest one to describe graphically.

25.2.1 Overlap Coefficient δ

Different measures use different concepts of closeness or differential. Obviously, no one measure is capable of summarizing information in all possible situations. One such measure, δ, also known as the overlap coefficient (Inman and Bradley (1989)) is based on the concept of common area under two distributions. In literature, the symbol Δ is used commonly to indicate the overlap coefficient. However, to maintain consistency of notations used for the distribution-fitting in the remaining chapters of this book, we will use the symbol $\delta = \delta(f_1, f_2)$ to indicate the overlap of two density functions f_1 and f_2.

Let $f_1(x)$ and $f_2(x)$ be the probability density functions of random variable X, defined on the real line for two different populations. The overlapping coefficient δ is the area under two functions simultaneously, defined as,

$$\delta = \int \min\left[f_1(x),\, f_2(x)\right]\, dx. \tag{25.2.1}$$

It indicates closeness between two distributions. This measure takes values between 0 and 1 ($0 \leq \delta \leq 1$). A value of δ closer to one is an indication of a better integration of two populations, whereas a value closer to zero indicates very little commonality between two populations with densities f_1 and f_2. The measure is equal to 1 when f_1 and f_2 coincide, and is equal to 0 when they do not overlap, that is, when the product, $f_1(x)f_2(x)$, is equal to 0 for all values of x. Mathematically, the measure of overlap can also be shown to be equal to

$$\delta = 1 - \frac{1}{2}\int |f_1(x) - f_2(x)|\, dx. \tag{25.2.2}$$

This index has also been referred to as Weitzman's measure (Mulekar and Mishra (1994)) and Czekanowski's proportional similarity index (Mimoto and Zitikis (2009)). Several different terms have been used to denote this measure. Bradley (1985), Gower (1985), Inman and Bradley (1989), and Mulekar and Mishra (1994) used the term OVL to refer to the overlap measure δ. Mimoto and Zitikis (2009) denoted it by $OV(f, g)$ to indicate the area of overlap of the two densities. Anderson, Ge, and Leo (2005) referred to δ as a measure of convergence. They also referred to the complement of δ, i.e., $1 - \delta$, as a measure of alienation between two groups. In the context of medical data, δ has been called the **proportion of similar responses** (Rom and Hwang (1996) and Stine and

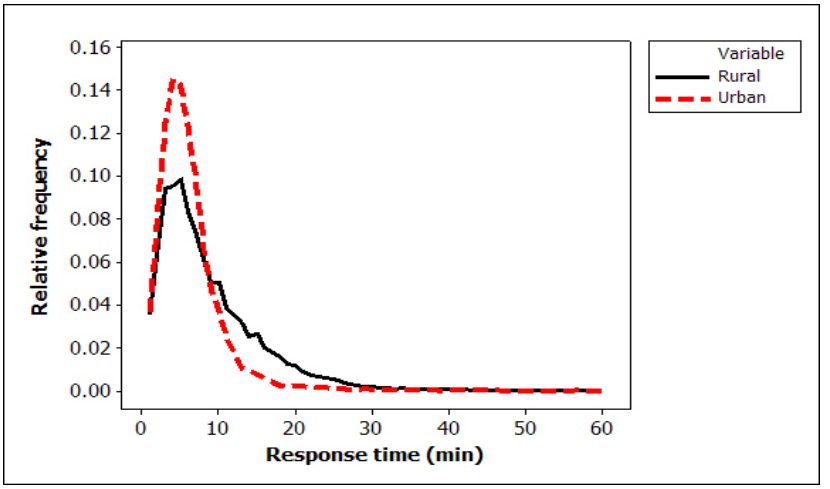


Figure 25.2–1. Overlap of response times in urban and rural Alabama.

Heyse (2001)). Harner and Whitmore (1977) referred to it as the **density overlap measure**. The similarity measure δ is also known as the **Renkonen's index** (Wolda (1981), Gerrard and Barbour (1986) and Schatzmann, Gerrard, and Barbour, (1986)) in reference to research by Renkonen (1938) and the **Bray-Curtis index** (Bray and Curtis (1957)) referring to scientists Bray and Curtis who revitalized the use of δ in practice. Under certain conditions the measure δ is also related to the Mahalanobis distance and the Kullback discrimination information (Inman and Bradley (1989) and (1994), and Mulekar and Mishra (1994)).

Example 1.
The Trauma Center at the University of South Alabama collected data from the emergency medical calls from vehicular accidents that occurred in 2001–02. Although geographically the majority (about 81% counties) of Alabama is rural, about 75% of the calls were from urban areas. The response times ranged from 1 minute to 186 minutes, making both rural and urban area distributions of response times extremely right-skewed. This overpowering skewness made it difficult to visually observe differences in two distributions for the lower response times where most of the data is located. As a solution to this problem and to obtain a visual aid of the overlap of two distributions, for demonstration purposes, the data was truncated at 60 min. Figure 25.2–1 shows the relative frequency distributions of response times (in minutes) for rural and urban vehicular accident locations. It shows that lower response times (at most 10 min.) are more likely for the urban area accidents and higher response times (over 10 min.) are more likely for the rural area accidents. The overlap between the two distributions was estimated to be $\hat{\delta} = 0.79$.

Computing the overlap coefficient. The overlap coefficient for a given data

can be computed as follows:

- Suppose for both groups, the data are available at points $y_1, y_2, \ldots, y_n$, such as a grouped frequency distribution. If the points are equidistant or the grouped frequency distribution has equal class lengths, then $y_i - y_{i-1} = \varepsilon$ is constant for all $i = 2. \ldots, n$.

- Compute the relative frequency distribution for each group, say rf_1 and rf_2.

- Compute $\min(rf_1, rf_2)$ for each class, $i = 2. \ldots, n$.

- Because
$$\lim_{\varepsilon \to 0} \left\{ \varepsilon \sum_{i=1}^{n} \min(rf_1, rf_2) \right\} = \delta,$$
for some small value of ε, compute the overlap measure as
$$\hat{\delta} = \varepsilon \sum_{i=1}^{n} \min(rf_1, rf_2).$$

Note that for a frequency distribution with unequal class sizes, an adjustment should be made by taking into account the class sizes. Then compute the overlap coefficient as,

$$\hat{\delta} = \sum_{i=1}^{n} \varepsilon_i \left[\min(rf_1, rf_2) \right],$$

where ε_i is the length of the i-th class.

Example 2.
Consider the response times for urban area accidents from Example 1. Figure 25.2–2 shows the distribution of response times with a Weibull fit (scale = 6.86 and shape = 1.63) superimposed on it indicated by the curve. The Cramer-von Mises W test used to assess the goodness of the Weibull fit resulted in $p = 0.01$, indicating that the Weibull fit is not good at a 5% level of significance. However the overlap coefficient calculated as,

$$\hat{\delta} = \sum_{i=1}^{n} \min \left[rf_i, \varepsilon_i \, \text{Weibull}(6.86, 1.63) \right],$$

with $\varepsilon_i = 1$, resulted in $\hat{\delta} = 0.87$. Although the goodness-of-fit test rejected the Weibull fit at a 5% level (which may be due to a large number of data points, $n = 11,164$), there is a considerable amount of overlap between the data distribution and the Weibull fit.

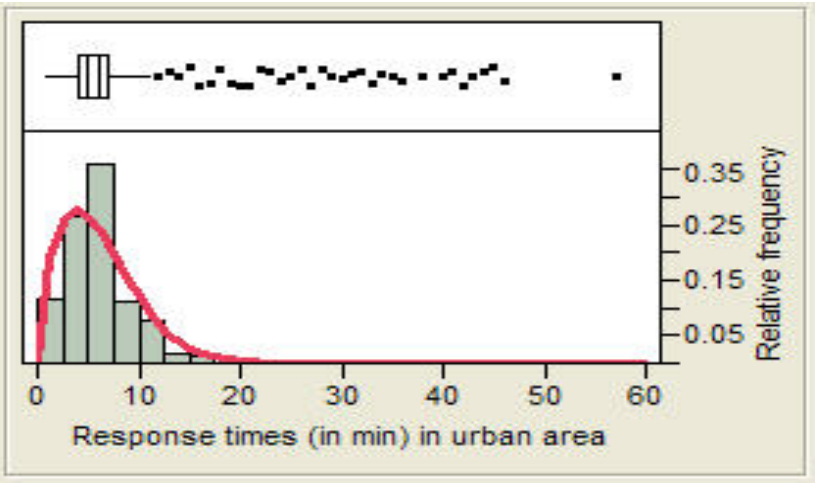


Figure 25.2–2. Response times with Weibull fit.

25.2.2 Matusita's Measure ρ

The second measure studied here is known as the Matusita's measure, ρ, which is defined as,

$$\rho = \int \sqrt{f_1(x) f_2(x)}\, dx. \tag{25.2.3}$$

This measure is based on the distance between two functions (Matusita (1955)). It also takes values in the interval [0, 1]. Matusita actually developed a discrete version of ρ, which is also known as the **Freeman-Tukey measure (*FT*)**. This measure is related to the Hellinger distance (Rao (1963) and Beran (1977)). It is also sometimes referred to as the **Hellinger distance** because Hellinger first proposed it in 1904. Later Bhattacharyya (1943) defined it as,

$$||F_1 - F_2||_r = \left| \int \left[f_1^{1/r}(x) - f_2^{1/r}(x) \right]^r dx \right|^{1/r} \quad \text{for } r \geq 1. \tag{25.2.4}$$

For $r = 1$, the relation $||F_1 - F_2||_2^2 = 2(1 - \rho)$ exists where F_1 and F_2 are the cumulative distribution functions corresponding to the densities f_1 and f_2, and ρ is the Matusita's measure defined in (25.2.3). Matusita (1955) actually used $(1 - \rho)$ as a measure of dissimilarity, but in the ecological literature, ρ has been referred to as **Matusita's measure**. The dual notion of affinity between F_1 and F_2 is extended to measure the closeness between individual members of a finite family of distributions. Smith and Zaret (1982) studied the discrete version of ρ.

25.2.3 Morisita's Measure λ

Morisita (1959) proposed an index of similarity between communities. Consider an ecological study involving two populations from each of which a random sample

is taken. Suppose a total S number of species are represented in these two samples. Let x_{ij} be the frequency of species j in sample i ($i = 1, 2$ and $j = 1, 2, \ldots S$). Then Morisita (1959) proposed the index

$$C_\lambda = \frac{2\sum_{j=1}^{S} x_{1j}x_{2j}}{(\lambda_1 + \lambda_2)\prod_{i=1}^{2}\left[\sum_{j=1}^{S} x_{ij}\right]}, \tag{25.2.5}$$

where λ_i is the Simpson's index of diversity for population i ($i = 1, 2$) given by,

$$\lambda_i = \frac{\sum_{j=1}^{S} x_{ij}(x_{ij} - 1)}{\left[\sum_{j=1}^{S} x_{ij}\right]\left[\sum_{j=1}^{S} x_{ij} - 1\right]}. \tag{25.2.6}$$

Horn (1966) interpreted C_λ as the probability that two individuals drawn randomly from two populations will both belong to the same species, relative to the probability of randomly drawing two individuals of the same species from those populations alone. He also showed that for equal sample sizes, using an estimate of λ_i for sampling with replacement, C_λ can be simplified to,

$$C_\lambda = \frac{2\sum_{j=1}^{S} x_{1j}x_{2j}}{\sum_{i=1}^{2}\sum_{j=1}^{S} x_{ij}^2} \tag{25.2.7}$$

This formulation is also appropriate when data is recorded as the proportions of the respective samples. A continuous counterpart of this index for discrete data is given by,

$$\lambda = \frac{2\int f_1(x)f_2(x)\,dx}{\int [f_1(x)]^2\,dx + \int [f_2(x)]^2\,dx}. \tag{25.2.8}$$

In biological sciences, measure λ is expressed in terms of "crowding" or "probability encounter" of two species in their resource utilization. This affinity measure has been used as an alternative to ρ. It is also more suitable than ρ for inference when the corresponding F_1 and F_2 admit square integrable densities.

25.2.4 MacArthur-Levins' Measure α^*

To study a pattern of resource use by two different species ($i = 1, 2$), Levins (1968) and MacArthur (1972) suggested using

$$\alpha_{12} = \frac{\sum_{j=1}^{n} x_{1j}x_{2j}}{\sum_{j=1}^{n} x_{1j}^2}, \tag{25.2.9}$$

where resource J is divided into n classes, and x_{ij} measures the number of individuals of species i in interval j of the resource. This original measure is asymmetric in nature, i.e., α_{12} is not necessarily equal to α_{21}, due to the term in the denominator which is a property of the i-th species. For example, it measures how species A is similar in resource utilization to species B without any indication of how species B performs compared to species A. Pianka (1973) modified this asymmetric measure to obtain a symmetric measure by taking a geometric mean of two asymmetric measures and thereby quantifying the amount of similarity between two populations regardless of order. Later, Slobodchikoff and Schulz (1980) extended it to a continuous case to get,

$$\alpha = \sqrt{\alpha_{12}\alpha_{21}} = \frac{\int f_1(x)f_2(x)\,dx}{\sqrt{\int [f_1(x)]^2\,dx \int [f_2(x)]^2\,dx}}. \tag{25.2.10}$$

Slobodchikoff and Schulz (1980) and May (1975) studied measure α^*, which reduces to ρ under certain configurations which are discussed later.

25.2.5 Common Properties

All of these four measures of similarity have some common properties:

1. They all take values between 0 and 1, i.e., $0 \leq \delta,\ \lambda,\ \rho,\ \alpha^* \leq 1$.

2. There is symmetry with respect to the two densities in the sense that all the measures are symmetric, i.e., measure(f_1, f_2) = measure(f_2, f_1). In other words, as in the correlation coefficient, they measure the amount of similarity regardless of order. The amount of similarity that one population has with respect to the other is the same as that the later population has with respect to the first one.

3. All of the measures are invariant under the linear transformations

$$Y = aX + b,\ a \neq 0.$$

25.3 Measures of Similarity for Exponential Populations

Comparison of survival functions is a problem commonly faced in survival analysis (Mantel (1966), Peto and Peto (1972), and Lin and Wang (2004)). One possible approach to comparison is to quantify and estimate the amount of similarity or dissimilarity between two survival distributions. Since non-rejection of the null hypothesis of equality of survival rates for two groups does not imply the mean lifetimes of two groups are the same, the estimation of amount of similarity between lifetime distributions of two groups might in fact provide more information about the behavior of two groups.

The simplest and most commonly used distribution in survival and reliability analysis is the one-parameter exponential distribution. Suppose $f_i(x;\, \theta_i)$ indicate two exponential populations with respective hazard rates $\theta_i > 0$ $(i = 1,\, 2)$, that is

$$f(x;\, \theta_i) = \theta_i \exp(\theta_i x), \text{ for } x \in (0,\, \infty).$$

Figure 25.3–1 shows the overlap of two exponential densities. A single crossing point, x^*, is given by,

$$x^* = \frac{\ln(\theta_1/\theta_2)}{\theta_1 - \theta_2} \tag{25.3.1}$$

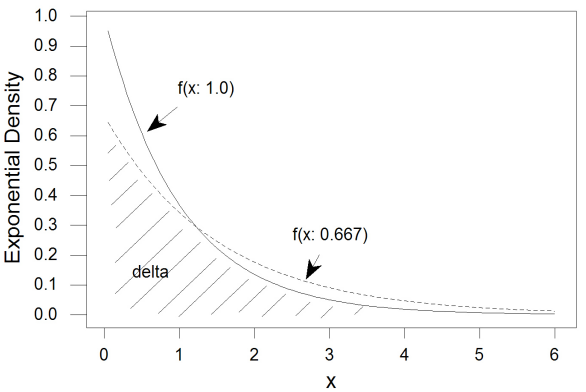


Figure 25.3–1. The overlap of two exponential densities.

Mulekar, Gonzales, and Aryal (2008) showed that the similarity measures discussed in Section 25.2 for exponential populations are given by,

$$\delta = \begin{cases} 1 - \left(\frac{\theta_1}{\theta_2}\right)^{\theta_1/(\theta_2-\theta_1)} + \left(\frac{\theta_1}{\theta_2}\right)^{\theta_2/(\theta_2-\theta_1)} & \text{if } \theta_2 > \theta_1 \\ 1 - \left(\frac{\theta_1}{\theta_2}\right)^{\theta_2/(\theta_2-\theta_1)} + \left(\frac{\theta_1}{\theta_2}\right)^{\theta_1/(\theta_2-\theta_1)} & \text{if } \theta_2 \leq \theta_1 \end{cases}, \tag{25.3.2}$$

$$\rho = \frac{2\sqrt{\theta_1\theta_2}}{\theta_1+\theta_2} \quad \text{and} \quad \lambda = \frac{4\theta_1\theta_2}{(\theta_1+\theta_2)^2}. \tag{25.3.3}$$

The overlap coefficient, δ, is shown graphically in Figure 25.3–1. Note that ρ can be described as a ratio of the geometric mean to the arithmetic mean of two hazard rates. From result (25.3.3), the relation between the two measures, $\rho^2 = \lambda$, is obvious. For two exponential populations it can be easily shown that $\alpha^* = \rho$. Thus, only results for ρ are described for similarity between exponential populations. Let $D = \theta_1/\theta_2$, the ratio of hazard rates, then these measures can be shown to be functions of D as follows

$$\delta = \begin{cases} 1-(1-D)D^{D/(1-D)} & \text{if } D < 1 \\ 1-(D-1)D^{D/(1-D)} & \text{if } D \geq 1, \end{cases} \tag{25.3.4}$$

$$\rho = \frac{2\sqrt{D}}{D+1} \quad \text{and} \quad \lambda = \frac{4D}{(D+1)^2}. \tag{25.3.5}$$

It is intuitive that in the limit as D goes to 0, all these measures equal 0. It can also be seen easily that D gets closer to zero, if θ_1 is considerably smaller than θ_2, i.e., when very little similarity exists between the two populations. The calculation of overlapping measure for two exponential densities is described in Example 3.

Example 3.
Consider the two exponential densities, $f_1(x;\, 1.0)$ and $f_2(x;\, 0.6667)$, as shown in Figure 25.3–1. Then the crossover point of two densities is given by $x^* = 1.2164$. Using this crossover point and the cumulative distribution function of the exponential, $F(x_0) = 1 - \exp(-\theta x_0)$, the overlap measure is calculated as,

$$\begin{aligned} \delta &= \int_0^{1.2164} f_2(x;\, 0.6667)\, dx + \int_{1.2164}^{\infty} f_1(x;\, 1.0)\, dx \\ &= [1-\exp(-1.2164(0.6667))] + 1 - [1-\exp(-1.2164(0.1))] \\ &= 0.8517. \end{aligned}$$

Using $D = 1.5$, we can also calculate δ by substitution in (25.3.4) as,

$$\delta = 1-(D-1)D^{D/(1-D)} = 1-(1.5-1)(1.5)^{1.5/(1-1.5)} = 0.8519.$$

The very insignificant difference in two values is due to differing rounding off in calculations.

How are the values of these three measures in relation to each other? As seen from Figure 25.3–2, λ attains values closer to δ, for a smaller ratio of hazard

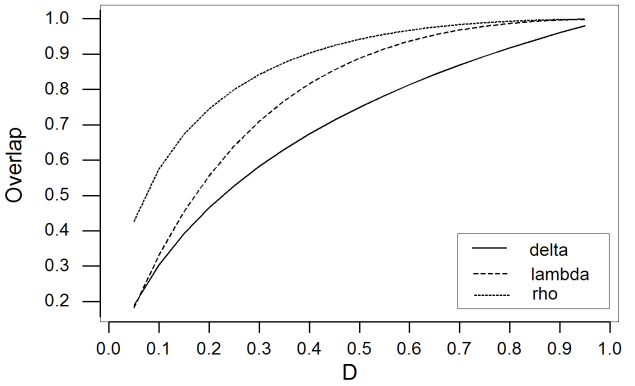


Figure 25.3–2. Measures of similarity as functions of D for exponential populations.

rates. On the other hand, it becomes closer to ρ as the ratio of hazard rates gets closer to 1. For all the D values in the range of 0.2 to 0.8, differences of as much as ± 0.1 are observed between the values of these three measures. In addition to the general properties for all overlap measures described in Section 25.2.4, these overlap measures for exponential populations exhibit the following properties for all values of D.

- $\lambda \leq \rho = \sqrt{\lambda}$.
- $D = 1 \Leftrightarrow \delta = \rho = \lambda = 1$.
- $\lim_{D \to 0} \delta = \lim_{D \to 0} \rho = \lim_{D \to 0} \lambda = 0$.
- Symmetry with respect to D: All the measures are "symmetric" in D around 1, i.e., $\delta(D) = \delta(1/D)$, $\rho(D) = \rho(1/D)$, and $\lambda(D) = \lambda(1/D)$. This "symmetry" is not equivalent to the mirror image symmetry. It is a multiplicative equivalent of the "even functions," where typically a function is considered an even function in the additive sense.
- Monotonicity property: All these measures are monotonically increasing with respect to D for $0 < D < 1$, and monotonically decreasing with respect to D for $D > 1$. The rate of increase changes depending on the measure and value of D.

Although these similarity measures are used in practice by ecologists, economists, and engineers, their sampling distributions are unknown. Bar-Hen and Daudin (1998) proposed the chi-square distribution to model asymptotic ($n_1 \to \infty$ and $n_2 \to \infty$) distribution of Matusita's distance under the null hypothesis of equality of parameters for the two populations, i.e., $H_0 : \theta_1 = \theta_2$. This result may

be useful for hypothesis testing but is not useful for the estimation of sampling distributions because in practice two populations with equal parameters are not observed very often.

Mulekar, Gonzales, and Aryal (2008) studied estimation of these measures under exponential assumption. They derived expressions for estimating approximate bias and mean squared errors for these measures using the Taylor series expansion method. They showed that the overlap estimates obtained using

$$\hat{D} = \frac{\min(\bar{x}_1, \bar{x}_2)}{\max(\bar{x}_1, \bar{x}_2)}$$

are positively biased for $D < 0.5$ and negatively biased for $D \geq 0.5$. Both the bias and MSE decreased steadily with increasing sample size.

25.4 Measures of Similarity for Normal Populations

Estimating measures of similarity for two normal densities is a slightly more complicated process because the normal is a two-parameter family of distributions as compared to the one-parameter exponential family studied here. Thus, we need to take into account different combinations of parametric relations.

25.4.1 The Equal Means Case

Although the problem of estimating the overlap of two populations and making inferences about it has received considerable attention (Maurer (1982), Mishra, Shah, and Lefante (1986), Jarrett, Slow, and Morgan (1985), and Lu, Smith, and Good (1989)), most of the results assume homogeneity of variances. According to Lu, Smith, and Good (1989) most of the methods are robust with respect to the condition of normality but heterogeneity of variances is often a more serious problem. Mulekar and Mishra (1994) studied inferential procedures for these measures when sampling from normal populations with different variances but equal means.

Let $f_i(x;\, \mu,\, \sigma_i)$ be the density functions of two normal distributions with the same expectation μ and different variances σ_i^2 $(i = 1, 2)$, respectively. Define $C = \sigma_1/\sigma_2$ $(C \geq 0)$ as the ratio of standard deviations. Under the equal means condition, the four similarity measures that are of interest to us are:

$$\rho = \alpha^* = \sqrt{\frac{2C}{1+C^2}}, \tag{25.4.1}$$

$$\lambda = \frac{2\sqrt{2}}{\sqrt{1+C^2}} \left(\frac{C}{1+C} \right), \tag{25.4.2}$$

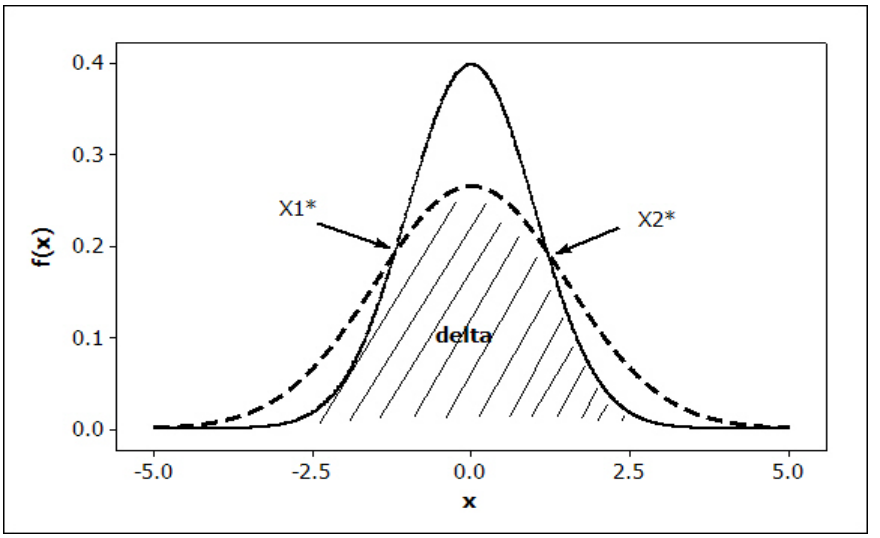


Figure 25.4–1. The overlap of two normal densities with equal means.

and

$$\delta = \begin{cases} 1 - 2\Phi(b) + 2\Phi(Cb) & \text{if } 0 < C < 1 \\ 1 + 2\Phi(b) - 2\Phi(Cb) & \text{if } C \geq 1, \end{cases} \tag{25.4.3}$$

where

$$b = \sqrt{-\ln\left[\frac{C^2}{1-C^2}\right]}$$

and $\Phi(\cdot)$ is the distribution function of the standard normal variate. The points of intersection of the two distributions are given by X_1^*, $X_2^* = \mu \pm \sigma_2 Cb$ (see Figure 25.4–1).

Just as in the exponential case, $\alpha^* = \rho$ in the equal means case for normal populations. Therefore, results will be presented for ρ only. Note that all these similarity measures become independent of the population mean μ and depend only on the population variances. In other words, all these similarity measures are completely determined by the ratio of population variances. The numerical value of similarity measures is zero if and only if $C = 0$ or $C = \infty$. Also, these measures take a numerical value of 1 if and only if $C = 1$. All four measures possess properties of reciprocity, invariance, and piecewise monotonicity.

- Symmetry with respect to C: All the measures are "symmetric" in C around 1, i.e., $\delta(C) = \delta(1/C)$, $\rho(C) = \rho(1/C)$, $\lambda(C) = \lambda(1/C)$. As in the case of the exponential, this "symmetry" is not equivalent to the mirror-image symmetry. It is a multiplicative equivalent of the "even functions," where typically a function is considered an even function in the additive sense.

- Monotonicity property: All these measures are monotonically increasing with respect to C for $0 \leq C \leq 1$, and monotonically decreasing in C for $C > 1$.

Example 4.
Consider two normal densities, $f_1 = N(0, 1)$ and $f_2 = N(0, 4)$. Using $C = 0.5$ and $b = 1.0481$, the points of intersection are given by,

$$0 \pm 2(0.5)\sqrt{-\ln[0.5^2/(1-0.5^2)]},$$

i.e., $x_1^* = -1.0481$ and $x_2^* = 1.0481$. Then, using the `Normalcdf` function on a TI-84, we get

$$\begin{aligned}\delta &= \int_{-\infty}^{-1.0481} f_1(x)\,dx + \int_{-1.0481}^{1.0481} f_2(x)\,dx + \int_{1.0481}^{\infty} f_1(x)\,dx \\ &= 0.1473 + 0.3998 + 0.1473 \\ &= 0.6944.\end{aligned}$$

We can get the same result using (25.4.3) as follows:

$$\begin{aligned}\delta &= 1 - 2\Phi(1.0481 + 2\Phi(0.5(1.0481)) \\ &= 1 - 2\Phi(0.8527) + 2\Phi(0.6999) \\ &= 0.6944.\end{aligned}$$

Mulekar and Mishra (1994) also studied the relationship of each of these similarity measures to the ratio C for $0 < C < 1$. Morisita's measure is always between δ and ρ with δ being the smaller. For smaller C values, λ assumes values close to δ. As the value of C increases, the difference between λ and ρ diminishes rapidly. This observation was supported by their Lemma 2. In fact, these relationships look very similar to those described in Figure 25.3–2 for the exponential populations.

Approximate expressions for bias and variance of the estimates of similarity measures were also developed by Mulekar and Mishra (1994) using the Taylor series expansion. They also computed estimates of similarity measures from samples simulated using the Alabama Supercomputer to corroborate their results and study sampling distributions. Because all measures of similarity are independent of mean for the equal means case, the simulation study included populations with $\mu = 0$ and $C = 0.1(0.1)0.9$. From each population 1,000 samples of 10, 25, 50, 100, 200, and 500 observations were generated.

In general, the actual measures were found to be underestimated. Only for very small values of C and small sample sizes, they were observed to be overestimated. The amount of upward bias was quite negligible compared to the downward bias and the bias approached zero very rapidly with increasing sample

size. In fact, for samples of size 50, the bias was fairly close to 0 for most practical applications. Matusita's measure was less biased than the other two and $\hat{\delta}$ was the most biased estimator. However, the bias of $\hat{\delta}$ was the faster in approaching 0 than that of the other two measures, and that of $\hat{\rho}$ the slower. In general, all the estimators of similarity measures obtained by substituting the estimates of population variances, i.e.

$$\hat{C} = \sqrt{\frac{S_1^2}{S_2^2}},$$

in equations (25.4.1) through (25.4.3) are biased, but the bias decreases as sample size increases. Even though the amount of bias is different for different measures, the bias by itself cannot be used as a criterion for selection of a measure. The amount of variance and the rate of change of bias with respect to the sample size need also to be taken into account. When sampling from populations with quite different variances, the estimates of similarity measures tend to be more biased. As C got close to 1, the sampling distributions of $\hat{\rho}$ and $\hat{\lambda}$ got closer to each other and the variability decreased.

Mulekar and Mishra (2000) studied the confidence interval estimation of these overlap measures computed using jackknife, bootstrap, transformation, and Taylor series approximation methods. They conducted a Monte Carlo study to compare the effectiveness of the four methods of constructing confidence intervals for the similarity measures on AREN (Alabama Research and Education Network) C94/264. They generated samples from two normal densities with equal means. One thousand sets, each containing two independent samples of sizes 10, 25, 50, 100, and 200 were generated from the normal populations with $C = 0.05(0.05)0.95$. For these configurations, they computed 90%, 95%, and 99% confidence intervals using each of the four methods, from each of the samples. Then they recorded the coverage probability (computed as the number of such intervals containing the true overlap coefficient value divided by 1,000) and the average length of intervals.

Although almost free of bias, the coverage probabilities clearly showed that the Jackknife method is not suitable for estimating overlap coefficients using confidence intervals. The Taylor series and transformation methods worked well except for values of C close to 1 and very small sample sizes. When comparing these estimation procedures with respect to the inclusion probability, the jackknife procedure was not recommended for the estimation of ρ, λ, and δ. All three measures displayed a similar pattern of behavior for values of C close to 1. As the sample size increased, the coverage probability got closer to the confidence level. In fact, coverage probabilities were very good for sample sizes as low as 40. Comparison of the average lengths of confidence intervals showed similar patterns for λ and ρ. However, in general the Jackknife, Taylor series, and transformation technique intervals for ρ are the shortest and those for δ are the longest. But for the bootstrap method this trend was reversed. Of the four methods, the jackknife

gave the shortest intervals but also had lowest inclusion probability. When using the bootstrap estimation method, the length of confidence intervals rose sharply (getting almost close to 1) for $C > 0.5$ and rendering these intervals useless in practice. Taking into account both the inclusion probabilities and the length of intervals, the bootstrap seemed to work best for $C \leq 0.5$, whereas the Taylor series and transformation methods were recommended for all C values in the interval (0,1) provided n is at least 40.

25.4.2 The Equal Variances Case

In equal variances case, i.e., when $\sigma_1^2 = \sigma_2^2$, there is only one point of intersection at $(\mu_1 + \mu_2)/2$, provided of course $\mu_1 \neq \mu_2$ (see Figure 25.4–2). Then the overlapping coefficient is given by Bradley (1985) as

$$\delta = 2\Phi\left(\frac{-|B|}{2}\right), \tag{25.4.4}$$

where $B^2 = (\mu_2 - \mu_1)^2/\sigma_1^2$ is the Mahalanobis distance measure. The relationships of δ with the Mahalanobis distance and the Kullback discrimination information is noted by Inman and Bradley (1989, 1994) and Mulekar and Mishra (1994).

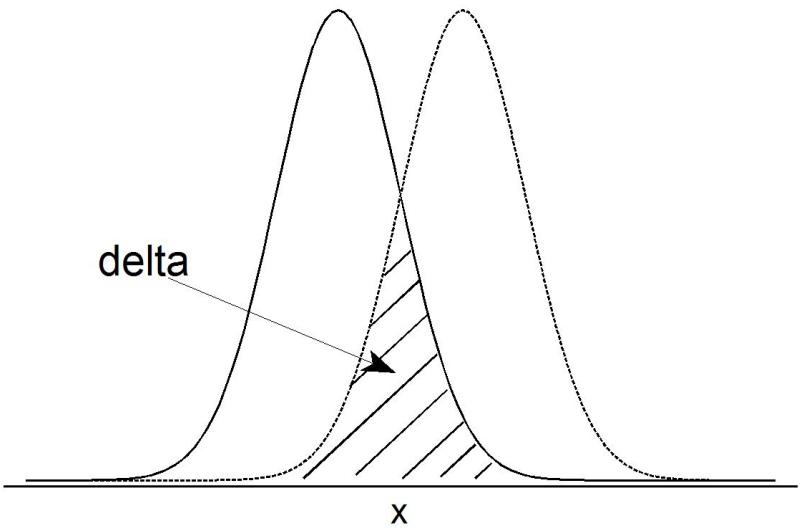


Figure 25.4–2. The overlap of two homogeneous normal densities.

25.4.3 The General Case

Let $f_i(x;\, \mu_i, \sigma_i^2)$ be the density functions of two normal distributions with the expectations μ_i and variances σ_i^2 $(i = 1, 2)$, respectively. Let us consider the estimation of overlap measure, δ, for two normal populations.

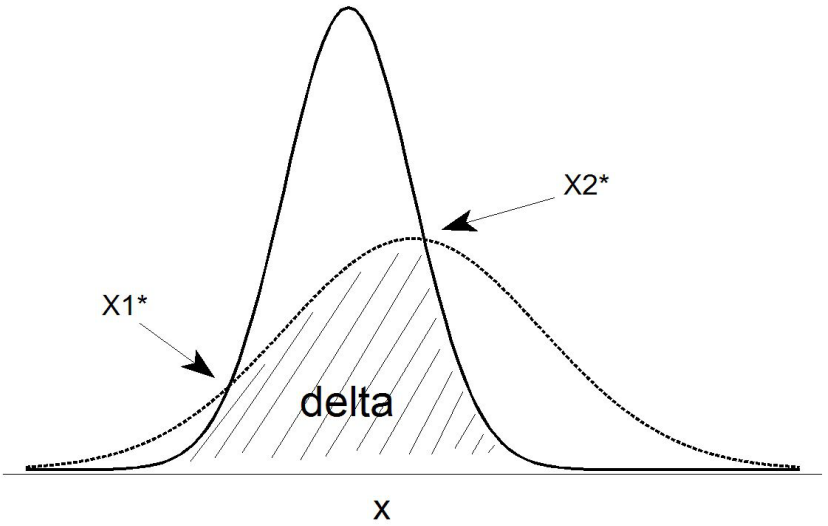


Figure 25.4–3. The overlap of two heterogeneous normal densities.

Figure 25.4–3 shows the overlap, δ, of two heterogeneous normal densities with two crossing points. The density functions of two heteroscadastic normal distributions with different means intersect at two different points given by,

$$K_j = \frac{(\mu_1\sigma_2^2 - \mu_2\sigma_1^2) \pm \sigma_1\sigma_2\sqrt{(\mu_1 - \mu_2)^2 + (\sigma_2^2 - \sigma_1^2)\ln(\sigma_2^2/\sigma_1^2)}}{\sigma_2^2 - \sigma_1^2}, \qquad (25.4.5)$$

for $j = 1, 2$, provided $\sigma_1^2 \neq \sigma_2^2$.

As shown in Figure 25.4–3, Inman and Bradley (1989) proposed using the crossover points $X_1^* = \min(K_1, K_2)$ and $X_2^* = \max(K_1, K_2)$ to compute the overlap measure as

$$\begin{aligned} \delta &= 1 + F_1(X_1^*) - F_2(X_1^*) - F_1(X_2^*) + F_2(X_2^*) \\ &= 1 + \Phi\left(\frac{X_1^* - \mu_1}{\sigma_1}\right) - \Phi\left(\frac{X_1^* - \mu_2}{\sigma_2}\right) \\ &\quad - \Phi\left(\frac{X_2^* - \mu_1}{\sigma_1}\right) + \Phi\left(\frac{X_2^* - \mu_2}{\sigma_2}\right), \end{aligned} \qquad (25.4.6)$$

where $F_i(x)$ is the cumulative distribution function of $f_i(x; \mu_i, \sigma_i^2)$ $(i = 1, 2)$. However, Mulekar (2009) showed that the condition $\sigma_1^2 < \sigma_2^2$ is necessary for using equation (25.4.6) to compute δ. If this condition is not satisfied, the resulting value of δ falls in the interval $(1, 2)$. Mulekar (2009) proposed using the formula (25.4.7) to avoid getting an invalid value for the overlap measure of similarity,

δ. When the parameters of normal densities are unknown, replace them by their unbiased estimators to obtain an estimate of δ, i.e., use $\bar{x}_i$ and s_i^2, respectively, in place of μ_i and σ_i^2 $(i = 1, 2)$.

$$\delta = \begin{cases} 1 + \sum_{j=1}^{2} \sum_{j'=1}^{2} (-1)^{j+j'} \Phi\left(\dfrac{X_j^* - \mu_{j'}}{\sigma_{j'}}\right) & \text{if } \sigma_1^2 < \sigma_2^2 \\ 2\Phi\left(\dfrac{-|\mu_1 - \mu_2|}{2\sigma_1}\right) & \text{if } \sigma_1^2 = \sigma_2^2 \\ 1 + \sum_{j=1}^{2} \sum_{j'=1}^{2} (-1)^{j-j'+1} \Phi\left(\dfrac{X_j^* - \mu_{j'}}{\sigma_{j'}}\right) & \text{if } \sigma_1^2 > \sigma_2^2. \end{cases} \tag{25.4.7}$$

Mulekar and Mishra (1997) studied estimation of three measures of similarity for the general heteroscadastic case. They gave the following results:

$$\rho = \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} \exp\left[\frac{-(\mu_1 - \mu_2)^2}{4(\sigma_1^2 + \sigma_2^2)}\right], \tag{25.4.8}$$

$$\lambda = \frac{2\sqrt{\sigma_1\sigma_2}}{\sigma_1 + \sigma_2} \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} \exp\left[\frac{-(\mu_1 - \mu_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right], \tag{25.4.9}$$

and

$$\alpha^* = \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} \exp\left[\frac{-(\mu_1 - \mu_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right]. \tag{25.4.10}$$

It is easily seen that these 3 measures are related to each other. Each measure can be expressed as a product of two terms, one a function of only scale parameters and the other a function of location-scale parameters. The measures ρ and α^* have the same scale function but differ slightly in location-scale function. The coefficients λ and α^* differ by a constant multiple, which is a ratio of geometric mean and arithmetic mean of population standard deviations. Note that the scale component disappears when $\sigma_1^2 = \sigma_2^2$ and the location-scale component disappears when $\mu_1 = \mu_2$.

Let $d = \mu_1 - \mu_2$ be the difference in population means. Mulekar and Mishra (1997) proposed a reparametrization from

$$\omega = \left\{(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) : \mu_i \in (-\infty, \infty) \text{ and } \sigma_i^2 \in (0, \infty)\right\},$$

for $i = 1, 2$ to

$$\omega_R = \{(A, B) : A \in (0, \infty) \text{ and } B \in (-\infty, \infty)\},$$

such that $A = \sigma_2/\sigma_1 = 1/C$ and $B = d/\sigma_1$. This reparametrization allows a reduction in the number of parameters involved in defining the coefficients.

The results for homogeneous variances can be obtained by setting $A = 1$ (or $C = 1$). The results of Mulekar and Mishra (1994) for the equal means case can be corroborated by setting $B = 0$. These similarity measures under the new reparametrization scheme become,

$$\begin{aligned}\lambda &= \frac{2\sqrt{2}}{\sqrt{1+A^2}}\left(\frac{A}{1+A}\right)\exp\left[\frac{-B^2}{2(1+A^2)}\right] \\ &= \frac{2C\sqrt{2}}{\sqrt{1+C^2}}\left(\frac{1}{1+C}\right)\exp\left[\frac{-B^2C^2}{2(1+C^2)}\right], \qquad (25.4.11)\end{aligned}$$

$$\rho = \sqrt{\frac{2A}{1+A^2}}\exp\left[\frac{-B^2}{4(1+A^2)}\right] = \sqrt{\frac{2C}{1+C^2}}\exp\left[\frac{-B^2C^2}{4(1+C^2)}\right], \qquad (25.4.12)$$

$$\alpha^* = \sqrt{\frac{2A}{1+A^2}}\exp\left[\frac{-B^2}{2(1+A^2)}\right] = \sqrt{\frac{2C}{1+C^2}}\exp\left[\frac{-B^2C^2}{2(1+C^2)}\right]. \qquad (25.4.13)$$

It is easily seen that the above measures exhibit the following relation.

$$\lambda = \left(\frac{2\sqrt{A}}{1+A}\right)\alpha^* = \left(\frac{\sqrt{2(1+A^2)}}{1+A}\right)\rho^2,$$

which is equivalent to

$$\lambda = \left(\frac{2\sqrt{C}}{1+C}\right)\alpha^* = \left(\frac{\sqrt{2(1+C^2)}}{1+C}\right)\rho^2. \qquad (25.4.14)$$

Equation (25.4.14) leads to the following relations.

- $\rho^2 \le \lambda \le \alpha^* \le \rho$.
- $\rho^2 = \lambda = \alpha^* \Leftrightarrow \sigma_1^2 = \sigma_2^2$.
- $\alpha^* = \rho \Leftrightarrow \mu_1 = \mu_2$.

For the proof of these relations, refer to Mulekar and Mishra (1997). In the extreme cases, i.e., if $A \to \infty$ (or $A \to 0$) or $|B| \to \infty$, then all the three measures tend to zero in limit. These reparametrized measures also hold the properties of symmetry, reciprocity, and invariance under linear transformation.

- Symmetry with respect to B: The measures are symmetric with respect to B, i.e., $\lambda(A, B) = \lambda(A, -B)$, $\rho(A, B) = \rho(A, -B)$, $\alpha^*(A, B) = \alpha^*(A, -B)$ for all $B \in (-\infty, \infty)$.
- Reciprocity: The reciprocity property with respect to A holds for all $A \in (0, \infty)$ if and only if $B = 0$, i.e., measure$(A, 0)$ = measure$(A^{-1}, 0)$.
- Invariance: The coefficients are invariant under one-to-one differentiable transformations, $Y_i = g(X_i)$.

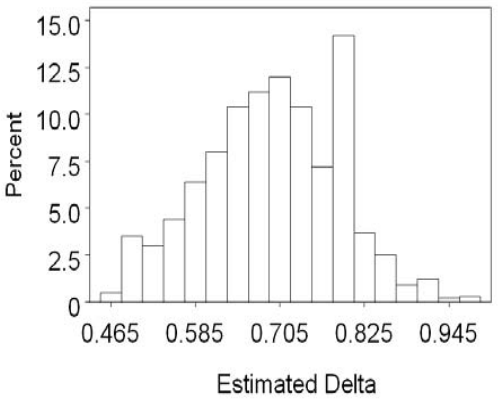
25.5 Sampling Distributions: Exponential Populations Case

To obtain more information about the nature of the sampling distributions of measures of similarity, a simulation study was conducted using the Alabama Supercomputer. One thousand pairs of independent random samples were generated from each pair of populations defined using the ratio of hazard rates $D = 0.05(0.05)0.95$. The samples of sizes $n = 10$, 25, 50, 100, and 200 were obtained from each pair of population. The similarity measures δ, ρ, and λ were estimated from each pair of samples. Using these 1,000 estimates of similarity measures from each population, a frequency distribution of estimated measure of similarity was created. This process was repeated for all three measures over all the configurations studied. Just as an example, the histograms of frequency distributions of two cases each for three similarity measures δ, ρ, and λ, respectively, are given in Figures 25.5–1 through 25.5–3, covering a wide range of sample sizes and parametric configurations.

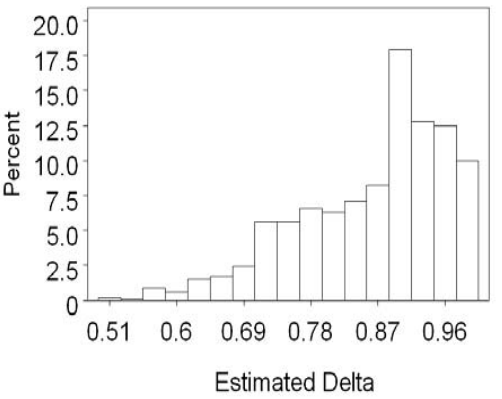
Further inspection of histograms of the similarity-measure estimates showed a variety of shapes of distributions, from fairly symmetric and mound-shaped to extremely skewed. Two examples of sampling distributions of $\hat{\delta}$ are displayed in Figure 25.5–1. Figure 25.5–1(a) shows a mound-shaped distribution of $\hat{\delta}$, whereas Figure 25.5–1(b) shows a very strong left-skewed distribution. On the other hand, the distribution of $\hat{\rho}$ displayed in Figure 25.5–2(a) is left-skewed and that in Figure 25.5–2(b) is fairly symmetric and mound-shaped. Although mound-shaped, normality in Figure 25.5–2(b) might be questionable due to more than the usual frequency of central values. Figure 25.5–3(a) and Figure 25.5–3(b) show the sampling distributions of $\hat{\lambda}$. Both are markedly left-skewed. Even for sample size as high as $n = 200$, the sampling distribution failed to approach symmetry and normality.

Sampling distributions were created and plotted for 95 different configurations for each of the three similarity measures described earlier. A program called RVL (Reagan and Mishra (1992)) was used to fit distributions to the data. RVL fits distributions to data using the maximum likelihood estimation technique, and ranks the fits based on likelihood values (Hogg, Uthoff, Randles, and Davenport (1972)). Depending on the shape of the sampling distribution, several different continuous distributions such as beta, gamma, normal, Erlang, lognormal, and exponential, were selected for a possible fit. The fit was assessed using goodness-of-fit tests such as the Kolmogorov-Smirnov and Anderson-Darling tests. The likelihood values were used to rank the distributions from the possible fits. In most situations, the two-parameter beta distribution was the best fit observed, i.e., it ranked first. In a few situations where beta fit was not ranked first, it came remarkably close to the first-ranked distribution in fit. The results were corroborated by fitting distributions using SAS and estimating the parameters.

Two examples of fits are shown, Figures 25.5–4 and 25.5–5. Figure 25.5–4

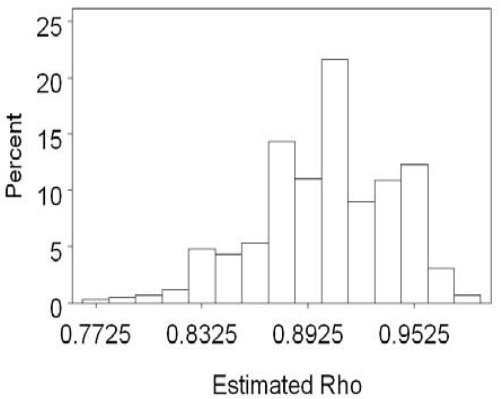


(a) $n_1 = n_2 = 25$ and $D = 0.4$

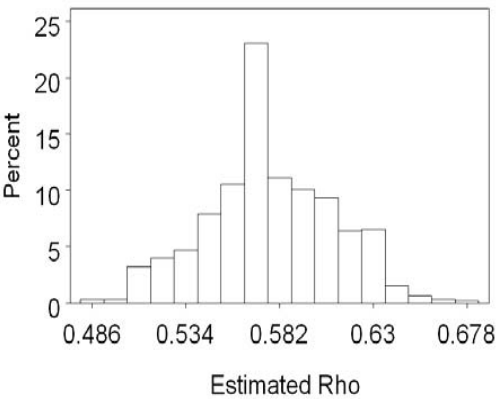


(b) $n_1 = n_2 = 10$ and $D = 0.95$

Figure 25.5–1. The sampling distribution of $\hat{\delta}$.

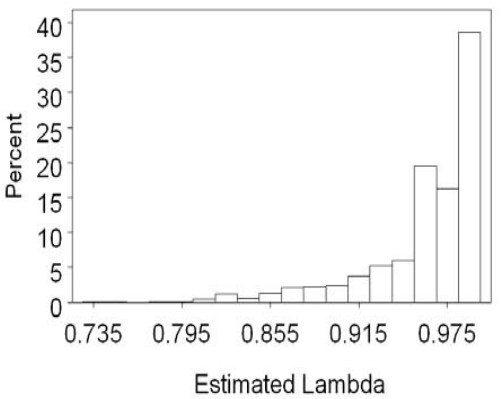


(a) $n_1 = n_2 = 50$ and $D = 0.4$

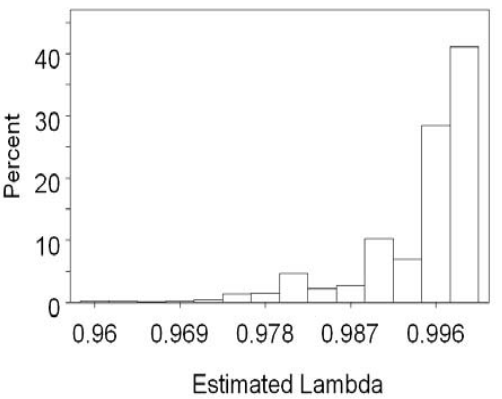


(b) $n_1 = n_2 = 100$ and $D = 0.1$

Figure 25.5–2. The sampling distribution of $\hat{\rho}$.



(a) $n_1 = n_2 = 25$ and $D = 0.4$



(b) $n_1 = n_2 = 10$ and $D = 0.95$

Figure 25.5–3. The sampling distribution of $\hat{\lambda}$.

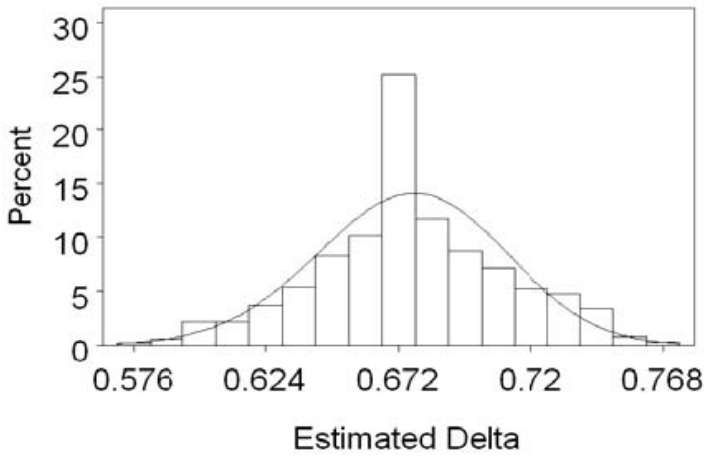


Figure 25.5–4. The distribution of $\hat{\delta}$ ($n_1 = n_2 = 200$ and $D = 0.4$) with a beta fit.

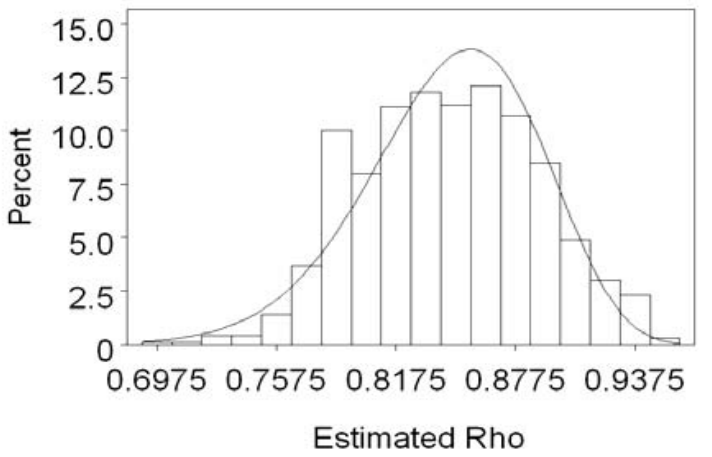


Figure 25.5–5. The distribution of $\hat{\delta}$ ($n_1 = n_2 = 50$ and $D = 0.3$) with a beta fit.

shows the sampling distribution of $\hat{\delta}$ for $n = 200$ and $D = 0.4$. The distribution seems to be fairly symmetric and mound-shaped, except for the exceptionally high frequency of the middle group. The superimposed curve shows the fitted beta distribution. Figure 25.5–5 shows the sampling distribution of $\hat{\rho}$ for $n = 50$ and $D = 0.3$, with the superimposed beta distribution. The sampling distribution is mound-shaped but slightly left-skewed.

The two-parameter beta is one of the most versatile density functions and therefore can be used to describe many different layouts of data. If Y_1^2 and Y_2^2 are independent random variables and Y_i^2 are χ^2 distributed with ν_i degrees of freedom ($i = 1, 2$), then the beta distribution arises naturally as the distribution of $Y_1^2/(Y_1^2 + Y_2^2)$ (Johnson, Kotz, and Balakrishnan (1995)). The probability

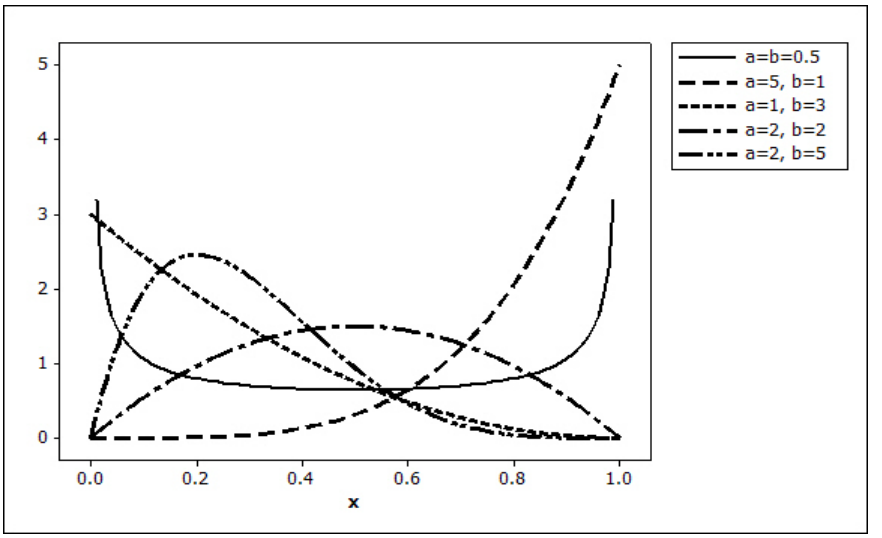


Figure 25.5–6. Beta density functions.

density function of the beta distribution is given by,

$$f_Y(y;\, a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} y^{a-1}(1-y)^{b-1} \text{ for } 0 \le y \le 1 \text{ and } a,\, b > 0, \qquad (25.5.1)$$

where $\Gamma(\cdot)$ is the gamma function. Figure 25.5–6 shows different shapes of the beta density function for different parametric configurations, and the choice of beta distribution is quite obvious from it.

Each similarity measure is a function of D, which is a ratio of two hazard rates and D can be estimated as $\bar{X}_1/\bar{X}_2$, where $\bar{X}_i$ $(i = 1, 2)$ are the means of independent random samples from two populations (Mulekar, Gonzales, and Aryal (2008)). We also know that $\bar{X}_i$ $(i = 1, 2)$ follow the χ^2 distribution with 2 degrees of freedom (Johnson, Kotz, and Balakrishnan (1995)). Taking into account the properties of the beta distribution, looking at different shapes of simulated sampling distributions, and considering the fact that all similarity measures take values in the interval [0, 1] and are functions of a ratio of hazard rates, it seems reasonable that the sampling distributions of overlap coefficients are approximated well by the beta distribution.

Now that we know that the beta family of distributions provides a good fit for the sampling distributions of all three similarity measures, the next question that needs to be answered is: "Which specific distributions from this family are useful?" In other words, can we estimate the parameters of the distribution knowing the sample size and the ratio of hazard rates?

Estimating Parameters of Sampling Distributions. Estimating sampling distribution of a similarity measure through simulation is a simple but time-consuming task. In a practical situation though, decisions are made based on

a single pair of independent random samples without any possibility of multiple samples to establish a sampling distribution. Can we determine the nature of sampling distribution for known population characteristics? Now the problem reduces to that of determining if the parameters a and b of the beta distribution defined in (25.5.1) are functions of known characteristics of populations involved in the study as well as sample sizes. And, if yes, then what is the nature of such relation?

In order to find a solution to this problem, values of the parameters a and b that are estimated from the fits for all different configurations, were plotted (see Figure 25.5–7 and Figure 25.5–8) for different sample sizes considered in this study. Figure 25.5–7 shows a fairly linear relation between a and D for the different sample sizes studied. Based on the slight curvature on both ends for the larger sample sizes, it could also be argued that a polynomial (maybe cubic) relation between a and D might be more appropriate for larger sample sizes which becomes increasingly linear with decreasing sample size. On the other hand, Figure 25.5–8 clearly shows an exponential relation between b and D, where the rate of change increases with the increasing sample size.

Figure 25.5–9 and Figure 25.5–10 show the three-dimensional views of relations between estimated values of a and b, respectively, and the values of (D, n) used in the simulations. Interaction between D and n is also evident from these plots.

A similar technique was used for similarity measures ρ and λ to establish the existence of relations between (a, b) and (D, n). Although different similarity measures showed different nature of relations, the existence of relations between beta parameters and the parameters of overlapping populations was obvious. Now the problem reduces to that of determining functions g_1 and g_2 such that $a = g_1(D, n)$ and $b = g_2(D, n)$, where a and b are parameters of beta distribution defined in (25.5.1).

To take into account the curvature, $\ln(a)$ and $\ln(b)$ were modeled as functions of D and n, or their combinations, i.e.,

$$\ln(a) = g_1(D, n) \quad \text{and} \quad \ln(b) = g_2(D, n).$$

The JMP (SAS Institute, Cary, North Carolina) software was used for model fitting. Variables D and n along with some of their functions such as $\sqrt{D}$, D/n, D^2, D^3, etc. were included in the possible pool of explanatory variables for modeling the beta parameters. A stepwise regression technique was used for selection of possible explanatory variables. The values of the coefficient of determination (R^2) and the mean-squared error (MSE) were used to determine the goodness-of-fit of model, and for the final model comparison and selection. The resulting explanatory variables selected to model beta parameters for each measure of similarity are listed in Table 25.5–11. All the MSE values are very low and fairly comparable, except for estimating parameter b for the sampling distribution of $\hat{\delta}$. However, it is not enough to cause a concern about the fit. Also, the same

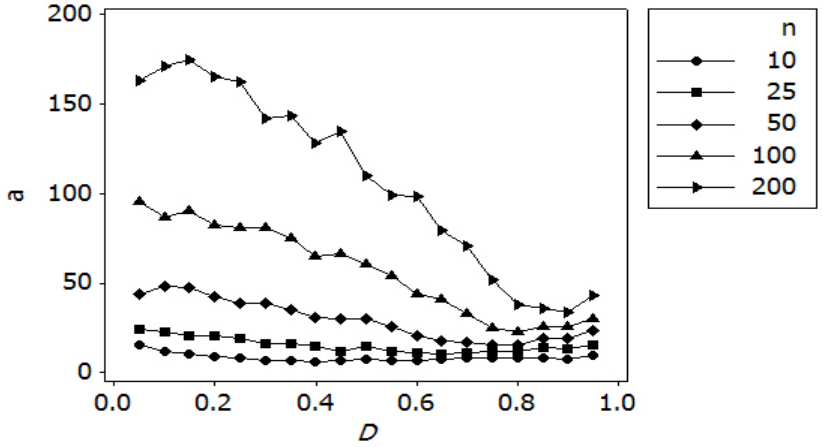


Figure 25.5–7. Scatterplot of a against D for different values of n for δ.

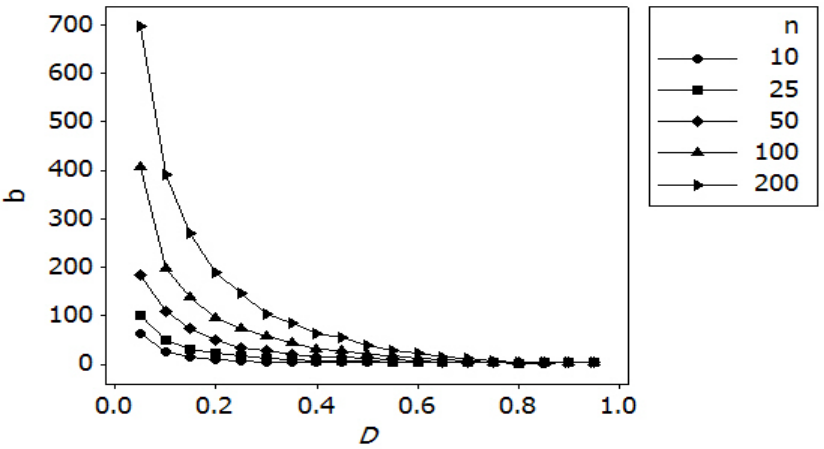


Figure 25.5–8. Scatterplot of b against D for different values of n for δ.

model gives the highest R^2 value. Results in Table 25.5–11 show that at least 97% of the variation in parameter values can be explained using functions of the sample size and hazard ratio, clearly indicating that the developed models fit the simulated data very well.

Using the sample size and hazards ratio, we propose the following relations to estimate the parameters of beta distribution to obtain estimated sampling distributions.

For δ:

$$\begin{aligned} \hat{a} &= n^{1.08} \exp\left[-0.3719 + \frac{9.9D}{n} - 4.69D^2 + 4.04D^3 - 0.00628nD^3\right] \\ \hat{b} &= n^{1.21} \exp\left[1.98 - 6.75D^{1/2} + \frac{20.3D}{n} - 0.0102nD^3\right] \end{aligned} \tag{25.5.2}$$

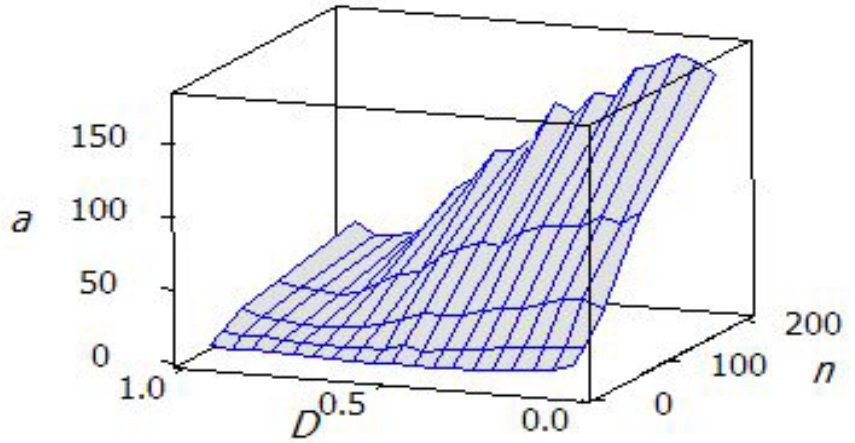


Figure 25.5–9. A 3-dimensional plot of a against D and n for δ.

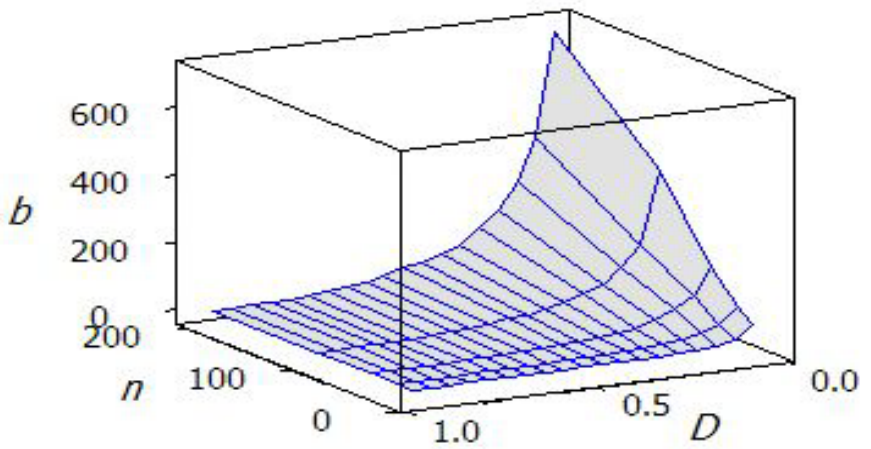


Figure 25.5–10. A 3-dimensional plot of b against D and n for δ.

Table 25.5–11. Assessment of fit for different models.

Similarity measure	Beta parameter	Explanatory variables	R^2	MSE
δ	$\ln(a)$	$\ln(n)$, D/n, D^2, D^3, nD^3	97.7	0.023
	$\ln(b)$	$\sqrt{D}$, $\ln(D)$, D/n, nD^3	98.2	0.054
ρ	$\ln(a)$	$\ln(n)$, D^3, $\sqrt{D}$	96.9	0.030
	$\ln(b)$	$\ln(n)$, D/n, nD^2, n^2D^2, nD^3, $\sqrt{D}$	99.0	0.028
λ	$\ln(a)$	$\ln(n)$, D, D^3	97.1	0.028
	$\ln(b)$	$\ln(n)$, nD^2, D/n, $\ln(D)$	98.1	0.052

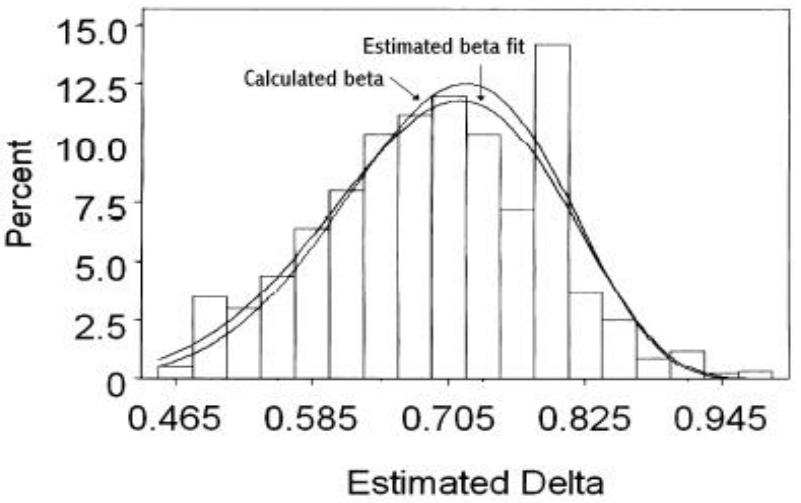


Figure 25.5–12. Beta distribution given by equation (25.5.2) and the fitted distribution using simulation.

For ρ:

$$\begin{aligned}\hat{a} &= n^{0.862}\exp\left[1.4 - 1.51D^{1/2} + 2.08D^3\right] \\ \hat{b} &= n^{1.06}\exp\left[2.35 - 8.06D^{1/2} + \frac{14.9D}{n} + 0.000048n^2D^2 \right. \\ &\qquad \left. + 2.49D^3 - 0.0257nD^3\right] \end{aligned} \tag{25.5.3}$$

For λ:

$$\begin{aligned}\hat{a} &= n^{0.864}\exp\left[0.0964 - D + 2.15D^3\right] \\ \hat{b} &= \frac{n^{1.17}}{D^{1.72}}\exp\left[-4.58 + \frac{18.5D}{n} - 0.0118nD^2\right] \end{aligned} \tag{25.5.4}$$

Example 5.
Suppose we want to estimate the sampling distribution of $\hat{\delta}$ using a pair of random independent samples of size 25 each from two exponential distributions such that $D = 0.4$. From Equations (25.5.2) we get

$$\hat{a} = 15.82 \quad \text{and} \quad \hat{b} = 6.78.$$

In other words, when samples of size 25 are taken from a pair of exponential populations related by ratio, $D = 0.4$, the sampling distribution of $\hat{\delta}$ is estimated to be beta(15.82, 6.78).

The fitted distribution ($n = 25$ and $D = 0.4$), estimated from the simulation study for this configuration, is beta(14.16, 6.26). Figure 25.5–12 shows both the beta densities, one estimated using (25.5.2) and the other estimated by fitting

distributions to simulated data, superimposed on the histogram of simulated values. The distribution estimated using the above-proposed equations predicts the sampling distribution very well.

In fact, the overlap coefficient calculated to measure the similarity between the fitted beta distribution and the one estimated using (25.5.2) was equal to 0.98, indicating a very good estimate by the proposed equations.

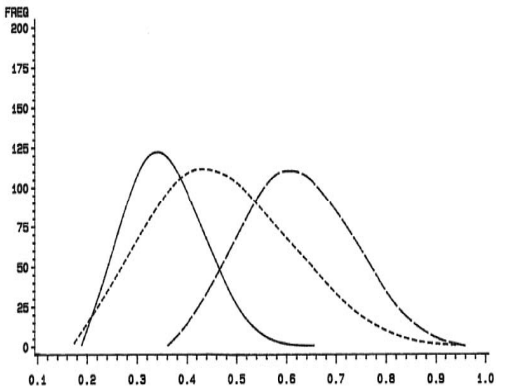
25.6 Sampling Distributions: Normal Populations Case

The normal distribution is the most widely-used distribution to model data (Johnson, Kotz, and Balakrishnan (1995)). In this section, we plan to estimate the sampling distributions of the overlap measure and the other three measures of similarity for two normal populations.

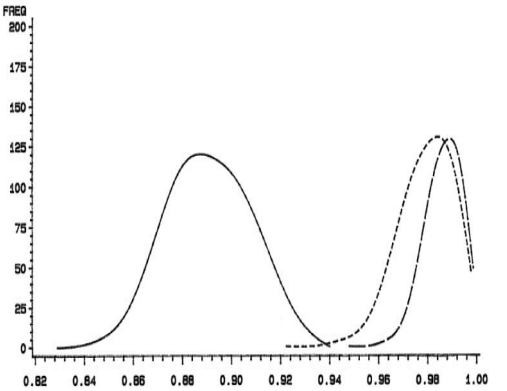
Mulekar and Mishra (1994) conducted a Monte Carlo study using the Alabama Supercomputer X-MP/216 under the assumption of equal means. Although the main objective of this study was to evaluate the performance of approximations to bias and variance of four similarity measures, they also took a passing look at sampling distributions of them. Random samples were generated from two standard normal populations using IMSL subroutines RNNOR and then the random variates were scaled using SSCAL. Since all four coefficients were shown to be independent of the common population mean μ and invariant under linear transformations, pairs of normal populations such that $\mu = 0$ and $C = 0.1(0.1)0.9$, were used in this study. From each population 1,000 samples of 10, 25, 50, 100, 200, and 500 observations were generated. For each pair of samples $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\delta}$ were computed.

The sampling distributions of these estimates for different configurations are shown in Figure 2 of Mulekar and Mishra (1994). They noted that when the two sampled populations have variances that are quite different from each other, the estimates of similarity measures tend to be more biased and the sampling distributions show larger variability. As C gets close to 1, the sampling distributions of $\hat{\rho}$ and $\hat{\lambda}$ get closer to each other and the variability decreases. From the definitions of similarity measures, it is not a surprise that the sampling distributions of $\hat{\rho}$ and $\hat{\lambda}$ behaved similarly, whereas that for $\hat{\delta}$ did not follow the others. The sampling distribution for $\hat{\delta}$ was found to be bell-shaped for all the configurations investigated in this study whereas the shapes of sampling distributions of $\hat{\rho}$ and $\hat{\lambda}$ varied depending on the parametric configuration and sample size. Some of the sampling distributions are shown in Figure 25.6–1. In this study, only comparisons of shapes of sampling distributions for different measures and different sample sizes were done. No attempt was made to estimate the sampling distributions.

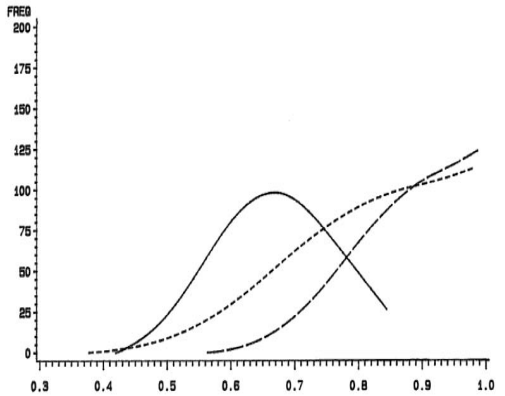
To study the sampling distributions of these similarity measures for the gen-



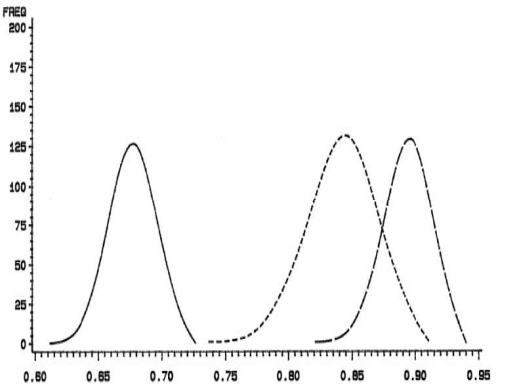
(a) $n = 10$, $C = 0.2$



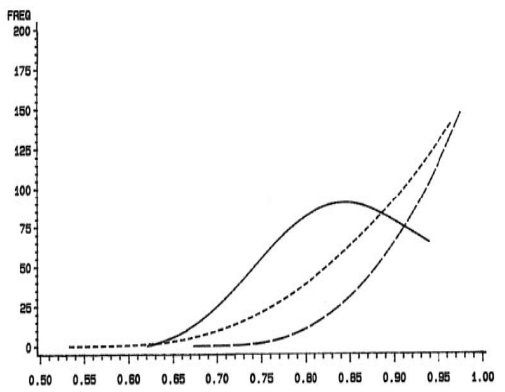
(d) $n = 200$, $C = 0.2$



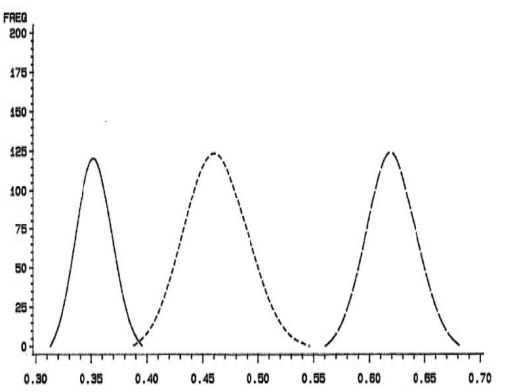
(b) $n = 10$, $C = 0.5$



(e) $n = 200$, $C = 0.5$



(c) $n = 10$, $C = 0.8$



(f) $n = 200$, $C = 0.8$

—— δ - - - - λ – – – – ρ

Figure 25.6–1. Sampling distributions of similarity measures for the equal means case.

eral case, another simulation study was conducted. Using the Alabama Supercomputer, 1,000 pairs of independent random samples were generated from each pair of populations for configurations $A = 0.25(0.25)2.0$ and $B = 0.2(0.2)0.8$. Samples of size $n_1 = n_2 = n = 6$, 10, 25, 50, 100, 200, 500 were obtained from each population.

The program generated 1,000 pairs of samples from standard normal populations using IMSL subroutine RNNOR, and then scaled and shifted them to obtain deviates from the normal distributions with mean μ_i and variance σ_i^2 $(i = 1, 2)$. The similarity measures were estimated from each pair of samples and frequency distributions of estimates were created for each measure and each parametric configuration. Four examples of these distributions for each measure are given in Figures 25.6–2 through 25.6–5. The distributions are generally irregular, with their shapes ranging from symmetric and mound-shaped to extremely left-skewed.

Figure 25.6–2 gives the sampling distributions of $\hat{\delta}$. The sampling distribution shown in Figure 25.2–2 with $n = 6$, $A = 0.5$, and $B = 0.2$ exhibits a left-skewed nature with a slight hump, almost like a half-normal. On the other hand, the one for $n = 25$, $A = 0.25$, and $B = 0.2$ is mound-shaped and fairly symmetric. However, normality of this distribution might be questionable because of the sparse observations in the left tail of the distribution. Even a sample of size as large as 200 did not lead to a normal-like, symmetric sampling distribution.

Figures 25.6–3 through 25.6–5, respectively, show examples of sampling distributions of $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\alpha^*}$. Considering different shapes of these sampling distributions, just like in the case of the exponential, it seems that a very versatile function is needed to describe them. Once again, the beta distribution came through understandably.

The process of fitting distributions with SAS and selecting the best possible fit was followed as described in Section 25.5 for the exponential populations. The estimated parameters of the best possible fit were recorded for each configuration studied. Two examples of the sampling distributions with a beta fit for $\hat{\delta}$ for configurations $(n, A, B) = (10, 0.25, 0.6)$ and $(500, 1.5, 0.2)$, respectively, are shown in Figure 25.6–6 and Figure 25.6–7. Both of them show that the beta distribution fits the sampling distributions well.

Figures 25.6–8 through 25.6–10 show one example each of the sampling distributions for $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\alpha^*}$ with the beta fit superimposed on it. The distribution of $\hat{\rho}$ is fairly symmetric except for only one bin with an unusually large number of measurements. The distribution of $\hat{\lambda}$ is right-skewed. The distribution of $\hat{\alpha^*}$ is also right-skewed, almost like an exponential distribution.

Estimating Parameters of Sampling Distributions. As was the case for the exponential, for the normal distributions also, the beta distribution was found to fit the sampling distributions well. Again, in practice, decisions will be made based on a pair of samples and not from 1,000 simulated samples. Thus, the relationships between the beta parameters, (a, b), and the possible predictors,

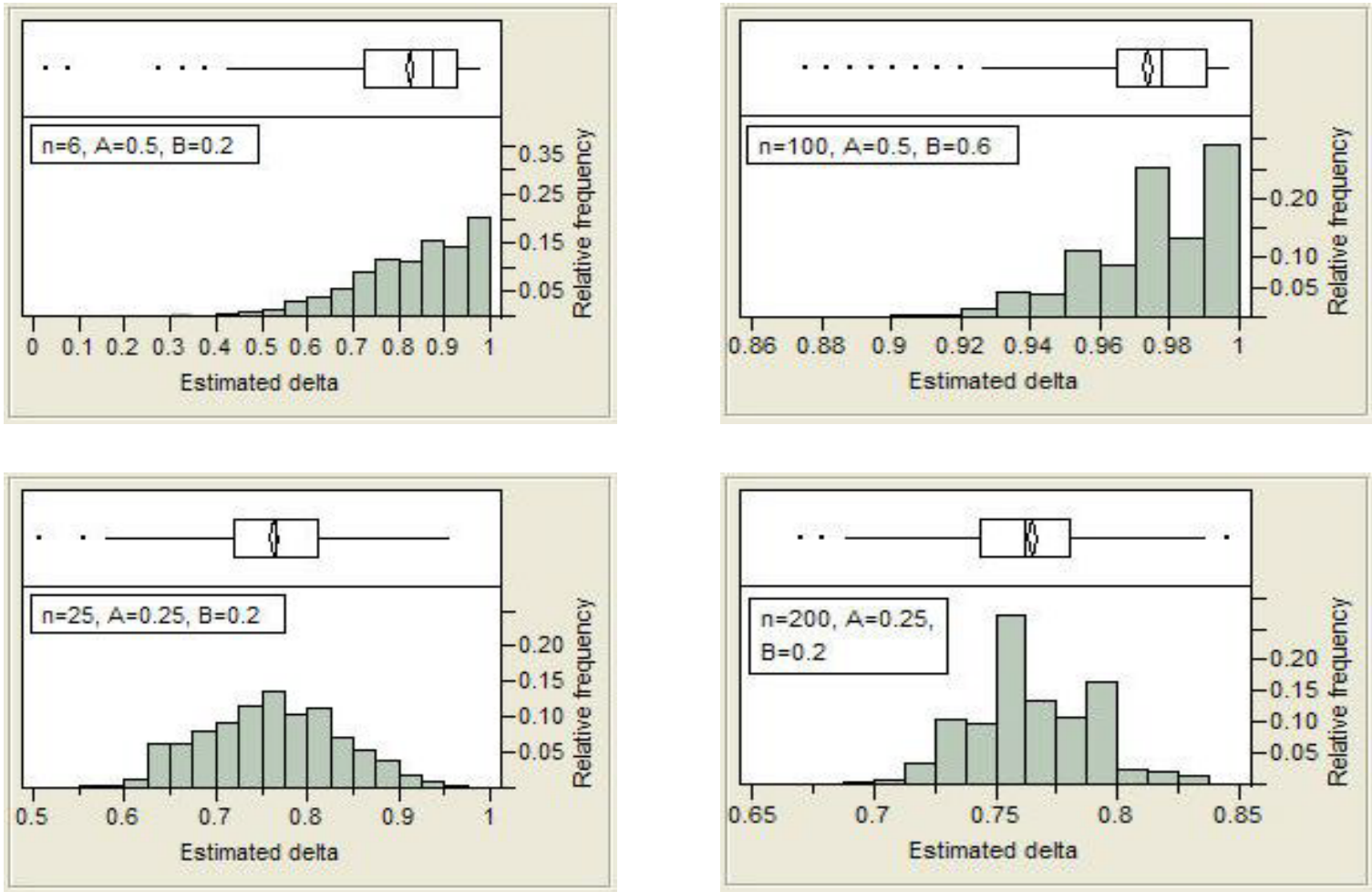


Figure 25.6–2. The sampling distribution of $\hat{\delta}$ for four different configurations.

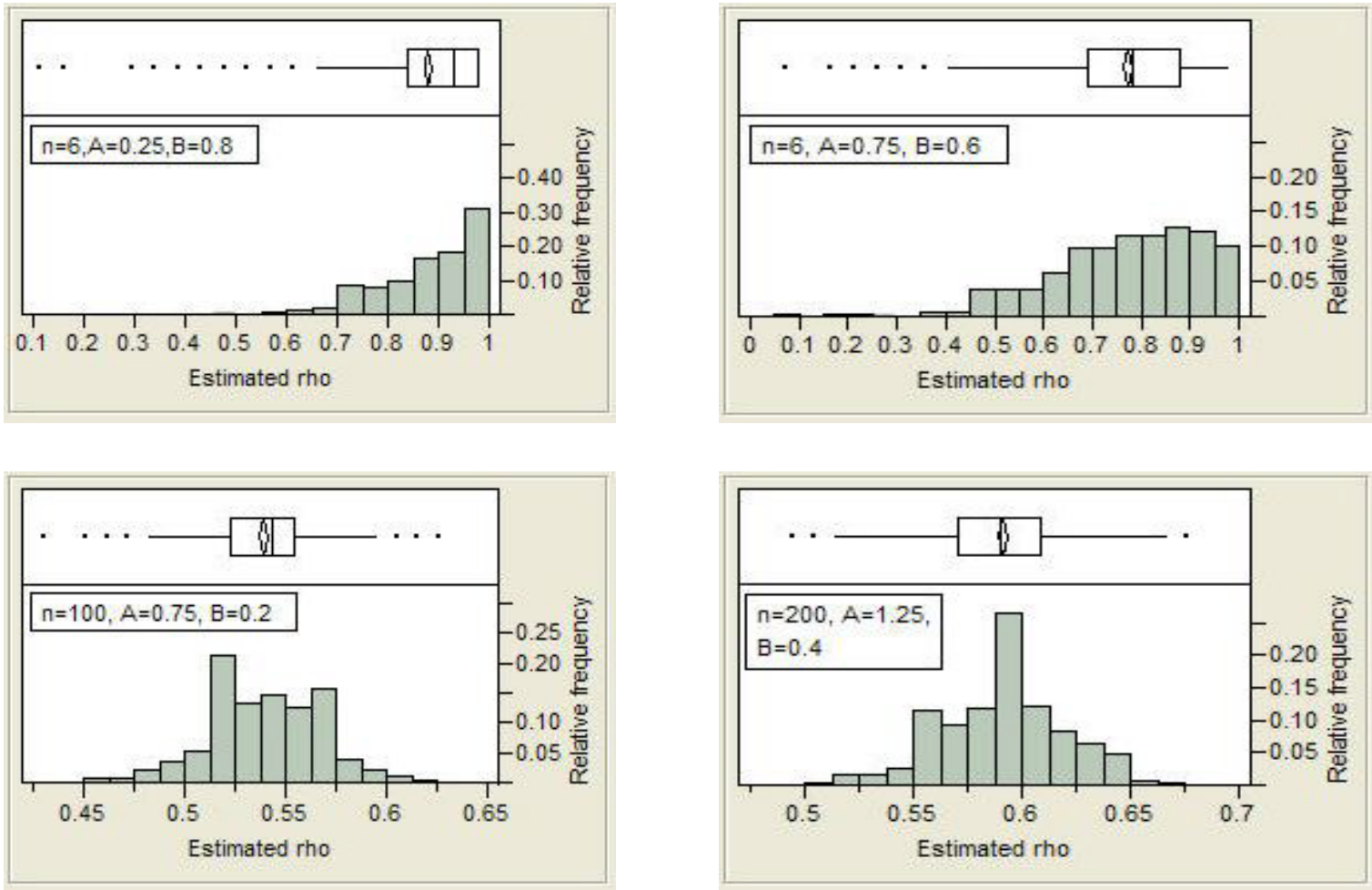


Figure 25.6–3. The sampling distribution of $\hat{\rho}$ for four different configurations.

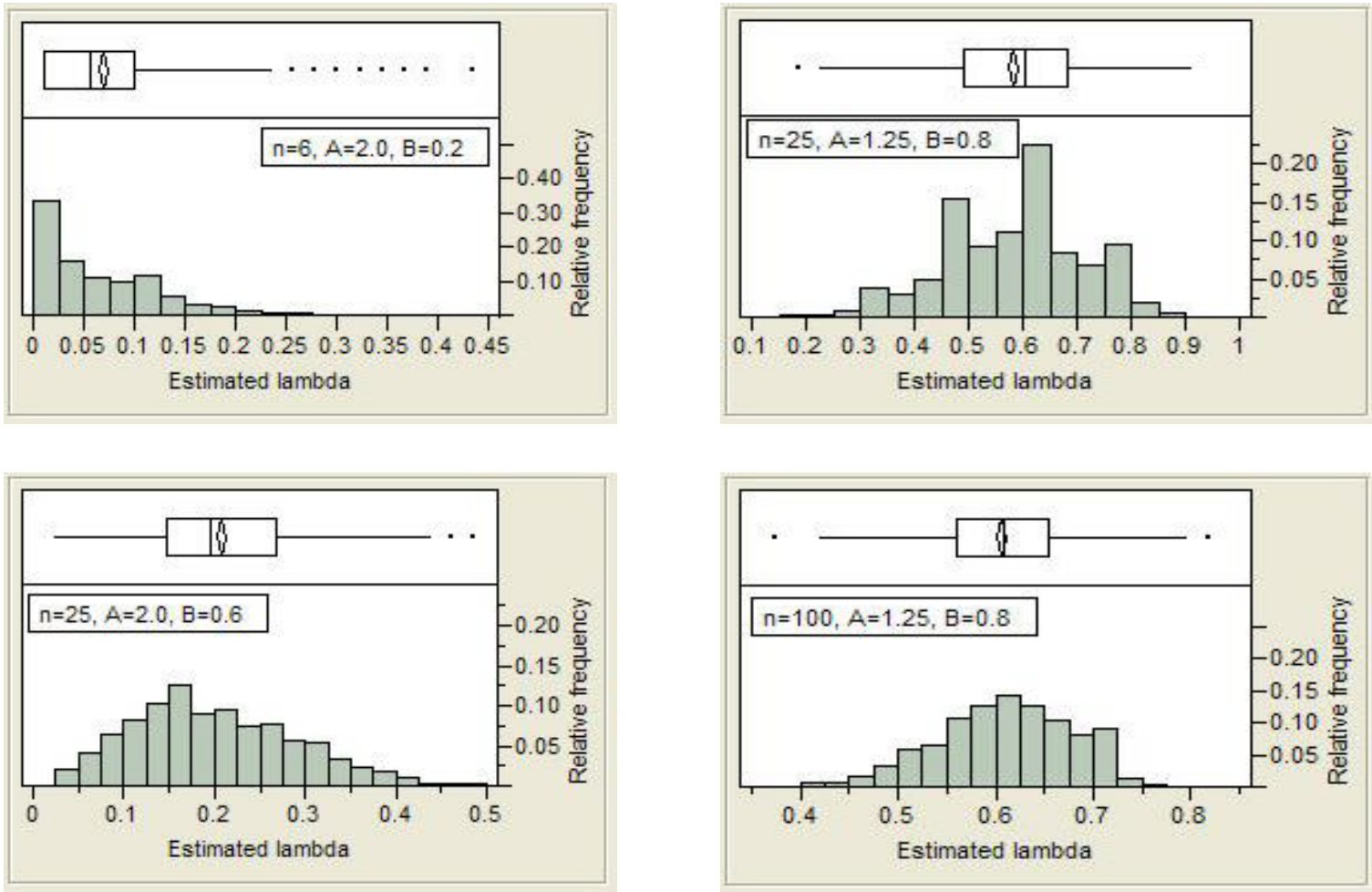


Figure 25.6–4. The sampling distribution of $\hat{\lambda}$ for four different configurations.

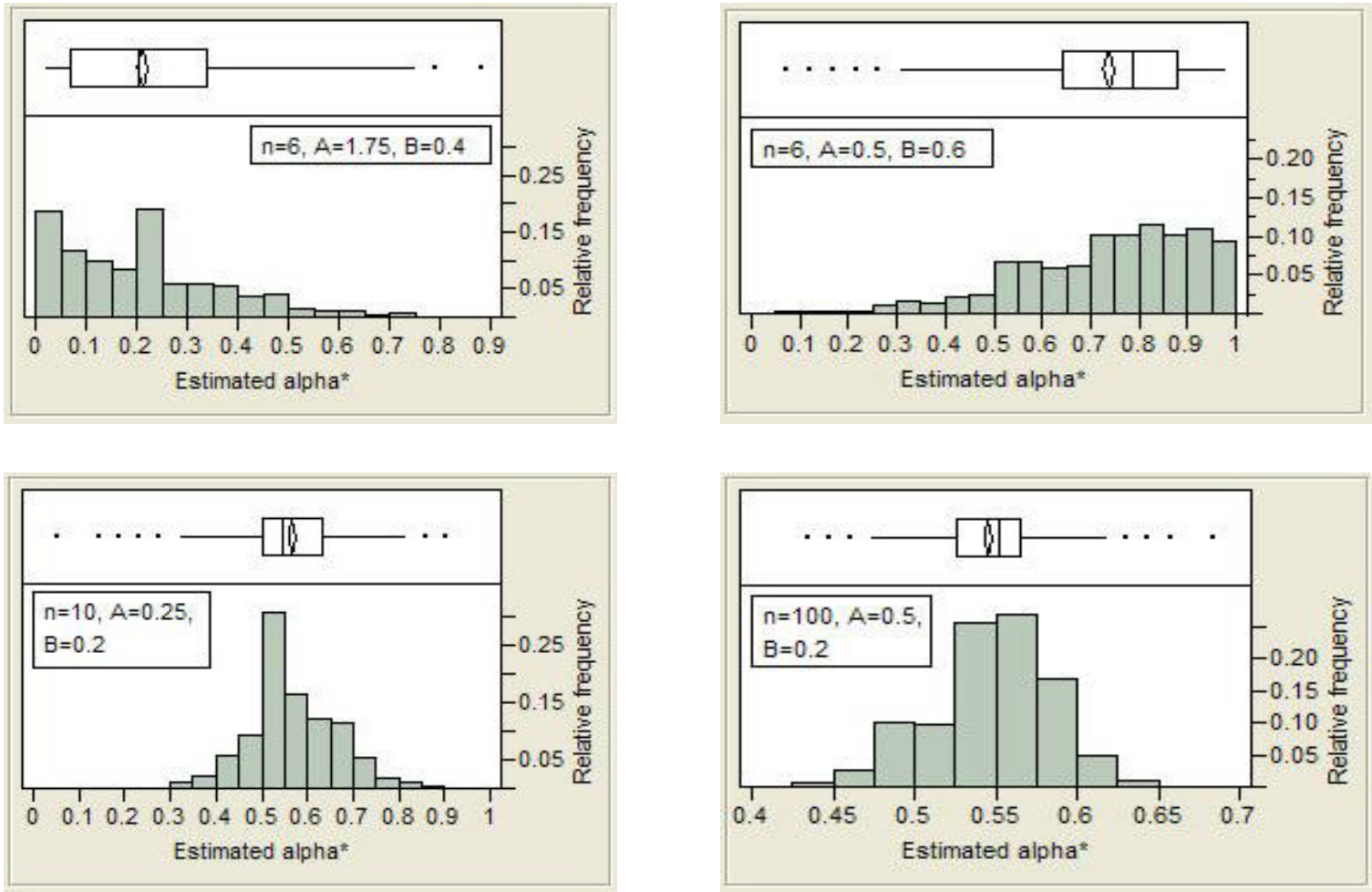


Figure 25.6–5. The sampling distribution of $\hat{\alpha}^*$ for four different configurations.

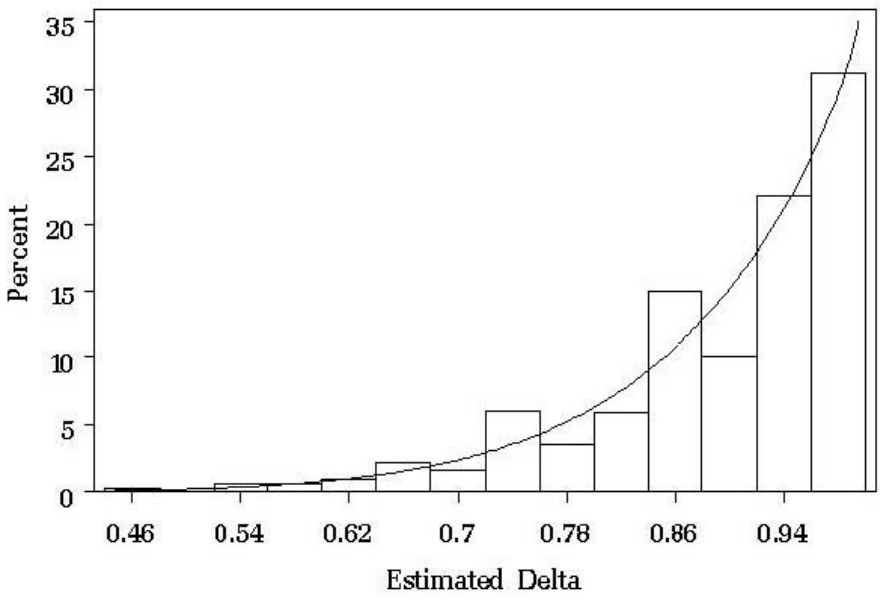


Figure 25.6–6. Sampling distribution for $\hat{\delta}$ with a beta fit for $n_1 = n_2 = 10$, $A = 0.25$, and $B = 0.6$.

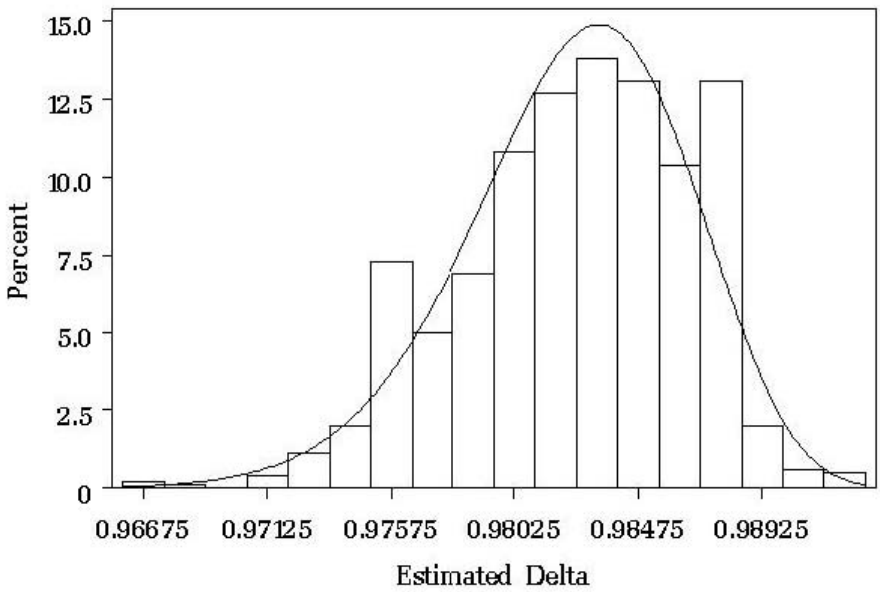


Figure 25.6–7. Sampling distribution for $\hat{\delta}$ with a beta fit for $n_1 = n_2 = 500$, $A = 1.5$, and $B = 0.2$.

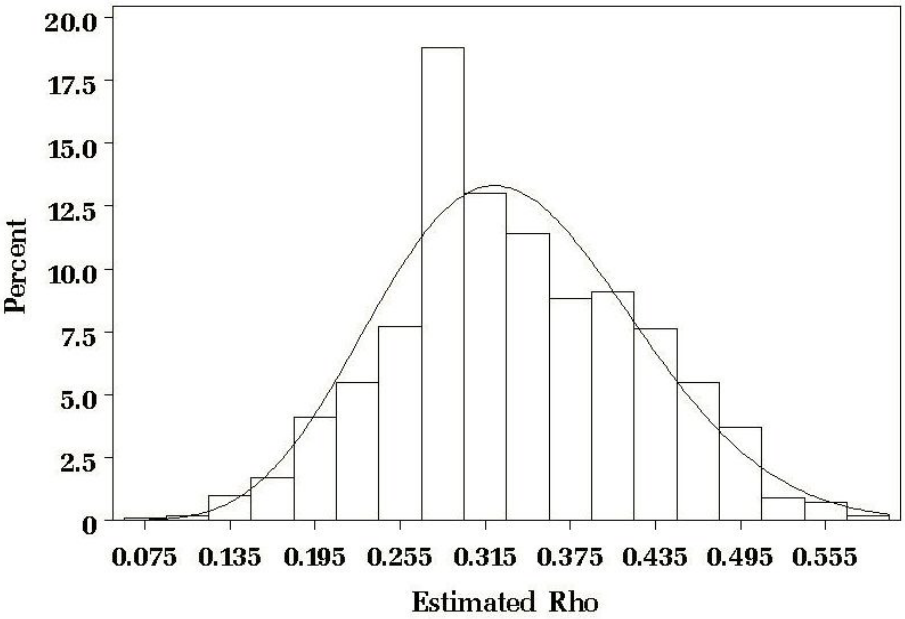


Figure 25.6–8. Sampling distribution for $\hat{\rho}$ with a beta fit for $n_1 = n_2 = 25$, $A = 2.0$, and $B = 0.4$.

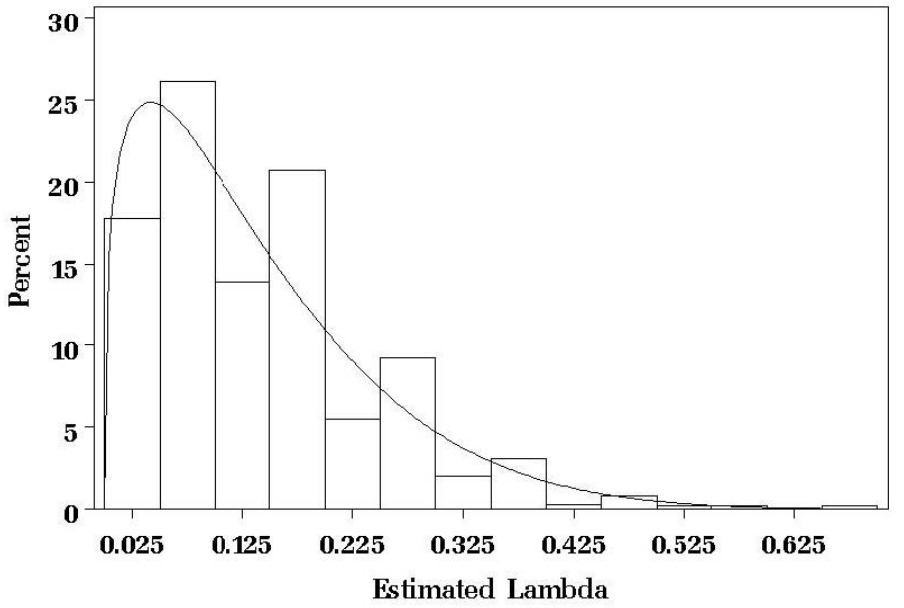


Figure 25.6–9. Sampling distribution for $\hat{\lambda}$ with a beta fit for $n_1 = n_2 = 6$, $A = 0.5$, and $B = 0.2$.

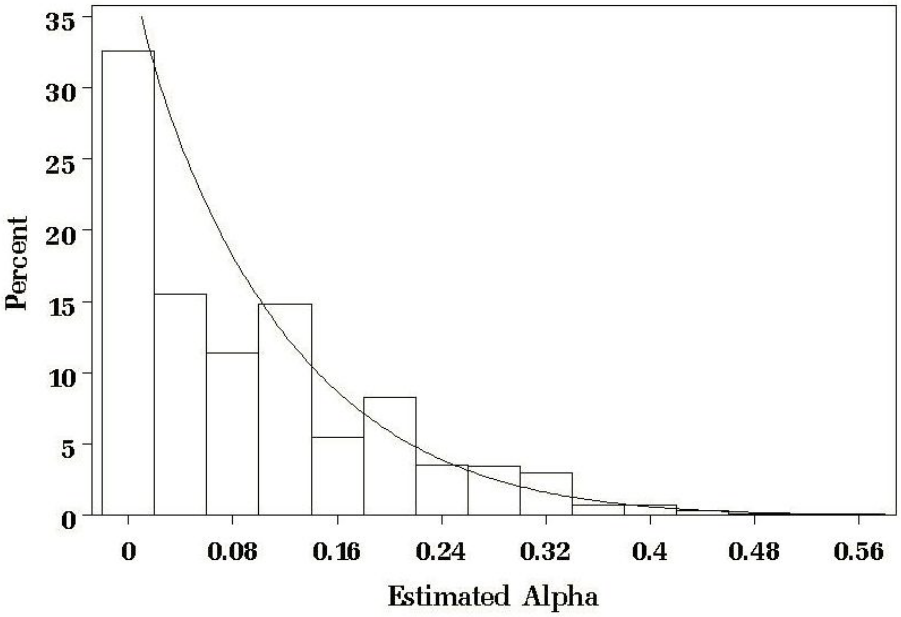


Figure 25.6–10. Sampling distribution for $\hat{\alpha^*}$ with a beta fit for $n_1 = n_2 = 6$, $A = 2.0$, and $B = 0.2$.

(n, A, B), whose values will be known, are investigated. In order to determine if the beta parameters (a, b) can be described as functions of (n, A, B), all the values of (a, b) estimated from fitted distributions were plotted. Figures 25.6–11 through 25.6–18 show the three-dimensional plots of $\hat{a}$ and $\hat{b}$ as functions of (n, A) or (n, B). These plots show non-linear relations as well as interaction effects in some cases.

To account for non-linearity, a natural log transformation of the beta parameters was used and to model some of the noted features, including interactions, a pool of possible predictors that included terms such as

$$\sqrt{n}\,\sqrt{A},\ \sqrt{B},\ n,\ A,\ B,\ n^2,\ A^2,\ B^2,\ nAB,\ A^2B,\ A^2B^2,$$

etc., was used. A stepwise-regression technique setting a five percent error rate was employed for variable selection. The coefficient of determination and the mean squared error values were used to assess the model fit. The results are summarized in Table 25.6–19. The resulting R^2 values ranged from 81.27% to 99.23% indicating that the selected model explains a considerably large chunk of variation in $\ln(a)$ and $\ln(b)$ values. The MSE values were fairly low for all the models, which means these functions of (n, A, B) fit the simulated data well.

Using the common sample size, n, $A = \sigma_2/\sigma_1 (= 1/C)$ and $B = (\mu_1 - \mu_2)/\sigma_1$, we propose the following relations to estimate the parameters of beta distribution to obtain the estimated sampling distributions:

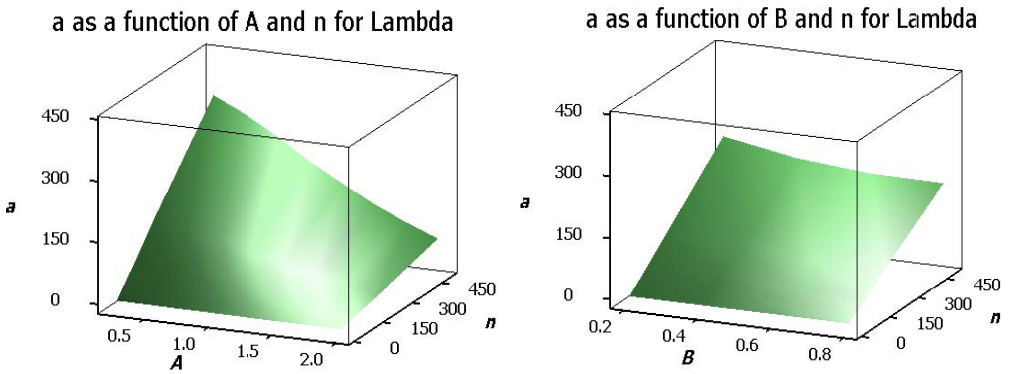


Figure 25.6–11. 3-dimensional plots of estimated beta parameter a as functions of sample size and values of A or B for similarity measure λ.

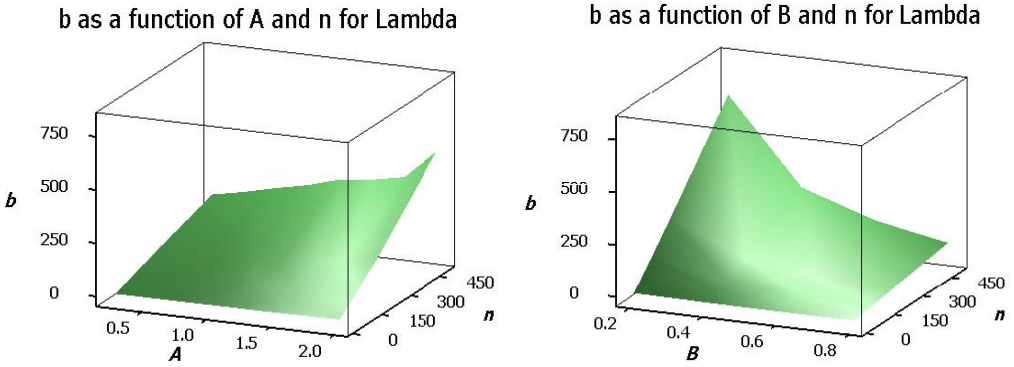


Figure 25.6–12. 3-dimensional plots of estimated beta parameter b as functions of sample size and values of A or B for similarity measure λ.

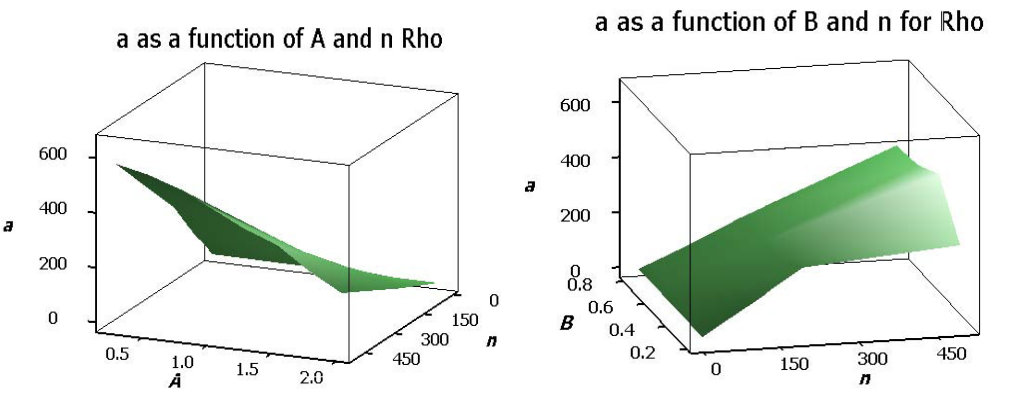


Figure 25.6–13. 3-dimensional plots of estimated beta parameter a as functions of sample size and values of A or B for similarity measure ρ.

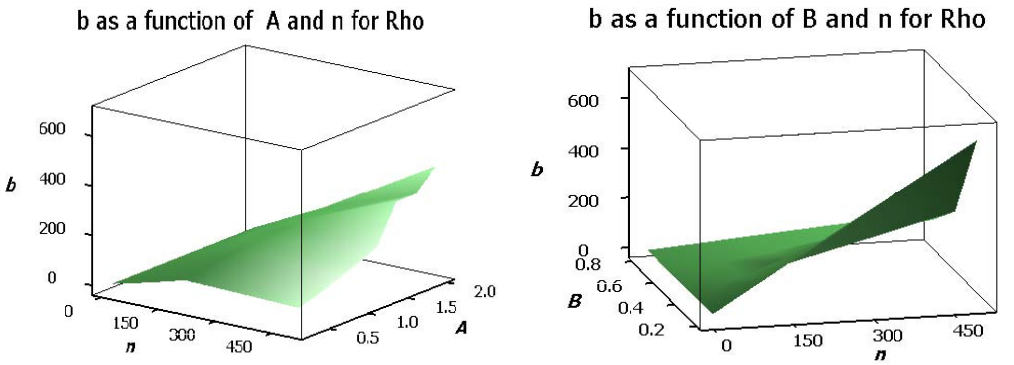


Figure 25.6–14. 3-dimensional plots of estimated beta parameter b as functions of sample size and values of A or B for similarity measure ρ.

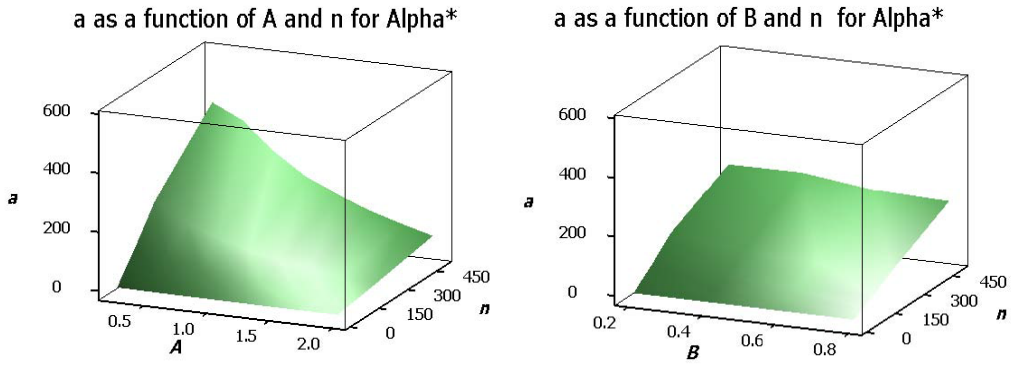


Figure 25.6–15. 3-dimensional plots of estimated beta parameter a as functions of sample size and values of A or B for similarity measure α^*.

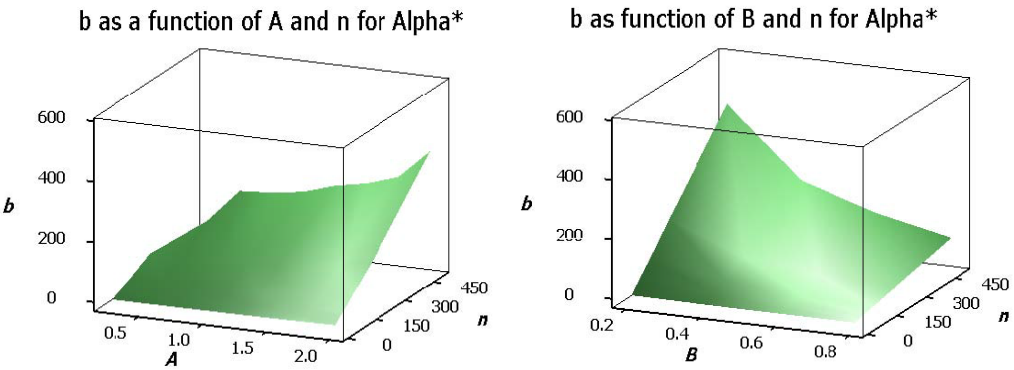


Figure 25.6–16. 3-dimensional plots of estimated beta parameter b as functions of sample size and values of A or B for similarity measure α^*.

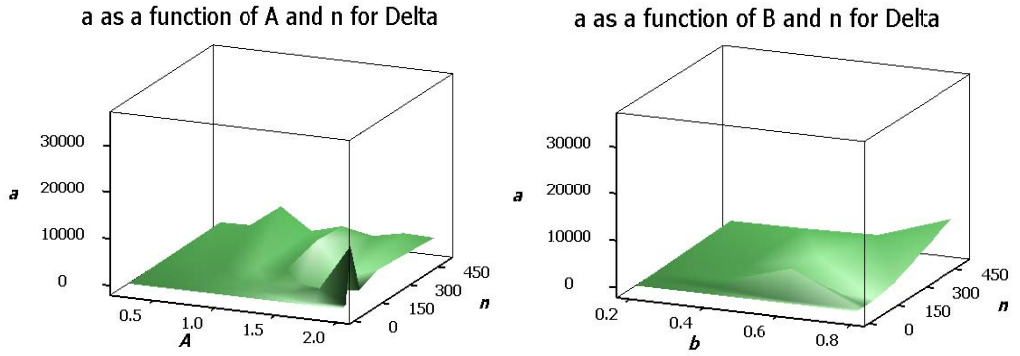


Figure 25.6–17. 3-dimensional plots of estimated beta parameter a as functions of sample size and values of A or B for similarity measure δ.

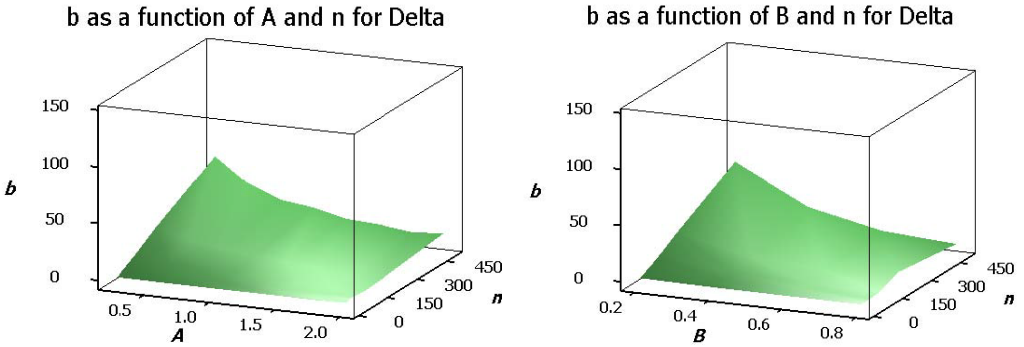


Figure 25.6–18. 3-dimensional plots of estimated beta parameter b as functions of sample size and values of A or B for similarity measure δ.

Table 25.6–19. Assessment of fit for different models.

Similarity measure	Beta parameter	Explanatory variables	R^2	MSE
δ	$\ln a$	$\sqrt{n}$, n, A^2, nAB, A^2B, A^2B^2	81.27	0.976
	$\ln b$	$\sqrt{n}$, $\sqrt{A}$, $\sqrt{B}$, AB, nAB, n^2B	83.25	0.517
ρ	$\ln a$	$\sqrt{n}$, $\sqrt{B}$, n, n^2, A^2, A^2B	99.21	0.144
	$\ln b$	$\sqrt{n}\sqrt{B}$, n, n^2, A^2, AB	98.27	0.218
λ	$\ln a$	$\sqrt{n}$, $\sqrt{A}$, n^2, A^2, nB, n^2B	97.37	0.266
	$\ln b$	$\sqrt{n}$, $\sqrt{B}$, n^2, nB, AB, n^2B	98.22	0.229
α^*	$\ln a$	$\sqrt{n}$, $\sqrt{B}$, n, A, n^2, A^2B	99.23	0.146
	$\ln b$	$\sqrt{n}$, $\sqrt{A}$, $\sqrt{B}$, n, n^2, AB	98.39	0.208

For $\boldsymbol{\delta}$:

$$\begin{aligned}\hat{a} &= 1.04\exp\Big[0.651\sqrt{n} - 0.0202n + 0.0098nAB - 0.261A^2 \\ &\qquad + 4.70A^2B - 6.36A^2B^2\Big] \qquad (25.6.1)\\ \hat{b} &= 62.94\exp\Big[0.215\sqrt{n} - 2.5\sqrt{A} + 4.93\sqrt{B} + 1.75AB \\ &\qquad - 0.005nAB - 10^{-5}n^2B\Big]. \qquad (25.6.2)\end{aligned}$$

For $\boldsymbol{\rho}$:

$$\begin{aligned}\hat{a} &= 2.2657\exp\Big[0.829\sqrt{n} - 1.05\sqrt{B} - 0.0362n + 0.00002n^2 \\ &\qquad - 0.4693A^2 + 0.035A^2B\Big] \qquad (25.6.3)\\ \hat{b} &= 42.1\exp\Big[0.6764\sqrt{n} - 7.43\sqrt{B} - 0.02716n + 0.00002n^2 \\ &\qquad - 0.176A^2 + 2.088AB\Big]. \qquad (25.6.4)\end{aligned}$$

For $\boldsymbol{\lambda}$:

$$\begin{aligned}\hat{a} &= 2.8557\exp\Big[0.3834\sqrt{n} - 0.55\sqrt{A} - 0.00001n^2 - 0.292A^2 \\ &\qquad - 0.00645nB + 10^{-6}n^2B\Big] \qquad (25.6.5)\\ \hat{b} &= 31.88\exp\Big[0.3972\sqrt{n} - 5.88\sqrt{B} - 0.00001n^2 - 0.01327nB \\ &\qquad + 0.00002n^2B + 1.604AB\Big]. \qquad (25.6.6)\end{aligned}$$

For $\boldsymbol{\alpha^*}$:

$$\begin{aligned}\hat{a} &= 3.7449\exp\Big[0.7502\sqrt{n} - 1.127\sqrt{B} - 0.03136n + 0.00002n^2 \\ &\qquad - 1.606A + 0.469A^2B + 1.3204\Big] \qquad (25.6.7)\\ \hat{b} &= 39.34\exp\Big[0.6264\sqrt{n} - 0.61\sqrt{A} - 6.81\sqrt{B} - 0.02422n \\ &\qquad + 0.00001n^2 + 2.106AB\Big]. \qquad (25.6.8)\end{aligned}$$

Example 6.
Suppose we are interested in estimating the sampling distribution of $\hat{\alpha^*}$. A pair of independent random samples of sizes $n_1 = n_2 = n = 50$ is to be taken from two populations such that $A = 1.75$, and $B = 0.8$. Solving equations (25.6.7) and (25.6.8) we get,

$$\hat{a} = 11.44606 \quad \text{and} \quad \hat{b} = 19.4144. \qquad (25.6.9)$$

Therefore, the sampling distribution of $\hat{\alpha^*}$ is estimated to be beta(11.44, 19.41). The fit is shown in Figure 25.6–20.

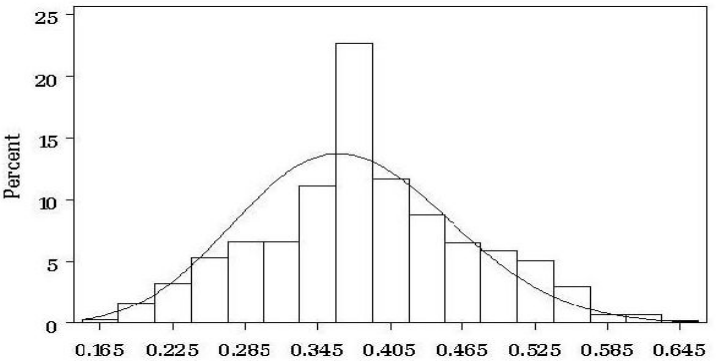


Figure 25.6–20. Sampling distribution of $\hat{\alpha^*}$ with estimated beta using equations (25.6.7) and (25.6.8).

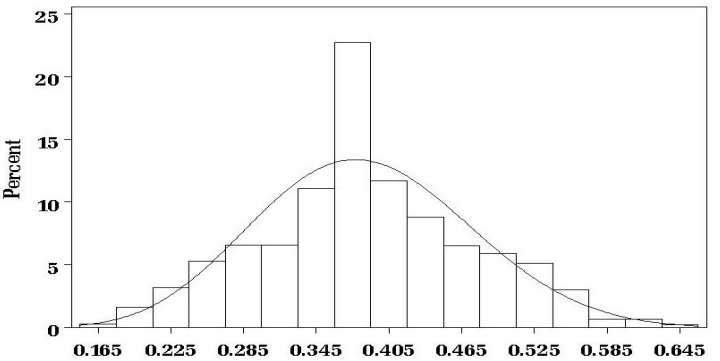


Figure 25.6–21. The sampling distribution of $\hat{\alpha^*}$ with its fitted beta distribution.

On the other hand, the estimated parameter values from the beta fit to the 1,000 simulated values are

$$\hat{a} = 11.54347 \quad \text{and} \quad \hat{b} = 18.3939. \tag{25.6.10}$$

The estimated sampling distribution of $\hat{\alpha^*}$ from the simulation is beta(11.44, 19.41). The fit is shown in Figure 25.6–21. The closeness of estimated parameter values (given by (25.6.9)) to that from simulation (given by (25.6.10)), indicate a fair amount of accuracy in the estimation of parameters of sampling distributions.

The amount of overlap as estimated by δ between the sampling distribution from simulation and the beta distribution estimated by parameters from (25.6.10) is 0.95, and that between the two beta distributions (fitted and estimated) is 0.93.

25.7 Conclusions

In conclusion, the excellent fit of beta distribution for all four similarity measures under two different distributions (one symmetric and one very skewed) indicates

that beta is one possible distribution that can be used to model the sampling distribution of the overlap coefficient and the other three measures of similarity discussed in this chapter. However, it in no way implies that the beta is the only distribution for this purpose. In fact in some of the simulated sampling distributions, periodically a bin with larger than usual frequency (compared to the other neighboring bins) has resulted, such as the one shown in Figure 25.6–21. There could be different possible reasons for it.

- It may be due to sampling variation and if the same situation is simulated again, such a peak may not appear.
- It may be due to choice of bin length. As shown by Figures 26.5–22(a) through 25.6–22(d). All of those four histograms for the same data show slightly different characteristics due to different choices of bins and bin lengths. Figures 26.5–22(a) and 26.5–22(b) show almost symmetric and mound-shaped histograms. Figure 26.5–22(c) shows a fairly symmetric histogram with two peaks and Figure 26.5–22(d) shows a slightly asymmetric histogram with one long bar in the center. However, the modified boxplot for all of them remains the same, almost symmetric with a few outliers on both ends.

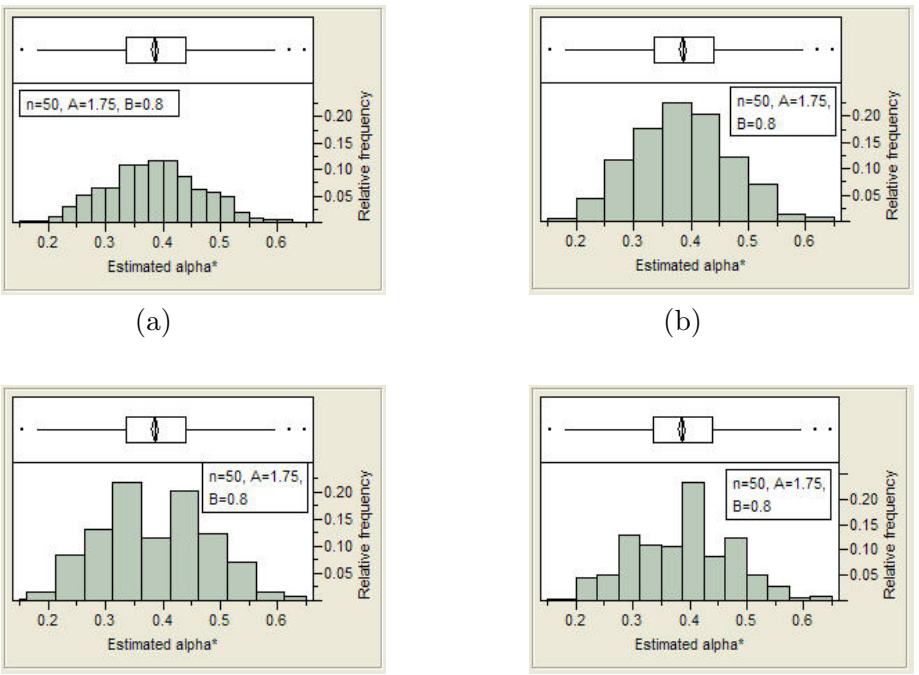


Figure 25.6–22. The sampling distribution of $\hat{\alpha}^*$ using different bin configurations.

- It may also be due to the choice of software used for making graphs. Each software has a different method of selecting bin lengths and boundaries. Histogram in Figure 25.6–21 was generated using Minitab, whereas those in Figure 25.6–22 were generated using JMP.

- Another possible explanation may be that under certain circumstances some other more peaked distribution might provide a better fit.

If occurrence of unusual characteristics in some histograms is due to bin length or boundaries, then it will not affect the fit of the distribution because the fit as well as goodness-of-fit are determined based on the data and not on the histogram bins as long as the same program is used for all distribution fittings (as we did). Use of different programs could result in slightly different fits. Further improvements to the estimation of sampling distributions of measures of similarity are currently being investigated.

Acknowledgments

This project was partially funded through grants from National Aeronautics and Space Administration (NASA), University Committee on Undergraduate Research (UCUR), and National Science Foundation (NSF). Madhuri Mulekar was supported in part by funding from NSF (DMS-9322184). Jeff Boone was supported by funding from NSF (DUE-0422121). Subhash Aryal was supported by grants from NASA and UCUR.

Both Jeff Boone and Subhash Aryal participated in this project when they were at the Department of Mathematics and Statistics, University of South Alabama, Mobile, Alabama. Currently Jeff Boone is at the University of Alabama, Tuscaloosa, Alabama and Subhash Aryal is at the University of North Texas, Fort Worth, Texas.

We would like to thank referees for valuable suggestions for improving this chapter. We would also like to extend our appreciation to Dr. Cynthia Schneider for consistency checks and suggestions for improvement.

References for Chapter 25

Anderson, G. (2004). "Toward an empirical analysis of polarization," *Journal of Econometrics,* 122, 1–26.

Anderson, G., Ge, Y., and Leo, T.W. (2005). "Distributional overlap: Simple, multivariate, parametric and non-parametric tests for alienation, convergence and general distributional difference issues," *Technical Report*, Department of Economics, University of Toronto, Toronto.

Bar-Hen, A. and Daudin, J.J. (1998). "Asymptotic distribution of Matusita's distance: Application to the location model," *Biometrika,* 85, 477–481.

Beran, H. (1977). "Minimum Hellinger's distance estimates for parametric models," *Annals of Statistics,* 5, 455–463.

Bhattacharyya, A. (1943). "On a measure of divergence between two statistical populations defined by their probability distributions," *Bulletin of Calcutta Mathematical Society,* 35, 99–107.

Bradley, E.L. (1985). "Overlapping coefficient," In *Encyclopedia of Statistical Sciences,* Vol. 6, eds: Kotz S., Johnson N.L., Read C.B., Wiley, New York, pp. 546–547.

Braun, D. (1988). "Multiple measurements of U.S. income inequality," *The Review of Economics and Statistics,* 70, 398–405.

Bray, J.R. and Curtis, J.T. (1957). "An ordination of the upland forest communities of southern Wisconsin," *Ecological Monographs,* 27, 325–349.

Chung, M. Y., Epperson, B.K., and Chung, M.G. (2002). "Genetic structure of age classes in Camellia japonica (theaceae)," *Evolution,* 57, 62–73.

Clemons, T.E. and Bradley, E.L., Jr. (2000). "A nonparametric measure of the overlapping coefficient," *Computational Statistics & Data Analysis,* 34, 51–61.

Cleveland, W.B. and Lachenbruch, P.A. (1974). "A measure of divergence among several populations," *Communications in Statistics,* 3, 201–211.

Czekanowski, J. (1909). "Zur Differentialdiagnose der Neandertalgruppe," *Korrespondenzblatt der Deutschen Gesellschaft für Anthropologie,* 40, 44–47.

Czekanowski, J. (1932). "'Coefficient of racial likeness' und 'durchschnittliche Differenz'," *Anthropologischer Anzeiger,* 9, 227–249.

Czekanowski, J. (1962). "The theoretical assumptions of Polish anthropology and the morphological facts," *Current Anthropology,* 3, 481–494.

Dixon, P.M., 1993. "The Bootstrap and the Jackknife: Describing the recision of ecological indices," In: Scheiner, S.M., Gurevitch J. (Eds.), *Design and Analysis of Ecological Experiments,* Chapman & Hall, New York, 290–318.

Federer, W.T., Powers, L., and Payne, M.G. (1963). "Studies on statistical procedures applied to chemical genetic data from sugar beets," Technical Report No. 77, *Agricultural Experimentation Station,* Colorado State University.

Feinsinger, P., Spears, E.E., and Poole, R.W. (1981). "A simple measure of niche breadth," *Ecology,* 62, 27–32.

Field, J.G. and McFarlane, G. (1968). "Numerical methods in marine ecology. 1. A quantitative similarity analysis of rocky shore samples in False Bay, South Africa," *Zoologica Africana,* 3, 119–137.

Gastwirth, J.L. (1975). "Statistical measures of earnings differentials," *American*

Statistician, 29, 32–35.

Gerrard, R. and Barbour, A.D. (1986). "Measures of niche overlap, II," *IMA Journal of Mathematics Applied in Medicine and Biology,* 3, 115–127.

Gower, J.C. (1985). "Measures of similarity, dissimilarity, and distance," in: *Encyclopedia of Statistical Sciences,* Volume 5 (Kotz S., Johnson N.L., Read C.B., eds.), Wiley, New York, 397–405.

Hanula, J.L. and Engstrom, R.T. (2000). "Comparison of red-cocked woodpecker (*Picoides borealis*) nestling diet in old-growth and old-field longleaf pine (*Pinus palustris*) habitats," *The American Midland Naturalist,* 144, 370–376.

Harner, E.J. and Whitmore, R.C. (1977). "Multivariate measures of niche overlap using discriminant analysis," *Theoretical Population, Biology,* 12, 21–36.

Hogg, R.V., Uthoff, V.A., Randles, R.H., and Davenport, A.S. (1972). "Selecting the underlying distribution and adaptive estimation," *Journal of the American Statistical Association,* 67, 597–600.

Horn, H.S. (1966). "Measurement of 'overlap' in comparative ecological studies," *American Naturalist,* 100, 419–424.

Hurlbert, S.H. (1978). "The measurement of niche overlap and some relatives," *Ecology,* 59, 67–77.

Ichikawa, M. (1993). "A meaning of the overlapped area under probability density curves of stress and strength," *Reliability Engineering & Systems Safety,* 41, 203–204.

Inman, H.F. (1981). "Migration in the Cotton South: The Geographic Mobility of Alabama Farmers, 1850–1860," Unpublished Thesis, Department of Biostatistics, University of Alabama at Birmingham, Birmingham, AL.

Inman, H.F. and Bradley, E.L. (1989). "The overlapping coefficient as a measure of agreement between probability distributions and point estimation of the overlap of two normal densities," *Communications in Statistics — Theory & Methods,* 18, 3851–3874.

Inman, H.F. and Bradley, E.L. (1994). "Hypothesis tests and confidence interval estimates for the overlap of two normal distributions with equal variances," *Environmetrics,* 5, 167–189.

Jarrett, H.G., Slow, S., and Morgan, B.J.T. (1985). "Density function approximation with particular reference to the distribution of overlap of circles," *Australian Journal of Statistics,* 27, 36–43.

Johnson, N.L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions — Vol. 2,* 2nd ed., Wiley Publications.

Karian, Z.A. and Dudewicz, E.J. (2003). "Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm," *Journal of the Iranian Statistical Society,* 2, 171–187.

Karoly, L.A. (1990). "Changes in the distribution of individual earnings in the United States," *The Review of Economics and Statistics,* 74, 107–115.

Kistler, S., Stephenson, A.G., and Benninghoff, W.S. (1979). "Description of understory development in a tree plantation with a new method of data structuring," *Vegetatio,* 40, 185–191.

Kokko, H., Mackenzie, A., Reynolds, J.D., Lindstrom, J., and Sutherland, W.J. (1999). "Measures of inequality are not equal," *The American Naturalist,* 154, 358–382.

Levins, R. (1968). *Evolution in Changing Environments,* Princeton University Press, Princeton, New Jersey, USA.

Lin, X. and Wang, H. (2004). "A new testing approach for comparing the overall homogeneity of survival curves," *Biometrical Journal,* 46, 489–496.

Lu, R.P., Smith, E.P., and Good, I.J. (1989). "Multivariate measures of similarity and niche overlap," *Theoretical Population Biology,* 35, 1–21.

MacArthur, R.H. (1972). *Geographical Ecology,* Harper and Row, New York.

MacArthur, R.H. and Levins, R. (1967). "The limiting similarity: Convergence and divergence of coexisting species," *Auwricall Nilturalist,* 101, 177–385.

Mahalanobis, P. C. (1936). "On the Generalized Distance in Statistics," *Proc. Natl. Inst. Sci. India,* 2, 49–55.

Mantel, N. (1966). "Evaluation of survival data and two new rank order statistics arising in its consideration," *Cancer Chemotherepy Reports,* 50, 163–170.

Matusita, K. (1955). "Decision rules based on the distance for problems of fit, two samples, and estimation," *Annals of Mathematical Statistics,* 26, 631–340.

Maurer, B. A. (1982). "Statistical inference for MacArthur-Levins niche overlap," *Ecology,* 63, 1712–1219.

May, R.M. (1975). "Some notes on estimating the competition matrix. α, *Ecology,* 56, 737–741.

Mimoto, N. and Zitikis, R. (2009). "Czekanowski's index of overlap, its L_p- type extension and bias reduction," *American Journal of Mathematical and Management Sciences,* 29, 229–261.

Minami, M. and Shimizu, K. (1999). "Estimation of similarity measures for multivariate normal distributions," *Environmental and Ecological Statistics,* 6, 229–248.

Mishra, S.N., Shah, A.K., and Lefante, J.J., Jr. (1986). "Overlapping coefficient: the generalized t approach," *Communications in Statistics — Theory and Methods,* 15, 123–128.

Mizuno, S., Yamaguchi, T., Fukushima, A., Matsuyama, Y., and Ohashi, Y. (2005). "Overlap coefficient for assessing the similarity of pharmacokinetic data

between ethnically different populations," *Clinical Trials,* 4, 174–181.

Moore, R.E. (1996). "Ranking income distributions using the geometric mean and a related general measure," *Southern Economic Journal,* 63, 69–75.

Morisita, M. (1959). "Measuring interspecific association and similarity between communities," *Memoirs of the Faculty of Kyushu University, Series E, Biology,* 3, 65–80.

Mueller, L.D. and Altenberg, L. (1985). "Statistical inference on measures of niche overlap," *Ecology,* 66, 1204–1210.

Mulekar, M.S. (2006). "Selecting the most diverse population using Gini measure of income inequality," *Statistica & Applicazioni,* Vol. IV, No. 2, 91–104.

Mulekar, M.S. (2009). "A necessary condition for the estimation of the overlap of normal populations," *American Journal of Mathematical and Management Sciences,* Submitted.

Mulekar M.S., Gonzales, S., and Aryal S. (2008). "Bias and variance estimates for the overlap of two exponential populations," *American Journal of Mathematical and Management Sciences,* 28, 61–79.

Mulekar, M.S. and Mishra, S.N. (1994). "Overlap coefficients of two normal densities: Equal means case," *Journal of the Japan Statistical Society,* 24, 169–180.

Mulekar, M.S. and Mishra, S.N. (1997). "Inference on measures of niche overlap for heteroscadastic normal populations," *American Journal of Mathematical and Management Sciences,* 17, 163–185.

Mulekar, M.S. and Mishra, S.N. (2000). "Confidence interval estimation of overlap: Equal means case," *Computational Statistics and Data Analysis,* 34, 121–137.

Ning, W., Gao, Y., and Dudewicz, E.J. (2008). "Fitting mixture distributions using generalized lambda distributions and comparison with normal mixtures," *American Journal of Mathematical and Management Sciences,* 28, 81–99.

Peto, R. and Peto, J. (1972). "Asymptotically efficient rank-invariant test procedures," *Journal of Royal Statistical Procedures, Series A,* 135, 185–206.

Pianka, E.R. (1973). "The structure of lizard communities," *Annual Review of Ecology and Systematics,* 4, 53–74,

Rao, C. R. (1963). "Criteria of estimation in large samples," *Sankhyā, Series A,* 25, 189–206.

Reagan, R.S. and Mishra, S.N. (1992). "Object Oriented Software for Automatic Inference," *Proceedings of the 1992 Winter Simulation Conference,* ed. J.J. Swain, D. Goldsman, R.C. Cain, and J.R. Wilson, 211–220.

Reiser, B. and Faraggi, D. (1999). "Confidence intervals for the overlapping coefficient: The normal equal variance case," *The Statistician,* 48, 413–418.

Renkonen, O. (1938). "Statistisch-ökologische untersuchungen über die terrestische käferwelt der finnischen bruchmoore," *Annales Societatis Zoologicae-Botanicae Fennicae "Vanamo"*, 6, 1–231.

Ricklefs, R.E. and Lau, M. (1980). "Bias and dispersion of overlap indices: Results of some Monte Carlo simulations," *Ecology,* 61, 1019–1024.

Rom, D.M. and Hwang, E. (1996). "Testing for individual and population equivalence based on the proportion of similar responses," *Statistics in Medicine,* 15, 1489–1505.

Sale, P.F. and Guy, J.A. (1992). "Persistence of community structure: What happens when you change taxonomic scale?," *Coral Reefs*, 11, 147–154.

Schatzmann, E., Gerrard, R., and Barbour, A.D. (1986). "Measures of niche overlap, I.," *Mathematical Medicine and Biology,* 3, 99–113.

Slobodchikoff, C.N. and Schulz, W.C. (1980). "Measures of niche overlap," *Ecology,* 1051–1055.

Smith, E.P. (1982). "Niche breadth, resource availability, and inference," *Ecology,* 63, 1675–1681.

Smith, E.P. and Zaret T.M. (1982). "Bias in estimating niche overlap," *Ecology,* 61, 1051–1055.

Sneath, P.H.A. (1977). "A method for testing the distinctness of clusters: A test of the disjunction of two clusters in Euclidean space as measured by their overlap," *Mathematical Geology,* 9, 123–143.

Stine, R.A. and Heyse, J.F. (2001). "Non-parametric estimates of overlap," *Statistics in Medicine,* 20, 215–236.

Weitzman, M.S. (1970). "Measures of Overlap of Income Distributions of White and Negro Families in the United States," *Technical Report No. 22,* Department of Commerce, Bureau of Census, Washington, DC.

Wolda, H. (1981). "Similarity indices, sample size and diversity," *Oecologia,* 50, 296–302.

PART VII: Applications

Chapter 26 Fitting Statistical Distributions Functions to Small Datasets

Chapter 27 Mixed Truncated Random Variable Fitting with the GLD, and Applications in Insurance and Inventory Management

Chapter 28 Distributional Modeling of Pipeline Leakage Repair Costs for a Water Utility Company

Chapter 29 Use of the Generalized Lambda Distribution in Materials Science, with Examples in Fatigue Lifetime, Fracture Mechanics, Polycrystalline Calculations, and Pitting Corrosion

Chapter 30 Fitting Statistical Distributions to Data in Hurricane Modeling

Chapter 31 A Rainfall-Based Model for Predicting the Regional Incidence of Wheat Seed Infection by *Stagonospora nodorum* in New York

Chapter 32 Reliability Estimation Using Univariate Dimension Reduction and Extended Generalized Lambda Distribution

Chapter 33 Statistical Analyses of Environmental Pressure Surrounding Atlantic Tropical Cyclones

Chapter 34 Simulating Hail Storms Using Simultaneous Efficient Random Number Generators

Chapter 26

Fitting Statistical Distribution Functions to Small Datasets

Jerry L. Alderman
Engineering Fellow,
Raytheon Missile Systems,
Tucson, AZ, USA
E-mail: jerryalderman@comcast.net

Allan T. Mense
Principal Engineering Fellow,
Raytheon Missile Systems,
Tucson, AZ, USA
E-mail: Allan_T_Mense@raytheon.com

This chapter reviews some of the problems typical of data analyses in industry. In particular, this chapter will investigate problems associated with fitting a probability density function (pdf) to data when only a small sample of data is available (Alderman and Mense (2007)). This is never a matter of choice, but is often necessitated by engineering practice, e.g., when a design may still be changing or at any time that data collection may be limited by availability or cost. Having hundreds of data points is best, but in practice this may not be possible.

26.1 Introduction

Karian and Dudewicz, in their Auburn Conference paper (Karian and Dudewicz, (2006)), gave the prescription for pdf fitting. They said, "At least three major steps must be undertaken when fitting a statistical distribution to data."

- First, a decision needs to be made about what type (or family) of distributions to consider.
- Second, it must be decided what fitting method to use (e.g., method of moments, maximum likelihood, least squares).
- Third, based on the two prior decisions, specific computational schemes must be invoked to estimate the parameters associated with the fit.

In accordance with the above prescription, the following choices have been made.

- First, the Johnson Family of distributions (Johnson (1949)) has been chosen for fitting data, but for completeness we compare the Johnson family to other distributions. The Johnson family has been chosen for several reasons, including that it is a four-parameter family that will allow for better fitting to a variety of data. Also, the Johnson family maps the user data into a unit normal distribution and since, by definition, the parameters of a unit normal are known, the goodness-of-fit (GOF) tests such as the Kolmogorov-Smirnov (KS) test are valid in this unit normal space.
- Second, in choosing the Johnson family, the moments matching method for determining estimators for the distribution parameters has been chosen. Specifically, the use of the mean, variance, coefficient of skewness, and coefficient of kurtosis of the data has been taken to determine the four parameters in the Johnson distributions.
- Third, the technique for determining the Johnson parameter estimators is taken from an algorithm published by Hill, Hill, and Holder (1976) and used in a tool titled JFIT, which will be discussed later. When fitting data to other distributions, a genetic algorithm search code developed by the first author has been used. This tool is labeled RAPID and is discussed elsewhere (Alderman, Mense, and Runger (2010)).

Some unusual situations can arise when fitting a pdf to data, and for each situation there are usually many approaches to solving the problem. For example, if there are only two parameters in a distribution, there is little possibility that steep slopes and wide tails with large skewness can be adequately handled. A three-parameter distribution such as a three-parameter Weibull may be useful in that situation; however, a host of four-parameter distributions can also be used, i.e., Pearson distributions, Johnson distributions, generalized lambda

distributions (GLD), generalized kappa distributions, and four-parameter beta distributions. The choice of the distribution family may be based on a variety of issues, such as:

- unimodal versus multimodal data,
- bell-shaped versus highly skewed data,
- data peaked at one or both ends of the data range, i.e., J-shaped or U-shaped pdf,
- data with extreme outliers,
- truncated (bounded) versus unbounded,
- large sample ($n > 30$) versus very small sample ($n < 15$),
- piece-wise fitting of data using multiple distributions (renormalized).

Many of the choices that must be made in fitting a pdf to data depend a great deal on how the resultant pdf will be used after being determined. If it is important to have an accurate fit to the tail(s) of the data, then distributions that weight the tail will be better choices. If the tails are not important, or it is believed the tails are outliers that can be ignored, then fitting the bulk of the data is more important. These considerations will be addressed in turn as the various datasets are described and fitted with pdfs.

A couple of caveats are in order. No biological, pharmacological, or medical datasets are included in our datasets. Some of these datasets have bizarre behaviors due to the large number of covariates that may be present in the experimental population. Also, data often cannot be dropped from a medical test due to federally mandated testing procedures. In such cases, separate distributions must be used to fit the various parts of the data and then renormalize the composite distribution.

All analysis tools developed by the authors and discussed in this chapter are Microsoft® Excel™-based tools and are available as downloads from the authors' Web site, http://statisticaldesignmethods.com, or on a CD-ROM that will accompany the authors' upcoming text, *Statistical Design Methods* (Alderman, Mense, and Runger, (2010)).

The procedure used in this chapter begins by presenting a set of data using descriptive statistics. The tools are the histogram, the dotplot, and the box and whisker plot. Calculations of the major inferential statistics will then be performed, resulting in numerical values for the median, mean, variance, coefficient of skewness, coefficient of kurtosis, and the standard error of the mean, including the classical confidence intervals about the mean and median. Once the moments of the data are known, a Raytheon developed tool called JFIT© is applied. This tool fits the data to the Johnson family of distributions (Johnson (1949)) and is

described in more detail below. The Kolmogorov-Smirnov (KS) statistic is then calculated for the Johnson distribution chosen by JFIT© and compared to four other data-fitting techniques. The additional fitting tools divide into two commercial tools (EasyFit™ and Crystal Ball™), and two Raytheon tools (GAFit and GLFit). Six datasets are analyzed to show a range of typical small sample size data the authors analyze on a day-to-day basis.

Before beginning the fitting process, a few words of caution are appropriate. Make sure all the data are correct, e.g., no data entry errors, no poorly setup experiments, thorough and careful experimental techniques have been used. Any violation of these standards could produce data that will cause poor estimation of the pdf parameters. This admonition seems obvious, but researchers may think the data they take or find to analyze has been correctly taken and registered in a database. This is not always true and data must be double-checked before they are used to fit to a pdf.

Excellent analysis of bad data is not an acceptable statistical best practice. Even one incorrect outlier in a small dataset may severely bias the parameter estimators and produce an inaccurate pdf. At Raytheon, an Excel™-based tool called DCAT has been developed that allows for quick analyses and discovery of outliers (Alderman, Mense, and Runger (2010)).

As most data analysts know, outliers turn out to be either the least important or most important data in a dataset. Outliers may be due to 1) actual differences in the physics determining the value of the random variable, 2) differences in the setup procedure for testing (no two lab technicians set up equipment in the same way), 3) use of different measurement equipment not well calibrated to the same standard, 4) coarseness of measurement instrument, or 5) mistakes in reporting. The fixes for each of the problems 2) through 5) are different, but usually require that the outlier data be discarded from the sample. If the outlier is real and comes from a carefully run experiment, then it does reflect the large variability of the population. It cannot be ignored, but it will skew the shape of the pdf. Examine the data descriptively before applying inferential analyses.

The authors have put together a database of small sample tests (SST) and use this database to illustrate some of the issues that arise when small samples are fitted to a pdf. Our attempt has been to give a wide variety of types of datasets and show how various fitting techniques handle these various types. The SST database has 73 datasets ranging from 27 to 32 data points each. The datasets come from electrical tests of real hardware collected from a variety of missile systems and subsystems in various stages of development and do not represent the constituent parts of any composite assembly. Each dataset should be considered independently. These datasets will be referred to as SST1, ..., SST73; (a full collection of all datasets is available from the authors). Six of these SST datasets (SST1, ..., SST5, and SST47) have been chosen to illustrate the various problems of fitting pdfs to data. The datasets in the SST collection represent single variable data.

26.2 Analysis Techniques

As mentioned earlier, two commercial pdf fitting tools (Crystal Ball™, and EasyFit™) and three Raytheon internally developed tools (JFIT©, GAFit, and GLFit) are examined for fitting data to pdfs. Trial versions of Crystal Ball™ and EasyFit™ can be downloaded from the Web at

`www.oracle.com/appserver/business-intelligence/crystalball/index.html`

and `www.mathwave.com`. The Raytheon tools JFIT©, GAFit, and GLFit will be supplied on a CD-ROM that accompanies our forthcoming text *Statistical Design Methods for Engineers* (Alderman, Mense, and Runger (2010)) or can be downloaded from the authors' Web site `statisticaldesignmethods.com`. The Raytheon programs JFIT©, GAFit, and GLFit are all Excel™-based. JFIT© and GAFit each fit data to the Johnson family of distributions. The Johnson family of distributions was chosen by the authors for three reasons:

- It gives reasonably good fits to most of the practical problems the authors encounter in an industrial setting, which is principally because it has four parameters.
- It produces a distribution that can be mapped into a standard normal distribution, which means that Monte Carlo sampling of the fitted distribution can be done by simply sampling from a unit-normal distribution. This is easy to do in Excel and other statistics programs.
- Moments matching for determining the four-parameter estimators is implemented in the Raytheon tool JFIT© using an algorithm developed by Hill, Hill, and Holder (1976) and in the commercial tool EasyFit™ by MathWave Technologies. This makes fitting easy. *Note: There were errors in the original FORTRAN code by Hill et al., so it had to be cleaned up. It has been rewritten in Visual Basic.*

A non-pragmatic reason for using moments matching is that it has a certain amount of philosophical appeal. It should be noted that matching moments seldom produces the best fit to the data when a GOF test such as the KS or Anderson-Darling (AD) test is applied, which will be addressed later in this chapter. The authors have found from day-to-day, practical experience that matching moments seems to be adequate for many pdf fitting situations when unimodal distributions are required. Multimodal distribution fitting requires a whole discussion of its own, and is touched upon later in this chapter.

Another fitting tool, called GAFit, developed at Raytheon and available to the reader (Alderman, Mense, and Runger (2010)), takes the same Johnson family of distributional forms, but chooses the parameters in a different manner. Instead of using the method of moments, GAFit searches the parameter space; to find a fit that minimizes the KS statistic. A typical search would take minutes to hours to find an acceptable solution using a 2GHz PC with Windows XP and 2GB of

SDRAM memory. The GAFit tool is a modification of the RAPID tool. There are better methods available for searching the parameter space, however, this genetic algorithm (GA) technique is simple to use and provides solutions quickly enough to meet engineers' needs. What more could an applied statistician want?

The GLFit tool developed at Raytheon also uses the RAPID algorithms and generates fits to the four-parameter lambda distributions (Karian and Dudewicz (2000)). Because the lambda distribution directly maps quantile values into random variable values, it is more convenient to use a different GOF measure. Several GOF measures have been tried for the GLFit tool, one measure is the total root mean square (rms) error between the fitted cumulative distribution function (cdf) and the actual data values. Another measure is to try and fit the first four central moments of the data. A third is to simply use the KS score. There is some inherent ambiguity in using the rms error method since the experimental probabilities are only known to within an approximate range of $1/n$ (for a dataset with n points), but there has been no adjustment for this fact in the work presented here.

The details of how the fitting is done in the commercial software Crystal Ball$^{\text{TM}}$ and EasyFit$^{\text{TM}}$ are discussed in the help files for these programs. Each of these tools has a family of distributions from which to choose and each offers a variety of fitting metrics of which the KS statistic is one. The Easy Fit$^{\text{TM}}$ tool includes Johnson distributions and 40 other distributions. The Crystal Ball$^{\text{TM}}$ tool does not include the Johnson family and has 14 different distributions for fitting however its primary purpose is to carry out Monte Carlo calculations, not to fit data to distributions. The technique used in the EasyFit$^{\text{TM}}$ application to fit the Johnson family of distributions is based on the same Hill, Hill, and Holder algorithm used in the Raytheon JFIT© tool. The commercial codes give comparisons for GOF using KS, Anderson-Darling (AD), and chi-squared tests.

26.3 The Johnson Family of Distributions and the JFit and GAFit Tools

As discussed above, the Johnson family is a good choice for matching many sets of univariate data. This section will discuss the Johnson family, but the definitive work is by Johnson (Johnson (1949)), who did his Ph.D. thesis work in this area. *Note: the parameters are ε, which is the location parameter for X, λ, which is the scale parameter for X, and γ and η, which are shape parameters.*

For the bounded Johnson distribution (JSB), the variable mapping uses

$$Z = \gamma + \eta \ln\left(\frac{X-\varepsilon}{\lambda+\varepsilon-X}\right), \tag{26.3.1}$$

where Z is distributed as a unit-normal variable. The minimum value that X can have is ε, and the maximum value is $\lambda+\varepsilon$. The JSB has the following functional

form for its pdf.

$$f(x) = \frac{\eta}{\lambda\sqrt{2\pi}\zeta(1-\zeta)} \exp\left\{-\frac{1}{2}\left(\gamma + \eta \ln\left(\frac{\zeta}{1-\zeta}\right)\right)^2\right\}, \tag{26.3.2}$$

where $\zeta = (x - \varepsilon)/\lambda$. The cdf is given by

$$F(x) = \Phi\left(\gamma + \eta \ln\left(\frac{\zeta}{1-\zeta}\right)\right), \tag{26.3.3}$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_0^y e^{-t^2/2}\, dt \tag{26.3.4}$$

is the unit-normal cdf.

For the unbounded Johnson distribution (JSU), the variable mapping uses

$$Z = \gamma + \eta \sinh^{-1}\left(\frac{X - \varepsilon}{\lambda}\right), \tag{26.3.5}$$

where Z is distributed as a unit normal variable. The JSU has the following functional form for the pdf:

$$f(x) = \frac{\eta}{\lambda\sqrt{2\pi(1+\zeta^2)}} \exp\left\{-\frac{1}{2}\left(\gamma + \eta \sinh^{-1}(\zeta)\right)^2\right\}. \tag{26.3.6}$$

The cdf of the JSU is given by $F(x) = \Phi\left(\gamma + \eta \sinh^{-1}(\zeta)\right)$.

The Johnson normal distribution (JSN) has the following functional form for its pdf:

$$f(x) = \frac{\eta}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\gamma + \eta x)^2\right\}, \tag{26.3.7}$$

with its cdf given by $F(x) = \Phi(\gamma + \eta x)$. Most users will recognize the normal transformation if one defines $\gamma = -\mu/\sigma$ and $\eta = 1/\sigma$, so that $Z = (x - \mu)/\sigma$.

For its pdf, the Johnson lognormal distribution (JSL) has the form,

$$f(x) = \frac{\eta}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\gamma + \eta \ln(x))^2\right\}, \tag{26.3.8}$$

and its cdf has the form, $F(x) = \Phi(\gamma + \eta \ln(x))$.

Which distribution to use is based upon the values of the coefficient of skewness ($\beta_1^{1/2}$) and the coefficient of kurtosis, β_2. Johnson (1949) outlines how the moments matching method might be used to determine the four parameter values (three for JSL and two for JSN) for the JSU and JSL distributions. He then acknowledges that using the method of moments to determine the parameters for the bounded distribution is complex and other methods can be used, such as

matching selected quantiles. However, Hill, Hill and Holder, (1976) did develop a numerical algorithm that does a reasonable job of fitting using the moment-matching method for all four members of the Johnson family. Other than the code (FORTRAN) being convoluted and because the iteration scheme to fit the JSB distribution sometimes does not converge, the numerical scheme is fast and accurate. This scheme has been incorporated into a tool called JFIT© and is available from the authors or at their Web site.

A second tool, called GAFit, developed by the first author, is based on using a genetic algorithm technique to search the parameter space of any given Johnson family distribution. The criterion for determining the parameter values is to search for the *best* fit to the data as measured by the KS score. Other scores such as AD or RuM could also be used. This tool is Excel™-based and runs rather slowly on a 2GHz PC but it almost always finds a better fit to the data. The question that arises and which will be addressed later in this chapter, is: How good a fit to the data does one need? GOF will be discussed in more detail below.

26.4 The Empirical Distribution Function (edf)

The empirical distribution is used in all GOF tests. It is produced as follows: Consider a sample of n data points. Sort the data values from smallest to largest. Label the smallest value $j = 1$, next smallest $j = 2$, etc. Build from this sorted dataset a cumulative distribution in which the data value, say x_j, is placed along the horizontal (abscissa) axis and plot the value j/n along the corresponding vertical (ordinate) axis. This is the plot of an edf. It appears as a series of steps with each step occurring at a given data value. Figure 26.4–1 illustrates such an edf with an accompanying smooth curve, which is the cdf fit using the KS score as a goodness-of-fit.

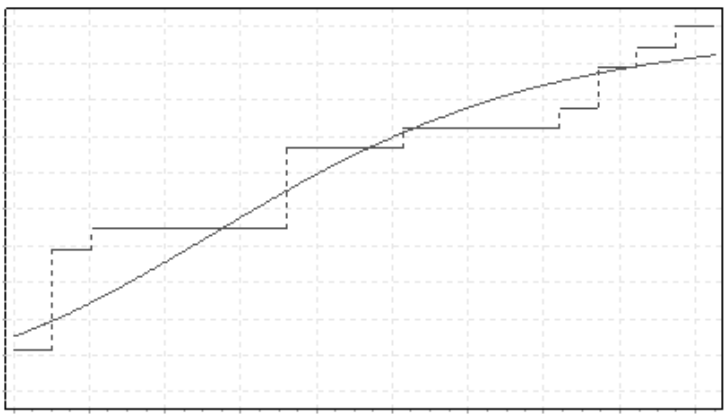


Figure 26.4–1. Example of an edf and cdf from fitting a dataset.

26.5 Goodness-of-Fit (GOF)

The most common question asked by users is, "How good is the fit of the pdf to the data?" Unfortunately, there is no agreed upon answer. The pragmatic answer is more likely based on what the users' needs are. How is the pdf going to be used after fitting it to the data? If only descriptive statistical information is required, then simply using a visual comparison is sufficient. In fact, even if the pdf is to be used for further analyses, one typically must choose where in the range of data values the user wishes the fit to be numerically closest to the data. This will be shown with examples later in this chapter.

If the pdf does not visually compare well to a histogram of the data then few users will be happy with that fit. However, it should be remembered that small samples of data may produce vastly different distributional fits than the population from which the sample is chosen. This can be demonstrated by taking a sample of 10 or 20 points from a normal distribution. When trying to make these points fit, it is found that many times the best fit is not a normal distribution but is instead a 3-parameter Weibull or a gamma distribution. If the form of the distribution is going to be used to deduce the physics that produced the population, one must exercise some care. Before one goes fishing for a distribution, it is important to understand the physical model, or to have in mind a physical model that gave rise to the data. In this regard, Wilkinson (1999) gives some sage advice: "... some researchers who use statistical methods pay more attention to goodness of fit than to the meaning of the model. It is not always convenient to remember that the right model for a population can fit a sample of data worse than a wrong model — even a wrong model with fewer parameters. Statisticians cannot rely on statistical diagnostics to save them, especially with small samples. They must think about what the models mean, regardless of fit, or they will promulgate nonsense."

If there is reason to have a particular physical model in mind, then even if the sample data is not the best fit, one should still use that hypothesized model unless further analysis (not necessarily statistical) leads the user in another direction. With this caution in mind, there are two major types of GOF tests (D'Agostino and Stephens (1984)) and the authors have developed a third type (Mense and Alderman (2008)). The first technique, characterized by the KS test, is a supremum test [e.g. $D = \max(i/n - F(x_i),\, F(x_i)(i-1)/n)$] in which the maximum separation is measured between the edf data points $(i/n,\, F(x_i))$ and the cdf that is being hypothesized as the best fit. If the hypothesized distribution is completely known, i.e., not using test data to estimate the distribution parameters, then the KS score is *distribution free*. If, however, test data is used to calculate the parameters of the hypothesized distribution, then the test is distribution dependent and does not have a specific distribution for the test statistic.

A metric that we find most useful and which is in the same vein as the "eyeball test" is called the **xad metric** (for maximum absolute difference). The xad is

the absolute value of the maximum difference between the empirical cdf and the fitted cdf, i.e., $\max |i/nF(x_i)|$. Fits that look good to the eye will return a small value for xad and the converse is also true. Note however that this may not be so for the KS, which compares the fitted cdf at one point to the empirical cdf at two adjacent points. Thus, any time there is a data bin that is significantly over-populated or under-populated compared to its nearest neighbors, the KS score can be significantly larger than the xad score. The net result is that fits which look pretty good always have low xad values but may not have low KS values. To avoid rejecting fits that have a low xad score but have significantly higher KS scores we typically select the best fit as the one with lowest xad value. In short, we do not use the KS score for selecting the best fit but only compute it for reference after the best fit has been determined.

The second technique is a form of a weighted mean squared approach where the difference between the edf and cdf is squared, weighted, and then integrated over the postulated cdf. The Anderson-Darling and Cramer-von Mises tests are examples of this approach. See the Appendix at the end of this chapter for derivations, which are included because they are hard to find in the literature.

There is also a newer, easy-to-compute, and useful GOF test called the **RuM-test** (Mense and Alderman (2008)) that looks at how the experimental data maps into the cdf of the hypothesized distribution. The RuM-test (Mense and Alderman (2008)) was developed by the authors to fill a need for an easy-to-use measure of GOF for small datasets. The test statistic for the RuM-test is given by $\text{Ru} = -2\sum_{i=1}^{n} \ln(P(x_i))$, where $P(x_i) = 1 - F(x_i)$ is called the **tail probability**. $F(x_i)$ is the cumulative distribution function (cdf) for the known test distribution evaluated at the i-th data value, x_i. If H_0 is true, i.e., the data is consistent with the proposed distribution, the RuM-test statistic is distributed as a chi-squared random variable with $2n$ degrees of freedom. The RuM-test is different from the more popular GOF tests in that it does not look at differences or distance measures between an edf and the test distribution, $F(x)$.

Consider the following situation: let x_i, $i = 1, \ldots, n$ be a data sample. The null hypothesis is H_0: The sample data is consistent with the test distribution $F(x)$ whose pdf is $f(x)$ and whose tail distribution is defined by $P(x) = 1 - F(x)$. If the sample data is consistent with the test distribution, then substitution of the sample data into $F(x)$, i.e., $F(x_1)$, $F(x_2)$, $\ldots$, $F(x_n)$, will produce a distribution that is close to $U(0, 1)$, the uniform distribution on (0, 1). However, if the sample data does not come from a distribution that is consistent with $F(x)$, then the values $F(x_1)$, $F(x_2)$, $\ldots$, $F(x_n)$ will not distribute as $U(0, 1)$. The same can be said about $P(x)$, the tail distribution, since $F(x) \to U(0, 1)$ clearly means that $P(x) \to U(0, 1)$. The test is based on this departure from a $U(0, 1)$ distribution, hence the chi-squared statistic with $2n$ degrees of freedom. The convenience comes about because the critical values for the chi-squared distribution are easily accessible (e.g., in Excel$^{\text{TM}}$) and one need not carry around critical value tables for KS or AD statistics. The RuM-test is particularly helpful in determining if

the data does not come from the postulated distribution, i.e., H_0 is false.

A description of the RuM-test appears in the Appendix at the end of this chapter. For GOF measures, the focus has been principally on the KS statistic. Other tests will be used as they are needed to demonstrate closer fits to the data.

As a side note, Crystal Ball^{TM} and EasyFit^{TM} have many of the same pdf families available in their libraries, yet do not normally rank the pdf's GOF in the same order. For example, both commercial programs calculate the KS statistic. They each quote different values for this statistic given the exact same data. Crystal Ball^{TM} argues the user should use the GOF measures scores as a relative measure, not an absolute measure. Therefore, Crystal Ball ^{TM} does not give critical values for the GOF measures statistics. EasyFit^{TM} does a more thorough job and does list acceptance criteria. The RuM-test uses the critical values from a chi-squared distribution as will be demonstrated. For the interested user, there is a National Institute of Standards and Testing (NIST) Web site, http://www.itl.nist.gov/div898/handbook/eda/section3/eda35g.htm, that offers guidance on how the KS statistic should be computed, but it seems clear that practitioners have not uniformly implemented the NIST's advice.

26.6 Example 1: The SST1 Dataset

The analysis of the SST1 dataset, shown in Table 26.6–1, proceeds as follows: The data is analyzed in its entirety with discussions of all the techniques and graphical representations that are used. The subsequent datasets will be presented with only discussions of the major differences or difficulties encountered in analyses.

The first step in data analysis is to form descriptive statistics. An often used visualization is the dotplot using all 29 data points. This is shown in Figure 26.6–2 and it immediately tells the analyst that one has a positively skewed distribution. The data points at the highest values should be examined for authenticity as per earlier remarks on using good data.

Table 26.6–1. The SST1 dataset: Electrical voltage measurements (in volts) of a subsystem used in a missile systems test.

0.05	0.05	0.19	0.16	0.12	0.15	0.16	0.34	0.09	0.3
0.12	0.35	0.05	0.06	0.07	0.05	0.08	0.03	0.11	0.05
0.05	0.42	0.22	0.07	0.3	0.18	0.09	0.07	0.12	

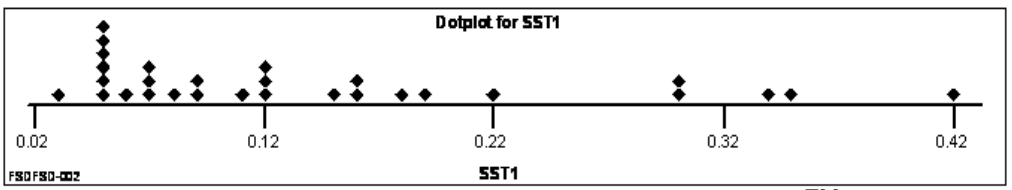


Figure 26.6–2. The SST1 dataset as a dotplot (Minitab^{TM}).

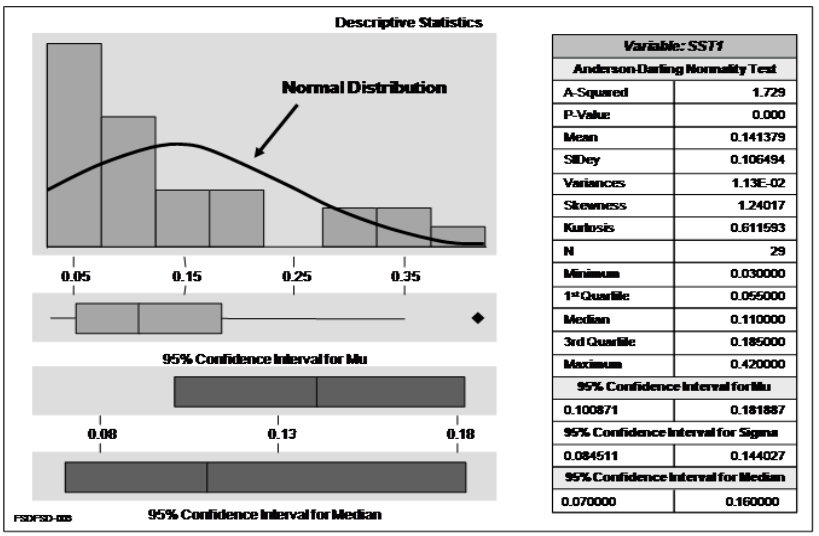


Figure 26.6–3. Descriptive statistics of SST1 data including a histogram, a box and whisker plot, the mean and median of the distribution with 95% confidence intervals and a normal distribution for reference purposes. Note: the normal is not a good fit to this data.

Note the grouping at lower x values. This grouping will determine the **center of mass** of the distribution, but the data points at higher values will more acutely affect the other principal moments of the distribution (variance, skewness, and kurtosis). It is surprising how many nonstatisticians simply assume all variability in data can be modeled by a normal distribution. If a normal distribution were to be fit to this data, much of the structure of the pdf would be lost. A normal distribution is not a good fit (see overlay in Figure 26.6–3). Note that the coefficient of kurtosis calculated by Mintab^{TM} and Excel^{TM} is actually the coefficient of excess kurtosis, not the actual kurtosis. Coefficient of excess kurtosis = coefficient of kurtosis -3.

There may be a temptation to try and fit a bimodal distribution to this data. That temptation should be resisted for two reasons: first, we know the experiment from which this data originated and there is no physical reason to suspect bimodality and second, there are only 29 data points and they could have been descriptively binned to look unimodal with a large positive skewness. Another useful plot compares the edf, and a proposed cdf for the SST1 data. This is shown in Figure 26.6–4.

How good is this fit? EasyFit^{TM} reports a KS score of 0.1233, which comes from calculating a statistic $D = \max(D+, D-) = 0.1233$. The KS statistic is well below the critical value, which is approximately 0.25 (D'Agostino and Stephens, 1984). Using the KS score, we cannot reject the fit to a JSB as being inadequate or inconsistent with data. In this instance, the EasyFit^{TM} commercial tool chose the

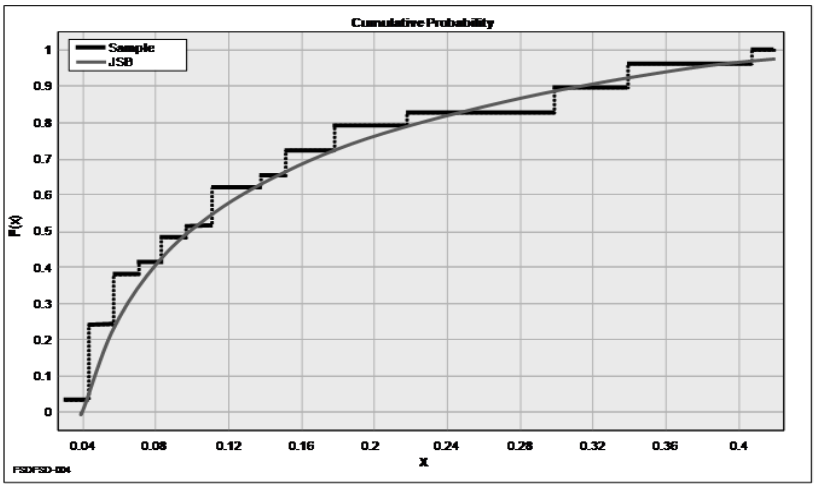


Figure 26.6–4. A fitted cumulative probability overlaid with an edf for the SST1 dataset (EasyFit^{TM}).

Johnson distribution that the Raytheon tool JFIT© would have selected. This is seldom the case for most datasets. As a reminder, the EasyFit^{TM} and the JFIT© tools both use the moments-matching method to determine the estimators of the Johnson distribution parameters. No matter which tool is selected, the Johnson fits generated by GAFit are always better (measured by the KS statistic) than those from the JFIT©, EasyFit^{TM}, or Crystal Ball^{TM}. However, the parameter estimates are more tedious to produce as the genetic algorithm takes minutes to converge on the correct parameter estimates. An improved algorithmic approach would be useful for batch fitting datasets such as the SST database.

As a practical matter, the Johnson distributions (using moments matching) have worked well for the typical engineering datasets encountered by the authors over the years. One can always invent datasets that are not well fitted by any particular distribution, hence the commercial market for EasyFit^{TM} and Crystal Ball^{TM}. Using the *eyeball* technique for estimating GOF is often the best method. The reason is simple; the user of the data and the fitted pdf knows which part of the fitted data is the most important in the application being studied. Therefore, the user can see if the fit is close in that region of data. If it is important to know the tails of a distribution accurately, then a test that fits better in the tails can be used. Certainly AD would be a better GOF test for these cases. Sometimes the root-sum-squared of the differences or the sum of absolute values of the differences is a good measure, particularly when the fitted distribution can only be represented in quantile form, i.e., $x = g(F)$ where F is the cdf of the distribution.

As mentioned, there is no universally accepted best numerical measure of GOF. The advantage of something like the xad or KS test is that small values are definitely indicative of good fits because the fitted cumulative has to hug the

empirical distribution for this to happen. Unfortunately, the KS test may not be the most powerful test to discriminate one fit from another. To improve this, the RuM-test might be used.

Another graphical technique that has been useful is plotting a histogram of the data and then taking the distribution whose fit one is determining and integrate the pdf over the bin size for each bin of the histogram. This gives a probability to match with the probability from each bin of the histogram. This is shown in Figure 26.6–5 for the SST1 data. Each point (triangle) on the plot in Figure 26.6–5 was generated by integrating the fitted pdf over the bin range and therefore represents a probability that can be compared directly to the histogram probabilities. The probabilities shown in Figure 26.6–5 are considered to be a fairly good fit for such a small sample size (29 points). The fitted pdf is shown in Figure 26.6–6. This pdf was generated using JFit. The form of the Johnson cdf and pdf that arose from this data is of the JSB form that was given in equations (26.2.3) and (26.2.2), respectively.

How good is this fit? From our experience the moments method gives a reasonably good fit. Also, if one applied the KS test, the test statistic would be D = 0.123 and the critical D value at the 95% confidence level is 0.252. Therefore, based on the KS test, the JSB is a good fit. However, if one applies the AD GOF test, the test statistic labeled A2 = 4.1, and the 95% confidence critical value is 2.5. So, the JSB shown in Figure 26.6–6 would be rejected.

The best AD fit, according to EasyFit^{TM}, is the fatigue life distribution, a three-parameter distribution, given by

$$f(x) = \frac{\sqrt{(x-\gamma)/\beta} + \sqrt{\beta/(x-\gamma)}}{2\alpha(x-\gamma)} \phi\left[\frac{1}{\alpha}\left\{\sqrt{(x-\gamma)/\beta} - \sqrt{\beta/(x-\gamma)}\right\}\right] \quad (26.6.1)$$

and

$$F(x) = \Phi\left[\frac{1}{\alpha}\left\{\sqrt{(x-\gamma)/\beta} - \sqrt{\beta/(x-\gamma)}\right\}\right]. \quad (26.6.2)$$

The AD score for this distribution is A2 = 0.372, which is well below the critical value of 2.5 for a 95% confidence level. The actual plot of the cdf versus edf is shown in Figure 26.6–7.

It can be seen in Figure 26.6–7, that the cdf fits better as values approach the lower and upper values of the variable, i.e., as the cdf approached zero and one. This is the intent of the AD test, i.e., to weight more heavily the tails of the pdf than the central region. A comparison of the two pdfs is shown in Figure 26.6–8. Keep in mind that the JSB preserves the first four central moments of the dataset and also just happens to have the lowest and best KS score. The fatigue-life distribution (shown in Figure 26.6–7) has the best AD score. Which fit does one pick as the best? The answer lies with how one intends to make use of the fitted distribution.

It is not clear that one pdf is any better than another from a visual examination of the pdfs themselves and comparing this to the histogram of the data.

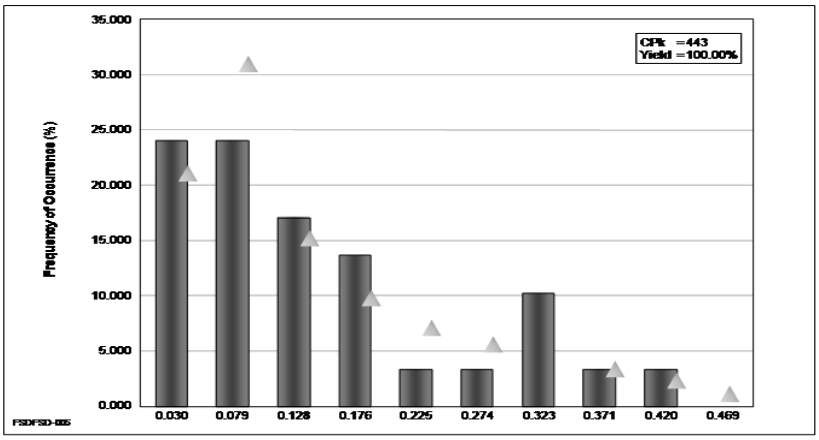


Figure 26.6–5. The SST1 histogram with fitted probabilities for each bin.

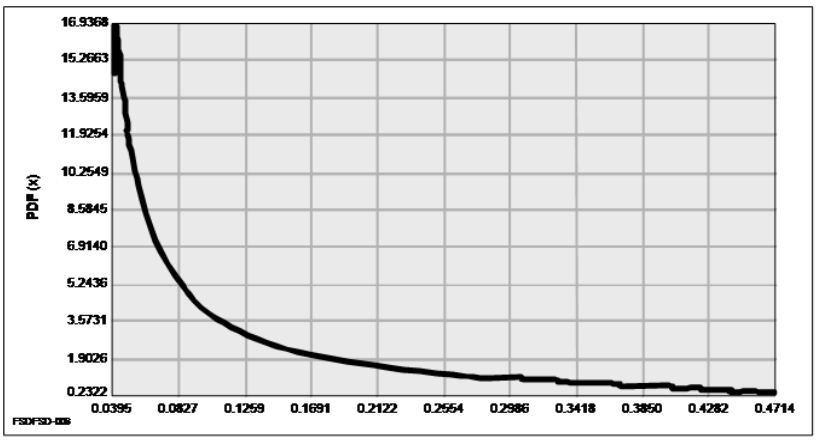


Figure 26.6–6. The pdf produced by JFIT© using the SST1 data from Table 26.6–1.

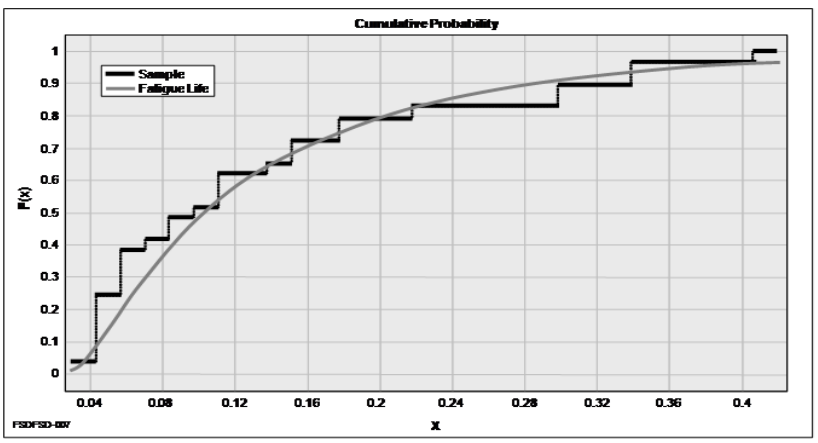


Figure 26.6–7. Fatigue life distribution fit to the SST1 data (EasyFit™). The parameter estimates are $\alpha = 0.9422$, $\beta = 0.0854$, $\gamma = 0.0179$.

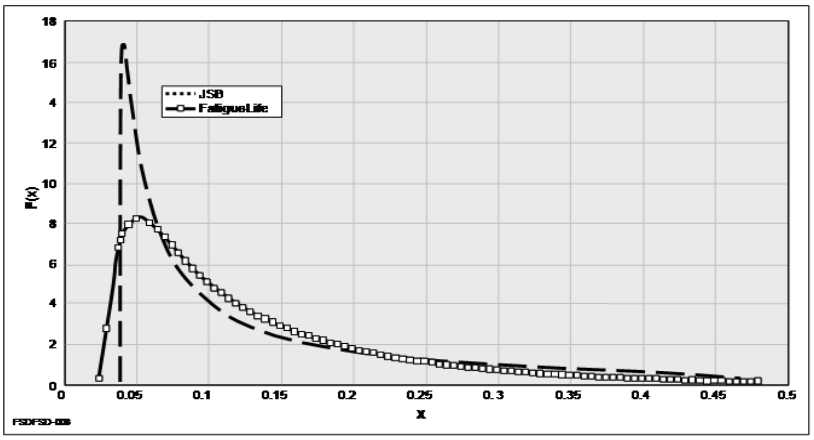


Figure 26.6–8. Comparison of pdf for fatigue life and JSB, both fitted to SST1 data.

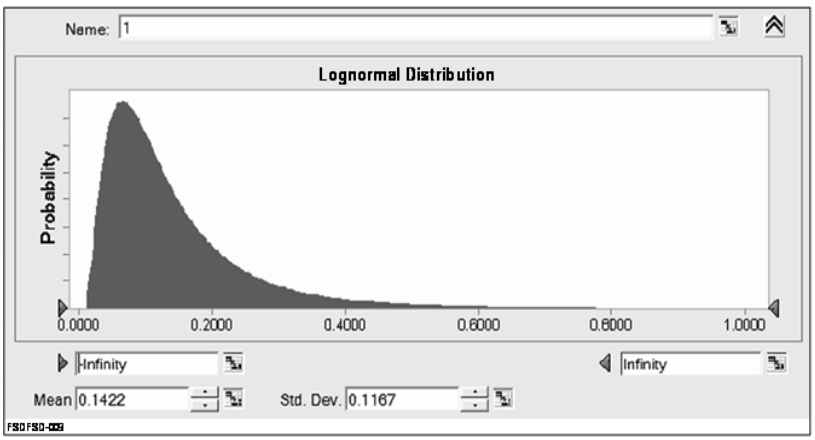


Figure 26.6–9. The lognormal fit to the SST1 data (Crystal Ball^{TM}) using the KS GOF metric.

Indeed, either distribution would be fine unless one is particularly sensitive to the cutoff at the lowest data values and the higher peaking in the pdf near the lower limit of x. The JSB exhibits a cutoff at $x = \varepsilon = 0.0392$. If the user believes there is not a real cutoff, then using a bounded distribution may not be a good choice. As much as users would like to draw definitive conclusions about which distribution is best, this example shows there is often no clear choice. *Caveat duco* (let the calculator beware). By contrast, Crystal Ball^{TM} would pick the log-normal distribution using the KS test (mean = 0.1422 and standard deviation = 0.1167) and pick the gamma distribution using the AD test (location = 0.0295, scale = 0.1093, shape = 1.02356). The lognormal distribution fit is shown in Figure 26.6–9.

Comparing the various fits from the commercial tools and JFIT© (Figure

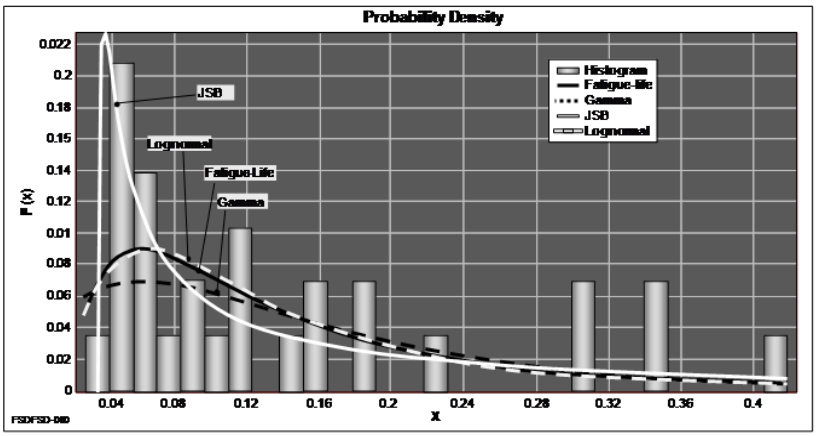


Figure 26.6–10. SST1 data comparison of pdf fits. JSB (best KS fit in EasyFit^{TM} and JFIT© choice), lognormal distribution (best KS fit using Crystal Ball^{TM}), fatigue-life distribution (best AD fit using Easy Fit^{TM}), and gamma distribution (best AD fit using Crystal Ball^{TM}).

26.6–10) gives some insight into the problem of deciding which fit is best. Use theory to guide the decision whenever possible and knowledge of how the pdf will be used if no theoretical guidance is at hand.

The differences between EasyFit^{TM} and Crystal Ball^{TM} selections for best fit are due to the fact that EasyFit^{TM} has more than 50 choices within its library while Crystal Ball^{TM} has only 13. We now consider two other methods for fitting data to pdfs that are not included in commercial tools, yet both of them are very useful.

26.6.1 Johnson Family Distributions that Minimize KS Scores

The first method is to use the functional form of the Johnson family of distributions, but instead of choosing the four parameters in a manner that preserves the four central moments of the data, one simply chooses the parameters that minimize user-defined metrics such as xad or KS. This is done using the Raytheon tool GAFit for Genetic Algorithm Fitting. Simply speaking, this tool does a genetic algorithm search over all the allowable values for the four Johnson fitting parameters for each of the four types of distributions in the Johnson family (e.g., bounded, unbounded, lognormal, and normal). The parameter estimates are chosen on the basis of the lowest KS score. The results of performing this search are shown in Figure 26.6–11. The results appear more in line with the fatigue life distributional fit.

These results may appear confusing at first. Earlier it was stated that a Johnson fit to the SST1 data using the moments-matching technique also gave the best KS score (according to EasyFit^{TM}). This is true; however, EasyFit^{TM}

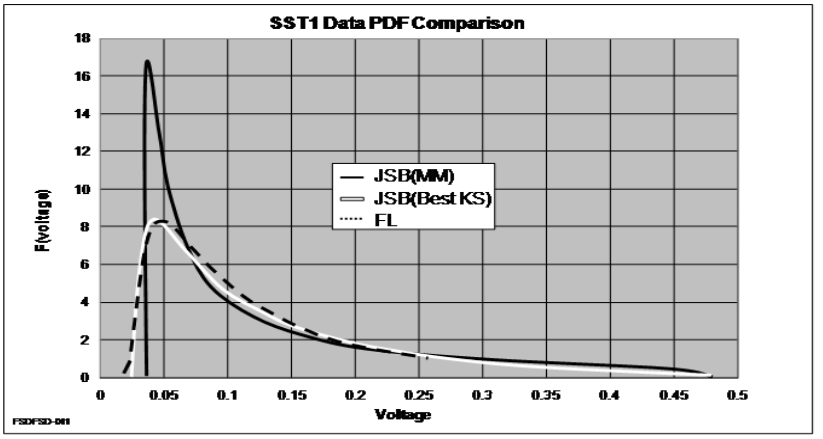


Figure 26.6–11. Plot of three possible fits to the SST1 data. Note the closeness of FL and GAF it to JSB.

does not try to find the Johnson parameter estimators that minimize the KS score. It just so happens that among the 50 or so distributions that EasyFit^{TM} uses to compare to the data the JSB gave the lowest KS score of any of the possible distributions. The parameter estimates that come from the GAFit are ($\gamma = 1.39170$, $\eta = 0.75359$, $\varepsilon = 0.57820$, $\lambda = 0.02929$). From Figure 26.6–11, it is easy to assert that two different fits give approximately the same shape and the Johnson fit using the moments-matching method does not give as good a fit. This is true using the KS and AD scores but the differences may not be as big as shown in the graphics.

This may be the best place to pause and reflect on reasons for using the method of moments. The reasons are: 1) It does produce good fits to data for many situations, 2) there is some theoretical appeal to having the same first four moments as the data, and 3) the structural differences in fitting only occur near the lower boundary of the data. Again, it is in the eyes of the user as to which distribution one chooses.

26.6.2 The Four-Parameter Generalized Lambda Distribution

The second method that will be demonstrated is the use of the four-parameter generalized lambda distribution (GLD). This particular distribution was chosen due to its widespread use and the availability of literature on its application (Karian and Dudewicz (2000)). Other four-parameter families exist, most notably the generalized kappa distribution and the four-parameter beta distribution (for bounded data), but the GLD is sufficiently broad to cover the types of data of interest in this chapter. There are two somewhat different functional forms for

the GLD. The first is the RS form, due to Ramberg and Schmeiser (1974), where

$$x = \lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right), \tag{26.6.3}$$

$$f(x) = \frac{du}{dx} = \frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}}. \tag{26.6.4}$$

The second form, the FKML form, due to Freimer, Kollia, Mudholkar, and Lin (1988) has

$$x = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4}}{\lambda_4}\right), \tag{26.6.5}$$

$$f(x) = \frac{du}{dx} = \frac{\lambda_2}{u^{\lambda_3 - 1} + (1-u)^{\lambda_4 - 1}}. \tag{26.6.6}$$

The parameter $\lambda_2 > 0$ since we must have $f(x) > 0$. Karian and Dudewicz (2000) use the RS form, which is the easiest to manipulate algebraically but one must pay attention to the regions of lambda values to guarantee convergence since it is not a valid pdf for all values of (λ_1, λ_2, λ_3, λ_4). The FKML formulation is valid for all real lambda values and is used by King and MacGillivray (2006) but is complex to work with since many subcases need to be considered.

As with all parameter estimators, there are many methods for finding a suitable set of lambda estimates. One method is to find expressions for the four moments (mean, variance, coefficient of skewness, and coefficient of kurtosis) and then match those to the four moments of the data. This is what is done by the Hill, Hill, and Holder algorithm used with the Johnson family of distributions. For the lambda distribution, the expressions for the moments are given below.

$$\alpha_1 = \lambda_1 + \frac{1}{\lambda_2}\left[\frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4}\right] = \lambda_1 + \frac{A}{\lambda_2}.$$

This expression shows the definition of A in terms of λ_1 and λ_2.

Similarly, the expression for the variance is given by

$$\alpha_2 = \sigma^2 = \frac{B - A^2}{\lambda_2^2}, \tag{26.6.7}$$

$$B = \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(\lambda_3 + 1,\, \lambda_4 + 1), \tag{26.6.8}$$

where $\beta(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the complete beta function.

The coefficient of skewness is given by

$$\alpha_3 = \frac{C - 3AB + 2A^2}{\lambda_2^3 \sigma^3},$$

where

$$C = \frac{1}{1+3\lambda_3} - \frac{1}{1+3\lambda_4} - 3\beta(2\lambda_3 + 1, \lambda_4 + 1) + 3\beta(\lambda_3 + 1, 2\lambda_4 + 1).$$

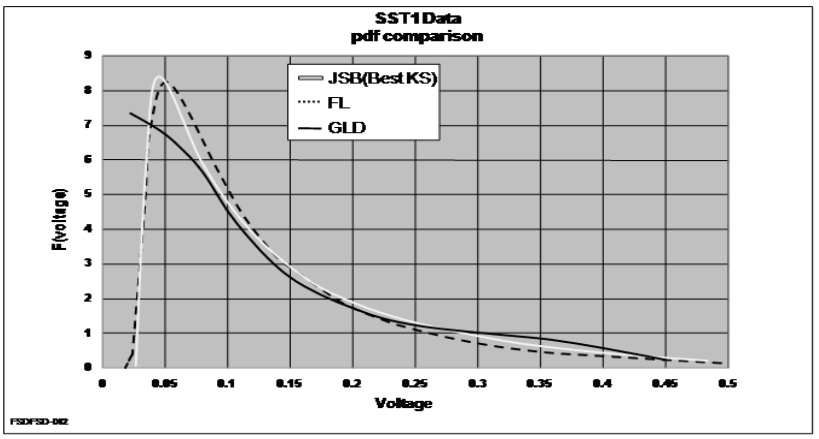


Figure 26.6–12. Comparison of four fitting alternatives to the SST1 data.

The coefficient of kurtosis has the expression

$$\alpha_4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \sigma^4},$$

where

$$D = \frac{1}{1+4\lambda_3} + \frac{1}{1+4\lambda_4} - 4\beta(3\lambda_3+1, \lambda_4+1) + 6\beta(2\lambda_3+1, 2\lambda_4+1) - 4\beta(\lambda_3+1, 3\lambda_4+1).$$

Using these expressions and matching them to the numerical values from the data, it is not difficult to produce a numerical scheme that will determine the parameters λ_1 through λ_4. Another method is to specify certain quantiles where one wants the distribution to match the data values, e.g., 10th, 25th, 50th, and 90th percentiles. Another method is to minimize the sum of the squares of the differences between the lambda distribution and the edf from the data. Alternatively, one could sum the absolute differences instead of the squares and then be more robust to outliers in the data. Applying the KS or AD test would also be fine, though slightly tricky. Several of the aforementioned methods require some form of parameter search routine. Again, the authors and Raytheon use the RAPID genetic algorithm search engine to perform these operations.

The results of fitting SST1 data to a GLD using the moments-matching method to find the lambda parameters is shown in Figure 26.6–12 where it is compared to the fatigue life and JSB.

The GAFit and FL fits minimize the KS score and the AD score, respectively. To validate the GLD fit, 200,000 Monte Carlo runs were performed on the GLD equation using the lambda estimates taken from matching moments (i.e., $\lambda_1 = 0.2392$, $\lambda_2 = 4.6208$, $\lambda_3 = 5.6294$, $\lambda_4 = 0.6301$). Crystal Ball^{TM} was used and sampled a uniform (0,1) distribution for u and solved the quantile expression for x. The values of x are binned in Figure 26.6–13. Note the GLD pdf does not go

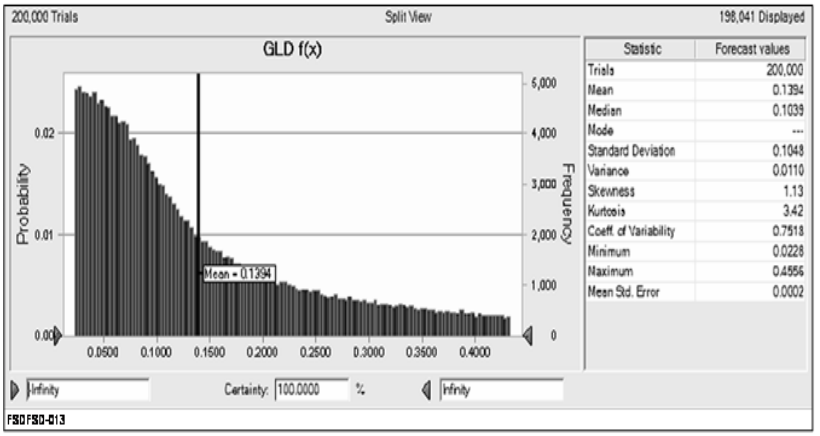


Figure 26.6–13 Monte Carlo simulation for the GLD distribution that matched the moments of the SST1 data (Crystal Ball^{TM}).

to zero at the lowest data values. Most other fitted distributions do go to zero. This may bode well for the lambda distribution being able to fit a larger variety of datasets.

26.7 Example 2: The SST2 Dataset

The SST2 dataset, given in Table 26.7–1, presents some unique challenges. We start with the dotplot in Figure 26.7–2 and cover the descriptive statistics further in Figure 26.7–3.

There is a large amount of data stacked up at a $x = 0$, with no negative data values, and this particular set of measurements would not have had any negative values. Taking the data and looking at a histogram and a box and whisker plot as shown in Figure 26.7–3 begins to show the potential difficulty as to how best to represent the data. Note that the normal distribution is clearly a bad choice for this data.

This is a difficult dataset to which to fit a pdf. There are a couple of general approaches: one can simply plow ahead and find a suitable single multi-parameter distribution using the techniques applied to SST1. A second technique might be to use a mixed discrete plus continuous distribution which would require assigning a probability mass function to the $x = 0$ point and then fitting the remainder of the data to a continuous pdf. Both options will be explored.

If one simply used the KS score as a guide, the best fit to this SST2 data will be a uniform distribution. The edf to cdf plot is shown in Figure 26.7–4 and the resulting pdf, with a histogram of the data, is shown in Figure 26.7–5. It must be noted that the KS scores for this fit are so bad that only if one expanded the confidence interval to $\alpha = 0.01$ (99% confidence level) the null hypothesis would not be rejected. Nevertheless, this type of data does occur and the question is:

Table 26.7–1. The SST2 dataset with $n = 30$ points, 16 of which are zero.

0.0000	0.0800	0.0000	0.0900	0.0000	0.0500	0.0200	0.0000
0.1000	0.0000	0.1000	0.0000	0.0000	0.0200	0.0000	0.0800
0.0000	0.0800	0.0100	0.0000	0.0900	0.0000	0.0100	0.0000
0.0000	0.1100	0.0000	0.0000	0.0000	0.0100		

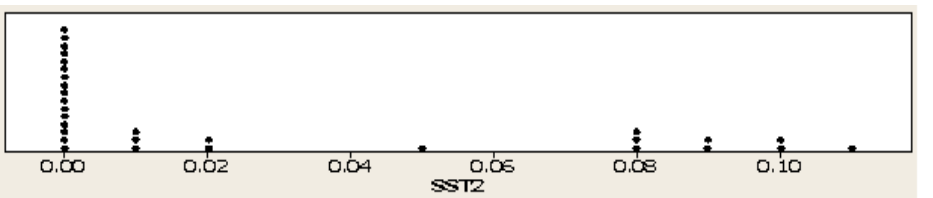


Figure 26.7–2. The SST2 dataset as a dotplot (Minitab^{TM}).

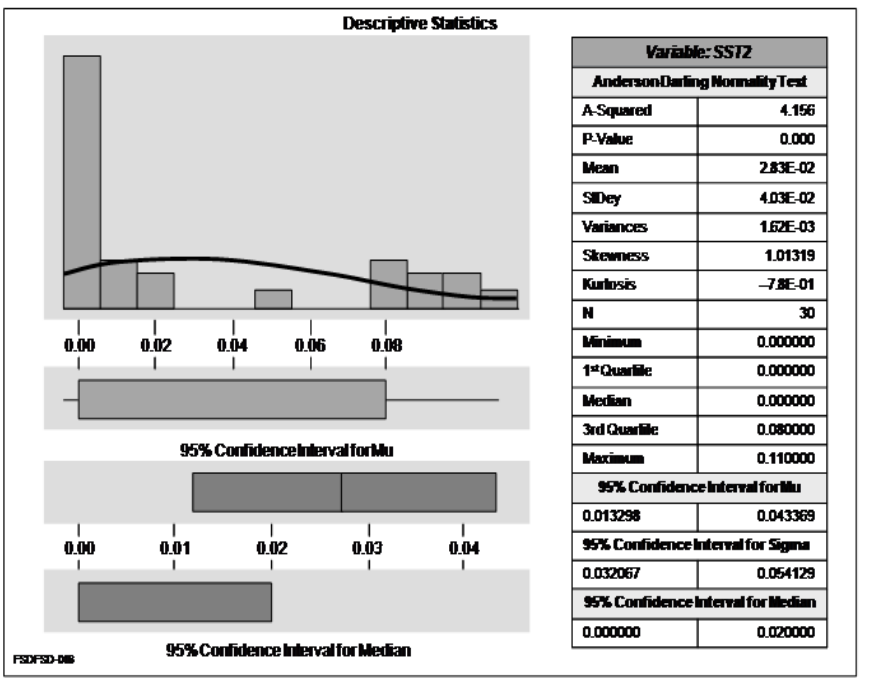


Figure 26.7–3. The SST2 data. The separation between the mean and the median indicates a large skewness (Minitab^{TM}). A normal distribution is clearly a bad fit to this data.

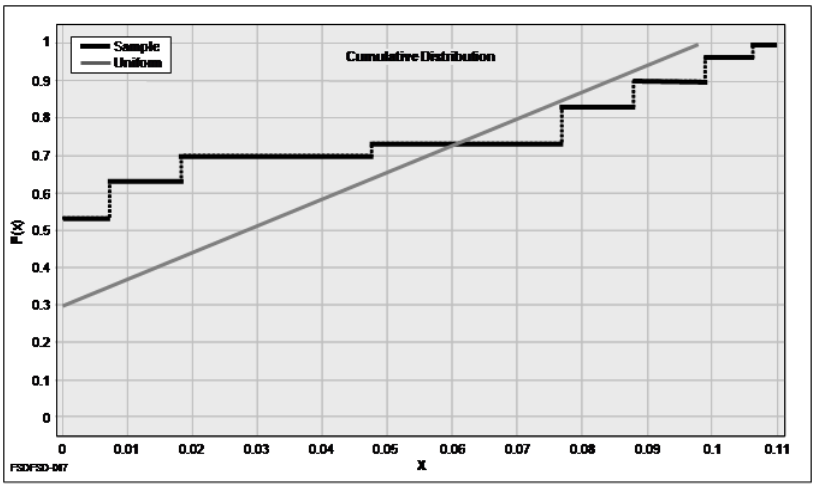


Figure 26.7–4. The edf versus cdf plot for a uniform pdf compared to the actual data edf (EasyFit^{TM}). The KS test indicates rejection of the uniform distribution (even though it is the best fit) unless the confidence interval is expanded to 99%.

how poor a fit is acceptable?

Understanding that a user would not take this to be a good fit, one can try the JFit routine. The results are shown in Figure 26.7–6.

The JSB tries to match the data at both ends of the dataset. The parameters from JFIT© are $\gamma = 0.6156$, $\eta = 0.1256$, $\lambda = 0.1012$, $\varepsilon = 0.0006$. Figure 26.7–7 shows the comparison of the fit with a histogram of the data. There is less probability under the right peak than the left peak in the pdf.

If one were to use the AD test, it would show that the best fit is a rather obscure distribution called the **phased bi-Weibull**. This is shown in Figure 26.7–8. This distribution has the best AD score and the test statistic is not rejected even at an $\alpha = 0.20$, but it would not be an obvious choice for users. The step change in the pdf (Figure 26.7–8) seems to make little sense unless the data is actually more discrete than continuous. The problem with this fit is that it does not account well for the frequencies at higher x-values. The gamma distribution does not necessarily do a better job (see Figure 26.7–9).

Generalized Lambda Distribution. Can the GLD perform better with this type of data? Figure 26.7–10 compares the pdfs for the JSB and the GLD, both of which use the moments-matching technique to fit the data with the GAFit, which is also the Johnson distribution but uses parameters that minimize the KS statistic and the gamma distribution, which is one of the best fits using the AD test and the GLD fit. If the user believes the higher x values are indeed real and do reflect the data correctly and if the distributions shown up to this point do not seem satisfactory, the user has two choices: use the edf and carry the data

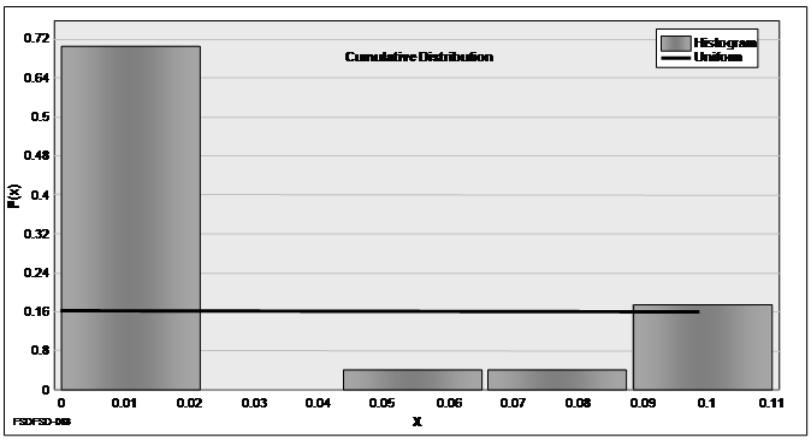


Figure 26.7–5. Uniform distribution fit to SST2 data. Best KS score clearly would not be the choice of most users.

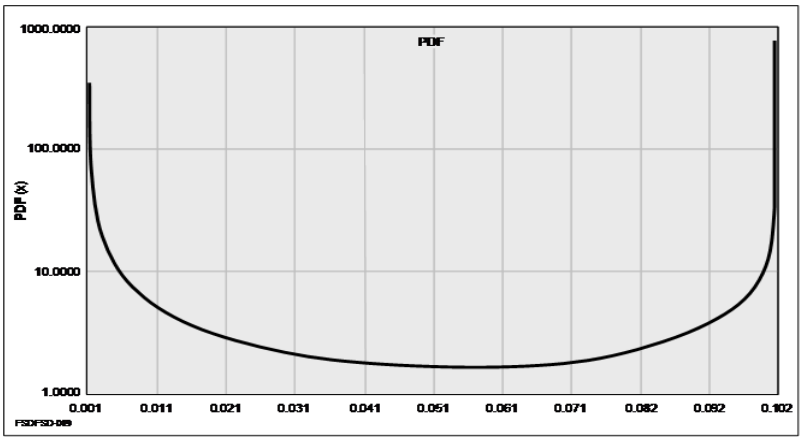


Figure 26.7–6. SST2 data fit to a JSB by matching the first four moments using JFIT©.

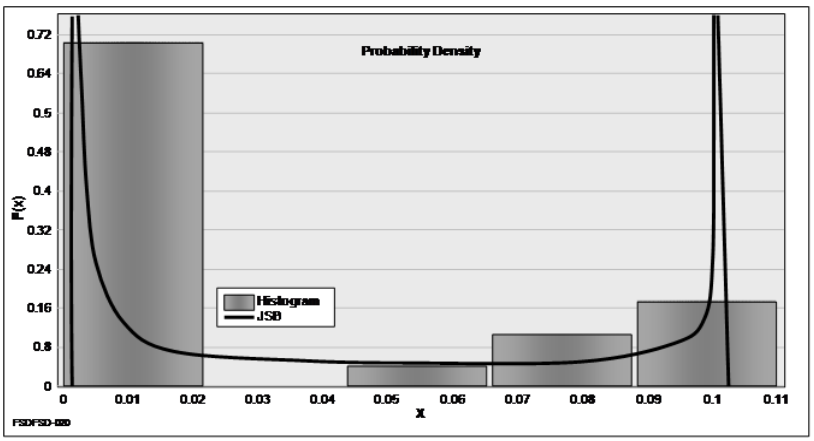


Figure 26.7–7. JSB to fit to the SST2 data from EasyFit™.

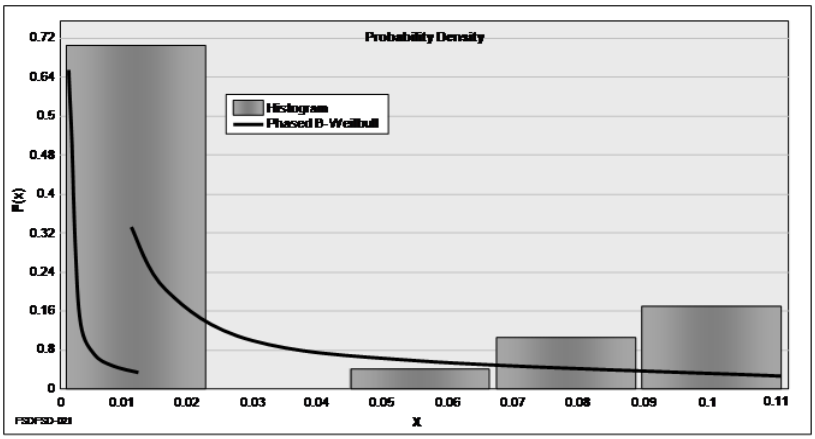


Figure 26.7–8. The phased bi-Weibull distribution fit to the SST2 dataset.

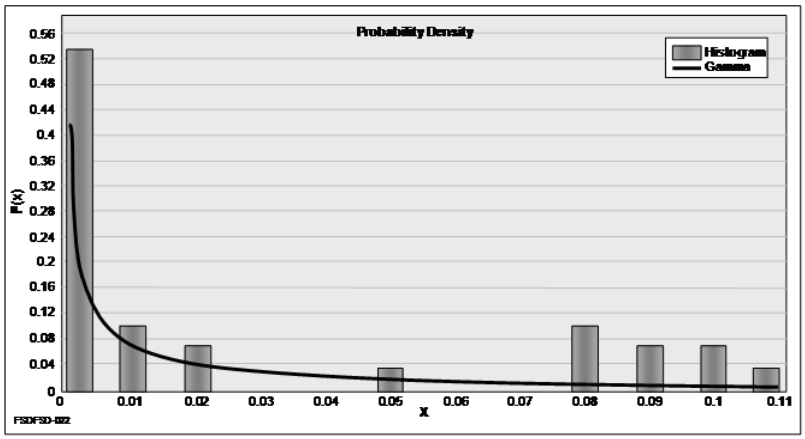


Figure 26.7–9. SST2 data fit to a gamma distribution ($k = 0.49516$, $\beta = 0.05722$).

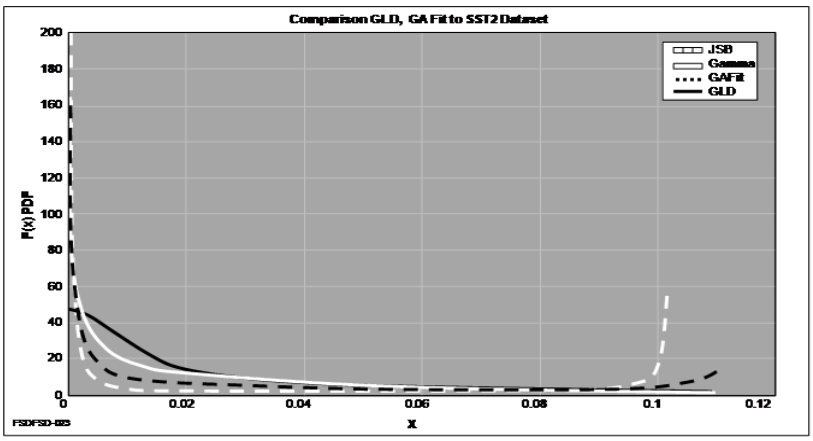


Figure 26.7–10. The GLD tries to fit the data near $x = 0$ but misses the data near $x = 0.1$.

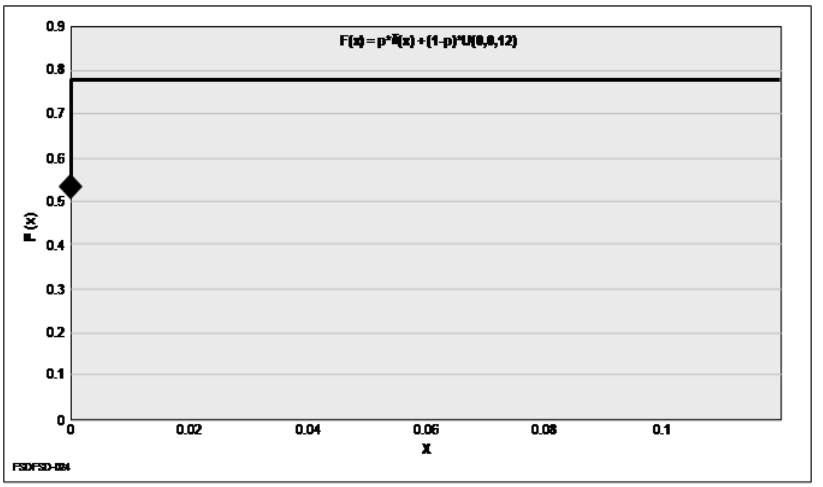


Figure 26.7–11. SST2 dataset fit to a mixed discrete and uniform distribution. Note the point mass function value at $x = 0$ equals the frequency of occurrence of the $x = 0$ data values in the sample of $n = 36$.

in that form or look at mixed distribution models. The edf has the undesirable property that it does not smooth the data at all and this may affect the user's purpose for fitting the data in the first place.

Mixed Distribution Models. There are some other possibilities. One could extract from the data all the $x = 0$ values and then fit the remainder of the data. This produces a mixed discrete plus continuous distribution. The construction of such a distribution is as follows: use the functional form $f(x) = p\delta(x)+(1-p)g(x)$ where $\delta(x)$ is the Dirac delta function. Using the SST2 data, one notes the occurrence of the $x = 0$ data is 16 occurrences in 30 tests. Using this, the user can construct the pdf $f(x)$ below which is shown in Figure 26.7–11.

$$f(x) = (8/15)\delta(x)+(7/15)*(25/3)U(0+, 0.12) = (8/15)\delta(x)+(7/9)U(0+, 0.12).$$

$U(0+, 0.12)$ is a uniform distribution that is nonzero over the region $0 < x \leq 0.12$.

This distribution is suitable as long as the sampling method one would use, if a Monte Carlo analysis were to be performed, could handle the point mass function at $x = 0$. Almost all Monte Carlo routines can handle this. This mixed model would be the pdf of preference to the authors, though the JSB selection may be sufficient for many practical tasks.

26.8 Example 3: The SST3 Dataset

Data SST3 is given in Table 26.8–1. From its dotplot (Figure 26.8–2), the data SST3 appears to be well behaved. Using the KS test to find an appropriate fit is sensible. In performing an analysis using EasyFit^{TM} (see Figure 26.8–3), one finds the best KS score belongs to a Burr distribution which is not well known.

Table 26.8–1. The SST3 dataset.

19.36	19.57	19.49	19.52	19.43	19.36	19.57	19.51
19.55	19.43	19.46	19.48	19.46	19.46	19.48	19.47
19.51	19.49	19.51	19.52	19.43	19.42	19.33	19.40
19.29	19.45	19.46	19.34	19.40	19.30		

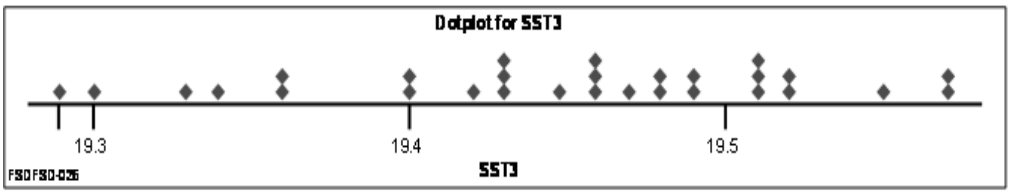


Figure 26.8–2. The SST3 dataset as a dotplot (Minitab^{TM}).

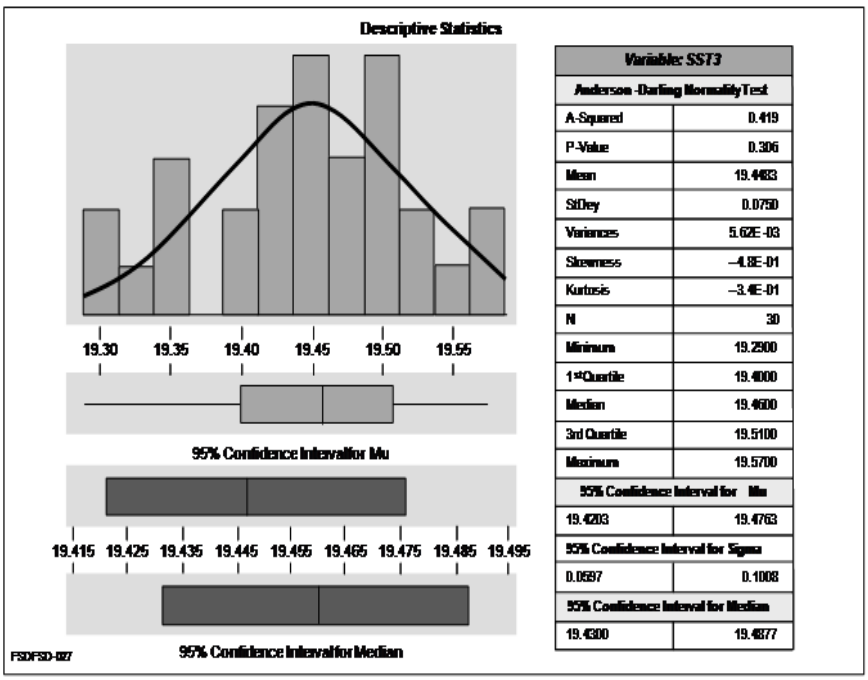


Figure 26.8–3. The SST3 data. Box and whisker plot shows reasonable dispersion about the median. A normal distribution might be an acceptable fit to this data.

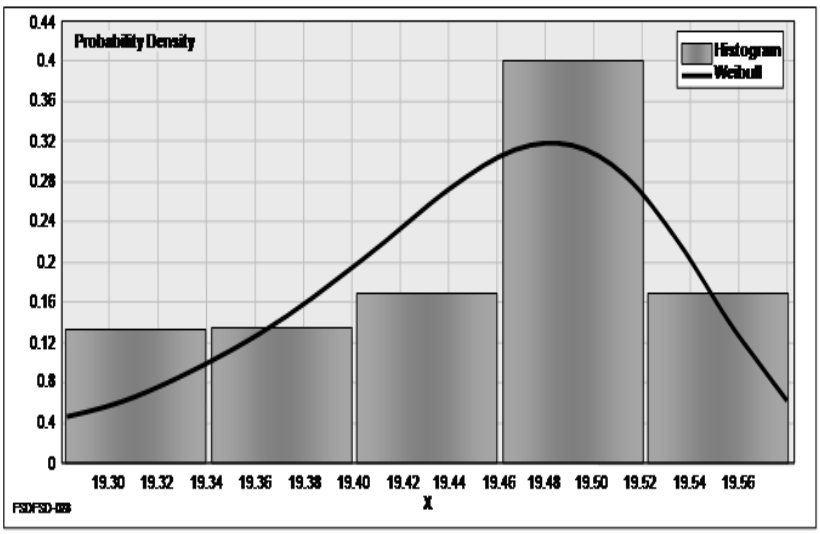


Figure 26.8–4. The SST3 data histogram versus the two-parameter Weibull, which is the best KS fit to data.

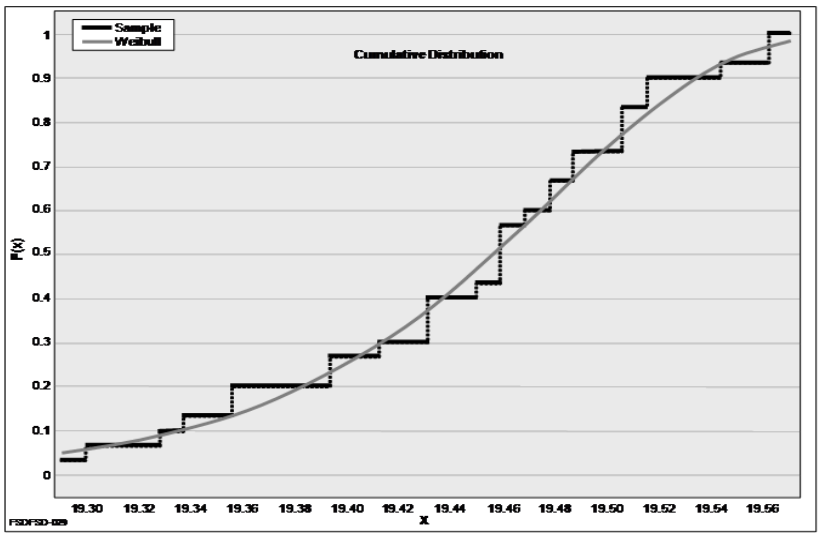


Figure 26.8–5. The SST3 data showing the Weibull edf versus cdf to a Weibull distribution.

An adequate fit using the KS score is the 2-parameter Weibull. To get a sense of the quality of the three-parameter Weibull fit, we plot the data histogram with the Weibull distribution that yields the best KS statisitc (see Figure 26.8–4). In the same vein, we also look at the edf and the cdf of the Weibull distribution (Figure 26.8–5).

26.9 Example 4: The SST4 Dataset

Two rather dichotomous sets of data appear in this dataset (given in Table 26.9–1). The dotplot of Figure 26.9–2 shows the nature of the problem.

One should first examine this dataset to understand the data points above $x = 0.3$. Make sure that these all had the same external factors influencing the measurements and that the measurement equipment was the same for all measurements, and set up by the same operators. If the data was measured using different equipment, and the measurement devices were not calibrated to one another, then the ensuing pdf will not be a good representation of the population from which the data was drawn. The histogram, box and whisker plot, and the point estimates for the data, are shown in Figure 26.9–3.

If one fits this data to a single distribution, one obtains using EasyFit^{TM} the Frechet three-parameter distribution. That distribution has the following form

$$\begin{aligned} f(x) &= \frac{\alpha}{\beta}\left(\frac{\beta}{x-\gamma}\right)^{\alpha+1} e^{-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}}, \\ F(x) &= e^{-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}}, \end{aligned}$$

where $\gamma < x < \infty$. This distributiuon appears to be an inverted Weibull distribution (but it is not). The resulting pdf is shown in Figure 26.9–4 and the edf versus cdf is shown in Figure 26.9–5. Trying to use JFIT© produces a JSB whose pdf is shown in Figure 26.9–6.

The JSB cuts off before the data does, so this cannot faithfully represent the range of data. This is a major drawback in using the moments matching method to find the parameter estimators. The JSB with these parameter values is an unacceptable pdf. How about using GAFit to try to optimize parameters to give the best KS score? How about using the GLD distribution? What else can one do? This will be explored later in this chapter when modeling a bimodal distribution is addressed explicitly.

Table 26.9–1. The SST4 dataset.

−0.16	−0.1	−0.07	−0.12	−0.1	−0.12	−0.08	0.7
−0.13	0.34	−0.1	−0.1	−0.1	−0.1	−0.12	0.01
0.56	0.6	0.55	0.46	−0.1	−0.11	−0.09	−0.08
−0.19	0.46	0.57	0.63	−0.03	−0.16		

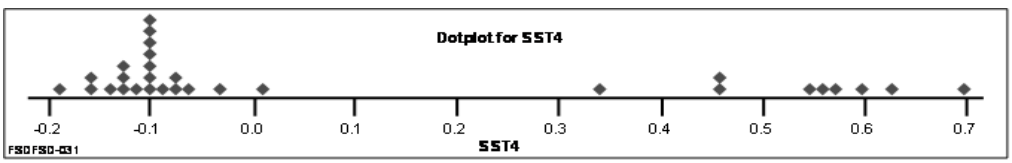
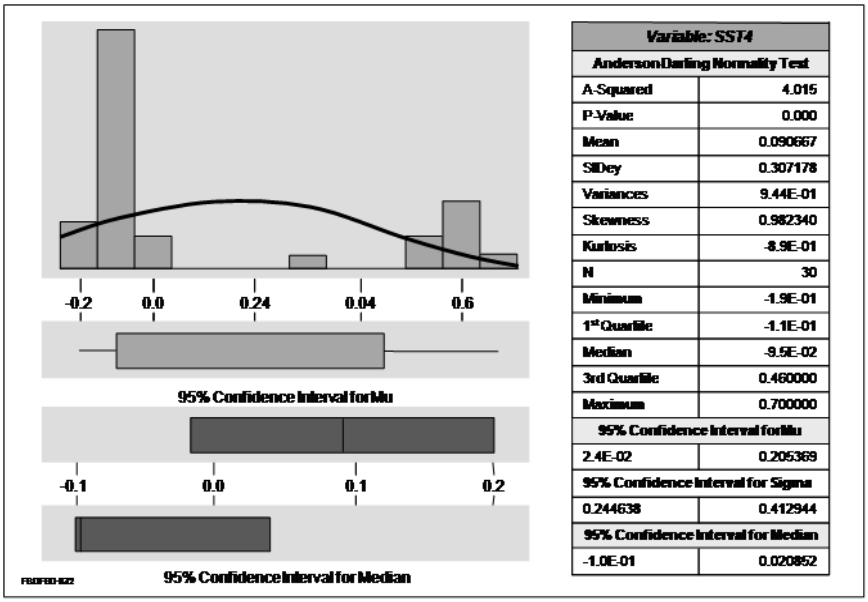


Figure 26.9–2. The SST4 dataset as a dotplot (Minitab^{TM}).



Variable: SST4	
Anderson Darling Normality Test	
A-Squared	4.015
P-Value	0.000
Mean	0.090667
StDev	0.307178
Variances	9.44E-01
Skewness	0.982340
Kurtosis	-8.9E-01
N	30
Minimum	-1.9E-01
1st Quartile	-1.1E-01
Median	-9.5E-02
3rd Quartile	0.460000
Maximum	0.700000
95% Confidence Interval for Mu	
2.4E-02	0.205369
95% Confidence Interval for Sigma	
0.244638	0.412944
95% Confidence Interval for Median	
-1.0E-01	0.020852

Figure 26.9–3. The histogram and box and whisker plots of the SST4 data show the extent of the dispersion of the data. The mean being so far from the median is characteristic of problems in the data (or measurement of the data).

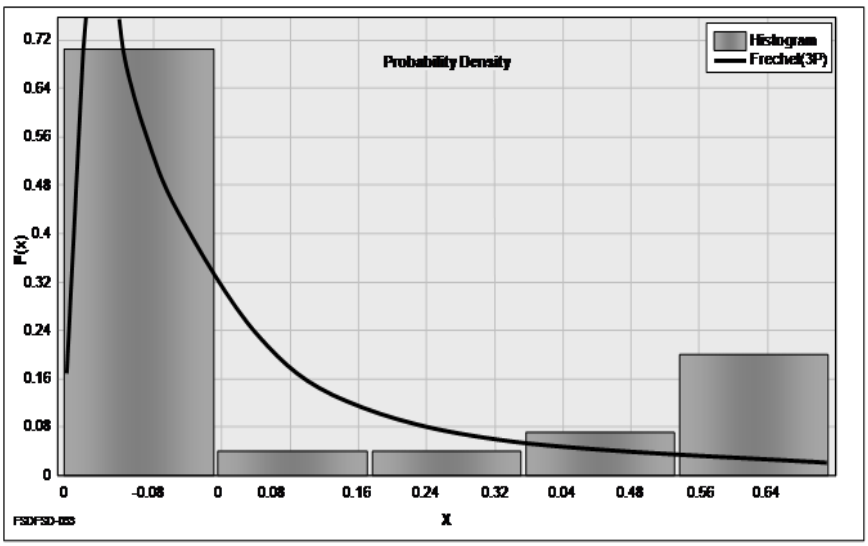


Figure 26.9–4. Frechet pdf fit to the SST4 dataset. This distribution has the best KS score and the best AD score. ($\alpha = 1.3857$, $\beta = 0.14047$, $\gamma = -0.23292$). It does not register the peaked data above $x = 0.3$ and for that reason may not be acceptable if that data is real.

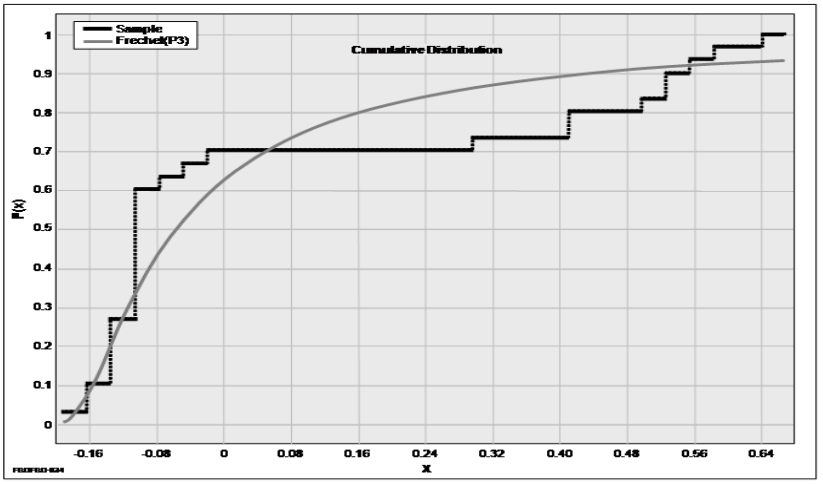


Figure 26.9–5. The SST4 data fit to the Frechet distribution showing how poor the fit becomes in the central region.

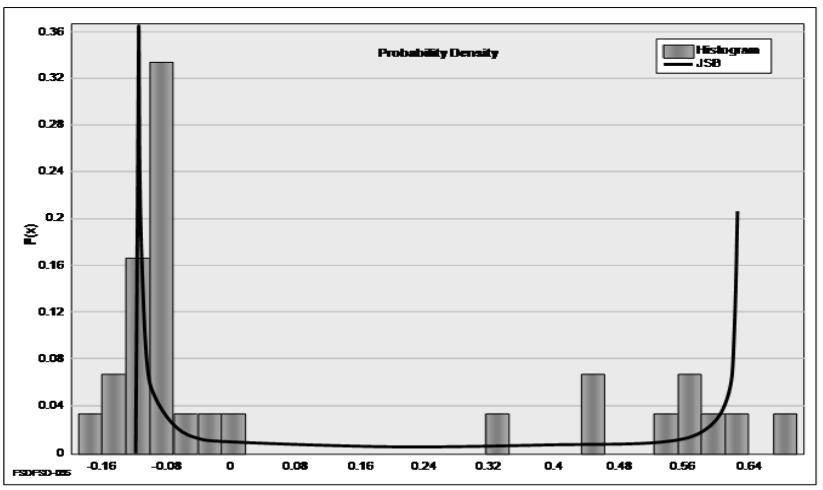


Figure 26.9–6. A JSB fit with $\gamma = 0.5929$, $\eta = 0.09739$, $\lambda = 0.74683$, $\varepsilon = -0.11817$.

26.10 Example 5: The SST5 Dataset

This dataset, whose listing is given in Table 26.10–1, needs further analysis of its integrity. There are two extreme outliers at 10.17 and these turn out to be bad data. This can be clearly seen in the dotplot (Figure 26.10–2) and in the histogram (Figure 26.10–3).

If one blindly went ahead and tried fitting a pdf to the full SST5 dataset, the best fit using the KS test is the Wakeby Distribution. The resulting pdf is shown

Table 26.10–1. The SST5 dataset.

9.8	9.83	10.17	9.82	9.75	9.81	9.78	9.8
9.82	9.75	9.81	9.79	9.75	9.75	9.84	9.75
9.77	9.74	9.73	9.82	9.74	9.78	9.76	9.8
9.8	9.83	10.17	9.76	9.78			

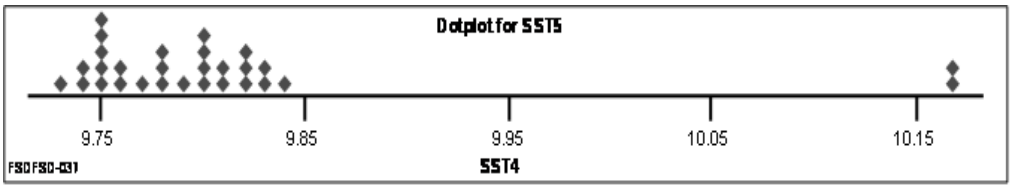


Figure 26.10–2. The SST5 dataset as a dotplot (Minitab^{TM}).

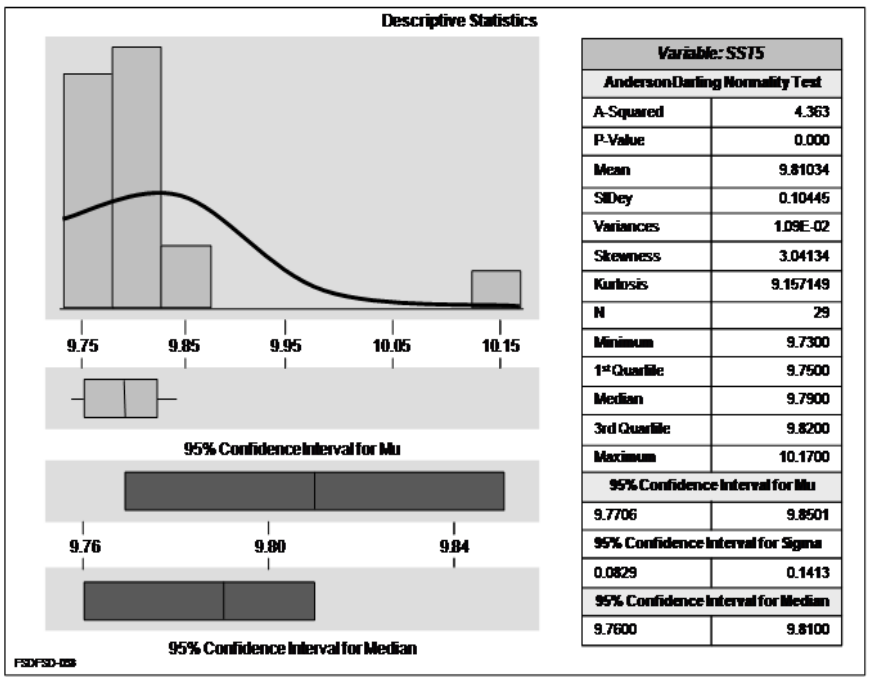


Figure 26.10–3. The histogram and box and whisker plots of SST5. Note the variance and skewness and how the 95 percent confidence intervals are affected by the outliers at $x = 10.17$.

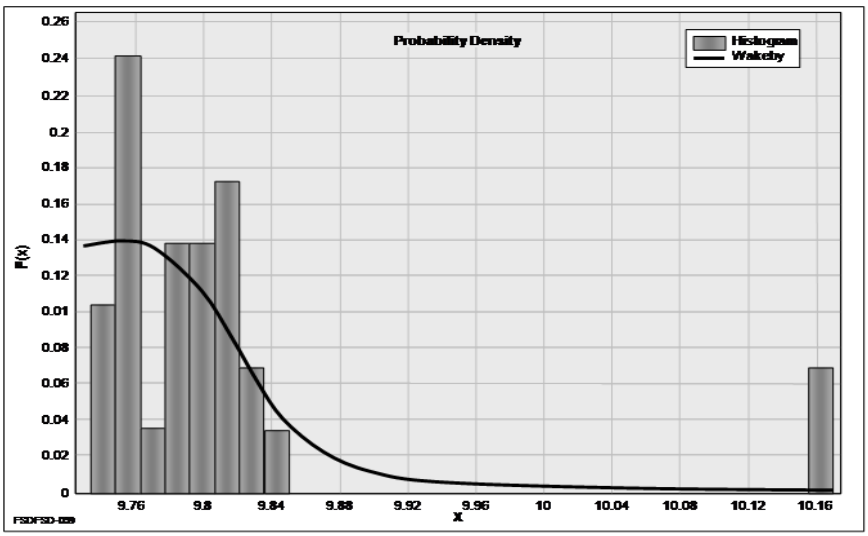


Figure 26.10–4. The Wakeby distribution fit to SST5 data that includes the two outliers.

in Figure 26.10–4. The functional form of a Wakeby distribution is shown below. It can only be represented in quantile form.

$$x = \zeta + \frac{\alpha}{\beta}\left(1 - (1-F)^{\beta}\right) - \frac{\gamma}{\delta}\left(1 - (1-F)^{-\delta}\right).$$

Differentiating once would give the pdf or $f(x) = dF/dx$.

If one were to exclude the two outliers as bad data, then fit the remaining 27 data points, the best fitting distribution using the KS test is the JSB, whose pdf is shown in Figure 26.10–5 and whose cdf is shown in Figure 26.10–6.

It would not, however, be a stretch to model this data with a uniform distribution or, if one did not want a bounded distribution, a good fit to the data (without outliers) would be a log-logistic distribution, with the follwoing pdf and cdf, respectively.

$$\begin{aligned} f(x) &= \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-2}, \\ F(x) &= \left(1+\left(\frac{\beta}{x}\right)^{\alpha}\right)^{-1}. \end{aligned}$$

The log-logistic fit to SST5 is shown in Figure 26.10–7.

The user of the data must determine if a bounded distribution is suitable for the data's intended purpose. Bounded distributions may be the best fits to data but they may not be the correct fit to the data as they will not allow for data outside the computed boundaries. The user must be encouraged to

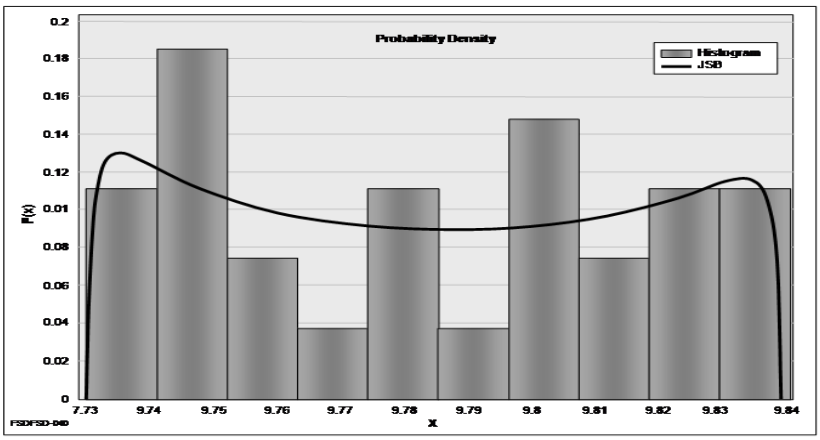


Figure 26.10–5. The SST5 data without outliers, JSB is the best KS fit.

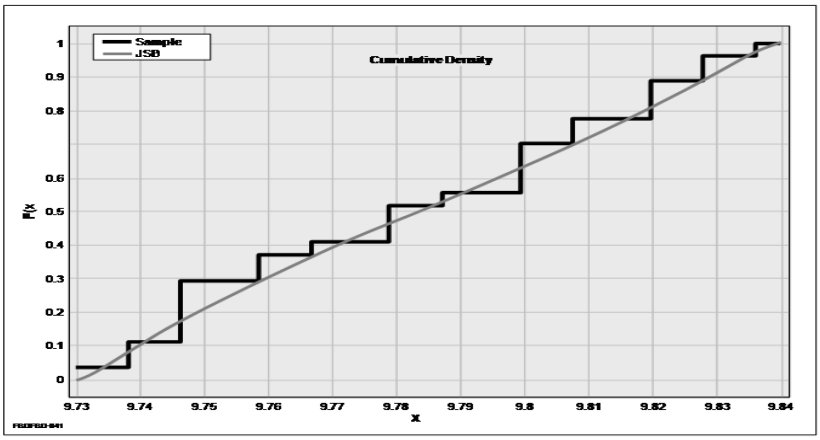


Figure 26.10–6. The edf of SST5, without outliers, with the fitted JSB cdf.

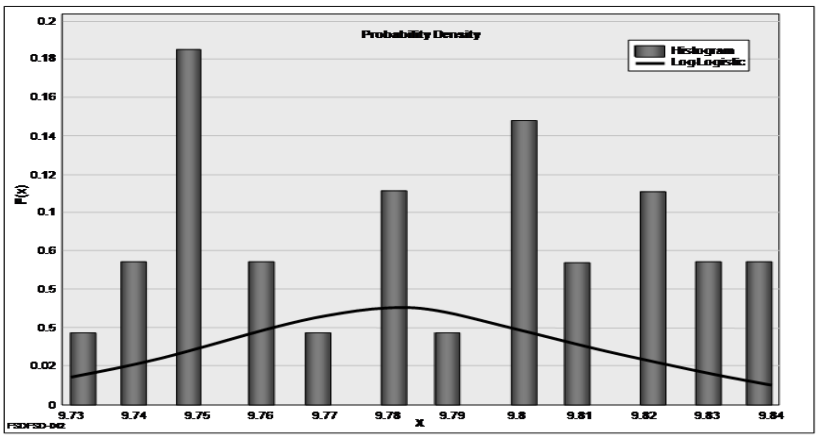


Figure 26.10–7. Log-logistic fit to the SST5 data (without outliers).

share the theoretical underpinnings of the population from which the data is drawn. Statistics is not a fishing operation. Fitting data to pdfs is done to add clarity, not to disguise outcomes. Therefore, the statistician needs to be wary of imposing bounded distributions onto data with unbounded population values or less bounded population values than the fitted pdf would imply.

This example shows the nature of a problem that has potentially bad data and what the analyst might do when the bad data is removed. The fitting problems do not automatically go away.

26.11 Example 6: The SST47 Dataset

The SST47 data, listed in Table 26.11–1, shows bimodality, as can be seen in the dataplot of Figure 26.11–2. Similarly, the histogram and box and whiskers plot show the same situation (see Figure 26.11–3).

Fitting this data produces some possible choices. EasyFit^{TM}, using the KS test as a criterion, ranks the top five of six pdf fits as having three parameters (as opposed to two parameters), which means the more parameters the better. The JFIT$^{©}$ tool produces a moments matching fit to a JSB with parameter estimators given by ($\gamma = 0.6026$, $\eta = 0.0796$, $\lambda = 0.7658$, $\varepsilon = 0.1383$) (see Figure 26.11–4).

The EasyFit^{TM} tool produces the best KS fit as a log-logistics distribution as seen in Figure 26.11–5.

The pdf and cdf of the three-parameter log-logistic distribution are, respectively,

Table 26.11–1. The SST47 dataset.

−0.15	−0.11	−0.11	−0.15	−0.14	−0.12	−0.1	−0.69
−0.14	0.3	−0.12	−0.12	−0.12	−0.15	−0.17	−0.02
0.6	0.62	0.55	0.45	−0.14	−0.13	−0.12	−0.12
−0.2	0.46	0.57	0.62	−0.08	−0.17		

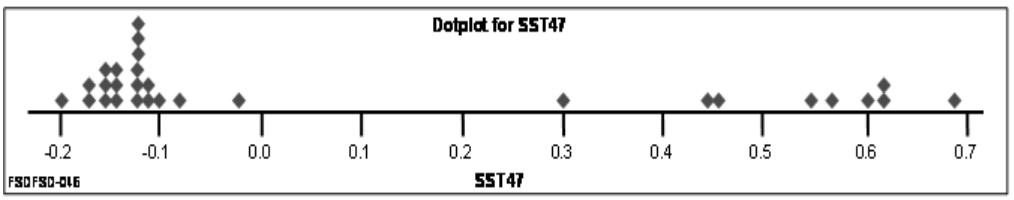
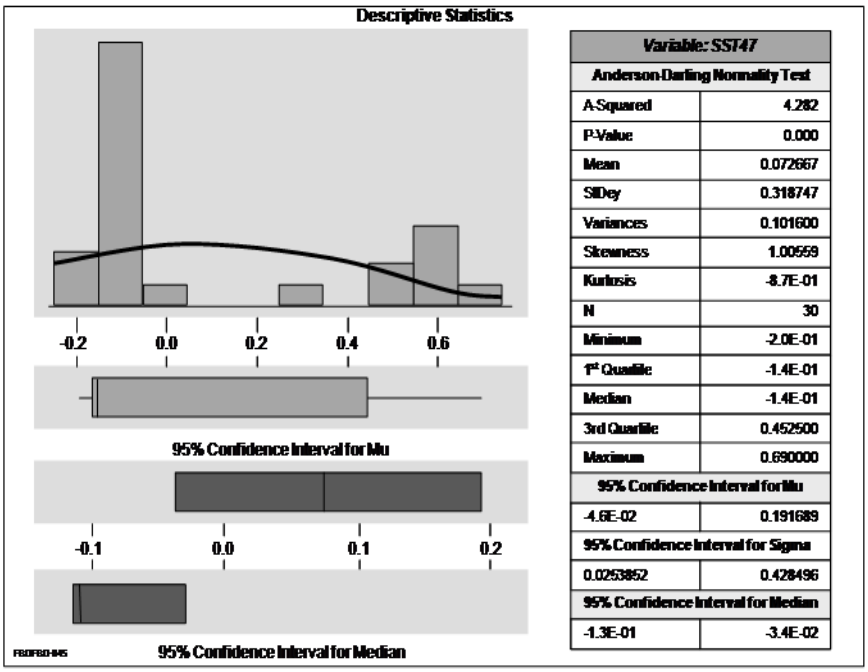


Figure 26.11–2. The SST47 dataset as a dotplot (Minitab^{TM}).



Variable: SST47	
Anderson-Darling Normality Test	
A-Squared	4.282
P-Value	0.000
Mean	0.072667
StDev	0.318747
Variances	0.101600
Skewness	1.00559
Kurtosis	-8.7E-01
N	30
Minimum	-2.0E-01
1st Quartile	-1.4E-01
Median	-1.4E-01
3rd Quartile	0.452500
Maximum	0.690000
95% Confidence Interval for Mu	
-4.6E-02	0.191689
95% Confidence Interval for Sigma	
0.0253852	0.428496
95% Confidence Interval for Median	
-1.3E-01	-3.4E-02

Figure 26.11–3. The histogram and box and whisker plots of SST47. Notice the large confidence interval for the mean and the smaller confidence interval for the median.

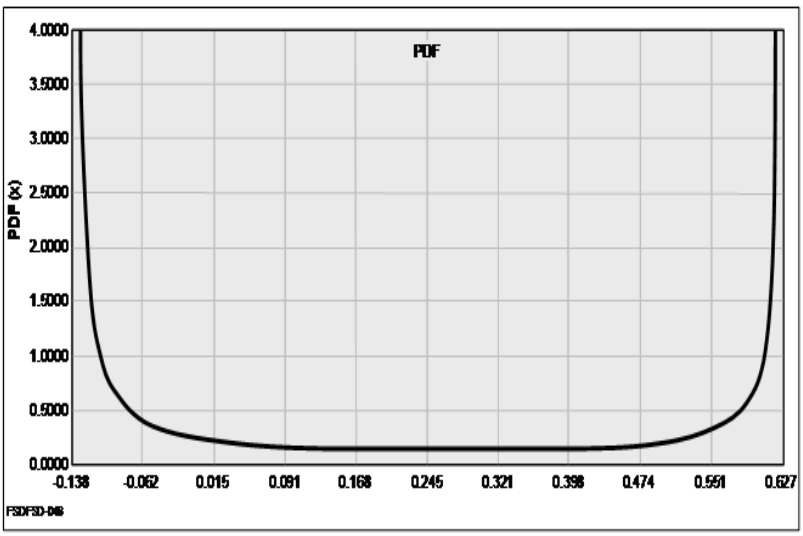


Figure 26.11–4. The SST47 data fitted by JFIT©. This is a JSB distribution. Note a fit by a bounded distribution may not be acceptable to the user.

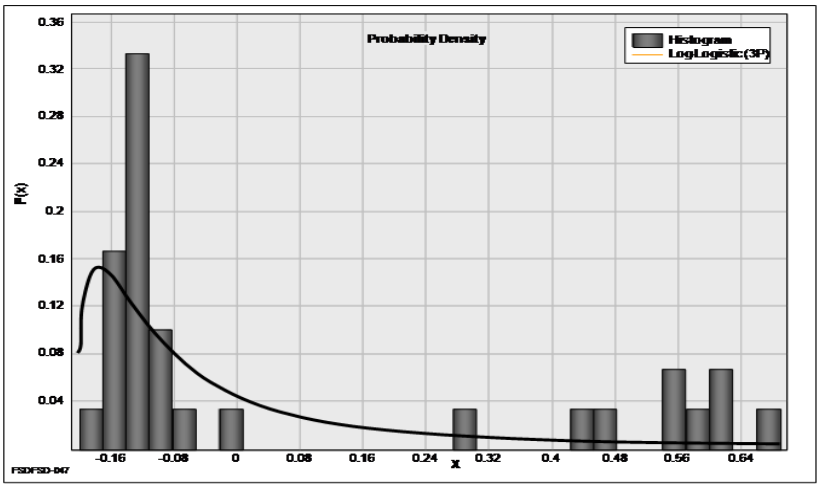


Figure 26.11–5. The SST47 data fitted with a three-parameter log-logistic (unbounded) distribution. The parameters for this fit are: $\alpha = 1.3258$, $\beta = 0.12302$, $\gamma = -0.20192$

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-2}$$
$$F(x) = \left(1+\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right)^{-1}.$$

The range of x values is cut off on the low side by the parameter λ, i.e., $\lambda < x < \infty$. The problem again is that the fit does not seem to account well for the second bunching of data near $x = 0.56$. This is a recurring theme in small sample sizes and one needs to examine the theoretical underpinnings of the population from which the data was drawn to answer questions about GOF. We will look in a later section at using a bimodal model to fit such datasets.

26.12 Example 7: The SST29 Dataset

This example, dataset SST29 that is given in Table 26.12–1, will use a JSU distribution to fit the data using JFIT©. Figure 26.12–2 gives a histogram, with a superimposed normal distribution, and other basic statistics related to this dataset.

Analyzing this data produces fits that are better than a normal distribution. The best KS score is for a Cauchy distribution, which also gives the second best AD score. This data also produces a good fit using a JSU distribution. These are both shown in Figure 26.12–3.

The Cauchy distribution has the form

Table 26.12–1. The SST29 dataset.

4.184	4.379	4.168	4.182	4.096	4.337	4.148	4.025
4.171	4.025	4.244	4.181	4.152	4.187	4.268	4.186
4.315	4.267	4.316	4.165	4.177	4.137	4.196	4.19
4.141	4.073	4.109	4.068	4.13			

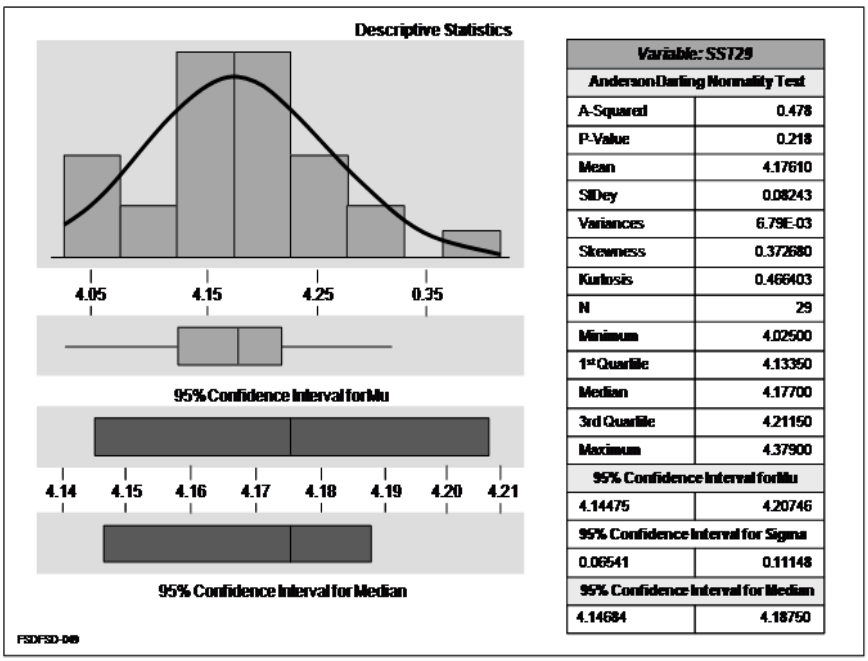


Figure 26.12–2. The histogram and box and whisker plots of SST29, which shows normal-like behavior.

$$f(x) = \left(\pi\sigma\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)\right)^{-1}$$
$$F(x) = \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\sigma}\right)+0.5,$$

for $-\infty < x < \infty$.

While the Cauchy certainly appears more accurate to the eye, the JSU cannot be rejected even at a confidence level of 80%. The JSU (ranks 12th on KS score and 7th on the AD score) seems to be adequate unless the central peak is important to the user or if there is a theoretical reason for choosing the Cauchy distribution.

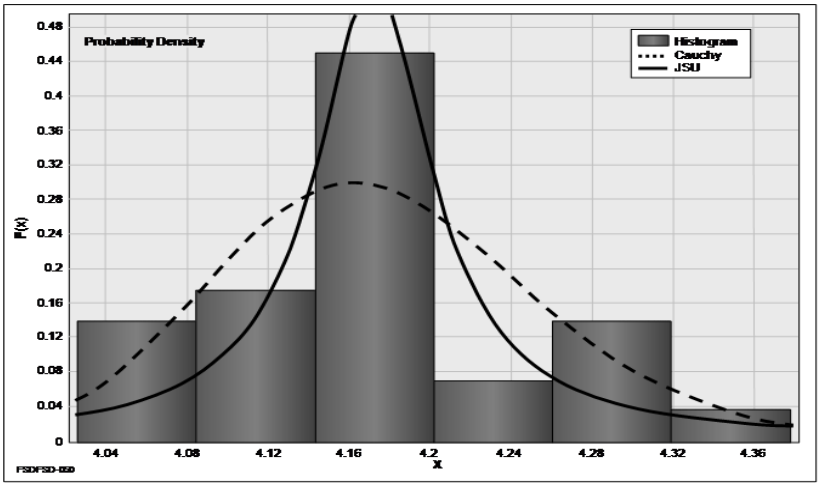


Figure 26.12–3. The SST29 data fitted with a Cauchy distribution with parameters $\sigma = 0.03575$ and $\mu = 4.1722$. The flatter curve is the JSU distribution with parameters $\gamma = -2.0829$, $\eta = 3.9751$, $\lambda = 0.27726$, $\varepsilon = 4.0192$.

26.13 Example 8: Solar Flux (Top 95% Amplitude) at 10.7 MHz

This dataset is for solar cycles 23, 24, and the start of 25. Taking monthly averages produces 189 data values with a pdf best fit by a JSB with parameters ($\gamma = 0.33748$, $\eta = 0.48935$, $\lambda = 173.9$, $\varepsilon = 71.399$) and is shown below in Figure 26.13–1.

If one now draws a random sample of size 30 from this distribution and fits that sample data to a pdf one also gets a JSB with a pdf that appears in Figure 26.13–2.

It can clearly be seen that the fit seems to gloss over a peak near the center of the data. The parameters are found to be $\gamma = 0.14075$, $\eta = 0.29345$, $\lambda = 0.145.37$, $\varepsilon = 77.099$. These certainly do not appear close to the population parameters, but it is hard to tell without knowing how the estimators are themselves distributed. Note the spike in the center. It is actually a fluctuation that can easily occur, so one must not give too much attention to it. Another sample of 30 would smooth out this bump. A t-test run on the mean values from the population and the $n = 30$ sample indicated the null hypothesis (population mean = sample mean) could not be rejected. An F test on the variances also showed no rejection. The skewness of the sample was 0.2 and for the population it was 0.4, which was twice as great. So, the sample does not fairly represent the skewness of the population, but this is common with small sample sizes. Nevertheless, the sample does not look like the population and if there were no theoretical basis for the data, one might suspect multimodal data.

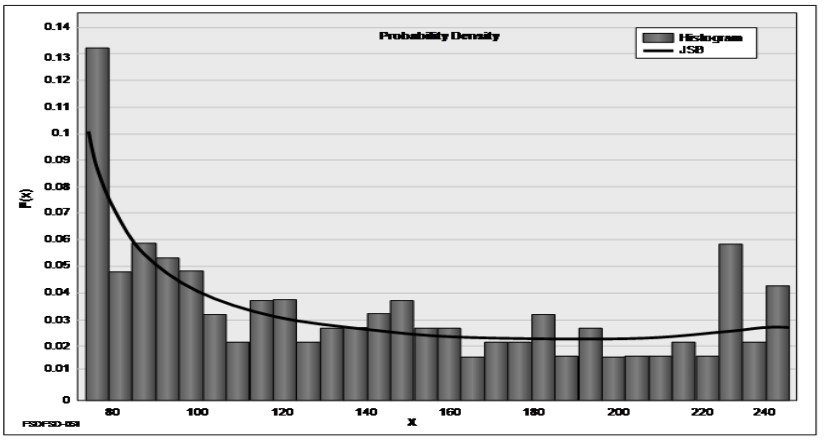


Figure 26.13–1. Solar flux measurements. The JSB was the best KS, AD, and chi-squared fit to the data.

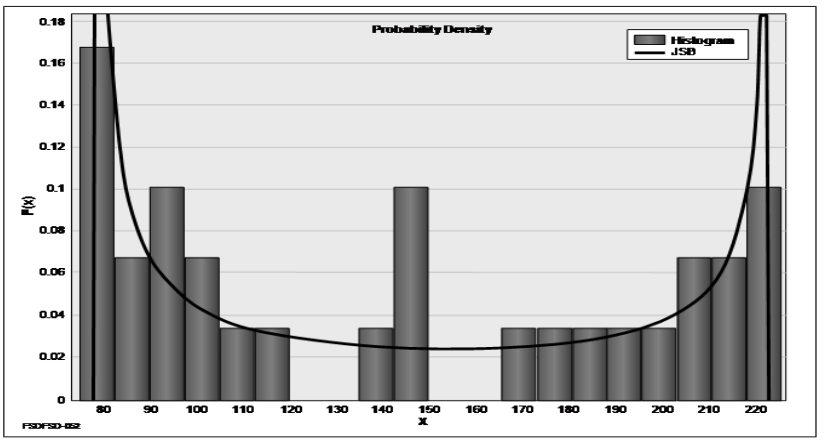


Figure 26.13–2. Solar flux data when a sample of 30 values was chosen randomly from 189 data values.

26.14 A Bimodal Fit to Data

Examples 2 and 4 in this chapter elicit a temptation to model their pdf using some form of bimodal distribution. One such distribution is given by the following functional form called the **AMRbeta formulation**.

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1} + \varepsilon x^{\gamma-1}(1-x)^{\delta-1}}{B(\alpha,\, \beta) + \varepsilon B(\gamma,\, \delta)}$$

One typically finds that at least five parameters are needed to faithfully reproduce a bimodal distribution. There are distributions such as the JSB and the beta distribution that try to fit bimodal data with four parameters, but none can fit it as well as the above form. This form uses two beta distributions weighted

Table 26.14–1. Bimodal dataset.

0.000	0.146	0.375	0.896	0.008	0.146	0.583	0.950
0.063	0.271	0.771	0.979	0.065	0.333	0.854	1.000

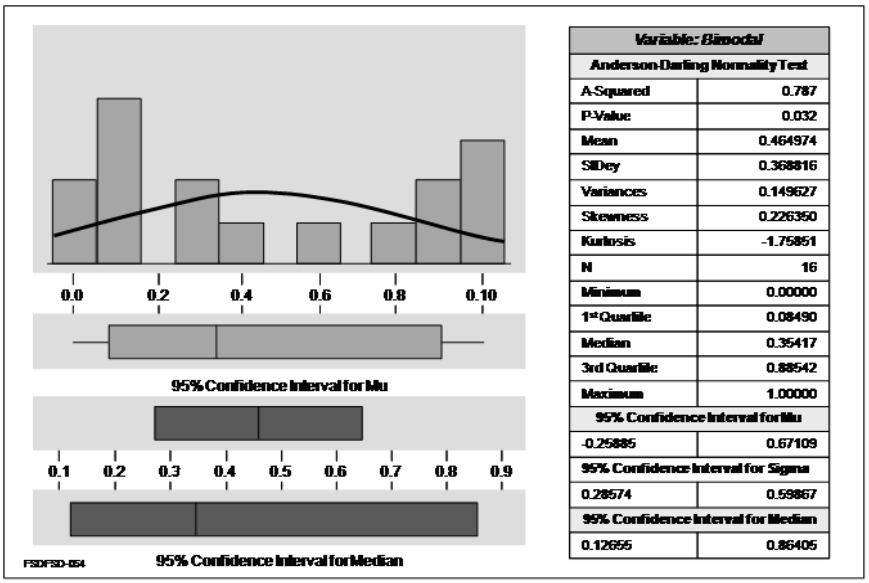


Figure 26.14–2. The histogram and box and whisker plots of the bimodal data.

and added together to produce a bounded distribution. One could form the same type of bimodal distribution using two normal or two gamma distributions, or mixtures of both. Rather than enter into a long philosophical discussion, the point can best be illustrated by an example. Consider the dataset given in Table 26.14–1 that has been normalized to the interval of interest and ordered from smallest to largest values of the variable x, where $0 \leq x \leq 1$.

This set of data can be used to fit the bimodal distribution. The data is shown in Figure 26.14–2 using the RAPID tool and with the KS score as the goal to minimize. The result is shown in Figure 26.14–3.

The parameter values that provide a good fit have a broad range of acceptable values. This is not uncommon for small datasets being fitted by a large number of parameters. Using this data, and the fitted bimodal distribution in the histogram in Figure 26.14–4 where the black squares represent the value of the probability found by integrating the fitted distribution over the data bin size, it is shown that this is a good fit.

Parameter search routines are widely used. The way the search is used is varied but the tool used to perform the search here was the Raytheon developed tool called **RAPID**. As stated earlier in this chapter, the tool uses a genetic

algorithm combined with a gradient search to find the best fit to any given set of goals under any given set of constraints. Fitting data with multiple modes has to be done carefully. As one continues to add parameters to the form of the fitting distribution, one can always get better and better fits using the same criterion for success, e.g., minimum KS score. One needs to be convinced that it is important to fit both modes and they are real, not just a manifestation of data sampling fluctuation. If the analyst is convinced this is the case then the above form can be effectively used and it produces useful results. Certainly other forms are possible.

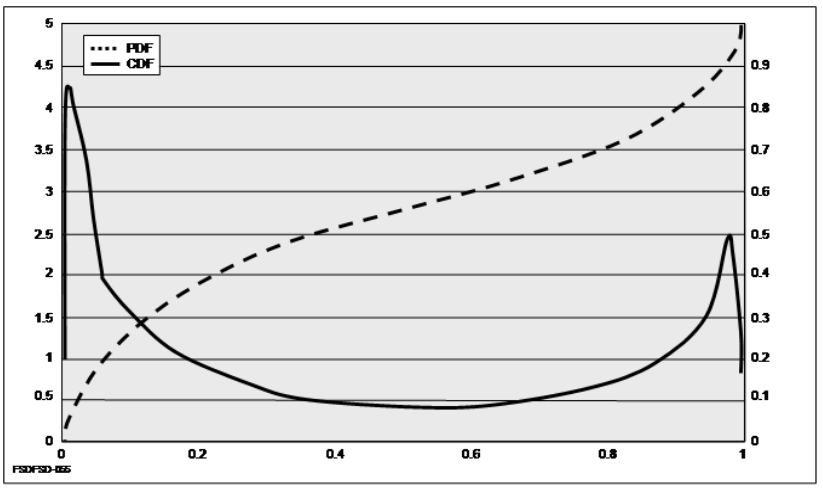


Figure 26.14–3. Sixteen data points were used to perform the KS tests for each possible value of the five parameters. The values that achieved the lowest KS score were $\alpha = 0.6868$, $\beta = 3.6225$, $\gamma = 2.8320$, $\delta = 0.6330$, $\varepsilon = 0.6158$.

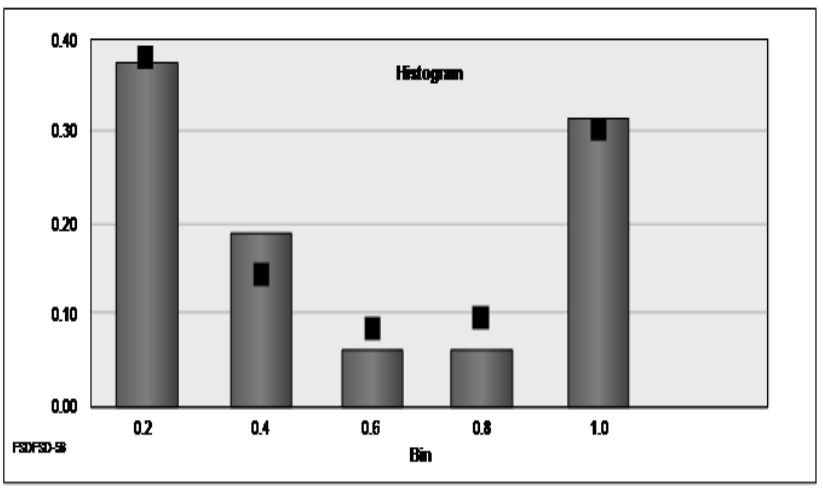


Figure 26.14–4. KS score was 0.0683 for the parameters values quoted above.

26.15 Comparisons

How do the various tools compare to a standard measure of GOF? Using the KS score as the comparison, Table 26.15–1 shows how the fits compare. Solutions in the JFIT© column of Table 26.15–1 marked by an asterisk should probably have been labeled as "no fits" as is done in the EasyFit™ application. For each case, the coefficients assigned by the Raytheon use of the Hill, Hill, and Holder algorithms produce mappings in regions where the Johnson transformation equations are not defined (mappings that would require the logarithm of a negative number, or would produce a divide by zero error for example). So, for the six test datasets selected, the JFIT solution is only valid for one (SST3), but as shown in the last column of the table, there are legitimate Johnson fits for each of the six sample cases.

Table 26.15–1. Comparison of pdf types and KS values (in parentheses) reported by Crystal Ball©, EasyFit™, JFIT©, and GAFit for datasets SST1 through 5, and for dataset SST47.

	Crystal Ball	EasyFit™	JFIT©	GAFit
SST1	Log Normal (.115)	JSB (0.89)	JSB* (0.1234)	JSB (0.103)
SST2	Logistic (.295)	Uniform (.265)	JSB* (0.53)	JSB (0.27)
SST3	Extreme Value (.070)	Weibull (.060)	JSB (0.086)	JSB (0.079)
SST4	Gamma (.239)	Frechet (.190)	JSB* (.33)	JSB (0.19)
SST5	Extreme Value (.168)	Wakeby (.099)	JSB* (0.34)	JSB (0.20)
SST47	Gamma (.255)	Log-Logistic (.195)	JSB* (0.3)	JSB (0.24)

Table 26.15–2 gives the parameter values of the JFIT©/EasyFit™ JSB fits that were obtained for the SST1 dataset and Table 26.13–3 shows the four coefficient values for fitting the generalized lambda distribution to each of the six selected datasets. The coefficients for each dataset were generated by using the Raytheon genetic algorithm optimization tool RAPID and were selected to produce the least error in replicating the data from the observed cumulative probabilities. There is inherent uncertainty in this since, with n data points, the observed cumulative probabilities cannot be determined closer than $1/n$. No consideration of this fact was given in the RAPID application of the lambda distribution fitting. Instead, using $(j-1)/n$ as the cumulative probability for the j-th ordered data point, the fitted variable value is compared to the observed value and the sum of squares of these differences was used as the GOF indicator.

Before closing out this chapter, there is one more topic of importance that needs to be addressed: fitting routines to perform model selection.

Table 26.15–2. Comparing JSB fit coefficients from JFIT/EasyFit™ that use moments matching to find parameter estimators with the same parameters chosen using GAFit that seeks to find the lowest KS score.

	JFIT©/EasyFit™ JSB Coefficient Values for SST1	GAFit JSB Coefficient Values for SST1
Gamma	1.0171	1.408
Eta	0.5469	0.8449
Lambda	0.4450	0.5550
Epsilon	0.0392	0.0226
KS Value	0.500	0.103

Table 26.15–3. Generalized lambda distribution coefficients that minimize KS scores for datasets 1 through 5 and 47.

	λ_1	λ_2	λ_3	λ_4
SST1	.2225	.1282	.2333	.5621
SST2	.0550	18.1818	5.1335	.3825
SST3	19.430	7.1429	2.0100	1.9248
SST4	−.1234	.1052	1.8477	1.2495
SST5	9.7834	2.5836	2.2626	19.5172
SST47	−.1990	.9919	3.2247	47.6130

26.16 Model Selection

A good account of this problem is presented in Chapter 4 of Sivia (1996) and will be paralleled here. Up to this point, we have been concerned with parameter estimation based on given data. A challenging problem is how one might decide between two or more models of a process that are represented by the data. This could be as simple as deciding if a Gaussian or a Lorentzian distribution is the better choice. There could be important theoretical implications depending on which model was chosen. This is not an easy problem, but here is an easy statement of the problem that dates back to the work of Jeffreys (1939) and the presentation by Gull (1988).

This example is called the "Story of Mr. A and Mr. B." Mr. A has a theory. Mr. B also has a theory, but with an adjustable parameter labeled λ. Whose theory should one prefer based on a given data D? Suppose there is noise in the data. Mr. A contends that this noise, in the measurement of y against x, is

described by $y = 0$. Mr. B believes that the data is best understood for some $y = \alpha$ where $\alpha \neq 0$, but he is not sure of the value of α. There could also be a Mr. C who believes the data is best understood by a model $y = ax + b$, but for ease of understanding let's just look at Mr. A and Mr. B.

This problem is one in which Bayesian probability techniques are applied. Sivia's book is on that subject. To try to solve this problem, one might compare the so-called posterior ratio = $\text{prob}(A|D, I)/\text{prob}(B|D, I)$ where D represents the data taken in the experiment and I is other information that may effect the results (sometimes I is called **covariate data**). If this ratio is much greater than one then we prefer A's theory. If the ratio is much less than one, we prefer B's theory. If the ratio is close to one then we cannot decide using the information at hand. To estimate the odds for this ratio one computes

$$\frac{\text{prob}(A|D, I)}{\text{prob}(B|D, I)} = \frac{\text{prob}(D|A, I)}{\text{prob}(D|B, I)} \times \frac{\text{prob}(A|I)}{\text{prob}(B|I)}.$$

This is Bayes' Theorem applied to both the numerator and denominator of the ratio. This formulation already says the ratio may depend on our previously held beliefs (prior distribution) as to the outcome. To be fair, however, one might take the ratio on the far right of the above equation to be unity just to keep from biasing the outcome. It could also be based on outcomes from previous experiments.

To assign values to the probabilities that come from the experimental data, i.e., $\text{prob}(D|A, I)$ and $\text{prob}(D|B, I)$, we need to compare the data with predictions of A and B. The larger the mismatch between theory and data, the lower the corresponding probability. The calculation is easy for Mr. A. It is not quite so easy for Mr. B. because he has a parameter λ without a value. In Bayesian statistics, one applies the sum and product rules of probability and arrives at the following expression for Mr. B.

$$\text{prob}(D|B, I) = \int \text{prob}(D, \lambda|B, I)\, d\lambda = \int \text{prob}(D|\lambda, B, I) \times \text{prob}(\lambda|B, I)\, d\lambda.$$

It is seen that the first term in the integrand is the likelihood function and the second term is the prior distribution for λ given theory B and other information I. The onus is now on the theorist to articulate his knowledge, or ignorance, i.e., state the prior, before he has access to the data.

To proceed further analytically requires some simplifying assumptions and Sivia proceeds to do so at some length. The point to be made here is that one can substitute any fitted or assumed pdf for $\text{prob}(\lambda|B, I)$, the prior, and then multiply this prior by any likelihood function. The integration usually needs to be performed numerically. With this information one may indeed be able to determine the correct model by calculating $\text{prob}(D|B, I)$ and comparing it to the prediction from Mr. A., $\text{prob}(D|A, I)$. One must be careful in reversing the process whereby a given theory is better supported by the data and infers a particular prior distribution.

26.17 Summary

Several techniques for selecting a suitable pdf because of a small sample of data have been shown by example and discussed. Appropriate admonitions have been made with regard to the use of real and accurate data. Two commercial software tools have been used in the pdf fitting process and compared to several tools developed by the authors. The EasyFit^{TM} tool seems to be very good and is (as of September 2009) reasonably priced. The cost for the Standard Edition, for a single user is $349, $499 for the Professional Edition and the academic license price is $89, if purchased through http://www.mathware.com.

Comparisons of Johnson, generalized lambda, and several other distributions have been performed and shown to have quantitatively different forms particularly when using the moments-matching method versus minimizing the KS score. Benchmark values for Johnson and GLD distributions were provided to allow novices to develop their own fitting algorithms.

References for Chapter 26

Alderman, J.L. and Mense, A.T. (2007). "Applications of Johnson PDF Fitting," Proc. of the Symposium on Fitting Distributions to Data, Auburn, AL, *Amer. J. Mathematical & Management Sciences,* Vol. 27, No. 3&4, 461–478.

Alderman, J., Mense, A.T., and Runger, G. (2010). *Statistical Design Methodology*, submitted for publication.

D'Agostino, R.B. and Stephens, M.A. (1984). *Goodness of Fit Techniques,* John Wiley & Sons, New York.

Dudewicz, E. J. and Mishra, S. N. (1988). *Modern Mathematical Statistics*, Wiley Series in Probability and Statistics, John Wiley & Sons, New York.

Dudewicz, E. J., Zhang, C. X., and Karian, Z. A. (2005). "The Completeness and Uniqueness of Johnson's System in Skewness-Kurtosis Space," *Communications in Statistics*, Vol. 33, No. 9, 2097–2116.

Freimer, M., Mudholkar, G. S., Kollia, G., and Lin, C. T. (1988). "A study of the generalized Tukey lambda family," *Communications in Statistics – Theory and Methods*, 17, 3547–3567.

Gilchrist, W.G. (2000). *Statistical Modeling with Quantile Functions*, Chapman & Hall/CRC.

Gull, S.F. (1988). "Bayesian inductive inference and maximum entropy," in *Maximum Entropy and Bayesian Methods in Science and Engineering,* Vol. 1, (ed. G.J. Erickson and C.R. Smith), Kluwer Press, Dordrecht.

Hahn, G.J. and Shapiro, S.S. (1967). *Statistical Models in Engineering*, John

Wiley & Sons, New York.

Hill, I. D., Hill, R., and Holder, R. L. (1976). "Algorithm AS99, fitting Johnson curves by moments," *Journal of the Royal Statistical Society, Series C, Applied Statistics*, 25, 180–189.

Jeffreys, H. (1939). *Theory of Probability*, Clarendon Press, London.

Johnson, N.L. (1949). "Systems of frequency curves generated by methods of translation," *Biometrica,* 36, 149–176.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* CRC Press, Boca Raton, Florida.

Karian, Z. A. and Dudewicz, E. J. (2007). "Computational Issues in Fitting Statistical Distributions to Data," Proc. of the Symposium on Fitting Statistical Distributions to Data at Auburn University, Auburn, Alabama, *American Journal Mathematical and Management Sciences,* Vol. 27, 319–349.

King, R.A.R., MacGillivray, H.L. (2007). "Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals," Proc. of the Symposium on Fitting Statistical Distributions to Data at Auburn University, Auburn, Alabama, *American Journal Mathematical and Management Sciences,* Vol. 27, 441–460.

Mense, A.T. and Alderman, J. (2008). "The RuM-test: A useful goodness-of-fit test," RMS internal report, to be submitted for publication.

Ramberg, J. S. and Schmeiser, B. W. (1974). "An approximate method for generating asymmetric random variables," *Communications of the Association for Computing Machinery* 17, 78–82.

Raytheon tool, JFIT.xls^{TM}. Software based on FORTRAN code from Johnson (1949) and Dudewicz Zhang, and Karian (2005). This code can be found on the Statlib Web Site hosted by the faculty at the Department of Statistics at Carnegie Mellon University. The Statlib Web address is `http://lib.statlcmu.edu/`. The direct address for the Hill, Hill, and Holder paper is `http://lib.stat.cmu.edu/apstat/99`.

Ross, S.M. (1997). *Simulation*, 2nd edition, Academic Press, New York.

Sivia, D.S. (1996). *Data Analysis: A Bayesian Tutorial*, Oxford Science Publications, Oxford University Press.

Wilkinson, L. (1999). *The Grammar of Graphics,* Springer, New York.

Appendix for Chapter 26

A.1 Goodness-of-Fit Scores (D'Agostino and Stephens (1984))

The measure used for goodness of fit **for both the Anderson-Darling and Cramer-von Mises (CVM) tests is** given by

$$Q = n \int (F_n(x) - F(x))^2 \Psi(x)\, dF(x)$$

and can be seen as a functional measure of distance between the empirical distribution $F_n(x)$ and the trial distribution $F(x)$, weighted by some as-yet-arbitrary function $\Psi(x)$. **For the AD statistic** $\Psi(x) = [F(x)(1 - F(x))]^{-1}$**. The CVM statistic uses** $\Psi(x) = 1$. Thus, for the AD we have

$$Q = n \int (F_n(x) - F(x))^2 \Psi(x)\, dF(x) = n \int \frac{(F_n(x) - F(x))^2}{F(x)(1 - F(x))}\, dF(x).$$

and for the CVM, we have

$$Q = n \int (F_n(x) - F(x))^2 \Psi(x)\, dF(x) = n \int \frac{(F_n(x) - F(x))^2}{1}\, dF(x).$$

The empirical distribution is a step-wise function increasing by steps (at the sample data values) from zero to one at $F_n(x_n)$, e.g., $F_n(x_1) = 1/n$, $F_n(x_j) = j/n$. $F(x)$ is the cdf of the test distribution and is presumed to be continuous for the rest of the derivation. Due to the finite steps in $F_n(x)$, the integral can be broken into separate intervals.

Derivation of the AD Criterion.

Let

$$Q = Q_0 + \sum_{j=1}^{n-1} Q_j + Q_n$$

where

$$Q_0 = n \int_0^{F_1} \frac{F^2}{F(1-F)}\, dF = n \int_0^{F_1} \frac{F}{(1-F)}\, dF = -nF_1 - n \ln(1 - F_1)$$

and $F_j = F(X = x_j)$, i.e., evaluate $F(x)$, the test distribution, at the data value x_j. The interval Q_n is given by,

$$Q_n = n \int_{F_n}^1 \frac{(1-F)^2}{F(1-F)}\, dF = n \int_0^{F_1} \frac{(1-F)}{F}\, dF = -nF_n - n \ln(1 - F_n) - n,$$

where $F_n = F(x_n) \neq 1$, in general.

The intermediate integrals are given by $Q_{(j,\,j+1)}$ and become,

$$\begin{aligned} Q_{(j,\,j+1)} &= n\int_{F_j}^{F_{j+1}} \left(\frac{j/n - F}{F(1-F)}\right)^2 dF \\ &= -\frac{1}{n}\left\{(n-j)^2\left[\ln(1-F_{j+1}) - \ln(1-F_j)\right] + j^2\left[\ln(F_j) - \ln(F_{j+1})\right]\right\} \\ &- n(F_{j+1} - F_j). \end{aligned}$$

Note: When the last term of the above equation is summed from $j = 1$ to $j = n-1$ and then added to the similar terms in Q_0 and Q_n, the result is a complete cancellation, yielding,

$$Q = A^2 = -n - \frac{1}{n}\sum_{j=1}^{n}(2j-1)\left\{\ln(F_j) + \ln(1 - F_{n+1-j})\right\}.$$

This is the classic Anderson-Darling goodness-of-fit statistic (a proof of it follows).

Proof:

$$\begin{aligned} A^2 &= \int_{F_0}^{F_1} \frac{(F_n - F)^2}{F(1-F)}\,dF \\ \frac{A^2}{n} &= \int_0^{F_1} \frac{F}{1-F}\,dF + \int_{F_n}^{1} \frac{1-F}{F}\,dF \\ &+ \sum_{j=1}^{n-1}\int_{F_j}^{F_{j+1}} \frac{(j/n - F)^2}{F(1-F)}\,dF \\ &= -F_1 - \ln(1-F_1) + F_n - \ln(F_n) - 1 + \sum_{j=1}^{n-1} Q_j. \end{aligned}$$

Note that in the above we have $F_j = F(X = x_j) = \Pr(X \le x_j)$. Now,

$$\begin{aligned} Q_j &= \int_{F_j}^{F_{j+1}} \frac{(j/n - F)^2}{F(1-F)}\,dF \\ &= F_j - F_{j+1} + \left\{\left(\frac{n-j}{n}\right)^2\left[\ln(1-F_j) - \ln(1-F_{j+1})\right]\right\} \\ &+ \left\{\left(\frac{j}{n}\right)^2\left[\ln(F_{j+1}) - \ln(F_j)\right]\right\}. \end{aligned}$$

Noting that $-F_1 + \sum_{j=1}^{n-1}(F_j - F_{j+1}) + F_n = 0$, we have

$$\begin{aligned} \frac{A^2}{n} &= -\ln(1-F_1) - \ln(F_n) - 1 + \sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^2\left[\ln(F_{j+1}) - \ln(F_j)\right] \\ &+ \sum_{j=1}^{n-1}\left(\frac{n-j}{n}\right)^2\left[\ln(1-F_j) - \ln(1-F_{j+1})\right]. \end{aligned}$$

The observations

$$\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^2[\ln(F_{j+1})-\ln(F_j)]-\ln(F_n)=-\frac{1}{n^2}\sum_{j=1}^{n}(2j-1)\ln(F_j)$$

and

$$\sum_{j=1}^{n-1}\left(\frac{n-j}{n}\right)^2[\ln(1-F_j)-\ln(1-F_{j+1})]-\ln(1-F_1)$$

$$=-\frac{1}{n^2}\sum_{j=1}^{n}(2j-1)\ln(F_{n+1-j}),$$

lead us to the conclusion

$$A^2=-n-\frac{1}{n}\sum_{j=1}^{n}(2j-1)\left\{\ln(F_j)+\ln(1-F_{n+1-j})\right\}.$$

Description of the RuM-Test.
The RuM-test (Mense and Alderman, (2008)) was developed by the authors to fill a need for an easy-to-use measure of GOF for small datasets. The test statistic for the RuM-test is given by

$$Ru=-2\sum_{i=1}^{n}\ln(P(x_i)),$$

where $P(x_i)=1-F(x_i)$ is called the **tail probability**. $F(x_i)$ is the CDF for the known test distribution evaluated at the i-th data value, x_i. If H_0 is true, i.e., the data is consistent with the proposed distribution, the RuM-test statistic is distributed as a chi-squared random variable with $2n$ degrees of freedom. The RuM-test is different from the more popular GOF tests in that it does not look at differences or distance measures between an edf and the test distribution, $F(x)$. Consider the following situation: Let x_i, $i=1,\ldots,n$ be sample data. The null hypothesis is H_0: The sample data is consistent with the test distribution $F(x)$ whose probability density function (pdf) is $f(x)$ and whose tail distribution is defined by $P(x)=1-F(x)$. If the sample data is consistent with the test distribution, then substitution of the sample data into $F(x)$, i.e., $F(x_1)$, $F(x_2)$, $\ldots$, $F(x_n)$ will produce a distribution that should be close to $U(0,1)$. However, if the sample data does not come from a distribution that is consistent with $F(x)$, then the values $F(x_1)$, $F(x_2)$, $\ldots$, $F(x_n)$ will not distribute as $U(0,1)$. The same can be said about $P(x)$, the tail distribution, since $F(x)\rightarrow U(0,1)$ means $P(x)$ is also $U(0,1)$. The test is based on this departure from a $U(0,1)$ distribution, hence the chi-squared statistic. The convenience comes about because the critical values for the chi-squared distribution are easily accessible (e.g., in Excel$^{\text{TM}}$) and one need not carry around critical value tables for KS or AD statistics. The RuM-test is particularly helpful in determining if the data does not come from the postulated distribution, i.e., H_0 is false.

A.2 Adding New Data

The following gives recursion relations for calculating mean, variance, skewness and kurtosis from sequentially produced data. Sometimes in simulations one needs to know when to stop simulating, and this can be done by continuously updating the moments of the data and then stopping when a given set of conditions on the moments, for example, are met.

The algorithmic approach to Johnson fitting used in JFIT© and EasyFit™, following the work of N.L. Johnson, uses the first four standard central moments of the data to produce the fit. For small datasets, the addition of a single new point can change one or more of these moment values enough to change the fit type or assigned coefficient values.

The formulas for recursive calculation of the mean and variance are fairly easy to derive. The formulas for the skewness and kurtosis, however, are quite complicated and are not readily found in the literature. The formulas are listed below. A reference for the mean and variance formulas is Ross (1997). The higher moments were derived by the authors using Mathematica™.

$$
\begin{aligned}
m &= \bar{x}_n = \frac{1}{n}\sum_{j=1}^{n} x_j, \\
V &= \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x}_n)^2, \\
S &= \frac{n}{(n-1)(n-2)V^{3/2}}\sum_{j=1}^{n}(x_j - \bar{x}_n)^3, \\
K &= \frac{n(n+1)}{(n-1)(n-2)(n-3)V^2}\frac{3(3n-5)}{(n-2)(n-3)}\sum_{j=1}^{n}(x_j - \bar{x}_n)^4,
\end{aligned}
$$

$$
m_{n+1} = m + \frac{x_{n+1} - m}{n+1},
$$

$$
V_{n+1} = \frac{1}{n}\sum_{j=1}^{n+1}(x_j - m_{n+1})^2 = \left(1 - \frac{1}{n}\right)V + (n+1)(m_{n+1} - m)^2
$$

$$
S_{n+1} = \frac{(n+1)^2(n-2)SV^{3/2} + n\left[n(x_{n+1}-m)^2 - 3(n+1)V\right](x_{n+1}-m)}{n\left[n(x_{n+1}-m)^2 + (n^2-1)V^{1/2}\right]\sqrt{\frac{n-1}{n}V + \frac{(x_{n+1}-m)^2}{(n+1)}}},
$$

$$
\begin{aligned}
K_{n+1} &= \frac{(n+1)^2(n+2)\left[(n-2)(n-3)K + 9n - 15\right]V^2}{(n-2)\left[(n^2-1)V + n(x_{n+1}-m)^2\right]^2} \\
&- \frac{9n-6}{(n-1)(n-2)} - \frac{4(n+1)^2(n+2)(x_{n+1}-m)SV^{3/2}}{\left[(n^2-1)V + n(x_{n+1}-m)^2\right]^2}
\end{aligned}
$$

$$+ \quad \frac{6n(n+1)(n+2)(x_{n+1}-\bar{x}_n)^2 s_n^2}{(n-2)\left[(n^2-1)s_n^2 + n(x_{n+1}-\bar{x}_n)^2\right]^2}$$

$$+ \quad \frac{n(n+2)(n^4+1)(x_{n+1}-\bar{x}_n)^4}{(n+1)(n-1)(n-2)\left[(n^2-1)s_n + n(x_{n+1}-\bar{x}_n)^2\right]^2}$$

Chapter 27

Mixed Truncated Random Variable Fitting with the GLD, and Applications in Insurance and Inventory Management

M. Kumaran
Department of Statistical Sciences
Kannur University 670567
Kerala, India
mkdsskannur@gmail.com

V.T. Beena
Govt. College for Women
Trivandrum 695014
Kerala, India
beenamadhu@rediffmail.com

Let X be a unimodal continuous random variable with support (α, β) and $Y = \min(X, r)$, $\alpha < r < \beta$, α and β being any real numbers. Then Y is a mixed type random variable, continuous within the range $\alpha < y < r$ and discrete at $Y = r$, and we call Y a **mixed type of truncated random variable**. In this chapter, we derive explicit expressions for the moments of Y for any fixed r and the product moments of arbitrary higher order of the variables X and Y. The results are derived by assuming that the distribution of the random variable is a member of the generalized lambda distribution family. **Applications** of the results **to find the optimal deductibles in the purchase of an automobile insurance**

policy, and the **optimum order size maximizing the expected utility of an investor in an inventory model**, are given. The method employed is independent of any specific distributional assumptions, so it can be used for all unimodal continuous distributions.

The general expression derived in this chapter can be used to determine the moments of any mixed type truncated random variable, using its GLD representation. The results are potentially useful for analyzing more complex statistical problems involving mixed type of truncated random variable.

27.1 Introduction

Truncated random variables occur in many areas, such as stochastic inventory analysis, portfolio analysis, management modeling, statistical quality control, meteorology, and other fields. Usually **two types of truncation occur**. **In the first type**, as in Johnson, Kotz, and Balakrishnan (1994) and (1995), the random variable on (α, β) is restricted to a range T, which is a subset of (α, β). Usually T is taken as (α, b), (a, β), or (a, b) where $\alpha < a < b < \beta$. The probability density function (**pdf**) of the truncated random variable becomes a weighted function of its original pdf. Thus if X is a continuous random variable with pdf $f(x)$, then the pdf of the truncated random variable is given by

$$h(x) = \begin{cases} \dfrac{f(x)}{\int_T f(y)dy}, & x \in T \\ 0, & \text{otherwise.} \end{cases}$$

In the second type, the range of the random variable is usually restricted to either (α, b) or (a, β), $\alpha < a < b < \beta$ and the entire probability mass corresponding to the truncated part of the range is assumed to be concentrated at the point of truncation, rendering the resulting random variable a mixed type. Due to the mixed nature of the resulting random variable, we call it the **mixed type of truncated random variable**. This type of truncated random variable has not been extensively discussed in the literature.

The mixed type of truncation arises in several cases of stochastic modeling. One typical situation is in **stochastic inventory management problem**, where the inventory holder can make a maximum supply of only what is held in stock. Consider a single period single item inventory problem in which the system can hold a maximum of H units of commodity in stock. Let $X > 0$ denote the stochastic demand for the commodity during the period with pdf $f(x)$ and let Y denote the actual sales, then

$$Y = \begin{cases} X, & \text{if} \quad X < H \\ H, & \text{if} \quad X \geq H. \end{cases} \tag{27.1.1}$$

Obviously, Y is a mixed type random variable, continuous in $(0, H)$ and discrete at $Y = H$. Another situation is the case of **automobile insurance purchasing**, where the policyholder voluntarily agrees to deduct a certain amount from his claim in return for a reduction in premium. Let $X > 0$ denote the stochastic loss with pdf $f(x)$, and H the deductible amount that the customer sets so as to enjoy a savings equivalent to the reduction in premium. Then the actual loss Y is a random variable as defined in (27.1.1). Similarly, consider the case of a **dam/reservoir with capacity** H. The inflow of water X during a fixed period of time is a random variable, $X > 0$. If Y denotes the quantity of water stored in the dam then, Y is also the mixed type of truncated random variable defined in (27.1.1).

Several such practical cases, where the above type of random variable emerges can be seen. In each of the above cases, the variable Y can be expressed as $Y{=}\min(X, r)$, $\alpha < r < \beta$. If $f(x)$ is the pdf of the random variable X defined over the range (α, β), α and β being any two reals, $\alpha < \beta$, then the pdf of Y described above is given by

$$g(y) = \begin{cases} f(x), & y < r \\ 1 - P[X \geq r], & y = r. \end{cases} \tag{27.1.2}$$

Hence, we note that, though the random variable X is subjected to truncation, it does not belong to the traditional form of truncation as discussed in Johnson, Kotz, and Balakrishnan (1994) and (1995). The basic difference is that, in this form, the truncated variable becomes a mixed type random variable with one part of its range being continuous and the remaining part discrete. Because of this mixed nature of the random variable, it is not easy to represent the distribution of Y in any of the standard forms generally used for analysis. Hamberg and Matlack (1968) use the mixed type of truncated random variable for the normative analysis of the insurance purchase of vehicles. Marshall and Olkin (1968) consider the optimal use of screening variables in the classification problems, using the above type of variables. DeGroot (1968) uses such random variables in optimal stopping rules in sampling and Gould (1969) uses truncated random variables to determine the optimal deductibles in insurance purchasing. Winkler, Roodman, and Britney (1972) have derived the partial moments of the mixed type of truncated normal random variable.

In this chapter, the characteristics of the mixed type of truncated random variable are discussed using its generalized lambda distribution (GLD) representation. A brief review of the GLD family, its moments, partial moments, and related results are described in Section 27.2. In Section 27.3, we derive explicit expressions for the moments of the mixed type of truncated random variable and the product moments between the mixed type and the original variable. Applications of the results to find the optimal deductibles in the purchase of an automobile insurance policy and the optimum order size maximizing the expected utility of an investor in a newsboy model are described in Section 27.4.

27.2 The Generalized Lambda Distribution and its Partial Moments

The GLD is a four-parameter family of distributions containing a large number of distributions of a variety of shapes. The GLD, originally proposed by Ramberg and Schmeiser (1972), is an extremely flexible four-parameter family of distributions capable of providing a good fit to a wide variety of distributional forms. The members of the GLD family are represented in terms of their quantile function, which is given by

$$x = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}, \quad 0 \le p \le 1, \tag{27.2.1}$$

where $p = P(X \le x)$. The parameters λ_1 and λ_2 are, respectively, location and scale parameters, while λ_3 and λ_4 determine the skewness and kurtosis of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The support of the random variable is $[\lambda_1 - \frac{1}{\lambda_2}, \lambda_1 + \frac{1}{\lambda_2}]$ when $\lambda_3 > 0, \lambda_4 > 0$ and $(-\infty, \infty)$ when $\lambda_3 < 0, \lambda_4 < 0$. When $\lambda_3 < 0, \lambda_4 = 0$, the support is $(-\infty, \lambda_1 + 1/\lambda_2)$ and $(\lambda_1 - 1/\lambda_2, \infty)$ if $\lambda_3 = 0, \lambda_4 < 0$. When $\lambda_3 = \lambda_4$ the members of the family will be symmetric about λ_1. Karian and Dudewicz (2000) have described in detail the various characteristics of the GLD and its extensions.
The k^{th} order moment about λ_1 of a member of the GLD family is given by

$$E[X - \lambda_1]^k = \lambda_2^{-k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \beta(\lambda_3(k-i) + 1, \lambda_4 i + 1),$$

where $k = 1, 2, \ldots$, and $\beta(a, b)$ is the beta function defined by

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx, \ a > 0, b > 0.$$

The k^{th} order moment exists if and only if $\min(\lambda_3, \lambda_4) > (-1/k)$.

The versatile nature of the GLD family has attracted practitioners to use this family for modeling. However, the nonavailability of a closed-form expression for its pdf and the problem of estimating parameters have been a major hurdle in its application. Several methods of estimation of parameters of the GLD are available: the method of moments by Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979), the method of least squares by Öztürk and Dale (1985), the starship estimation method by King and MacGillivray (1999), the controlled randomization approach by Lakhany and Mausser (2000), and the discretized approach by Su (2005).

The **partial moments** of a random variable are the expectation of the random variable evaluated over a part of its support (Winkler, Roodman, and Britney (1972)). Consider a continuous random variable X with pdf $f(x)$ defined over

the interval (α, β), α and β being any two reals with $\alpha < \beta$. Then the **k^{th} order partial moment of X is defined as**

$$\int_{x \le r} x^k f(x)dx \quad \text{or} \quad \int_{x \ge r} x^k f(x)dx, \quad k = 1, 2, \ldots, \quad \alpha < r < \beta.$$

In stochastic decision-making problems, partial moments occur very frequently. However, closed-form expressions for the partial moments are not available for most commonly used distributions. Kumaran and Achary (2005) have derived the partial moments and their properties for the GLD family. A few of the results that we require in the next section are as follows:

1. **The k^{th} order partial moment of a random variable X** with pdf $f(x)$ denoted by $M_k(r)$ is given by

$$M_k(r) = \int_{x \ge r} x^k f(x)dx, \quad \alpha < r < \beta, \quad k = 1, 2, \ldots . \tag{27.2.2}$$

For any member of the GLD family, $M_k(r)$ is given by

$$M_k(r) = \sum_{j=0}^{k} \binom{k}{j} {\lambda_1}^{k-j} {\lambda_2}^{-j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{-i} \beta_{F_r}(1 + \lambda_4 i, 1 + \lambda_3(j - i)), \tag{27.2.3}$$

where

$$\beta_x(m, n) = \int_0^x y^{m-1}(1 - y)^{n-1} dy$$

is the incomplete beta function, $F_r = 1 - p_r$, and $p_r = P[X \le r]$. For all $r > 0$, $M_k(r)$ is a monotonic decreasing function of r.

2. **The k^{th} order partial moment about r of a random variable X** with pdf $f(x)$ is given by

$$L_k(r) = \int_{x \ge r} (x - r)^k f(x)dx. \tag{27.2.4}$$

For the GLD family, $L_k(r)$ is given by

$$L_k(r) = \sum_{j=0}^{k} \binom{k}{j} (\lambda_1 - r)^{k-j} {\lambda_2}^{-j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{i} \beta_{F_r}(1 + \lambda_4 i, 1 + \lambda_3(j - i)). \tag{27.2.5}$$

3. **$L_k(r)$** exists whenever the k^{th} order moment of X exists and is a non-negative monotonic decreasing, strictly convex, continuous function of r, possessing continuous derivatives of all orders up to $(k + 1)$.

4. **The two functions in (27.2.2) and (27.2.4) above are related through**

$$L_k(r) = \sum_{j=0}^{k} \binom{k}{j} (-r)^{k-j} M_j(r) \tag{27.2.6}$$

and

$$M_k(r) = \sum_{j=0}^{k} \binom{k}{j} r^{k-j} L_j(r). \tag{27.2.7}$$

5. **The s^{th} order derivative of $L_k(r)$ is given by**

$$\frac{d^s}{dr^s} L_k(r) = (-1)^s k^{[s]} L_{k-s}(r), \quad s = 1, 2, \ldots, k, \quad k = 1, 2, \ldots, \tag{27.2.8}$$

where $m^{[n]} = m(m-1)\cdots(m-n+1)$.

$$\frac{d}{dr} L_0(r) = -f_r \text{ where } f_r \text{ is the value of } f(x) \text{ at } x = r \tag{27.2.9}$$

$$\frac{d}{dr} L_1(r) = -F_r. \tag{27.2.10}$$

27.3 Moments of Mixed Type of a Truncated Random Variable

Consider a continuous random variable X on the interval (α, β), α and β being any two reals with $\alpha < \beta$, with pdf $f(x)$. An associated random variable Y defined as

$$Y = \min(X, r), \quad \alpha < r < \beta \tag{27.3.1}$$

is a mixed type random variable truncated on the upper tail. The pdf of Y is

$$g(y) = \begin{cases} f(x), & y < r \\ 1 - p_r, & y = r. \end{cases} \tag{27.3.2}$$

Similarly, the variable X can be regarded as being truncated on the lower tail with associated random variable Z defined by

$$Z = \max(X, s), \quad \alpha < s < \beta. \tag{27.3.3}$$

The pdf of Z is

$$h(z) = \begin{cases} p_s, & y = s \\ f(x), & y > s. \end{cases} \tag{27.3.4}$$

Theorem 27.3.5: The k^{th} order raw moment $\mu'_k(Y)$ of Y is given by

$$\mu'_k(Y) = \mu'_k - \sum_{j=1}^{k} \binom{k}{j} r^{k-j} L_j(r), \quad k = 1, 2, \ldots, \tag{27.3.6}$$

where $L_j(r)$ is the j^{th} order partial moment about r of X as defined in (27.2.4) and $\mu'_k = \int_x x^k f(x)dx$ is the k^{th} order raw moment of X.

Proof: By its definition Y is a mixed random variable with pdf given by (27.3.2). So its k^{th} order raw moment is given by

$$\begin{aligned}
\mu'_k(Y) &= E(Y^k) \\
&= \int_{x<r} x^k f(x)dx + r^k P[X \geq r], \quad \text{by (27.3.1)} \\
&= \int_{x<r} x^k f(x)dx + r^k(1 - p_r), \quad \text{by (27.3.2)} \\
&= \int_x x^k f(x)dx - \int_{x\geq r} x^k f(x)dx + r^k(1 - p_r) \\
&= \mu'_k - M_k(r) + r^k(1 - p_r), \quad \text{by (27.2.2)} \\
&= \mu'_k - \sum_{j=1}^{k} \binom{k}{j} r^{k-j} L_j(r), \quad \text{by (27.2.7)}
\end{aligned}$$

which proves the theorem.

The k^{th} order central moments of Y can be obtained by first computing the $\mu'_k(Y)$'s and then applying the relationship

$$\mu_k(Y) \quad = \textstyle\sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu'_{k-j}(Y)[\mu'_1(Y)]^j.$$

Theorem 27.3.7: The central product moment of order (m, n) of (Y, X) is given by

$$\begin{aligned}
\mu_{m,n} &= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \mu_{n+i} - \sum_{j=0}^{n+i} \binom{n+1}{j} (r-\mu)^{n+i-j} L_j(r) \right. \\
&+ \left. \sum_{k=0}^{n} \binom{n}{k} (r-\mu)^{n+i-k} L_k(r) \right\},
\end{aligned} \tag{27.3.8}$$

where m and n assume values $1, 2, \ldots$, and μ_k is the k^{th} order central moment of X.

Proof: By definition

$$\mu_{m,n} = E[(Y - E(Y))^m (X - E(X))^n].$$

Putting $k = 1$ in (27.3.6), we have, $E(Y) = \mu_1^{'} - L_1(r)$ and $\mu_1^{'} = E(X) = \mu$, so that

$$\begin{aligned}
\mu_{m,n} &= E[\{Y - \mu + L_1(r)\}^m (X - \mu)^n] \\
&= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} E[(Y - \mu)^i (X - \mu)^n] \\
&= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \int_{x<r} (x - \mu)^{n+i} dF(x) \right. \\
&+ \left. (r - \mu)^i \int_{x \geq r} (x - \mu)^n dF(x) \right\} \\
&= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \int_{x} (x - \mu)^{n+i} dF(x) - \int_{x \geq r} (x - \mu)^{n+i} dF(x) \right. \\
&+ \left. (r - \mu)^i \int_{x \geq r} (x - r + r - \mu)^n dF(x) \right\} \\
&= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \mu_{n+i} - \int_{x \geq r} (x - r + r - \mu)^{n+i} dF(x) \right. \\
&+ \left. (r - \mu)^i \int_{x \geq r} (x - r + r - \mu)^n dF(x) \right\} \\
&= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \mu_{n+i} - \sum_{j=0}^{n+i} \binom{n+i}{j} (r - \mu)^{n+i-j} \int_{x \geq r} (x - r)^j dF(x) \right. \\
&+ \left. (r - \mu)^i \sum_{k=0}^{n} \binom{n}{k} (r - \mu)^{n-k} \int_{x \geq r} (x - r)^k dF(x) \right\}
\end{aligned}$$

i.e.,

$$\begin{aligned}
\mu_{m,n} &= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \mu_{n+i} - \sum_{j=0}^{n+i} \binom{n+i}{j} (r - \mu)^{n+i-j} L_j(r) \right. \\
&+ \left. \sum_{k=0}^{n} \binom{n}{k} (r - \mu)^{n+i-k} L_k(r) \right\},
\end{aligned}$$

which proves the theorem.

When $m = n = 1$ in (27.3.8), we get

$$\mu_{1,1} = \mathrm{Cov}(Y, X) = \mu_2 - (r - \mu)L_1(r) - L_2(r). \tag{27.3.9}$$

Similarly, for $m = 2$ and $n = 0$, we get

$$\mu_{2,0} = V(Y) = \mu_2 - L_2(r) - 2(r - \mu)L_1(r) - [L_1(r)]^2. \tag{27.3.10}$$

Theorem 27.3.11: The derivative of $\mu_{m,n}$ with respect to r is given by

$$\begin{aligned}
\frac{d}{dr}\mu_{m,n} &= \sum_{i=0}^{m} \binom{m}{i} [L_1(r)]^{m-i} \left\{ \sum_{j=0}^{n+i} \binom{n+i}{j} (r-\mu)^{(n+i-j-1)} \right. \\
&\times \left[(r-\mu) j L_{j-1}(r) - (n+i-j) L_j(r) \right] + \sum_{k=0}^{n} (r-\mu)^{(n+i-k-1)} \\
&\times \left. \binom{n}{k} [L_k(r)(n+i-k) - (r-\mu)kL_{k-1}(r)] \right\} - \sum_{i=0}^{m} \binom{m}{ii} (m-i) \\
&\times [L_1(r)]^{(m-i-1)} F_r \left\{ \mu_{n+i} - \sum_{j=0}^{n+i} \binom{n+i}{j} (r-\mu)^{n+i-j} L_j(r) \right. \\
&+ \left. \sum_{k=0}^{n} \binom{n}{k} (r-\mu)^{n+i-k} L_k(r) \right\},
\end{aligned} \tag{27.3.12}$$

where m and n assume values $1, 2, \ldots$.

The corresponding results for the variable Z truncated below can be obtained in a similar way, where the partial moments $M_k(r)$ and $L_k(r)$ are to be taken on the lower tail instead of the upper tail.

The method described above is illustrated in the case of GLD representation of the $N(10, 10)$ distribution [i.e., GLD$(10, 0.01975, 0.1349, 0.1349)$]. Table 27.3–1 contains the mean and central moments of the mixed type of truncated random variable Y and the correlation coefficients between the random variable X and truncated random variable Y at various points of truncation r. As the point of truncation increases, the mean as well as the central moments also increase and ultimately coincide with that of the original random variable. Similarly, correlation between X and Y also increases with r.

Table 27.3–1. The mean and central moments of mixed type of truncated random variable Y and correlation coefficient between Y and X, $L \sim \text{GLD}(10, 0.01975, 0.1349, 0.1349)$.

r	$\mu^{'}(Y)$	$\mu_2(Y)$	$\mu_3(Y)$	$\mu_4(Y)$	ρ
3	1.5726	12.2562	−129.791	2086.79	0.693701
5	3.0257	17.1550	−180.827	3011.65	0.746634
7	4.3380	23.1900	−238.681	4173 .00	0.794554
9	5.4973	30.2847	−298.346	5550.97	0.836904
11	6.4973	38.2365	−352.957	7108.96	0.873410
13	7.3380	46.7289	−395.059	8807.92	0.904078
15	8.0257	55.3700	−418.348	10619.90	0.929157
17	8.5726	63.7488	−419.286	12532.50	0.949093
19	8.9950	71.4960	−398.001	14539.40	0.964475
21	9.3116	78.3324	−358.139	16624.50	0.975973
23	9.5418	84.0956	−305.787	18747.30	0.984285
25	9.7040	88.7418	−247.923	20840.50	0.990084
27	9.8149	92.3262	−190.944	22820.30	0.993980
29	9.8881	94.9733	−139.664	24604.00	0.996494
31	9.9348	96.8439	−96.916	26128.00	0.998049
33	9.9636	98.1070	−63.672	27358.90	0.998967
35	9.9805	98.9204	−39.470	28296.20	0.999482
37	9.9901	99.4182	−22.967	28967.00	0.999756
39	9.9953	99.7063	−12.445	29416.50	0.999893
41	9.9979	99.8631	−6.198	29697.10	0.999957
43	9.9992	99.9425	−2.771	29858.80	0.999984
45	9.9997	99.9794	−1.055	29943.90	0.999995
47	9.9999	99.9947	−0.290	29983.70	0.999998
49	10	100	−0.004	29999.50	1

27.4 Applications

As indicated in Section 27.1, there are several practical cases wherein the mixed type of truncated random variable arises. Two such cases, the choice of optimum deductible in insurance purchasing and the determination of optimum order size in single period inventory management, are illustrated below.

27.4.1 Optimum Deductible in Insurance Purchasing

"Insurance" is indemnifying a loss, and the insured aims at maximum indemnity limit with minimum premium. Consider an insurance purchase, where the policyholder voluntarily agrees to deduct a certain amount from his claim in return for a reduction in premium. In the selection of deductible collision insurance, the insured weighs the advantage of a reduction in premium for choosing a larger deductible against the disadvantage of the likelihood of sustaining a larger loss due to collision. The purchase of collision insurance for private passenger vehicles has a deductibility provision that, on the one hand, waives the liability of the insurance company for losses less than the deductible and, on the other hand, allows the insured a saving which equals the reduction in the premium. The vehicle owner's choice of the amount of deductible may be influenced by factors such as the value of the vehicle, the driving record of the regular operators of the vehicle, premium for different amounts of deductible, willingness to assume risk, personal financial condition, and so on. His ultimate aim is to minimize the total expected cost or maximize the utility from purchasing an insurance policy.

Deductible policies have received more attention in the literature in view of their apparently superior risk-reducing characteristics compared with other insurance arrangements. Schkade and Menefee (1966) have suggested a total expected cost minimizing model for selection of a deductible in collision insurance. Doherty and Schlesinger (1983) have examined the choice of a deductible insurance contract in the presence of un-insurable background risk. Different aspects of insurance demand related to deductible insurance are discussed by Blazenko (1985) and Meyer and Ormiston (1999). **A procedure for determining the optimum deductible using a quadratic utility function of wealth is described below**.

Consider the problem of a vehicle owner who desires to choose a deductible D while purchasing a collision insurance policy for a vehicle of value V. Let p be the probability that, during the period of one year, the automobile will be damaged due to collisions resulting in a loss L, where L is a random variable with pdf $f(L)$, $0 \leq L \leq V$. Then the actual loss is

$$Y = \begin{cases} L, & \text{if} \quad L < D \\ D, & \text{if} \quad L \geq D. \end{cases}$$

i.e., $Y = \min(L, D)$, which is a mixed type of random variable truncated on the

upper tail of the distribution of L. Hence, the characteristics of this variable can be found easily using the results in Section 27.3.

If the vehicle is damaged in a collision and the loss is less than the deductible, the full amount of the loss is sustained by the insured. If the collision loss exceeds the deductible, the individual is indemnified for the amount by which the loss exceeds the deductible. Then, the expected actual loss in the event of a collision is

$$\begin{aligned} E(Y) &= \int_0^D Lf(L)dL + D\int_D^V f(L)dL \\ &= \mu_1' - L_1(D), \quad \text{by (27.3.6).} \end{aligned} \tag{27.4.1}$$

The variance of the actual loss is

$$V(Y) = \mu_2 - L_2(D) - 2(D-\mu_1')L_1(D) - [L_1(D)]^2, \quad \text{by (27.3.10),}$$

where μ_1' and μ_2 are the mean and variance of L.

The investor seeks to avert the risk of substantial collision loss by the purchase of deductible collision insurance. Suppose that deductible insurance is available at any deductible level, $D \in (0, V)$. Also, it is assumed that each individual has a utility of wealth function $U(A)$ which he maximizes in the Von Neumann-Morgenstern sense and which has the following properties:

$$U'(A) = \frac{dU}{dA} > 0 \quad \text{and} \quad \frac{d^2U}{dA^2} < 0, \tag{27.4.2}$$

where A is the initial wealth (before purchase of the insurance policy). The first condition says that the individual prefers more wealth to less, and the second is the condition for diminishing marginal utility of wealth. The consumer is assumed to be confronted with a quoted relationship $R(D)$, between the annual premium R, and the deductible D. The function $R(D)$ is referred to as the **premium schedule**, and is assumed as continuous and differentiable with $\frac{d}{dD}[R(D)] < 0$. In an attempt to determine the optimum deductible by maximizing the utility, Gould (1969) has developed bounds for the slope of the premium schedule. Doherty and Schlesinger (1983) have assumed a conventional form of the premium schedule $R(D)$, which entails an actuarially fair price with proportional loading as

$$R(D) = (1+\gamma)\int_D^V (L-D)f(L)\,dL$$

where $\gamma \geq 0$ is a loading factor which represents the processing charge per unit of premium. Using partial moment notation, we get,

$$\begin{aligned} R(D) &= (1+\gamma)L_1(D), \quad \text{by (27.2.4)} \quad \text{so that} \\ \frac{dR}{dD} &= -(1+\gamma)L_0(D), \quad \text{by(27.2.8).} \end{aligned} \tag{27.4.3}$$

For low values of deductibles, the annual premium will be quite high because the company assumes a larger liability. As the size of deductible increases, the annual premium declines and a policy with deductible equal to the value of the insured item V will have a zero premium, that is $R(V) = 0$. Suppose the individual has selected a deductible and suffers a collision loss less than the selected deductible, the utility of the individual is $U(A-L-R(D))$. If the collision loss is greater than the deductible, the individual's utility is $U(A-D-R(D))$, which is independent of L. Then the expected utility for a given deductible is

$$\begin{aligned} E[U(D)] &= (1-p)U(A-R(D)) + p\left[\int_0^D U(A-L-R(D))f(L)dL \right. \\ &+ \left. \int_D^V U(A-D-R(D))f(L)dL\right]. \end{aligned} \tag{27.4.4}$$

Suppose we choose a quadratic utility function $U(A) = A - \frac{B}{2}A^2$, $0 \le B \le \frac{1}{A}$. As B approaches zero, the utility function becomes linear in wealth. This situation represents risk indifference and would be applicable to an individual who is indifferent towards risk. Such an individual will seldom insure. Similarly, as B approaches $1/A$, the situation depicts extreme risk aversion with negative marginal utility. Under the above choice of the utility function, the expected utility (27.4.4) becomes

$$\begin{aligned} E[U(D)] &= (A-R(D)) - \frac{B}{2}(A-R(D))^2 - p\left[E(Y) + \frac{B}{2}E(Y^2)\right] \\ &- pB(A-R(D))E(Y). \end{aligned} \tag{27.4.5}$$

We select D so as to maximize $E[U(D)]$, for which a necessary condition is

$$\frac{d}{dD}[E[U(D)] = 0.$$

Using the results in Section 27.3 (particular cases of (27.2.8) and (27.3.6)), we have the derivatives of $E(Y^2)$ and $E(Y)$ as

$$\frac{d}{dD}[E(Y^2)] = 2DL_0(D)$$

and

$$\frac{d}{dD}[E(Y)] = L_0(D).$$

Thus, **the optimum deductible is the solution of the equation**

$$(1+\gamma)[1-B(A-R(D))+pB(\mu_1'-L_1(D))]+pB(A-R(D)-D)-p=0. \tag{27.4.6}$$

Schkade and Menefee (1966) have determined the value of D by minimizing the total expected cost (TEC), where

$$TEC = pE(Y) + R(D). \tag{27.4.7}$$

Due to the difficulty in computing $E(Y)$, they considered (27.4.7) by replacing $E(Y)$ by D.

To illustrate the choice of the optimum deductible by maximizing the expected utility, we consider the following **examples**.

Example 1:
A private car owner who desires to purchase a deductible insurance policy for his car with original price of Indian Rupees (Rs.) $300,000$ for a premium of Rs. 6000. Let the parameters of the quadratic utility function be A $= 6500$ and B$= \frac{1}{7000}$. The distribution describing the accidents is assumed to follow a point binomial with p as the probability of accidents leading to a loss L. The loading factor is taken as $\gamma = 0.001$. L is assumed to have a gamma distribution with mean Rs. 6000 having GLD representation as GLD$(6000, 0.0614, 0.0543, 0.1297)$. The optimum value of D is computed for various values of p ranging from 0.1 to 0.9 by solving (27.4.6). The corresponding premium $R(D)$ and the optimum utility are computed using (27.4.3) and (27.4.5), respectively. In order to compare the results, the corresponding values of D, $R(D)$ and the minimum total expected cost are computed under the cost minimization method using (27.4.7). Figure 27.4–1 illustrates the behavior of expected utility, total expected cost, and premium for various values of the deductible at $p = 0.2$. The figure shows that similar to the total expected cost curve possessing a unique minimum, the expected utility curve also possesses a unique maximum. Also, the premium decreases with increase in the deductible.

Table 27.4–2 presents the optimum deductible D and the premium $R(D)$ corresponding to selected values of p obtained under the two methods. The table reveals that with increase in the chance of collision damage p, the optimum deductible decreases under both methods. While the optimum premium and the expected cost increase, the optimum utility decreases with increase in p. The last column of the table reveals that the optimum deductible under the two methods are equal for $p = 0.5$ and they differ for $p \neq 0.5$. As the value of p moves away from 0.5, the differences in D increases. It may also be noted that under the cost minimization method, the decrease in the value of D and increase in the value of $R(D)$ are more or less proportionate for the successive values of p. However, under the utility maximization method, D and $R(D)$ exhibit changes of high magnitude for values of p on either extremes. The illustration reveals that, the choice of deductible largely depends on the investor's knowledge of the collision damage risk. The low levels of risk not only lowers the optimum cost but also results in a high level of utility (higher consumer satisfaction). Another important aspect revealed by this example is that whenever the collision damage

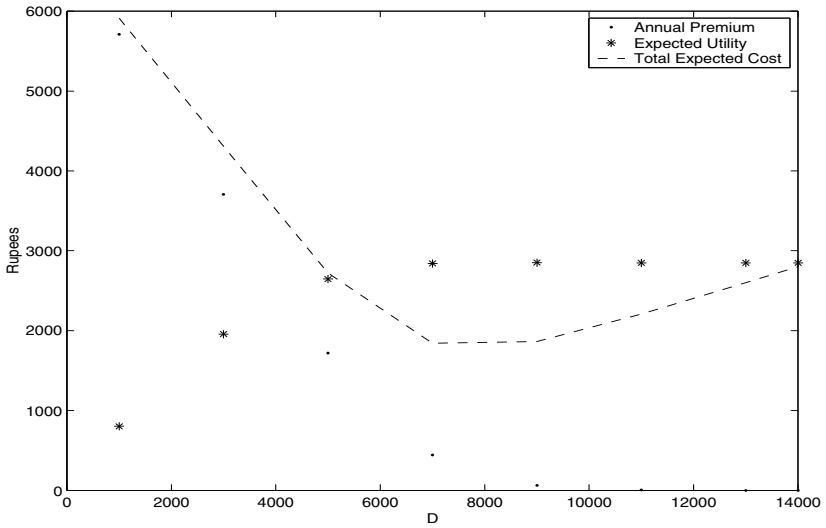


Figure 27.4–1. Annual premium, expected utility, and expected total cost for different D with $p = 0.2$, $L \sim \text{GLD}(6000, 0.0614, 0.0543, 0.1297)$.

Table 27.4–2. The optimum deductible in insurance purchasing, $L \sim \text{GLD}(6000, 0.0614, 0.0543, 0.1297)$.

	Utility Maximization			Total Cost Minimization			
Damage risk p	Opt. D (Rs.)	R(D) (Rs.)	E[U(D)] (Rs.)	Opt. D1 (Rs.)	R(D1) (Rs.)	TEC (Rs.)	$\frac{D-D1}{D1}$ $\times 100$
0.1	10500	9	3165	8500	107	957	23.53
0.2	8250	139	2852	7750	227	1777	6.45
0.3	7500	286	2546	7250	357	2532	3.45
0.4	7000	441	2245	6750	541	3241	3.70
0.5	6500	657	1952	6500	657	3907	0
0.6	6000	941	1667	6250	790	4540	−4.00
0.7	5500	1297	1394	5750	1110	5135	−4.35
0.8	4250	2438	1145	5500	1297	5697	−22.73
0.9	1000	5710	1005	5000	1718	6218	−80.00

Table 27.4–3. The optimum deductible in insurance purchasing, $L \sim \text{GLD}(800, 0.00061, 0.05042, 0.1554)$.

	Utility Maximization			Total Cost Minimization			
Damage risk p	Opt. D ($)	R(D) ($)	E[U(D)] ($)	Opt. D1 ($)	R(D1) ($)	TEC ($)	$\frac{D-D1}{D1}$ $\times 100$
0.1	1700	1	452	1250	15	140	36.00
0.2	1225	18	409	1125	33	258	8.89
0.3	1075	44	368	1025	58	365	4.88
0.4	975	75	328	950	85	465	2.63
0.5	900	107	290	900	107	557	0
0.6	800	166	254	875	120	645	−8.57
0.7	700	243	221	800	166	726	−12.50
0.8	475	463	195	725	222	843	−34.48
0.9	150	794	194	675	264	872	−77.78

risk is around 0.5, it is immaterial which method is adopted for the choice of the optimum deductible. Otherwise, it is better to choose the utility maximization method as it leads to a higher deductible for low risks and a lower deductible for higher risks as compared to the other method.

Example 2:
Consider the choice of optimum deductible D and the premium $R(D)$, in the case of purchase of an insurance policy for a car worth \$25,000 with a premium of \$800, under the utility maximization and cost minimization methods. Let the initial wealth of the insured be \$900 and the parameters of the quadratic utility function be A = 900 and B= $\frac{1}{1000}$. The loss function is taken as the GLD representation of a gamma distribution with mean \$800 [i.e., $\text{GLD}(800, 0.00061, 0.05042, 0.1554)$]. The optimum deductible D, the corresponding premium $R(D)$, the optimum utility and the minimum total expected cost are tabulated in Table 27.4–3. The table shows the risk-deductible tradeoff for different values of p under both methods. The last column of the table indicates that utility maximization leads to higher deductibles for low risk and lower deductibles for higher risk compared to the cost minimization method.

Thus, both the examples reveal that the utility maximizer is more rational and responsive to the knowledge of the collision damage risk than the cost minimizer.

27.4.2 Newsboy Model: Solution under Utility Maximization

Another typical area of realization of the mixed type of truncated random variable described in Section 27.3 is in the modeling of the **single period single item stochastic inventory management problem**, which is popularly known as the newsboy model. Lau (1980) has discussed the problem under alternative optimization objectives. Kumaran and Achary (2000) have discussed the newsboy problem under balking.

The newsboy model can be described as follows: A firm orders for a fixed quantity of items which is delivered at the beginning of the period. The demand for the item is random and is met from the stock held. All demands in excess of the stock are regarded as loss. The leftover stock at the end of the period is disposed of at a lower rate. The firm has to order enough so that the full potential for profit may be realized, but not too much so as to avoid losses due to excess stock. There are many inventory problems under this category with the major distinguishing feature of a tradeoff between the consequences of ordering too much or too little. Details are available in Hadley and Whitin (1963), Sivazlian and Stanfel (1975).

Consider a newsboy model inventory problem. Let H denote the quantity ordered at the beginning of the period at a purchase cost of C_1 per unit. Depending on the demand, items are sold at a price of C_2 per unit. Unsold items leftover at the end of the period are disposed of at a lower price of C_3 per unit. C_4 is the penalty cost per unit representing the allowance for excess demand which cannot be met from the available stock. The demand, being uncertain, is represented by a continuous random variable X having pdf $f(x)$, with mean μ and standard deviation σ. Since the stock held H is fixed, the actual sale Y will be a random variable related to the demand random variable X as

$$Y = \begin{cases} X, & \text{if demand} \quad X < H \\ H, & \text{if demand} \quad X \geq H, \end{cases}$$

i.e., $Y = \min(X, H)$. Thus the variable Y is a mixed type of truncated random variable.

In the classical example, the investor's preferences, such as the taste towards risk, are not taken into account even though the tastes, preferences, and attitudes of the investor have a vital role in determining his expected return. Maximizing the expected utility should lead to the choice of actions consistent with preferences as it enables the specification and control of certain types of investor's risk and preferences. Due to the non-availability of closed-form expressions for the single and product moments of X and Y, the analysis of the model, based on utility considerations was not made under the general case. Lau (1980) has discussed the model under alternative optimization objectives, assuming the Schmeiser-Deutsch distribution for demand. Using the results in Section 27.3, we consider the maximization of the expected utility of the investor instead of maximization of the profit, assuming the distribution of demand as a member of the GLD.

Let the investor's wealth be denoted as w and his income/profit, which is a random variable, be denoted as G. Let $U(w)$ be the utility function, then the expected utility of the project is given by Kottas, Lau, and Lau (1978) as

$$\begin{aligned} E[U(w+G)] &= U(w) + \frac{1}{2!}U^{(2)}(w)\mu_2(G) + \frac{1}{3!}U^{(3)}(w)\mu_3(G) \\ &+ \frac{1}{4!}U^{(4)}(w)\mu_4(G) + \ldots + \frac{1}{k!}U^{(k)}(w)\mu_k(G) + \ldots, \end{aligned} \quad (27.4.8)$$

where $\mu_k(G)$ is the k^{th} order moment of G and $U^{(k)}(w)$ is the k^{th} order derivative of $U(x)$ at $x = w$. Equation (27.4.8) states that the expected utility is a function of all those risks associated with the successive higher-order moments of the profit distribution. Hence, to consider the expected utility of profit, we require the higher-order moments of the profit distribution, which can be expressed in terms of the product moments $\mu_{m,n}(Y, X)$ as given by (27.3.8).

The expected utility in the classical newsboy model described above is derived as follows. In terms of X and Y, the profit function can be written as

$$\begin{aligned} G(H) &= C_2Y + C_3(H - Y) - C_4(X - Y) - C_1H \\ &= (C_2 - C_3 + C_4)Y + (C_3 - C_1)H - C_4X. \end{aligned} \quad (27.4.9)$$

Clearly, $G(H)$ also is a random variable. Then the expected profit is

$$E[G(H)] = (C_2 - C_3 + C_4)E(Y) + (C_3 - C_1)H - C_4E(X), \quad (27.4.10)$$

$$\begin{aligned} G(H) - E[G(H)] &= (C_2 - C_3 + C_4)[Y - E(Y)] - C_4[X - E(X)] \\ &= a[Y - E(Y)] + b[X - E(X)], \end{aligned}$$

where $a = C_2 - C_3 + C_4$, $b = -C_4$. The k^{th} order central moment of the profit function $G(H)$ is

$$\begin{aligned} \mu_k(G) &= E[G(H) - E[G(H)]]^k \\ &= \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \mu_{j,k-j}(Y, X), \end{aligned} \quad (27.4.11)$$

where $\mu_{m,n}(Y, X)$ is given by (27.3.8). Also the first-order derivative of $\mu_k(G)$ with respect to H is given by

$$\frac{d}{dH}\mu_k(G) = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \frac{d}{dH}\mu_{j,k-j}(Y, X), \quad (27.4.12)$$

where the derivative of $\mu_{m,n}(Y, X)$ is given by (27.3.12). Hence, the expected utility of the profit in the case of the newsboy problem with an investment w and profit G is given by

$$E[U(w+G)] = U(w) + \sum_{i=2}^{\infty} \frac{U^{(i)}(w)}{i!} \sum_{j=2}^{i} \binom{i}{j} a^j b^{i-j} \mu_{j,i-j}(Y, X). \quad (27.4.13)$$

Since we got the explicit expression for the expected profit, we can derive the optimal order size H so as to maximize the expected utility. The value of H that maximizes the expected utility is obtained by solving the equation,

$$\frac{d}{dH}E[U(w+G)] = 0,$$

i.e.,

$$\sum_{i=2}^{\infty} \frac{U^{(i)}(w)}{i!} \sum_{j=2}^{i} \binom{i}{j} a^j b^{i-j} \frac{d}{dH}\mu_{j,i-j}(Y,X) = 0, \tag{27.4.14}$$

where $\frac{d}{dH}\mu_{j,i-j}(Y,X)$ is given by (27.3.12). For more details, see Sivazlian and Stanfel (1975).

Thus, the optimum order size, maximizing the expected utility of the investor in a newsboy model which is a hitherto overlooked one, can be computed using the moments and product moments of the mixed type of truncated random variable, assuming the GLD representation of the stochastic demand variable.

For the purpose of comparison of the optimal order size as given by the above procedure, its value under the classical newsboy procedure is given below. The expected profit $Z(H)$ is written as (Hadley and Whitin (1963))

$$Z(H) = (C_2 - C_3)\mu - (C_1 - C_3)H - (C_2 + C_4 - C_3)L_1(H), \tag{27.4.15}$$

where $L_1(H) = \int_{x \geq H} (x - H) f(x) dx$ is the average demand lost under the policy and is generally referred to as the loss function. In the classical approach, the decision variable H is determined so as to maximize the expected profit. Thus the optimal order size H is given by

$$P[X \geq H] = C_r, \tag{27.4.16}$$

where

$$C_r = \frac{C_1 - C_3}{C_2 + C_4 - C_3}.$$

C_r is called the **critical ratio**. Thus the optimal order size H must be selected so that the probability of running out of stock is equal to C_r and the policy is called a **critical ratio policy**.

To illustrate the method, we have to choose a utility function. The utility function is a subjective entity that assumes different functional forms. Pratt (1964), Zeckhauser and Keeler (1970) have suggested several simple utility functions. Here, we have used the utility function $U(w)$=log(w), which is consistent with the behavior of a risk averse investor who prefers more to less and whose percentage wealth invested in risky assets remains constant as wealth increases. When the wealth is more, $U^{(k)}(w)$, the k^{th} order derivative of log(w), will have k^{th} power of w in the denominator and hence make a negligible contribution for large

values of k. Hence, neglecting derivatives of order greater than four, equation (27.4.13) for expected utility, in the case of the newsboy model, becomes

$$E[U(w+G)] = U(w) + \sum_{i=2}^{4} \frac{(-1)^{(i-1)}}{iw^i} \sum_{j=2}^{i} \binom{i}{j} a^j b^{i-j} \mu_{j,i-j}(Y,X). \qquad (27.4.17)$$

Then equation (27.4.14) for determining the optimal order size H reduces to

$$\sum_{i=2}^{4} \frac{(-1)^{(i-1)}}{iw^i} \sum_{j=2}^{i} \binom{i}{j} a^j b^{i-j} \frac{d}{dH} \mu_{j,i-j}(Y,X) = 0, \qquad (27.4.18)$$

where the derivative of the product moment is given by (27.3.12). Equation (27.4.18) is a nonlinear equation which can be solved by using standard procedures such as Newton-Raphson method.

For the numerical illustration of the procedure in Section 27.4.2, we consider the problem of a vegetable vendor who starts his daily sales with the delivery of the quantity ordered in the morning. Computation of the optimal order size H under the principle of maximization of the average utility of the profit is done by solving (27.4.18), assuming the GLD representation of $N(1000, 100)$ distribution for the demand [i.e., GLD$(1000, 0.001975, 0.1349, 0.1349)$] and using the logarithmic utility function. Purchase cost is taken as the initial wealth. For the purpose of comparison, the optimal order size H under the classical procedure, optimizing the average profit, is obtained by solving (27.4.15). The behavior of the expected profit and expected utility with respect to the order size are exhibited in Figure 27.4–4 and Figure 27.4–5, respectively for specified values of the cost components. These figures indicate that both profit and utility functions are convex. Figure 27.4–5 reveals that the expected utility is more or less stable for order size less than the optimum while it steadily decreases for order size greater than the optimum. The optimal values of H under both the methods for varying sets of cost components are given in Table 27.4–6.

The entries in Table 27.4–6 reveal that the optimal order size and the expected profit under the utility maximization case are always less than the corresponding values under profit maximization. This is because, in the case of utility maximization, the investor has an opportunity to make a better assessment of the real-world situation by incorporating his preferences. This enables the investor to fix the risk-uncertainty-preference adjusted optimum order size that may maximize the whole bundle of satisfaction. It is interesting to note that the profit per unit in the case of utility maximization is slightly higher than those in the profit maximization case. However, this is not customary in all cases. The table reveals that the optimum order size under profit maximization is nearly 10 to 30% more than that under utility maximization for a given set of costs. As far as the vendor is concerned, handling more stock implies entertaining more risk. Hence, our study reveals that a vendor who is a pro-risk-taker is to earn maximum profit

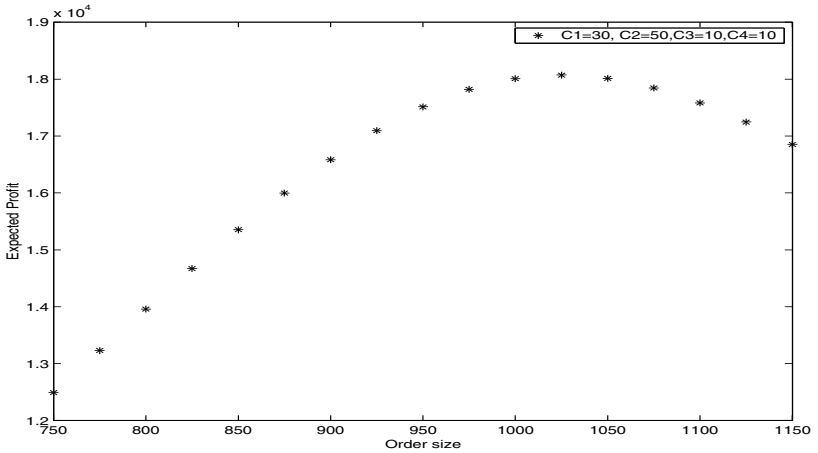


Figure 27.4–4. Order size and expected profit under profit maximization.

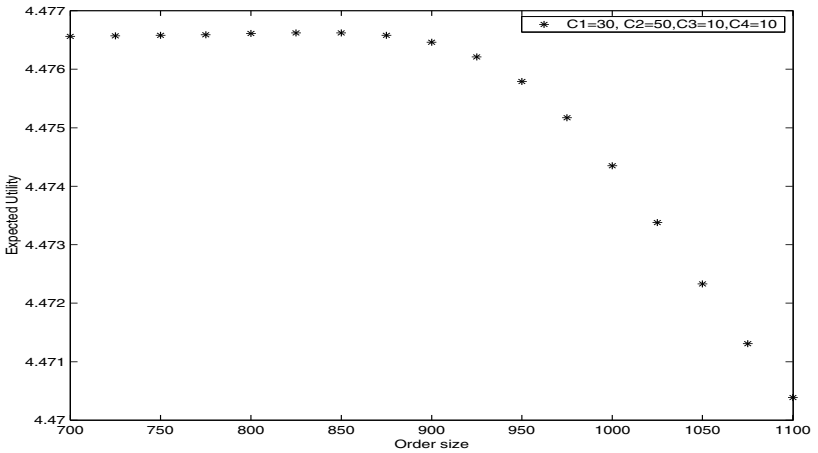


Figure 27.4–5. Order size and expected utility under utility maximization.

by handling a stock very close to the average demand during the period, while a risk averse investor can derive full satisfaction by handling a stock nearly 20% less than the average demand. So a vendor can adopt one of these methods based on his preferences.

It is true that the profit maximization and utility maximization are not strictly comparable as profit is a real characteristic and utility is abstract. However, entries in Table 27.4–6 give an indication that, with a suitable choice of the utility function, it is possible to attain more or less the same level of profit/unit under utility consideration as that under profit maximization. The advantage of using utility consideration is that it allows the investor to arrive at a decision consistent with his preferences (which cannot be accounted otherwise). The relevance of this procedure entirely depends on the choice of the utility function capable of

Table 27.4–6. The optimum order size and corresponding profit under expected utility maximization–utility function $U(w) = log(w)$ demand $\sim$ GLD$(1000, 0.001975, 0.1349, 0.1349)$.

Profit maximization			Cost factors				Utility maximization		
opt. size	maximum profit (Rs.)	profit/ unit (Rs.)	C_1	C_2	C_3	C_4	opt. size	maximum profit (Rs.)	profit/ unit (Rs.)
1000	8406.96	8.41	30	40	10	10	885	7450.59	8.42
985	8627.80	8.76	30	40	10	5	810	7110.82	8.78
955	8365.38	8.76	30	40	10	0	790	6820.66	8.63
985	8225.03	8.35	30	40	5	10	855	6951.01	8.13
975	8071.25	8.28	30	40	0	10	855	6934.45	8.11
1015	13225.00	13.03	30	45	10	10	855	11226.00	13.13
1000	13407.00	13.41	30	45	10	5	810	11155.20	13.77
980	13628.10	13.91	30	45	10	0	725	10872.10	15.00
1000	13008.70	13.01	30	45	5	10	855	11209.50	13.11
990	12823.50	12.96	30	45	0	10	835	10760.10	12.89
1030	18069.00	17.54	30	50	10	10	855	15484.50	18.11
980	18121.80	18.49	30	50	10	5	790	14720.70	18.63
1000	18407.00	18.41	30	50	10	0	710	14198.10	20.00
1010	17823.50	17.65	30	50	5	10	835	14935.10	17.89
1000	17610.40	17.61	30	50	0	10	815	14373.60	17.64
1035	22938.10	22.16	30	55	10	10	835	19110.10	22.89
1025	23071.20	22.51	30	55	10	5	770	18081.80	23.48
1015	23225.00	22.88	30	55	10	0	745	18617.60	24.99
1020	22662.40	22.22	30	55	5	10	815	18448.60	22.64
1015	22419.50	22.09	30	55	0	10	805	18111.20	22.50

representing the true mind-set of the investor. The issue of the choice of a utility function is not addressed in this chapter since our attempt is only to derive the expected utility of profit/investment and its derivative based on any polynomial or transcendental utility function. The newsboy example is given simply to illustrate the matter of how to proceed with the analysis for a given utility function.

27.5 Conclusion

The general expression derived in this chapter can be used to determine the moments of any mixed type of truncated random variable, using its GLD representation. The results are potentially useful for analyzing more complex statistical problems involving the mixed type of truncated random variable.

Acknowledgments

The authors gratefully acknowledge the valuable comments and suggestions of the referee and the Editors, E.J. Dudewicz and Z.A. Karian, helped to improve the quality of the chapter.

References for Chapter 27

Blazenko, G. (1985). "The design of an optimal insurance policy: Note," *American Economic Review*, 75(1), 253–255.

DeGroot, M. H. (1968). "Some problems of optimal stopping," *Journal of the Royal Statistical Society*, B, 30, 108–122.

Doherty, A. N. and Schlesinger, H. (1983). "The optimal deductible for an insurance policy when initial wealth is random," *Journal of Business*, 56(4), 555–565.

Gould, J. P. (1969). "The expected utility hypothesis and the selection of optimal deductibles for a given insurance policy," *The Journal of Business*, 42, 143–151.

Hadley, G. and Whitin, T. M. (1963). *Analysis of Inventory Systems.* Prentice-Hall, New Jersey.

Hamburg, M. and Matlack, W. F. (1968). "Maximizing insurance buyer's utility," *Management Science*, 14, 294–301.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions,* Vol. I, John Wiley & Sons, New York.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions,* Vol. II, John Wiley & Sons, New York.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* CRC Press, New York.

King, R. and MacGillivray, H. L. (1999). "A starship estimation method for the generalized lambda distributions," *Australian & New Zealand Journal of Statistics*, 41(3), 353–374.

Kottas, J. F., Lau. A. H., and Lau. H. S. (1978). "A general approach to stochastic planning models: An overview," *The Accounting Review*, 53(2), 389–400.

Kumaran, M. and Achary, K. K. (2000). "Newsboy model with balking," *Indian Association of Productivity, Quality and Reliability Transactions*, 25(2), 83–96.

Kumaran, M. and Achary, K. K. (2005). "Partial moments of Generalized Lambda Distribution," *American Journal of Mathematical and Management Sciences*, 25, 43–64.

Lakhany, A. and Mausser, H. (2000). "Estimating the parameters of the Generalized Lambda Distribution," *Algo Research Quarterly*, 3(3), 47–58.

Lau, H. S (1980). "The newsboy problem under alternative optimization objectives," *Journal of the Operational Research Society*, 31, 525–535.

Marshall, A. W. and Olkin, I. (1968). "A general approach to some screening and classification problems," *Journal of the Royal Statistical Society*, B, 30, 407–435.

Meyer, Jack and Ormiston, B. Michael. (1999). "Analyzing the demand for deductible insurance," *Journal of Risk and Uncertainty*, 18 (3), 223–230.

Öztürk, A. and Dale, R. (1985). "Least squares estimation of the parameters of the Generalized Lambda Distribution," *Technometrics*, 27(1), 81–84.

Pratt, J. W. (1964). "Risk aversion in the small and in the large," *Econometrika*, 32, 122–136.

Ramberg, J. S. and Schmeiser, B. W. (1972). "An approximate method for generating symmetric random variables," *Communications of the ACM*, 15(11), 987–990.

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting data," *Technometrics*, 21(2), 201–214.

Schkade, L. L. and Menefee, H. G. (1966). "A normative model for deductible collision insurance selection," *The Journal of Risk and Insurance*, 33(3), 427–435.

Sivazlian, B. D. and Stanfel, L. E. (1975). *Analysis of Systems in Operations Research.* Prentice-Hall, New Jersey.

Su, S. (2005). "A discretized approach to flexibly fit Generalized Lambda Distributions to data," *Journal of Modern Applied Statistical Methods*, 4(2), 408–424.

Winkler, R. L., Roodman, G. M., and Britney, R. R. (1972). "The determination of partial moments," *Management Science*, 19(3), 290–296.

Zeckhauser, R. and Keeler, E. (1970). "Another type of risk aversion," *Econometrica*, 38(5), 661–665.

Chapter 28

Distributional Modeling of Pipeline Leakage Repair Costs for a Water Utility Company[1]

Steve Su

School of Mathematics and Statistics,
University of Western Australia,
M019, 35 Stiring Highway, Crawley, 6009, WA, Australia
E-mail: allegro.su@gmail.com

A fundamental problem of measuring the uncertainty and risk of asset repair costs in practice is addressed in this chapter. By fitting generalized lambda distributions (GLD) to empirical data using the starship and numerical maximum likelihood methods, this chapter examines the impact of factors such as time of pipe leakage, age of pipe, pipe diameter, pipe location, and pipe length on the empirical probability distributions of pipe repair costs. This provides important management information regarding the impact that different factors have on the range and the likelihood of extreme costs. Additionally, the use of GLD can be used in conjunction with Monte Carlo simulations and other strategies to set monthly prices in a rational manner, allowing the water utility company to breakeven their basic leak repair costs in the long term within a specified level of confidence.

[1]This chapter appeared as a paper, under the same title, in *American Journal of Mathematical and Management Sciences*, Vol. 27, (2007), pp. 479–498, and won the Thomas L. Saaty prize in 2008.

Copyright © 2007 by American Sciences Press, Inc., Syracuse, New York 13224–2104. Reprinted by permission.

28.1 Introduction

This chapter introduces a new way of modeling cost distribution, owing to the recent advancement of computer power and the development of quantile distribution theory particularly in generalized lambda distributions (Freimer, Kollia, Mudholkar, and Lin (1988), Okur (1988), King and MacGillivray (1999), Karian and Dudewicz (2000), Lakhany and Massuer (2000), Su (2007b), and Su (2007a)). The advancement of computer power is especially important to facilitate the use of these distributions in fitting the dataset because these distributions are only defined explicitly by their quantiles, hence numerical methods need to be used.

The use of generalized lambda distributions is advantageous in several ways. These distributions have a wide range of shapes and can model many well-known distributions such as Weibull, Uniform, Cauchy, Normal, and Log Normal, to name a few. This eliminates the difficulty in choosing the appropriate distributions for the dataset. Also, due to its flexibility in modeling a range of different distributions, it is possible to directly model the underlying process, rather than relying on the central limit theorem and the mean of the process, as in the case of traditional statistical analysis.

The basic approach to modeling costs as probability distributions has appeared in business literature such as in Statistical Activity Cost Theory (Willett (1987), (1988), and (1991) and Falta and Wolff (2004)). Statistical activity cost analysis had been used in a variety of different contexts (Su (2005b), (2006), and (2007c)). Flexible and reliable automated methods for estimating distributions of costs of the kind described in this chapter are essential to the development of reliable decision support systems in engineering asset management.

This chapter provides a statistical cost analysis of the XYX pipeline data across different regions in Queensland, Australia during 2000–2005 for a water utility company. Due to confidentiality, the exact identification and the full analysis of the type of pipeline are not given. However, this chapter will demonstrate how it is possible to use the generalized lambda distribution in investigating factors (pipe length, age, diameter, time of leak, location) which influence the pipeline repair costs. Furthermore, this chapter demonstrates how it is possible to use the information from generalized lambda distributions to set water prices to breakeven its repair costs in the long run as required by regulations for most water providers around the world.

28.2 Generalized Lambda Distributions

The key statistical tool in this chapter is the use of generalized lambda distributions to model the statistical distribution of repair costs. The use of GLD and its fitting methods to empirical data have appeared in statistical literature (Ramberg and Schmeiser (1974), Ramberg, Tadikamalla, Dudewicz, and MykYtka (1979), Öztürk and Dale (1985), Freimer, Kollia, Mudholkar, and Lin (1988); Okur

(1988); King and MacGillivray (1999); Karian and Dudewicz (2000), Lakhany and Massuer (2000), and Su (2005a), (2007b), and (2007a)). Due to the wide range of shapes GLD possesses, for example: U shaped, bell shaped, triangular, and exponentially shaped distributions, it has been used in a variety of analyses. These include Monte Carlo simulations (Hogben (1963)), the modeling of empirical distributions (Ramberg, Tadikamalla, Dudewicz, and Mykytka (1979), Okur (1988)), and in the sensitivity analysis of robust statistical methods (Shapiro, Wilk, and Chen (1968)).

28.2.1 Basic Theory

There are two commonly known different GLDs. The Ramberg-Schmeiser (1974) (RS) GLD is an extension of Tukey's lambda distribution (Hastings, Mosteller, Tukey, and Winsor (1947)). It is defined by its inverse distribution function

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}. \tag{28.2.1}$$

In Expression 28.2.1, $0 \leq u \leq 1$, $\lambda_2 \neq 0$ and λ_1, λ_2, λ_3, λ_4 are, respectively, the location, scale, and shape parameters of generalized lambda distribution GLD(λ_1, λ_2, λ_3, λ_4). In particular, Karian, Dudewicz, and MacDonald (1996) noted that GLD is defined if and only if

$$\frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}} \geq 0 \text{ for } u \in [0,1]. \tag{28.2.2}$$

Another distribution known as FMKL GLD also exists, due to the work of Freimer, Kollia, Mudholkar, and Lin (1988). The FMKL GLD can be written as:

$$F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2} \tag{28.2.3}$$

Under (28.2.3), $0 \leq u \leq 1$ and λ_1, λ_2, λ_3, λ_4 are consistent with the interpretations in RS GLD: λ_1, λ_2 are the location and scale parameters and λ_3, λ_4 are the shape parameters. If $\lambda_3 = \lambda_4 = 0$, then the FMKL GLD has the limiting form

$$F^{-1}(u) = \lambda_1 + \frac{\ln(u) - \ln(1-u)}{\lambda_2} \tag{28.2.4}$$

The fundamental motivation for the development of FMKL GLD is that the distribution is proper over all λ_3 and λ_4 (Freimer, Kollia, Mudholkar, and Lin (1988)). The only restriction on FMKL GLD is $\lambda_2 > 0$.

28.2.2 Fitting Methods

The fitting of both RS and FMKL GLD to empirical data always requires numerical optimization as GLDs are defined by their inverse quantile functions. It in-

volves a careful consideration of initial values for optimization, a suitable criterion to optimize and ensuring the eventual solution remains a valid probability distribution. This chapter will use the starship method or goodness-of-fit method (King and MacGillivray (1999)) as well as using the numerical maximum likelihood (ML) developed by the author (Su (2007b)). Overall, in comparison to the starship method, the ML fits have closer mean, variances, skewness and kurtosis to that of empirical data, as well as satisfying the bootstrap Kolmogorov-Smirnov goodness-of-fit test.

As an alternative method, Su (2005a) developed a customized, discretized approach (DS) of fitting by matching the frequency of the data from a particular histogram bin class with the frequency from a feasible GLD. This is a very flexible method in which the same data can be fitted by several different credible distributions, depending on the bin-width representation of the histogram. Along with the discretized approach, Su (2005a) also proposes an optimization scheme which can suppress or highlight the tails of the distribution by applying and removing a set of weights. The advantage of this method is that it allows the user to conduct sensitivity analysis by capturing different features of the empirical distribution.

As an example, consider the following distribution fits of the entire individual pipe repair cost data shown in Figure 28.2–1. The weighted RPRSDS (Revised percentile RS discretized approach) and RMFMKLDS (Revised method of moment FMKL discretized approach) are in Figure 28.2–1 (a). Figure 28.2–1 (b) represents the unweighted discretized methods, namely RPRSnwDS and RMFMKLnwDS. The maximum likelihood strategies RPRSML, RMFMKLML, and starship method are in Figure 28.2–1 (c).

The weighted approach has a tendency to suppress the tails of the distribution as opposed to the unweighted approach. In the light of a cost analysis, the weighted approach may be considered as an optimistic approach where the probability of high costs is suppressed. This can happen, for example, when the organization takes a new insurance policy to cover these higher costs so it is no longer necessary to account for them or a major recent revamp on the asset means high repair costs are less likely. Conversely, the unweighted approach can be seen as a more pessimistic approach. By not suppressing the tails, the probability of occurrences for these higher costs will be greater than the weighted approach. Although not shown here, the shape of both the discretized unweighted and weighted approaches can be altered by different number of bin classes of the histogram, allowing users to choose a distribution that models different features of the dataset.

The maximum likelihood and starship methods are the definitive fits of the dataset and will be used in this chapter. It provides the generally acceptable fits to empirical data using standard statistical criteria. As there is currently no single method that would work the best in every situation, the best fitted distribution chosen for a dataset in this chapter will be determined by fitted GLD which best matches the moments, i.e., the mean, variance, skewness, and kurtosis of the

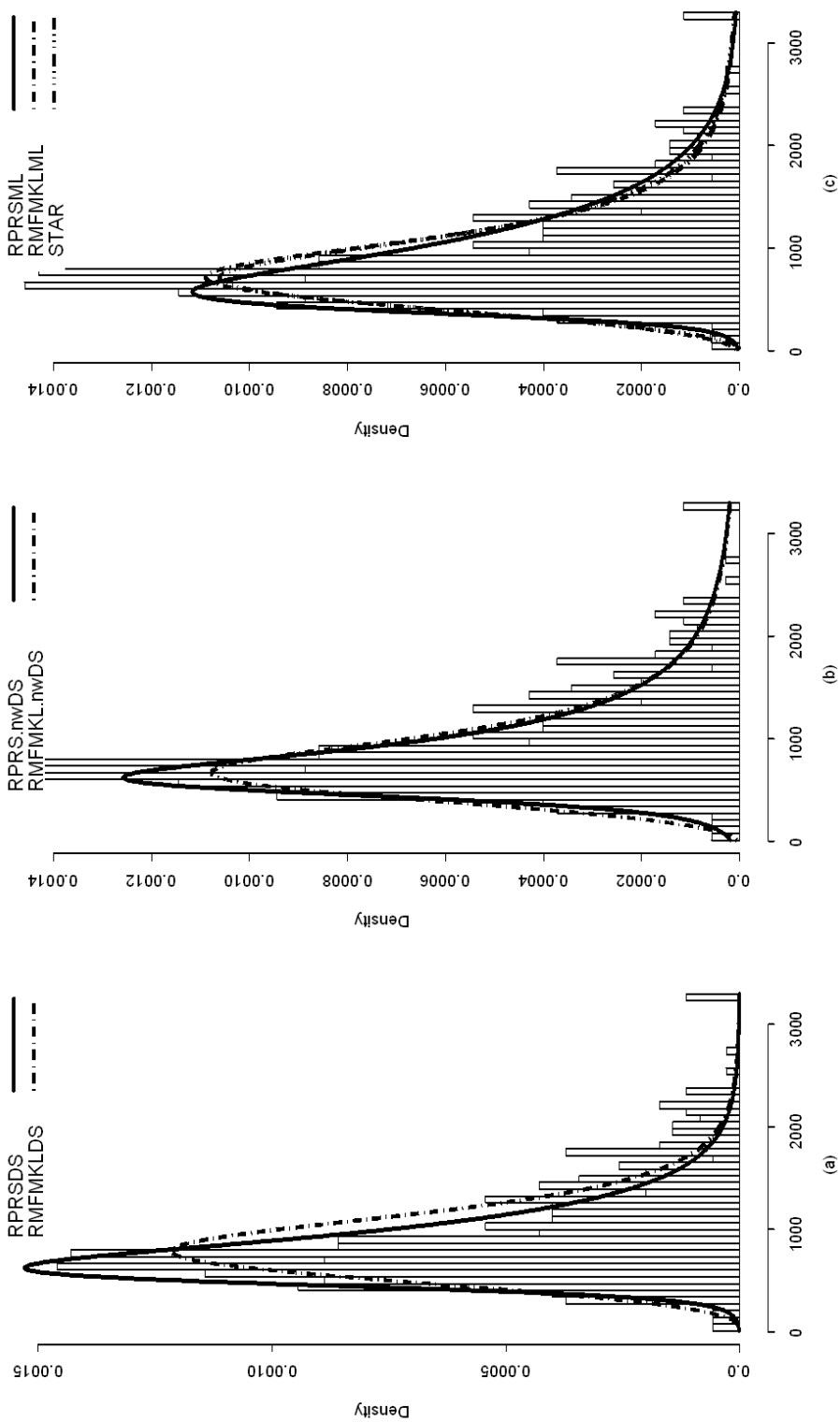


Figure 28.2–1. Demonstration showing different distribution fitting strategies on the empirical distribution of individual repair data.

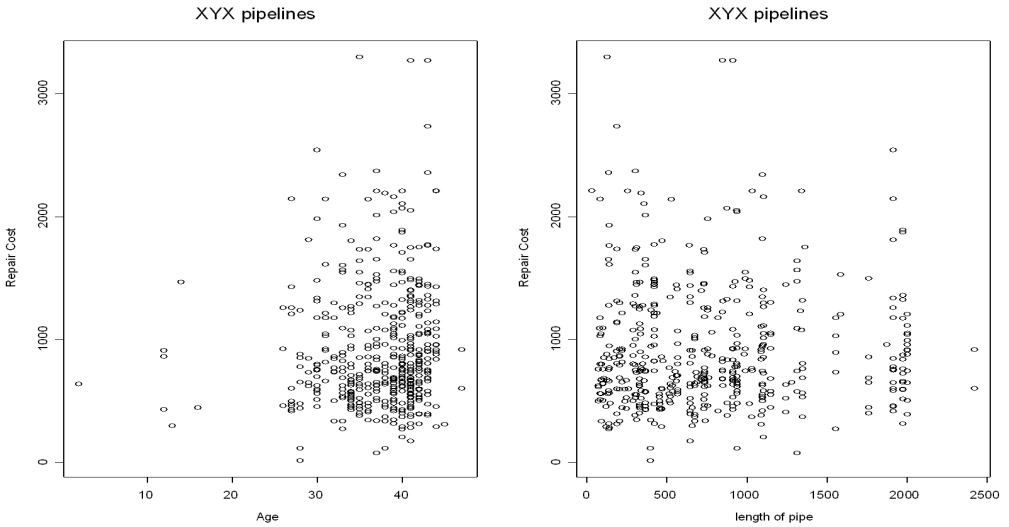


Figure 28.3–1. The relationship between pipe repair costs with age and length of pipe.

empirical data. This criterion is appropriate for this research since traditional cost analysis primarily concentrates on the mean and variance of the data. For this reason this criterion is used rather than the alternative bootstrap Kolmogorov-Smirnov (KS) test, which is a general purpose goodness-of-fit procedure. The use of bootstrap KS or a goodness-of-fit statistical test would obviously tend to favor the starship GLD fitting scheme but the criterion of moment matching will often choose an alternative method of fitting. This is useful as a further check, as sometimes the goodness-of-fit results are very similar between the different fits. To date, there are no definitive diagnostic tools in the statistical literature for the assessment of the quality of fit for quantile distributions such as the GLD. A further discussion regarding the choice of the right distribution is given in Section 28.5.

28.3 Factors Influencing the Costs of Pipe Repairs

As an exploratory analysis, the following graphs examine whether a relationship exists between repair costs and season (leakage month), age of pipe, pipe diameter, pipe location, and pipe length.

28.3.1 Continuous Data: Pipe Length and Age of Pipe

Figure 28.3–1 shows that there is no association between cost of repairs and length of pipe or age of pipe. However a clear pattern in the age of pipe graph is that most of the repairs took place after 25 years, suggesting this is a critical age at which a regular check of the pipeline may be worthwhile.

28.3.2 Discrete Data: Pipe Diameter, Pipe Location, and Month

The two main exploratory investigations of interest are the influence of pipe diameter and months on the level of repair costs, as well as the effect of month and pipe location on the repair costs. These are shown in Figures 28.3–2 and 28.3–3. A clear pattern in Figure 28.3–2 is that most of the repairs seem to occur at the lower triangle of the graph, suggesting there are relatively few large and expensive pipe repairs. However, there is no discernable pattern to suggest pipe breakdowns tend to happen more often for a particular pipe diameter in a particular month. In Figure 28.3–3 the numbers in the graph represent the age of pipe at time of leak and these graphs suggest that the leakages for each area are fairly evenly spread out throughout the year, with the largest number of leakages in B and F. This is possibly due to the pipes in these areas being generally older compared to others. This result, along with Figure 28.3–1, seems to suggest that older pipes (of age) would need more repairs. However, the cost range for each repair for pipes > 25 years of age is quite similar and the individual repair cost does not appear to increase with the age of pipe.

28.3.3 GLD Fits on the Repair Costs in Relation to Pipe Diameter, Pipe Location, and Month

Figures 28.3–4 and 28.3–5 show the GLD fits for the repair costs for each month as well as for different categories of pipe diameter and pipe location. The actual parameters, method of fitting, and type of distribution used are shown in Table 28.3–6. For pipe diameter, not every single category is examined and some categories are combined if there is very little data and the differences between categories is not great. It is quite possible to fit distributions even when there is relatively less data, for example in 750+ pipe diameter data, there were only five repair costs entries. However the accuracy of the resulting fit as a representation of the long run cost behavior would obviously be compromised. For the pipe locations, only the six most frequent locations for pipe repairs are considered. All fits shown in Figures 28.3-4 and 28.3-5 are very good approximations to the empirical data and Table 28.3–7 shows the first four moments of the empirical data (designated by $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, $\hat{\alpha}_4$) and the moments (designated by α_1, α_2, α_3, α_4) of the resulting GLD fits.

The GLD fits by month in Figure 28.3–4 are grouped based on the similarity of distributional shape. The exceptions are May, June, October, and December, which have quite different GLD fits. It may be possible to find reasons why the repair cost distributions are similar in the months of January and September, for example. Figure 28.3–4 is also particularly useful to understand the range of variability that can be expected for individual repairs in each month. For example, the repair costs in June tends to be less variable and lower than that of May, October, and December. A further investigation as to why this may be the case could lead to a reduction in the variability and level of costs. For example,

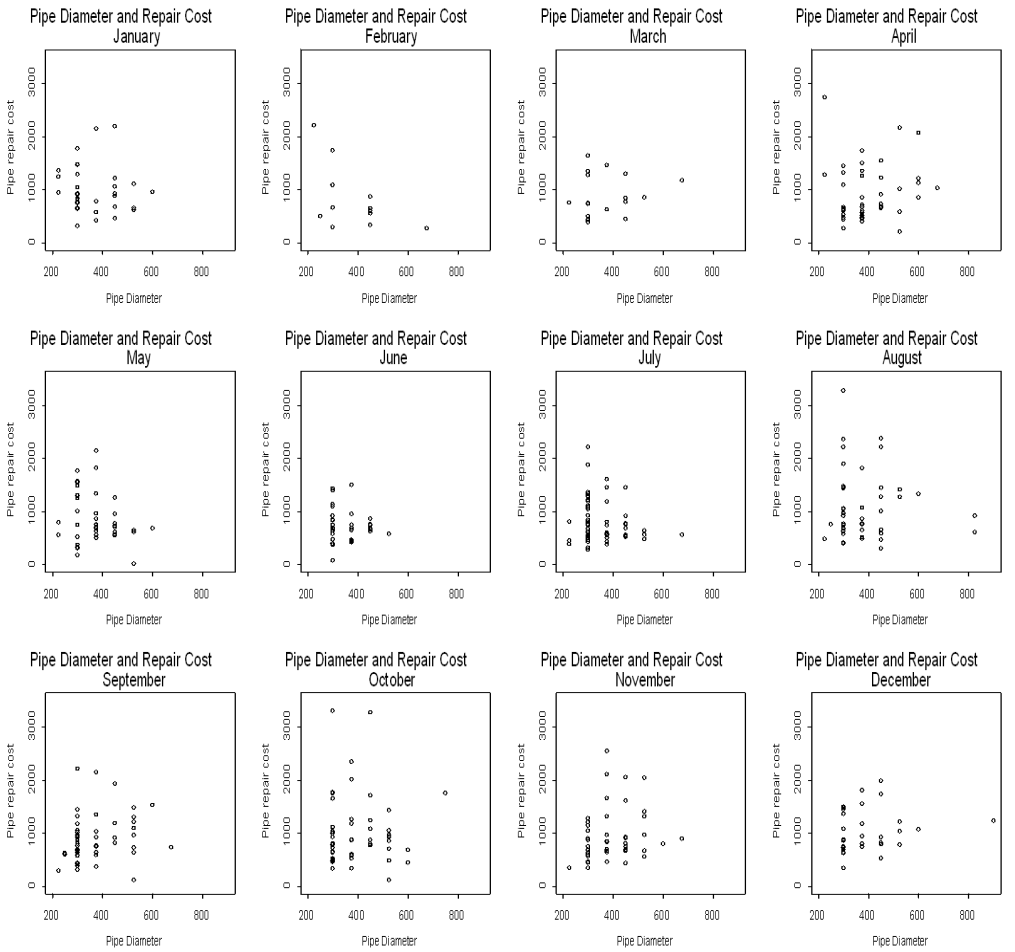


Figure 28.3–2. Graphs showing the relationship between repair costs, pipe diameter, and month.

if it can be found that the only difference in the pipe leak repairs between May and June is the repair method, then the method used in June can be employed to reduce both the variability and level of costs in May and possibly other months as well.

Figure 28.3–5 shows the GLD cost distribution fits by location and pipe diameter. For locations, the cost distributions for regions A, E, and F are somewhat similar. Regions B, C, and D however, appear to have very different cost distributions. The cost distribution for region B appears to have a lower range and mean value of repair costs than all other regions and the management practice in that area may shed some light as to how this can be applied to other regions. It is worth mentioning that the GLD fits to B have nearly twice the variance of

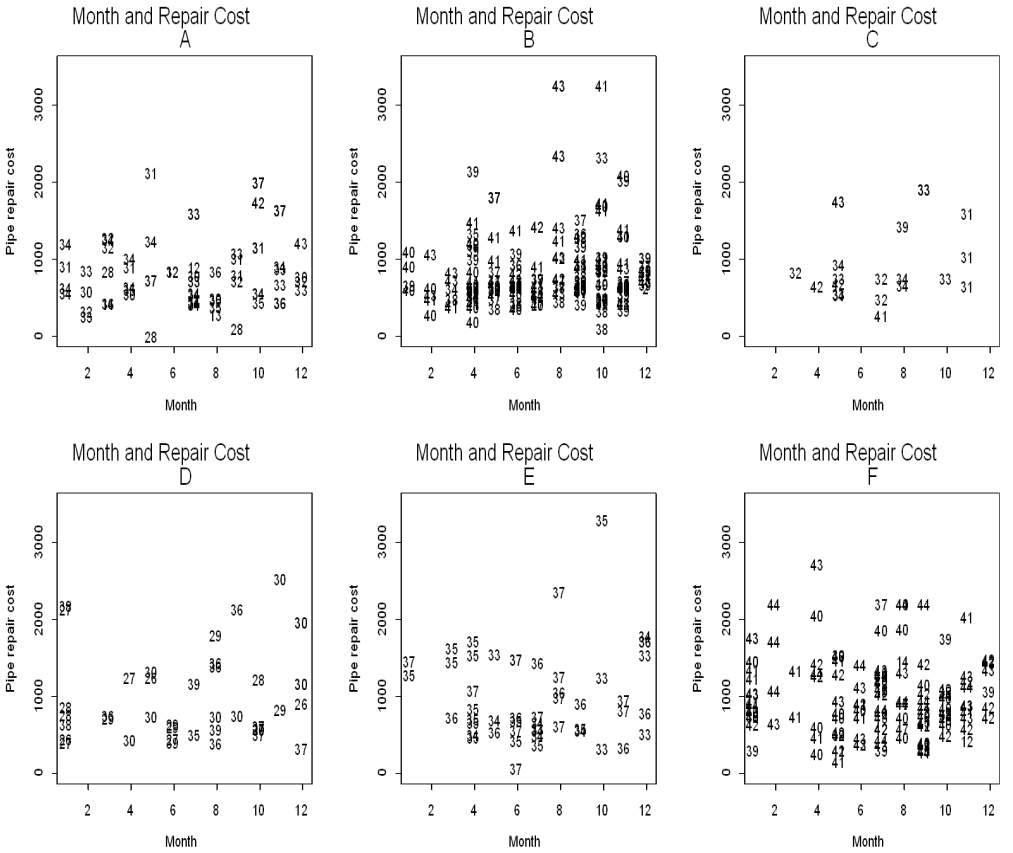


Figure 28.3–3. Graphs showing the relationship between repair costs, pipe repair location, and month; the numbers in the graph represent the age of pipe when it leaks.

the empirical data and this is an important reminder that these GLDs are fitted using maximum likelihood method rather than the moment-matching method, which often acts as a compromise between goodness-of-fit and moment-matching fitting methods.

The other part of Figure 28.3–5 shows the influence of pipe diameter on repair costs. A trend evident in the graph is that as the pipe diameter increases, there is a tendency for the repair cost distribution to shift to the right, indicating it is more costly to fix larger pipes. The distribution fit for 750+ only used five observations, so the credibility of this fit can be somewhat questionable, but already it is showing this is one of the most expensive repairs in Figure 28.3–5. Table 28.3–8 gives the water prices that need to be set to recover the cost of repairing pipeline leakages as well as the actual costs for the years from 2000 to 2004.

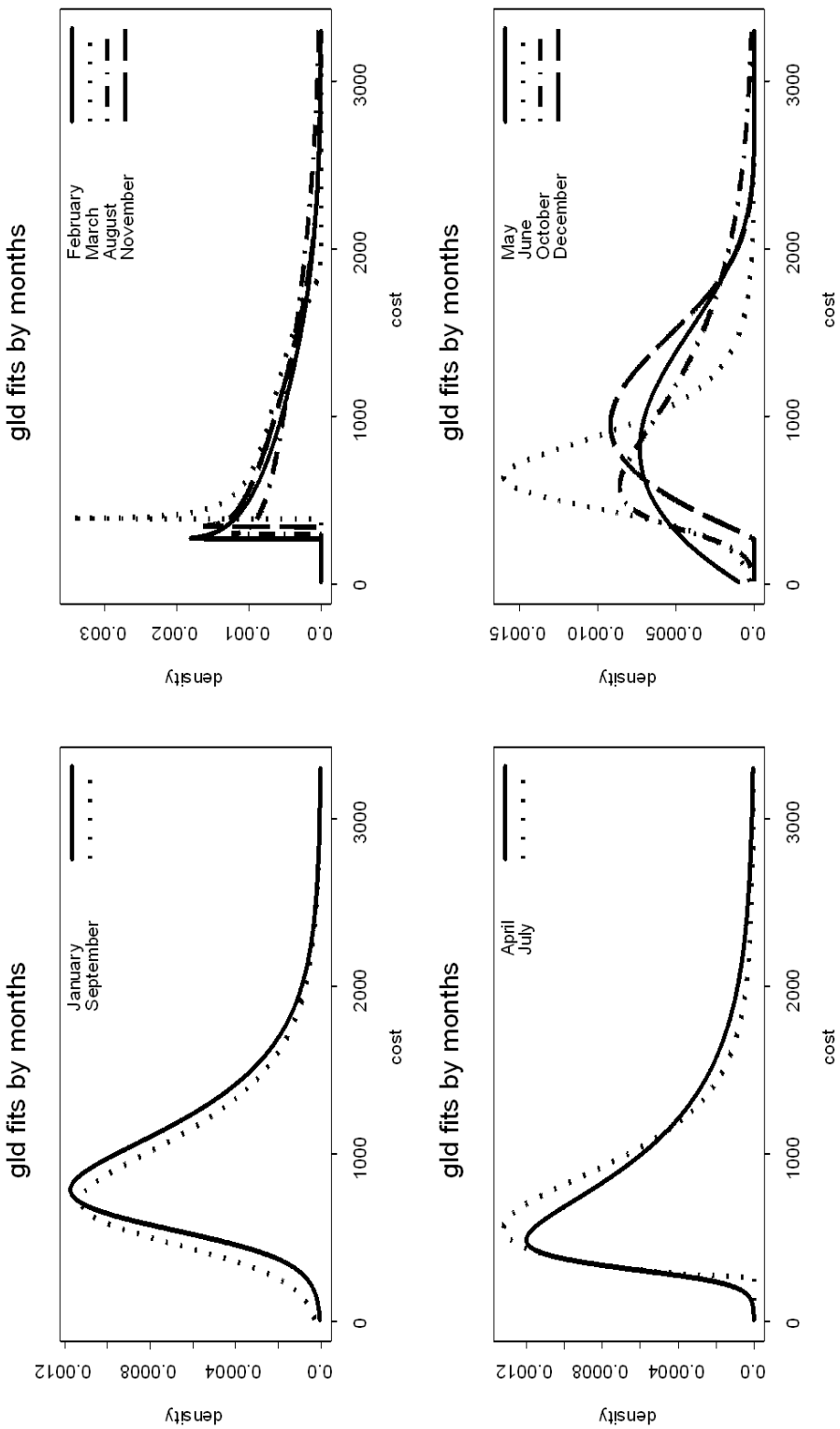


Figure 28.3–4. GLD fits on the repair cost data by month.

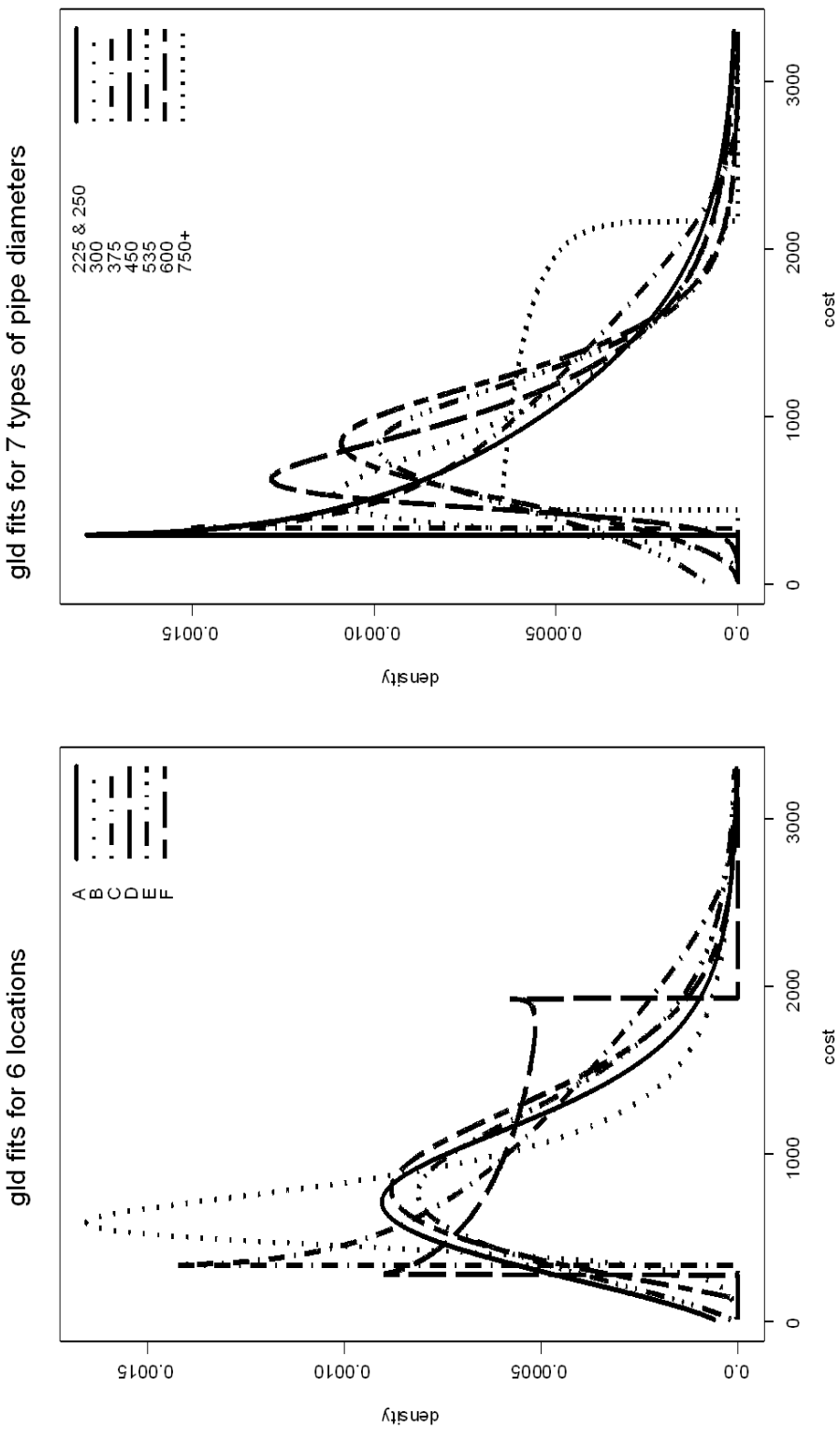


Figure 28.3–5. GLD fits on the repair cost data by locations and pipe diameters.

Table 28.3–6. Parameters of GLD fits for Figure 28.3–4 and Figure 28.3–5.

Month	λ_1	λ_2	λ_3	λ_4	Type	Method of fitting
January	736.958114	0.000000	-0.000008	-0.000032	RS	ML
February	622.035572	0.001856	1.538179	0.109800	FMKL	ML
March	1229.688454	0.001189	1.496845	0.204101	RS	ML
April	421.744830	0.000000	-0.000001	-0.000009	RS	ML
May	891.400677	0.002278	0.463107	0.229644	FMKL	ML
June	616.765022	-0.000192	-0.018850	-0.042173	RS	ML
July	681.703801	0.004208	0.572078	-0.171472	FMKL	ML
August	848.118625	0.001363	1.334032	0.129819	FMKL	ML
September	675.363247	-0.000084	-0.009690	-0.029951	RS	ML
October	479.005691	0.000108	0.006257	0.077604	RS	ML
November	740.632562	0.001979	1.270721	0.138411	FMKL	ML
December	1026.940898	0.002933	0.446491	0.167831	FMKL	ML

Location	λ_1	λ_2	λ_3	λ_4	Type	Method of fitting
A	800.624888	0.003235	0.341137	-0.091306	FMKL	ML
B	566.337598	-0.001257	-0.065757	-0.326125	RS	ML
C	786.228607	0.001444	1.544479	0.287967	FMKL	ML
D	904.476431	0.000907	1.767742	1.078022	FMKL	ML
E	879.824068	0.002858	0.386535	-0.114317	FMKL	ML
F	896.785601	0.002928	0.445704	0.001095	FMKL	ML

Diameter	λ_1	λ_2	λ_3	λ_4	Type	Method of fitting
225 & 250	644.292238	0.001970	1.431690	-0.013092	FMKL	ML
300	440.997635	0.000094	0.004273	0.051952	RS	ML
375	761.102089	0.001569	1.491811	0.289815	FMKL	ML
450	573.617953	-0.000215	-0.010147	-0.092574	RS	ML
535	850.283879	0.003777	0.152752	-0.029614	FMKL	ML
600 & 675	894.946881	0.003990	0.269125	-0.052896	FMKL	ML
750+	1212.078119	0.001311	0.995333	0.799794	FMKL	Starship

Table 28.3–7. Comparing the mean, variance, skewness, and kurtosis (α_1, α_2, α_3, α_4) of GLD fits and their corresponding data for Figures 28.3–4 and 28.3–5.

	Data				Fitted GLD			
Repair Data Category	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	α_1	α_2	α_3	α_4
Jan.	1013.40	194475.96	0.99	3.56	1015.95	196920.82	1.30	**6.28**[1]
Feb.	832.18	307896.32	1.27	3.67	895.23	291573.39	1.20	4.36
March	864.31	132675.42	0.57	2.21	868.10	136440.45	0.56	2.28
April	899.76	261341.80	1.44	5.06	888.60	285466.38	1.70	**7.98**[1]
May	949.38	252025.30	0.58	2.35	948.36	244922.13	0.33	2.51
June	745.85	95354.37	0.75	3.60	746.08	98090.89	1.10	6.34
July	813.09	160601.18	1.22	4.08	817.36	187429.54	2.58	22.24
Aug.	1176.77	538251.06	1.24	3.93	1183.27	540301.84	1.08	3.93
Sept.	921.64	213209.44	1.01	3.72	926.75	213994.73	1.34	6.79
Oct.	1095.83	432779.08	1.35	4.95	1087.14	434846.88	1.40	5.57
Nov.	959.63	255264.62	1.19	3.79	961.94	255222.46	1.04	3.78
Dec.	1085.73	170322.06	0.60	2.51	1083.18	166416.15	0.44	2.76
A	911.18	250615.52	0.79	2.82	910.32	268173.51	1.37	7.48
B	885.91	266081.70	2.20	9.13	895.26	461204.66	64.38	NA
C	1018.09	340894.58	1.00	2.92	1051.72	314264.48	0.79	2.87
D	1039.37	277422.75	0.63	1.87	1036.71	233025.65	0.17	1.80
E	1020.50	349381.57	1.32	5.58	1022.51	358741.71	1.63	9.51
F	1000.57	236095.97	0.90	3.75	1001.71	237259.04	0.98	4.56
225 & 250	880.13	355503.99	1.83	5.79	949.89	382295.55	1.67	7.09
300	925.59	270988.19	1.70	7.38	918.91	279631.74	1.51	6.19
375	980.11	281393.46	0.96	2.84	999.52	269525.64	0.78	2.84
450	1002.91	284619.78	1.56	5.60	1000.33	308616.11	2.42	14.33
535	893.76	197833.03	0.53	3.74	893.45	197539.66	0.62	4.41
600 & 675	965.87	165797.62	0.80	3.92	962.10	167219.32	0.99	5.39
750+	1249.65	206977.53	-0.13	1.48	1253.61	233855.14	0.10	1.85

[1]The numbers in bold (6.28 and 7.98) are estimated from 5 million simulations, since there is a calculation under/over-flow problem in calculating the theoretical moments. Depending on the computer software used and the numerical precision deployed, some differences in these results may be observed.

Table 28.3–8. Setting water prices to recover cost of pipe leakage repairs in the long run (units are $ for all columns except month (Mo.)).

Mo.	Conservative water prices to be set	Water prices to be set	2000 Actual results	2001 Actual results	2002 Actual results	2003 Actual results	2004 Actual results	Mean cost over five years
1	12,996.52	7,221.78	1,449.41	15,968.18	3,389.53	12,047.49	4,641.18	7,499.16
2	5,912.55	2,703.17	595.05	—	605.36	1,802.76	7,815.20	2,163.67
3	6,208.80	3,694.64	—	5,181.74	5,876.67	2,930.90	3,296.81	3,457.22
4	19,394.18	8,659.16	1,838.32	8,077.53	14,793.51	12,597.79	8,580.77	9,177.58
5	18,198.04	10,544.15	8,077.96	6,912.45	5,201.84	6,322.77	22,852.55	9,873.51
6	10,507.32	6,173.14	860.46	5,059.25	6,404.69	10,747.10	8,254.29	6,265.16
7	22,402.65	11,371.09	13,759.42	9,226.37	11,034.94	11,675.38	9,594.12	11,058.05
8	23,699.55	10,892.78	3,316.81	8,078.53	882.25	8,957.59	30,542.77	10,355.59
9	19,671.84	10,426.46	3,614.57	11,545.04	9,479.73	12,253.96	15,639.90	10,506.64
10	28,719.21	13,006.88	8,510.64	9,367.81	13,272.31	13,352.53	21,246.33	13,149.92
11	19,545.12	9,855.69	1,168.34	7,531.97	10,978.09	8,593.71	21,628.63	9,980.15
12	12,686.41	8,125.41	7,385.01	1,073.51	8,792.79	8,871.73	12,963.11	7,817.23
Tot.	**199,942.**	**102,674.**	**48,531.**	**72,054.**	**90,711.**	**110,153.**	**167,055.**	**101,303.**

Note: Last row (Tot.) truncated at the decimal point due to space limitations.

28.3.4 GLD Fits to Repair Costs on Pipelines with Diameter 300 in LOCA and LOCB Regions

It is often important to explore the effect of joint factors on the distribution of costs. In the dataset provided by the water utility company, there were very few data to allow meaningful GLD fits on the distribution of costs for location by each month. The only reliable joint factors analysis with more than 50 data points are pipelines with diameter 300 in LOCA and LOCB regions. The resulting fits use the FMKL GLD with parameters (600.269459730, 0.002475292, 1.297470746, -0.078343372) and (925.496939042, 0.002769834, 0.449719726, 0.131730870) for LOCA and LOCB regions, respectively. These results and goodness-of-fit comparisons are shown in Figure 28.3–9 and Table 28.3–10. In this case, use of the maximum likelihood method with FMKL GLD provides a very convincing fit to the data. Interestingly, while the mean repair cost for LOCA 300 pipes is lower than LOCB 300 pipes, the repair costs are more variable. The two distributions are also quite different in their shapes, suggesting the fundamental nature of repairs or the way repairs are conducted may be quite different between the two regions. Figure 28.3–11 shows the data histograms with the superimposed fitted density functions for LOCA and LOCB. It also gives quantile plots for these fits.

28.3.5 Remarks

Traditional statistical tests such as t-tests and f-tests or even non-parametric methods can only give inferences based on the population mean or median. Furthermore, even if a difference was found using these statistical tests, they do not tell the user how the datasets are different. With GLD it is possible to make statements about the actual distribution of costs, see how they are different graphically, without having to make any statement about a single point estimate. This is an important consideration as it is often the case where a business would like to know the probability that the actual cost will exceed $5,000 rather than the probability their average cost will exceed $5,000.

28.4 Rationally Setting Water Prices to Break Even: A Case of Pipe Repair Costs

28.4.1 Global Pipe Repair Cost Example

This section illustrates the use of GLD to set monthly prices at a level necessary to breakeven the pipe repair costs in the long term. The basic strategy is simple, it involves choosing a price from the distribution fits, along with number of repairs per month that would recover the long-term repair costs without making an excessive profit or loss.

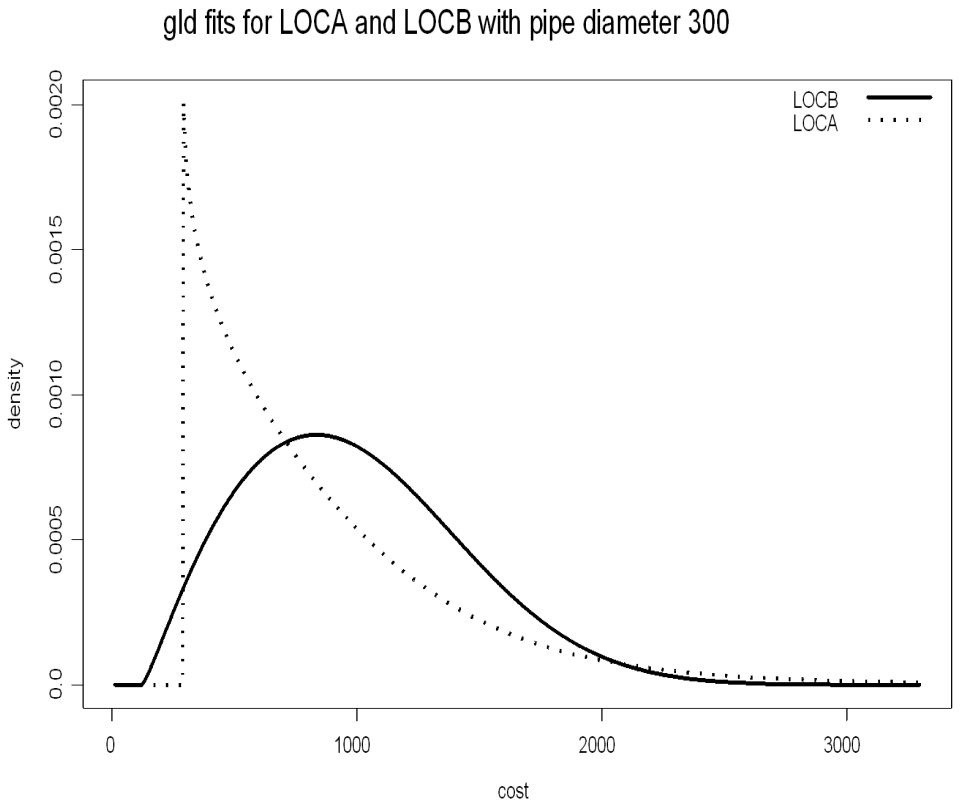


Figure 28.3–9. GLD fits on the joint factors, pipe diameter by region.

Table 28.3–10. Comparing the mean, variance, skewness, and kurtosis of GLD fits and its corresponding data for Figure 28.3–9.

	LOCB 300		LOCA 300	
	Data	RMFMKL ML	Data	RMFMKL ML
Mean	989.000	995.000	849.078	862.760
Variance	199000.000	199000.000	317965.909	312272.226
Skewness	0.584	0.537	2.567	2.044
Kurtosis	3.000	2.970	10.365	10.429

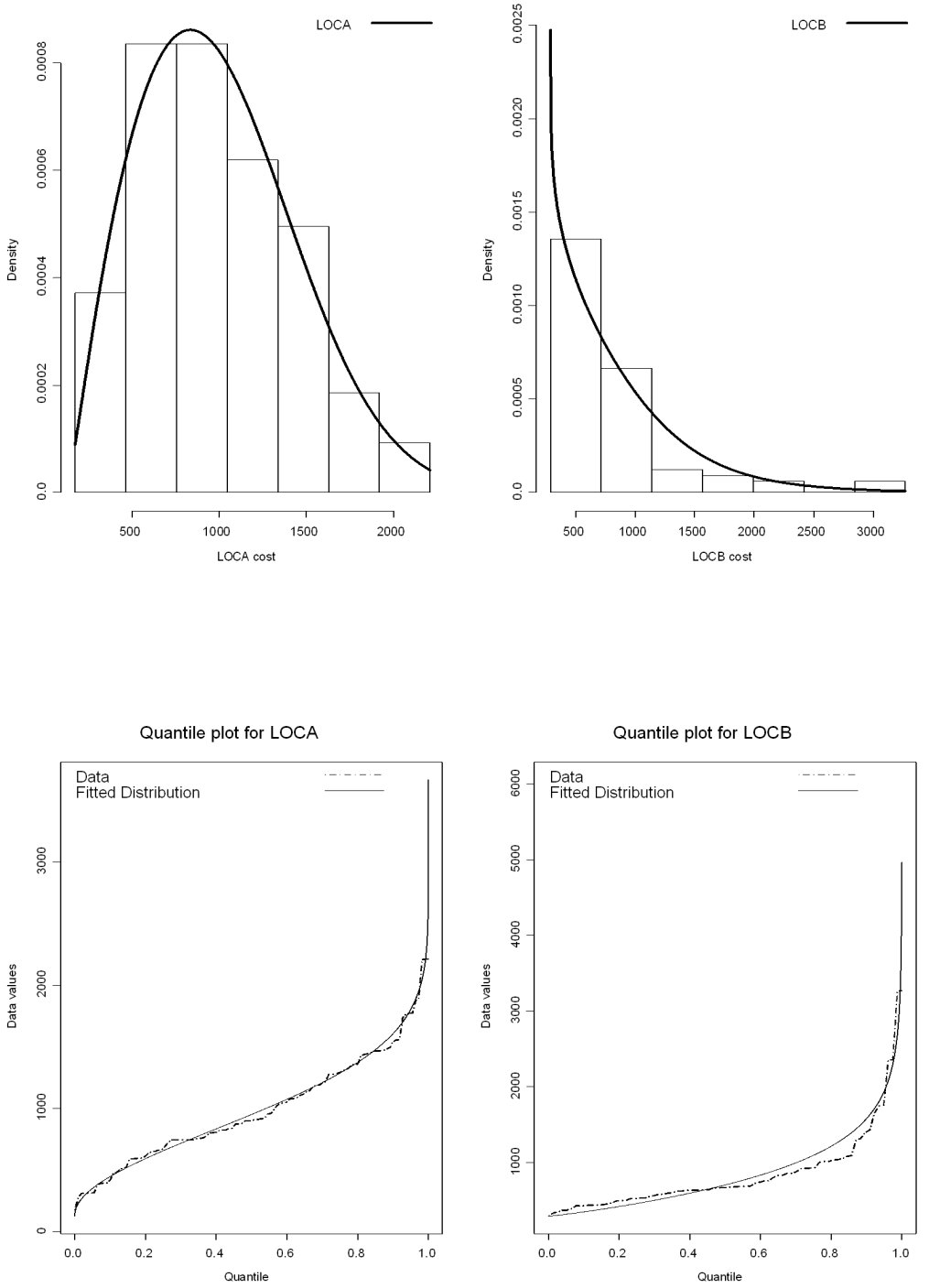


Figure 28.3–11. LOCA and LOCB GLD fits on the datasets.

Table 28.4–1 Mean monthly number of leakages.

January	February	March	April
7	3	4	10
May	June	July	August
10	8	14	9
September	October	November	December
11	12	10	7

The average number of repairs per month during 5 January 2000 to 16 December 2004 are shown in Table 28.4–1. A reasonable strategy is to model these repairs as Poisson random variables, without any further knowledge regarding the occurrence of these pipe leakages. It is also assumed that pipe leakages are fixed as soon as these are identified within the month and that there are no cross-over works to fix pipe leakages occurring in another month. The monthly distributions of repair costs are also assumed to be stable with negligible inflation effects.

Under these assumptions, it is possible now to generate randomly the total monthly costs. The number of leakages in each month is governed by the Poisson distribution with mean specified in Table 28.4–1 and this is used to generate the number of monthly costs from the relevant GLD. This will simulate the monthly costs for repairing the pipe leakages.

For simplicity, costs at a specified quantile are chosen from the monthly fitted GLD. These figures are then multiplied by the corresponding mean monthly number of leakages to give the monthly repair figures. To check that the organization breaks even in the long term, these monthly budgeted figures are subtracted from the simulated monthly costs over 10,000 simulation runs and the average value is reported. It is also possible to choose the required quantile by an optimization scheme; however this is outside the scope of this chapter.

In this chapter, a reasonable level of costs was found to be at the 60% quantile of the monthly GLD fits, giving an average of approximately $-\$1,370.43$, or a gain of \$1370.43. An analysis using the theoretical mean of GLD will give on average a loss of approximately $\$1,161.93$. In the interest of ensuring the long run sustainability of the water utility operations it is prudent to choose the option with a slight gain rather than loss. It is also easier to distribute the gain by way of refund rather than ask the customers to pay excess fees in order to breakeven.

Using this simple strategy, the set of water prices required for the water utility to recover its water pipe leakage repair costs are in Table 28.3–8. The total annual amount to be charged is \$102,674.32.

This analysis does not account for the fact that the total costs for the pipeline repairs are increasing each year with a very dramatic increase in 2004. If the cur-

rent trend continues, then it will be necessary to remodel with a higher coverage, perhaps at 95% of the monthly GLD fits. These set of prices are listed as Conservative water prices to be set in Table 28.3–8. The "Conservative water prices to be set" can be justified on the grounds that these are the prices necessary to recover 95% of the monthly repair costs and it is in keeping with the increasing costs of pipeline repairs.

28.4.2 Individual Pipe Repair Cost Example

It is often desirable to give confidence intervals on the cost estimation for the actual pipeline repairs. This section models the distribution of the yearly individual pipe repair costs. To ensure sufficient data for statistical analysis, only pipelines aged between 26–44 years with diameter "225," "300," "375," "450," "525," or "600" can be considered. For illustrative purposes, the annual cost distributions for the following individual pipelines are shown.

- B-M14-P001
- B-MC-P026
- B-M04_01_ORG-P002
- E-WB11-P001

To model the distribution of the yearly cost of each individual pipe it is necessary to know the factors that influence the number and cost of failures. Figure 28.3–5 shows that pipe diameter appears to influence the cost per repair for individual pipes. It can be demonstrated in Figure 28.4–2 that the number of failures is possibly influenced by the pipe age and there is no relationship between pipe length and the number of failures.

The above analyses suggest a reasonable strategy to find the annual repair cost for a particular pipe type is to model this with respect to their age and pipe diameter. This strategy assumes that both age and pipe diameters provide sufficient information with regard to repair cost for a specific pipe type. The steps of modeling are as follows:

1. Tabulate number of failures by age. For each age, evaluate the failure rate by dividing the number of failures over the total number of operational pipes. This is the rate to be used in the Poisson distribution model to determine the number of failures at a particular age.

2. Fit generalized lambda distributions using maximum likelihood estimation for repair costs by different pipe diameters. In this analysis, all the GLD distribution fits have very convincing KS goodness-of-fit results and are again chosen on the basis of the closest match to the first four moments of the data. This is shown in Figures 28.4–3 and 28.4–4.

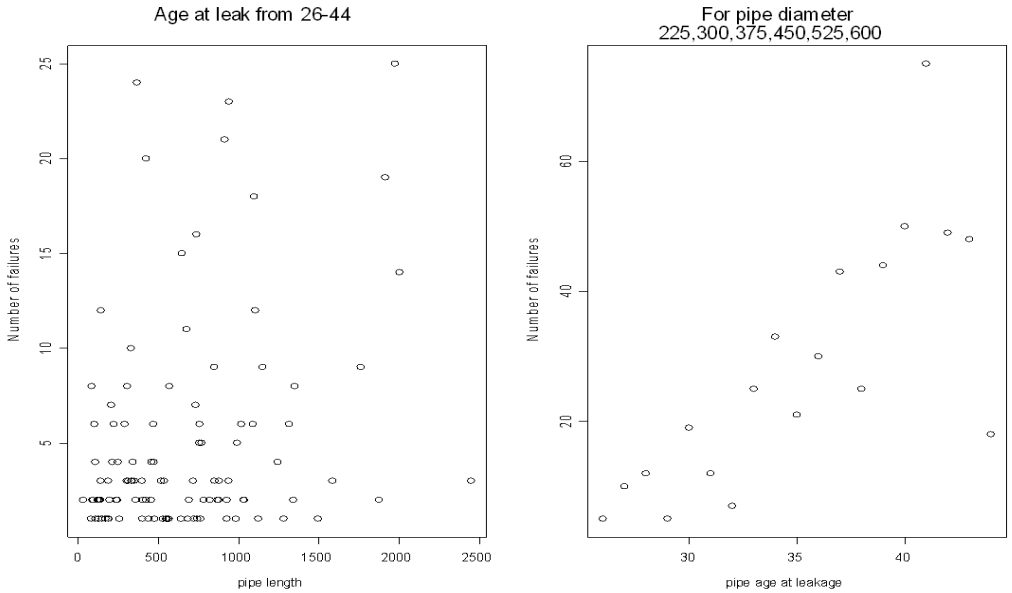


Figure 28.4–2. The relationship between pipe age, pipe length, and number of failures.

3. The total number of failures (f) is sampled from step 1. Consequently, a sample of f costs is taken from the distribution in step 2. The sum of these costs determines a single simulation result of total yearly repair cost for that pipe.

4. Repeat step 3 a thousand times to get a distribution of the total yearly cost of repairs for a particular pipe. The fitted distribution (again using maximum likelihood estimation) will be chosen on the basis of best approximation to the first four moments of the data.

The resulting cost distributions for four pipes are shown in Figures 28.4–5 through 28.4–12. From the fitted distributions it is possible to derive the 95% confidence interval for the actual cost under each of the different pipe configurations. This is given in Table 28.4–13.

This technique can also facilitate cost forecasting, as an example, the forecast for E-WB11-P001 beyond 37 years is given in Table 28.4–14.

Note that these costs will only incur if there is a leakage. A full interpretation of these results must be done in conjunction with probability of failure. The derived probability of failure in this analysis used the Poisson distribution and its corresponding empirical probability of failure is given in Table 28.4–15.

As an example, to predict the failure of annual E-WB11-P001 repair cost at age 38, there is about 80% chance that we would not incur any leakage cost. In the unfortunate event of a leakage, with 95% confidence, the actual cost will range from $261 to $3,700.

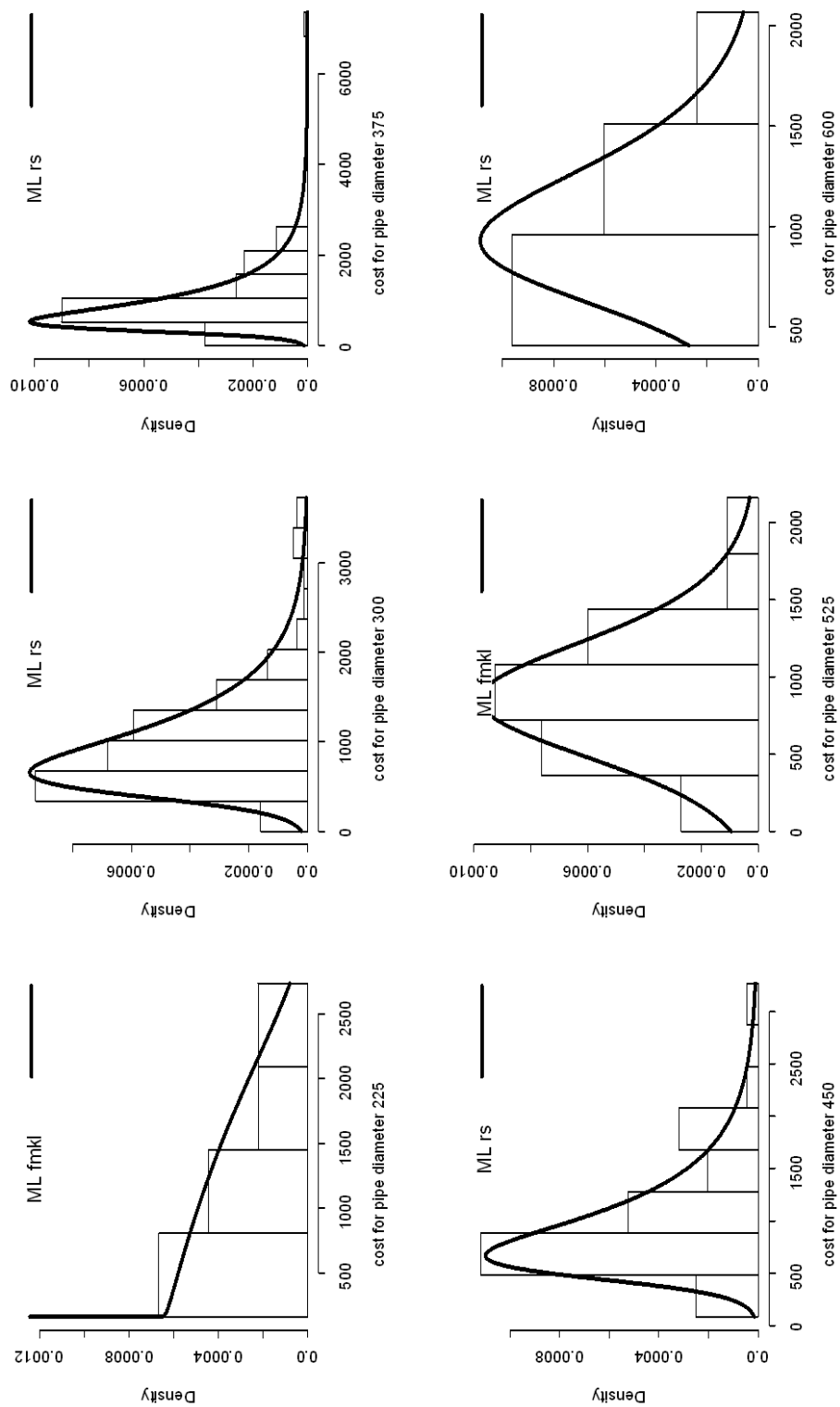


Figure 28.4–3. The cost density functions of a single repair by pipe diameter.

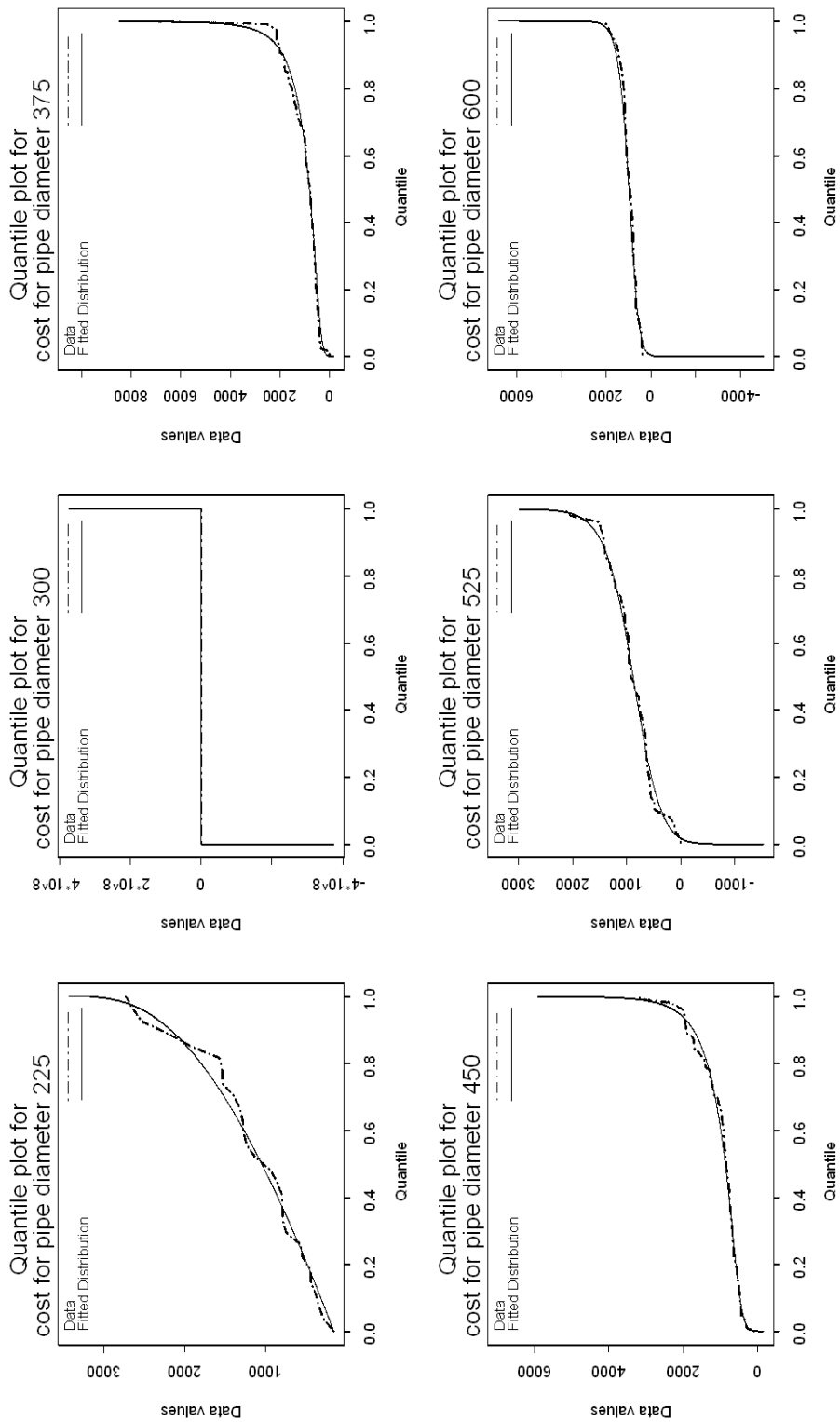


Figure 28.4–4. The cost distributions of a single repair by pipe diameter.

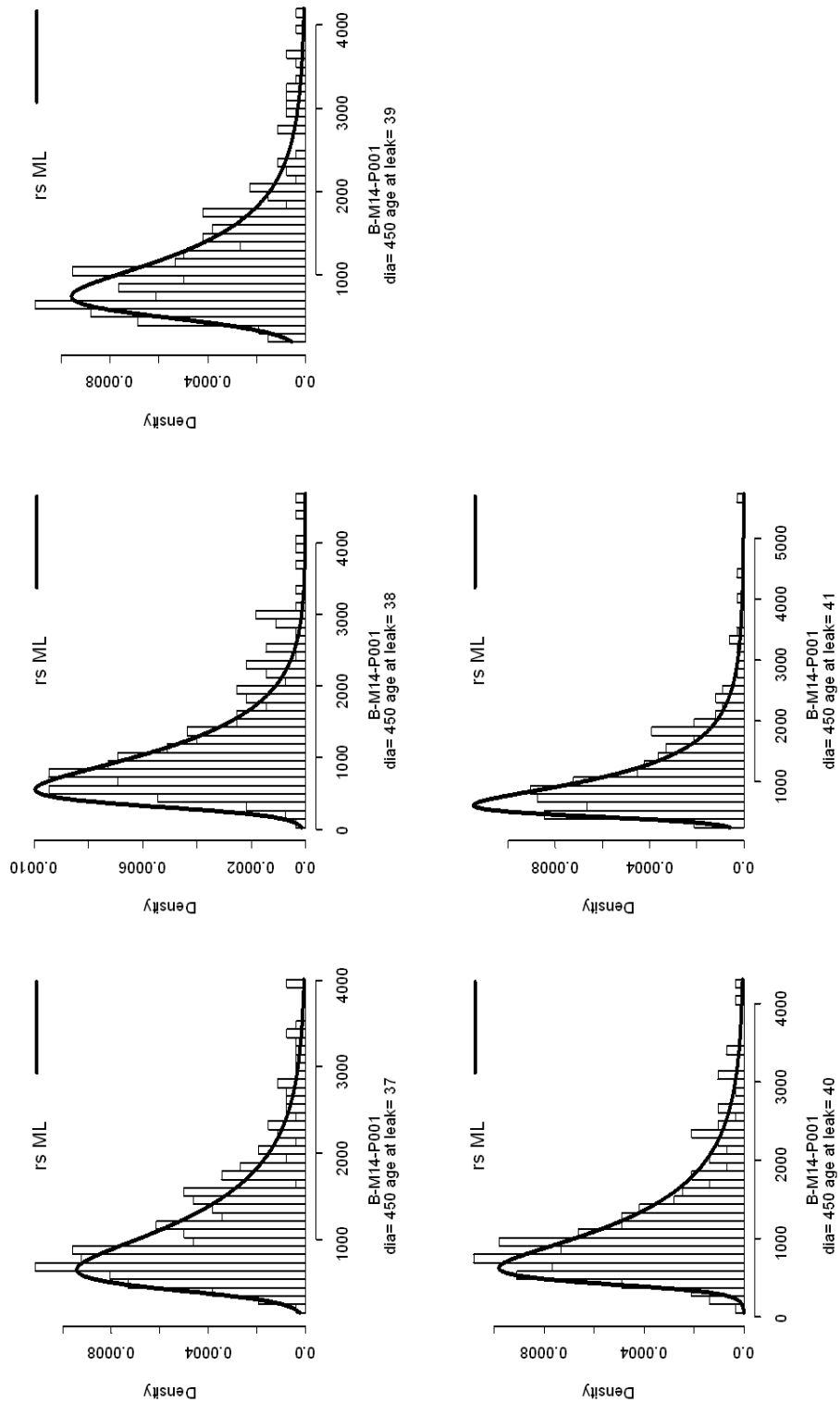


Figure 28.4–5. Density functions of yearly cost for B-M14-P001.

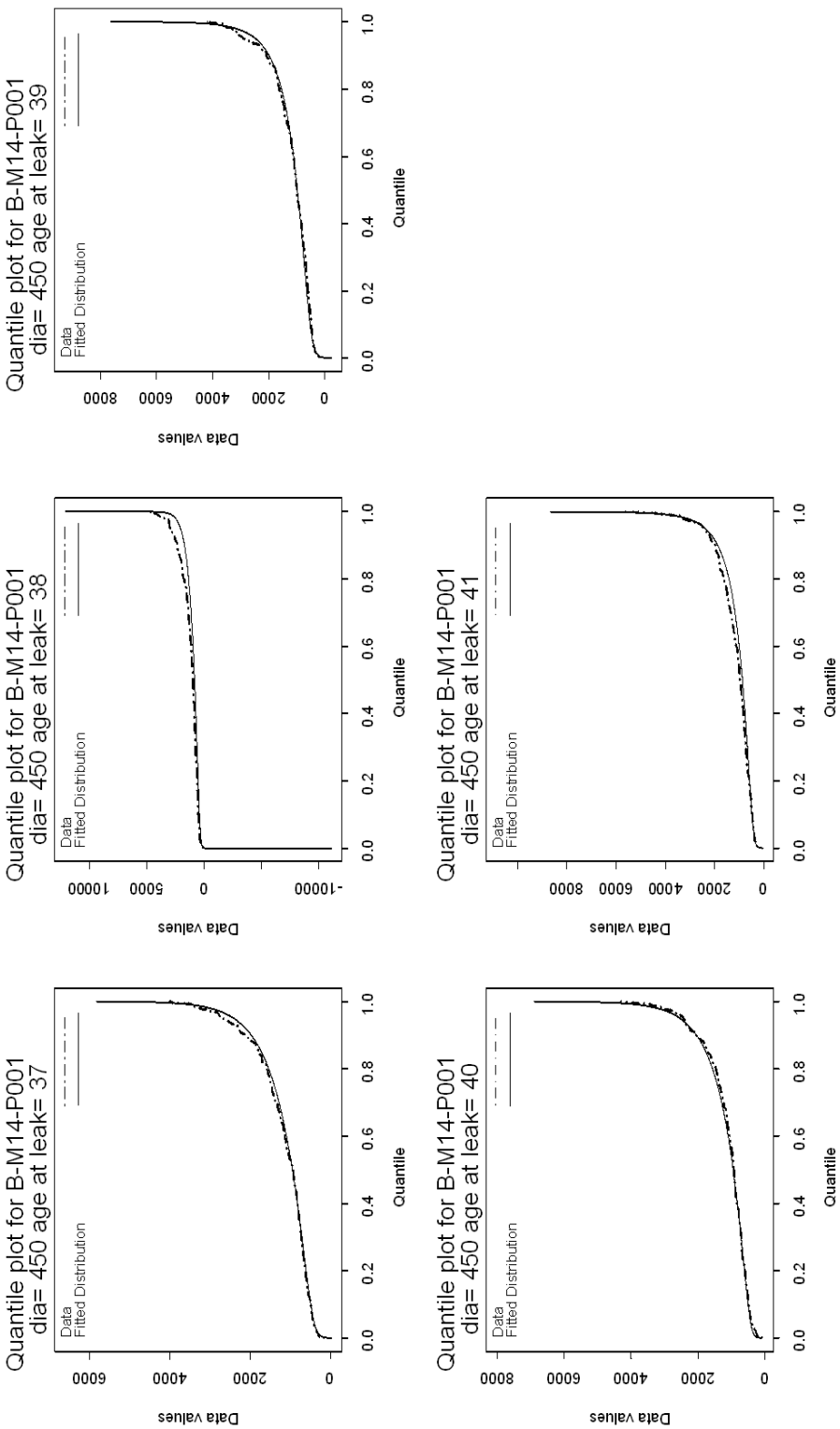


Figure 28.4–6. Yearly distribution of cost for B-M14-P001.

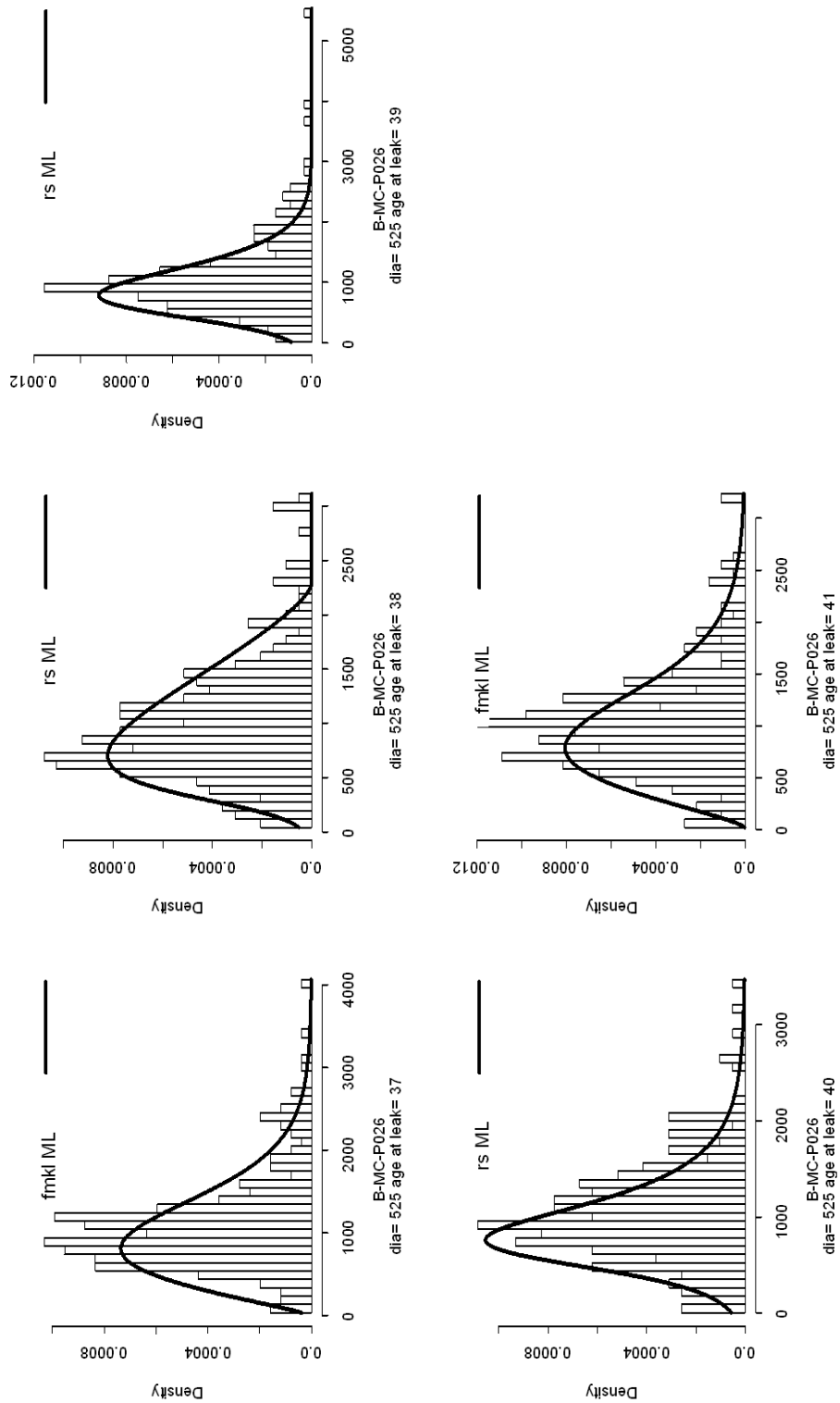


Figure 28.4–7. Density functions for yearly cost for B-MC-P026.

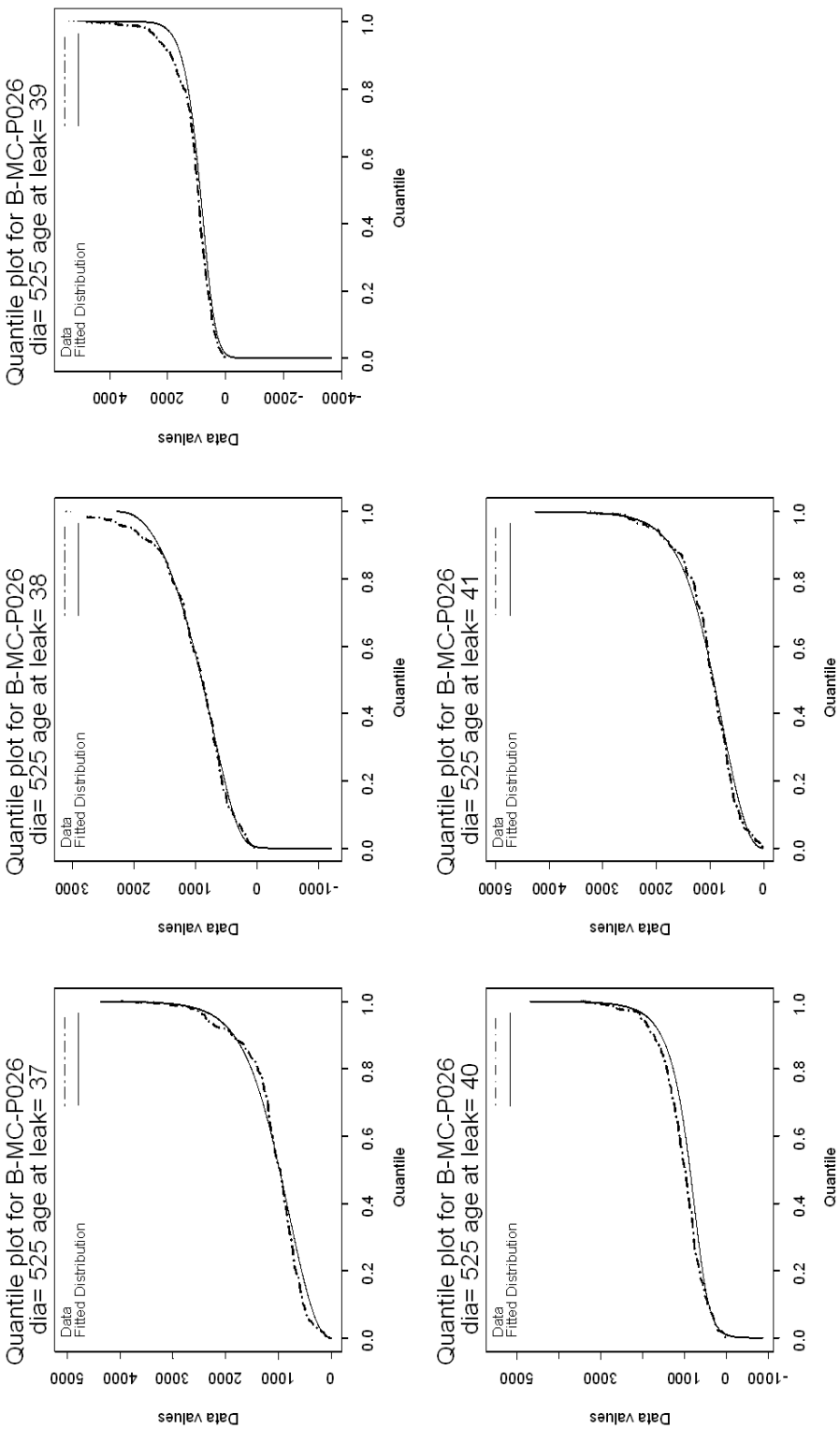


Figure 28.4–8. Yearly distribution of cost for B-MC-P026.

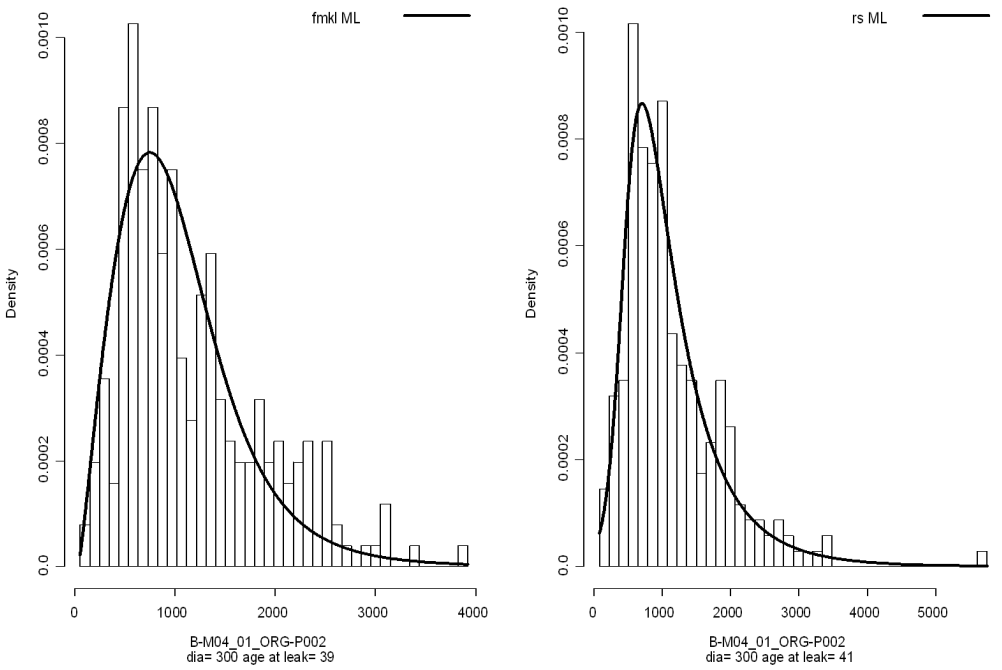


Figure 28.4–9. Density functions for yearly cost for B-M04_01_ORG-P002.

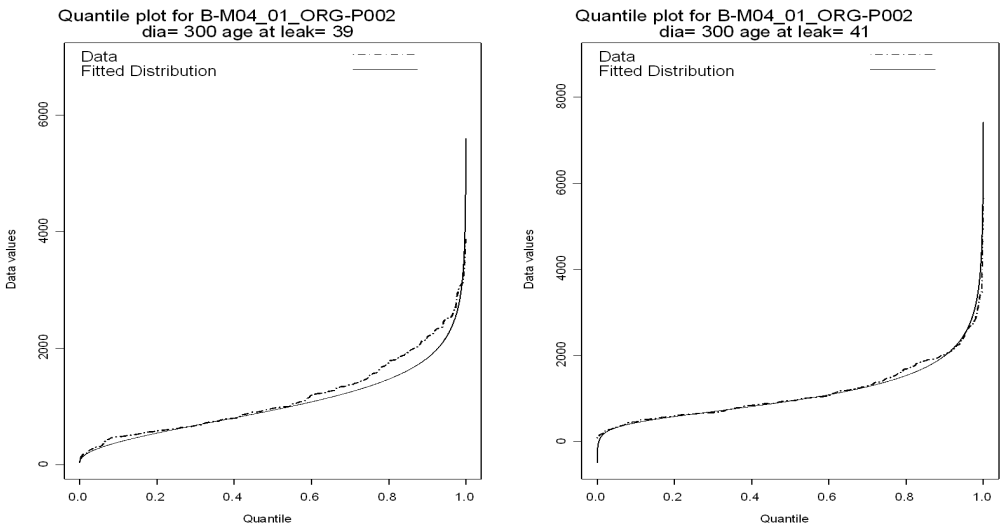


Figure 28.4–10. Yearly distribution of cost for B-M04_01_ORG-P002.

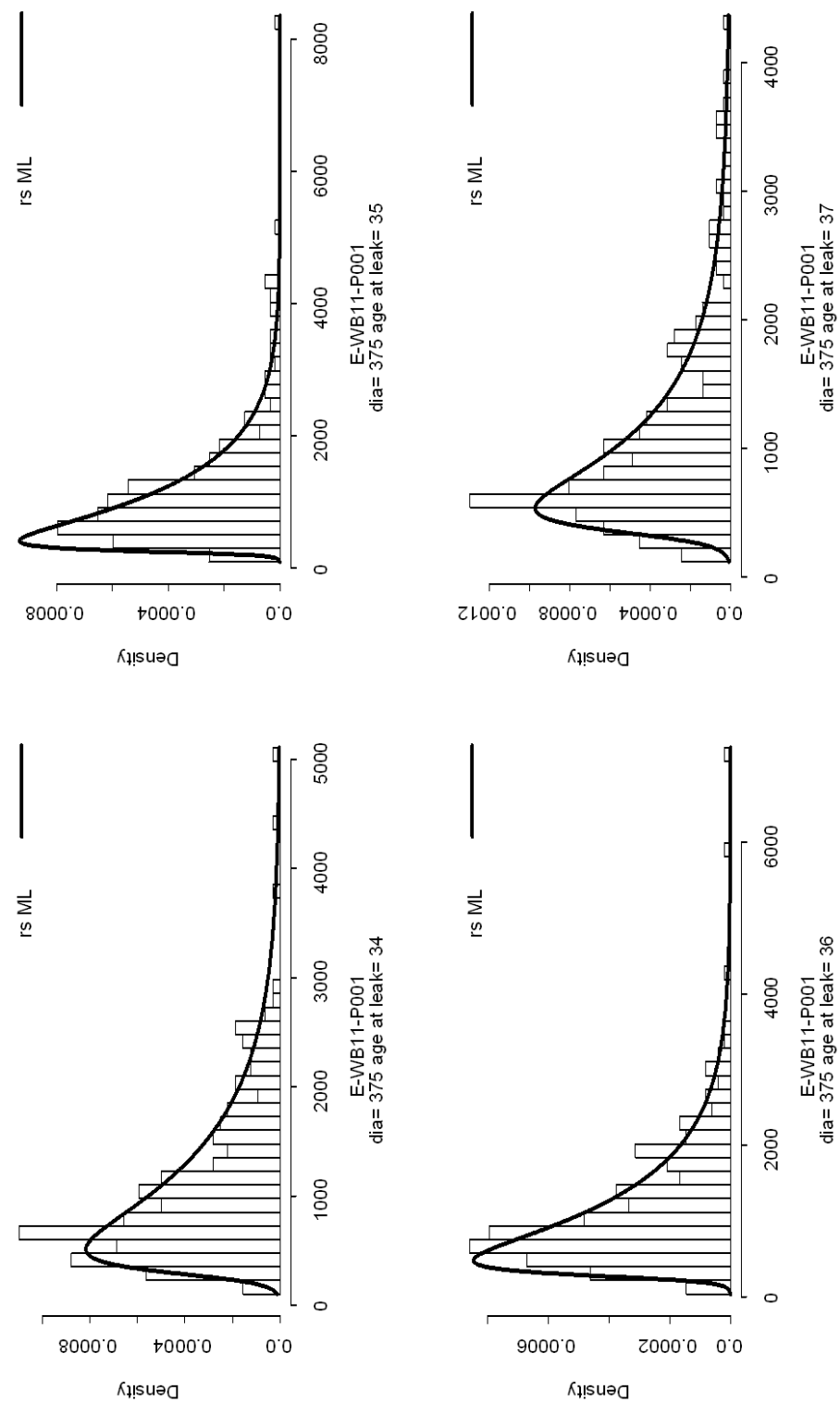


Figure 28.4–11. Density functions for yearly cost for E-WB11-P001.

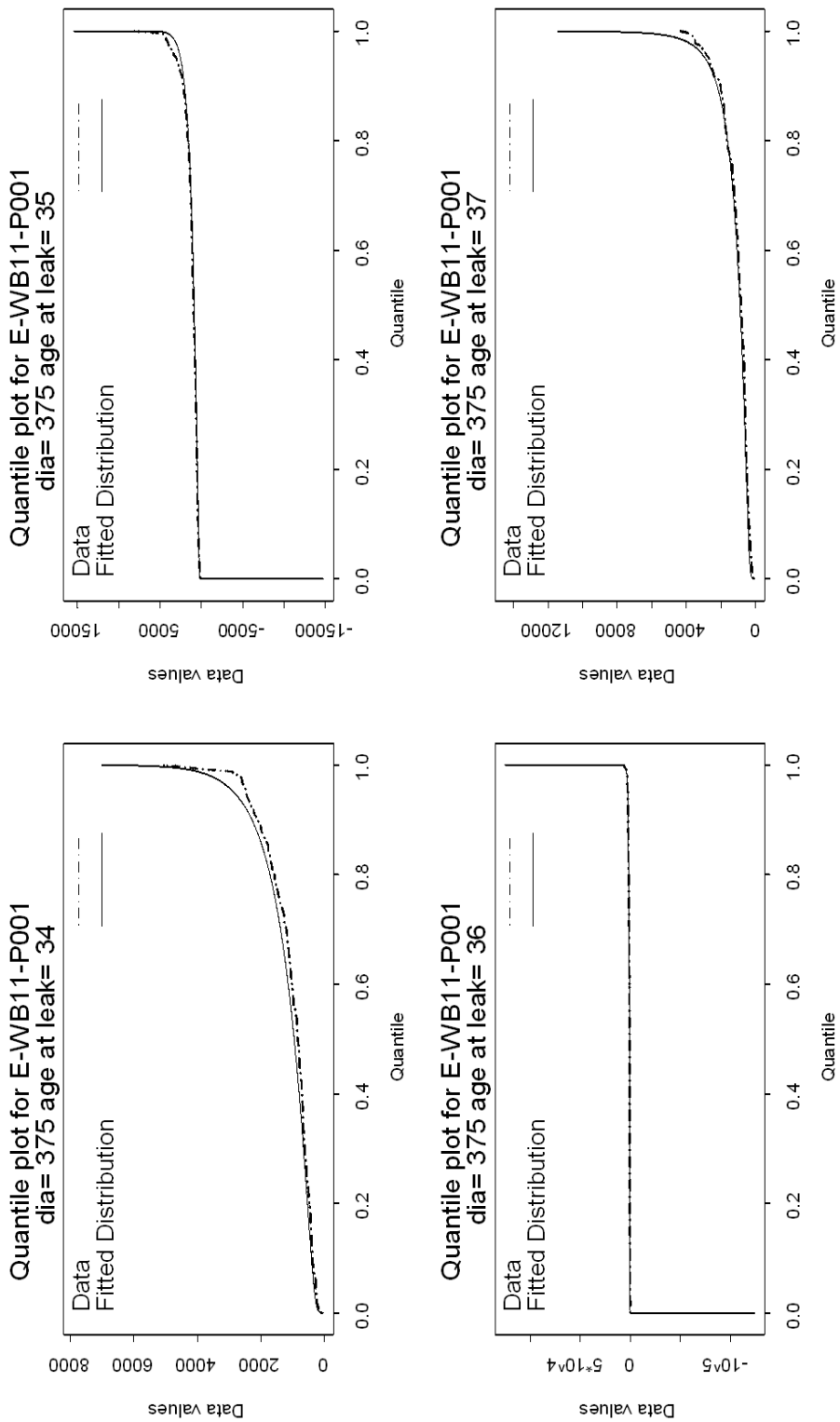


Figure 28.4–12. Yearly distribution of cost for E-WB11-P001.

Table 28.4–13. 95% confidence interval for annual pipe repair cost.

Pipe	Pipe Diameter	Pipe Age	Estimated Lower Bound	Estimated Upper Bound
B-M14-P001	450	37	$349.09	$2,857.26
B-M14-P001	450	38	$249.51	$2,379.95
B-M14-P001	450	39	$347.53	$2,994.13
B-M14-P001	450	40	$380.83	$3,056.79
B-M14-P001	450	41	$374.87	$2,959.28
B-MC-P026	525	37	$199.71	$2,475.67
B-MC-P026	525	38	$203.67	$1,901.54
B-MC-P026	525	39	$104.22	$1,983.54
B-MC-P026	525	40	$163.36	$2,168.10
B-MC-P026	525	41	$218.58	$2,340.47
B-M04_01_ORG-P002	300	39	$213.69	$2,589.86
B-M04_01_ORG-P002	300	41	$230.64	$3,005.59
E-WB11-P001	375	34	$281.34	$3,409.98
E-WB11-P001	375	35	$264.75	$2,977.74
E-WB11-P001	375	36	$276.70	$3,332.77
E-WB11-P001	375	37	$321.63	$3,677.75

28.4.3 Other Considerations

The above analysis does not consider administration costs and salaries that are necessary for the running of the water utility organization. Also it excludes considerations such as inflation. However these can be easily incorporated once a base figure has been derived. As more data become available over time, it will be possible to derive more refined water prices, for example, a set of different water prices for different customers in different regions.

Table 28.4–14. 95% confidence interval forecast for annual E-WB11-P001 repair cost.

Fitting Method	Age	Lower Bound	Upper Bound
RS ML	38	$282.71	$3,514.47
FMKL ML	38	$262.22	$3,697.95
FMKL STAR	38	$261.31	$3,557.26
RS ML	39	$249.31	$3,113.42
FMKL ML	39	$191.67	$3,054.41
FMKL STAR	39	$215.28	$3,152.06
RS ML	40	$215.10	$2,572.93
FMKL ML	40	$176.43	$2,589.64
FMKL STAR	40	$189.68	$3,114.51
RS ML	41	$259.72	$3,799.14
FMKL ML	41	$228.90	$3,879.88
FMKL STAR	41	$221.39	$3,569.67
RS ML	42	$177.46	$3,548.71
FMKL ML	42	$169.83	$3,836.25
FMKL STAR	42	$185.52	$5,172.59
RS ML	43	$258.52	$2,734.90
FMKL ML	43	$225.26	$2,740.38
FMKL STAR	43	$223.46	$3,060.10
RS ML	44	$221.42	$2,084.86
FMKL ML	44	$199.52	$2,185.55
FMKL STAR	44	$197.66	$2,325.58

Table 28.4–15. Probability of no pipeline leakage.

Age at Leak	Probability of No Leakage	Empirical Probability of No Leakage	Age at Leak	Probability of No Leakage	Empirical Probability of No Leakage
1	NA	1	25	0.84	0.88
2	0.78	0.75	26	0.82	0.87
3	NA	1	27	0.8	0.89
4	0.89	0.89	28	0.76	0.81
5	0.75	0.86	29	0.83	0.89
6	0.75	0.71	30	0.73	0.8
7	0.65	0.86	31	0.78	0.87
8	0.87	0.86	32	0.75	0.83
9	NA	1	33	0.75	0.83
10	NA	1	34	0.76	0.84
11	0.81	0.79	35	0.76	0.83
12	0.81	0.86	36	0.75	0.84
13	0.93	0.93	37	0.66	0.82
14	0.93	0.93	38	0.8	0.87
15	NA	1	39	0.75	0.87
16	0.85	0.83	40	0.73	0.86
17	NA	1	41	0.63	0.8
18	0.92	0.91	42	0.66	0.84
19	0.8	0.81	43	0.64	0.8
20	0.89	0.91	44	0.64	0.74
21	0.89	0.91	45	0.37	0
22	0.72	0.72	46	NA	1
23	0.81	0.81	47	0.14	0
24	0.85	0.85			

28.5 Further Considerations in Choosing the GLD for Analysis

This chapter examines the quality of distribution fit through the use of resample KS tests, histograms and qq plots, provided that the fit is reasonably adequate, the generalized lambda distribution that matches most closely to the first four moments of the empirical data is used for the analysis. The main reason for this approach is to allow consistency in analysis between GLD and other traditional analysis using the moments of the data and simulation studies using data directly. Essentially, the choice of which GLD and which fitting method should be employed is dependent on the research question of interest. For example, when it is desirable to find the best overall data set fit, perhaps resample KS tests and qq plots should primarily be used to make these decisions. Currently, there is no universally accepted way of determining the overall goodness-of-fit for GLD and this is an important area for future research for GLD researchers.

28.6 Conclusion

This chapter demonstrates how the flexible use of generalized lambda distributions can provide information to assist an organization in decision making. This provides a significant improvement over the traditional statistical analysis since a direct statistical distribution can be fitted to the actual data. Consequently, it is no longer necessary to interpret the mean of the data as it is now possible to make a direct inference on the actual data. Overall, this chapter provides a valid alternative to traditional statistical analysis; it is, however, much more intuitive, direct, and easy to understand and it is customized for individual dataset, requiring neither transformations nor assumptions of normality.

Acknowledgments

The author is grateful to Steve Pudney, Yong Sun, and Lin Ma for their comments and the CIEAM for providing the dataset, as well as to Roma Keswani and Lillias Nairn for their administrative assistance.

References for Chapter 28

Falta, M. and Wolff, R. (2004). "Recent developments of statistical approaches in aspects of accounting: A review," *International Statistical Review* 72(3), 377–396.

Freimer, M., Kolia, G., Mudholkar, G. S., and Lin, C.T. (1988). "A study of the generalised Tukey lambda family," *Communications in Statistics–Theory and Methods*, 17, 3547–3567.

Hastings, J. C., Mosteller, F., Tukey, J. W., and Watson, C. P. (1947). "Low moments for small samples: A comparative study of order statistics," *The Annals of Statistics,* 18, 413–426.

Hogben, D. (1963). *Some Properties of Tukey's Test for Non-Additivity,* Ph.D Thesis, Rutgers – The State University of New Jersey.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalised Bootstrap Methods,* New York, Chapman and Hall.

Karian, Z. A., Dudewicz, E. J., and McDonald, P. (1996). "The extended generalized lambda distribution systems for fitting distributions to data: History, completion of theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics–Computation and Simulation,* 25(3), 611–642.

King, R. and MacGillivray, H. (1999). "A starship estimation method for the generalised lambda distributions," *Australia and New Zealand Journal of Statistics,* 41(3), 353–374.

Lakhany, A. and Massuer, H. (2000). "Estimating the parameters of the generalised lambda distribution," *Algo Research Quarterly*, December 2000, Vol. 3, 47–58.

Okur, M. (1988). "On fitting the generalised lambda distribution to air pollution data," *Atmospheric Environment,* 22, 2569–2572.

Öztürk, A. and Dale, R. (1985). "Least squares estimation of the parameters of the generalised lambda distribution," *Technometrics,* 27, 8–84.

Ramberg, J. and Schmeiser, B. (1974). "An approximate method for generating asymmetric random variables," *Communications of the Association for Computing Machinery,* 17, 78–82.

Ramberg, J. S., Tadikamalla, P. R. Dudewicz, E. J., and Mykytka, E. F. (1979). "A probability distribution and its uses in fitting the data," *Technometrics,* 21, 201–214.

Shapiro, S. S, Wilk, M. B., and Chen, H. J. (1968). "A comparative study of various tests of normality," *Journal of the American Statistical Association,* 63, 1343–1372.

Su, S. (2005a). "A discretized approach to flexibly fit Generalized Lambda Distributions to data," *Journal of Modern Applied Statistical Methods,* (November), 408–424.

Su, S. (2005b). "To match or not to match?," *British Accounting Review,* 37(1), 1–21.

Su, S. (2006). "To match or not to match revisited," *The ICFAI Journal of Accounting Research,* V. 1, 7–19.

Su, S. (2007a). "Fitting single and mixture of Generalized Lambda Distributions

to data via discretized and maximum likelihood methods: GLDEX in **R**," *Journal of Statistical Software,* 21(9), 1–7.

Su, S. (2007b). "Numerical maximum log likelihood estimation for Generalized Lambda Distributions," *Computational Statistics and Data Analysis,* 51(8), 3983–3998.

Su, S. (2007c). "Optimal smoothing of profit via overhead allocation," *Abacus,* 43(2), 136–155.

Willett, R. (1987). "An axiomatic theory of accounting measurement," *Accounting and Business Research,* 18, 155–171.

Willett, R. (1988). "An axiomatic theory of accounting measurement–Part II," *Accounting and Business Research,* 19, 79–91.

Willett, R. (1991). "Transaction theory, stochastic processes and derived accounting measurement," *Abacus*, September 1991(b), 117–134.

Chapter 29

Use of the Generalized Lambda Distribution in Materials Science, with Examples in Fatigue Lifetime, Fracture Mechanics, Polycrystalline Calculations, and Pitting Corrosion

B. Fournier: CEA, DEN/DANS/DMN/SRMA/LA2M,
Bat. 453, 91191 Gif sur Yvette Cedex, France
benjamin.fournier@cea.fr

N. Rupin: EDF R&D, Départ. Matériaux et Mécanique des composants,
avenue des Renardiéres, Ecuelles, 77818 Moret-sur-Loing Cedex, France

M. Bigerelle: Laboratoire Roberval, FRE 2833, UTC/CNRS,
Centre de Recherches de Royallieu, BP20529, 60205 Compiégne France

D. Najjar: LMPGM Labor. de Métallurgie Physique et Génie des Matériaux,
CNRS UMR 8517, Equipe Caractérisation et Propriétés de la Périsurface,
ENSAM, 8 Boulevard Louis XIV, 59046 Lille Cedex, France

A. Iost: LMPGM Laboratoire de Métallurgie Physique et Génie des Matériaux,
CNRS UMR 8517, Equipe Caractérisation et Propriétés de la Périsurface,
ENSAM, 8 Boulevard Louis XIV, 59046 Lille Cedex, France

Till Vallée: College of Engineering and Architecture
University of Applied Sciences Fribourg
Boulevard de Pérolles 80, CH 1705 Fribourg, Switzerland

As in many scientific fields, statistical analyses are becoming more and more used in mechanics and materials sciences. Most materials are intrinsically heterogeneous. This heterogeneity can lie at very different scales. At the atomic scale, for crystalline materials (e.g., metals), additional elements (impurities or voluntary added elements, like carbon in iron to obtain steel) perturb the atomic arrangement either by substitution or insertion (François, Pineau, and Zaoui (1998)). Locally, vacancies (one atom missing) are also point defects modifying the perfect crystalline arrangement. Linear defects, named dislocations (see Friedel (1956) and Nabarro (1952)), are another type of heterogeneity, responsible for plastic (irreversible) deformation. These dislocations are an essential feature, e.g., for forming materials, but their arrangement does not obey a definite order and can change during the "life" of a material (Polak, Petrenec, and Man (2005)). Most metallic materials are polycrystalline materials, which means that the crystallographic arrangement is different depending on the grain orientation (Sutton and Balluffi (1995)). These grains are separated by grain boundaries which are surfaces where the atomic arrangement is not respected. At that scale, both the shape, the size, and the orientation of grains can only be described by statistical distributions (Kocks, Tomé and Wenk (1998)).

To understand and model the macroscopic mechanical behavior of these materials, it is necessary to account for these various heterogeneities and then run calculations on representative volume elements (Cailletaud, Forest, Jeulin, Feyel, Galliet, Mounoury, and Quilici (2003)), either by direct local calculations or by considering a homogeneized mechanical behavior (Bornert, Bretheau, and Gilormini (2000)).

In addition to mechanical behavior, the whole field of damage and fracture is directly concerned by statistical considerations. **This chapter will focus on three specific fields linked to damage of materials: fatigue, brittle fracture, and extreme values analysis (e.g., corrosion, wear analyses, etc.).** In each field, two examples will detail how the generalized lambda distribution family (GLD) can be used and how its use can improve the usual statistical treatments that need to be performed. The GLD is not described since a complete description can be found in Karian and Dudewicz (2000).

29.1 Fatigue Lifetimes

In mechanical design, fatigue strength is often one of the most important considerations. The generic term of fatigue holds for cyclic loadings, or, more generally, loadings with an amplitude varying with time. Such mechanical loadings appear in almost all components. Due to its delayed nature, failure by fatigue is always difficult and expensive to test and understand. Fatigue fracture always proceeds by three different stages. Firstly, a crack is most often initiated at the surface

of the material, either on an existing defect (machining flaw, large inclusion, etc.) or on a defect created by the plastic deformation cumulating through cycles (Man, Petrenec, Obrtlik, and Polak (2004)). Secondly, one or several cracks become large enough to propagate through the material. Finally, either one crack becomes large enough to lead to the final fracture or several cracks coalesce.

Initiation and the first stages of crack propagation are intrinsically stochastic. The lower the loading the larger the lifetime scatter. As mentioned above, initiation depends on the probability of occurrence of surface defects, of their size, their nature. The early stages of crack propagation are influenced by the local plasticity behavior and the microstructure (grain boundaries, precipitates, density of dislocations, etc.).

The following examples tackle two aspects of fatigue lifetime (the number of cycles before fracture). We first give a description of the fatigue lifetime distribution as a GLD and then use the GLD to describe crack initiation and crack propagation and their influence on the fatigue lifetime.

29.1.1 Lifetime Distributions

In addition to the intrinsic stochastic nature of fatigue lifetime, tests carried out on different sample geometries, on different testing machines, and by different research teams always add some scatter to the lifetime distribution. However, as fatigue tests are time consuming and expensive, people usually collect all the available lifetime data of the literature in a single lifetime distribution. As far as the authors know, there is, until now, no theory available to describe both this intrinsic lifetime scatter and the scatter coming from testing procedures. Therefore, there is no *a priori* reason to use a specific type of statistical distribution to describe fatigue lifetimes. Actually, both two-parameter Weibull distributions (Soboyejo, Shen, Lou, Mercer, Sinha, and Soboyejo (2002), Hall and Strutt (2003), Ferreira, Balthazar, and Araujo (2003), Lewis, Janna, and Carroll (2003), Petryna, Pfanner, Stangenberg, and Kratzig (2003), Pauchard, Grosjean, and Campion-Boulharts (2002), Diao (1995)) and Gaussian distributions (Bastenaire (1971), Liao, Shi, and Xiong (2001)) are most frequently used to describe lifetime distributions, even though three-parameter Weibull (Caron and Ehrlacher (1999)), multi-modal Weibull distribution (Nagode and Fajdiga (1998a) and (1998b)) or lognormal distributions (Liao, Shi, and Xiong (2001), Gope (1999), Ferry, Perreux, Varchon, and Le Bras (1997), Nikbin, Yatomi, Wasmer, and Webster (2003)) can also be found in the literature.

In the example that we consider here, fatigue lifetime data of a 9%Cr steel (named P91) tested at 550°C are reported. The references from which these data are taken are detailed in Fournier (2007). These tests were performed under total strain control, following the loading scheme presented in Figure 29.1–1. The strain range applied is denoted $\Delta\epsilon_t$ and expressed in %. The evolution of fatigue lifetimes with the strain range are represented in Figure 29.1–2. The lower the

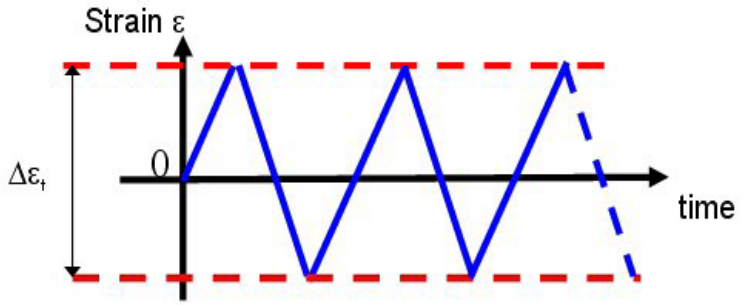


Figure 29.1–1. Scheme of a fatigue loading.

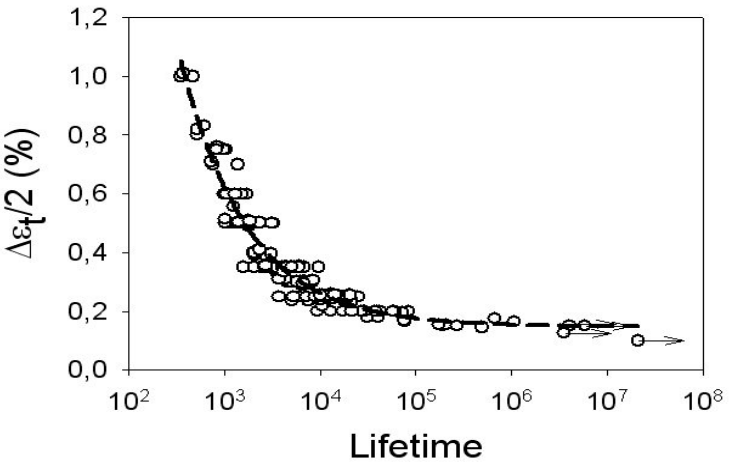


Figure 29.1–2. Experimental lifetime as a function of the strain amplitude at 550°C for a 9%Cr steel.

loading, the larger the lifetime. But in addition, for low strain amplitudes, the fatigue lifetime scatter is found to increase. To look more precisely at the lifetime distributions, the four strain levels ($\Delta\epsilon_t/2 = 0.2, 0.25, 0.35$ and 0.5%) for which the largest number of data are available are studied in more detail.

Table 29.1–3 gives the number of available data and the first four moments of these four distributions. **In fatigue, mainly two usual distributions are considered:** on the one hand the Gaussian distribution, and on the other hand the Weibull distribution. Both the skewness and the kurtosis values show that the Gaussian distribution cannot account for the assymetry and the flatness of these distributions. In addition, both the Gaussian and the Weibull distribution have either one or two infinite bounds. This means that for the Gaussian distribution negative values are possible, which is a physical nonsense. The infinite right tail has also no experimental support, especially for such high loadings, where

Table 29.1–3. Descriptive statistics of the four lifetime distributions.

$\frac{\Delta\epsilon_t}{2}(\%)$	Number of data	mean	standard error	skewness	kurtosis
0.2	15	37460	20471	0.67	2.86
0.25	38	13163	4486	0.19	3.13
0.35	25	4295	1884	1.02	3.92
0.5	44	1739	488	1.01	4.26

generalized plasticity is observed.

Figure 29.1–4 shows a cumulative distribution function (cdf) for the data (the empirical cdf) along with the cdf's of the Gaussian, Weibull, and GLD fit. In all cases, the GLD provides a better qualitative adequacy to the experimental distribution. More specifically, in the case of the lowest strain amplitude, more than 2% of the values predicted by a Gaussian fit are negative, which cannot be considered as negligible. For $\Delta\epsilon_t/2 = 0.35\%$, the experimental distribution is highly asymmetric with a very short right tail. This feature is clearly not accounted by Gaussian or Weibull modeling. Table 29.1–5 reports the bounds of the GLD support and the comparison of its adequacy with Gaussian and Weibull modeling using the Kolmogorov-Smirnov distance. Except at $\Delta\epsilon_t/2 = 0.5\%$ the GLD is the closest approximation to the experimental data. And at $\Delta\epsilon_t/2 = 0.5\%$ the adequacy remains very comparable to the Gaussian one. More importantly, the GLD always provides a bounded support. Therefore, no physically impossible infinite lifetimes are predicted. However at $\Delta\epsilon_t/2 = 0.5\%$, the lower bound of the GLD is negative, which actually is a physical nonsense. These negative values correspond to a cumulated probability of less than 10^{-7}. This could therefore be considered negligible. However, it remains possible to constraint the GLD bounds as shown in Fournier (2007).

29.1.2 Crack Initiation and Crack Propagation

Even more important than lifetime distribution description, researchers and practioners are looking for models that are able to predict this lifetime distribution for a given loading. When the loading remains low enough, elastic parameters such as the stress intensity factor K can be used to describe crack propagation. This example (Bigerelle, Najjar, Fournier, Rupin, and Iost (2006)) makes use of the GLD to describe the dispersion of the crack propagation parameters and demonstrate that the crack initiation dispersion must also be taken into consideration to account for actual lifetime distributions.

Figure 29.1–6 shows experimental results of crack propagation tests. Here a is the crack length and N is the number of fatigue cycles. This figure illustrates the

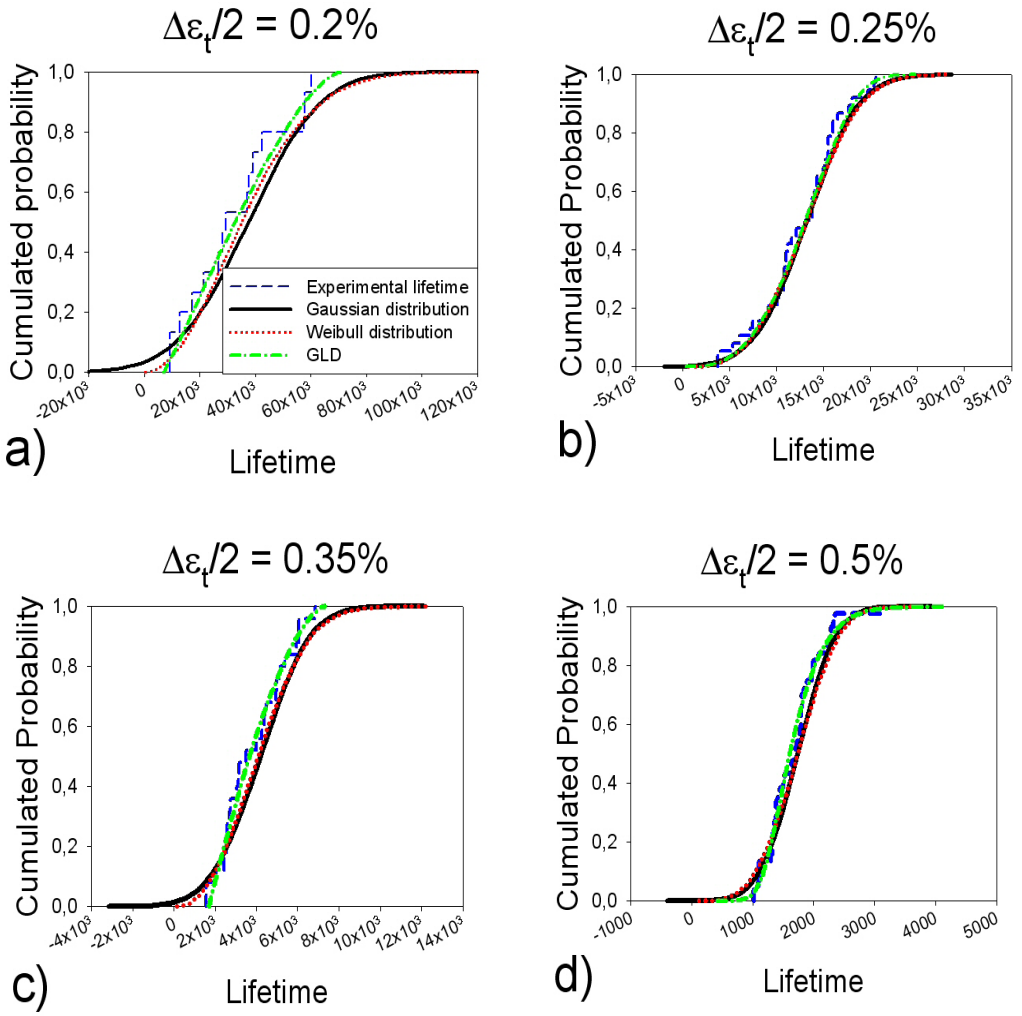


Figure 29.1–4. Experimental fatigue lifetime distributions modeled by a Gaussian law, a Weibull law, and a GLD for various strain amplitudes $\Delta\epsilon_t/2$.

Table 29.1–5. Support of the GLD distribution and comparison of the Kolmogorov-Smirnov (KS) distance with that obtained by Gaussian and Weibull fitting.

$\frac{\Delta\epsilon_t}{2}(\%)$	Left bound	Right bound	KS GLD	KS Gauss	KS Weibull
0.2	7018	72409	0.1148	0.1342	0.1375
0.25	81	28883	0.0719	0.0924	0.0930
0.35	1694	7305	0.1202	0.1510	0.1571
0.5	-3310	6098	0.1071	0.1105	0.1005

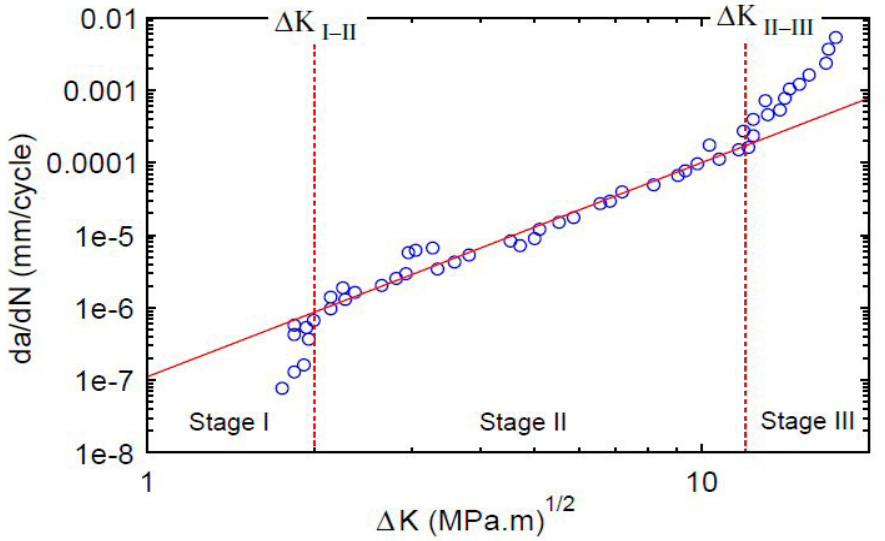


Figure 29.1–6. Result of the fatigue crack growth on an aluminum alloy obtained by Krausz, Zhiven, and Krausz (1992).

fact that the behavior of the fatigue crack propagation of engineering alloys can be generally divided into three stages. In stage II, stable fatigue crack growth conditions prevail and the fatigue crack growth rate (FCGR) is given by the well-known Paris-Erdogan relationship (Paris and Erdogan (1963))

$$\frac{da}{dN} = C\Delta K^n, \tag{29.1.1}$$

where C and n are two parameters depending on the material and on the experimental conditions. In Bigerelle and Iost (1999), the Paris coefficients and the interval limits $[\Delta K_{I-II},\ \Delta K_{II-III}]$ of stage II were determined combining the Bootstrap analysis and the linear regression method in the case of the data presented in Figure 29.1–6. The values of ΔK_{I-II} and ΔK_{II-III} were statistically determined by taking into account the dispersion along the linear relationship observable on Figure 29.1–6.

29.1.2.1 Distributions of C and n

The Bootstrap resampling technique was used to generate the correlated distributions of C and n. Figure 29.1–7 show these distributions and their modeling by a Gaussian and a GLD distribution. Table 29.1–8 gives the values of the χ^2 goodness-of-fit criterion. Considering the very large number of classes, these values can be compared between a Gaussian and a GLD fit. It shows that the GLD offers a better adequacy than the Gaussian distribution.

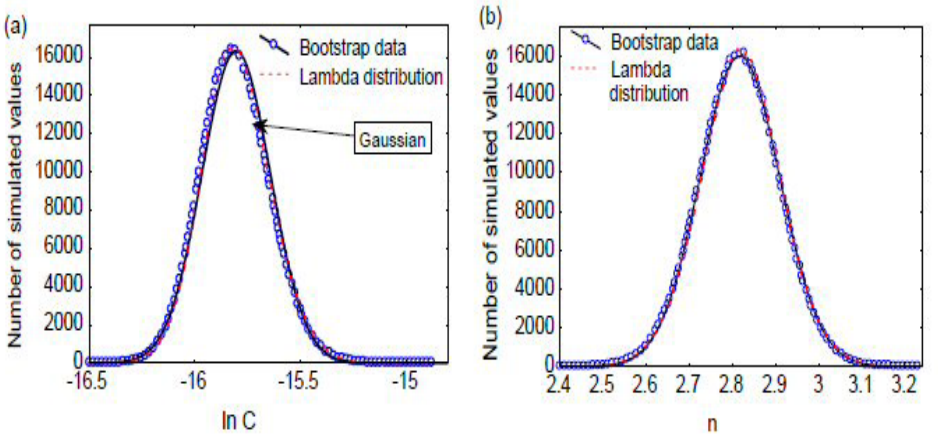


Figure 29.1–7. Bootstrap distributions of (a) $\ln(C)$ and (b) n and their modeling by a Gaussian and a GLD.

Table 29.1–8. Comparison of the goodness-of-fit for both a Gaussian and a GLD modeling using χ^2 values.

	GLD	Gaussian
$\ln(C)$	82	128
n	38	45

29.1.2.2 Application to Lifetime Prediction

Once all the various materials and structural inputs are obtained, the fatigue lifetime prediction (FLP) can be assessed by integrating the Paris relationship. As C and n do not depend on the initial crack length (a_i), a simple closed-form equation can be obtained for the FLP. By applying the Irwin criterion, the expression of the stress intensity factor $K = \alpha\tau\sqrt{\pi a}$ can be obtained, where a is the crack length, τ is the loading stress, α is a geometry-load factor related to the crack shape, the structure and the loading type. Then the integration of the Paris equation gives

$$N_f = \int_0^{N_f} dN = \int_{a_i}^{a_c} \frac{da}{C\Delta K^n}, \tag{29.1.2}$$

where a_c is the critical crack length to fracture. This leads to the FLP

$$N_f = \frac{1}{C(\Delta\tau\alpha\sqrt{\pi})^n} \int_{a_i}^{a_c} \frac{da}{a^{n/2}}, \tag{29.1.3}$$

from which one obtains

$$N_f = \frac{1}{C\left(\Delta\tau\alpha\sqrt{\pi}\right)^n\left(\frac{n}{2}-1\right)}\left[\left(\frac{1}{a_i}\right)^{n/2-1} - \left(\frac{1}{a_c}\right)^{n/2-1}\right] \text{ for } n \neq 2, \quad (29.1.4)$$

where $\Delta\tau$ is the stress range during the fatigue cycle.

29.1.2.3 Engineering Application

In a first approach, the initial crack length will be supposed to be known. Let us consider for example an infinite 7 cm diameter (ϕ) and 1 cm thick (e) hollow cylinder loaded with a cyclic differential pressure of 20 MPa ($\Delta\tau$). The maximal initial crack's length located on the inner surface of the cylinder (a_i) is supposed to be 1.5 mm in length. The cylinder, made of an aluminum alloy (Krausz, Zhiven, and Krausz (1992)) for which the distribution of the parameters of the Paris-Erdogan relationship were obtained, is considered to be perforated as soon as the crack reaches the outer surface. Taking $\alpha = 0.71$, an elementary mechanical analysis allows us to know that ΔK varies between 3.4 and 8.8 MPa$\sqrt{m}$, so that the crack propagation can be modeled by the Paris relationship because these ΔK values that we have are within the range $[\Delta K_{I-II}, \Delta K_{II-III}]$.

Resampling from the GLD obtained to model C and n enables us to generate the FLP distribution shown in Figure 29.1–9. Here again the lifetime distribution is better modeled by a GLD than by usual statistical laws as checked by the χ^2 goodness-of-fit test (Table 29.1–10).

To tackle the influence of the initial crack length a_i, this initial crack is assumed to come from a specific process (notch, threading, groove, etc.) that can be modeled by a Gaussian distribution of standard error σ. Let us now launch the same calculations with a_i randomly sampled from this Gaussian distribution. Figure 29.1–11 shows that the larger the initial crack scatter, the larger the scatter and the asymmetry of the FLP PDF. In fact, the scatter of the FLP PDF presents an exponential growth when the scatter of the initial crack length increases. Even the mean value of the FLP PDF increases with the initial crack length scatter.

All the distributions of Figure 29.1–11 do come from a single process (crack initiation and propagation modeled by the same equations), only the parameters of the process change. However, they present various shapes and none of the usual distributions used in the study of fatigue (Gaussian, lognormal, Weibull) can account for all these various shapes. On the contrary, the adaptability of the GLD enables one to offer a very accurate fit in all theses cases (Bigerelle, Najjar, Fournier, Rupin and Iost (2006)).

These two examples have shown that in the study of fatigue, when no theoretical reason dictates the choice of the statistical distribution under study, the GLD family of distributions, thanks to its versatility and its bounded character, is determined to be very useful both in a simple lifetime distribution description and in a prediction study.

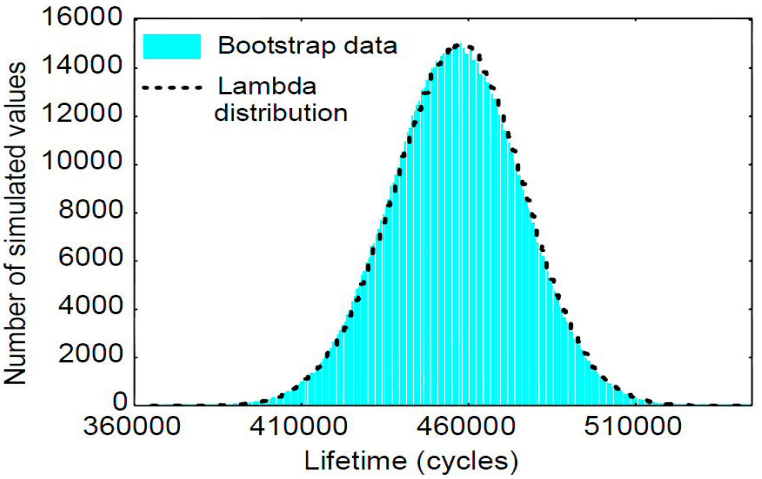


Figure 29.1–9. FLP and the corresponding GLD fit.

29.1–10. Comparison of the goodness-of-fit for the FLP using χ^2 values.

	GLD	Gaussian	lognormal	Weibull
χ^2 value	14	208	1570	39

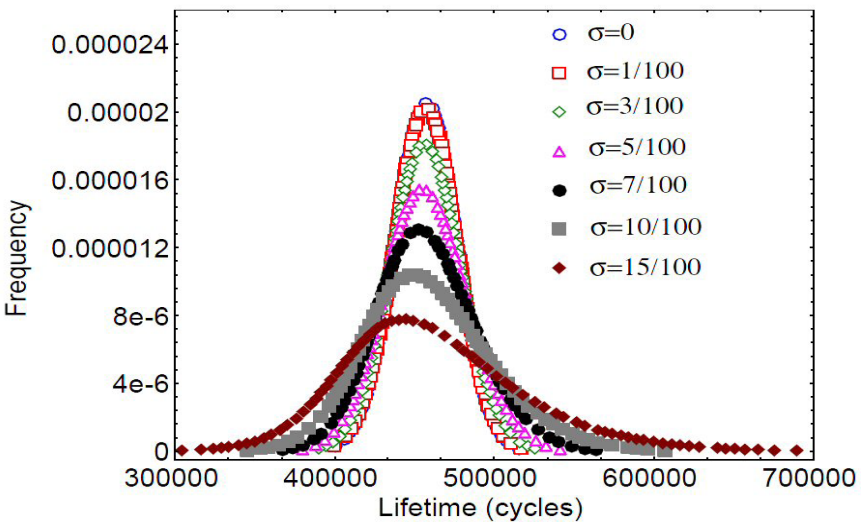


Figure 29.1–11. FLP for various values of the initial crack scatter.

29.2 Fracture Mechanics

Brittle fracture is definitely one of the areas where the intrinsically stochastic nature of the phenomenon has been pointed out for some time (Griffith (1921), Pierce (1926), Weibull (1939)). Contrary to the ductile mechanisms, where large deformations and damage evidences can first be observed before the final fracture, the brittle process is brutal and does not usually give any symptoms before the final fracture. This, along with the fact that experiments always show a very large scatter in mechanical strength for brittle material, led to the statistic theory of brittle fracture (Weibull (1939)).

This theory is based on the weakest link assumption, first proposed for ceramics and cleavage fracture. The fracture strength (the stress for which the material experiences brittle fracture) is statistically distributed because the fracture phenomenon starts at particular points of the microstructure (specific grain boundaries, surface indentation, inclusions, etc.) that are statistically distributed. The fracture usually starts at the weakest point of the material and then propagates almost instantaneously. This theory is based on two fundamental hypotheses. First of all, the material can be considered as the juxtaposition of perfectly independent (in terms of fracture) elements. Second, the fracture of the weakest element leads to the fracture of the whole material. These assumptions must be carefully checked and do not apply for all materials (François, Pinean, and Zaoui (1998)).

When these assumptions are verified, a failure probability, depending on the applied stress σ is given to all elements. When denoting this cumulated probability $P_0(\sigma)$, the survival probability is $1 - P_0(\sigma)$ for a given element. Therefore, the failure probability for N independent elements is

$$P_F(\sigma) = 1 - (1 - P_0(\sigma))^N \tag{29.2.1}$$

There is no *a priori* knowledge about the $P_0(\sigma)$ function. Equation (29.2.1) can also be written as

$$P_F(\sigma) = 1 - \exp\left[-N \ln\left(\frac{1}{1 - P_0(\sigma)}\right)\right] \tag{29.2.2}$$

If the elementary volume element is noted V_0 and V the total volume, then $N = V/V_0$. This formula thus shows that the fracture probability increases with the volume, as empirically observed. This can by explained by the fact that, when the volume increases, the probability to find a very weak element increases. Finally, this equation can be rewritten as

$$P_F(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0} f(\sigma)\right], \tag{29.2.3}$$

where

$$f(\sigma) = \ln\left(\frac{1}{1 - P_0}\right).$$

However, most often, components having a complex geometry are not subjected to an homogeneous stress. Even for testing samples, where geometry does not modify the stress distribution, the microstructure (grain boundaries, local crystallographic orientations, inclusions, etc.) leads to a highly heterogeneous stress field. This means that the previous equation must also take into account the stress distribution and consider each elementary volume subjected to a stress coming from this distribution.

The two following examples illustrate that, on the one hand, for $P_F(\sigma)$ the Weibull distribution is not always the most adequate and, on the other hand that the GLD can model very accurately the stress distribution required to use equation (29.2.3) for a complex component.

29.2.1 Fracture of Adhesively Bonded Joints Composed of Pultruted Adherends

If the Weibull distribution corresponding to the weakest link assumption has been verified on a wide range of materials, the quality of the fracture stress fitting gets poorer towards the higher strengths (upper tail) (see Yasuhiro, Takahiro, and Kazuko (2003), and Taylor (1994)). Weibull plots of experimental strength data for various materials (Bazant and Pang (2004)), including fibrous composites (Durham and Padgett (1997)), rather than following a straight line, often have a curvature or exhibit a kink after which the slope decreases. This suggests that other statistical models might be more appropriate (Park and Padgett (2005)). The weakest link assumption is based on independent elements arranged in series. For fiber materials, several authors (e.g., Daniels (1945) and Coleman (1958)) have proposed models based on elements in parallel. For most materials, the truth probably lies somewhere in between. Recently, Bazant and Pang (2004) have proposed a model based on both parallel and series elements, leading to a grafted pdf.

The present example deals with adhesively bonded joints composed of pultruted glass fiber-reinforced polymer (GFRP) and is extracted from a study recently published (Vallée, Keller, Fourstey, Fournier, and Correia (2009)). Quasi-static tensile tests were carried out up to failure on these joints, shown in Figure 29.2–1. The experimental strenghts are gathered in Table 29.2–2. The actual normal and shear stresses were calculated at all points of the joint thanks to finite element computations carried out with the Ansys 8.0 software. Even though the actual maximum stress depends on the joint thickness, it was clearly demonstrated by Vallée, Keller, Fourstey, Fournier, and Correia (2009) that a size effect, coming from the material and not from the stress field must be taken into account to explain results of Table 29.2–2. The failure initiated in the GFRP adherends close to the fillet region by combined tensile normal and shear stresses. To determine the shear-tensile-interaction strength, 55 tests were carried out on specific coupons.

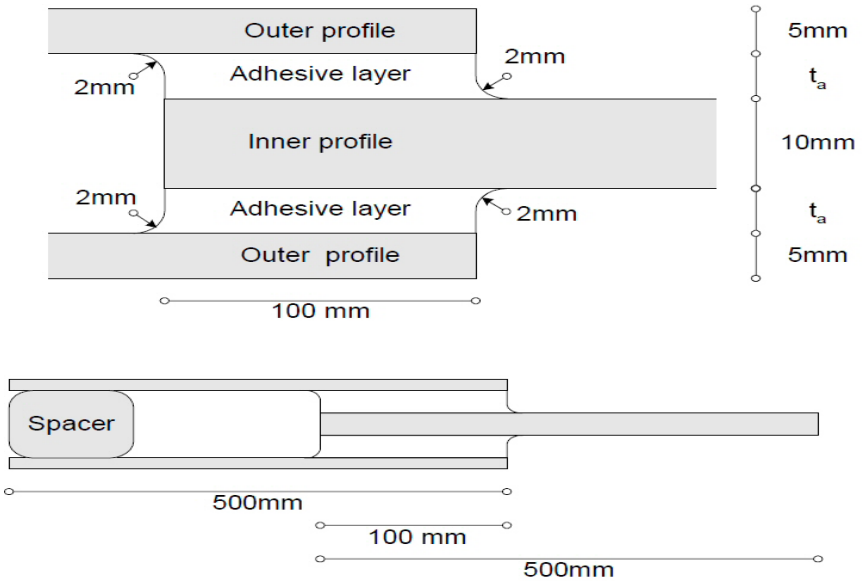


Figure 29.2–1. Scheme of the adhesively bonded joints tested.

From these tests a shear-tensile-interaction failure criterion was defined. Finally, an equivalent stress σ_Q defined as a quadratic combination of the normal stress and the shear stress is calculated. Figure 29.2–3 shows within a Weibull plot the failure probability as a function of this equivalent stress. It appears that, contrary to the straight line expected from a weakest link process, a clear curvature and a kink at the upper tail can be observed.

To model this strength distribution, 6 methods were compared. The usual two parameters of the Weibull distribution was compared to n-folded Weibull distributions (n being 2 and 3), to a Grafted distribution, to a Birnbaum-Saunders distribution, and to a GLD. Results are presented in Figure 29.2–4. It appears, that this experimental distribution is clearly not a Weibull distribution considering its curvature within a Weibull plot. A 3-folded distribution can very correctly render the global shape of the experimental distribution and the two tails. A 2-folded or a 3-folded Weibull distribution could reflect the existence of 2 or 3 types of brittle fracture mechanisms linked to several populations of defects. However, up to now, no experimental evidence can support this assumption. It appears in Figure 29.2–4 that the GLD offers the best fit for the upper tail of the distribution and a good approximation of the global distribution (except for the lower tail, but this has no direct influence on the following strength prediction). Both Grafted and Birnbaum-Saunders distributions seem less adequate. Finally, with only 4 parameters, the GLD offers a quite correct modeling of the experimental data. Only the 3-folded Weibull distribution offers a better fit, however, it corresponds to 8 parameters and does not present more physical justification than the GLD.

Table 29.2–2. Experimental strength measured during tensile tests.

t_0 (mm)	Number of specimens	Experimental strength (kN)
5	3	111±7.3
10	3	91.2±9.5
15	3	88.1±2.0
20	3	67.8±4.3
25	3	70.6±1.8
30	3	63.8±6.8
35	3	58.7±1.8

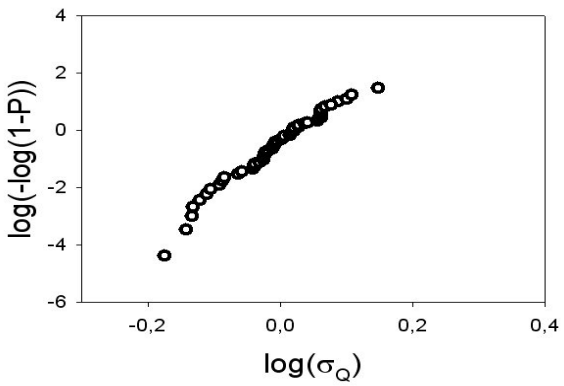


Figure 29.2–3. Distribution of the experimental fracture stresses in a Weibull plot.

Based on these analytical distributions coupled to Finite Element (FE) calculations, the failure strength of the whole joint can be predicted. Figure 29.2–5 shows the results obtained with these different distributions. It clearly appears that the GLD modeling offers the best support for the joint strength calculation. The fact that the GLD leads to a better strength prediction, whereas a 3-folded Weibull distribution looks quite adequate also for the previous modeling, is probably linked to the fact that the GLD is bounded towards large stresses, which is not the case for Weibull distributions. However it remains relatively conservative compared to the actual joint strength, but this can be explained by local plasticity phenomena (not taken into account in the FE calculations) and also by the fact that FE calculations, were carried out using a homogeneized mechanical behavior for the GFRP adherends, whereas, as shown in Figure 29.2–6, this material is definitely heterogeneous. However, the GLD modeling remains the best approximation of available data.

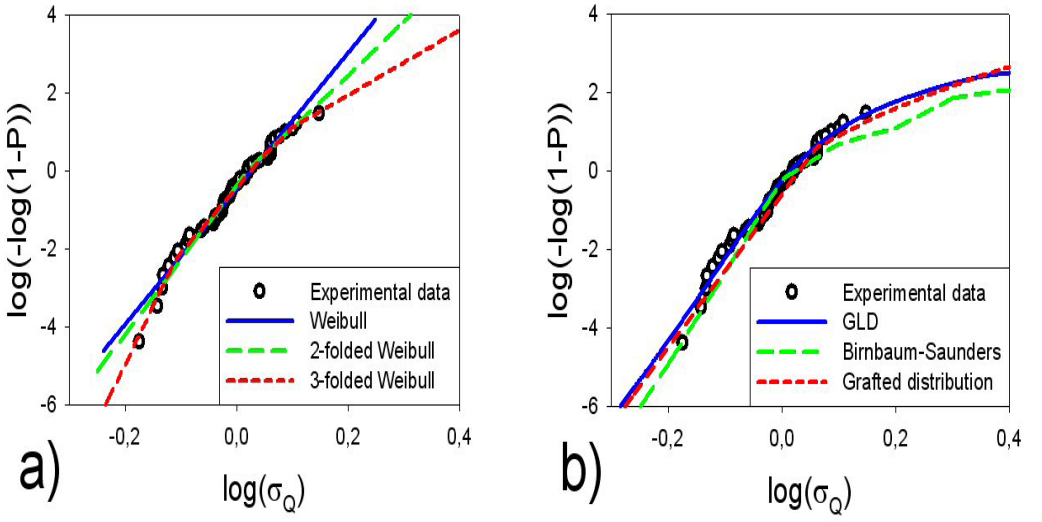


Figure 29.2–4. Comparison between the experimental data, plotted within a Weibull plot, with a) a Weibull, a 2-folded Weibull, and a 3-folded Weibull distribution and b) a GLD, a Birnbaum-Saunders and a Grafted distribution.

29.2.2 Modeling the Nuclear Reactor Pressure Vessel Steel Brittle Fracture

Reactor Pressure Vessel (RPV) steels of nuclear power plants show a strong temperature dependence of the fracture toughness in the range of the ductile to brittle transition temperature (DBTT). As a matter of fact, a convincing demonstration that the onset of brittle fracture cannot be reached under in-service conditions must be proposed to nuclear safety authorities.

Predicting this brittle fracture onset was one of the objectives of the European PERFECT project. The present example is taken from a study recently carried out within this framework (Vincent, Libert, Marini, and Rey (2009)).

In these RPV steels, brittle fracture has been identified to occur on carbides present within the microstructure. The choice was made to consider a Representative Elementary Volume V_0 as the minimum volume for which there is, on average, at least one carbide. As mentioned earlier, in equations (29.2.1), (29.2.2), and (29.2.3), there is no *a priori* knowledge about the probability distribution P_0. This elementary failure probability depends of course on the volume under study, but also on the local stress distribution and on the size of the carbide(s) within this elementary volume. The distribution of the local (microscopic) stress (σ) of course depends on the macroscopic stress applied (Σ) but also on the local crystallographic orientation.

Finally, the elementary failure probability can be written as

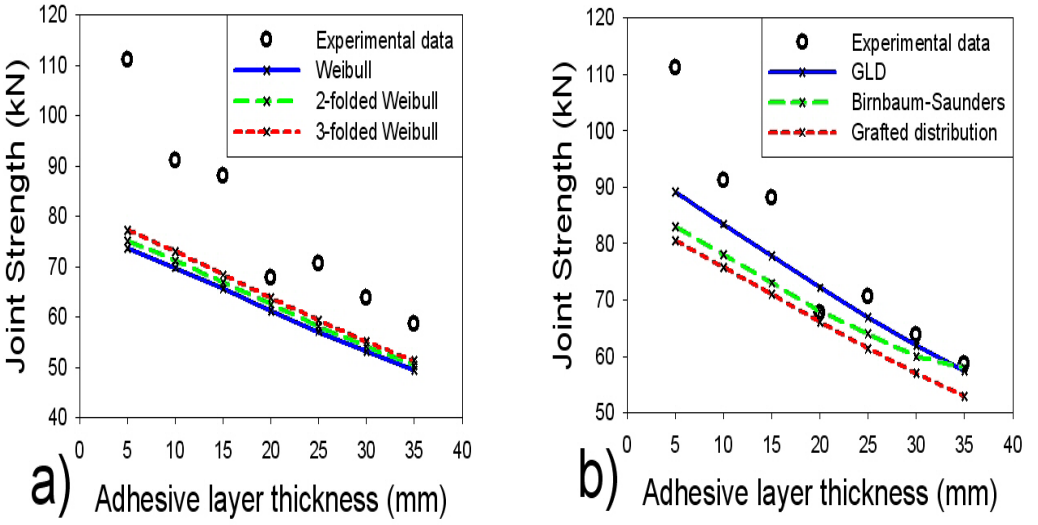


Figure 29.2–5. Comparison between the experimental strength with strength predictions based on a) a Weibull, a 2-folded Weibull, and a 3-folded Weibull distribution and b) a GLD, a Birnbaum-Saunders, and a Grafted distribution.

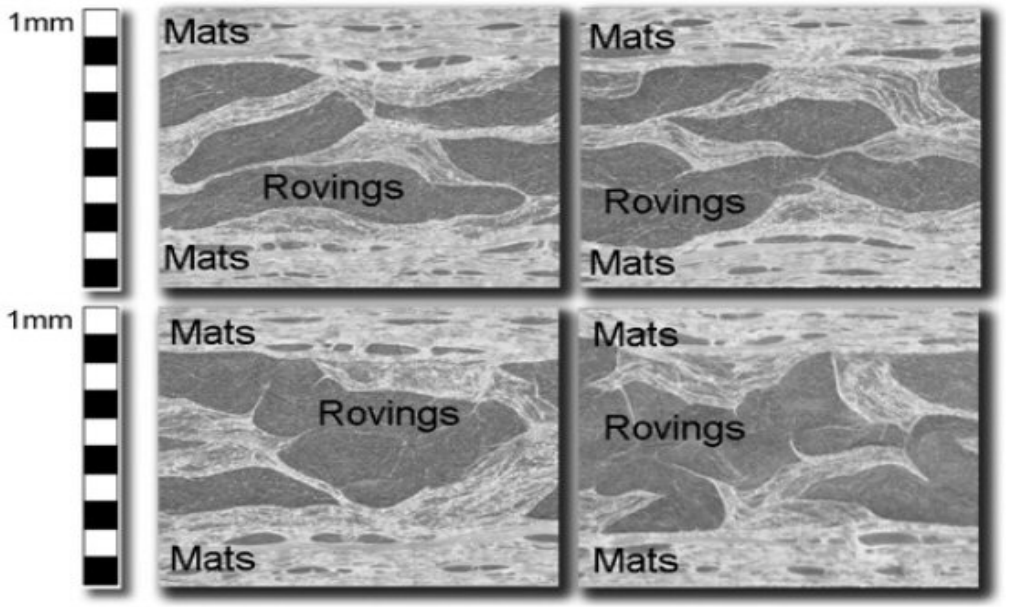


Figure 29.2–6. Fiber architecture of 10-mm thick pultruted material (inner adherends).

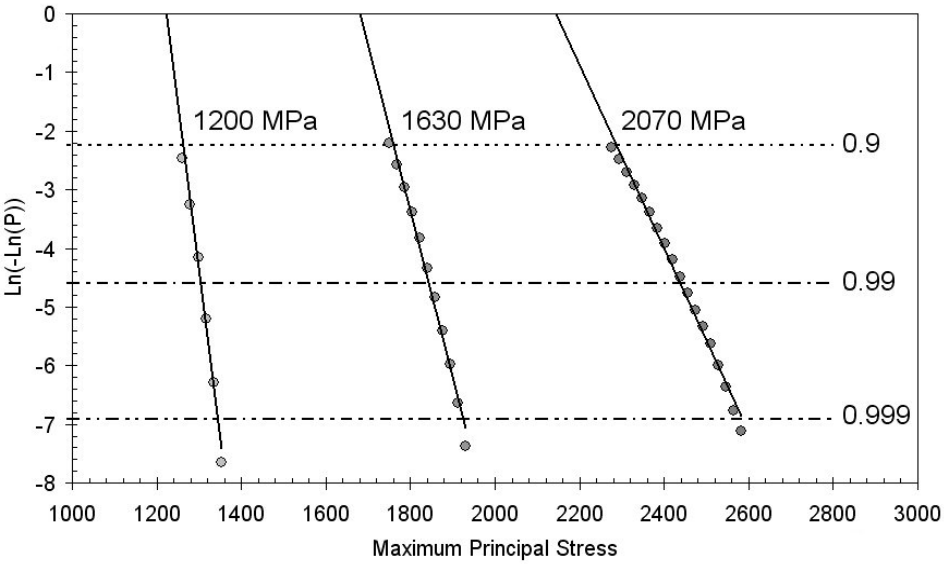


Figure 29.2–7. Right tail of the experimental local stress distributions (with three different macroscopic stress values) fitted by a Gumbel law.

$$P_0(V_0, \Sigma) = \int_0^{+\infty} \frac{dF(r)}{dr} P\left(\sigma\left(\Sigma\right) > \sigma_f(r)\right) dr, \tag{29.2.4}$$

where $F(r)$ is the distribution of the carbides size r, and $\sigma_f(r)$ is the critical fracture stress of these carbides, depending on their size. Both the distributions $\sigma(\Sigma)$ and $F(r)$ could be modeled by a GLD. However, in the present example, only the distribution of stresses will be studied. In the approach proposed by Vincent, Libert, Marini, and Rey (2009), the local stress distribution is obtained by Finite Element calculations of aggregates of grains using crystalline plasticity behavior. Once this "experimental" distribution is obtained it must be approximated by an analytical statistical law. It is usually assumed that only the large stress values are important since failure occurs when a combination of a large particle and a large stress exists within a given elementary volume. The emphasis is then placed on the right tail of the distribution. The authors choose to model the stress distribution using a Gumbel distribution fitted on values higher than the 90th percentile of the whole distribution. This is illustrated in Figure 29.2–7 for three values of the macroscopic stress.

It works reasonably well, even though when the macroscopic stress increases, the "experimental" distribution shows a significant curvature in the Gumbel plot. However, when one looks at the whole distribution, the Gumbel fit is quite poor. It is thus proposed to use a GLD to model accurately both the whole distribution and its right tail. Two fitting approaches were used. On the one hand, the usual

method of moments was used to fit the whole distribution. On the other hand, to lay the emphasis on the right tail, a new method is derived. It is quite similar to the method proposed by Fournier, Rupin, Bigerelle, Najjar, Iost and Wilcox (2007b) based on the minimization of the Kolmogorov-Smirnov distance. Here, only data larger than the 90th percentile (p_{90}) are considered. A distance D similar to a least square error is minimized within a (λ_3, λ_4) space. The equation for D is

$$D(\lambda_3, \lambda_4) = \frac{1}{N} \sum_{x \geq p_{90}} \left(\hat{F}(x) - F_{exp}(x) \right), \tag{29.2.5}$$

where N is the number of the experimental values larger than p_{90}, $\hat{F}(x)$ is the GLD approximation of the cdf of x and $F_{exp}(x)$ is the "experimental" cdf of x. $\hat{F}(x)$ can only be evaluated when the four GLD parameters are given. To obtain λ_1 and λ_2, the philosophy proposed by Fournier, Rupin, Bigerelle, Najjar, Iost and Wilcox (2007b) and Karian and Dudewicz (2000) is used and the following equations (derived from the percentile method) are used.

$$\lambda_2 = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\hat{\rho}_2}, \tag{29.2.6}$$

$$\lambda_1 = \hat{\rho}_1 - \frac{\left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}}{\lambda_2}, \tag{29.2.7}$$

with

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{29.2.8}$$

$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{29.2.9}$$

where u is chosen by the investigator such that $u \in]0, 0.25[$ and $\hat{\pi}_p$ is an estimate of the p-th quantile.

All the other constraints detailed in Fournier, Rupin, Bigerelle, Najjar, Iost and Wilcox (2007a) are also used. In the following the results obtained with this method will be denoted TLSM (standing for Tail Least Square Method) in comparison with the MM (Method of Moments) results.

Tables 29.2–8 and 29.2–9 give the GLD parameters obtained for the three macroscopic stress values with the TLSM and MM methods. It appears that, if the GLD parameters obtained with these two methods can be significantly different, the actual distributions bounds remain similar.

Figure 29.2–10 shows the comparison between the experimental cdf, the Gumbel cdf and the TLSM and MM GLD cdf, both for the whole distributions and with a focus on the right tail. It highlights the fact that the GLD obtained using MM fits reasonably well both the whole distributions and their tails, whereas the GLD obtained using TLSM offers a good fit at the distribution tails and gives

Table 29.2–8. GLD parameters and GLD support obtained with the MM method.

Macroscopic stress (MPa)	λ_1	λ_2	λ_3	λ_4	lower bound	upper bound
1200	1193.7	0.0013	0.1501	0.1540	985	1402
1630	1631.6	0.0024	0.1705	0.1786	1214	2048
2070	2064.1	0.0048	0.1367	0.1935	1294	2833

Table 29.2–9. Parameters and GLD support obtained with the TLSM method.

Macroscopic stress (MPa)	λ_1	λ_2	λ_3	λ_4	lower bound	upper bound
1200	1194.4	0.0041	0.0538	0.1356	950	1438
1630	1630.6	0.0024	0.1212	0.1628	1213	2046
2070	2056.7	0.0011	0.0960	0.1305	1147	2965

a correct central tendency (50th percentile). However it is clearly inaccurate for percentiles lower than 50.

On Figure 29.2–11, only the tails of the distributions are compared within a Gumbel plot. It appears that for macroscopic stresses of 1200MPa and 1630MPa, the three fits are quite equivalent at the distributions tails. However at $\Sigma =$ 2070MPa, the curvature of the experimental distribution is correctly rendered only by the GLD and the TLSM gives a significantly better adequacy than the MM. For this stress level, the three fits strongly diverge in the further right tail extrapolations. Therefore, in this particular example, the GLD offers a better, or at least a comparable, fit to the stress distributions. These fits are superior at the tails and provide a global adequacy to the whole distribution that is much better than the usual Gumbel distribution.

In these two examples, adhesively bonded joints and steel used in nuclear pressure vessels, the GLD appears to offer both a global adequacy to the experimental distributions but, most importantly, it is bounded on a finite support. Indeed, if the Gumbel or Weibull distributions can rely on a theoretical background (weakest link assumption), they do have an infinite right tail. This is physically inconsistent for both fracture strength and local stresses distributions. The most important improvement is that the bounded character of distribution can help in defining the minimum value of fracture resistance (toughness), which is of great practical importance.

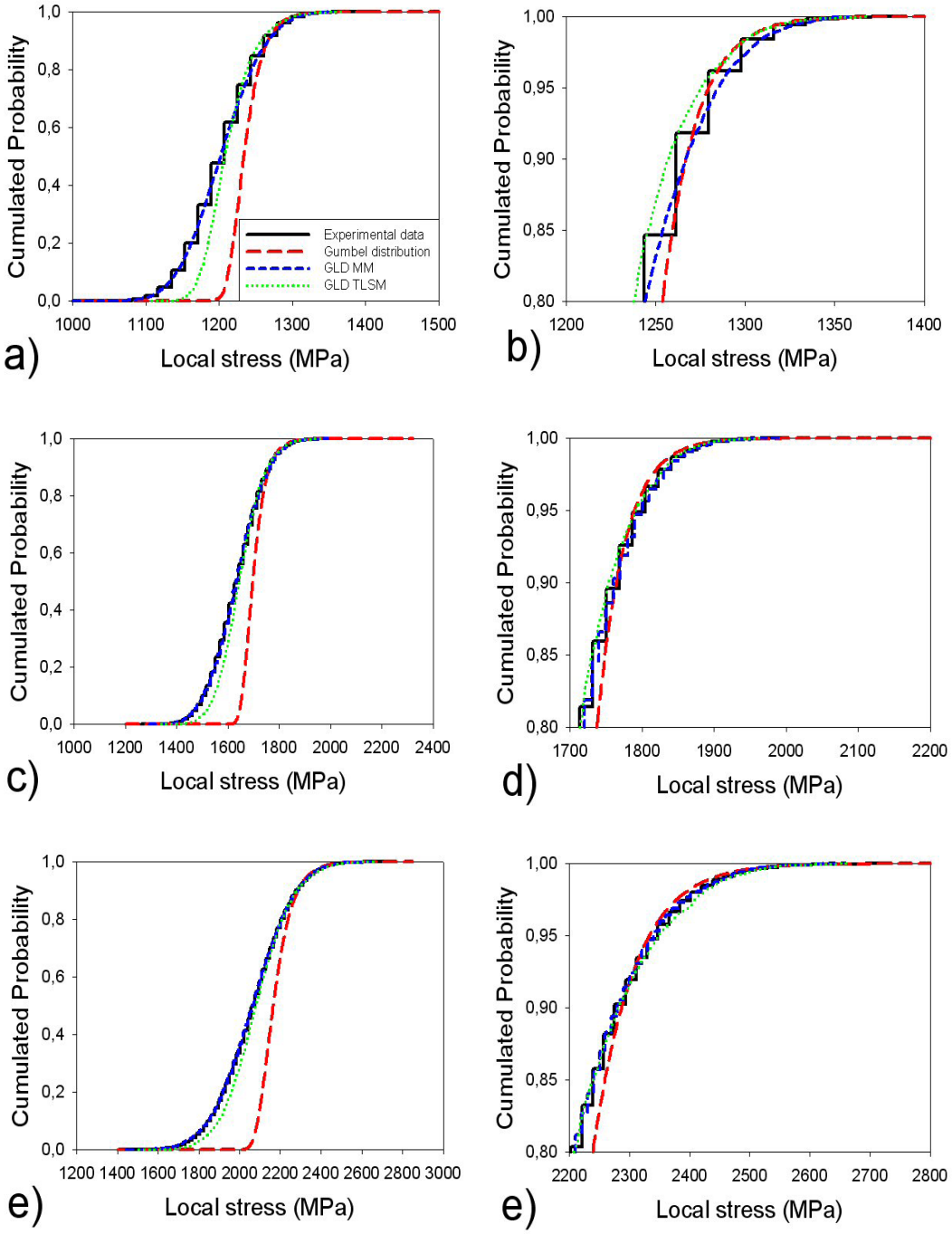


Figure 29.2–10. For the local stresses distributions, comparison between a Gumbel modeling with a GLD modeling obtained either with the Method of Moments (MM) or with the Tail Least Squares Method (TLSM), for a macroscopic loading of a) and b) 1200MPa, c) and d) 1630MPa, e) and f) 2070MPa.

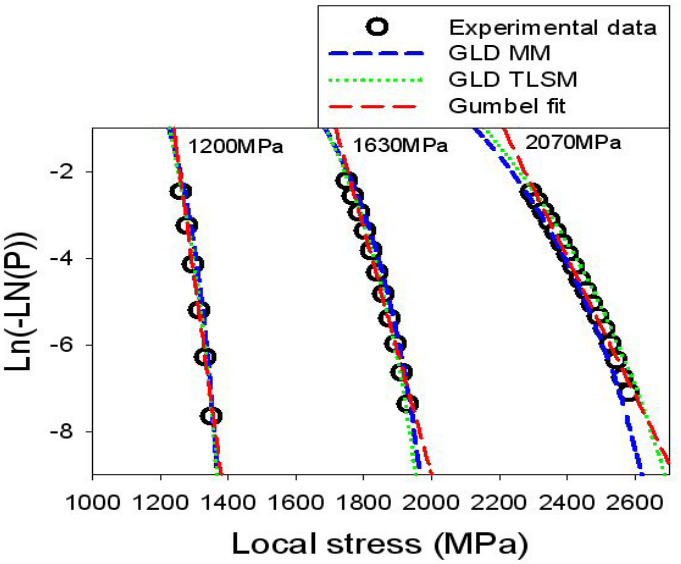


Figure 29.2–11. For the local stresses distributions, comparison between a Gumbel modeling with a GLD modeling obtained either with the Method of Moments (MM) or with the Tail Least Squares Method (TLSM) at the distributions tail.

29.3 Extreme Values

The previous examples dealt with the description of statistical data using a GLD. However, as in many different fields, in mechanics of materials, one of the key issues today is related to prediction and forecasting. These extrapolations can be either time extrapolations or scale extrapolations. For example in the nuclear energy industry, the next generation of power plants will be designed for at least 60 years, which means that designers must ensure that components will be functional and safe for 60 years, whereas, experimental data (creep, fatigue, corrosion, etc.) can of course not correspond to the same duration. Therefore modeling, based on the physical understanding of these "short" (about several years) duration tests, must enable researchers and designers to extrapolate the lifetime of materials to up to 60 years. In terms of scale extrapolation, one can think of maintenance operations of large-scale components.

Let us consider a several thousands kilometers long pipeline. These components are subjected to fatigue loadings, creep loadings, and highly corrosive environments. This means that damage inspections must be carried out quite often to observe whether existing corrosion pits, fatigue cracks, or other types of damage might soon (before the next scheduled inspection) cause fracture (linked to damage propagation concepts). In these cases, the measured damage must remain below a critical damage value (a pit depth, a crack length, etc.). However,

for such large-scale components, accurate observations cannot be carried out on the whole component. Therefore, only limited areas are inspected leading to finite samples of the whole damage distribution. To assess whether this critical damage value is exceeded on the whole component, an extreme value analysis must be carried out.

More generally, the question of damage (brittle fracture, fatigue lifetime, creep failure, roughness, corrosion, etc.) always necessitates extreme values analysis or, at least, a good description of the distribution queue. The distribution under study can deal with defects (crack, scratch, pits, etc.) size, with loading amplitude (e.g., unexpected overload, accidental conditions), or with material microstructure (inclusion or precipitate size, grain size, texture, etc.).

Several approaches can be used to deal with extreme values. The most usual was developed by Gumbel (1954) half a century ago and remains widely used under the name of Generalized Extreme Values (GEV) method (Galambos, Lechner, and Simiu (1994) and Castillo (1988)). Roughly speaking, this method consists of studying several samples of limited size. In each sample only the largest (or smallest) value is taken into account. The asymptotic extreme value distribution (Gumbel, Weibull, or Fréchet) is then fitted on this extreme values sample. This method is very easy to implement but it can be time consuming (in order to obtain the largest value of a sample, all the other values must have been measured). In addition, two main criticisms can be raised against this approach. First, the GEV distributions remain asymptotic results. Second, there is a huge loss of information: only the largest value is considered, whereas the whole distribution was obtained during the data acquisition step.

Another approach for extreme value analysis, which partially answers these two drawbacks, is based on the Generalized Pareto Distribution (GPD). Here the idea is to consider all the data higher (or lower) than a given threshold value (Shi, Atkinson, Sellars, and Anderson (1999), Shi, Atkinson, Sellars, Anderson, and Yates (2001)). Finally, a third approach aimed at considering the whole sample distribution has been proposed. It consists of fitting the whole distribution of data with a specific emphasis on the adequacy of tails. In this purpose, an adequacy test, based on bootstrap resampling, was proposed and named the Extreme Tail test (ET) (Diebolt, Garrido, and Girard (2003) and Diebolt, Garrido, and Trottier (2003)). Once this distribution has been obtained, either resampling techniques or an analytical relationship enables the extreme values to be predicted.

The purpose of this section is not to compare these three methodologies, but to illustrate how the GLD can be used within the framework of the third method described above. Indeed, considering their high flexibility and the fact that several different sets of GLD parameters can be considered as acceptable to describe a given dataset, depending on the fitting criterion used, the GLD can model accurately the distribution tail with a reasonable description of the distribution center. In addition, one of the main advantages of the GLD is that it can be bounded, which is not the case with the Gumbel distribution, whereas

the physical mechanisms under study are bounded (e.g., a pit cannot have an infinite depth, a grain cannot have an infinite diameter). Three applications are presented below, each dealing with the use of the GLD to model extreme values and tails.

29.3.1 Pitting Corrosion

Corrosion is responsible for the destruction of 25% of the world's annual production of steel, which means about 5 tons per second. This simple fact illustrates that, in addition to being a scientific challenge, the study and understanding of corrosion processes is a true economic issue.

Generalized corrosion is usually described as a uniform degradation of a surface, whereas the localized corrosion occurs preferentially on specific parts (inclusions, defects, slip bands, etc.) of a material. This study is focused on one type of localized corrosion named **pitting corrosion**. This kind of phenomenon occurs mainly on passive alloys for which a protective layer prevents generalized corrosion from occurring. Due to a very aggressive environment, combined or not with a mechanical stress or strain, this protective layer can be locally damaged. On this damaged area, that can be of a very small size, corrosion pits can initiate and then grow. This will lead to the formation of pits at the material's surface having very different shapes (see Figure 29.3–1). Even if, as is usually the case, these pits have a relatively limited depth, they can act as crack initiation sites (for example, under fatigue loadings), and finally lead to the fracture of the component. As in most damage mechanics applications, fracture will initiate on one of the most severe defects. As in-service inspections are usually performed on a limited fraction of the component surface (it would be impossible to look for pits on a whole pipeline!), the experimental data are then used for an extreme value analysis. Most usually, this analysis is performed with the well-known Gumbel methodology.

29.3.1.1 Experimental Procedure and Distribution

An accelerated corrosion test, consisting of a salt fog spray with cyclic wet and dry periods was carried out on a sample (rectangular area of 50cm^2 and 1.5mm in thickness) of AISI 409 steel used in the automotive industry. Optical microscopy was used to measure the experimental pit depth on 195 pits. The corresponding histogram is plotted in Figure 29.3–2. The global shape looks like a lognormal distribution and this assumption is not rejected by a chi-square test of adequacy at the 5% confidence level. This would enable a Gumbel extreme value procedure to be used. However, as mentioned above, the Gumbel distribution has an infinite right tail, which means that pits with an infinite depth have a small but finite probability of occurring when this methodology is used. This is of course not physically possible. Therefore, an alternative method based on the GLD is proposed below.

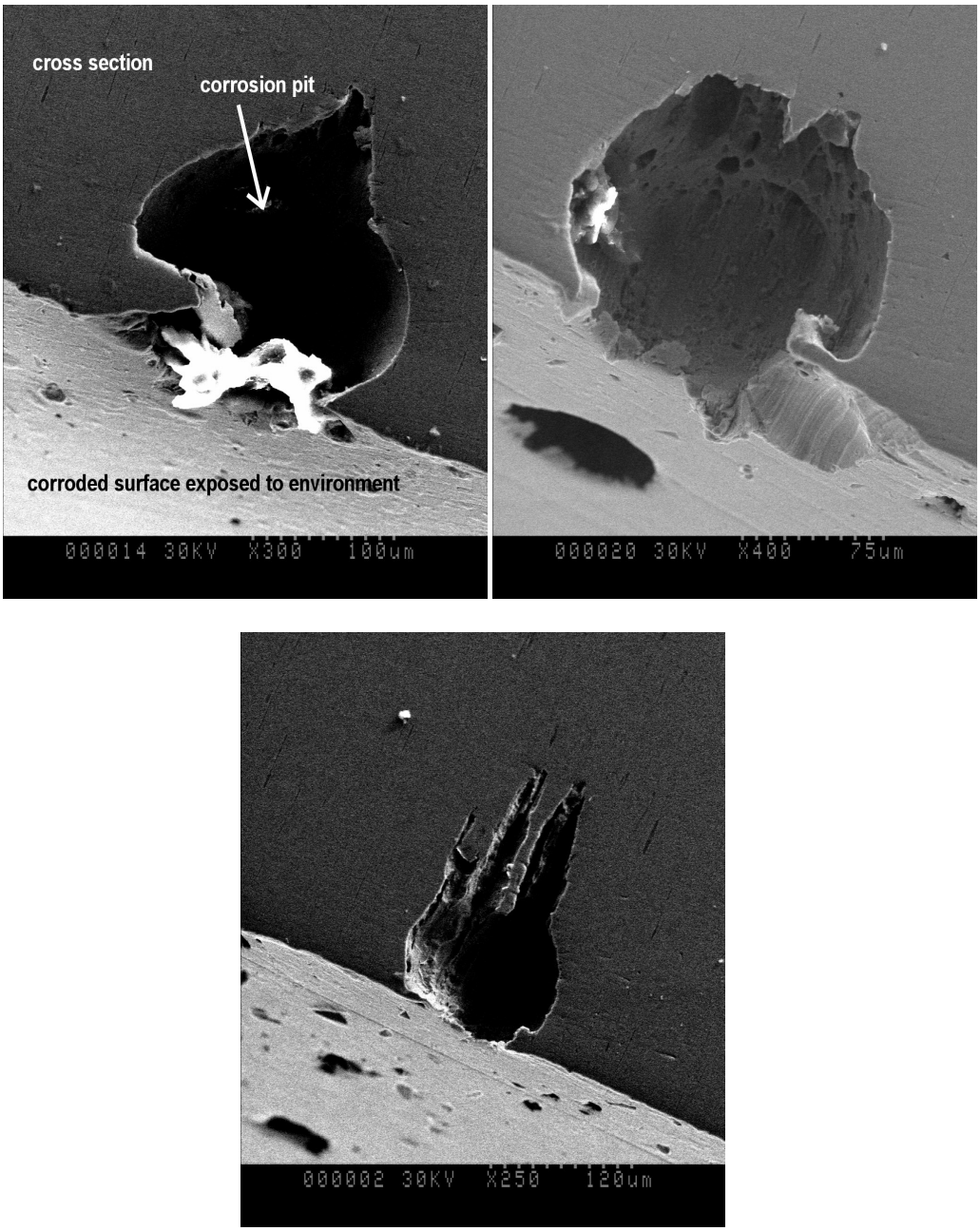


Figure 29.3–1. Scanning electron microscopy observations of cross sections of a stainless ferritic steel sample subjected to an electrochemical corrosion test.

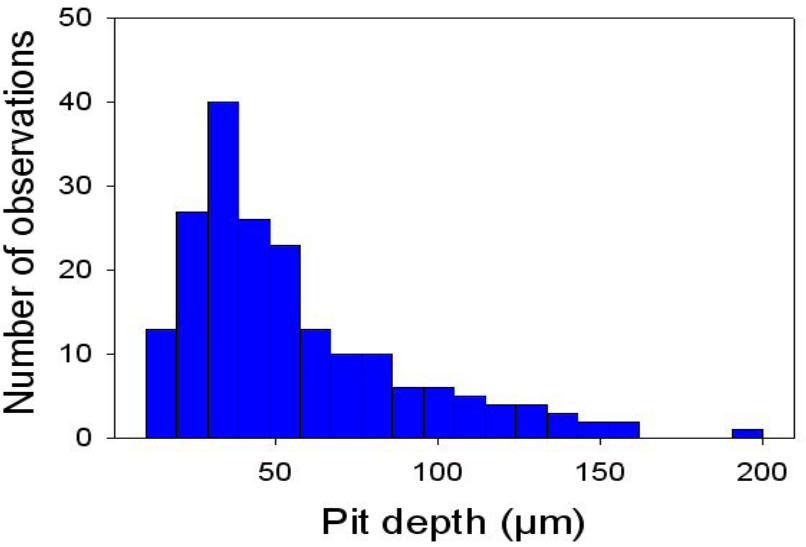


Figure 29.3–2. Pit depths histogram.

29.3.1.2 Fit of a GLD Distribution

First of all, a GLD having a bounded support is fitted on the experimental data. A percentile method was used in this case with $u = 0.01$. The GLD parameters found that best approximate this experimental dataset are $\lambda_1 = 22.3$, $\lambda_2 = 0.0016$, $\lambda_3 = 0.0034$, and $\lambda_4 = 0.055$. Figure 29.3–3 shows the GLD cumulative probability function and probability density function. The adequacy is not rejected with the chi-square test at the 5% confidence level.

With these parameters the upper bound of the GLD is 647 μm, which is lower than the thickness of the plate and thus physically relevant. It can be noticed that the lower bound is negative, which has no physical meaning. The negative part of the distribution represents, however, less than 0.0023% of the whole distribution.

29.3.1.3 Extreme Values Prediction

In order to predict the extreme values of pit depth, a Monte-Carlo method (Efron and Tibshirani (1993)) was used. Starting from the GLD fitted above, 1,000,000 random samples of size 195 were generated using the following equation.

$$P(\nu) = 22.3 + \frac{\nu^{0.0034} - (1-\nu)^{0.055}}{0.0016}, \tag{29.3.1}$$

where ν is a uniformly distributed random number between 0 and 1. From all these bootstrap datasets only the maximum value is retained and the resulting distribution is shown in Figure 29.3–4. From this empirical distribution, the mean and the 90% confidence interval are estimated. The mean extreme pit depth is

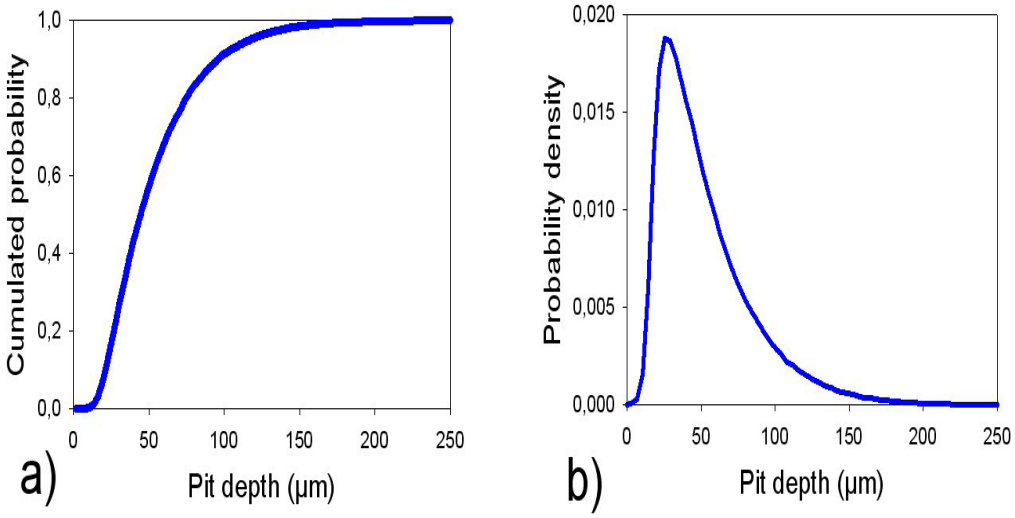


Figure 29.3–3. Cdf and pdf of the fitted GLD.

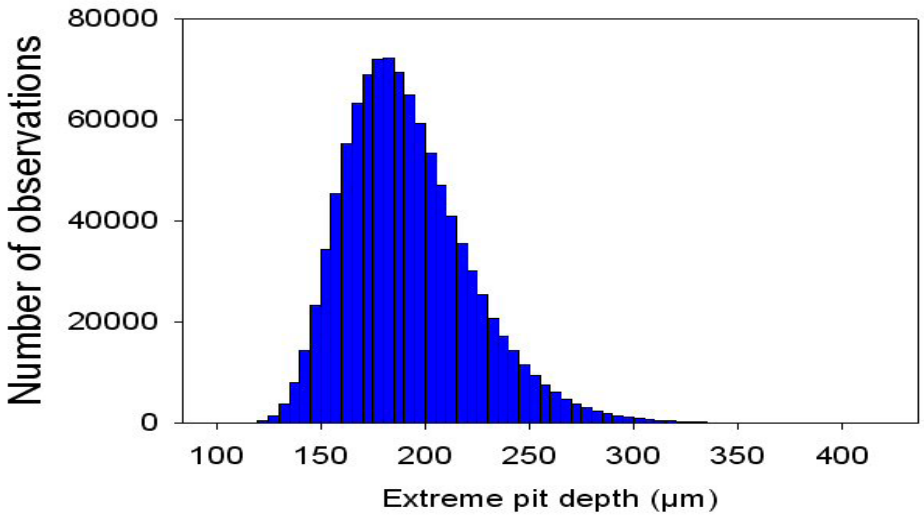


Figure 29.3–4. Empirical probability density function of the 1,000,000 extreme pit depths extracted from the B = 1,000,000 simulated Monte Carlo datasets.

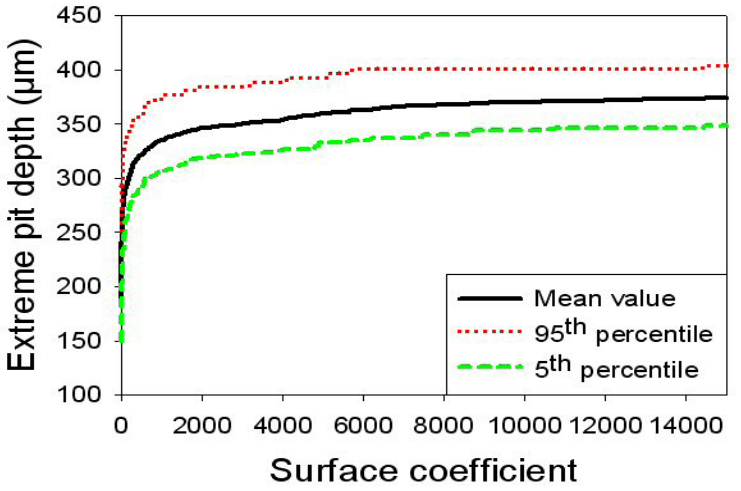


Figure 29.3–5. Prediction of the mean extreme pit depth and its 90% confidence interval for surfaces up to 15,000 times larger than the initial one.

then 190μm and the 90% confidence interval $[150\mu m, 250\mu m]$. This is relevant with the experimental pit depth obtained by microscopy (see Figure 29.3–2).

The question is how to extrapolate this extreme value prediction to larger surfaces. Assuming that the number of pits is proportional to the exposed surface area, the same methodology can be applied to generate bootstrap samples containing 1,950, 19,500, 195,000,... corrosion pits, corresponding to surfaces 10, 100, 1000,... times larger. Figure 29.3–5 shows that the mean of the maximum pit depth increases with the exposed area, however, even for a 10,000 times larger surface, the mean maximum pit depth remains approximately half the actual GLD upper bound. One can also notice that the confidence interval is quite stable, which is encouraging for the stability and reliability of the method, even for large-scale extrapolations.

29.3.2 Roughness

Depending on the application, the roughness of a component can be expected to be either as small as possible (polished surfaces) or close to a necessary value to obtain a given property (adhesion, adherence, wettability etc.). In mechanical applications, most usually, roughness is required to remain limited, because surface relief can act as a crack initiation site under mechanical loading. In this section, the case of the roughness of a metallic femoral head retrieved from a patient is tackled. After surgery, the surface of this articular component shows two severely worn regions by abrasion, that can be seen by the naked eye, as well as a region having a bright finish aspect. It should be mentioned, first, that it is increasingly suspected that abrasive wear of metallic femoral heads of Charnley-

type total hip prostheses (metallic or ceramic femoral head articulating against a UHMWPE acetabular cup) is due to foreign third bodies (like particles of bone or hydroxyapatite, polymeric debris of cement, metallic beads or fibers) that have been often found embedded into the surface of UHMWPE (Ultra High Molecular Weight Polyethylene) acetabular cups and in the peri-prosthetic tissues. Second, scratching of metallic femoral heads of Charnley type hip prostheses is thought to be a major factor affecting the wear rate of the UHMWPE counterface leading to an excessive liberation of wear debris at the origin of osteolysis phenomena and premature retrieval of the medical devices. Third, the assumption of scratching by foreign hard bodies is particularly of paramount importance regarding contemporary prostheses with modular interfaces and/or porous metal coatings that are likely to liberate hard metal debris (see Najjar, Bigerelle, Migaud, and Iost (2005), Najjar, Bigerelle, Migaud, and Iost (2006), Hall, Siney, Unsworth, and Wroblewski (1997), Brown, Stewart, Nieman, Pederson, and Callaghan (2002), McNie, Barton, Ingham, Tipper, Fischer, and Stone (2000a), Tipper, Ingham, Hailey, Besong, Fisher, Wroblewski, and Stone (2000), Elfick, Smith, Green, and Usworth (2001), Bowsher and Shelton (2000), McNie, Barton, Ingham, Tipper, Fischer, and Stone (2000b), Najjar, Behnamghader, Iost, and Migaud (2000)).

In the case of the femoral head analyzed in this study, it is therefore assumed that only the region having a bright finish aspect suffers a "normal wear behavior." To study, as a first step, the normal wear behavior between the femoral head and the acetabular cup, only this region is considered. Two questions can be asked about this region: "does the roughness of this region remains acceptable after an explantation regarding the lifetime of the prosthesis?" To answer this question, one can check in a first approach if the Ra (arithmetic roughness parameter) is lower than the limit of $0.05\mu m$, which corresponds to the value recommended by the ISO 7206-2 standard for a metallic femoral head before its implantation (ISO 7206–2 (1996)). If the answer to this question is positive, the extrapolation of the Ra distribution to the whole head surface should help in answering the second question: "Assuming the case of a normal wear behavior affecting the overall surface of the femoral head (hypothetical case of the absence of any kind of foreign third body embedded in the acetabular cup), would this component still follow the ISO 7206-2 standard requirements?"

29.3.2.1 Clinical Information and Experimental Data

The analyzed femoral head and the UHMWPE counterface were retrieved 4 years and 5 months after the implantation because of an osteolysis phenomenon detected on the survey radio of a patient (Najjar Bigerelle Migaud, and Iost (2005) and (2006)). Such a short lifetime is well under that usually expected for Charnley hip prostheses (10 to 15 years) which have been the most frequently implanted ones for the past forty years, all over the world. It is worth noting that metallic and polyethylene debris were also detected in the peri-prosthetic tissues. Dur-

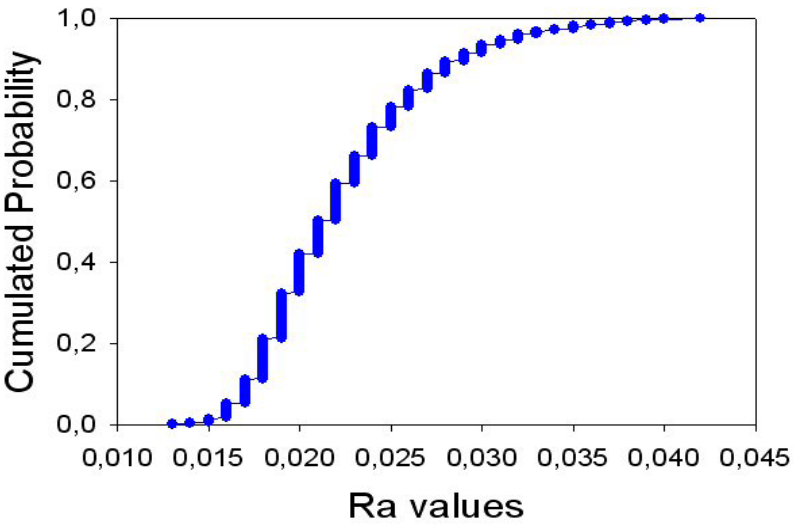


Figure 29.3–6. Distribution of the Ra values.

ing the period of implantation, the metallic femoral head articulated against an UHMWPE actabular cup inserted in a Harris-Galante Mark I metallic shell with a surface treatment consisting of metallic fibers. After revision surgery, one of these fibers was found embedded into the surface of the UHMWPE counterface. This hard foreign body that migrated at the interface of friction is suspected to be the dominant cause of the two severely scratched regions detected at the surface of the femoral head by visual inspection. These two severely worn regions cover approximately 70% of the overall surface of the femoral head and the region of interest for this study, which has a bright finish aspect, covers the rest.

Roughness measurements were only carried out at the surface of the femoral head in this region of interest assumed to have suffered a normal wear behavior. A three-dimensional contacting profilometer (KLA Tenco P10) having $2\mu m$ stylus radius was used and 800 profils of $600\mu m$ length were measured; leading to 800 values of Ra computed, thanks to a dedicated algorithm.

The Ra distribution is plotted in Figure 29.3–6. The mean Ra value is 0.022 μm with a standard deviation of $0.005\mu m$ and the experimental extreme value is $0.042\mu m$. Gaussian modeling is clearly inaccurate (the distribution is not symmetric) and rejected by the chi-square goodness-of-fit test at the 5% confidence level.

29.3.2.2 Modeling the Ra Distibution by a GLD

Here a method of moments is used. The experimental moments are:

$$\text{mean} = 0.0223, \qquad \text{standard deviation} = 0.0047,$$

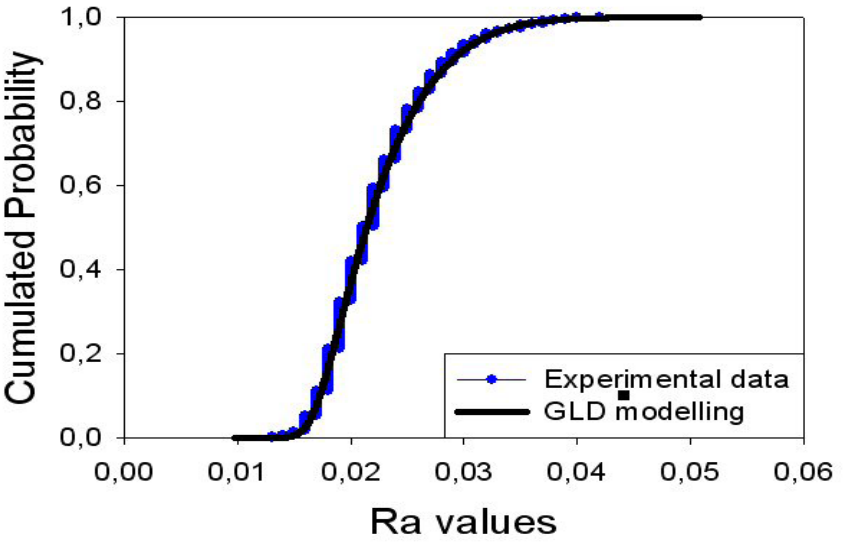


Figure 29.3–7. GLD modeling of the Ra distribution.

$$\text{skewness} = 1.043, \qquad \text{kurtosis} = 4.152.$$

The GLD parameters obtained are:

$$\lambda_1 = 0.01814, \quad \lambda_2 = 26.37, \quad \lambda_3 = 0.01759, \quad \lambda_4 = 0.1469.$$

The upper bound of the GLD support is 0.056 μm which is larger than the maximum Ra authorized by the ISO 7206-2 standard. Even if the lower bound is negative: $-0.0198\mu m$, which is physically inconsistent, only less than $10^{-14}\%$ of the values are actually lower than 0. In addition, this GLD approximation is not rejected by the chi-square goodness-of-fit test at the 5% confidence level. Figure 29.3–7 presents the GLD fit of the experimental Ra distribution.

29.3.2.3 Extreme Values Related to the Bright Area

The experimental extreme value of the 800 Ra measurements is $0.042\mu m$. This distribution has been modeled by a GLD. One can therefore compute the probability for 800 random values to produce an extreme higher than the $0.05\mu m$ authorized roughness. This is done by sampling 800 values from the fitted GLD and only storing the maximum value. This step is repeated 10^6 times. One finally has a distribution of 10^6 extreme values for samples of 800 Ra measurements. This distribution is plotted in Figure 29.3–8.

The mean of this distribution is $0.042\mu m$, which means that, for the extreme value modeling the proposed GLD is unbiased. In addition, the probability to find an extreme value higher than $0.05\mu m$ is of 0.35%. This leads to the following

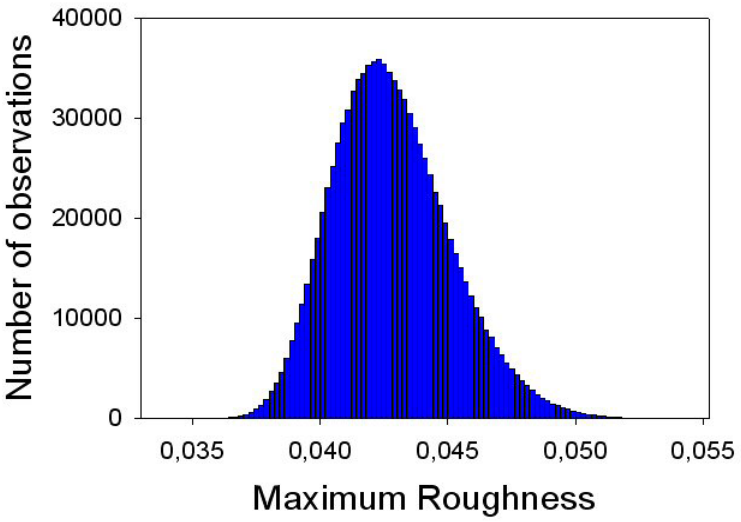


Figure 29.3–8. Distribution of the maximum Ra values of the 10^6 sampling of 800 values.

conclusion: for a surface equivalent to 30% of the femoral head and presenting a bright appearence (the assumption is made that this corresponds to normal wear behavior between the acetabular cup and the femoral head), there is less than 0.35% probability of finding a surface roughness value higher than that mentioned in the ISO 7206-2 standard.

29.3.2.4 Extreme Values of the Whole Femoral Head in Case of Normal Wear

In order to test whether a normal wear behavior of the same duration would lead to an acceptable roughness on the whole femoral head, extreme value extrapolations are now carried out. If one considers that the GLD distribution fitted on the 800 Ra data is valid for the whole head when no third body exists, the same methodology of scale extrapolation as that used for pitting corrosion can be applied. From the GLD 100,000 samples of $800*100/30 = 2666$ data (corresponding to the whole surface) are generated randomly. This leads to a distribution of 100,000 extreme values. The mean extreme value obtained is $0.045\mu m$, the bounds of the 95% confidence interval are [0.0417, 0.0491] and the probability of finding an extreme value higher than $0.05\mu m$ is 0.97%.

This result means that, in less than 1% of cases, the normal wear behavior should remain acceptable after 4 years and 5 months according to the ISO 7206-2 standard. As a consequence, it also implies that the embedded metallic fiber detected after surgery is very probably a major variable in explaining the reduced lifetime of the retrieved prosthesis considered in this study. Indeed, it demonstrates, from a general point of view, how important it is to impede for-

eign bodies to access the interface of friction between articular components with a view to preserving the integrity of their surfaces as long as possible to improve the lifetime of Charnley hip prostheses.

29.4 Conclusion

In the six previous examples of the previous sections dealing with well-known fields of mechanics of materials, GLDs were used either with a descriptive goal or with a predictive objective. A careful fitting procedure is always needed to obtain relevant results. However, it appears that the GLDs offer a quite effective and versatile tool thanks to the following interesting properties.

- The GLD can be definite on a finite support. This leads to the fact that, whatever the quantity under study (grain size, stress field, fatigue lifetime, etc.), the probability of encountering infinite and physically inconsistent values is null. As suggested in Fournier, Bigerelle, Najjar, Iost, and Wilcox (2007), specific fitting procedures can be written to obtain a GLD with precise bounds, if these values are somehow available. This feature is of particular interest when extreme values extrapolations are being carried out.

- The GLD family is very versatile. This means that the GLD distributions can adopt various shapes. For a given phenomenon, e.g. fatigue lifetime, when a simple change of the parameters leads to a complete modification of the final distribution, it is quite convenient to be able to describe all these shapes with a single family of distributions, especially when no *a priori* theoretical knowledge is available to describe this phenomenon.

- As shown in the examples, the GLD is able to offer a correct fit for both the central tendencies and the tails of the distributions. Here again, this a quite important feature when dealing with extreme values considerations, or rare events (e.g. brittle fracture).

References for Chapter 29

Basternaire, F. (1971). "Aspects probabilistes et statistiques de la rupture par fatigue," *Revue Française de Mécanique,* 37, 25–31.

Bazant, Z. P. and Pang, S. D. (2004). "Mechanics-Based Statistics of Failure Risk of Quasibrittle Structures and Size Effect on Safety Factors," *Proc. Natl. Acad. Sci.,* 103, 9434–9439.

Bigerelle, M. and Iost, A. (1999). "A bootstrap analysis of fatigue crack growth

rate: application on the Paris relationship and to lifetime prediction," *International Journal of Fatigue,* 21, 299–307.

Bigerelle, M., Najjar, D., Fournier, B., Rupin, N., and Iost, A. (2006). "Application of lambda distributions and bootstrap analysis to the prediction of fatigue lifetime and confidence intervals," *International Journal of Fatigue,* 28, 223–236.

Bornert, M., Bretheau, T., and Gilormini, P. (2000). *Homogénéisation en Mécanique des Matériaux 1*, Hermes science, Librairie Lavoiser.

Bowsher, J. G. and Shelton, J. C. (2000). "A hip simulator study of the influence of patient activity level on the wear of crosslinked polyethylene under smooth and roughened femoral conditions," *Wear,* 259, 167–179.

Brown, T. D., Stewart, K. J., Nieman, J. C., Pedersen, D. R. and Callaghan, J. J. (2002). "Local head roughening as a factor contributing to variability of total hip wear: A finite element analysis," *J. Biomech. Eng.,* 124, 691–698.

Cailletaud, G., Forest, S., Jeulin, D., Feyel, F., Galliet, I., Mounoury, V., and Quilici, S. (2003). "Some elements of microstructural mechanics," *Computational Materials Science,* 27, 351–374.

Caron, J. F. and Ehrlacher, A. (1999). "Modelling of fatigue microcracking kinetics in crossply composites and experimental validation," *Compos. Sci. Technol.,* 59, 1349–1359.

Castillo, E. (1988). *Extreme Value Theory in Engineering,* Academic Press/Elsevier Science.

Coleman, B. D. (1958). "On the strength of classical fibers and fibers bundle," *J. Mech. Phys. Solid,* 7, 60–70.

Daniels, H. E. (1945). "The statistical theory of the strength of bundles of threads I," *Proc. R. Soc. A,* 183, 405–435.

Diao, X. (1995). "A statistical equation of damage evolution," *Eng. Fract. Mech.,* 52, 33–42.

Diebolt, J., Garrido, M., and Girard, G. (2003). "Asymptotic normality of the ET method. Application to the ET test," *C. R. Acad. Sci. Paris, Ser. I,* 337, 213–218.

Diebolt, J., Garrido, M. and Trottier, C. (2003). "Improving extremal fit: A Bayesian regularization procedure," *Reliability Engineering and System Safety,* 82, 21–31.

Durham, S. D. and Padgett, W. J. (1997). "Cumulative damage models for system failure with application to carbon fibers and composites." *Technometrics,* 39, 34–44.

Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap,* Chapman

Hall, New York.

Elfick, A. P. D., Smith, S. L., Green, S. M., and Unsworth, A. (2001). "The quantitative assessment of UHMWPE wear debris produced in hip simulator testing: The influence of head material and roughness, motion and loading," *Wear*, 249, 517–527.

Ferreira, J. L. A., Balthazar, J. C. and Araujo, A. P. N. (2003). "An investigation of rail bearing reliability under real conditions of use," *Eng. Fail. Anal.,* 10, 745–758.

Ferry, L., Perreux, D., Varchon, D., and Le Bras, J. (1997). "Determination of sample size for estimation of fatigue life by using Weibull or lognormal distribution," *Compos Sci. Technol.,* 57, 1281–1288.

Fournier, B. (2007). "Fatigue-fluage des aciers martensitiques à 9-12%Cr.," Comportement et endommagement, PhD thesis, Ecole des Mines de Paris.

Fournier, B., Rupin, N., Bigerelle, M., Najjar, D., Iost, A., and Wilcox, R. (2007). "Estimating the parameters of a Generalized Lambda Distribution," *Computational Statistics and Data Analysis,* 251, 2813–2835.

François, D., Pineau, A., and Zaoui, A. (1998). *Mechanical Behaviour of Materials II,* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Friedel, J. (1956). *Les dislocations*, Monographies de chimie physique, Gauthier-Villars, Paris, France.

Galambos, J., Lechner, J., and Simiu, E. (1994). "Extreme Value Theory and Applications," Proceedings of the Conference on Extreme Value Theory and Applications. Gaithersburg, Maryland, 1993, Kluwer Academic, Dordrecht.

Gope, P. C. (1999). "Determination of sample size for estimation of fatigue life by using Weibull or lognormal distribution," *International Journal of Fatigue,* 21, 745–752.

Griffith, A. A. (1921). "The phenomena of rupture and flow in solids," *Philos. Trans. Roy. Soc. Lond. Series A,* Containing Papers of a Mathematical or Physical Character, 221, 163–198.

Gumbel, E.J. (1954). "Statistical theory of extreme values and some practical applications," *Applied Mathematics Series 33,* National Bureau of Standards, Washington, D.C.

Hall, P.L, and Strutt, E. (2003). "Probabilistic physics-of-failure models, for component reliabilities using Monte Carlo simulation and Weibull analysis: A parametric study. Reliab," *Eng. Sys. Saf.,* 80, 233–42.

Hall, R. M., Siney, P., Unsworth, A., and Wroblewski, B.M. (1997). "The effect of surface topography of retrieved femoral heads on the wear of UHMWPE sockets,"

Med. Eng. Phys., 19, 711–719.

ISO7206-2 (1996). "Implants for surgery – partial and total hip joint prostheses – part 2: Articulating surfaces made of metallic, ceramic and plastics materials," Geneva, Switzerland.

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Method,* CRC Press, Boca Raton, FL.

Kocks, U. F., Tomé, C. N., and Wenk, H. R. (1998). *Texture and Anisotropy,* Cambridge University Press, Cambridge, United Kingdom.

Krausz, A., Zhiven, X. W., and Krausz, K. (1992). "On the constitutive law of environment assisted fatigue: the physical meaning of the Paris type equations," *Zeitschrieft der Metallkunde,* 83, 356–63.

Lewis, G., Janna, S., and Carroll, M. 2003). "Effect of test frequency on the in-vitro fatigue life of acrylic bone cement," *Biomaterials,* 24, 1111–1117.

Liao, M., Shi, G., and Xiong, Y. (2001). "Analytical methodology for predicting fatigue life distribution of fuselage splices," *International Journal of Fatigue,* 23, 177–185.

Man, J., Petrenec, M., Obrtlik, K., and Polak, J. (2004). "AFM and TEM study of cyclic slip localization in fatigued ferritic X10CrAl24 stainless steel," *Acta Materialia,* 52, 5551–5561.

McNie, C. M., Barton, D. C., Ingham, E., Tipper, J. L., Fischer, J., and Stone, M.H. (2000a). "The prediction of polyethylene wear rate and debris morphology produced by microasperities on femoral heads," *J. Mater. Sci. Mater. Med.,* 11, 163–174.

McNie, C. M., Barton, D. C., Ingham, E., Tipper, J. L., Fischer, J., and Stone, M.H. (2000b). "Modelling of damage to articulating surfaces by third body particles in total joint replacements," *J. Mater. Sci. Mater. Med.,* 11, 569–578.

Nabarro, F. R. N. (1952). "The mathematical theory of stationary dislocations," *Philosophical Magazine,* 1, 269–395.

Nagode, M. and Fajdiga, M. (1998a). "A general multi-modal probability density function suitable for the rainflow ranges of stationary random processes," *International Journal of Fatigue,* 20, 211–223.

Nagode, M. and Fajdiga, M. (1988b). "On a new method for prediction of the scatter of loading spectra," *International Journal of Fatigue,* 20, 271–277.

Najjar, D., Behnamghader, A., Iost, A., and Migaud, H. (2000). "Influence of a foreign body on the wear of metallic femoral heads and polyethylene acetabular

cups of total hip prostheses," *J. Mater. Sci.,* 35, 4583-4588.

Najjar, D., Bigerelle, M., Migaud, H., and Iost, A. (2005). "Identification of mechanism of scratch generation on a retrieved metallic femoral head," *Wear,* 258, 240–250.

Najjar, D., Bigerelle, M., Migaud, H., and Iost, A. (2006). "About the relevance of roughness parameters used for characterizing worn femoral heads," *Triboloogy International,* 39, 1527–1537.

Nikbin, K., Yatomi, M., Wasmer, K., and Webster, G. (2003). "Probabilistic analysis of creep crack initiation and growth in pipe components," *International Journal of Pressure Vessels and Pipping,* 80, 85–595.

Paris, P. C. and Erdogan, F. (1963). "A critical analysis of crack propagation laws," *J. Basic Eng. ASME, Series D,* 55, 528–534.

Park, C. and Padgett, W. J. (2005). "Newcumulative damage models for failure using stochastic processes as initial damage," *IEEE Trans. Reliab.,* 54, 34-44.

Pauchard, V., Grosjean, F., Campion-Boulharts, H. and Chateauminois, A. (2002). "Applications of a stress-corrosion-cracking model to an analysis of durability of glass/epoxy composites in wet environments," *Composites Science and Technology*, 62, 493–498.

Petryna, Y. S., Pfanner, D., Stangenberg, F., and Krätzig, W. B. (2003). "Reliability of reinforced concrete structures under fatigue," *Reliability Engineering System Safety*, 77, 253–261.

Pierce, F.T. S. (1926). "Tensile tests for cotton yarns v. "the weakest link" – theorems on the strength of long composite specimens," *Journal of the Textile Institute, Transactions*, 17, 7355–7368.

Polak, J., Petrenec, M., and Man, J. (2005). "Dislocation structure and surface relief in fatigued metals," *Materials Science & Engineering A, Structured Materials* Vols. 400–401, 405–408.

Shi, G., Atkinson, H., Sellars, C., and Anderson, C. (1999). "Application of the generalized Pareto distribution to the estimation of the size of the maximum inclusion in clean steels," *Acta Materialia,* 47, 1455–1468.

Shi, G., Atkinson, H., Sellars, C., Anderson, C., and Yates, J. (2001). "Computer simulation of the estimation of the maximum size in clean steels by the generalized Pareto distribution method," *Acta Materialia,* 49, 1813–1820.

Soboyejo, W. O., Shen, W., Lou, J., Mercer, C., Sinha, V., and Soboyejo, A. B. O. (2002). "A probabilistic framework for the modelling of fatigue in a lamellar XD gamma titanium aluminide alloy," *International Journal of Fatigue,* 24, 69–81.

Sutton, A. P. and Balluffi, R. W. (1995). *Interfaces in crystalline materials*,

Oxford Science Publications, University Press, Inc., New York.

Taylor, H. M. (1994). "The Poisson-Weibull flaw model for brittle fiber strength", Extreme Value Theory and Applications: Proceedings of the Conference on Extreme Value Theory and Applications, Gaithersburg, Maryland, 1993, Vol. I, J. Galambos, J. Lechner and E. Simin, Editors, 43–60, Kluwer Academic, Publishers, Dordrecht, The Netherlands.

Tipper, J. L., Ingham, E., Hailey, J. L., Besong, A. A., Fisher, J., Wroblewski, B. M., and Stone, M. H. (2000). "Quantitative analysis of polyethylene wear debris, wear rate and head damage in retrieved Charnley hip prostheses," *J. Mater. Sci. Mater. Med.,* 11, 117–124.

Vallée, T., Keller, T., Fourestey, G., Fournier, B., and Correia, J. (2009). "Adhesively bonded joints composed of pultruded adherends: Considerations at the upper tail of the material strength statistical distribution," *Probabilistic Engineering Mechanics,* 24, 358–366.

Vincent, L., Libert, M., Marini, B., and Rey, C. (2009). "Towards a modelling of RPV steel brittle fracture using crystal plasticity computations on polycrystalline aggregates," submitted to *Journal of Nuclear Materials.*

Weibull, W. (1939). "A statistical theory on the strength of materials," *Ingenioersvetenskapsakad. Handl.,* 151, 1–45.

Yasuhiro, M., Takahiro, K., and Kazuko, M. (2003). "Probabilistic models of combinations of stochastic loads for limit state design," *Struct Safety,* 25, 69–97.

Chapter 30

Fitting Statistical Distributions to Data in Hurricane Modeling[1]

Mark E. Johnson
Department of Statistics
University of Central Florida
Orlando, FL 32816-2370, USA
mejohnso@mail.ucf.edu
Charles C. Watson, Jr.
Watson Technical Consulting
1204 E. 49th Street
Savannah, GA 31404-4014, USA
cwatson@methaz.org

Fitting probability distributions to hurricane-related data is an essential activity in hurricane planning, designing structures, and catastrophe modeling applications. The recent devastating hurricane seasons and the resultant debates over design criteria and computer models using analyses based on these fits motivate a closer examination of this issue. The primary objective in this chapter is to describe the background and applications of historical hurricane data fitting, the operational aspects of which have dictated adjustments to the standard methodology. The emphasis here is on the interaction between data quality and dynamics, the need for rapid but stable assessments of that data, and statistical fitting methodologies. Validation and applications are discussed, along with an analysis of the return periods of damage in the New Orleans area.

[1]This chapter appeared, under the same title, in the *American Journal of Mathematical and Management Sciences*, (2007). Vol. 27, Nos. 3 & 4, pp. 479–498. Copyright © 2007 by American Sciences Press, Inc., Syracuse, New York, 13224-2104. Reprinted by permission.

30.1 Introduction

The 2005 Atlantic hurricane season was record breaking in terms of number of storms, extent of damage, and impact on the insurance industry. The financial hits taken by insurance and re-insurance companies have generated a flurry of commentary on catastrophe models, which are the basis for establishing residential premiums for wind perils from hurricanes. Rising water damage such as storm surge or riverine flooding is not covered by traditional insurance policies. However, since damage by wind or rain intrusion is covered, arguments and legal actions have started over not only the extent of damage but the timing and cause of the damage (if the structure failed by wind before failing from water, the private company pays; if the reverse, federal flood insurance covers the loss). Since many flooded areas were outside federal flood zones, which are in theory based on a 100-year return period event, homeowners are left with massive out-of-pocket losses. The following comments from Matthias Weber are (see Best's Review (2005)) typical:

> Everyone's a little bit disappointed with the fact that the modeled losses were significantly smaller than the actual losses, and $\cdots$ people are saying the models aren't reliable $\cdots$ what industry needs to do is demand improvement of computer models. (from Best's Review, (2005)).

Of course, when financial losses or loss of life are staggering, design criteria and computer models offer an inviting target for venting. In their December 2005 issue, *Best's Review* noted that AIR Worldwide Corp., a commercial catastrophe modeler, had cited the quality of exposure data as a more pertinent explanation for the discrepancies between modeled and actual losses. There is clearly intense interest in computer modeling of hurricanes and likewise interest in data sources and ultimately distribution fitting.

Since 1996, the authors have been working jointly on fitting distributions to hurricane phenomena or their effects in the context of operational, funded projects. Our initial joint effort (Watson and Johnson (1997)) concerned fitting extreme value distributions to wind and storm surge data at Montego Bay, Jamaica. In this endeavor both statistical expertise and numerical modeling and geophysics were given equal emphasis and an effective team was created. This effort was quickly expanded to the full Atlantic Basin and ultimately worldwide, as described subsequently.

Early on we realized that the typical paradigm in statistics of distribution fitting to historical datasets at the analyst's leisure was untenable. Although we continue to observe re-analyses of Fisher's iris data, Brownlee's stack loss data, 1986 baseball salary data, and other classical datasets (e.g., Friedman and Meulman, (2004)), in the realm of hurricane science we are confronted with terabytes of new data sources requiring automated fitting procedures. Not only does every

year bring in new volumes of data, but even the historical datasets are not necessarily static. Hurricane Andrew in 1992 has undergone extensive re-analyses, and its maximum wind speed as given in the widely used HURDAT database (Jarvinen, Neuman, and Davis (1984), as updated annually by the Tropical Prediction Center, TPC) was increased by 20 knots. Hence, fitting methodologies must be robust to such perturbations to the databases. This environment has a major impact on the focus of our statistical concerns, a point further discussed in Iman, Johnson, and Watson (2006). Some retrospective statistical analyses can be necessitated on a time available basis, but for the most part we are confronted with continual pressures to advise decision makers. In our applications, the criterion of goodness-of-fit tends to be trumped by the goodness of our predictions (which are indirectly supported by the distribution fits and other analyses). In spite of the occasional stresses owing to time constraints, it is nevertheless gratifying to see the fruits of these labors having tangible impacts in the field.

This chapter first reviews the basics of hurricane modeling necessary to generate datasets for analyses. Note that buoy-based and ground-based anemometers (wind speed gauges) are insufficient to characterize historical events for damage forecasts and estimating insurance premiums. Wind gauges tend to fail during extreme events and their sparse placement precludes detailed analysis even if the gauges survive to record peak winds. Satellite Remote Sensing offers tremendous opportunities, but this data is available only for the last decade. Thus, wind field models are necessary to determine likely wind speeds across an entire region. With annual maximum winds available at any given site, Section 30.3 considers the fitting of extreme value distributions for massive numbers of individual sites. Graphical summaries and predictive performance indicate that the two-parameter Weibull distribution is more than adequate for our purposes. Section 30.4 documents several examples of datasets used in our work where data preparation issues dominate. Data quality is critical in order to avoid optimal fits to irrelevant or tainted data. Section 30.5 focuses on basin-wide distribution fitting performance, a critical aspect to avoid being duped on the results at one favorite site, while in Section 30.6 we apply our analyses to the debate over the design criteria for the New Orleans flood protection system. We conclude the chapter with a call for statisticians to consider operational projects that test their mettle, and for practitioners in other fields to more closely incorporate such expertise as an inherent part of all phases of the analysis.

30.2 Hurricane Modeling Basics

Historical wind speed observations, damage estimates, and insurance loss data provide insight into the impacts of wind perils but do not provide direct mechanisms for estimating future losses or future hurricane activity in general. Watson and Johnson (2004) describe hurricane catastrophe modeling as consisting of five

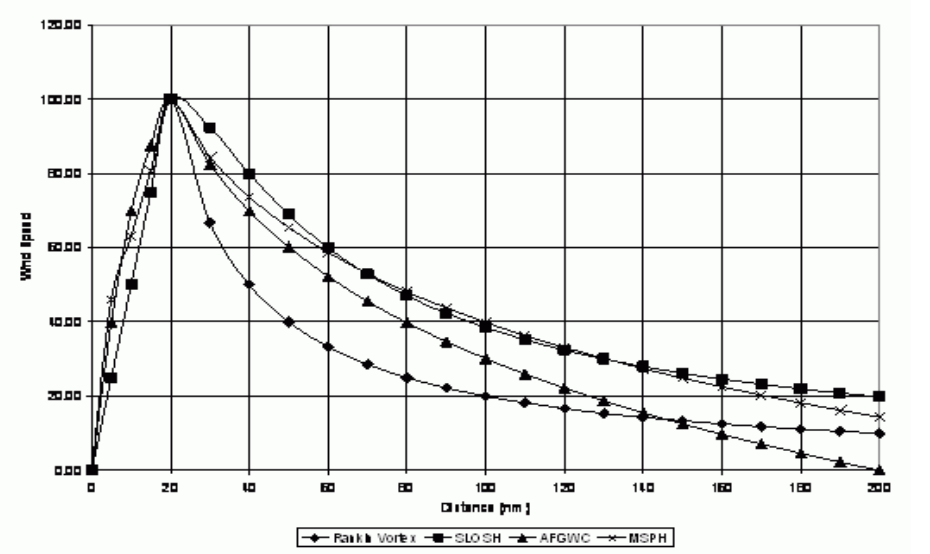


Figure 30.2–1. Representative wind profiles for parametric wind models.

basic components:

- Wind field,
- Damage functions (vulnerability),
- Insurance models,
- Frequency of events,
- Input databases.

The hurricane field itself can be mathematically modeled as a function of such characteristics as the minimum central pressure, the far field pressure, the maximum wind velocity (at the eye wall), the radius of maximum winds, forward speed, and so forth. More complex wind fields can be devised using additional input parameters. Figure 30.2–1 provides a set of four cross-section snapshots of a hurricane. When combined with forward velocity, these functions produce realistic asymmetric wind fields. The simplest function is a solid of rotation (the "Rankin Vortex"). Despite its simplicity, it performs well with some historical storms. The SLOSH wind field is used by the National Weather Service in its storm surge modeling. The AFGWC wind field is used in wave models, while the Standard Project Hurricane (MSPH) is used for a wide variety of design applications. Even models that appear to produce nearly identical winds can produce different damage levels due to the nonlinear nature of damage functions.

Damage functions can be very sophisticated and tend to be the most proprietary aspects of commercial models. A simple but sometimes effective damage

Table 30.2–2. Wind (in knots) and damage percentage for wood frame structure at 30 nm.

	Wind	Damage
Rankin	66.67	22.26%
SLOSH	92.31	59.09%
AFGWC	82.39	42.02%
MSPH	84.32	45.05%

function yields no damage for wind speeds less than a nominal wind speed (for example, 30 mph), complete damage for wind speeds in excess of 150 mph, and a cubic function in between. The insurance component handles deductibles and converting physical damage to dollar losses (for example, in some jurisdictions 51% damage equates to a total loss due to building code requirements). Table 30.2–2 shows the difference in computed damage using the X-Cubed damage function for the wind models used in Figure 30.2–1 at a distance of 30 nautical miles (1 nautical mile = 1.15 statute miles and 1 knot = 1 nm/hr). As can be seen, small differences in wind speed produce significant differences in damage.

For the purposes of this chapter, the damage and insurance components are not central to the discussion. Moreover, in a recently completed study, Watson and Johnson (2006) refined their earlier assessment of the relative contributions to loss costs to be dominated by meteorology. In particular, we examined nine distinct wind field models, four friction functions, and nine damage functions.

Given a hurricane simulation model, the next step in loss estimation is to accumulate damage over a single hurricane (to compare modeled to actual losses) or a collection of hurricanes (to determine long-term insurance rates). Incidentally, our approach has been to simulate the complete historical storm set (1346 storms from 1851 through 2005) without any smoothing of tracks or intensities. Our smoothing comes into play at individual sites by recording the annual maximum winds and then fitting an extreme value distribution to the collection of values (Section 30.3). This approach has been operational in Florida since its initial construction in 1999 to support local mitigation measures for the state emergency management agency. This project was at a high resolution (30 meter grid cells), which led to consideration of 3 billion sites in the state. Residential loss costs are then computed by overlaying the housing tax parcel data on the basic grid.

The fifth component of hurricane catastrophe modeling involves the supporting databases. Topography and bathymetry are required for the region of interest. Great strides have been made in this regard. Digital terrain databases can be constructed from satellite data. The bathymetry which comes into play in wave and storm surge calculations can be gleaned adequately to depths of 50 ft. Land

use and land cover data are also obtained from remote sensing and then classified to determine the frictional impacts on wind.

30.3 Distributional Fitting at Individual Sites

Johnson and Watson (1999) have found that the two-parameter Weibull distribution is quite suitable for fitting annual maximal winds at individual sites. This distribution has density function:

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right], \quad x \geq 0,$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. Early on we considered other extreme value or reliability-type distributions (for example, lognormal and inverse Gaussian). The lognormal distribution provided upper-tail behavior that was meteorologically unrealistic (the tails are too heavy suggesting 200 mph wind speeds throughout the Atlantic over a 100-year time frame). The inverse Gaussian was slightly worse in prediction than the Weibull and was not as convenient for generating prediction limits, to be described shortly. Its prediction performance was also substandard compared to the Weibull.

Upon selecting the Weibull distribution, we recognized that we would be fitting it throughout multiple hurricane basins at an enormous number of sites. Hence, it was critical to develop an estimation strategy that was virtually foolproof. We found that an initial detailed grid of values for alpha (beta can be obtained in closed form once alpha is estimated) was much safer than using the fast Newton-Raphson iterative scheme but which occasionally is an unreliable numerical method (diverges).

Figure 30.3–1 shows the distributional fits at Taipei, Taiwan. Viewed graphically, the Weibull fit is a clear winner at this site, and the χ^2 goodness-of-fit statistic is superior to the corresponding statistic for both the lognormal and Inverse Gaussian fits. We caution against drawing too many conclusions for the analysis at a single site. For any given site, sometimes the Weibull would prove superior, and sometimes perhaps the lognormal. However, when thousands of sites were considered the Weibull was superior in predictive ability, despite the lognormal having fairly comparable χ^2 values. In fairness, the χ^2 is an omnibus test and our major concern is with the tail behavior (extreme events) for which there are not many events to assist in the distribution discrimination. We found that the lognormal distribution was much more sensitive to the extreme event at a given location, also causing a high bias. This indicates that it is important not to lose sight of the overall objective in the quest for optimization of a narrow statistical objective.

Return periods are well understood by the emergency management and government agencies that have funded our projects. To illustrate, the hundred-year

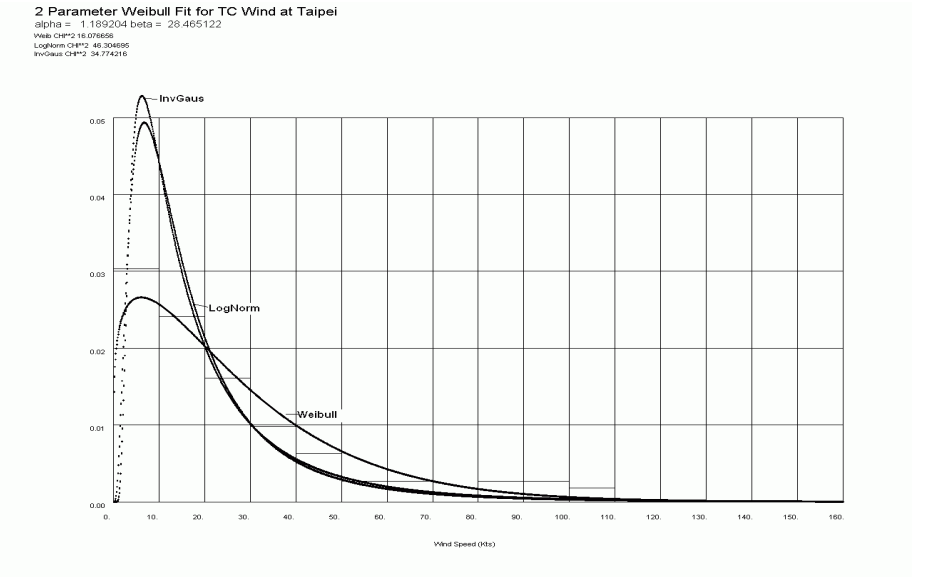


Figure 30.3–1. Fit for tropical cyclone wind at Taipei, Taiwan.

wind is roughly the maximum wind that we expect to see in the next hundred years. Following Simiu and Scanlan (2003), more formally, the hundred-year wind is the wind speed such that there is a 1 in 100 chance of exceeding it in the next year (the Atlantic hurricane season is officially June 1 through November 30, although hurricanes have occurred in every calendar month over the past 150-plus years). Return periods matter since structure engineers and architects typically design buildings and infrastructure against various extreme events, such as 100-year wind storms.

Upon fitting a Weibull distribution via maximum likelihood to a site's annual maximal winds, various return-period winds can be readily extracted. The 90th percentile represents the 10-year event, the 96th percentile the 25-year event and the 99th percentile corresponds to the commonly used 100-year return period. If X represents the random variable for the fitted Weibull distribution, then the value of c satisfying $P(X < c) = 0.99$ corresponds to the 100-year return period maximum wind. The 50th percentile is our best guess of next years expected maximal wind. These percentiles extracted from the fitted distribution do not take into account the uncertainty in parameter estimation. The percentiles offer our best estimate of the return periods but our decision-maker clients wanted to build in further mechanisms for accounting for the uncertainty owing to estimation. This issue could be avoided by merely using the 155 years' worth of annual maxima at each site and thus rely totally on the historical storm record. There are, however, some areas that have not experienced any direct hits of hurricanes yet it is meteorologically possible or even probable. Given the relatively small swath of the most intense winds (generally only 30 to 60 miles wide), of over such

Table 30.3–2. Prediction limits for Taipei, Taiwan.

YEAR	50%	75%	90%	95%	99%
10 YR	58.2	61.2	63.9	66.0	70.4
25 YR	77.1	81.6	86.7	90.5	104.5
50 YR	90.6	97.0	105.1	111.3	130.3
100 YR	103.2	112.9	124.1	133.2	157.7

a short period it is possible for a site to have fallen in between major damage swaths. Some proprietary models address these spatial gaps by fitting distributions to the historical storm tracks and intensities and then simulate 100,000 years or more of events to fill out the regions. This approach is computationally onerous in order to cover those areas that have not experienced the rare Category 5 storms. Our approach is to capture the parameter estimation uncertainty by sampling from the asymptotic distributions of alpha and beta (easily and adequately modeled using the Fisher Information matrix and the normal asymptotic distribution), generating 10,000 realizations of fitted Weibull distributions and then determining percentiles from these samples. For example, the 10,000 upper 90th percentiles at a given site are computed and sorted. We use the 9500th ordered value as an upper 95% prediction limit for the 10-year return period.

Table 30.3–2 shows the prediction limits computed from the Weibull distribution shown in Figure 30.3–1. This information allows the planner to build in an appropriate level of protection against uncertainty in the parameter estimates. In our work in the Caribbean and elsewhere, some standard guidance has emerged. For easily replaced infrastructure such as coastal recreational facilities, the 25-year 50% value is used. For single-family homes, which have typical design (and mortgage!) lifetimes of 30 to 50 years, the 50-year 50% or 75% limit can be used, while the 100 year is typically used for larger multifamily use. Facilities important for economic recovery or shelters often use the 100-year 75% and 90% values. Critical lifeline infrastructure (hospitals, etc.) should use the 100 year 95% prediction limit.

30.4 Data Quality and Preparation Issues

The data fitting efforts described in the earlier sections are predicated upon some key datasets that continue to evolve (and hopefully improve!) over time. Of particular interest is the dataset known in the field as HURDAT (Jarvinen, Newman, and Davis (1984)). One of the author's (MEJ) initial exposure to the database was in 1996 while on sabbatical at the National Hurricane Center in Coral Gables, Florida. The Associate Director at the time, Jerry Jarrell provided a copy of the dataset on a 3.5 inch floppy which turned out to be very difficult to decipher. Eventually it was discovered that the data was stored in a Wordperfect File! Al-

Table 30.4–1. 100 Year MLE sind speed (knots) from various subsets of HURDAT.

	Miami	Tampa	Ft. Meyers
1851–2003	104	88	108
1871–2003	108	85	107
1886–2003	107	86	104
1900–2003	118	82	93
1950–2003	92	66	92

though it was helpful to have the dataset, it was astonishing that this wealth of information resided in such a peculiar format. At that time, there were also found a number of obvious typos, incorrect entries, or coded entries that were inconsistent. For example, a missing central pressure value was sometimes given as 999, which also is a legitimate central pressure!

Fortunately, since 1996, the Hurricane Research Division has made great strides to revise and to clean the database as well as to extend the time back to 1851 (Landsea, Anderson, Charles, Clark, Dunion, Fernández-Partagás, Hungerford, Neuman, and Zimmer (2004a)). Certain decades of data have also been revisited and adjustments made based on consensus of a review committee of experts. This effort is not without some controversy, as the group increased the maximum wind speed of Hurricane Andrew (1992) by 20 knots while not changing the corresponding minimum central pressure value, causing considerable confusion in the catastrophe modeling community. It should also be noted that when we began our work in 1996, there were some groups that advocated the use of only hurricane data obtained after the advent of routine aircraft reconnaissance in 1948. Other applications included only data since 1900. For both of these limited datasets and especially for the 50 years set since 1948, the estimation of extreme percentiles corresponding to the 100-year return period, for example, are greatly inhibited. Our own examination of the data prior to 1948 did not demonstrate strong inter-decadal trends and we have opted to use all of the HURDAT data that is available. We were one of the first groups to advocate the 1886–present HURDAT dataset, and when the 1850–present time frame became available, upon further examination again opted to use the full dataset.

Table 30.4–1 provides a summary on the return periods for three south Florida cities determined from various subsets of HURDAT. Not surprisingly, the full HURDAT data set provides the most stable estimates of wind return periods. In particular, the inclusion of results from 1886 down to 1851 provides tighter bands on the prediction intervals without a notable overall shift. The table also demonstrates that our values for Southwest Florida tend to be larger than those observed by groups employing data for the 1900–present data. There was a cluster of intense hurricane activity in the late 19th century, which meteorologists point to in the context of intra-decadal variability. With consideration of the full dataset,

Hurricane Charley of 2004 (which struck Punta Gorda near Sarasota, Florida) does not appear quite so unusual compared to the historical record.

HURDAT has several advantages: it is widely available, is in an easily decoded format, and is routinely updated. Unfortunately, it has a number of aspects that make it problematic. On aspect of HURDAT which demonstrates fairly convincingly the lack of statistical expertise input to its development is the rounding or truncation (it is unclear which technique is used) of positions to the nearest 0.1 degree. Such precision was common in the 1960s when 16 byte integer arithmetic was more common and every digit was an effort to retain. Since 0.1 degree is about 6 nautical miles, this rounding introduces undue error and an additional source of variation. The counterpart group of HRD in the Pacific reports locations to the nearest 0.01 degree, which is an improvement. The rounding phenomenon is not confined to spatial location. Maximum wind speeds in HURDAT are rounded (or truncated) to the nearest 5 knots on the pretext that wind speeds are impossible to measure very accurately. However, in the spirit of measurement systems analysis source of variation studies, rounding introduces yet another source of variation to the dataset. Additionally, with damage proportional to the square or cube of wind speed, the rounding procedure can adversely propagate into the damage calculation. A further complication is that missing values are sometimes estimated from simple relationships rather than flagged as missing. Thus, central pressure may be estimated from winds and vice versa without the data point being flagged as being computed rather than observed. This precludes the use of advanced data mining techniques that are likely to shed light on the problem areas and opportunities rather than simply obfuscating them.

Although the HURDAT dataset has a relatively long record (157 years at present), it does not include vital additional variables characterizing hurricane behavior such as the radius of maximum winds, the far field pressure, and radius to far field pressure. The far field pressure is known to be influential on the sensitivity and uncertainty of catastrophe models (Iman, Johnson, and Watson (2005a) and (2005b)). One dataset available to assess far field pressure involves comprehensive archive (CARQ) records from the Automated Tropical Cyclone Forecast (ATCF) system database (Sampson and Schrader (2000)). The ATCF system is used for operational forecasting at the National Hurricane Center, the Joint Typhoon Warning Center, and other agencies. Our initial consideration of these records discovered anomalies with specific entries and some obvious errors. For example, there were a few examples where the radius to maximum winds was greater than the radius to far field pressure or there was a negative pressure gradient. Although this dataset represents a superb set of detailed data on recent events (1990 to present), the dataset is not a panacea. Our work has been completed and submitted to a meteorological journal, as the substance is of passing interest to statisticians and the statistical content is quite elementary. The implications here are that data quality is essential for providing suitable inputs to catastrophe models. The ATCF CARQ records suffer from some similar prob-

lems as HURDAT in terms of structure. Watson and Johnson (2006) note the preponderance of even pressure values (1010 and 1012 rather than 1009 and 1011 mb). This is due to the tendency of the human analysts to use even numbers (a technique taught in most university level weather analysis classes), and offers an additional source of variation to the data that can further propagate depending on the analysis. It is vital for the statistician to understand and to be involved in the underlying decision-making process when data bases are designed.

For another example, in the ATCF system positions in the real-time position estimate file are recorded to the nearest 0.01 degree, yet as noted earlier are truncated to the nearest 0.1 degree in the "best fit" and HURDAT files. A meteorologist would say "we can't reliably measure storm positions to the nearest 0.1 degree some times, and the older data is worse than that, so why bother to record it with greater accuracy?" The statistician would advocate retaining the accuracy to avoid additional uncertainty as well as to allow the potential for improvement in measurement accuracy.

30.5 Predictive Ability and Performance

In light of the data quality issues outlined above, and the simplicity of the statistical approach, one might ask how well the ultimate predictions made by the approach outlined in Section 30.3 perform. Here we considered two issues: stability and accuracy. Stability is important as one would hope that as new data is added to the process, the original estimates would not change substantially. Imagine designing a structure for what was thought to be a 100-year event, and then discovering the next year that the design storm actually occurs every 30 years (on average). As noted earlier the HURDAT dataset has undergone many changes over time. At the onset of the Montego Bay study, the dataset extended from 1886 to 1996. In addition to annual updates, in 2001 the dataset was extended to 1871, while the current version runs from 1851 through 2005 (Landsea, Anderson, Charles, Clark, Dunion, Fernández-Partagás, Hungerford, Neumann, and Zimmer (2004a)). Not only have additional historical years been added, changes have been made to the track and intensity of storms already in the dataset, most famously the 2004 "upgrade" of Hurricane Andrew from Category 4 to Category 5 (Landsea, Franklin, McAdie, Beven, Gross, Pasch, Rappaport, Dunion, and Dodge (2004b)). Given the dynamic nature of the database, techniques used to compute return periods for lifeline infrastructure must be resilient and insensitive to such changes while maintaining accuracy. Table 30.5–1 shows our original 1996 10-, 25-, 50-, and 100-year Maximum Likelihood Estimates for Montego Bay, Jamaica, along with the estimates using the update 1871–2002 and 1851 to 2005 datasets, as well as the observed 0.9, 0.96, 0.98, and 0.99 percentile values from the full 1851–2005 dataset. In addition to database quality improvements, the 1851–2005 analysis used a multi-model ensemble approach to making the wind

Table 30.5–1. Stability of Weibull approach to dataset evolution.

Return Period	1886–1996	1871–2005	1851–2005	Observed
10	38.9	36.4	34.4	35.4
25	66.6	66.8	64.2	68.4
50	89.5	89.2	88.0	89.6
100	99.9	99.7	93.6	94.1

estimates (Watson and Johnson (2004)).

In light of the 5-knot rounding/truncation issues discussed in Section 30.4, the stability of the values is remarkable. The close agreement between the forecast and observed proportions is initially comforting, but an analysis for a single site is not diagnostic and could in fact be misleading: the close agreement might be evidence of over-fitting.

A better performance metric is to compare the predictive ability of the technique over many sites using automated techniques. Using the prediction limit technique described in Section 30.2 and the longer datasets created by TPC, we have an opportunity to construct a test on a basin-wide basis. First, we created a 6-nautical mile grid over the Atlantic Basic from 15N to 45N, and 60W through 100W (135,000 points) to span the area impacted by hurricanes. Next, we segmented the historical dataset into two parts. The period from 1851 through 1955 was used to fit the Weibull distribution at each site. Fifty-year wind speeds were computed at 50%, 75%, 90%, and 95% prediction limits for the 27,106 land cells in this grid that experienced at least one hurricane force wind event between 1851 and 1955. Finally, we compared the number of sites that exceeded the various prediction limits during the fifty year period 1956–2005, the second segmentation of HURDAT. Thus, the dataset used for fitting and the dataset used for verification were independent; it was as if we were in 1955 trying to forecast the next 50 years. The results of the analysis are presented in Table 30.5–2. The agreement is very satisfying and confirms that the use of the Weibull distribution is reasonable. Any other distributional fit deemed "superior" would be going after the difference column, which is nearly negligible from a practical standpoint.

There are a number of issues which are raised by Table 30.5–2. The difference column although small in practical magnitude, is "large" if viewed in the context of a binomial reference distribution. A difference of 0.0095 for the 95% prediction limit is statistically significant, if not practically significant for a large sample size of 27,108. The binomial model is at best a rough guide, however, since it is doubtful that adjacent sites are independent. One strong but relatively rare event can impact a large number of sites.

The difference between observed and predicted evidenced in Table 30.5–2 could be attributable to a number of sources. An interesting possibility is that

Table 30.5–2. Cross-validation study of 50-year prediction limits.

Percentage of Sites within 50-Year Prediction Limit		
Predicted	Observed	Difference
50%	49.52%	0.48%
75%	74.44%	0.56%
90%	90.91%	0.91%
95%	95.95%	0.95%

we are detecting natural or anthropogenic climate change signals. It is well established that hurricane activity is influenced by global climate patterns such as the El Niño/Southern Oscillation (ENSO) (Watson (2002)). The ENSO is classified into three modes: warm phase (El Niño), normal, or cold phase (La Niña) according to temperatures in the eastern South Pacific. Historically, normal years constitute about 40% of years, while the remaining 60% are evenly split between El Niño and La Niña years. By segmenting the overall (1851–2005) data-set according to the state of the ENSO, we can assess the impact of ENSO on hurricane frequencies on a site-by-site basis. This site-by-site aspect is important as we have found that while a warm phase ENSO tends to suppress hurricane formation basin-wide, it actually increases the probability for hurricane-force winds at some sites. In other words, even though there are fewer storms, they tend to traverse different areas than in other years. Table 30.5–3 shows the impact of ENSO on Montego Bay, Jamaica. Montego Bay is an astonishing 7.5 times more likely to experience a "100-year event" during a cold phase La Niña year than the overall historical average (computed as 1.0). In contrast, during a warm phase of an El Niño year, Montego Bay is only 1/5 as likely to experience a "100 year event." Compare the Montego Bay results with the results for Miami, Florida, in Table 30.5–4 where the risk is more consistent, but biased somewhat against cold phase (La Niña) years. Since it appears that 2006 will be a cold phase year, Jamaica should take special care in its hurricane preparations this year.

We first applied this type of analysis to the storage of an oil platform in transit through the Caribbean during the 1998 hurricane season. The client, being self-insured for the full value of the billion dollar asset, obviously wanted to find a safe harbor to sit out the season. (Incidentally, for the site we recommended, the platform did not experience tropical storm winds or higher.) Starting in 2005, our research team has begun applying this approach to forecast annual probabilities of hurricane-force winds on a city-by-city basis. We are conducting further research in this area with respect to other climate signals, such as the Multi-Decadal Oscillation, the North Atlantic Oscillation, as well as for potential anthropogenic climate change signals. The human climate change debate aside,

Table 30.5–3. Excerpt of output of the TAOS Statistical Analysis Program for Montego Bay.

```
100-year 50% value for all years:   82.7 kts
  probability of exceeding in normal year: 0.00638   0.6
  probability of exceeding in elnino year: 0.00165   0.2
  probability of exceeding in lanina year: 0.07516   7.5
```

Table 30.5–4. Excerpt of output of the TSAP program for Miami, Florida.

```
100-year 50% value for all years:  104.1 kts
  probability of exceeding in normal year: 0.01222   1.2
  probability of exceeding in elnino year: 0.01154   1.2
  probability of exceeding in lanina year: 0.00899   0.9
```

understanding that annual probabilities are not identical from year to year but are predictable given various natural climate signals (available in the Northern Hemisphere spring) allows such variability to be incorporated in the short-term planning process.

Given the demonstrated predictive ability from the basin-wide tests, we are reasonably confident that our return-period predictions are valid. Next we apply the analysis to the recent devastation of the city of New Orleans.

30.6 Case Study of the New Orleans Levees: Wrath of Mother Nature or Ordinary Extreme Event

The collapse of the New Orleans levees drew considerable attention to return periods through the simple concept of Saffir-Simpson categories, given that higher-category storms occur less often. Initially, the Army Corps of Engineers declared that the levees were "overtopped" since they were designed for a Category 3 storm, but Katrina was a Category 5 event. Subsequently, with the realization that Katrina was not a category 5 at landfall the defense was modified to "not a category 4." As it became clear that in fact the levees collapsed due to internal failures, the story changed again (Marshall (2005), Warrick and Hsu (2005)). In terms of the rebuilding of New Orleans, decision makers should consider a return period concept on a phenomenon-by-phenomenon basis to assess the likelihood of a recurrence rather than the overly simplistic Saffir-Simpson categories, because damaging phenomena produced by a hurricane (wind, waves, and storm surges) interact in highly nonlinear and spatially diverse ways.

How strong an event was Katrina, and how often should the New Orleans area experience impacts similar to those experienced in late August 2005? The

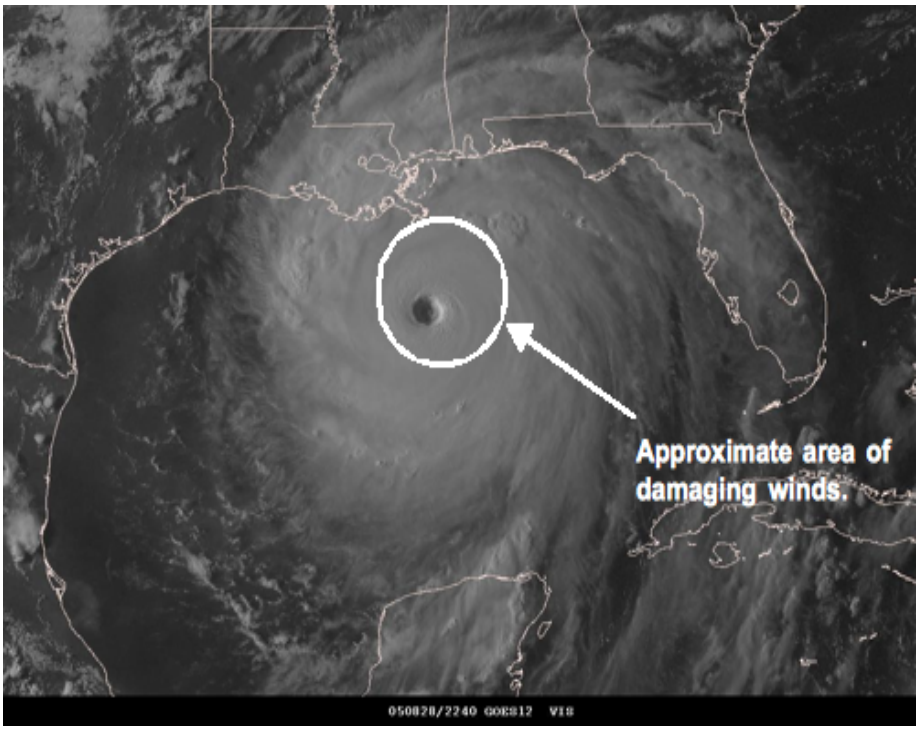


Figure 30.6–1. Hurricane Katrina, 28 August 2005, 2240UT.

answer to this question will likely receive continued scrutiny, especially if the answer and the outcome become even more politicized. Hurricanes are classified according to peak wind speed but as is readily seen from the wind profiles shown in Figure 30.2–1 the phenomenon is not one-dimensional. The damaging wind swath for most storms is actually quite narrow, especially when compared to the overall size of the storm. Figure 30.6–1 shows Hurricane Katrina the day before landfall, where the region of damaging winds is an area around the epicenter that represents about 5% of the area that is under cloud cover. A satellite depiction of a storm can be highly misleading as the outer cloud shield extends two to three times the radius of the damaging winds. Even within the damaging wind zone winds are highly variable. While the peak winds at landfall in Mississippi were in the 120 to 130 mile per hour range, winds in New Orleans were far less. The wind speed at NASAs Michoud assembly plant, located east of the city and thus closer to the storm track, was only 96 mph. The NOAA buoy over Lake Pontchartain, north of the city, recorded a peak wind of 78mph (Knabb, Rhomer, and Brown (2005)). A detailed inspection of damage as seen on high resolution aerial photography conducted by one of the authors (Watson) showed minimal direct wind damage to most single-family structures, indicating winds in the 70 to 80 mph range. Most damage was due to either flooding or falling trees. The dramatic destruction to mid- and high-rises in the downtown area, as well as

damage to the Super Dome, appeared to be due to a combination of debris and poor construction.

The question of storm surge and the failure of the New Orleans flood protection system is complex from a return period standpoint. Engineering studies are currently underway, and there are both political and scientific disputes in progress over both design criteria and quality of construction (van Heerden, (2005)). Our analyses indicate that for the New Orleans area, Katrina's impacts were well within what should have been anticipated. The peak 80 mph winds experienced in New Orleans have a return period of approximately 85 years, using the methodology described earlier. As reported in an investigation by PBS's NOVA program, the levee system failed at a peak surge of 10.5 feet, below the rated value of 14 feet mean sea level (although the ongoing subsidence of the New Orleans area reduced the margin by perhaps as much as 2 feet). Using our methodology, a 10.5 ft surge at the 17th Street Canal is a 79-year event. It should be noted that the Corps did not use return periods in their design, instead designing to a category 3 storm (111 to 130 mph winds, surges of 9 to 12 feet).

New Orleans presents a unique challenge in that because of Lake Pontchartrain, wind and surge return periods are more disconnected than is evident in other areas. The 100-year wind for the city is approximately 115 mph, a Category 3 storm, but the 100-year storm surge is nearly 20 feet (19.6 ft). Thus, leaving aside issues of construction quality, not taking into account the complexity of the area, and assuming storm surge return periods tracked wind return periods may have contributed to an under-design of the flood protection system.

30.7 Conclusions

The science of statistics has direct implications for both lives and property. Data used in these analyses occasionally suffer from the oft quoted computer science adage, "Garbage In, Garbage Out." This is nowhere more true than in the field of tropical cyclone planning and response, where decisions involving trillions of dollars of property and millions of lives depend on statistical projections derived from dynamic databases of often suspect accuracy. We have argued indirectly in this chapter that the choice of data to be fit is a critical and possibly overlooked concern. Moreover, even with a relevant dataset being identified, further data preparation and quality control are necessary to avoid radically different results caused by subsequent re-fitting of the evolving datasets. Techniques that may work well for the leisurely analysis of datasets that have been carefully screened and vetted may not be appropriate. Thus, statisticians must avoid overly theoretical approaches and concentrate on solving the problem of interest to the client, even if this means not using the most glamorous approach. Our primary example of distribution fits involved the two-parameter Weibull distribution, which was shown to perform admirably for the purpose of predicting return periods for perils

both at the site and basin level. Finally, convincing specialists in other fields to include statistical considerations early in the process is essential. Their involvement in the analysis process from data collection to presentation can result in an improved product for critical decision-making purposes.

Acknowledgments

This work was supported in part by the Organization of American States (with funding from the U.S. Agency for International Development), the National Oceanic and Atmospheric Administration, the Florida State Board of Administration, the Florida Department of Community Affairs, and the North Carolina Department of Insurance. The views expressed in this chapter are entirely the authors' and may not reflect the official positions of the funding agencies. The authors appreciate the comments of the Editor and reviewers.

References for Chapter 30

Best's Review (2005). "Katrina exposes falliability of catastrophe models," briefing, December 1, 2005, *Best's Review.*

Florida Commission on Hurricane Loss Prediction Methodology (2007). *Report of Activities as of November 1, 2007*, Florida State Board of Administration, Tallahassee, FL.

Friedman, J. H. and Meulman, J. (2004). "Clustering objects on subsets of attributes (with discussion),", *Journal of the Royal Statistical Society,* Series B, 66, 815–849.

Iman, R. L., Johnson, M. E., and Watson, C., Jr. (2005a). "Sensitivity analysis for computer model projections of hurricane losses," *Risk Analysis,* 25, 1277–1297.

Iman, R. L., Johnson, M. E., and Watson, C., Jr. (2005b). "Uncertainty analysis for computer model projections of hurricane losses," *Risk Analysis,* 25, 1299–1312.

Iman, R. L., Johnson, M. E., and Watson, C., Jr. (2006). "Statistical aspects of forecasting and planning for hurricanes," *The American Statistician,* 60, 1–17.

Jarvinen, B. R., Newman, C., and Davis, M. (1984). "A tropical cyclone data tape for the North Atlantic Basin, 1886–1983: Contents, limitations, and uses," NOAA Technical Memorandum NWS NHC 22, Coral Gables, FL.

Johnson, M. E. (1997). "Caribbean Storm Surge Return Periods," Organization of American States Caribbean Disaster Mitigation Project Workshop, Kingston, Jamaica, October 31, 1997.

Johnson, M. E. and Watson, C., Jr. (1999). "Hurricane Return Period Estimation," 10th Symposium on Global Change Studies, Dallas, TX, 478–479.

Knabb, R. D., Rhome, J.R., and Brown, D.P. (2005). Tropical Cyclone Report: Hurricane Katrina, 23–30 August 2005, National Hurricane Center.

Landsea, C., Anderson, C., Charles, N., Clark, G., Dunion, J., Fernández-Partagás, J., Hungerford, P., Neumann, C., and Zimmer, M. (2004a). The Atlantic hurricane database re-analysis project: Documentation for the 1851–1910 alterations and additions to the HURDAT database. *Hurricanes and Typhoons: Past, Present and Future*, R. J. Murnane and K.-B. Liu, Editors, Columbia University Press, 177–221.

Landsea, C., Franklin, J., McAdie, C., Beven, J., II, Gross, J., Pasch, R., Rappaport, E., Dunion, J., and Dodge, P. (2004b). "A re-analysis of Hurricane Andrew's (1992) intensity," *Bulletin of the American Meteorological Society,* 85, 1699–1712.

Marshall, R. (2005). "17th Street Canal levee was doomed: Report blames corps," *The Times-Picayune* (New Orleans), November 30, 2005.

Sampson, C. R., and Schrader, A. J. (2000). "The automated tropical cyclone forecasting system (Version 3.2)," *Bulletin of the American Meteorological Society,* 81, 1231–1240.

van Heerden, I. (2005). "Failure of Levee Systems Surrounding Greater New Orleans during Hurricane Katrina, Preliminary Assessment," testimony before the U.S. Senate Committee on Homeland Security and Governmental Affairs, November 2, 2005.

Warrick, J., and Hsu, S. (2005). "Levee's construction faulted in New Orleans flood inquiry," *The Washington Post,* November 3, 2005, Page A03.

Watson, C., Jr. (1995). "The arbiter of storms: A high resolution, GIS based storm hazard model," *National Weather Digest,* 20, 2–9.

Watson, C., Jr. (2002). "Implications of Climate Change for Modeling Coastal Hazards, Solutions for Coastal Disaster," *2002 Conference Proceedings,* American Society of Civil Engineers, Reston, VA.

Watson, C., Jr. and Johnson, M. (1997). "Return Period Estimation for Montego Bay," presentation at Workshop of organization of American States, Kingston, jamaica, October 31, 1997.

Watson, C., Jr and Johnson, M. (2004). "Hurricane loss estimation models, opportunities for improving the state of the art," *Bulletin of the American Meteorological Society,* 85, 1713–1726.

Watson, C., Jr. and Johnson, M. (2006). "An Assessment of Computer Generated Loss Costs in Florida," Report to the Florida Commission on Hurricane Loss Projection Methodology, Tallahassee, FL.

Chapter 31

A Rainfall-Based Model for Predicting the Regional Incidence of Wheat Seed Infection by *Stagonospora nodorum* in New York[1]

Denis A. Shah
4662 Vrooman Drive
Lewiston, NY 14092
quinnconsulting@verizon.net

Gary C. Bergstrom
Department of Plant Pathology
Cornell University
Ithaca, NY 14853
gcb3@cornell.edu

Our goal was to develop a simple model for predicting the incidence of wheat seed infection by *Stagonospora nodorum* across western and central New York in any given year. The distribution of the incidence of seed infection by *S. nodorum* across the region was well described by the beta-binomial probability distribution

[1]This chapter appeared, under the same title, in *Phytopathology*, Vol. 92, No. 5, pp. 511–518. Copyright © The Amrican Phytopatological Society, St. Paul, MN. Reprinted by permission.

(parameters p and θ). Mean monthly rainfalls in May and in June across western and central New York were used to predict p. The binary power law was used to predict θ. The model was validated with independent data collected from New York. The predicted distribution of seed infection incidence was not statistically different from the actual distribution of the incidence of seed infection.

31.1 Introduction

Much careful mathematical attention has been paid to describing plant disease epidemics at the field scale, whereas the description of epidemics at the regional scale has remained mostly qualitative or summarized as surveys. Unique combinations of inoculum availability, host susceptibility, and microscale environmental conditions lead to variability in epidemics among fields. However, there is an added component of variability in epidemics because of factors (mainly environmental) occurring at the mesoscale. Mesoscale factors may account for much of the variability in epidemics occurring across a region, so that one year is not described as an epidemic year, whereas the following year may see more severe epidemics as a whole. Quantifying the variability in epidemics at the mesoscale level is a necessary step toward predicting the risk of severe epidemics across a region.

Stagonospora nodorum blotch (SNB) is a common disease of wheat (*Triticum aestivum* L.) caused by the fungus *Stagonospora nodorum* (Berk.) Castellani and E. G. Germano (syn. *Septoria nodorum* (Berk.) Berk. in Berk. and Broome; teleomorph *Phaeosphaeria nodorum* (E. Müller) Hedjaroude, syn. *Leptosphaeria nodorum* E. Müller). Winter wheat in New York is grown mainly in the western and central counties. Agronomic practices in New York preclude infected wheat debris as an important source of *Stagonospora nodorum* inoculum, because the crop is grown in a two year or greater rotation with corn, alfalfa, oat, soybean, and vegetable crops (all nonhosts to *S. nodorum*) on relatively small, spatially discontinuous fields. The teleomorph has never been positively identified in New York. Seed infection by *S. nodorum* is frequent in New York wheat seeds (Shah and Bergstrom (1993)) and exceeds 50% in some seed lots. Seed transmission occurs at high frequency (70% or greater) to the coleoptiles (Shah and Bergstrom (2000)), which may be partially exposed above the soil line. Transmission also occurs to the first leaves at frequencies of 2 to 37% (Shah and Bergstrom (2000)), and possibly occurs to other leaves as well. Therefore, infected seeds can be a source of primary inoculum for SNB in New York.

Rainfall is crucial to SNB development because pycnidiospores of *S. nodorum* are dispersed in splash droplets. For there to be any chance of seed infection, pycnidiospores must be splashed upwards from infected foliage to the wheat spikes, which are held above the canopy by the peduncle. Because vertical splashing occurs over a few centimeters (Pietravalle, van den Bosch, Welham, Parker, and

Lovell (2001)), the pathogen must be sporulating at a height in the canopy that would enable spore-bearing splash droplets to reach the spikes. Thus, rainfall during stem elongation (so that SNB may progress vertically in the developing canopy) as well as rainfall during the seed developmental stages (for splashing onto the spikes) may both be important factors affecting the incidence of wheat seed infection by *S. nodorum*.

Certified wheat seed lots in New York are harvested in July and are sown by grain producers as soon as 6 weeks later, allowing little time for the completion of seed assays for the incidence of *S. nodorum*. Our goal was to develop a simple model that could be used to predict the regional distribution in the incidence of wheat seed infection by *S. nodorum* shortly after harvest, or even before, without having to wait on the results of seed assays. The objective of this research was to model the relationship between the incidence of wheat seed infection by *S. nodorum* and rainfall across western and central New York.

31.2 Materials and Methods

The incidence of seedborne *S. nodorum* in New York winter wheat has been surveyed since 1990. Samples of seeds from lots produced in New York and submitted for certification testing were obtained from the New York State Seed Testing Laboratory (New York Agricultural Experiment Station, Geneva). Testing of wheat seed lots for infection by *S. nodorum* is not part of the certification process. Therefore, samples submitted to the seed testing laboratory may be regarded as random with respect to *S. nodorum* infection; moreover, lots with high levels of seed infection by *S. nodorum* are often healthy otherwise (Shah, Bergstrom, and Sorrells (2002)). The number of lots assayed varied from year to year (Table 31.2–1), depending on how many samples were received from the testing laboratory. Samples were stored at 5°C until they could be assayed for seed infection (usually within 1 to 4 months). The incidence of seed infection by *S. nodorum* does not decrease during this length of time at low temperatures (Cunfer (1981)). The incidence of seed infection by *S. nodorum* was determined by assaying a minimum of 100 arbitrarily selected seeds per lot on a selective medium (Manandhar and Cunfer (1991)).

31.2.1 Probability Distribution for Seed Infection Incidence

When several seed lots from a region are tested, the mean seed infection incidence across all lots is usually reported and sometimes the range (which may be fairly wide or narrow) is also reported. However, there is the chance that some incidences will not be represented by the lots sampled from the region. Also, the mean and range do not indicate the likelihood of sampling a lot with a specified seed infection incidence. Probability distributions describing the incidence of infection not only convey the same information as summary statistics, such as the

Table 31.2–1. Incidence of seed infection by *Stagonospora nodorum* in winter wheat seed lots produced in western and central New York, 1990 to 1999, and statistics for assessing the fit of the beta-binomial distribution to the data[a].

		% of Seeds infected[c]			Goodness-of-fit test		Beta-binomial parameters	
Year[b]	No. of seed lots	Mean	Range	D[d]	P_{gof}[e]	df[f]	p	θ
1990	50	22.5	1–71	38.1	0.36	3	0.23	0.19
1991	50	2.4	0–19	11.0	0.05	1	0.02	0.05
1994	24	11.7	0–50	19.0	0.17	1	0.12	0.17
1995	70	4.5	0–42	22.0	< 0.01[g]	NA	0.05	0.12
1996	94	25.2	2–77	18.6	0.05	4	0.26	0.16
1998	26	10.4	0–40	12.0	0.12	1	0.10	0.10
1999	24	1.6	0–15	6.5	> 0.05[g]	NA	0.02	0.04

[a] No data on seed infection were available for the years 1992, 1993, and 1997.

[b] Data from 1990 and 1991 were published previously (Shah and Bergstrom (1993)).

[c] Percentage of seeds infected was determined by assaying samples of at least 100 seeds per lot on a selective medium (Manandhar and Cunfer (1991)).

[d] Index of dispersion. $P < 0.0001$ in all years, meaning that the incidence of seed infection by *S. nodorum* across the western and central New York region was overdispersed.

[e] P-value associated with the χ^2 goodness-of-fit (gof) test for fit of the beta-binomial distribution to the data.

[f] Degrees of freedom associated with the χ^2 goodness-of-fit test for fit of the beta-binomial distribution to the data. Degrees of freedom are not applicable (NA) to the dithered Kolmogorov-Smirnov goodness-of-fit test, which is a nonparametric test.

[g] P-value is for the dithered Kolmogorov-Smirnov goodness-of-fit test (Washburn (1999)).

mean, but also offer the advantage of allowing one to estimate the chance of a given seed infection incidence for a particular seed lot.

Seed infection status is a binary variable (either an individual seed is infected by *S. nodorum* or it is not). Therefore, the binomial distribution is a reasonable starting point in fitting a discrete probability distribution to the incidence of wheat seed infection by *S. nodorum*. This probability distribution assumes that the mean incidence of infection is constant for all lots. However, differences in local environment together with differential cultivar susceptibility to seed infection by *S. nodorum* (Shah, Bergstrom, and Sorrells (2000)) could result in a nonconstant mean incidence of seed infection across lots. Therefore, the variation in seed infection incidence among seed lots (e.g., across western and central New York) would be greater than the expected variation assuming a binomial distribution fit the incidence of seed infection. This extra variation is referred to as overdispersion (Collett (1991)). A first step is to test whether datasets show overdispersion in the incidence of seed infection. Datasets were tested for overdispersion by calculating the binary form of the index of dispersion, D, (Madden and Hughes (1995)) for the number of seed positive for *S. nodorum* per lot. If $D > 1$, then seed infection incidence may be overdispersed (contains variation in excess of that expected with the binomial distribution). The statistic $(N-1)D$ has a χ^2 distribution (with $N-1$ df, where N is the number of seed lots in the sample) under the null hypothesis that incidence is distributed binomially, which allows a formal test for overdispersion (Madden and Hughes (1995)).

The beta-binomial distribution is suitable for modeling overdispersed disease incidence data (Madden, Hughes, and Ellis (1995), Smith, Page, Holt, and Kyetere (2000), Turechek and Madden (1999a), and Zarnoch, Anderson, and Sheffield, (1995)). The distribution may be expressed in several formats (Madden and Hughes (1995)), with one form being

$$P(X = x) = \binom{n}{x} \frac{\text{Beta}\{x + p/\theta,\, n - x + [(1-p)/\theta]\}}{\text{Beta}(p/\theta,\, (1-p)/\theta)}, \tag{31.2.1}$$

where, in the present context, $P(X = x)$ is the probability that the random variable X has the value x, x is the number of seeds out of n infected by *S. nodorum*, and $\text{Beta}(a, b)$ is the beta function with parameters a and b. The parameter p is the expected value of the probability of a seed being infected, and θ is an index that increases with the amount of overdispersion. Whereas mean incidence of infection is assumed to be a constant for all lots with the binomial distribution, the beta-binomial allows for a mean incidence that varies from lot to lot.

The beta-binomial distribution was fitted to the incidence of seed infection by *S. nodorum* in each year. The χ^2 goodness-of-fit (gof) test was used to assess the fit of the distribution to the observed data. For χ^2 gof tests, seed infection incidence levels were pooled into bins (e.g., 0 to 5%) so that expected frequencies were greater than five. The χ^2 gof test was not possible for the 1999 data because

there were insufficient degrees of freedom after pooling (Hughes and Madden (1993)). Evaluating gof with the 1995 dataset was problematic because different bin sizes led to different interpretations of gof with the χ^2 test, illustrating an inherent weakness of the test. Therefore, for the 1995 and 1999 datasets, a dithered Kolmogorov- Smirnov (K-S) gof test was used (Washburn (1999)). This test is based on the traditional K-S gof test for continuous distributions, but the test statistic is modified so that the tables for continuous distributions may be used for discrete distributions as well. A disadvantage of the K-S test is that software is not available for obtaining exact p-values as with the χ^2 gof test; goodness-of-fit is assessed by comparing the test statistic to critical levels that are generally tabulated for only a few significance values. Tests for overdispersion and χ^2 gof tests were done with the BBD software program (Madden and Hughes (1994)). The dithered K-S gof test was done in Microsoft Excel 98 (Microsoft Corp., Redmond, WA).

31.2.2 Relating Mean Seed Infection to Rainfall

Wheat seeds are susceptible to infection by *S. nodorum* from flowering until the end of the dough period (Cunfer and Johnson (1981)). Inoculum availability and temperature are unlikely limiting factors for seed infection; SNB is a polycyclic disease, seed infection occurs toward the end of the epidemic, and temperatures during seed development are favorable to the pathogen's growth (Djurle and Yuen (1991)).

We started with the premise that rainfall during stem elongation, and then rainfall from flowering to the end of seed maturity, were the two most important variables governing the incidence of wheat seed infection by *S. nodorum*. Data were not available on the locations of seed production fields or the dates of the beginning of stem elongation and flowering. However, in New York, the beginning of stem elongation generally coincides with the first week of May, and flowering starts about the first week of June. Seed development and maturation continues until the end of June. Seed infection should be a function of vertical splash droplet dispersal during rain. Yet commonly available meteorological measurements such as rainfall amount, duration, or rate are not accurate predictors of splash height distribution, and rainfalls with similar characteristics may differ considerably in the splashing they cause (Shaw (1991)). Thus, estimating the daily progression of SNB vertically in the canopy using daily rainfall measurements is a difficult task. In light of these limitations, and in keeping with our goal of developing a simple, practical model, we interpreted monthly rainfall as a summation of all possible dispersal events in that time period, regardless of their magnitude. Therefore, as a very general simplification, we used total rainfall during May as an estimate of the amount of rain on the crop during stem elongation, and total rainfall during June as an estimate of the amount of rain during seed development. We did not know the actual amount of rain received in May and June by the fields from

which the seed lots were harvested, because we did not know the locations of those fields. However, these amounts can be estimated if we know the variability in the amount of rain received in May and June across the western and central New York region.

We obtained the rainfall data recorded at weather stations across the 16 western and central New York counties, which together account for over 90% of winter wheat acreage. The number of stations with available data varied between 41 and 54. The rainfalls recorded at these stations were regarded as a sample of all rainfall events that occurred throughout the region. We assumed that the samples were representative of the total distribution in rainfall across the region, and that probability models could describe this distribution. The gamma distribution is a probability model often used to describe rainfall (Wilks (1995)), and was fitted to rainfall totals in May and June. The distribution appears in slightly different forms depending on the text consulted. We fitted gamma (Φ, ω), where the probability density function is defined as

$$f(x) = \frac{w^{-\Phi} x^{\Phi-1} e^{-x/w}}{\Gamma(\Phi)} \quad \text{if } x > 0, \text{ and } f(x) = 0 \text{ otherwise,} \tag{31.2.2}$$

where x is monthly rainfall (millimeters), Φ is the shape parameter ($\Phi > 0$), and ω is the scale parameter ($\omega > 0$). The VTFIT software program (R. Cooke, University of Illinois, Urbana) was used to fit the distributions. The K-S gof test was used to determine the adequacy of the fit (Conover (1980)).

Preliminary plots were made of the distributions in the incidence of seed infection overlayed with the distributions in May and June rainfall. Visual inspection of the plots indicated that mean seed infection incidence (np) might be related to mean rainfall in May and mean rainfall in June. Multiple regression equations were used to relate np to mean rainfall in both May and June. Different equations with May and June rainfall as the predictor variables were tried. The form of the equation that predicted np best was

$$np_{ij} = \lambda_0 + \lambda_1 \text{May}_i + \lambda_2 \ln(\text{June}_j) + \varepsilon_{ij}, \tag{31.2.3}$$

where May_i is the mean rainfall (millimeters) in May, June_j is the mean rainfall in June (millimeters), λ_0, λ_1, and λ_2 are regression parameters to be estimated, and ε_{ij} is the associated error term. Since we were interested in modeling percent seed infection, n was set at 100. Mean monthly rainfall was estimated by the sample mean, since it is equivalent to the moment estimate of the mean of the gamma distribution (Law and Kelton (2000)).

Data on the incidence of wheat seed infection by *S. nodorum* in New York were available for 7 years only (Table 31.2–1). Because least square parameter estimates have a bias inversely proportional to sample size, there is reason for concern with such a small dataset. As a solution, we generated datasets for the regression analysis by a method known as the generalized bootstrap (Karian and

Dudewicz (2000)). It differs from the familiar bootstrap (Efron and Tibshirani (1993)) in that samples are generated from probability distributions fitted to the original data rather than sampling from the observed data itself. The bootstrap is not capable of generating samples containing observations that may be rare because it is based on the actual observed data, but this drawback was overcome by the generalized bootstrap, which is based on the probability distribution assumed to describe the population from which the actual sampled data were obtained. For the multiple regression, we generated 1,000 generalized bootstrap samples for each year (7,000 total samples) from the probability distributions fitted to seed infection incidence and rainfall. Mathematica (version 3.0, Wolfram Research, Inc., Champaign, IL) notebooks were written to generate the generalized bootstrap samples. Regression modeling was done in SAS (version 6.12, SAS Institute Inc., Cary, NC).

31.2.3 Binary Power Law

Once the parameter p of the beta-binomial distribution was predicted, the next step was to obtain a prediction of θ. In phytopathosystems examined to date, the binary power law (Hughes and Madden (1992)) has been a good descriptor of the relationship between the observed variance in disease incidence and the expected variance, assuming incidence is described by the binomial distribution (Madden and Hughes (1995), Madden, Nault, Murral, and Apelt (1995), Turechek and Madden (1999b)). The binary power law and the beta-binomial distribution are linked, which allows the parameter θ of the beta-binomial to be estimated from the binary power law relationship (Madden and Hughes (1999)).

We first determined whether the binary power law held for the incidence of seed infection by *S. nodorum*. Again, the generalized bootstrap was used to generate 1,000 samples from each of the 7 years, using the fitted beta-binomial distributions to incidence. The binary power law equation relating the variance in the observed number of infected seeds per sample to the variance expected with the binomial distribution is as follows:

$$\nu_{obs} = A_x \nu_{bin}^{b}, \tag{31.2.4}$$

where ν_{obs} is the observed variance in the number of seeds infected by *S. nodorum* per lot, and ν_{bin} is the variance assuming the number of infected seeds per lot is distributed binomially. A_x and b are parameters that can be estimated by linear regression after linearizing equation (31.2.4) by taking logarithms. If $b = 1$ and $A_x > 1$ (or $\ln(A_x) > 0$), then θ does not vary with p. If both A_x and b are greater than 1, then θ varies with p, which means that the degree of overdispersion among seed lots changes with the mean incidence of *S. nodorum* (Madden and Hughes (1999)).

If the binary power law relationship holds, then the beta-binomial parameter

θ can be estimated from

$$\theta = \frac{a - f(p)/n}{f(p) - a}, \tag{31.2.5}$$

where $f(p) = [p(1-p)]^{1-b}$ and $a = A_x n^{b-2}$ (Turechek and Madden (1999b)).

31.2.4 Model Verification and Validation

The most definitive test of validity is to establish that the model's output is close to the output expected from the real system (Law and Kelton (2000)). The model was first checked to make sure it was giving an output (predicted p and θ of the beta-binomial) that was realistic. Plots of the actual distribution of seed infection incidence were compared with the predicted distributions for the 7 years in Table 31.2–1. As a further check, the model was run with rainfall input from 1980 to 1988, 1992, 1993, and 1997. Seed infection data were not available for those years, but we wanted to make sure that the model's output probability distributions for seed infection incidence were within the limits of what would have been expected given the rainfall distribution in western and central New York in May and June.

Validation is as much an art as it is a science (Kleijnen (1995a)). No standard theory on validation exists, but there is a set of general guidelines (Balci (1997)). There is strong interest in statistical validation techniques because they are reproducible and objective (Kleijnen (1995b)). In reality, the amount of data available for any modeling project varies greatly, and the availability of data on the real system determines which type of statistical technique can be used for validation (Kleijnen (1995b)).

Statistical Validation. One independent dataset, consisting of the incidence of seed infection by *S. nodorum* determined for 279 lots produced in New York in 2000, was used for validation purposes. Mean monthly rainfall in May and June 2000, determined from the records at 38 weather stations across western and central New York, were used to predict the beta-binomial parameters p and θ. The predicted distribution of seed infection incidence was compared with the actual data by the χ^2 gof test.

However, rainfall recorded at the 38 stations across western and central New York in 2000 is only one (assumed representative) sample of the distribution of rainfall in May and June. If rainfall had been recorded at different locations and used as model input, would the predicted parameters of the beta-binomial distribution still describe the actual data? To answer this question, we used a confidence interval approach. First, the beta-binomial distribution was fitted to the year 2000 data. The fit of the distribution was assessed by the χ^2 gof test. The gamma distribution was then fitted to rainfall recorded in May and June 2000 at the 38 stations across central and western New York. The χ^2 gof test was used to assess the adequacy of the fit of the gamma distribution. We did not know the actual rainfall during stem elongation and during seed development at the fields represented by the 279 sampled lots. We, however, assumed that the rainfalls at

those fields and at the weather stations were realizations of the same population described by gamma distributions. One thousand generalized bootstrap samples were generated, each of size 38, using the fitted gamma distributions to May and June rainfall totals in 2000. Mean monthly rainfalls were calculated from each of the 1,000 samples, and served as the model's input rainfall variables. The predicted parameters ($[p_i, \theta_i], i = 1, \ldots, 1,000$) of the beta-binomial distribution describing seed infection incidence were used to calculate 95% confidence intervals for $(p_i - p_a)$ and $(\theta_i - \theta_a)$, where p_a and θ_a are the parameters of the beta-binomial distribution fitted to the actual seed infection incidence data in 2000. If zero is included in the confidence interval, then we conclude that the difference between the observed and predicted parameters is not significantly different at the 95% level. If zero is not included in the confidence interval, this does not necessarily invalidate the model as a practical representation of the system; it depends on the magnitude of the confidence interval (Law and Kelton (2000)).

31.3 Results

31.3.1 Probability Distribution for Seed Infection Incidence

There was significant overdispersion in all datasets tested ($P < 0.0001$). The beta-binomial distribution was a reasonable fit to the observed data, except in 1995 (Table 31.2–1). The lack of fit in 1995 was mainly due to overprediction by the beta-binomial at zero incidence of infection.

31.3.2 Relating Mean Seed Infection to Rainfall

The gamma distribution fit monthly rainfall across western and central New York in May and in June ($P > 0.05$). Plots of the beta-binomial distributions fitted to seed infection incidence were overlayed with the fitted gamma distributions describing rainfall in May and in June (Figure 31.3–1). Although the distributions of seed infection incidence were similar in some years, the underlying rainfall distributions could be quite different (Figure 31.3–1).

The patterns revealed by the plots led to the hypothesis that the mean incidence of seed infection (np) may be related to the combined effects of mean rainfall in May and June. Equation (31.2.3) fit the generalized bootstrap samples very well ($R^2 = 0.91$) and was therefore considered suitable for representing the relationship between mean seed infection incidence and rainfall. Estimates for λ_0, λ_1, and λ_2 were -26.10, 0.19, and 5.34, respectively. All three parameters were significantly different from zero ($P < 0.0001$).

31.3.3 Binary Power Law

Once the beta-binomial parameter p could be specified from rainfall, the next step was to estimate the parameter θ. Equation (31.2.4) was fitted by least

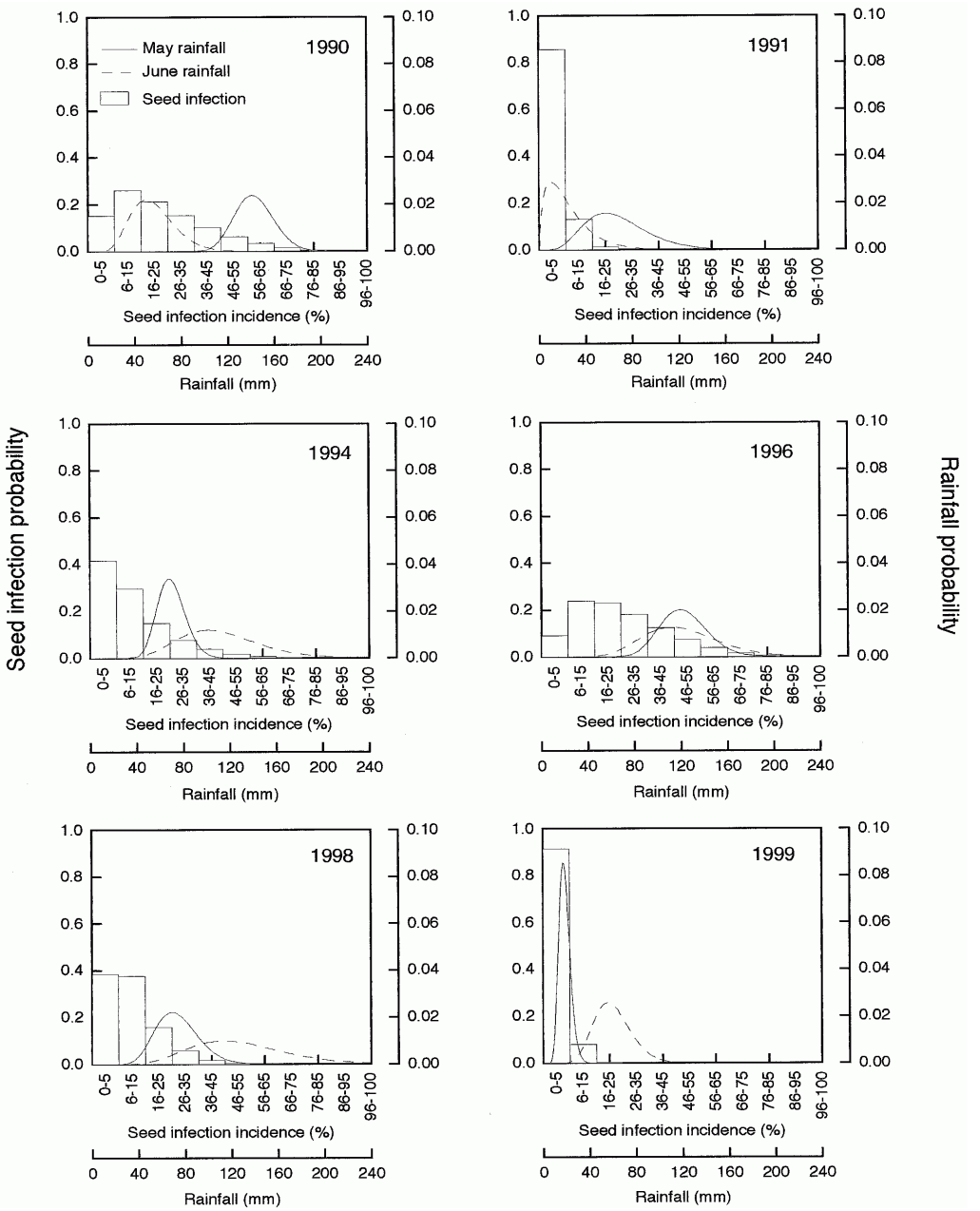


Figure 31.3–1. Fitted distributions describing the incidence of wheat seed infection by *Stagonospora nodorum* and total rainfall in May and June across 16 western and central New York counties in 6 years. The beta-binomial distribution was fitted to percent seed infection and gamma distributions to total rainfall in May and in June. The y axes represent probabilities of a given seed infection incidence range (x axis) or total monthly rainfall (offset x axis).

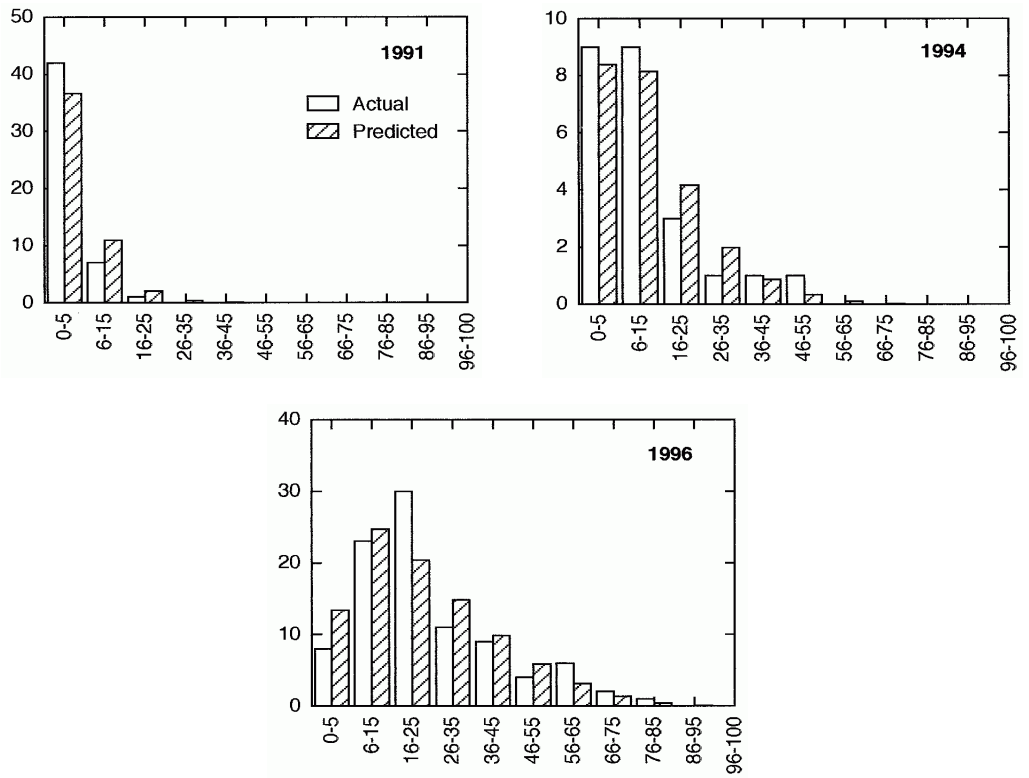


Figure 31.3–2. Actual and model predicted frequencies of the incidence of seed infection by *Stagonospora nodorum* in winter wheat seed lots across western and central New York in 1991, 1994, and 1996. Predicted frequencies were obtained from model derived parameters (p, θ) of the beta-binomial distribution. Predicted parameters were (0.041, 0.069), (0.125, 0.133), and (0.230, 0.181) for 1991, 1994, and 1996, respectively.

squares regression to generalized bootstrap samples of seed lots. The regression was significant ($R^2 = 0.95$). Estimates of $\ln(A_x)$ and b were 1.28 and 1.52, respectively. The t tests indicated that $\ln(A_x)$ and b were significantly greater than 0 and 1, respectively ($P < 0.0001$), indicating that the parameter θ varied with p. The parameter θ could now be predicted from equation (31.2.5).

31.3.4 Model Verification and Validation

The actual and predicted distributions of seed infection incidence for the 7 years in Table 31.2–1 were quite close (Figure 31.3–2), except for 1995. The model did give a negative value in 1999 for p ($= -0.0066$). The model's output probability distributions for 1980 to 1988, 1992, 1993, and 1997 appeared reasonable and corresponded to what would be expected given the rainfall patterns in those years (Figure 31.3–3).

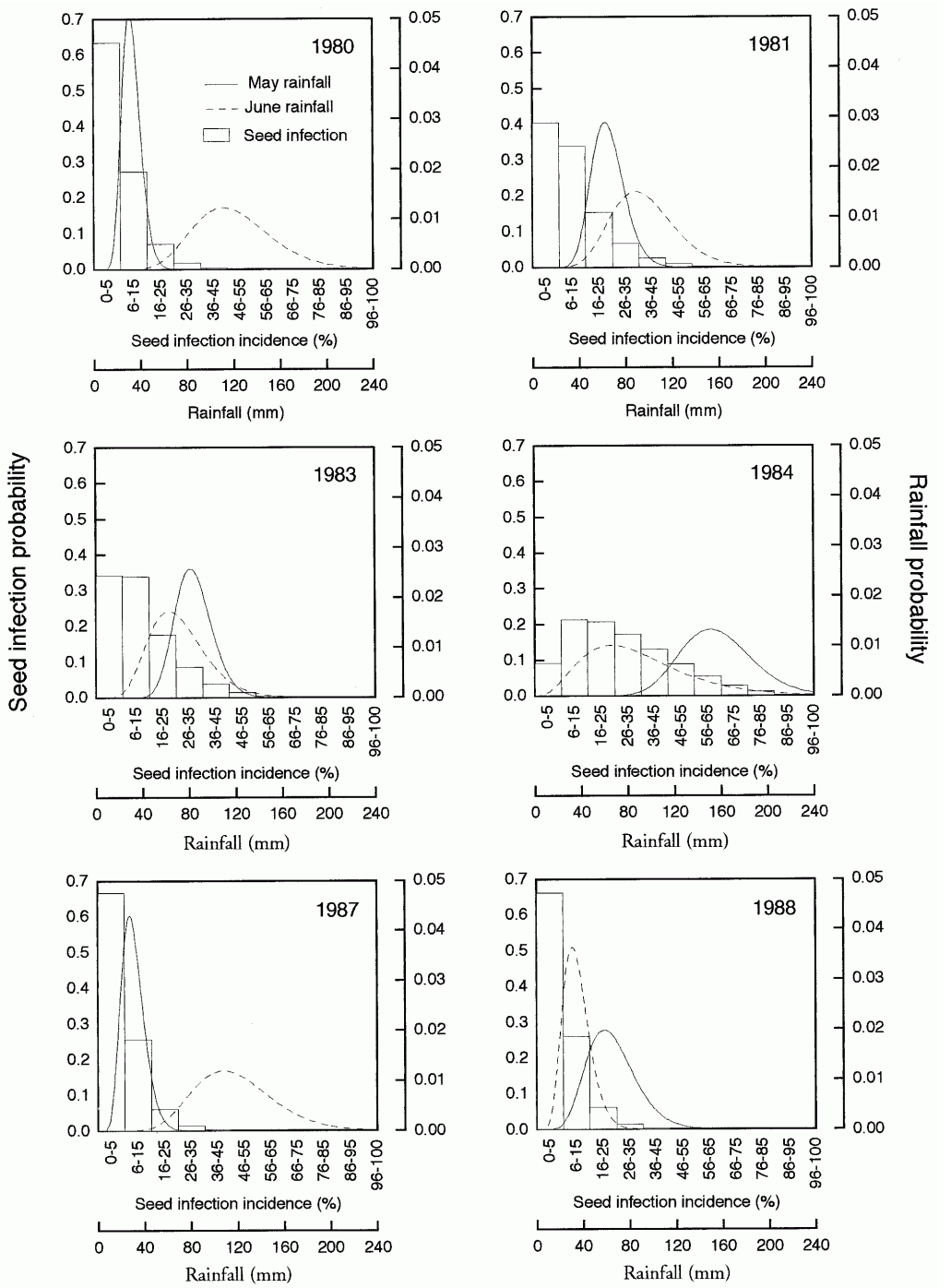


Figure 31.3–3. Model derived beta-binomial distributions for the incidence of wheat seed infection by *Stagonospora nodorum* shown with gamma distributions fitted to monthly total rainfall in May and in June across 16 western and central New York counties, 1980 to 1988.

Statistical Validation. Model-predicted beta-binomial parameters (p, θ), based on mean rainfall in May and June across western and central New York in 2000, were (0.233, 0.182). The distribution of seed infection incidence based on the model-predicted parameters was not significantly different from the actual distribution in seed infection incidence ($P = 0.50$, χ^2 gof test, 23 df). Actual and model-predicted frequencies for seed infection incidence for the 279 sampled lots are shown in Figure 31.3–4.

The beta-binomial distribution described actual seed infection incidence in 2000 ($P = 0.57$). The parameters p_a and θ_a were estimated at 0.245 and 0.127, respectively. The gamma distribution fit monthly rainfall in May ($P = 0.58$, 6 df) and June ($P = 0.45$, 7 df) in 2000. Fitted parameters (Φ, ω) were (38.34, 3.18) for total rainfall in May and (35.32, 3.72) for total rainfall in June.

The 95% confidence interval for $(p_i - p_a)$ was $(-0.025, 0.001)$, and was (0.050, 0.060) for $(\theta_i - \theta_a)$. Thus, model-derived predictions of p are close to, and include, the actual value, because $0 \in (p_i - p_a)$. Model predictions for θ are, however, positively biased. Nevertheless, the bias is fairly small and its range is very narrow.

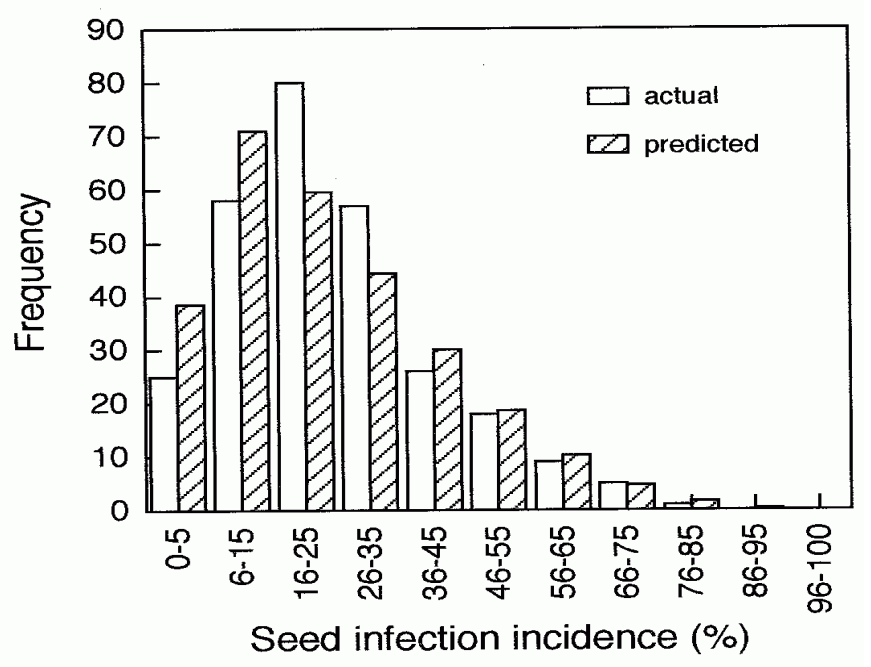


Figure 31.3–4. Actual and model-predicted frequencies of the incidence of wheat seed infection by *Stagonospora nodorum* for 279 seed lots produced across western and central New York in 2000. Model-derived beta-binomial parameters (p, θ) were (0.233, 0.182).

31.4 Discussion

Our objective was to develop a simple model for predicting the distribution in the incidence of wheat seed infection by *S. nodorum* across western and central New York from rainfall. To do this, we first needed a probability distribution that adequately described seed infection levels across the region. The beta-binomial distribution was a suitable probability distribution for this purpose. The beta-binomial distribution is described by two parameters (p, θ). Mean rainfalls in May and in June across western and central New York were used to predict p. Next, the binary power law was used to predict θ. The predicted distribution of seed infection incidence in 2000 was not statistically different from the actual distribution of the incidence of seed infection that year.

It was possible to explain the distributions in the incidence of seed infection by the combined effects of rainfall during May and June (which were assumed to correspond to rainfall during stem elongation and then after flowering). For example, the distributions in seed infection incidence in 1991 and 1999 were quite similar. In 1991, it would appear that there was enough rain in May for the upward movement of SNB in the canopy, but very low rainfall in June would not have allowed for much splashing to occur from the canopy to the spikes during seed development. The situation appeared to have been different in 1999. In this year, there was very little rain in May, and so SNB would not have had many chances for progressing vertically in the canopy. Therefore, although rainfall in June was similar to other years in which there was high seed infection (e.g., 1990), the lack of vertical disease progression during stem elongation would have meant that the spikes were mainly beyond the reach of splash dispersal of *S. nodorum* pycnidiospores.

The incidence of seed infection by *S. nodorum* in one year was not related to the incidence in the previous year. For example, 1999 was a year with very low seed infection levels, but 2000 was a year in which seed infection incidences were relatively high, again supporting our assumption of rainfall as a major determinant of seed infection by *S. nodorum* in New York. Assuming that seedborne inoculum is the primary source for epidemics of SNB in New York and given that SNB is polycyclic, it follows that very low levels of initial inoculum (i.e., seed infection) can start potentially serious epidemics. Attempting to control SNB epidemics initiated from seedborne inoculum by treating seeds with fungicides against *S. nodorum* may be futile, not because of fungicidal inefficiency but because a few escapes may be sufficient to start the epidemic. Counterintuitively, the only time a seed treatment may actually reduce foliar epidemics substantially would be in years in which the incidence of seed infection in the sown seed was already very low, as in 1999. Under such circumstances, seed treatment against *S. nodorum* may come closest to true eradication of seedborne inoculum. This hypothesis requires testing.

We now have a way of estimating the probability of a given seed infection

incidence level for a particular seed lot produced in western and central New York in any year. Furthermore, the long-term probabilities associated with any seed infection incidence can now be determined by driving the model with historic weather data. Results presented in this paper indicated that in most years the majority of lots contained less than 10% of seeds infected by *S. nodorum.* Also, a probability distribution describing the incidence of seed infection by S. nodorum is important in modeling seed to seedling transmission in the epidemiology of SNB.

One drawback of the model was that it predicted a negative value for mean probability of seed infection in 1999. Yet, seed infection incidences in that year were among the lowest we had observed since beginning routine seed assays. Viewed another way, the model was correct in predicting that seed infection incidences would have been very low in 1999. As more data on seed infection levels and rainfall are gathered, they should be incorporated into the model to allow more accurate estimation of regression and binary power law parameters. Other predictors of mean probability of seed infection, such as the number of days with rain during stem elongation and flowering, could be investigated. The model's structure should make future modifications a fairly easy task.

Ultimately, one may be interested in site-specific predictions of the incidence of seed infection. Site-specific predictions of seed infection would require estimates of the dates of key phenological stages as well as data on rain intensity and duration in individual seed production fields. A geographical information system may be of value in implementing such a model once the parameters have been established.

References for Chapter 31

Balci, O. (1997). "Principles of simulation model validation, verification, and testing." *Trans. Soc. Comput. Simul.* 14:3–12.

Collett, D. (1991). *Modelling Binary Data,* Chapman & Hall, London.

Conover, W. J. (1980). *Practical Nonparametric Statistics,* 2nd ed. John Wiley & Sons, New York.

Cunfer, B. M. (1981). "Survival of *Septoria nodorum* in wheat seed," *Trans. Br. Mycol. Soc.,* 77:161–164.

Cunfer, B. M., and Johnson, J. W. (1981). "Relationship of glume blotch symptoms on wheat heads to seed infection by *Septoria nodorum*," *Trans. Br. Mycol. Soc.,* 76:205–211.

Djurle, A., and Yuen, J. E. (1991). "A simulation model for *Septoria nodorum* in winter wheat," *Agric. Syst.,* 37:193–218.

Efron, B., and Tibshirani, R. J. (1993). *An Introduction to the Bootstrap,* Chapman & Hall, New York.

Hughes, G., and Madden, L. V. (1992). "Aggregation and incidence of disease," *Plant Pathol.,* 41:657–660.

Hughes, G., and Madden, L. V. (1993). "Using the beta-binomial distribution to describe aggregated patterns of disease incidence," *Phytopathology* 83:759–763.

Karian, Z. A., and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods,* CRC Press, Boca Raton, FL.

Kleijnen, J. P. C. (1995a). "Case study: Statistical validation of simulation models," *Eur. J. Oper. Res.,* 87:21–34.

Kleijnen, J. P. C. (1995b). "Verification and validation of simulation models," *Eur. J. Oper. Res.,* 82:145–162.

Law, A. M., and Kelton, W. D. (2000). *Simulation Modeling and Analysis,* 3rd ed. McGraw-Hill Publishing, Boston.

Madden, L. V., and Hughes, G. (1994). "BBDComputer software for fitting the beta-binomial distribution to disease incidence data," *Plant Dis.,* 78:536–540.

Madden, L. V., and Hughes, G. (1995). "Plant disease incidence: Distributions, heterogeneity, and temporal analysis," *Annu. Rev. Phytopathol.,* 33:529–564.

Madden, L. V., and Hughes, G. (1999). "Sampling for plant disease incidence," *Phytopathology* 89:1088–1103.

Madden, L. V., Hughes, G., and Ellis, M. A. (1995). "Spatial heterogeneity of the incidence of grape downy mildew," *Phytopathology* 85:269–275.

Madden, L. V., Nault, L. R., Murral, D. J., and Apelt, M. R. (1995). "Spatial pattern analysis of the incidence of aster yellows disease in lettuce," *Res. Popul. Ecol.,* 37:279–289.

Manandhar, J. B., and Cunfer, B. M. (1991). "An improved selective medium for the assay of *Septoria nodorum* from wheat seed," *Phytopathology* 81:771–773.

Pietravalle, S., van den Bosch, F., Welham, S. J., Parker, S. R., and Lovell, D. J. (2001). "Modelling of rain splash trajectories and prediction of rain splash height," *Agric. For. Meteorol.,* 109:171–185.

Shah, D., and Bergstrom, G. C. (1993). "Assessment of seedborne *Stagonospora nodorum* in New York soft white winter wheat," *Plant Dis.,* 77:468–471.

Shah, D. A., and Bergstrom, G. C. (2000). "Temperature dependent seed transmission of *Stagonospora nodorum* in wheat," *Eur. J. Plant Pathol.,* 106:837–842.

Shah, D. A., Bergstrom, G. C., and Sorrells, M. E. (2000). "Differential seed infection of wheat cultivars by *Stagonospora nodorum," Plant Dis.,* 84:749–752.

Shah, D. A., Bergstrom, G. C., and Sorrells, M. E. (2002). "Relationship between wheat seed infection by *Stagonospora nodorum* and seed weight," *Seed Sci. Technol.*, 30:339–346.

Shaw, M. W. (1991). "Variation in the height to which tracer is moved by splash during natural summer rain in the UK," *Agric. For. Meteorol.,* 55:1–14.

Smith, M., Page, W., Holt, J., and Kyetere, D. (2000). "Spatial dynamics of maize streak virus disease epidemic development in maize fields," *Int. J. Pest Manag.,* 46:55–66.

Turechek, W. W., and Madden, L. V. (1999a). "Spatial pattern analysis of strawberry leaf blight in perennial production systems," *Phytopathology* 89:421–433.

Turechek, W. W., and Madden, L. V. (1999b). "Spatial pattern analysis and sequential sampling for the incidence of leaf spot on strawberry in Ohio," *Plant Dis.,* 83:992–1000.

Washburn, A. (1999). "A dithered K-S statistic," *Commun. Stat. Theory Methods,* 28:1535–1546.

Wilks, D. S. (1995). *Statistical Methods in the Atmospheric Sciences,* Academic Press, San Diego.

Zarnoch, S. J., Anderson, R. L., and Sheffield, R. M. (1995). "Using the beta-binomial distribution to characterize forest health," *Can. J. For. Res.,* 25:462–469.

Chapter 32

Reliability Estimation Using Univariate Dimension Reduction and Extended Generalized Lambda Distribution[1]

Erdem Acar
Department of Mechanical Engineering
TOBB University of Economics and Technology
Söğütözü, Ankara 06560, Turkey
E-mail: acar@etu.edu.tr

Masoud Rais-Rohani
Department of Aerospace Engineering
Center for Advanced Vehicular Systems
Mississippi State University
Mississippi State, MS 39762 USA
E-mail: masoud@ae.msstate.edu

Christopher D. Eamon
Department of Civil Engineering
Lawrence Technological University
Southfield, MI 48075, USA
E-mail: ceamon@ltu.edu

[1]This chapter appeared, under the same title, in the *International Journal of Reliability and Safety*, Vol. 4, Nos. 2/3, 2010. Copyright © Inderscience Publishers, P.O.Box 735, Olney Bucks, MK46 5WB, UK. Reprinted with permission.

This chapter presents an analytical approach for structural reliability analysis without requiring the calculation of the most probable point of failure. Initially, the primary statistical moments of a multi-dimensional performance function are estimated using the Univariate Dimension-Reduction (UDR) methodology based on additive decomposition of the limit state function. Through moment matching, the UDR-based estimated moments are then used to fit the parameters of Extended Generalized Lambda Distribution (EGLD), and finally the probability of failure is calculated. To evaluate the accuracy and efficiency of the UDR + EGLD approach in comparison to the traditional First-Order Reliability Method (FORM) and direct Monte Carlo Simulation (MCS), five example problems involving nonlinear limit state functions are examined. The results show that UDR + EGLD offers nearly the same level of accuracy as MCS with superior efficiency to FORM. However, UDR + EGLD appears to have tail sensitivity, which limits its application to problems with moderate levels of reliability.

32.1 Introduction

In structural reliability, a performance (limit-state) function is used as the criterion to separate the safe and failure regions of the random-variable space. For problems involving two or more random variables, the failure probability calculation requires the solution to the multiple integral of the joint Probability Density Function (PDF) of all the random variables over the failure region. However, since for most realistic problems the analytical integration of the multidimensional joint PDF is not possible, many approximate analytical and simulation methods have been developed for reliability estimation.

The analytical methods offer efficient estimation of reliability. The First- and Second-Order Reliability Methods (FORM and SORM), which are based on the first- and second-order expansions of the limit-state function at the Most Probable failure Point (MPP) are the most popular analytical methods (Hasofer and Lind (1974); Rackwitz and Fiessler (1978); Der Kiureghian, Lin, and Hwang (1987)).

Conversely, the simulation-based methods such as the direct Monte Carlo Simulation (MCS) (Rubinstein (1981)) and its more advanced variants including importance sampling (Melchers (1989)), adaptive importance sampling (Wu (1994)) and directional simulation (Nie and Ellingwood (2000)) can provide very accurate estimates of structural reliability. However, since for many complex engineering problems the evaluation of the performance function requires the use of high-fidelity models and solution techniques (e.g. finite element analysis), the large number of analyses required by simulation-based methods can greatly increase the computational cost of reliability analysis.

Reliability estimation for problems involving a large number of random variables can be performed more efficiently by using Monte Carlo-based techniques than analytical methods when advanced sampling methods are used. These ad-

vanced sampling methods include Markov chain subset simulation (Au and Beck (2001)), subset simulation with splitting (Ching, Au, and Beck (2005)), spherical subset simulation (Katafygiotis and Cheung (2003)), auxiliary domain (Katafygiotis and Cheung (2004)), line sampling (Pradlwarter, Schueller, Koutsourelakis, and Charmpis (2007)). An excellent overview of methods for reliability estimation for structural problems involving large number of random variables can be found in Schueller and Pradlwarter (2007).

With the growing interest in reliability-based design optimization and its application to complex structural systems (Rais-Rohani, Solanki, and Eamon (2006); Thompson, Eamon, and Rais-Rohani (2006)), there is a strong need for the development of efficient and accurate reliability analysis methods.

Here, an alternative approach for failure probability calculation is presented. The proposed analytical technique combines an efficient reduced-order integration method for evaluation of statistical moments with a flexible distribution fitting technique for modeling the PDF or the Cumulative Distribution Function (CDF) of the multidimensional performance function. This general approach was followed by Hong (1996) and Zhao and Ono (2001). Hong (1996) proposed a non-linear system of equations for point estimation of failure probability in combination with the Johnson distribution and GramCharlier series. Zhao and Ono (2001) proposed a point estimation method using the Rosenblatt transformation and a $k \times n$ point concentration, where k is the number of quadrature points and n is the number of variables. In this study, the distribution fitting technique considered is the four-parameter Extended Generalized Lambda Distribution (EGLD) (Freimer, Mudholkar, Kollia, and Lin (1988); Karian, Dudewicz, and McDonald (1996)) in which the unknown parameters are found by using the four statistical moments of the performance function. To the authors' knowledge, this study is the first attempt to use EGLD in reliability assessment.

Calculation of the statistical moments of a multidimensional performance function requires evaluation of multidimensional integrals. These integrals are usually simpler to compute than the failure probability integral because the integration is not restricted to the failure domain, but rather it covers the entire random-variable space. In this chapter, the UDR method (Rahman and Xu (2004)) is used for the calculation of the statistical moments.

The advantage of the UDR + EGLD approach over other analytical methods is that it does not require the calculation of MPP; moreover, it requires considerably fewer function evaluations for calculation of CDF in comparison to other simulation-based or analytical methods. The disadvantage of the UDR + EGLD approach is in its tail sensitivity. Even very small errors in the probability distribution estimation can lead to significant errors in the failure probability prediction (Ben-Haim (1996), Chapter 7). This shortcoming limits the accuracy of this approach to problems with moderate reliability ($Re < 0.99$) for which FORM and other reliability index-based methods provide inadequate accuracy. Furthermore, the computational savings offered by UDR + EGLD over direct simulation

methods such as MCS can be substantial, although for simple analytical limit states, this cost reduction may not be important.

The next section discusses calculation of the statistical moments via UDR, followed by discussion of probability distribution fitting via EGLD. Then, the characteristics of the proposed approach, such as efficiency and accuracy, are illustrated using five different non-linear performance functions with two to eight random variables.

32.2 Calculation of Statistical Moments

For an N-dimensional, differentiable, real-valued performance function $Y(\boldsymbol{X})$ that depends on random-variable vector, $\boldsymbol{X}=\{X_1, X_2 \ldots, X_n\}^T \in \Re^N$, the l-th statistical moment, m_l is calculated as:

$$m_l = E\left[Y^l(\boldsymbol{X})\right] = \int_{\Re^N} y^l(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\, \boldsymbol{dx}, \tag{32.2.1}$$

where E is the expectation operator and $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint PDF of $\boldsymbol{X}$. Note that the statistical moments calculated using equation (32.2.1) are the moments about the origin as opposed to central moments which are around the mean values of the random variables. Using the additive decomposition approach of Rahman and Xu (2004), the multidimensional function $Y(\boldsymbol{X})$ is approximated using multiple one-dimensional functions as:

$$\hat{Y}(\boldsymbol{X}) = \sum_{j=1}^{N} Y(\mu_1, \ldots, \mu_{j-1}, X_j, \mu_{j+1}, \ldots, \mu_N) - (N-1)y(\mu_1, \ldots, \mu_N), \tag{32.2.2}$$

where each term in the summation is a one-dimensional random function that depends on the j-th random variable, X_j whereas $y(\mu_1, \ldots, \mu_N)$ is the deterministic value of $Y(\boldsymbol{X})$ calculated at the mean values of all the random variables, μ_j with $j = 1, \ldots, N$.

Using the additive decomposition in equation (32.2.2) as part of the UDR method developed by Rahman and Xu (2004) and applying the binomial formula, the l-th moment of $\hat{Y}(\boldsymbol{X})$ can be approximated as:

$$m_l \approx E\left[\hat{Y}^l(\boldsymbol{X})\right] = \sum_{i=0}^{l} \binom{l}{i} S_N^i \left[-(N-1)y(\mu_1, \ldots, \mu_N)\right]^{l-i}, \tag{32.2.3}$$

where the S_j^i, $j = 1, \ldots, N$ term represents the recursive equation defined as:

$$\begin{aligned} S_1^i &= E[Y^i(X_1, \mu_2, \ldots, \mu_N)]; \quad i = 1, \ldots, l \\ S_2^i &= \sum_{k=0}^{i} \binom{i}{k} S_1^k E[Y^{i-k}(\mu_1, X_2, \mu_3, \ldots, \mu_N)]; \quad i = 1, \ldots, l \end{aligned}$$

$$\vdots$$

$$S_N^i = \sum_{k=0}^{i} \binom{i}{k} S_{N-1}^k E[Y^{i-k}(\mu_1, \ldots, \mu_{N-1}, X_N)]; \quad i = 1, \ldots, l. \tag{32.2.4}$$

Here $S_0^i = 1$ for $i = 1, \ldots, l$. The univariate moments,

$$E\left[Y^m(\mu_1, \ldots, \mu_{j-1}, X_j, \mu_{j+1}, \ldots, \mu_N)\right],$$

with $m = i - k$, equation (32.2.4) are calculated using the formula:

$$E\left[Y^m(\mu_1, \ldots, \mu_{j-1}, X_j, \mu_{j+1}, \ldots, \mu_N)\right]$$
$$= \int_{-\infty}^{\infty} Y^m\left(\mu_1, \ldots, \mu_{j-1}, x_j, \mu_{j+1}, \ldots, \mu_N\right) f_{X_j}(x_j)\, dx_j, \tag{32.2.5}$$

where $f_{X_j}(x_j)$ is the marginal probability distribution of the j-th random variable. While performing the numerical integration of equation (32.2.5), we used finite limits for the upper and lower bounds as ± 10 standard deviation away from the mean, respectively. Since the proposed method is suitable for moderate reliabilities ($Re < 0.99$), the use of ± 10 standard deviation away from the mean is acceptable. To use equation (32.2.5), the random variables have to be independent. However, when there is dependency among the random variables, a multivariate transformation such as that of Rosenblatt can be used to transform the dependent random variables into a set of independent standard Gaussian random variables prior to the application of the dimension-reduction procedure. It must be noted here that the Rosenblatt transformation has its limitations in that it requires the conditional marginal density of random variables.

The main advantages of the approximations in equations (32.2.3) and (32.2.5) over the exact formulation in equation (32.2.1) are twofold: the replacement of a complicated multidimensional integral with multiple simpler one-dimensional integrals, and approximation of moments of performance function by moments of univariate function.

The UDR method has also been successfully applied by others (Huang and Du (2006); Youn, Xi, Wells, and Wang (2006)) in estimation of statistical moments. Depending upon the complexity of the performance function and its formulation (explicit or implicit), the one-dimensional integrals can be evaluated analytically or numerically using, for instance, the moment-based quadrature rule (Rahman and Xu (2004)). However, Youn, Xi, Wells, and Wang (2006) discovered that for some large-scale non-linear systems, the moment-based quadrature rule may lead to unstable moment predictions. To remedy this problem, they approximate the performance function using a response surface model, which is then integrated using a numerical technique.

The total number of limit-state function evaluations required by the UDR method depends on the number of quadrature points (M) and distribution types

of the input random variables. For a problem with N random variables, if the input random variables have symmetric distributions (e.g., normal) and the same number of quadrature points are used for all variables, then $(M-1) \times N + 1$ total function evaluations are required. If the input random variables follow asymmetric distributions (e.g., log-normal), then $MN + 1$ function evaluations are required. Usually five quadrature points (i.e., $M = 5$) work quite well for most problems (Youn, Xi, Wells, and Wang (2006)). For mildly non-linear functions, more function evaluations may be needed for accuracy.

32.3 Estimating the Distribution of Performance Function

The distribution of a continuous random variable, such as an N-dimensional performance function, can be approximated using several approaches including the Johnson and Pearson families of distributions, saddlepoint approximations, and the generalized beta and lambda distributions. Usually the first four statistical moments, which define the mean, variance, skewness, and kurtosis of the distribution, respectively, are used in calculating the unknown fitting parameters in each distribution fitting model.

Johnson's distribution (Johnson (1949)) is a four-parameter model based on a transformation of the standard normal variate. It includes three distinct forms (i.e., unbounded, bounded, and log-normal) in addition to the normal form that is a special case of the unbounded form. The main disadvantage of using the Johnson distribution is that it is often difficult to determine the four unknown parameters from the moments of the sample data (Johnson, Kotz, and Balakrishnan (1995)).

In the Pearson family of distributions, one of five different types of continuous distributions is usually used depending on the limited or unlimited range of distribution in one or both directions as well as the skewness or symmetry of the distribution. The major drawback of the Pearson family of distributions is that it can lead to unstable results near the boundaries of different regions corresponding to different distribution types (Youn, Xi, Wells, and Wang (2006)).

Saddlepoint approximations are another powerful tool for approximating distributions that rely on the Cumulant Generating Function (CGF) of the continuous random variable. The CGF of the random variable is approximated using the first few statistical moments. The main shortcoming of this approach is that singularities can arise during the numerical procedure (Huang and Du (2006)).

Here, we use the EGLD (Karian, Dudewicz and McDonald (1996)), which is based on the Generalized Lambda Distribution (GLD) combined with the Generalized Beta Distribution (GBD). EGLD represents a fairly flexible distribution fitting model, and it does not suffer the difficulties encountered with the other above-mentioned approaches. However, as in the case of other distributions, it

suffers from the tail sensitivity problem.

32.3.1 Overview of GLD and GBD

The GLD is most easily defined by its inverse CDF, which takes a simple explicit form given as (Freimer, Mudholkar, Kollia, and Lin (1988)):

$$F_\lambda^{-1}(y) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{y^{\lambda_3}-1}{\lambda_3} - \frac{(1-y)^{\lambda_4}-1}{\lambda_4}\right), \tag{32.3.1}$$

where λ_1 is the location parameter, λ_2 is the scale parameter, λ_3 and λ_4 define the shape characteristics ($\lambda_3 = \lambda_4$ for a symmetric distribution) and y is the CDF value ($0 \le y \le 1$). The PDF of GLD is defined as:

$$f_\lambda(x) = \frac{\lambda_2}{y^{\lambda_3-1} + (1-y)^{\lambda_4-1}}, \quad x = F_\lambda^{-1}(y). \tag{32.3.2}$$

The mean (μ), variance (σ^2), skewness (α_3), and kurtosis (α_4) of the GLD are defined in terms of its distribution parameters (Lakhany and Mausser, (2000)) as:

$$\alpha_1 = \mu = \nu_1, \tag{32.3.3}$$

$$\alpha_2 = \sigma^2 = \frac{\nu_2 - \nu_1^2}{\lambda_2^2}, \tag{32.3.4}$$

$$\alpha_3 = \frac{\nu_3 - 3\nu_1\nu_2 + 2\nu_1^3}{(\nu_2 - \nu_1^2)^{3/2}}, \tag{32.3.5}$$

$$\alpha_4 = \frac{\nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4}{(\nu_2 - \nu_1^2)^2}, \tag{32.3.6}$$

where ν_1 through ν_4 are functions of only λ_3 and λ_4 and are defined as:

$$\nu_1 = \frac{1}{(1+\lambda_3)\lambda_3} - \frac{1}{(1+\lambda_4)\lambda_4}, \tag{32.3.7}$$

$$\nu_2 = \frac{1}{(1+2\lambda_3)\lambda_3^2} + \frac{1}{(1+2\lambda_4)\lambda_4^2} - \frac{2\beta(1+\lambda_3, 1+\lambda_4)}{\lambda_3\lambda_4}, \tag{32.3.8}$$

$$\begin{aligned}\nu_3 &= \frac{1}{(1+3\lambda_3)\lambda_3^3} - \frac{1}{(1+3\lambda_4)\lambda_4^3} \\ &- \frac{3\beta(1+2\lambda_3, 1+\lambda_4)}{\lambda_3^2\lambda_4} + \frac{3\beta(1+\lambda_3, 1+2\lambda_4)}{\lambda_3\lambda_4^2},\end{aligned} \tag{32.3.9}$$

$$\begin{aligned}\nu_4 &= \frac{1}{(1+4\lambda_3)\lambda_3^4} + \frac{1}{(1+4\lambda_4)\lambda_4^4} - \frac{4\beta(1+3\lambda_3, 1+\lambda_4)}{\lambda_3^3\lambda_4} \\ &+ \frac{6\beta(1+2\lambda_3, 1+2\lambda_4)}{\lambda_3^2\lambda_4^2} - \frac{4\beta(1+\lambda_3, 1+3\lambda_4)}{\lambda_3\lambda_4^3},\end{aligned} \tag{32.3.10}$$

with the function $\beta(a, b)$ determined using:

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \quad \text{for } a, b > 0. \tag{32.3.11}$$

The GLD distribution moments can be related to the statistical moments (m_l) via:

$$\mu = m_1, \tag{32.3.12}$$
$$\sigma^2 = m_2 - m_1^2, \tag{32.3.13}$$
$$\alpha_3 = \frac{m_3 - 3m_1m_2 + 2m_1^3}{\sigma^3}, \tag{32.3.14}$$
$$\alpha_4 = \frac{m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4}{\sigma^4}. \tag{32.3.15}$$

The distribution fitting (i.e., the estimation of the distribution parameters λ_1, λ_2, λ_3 and λ_4) is performed by matching the skewness and kurtosis of GLD with the corresponding moments of the performance function obtained using UDR as discussed previously in Section 32.2.

Even though GLD is a flexible distribution that can model a wide variety of distributions, its accuracy is not guaranteed in some regions in the skewness-kurtosis plane. Karian, Dudewicz, and McDonald (1996) proposed using GBD in the regions where GLD is not defined, as shown in Figure 32.3–1. The resulting modified distribution is known as the extended generalized lambda distribution or EGLD. Here, in the region of the plane where GLD and GBD coexist, we use a combination (average) of both distributions.

Karian, Dudewicz, and McDonald (1996) formulated GBD with its PDF defined as:

$$f_\beta(x) = C\beta_2^{-(\beta_3+\beta_4+1)}(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}, \tag{32.3.16}$$

where

$$C = \frac{\Gamma(\beta_3+\beta_4+2)}{\Gamma(\beta_3+1)\Gamma(\beta_4+1)} \tag{32.3.17}$$

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt. \tag{32.3.18}$$

The mean (μ), variance (σ^2), skewness (α_3) and kurtosis (α_4) of the GBD are defined in terms of its distribution parameters (Karian, Dudewicz, and McDonald (1996)) as:

$$\alpha_1 = \mu = \beta_1 + \frac{\beta_2(1+\beta_3)}{B_2}, \tag{32.3.19}$$

$$\alpha_2 = \sigma^2 = \frac{\beta_2^2(1+\beta_3)(1+\beta_4)}{B_2^2B_3}, \tag{32.3.20}$$

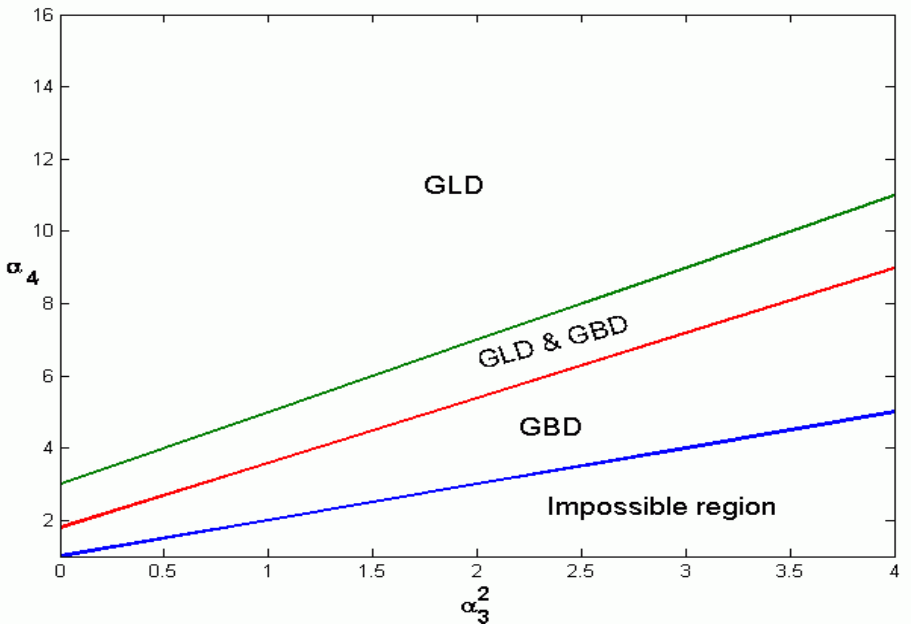


Figure 32.3–1. Regions in the skewness-kurtosis plane.

$$\alpha_3 = \frac{2(\beta_4 - \beta_3)\sqrt{B_3}}{B_4\sqrt{(1+\beta_3)(1+\beta_4)}}, \tag{32.3.21}$$

$$\alpha_4 = \frac{3B_3(\beta_3\beta_4 B_2 + 3\beta_3^2 + 5\beta_3 + 3\beta_4^2 + 5\beta_4 + 4)}{B_4 B_5(1+\beta_3)(1+\beta_4)}, \tag{32.3.22}$$

where

$$B_i = \beta_3 + \beta_4 + i, \quad \text{for } i = 1, \ldots, 5. \tag{32.3.23}$$

Similar to the estimation of GLD parameters, the GBD parameters are also obtained by matching the skewness and kurtosis of GBD with those of the performance function obtained through UDR.

32.3.2 Estimation of EGLD Parameters

In equations (32.3.3) through (32.3.15), we notice that the distribution parameters λ_1 and λ_2 can be defined in terms of λ_3 and λ_4 along with the first two moments, m_1 and m_2. To find λ_3 and λ_4, the skewness and kurtosis of GLD and the ones of the performance function are matched (i.e., the differences between them are minimized). The skewness and kurtosis of the performance function, $Y(\boldsymbol{X})$ as calculated via UDR are defined as:

$$\alpha_3^* = \frac{\int_{-\infty}^{\infty}(Y-\mu^*)^3 f(Y)\,dY}{(\sigma^*)^3} \tag{32.3.24}$$

$$\alpha_4^* = \frac{\int_{-\infty}^{\infty}(Y-\mu^*)^4 f(Y)\,dY}{(\sigma^*)^4}, \tag{32.3.25}$$

where μ^*, σ^* are the mean and standard deviation of the performance function, respectively, as estimated by UDR. The GLD distribution parameters λ_3 and λ_4 can then be computed by solving the following optimization problem: Find λ_3 and λ_4 that would minimize

$$\left(\frac{\alpha_3(\lambda_3,\,\lambda_4)}{\alpha_3^*}-1\right)^2+\left(\frac{\alpha_4(\lambda_3,\,\lambda_4)}{\alpha_4^*}-1\right)^2, \tag{32.3.26}$$

such that λ_3 and $\lambda_4 > -0.25$. Note that α_3 and α_4 are the skewness and kurtosis of GLD, whereas α_3^* and α_4^* are the skewness and kurtosis of the performance function as estimated through UDR. The constraints in equation (32.3.26) require that the moments be finite. Once λ_3 and λ_4 are determined, the other two parameters (λ_1 and λ_2) can be found from:

$$\lambda_1 = \mu^* - \frac{\nu_1}{\lambda_2} \tag{32.3.27}$$

$$\lambda_2 = \frac{1}{\sigma^*}\sqrt{\nu_2-\nu_1^2}. \tag{32.3.28}$$

The calculation of the distribution parameters of GBD is similar to that of the GLD. In this case, we solve the following unconstrained optimization problem: Find β_3 and β_4 that would minimize

$$\left(\frac{\alpha_3(\beta_3,\,\beta_4)}{\alpha_3^*}-1\right)^2+\left(\frac{\alpha_4(\beta_3,\,\beta_4)}{\alpha_4^*}-1\right)^2. \tag{32.3.29}$$

Notice that there are no constraints on the values of β_3 or β_4 since GBD has finite moments regardless of the values of β_3 and β_4. Once β_3 and β_4 are calculated, the other two parameters (β_1 and β_2) can be computed from:

$$\beta_1 = \mu^* - \frac{\beta_2(1+\beta_3)}{B_2} \tag{32.3.30}$$

$$\beta_2 = \sqrt{\frac{B_2^2 B_3 (\sigma^*)^2}{(1+\beta_3)(1+\beta_4)}}. \tag{32.3.31}$$

The EGLD is very flexible because it can be adjusted to describe a wide variety of unimodal (i.e., single-peaked) distributions. More information on the efficiency and accuracy of distribution fitting via GLD, GBD, and EGLD can be found in Karian, Dudewicz, and McDonald (1996), Karian and Dudewicz (2000), Lampasi, Nicola, and Podesta (2005), King and MacGillivray (2006) and Fournier, Rupin, Bigerelle, Najjar, Iost, and Wilcox (2007).

32.3.3 Probability of Failure Estimation Using EGLD

Once the distribution parameters of EGLD are calculated, the probability of failure can be computed in a single step as:

$$P_f = F_\lambda(x = 0) \text{ or } P_f = F_\beta(x = 0). \tag{32.3.32}$$

The use of F_λ or F_β for probability of failure estimation depends on the values of the skewness and kurtosis of the performance function (Figure 32.3–1) as noted earlier.

32.4 The UDR + EGLD Algorithm

Given an N-dimensional performance function $Y(\boldsymbol{X})$ and statistical properties of the associated random-variable vector $\boldsymbol{X} = \{X_1, \ldots, X_N\} \in \Re^N$, use:

1. Equation (32.2.1) to find the required moments of all random variables.

2. Equation (32.2.3) to find the first four statistical moments of $Y(\boldsymbol{X})$ based on the recursive formula in equation (32.2.4). If the random variables are dependent, use the Rosenblatt transformation to obtain a set of independent standard Gaussian random variables before using equation (32.2.3).

3. Equations (32.3.24) through (32.3.26) to solve for the parameters λ_3 and λ_4 and/or β_3 and β_4 by minimizing the differences in the third and fourth moments of GLD and/or GBD and the corresponding moments of the performance function as calculated via UDR.

4. Equations (32.3.27) and (32.3.28) and/or (32.3.30) and (32.3.31) to solve for the remainder of the EGLD parameters (λ_1 and λ_2 and/or β_1 to β_2).

5. The approximate CDF or PDF per equation (32.3.32) to find the failure probability.

In this procedure, the accuracy of the reliability estimation depends on the accuracies of the estimated statistical moments and probability distribution. The example problems in the next section are used to measure the accuracy and efficiency of the proposed approach.

Although the UDR + EGLD approach is straightforward and easy to implement, it has the following restrictions:

1. It is applicable to problems with moderate reliability levels ($Re < 0.99$). For $Re > 0.99$, there will be a loss of accuracy due to the tail sensitivity of the fitted distribution.

2. It is suitable for problems with smooth, unimodal, scalar limit-state functions that are additive in nature and do not have-higher order mixed terms. Otherwise, the UDR method and thereby the proposed approach could be quite inaccurate.

3. The use of UDR for estimation of statistical moments requires an independent set of random variables; otherwise, the dependent variables have to be transformed into an independent set prior to using the UDR procedure.

4. There is no theoretical proof for formulation invariance as altering the formulation of the performance function can potentially alter the values of EGLD parameters. Therefore, this possibility should be considered when using EGLD for distribution fitting.

32.5 Example Problems

Five illustrative example problems are considered. Examples 1 through 3 consider analytical functions involving two random variables, whereas Examples 4 and 5 describe structural mechanics problems with seven and eight random variables, respectively. For the assessment of accuracy and efficiency, the results obtained from the UDR + EGLD approach are compared with those from MCS and FORM. Moreover, to investigate the possibility of formulation variance, the performance function in Example 5 is modified and analyzed in Example 6. Here, the number of function evaluations as opposed to CPU time is used for computational efficiency as it represents a more accurate and unbiased metric.

32.5.1 Example 1: A Concave Function

The performance function is expressed in terms of two normally distributed random variables X_1 and X_2 as:

$$Y_1 = \frac{1}{10}\left[\exp(0.8X_1 - 1.2) + \exp(0.7X_2 - 0.6) - 5\right] \tag{32.5.1}$$

The mean values and standard deviations of both random variables are set at 3 and 0.8, respectively. Table 32.5–1 provides the summary of results, which include the predicted values for the first four moments of Y_1 as well as the failure probability. The MCSs are repeated 100 times with different samples and the coefficients of variations of the statistical moments and failure probabilities are computed. Notice that the coefficients of variation of the higher moments are larger than those of the lower moments, as expected. Notice also that the computed value of the coefficient of variation of the failure probability is the same as the analytical prediction, i.e.,

$$\text{c.o.v} = \sqrt{1 - P_f/N_{MCS}P_f} = \sqrt{1 - 0.1123/10^6(0.1123)} = 0.3\%.$$

Table 32.5–1. Summary of results for Example 1. The numbers in the parentheses denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of function evaluations	μ	σ	α_3	α_4	Failure probability	% error
MCS	1,000,000	0.4316 (0.1%)	0.4304 (0.1%)	1.589 (0.9%)	8.120 (2.8%)	0.1123 (0.3%)	–
UDR + EGLD	$6 \times 2 + 1 = 13$	0.4316	0.4305	1.590	8.129	0.1103	−1.8
FORM	21	–	–	–	–	0.1456	30.1

Note: The mean, standard deviation, skewness, and kurtosis (μ, σ, α_3, α_4) are calculated using equations (32.3.12) through (32.3.15), respectively.

The predicted values of statistical moments as well as failure probability are very close to the MCS predictions, which indicate the level of accuracy in the UDR + EGLD approach. By contrast, the FORM predicted failure probability has a significant error. Since the statistical moments are not part of FORM analysis, they are not provided in this or any of the other examples. Judging by the number of function evaluations, UDR + EGLD is much more efficient than FORM for this problem. Also, the number of samples to be used in MCS for the same accuracy as UDR + EGLD can be computed by

$$N_{MCS} = 1 - \frac{P_f}{e^2_{MCS} P_f}.$$

For this example, around 25,000 MCS samples are needed. Note that seven quadrature points ($M = 7$) are used in this example for both input variables due to the high non-linearity of the limit-state function. In addition, the PDF of Y_1 as obtained by MCS is shown in Figure 32.5–2 together with that estimated by the UDR + EGLD procedure. The two distributions compare fairly well.

32.5.2 Example 2: A Non-Linear Function

In this example, the non-linear performance function, Y_2 depends on two normally distributed random variables, X_1 and X_2, and is expressed as:

$$Y_2 = \frac{(X_1 + X_2 - 5)^2}{30} + \frac{(X_1 - X_2 - 12)^2}{140} - 0.8. \tag{32.5.2}$$

The random variables have mean values of 1.5 and 0.0, respectively, with standard deviations of 1.0. The results in Table 32.5–3 indicate that the UDR + EGLD predictions compare fairly well with those of MCS. The FORM predicted failure probability has over 20% error, while the UDR + EGLD prediction has an error of less than 1%. The number of samples to be used in MCS for the same accuracy as UDR + EGLD is around 115,000 for this example. Figure 32.5–4 shows that the PDF of Y_2 as obtained by the UDR + EGLD procedure is in good agreement

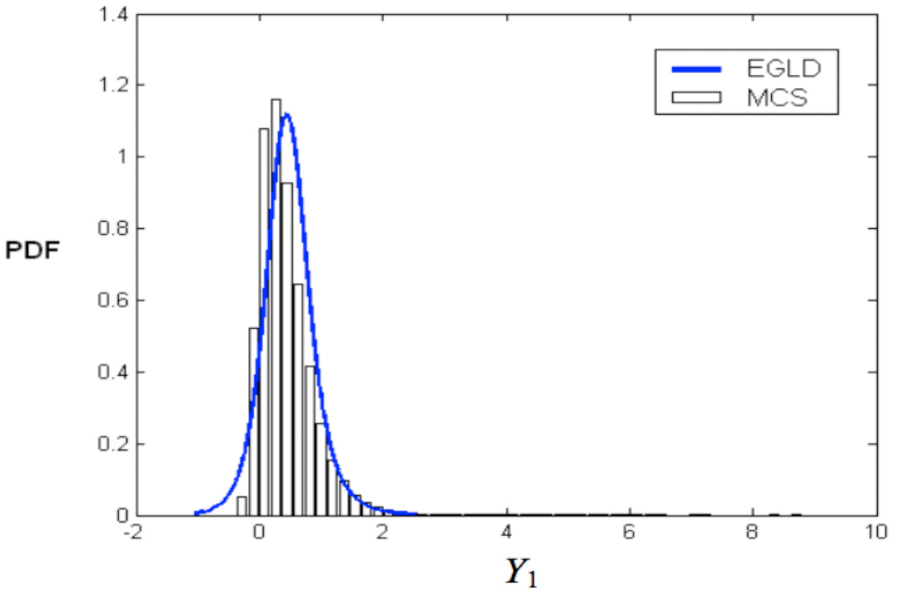


Figure 32.5–2. Comparison of PDF of Y_1 as obtained via UDR + EGLD and MCS.

with that based on MCS. Note that five quadrature points are used here for both input variables since the limit-state function has moderate non-linearity.

Table 32.5–3. Summary of results for Example 2. The numbers in the parenthesis denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of function evaluations	μ	σ	α_3	α_4	Failure probability	% error
MCS	1,000,000	0.4768 (0.1%)	0.4040 (0.1%)	0.7558 (0.5%)	3.924 (0.4%)	0.0970 (0.3%)	–
UDR + EGLD	$4 \times 2 + 1 = 9$	0.4768	0.4006	0.5971	3.478	0.0979	0.9
FORM	21	–	–	–	–	0.1196	23.1

32.5.3 Example 3: Performance Function with Infinite MPP

One of the advantages of UDR + EGLD, as mentioned earlier, is that it does not require an MPP search, and as such, it is not susceptible to problems encountered by FORM and other reliability methods that do require the MPP location. In this example, we consider a simple non-linear performance function of the form:

$$Y_3 = X_1^2 + X_2^2 - 1. \tag{32.5.3}$$

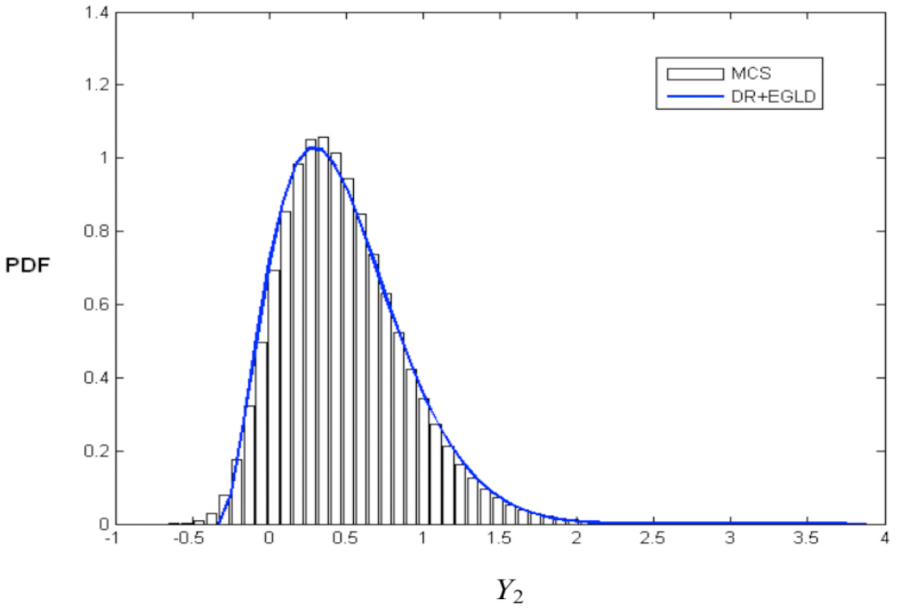


Figure 32.5–4. Comparison of PDF of Y_2 as obtained via UDR + EGLD and MCS.

Both random variables (X_1 and X_2) have standard normal distributions. The performance function in equation (32.5.3) represents the equation of a circle in the standard normal plane. The results in Table 32.5–5 show the accuracy and efficiency of UDR + EGLD and its superiority to FORM. The number of samples to be used in MCS for the same accuracy as UDR + EGLD is around 15,000 for this example. Figure 32.5–6 shows the exact and estimated PDF of Y_3. Seven quadrature points are used here for both input variables due to the high non-linearity of the limit-state function.

Table 32.5–5. Summary of results for Example 3. The numbers in the parentheses denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of function evaluations	μ	σ	α_3	α_4	Failure probability	% error
MCS	1,000,000	1.000 (0.2%)	2.000 (0.1%)	2.000 (0.4%)	8.997 (0.9%)	0.3935 (0.1%)	–
UDR + EGLD	$6 \times 2 + 1 = 13$	1	2	2	9	0.3974	1.0
FORM	41	–	–	–	–	0.8413	113.7

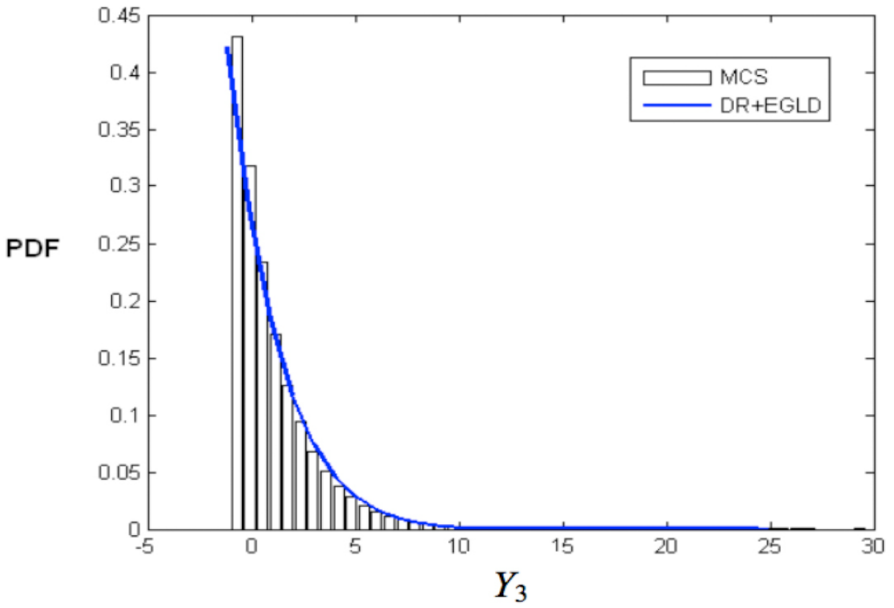


Figure 32.5–6. Comparison of PDF of Y_3 as obtained via UDR + EGLD and MCS.

32.5.4 Example 4: Vehicle Crash

A vehicle side impact problem is considered in this example where crashworthiness is measured based on the intrusion velocity of the door at the B-pillar location. An explicit relationship for intrusion velocity, in terms of seven random variables (X_1 through X_7) is given as (Youn, Xi, Wells and Wang (2006)):

$$\begin{aligned} V &= 0.75 - 0.489X_1X_4 - 0.843X_2X_3 + 0.0432X_5X_6 \\ &- 0.0556X_5X_7 - 0.000786X_7^2. \end{aligned} \tag{32.5.4}$$

For this problem, the failure occurs when the intrusion velocity has a positive value. Therefore, the limit-state function for this problem is:

$$Y_4 = -V. \tag{32.5.5}$$

The statistical properties of the random variables, taken from Youn, Xi, Wells, and Wang (2006), are provided in Table 32.5–7. Notice that out of seven random variables, only two are normal, while the remaining ones follow beta or uniform distributions. The parameters of the normal distribution are the mean and the standard deviation, respectively. The distribution parameters of the beta distribution are the mean, standard deviation and the two shape parameters, respectively; whereas the parameters of the uniform distribution are the lower and the upper bounds, respectively.

The results for this example are shown in Table 32.5–8. Although the number of random variables has increased, the UDR + EGLD procedure still provides a high degree of accuracy and efficiency in both calculation of statistical moments and failure probability. The number of samples to be used in MCS for the same accuracy as UDR + EGLD is around 29,000 for this example. The MCS and the UDR + EGLD predicted PDF plots are shown in Figure 32.5–9. For this example, five quadrature points are used for all random variables since the non-linearity of the limit-state function is moderate. Four of the random variables are symmetric and three of them are asymmetric. Thus, the total number of function evaluations is 33.

Table 32.5–7. Statistical properties of the random variables in Example 4.

Random variable	Distribution type and parameters
X_1	Normal (1, 0.05)
X_2	Normal (1, 0.05)
X_3	Beta (1, 0.05, 5, 36.44)
X_4	Beta (1, 0.05, 5, 36.44)
X_5	Beta (1, 0.05, 5, 36.44)
X_6	Uniform (−17.3205, 17.3205)
X_7	Uniform (−17.3205, 17.3205)

Table 32.5–8. Summary of results for Example 4. The numbers in the parentheses denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of function evaluations	μ	σ	α_3	α_4	Failure probability	% error
MCS	1,000,000	0.6604 (0.3%)	0.7120 (0.2%)	0.1628 (2.8%)	2.431 (0.3%)	0.1951 (0.2%)	–
UDR + EGLD	33	0.6605	0.7111	0.1611	2.414	0.1975	1.2
FORM	74	–	–	–	–	0.2656	35.4

32.5.5 Example 5: I-Beam Design Problem

In this example, a simply supported I-beam (Figure 32.5–10) under a concentrated load as discussed in Huang and Du (2006) is examined. The performance function is taken as the difference between resistance, S and load effect in terms of maximum normal stress, $\sigma_{\max}$ due to bending given by:

$$Y_5 = S - \sigma_{\max}, \tag{32.5.6}$$

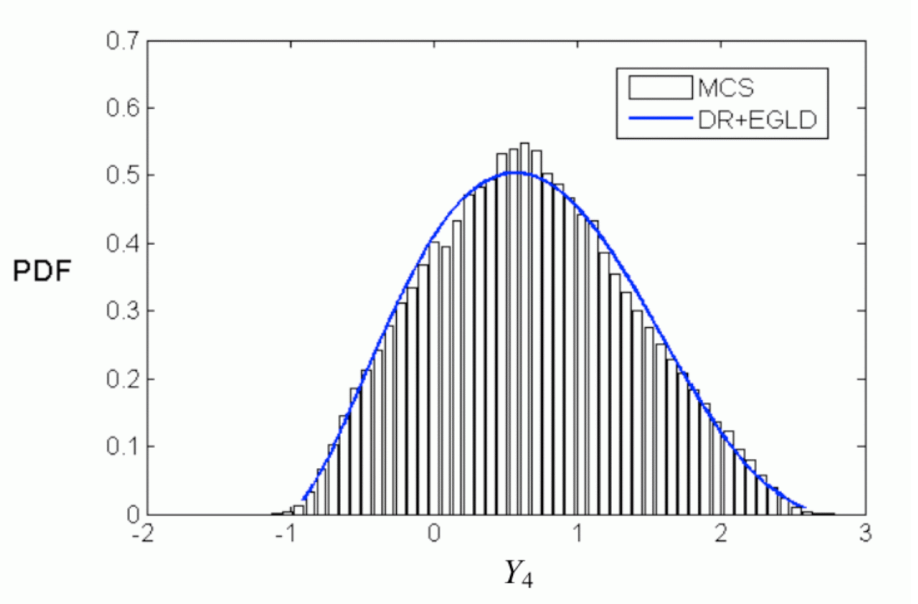


Figure 32.5–9. Comparison of PDF of Y_4 as obtained via UDR + EGLD and MCS.

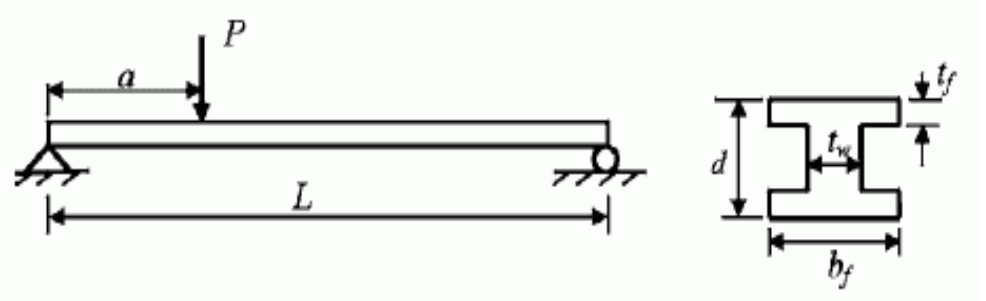


Figure 32.5–10. The cross section and loading on the I-beam.

where

$$\sigma_{\max} = \frac{Pa(L-a)d}{2LI} \tag{32.5.7}$$

$$I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12}. \tag{32.5.8}$$

There are eight random variables in this problem with the corresponding statistical characteristics provided in Table 32.5–11. The solution results for this example are given in Table 32.5–12 with PDF plots shown in Figure 32.5–13. The number of samples to be used in MCS for the same accuracy as UDR + EGLD is around 75,000 for this example. For UDR, five quadrature points are used for all input variables due to the moderate non-linearity of the limit-state function.

Table 32.5–11. Statistical properties of the random variables in Example 5.

Random variable	Distribution type and parameters
P	Normal (6070, 200)
L	Normal (120, 6)
A	Normal (72, 6)
S	Normal (170000, 4760)
D	Normal (2.3, 1/24)
b_f	Normal (2.3, 1/24)
t_w	Normal (0.16, 1/48)
t_f	Normal (0.26, 1/48)

Table 32.5–12. Summary of results for Example 5. The numbers in the parentheses denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of func. evaluations	μ	σ	α_3	α_4	Failure prob.	% error
MCS	1,000,000	1.982×10^4 (0.1%)	1.792×10^4 (0.1%)	0.1136 (2.7%)	3.337 (0.3%)	0.1290 (0.2%)	–
UDR + EGLD	$4 \times 8 + 1$ 33	1.980×10^4	1.773×10^4	0.1060	3.102	0.1293	0.3
FORM	141	–	–	–	–	0.1442	11.7

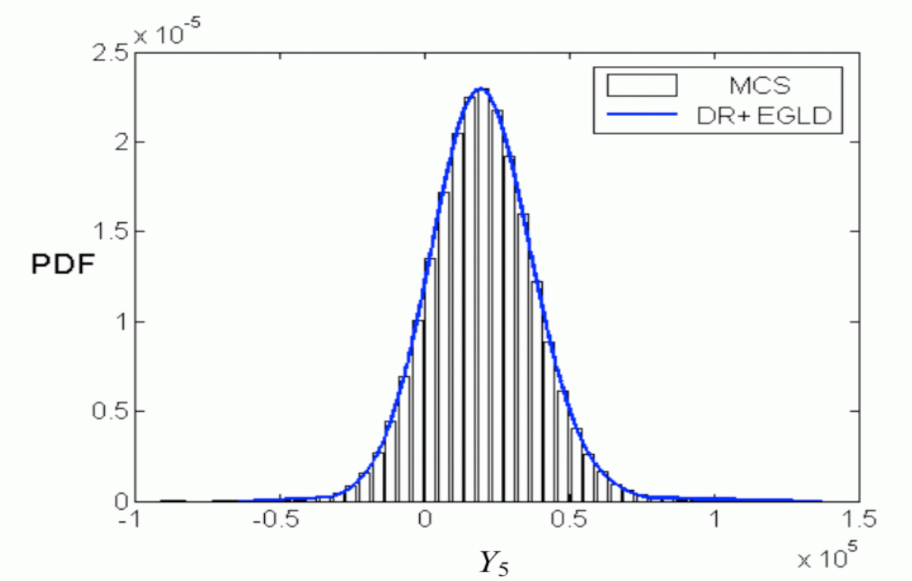


Figure 32.5–13. Comparison of PDF of Y_5 as obtained via UDR + EGLD and MCS.

32.5.6 Example 6: Checking for Formulation Invariance

To determine whether the UDR + EGLD predictions are impacted as a result of modifying the performance function equation, we considered an equivalent formulation for Y_5 in the problem above, given as:

$$Y_6 = \frac{S}{\sigma_{\max}} - 1. \tag{32.5.9}$$

The results in Table 32.5–14 indicate that even though the reformulation of the performance function affected the predicted value of failure probability, the error in the UDR + EGLD prediction is still fairly low in comparison to FORM as well as MCS conducted with 1000 samples. Figure 32.5–15 shows that the PDF of Y_6 as estimated by the UDR + EGLD procedure compares well with that obtained by MCS. The number of samples to be used in MCS for the same accuracy as UDR + EGLD is around 6000 for this example.

Table 32.5–14. Summary of results for Example 6. The numbers in the parentheses denote the coefficient of variation of the relevant quantity computed over 100 different samples.

	No. of function evaluations	μ	σ	α_3	α_4	Failure probability	% error
MCS	1,000,000	0.1480 (0.1%)	0.1474 (0.2%)	1.221 (0.7%)	9.725 (0.6%)	0.1290 (0.2%)	–
UDR + EGLD	$4 \times 8 + 1$ 33	0.1466	0.1349	0.3424	3.308	0.1333	3.3
FORM	141	–	–	–	–	0.1442	11.7

32.5.7 Effect of Probability Level

To investigate the effect of the level of failure probabilities on the accuracy of the UDR + EGLD approach, the mean value of the load is progressively reduced, thereby the failure probability level is reduced and the accuracy of the proposed method is evaluated. As shown in Table 32.5–16, when the mean load is reduced by 5%, the predicted failure probability is reduced from 0.1293 to 0.0513, which are slightly different than the MCS predictions. Further reductions of the mean value of the applied load result in reduced failure probabilities and increased errors in the UDR + EGLD predictions. Based on the results given in Table 32.5–16 and our experience with other examples, we conclude that UDR + EGLD is suitable for moderate reliability levels ($Re < 0.99$).

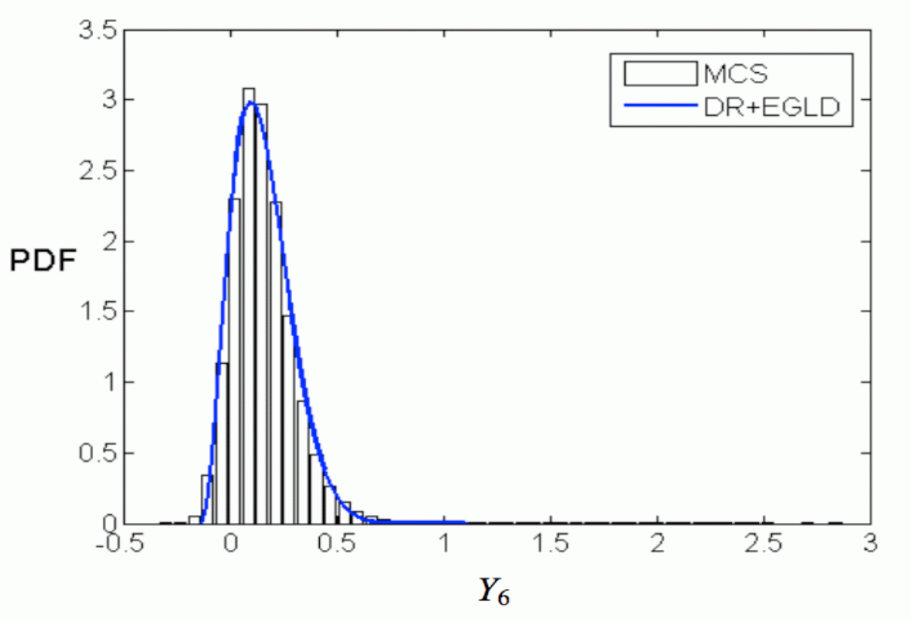


Figure 32.5–15. Comparison of PDF of Y_6 as obtained via UDR + EGLD and MCS.

Table 32.5–16. Effect of failure probability level on the accuracy of UDR + EGLD for Example 5. The sample size of MCS is increased to 100 millions.

$P_{mean}/(P_{mean})_{orig}$	$P_f(MSC)$	$P_f(UDR+EGLD)$	% error
1.00	0.1291	0.1293	0.2
0.95	0.0524	0.0513	2.1
0.90	0.0162	0.0145	10
0.85	0.0037	0.0025	32

32.6 Concluding Remarks

Given an N-dimensional, differentiable, unimodal performance function along with the statistical properties of the underlying random variables, the proposed approach uses the UDR technique for estimation of the four primary statistical moments, which are in turn used for finding the unknown parameters in the EGLD for probability distribution fitting of the performance function.

The accuracy and efficiency of the proposed (UDR + EGLD) approach for estimations of statistical moments and failure probability were examined with the help of three mathematical functions and two engineering design problems involving non-linear performance functions with two to eight random variables.

In comparison to direct MCS and FORM, the UDR + EGLD approach was shown to be fairly accurate in estimating the failure probability while requiring significantly fewer performance function evaluations. Even in problems where the FORM prediction of failure probability was extremely erroneous, the UDR + EGLD approach proved to be very accurate and efficient. Going over the procedures of the UDR method, one can see that it can handle implicit limit states. Therefore, no additional effort is spent for including examples that have implicit limit-state functions.

Although UDR + EGLD has some limitations as articulated in this paper, the results are promising, prompting further investigation of this approach in problems involving non-smooth limit-state functions that depend on much larger sets of random variables.

Acknowledgments

The funding provided for this study by the US Department of Energy Grant No. DE-FC26-06NT42755 is gratefully acknowledged. The authors also like to acknowledge the support of Center for Advanced Vehicular Systems at Mississippi State University.

References for Chapter 32

Au, S.K. and Beck, J. (2001). "Estimation of small failure probabilities in high dimensions by subset simulation," *Probabilistic Engineering Mechanics,* Vol. 16, pp. 263–277.

Ben-Haim, Y. (1996). *Robust Reliability in the Mechanical Sciences,* Springer, Berlin.

Ching, J., Au, S.K., and Beck, J. (2005). "Reliability estimation for dynamical systems subject to stochastic excitation using subset simulation with splitting," *Computer Methods in Applied Mechanics and Engineering,* Vol. 194, Nos. 12–16, pp. 1557–1579.

Der Kiureghian, A., Lin, H.Z., and Hwang, S.J. (1987). "Second-order reliability approximations," *Journal of Engineering Mechanics,* Vol. 113, No. 8, pp. 1208–1225.

Fournier, B., Rupin, N., Bigerelle, M., Najjar, D., Iost, A., and Wilcox, R. (2007). "Estimating the parameters of a generalized lambda distribution," *Computational Statistics & Data Analysis,* Vol. 51, pp. 2813–2835.

Freimer, M., Mudholkar, G., Kollia, G., and Lin, C. (1988). "A study of the generalized Tukey lambda family," *Communications in Statistics: Theory and Methods,* Vol. 17, No. 10, pp. 3547–3567.

Hasofer, A.M. and Lind, N. (1974). "An exact and invariant first-order reliability format," *Journal of Engineering Mechanics,* Vol. 100, No. EM1, pp. 111–121.

Hong, H.P. (1996). "Point-estimate moment-based reliability analysis," *Civil Engineering System,* Vol. 13, pp. 281–294.

Huang, B. and Du, X. (2006). "Uncertainty analysis by dimension reduction integration and saddlepoint approximations," *Journal of Mechanical Design,* Vol. 126, No. 1, pp. 26–33.

Johnson, N.L. (1949). "Systems of frequency curves generated by methods of translation," *Biometrika*, Vol. 36, pp. 149–176.

Johnson, N.L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions, Vol. 2,* 2nd ed., John Wiley, New York.

Karian, Z.A. and Dudewicz, E.J. (2000). *Fitting Statistical Distributions to Data: The Generalized Lambda Distribution and the Generalized Bootstrap Methods,* CRC Press, Boca Raton, Florida.

Karian, Z.A., Dudewicz, E.J., and McDonald, P. (1996). "The extended generalized lambda distribution system for fitting distributions to data: History, completion of theory, tables, applications, the 'final word' on moment fits," *Communications in Statistics – Computation and Simulation,* Vol. 25, No. 3, pp. 611–642.

Katafygiotis, L.S. and Cheung, S.H. (2003). "An Efficient Method for Calculation of Reliability Integrals," Fifth International Conference on Stochastic Structural Dynamics (SSD03), Hangzhou, China.

Katafygiotis, L.S. and Cheung, S.H. (2004). "Auxiliary Domain Method for Solving Non-Linear Reliability Problems," *Proceedings of the 9th ASCE specialty conference on probabilistic mechanics and structural reliability (PMC 2004)*, Albuquerque, New Mexico, USA.

King, R.A.R. and MacGillivray, H.L. (2006, March). "Fitting the Generalized Lambda Distribution with Location and Scale-Free Shape Functionals," *Proceedings of the Symposium on Fitting Statistical Distributions to Data,* Auburn, Alabama.

Lakhany, A. and Mausser, H. (2000). "Estimating the parameters of the generalized lambda distribution," *Algo Research Quarterly,* Vol. 3, No. 3, pp. 47–58.

Lampasi, D.A., Nicola, F.D., and Podesta, L. (2005, May). "The Generalized Lambda Distribution for the Expression of Measurement Uncertainty," IEEE Instrumentation and Measurement Conference, Ottawa, Canada.

Melchers, R.E. (1989). "Importance sampling in structural systems," *Structural Safety,* Vol. 6, pp. 3–10.

Nie, J. and Ellingwood, B.R. (2000). "Directional methods for structural reliability analysis," *Structural Safety,* Vol. 22, pp. 233–249.

Pradlwarter, H.J., Schueller, G.I., Koutsourelakis, P.S., and Charmpis, D.C. (2007). "Application of line sampling simulation method to reliability benchmark problems," *Structural Safety,* Vol. 29, No. 3, pp. 208–221.

Rackwitz, R. and Fiessler, B. (1978). "Structural reliability under combined load sequences," *Journal of Computers and Structures,* Vol. 9, pp. 489–494.

Rahman, S. and Xu, H. (2004). "A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics," *Probabilistic Engineering Mechanics,* Vol. 19, pp. 393–408.

Rais-Rohani, M., Solanki, K., and Eamon, C. D. (2006). "Reliability-Based Optimization of Lightweight Automotive Structures for Crashworthiness," 11th AIAA/ ISSMO Multidisciplinary Analysis and Optimization Conference, Portsmouth, VA.

Rubinstein, R.Y. (1981). *Simulation and the Monte Carlo Method,* Wiley, New York, NY.

Schueller, G.I. and Pradlwarter, H.J. (2007). "Benchmark study on reliability estimation in higher dimensions of structural systems: An overview," *Structural Safety,* Vol. 29, No. 3, pp. 167–182.

Thompson, M., Eamon, C.D., and Rais-Rohani, M. (2006). "Reliability-based optimization of composite bridge decks," *ASCE: Journal of Structural Engineering,* Vol. 132, No. 12, pp. 1898–1906.

Wu, Y.T. (1994). "Computational methods for efficient structural reliability and reliability sensitivity analysis," *American Institute of Aeronautics and Astronautics Journal,* Vol. 32, No. 8, pp. 1717–1723.

Youn, B.D., Xi, Z., Wells, L.J. and Wang, P. (2006, September). "Enhanced Dimension Reduction (eDR) Method for Sensitivity-Free Uncertainty Quantification," AIAA Paper 2006-6977, 11th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Portsmouth, Virginia.

Zhao, Y.G. and Ono, T. (2001). "Moment methods for structural reliability," *Structural Safety,* Vol. 23, pp. 47–75.

Chapter 33

Statistical Analyses of Environmental Pressure Surrounding Atlantic Tropical Cyclones

Mark E. Johnson
Department of Statistics
University of Central Florida
Orlando, FL 32816-2370 USA
mejohnso@mail.ucf.edu

Charles C. Watson, Jr.
Watson Technical Consulting
1204 E. 49th Street
Savannah, GA 31404-4014 USA
cwatson@methaz.org

Johnson and Watson (2007) addressed many issues in fitting statistical distributions arising in hurricane catastrophe modeling, including data quality in support of estimating key modeling parameters. One of these parameters is environmental pressure, which is commonly presumed and set to be the standard sea level pressure of 1013 millibars. A detailed statistical analysis of the historical Atlantic record suggests that this value is too high and thereby introduces potential distortions in parametric functions and biases in hurricane catastrophe models that incorporate far-field pressure.

33.1 Introduction

A critical relationship in parametric models of tropical cyclones is the relationship between the pressure drop, ΔP (environmental pressure minus the minimum central pressure) and the maximum winds, Vmax. Simple relationships form the basis for establishing storm intensities used in catastrophe models applied by the insurance industry for setting premium rates, establishing reserves, and so forth. Typically, modelers generate distributions of minimum central pressure as part of simulating thousands of years of hypothetical storms. In the public documentation reviewed by the Florida Commission on Hurricane Loss Prediction Methodology (FCHLPM), a constant environmental pressure of 1013.25 mb or its rounded value of 1013 mb is commonly indicated. While this value represents the accepted average standard sea level pressure, it is well-known that sea level barometric pressure is anything but constant – after all, the basis of classical weather forecasting uses the drop in atmospheric pressure as a precursor of inclement weather. What is needed for hurricane catastrophe modeling is a physically reasonable value for environmental pressure that produces an appropriate pressure difference ΔP and maximum wind speed when applied within wind field models. Conventional wisdom has been that a one or two millibar difference would make no difference in loss calculations; however, as demonstrated in Watson and Johnson (2004), a single millibar difference in environmental pressure can induce a 5% difference in residential loss costs. Moreover, the argument that environmental pressure is difficult to measure to within 1 or 2 millibars does not release the statistician from investigating general biases.

A major purpose of this paper is to dispel the notion that 1013 mb is an appropriate, constant value for environmental pressure, based on a detailed analysis of the Atlantic hurricane record as given in HURDAT and in the more recent records found in the Hurricane Research Divisions (HRD) Atlantic Tropical Cyclone Forecasting (ATCF) Systems data (Sampson and Schrader (2000)). Additionally, far field pressure variability in relation to other potentially correlated variables such as latitude and time of year is examined.

33.2 Historical Data Sources

Barometric pressure since the time of Torricelli has been understood to be an indicator of climatic conditions. The basic value of atmospheric pressure at sea level is an ingrained baseline constant that can be chanted by most high school physics students. However, for hurricane catastrophe modeling, the long term, placid weather context is not the situation of interest. Rather, what is needed is a physically reasonable value to compute the pressure gradient that is the driving force behind the winds in the system. This perspective affords the possibility that environmental pressure is not a global constant but in fact could vary with latitude, time of year, and even in conjunction with the evolution of a specific

Table 33.2–1. Data from the Atlantic Basin CARQ records.

Identification code	(aalxxyyyy, where a signifies the objective guidance file, basin = al for Atlantic, xx = storm # in year yyyy)
Year of the event	(1990–2004)
Day in the year	(1–366)
Longitude	(degrees West, storm center)
Latitude	(degrees North, storm center)
Vmax	(maximum observed wind, n mi)
Pmin	(central pressure, mb)
Rmax	(radius of maximum winds, n mi)
Penv	(environmental or far field pressure, mb)
Renv	(radius of environmental winds, n mi)
Direction	(direction or heading, 0 = N, 90 = E, etc.)
Speed	(VT, forward speed, kt)
Delta pressure	(Penv Pmin, mb)

tropical cyclone.

Our main source of model input data for pressure analyses is the ATCF/CARQ (Automated Tropical Cyclone Forecast system; Comprehensive Archive Records see Sampson and Schrader, 2000). As the CARQ data are input feeds to real time track and intensity forecasting models, there is a tremendous incentive to set these values correctly or their errors will propagate into erroneous track forecasts. While post-season best-fit tracks such as HURDAT may be useful for many applications, these best fit tracks have small storm motions smoothed out, and more pertinently, lack many of the required variables needed for full storm simulation such as far field pressure, radius of maximum winds, and so forth. Furthermore, if at season's end, the data is "quality-controlled" there may be an inherent attraction to the traditional 1013 value for environmental pressure.

The ATCF CARQ data used here consists of 5570 records over the fifteen-year period (1990–2004) in which this detailed set of information has been collected and assembled. The dataset comprises 244 distinct events – namely, tracked, tropical cyclones. Although the 15-year time frame may seem short compared to the 153 years of HURDAT, the two datasets will be shown to be consistent with respect to other overlapping parameters. The fields that were extracted from the Atlantic Basin CARQ records are shown in Table 33.2–1.

The use of historical environmental pressure data as presented by Schwerdt, Ho, and Watkins (1979) with a later update by Ho, Su, Hanevich, Smith, and Richards (1987) was examined. The average far field pressure (Penv) of the combined Gulf Coast and Atlantic Coast regions across 131 events is 1012.6 mb, which rounds up to 1013 mb. Although the initial study noted no strong latitude effect, there is in fact a significant difference between the values for the two coastal regions (1013.73 mb for the East Coast average; 1011.85 mb for the Gulf

Coast). Moreover, from an implementation standpoint, the standard project hurricane wind field described in NWS 23 uses a constant 1008 mb for the far field pressure. In spite of the implicitly recommended value of 1008 mb for catastrophe models, researchers continue to retain 1013 mb (Powell, Soukup, Cocke, Gulati, Morisseau-Leroy, Dorst, and Axe (2005)).

33.3 Distribution of Environmental Pressure

As mentioned previously, environmental pressure is treated as a "constant" input (typically 1013 mb) in various hurricane loss estimation models. This section examines the suitability of this constant in particular and the use of a constant value more generally.

A total of 6387 environmental pressure (Penv) values corresponding to 244 tracked events are summarized in Figure 33.3–1. The sample mean is 1011.5751, which is exceedingly far from 1013 (with a standard error of 0.035). The sample median of 1012 may not seem so far from 1013, but this is partly due to the resolution of the data given to 1 mb. Some light can be shed by considering a sorted list of Penv values and noting the following:

Penv	Percentages of Observations Less than or Equal to Penv
1009	17%
1010	40%
1011	46%
1012	72%
1013	77%

The central part of the distribution is dominated by the value 1012 (45.81% to 71.75%) while the traditional value of 1013 is effectively anchored to the upper quartile of the distribution. The "spikey-ness" of the histogram is attributable to an apparent attraction to even numbers (4895 of 6387 or 77% of the Penv values are even; excluding 1012, which is near the middle of the distribution, the percentage even of the remaining values is 68%).

For hurricane modeling applications (especially related to insurance loss estimation), a natural concern could be the inclusion in the Atlantic Basin data set of "fish storms" or hurricanes that have experienced extra-tropical transitions and pose little or no threat to insured property following a northeast re-curvature. Thus, a reduced set of high impact events was examined by considering records in the Florida "rectangle" (24°N to 31.5°N and 78°W to 88°W). Corresponding summary statistics on the reduced dataset of 397 observations is provided in Figure 33.3–2. The mean is slightly lower than for the full dataset, and there is continued evidence against using the value 1013.

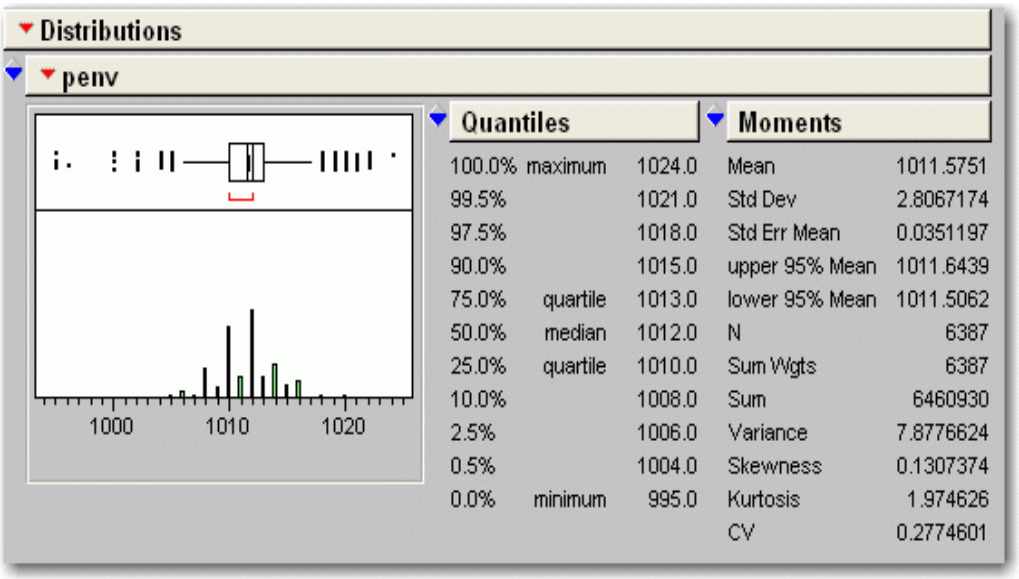


Figure 33.3–1. Atlantic basin Penv results.

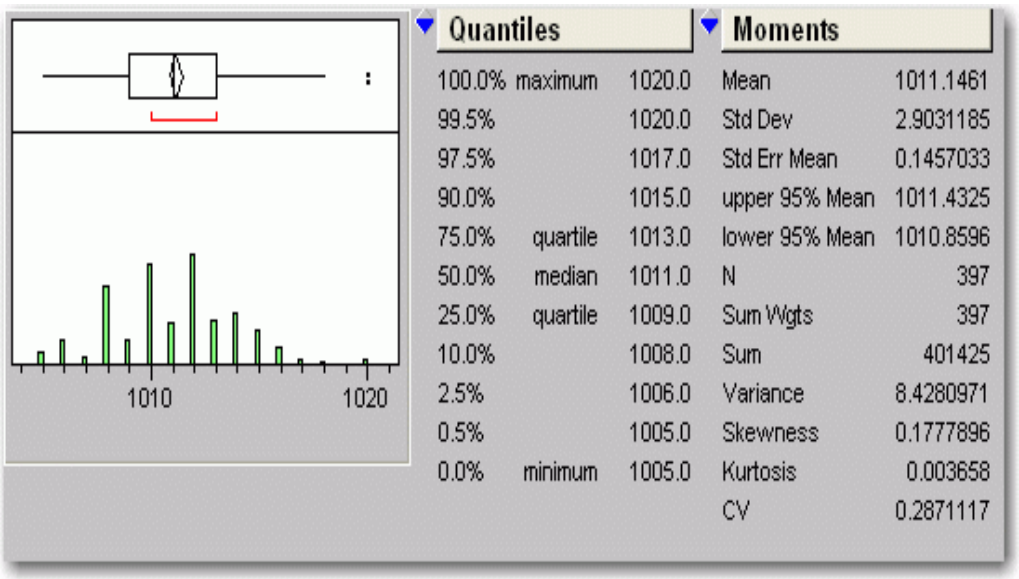


Figure 33.3–2. Florida specific Penv summary statistics.

Of further interest are those environmental pressures related to genuine active events. The database consists of storms both at their onset and at their dissipation. Figure 33.3–3 considers the environmental pressure distributions for records having a maximum wind speed of at least 50 kt. Here it appears that a

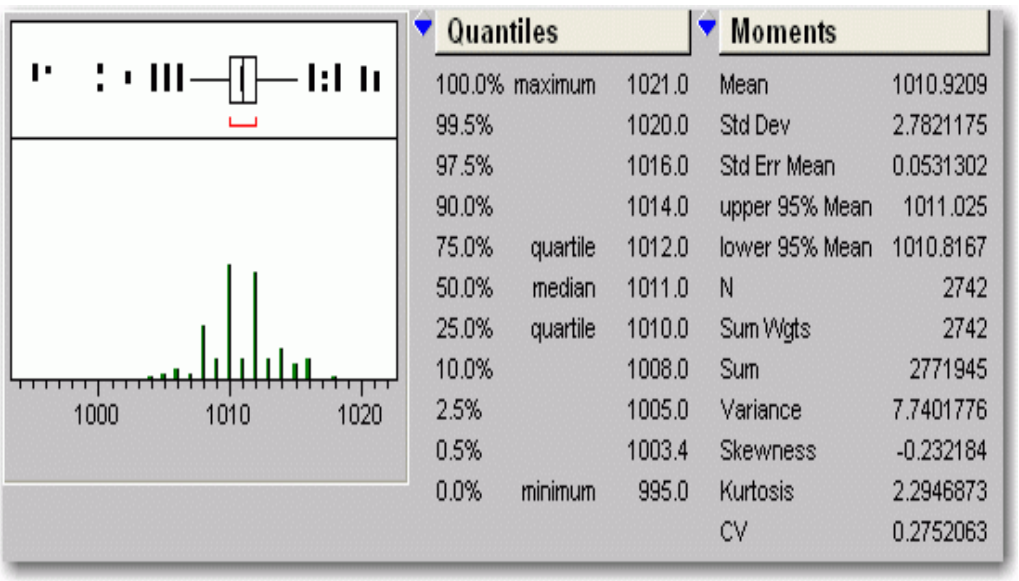


Figure 33.3–3. Environmental pressure data restricted to Vmax greater than or equal to 50 kt.

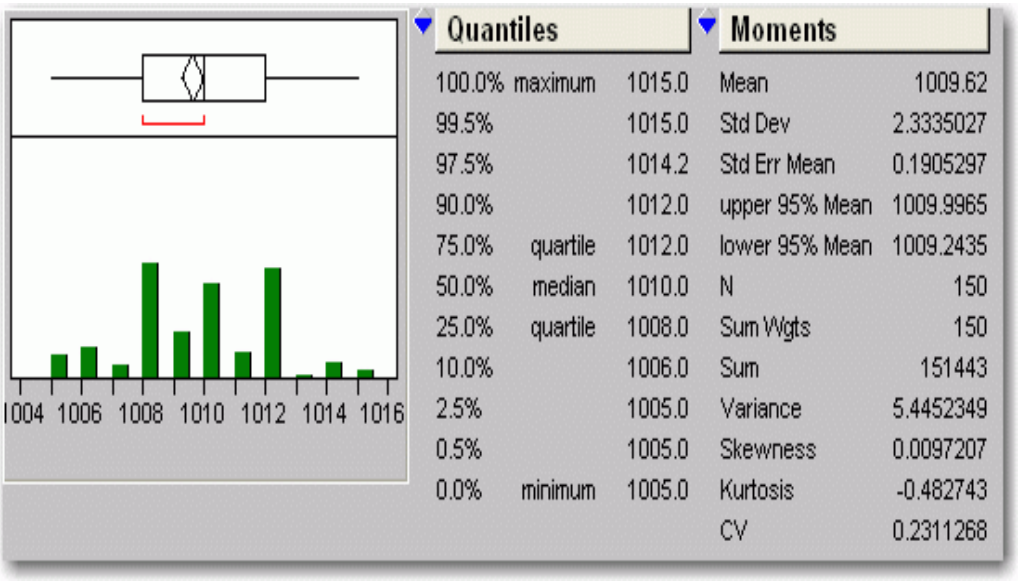


Figure 33.3–4. Environmental pressure for Florida and events with Vmax $\geq$ 50 kt.

value of 1011 could be supported. The further restriction to the Florida region leads to further reductions in the central part of the distribution, as evidenced in Figure 33.3–4.

33.4 Environmental Pressure for Specific Storms

From the previous section, it appears that a universal, constant value of environmental pressure is untenable based on consideration of the ATCF/CARQ dataset. A natural follow-on question is whether a constant value would be suitable in the simulation of a single event. In other words, in a simulation context (as in catastrophe model applications) one could draw a realization from the probability distribution of environmental pressure and then hold the value constant for the duration of an event. In this section the extent to which this approach is supported by the ATCF/CARQ data is examined. Specifically, for individual events, how variable is the environmental pressure?

A useful summary statistic to address the variability of environmental pressure is the range of values observed in the course of each of the 244 storms. There were only 28 of 244 storms for which Penv did not vary at all in the course of the event as given in the dataset. As seen in Figure 33.4–1, the range of values of Penv is typically around 4 mb with 2 to 6 mb constituting at least 50% of the values. It might be conjectured that attention to Penv and its determination is greater for stronger events. Figure 33.4–2 considers records having Vmax at least 50 kt. As expected, the distribution of ranges of Penv is a bit tighter and is centered around 3 mb with 50% of the values from 1 to 5 mb. Three representative events are depicted in Figure 33.4–3.

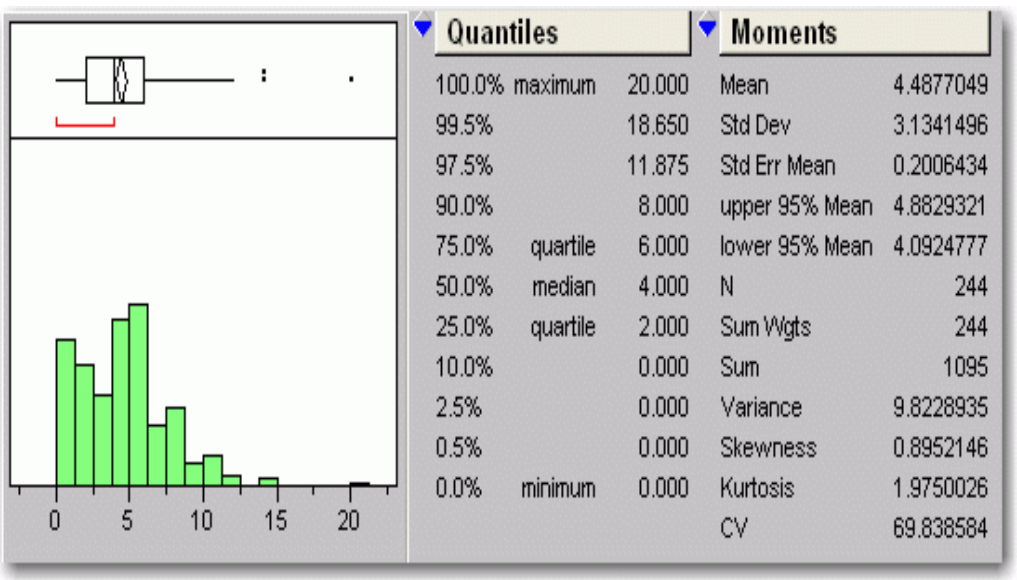


Figure 33.4–1. Environmental pressure ranges for individual storms.

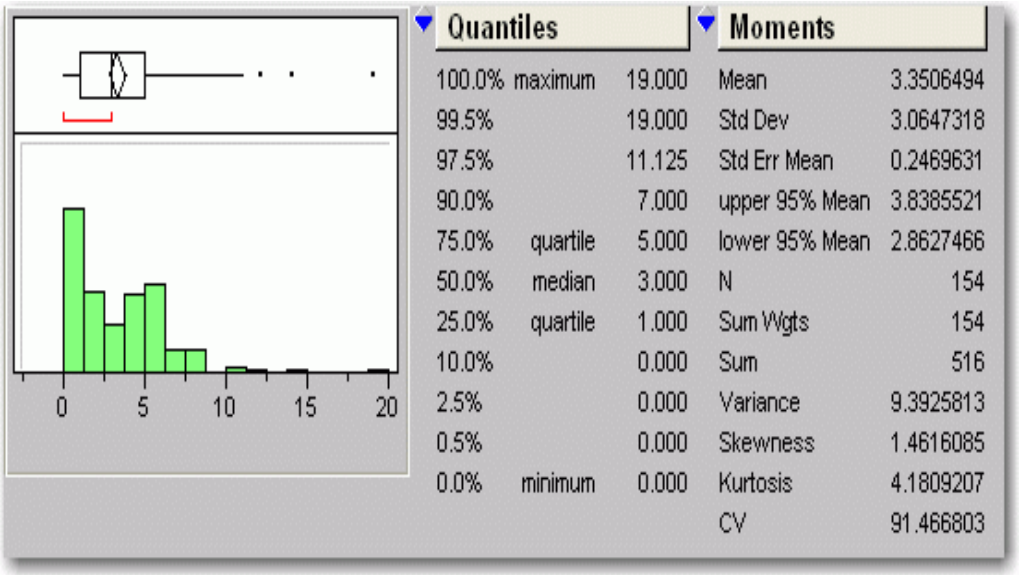


Figure 33.4–2. Environmental pressure ranges for individual storms with Vmax $\geq$ 50 kts.

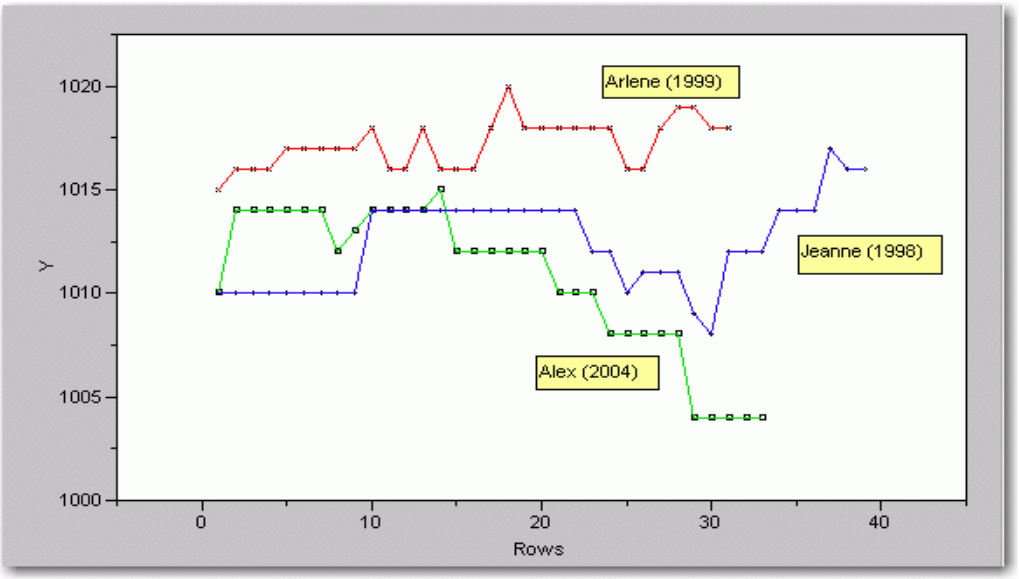


Figure 33.4–3. Environmental pressure history for three representative events.

33.5 Relationship of Penv to Latitude, Time of Year, Pmin

In an attempt to explain the variability of Penv across and within storms, it is natural to consider potentially correlated variables such as latitude, time of year, and central pressure (Pmin). Figures 33.5–1, 33.5–2, and 33.5–3 show the very weak nature of these relationships. Consequently, modeling Penv as a function of these covariates is not supported by the data. The largest R^2 value is 0.10, which is not practically significant.

Of additional interest in Figure 33.5–3 are the Pmin values for the lowest pressure events. Aside from one event with Penv = 1012 mb, the next 12 events have Penv from 1006 to 1008. In other words, catastrophe modelers should not incorporate Penv as an add-on random variable to be simulated with mean 1013, as this could greatly inflate the intensity of the strongest simulated storms.

Some authors note a weak relationship between latitude and the pressure gradient (Vickery, Skerlj, and Twisdale (2000)). Strong storms tend to occur at the lower latitudes and most hurricanes weaken as they move north (unless they undergo extratropical transitions, in which case they could re-intensify beyond their strength in the tropical phase; Hart and Evans (2001). Although latitude is conveniently available in the course of a storms evolution, it does not have a strong physical basis not already incorporated in the Coriolis parameter, sea surface temperature, or other phenomena that exerts its influence hand in hand with latitude. Latitude can be viewed as an omnibus variable for other physically driven effects.

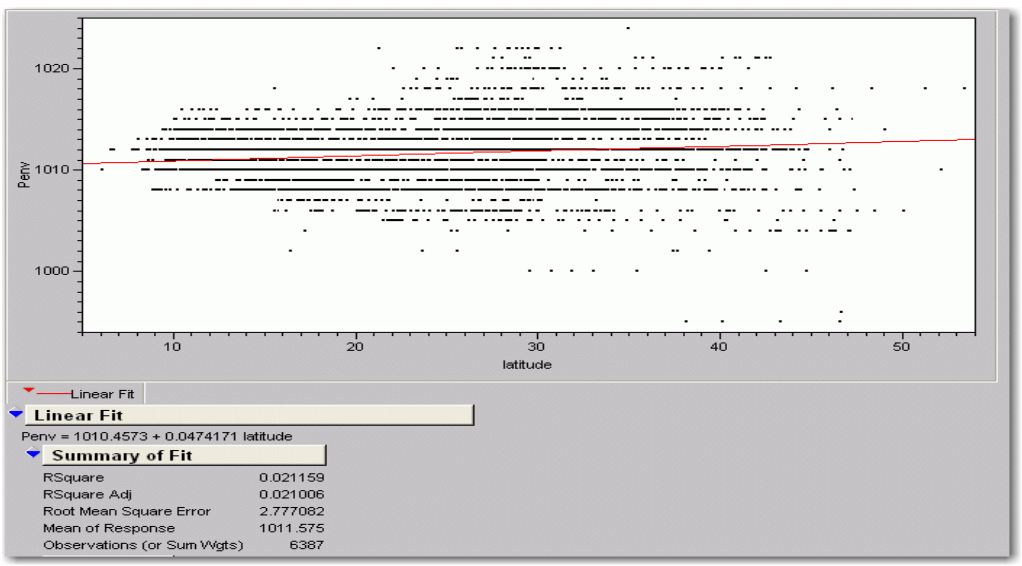


Figure 33.5–1. Environmental pressure versus latitude from ATCF/CARQ data.

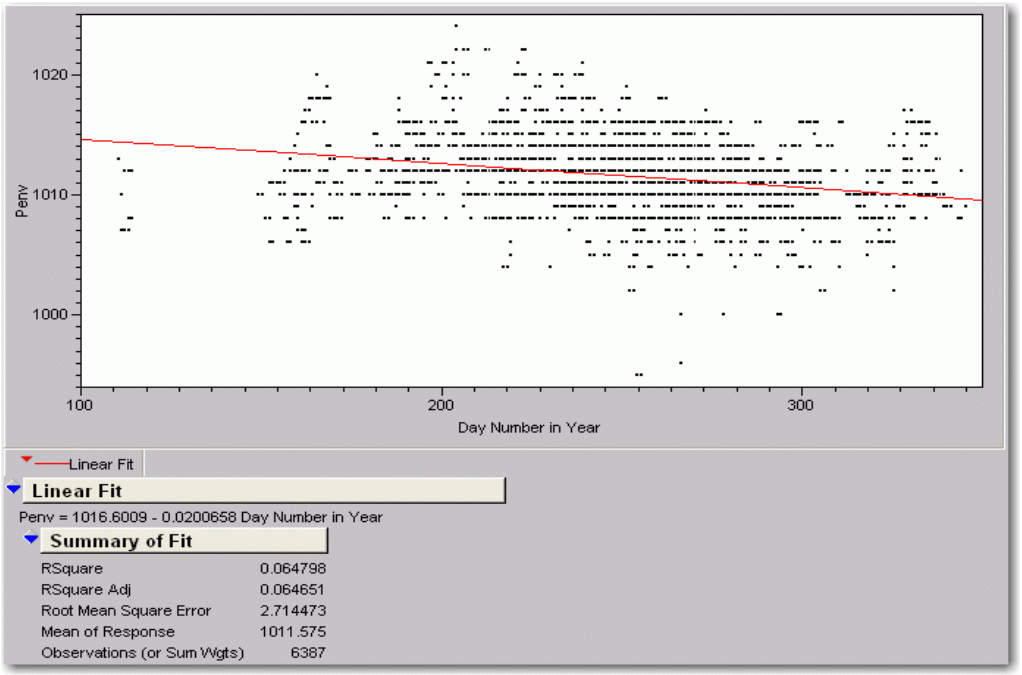


Figure 33.5–2. Environmental pressure versus day number in the year.

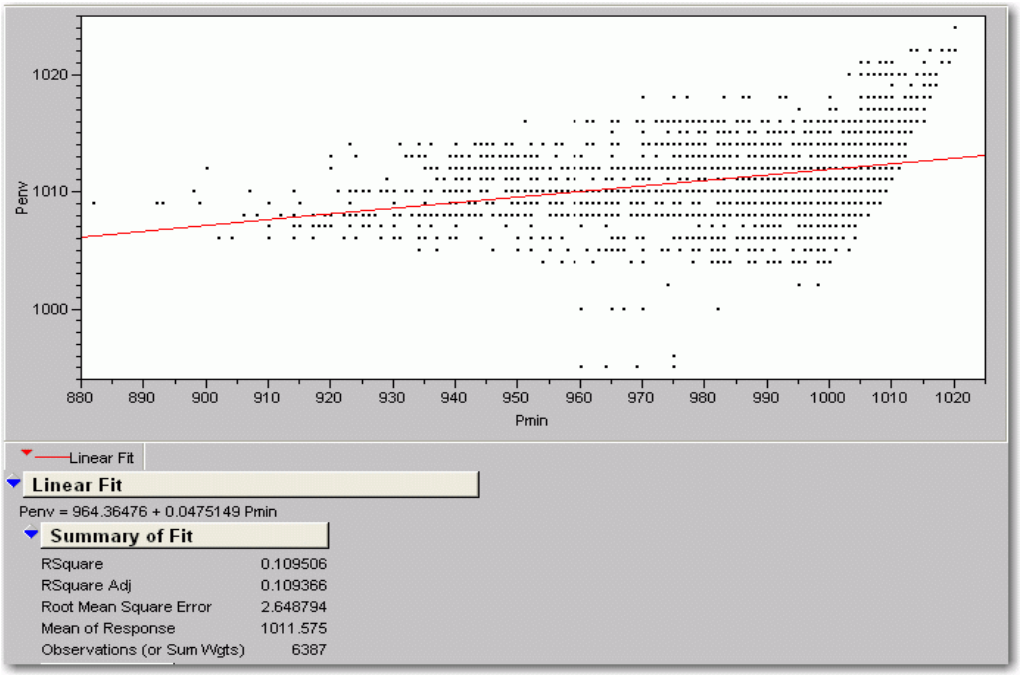


Figure 33.5–3. Environmental pressure versus corresponding minimum pressure.

33.6 Extrapolation to the Full Historical Record

A potential limitation of the present study is the limited time frame of 15 years (1990–2004) for the ATCF/CARQ records. Although detailed environmental pressure data is not more broadly available, the suitability of the present data can be examined at least indirectly by comparing common variables in ATCF/CARQ and HURDAT. Common variables include minimum central pressure (Pmin) and maximum wind speed (Vmax). If these datasets yield comparable distributions, then there is not evidence to suggest a difficulty (relative bias) with the ATCF/CARQ dataset. Obviously, this lack of evidence does not prove the viability of using ATCF/CARQ data, but it closes one door on uncertainty associated with the database. This being said, it should be noted *a priori* that one would not expect the distributions across the two datasets to be identical. In particular, the HURDAT dataset records maximum winds at a resolution of 5 mph and secondly, the earlier years of HURDAT do not necessarily include weak storms or hurricanes at the early stages of development or dissipation. Thus, one would expect that the storms in the HURDAT set will in general be somewhat stronger than those of ATCF/CARQ. The common variables Vmax and Pmin from HURDAT and ATCF/CARQ are considered in turn.

Figures 33.6–1 and 33.6–2 consider the full Atlantic Basin for Vmax. As expected, the HURDAT dataset has somewhat higher Vmax values than those found in ATCF/CARQ but the discrepancy (about 5 kt for the Basin, and 9 kt for Florida) is consistent with HURDAT data resolution and the focus on stronger storms. The quantiles (especially 10% and 90%) are comfortably close. Figures 33.6–3 and 33.6–4 consider the Florida rectangle restriction with Vmax. The restriction to the Florida area does not generate any particular issues with the two datasets.

Figures 33.6–5 and 33.6–6 consider the full Atlantic Basin for Pmin. The results for Pmin are consistent in direction with the Vmax values (ATCF/CARQ having higher-on-average Pmin values). Figures 33.6–7 and 33.6–8 consider the Florida rectangle restriction with Pmin. On the basis of these comparisons, there is not sufficient evidence to suspect the suitability of the ATCF/CARQ data with regard to the environmental pressure data.

In addition to basic graphical and numerical summaries of the environmental and minimum pressure, distribution summaries could also be considered. The generalized lambda distribution (Karian and Dudewicz (2000)) is an attractive choice as a flexible family of distributions. The generalized lambda distribution was fit to the Pmin data from Figure 33.6–8 using the starship method (King and MacGillivray (1999)) as implemented in R (King (2009) and R Development Core Team (2009)). As can be seen from Figure 33.6–9, the fit is excellent. Such a fit could be further used in simulation studies to sample storms having intensities comparable to what have been observed in the Florida region.

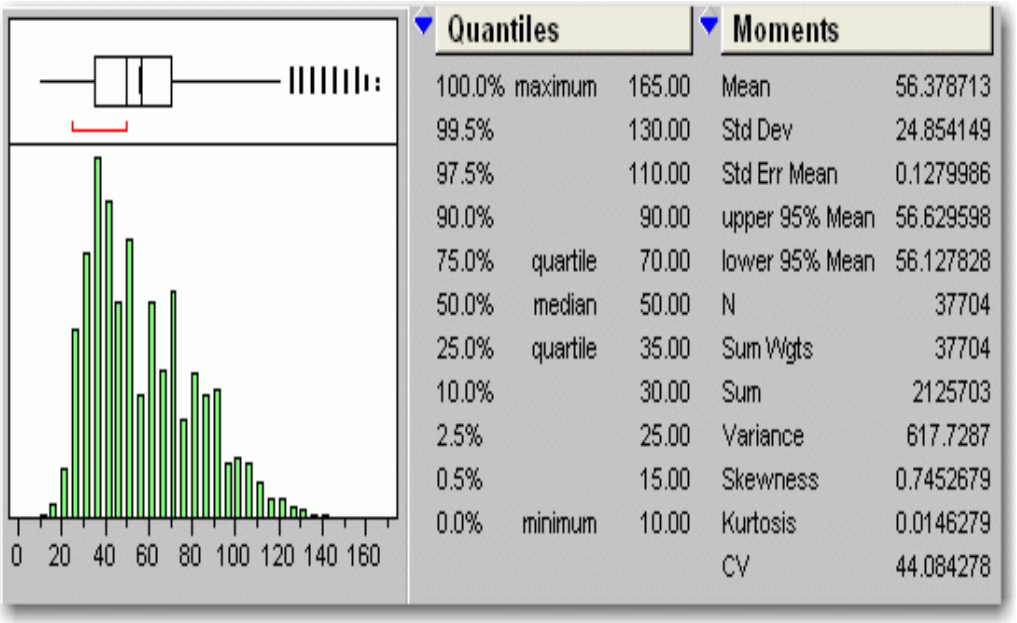


Figure 33.6–1. HURDAT, Vmax distribution.

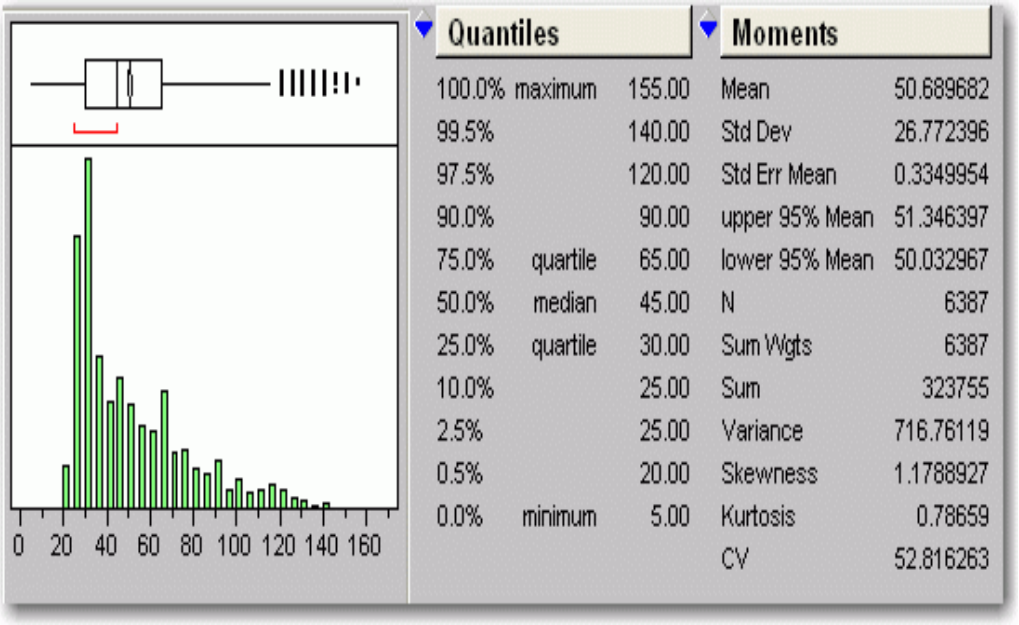


Figure 33.6–2. ACTF/CARQ, Vmax Distribution.

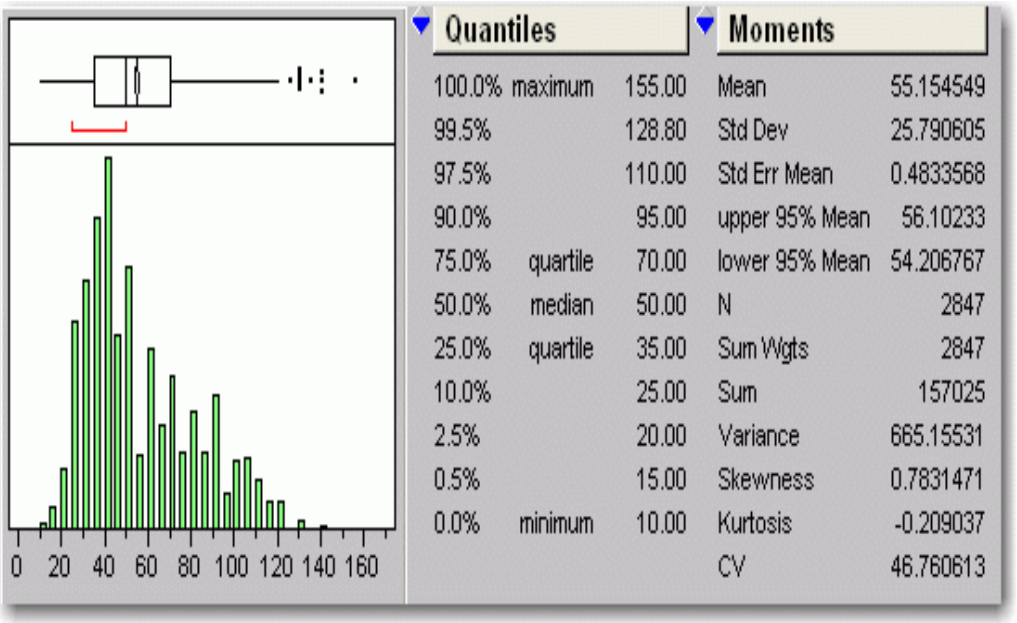


Figure 33.6–3. Florida HURDAT, Vmax distribution.

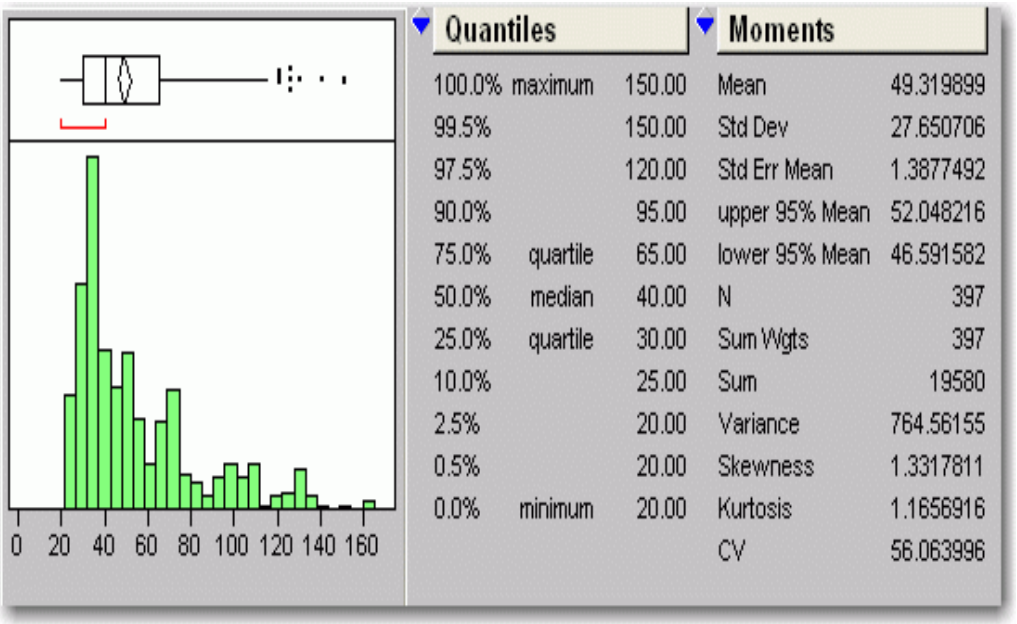


Figure 33.6–4. Florida ATCF/CARQ Data, Vmax Distribution.

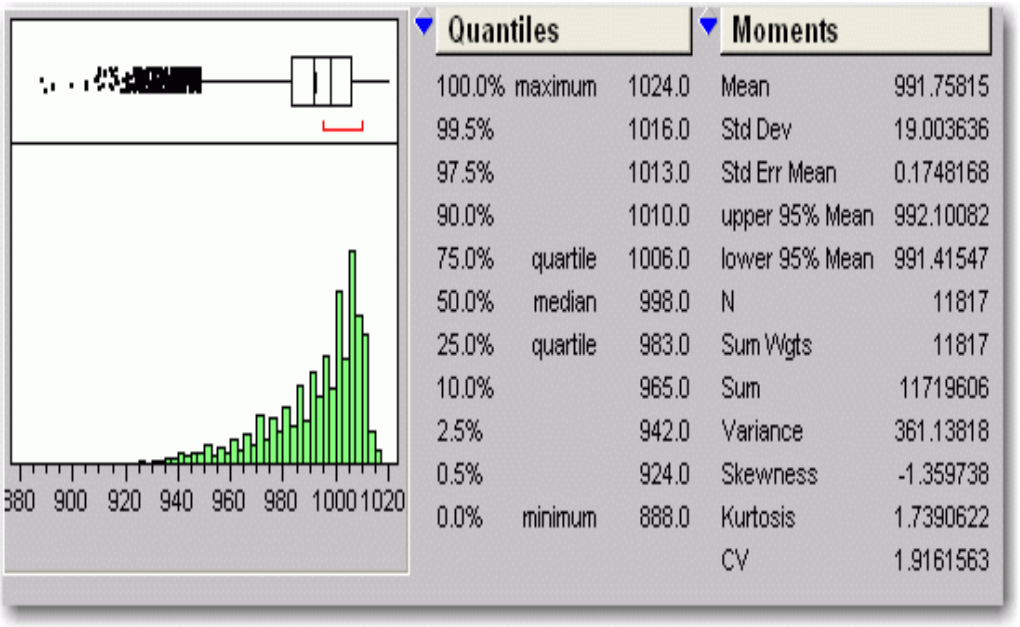


Figure 33.6–5. HURDAT, Pmin distribution.

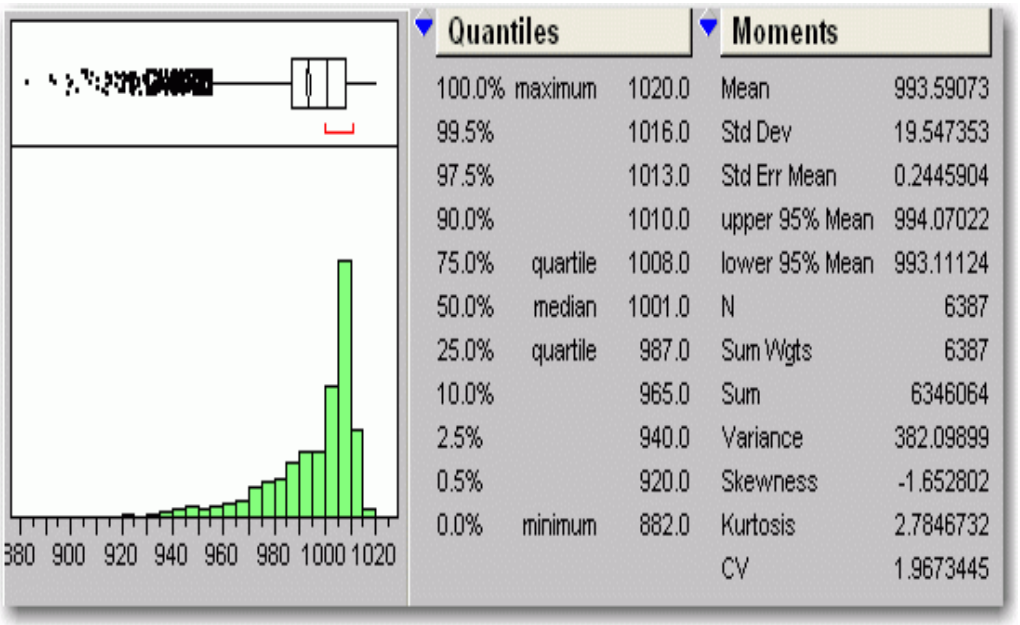


Figure 33.6–6. ATCF data, Pmin distribution.

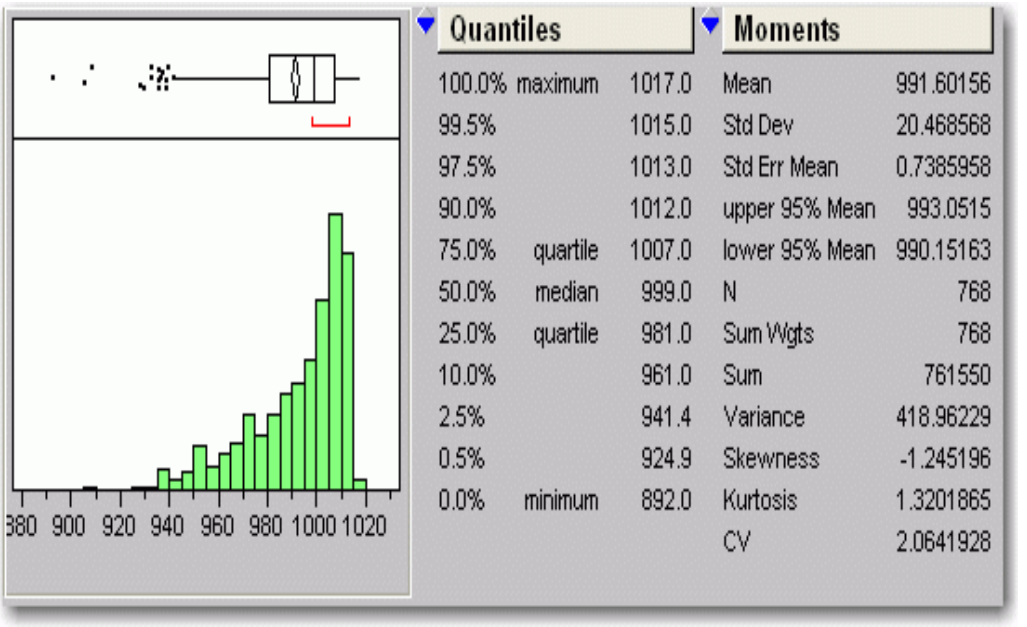


Figure 33.6–7. Florida HURDAT data, Pmin distribution.

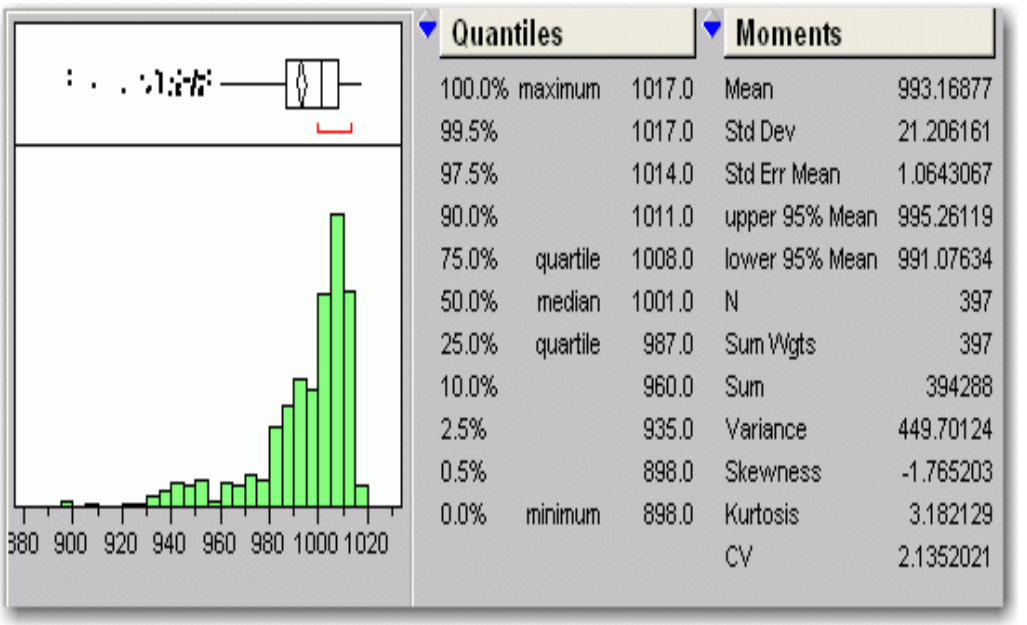


Figure 33.6–8. Florida HURDAT data, Pmin distribution.

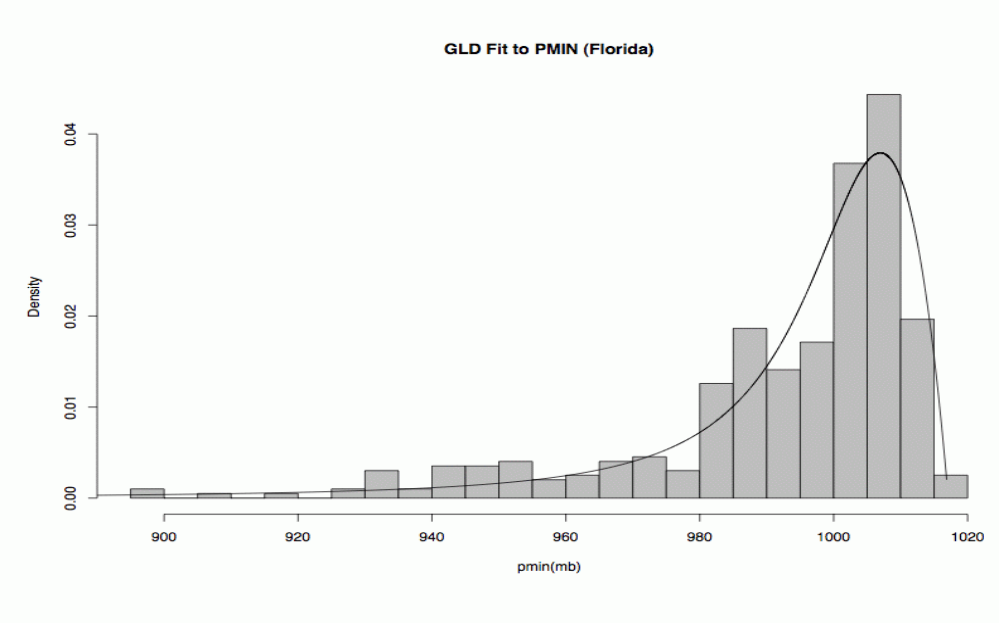


Figure 33.6–9. GLD fit to Pmin (Florida).

33.7 Wind Field Impacts

Watson and Johnson (2004) provided empirical evidence that a one mb deviation in the far field pressure could induce substantial alterations (5–10 percent in some areas) to estimated insured loss costs. In the context of a formal submission to the Florida Commission on Hurricane Loss Projection Methodology, Applied Insurance Research, Inc. (AIR, 2003) noted the considerable impact of Penv on loss cost estimation. This modeler per the Commission standards followed the sensitivity and uncertainty analyses described in a realistic context by Iman, Johnson, and Watson (2005a) and (2005b). The severe loss costs impacts evidenced in the analyses would appear to be symptomatic of impacts on the underlying wind fields, some of which are driven by Pmin and Penv. To bring the present environmental pressure study back to wind speeds (the true damage source rather than the pressure gradient), the Vmax values in the ATCF/CARQ dataset are examined in further detail versus the winds that could be modeled from standard wind speedpressure relationships. The purpose of this section is to assess the variability induced by Penv on Vmax (and thus on overall wind speeds) as used in basic wind field models, as commonly employed in catastrophe modeling.

One commonly used function that relates Vmax to Pmin and Penv (via the pressure gradient ΔP =Penv – Pmin) is given in the Standard Project Hurricane (SPH) project Schwerdt, Ho and Watkins (1979), as follows:

$$\text{Vmax} = 73[\Delta P \times 29.98/1013]^{1/2} = 12.558[\Delta P]^{1/2}$$

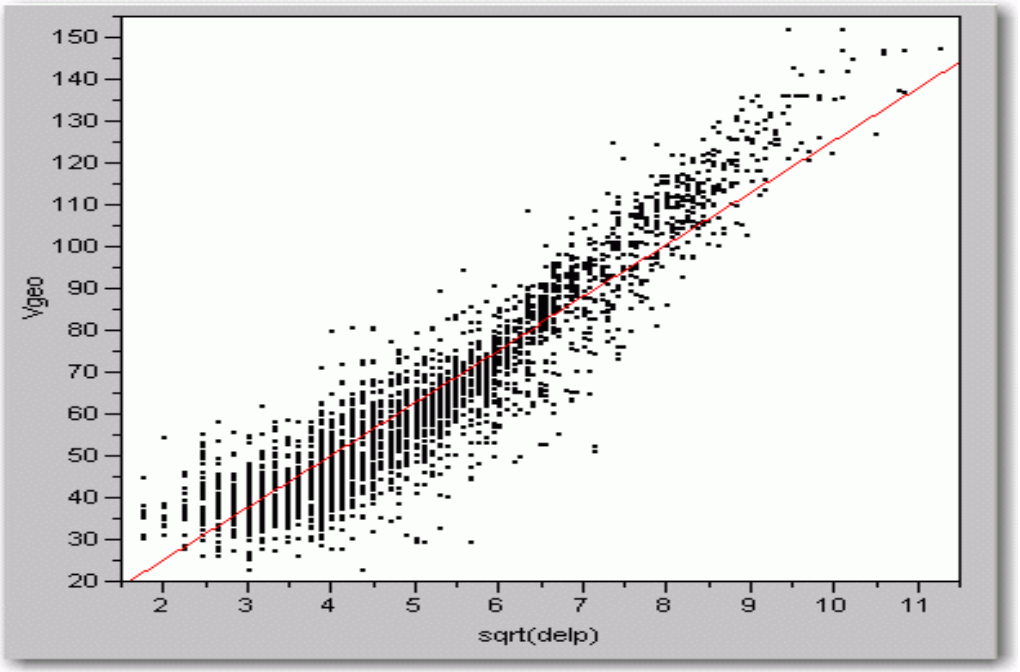


Figure 33.7–1. Linear fit that mimics the SPH development.

This function is referred to as **V classic**. As noted in the introduction, the pressure gradient ΔP is regularly determined by computing 1013 - Pmin.

Since V classic was derived from data collected on U.S. hurricane landfalls in the 1960s and 1970s, it first seemed prudent to use the ATCF/CARQ records for possible adjustment to the estimated slope. To mimic the development of the standard project hurricane model, the original dataset was restricted to over-water records with Vmax at least 40 kt and minimum pressure at or below 1007 mb. The application area is residential wind damage which motivates the 40kt value corresponding to the onset of damage. The original SPH development used a far field pressure of 1008 mb. Ultimately, these constraints provide 3321 records across 110 events. The fit to the selected data is given in Figure 33.7–1.

The fitted line in the Figure is Vgeo $= 0 + 12.534\sqrt{\Delta P}$, where Vgeo is the geostrophic wind (vmax $- V_T/2$), and the "0" indicates that the regression is through the origin (again, a carryover from the SPH development). The consistency of the slope estimates are striking. The slope estimate from Vgeo is within 0.024 of V classic. On the other hand, consideration of the fit in Figure 33.7–1 suggests a curvilinear structure, which will be investigated in a separate publication.

Returning to the impact of various uses of environmental pressure, consider the following three maximum velocity expressions:

$$\text{Vpenv} \quad = \quad 12.534\sqrt{\text{Penv} \ - \ \text{Pmin}} + V_T/2$$

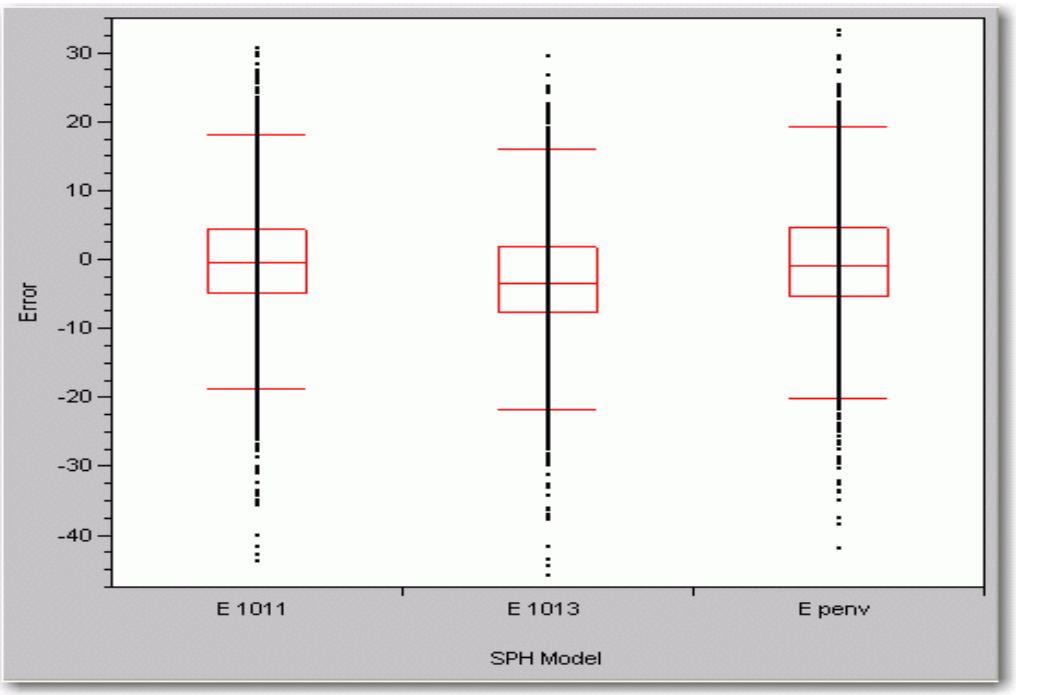


Figure 33.7–2. Residuals for three versions of SPH wind functions.

$$\begin{aligned} \text{V1011} &= 12.534\sqrt{1011 - \text{Pmin}} + V_T/2 \\ \text{V1013} &= 12.534\sqrt{1013 - \text{Pmin}} + V_T/2. \end{aligned}$$

The ATCF/CARQ records provide the ground-truth vmax. The differences between each velocity function and vmax provide a basis for comparisons. Figure 33.7–2 provides a box plot of the residuals for each comparison to vmax.

Epenv uses the actual Penv values that are available for each record. E1011 uses a constant value of 1011 (effectively the average of all of the Penv entries in the ATCF/CARQ dataset). The impact of using the "wrong" but historical value of 1013 is about a 3kt increase in average wind speed. The standard deviations are close, with Vpenv being slightly superior. The following summary statistics correspond to the residuals in Figure 33.7–2.

Penv	Bias (Vi-vmax)	Standard Deviation
actual	0.54	8.39
1011	0.40	8.46
1013	3.08	8.49

The results in this section support the original observation that, if a constant far field pressure were to be used, 1013 mb is an inferior choice. In particular,

the overall average of 1011 mb performs very well compared to having the actual environmental pressure.

33.8 Conclusions

The results of this paper demonstrate that the environmental pressure associated with tropical cyclones is rife with uncertainty that is inherent rather than explainable by other factors such as latitude, time of year, or minimum central pressure. Thus, catastrophe models that employ environmental pressure as a key input constant or variable must be scrutinized carefully. "Correcting" environmental pressure by using latitude or time of year is unsupported through the analysis of the ATCF/CARQ data set. Full disclosure of the handling of environmental pressure is required while recognizing that it has the potential to be a source of bias and could distort the uncertainty in model results. Using actual environment pressure provided at best marginal improvement in the sense of variability compared to a constant value of 1011 mb.

References for Chapter 33

Hart, R. and Evans, J. L. (2001). "Climatology of the extratropical transition of Atlantic tropical cyclones," *Journal of Climate,* 14, 546–564.

Ho, F. P., Su, J. C., Hanevich, K. L., Smith, R. J., and Richards, F. P. (1987). "Hurricane Climatology for the Atlantic and Gulf Coasts of the United States," *NOAA Technical Report NWS 38*, Washington, D.C., FEMA.

Holton, J. (1992). *An Introduction to Dynamic Meteorology*, Academic Press, New York.

Iman, R. L., Johnson, M.E., and Watson, C., Jr., (2005a). "Sensitivity analysis for computer model projections of hurricane losses," *Risk Analysis,* 25, 1277–1297.

Iman, R. L., Johnson, M. E., and Watson, C., Jr. (2005b). "Uncertainty analysis for computer model projections of hurricane losses," *Risk Analysis,* 25, 1299–1312.

Johnson, M. E. and Watson, C., Jr. (2007). "Fitting statistical distributions to data in hurricane modeling," *American Journal of Mathematical and Management Sciences,* 27, 479–498.

Karian, Z.A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods*, CRC Press/ Chapman and Hall, Boca Raton.

King, R. (2009). "GLD: Estimation and use of the generalized (Tukey) lambda distribution," R package version 1.8.4. http://CRAN.R-project.org/package=gld.

King, R.A.R. and MacGillivray, H. L. (1999). "A starship method for fitting the generalized lambda distributions," *Australian and New Zealand Journal of Statistics,* 41, 353–374.

Powell, M. D., Soukup, G., Cocke, S., Gulati, S., Morisseau-Leroy, N., Dorst, N., and Axe, L. (2005). "State of Florida hurricane loss projection model: Atmospheric science component," *Journal of Wind Engineering and Industrial Aerodynamics,* 93, 651–674.

R Development Core Team (2009). "R: A language and environment for statistical computing," R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org.

Sampson, C. R., and Schrader, A. J., (2000). "The automated tropical cyclone forecasting system (Version 3.2)," *Bulletin of the American Meteorological Society,* 81, 1231–1240.

Schwerdt, R., Ho, F., and Watkins, R. (1979). "Meteorological Criteria for Standard Project Hurricane and Probable Maximum Hurricane Wind Fields, Gulf and East Coasts of the United States," *NOAA Technical Report NWS 23*, Silver Spring, MD, National Weather Service.

The AIR Hurricane Model (2003). Submission to the Florida Commission on Hurricane Loss Projection Methodology.

Vickery, P. J., Skerlj, P. F., and Twisdale, L. A. (2000). "Simulation of hurricane risk in the U.S. using empirical track model," *Journal of Structural Engineering,* 126, 1222–1237.

Watson, C. Jr., and Johnson, M.E. (2004). "Hurricane loss estimation models, opportunities for improving the state of the art," *Bulletin of the American Meteorological Society,* 85, 1713–1726.

Chapter 34

Simulating Hail Storms Using Simultaneous Efficient Random Number Generators

Ángel Adolfo Olmos
Professor, Facultad de Informatica
Universidad Nacional del Comahue
Neuquén (8300), Argentina
aolmos@neunet.com.ar

Jorge Luis Romeu
Research Professor, Dept. of Mech. & Aerospace Engineering
Syracuse University
Syracuse, New York 13244 USA
romeu@cortland.edu

We begin with a description of hail precipitation system models for the Nord-Patagonia region "Alto Valle de Río Negro y Neuquén" in Argentina. We then discuss the number of hail storms, the types of most frequent storms, and the economic loss caused to the production of apples and pears by these storms. Next, we discuss the observed performance of efficient random number generators (RNGs), as opposed to the performance of inefficient random number generators, when used in the simultaneous production of multiple flows of pseudorandom numbers used to represent the different intervening variables in this simulation model. Finally, we conclude that the model results from the efficient pseudorandom number generators agree satisfactorily with the behavior of the real system. As shown in Section 34.6, results obtained by using inefficient RNGs depart significantly from the expected values of systems performance measures.

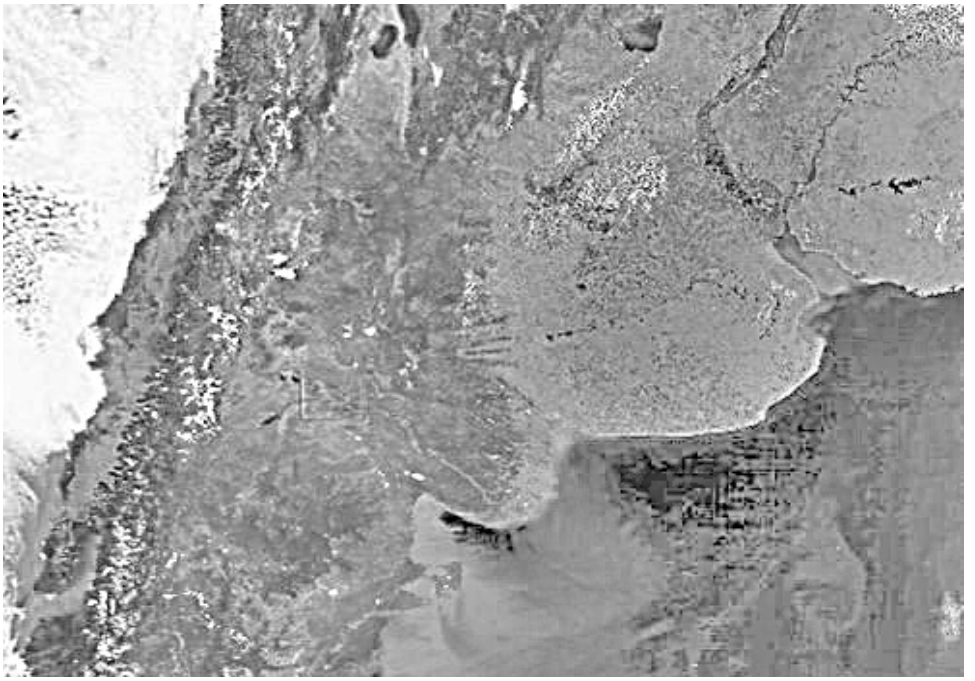


Figure 34.1–1. High Valley of Río Negro and Neuquén (in box; to lower left of center of image; to right is coastal Argentina, to left is Chile, to upper right is Buenos Aires).

Sections of this chapter are: Simulation of a Hail Precipitation System (Section 34.1), Models (Section 34.2), Conclusions (Section 34.3), Tests of Simultaneous Generators (Section 34.4), Introduction of Several Generators into a Simulation (Section 34.5), and Statistical Analysis (Section 34.6).

34.1 Simulation of a Hail Precipitation System

The consequences of hail storms are very important for the economy of the "Alto Valle de Río Negro y Neuquén" region, located in the Nord-Patagonia part of Argentina. Figures 34.1–1 through 34.1–6 show the region in Argentina and South America, the localities affected in this study, hail stones of a particular storm on November 21, 1994 and in a Laboratory of the National Weather Service, Argentina, marks on apples produced by hail, and laboratory Impactometers using polystyrene plates with aluminum foil. (All figures are from Lassig (2006).) With these figures to put the location and problem in visual prespective, we now move on to simulation of models of hail.

From information gathered between 1966 and 1986, an inter-disciplinary group of the Universidad Nacional del Comahue studied (Olmos (2006)) the historical behavior of the hail system in this region. This group developed simulation models



Figure 34.1–2. Localities in the High Valley of Río Negro and Neuquén Region.



Figure 34.1–3. Hail in the city of Neuquén (November 21, 1994).

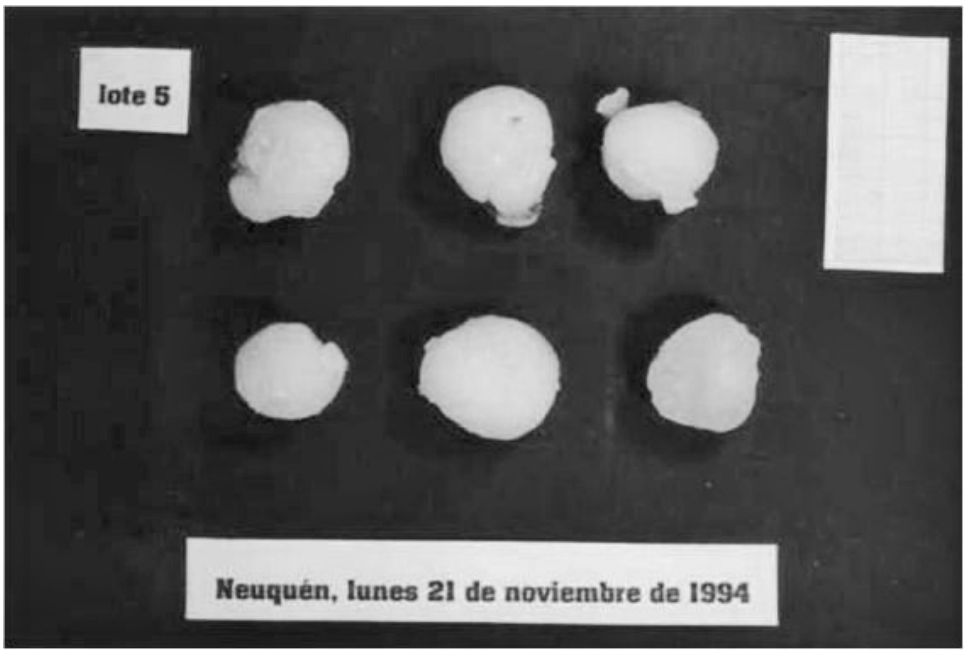


Figure 34.1–4. Hailstones from Figure 34.1–3, now in the Cloud Physics Laboratory, National Weather Service, Argentina.



Figure 34.1–5. Marks on apples produced by a hail storm. Note: hand is holding leaf up, so you can view the apple below it, with damage to its left side (dark mark).

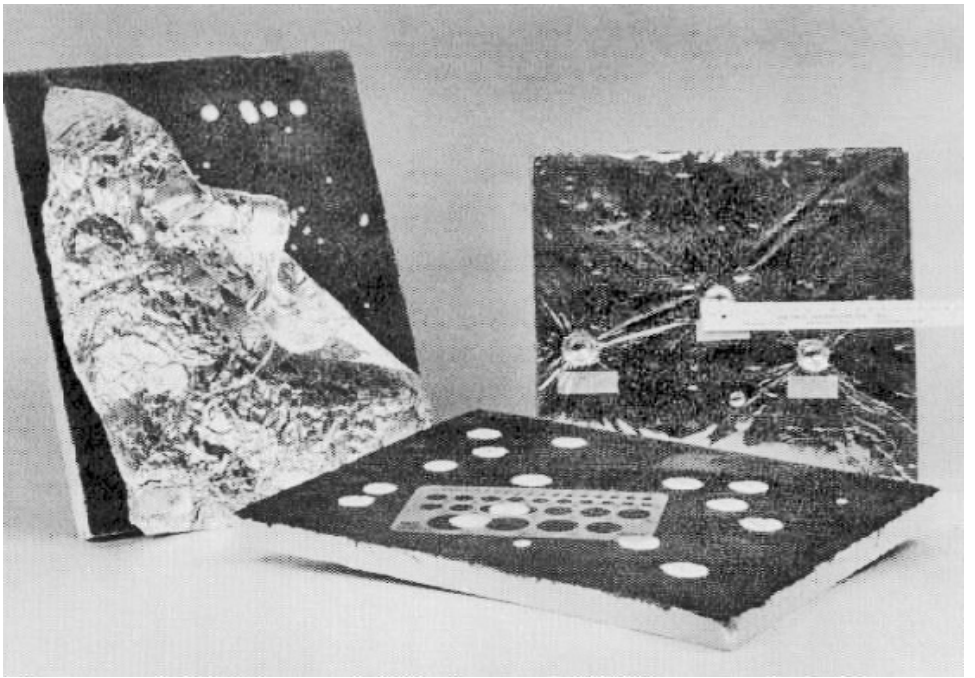


Figure 34.1–6. Impactometers of polystyrene plates with aluminum foil.

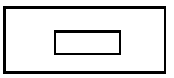
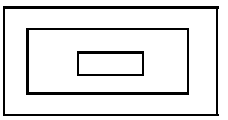
identifying such variables as the fruit potential, the quantity and types of hail storms, the phenology state of fruit-bearing trees, and several economic variables such as market prices.

We define "phenology state of fruit-bearing trees" as each of the variation and phenomenological stages that a fruit tree experiences during its lifecycle. Therefore, it won't be possible to pass to the following "phenology state" until the plant has gone through its prefixed variations. For example, in the apple tree there are two consecutive phenology states: the "beginning of plant flowering," and the "full flowering." We have to add that the duration of each phenology state will be clearly influenced by environmental conditions (temperature, light, humidity) and will differ for each plant and their varieties.

In the hail system, the analyzed components were: storms and their locations (with their respective attributes), the types of storms according to the surface that they affect, the meteorological characteristics, and the fruit potential represented by apples and pears.

The random variables used in the simulation model represent: the number of storms occurring in a given month (0 to 4), according to four types of storms (I to IV), according to a conceptual scheme based on analogies with a model studied by the French, the Swiss, and the Italians (European model of Grossversuch, Switzerland, decade of the 1970s, see Table 34.1–7). References for the European model include Admirat (1977) and Admirat, Goyer, Wojtiw, Carte, Roos, and Lozowki (1985).

Table 34.1–7. The European model.

Type	Diameter	Dimensions	Surface
I	3–5 mm	3–5 Km	< 25 Km^2
II	5–l5 mm	5–10 Km	25–50 Km^2
			
III	20–30 mm	10–20 Km	50–100 Km^2
			
IV	two cells 20–30 mm	50–100 Km	> 100 Km^2

In the European model the "form," the "nature," and the "dynamics" of the cells of the hail storms are defined. From the comparative study between numerous observations made in Napf (Switzerland), Alberta (Canada), and Transvaal Highval (South Africa) a conceptual scheme for explaining these phenomena was obtained. These show many analogies with hail storms in our region of Alto Valle de Río Negro y Neuquén en Argentina (see Admirat (1977) and Admirat, Goyer, Wojtiw, Carte, Roos, and Lozowki (1985)).

For the determination of the forms and structures of the hail surfaces, one takes into account the special apportionment of their densities, amounts, diameter, mass, and kinetic energy. One then concludes on schematizing them through rectangular and elongated forms that represent the direction of the storm trajectories (see Table 34.1–7).

The largest spread of hail is located in the central zone of the rectangle, while the lower values are situated in its periphery. For example, storms of Type II correspond to hail that ranges between 5 and 15 mm in diameter, the dimensions of the rectangle range between 5 and 10 kms, and the largest concentration is illustrated by drawing a small rectangle, while the smallest amount is located in the largest rectangle that is concentric with the first one. In Type III storms, the concentration phenomenon is stressed in the form of three concentric and rectangular zones.

Regarding the surfaces, storms can be as large as 25 Km^2 (Type I); between 25 and 50 Km^2 (Type II); between 50 and 100 Km^2 (Type III); and surfaces larger than 100 Km^2 (Type IV). Type IV (repetition of stage three) may be illustrated by drawing two consecutive cells of Type III, with lateral dimensions between 50

and 100 Kms, of the total rectangle that contains it.

The affected localities, according to the affected areas and the density of the hail, are the 21 localities shown in Figure 34.1–2 in the two provinces mentioned before. Three essential concepts upon which the mentioned European model is based are:

1. Hail is not uniformly distributed in the space.
2. The types of hailfalls (I, II, and II) only constitute three aspects of the unique form that evolves from an original state (Type I) to a terminal state (Type III).
3. The evolution is continuous from states one to three and state four (for very big storms) is a repetition of state three.

Some hypotheses that were considered to be basic are:

1. The variables "number of storms per month" and "locality"" are statistically independent.
2. The occurrences of storms are random variables with the properties of Markov processes.
3. The same storm can occur in several localities.
4. The number of affected localities is a random variable, according to the type of storms.

34.2 Models Used

The simulation process is described in the scheme shown in Figure 34.2–1. In this Figure, the test on "Type of Storm" refers to the storms of Type I. If the test is not satisfied (e.g., storms are of Types II, III, and IV), one considers that the surface affected by the storm exceeds the limits of a single locality. Therefore, it will affect other neighboring localities, in the East-West direction, according to one of the basic model hypotheses.

For assigning the surfaces affected by hail, one incorporates to the model the calculations of mean surfaces and the damage percentages, according to each type of storm. Consider, for example, a storm of Type II. The Damage D2 that it provokes may be described by such that is integrated by two affected surfaces: one in the center of the rectangular form (S22) and the other in its periphery (S21). In the central part, the damage suffered is greater (30%), whereas in the periphery it is minor (5%). Therefore, D2 = 0.3*S22 + 0.05*S21.

Using the historical data from the Appendix at the end of this chapter, for each locality, we have calculated the total damages (in dollars) of the fruits (apples),

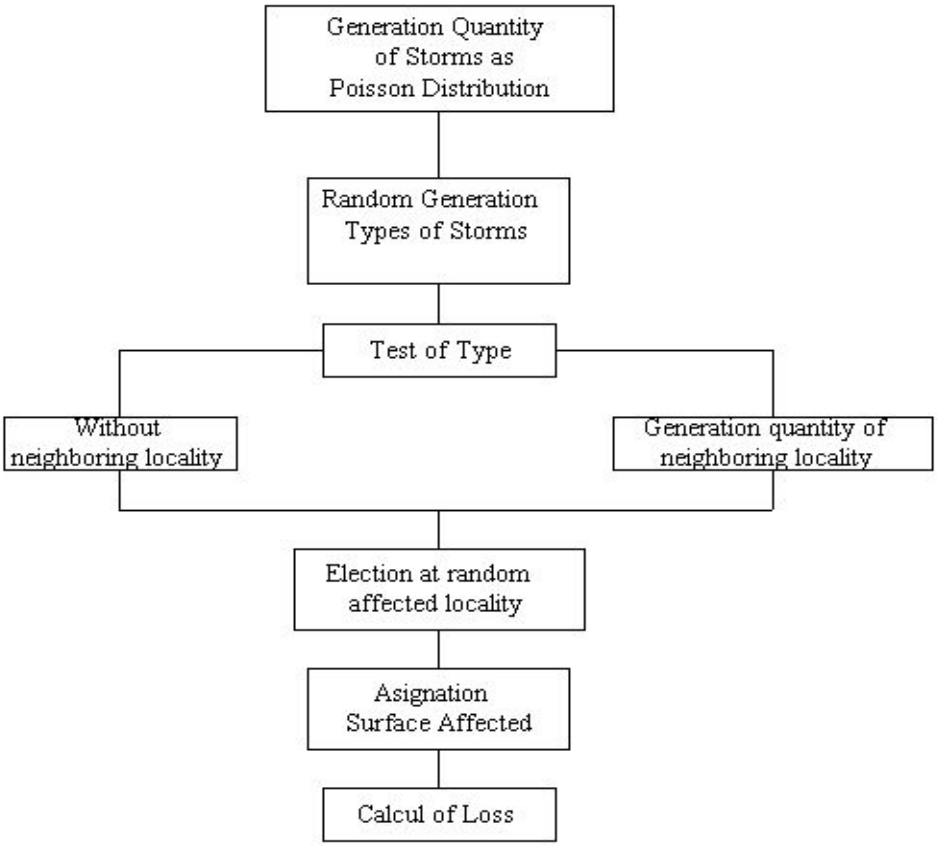


Figure 34.2–1. The simulation process.

taking into consideration the damaged surface, the percent of total damage, and the yield of apples.

To generate a random quantity of storms, and their type, we analyzed the frequencies provided by historical data in twenty seasons of hail storms (1966–1986) in Alto Valle de Río Negro and Neuquén (see the Appendix at the end of this chapter for hail storm data for the region of Río Negro and Neuquén (1967–1986)). Every hail "storm season" is considered to last from October of any year, until March of the following year. Consequently, the twenty campaigns considered range between the years of 1966–67 and 1985–86.

In the first models we included the drawing of a number of neighboring localities to the locality affected. Subsequently, this characteristic was determined with greater precision by incorporating information on measures of size of the localities and on economic coefficients, determined by surface units cultivated in each of them. Finally, according to the type of storm, the affected areas were randomly calculated, and from them, the corresponding economic evaluation of damages in apples and pears.

As is well-known, the type of storm relates to the extension of the storm. Hence, also to the size and density of the hail. Our main objective is to estimate the average number of storms per season, for each type, developing confidence intervals for the above-mentioned parameters, which were compared to the simulation results, in order to validate this artificial information.

Twenty seasons of hail were generated and the quantities for storms of each type were counted. With this information, the average number of storms was calculated. From previously implemented studies, the quantity of storms for each type, in the region, was known. The statistical fits were satisfactorily close, using a Poisson distribution with different parameters, for every month of the season October–March.

In particular, variable "amount of hail storms" was studied in the period October–March, using the historical data available. To determine whether the data in the sample fitted any known statistical distribution, the Kolmogorov-Smirnov (KS) test was implemented. The null hypothesis (H_0) established that the data observed came from a population distributed according to the Poisson, with variable λ parameters: October, 0.55 storms/month; November, 0.65; December, 0.7; January, 0.95; February, 0.25; and March, 0.231.

Taking into consideration that the critical value D of the K-S table, adopting a 20% common significance level for the six months was always greater than 20%, we do not reject H_0 and, therefore, assume that the observed data come from a population that follows the Poisson law.

The economic losses were also estimated using the model of historical prices of the fruit. For the calculation of the above-mentioned losses, we considered price fluctuations of the fruits at varying production levels.

The average value of said losses was of US0.11 for a Kg of pears, and of US0.14 for a Kg of apples. The total losses, by season (in millions of dollars) were distributed in successive intervals: 0–1, 1–2, ..., 15–16. The results of the simulation yielded a frequency of 43% for the losses between 0 and 1 million dollars; and 2% for the losses between 15 and 16 million, for example. Thence, an average value of 3 million dollars per season is obtained. Such value, in reality, will be incremented by the costs of the commercialization chain, the level of jobs, etc.

34.3 Conclusions

1. The economic damages fluctuate among seasons even without the occurrence of storms. Consequently, economic losses in the millions of dollars were caused to the production of apples and pears, even without the occurrence of storms.

2. The number of hail storms, during the October–March season, follows approximately the Poisson distribution.

3. Among the types of storms we can distinguish two subgroups: one formed by storm Types I arid II, and the other group by the Types III and IV.

The subgroups I and II yield around 85% of the hailfalls, being the type I the most frequent (49%). The second subgroup, of storm Types III and IV, yield 11% and 4% of the hailfalls, respectively.

We have the historical sample data (counts) for twenty seasons between 1966 and 1986, affecting the area of the Alto Valle of Río Negro and Neuquén. To each experiment (simulation) the duration of the season has been accordingly assigned, for we want to use the historical model to obtain annual predictions.

At the present time, farmers are, in some cases, beginning to use anti-hail cannons, which disperse and detour the storms. In other cases, they use systems of nets of protective meshes, which diminish the effects that the hail phenomenon causes to the region's crops.

34.4 Tests of Simultaneous Generators

In a simulation process it is generally insufficient to have a single pseudorandom number generator. We cite some reasons supporting this assertion.

According to Law and Kelton (1991), page 619, "... if there are several different random-number generators operating simultaneously, we could 'dedicate' a generator to producing the random numbers for each particular type of input random variate." Another reason, according to the same author (page 423), "... using separate streams for separate purposes facilitates reproducibility and comparability of simulation results."

In many applications, several independent sequences or flows of random numbers are required.

With the purpose of producing multiple and independent flows, a number of seeds that are suitably spread among the sequences of numbers provided by the same generator are calculated. Such seeds determine the initial points of pseudorandom number sequences, which are thence endowed with the property of independence.

Some computer packages possessing the same RNG (Random Number Generator), can generate a given number of sequences from different seeds. Nevertheless, this property can become limited in the face of larger pseudo-random number flow requirements for a given application.

On the other hand, some Simulation Languages are more flexible, since different main program task blocks allow the access to different RNGs, which increases the number of independent sequences available to the model.

Our approach consists of using several generators that have performed satisfactorily in empirical and theoretical statistical tests, and

of entropy tests included in the system GENTSTPC.[1] Then, we use these as sources for the generation of every random variable in the simulation model studied. This includes the generation of independent sequences, from seeds that the algorithms determine, for every flow, according to the user specification. The required quantity can reach, in GENTSTPC, up to ten thousand numbers in every flow.

34.5 Introduction of Several Generators into a Simulation

An aspect that complements the characterization of a good generator is the positive results of their use (Hellekalek (1998)). Tests implemented with different simultaneous generators were fundamental to verify the advantages of some and the inefficiency of others, having been used to simulate the behavior of the different variables.

With this purpose in mind, we consider of major interest to choose the simulation of a system, whose historical behavior has already been studied, such as the hail system previously described. In this application, the main model variables considered are: number of storms, types of storms by season, number of storms for each of the localities in the analyzed region, and seasonal economic loss for two types of fruit trees.

In Figure 34.5–1, The process of simulation is summarized by simultaneous generators. We compare six RNGs whose efficiencies were previously verified, six inefficient RNGs, and six RN_i $(i = 1, 6)$ generators from the simulation language used (GPSS World). From GPSS:

http://www.minutemansoftware.com/reference/r4.htm#4.15

"The GPSS World pseudo-random number generation algorithm is based on Lehmers multiplicative congruential algorithm, with a maximal period. The algorithm produces pseudo-random numbers in the open interval 0 to 2,147,483,647 and it generates 2,147,483,646 unique random numbers before repeating itself. There is an additional shuffling step used by GPSS World." And also: "... random number streams are generated by a maximal period 32 bit multiplicative congruential algorithm. The period is $2^{31} - 2$ and does not include 0."

The utilization of other alternative models in GPSS World, with the RNGs of this package has allowed us to arrive at results that are analogous to those obtained via conventional languages that use generators of the type URNxx, already shown to be efficient. This has also been the case in the hail system application.

[1]Software package used to generate, test, and store files of pseudo-random numbers. This package is part of the Ph.D. dissertation of Dr. A. A. Olmos, UNAM, Mexico, 2006, and is available online in the School of Engineering of said institution: http://www.ingenieria.unam.mx/bibliotecas/.

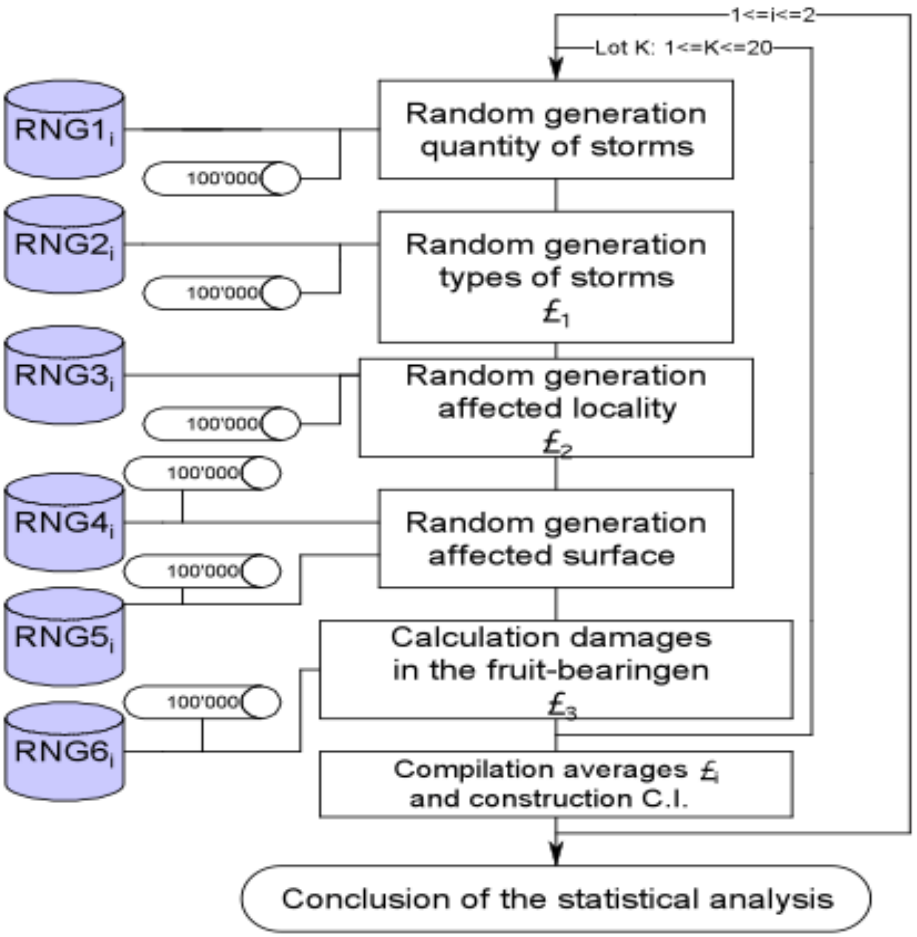


Figure 34.5–1. Process of simulation with simultaneous generators.

The efficient RNGs selected were: URN13, to generate "types of storms"; URN12, for the "number of storms"; URN22, for affected "localities"; and URN01, URN35, and URN39, to generate three quantities that represent the affected "surfaces." In the selection of inefficient RNGs, the above-mentioned random variables were generated, respectively from URN40, URN08, URN34, URN05, URN06, and URN21.

The twelve URNxx generators mentioned before have been studied and tested, according to several different empirical statistical tests, using the TESTRAND package of Dudewicz and Ralley (1981). After the initial tests implemented, these were expanded by using those in Karian and Dudewicz (1991); and more recently, by the GENTSTPC system (Olmos (2006)). Results obtained, according to the classification of efficient and inefficient RNGs, coincide with those in the sources mentioned, and to which entropy tests were also added.

34.6 Statistical Analysis

For the statistical analysis of results, we considered the system in stationary state (steady-state). And with the purpose of reducing the correlation between the sequences of randomly generated variables, we implement the technique of a single "long run," divided into sub-sequences, for the gathering of a large number of observations.

To estimate the parameters (θ) that represent the different measures of performance of the system in stationary state, we consider the definition of (θ) (Banks, Carson, and Nelson (1996)).

$$\theta = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{L} X_i.$$

It is expected that the averages of the samples of the sequences X_i, produced by different random numbers, converge to this value. In our application we fixed the length of the "single long run" at $n = 2,000,000$. Having chosen the method of "batch means," a number of K batches, $K = 20$ of the same size L, results from dividing the number N of runs ($n = 2,000,000$) by L ($= 100,000$).

Then, for the calculation of the average of every batch L_j, we use:

$$\bar{X}_j = \frac{1}{n} \sum_{i=1}^{L} X_i, \quad \text{for } j = 1, \ldots, K,$$

where $\bar{X}$ represents the total average of the averages $\bar{X}_j$. Simultaneously we calculated, for each performance measure, the corresponding confidence intervals (CI) that include the averages, for a given confidence coefficient, which represents the probability of including in such CI, the unknown real values of the averages.

The selection of the number K of batches coincides with the values proposed by Schmeiser (1982), since in this matter there is no single accepted criterion. The simulation stages, developed for the application of the method of "batch means" in the already mentioned Figure 34.5–1, include two cycles. And since there are both, the efficient and the inefficient RNGs, we adopt the notation RNGl_i, RNG2_i, ..., RNG6_i for the six generators.

The files indicated with 100,000 are the sequences of pseudorandom numbers available in each of the L_k batches ($k = 1, 20$) of the method implemented.

The variables, whose performance measures have been selected in this model, are: the average number θ_1 of storms of the types I to IV, by season, the average number, θ_2, of storms for locality (from 1 to 21, see Figure 34.1–2), and the average economic loss, θ_3, for apples and pears in the region. The first one, θ_1, turns out to be broken in four values, according to the types of storm, from which it is possible to infer the percentage of number of storms for each type (see Tables 34.6–1 and 34.6–2). θ_2 is calculated for each of the localities and the estimations of the parameter, θ_3, appear in the Figure 34.6–3. The results in Figure 34.6–1

Table 34.6–1. Types of storms, efficient generators (i.e., generators which have passed extensive testing).

Type	I	II	III	IV
%	45	38	14	3

Table 34.6–2. Types of Storms, inefficient generators (i.e., generators which have failed extensive testing).

Type	I	II	III	IV
%	46	34	19	1

were obtained by efficient RNG, whereas those of Figure 34.6–2 correspond to the not efficient RNG.

From the statistical analysis carried out, and having estimated the average θ_1, with a confidence coefficient of 95%, **we conclude that the results contributed by the efficient RNG, agree satisfactorily with the behavior of the real system.** The efficiency of the GPSS World generators is what the first author has seen in his simulations. The RNGs of GPSS World were compared in other models with URNs that passed the TESTRAND and GENTSTPC tests. Almost no differences in average values were obtained for the variables considered. Of course this does not constitute statistical proof of efficiency of the GPSS World generators. The efficiency and inefficiency noted in Figures 34.6–1 and 34.6–2 would need to be the subject of statistical analysis, comparing the results of Figures 34.6–1 and 34.6–2 with the historical data, given later in this chapter, to provide a statistical basis for this observation. The applications to hail damage, and to random number generator evaluation and simultaneous use in simulations, are the primary goals of this chapter, so such a statistical analysis is a topic for future work.

In effect, 45% of the storms, on average, are of Type I, 38% of Type II, and Types III and IV reach 14% and 3%, respectively. These values coincide well, in percentage, with those from, RNG (RN_i) of the language of simulation used independently with comparison purposes.

On the other hand, there are considerable differences in all the percentages of types of storms for the inefficient RNG (Table 34.6–2).

A similar statistical analysis to estimate θ_3, allowed us to observe serious discrepancies among economic loss values measured in monetary units, for the damages from the hail storms, in cultures of apples and pears, when simulators used efficient and inefficient generators. Due to space limitations, details are not shown here beyond a brief summary in Figure 34.6–3 and following. The

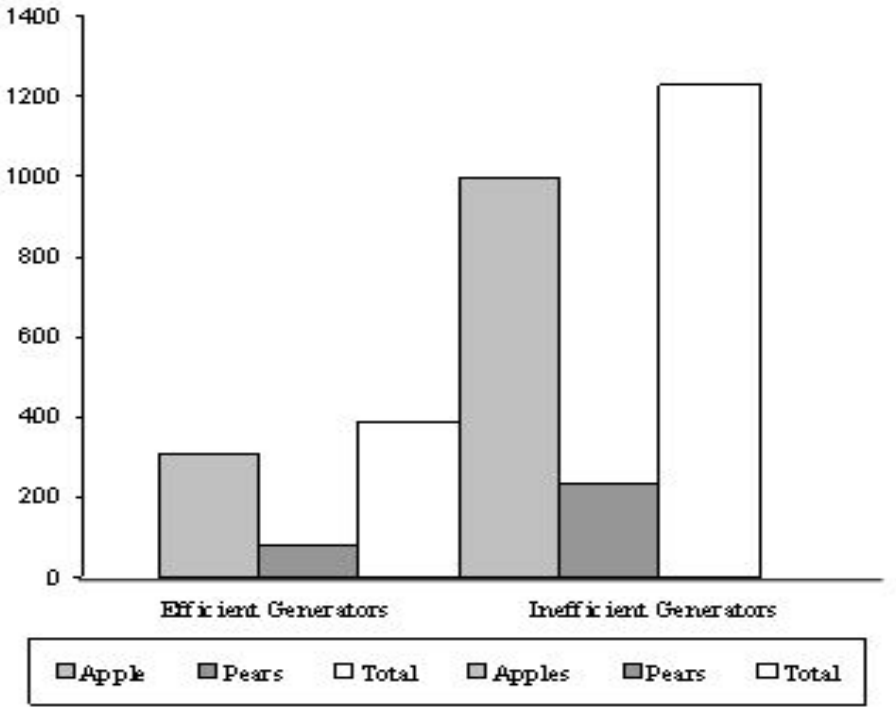


Figure 34.6–3. Economic loss: efficient and inefficient generators.

results corresponding to the averages of storms, according to localities, θ_2, in the analyzed region, do not allow to mark considerable differences between the results obtained from the diverse generators.

References for Chapter 34

Admirat, P. (1977). "The Measurement of Hail at Ground Level," C.N.R.S., Grenoble, France.

Admirat, P., Goyer, G., Wojtiw, L., Carte, E., Roos, D., and Lozowki E. (1985). "A Comparative Study of Hailstorms in Switzerland, Canada and South Africa," *Journal of Climatology,* 5, 35–51.

Banks, J., Carson, J. S., and Nelson, B. L. (1996). *Discret-Event System Simulation,* 2nd. ed., Prentice-Hall, New Jersey.

Bernhofen, L., Dudewicz, E., Levendovszky, J., and Van der Meulen, E. (1996). "Ranking of the best random number generators via entropy-uniformity theory," *American Journal of Mathematical and Management Sciences,* 16, 49–88.

Dudewicz, E. J. and Ralley, T. (1981). *The Handbook of Random Number Generation and Testing with TESTRAND Computer Code,* American Series in Mathe-

matical and Management Sciences, American Sciences Press, Inc., Syracuse, New York.

Dudewicz, E. J., van der Meulen, E. C., SriRam, M. G., and Teoh, N. K. W. (1995). "Entropy-based random number evaluation," *American Journal of Mathematical and Management Sciences,* 15, 115–153.

Dudewicz, E. J. and Van der Meulen, E. C. (1981). "Entropy-based tests of uniformity," *Journal of the American Statistical Association,* 76, 967–974.

Hellekalek, P. (1998). "Good random number generators are (not so) easy to find," *Mathematics and Computers in Simulation* 46, 485–505.

Karian, Z. A. and Dudewicz, E. J. (1991). *Modern Statistical, Systems, and GPSS Simulation,* Computer Science Press, W.H. Freeman, New York.

Knuth, D. (1981). *The Art of Computer Programming: Seminumerical Algorithms,* Vol. 2, Addison-Wesley.

Lassig, J. L. (2006). *Precipitaciones de granizo en el Alto Valle de Río Negro y Neuquén, su influencia en la fruticultura regional* [Hail in the High Valley of Rio Negro and Neuquén, its influence in regional fruit-growing], Ph.D. Dissertation, University of Buenos Aires, Department of Atmospheric Sciences and Oceans.

Law, A. M. and Kelton, W. D. (1991). *Simulation Modeling and Analysis,* McGraw-Hill.

L'Ecuyer, P., Compagner, A. and Cordeau, J-F. (1997). "Entropy Tests for Random Number Generator," Dépt. d'Informatique et de Recherche Opérationnelle, Université de Montréal, Canada. (http://www.iro.umontreal.ca/ lecuyer/myftp/papers/).

L'Ecuyer, P. (2001). "Software for Uniform Random Number Generation: Distinguishing the Good and the Bad," Proceedings of the 33rd conference on Winter simulation, Arlington, Virginia (December 09–12, 2001).

Olmos, A. A. (2006). *Generación de números pseudoaleatorios eficientes en microcomputadores,* thesis of doctorate in engineering, Escuela de Posgrado, Facultad de Ingeniería, Universidad Nacional Autónoma de México (UNAM), México. The dissertation is available from the School of Engineering, UNAM. Interested parties should contact UNAM Library via http://www.ingenieria.unam.mx/bibliotecas/. A copy can be provided gratis online in a simple manner[2]

Olmos, A. A. (1988). Project manager of research: *Simulación de Modelos en precipitaciones de granizo,* Report: *Simulaci on de granizadas en el Alto Valle de Río Negro y Neuquén según localidad, tipos de tormentas, mes y superficie afectada.*

[2]Historical note of the Editors: The National Autonomous University of Mexico (Universidad Nacional Autónoma de México (UNAM)) is the largest one-campus university in the Americas. It has 137,000 students, plus 23,000 M.S. and Ph.D. students, and graduates 540 Ph.D.s per year. It was founded in 1910, and is celebrating its 100th anniversary in 2010. It should not be confused with the Roman Catholic Pontifical University of Mexico, which was founded in 1551, but closed in 1867. UNAM is a liberal alternative to the older PUM (which no longer exists).

Serie Cuadernos de Investigación No. 9, Facultad de Economía y Administración, Universidad Nacional del Comahue.

Schmeiser, B. (1982). "Batch size effects in the analysis of simulation output," *Operations Research*, Vol. 30, No. 3, pp. 556–568.

Appendix for Chapter 34

Data on hail storms[3] of the Río Negro and Neuquén region, 1967–1986.

Locality	Day	Mo.	Year	Damaged surface km^2	% Total damage	Surface in apples	Yield per apple	Total fruit damage (US $)
C.SALTOS	14	1	1966	50	13	0.4129	31110	770719.14
CIPOLLETTI	14	1	1966	1	30	0.3746	29480	23853.32928
V.ALEGRE	14	1	1966	13	80	0.4808	31810	1439494.622
ALLEN	29	3	1966	30	0	0.3235	25990	0
CENTENARIO	29	3	1966	24	60	0.4246	30410	1338725.076
CIPOLLETTI	29	3	1966	55	60	0.3746	29480	2623866.221
F.ORO	29	3	1966	46	0	0.3036	21970	0
GUERRICO	29	3	1966	28	40	0.3235	25990	678002.1696
NEUQUEN	29	3	1966	34	60	0.2661	27730	1083820.617
ROCA	29	3	1966	12	0	0.2811	19340	0
REGINA	24	11	1966	4	2	0.3992	23320	8936.97024
CIPOLLETTI	26	11	1966	28	20	0.3746	29480	482367.3254
REGINA	13	1	1968	1	0	0.3992	23320	0
ALLEN	13	10	1968	16	40	0.3235	25990	387429.8112
F.ORO	13	10	1968	2	30	0.3036	21970	28814.79744
GUERRICO	13	10	1968	13	60	0.3235	25990	472180.0824
J.J.GOMEZ	13	10	1968	1	1	0.2811	19340	652.37688
CHICHINALES	13	10	1968	6	90	0.2959	15700	246404.2152
GODOY	13	10	1968	4	90	0.3499	21210	262419.9614
CIPOLLETTI	20	10	1968	8	0	0.3746	29480	0
F.ORO	20	10	1968	5	0	0.3036	21970	0
ALLEN	13	11	1968	2	0	0.3235	25990	0
GUERRICO	13	11	1968	6	0	0.3235	25990	0
J.J.GOMEZ	13	11	1968	2	0	0.2811	19340	0
ALLEN	19	11	1968	1	0	0.3235	25990	0
C.SALTOS	26	11	1968	2	10	0.4129	31110	30828.7656
REGINA	26	11	1968	1	0	0.3992	23320	0
ROCA	26	11	1968	14	3	0.2811	19340	27399.82896
ALLEN	26	11	1968	36	3	0.3235	25990	108964.6344
CIPOLLETTI	26	11	1968	20	3	0.3746	29480	79511.0976
F. ORO	26	11	1968	20	3	0.3036	21970	48024.6624
GUERRICO	26	11	1968	24	3	0.3235	25990	72643.0896
GODOY	14	4	1969	5	0	0.3499	21210	0
REGINA	14	4	1969	22	0	0.3992	23320	0
C.SALTOS	13	11	1969	5	0	0.4129	31110	0
CIPOLLETTI	13	11	1969	5	0	0.3746	29480	0
CHICHINALES	27	11	1969	14	10	0.2959	15700	78046.584
GODOY	27	11	1969	15	10	0.3499	21210	133584.822
REGINA	27	11	1969	23	30	0.3992	23320	462488.2099

[3]Source: Lassig J. L., Precipitaciones de granizo en el Alto Valle de Río Negro y Neuquén, su influencia en la fruticultura regional [Hail in the High Valley of Río Negro and Neuquén, its influence in regional fruit-growing], Ph.D. Dissertation, University of Buenos Aires, Department of Atmospheric Sciences and Oceans, April 2006.

Data on hail storms (continued).

Locality	Day	Mo.	Year	Damaged surface km^2	% Total damage	Surface in apples	Yield per apple	Total fruit damage (US $)
C.SALTOS	20	1	1970	15	0	0.4129	31110	0
ALLEN	27	1	1970	24	80	0.3235	25990	1460933.246
GODOY	27	1	1970	1	0	0.3499	21210	0
CIPOLLETTI	7	1	1970	8	10	0.3746	29480	106014.7968
C.SALTOS	3	1	1970	24	90	0.4129	31110	2725262.879
CENTENARIO	3	1	1970	24	90	0.4246	30410	2739428.166
CIPOLLETTI	3	1	1970	12	70	0.3746	29480	768607.2768
NEUQUEN	3	1	1970	11	30	0.2661	27730	175323.9233
ALLEN	28	1	1970	28	30	0.3235	25990	508501.6272
C.CORDERO	16	2	1971	18	0	0.4129	31110	0
C.GRANDE	16	2	1971	1	0	0.3544	31120	0
C.SALTOS	16	2	1971	9	0	0.4129	31110	0
CHICHINALES	1	10	1971	1	0	0.2959	15700	0
GODOY	1	10	1971	1	0	0.3499	21210	0
ROCA	16	12	1971	1	0	0.2811	19340	0
ALLEN	4	1	1972	9	1	0.3235	25990	9080.3862
CIPOLLETTI	4	1	1972	9	1	0.3746	29480	11926.66464
F.ORO	4	1	1972	27	1	0.3036	21970	21611.09808
ROCA	4	1	1972	1	1	0.2811	19340	652.37688
GODOY	21	1	1972	14	70	0.3499	21210	602615.9748
REGINA	21	1	1972	7	70	0.3992	23320	377959.3664
ALLEN	6	11	1972	8	50	0.3235	25990	242143.632
GUERRICO	6	11	1972	15	50	0.3235	25990	454019.31
C.GRANDE	6	11	1972	1	0	0.3544	31120	0
CENTENARIO	6	11	1972	4	0	0.4246	30410	0
NEUQUEN	6	11	1972	6	0	0.2661	27730	0
C.GRANDE	11	12	1972	1	0	0.3544	31120	0
NEUQUEN	11	12	1972	1	0	0.2661	27730	0
C.SALTOS	30	1	1973	1	0	0.4129	31110	0
C.SALTOS	5	10	1973	4	20	0.4129	31110	0
CIPOLLETTI	5	10	1973	25	70	0.3746	29480	1601265.16
NEUQUEN	5	10	1973	13	70	0.2661	27730	556373.0562
ROCA	18	10	1973	1	0	0.2811	19340	0
CENTENARIO	4	10	1973	1	20	0.4246	30410	20142.85416
CIPOLLETTI	4	10	1973	9	50	0.3746	29480	357799.9392
NEUQUEN	4	10	1973	1	20	0.2661	27730	11511.16668
HUERGO	5	1	1975	25	70	0.3709	17450	938469.725
MAINQUE	5	1	1975	11	55	0.2604	17680	200544.6643
CHICHINALES	3	1	1975	10	80	0.2959	15700	336343.612
REGINA	3	1	1975	15	80	0.3992	23320	1010994.758
NEUQUEN	21	10	1975	1	0	0.2661	27730	0
NEUQUEN	11	10	1976	10	0	0.2661	27730	0
CHICHINALES	28	10	1976	14	60	0.2959	15700	280967.7024

Data on hail storms (continued).

Locality	Day	Mo.	Year	Damaged surface km^2	% Total damage	Surface in apples	Yield per apple	Total fruit damage (US $)
CHICHINALES	29	1	1977	7	60	0.2959	15700	140483.8512
REGINA	29	1	1977	8	50	0.3992	23320	268109.1072
GUERRICO	13	12	1977	21	100	0.3235	25990	2245882.187
ALLEN	29	12	1977	35	80	0.3235	25990	2130527.651
CIPOLLETTI	29	12	1977	24	25	0.3746	29480	508871.0246
F.ORO	29	12	1977	15	30	0.3036	21970	216110.9808
GUERRICO	29	12	1977	11	30	0.3235	25990	199768.4964
GODOY	12	1	1978	3	70	0.3499	21210	129131.9946
REGINA	12	1	1978	22	70	0.3992	23320	1187872.294
CIPOLLETTI	12	1	1978	1	0	0.3746	29480	0
GUERRICO	16	12	1978	1	0	0.3235	25990	0
CIPOLLETTI	22	12	1978	31	2	0.3746	29480	82161.46752
F.ORO	22	12	1978	2	10	0.3036	21970	16008.2208
CERVANTES	1	1	1979	25	30	0.2497	13830	186480.954
STEFENELLI	1	1	1979	10	50	0.2811	19340	195713.064
ALLEN	4	3	1979	3	0	0.3235	25990	0
CERVANTES	3	3	1980	25	10	0.2497	13830	103600.53
HUERGO	3	3	1980	15	80	0.3709	17450	702881.463
MAINQUE	3	3	1980	17	30	0.2604	17680	169054.1798
ROCA	3	3	1980	35	50	0.2811	19340	684995.724
CHICHINALES	14	10	1980	1	70	0.2959	15700	26944.654
REGINA	18	11	1980	1	20	0.3992	23320	14522.57664
CERVANTES	31	1	1981	22	70	0.2497	13830	440647.5876
ROCA	31	1	1981	16	70	0.2811	19340	504504.7872
C.SALTOS	6	2	1981	6	0	0.4129	31110	0
ROCA	6	2	1981	16	0	0.2811	19340	0
CHICHINALES	28	12	1981	5	80	0.2959	15700	168171.806
GODOY	28	12	1981	4	20	0.3499	21210	46309.40496
REGINA	28	12	1981	21	30	0.3992	23320	422271.8438
CHICHINALES	29	12	1981	5	70	0.2959	15700	134723.27
REGINA	29	12	1981	2	70	0.3992	23320	107988.3904
MAINQUE	9	2	1982	11	70	0.2604	17680	293727.0336
CERVANTES	1	11	1982	8	80	0.2497	13830	200018.0899
HUERGO	1	11	1982	12	80	0.3709	17450	562305.1704
MAINQUE	1	11	1982	28	80	0.2604	17680	933296.9318
ALLEN	3	12	1982	5	80	0.3235	25990	304361.093
GUERRICO	3	12	1982	19	0	0.3235	25990	0
J.J.GOMEZ	3	12	1982	10	0	0.2811	19340	0
CHICHINALES	4	12	1982	1	10	0.2959	15700	5574.756
REGINA	12	12	1982	7	30	0.3992	23320	140757.2813
CHICHINALES	12	12	1982	4	40	0.2959	15700	57977.4624

Data on hail storms (concluded).

Locality	Day	Mo.	Year	Damaged surface km^2	% Total damage	Surface in apples	Yield per apple	Total fruit damage (US $)
NEUQUEN	25	1	1983	9	1	0.2661	27730	7969.26924
GUERRICO	22	2	1983	12	75	0.3235	25990	657823.5336
ROCA	22	2	1983	33	75	0.2811	19340	1169711.746
ROCA	1	3	1983	24	60	0.2811	19340	563653.6243
CERVANTES	8	12	1983	16	40	0.2497	13830	159130.4141
ROCA	8	12	1983	45	50	0.2811	19340	880708.788
ALLEN	10	12	1983	23	30	0.3235	25990	417697.7652
CIPOLLETTI	10	12	1983	20	50	0.3746	29480	795110.976
F.ORO	10	12	1983	15	60	0.3036	21970	432221.9616
CERVANTES	16	2	1984	14	40	0.2497	13830	139239.1123
CHICHINALES	16	2	1984	4	10	0.2959	15700	22299.024
GODOY	16	2	1984	8	30	0.3499	21210	56996.19072
HUERGO	16	2	1984	7	50	0.3709	17450	163099.566
MAINQUE	16	2	1984	20	30	0.2604	17680	198887.2704
REGINA	16	2	1984	28	50	0.3992	23320	938381.8752
CHICHINALES	28	12	1984	1	0	0.2959	15700	0
GODOY	1	1	1985	18	30	0.3499	21210	288543.2155
HUERGO	1	1	1985	18	30	0.3709	17450	251639.3304
REGINA	1	1	1985	9	30	0.3992	23320	180973.6474
MAINQUE	11	12	1985	3	0	0.2604	17680	0
ROCA	16	1	1986	13	60	0.2811	19340	305312.3798

PART VIII: Appendices

Appendix A	Programs and their Documentation
Appendix B	Table B 1 for GLD Fits: Method of Moments
Appendix C	Table C 1 for GBD Fits: Method of Moments
Appendix D	Tables D 1 through D 5 for GLD Fits: Method of Percentiles
Appendix E	Tables E 1 through E 5 for GLD Fits: Method of L-Moments
Appendix F	Table F 1 for Kappa Distribution Fits: Method of L-Moments
Appendix G	Table G 1 for Kappa Distribution Fits: Method of Percentiles
Appendix H	Table H 1 for Johnson System Fits in the S_U Region: Method of Moments
Appendix I	Table I 1 for Johnson System Fits in the S_B Region: Method of Moments
Appendix J	Table J 1 for p-Values Associated with Kolmogorov-Smirnov Statistics
Appendix K	Table K 1 Normal Distribution Probabilities

Appendix A

Programs and their Documentation

In this appendix we give a brief descriptions of the **R** programs that are available on the CD that is distributed with this book. These programs have been used for many, but not all, of the computations of the examples that appear in various chapters.

Section A.1 discusses the general search method that is used by the various fitting programs. Section A.2 describes the programs of general scope that are used to compute those statistics from datasets that are needed for fitting GLD, GBD, or kappa distributions. These programs are typically used by specific fitting programs that are described in Sections A.3 through A.5. However, all programs, including those mentioned in Section A.2, are also available for direct use. The programs of Section A.2 deal with the computations associated with fitting GLD distributions to datasets, with the **R** functions needed for moment, percentile, and L-moment fits given in separate subsections. Sections A.4 and A.5 describe the programs used in connection with GBD and kappa distribution fits, respectively and Section A.6 describes the programs (written in Maple) for fitting distributions from the Johnson system described in Chapter 16. In giving synoptic descriptions of the programs available on the CD that accompanies this book, Sections A.2 through A.7 use the same notations that were used when the fitting methods were introduced in the relevant chapters of this book (e.g., λ_1, λ_2, λ_3, λ_4 refer to the parameters of the GLD distribution). Section A.7 deals with the programs in connection with fitting the GLD–2 distribution (the bivariate GLD) to two datasets (see Chapter 9).

Section A.8 describes the directory structure and the files that are on the CD. Instructions of how to transfer these files to your PC are also included in this section.

A.1 General Computational Issues

Throughout this text, whenever we discussed the need to estimate parameters, we used graphic methods to develop insight into the possible location of solutions. We also provided specific algorithms for determining solutions from the tables that appear in subsequent appendices. By their nature, graphic solutions lack precision, although it is not wise to dispense with them altogether since they can not only guide the investigator's intuition, but can also uncover solutions that may have gone unnoticed.

The main disadvantage of graphic and table-driven solutions is their lack of accuracy. It makes sense, whenever possible, to estimate parameters, say α and β, through direct computation. The most efficient algorithms for finding solutions on surfaces start by searching for a solution at a "reasonable" point and use gradient vectors to move in the optimal direction to find a better solution. This is continued until a predetermined level of accuracy is achieved. Such algorithms work best if the surfaces are reasonably smooth, but even then, on some occasions, at some step within the algorithm the search can get misdirected and, once misdirected, a proper path may never be re-established.

An alternative to a gradient-based algorithms is a "brute force" approach where a rectangular grid is established in (α, β)-space and grid points are tested until one with acceptable error is found. Of course, if the original grid does not cover the area where there is a solution, the search will not succeed; thus, one needs to have a good idea of where to conduct the search. Even if the grid contains a solution, the search can fail because the grid may be too coarse to come across a point that is within the desired error limit. Making the grid overly fine in order to counteract this problem may not work because too fine a grid may result in an intolerably long computation time. For example, moving from a 10×10 grid to a 25×25 grid increases computation time by more than a factor of 6 and moving to a 250×250 grid would increase a computation time by a factor of 625. If the original 10×10 grid required 5 seconds of computation time, the 250×250 grid would require about 52 minutes of computation.

A more acceptable algorithm can be designed to read table entries in order to determine the proper area that needs to be covered by a grid. After all, the smaller this area, the fewer grid lines should be required for a close search. In addition, the grid search process itself could be improved. Suppose that a search for (α, β) is to be conducted on the rectangle $[A_1, A_2] \times [B_1, B_2]$. Instead of covering the search area by a 1000×1000 grid, the algorithm could start with a 10×10 grid (let's say with increments of $\Delta_1 = (A_2 - A_1)/10$ and and $\Delta_2 = (B_2 - B_1)/10$ between successive grid lines), find the best solution $(\alpha, \beta) = (a, b)$ within that grid, and restart a 10×10 grid search over the region

$$a - \Delta_1 \leq \alpha \leq a + \Delta_1, \quad b - \Delta_2 \leq \beta \leq b + \Delta_2,$$

with a suitably smaller set of Δ_1 and Δ_2. In this situation, Δ_1 and Δ_2 for the

second iteration would be

$$\Delta_1' = 2\Delta_1/10 = (A_2 - A_1)/50 \quad \text{and} \quad \Delta_2' = 2\Delta_2/10 = (B_2 - B_1)/50.$$

Note that this reduces the area of the search rectangle to 4% of its original area. If this "re-gridding" were repeated two more times, it would almost have the same effect as an initial 1000×1000 grid search. If we again assume that a 10×10 grid requires 5 seconds of computation time, this new approach would take about 15 seconds, whereas a 1000×1000 grid would need almost 14 hours.

Unfortunately, when the surfaces that represent α and β have sharp peaks, one of the rectangles in an interim search may not contain a solution. When these types of surfaces are encountered, one would need to make the successive rectangles somewhat larger by taking

$$a - k\Delta_1 \le \alpha \le a + k\Delta_1, \quad b - k\Delta_2 \le \beta \le b + k\Delta_2,$$

where k is larger than 1. In such cases, successive values of Δ_1' and Δ_2' would be $\Delta_1' = 2k\Delta_1/10$ and $\Delta_2' = 2k\Delta_2/10$. The gain from iterated searches is diminished when k gets large. When $k = 1.5$, each iteration reduces the area of the search rectangle to about 7% of the original area; when $k = 2$, the reduction is to 16% of the original area; when $k = 3$, the search rectangle is reduced to 36% of its original area. The search program that will be discussed in the following sections use tables to establish A_1, A_2, B_1, B_2 and use three to five re-griddings with k taken as 1 or 2.

A.2 General Functions

1. `FindAlphas(Data)`
   For a numeric vector, `Data`, this function returns $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$ the vector of the first four moments of `Data`.

2. `Percentile(Data, p)`
   The 100`p`-th percentile of the numeric vector `Data` is computed and returned.

3. `FindRhos(Data)`
   This function computes and returns $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4)$ of the numeric vector `Data`.

4. `FindLMoms(Data)`
   This function computes and returns the first four L-moments, $(\ell_1, \ell_2, \ell_3, \ell_4)$, of the numeric vector `Data`.

5. `Freq(Data, Breaks)`
   `Data` and `Breaks` are numeric vectors with `Breaks`$= (b_1, b_2, b_3, \ldots, b_k)$ in ascending order. This function returns a vector of the $k+1$ frequencies of `Data` in the intervals $(-\infty, b_1)$, $[b_1, b_2)$, $[b_2, b_3), \ldots, [b_{k-1}, b_k)$, $[b_k, \infty)$.

A.3 Functions for GLD Computations

1. `SupportGLD(Lambda)`
   For `Lambda`= (λ_1, λ_2, λ_3, λ_4), this function computes and returns the support of the GLD(λ_1, λ_2, λ_3, λ_4) distribution. It uses $\pm 10^{10}$ when $\pm\infty$ are indicated.

2. `pdfGLD(Lambda, p)`
   For `Lambda`= (λ_1, λ_2, λ_3, λ_4) and `p` a numeric value between 0 and 1, this procedure returns the value of the GLD(λ_1, λ_2, λ_3, λ_4) p.d.f. at `p`.

3. `cdfGLD(Lambda, t)`
   `Lambda` is a vector (λ_1, λ_2, λ_3, λ_4), and `t` is a number. The function returns $P[X \leq t]$ where X is the GLD(λ_1, λ_2, λ_3, λ_4) random variable.

4. `cdfVectGLD(Lambda, Breaks)`
   `Lambda` is a numeric vector, representing (λ_1, λ_2, λ_3, λ_4), and `Breaks` is a numeric vector, (b_1, b_2, $b_3, \ldots, b_k$), in ascending order. The vector of $k+2$ probabilities, (0, $P[X \leq b_1]$, $P[X \leq b_2]$, $\ldots$, $P[X \leq b_k]$, 1) is returned.

5. `KSGLD(Data, Lambda)`
   For the numeric vector, `Data`, and `Lambda` representing (λ_1, λ_2, λ_3, λ_4), this procedure returns the Kolmogorov-Smirnov statistics associated with fitting GLD(λ_1, λ_2, λ_3, λ_4) to `Data`.

6. `ChiSqGLD(Data, Lambda, Breaks)`
   This function partitions the numeric vector `Data` into frequency classes determined by the points in the ascending numeric vector `Breaks`. If `Breaks` = (b_1, b_2, $b_3, \ldots, b_k$) then the classes will be $(-\infty, b_1)$, $[b_1, b_2)$, $[b_2, b_3), \ldots,$ $[b_{k-1}, b_k)$, $[b_k, \infty)$. These frequencies are then compared to the expected frequencies based on the GLD distribution with its parameters specified by numeric vector `Lambda`=(λ_1, λ_2, λ_3, λ_4). The function returns the chi-square statistic and p-value associated with this fit.

7. `plotGLD(Data, L, Breaks)`
   Produces a superimposed plot of two items: the first is the p.d.f. of GLD(λ_1, λ_2, λ_3, λ_4) = GLD(`L`); the second is a histogram of the numeric vector `Data` with endpoints given by the ascending numeric vector `Breaks`.

A.3.1 Functions for GLD Moment-Based Fits

1. `ErrGLDM(a3, a4, l3, l4)`
   Using $\hat{\alpha}_3$ = `a3`, $\hat{\alpha}_4$ = `a4`, λ_3 = `l3`, λ_4 = `l4`, this function computes α_3 and α_4 associated with λ_3 and λ_4 and returns $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$.

2. `RefineSearchGLDM(Alphas, l3MinMax, l4MinMax, Grid, Iter)`
   `Alphas` is a vector of four entries representing $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, and $\hat{\alpha}_4$ of some dataset or distribution. A search for λ_1, λ_2, λ_3, λ_4 with minimal error (i.e., minimal value of $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$) is conducted. `l3MinMax` is a pair specifying the minimal and maximal values in the search of λ_3 and `l4MinMax` provides a similar constraint for λ_4. Each side of the rectangle designated by `l3MinMax` and `l4MinMax` is divided into `Grid` subintervals and every point in this grid is checked for optimality. When an optimal point is located, a new rectangle is formed by moving in all four directions by one grid interval and this process is repeated `Iter` times. The function returns a numeric vector consisting of λ_3, λ_4, and $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$ associated with this choice.

3. `FindLambdasM(Alphas)`
   `Alphas` is the numeric vector representing $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. This program sets initial values of λ_3 and λ_4 from the moment-based GLD fitting tables and `RefineSearchGLDM(a3, a4, l3MinMax, l4MinMax, Grid, Iter)` is invoked, with `Grid`= 100 and `Iter`= 4, in order to determine the numeric vector $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. If the search for (λ_3, λ_4) fails, `c(0,0,0,0)` is returned; if it is successful, a five-component vector consisting of the four λ's and the error associated with the fit is returned.

4. `FitGLDM(Data)`
   This function prints the α_1, α_2, α_3, α_4 of the numeric vector `Data`, the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the fitted GLD, the $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$ associated with the fit, the support of the GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, the minimum and maximum values of `Data` and the Kolmogorov-Smirnov statistic for the fit. The vector $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is returned (when no fit is found, `c(0,0,0,0)` is returned).

Example 1: To obtain a GLD fit that is based on moments for the dataset that was introduced in Section 3.5.2, we first enter the data by using the following **R** command. (Section A.6 will give an alternate, and simpler, way of accessing this particular dataset.)

```
D <- c(11.9, 16.7, 23.4, 25.8, 25.9, 27.5, 28.5, 31.1,
+  32.5, 35.4, 38.3, 38.5, 41.8, 42.9, 50.7, 52.3,
+  52.5, 52.6, 54.5, 54.7, 56.6, 56.7, 58.0, 60.8,
+  61.8, 62.3, 62.5, 62.6, 63.0, 67.7, 68.5, 69.7,
+  73.1, 76.0, 76.9, 77.7, 78.2, 80.3, 93.7, 101.0,
+  104.5, 105.4, 107.0)
```

Next, when `Fit1 <- FitGLDM(D)` is invoked, we obtain the following output.

```
          GLD MOMENT-BASED FIT

Alphas      :  57.24419 576.0741 0.2545719 2.52575
Lambdas     :  41.78965 0.01134493 0.09852547 0.3605804
Error       :  1.672183e-09
Support     :  ( -46.35541 , 129.9347 )
Data Min-Max:  11.9    107
K-S Stat.   :  0.1025044
```

This indicates that for this data, we have

$$(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4) = (57.24419, 576.0741, 0.2545719, 2.52575)$$

and the moment-based fit to this data is

$$\text{GLD}(41.78965, 0.01134493, 0.09852547, 0.3605804)$$

with $\max(|\alpha_3 - \hat{\alpha}_3|, |\alpha_4 - \hat{\alpha}_4|) = 1.672183 \times 10^{-9}$. The support of the fitted GLD is $(-46.35541, 129.9347)$ and the minimum and maximum of the data are, respectively, 11.9 and 107. The Kolmogorov-Smirnov statistic for this fit is 0.1025044. The value assigned to `Fit1` is the vector of λ's.

If, in addition to the Kolmogorov-Smirnov test, we want to have a chi-square test for this fit, we can define class boundaries by using `B <- c(30, 50, 60, 70, 85)` and then invoke the function `ChiSqGLD(D, Fit1, B)` to obtain

```
Breaks :
 30 50 60 70 85
Observed Frequencies:
 7 7 9 9 6 5
Expected Frequencies:
 5.56335 12.30325 6.538715 5.807501 6.677516 6.109668
Chi-Square = 5.608667 p-Value = 0.01787186
[1] 5.60866680 0.01787186
```

indicating that when the classes

$$(-\infty, 30),\ [30, 50,),\ [50, 60),\ [60, 70),\ [70, 85),\ [85, \infty)$$

are used, we get observed frequencies of

$$7,\ \ 7,\ \ 9,\ \ 9,\ \ 6,\ \ 5,$$

expected frequencies of

$$5.56335, 12.30325, 6.538715, 5.807501, 6.677516, 6.109668$$

for these classes. The resulting chi-square statistic and its associated p-value are, respectively, 5.60866680 and 0.01787186.

A.3.2 Functions for GLD Percentile-Based Fits

1. `ErrGLDP(r3, r4, l3, l4)`
   Using $\hat{\rho}_3 =$ `r3`, $\hat{\rho}_4 =$ `r4`, $\lambda_3 =$ `l3`, $\lambda_4 =$ `l4`, this function computes ρ_3 and ρ_4 associated with λ_3 and λ_4 and returns $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|)$.

2. `RefineSearchGLDP(Rhos, l3MinMax, l4MinMax, Grid, Iter)`
   Similar to `RefineSearchGLDM` above, this function zooms in to a (λ_3, λ_4) that produces a small $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|)$ and returns a numeric vector with 5 components: λ_1, λ_2, λ_3, λ_4 and the $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|)$ associated with it.

3. `FindLambdasPA(Rhos)`, `FindLambdasPB(Rhos)`, `FindLambdasPC(Rhos)`, `FindLambdasPD(Rhos)`, `FindLambdasPE(Rhos)`
   These five functions are similar to `FindLambdasM(Alphas)` of Section A.3.1. They attempt to find GLD fits with small $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|)$. The search for a good fit is conducted in Region A by `FindLambdasPA`, in Region B by `FindLambdasPB`, etc. These functions return `c(0,0,0,0)` for unsuccessful searches and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, along with the error when a search is successful.

4. `FitGLDP(Data)`
   After obtaining $(\rho_1, \rho_2, \rho_3, \rho_4)$, `FindLambdasPA`, ..., `FindLambdasPE` are invoked to obtain possibly as many as 5 fits. The output is similar to that of `FitGLDM`, repeated for each sucessful fit. A list of vectors, each representing the result (i.e., λ's) of a successful fit is returned. For example, if `L <- FitGDP(Data)` is used, then `length(L)` will give the number of sucessful fits and `L[[1]]`, `L[[2]]`, ..., will give the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for each fit.

Example 2: To illustrate the fitting of a GLD distribution using percentiles, we consider the same dataset of Example 1 and assume that the data has already been established as the vector `D`, as was done in Example 1. The command `Fit2 <- FitGLDP(D)` produces the following output.

```
          GLD PERCENTILE-BASED FITS

Rhos          :  56.7 72.24 0.7457709 0.4817276
Lambdas       :  42.80476 0.01412592 2.348305 13.38325
Error         :  2.147027e-11
Support       :( -27.98707 , 113.5966 )
Data Min-Max  : 11.9   107
K-S Stat.     : 0.1208249
```

The interpretation of this output is similar to that of Example 1 and a detailed explanation is unnecessary. However, we do point out that, following the

execution of `Fit2 <- FitGLDP(D)`, `Fit2` becomes a list of vectors (in this case, the list has only one entry, since only one percentile-based fit was discovered). The assigned value to `Fit2` can be obtained by accessing the first (and only) entry of this list through `Fit2[[1]]`. When this is done, the λ's of the fitted GLD, `42.80476 0.01412592 2.348305 13.38325` are produced. Had there been several fits, output similar to the one above would have been repeated for each fit. Moreover, the λ's of the additional fits would be accessible through `Fit2[[2]]`, `Fit2[[3]]`, ..., as appropriate.

A chi-square test can also be obtained for this fit. If the same class intervals that were defined in Example 1 are used, we would get, through `ChiSqGLD(D, Fit2[[1]], B)`, the output

```
Breaks :
 30 50 60 70 85
Observed Frequencies:
 7 7 9 9 6 5
Expected Frequencies:
 5.056573 11.29733 7.184509 5.072822 5.88518 8.503582
Chi-Square = 7.326362 p-Value = 0.006795046
[1] 7.326361712 0.006795046
```

with the same interpretations given in Example 1.

If we look at $(\hat{\rho}_3, \hat{\rho}_4) = (0.7457709, 0.4817276)$ for this data and try to locate it in the tables of Appendix D, we discover that we can only do this for Table D–1 and the entry of that table gives us a less precise solution of what we obtained through `FitGLDP` above. However, Table D–5 narrowly misses $(\hat{\rho}_3, \hat{\rho}_4) = (0.7457709, 0.4817276)$, indicating that it may be possible to find another fit with negative values of λ_3 and λ_4. To explore this possibility we employ `RefineSearchGLDP` as follows.

```
> R <- FindRhos(D)
[1] 56.7000000 72.2400000  0.7457709  0.4817276
> Fit <- RefineSearchGLDP(R, c(-0.2,-0.01), c(-0.2,-0.01), 50, 3)
> Fit
[1]  5.142605e+01 -6.275426e-03 -6.916448e-02 -1.139741e-01  3.442806e-05
```

This indicates that there is, indeed, another fit with the λ's specified above for which $\max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|) = 3.442806 \times 10^{-5}$. The following will allow us to obtain the Kolmogorov-Smirnov and chi-square statistics associated with this fit.

```
> L <- c(Fit[1], Fit[2], Fit[3], Fit[4])
> KSGLD(D,L)
[1] 0.08974355
```

```
> B <- c(30, 50, 60, 70, 85)
> ChiSqGLD(D, L, B)
Breaks :
 30 50 60 70 85
Observed Frequencies:
 7 7 9 9 6 5
Expected Frequencies:
 5.653485 11.15987 6.914676 5.921596 6.033998 7.31637
Chi-Square = 4.834099 p-Value = 0.02790222
[1] 4.83409888 0.02790222
```

A.3.3 Functions for GLD L-Moment Fits

1. `ErrGLDL(LMom, l3, l4)`
   `LMom` is a vector representing L-moments and `l3` and `l4` are λ_3 and λ_4, respectively, of a GLD distribution. Using $\hat{\tau}_3$ = `LMom[3]/LMom[2]`, $\hat{\tau}_4$ = `LMom[4]/LMom[2]`, this function computes τ_3 and τ_4 associated with `l3` and `l4` and returns $\max(|\hat{\tau}_3 - \tau_3|,\ |\hat{\tau}_4 - \tau_4|)$.

2. `RefineSearchGLDL(LMom, l3MinMax, l4MinMax, Grid, Iter)`
   Similar to `RefineSearchGLDM` of Section A.3.1, this function zooms in to a (λ_3, λ_4) that produces a small $\max(|\hat{\tau}_3 - \tau_3|,\ |\hat{\tau}_4 - \tau_4|)$ and returns a numeric vector with 5 components: λ_1, λ_2, λ_3, λ_4 and $\max(|\hat{\tau}_3 - \tau_3|,\ |\hat{\tau}_4 - \tau_4|)$ that is associated with it.

3. `FindLambdasL(LMom)`
   This function is similar to `FindLambdasM(Alphas)` of Section A.3.1 and it attempts to find GLD fits with small $\max(|\hat{\tau}_3 - \tau_3|,\ |\hat{\tau}_4 - \tau_4|)$. The search for a good fit is conducted in five different regions of the valid (λ_3, λ_4)-space. The function returns a list of vectors, each member of which represents the $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of a sucessful search in one of the five regions (if the search is unsuccessful in all five regions, it returns `c(0,0,0,0)`).

4. `FitGLDL(Data)`
   After obtaining $(\ell_1, \ell_2, \ell_3, \ell_4)$ (and from them $\hat{\tau}_3$ and $\hat{\tau}_4$), this function searches for (λ_3, λ_4) in eight regions of the valid (λ_3, λ_4)-space. The results (i.e., the λ's of all searches with $\max(|\hat{\tau}_3 - \tau_3|,\ |\hat{\tau}_4 - \tau_4|) < 10^{-4}$ are returned as a list of vectors. If all searches fail, `c(0,0,0,0)` is returned. For each "successful" fit, output is produced to indicate various specifications associated with the fit.

Example 3: Let us assume that the data of Example 1 has already been entered as the vector `D`. Invoking the command `Fit3 <- FitGLDL(D)` will produce the following output.

```
           GLD L-MOMENT BASED FITS

L-Moments:    57.24419 13.87708 0.635216 1.654307

Region [0.00001, 1]X[0.00001, 1]
    Lambdas     :  49.84717 0.008437425 0.1049347 0.1867759
    Error       :  8.742673e-07
    Support     :  ( -68.6724 , 168.3667 )
    Data Min-Max:  11.9    107
    K-S Stat.   :  0.09232132
Region [1, 8]X[1, 8]
    Lambdas     :  64.29084 0.01848038 6.438982 2.778549
    Error       :  1.727927e-07
    Support     :  ( 10.17940 , 118.4023 )
    Data Min-Max:  11.9    107
    K-S Stat.   :  0.06649415
Region [0.00001, 0.1]X[0.00001, 0.1]
    Lambdas     :  49.84706 0.00843754 0.1049360 0.1867799
    Error       :  7.674184e-08
    Support     :  ( -68.6709 , 168.365 )
    Data Min-Max:  11.9    107
    K-S Stat.   :  0.09232156
```

This output indicates that the searches were successful in the regions $0.00001 \leq \lambda_3, \lambda_4 \leq 1$, $1 \leq \lambda_3, \lambda_4 \leq 8$, and $0.00001 \leq \lambda_3, \lambda_4 \leq 0.1$. Note that in this case, searches in the first and third listed regions produce essentially the same fit. The details that are provided for each of the fits are similar to those encountered in Examples 1 and 2.

Again, using the same class intervals used in Examples 1 and 2, we can do a chi-square test for these fits with `ChiSq(D, Fit3[[1]], B)` and `ChiSqGLD(D, Fit3[[2]], B)`.

A.4 Functions for GBD Fits

1. `pdfGBD(Betas, x)`
   `Betas` is a numeric vector $(\beta_1, \beta_2, \beta_3, \beta_4)$ and `x` is a number. The function returns $f(x)$ where f is the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ p.d.f.

2. `cdfGLD(Beta, x)`
   For `Beta`= $(\beta_1, \beta_2, \beta_3, \beta_4)$ and `x` a numeric value, this procedure returns the value of $F(x)$ where $F(x)$ is the c.d.f. of the GBD$(\beta_1, \beta_2, \beta_3, \beta_4)$ distribution.

3. `KSGBD(Data, Beta)`

For the numeric vector, `Data`, this procedure returns the Kolmogorov-Smirnov statistics associated with fitting the GBD(β_1, β_2, β_3, β_4) distribution to `Data`.

4. `ChiSqGBD(Data, Beta, Breaks)`
   This function partitions the numeric vector `Data` into frequency classes determined by the points in the ascending numeric vector `Breaks`. If `Breaks` $= (b_1, b_2, b_3, \ldots, b_k)$ then the classes will be $(-\infty, b_1)$, $[b_1, b_2)$, $[b_2, b_3), \ldots,$ $[b_{k-1}, b_k)$, $[b_k, \infty)$. These frequencies are then compared to the expected frequencies based on the GBD distribution with its parameters specified by numeric vector `Beta`$= (\beta_1, \beta_2, \beta_3, \beta_4)$. The function returns the chi-square statistic and p-value associated with this fit.

5. `plotGBD(Data, B, Breaks)`
   Produces a superimposed plot of two items. The first of these is the p.d.f. of GLD(β_1, β_2, β_3, β_4) $=$GLD(`B`) and the second is a histogram of the numeric vector `Data` with endpoints given by the ascending numeric vector `Breaks`.

6. `ErrGBD(a3, a4, b3, b4)`
   Using $\hat{\alpha}_3 =$ `a3`, $\hat{\alpha}_4 =$ `a4`, $\beta_3 =$ `b3`, $\beta_4 =$ `b4`, this function computes α_3 and α_4 associated with β_3 and β_4 and returns $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$.

7. `RefineSearchGBD(Alphas, b3MinMax, b4MinMax, Grid, Iter)`
   `Alphas` is the vector of α's of some dataset or distribution. A search for β_3 and β_4 with minimal error (i.e., minimal value of $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$) is conducted. `b3MinMax` is a pair specifying the minimal and maximal values in the search of β_3 and `b4MinMax` provides a similar constraint for β_4. Each side of the rectangle designated by `b3MinMax` and `b4MinMax` is divided into `Grid` subintervals and every point in this grid is checked for optimality. When an optimal point is located, a new rectangle is formed by moving in all four directions by one grid interval and this process is repeated `Iter` times. The function returns a numeric vector consisting of β_3, β_4, and $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$ associated with this choice.

8. `FindBetasM(Alphas)`
   `Alphas` is the numeric vector of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Initial values of `b3MinMax` and `b4MinMax` are taken from the tables for fitting generalized beta distributions and `RefineSearchGBD(a3, a4, b3MinMax, b4MinMax, Grid, Iter)` is invoked with `a3=Alphas[3], a4=Alphas[4], Grid`$= 150$, `Iter` $= 4$. If the search for (λ_3, λ_4) fails, `c(0,0,0,0)` is returned; if it is successful, a five-component vector consisting of the four λ's and the error associated with the fit is returned.

9. `FitGBD(Data)`
   This function prints the α_1, α_2, α_3, α_4 of the numeric vector `Data`, the

β_1, β_2, β_3, β_4 computed from α_1, α_2, α_3, α_4, and $\max(|\hat{\alpha}_3 - \alpha_3|,\ |\hat{\alpha}_4 - \alpha_4|)$ associated with the fit. The support of GBD(β_1, β_2, β_3, β_4), the minimum and maximum values of `Data` and the Kolmogorov-Smirnov statistic for this fit are also part of the output. The vector (β_1, β_2, β_3, β_4) is returned (if no fit is found, `c(0,0,0,0)` is returned).

Example 4: We again use the data of Examples 1, 2, and 3 and assume that it has already been entered as the vector `D`. Invoking the command `Fit4 <- FitGBD(D)` will produce the following output.

```
           GBD MOMENT-BASED FIT

Alphas      : 57.24419 576.0741 0.2545719 2.52575
Betas       : -0.1826617 144.3947 2.050227 3.619296
Error       : 6.268119e-11
Support     : ( -0.1826617 , 144.212 )
Data Min-Max:   11.9 107
K-S Stat.   : 0.09611597
```

The interpretation of this output is similar to that of the previous examples. In this case, the vector (β_1, β_2, β_3, β_4) is assigned to `Fit4`, indicating the fit GBD(-0.1826617, 144.3947, 2.050227, 3.619296) to `Data`.

As we have done in previous examples, we partition the data into the classes that were used in Example 1 and perform a chi-square test through `ChiSqGBD(D, Fit4, B)` to get the output

```
Breaks :
 30 50 60 70 85
Observed Frequencies:
 7 7 9 9 6 5
Expected Frequencies:
 5.85771 11.75901 6.53322 5.984339 6.908355 5.95737
Chi-Square = 4.87313 p-Value = 0.02727798
[1] 4.87312999 0.02727798
```

A.5 Functions for the Kappa Distribution

1. `pdfKappa(P, x)`
   `P` is the numeric vector of the parameters of the kappa distribution and `x` is a number. This function returns the value of the kappa p.d.f. at `x`.

2. `cdfKappa(P, x)`
   `P` is the numeric vector of the parameters of the kappa distribution and `x` is a number. This function returns the value of the kappa c.d.f. at `x`.

3. `KSKappa(Data, P)`
   For the data, represented by the numeric vector `Data`, and the parameters of the kappa distribution, represented by `P`, this function returns the Kolmogorov-Smirnov statistics associated by the fit specified by `P` to `Data`.

4. `ChiSqKappa(Data, P, Breaks)`
   This function partitions the numeric vector `Data` into frequency classes determined by the points in the ascending numeric vector `Breaks`. If `Breaks` $= (b_1, b_2, b_3, \ldots, b_k)$ then the classes will be $(-\infty, b_1)$, $[b_1, b_2)$, $[b_2, b_3), \ldots,$ $[b_{k-1}, b_k)$, $[b_k, \infty)$. These frequencies are then compared to the expected frequencies based on the kappa distribution with parameters specified by numeric vector `P`. The function returns the chi-square statistic and p-value associated with this fit.

5. `plotKappa(Data, P, Breaks)`
   Produces a superimposed plot of two items: the first of these is the p.d.f. of the kappa distribution with parameters given by `P` and the second is a histogram of the numeric vector `Data` with endpoints given by the ascending numeric vector `Breaks`.

6. `KappaSupport(P)`
   This function returns, as a vector, the lower and upper end-points of the support of the kappa distribution with parameters specified by `P` (when needed, $\pm 10^{10}$ are returned instead of $\pm\infty$).

A.5.1 Fitting Kappa Distributions through L-Moments

1. `ErrKappaL(LMoms, h, k)`
   Using $(\ell_1, \ell_2, \ell_3, \ell_4)$ specified by the numeric vector `LMoms`, this function computes the τ_3 and τ_4 associated with `h` and `k` and returns $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|)$, where $\hat{\tau}_3$ and $\hat{\tau}_4$ are obtained form `LMoms`.

2. `RefineSearchKappaL(LMoms, hMinMax, kMinMax, Grid, Iter)`
   `LMoms` represents the $(\ell_1, \ell_2, \ell_3, \ell_4)$ of a dataset or a distribution. A search is conducted for the h and k parameters that have minimal error (i.e., minimal value of $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|)$). `hMinMax` is a pair specifying the minimal and maximal values in the search of h and `kMinMax` provides a similar constraint for k. Each side of the rectangle designated by `hMinMax` and `kMinMax` is divided into `Grid` subintervals and every point in this grid is checked for optimality. When an optimal point is located, a new rectangle is formed by moving in all four directions by one grid interval and this process is repeated `Iter` times. The function returns a numeric vector consisting of ξ, α, h, k, and $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|)$ associated with this choice.

3. `FindKappaL(LMoms)`
   By repeated applications of `RefineSearchKappaL`, this function searches

for the kappa distributions that are good fits for the specified L-moments, `LMoms`=$(\ell_1, \ell_2, \ell_3, \ell_4)$, in various regions of (h, k)-space. The parameters of the best of these fits (i.e., those for which $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|)$ is minimal) and the value of $\max(|\hat{\tau}_3 - \tau_3|, |\hat{\tau}_4 - \tau_4|)$ is returned as a vector (if no kappa distribution is found that matches `LMoms` then `c(0,0,0,0)` is returned).

4. `FitKappaL(Data)`
   This function determines a kappa distribution fit to the data specified by the numeric vector `Data`. Additional information related to the fit is also provided as part of the output (see example below).

Example 5: We again use the data that was used in the previous four examples and assume that it has been stored as the numeric vector `D`. When the command `Fit5 <- FitKappaL(D)` is used, we get the output

```
           KAPPA L-MOMENT BASED FITS

L-Moments   :  57.24419 13.87708 0.635216 1.654307
Tau3, Tau4  :  0.04577448 0.1192115
Parameters  : 48.43829 22.43768 -0.05470911 0.1813178
Error       :  4.176345e-07
Support     : ( -1e+10 , 172.1861 )
Data Min-Max: 11.9 107
K-S Stat.   : 0.09121104
```

and the assignment of the parameter values

$$(\xi, \alpha, h, k) = (48.43829, 22.43768, -0.05470911, 0.1813178)$$

to the vector `Fit5`. Note that the output indicates a support of the fitted kappa `( -1e+10 , 172.1861 )`, indicating an actual support of $(-\infty, 172.1861)$.

As in previous examples, we use the same partition on the data to perform a chi-square test through the use of `ChiSqKappa(D, Fit5, Breaks)` to get

```
Breaks :
 30 50 60 70 85
Observed Frequencies:
 7 7 9 9 6 5
Expected Frequencies:
 5.62651 11.69325 6.922399 6.222993 6.754224 5.780623
Chi-Square = 4.271406 p-Value = 0.03875885
[1] 4.27140583 0.03875885
```

indicating a chi-square statistic of 4.271406 with a p-value of 0.03875885.

A.5.2 Fitting Kappa Distributions through Percentiles

1. `ErrKappaP(Rhos, h, k)`
   Using $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4)$ specified by the numeric vector `Rhos`, this function computes the ρ_3 and ρ_4 associated with `h` and `k` and returns the value of $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|)$.

2. `RefineSearchKappaP(Rhos, hMinMax, kMinMax, Grid, Iter)`
   `Rhos` represents the $(\rho_1, \rho_2, \rho_3, \rho_4)$ of a dataset or a distribution. A search for the parameters h and k with minimal error (i.e., minimal value of $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|)$) is conducted. `hMinMax` is a pair specifying the minimal and maximal values in the search of h and `kMinMax` provides a similar constraint for k. Each side of the rectangle designated by `hMinMax` and `kMinMax` is divided into `Grid` subintervals and every point in this grid is checked for optimality. When an optimal point is located, a new rectangle is formed by moving in all four directions by one grid interval and this process is repeated `Iter` times. The function returns a numeric vector consisting of ξ, α, h, k and $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|)$ associated with this choice.

3. `FindKappaP(Rhos)`
   This function uses the tables for fitting kappa distributions through percentiles to determine the `hMinMax` and `kMinMax` that should be used when it invokes `RefineSearchKappaP` on various regions of (h, k)-space. The best of the results i.e., the one with minimal $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|)$ of these searches is returned, along with the value of $\max(|\hat{\rho}_3 - \rho_3|, |\hat{\rho}_4 - \rho_4|)$ for that fit.

4. `FitKappaP(Data)`
   This function determines a kappa distribution fit to the data specified by the numeric vector `Data`. In the event of a successful fit, information related to the fit, similar to that of previous examples, is also provided as part of the output.

Example 6: We again use the data that was used in the previous four examples and assume that it has been stored as the numeric vector `D`. When the command `Fit6 <- FitKappaL(D)` is used, we get the output

```
          KAPPA PERCENTILE-BASED FITS

Rhos    :  56.7 72.24 0.7457709 0.4817276

A Fit could not be obtained
```

indicating that `FitKappaP` is not able to find a fit. When we try to locate $(\hat{\rho}_3, \hat{\rho}_4) = (0.7457709, 0.4817276)$ in the tables for fitting a kappa distribution

through percentiles, we see that $(\hat{\rho}_3, \hat{\rho}_4)$ is located near the edge of the region covered by the percentiles of the kappa distribution. It seems, from the table entries near $(\hat{\rho}_3, \hat{\rho}_4)$, that a search for a solution with $-2 \le h \le -0.0001$ and $-2 \le k \le -0.0001$ may be productive. This can be put to the test with the commands

```
> Rhos <- FindRhos(D)
> RefineSearchKappaP(R, c(-2, -0.0001), c(-2, -0.0001), 50, 4)
```

which produce

```
[1] 57.81904163 15.44681892 -1.18688482 -0.16195815  0.01182395
```

indicating the fit $\kappa(57.81904163, 15.44681892, -1.18688482, -0.16195815)$ with a somewhat large $\max(|\hat{\rho}_3 - \rho_3|,\ |\hat{\rho}_4 - \rho_4|) = 0.01182395$.

A.6 Maple Programs for Johnson System Fits

1. `FitJnsnSL(Alphas)`
   `Alphas` is the list of α_1, α_2, α_3, α_4 and this program returns the list with ξ, λ, δ, γ as its components that represents the S_L fit for the α_1, α_2, α_3, α_4.

2. `FitJnsnSU(Alphas)`
   `Alphas` is the list of α_1, α_2, α_3, α_4 and this program returns the list with ξ, λ, δ, γ as its components that represents the S_U fit for the α_1, α_2, α_3, α_4.

3. `FitJnsnSB(Alphas)`
   `Alphas` is the list of α_1, α_2, α_3, α_4 and this program returns the list with ξ, λ, δ, γ, and `Error` as its components that represents a S_B fit for the α_1, α_2, α_3, α_4. Of the three programs, `JnsnSL, JnsnSU` and `JnsnSB`, the latter is computationally most difficult as it involves the evaluations of integral equations. The presence of `Error` $= \max(|\hat{\alpha}_3 - \alpha_3|, |\hat{\alpha}_4 - \alpha_4|)$ in its output allows the user to evaluate the quality of the fit.

4. `FreqCount(Data, CB)`
   This program takes for its inputs `Data`, a list representing a dataset and `CB`, a list of class boundaries, say $b_1, b_2, \ldots, b_k$. The program then returns the frequencies $f_1, f_2, \ldots, f_{k+1}$ of the dataset entries in the classes $(-\infty, b_1), [b_1, b_2), \ldots, [b_k, \infty)$.

5. `FitJohnson(Data)`
   `Data` is a list representing a dataset. Using the appropriate one of the programs `FitJnsnSL`, `FitJnsnSU`, and `FitJnsnSB`, this program fits a Johnson system distribution to `Data`.

6. `ChiSqJnsnSU(Data, J, CB)` and `ChiSqJnsnSB(Data, J, CB)`
   The input to these two programs is a dataset (as the list `Data`), the four Johnson system fit parameters (`J`), and the class boundaries (as described in `FreqCount` above). These two programs return, for S_U and S_B fits, respectively, the chi-square statistics and p-values for the fit.

7. `KSJnsnSU(Data, J)` and `KSJnsnSB(Data, J)`
   For S_U and S_B fits, respectively, these programs take input of data (the list `Data`) and the Johnson system fit parameters (the list `J`), and returns the Kolmogorov-Smirnov statistic associated with the fit.

8. `FindJnsnSB(Alphas)`
   This program can be used as an alternative to `FitJnsnSB`. It takes `Alphas`, the list of α_1, α_2, α_3, α_4 as its sole imput and returns the ξ, λ, δ, γ of the S_B fit.

A.7 The Maple Code for the Bivariate GLD

1. `PlackettPsi(Data1, Data2)`
   `Data1` and `Data2` are lists representing two datasets. This program returns Plackett's Ψ for `Data1` and `Data2`.

2. `GLD2PDFplot(Psi, L1, L2)`
   `L1` and `L2` are the λ_1, λ_2, λ_3, λ_4 for two datasets and `Psi` is Plackett's Ψ (see the program described above). This program produces a 3-dimensional graph of the GLD–2 fit based on `L1, L2`, and `Psi`.

3. `GLD2RAND(L1, L2, Psi, N)`
   This program will generate a random sample of size `N` from the bivariate distribution determined by the two sets of λ_1, λ_2, λ_3, λ_4 (represented by the two lists `L1` and `L2`) and Plackett's Ψ (`Psi`).

A.8 Content of the Attached CD

All the files on the CD were created in a Microsoft Windows environment and should pose no problems when they are transferred to any Microsoft Windows-based computer. The directory structure of the CD has the directory "Fit" for its root and everything on the CD is located in this directory, either as a file or as a file within a subdirectory of "Fit."

The file "BookData.txt" in "Fit" consists of the data used in Chapters 3 through 5 of this handbook. We have chosen to give a dataset a name such as "DataXYZ" to indicate that the data used in Section X.Y.Z can be accessed in **R** by this name. For example, if in an **R** session you wish to use the data that

was introduced in Section 3.5.2 under the name `D`, you can use the following two **R** commands to accomplish this.

```
> source("c:/Fit/R/BookDataR.txt")
> D <- Data352
```

Programs and other files are located in subdirectories "GLD," "GBD," "Kappa," and "Jnsn."

- **The "GLD" Subdirectory:**
  The **R** programs associated with fitting GLD distributions to data through the use of moments, percentiles, and L-moments are in the file "Code.txt" in this subdirectory. All these programs can be brought into an **R** session through the command
  `> source("c:/Fit/GLD/R/Code.txt")`.
  In addition to the file containing the programs, this subdirectory contains two groups of files:

  Group 1: "GLDM1," "GLDM2," $\cdots$ "GLDM31,"
  Group 2: "GLDPA1," "GLDPA2," $\cdots$ "GLDPA24;"
  "GLDPB1," "GLDPB2," $\cdots$ "GLDPB14;"
  "GLDPC1," "GLDPC2," "GLDPC3;"
  "GLDPD1;"
  "GLDPE1," "GLDPE2," $\cdots$ "GLDPE21."

  The files in Group 1 (these have GLDM designation followed by a number) are tables to assist in the fitting of GLD distributions through moments. The files of the second group are named GLDP, followed by an A, B, C, D or E, which is in turn followed by a number. The "P" in GLDP indicates that all these files contain tabular information associated with percentile-based GLD fits and the letter that follows "P" gives the specific region that is covered.

- **The "GBD" Subdirectory:**
  This subdirectory also contains a file "Code.txt" which has in it all the programs associated with finding GBD fits to data. In addition, this subdirectory has the files "GBD1," "GBD2," $\cdots$ "GBD24," are files that contain tabular information to facilitate the finding of GBD fits.

- **The "Kappa" Subdirectory:**
  In addition to "Code.txt," which is the collection of programs needed for fitting kappa distributions through L-moments and percentiles, this subdirectory has the files "KappaLT1," "KappaLT2," $\cdots$ "KappaLT79" needed for L-moment fits and the files "KappaPT1," "KappaPT2," $\cdots$ "KappaLP40" that are needed for percentile-based fits.

- **The "Jnsn" Subdirectory:**
  This subdirectory contains the Maple programs that were described in Section A.6 in the file "Code.txt." Additionally, the file "JnsnSBTbl.txt" contains the table entries that allow some of the programs to find fits in the S_B region of the Johnson system.

- **The "GLD–2" Subdirectory:**
  The Maple programs associated with fits of a bivariate GLD distribution (GDD–2) are located in this subdirectory.

It was assumed in the above discussion that those who wish to use the program from the CD have located these files, with the same directory structure that appears on the CD, on their C: drive. We suggest that users follow this scheme when copying files from the CD to their own hard drives. Some of the programs in the "Code.txt" files of the three subdirectories mentioned above need to open files that contain tabular information and the directory paths to these table files is embedded within the programs of the "Code.txt" files. If this recommended directory structure is not followed, then the path designations used by these programs must be appropriately adjusted.

In addition to what has been described here, there are other subdirectories that contain additional programs. For example, the subdirectory "GLD/Maple" contains Maple programs for obtaining GLD fits.

A.9 The R Code of the Programs for GLD Fits

```
FindAlphas <- function(Dat) {

    n <- length(Dat)
    a1 <- mean(Dat)
    a2 <- (n-1)*var(Dat)/n
    a3 <- (sum((Dat-a1)^3))/(n*a2^(1.5))
    a4 <- (sum((Dat-a1)^4))/(n*a2^2)
    return(c(a1,a2,a3,a4))
}


ErrGLDM <- function(a3, a4, l3, l4) {

    if (l3 == 0) {return(10000)}
    A <- 1/(1+l3) - 1/(1+l4)
    B <- 1/(1+2*l3) + 1/(1+2*l4) - 2*beta(1+l3, 1+l4)
    C <- 1/(1+3*l3) - 1/(1+3*l4) - 3*beta(1+2*l3, 1+l4) +
+ 3*beta(1+l3, 1+2*l4)
    D <- 1/(1+4*l3)+1/(1+4*l4) + 6*beta(1+2*l3, 1+2*l4) -
+ 4*beta(1+3*l3,1+l4) -4*beta(1+l3,1+3*l4)
```

```
    Alph3 <- (C-3*A*B+2*A^3)/((B-A^2)^(1.5));
    Alph4 <- (D-4*A*C+6*B*A^2-3*A^4)/((B-A^2)^2);

    Err3 <- abs(a3-Alph3); Err4 <- abs(a4-Alph4)
    if (Err3 > Err4) {Err <- Err3} else {Err <- Err4}
    return(Err)
}


RefineSearchGLDM <- function(Alphas, Pt1, Pt2, Grid, Iter) {

  a1 <- Alphas[1];  a2 <- Alphas[2]
  a3 <- Alphas[3];  a4 <- Alphas[4]
  s3 <- Pt1[1]; f3 <- Pt1[2]
  s4 <- Pt2[1]; f4 <- Pt2[2]
  for (k in 1:Iter) {
    Delta3 <- (f3 - s3)/Grid; Delta4 <- (f4 - s4)/Grid
    L3List <- s3 + Delta3*(0:Grid)
    L4List <- s4 + Delta4*(0:Grid)
    BestErr <- 10000
    BL3 <- Pt1[1]; BL4 <- Pt1[2]
    for (i in 1:Grid) {
        for (j in 1:Grid) {
            l3 <- L3List[i]; l4 <- L4List[j]
            NewErr <- ErrGLDM(abs(a3), a4, l3, l4)
            if (NewErr < BestErr) {
                BestErr <- NewErr
                BL3 <- l3; BL4 <- l4
            }
        }
    }
    s3 <- BL3-Delta3; f3 <- BL3+Delta3
    s4 <- BL4-Delta4; f4 <- BL4+Delta4
  }

    L3 <- BL3; L4 <- BL4
    a <- 1/(1 + L3) - 1/(1 + L4)
    b <- 1/(1+2*L3) +1/(1+2*L4) - 2*beta(1 + L3, 1 + L4)
    LL2 <- sqrt(b-a^2)
    if ( (L3 < 0) & (LL2 > 0) ) {LL2 <- -LL2}
    LL1 <- -a/LL2
    if (a3 < 0 ) {
        temp <- L3; L3 <- L4; L4 <- temp
        LL1 <- -LL1
    }
    L1 <- LL1*sqrt(a2) + a1; L2 <- LL2/sqrt(a2)
    return(c(L1, L2, L3, L4, BestErr))
}
```

```
FindLambdasM <- function(Alphas) {

    a1 <- Alphas[1]; a2 <- Alphas[2]; A3 <- Alphas[3]; a4 <- Alphas[4];
    a3 <- abs(A3)

    A3List <- 0.05*(0:61)
    if (a3 > 3.0) {return(c(0,0,0,0))}

    Filenames <- paste(c("c:/Fit/GLD/R/GLDM"), 1:31, sep="")
    j <- 2; iseven <- 1
    while (A3List[j] < a3) {j <- j+1; iseven <- -1*iseven}
    if (iseven > 0) {filename <- Filenames[j/2]}
    else {filenameA <- Filenames[(j-1)/2]
        filenameB <- Filenames[(j+1)/2]
    }
    if (iseven > 0) {
        source(filename)
        Rows <- length(T)/6
        A4List <- T[1+6*(0:(Rows-1))]
        if (a4 < A4List[1] | a4 > A4List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (A4List[i] < a4) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[6*(i-1) + 2], T[6*(i-1) + 3])
        Pt3 <- c(T[6*i + 2], T[6*i + 3])
        A4List <- T[4+6*(0:(Rows-1))]
        if (a4 < A4List[1] | a4 > A4List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (A4List[i] < a4) {i <- i+1}
        i <- i-1
        Pt2 <- c(T[6*(i-1) + 5], T[6*(i-1) + 6])
        Pt4 <- c(T[6*i + 5], T[6*i + 6])
        if ( (Pt2[1] == 0) | (Pt4[1] == 0) ) {return(c(0,0,0,0))}
    }
    else {
        source(filenameA)
        Rows <- length(T)/6
        A4List <- T[4+6*(0:(Rows-1))]
        if (a4 < A4List[1] | a4 > A4List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (A4List[i] < a4) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[6*(i-1) + 5], T[6*(i-1) + 6])
        Pt3 <- c(T[6*i + 5], T[6*i + 6])
        if ( (Pt1[1] == 0) | (Pt3[1] == 0) ) {return(c(0,0,0,0))}
```

```
        source(filenameB)
        Rows <- length(T)/6
        A4List <- T[1+6*(0:(Rows-1))]
        if (a4 < A4List[1] | a4 > A4List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (A4List[i] < a4) {i <- i+1}
        i <- i-1
        Pt2 <- c(T[6*(i-1) + 2], T[6*(i-1) + 3])
        Pt4 <- c(T[6*i + 2], T[6*i + 3])
    }
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NL <- RefineSearchGLDM(Alphas, PT1, PT2, 100, 4)
    return(NL)

}


SupportGLD <- function(L) {
    L1 <- L[1]; L2 <- L[2]; L3 <- L[3]; L4 <- L[4]
    if ( (L3 == 0) & (L4 == 0) ) {return(c(0,0))}
    if ( (L3 < -1) & (L4 > 1) ) {a <- -10^10; b <- L1+1/L2}
    if ( (L3 > 1) & (L4 < -1) ) {a <- L1-1/L2; b <- 10^10}
    if ( (-1 < L3) & (L3 < 0) & (L4 > 1) ) {a <- -10^10; b <- L1+1/L2}
    if ( (L3 > 1) & (-1 < L4) & (L4 < 0) ) {a <- L1-1/L2; b <- 10^10}
    if (L3 > 0 & L4 > 0) {a <- L1 - 1/L2; b <- L1 + 1/L2}
    if (L3 > 0 & L4 == 0) {a <- L1 - 1/L2; b <- L1}
    if (L3 == 0 & L4 > 0) {a <- L1; b <- L1 + 1/L2}
    if (L3 < 0 & L4 < 0) {a <- -10^10; b <- 10^10}
    if (L3 < 0 & L4 == 0) {a <- -10^10; b <- L1}
    if (L3 == 0 & L4 < 0) {a <- L1; b <- 10^10}
    return(c(a,b))
}

PrintSupportGLD <- function(L) {
    S <- SupportGLD(L)
    if ( (S[1] < -10^9) & (S[2] > 10^9) ) {
        cat("(-inf, inf)", "\n")
    }
    if ( (S[1] < -10^9) & (S[2] < 10^9) ) {
        cat("(-inf,", S[2],")", "\n")
    }
    if ( (S[1] > -10^9) & (S[2] > 10^9) ) {
        cat("(",S[1],", inf)", "\n")
    }
```

```
    if ( (S[1] > -10^9) & (S[2] < 10^9) ) {
        cat("(",S[1],",",S[2],")", "\n")
    }
}


FitGLDM <- function(dat) {
    cat("\n", "          GLD MOMENT-BASED FIT","\n\n")
    A <- FindAlphas(dat)
    cat("Alphas      : ",A,"\n")
    L <- FindLambdasM(A)
    cat("Lambdas     : ",c(L[1],L[2],L[3],L[4]),"\n")
    cat("Error       : ",L[5],"\n")
    cat("Support     :  "); PrintSupportGLD(L)
    cat("Data Min-Max: ", min(dat),"  ", max(dat),"\n")
    KS <- KSGLD(dat,c(L[1],L[2],L[3],L[4]))
    cat("K-S Stat.   : ", KS, "\n")
    return(c(L[1],L[2],L[3],L[4]))
}


plotGLD <- function(dat, L, breaks) {
    Y <- c(0.0001, 0.001*(1:9), 0.01*(1:99), .99+0.001*(1:9), .9999)
    x <- L[1] + (Y^L[3] - (1-Y)^L[4])/L[2]
    y <- L[2]/(L[3]*Y^(L[3]-1) + L[4]*(1-Y)^(L[4]-1))
    plot(x,y, type="l")
    hist(dat, breaks, prob=T, main="",add=TRUE)
}


pdfGLD <- function(L, p) {
    x <- L[1] + (p^L[3] - (1-p)^L[4])/L[2]
    y <- L[2]/(L[3]*p^(L[3]-1) + L[4]*(1-p)^(L[4]-1))
    return(c(x,y))
}


cdfGLD <- function(L, t) {
    L1 <- L[1]; L2 <- L[2]; L3 <- L[3]; L4 <- L[4]
    S <- SupportGLD(L);  a <- S[1];  b <- S[2]

    if (t <= a) {return(0)}
    if (t >= b) {return(1)}
    Left <- 0; Right <- 1; Err <- 10000
    LeftVal <- L1 - t + (Left^L3 - (1-Left)^L4)/L2
    RightVal <- L1 - t + (Right^L3 - (1-Right)^L4)/L2
    if (LeftVal < -10^(10)) {LeftVal <- -10^(10)}
    if (RightVal > 10^(10)) {RightVal <- 10^(10)}
```

```
    while (Err > 0.00001) {
        Mid <- (Left + Right)/2
        MidVal <- L1 - t + (Mid^L3 - (1-Mid)^L4)/L2
        if (MidVal*LeftVal < 0) {
            Right <- Mid; RightVal <- MidVal
        }
        else {
            Left <- Mid; LeftVal <- MidVal
        }
        Err <- abs(MidVal)
    }
    return(Mid)
}


cdfVectGLD <- function(L, Br) {
    k <- length(Br)
    Pr <- rep(0,(k+2))
    Pr[k+2] <- 1
    for (i in 2:(k+1)) {
        x <- Br[i-1]
        Pr[i] <- cdfGLD(L, x)
    }
    return(Pr)
}


EmpCDF <- function(dat, x) {
    Dat <- sort(dat); N <- length(Dat)
    m <- min(Dat); M <- max(Dat)
    if (x < m )  { Fx <- 0; return(Fx) }
    if (x >= M ) { Fx <- 1; return(Fx) }
    count <- 0; i <- 1
    while (Dat[i] <= x) { count <- count +1; i <- i+1}
    Fx <- count/N
    return(Fx)
}


empCDF <- function(dat, x) {
    Dat <- sort(dat); N <- length(Dat)
    m <- min(Dat); M <- max(Dat)
    if (x <= m ) { Fx <- 0; return(Fx) }
    if (x > M )  { Fx <- 1; return(Fx) }
    count <- 0; i <- 1
    while (Dat[i] < x) { count <- count +1; i <- i+1}
    Fx <- count/N
    return(Fx)
```

```
}


KSGLD <- function(dat, L) {
    N <- length(dat); Dat <- sort(dat)
    KS <- 0
    for (i in 1:N) {
        GLDF <- cdfGLD(L,Dat[i])
        A <- EmpCDF(Dat, Dat[i])
        B <- empCDF(Dat, Dat[i])
        D1 <- abs(GLDF - A);
        D2 <- abs(GLDF - B)
        D <- max(D1, D2)
        if (KS < D) {KS <- D}
    }
    return(KS)
}


ChiSqGLD <- function(Dat, L, Br) {
    Probs <- cdfVectGLD(L, Br)
    n <- length(Dat); k <- length(Br)
    EF <- rep(0,(k+1))
    for (i in 1:(k+1)) {
        EF[i] <- n*(Probs[i+1] - Probs[i])
    }
    OF <- Freq(Dat, Br)
    CS <- sum( (EF-OF)^2/EF )
    pval <- 1-pchisq(CS, k-4)
cat("Breaks : \n", Br, "\n")
cat("Observed Frequencies: \n", OF, "\n")
cat("Expected Frequencies: \n", EF, "\n")
cat("Chi-Square =", CS, "p-Value =", pval, "\n")
    return(c(CS,pval))
}


Freq <- function(Dat, Br) {
    n <- length(Dat); k <- length(Br)
    Count <- rep(0, (k+1))
    for (i in 1:n) {
        if (Dat[i] >= Br[k]) {
            Count[k+1] <- Count[k+1] + 1
        }
        Done <- FALSE
        for (j in 1:k) {
            if ( (Dat[i] < Br[j]) & (Done == FALSE) ) {
                Count[j] <- Count[j] + 1
```

```
                Done <- TRUE
            }
        }
    }
    return(Count)
}


Percentile <- function(Dat, P) {
    if ( (P<0) | (P>1) ) {
        stop("Second argument in Percentile must be between 0 and 1")
    }
    n <- length(Dat)
    if ( (P < 1/(n+1)) | (P > n/(n+1)) ) {
        stop("Percentile does not exist")
    }
    SortDat <- sort(Dat); F <- (n+1)*P
    r <- floor(F);
    ab <- F-r;
    Perc <- SortDat[r]+ ab*(SortDat[r+1]-SortDat[r])
    return(Perc)
}


FindRhos <- function (Dat) {

    P10 <- Percentile(Dat,0.1)
    P25 <- Percentile(Dat,0.25)
    P50 <- Percentile(Dat,0.5)
    P75 <- Percentile(Dat,0.75)
    P90 <- Percentile(Dat,0.9)
    Rho2 <- P90-P10
    Rho3 <- (P50-P10)/(P90-P50)
    Rho4 <- (P75-P25)/Rho2
    return(c(P50,Rho2,Rho3,Rho4))
}


ErrGLDP <- function(r3, r4, l3, l4) {

    if (l3 == 0) {return(10000)}
    Rho3 <- (0.9^l4 - 0.1^l3 + 0.5^l3 - 0.5^l4)/
+ (0.9^l3 - 0.1^l4 + 0.5^l4 - 0.5^l3)
    Rho4 <- (0.75^l3 - 0.25^l4 + 0.75^l4 - 0.25^l3)/
+ (0.9^l3 - 0.1^l4 + 0.9^l4 - 0.1^l3)
    Err3 <- abs(r3-Rho3); Err4 <- abs(r4-Rho4)
    if (Err3 > Err4) {Err <- Err3} else {Err <- Err4}
    return(Err)
```

```
}


RefineSearchGLDP <- function(Rhos, Pt1, Pt2, Grid, Iter) {

  r1 <- Rhos[1];  r2 <- Rhos[2];  R3 <- Rhos[3];  r4 <- Rhos[4]
  if (R3 > 1) { r3 <- 1/R3 } else {r3 <- R3}
  s3 <- Pt1[1]; f3 <- Pt1[2]
  s4 <- Pt2[1]; f4 <- Pt2[2]
  for (k in 1:Iter) {
    Delta3 <- (f3 - s3)/Grid; Delta4 <- (f4 - s4)/Grid
    L3List <- s3 + Delta3*(0:Grid)
    L4List <- s4 + Delta4*(0:Grid)
    BestErr <- 10000
    BL3 <- Pt1[1]; BL4 <- Pt1[2]
    for (i in 1:Grid) {
        for (j in 1:Grid) {
            l3 <- L3List[i]; l4 <- L4List[j]
            NewErr <- ErrGLDP(r3, r4, l3, l4)
            if (NewErr < BestErr) {
                BestErr <- NewErr
                BL3 <- l3; BL4 <- l4
            }
        }
    }
    s3 <- BL3-Delta3; f3 <- BL3+Delta3
    s4 <- BL4-Delta4; f4 <- BL4+Delta4
  }
    if (R3 > 1) {
        l3 <- BL4; l4 <- BL3
    }
    else {
        l3 <- BL3; l4 <- BL4
    }
    l2 <- (.9^l3 - .1^l4 + .9^l4 - .1^l3)/r2
    l1 <- r1 - (.5^l3 -.5^l4)/l2
  return(c(l1,l2,l3,l4, BestErr))
}


FindLambdasPA <- function(Rhos) {

    r1 <- Rhos[1]; r2 <- Rhos[2]; R3 <- Rhos[3]; r4 <- Rhos[4];
    if (R3 > 1) {r3 <- 1/R3} else {r3 <- R3}

    A1<-c(0.005+0.001*(0:7), 0.0135+.0015*(0:7), 0.026+.002*(0:7))
    A2<-c(.0425+.0025*(0:3), .053+.003*(0:3), .0655+.0035*(0:3),
+ .08+.004*(0:3))
```

```
    A3<-c(.0965+.0045*(0:3), .115+.005*(0:3), .136+.006*(0:3),
+ .161+.007*(0:3))
    A4<-c(.19+.008*(0:3), .223+.009*(0:3), .26+.01*(0:3), .301+.011*(0:3))
    R4List <- c(A1,A2,A3,A4, .346+.012*(0:23))
    if ( (r4 < 0.005) | (r4 > 0.622) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/GLD/R/GLDPA"), 1:24, sep="")
    j <- 1; jmod4 <- 0; FileNo <- 1
    while (R4List[j] < r4) {
        j <- j+1; jmod4 <- jmod4+1
        if (jmod4 == 5) {
            jmod4 <- 1; FileNo <- FileNo+1
        }
    }

    RL <- 9 # RL <-> Row Length
    if (jmod4 < 4) {filename <- Filenames[FileNo]}
    else {filenameA <- Filenames[FileNo]
        filenameB <- Filenames[FileNo+1]
    }
     if (jmod4 < 4) {
        source(filename)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1)+2+2*(jmod4-1)], T[RL*(i-1) + 3 + 2*(jmod4-1)])
        Pt3 <- c(T[RL*i + 2 + 2*(jmod4-1)], T[RL*i + 3 + 2*(jmod4-1)])
        Pt2 <- c(T[RL*(i-1) + 2 + 2*jmod4], T[RL*(i-1) + 3 + 2*jmod4])
        Pt4 <- c(T[RL*i + 2 + 2*jmod4], T[RL*i + 3 + 2*jmod4])
    }
    else {
        source(filenameA)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1) + 8], T[RL*(i-1) + 9])
        Pt3 <- c(T[RL*i + 8], T[RL*i + 9])
        source(filenameB)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
```

```
        i <- i-1
        Pt2 <- c(T[RL*(i-1) + 2], T[RL*(i-1) + 3])
        Pt4 <- c(T[RL*i + 2], T[RL*i + 3])
    }
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NB <- RefineSearchGLDP(Rhos, PT1, PT2, 100,5)
    return(c(NB[1],NB[2],NB[3],NB[4],NB[5]))
}


FindLambdasPB <- function(Rhos) {

    r1 <- Rhos[1]; r2 <- Rhos[2]; R3 <- Rhos[3]; r4 <- Rhos[4];
    if (R3 > 1) {r3 <- 1/R3} else {r3 <- R3}

    A1<-c(0.0965+0.004*(0:3), 0.115+.005*(0:3), 0.136+.006*(0:3))
    A2<-c(.161+.007*(0:3), .19+.008*(0:3), .223+.009*(0:3), .26+.01*(0:3))
    A3<-c(.301+.011*(0:3), .346+.012*(0:23))
    R4List <- c(A1,A2,A3)
    if ( (r4 < 0.0965) | (r4 > 0.622) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/GLD/R/GLDPB"), 1:14, sep="")
    j <- 1; jmod4 <- 0; FileNo <- 1
    while (R4List[j] < r4) {
        j <- j+1; jmod4 <- jmod4+1
        if (jmod4 == 5) {
            jmod4 <- 1; FileNo <- FileNo+1
        }
    }

    RL <- 9 # RL <-> Row Length
    if (jmod4 < 4) {filename <- Filenames[FileNo]}
    else {filenameA <- Filenames[FileNo]
        filenameB <- Filenames[FileNo+1]
    }
     if (jmod4 < 4) {
        source(filename)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1)+2+2*(jmod4-1)], T[RL*(i-1) + 3 + 2*(jmod4-1)])
        Pt3 <- c(T[RL*i + 2 + 2*(jmod4-1)], T[RL*i + 3 + 2*(jmod4-1)])
```

```
        Pt2 <- c(T[RL*(i-1) + 2 + 2*jmod4], T[RL*(i-1) + 3 + 2*jmod4])
        Pt4 <- c(T[RL*i + 2 + 2*jmod4], T[RL*i + 3 + 2*jmod4])
    }
    else {
        source(filenameA)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1) + 8], T[RL*(i-1) + 9])
        Pt3 <- c(T[RL*i + 8], T[RL*i + 9])
        source(filenameB)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt2 <- c(T[RL*(i-1) + 2], T[RL*(i-1) + 3])
        Pt4 <- c(T[RL*i + 2], T[RL*i + 3])
    }
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NB <- RefineSearchGLDP(Rhos, PT1, PT2, 100,5)
    return(c(NB[1],NB[2],NB[3],NB[4],NB[5]))
}


FindLambdasPC <- function(Rhos) {

    r1 <- Rhos[1]; r2 <- Rhos[2]; R3 <- Rhos[3]; r4 <- Rhos[4];
    if (R3 > 1) {r3 <- 1/R3} else {r3 <- R3}

    R4List <- .502 + .012*(0:11)
    if ( (r4 < 0.502) | (r4 > 0.634) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/GLD/R/GLDPC"), 1:3, sep="")
    j <- 1; jmod4 <- 0; FileNo <- 1
    while (R4List[j] < r4) {
        j <- j+1; jmod4 <- jmod4+1
        if (jmod4 == 5) {
            jmod4 <- 1; FileNo <- FileNo+1
        }
    }
```

```
    RL <- 9 # RL <-> Row Length
    if (jmod4 < 4) {filename <- Filenames[FileNo]}
    else {filenameA <- Filenames[FileNo]
        filenameB <- Filenames[FileNo+1]
    }
     if (jmod4 < 4) {
        source(filename)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1)+2+2*(jmod4-1)], T[RL*(i-1) + 3 + 2*(jmod4-1)])
        Pt3 <- c(T[RL*i + 2 + 2*(jmod4-1)], T[RL*i + 3 + 2*(jmod4-1)])
        Pt2 <- c(T[RL*(i-1) + 2 + 2*jmod4], T[RL*(i-1) + 3 + 2*jmod4])
        Pt4 <- c(T[RL*i + 2 + 2*jmod4], T[RL*i + 3 + 2*jmod4])
    }
    else {
        source(filenameA)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1) + 8], T[RL*(i-1) + 9])
        Pt3 <- c(T[RL*i + 8], T[RL*i + 9])
        source(filenameB)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt2 <- c(T[RL*(i-1) + 2], T[RL*(i-1) + 3])
        Pt4 <- c(T[RL*i + 2], T[RL*i + 3])
    }
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NB <- RefineSearchGLDP(Rhos, PT1, PT2, 200,3)
    return(c(NB[1],NB[2],NB[3],NB[4],NB[5]))
}
```

```
FindLambdasPD <- function(Rhos) {

    r1 <- Rhos[1]; r2 <- Rhos[2]; R3 <- Rhos[3]; r4 <- Rhos[4];
    if (R3 > 1) {r3 <- 1/R3} else {r3 <- R3}

    R4List <- .586 + 0.012*(0:3)
    if ( (r4 < .586) | (r4 > 0.622) ) {return(c(0,0,0,0))}
    filename <- "c:/texfiles/R/GLDPD1"
    j <- 1; jmod4 <- 0
    while (R4List[j] < r4) {
        j <- j+1; jmod4 <- jmod4+1
    }

    RL <- 9 # RL <-> Row Length
    source(filename)
    Rows <- length(T)/RL
    R3List <- T[1+RL*(0:(Rows-1))]
    if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
    i <- 1
    while (R3List[i] < r3) {i <- i+1}
    i <- i-1
    Pt1 <- c(T[RL*(i-1)+2+2*(jmod4-1)], T[RL*(i-1) + 3 + 2*(jmod4-1)])
    Pt3 <- c(T[RL*i + 2 + 2*(jmod4-1)], T[RL*i + 3 + 2*(jmod4-1)])
    Pt2 <- c(T[RL*(i-1) + 2 + 2*jmod4], T[RL*(i-1) + 3 + 2*jmod4])
    Pt4 <- c(T[RL*i + 2 + 2*jmod4], T[RL*i + 3 + 2*jmod4])
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NB <- RefineSearchGLDP(Rhos, PT1, PT2, 100,5)
    return(c(NB[1],NB[2],NB[3],NB[4],NB[5]))
}


FindLambdasPE <- function(Rhos) {

    r1 <- Rhos[1]; r2 <- Rhos[2]; R3 <- Rhos[3]; r4 <- Rhos[4];
    if (R3 > 1) {r3 <- 1/R3} else {r3 <- R3}

    A1 <- c(.005 + .001*(0:7), .0135+.0015*(0:7), .026 + .002*(0:7),
+ .0425+.0025*(0:3))
    A2 <- c(.053 + .003*(0:3), .0655+.0035*(0:3), .080 + .004*(0:3),
+ .0965+.0045*(0:3))
    A3 <- c(.115 + .005*(0:3), .136 + .006*(0:3), .161 + .007*(0:3),
+ .190 + .008*(0:3))
    A4 <- c(.223 + .009*(0:3), .260 + .010*(0:3), .301 + .011*(0:3),
```

```
+ .346 + .012*(0:11))
    R4List <- c(A1,A2,A3,A4, .486, .491, .496, .499)
    if ( (r4 < 0.005) | (r4 > 0.499) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/GLD/R/GLDPE"), 1:22, sep="")
    j <- 1; jmod4 <- 0; FileNo <- 1
    while (R4List[j] < r4) {
        j <- j+1; jmod4 <- jmod4+1
        if (jmod4 == 5) {
            jmod4 <- 1; FileNo <- FileNo+1
        }
    }

    RL <- 9 # RL <-> Row Length
    if (jmod4 < 4) {filename <- Filenames[FileNo]}
    else {filenameA <- Filenames[FileNo]
        filenameB <- Filenames[FileNo+1]
    }
     if (jmod4 < 4) {
        source(filename)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1)+2+2*(jmod4-1)], T[RL*(i-1) + 3 + 2*(jmod4-1)])
        Pt3 <- c(T[RL*i + 2 + 2*(jmod4-1)], T[RL*i + 3 + 2*(jmod4-1)])
        Pt2 <- c(T[RL*(i-1) + 2 + 2*jmod4], T[RL*(i-1) + 3 + 2*jmod4])
        Pt4 <- c(T[RL*i + 2 + 2*jmod4], T[RL*i + 3 + 2*jmod4])
    }
    else {
        source(filenameA)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt1 <- c(T[RL*(i-1) + 8], T[RL*(i-1) + 9])
        Pt3 <- c(T[RL*i + 8], T[RL*i + 9])
        source(filenameB)
        Rows <- length(T)/RL
        R3List <- T[1+RL*(0:(Rows-1))]
        if (r3 < R3List[1] | r3 > R3List[Rows]) {return(c(0,0,0,0))}
        i <- 1
        while (R3List[i] < r3) {i <- i+1}
        i <- i-1
        Pt2 <- c(T[RL*(i-1) + 2], T[RL*(i-1) + 3])
```

```
        Pt4 <- c(T[RL*i + 2], T[RL*i + 3])
    }
    if (Pt1[1]*Pt2[1]*Pt3[1]*Pt4[1] == 0) {return(c(0,0,0,0))}
    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    NB <- RefineSearchGLDP(Rhos, PT1, PT2, 100,5)
    return(c(NB[1],NB[2],NB[3],NB[4],NB[5]))
}


FindLambdasP <- function(Rhos) {
    i <- 1; L <- list()
    X <- FindLambdasPA(Rhos)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- FindLambdasPB(Rhos)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- FindLambdasPC(Rhos)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- FindLambdasPD(Rhos)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- FindLambdasPE(Rhos)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- RefineSearchGLDP(Rhos,c(-6,-0.000001),c(-6,-0.000001),50,4)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- RefineSearchGLDP(Rhos,c(-0.001,-0.00000001),
+ c(-0.001,-0.00000001),50,4)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- RefineSearchGLDP(Rhos,c(-0.001,-0.00000001),c(-1,-0.1),50,4)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- RefineSearchGLDP(Rhos,c(-1,-0.1),c(-0.001,-0.00000001),50,4)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X; i <- i+1}
    X <- RefineSearchGLDP(Rhos,c(0.000000001,0.04),
+ c(0.000000001,0.04),50,4)
    if ((X[2] != 0) & (X[5] < 0.0001)) {L[[i]] <- X }
    cat("\n\n"); return(L)
}


FitGLDP <- function(dat) {
    cat("\n", "          GLD PERCENTILE-BASED FITS","\n\n")
    R <- FindRhos(dat)
    cat("Rhos         : ",R)
    ListCount <- 0; L <- list(); F <- list()
    F <- FindLambdasP(R)
    NoFit <- length(F)
    for (k in 1:NoFit) {
```

```
        X <- F[[k]]
        if (X[2] != 0) {
            S <- SupportGLD(X)
            cat("Lambdas      : ",X[1:4],"\n")
            cat("Error        : ", X[5],"\n")
            cat("Support      :"); PrintSupportGLD(X)
            cat("Data Min-Max :", min(dat), " ", max(dat),"\n")
            KS <- KSGLD(dat, X)
            cat("K-S Stat.    :",KS, "\n\n")
            ListCount <- ListCount +1
            L[[ListCount]] <- X[1:4]
    }
  }
    return(L)
}


FindLMoms <- function(Data) {

    X <- sort(Data); n <- length(X)
    a1 <- 0; a2 <- 0; a3 <- 0;
    b0 <- mean(X)
    for (k in 2:n) {
        a1 <- a1 + (k-1)*X[k]
        a2 <- a2 + (k-1)*(k-2)*X[k]
        a3 <- a3 + (k-1)*(k-2)*(k-3)*X[k]
    }
    b1 <- a1/(n*(n-1))
    b2 <- a2/(n*(n-1)*(n-2))
    b3 <- a3/(n*(n-1)*(n-2)*(n-3))

    return(c(b0,-b0+2*b1,b0-6*b1+6*b2,-b0+12*b1-30*b2+20*b3))
}


ErrGLDL <- function(LMOM, l3, l4) {
    N3a <- l3^2*l4^3 + 17*l3^2*l4 + 6*l3^2 -l3*l4^3 - 17*l3*l4^2
    N3b <- -6*l3 - l3^3*l4^2 - 6*l4^2 + l3^3*l4 + 6*l4
    N3 <- N3a + N3b
    D3a <- l3*l4^2 + 6*l3*l4 + 2*l3 +l3^2*l4 + 2*l4
    D3b <- (l4+3)*(l3+3)
    D3 <- D3a*D3b

    N4a <- 48*l3 + 48*l4 +200*l3*l4 - 80*l3*l4^2 - 72*l4^2
    N4b <- 70*l3*l4^3+5*l3^2*l4^3 - 210*l3^2*l4^2 - 80*l3^2*l4 - 72*l3^2
    N4c <- 70*l3^3*l4 + 5*l3^3*l4^2 + 24*l4^3 + 24*l3^3 - 3*l3^2*l4^4
    N4d <- 2*l3*l4^4 + l3^3*l4^4 + 20*l3^3*l4^3 -
+ 3*l3^4*l4^2 + 2*l3^4*l4 + l3^4*l4^3
```

```
    N4 <- N4a + N4b + N4c + N4d
    D4 <- D3*(l3+4)*(l4+4)

    Err3 <- abs(N3/D3 - LMOM[3]/LMOM[2])
    Err4 <- abs(N4/D4 - LMOM[4]/LMOM[2])
    Err <- max(c(Err3,Err4))
    return(Err)
}


RefineSearchGLDL <- function(LMOM, L3MinMax, L4MinMax, Grid, Iter) {

    a <- L3MinMax[1]; b <- L3MinMax[2]
    c <- L4MinMax[1]; d <- L4MinMax[2]
    Err <- 10000; iter <-1
    while ( (Err > 0.000001) & (iter < Iter+1) ) {
        Delta3 <- (b-a)/Grid; Delta4 <- (d-c)/Grid
        for (i in (0:Grid)) {
            for (j in (0:Grid)) {
                l3 <- a + i*Delta3
                l4 <- c + j*Delta4
                NewErr <- ErrGLDL(LMOM, l3, l4)
                if (NewErr < Err) {
                    Err <- NewErr
                    L3 <- l3; L4 <- l4
                }
            }
        }
        iter <- iter + 1
        a <- L3 - 3*Delta3; b <- L3 + 3*Delta3
        c <- L4 - 3*Delta4; d <- L4 + 3*Delta4
    }
    A <- L3/((L3+1)*(L3+2)); B <- L4/((L4+1)*(L4+2))
    L2 <- (1/LMOM[2])*(A+B)
    C <- L2*(L3+1)
    D <- L2*(L4+1)
    L1 <- LMOM[1] - 1/C + 1/D
    return(c(L1,L2,L3,L4,Err))
}


FindLambdasL <- function(LMOM) {
    ListCount <- 0;  L <- list()
    LA <- RefineSearchGLDL(LMOM, c(0.00001,1), c(0.00001,1), 50, 8);
    LB <- RefineSearchGLDL(LMOM, c(1,12), c(0.00001,0.25), 100, 4);
    LC <- RefineSearchGLDL(LMOM, c(1,12), c(0.25,1), 100, 4);
    LD <- RefineSearchGLDL(LMOM, c(1,12), c(1,12), 100, 4);
    LE <- RefineSearchGLDL(LMOM, c(0.00001,0.25), c(1,12), 100, 4);
```

```
    LF <- RefineSearchGLDL(LMOM, c(0.25,1), c(1,8), 50, 8);
    LG <- RefineSearchGLDL(LMOM, c(-0.999,-0.00001),
+ c(-0.999,-0.00001), 50, 8);
    LH <- RefineSearchGLDL(LMOM, c(0.00001,0.1), c(0.00001,0.1), 50, 8);
    if ( (LA[5] < 0.001) & (LA[3]*LA[4] >0) ) { cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LA }
    if ( (LB[5] < 0.001) & (LB[3]*LB[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LB }
    if ( (LC[5] < 0.001) & (LC[3]*LC[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LC }
    if ( (LD[5] < 0.001) & (LD[3]*LD[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LD }
    if ( (LE[5] < 0.001) & (LE[3]*LE[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LE }
    if ( (LF[5] < 0.001) & (LF[3]*LF[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LH }
    if ( (LG[5] < 0.001) & (LG[3]*LG[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LG }
    if ( (LH[5] < 0.001) & (LH[3]*LH[4] >0) ) { #cat(ListCount,"\n")
        ListCount <- ListCount + 1; L[[ListCount]] <- LH }
    if (length(L) > 0) {cat(ListCount,"\n"); return(L)} else
        {return(c(0,0,0,0))}
}


FitGLDL <- function (Data) {
    DataL <- FindLMoms(Data)
    ListCount <- 0; L <- list()
    cat("\n", "          GLD L-MOMENT BASED FITS","\n\n")
    cat("L-Moments :   ",DataL,"\n")
    cat("Tau3, Tau4:   ",DataL[3]/DataL[2], DataL[4]/DataL[2],"\n\n")
    LA <- RefineSearchGLDL(DataL, c(0.00001,1), c(0.00001,1), 50, 6)
    if ( (LA[5] < 0.0001) & (LA[3]*LA[4] >0) )  {
        cat("Region [0.00001, 1]X[0.00001, 1]","\n")
        cat("    Lambdas     : ",LA[1:4],"\n")
        cat("    Error       : ", LA[5],"\n")
        cat("    Support     :  "); PrintSupportGLD(LA)
        cat("    Data Min-Max: ", min(Data),"  ", max(Data),"\n")
        KS <- KSGLD(Data,LA)
        cat("    K-S Stat.   : ",KS,"\n")
        ListCount <- ListCount +1
        L[[ListCount]] <- LA[1:4]
    }

    LB <- RefineSearchGLDL(DataL, c(1,8), c(0.00001,0.25), 50, 6)
    if ( (LB[5] < 0.0001) & (LB[3]*LB[4] >0) )  {
        cat("Region [1, 8]X[0.00001, 0.25]","\n")
        cat("    Lambdas     : ",LB[1:4],"\n")
```

```
    cat("    Error       : ", LB[5],"\n")
    cat("    Support     :  "); PrintSupportGLD(LB)
    cat("    Data Min-Max: ", min(Data), "  ",max(Data),"\n")
    KS <- KSGLD(Data,LB)
    cat("    K-S Stat.   : ",KS,"\n")
    ListCount <- ListCount +1
    L[[ListCount]] <- LB[1:4]
}

LC <- RefineSearchGLDL(DataL, c(1,8), c(0.25, 1), 50, 6)
if ( (LC[5] < 0.0001) & (LC[3]*LC[4] >0) )  {
    cat("Region [1, 8]X[0.25, 1]","\n")
    cat("    Lambdas     : ",LC[1:4],"\n")
    cat("    Error       : ", LC[5],"\n")
    cat("    Support     :  "); PrintSupportGLD(LC)
    cat("    Data Min-Max: ", min(Data), "  ",max(Data),"\n")
    KS <- KSGLD(Data,LC)
    cat("    K-S Stat.   : ",KS,"\n")
    ListCount <- ListCount +1
    L[[ListCount]] <- LC[1:4]
}

LD <- RefineSearchGLDL(DataL, c(1,8), c(1,8), 50, 8)
if ( (LD[5] < 0.0001) & (LD[3]*LD[4] >0) )  {
    cat("Region [1, 8]X[1, 8]","\n")
    cat("    Lambdas     : ",LD[1:4],"\n")
    cat("    Error       : ", LD[5],"\n")
    cat("    Support     :  "); PrintSupportGLD(LD)
    cat("    Data Min-Max: ", min(Data),"  ", max(Data),"\n")
    KS <- KSGLD(Data,LD)
    cat("    K-S Stat.   : ",KS,"\n")
    ListCount <- ListCount +1
    L[[ListCount]] <- LD[1:4]
}

LE <- RefineSearchGLDL(DataL, c(0.00001,0.25), c(1,8), 50, 6)
if ( (LE[5] < 0.0001) & (LE[3]*LE[4] >0) )  {
    cat("Region [0.0001, 0.25]X[1, 8]","\n")
    cat("    Lambdas     : ",LE[1:4],"\n")
    cat("    Error       : ", LE[5],"\n")
    cat("    Support     :  "); PrintSupportGLD(LE)
    cat("    Data Min-Max: ", min(Data),"  ", max(Data),"\n")
    KS <- KSGLD(Data,LE)
    cat("    K-S Stat.   : ",KS,"\n")
    ListCount <- ListCount +1
    L[[ListCount]] <- LE[1:4]
}
```

```
    LF <- RefineSearchGLDL(DataL, c(0.25, 1), c(1,8), 50, 6)
    if ( (LF[5] < 0.0001) & (LF[3]*LF[4] >0) )  {
        cat("Region [0.25, 1]X[1, 8]","\n")
        cat("    Lambdas     : ",LF[1:4],"\n")
        cat("    Error       : ", LF[5],"\n")
        cat("    Support     :  "); PrintSupportGLD(LF)
        cat("    Data Min-Max: ", min(Data),"  ", max(Data),"\n")
        KS <- KSGLD(Data,LF)
        cat("    K-S Stat.   : ",KS,"\n")
        ListCount <- ListCount +1
        L[[ListCount]] <- LF[1:4]
    }

    LG <- RefineSearchGLDL(DataL, c(-0.999,-0.00001),
+ c(-0.999,-0.00001), 50, 6)
    if ( (LG[5] < 0.0001) & (LG[3]*LG[4] >0) )  {
        cat("Region [-0.999, -0.00001]X[-0.999, -0.00001]","\n")
        cat("    Lambdas     : ",LG[1:4],"\n")
        cat("    Error       : ", LG[5],"\n")
        cat("    Support     :  "); PrintSupportGLD(LG)
        cat("    Data Min-Max: ", min(Data), "  ",max(Data),"\n")
        KS <- KSGLD(Data,LG)
        cat("    K-S Stat.   : ",KS,"\n")
        ListCount <- ListCount +1
        L[[ListCount]] <- LG[1:4]
    }

    LH <- RefineSearchGLDL(DataL,c(0.00001, 0.2), c(0.00001,0.2),100,6)
    if ( (LH[5] < 0.0001) & (LH[3]*LG[4] >0) )  {
        cat("Region [0.00001, 0.1]X[0.00001, 0.1]","\n")
        cat("    Lambdas     : ",LH[1:4],"\n")
        cat("    Error       : ", LH[5],"\n")
        cat("    Support     :  "); PrintSupportGLD(LH)
        cat("    Data Min-Max: ", min(Data), "  ",max(Data),"\n")
        KS <- KSGLD(Data,LH)
        cat("    K-S Stat.   : ",KS,"\n")
        ListCount <- ListCount +1
        L[[ListCount]] <- LH[1:4]
    }

    if (length(L) > 0) {return(L)} else {return(c(0,0,0,0))}
}
```

A.10 R Code of the Programs for Kappa Distribution Fits

```
FindKappaL <- function(LMom) {
```

```
    L1 <- LMom[1]; L2 <- LMom[2]; L3 <- LMom[3]; L4 <- LMom[4]
    Tau3 <- L3/L2; Tau4 <- L4/L2
    T3List <- -1 + 0.025*(1:79)
    if ( (Tau3 < -0.975) | (Tau3 > 0.975) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/Kappa/R/KappaLT"), 1:79, sep="")
    j <- 1
    while (T3List[j] < Tau3) {j <- j+1}
    j <- j-1
    filenameA <- Filenames[j]; filenameB <- Filenames[j+1]

    source(filenameA)
    Rows <- length(T)/5
    T4List <- T[1+5*(0:(Rows-1))]
    if ( (Tau4 < T4List[1]) | (Tau4 > T4List[Rows]) )
        {return(c(0,0,0,0))}
    i <- 1
    while (T4List[i] < Tau4) { i<- i+1}
    i <- i-1
    Pt1 <- c(T[5*(i-1)+4], T[5*(i-1)+5])
    Pt3 <- c(T[5*i+4], T[5*i+5])

    source(filenameB)
    Rows <- length(T)/5
    T4List <- T[1+5*(0:(Rows-1))]
    if ( (Tau4 < T4List[1]) | (Tau4 > T4List[Rows]) )
        {return(c(0,0,0,0))}
    i <- 1
    while (T4List[i] < Tau4) { i<- i+1}
    i <- i-1
    Pt2 <- c(T[5*(i-1)+4], T[5*(i-1)+5])
    Pt4 <- c(T[5*i+4], T[5*i+5])

    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))
    if ( (PT1[1]*PT1[2] > 0) & (PT2[1]*PT2[2] > 0) ) {
        F <- RefineSearchKappaL(LMom, PT1, PT2, 50, 3)
    }
    if ( (PT1[1]*PT1[2] < 0) & (PT2[1]*PT2[2] > 0) ) {
        F1 <- RefineSearchKappaL(LMom, c(PT1[1],-0.035), PT2, 50, 3)
        F2 <- RefineSearchKappaL(LMom, c(0.035,PT1[2]), PT2, 50, 3)
        if (F1[5] < F2[5]) {F <- F1} else {F <- F2}
    }
    if ( (PT1[1]*PT1[2] < 0) & (PT2[1]*PT2[2] < 0) ) {
        F1 <- RefineSearchKappaL(LMom, c(PT1[1],-0.035),
+ c(PT2[1],-0.035), 50, 3)
        F2 <- RefineSearchKappaL(LMom, c(PT1[1],-0.035),
```

```
+ c(0.035,PT2[2]), 50, 3)
        F3 <- RefineSearchKappaL(LMom, c(0.035,PT1[2]),
+ c(PT2[1],-0.035), 50, 3)
        F4 <- RefineSearchKappaL(LMom, c(0.035,PT1[2]),
+ c(0.035,PT2[2]), 50, 3)
        MinErr <- min(F1[5],F2[5],F3[5],F4[5])
        if (F1[5] == MinErr) {F <- F1}
        if (F2[5] == MinErr) {F <- F2}
        if (F3[5] == MinErr) {F <- F3}
        if (F4[5] == MinErr) {F <- F4}
    }
    if ( (PT1[1]*PT1[2] > 0) & (PT2[1]*PT2[2] < 0) ) {
        F1 <- RefineSearchKappaL(LMom, PT1, c(PT2[1],-0.035), 50, 3)
        F2 <- RefineSearchKappaL(LMom, PT1, c(0.035,PT2[2]), 50, 3)
        if (F1[5] < F2[5]) {F <- F1} else {F <- F2}
    }
    return(F)
}


FitKappaL <- function(Data) {
    LM <- FindLMoms(Data)
    cat("\n", "          KAPPA L-MOMENT BASED FITS","\n\n")
    cat("L-Moments   : ", LM,"\n")
    cat("Tau3, Tau4  : ", LM[3]/LM[2], LM[4]/LM[2], "\n")
    F <- FindKappaL(LM)
    if ( (F[1] == 0) & (F[2] == 0) & (F[3] == 0) & (F[4] == 0) ) {
        cat("A Fit could not be obtained \n"); return(c(0,0,0,0)) }
    cat("Parameters  :",F[1],F[2],F[3],F[4],"\n")
    cat("Error       : ", F[5],"\n")
    Supp <- KappaSupport(F)
    cat("Support     : (",Supp[1],",",Supp[2],")", "\n");
    cat("Data Min-Max:", min(Data), max(Data),"\n")
    S <- KSKappa(Data, F[1:4])
    cat("K-S Stat.   :", S,"\n")
    return(c(F[1],F[2],F[3],F[4]))
}


FindKappaP <- function(Rhos) {
    R1 <- Rhos[1]; R2 <- Rhos[2]; R3 <- Rhos[3]; R4 <- Rhos[4]
    R3List <- 0.05*(1:40)
    if ( (R3 < 0.05) | (R3 > 2.0) ) {return(c(0,0,0,0))}
    Filenames <- paste(c("c:/Fit/Kappa/R/KappaPT"), 1:40, sep="")
    j <- 1
    while (R3List[j] < R3) {j <- j+1}
    j <- j-1
    filenameA <- Filenames[j]; filenameB <- Filenames[j+1]
```

```
    source(filenameA)
    Rows <- length(T)/5
    R4List <- T[1+5*(0:(Rows-1))]
    if ( (R4 < R4List[1]) | (R4 > R4List[Rows]) )
        {return(c(0,0,0,0))}
    i <- 1
    while (R4List[i] < R4) { i<- i+1}
    i <- i-1
    Pt1 <- c(T[5*(i-1)+4], T[5*(i-1)+5])
    Pt3 <- c(T[5*i+4], T[5*i+5])

    source(filenameB)
    Rows <- length(T)/5
    R4List <- T[1+5*(0:(Rows-1))]
    if ( (R4 < R4List[1]) | (R4 > R4List[Rows]) )
        {return(c(0,0,0,0))}
    i <- 1
    while (R4List[i] < R4) { i<- i+1}
    i <- i-1
    Pt2 <- c(T[5*(i-1)+4], T[5*(i-1)+5])
    Pt4 <- c(T[5*i+4], T[5*i+5])

    PT1 <- c(min(Pt1[1],Pt2[1],Pt3[1],Pt4[1]),
+ max(Pt1[1],Pt2[1],Pt3[1],Pt4[1]))
    PT2 <- c(min(Pt1[2],Pt2[2],Pt3[2],Pt4[2]),
+ max(Pt1[2],Pt2[2],Pt3[2],Pt4[2]))

    if ( (PT1[1]*PT1[2] > 0) & (PT2[1]*PT2[2] > 0) ) {
        F <- RefineSearchKappaP(Rhos, PT1, PT2, 50, 3)
    }
    if ( (PT1[1]*PT1[2] < 0) & (PT2[1]*PT2[2] > 0) ) {
        F1 <- RefineSearchKappaP(Rhos, c(PT1[1],-0.005), PT2, 50, 3)
        F2 <- RefineSearchKappaP(Rhos, c(0.005,PT1[2]), PT2, 50, 3)
        if (F1[5] < F2[5]) {F <- F1} else {F <- F2}
    }
    if ( (PT1[1]*PT1[2] < 0) & (PT2[1]*PT2[2] < 0) ) {
        F1 <- RefineSearchKappaP(LMom, c(PT1[1],-0.005),
+ c(PT2[1],-0.005), 50, 3)
        F2 <- RefineSearchKappaP(LMom, c(PT1[1],-0.005),
+ c(0.005,PT2[2]), 50, 3)
        F3 <- RefineSearchKappaP(LMom, c(0.005,PT1[2]),
+ c(PT2[1],-0.005), 50, 3)
        F4 <- RefineSearchKappaP(LMom, c(0.005,PT1[2]),
+ c(0.005,PT2[2]), 50, 3)
        MinErr <- min(F1[5],F2[5],F3[5],F4[5])
        if (F1[5] == MinErr) {F <- F1}
        if (F2[5] == MinErr) {F <- F2}
```

```
        if (F3[5] == MinErr) {F <- F3}
        if (F4[5] == MinErr) {F <- F4}
    }
    if ( (PT1[1]*PT1[2] > 0) & (PT2[1]*PT2[2] < 0) ) {
        F1 <- RefineSearchKappaP(Rhos, PT1, c(PT2[1],-0.005), 50, 3)
        F2 <- RefineSearchKappaP(Rhos, PT1, c(0.005,PT2[2]), 50, 3)
        if (F1[5] < F2[5]) {F <- F1} else {F <- F2}
    }
    return(F)
}


FitKappaP <- function(Data) {
    Rhos <- FindRhos(Data)
    cat("\n", "          KAPPA PERCENTILE-BASED FITS","\n\n")
    cat("Rhos        : ",Rhos,"\n")
    F <- FindKappaP(Rhos)
    if ( (F[1] == 0) & (F[2] == 0) & (F[3] == 0) & (F[4] == 0) ) {
        cat("A Fit could not be obtained \n"); return(c(0,0,0,0)) }
    cat("Parameters  :",F[1],F[2],F[3],F[4],"\n")
    cat("Error       : ", F[5],"\n")
    Supp <- KappaSupport(F)
    cat("Support     : (",Supp[1],",",Supp[2],")", "\n");
    cat("Data Min-Max:", min(Data), max(Data),"\n")
    S <- KSKappa(Data, F[1:4])
    cat("K-S Stat.   :", S,"\n")
    return(c(F[1],F[2],F[3],F[4]))
}


ErrKappaL <- function(LMOM, h, k) {
    if ( (h > 0) & ( k > -1) ) {
        A1 <- gamma(k+1)*gamma(1/h)
        A2 <- 2*gamma(k+1)*gamma(2/h)
        A3 <- 3*gamma(k+1)*gamma(3/h)
        A4 <- 4*gamma(k+1)*gamma(4/h)
        B1 <- gamma(1+k+1/h)*(h^(k+1))
        B2 <- gamma(1+k+2/h)*(h^(k+1))
        B3 <- gamma(1+k+3/h)*(h^(k+1))
        B4 <- gamma(1+k+4/h)*(h^(k+1))
        b0 <- (1/k)*(1-A1/B1)
        b1 <- (1/(2*k))*(1-A2/B2)
        b2 <- (1/(3*k))*(1-A3/B3)
        b3 <- (1/(4*k))*(1-A4/B4)
    }
    if ( (h < 0) & (-1 < k) & (k < -1/h) ) {
        A <- gamma(k+1)/( (-h)^(k+1) )
        B1 <- gamma(-k-1/h)/gamma(1-1/h)
```

```
        B2 <- 2*gamma(-k-2/h)/gamma(1-2/h)
        B3 <- 3*gamma(-k-3/h)/gamma(1-3/h)
        B4 <- 4*gamma(-k-4/h)/gamma(1-4/h)
        b0 <- (1/k)*(1-A*B1)
        b1 <- (1/(2*k))*(1-A*B2)
        b2 <- (1/(3*k))*(1-A*B3)
        b3 <- (1/(4*k))*(1-A*B4)
    }
    l1 <- b0; l2 <- 2*b1 - b0
    l3 <- 6*b2 - 6*b1 + b0
    l4 <- 20*b3 - 30*b2 + 12*b1 - b0
    tau3 <- l3/l2; tau4 <- l4/l2
    Err3 <- abs(tau3 - LMOM[3]/LMOM[2])
    Err4 <- abs(tau4 - LMOM[4]/LMOM[2])
    Err <- max(Err3,Err4)
    return(Err)
}


RefineSearchKappaL <- function(LMOM, hMinMax, kMinMax, Grid, Iter) {

    a <- hMinMax[1]; b <- hMinMax[2]
    c <- kMinMax[1]; d <- kMinMax[2]
    Err <- 10000; iter <-1
    while ( (Err > 0.000001) & (iter < Iter+1) ) {
        Deltah <- (b-a)/Grid; Deltak <- (d-c)/Grid
        for (i in (0:Grid)) {
            for (j in (0:Grid)) {
                h <- a + i*Deltah
                k <- c + j*Deltak
                if ( (h<0) & (k<0) & (k > -1 -0.725*h) ) {
                     NewErr <- ErrKappaL(LMOM, h, k)
                     if (NewErr < Err) {
                          Err <- NewErr
                          K <- k; H <- h
                     }
                }
                if ( (h<0) & (k>=0) & (h > -1/k) ) {
                     NewErr <- ErrKappaL(LMOM, h, k)
                     if (NewErr < Err) {
                         Err <- NewErr
                         K <- k; H <- h
                     }
                }
               if ( (h>0) & (k > -1) ) {
                     NewErr <- ErrKappaL(LMOM, h, k)
                     if (NewErr < Err) {
                         Err <- NewErr
```

```
                        K <- k; H <- h
                    }
                }

            }
        }
        iter <- iter + 1
        a <- H - 2*Deltah; b <- H + 2*Deltah
        c <- K - 2*Deltak; d <- K + 2*Deltak
    }
    h <- H; k <- K
    if ( (h > 0) & ( k > -1) ) {
        A1 <- gamma(k+1)*gamma(1/h)
        A2 <- 2*gamma(k+1)*gamma(2/h)
        B1 <- gamma(1+k+1/h)*(h^(k+1))
        B2 <- gamma(1+k+2/h)*(h^(k+1))
        b0 <- (1/k)*(1-A1/B1)
        b1 <- (1/(2*k))*(1-A2/B2)
    }
    if (h < 0)  {
        A <- gamma(k+1)/( (-h)^(k+1) )
        B1 <- gamma(-k-1/h)/gamma(1-1/h)
        B2 <- 2*gamma(-k-2/h)/gamma(1-2/h)
        b0 <- (1/k)*(1-A*B1)
        b1 <- (1/(2*k))*(1-A*B2)
    }
    l1 <- b0; l2 <- 2*b1 - b0
    alpha <- LMOM[2]/l2
    xi <- LMOM[1] - alpha*b0
    return(c(xi, alpha, h, k, Err))
}


FindLMoms <- function(Data) {

    X <- sort(Data); n <- length(X)
    a1 <- 0; a2 <- 0; a3 <- 0;
    b0 <- mean(X)
    for (k in 2:n) {
        a1 <- a1 + (k-1)*X[k]
        a2 <- a2 + (k-1)*(k-2)*X[k]
        a3 <- a3 + (k-1)*(k-2)*(k-3)*X[k]
    }
    b1 <- a1/(n*(n-1))
    b2 <- a2/(n*(n-1)*(n-2))
    b3 <- a3/(n*(n-1)*(n-2)*(n-3))
    return(c(b0,-b0+2*b1,b0-6*b1+6*b2,-b0+12*b1-30*b2+20*b3))
}
```

```
ErrKappaP <- function(Rhos, h, k) {
    Q10 <-  (1 - ((1-0.1^h)/h)^k)/k
    Q25 <-  (1 - ((1-0.25^h)/h)^k)/k
    Q50 <-  (1 - ((1-0.5^h)/h)^k)/k
    Q75 <-  (1 - ((1-0.75^h)/h)^k)/k
    Q90 <-  (1 - ((1-0.9^h)/h)^k)/k
    Rho3 <- (Q50 - Q10)/(Q90 - Q50)
    Rho4 <- (Q75 - Q25)/(Q90 - Q10)
    Err3 <- abs(Rho3 - Rhos[3])
    Err4 <- abs(Rho4 - Rhos[4])
    Err <- max (Err3, Err4)
    return(Err)
}


RefineSearchKappaP <- function(Rhos, hMinMax, kMinMax, Grid, Iter) {

    a <- hMinMax[1]; b <- hMinMax[2]
    c <- kMinMax[1]; d <- kMinMax[2]
    Err <- 10000; iter <-1

    while ( (Err > 0.00001) & (iter < Iter+1) ) {
        Deltah <- (b-a)/Grid; Deltak <- (d-c)/Grid
        for (i in (0:Grid)) {
            for (j in (0:Grid)) {
                h <- a + i*Deltah
                k <- c + j*Deltak
                NewErr <- ErrKappaP(Rhos, h, k)
                if (NewErr < Err) {
                    Err <- NewErr
                    K <- k; H <- h
                }
            }
        }
        iter <- iter + 1
        a <- H - 3*Deltah; b <- H + 3*Deltah
        c <- K - 3*Deltak; d <- K + 3*Deltak
    }
    A <-  (1 - ((1-0.9^H)/H)^K)/K
    B <-  (1 - ((1-0.1^H)/H)^K)/K
    alpha <- Rhos[2]/(A-B)
    C <-  (1 - ((1-0.5^H)/H)^K)/K
    xi <- Rhos[1] - alpha*C
    return(c(xi, alpha, H, K, Err))
}
```

```
KappaSupport <- function(Params) {
    xi <- Params[1]; alpha <- Params[2]; h <- Params[3]; k <- Params[4]
    if ((h > 0) & (k > 0)) {S1 <- xi+alpha*((1-h^(-k)))/k
        S2 <- xi+alpha/k
    }
    if ((h > 0) & (k == 0)) {S1 <- xi+alpha*ln(h)
        S2 <- 10^9
    }
    if ((h > 0) & (k < 0)) {S1 <- xi+alpha*((1-h^(-k)))/k
        S2 <- 10^9
    }
    if ((h <= 0) & (k > 0)) {S1 <- -10^10
        S2 <- xi+alpha/k
    }
    if ((h <= 0) & (k == 0)) {S1 <- -10^10
        S2 <- 10^9
    }
    if ((h <= 0) & (k < 0)) {S1 <- xi + alpha/k
        S2 <- 10^9
    }
    return(c(S1,S2))
}


pdfKappa <- function (P, x) {
    S <- KappaSupport(P)
    if ( (x < S[1]) | (x > S[2]) ) { return(0)
    }
    else {
        xi <- P[1]; alpha <- P[2]
        h <- P[3]; k <- P[4]
        A <- k*(x - xi)/alpha
        B <- (1 - A)^(1/k - 1)
        f <- (B/alpha)*(cdfKappa(P,x))^(1-h)
#cat(" #1", xi,alpha,h,k,"\n", x, A,B,f, "\n")
        return(f)
    }
}


cdfKappa <- function(P,t) {
    S <- KappaSupport(P)
    if (t < S[1]) {return(0)}
    if (t > S[2]) {return(1)}
    xi <- P[1]; alpha <- P[2]
    h <- P[3]; k <- P[4]
    A <- k*(t - xi)/alpha
```

```
    B <- (1 - A)^(1/k)
    C <- 1 - h*B
    F <- C^(1/h)
#cat(" #2", xi,alpha,h,k,"\n", t, A,B,F, "\n")
    return(F)
}


KSKappa <- function(dat, P) {
    N <- length(dat); Dat <- sort(dat)
    FDat <- (1/N)*(1:N)
    KappaF1 <- cdfKappa(P,Dat[1])
    D <- KappaF1
    for (i in 1:(N-1)) {
        diff <- abs(FDat[i] - KappaF1)
        if (D < diff) {D <- diff}
        KappaF2 <- cdfKappa(P,Dat[i+1])
        diff <- abs(FDat[i] - KappaF2)
        if (D < diff) {D <- diff}
        KappaF1 <- KappaF2
    }
    diff <- abs(FDat[N] - KappaF1)
    if (D < diff) {D <- diff}
    return(D)
}


ChiSqKappa <- function(Dat, P, Br) {
    n <- length(Dat); k <- length(Br)
    K <- k+1; EF <- rep(0,K)
    OF <- Freq(Dat, Br)
    Probs <- rep(0, k)
    for (i in 1:k) { Probs[i] <- cdfKappa(P,Br[i]) }
    CumProbs <- c(0,Probs,1)
    for (i in (1:(K))) {
        EF[i] <- n*(CumProbs[i+1] - CumProbs[i])
    }
    CS <- sum( (EF-OF)^2/EF )
    pval <- 1-pchisq(CS, K-5)
cat("Breaks : \n", Br, "\n")
cat("Observed Frequencies: \n", OF, "\n")
cat("Expected Frequencies: \n", EF, "\n")
cat("Chi-Square =", CS, "p-Value =", pval, "\n")
    return(c(CS,pval))
}


plotKappa <- function(Dat,P,breaks) {
```

```
    Supp <- KappaSupport(P)
    M <- mean(Dat);  s <- sd(Dat)
    DataMin <- M-5*s; DataMax <- M+5*s
    xMin <- max(DataMin,Supp[1])
    xMax <- min(DataMax,Supp[2])
    Delta <- (xMax-xMin)/200
    x <- xMin + Delta*(0:200)
    y <- pdfKappa(P,x)
    plot(x,y, type="l")
    hist(Dat, breaks, prob=T, main="", add=TRUE)
}
```

A.11 Maple Code for Johnson System Fits

```
FindAlphas := proc (DL::list)
    local n, i, AH1, AH2, AH3, AH4, S;
    n:=nops(DL);
    AH1 := convert(DL, `+`)/n;
    S:= [seq((DL[i]-AH1)^2, i=1..n)];
    AH2 := convert(S, `+`)/n;
    S:= [seq((DL[i]-AH1)^3, i=1..n)];
    AH3 := convert(S, `+`)/(n*AH2^(3/2));
    S:= [seq((DL[i]-AH1)^4, i=1..n)];
    AH4 := convert(S, `+`)/(n*AH2^2);

evalf([AH1, AH2, AH3, AH4]);
end:


FitJohnson := proc(Data::list)
    local A, a3, a4, C, w, b2, F, d, Type, Supp;

    A := FindAlphas(Data);
    a3 := A[3]; a4 := A[4];
    C := (2+a3^2)/2; d := sqrt(((2+a3^2)^2)/4 - 1);
    w := (C+d)^(1/3) + (C-d)^(1/3) -1;
    b2 := w^4 +2*w^3 +3*w^2 - 3;
    if (A[4] > b2) then Type := SU; lprint(Type);
        F := FitJnsnSU(A);
    elif (A[4] < b2) then Type := SB; #lprint(Type);
        F := FitJnsnSB(A, 13, 7);
        Supp:=[F[1], F[1]+F[2]];
    else Type := SL; lprint(Type); F := FitJnsnSL(A);
    fi:
printf("\n             JOHNSON FIT --%3s \n",Type);
printf("Alphas :  %9.6f %9.6f %9.6f %9.6f \n", A[1],A[2],A[3],A[4]);
printf("Error  :   %e \n", F[5]);
printf("xi, lambda, delta, gamma : \n");
```

```
printf("          %9.6f %9.6f %9.6f %9.6f \n", F[1],F[2],F[3],F[4]);
printf("Support: (%9.6f %9.6f) \n", Supp[1], Supp[2]);
printf("Data Min-Max: %9.6f %9.6f \n\n", Min(Data), Max(Data));
    [F[1],F[2],F[3],F[4]]
end:


FitJnsnSL := proc(Alphas::list)
    local  eqn1, eqn2, mu, var, w0, O, d, g, muY, l, c, a4, varY;

    eqn1 := (sqrt(w-1))*(w+2);
    eqn2 := w^4+2*w^3 +3*w^2-3;
    mu := sqrt(w)*exp(-Omega);
    var:= w*(w-1)*exp(-2*Omega);
    w0 := fsolve(eqn1=Alphas[3], w, w=1..10);
    O:=ln(sqrt(w0*(w0-1)/Alphas[2] ));
    d := 1/sqrt(ln(w0));
    g := d*O;
    muY := subs({w=w0, Omega=O},mu);
    varY := subs({w=w0, Omega=O},var);
    l := sqrt(Alphas[2]/varY);
    c := Alphas[1]-l*muY;
    a4 := subs(w=w0, eqn2);
    printf("%8.6f, %8.6f, %8.6f, %8.6f, %8.6f, %8.6f, \\\\ \n",
       A[3],a4,c,l,d,g);
     evalf([c, l, d, g])
end:


FitJnsnSU:=proc(Alphas::list)
    local b1a, b1b, b1c, b1, b2a, b2b, b2c, b2,
        w3, w4, mu, var, lambd, c, delta, Gamma;

    b1a := sqrt(w3*(w3-1)); b1b := w3*(w3+2)*sinh(3*w4) + 3*sinh(w4);
    b1c := sqrt(2)*((w3*cosh(2*w4)+1)^(3/2));
    b1 := evalf(-b1a*b1b/b1c);
    b2a := (w3^2)*(w3^4 + 2*w3^3 +3*w3^2 -3)*cosh(4*w4);
    b2b := 4*(w3^2)*(w3+2)*cosh(2*w4) + 6*w3 +3;
    b2c := 2*((w3*cosh(2*w4)+1)^(2));
    b2 := evalf((b2a + b2b)/b2c);
    fsolve({b1=Alphas[3], b2=Alphas[4]}, {w3,w4});
    assign(%);
    mu := -sqrt(w3)*sinh(w4);
    var := 0.5*(w3-1)*(w3*cosh(2*w4)+1);
    lambd := sqrt(Alphas[2]/var):
    c := Alphas[1] - lambd*mu;
    delta := 1/sqrt(ln(w3)):
    Gamma := w4*delta:
```

```
    [c, lambd, delta, Gamma]
end:


FitJnsnSB := proc(Alphas::list, Grid::posint, Iter::posint,
                  PT1::list, PT2::list)
    local B1List, B2List, A3, A4, TA, TB, NRows, pt1, pt2,
          pt3, pt4, Pt1, Pt2, f, EYm, mu, EY2, EY3, EY4, s,
          B1, B2, s3, s4, f3, f4, BestErr, K, k, l, Delta3,
          Delta4, i, j, BB1, BB2, Err1, Err2, Err, mean, sd,
          lambd, c, InList, FileNo1, FileNo2, FileName1, FileName2;

if nargs < 5 then

    B1List := [seq(0.05*i, i=0..60)]:
    A3 := abs(Alphas[3]):
    if A3  < B1List[1] or A3 > B1List[nops(B1List)] then
        RETURN([0,0,0,0])
    fi:
    j:=0:
    for i from 1 to nops(B1List) do
        if A3 > B1List[i] then j:=j+1 fi:
    od:
    FileNo1:=j; FileNo2:=j+1:
    FileName1:="c:\\Fit\\Jnsn\\Maple\\JnsnSB"||FileNo1;
    FileName2:="c:\\Fit\\Jnsn\\Maple\\JnsnSB"||FileNo2;
    read(FileName1): TA:=T:
    read(FileName2): TB:=T:

    NRows:=nops(TA)/3;
    B2List:=[seq(TA[1+3*i], i=0..NRows-1)];
    if Alphas[4] < TA[1] or Alphas[4] > TA[3*NRows-2] then
        RETURN([0,0,0,0])
    fi:
    j:=1:
    while Alphas[4] > B2List[j] do j:=j+1 od:
    pt1:=[TA[3*(j-1)+2], TA[3*(j-1)+3]];
    pt2:=[TA[3*(j-2)+2], TA[3*(j-2)+3]];
    NRows:=nops(TB)/3;
    B2List:=[seq(TB[1+3*i], i=0..NRows-1)];
    if Alphas[4] < TB[1] or Alphas[4] > TB[3*NRows-2] then
        RETURN([0,0,0,0])
    fi:
    j:=1:
    while Alphas[4] > B2List[j] do j:=j+1 od:
    pt3:=[TB[3*(j-1)+2], TB[3*(j-1)+3]];
    pt4:=[TB[3*(j-2)+2], TB[3*(j-2)+3]];
    Pt1:=[min(pt1[1],pt2[1],pt3[1],pt4[1]),
```

```
        max(pt1[1],pt2[1],pt3[1],pt4[1])];
    Pt2:=[min(pt1[2],pt2[2],pt3[2],pt4[2]),
        max(pt1[2],pt2[2],pt3[2],pt4[2])];

else Pt1:=PT1: Pt2:=PT2 fi:
    A3:=abs(Alphas[3]):  A4:=Alphas[4]:
    f:=(1/sqrt(2*Pi))*(1+exp(-(z-Gamma)/delta))^(-m)*exp(-z^2/2):
    EYm:=int(f, z=-infinity..infinity):
    mu:=evalf(subs(m=1,EYm)):
    EY2:=evalf(subs(m=2,EYm)):
    EY3:=evalf(subs(m=3,EYm)):
    EY4:=evalf(subs(m=4,EYm)):
    s:=sqrt(EY2-mu^2):
    B1:=(EY3-3*EY2*mu+2*mu^3)/s^3:
    B2:=(EY4-4*EY3*mu+6*EY2*mu^2-3*mu^4)/s^4:
    s3 := Pt1[1];  f3 := Pt1[2];  s4 := Pt2[1];  f4 := Pt2[2];
    BestErr := 100000; K:=1;
    while BestErr > 0.0001 and K <= Iter do
        Delta3 := (f3-s3)/(Grid); Delta4 := (f4-s4)/Grid;
        for i from s3 to f3 by Delta3 do
            for j from s4 to f4 by Delta4 do
                BB1:=subs({delta=i,Gamma=j}, B1):
                BB2:=subs({delta=i,Gamma=j}, B2):
                Err1 := evalf(abs(BB1-A3)):
                Err2:=evalf(abs(BB2-A4)):
                Err := max(Err1, Err2):
                if Err < BestErr then
                    k:=i: l:=j: BestErr:=Err
                fi:
            od:
        od:
        K := K+1;
        s3 := k-3*Delta3;  f3 := k+3*Delta3;
        s4 := l-3*Delta4;  f4 := l+3*Delta4;
    od:
    mean := evalf(subs({delta=k, Gamma=l}, mu)):
    sd := evalf(subs({delta=k, Gamma=l}, s)):

    if (Alphas[3] < 0) then l:= -l; mean := 1- mean fi:

    lambd := sqrt(Alphas[2]/(sd^2));
    c := Alphas[1] - lambd*mean;
    evalf([c, lambd, k, l, BestErr])
end:


FreqCount := proc(Dat::list, CB::list)
    local i, j, n, k, Count, Done;
```

```
    n:=nops(Dat): k := nops(CB):
    Count:= array(1..k+1):
    for i from 1 to k+1 do
        Count[i] := 0
    od:
    for i from 1 to n do
        if (Dat[i] >= CB[k]) then Count[k+1]:=Count[k+1]+1 fi:
        Done := 0:
        for j from 1 to k do
            if ((Dat[i] < CB[j]) and Done=0) then
                Count[j] := Count[j] + 1:
                Done := 1
            fi:
        od:
    od:
    [seq(Count[r], r=1..k+1)]
end:


ChiSqJnsnSB := proc(Dat::list, J::list, CB::list)
    local A, d, c, g, l, SB1, SB2, SB, pdf, cdf,
          i, j, x, t, Probs, EF, OF, ChiSq, pVal;

    A:=FindAlphas(Dat):
    SB1:=d*l/(sqrt(2*Pi)*(x-c)*(l-x+c)):
    SB2:=(g+d*ln((x-c)/(l-x+c)))^2:
    SB:=SB1*exp(-0.5*SB2):
    pdf := subs({c=J[1], l=J[2], d=J[3], g=J[4]}, SB);
    cdf:=int(pdf, x=J[1]..t):
    Probs:=array(1..nops(CB)+2):
    Probs[1] := 0:  Probs[nops(CB)+2] := 1:
    for i from 2 to nops(CB)+1 do
        if CB[i-1] < J[1] then Probs[i] := 0
            elif CB[i-1] > J[1] + J[2] then Probs[i] := 1
            else Probs[i] := evalf(subs(t=CB[i-1],cdf))
        fi:
    od:
    EF:=[seq(nops(Dat)*(Probs[i+1]-Probs[i]), i=1..nops(CB)+1)];
    OF:=FreqCount(Dat, CB):

    ChiSq:=sum((EF[j]-OF[j])^2/EF[j], j=1..nops(CB)+1);
    pVal:=1-ChisquareCDF(nops(CB)-4,ChiSq);
    printf("Class Boundary :      "): lprint(CB);
    printf("Observed Frequencies: "); lprint(OF);
    printf("Expected Frequencies: "): lprint(evalf(EF,6)):
    printf("Chi-Square:           "): lprint(ChiSq):
    printf("p-Value:              "): lprint(pVal):
```

```
    [ChiSq, pVal]
end:


ChiSqJnsnSU := proc(Dat::list, J::list, CB::list)
    local N, A, d, c, g, l, SU1, SU2, SU3, SU, pdf, cdf,
          x, t, Probs, EF, OF, ChiSq, pVal;

    N:=nops(CB):
    A:=FindAlphas(Dat):
    SU1:=d/sqrt(2*Pi*((x-c)^2+l^2)):
    SU2:=(x-c+sqrt((x-c)^2+l^2))/l:
    SU3:=-0.5*(g+d*ln(SU2))^2:
    SU:=SU1*exp(SU3):
    pdf := subs({c=J[1], l=J[2], d=J[3], g=J[4]}, SU);
    cdf:=int(pdf, x=J[1]..t):

    Probs:=evalf([0,seq(subs(t=CB[i],cdf), i=1..nops(CB)),1]):
    EF:=[seq(nops(Dat)*(Probs[i+1]-Probs[i]), i=1..nops(CB)+1)];
    OF:=FreqCount(Dat, CB):
    ChiSq:=sum((EF[i]-OF[i])^2/EF[i], i=1..nops(CB)+1);
    pVal:=1-ChisquareCDF(nops(CB)-4,ChiSq);
    printf("Class Boundary :      "): lprint(CB);
    printf("Observed Frequencies: "); lprint(OF);
    printf("Expected Frequencies: "): lprint(evalf(EF,6)):
    printf("Chi-Square:           "): lprint(ChiSq):
    printf("p-Value:              "): lprint(pVal):
    [ChiSq, pVal]
end:


Fn:=proc(Data::list, u::numeric)
    local fn, i, SD;
    SD:=sort(Data): fn:=0:
    for i from 1 to nops(SD) do
        if u >= SD[i] then fn:= fn+(1./nops(SD)) fi:
    od:
    evalf(fn)
end:


KSJnsnSB:=proc(Data::list, J::list)
    local SD, SB1, SB2, SB, c, l, d, g, pdf, cdf, FN,
          Cdf, cdfY, i, diff1, diff2, m1, m2;

    SD:=sort(Data): FN:=[seq(Fn(SD,SD[i]), i=1..nops(SD))]:
    SB1:=d*l/(sqrt(2*Pi)*(x-c)*(l-x+c)):
    SB2:=(g+d*ln((x-c)/(l-x+c)))^2:
```

```
    SB:=SB1*exp(-0.5*SB2):
    pdf := subs({c=J[1], l=J[2], d=J[3], g=J[4]}, SB);
    cdf:=int(pdf, x=J[1]..t):
    Cdf := array(1..nops(SD)):
    for i from 1 to nops(SD) do
        if ((SD[i] >= J[1]) and (SD[i] <= J[1]+J[2])) then
            cdfY :=evalf(subs(t=SD[i],cdf)): Cdf[i] := cdfY
        elif (SD[i] < J[1]) then cdfY := 0: Cdf[i]:=cdfY
        else cdfY:=1: Cdf[i] := cdfY
        fi:
    od:
    diff1:=[seq(abs(Cdf[i]-FN[i]), i=1..nops(SD))];
    diff2:=[Cdf[1],seq(abs(Cdf[i]-FN[i-1]), i=2..nops(SD))];
    m1:=max(op(diff1)); m2:=max(op(diff2));
    max(m1,m2);
end:


KSJnsnSU:=proc(Data::list, J::list)
    local SD, SU1, SU2, SU3, SU, c, l, d, g, pdf, cdf, FN,
          Cdf, cdfY, i, diff1, diff2, m1, m2;

    SD:=sort(Data): FN:=[seq(Fn(SD,SD[i]), i=1..nops(SD))]:
    SU1:=d/sqrt(2*Pi*((x-c)^2+l^2)):
    SU2:=(x-c+sqrt((x-c)^2+l^2))/l:
    SU3:=-0.5*(g+d*ln(SU2))^2:
    SU:=SU1*exp(SU3):
    pdf := subs({c=J[1], l=J[2], d=J[3], g=J[4]}, SU);
    cdf:=int(pdf, x=-infinity..t):
    Cdf := array(1..nops(SD)):
    for i from 1 to nops(SD) do
        Cdf[i] := evalf(subs(t=SD[i],cdf)):
    od:
    diff1:=[seq(abs(Cdf[i]-FN[i]), i=1..nops(SD))];
    diff2:=[Cdf[1],seq(abs(Cdf[i]-FN[i-1]), i=2..nops(SD))];
    m1:=max(op(diff1)); m2:=max(op(diff2));
    max(m1,m2);
end:


FindJnsnSB := proc(Alphas::list)
    local f1, f2, m, EY1, EY2, EY3, EY4, M,
        V, b1, b2, Err, Error, d, g, dd, gg,
        ddd, ggg, dMin, dMax, gMin, gMax, Iter,
        dN, gN, dInc, gInc, i, j, m3, m4, k, c;

f1:=(1+exp(-(z-g)/d))^(-m);
f2:=exp(-z^2/2);
```

```
EY1:=(1/sqrt(2*Pi))*Int(subs(m=1,f1*f2), z=-infinity..infinity):
EY2:=(1/sqrt(2*Pi))*Int(subs(m=2,f1*f2), z=-infinity..infinity):
EY3:=(1/sqrt(2*Pi))*Int(subs(m=3,f1*f2), z=-infinity..infinity):
EY4:=(1/sqrt(2*Pi))*Int(subs(m=4,f1*f2), z=-infinity..infinity):
M:=EY1;
V:=EY2-M^2:
b1:=(EY3-3*EY2*M+2*M^3)/(V^(3/2)):
b2:=(EY4-4*EY3*M+6*EY2*M^2-3*M^4)/V^2:

Err:=infinity: Iter:=1;

dMin:=0.01: dMax:=5.0: dN:=5: dInc:=(dMax-dMin)/dN:
gMin:=-5.0: gMax:=5.0: gN:=5: gInc:=(gMax-gMin)/gN:

while Err > 0.000001 and Iter <= 10 do
    for i from 1 to dN+1 do
        for j from 1 to gN+1 do
            dd:=dMin+(i-1)*dInc:
            gg:=gMin+(j-1)*gInc:
            m3:=evalf(subs({d=dd, g=gg}, b1));
            m4:=evalf(subs({d=dd, g=gg}, b2));
            Error:=max(abs(m3-Alphas[3]), abs(m4-Alphas[4])):
            if Err > Error then
                ddd:=dd; ggg:=gg; Err:=Error:
            fi:
        od:
    od:
    dMin:=ddd-2*Err: dMax:=ddd+2*Err:
    gMin:=ggg-2*Err: gMax:=ggg+2*Err:
    dInc:=(dMax-dMin)/dN: gInc:=(gMax-gMin)/gN:
    lprint(Iter, ddd, ggg, Err);
    Iter:=Iter+1;
od:
k:=sqrt(Alphas[2]/evalf(subs({d=ddd, g=ggg},V)));
c:=Alphas[1]-k*evalf(subs({d=ddd, g=ggg},M));
[c,k,ddd,ggg]
end:
```

A.12 Maple Code for Bivariate GLD (GLD–2) Fits

```
PlackettPsi := proc(X::list, Y::list)
local n, i, C1, C2, C3, C4, MX, MY;
n := nops(X);
C1 := 0; C2 := 0; C3 := 0; C4 := 0;
MX := Percentile(X,0.5);
MY := Percentile(Y,0.5);
for i from 1 to n do
    if X[i] >= MX and Y[i] >= MY then C1 := C1+1
```

```
       elif X[i] >= MX and Y[i] < MY then C4 := C4+1
       elif X[i] <= MX and Y[i] > MY then C2 := C2+1
       else C3 := C3+1
    fi
od;
[C1, C2,C3,C4,(C1*C3)/(C2*C4)]
end:


GLD2PDFplot := proc(Psi::numeric, L1::list, L2::list)
local A, P1, P2, P, XX, fX, YX, fY, i,
      j, S, PL, n, top, bottom, PlotSet;
with(plots,[display, spacecurve]);
n := 25;
P1 := .001,.005,.012,.003,.02,.03,.042,.055,.075,
      .1,.13,.165,.205,.25,.3,.35,.4,.45,.5;
P2 := seq(1-P1[19-i],i=1..18);
P := [P1,P2];

XX:=[seq(L1[1]+ ( P[i]^L1[3] - (1-P[i])^L1[4] )/L1[2], i=1..n)]:
fX:=[seq(L1[2]/(L1[3]*P[i]^(L1[3]-1)+L1[4]*(1-P[i])^(L1[4]-1)),i=1..n)]:
YX:=[seq(L2[1]+ ( P[i]^L2[3] - (1-P[i])^L2[4] )/L2[2], i=1..n)]:
fY:=[seq(L2[2]/(L2[3]*P[i]^(L2[3]-1)+L2[4]*(1-P[i])^(L2[4]-1)),i=1..n)]:
if Psi = 1 then
    for i from 1 to n do
        for j from 1 to n do
            A[i,j] := [XX[i], YX[j], fX[i]*fY[j]];
        od
    od
    else
    for i from 1 to n do
      for j from 1 to n do
        top := Psi*fX[i]*fY[j]*(1+(Psi-1)*(P[i]+P[j]-2*P[i]*P[j]));
        S := 1+(P[i]+P[j])*(Psi-1);
        bottom := (S^2-4*Psi*(Psi-1)*P[i]*P[j])^(3/2);
        A[i,j] := [XX[i], YX[j], top/bottom]
      od
    od
fi;
PlotSet := {};
for i from 1 to n do
    PL := spacecurve([seq(A[i,j], j=1..n)], color=black);
    PlotSet := PlotSet union {PL};
od;
for j from 1 to n do
    PL := spacecurve([seq(A[i,j], i=1..n)], color=black);
    PlotSet := PlotSet union {PL};
od;
```

```
display(PlotSet);
end:


GLD2RAND := proc(LX::list, LY::list, psi::numeric, N::posint)
    local y, l1, l2, l3, l4, FInv, GInv, f, g, R,
          gInt, k, r, u, x, Fx, fx, Gy, X, i, Delta,
          gy, S, hNum, hDen, h, gPDF, j, iter;

l1+(y^l3-(1-y)^l4)/l2;
FInv:=subs({l1=LX[1],l2=LX[2],l3=LX[3],l4=LX[4]},%);
GInv:=subs({l1=LY[1],l2=LY[2],l3=LY[3],l4=LY[4]},%%);
l2/(l3*y^(l3-1)+l4*(1-y)^(l4-1));
f:=simplify(subs({l1=LX[1],l2=LX[2],l3=LX[3],l4=LX[4]},%));
g:=simplify(subs({l1=LY[1],l2=LY[2],l3=LY[3],l4=LY[4]},%%));
iter:=1000;
R:=array(1..N):  gInt:=array(1..iter);
    Gy:=[seq(i*(1.0/iter), i=0..iter)]:
    X:=[seq(subs(y=Gy[i],GInv), i=1..nops(Gy))]:
    Delta:=[seq(X[i+1]-X[i], i=1..nops(Gy)-1)]:
    gy:=[seq(subs(y=Gy[i],g), i=1..nops(Gy))]:

for k from 1 to N do
    r:=evalf(rand()/10^12):
    u:=evalf(rand()/10^12):
    x:=subs(y=r,FInv):
    Fx:=r; fx:=subs(y=r,f):
    S:=[seq(1+(Fx+Gy[i])*(psi-1), i=1..nops(Gy))]:
    hNum:=[seq(psi*fx*gy[i]*(1+(psi-1)*(Fx+Gy[i]-2*Fx*Gy[i])),
           i=1..nops(Gy))]:
    hDen:=[seq((S[i]^2-4*psi*(psi-1)*Fx*Gy[i])^(3/2),
           i=1..nops(Gy))]:
    h:=[seq(hNum[i]/hDen[i], i=1..nops(Gy))]:
    gPDF:=[seq(h[i]/fx, i=1..nops(Gy))]:
    gInt[1]:=Delta[1]*gPDF[1];
    for i from 2 to iter do
        gInt[i]:=gInt[i-1]+Delta[i]*gPDF[i];
    od:
    j:=1:
    while gInt[j] < u do j:=j+1 od:
    R[k]:=[x,(X[j-1]+X[j])/2]
od:
[seq(R[i], i=1..N)]:

end:
```

Appendix B

Table B–1 for GLD Fits: Method of Moments

For specified α_3 and α_4, Table B–1 gives the $\lambda_1(0,1)$, $\lambda_2(0,1)$, λ_3, and λ_4 for GLD fits in the shaded region of Figure 4.3–5. In order to give a sufficient number of significant digits, superscripts are used to designate factors of $\frac{1}{10}$. Thus, an entry of a^s designates $a \times 10^s$. For example, for $(\alpha_3,\ \alpha_4)$ close to $(0.15, 4.1)$, the table gives

$$(\lambda_1(0,1),\ \lambda_2(0,1),\ \lambda_3,\ \lambda_4) = (-0.7268^1, 0.1603^1, 0.8378^2, 0.9564^2).$$

The proper interpretation of these table entries is

$$(\lambda_1(0,1),\ \lambda_2(0,1),\ \lambda_3,\ \lambda_4) = (-0.07268, 0.01603, 0.008378, 0.009564).$$

With few exceptions, Table B–1 provides values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for which

$$\max_{1\le i\le 4} |\alpha_i - \hat{\alpha}_i| < 10^{-5}.$$

The exceptions occur when very small changes in λ_3 or λ_4 cause large variations in α_3 and α_4 — a situation that arises when λ_3 or λ_4 gets close to 0. When $|\lambda_3| < 10^{-2}$ or $|\lambda_4| < 10^{-2}$, we generally have

$$\max_{1\le i\le 4} |\alpha_i - \hat{\alpha}_i| < 10^{-4}.$$

In the rare instances where $|\lambda_3| < 10^{-4}$ or $|\lambda_4| < 10^{-4}$, we can only be assured of

$$\max_{1\le i\le 4} |\alpha_i - \hat{\alpha}_i| < 10^{-3}.$$

The entries of Table B–1 are from "The Extended Generalized Lambda Distribution (EGLD) System for Fitting Distributions to Data with Moments, II: Tables" by E.J. Dudewicz and Z.A. Karian, *American Journal of Mathematical and Management Sciences*, Vol. 16, 3 & 4 (1996), pp. 287–307, copyright ©1996 by American Sciences Press, Inc., 20 Cross Road, Syracuse, New York 13224-2104. Reprinted with permission.

$\hat{\alpha}_3 = 0.00$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
1.8	0	0.5773	1.0000	1.0000
1.9	0	0.5360	0.7315	0.7315
2.0	0	0.4952	0.5843	0.5843
2.1	0	0.4563	0.4839	0.4839
2.2	0	0.4197	0.4092	0.4092
2.3	0	0.3854	0.3507	0.3507
2.4	0	0.3533	0.3032	0.3032
2.5	0	0.3232	0.2637	0.2637
2.6	0	0.2950	0.2303	0.2303
2.7	0	0.2684	0.2016	0.2016
2.8	0	0.2434	0.1765	0.1765
2.9	0	0.2198	0.1545	0.1545
3.0	0	0.1974	0.1349	0.1349
3.1	0	0.1763	0.1174	0.1174
3.2	0	0.1563	0.1016	0.1016
3.3	0	0.1372	0.8724[1]	0.8724[1]
3.4	0	0.1191	0.7418[1]	0.7418[1]
3.5	0	0.1018	0.6222[1]	0.6222[1]
3.6	0	0.8527[1]	0.5122[1]	0.5122[1]
3.7	0	0.6951[1]	0.4106[1]	0.4106[1]
3.8	0	0.5442[1]	0.3166[1]	0.3166[1]
3.9	0	0.3996[1]	0.2291[1]	0.2291[1]
4.0	0	0.2610[1]	0.1476[1]	0.1476[1]
4.1	0	0.1279[1]	0.7140[2]	0.7140[2]
4.2	0	-0.6579[3]	-0.3630[3]	-0.3630[3]
4.3	0	-0.1231[1]	-0.6707[2]	-0.6707[2]
4.4	0	-0.2416[1]	-0.1302[1]	-0.1302[1]
4.5	0	-0.3558[1]	-0.1897[1]	-0.1897[1]
4.6	0	-0.4660[1]	-0.2460[1]	-0.2460[1]
4.7	0	-0.5723[1]	-0.2993[1]	-0.2993[1]
4.9	0	-0.7744[1]	-0.3978[1]	-0.3978[1]
5.1	0	-0.9636[1]	-0.4868[1]	-0.4868[1]
5.3	0	-0.1141	-0.5678[1]	-0.5678[1]
5.5	0	-0.1308	-0.6419[1]	-0.6419[1]
5.7	0	-0.1466	-0.7098[1]	-0.7098[1]
5.9	0	-0.1614	-0.7725[1]	-0.7725[1]
6.1	0	-0.1755	-0.8304[1]	-0.8304[1]
6.3	0	-0.1889	-0.8842[1]	-0.8842[1]
6.5	0	-0.2016	-0.9342[1]	-0.9342[1]
6.7	0	-0.2137	-0.9809[1]	-0.9809[1]
7.0	0	-0.2307	-0.1045	-0.1045
7.3	0	-0.2466	-0.1104	-0.1104
7.6	0	-0.2615	-0.1158	-0.1158
7.9	0	-0.2755	-0.1207	-0.1207
8.2	0	-0.2887	-0.1253	-0.1253
8.7	0	-0.3089	-0.1321	-0.1321
9.2	0	-0.3274	-0.1382	-0.1382
9.7	0	-0.3443	-0.1436	-0.1436
10.2	0	-0.3599	-0.1485	-0.1485
10.7	0	-0.3743	-0.1529	-0.1529
11.5	0	-0.3951	-0.1592	-0.1592
12.3	0	-0.4138	-0.1646	-0.1646
13.1	0	-0.4305	-0.1694	-0.1694
13.9	0	-0.4457	-0.1736	-0.1736
14.7	0	-0.4595	-0.1774	-0.1774

$\hat{\alpha}_3 = 0.05$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
1.9	-1.4668	0.2991	0.2295[1]	0.8558
2.0	-1.2292	0.3122	0.5052[1]	0.7602
2.1	-1.0128	0.3230	0.7990[1]	0.6698
2.2	-0.8022	0.3314	0.1128	0.5802
2.3	-0.5873	0.3361	0.1502	0.4881
2.4	-0.3748	0.3328	0.1876	0.3941
2.5	-0.2200	0.3164	0.2046	0.3148
2.6	-0.1431	0.2924	0.1973	0.2605
2.7	-0.1046	0.2673	0.1809	0.2212
2.8	-0.8253[1]	0.2429	0.1625	0.1903
2.9	-0.6818[1]	0.2196	0.1445	0.1645
3.0	-0.5835[1]	0.1975	0.1276	0.1425
3.1	-0.5110[1]	0.1765	0.1119	0.1232
3.2	-0.4554[1]	0.1565	0.9745[1]	0.1061
3.3	-0.4115[1]	0.1375	0.8412[1]	0.9081[1]
3.4	-0.3758[1]	0.1194	0.7182[1]	0.7700[1]
3.5	-0.3463[1]	0.1021	0.6045[1]	0.6444[1]
3.6	-0.3215[1]	0.8562[1]	0.4993[1]	0.5297[1]
3.7	-0.3003[1]	0.6986[1]	0.4015[1]	0.4243[1]
3.8	-0.2820[1]	0.5478[1]	0.3105[1]	0.3270[1]
3.9	-0.2661[1]	0.4033[1]	0.2256[1]	0.2369[1]
4.0	-0.2518[1]	0.2646[1]	0.1463[1]	0.1531[1]
4.1	-0.2395[1]	0.1315[1]	0.7184[2]	0.7504[2]
4.2	-0.2284[1]	0.3585[3]	0.1936[3]	0.2018[3]
4.3	-0.2183[1]	-0.1195[1]	-0.6386[2]	-0.6643[2]
4.4	-0.2091[1]	-0.2381[1]	-0.1259[1]	-0.1308[1]
4.5	-0.2010[1]	-0.3523[1]	-0.1845[1]	-0.1913[1]
4.6	-0.1935[1]	-0.4625[1]	-0.2400[1]	-0.2485[1]
4.7	-0.1867[1]	-0.5689[1]	-0.2926[1]	-0.3026[1]
4.8	-0.1803[1]	-0.6717[1]	-0.3425[1]	-0.3538[1]
5.0	-0.1691[1]	-0.8672[1]	-0.4351[1]	-0.4485[1]
5.2	-0.1593[1]	-0.1050	-0.5193[1]	-0.5343[1]
5.4	-0.1509[1]	-0.1223	-0.5961[1]	-0.6124[1]
5.6	-0.1434[1]	-0.1385	-0.6666[1]	-0.6839[1]
5.8	-0.1368[1]	-0.1538	-0.7315[1]	-0.7495[1]
6.0	-0.1309[1]	-0.1683	-0.7914[1]	-0.8101[1]
6.2	-0.1256[1]	-0.1820	-0.8471[1]	-0.8662[1]
6.4	-0.1208[1]	-0.1950	-0.8988[1]	-0.9183[1]
6.6	-0.1165[1]	-0.2074	-0.9470[1]	-0.9668[1]
6.8	-0.1125[1]	-0.2192	-0.9921[1]	-0.1012
7.1	-0.1072[1]	-0.2359	-0.1055	-0.1075
7.4	-0.1024[1]	-0.2515	-0.1112	-0.1132
7.7	-0.9820[2]	-0.2661	-0.1164	-0.1184
8.0	-0.9442[2]	-0.2798	-0.1212	-0.1232
8.3	-0.9101[2]	-0.2927	-0.1256	-0.1277
8.8	-0.8600[2]	-0.3126	-0.1323	-0.1343
9.3	-0.8169[2]	-0.3307	-0.1383	-0.1403
9.8	-0.7795[2]	-0.3474	-0.1436	-0.1456
10.3	-0.7465[2]	-0.3627	-0.1484	-0.1503
10.8	-0.7173[2]	-0.3768	-0.1527	-0.1547
11.6	-0.6769[2]	-0.3974	-0.1589	-0.1608
12.4	-0.6428[2]	-0.4158	-0.1643	-0.1661
13.2	-0.6134[2]	-0.4323	-0.1690	-0.1708
14.0	-0.5879[2]	-0.4473	-0.1732	-0.1750
14.8	-0.5656[2]	-0.4610	-0.1769	-0.1787

$\hat{\alpha}_3 = 0.10$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
1.9	-1.5049	0.2911	0.1562[1]	0.8296
2.0	-1.2708	0.3028	0.4121[1]	0.7373
2.1	-1.0645	0.3117	0.6711[1]	0.6520
2.2	-0.8721	0.3177	0.9412[1]	0.5700
2.3	-0.6878	0.3199	0.1219	0.4897
2.4	-0.5150	0.3163	0.1477	0.4116
2.5	-0.3700	0.3049	0.1646	0.3407
2.6	-0.2687	0.2863	0.1678	0.2831
2.7	-0.2047	0.2644	0.1607	0.2385
2.8	-0.1640	0.2417	0.1486	0.2033
2.9	-0.1366	0.2193	0.1346	0.1746
3.0	-0.1170	0.1977	0.1205	0.1503
3.1	-0.1026	0.1770	0.1067	0.1294
3.2	-0.9153[1]	0.1572	0.9365[1]	0.1111
3.3	-0.8271[1]	0.1383	0.8135[1]	0.9490[1]
3.4	-0.7555[1]	0.1203	0.6984[1]	0.8034[1]
3.5	-0.6961[1]	0.1031	0.5908[1]	0.6719[1]
3.6	-0.6461[1]	0.8666[1]	0.4903[1]	0.5523[1]
3.7	-0.6034[1]	0.7092[1]	0.3964[1]	0.4429[1]
3.8	-0.5665[1]	0.5585[1]	0.3085[1]	0.3422[1]
3.9	-0.5346[1]	0.4141[1]	0.2261[1]	0.2493[1]
4.0	-0.5058[1]	0.2755[1]	0.1488[1]	0.1631[1]
4.1	-0.4811[1]	0.1423[1]	0.7606[2]	0.8302[2]
4.2	-0.4588[1]	0.1432[2]	0.7577[3]	0.8235[3]
4.3	-0.4385[1]	-0.1089[1]	-0.5703[2]	-0.6175[2]
4.4	-0.4202[1]	-0.2275[1]	-0.1181[1]	-0.1274[1]
4.5	-0.4036[1]	-0.3418[1]	-0.1758[1]	-0.1891[1]
4.6	-0.3883[1]	-0.4522[1]	-0.2306[1]	-0.2474[1]
4.7	-0.3746[1]	-0.5587[1]	-0.2826[1]	-0.3023[1]
4.8	-0.3619[1]	-0.6616[1]	-0.3320[1]	-0.3543[1]
5.0	-0.3392[1]	-0.8574[1]	-0.4238[1]	-0.4504[1]
5.2	-0.3197[1]	-0.1041	-0.5074[1]	-0.5373[1]
5.4	-0.3027[1]	-0.1213	-0.5838[1]	-0.6162[1]
5.6	-0.2877[1]	-0.1376	-0.6540[1]	-0.6884[1]
5.8	-0.2744[1]	-0.1529	-0.7187[1]	-0.7547[1]
6.0	-0.2625[1]	-0.1674	-0.7785[1]	-0.8157[1]
6.2	-0.2519[1]	-0.1812	-0.8340[1]	-0.8722[1]
6.4	-0.2423[1]	-0.1942	-0.8857[1]	-0.9246[1]
6.6	-0.2335[1]	-0.2066	-0.9339[1]	-0.9734[1]
6.8	-0.2255[1]	-0.2184	-0.9790[1]	-0.1019
7.1	-0.2148[1]	-0.2351	-0.1042	-0.1082
7.4	-0.2053[1]	-0.2507	-0.1099	-0.1139
7.7	-0.1968[1]	-0.2653	-0.1151	-0.1192
8.0	-0.1892[1]	-0.2791	-0.1199	-0.1240
8.3	-0.1824[1]	-0.2920	-0.1244	-0.1284
8.8	-0.1723[1]	-0.3119	-0.1311	-0.1351
9.3	-0.1637[1]	-0.3301	-0.1370	-0.1411
9.8	-0.1562[1]	-0.3467	-0.1424	-0.1464
10.3	-0.1495[1]	-0.3621	-0.1472	-0.1511
10.8	-0.1437[1]	-0.3763	-0.1516	-0.1554
11.6	-0.1356[1]	-0.3969	-0.1578	-0.1616
12.4	-0.1287[1]	-0.4153	-0.1632	-0.1669
13.2	-0.1229[1]	-0.4318	-0.1679	-0.1716
14.0	-0.1177[1]	-0.4468	-0.1721	-0.1757
14.8	-0.1133[1]	-0.4605	-0.1759	-0.1794

$\hat{\alpha}_3 = 0.15$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
1.9	-1.5589	0.2826	0.7231[2]	0.8107
2.0	-1.3231	0.2934	0.3145[1]	0.7203
2.1	-1.1221	0.3010	0.5487[1]	0.6387
2.2	-0.9405	0.3056	0.7814[1]	0.5622
2.3	-0.7722	0.3066	0.1009	0.4891
2.4	-0.6175	0.3031	0.1215	0.4194
2.5	-0.4830	0.2939	0.1366	0.3553
2.6	-0.3760	0.2791	0.1435	0.2994
2.7	-0.2984	0.2604	0.1421	0.2533
2.8	-0.2436	0.2398	0.1350	0.2156
2.9	-0.2049	0.2187	0.1249	0.1846
3.0	-0.1766	0.1980	0.1135	0.1586
3.1	-0.1550	0.1778	0.1017	0.1362
3.2	-0.1384	0.1584	0.9006[1]	0.1167
3.3	-0.1251	0.1397	0.7885[1]	0.9958[1]
3.4	-0.1143	0.1219	0.6815[1]	0.8428[1]
3.5	-0.1053	0.1048	0.5803[1]	0.7052[1]
3.6	-0.9772[1]	0.8838[1]	0.4848[1]	0.5806[1]
3.7	-0.9124[1]	0.7268[1]	0.3947[1]	0.4669[1]
3.8	-0.8564[1]	0.5764[1]	0.3100[1]	0.3627[1]
3.9	-0.8082[1]	0.4320[1]	0.2301[1]	0.2667[1]
4.0	-0.7649[1]	0.2935[1]	0.1548[1]	0.1780[1]
4.1	-0.7268[1]	0.1603[1]	0.8378[2]	0.9564[2]
4.2	-0.6927[1]	0.3218[2]	0.1667[2]	0.1891[2]
4.3	-0.6618[1]	-0.9116[2]	-0.4682[2]	-0.5279[2]
4.4	-0.6344[1]	-0.2099[1]	-0.1069[1]	-0.1199[1]
4.5	-0.6093[1]	-0.3245[1]	-0.1639[1]	-0.1830[1]
4.6	-0.5863[1]	-0.4350[1]	-0.2180[1]	-0.2424[1]
4.7	-0.5653[1]	-0.5417[1]	-0.2694[1]	-0.2984[1]
4.8	-0.5460[1]	-0.6448[1]	-0.3184[1]	-0.3513[1]
5.0	-0.5116[1]	-0.8410[1]	-0.4095[1]	-0.4489[1]
5.2	-0.4820[1]	-0.1025	-0.4927[1]	-0.5371[1]
5.4	-0.4562[1]	-0.1198	-0.5688[1]	-0.6171[1]
5.6	-0.4336[1]	-0.1361	-0.6388[1]	-0.6902[1]
5.8	-0.4135[1]	-0.1514	-0.7033[1]	-0.7571[1]
6.0	-0.3955[1]	-0.1660	-0.7631[1]	-0.8188[1]
6.2	-0.3794[1]	-0.1798	-0.8186[1]	-0.8758[1]
6.4	-0.3649[1]	-0.1928	-0.8704[1]	-0.9286[1]
6.6	-0.3517[1]	-0.2053	-0.9187[1]	-0.9778[1]
6.8	-0.3396[1]	-0.2171	-0.9639[1]	-0.1024
7.1	-0.3234[1]	-0.2339	-0.1027	-0.1087
7.4	-0.3090[1]	-0.2495	-0.1084	-0.1145
7.7	-0.2962[1]	-0.2641	-0.1136	-0.1197
8.0	-0.2848[1]	-0.2779	-0.1185	-0.1246
8.3	-0.2744[1]	-0.2908	-0.1229	-0.1290
8.8	-0.2593[1]	-0.3108	-0.1297	-0.1357
9.3	-0.2462[1]	-0.3290	-0.1357	-0.1417
9.8	-0.2349[1]	-0.3457	-0.1411	-0.1470
10.3	-0.2249[1]	-0.3611	-0.1459	-0.1518
10.8	-0.2161[1]	-0.3753	-0.1503	-0.1561
11.6	-0.2039[1]	-0.3959	-0.1565	-0.1622
12.4	-0.1936[1]	-0.4144	-0.1620	-0.1676
13.2	-0.1847[1]	-0.4310	-0.1668	-0.1722
14.0	-0.1770[1]	-0.4460	-0.1710	-0.1764
14.8	-0.1703[1]	-0.4597	-0.1748	-0.1801

$\hat{\alpha}_3 = 0.20$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.0	-1.3871	0.2841	0.2124^{1}	0.7090
2.1	-1.1865	0.2909	0.4302^{1}	0.6298
2.2	-1.0104	0.2947	0.6384^{1}	0.5571
2.3	-0.8509	0.2952	0.8359^{1}	0.4888
2.4	-0.7062	0.2919	0.1013	0.4246
2.5	-0.5783	0.2841	0.1151	0.3652
2.6	-0.4710	0.2718	0.1233	0.3120
2.7	-0.3862	0.2558	0.1253	0.2661
2.8	-0.3217	0.2374	0.1221	0.2273
2.9	-0.2734	0.2180	0.1153	0.1948
3.0	-0.2368	0.1983	0.1065	0.1672
3.1	-0.2087	0.1789	0.9673^{1}	0.1435
3.2	-0.1865	0.1599	0.8659^{1}	0.1230
3.3	-0.1687	0.1416	0.7651^{1}	0.1049
3.4	-0.1542	0.1240	0.6668^{1}	0.8888^{1}
3.5	-0.1420	0.1070	0.5722^{1}	0.7449^{1}
3.6	-0.1318	0.9075^{1}	0.4819^{1}	0.6150^{1}
3.7	-0.1230	0.7512^{1}	0.3960^{1}	0.4968^{1}
3.8	-0.1154	0.6012^{1}	0.3145^{1}	0.3888^{1}
3.9	-0.1088	0.4570^{1}	0.2372^{1}	0.2896^{1}
4.0	-0.1030	0.3185^{1}	0.1640^{1}	0.1980^{1}
4.1	-0.9786^{1}	0.1853^{1}	0.9467^{2}	0.1132^{1}
4.2	-0.9323^{1}	0.5707^{2}	0.2894^{2}	0.3429^{2}
4.3	-0.8910^{1}	-0.6640^{2}	-0.3342^{2}	-0.3929^{2}
4.4	-0.8535^{1}	-0.1854^{1}	-0.9261^{2}	-0.1081^{1}
4.5	-0.8195^{1}	-0.3002^{1}	-0.1489^{1}	-0.1727^{1}
4.6	-0.7886^{1}	-0.4110^{1}	-0.2024^{1}	-0.2334^{1}
4.7	-0.7600^{1}	-0.5179^{1}	-0.2533^{1}	-0.2906^{1}
4.8	-0.7339^{1}	-0.6213^{1}	-0.3019^{1}	-0.3445^{1}
4.9	-0.7097^{1}	-0.7213^{1}	-0.3482^{1}	-0.3956^{1}
5.1	-0.6668^{1}	-0.9118^{1}	-0.4347^{1}	-0.4900^{1}
5.3	-0.6294^{1}	-0.1091	-0.5139^{1}	-0.5753^{1}
5.5	-0.5968^{1}	-0.1259	-0.5867^{1}	-0.6528^{1}
5.7	-0.5680^{1}	-0.1418	-0.6539^{1}	-0.7237^{1}
5.9	-0.5424^{1}	-0.1568	-0.7160^{1}	-0.7888^{1}
6.1	-0.5195^{1}	-0.1710	-0.7737^{1}	-0.8487^{1}
6.3	-0.4989^{1}	-0.1844	-0.8274^{1}	-0.9041^{1}
6.5	-0.4802^{1}	-0.1972	-0.8775^{1}	-0.9556^{1}
6.7	-0.4632^{1}	-0.2094	-0.9243^{1}	-0.1004
6.9	-0.4477^{1}	-0.2210	-0.9683^{1}	-0.1048
7.2	-0.4267^{1}	-0.2374	-0.1029	-0.1110
7.5	-0.4082^{1}	-0.2528	-0.1085	-0.1166
7.8	-0.3916^{1}	-0.2672	-0.1137	-0.1218
8.1	-0.3767^{1}	-0.2807	-0.1184	-0.1265
8.4	-0.3633^{1}	-0.2934	-0.1228	-0.1309
8.9	-0.3435^{1}	-0.3131	-0.1294	-0.1375
9.4	-0.3265^{1}	-0.3310	-0.1353	-0.1433
9.9	-0.3116^{1}	-0.3475	-0.1406	-0.1485
10.4	-0.2986^{1}	-0.3627	-0.1454	-0.1532
10.9	-0.2870^{1}	-0.3767	-0.1497	-0.1575
11.7	-0.2710^{1}	-0.3971	-0.1559	-0.1635
12.5	-0.2574^{1}	-0.4153	-0.1613	-0.1687
13.3	-0.2457^{1}	-0.4317	-0.1660	-0.1733
14.1	-0.2356^{1}	-0.4466	-0.1703	-0.1774
14.9	-0.2267^{1}	-0.4602	-0.1740	-0.1811

$\hat{\alpha}_3 = 0.25$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.0	-1.4649	0.2748	0.1050^{1}	0.7034
2.1	-1.2592	0.2814	0.3133^{1}	0.6252
2.2	-1.0840	0.2847	0.5060^{1}	0.5548
2.3	-0.9287	0.2850	0.6845^{1}	0.4898
2.4	-0.7893	0.2820	0.8436^{1}	0.4294
2.5	-0.6654	0.2753	0.9730^{1}	0.3734
2.6	-0.5584	0.2649	0.1062	0.3226
2.7	-0.4696	0.2512	0.1103	0.2776
2.8	-0.3983	0.2349	0.1099	0.2385
2.9	-0.3423	0.2172	0.1059	0.2050
3.0	-0.2985	0.1987	0.9957^{1}	0.1763
3.1	-0.2640	0.1802	0.9173^{1}	0.1515
3.2	-0.2364	0.1618	0.8313^{1}	0.1300
3.3	-0.2140	0.1440	0.7423^{1}	0.1110
3.4	-0.1956	0.1266	0.6532^{1}	0.9419^{1}
3.5	-0.1802	0.1099	0.5658^{1}	0.7916^{1}
3.6	-0.1672	0.9375^{1}	0.4810^{1}	0.6560^{1}
3.7	-0.1560	0.7821^{1}	0.3995^{1}	0.5331^{1}
3.8	-0.1463	0.6326^{1}	0.3214^{1}	0.4210^{1}
3.9	-0.1379	0.4888^{1}	0.2469^{1}	0.3182^{1}
4.0	-0.1305	0.3504^{1}	0.1760^{1}	0.2235^{1}
4.1	-0.1239	0.2172^{1}	0.1084^{1}	0.1360^{1}
4.2	-0.1180	0.8889^{2}	0.4408^{2}	0.5467^{2}
4.3	-0.1127	-0.3477^{2}	-0.1713^{2}	-0.2104^{2}
4.4	-0.1080	-0.1541^{1}	-0.7542^{2}	-0.9178^{2}
4.5	-0.1036	-0.2691^{1}	-0.1309^{1}	-0.1580^{1}
4.6	-0.9964^{1}	-0.3801^{1}	-0.1838^{1}	-0.2202^{1}
4.7	-0.9602^{1}	-0.4875^{1}	-0.2343^{1}	-0.2787^{1}
4.8	-0.9270^{1}	-0.5912^{1}	-0.2825^{1}	-0.3340^{1}
4.9	-0.8962^{1}	-0.6916^{1}	-0.3285^{1}	-0.3861^{1}
5.1	-0.8415^{1}	-0.8828^{1}	-0.4147^{1}	-0.4824^{1}
5.3	-0.7941^{1}	-0.1062	-0.4937^{1}	-0.5693^{1}
5.5	-0.7526^{1}	-0.1231	-0.5665^{1}	-0.6482^{1}
5.7	-0.7161^{1}	-0.1391	-0.6336^{1}	-0.7202^{1}
5.9	-0.6836^{1}	-0.1541	-0.6959^{1}	-0.7862^{1}
6.1	-0.6545^{1}	-0.1684	-0.7537^{1}	-0.8470^{1}
6.3	-0.6284^{1}	-0.1819	-0.8075^{1}	-0.9031^{1}
6.5	-0.6047^{1}	-0.1948	-0.8578^{1}	-0.9552^{1}
6.7	-0.5832^{1}	-0.2070	-0.9049^{1}	-0.1004
6.9	-0.5635^{1}	-0.2187	-0.9491^{1}	-0.1049
7.2	-0.5370^{1}	-0.2351	-0.1010	-0.1111
7.5	-0.5135^{1}	-0.2506	-0.1067	-0.1168
7.8	-0.4925^{1}	-0.2650	-0.1118	-0.1220
8.1	-0.4737^{1}	-0.2786	-0.1166	-0.1268
8.4	-0.4567^{1}	-0.2914	-0.1210	-0.1312
8.9	-0.4318^{1}	-0.3111	-0.1277	-0.1378
9.4	-0.4103^{1}	-0.3291	-0.1337	-0.1437
9.9	-0.3915^{1}	-0.3457	-0.1390	-0.1489
10.4	-0.3751^{1}	-0.3609	-0.1438	-0.1536
10.9	-0.3605^{1}	-0.3750	-0.1482	-0.1579
11.7	-0.3403^{1}	-0.3954	-0.1544	-0.1639
12.5	-0.3232^{1}	-0.4138	-0.1599	-0.1692
13.3	-0.3085^{1}	-0.4302	-0.1647	-0.1738
14.1	-0.2957^{1}	-0.4452	-0.1689	-0.1779
14.9	-0.2845^{1}	-0.4588	-0.1727	-0.1816

$\hat{\alpha}_3 = 0.30$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.0	-1.5501	0.2660	-0.1001[4]	0.7019
2.1	-1.3428	0.2721	0.1956[1]	0.6249
2.2	-1.1637	0.2755	0.3797[1]	0.5556
2.3	-1.0087	0.2759	0.5467[1]	0.4927
2.4	-0.8712	0.2732	0.6949[1]	0.4348
2.5	-0.7491	0.2675	0.8187[1]	0.3813
2.6	-0.6421	0.2586	0.9108[1]	0.3324
2.7	-0.5505	0.2467	0.9658[1]	0.2884
2.8	-0.4743	0.2323	0.9830[1]	0.2495
2.9	-0.4121	0.2162	0.9671[1]	0.2155
3.0	-0.3618	0.1991	0.9255[1]	0.1859
3.1	-0.3213	0.1816	0.8661[1]	0.1602
3.2	-0.2884	0.1640	0.7957[1]	0.1377
3.3	-0.2614	0.1467	0.7191[1]	0.1179
3.4	-0.2390	0.1298	0.6398[1]	0.1003
3.5	-0.2202	0.1133	0.5600[1]	0.8457[1]
3.6	-0.2042	0.9733[1]	0.4813[1]	0.7043[1]
3.7	-0.1905	0.8192[1]	0.4046[1]	0.5763[1]
3.8	-0.1786	0.6705[1]	0.3303[1]	0.4597[1]
3.9	-0.1682	0.5272[1]	0.2588[1]	0.3530[1]
4.0	-0.1591	0.3891[1]	0.1902[1]	0.2549[1]
4.1	-0.1510	0.2559[1]	0.1245[1]	0.1643[1]
4.2	-0.1438	0.1275[1]	0.6176[2]	0.8035[2]
4.3	-0.1381	0.7885[3]	0.3799[3]	0.4890[3]
4.4	-0.1314	-0.1158[1]	-0.5555[2]	-0.7058[2]
4.5	-0.1260	-0.2312[1]	-0.1103[1]	-0.1387[1]
4.6	-0.1212	-0.3427[1]	-0.1626[1]	-0.2026[1]
4.7	-0.1167	-0.4504[1]	-0.2126[1]	-0.2627[1]
4.8	-0.1127	-0.5545[1]	-0.2604[1]	-0.3193[1]
4.9	-0.1089	-0.6553[1]	-0.3062[1]	-0.3728[1]
5.1	-0.1022	-0.8474[1]	-0.3920[1]	-0.4713[1]
5.3	-0.9637[1]	-0.1028	-0.4709[1]	-0.5600[1]
5.5	-0.9129[1]	-0.1198	-0.5437[1]	-0.6404[1]
5.7	-0.8682[1]	-0.1358	-0.6110[1]	-0.7137[1]
5.9	-0.8285[1]	-0.1509	-0.6734[1]	-0.7809[1]
6.1	-0.7930[1]	-0.1653	-0.7314[1]	-0.8426[1]
6.3	-0.7611[1]	-0.1789	-0.7855[1]	-0.8997[1]
6.5	-0.7322[1]	-0.1918	-0.8361[1]	-0.9525[1]
6.7	-0.7059[1]	-0.2041	-0.8834[1]	-0.1002
6.9	-0.6819[1]	-0.2158	-0.9279[1]	-0.1047
7.2	-0.6496[1]	-0.2324	-0.9897[1]	-0.1111
7.5	-0.6210[1]	-0.2479	-0.1046	-0.1168
7.8	-0.5955[1]	-0.2624	-0.1098	-0.1221
8.1	-0.5726[1]	-0.2760	-0.1147	-0.1269
8.4	-0.5519[1]	-0.2889	-0.1191	-0.1313
8.9	-0.5216[1]	-0.3087	-0.1258	-0.1380
9.4	-0.4955[1]	-0.3269	-0.1319	-0.1439
9.9	-0.4728[1]	-0.3435	-0.1372	-0.1492
10.4	-0.4528[1]	-0.3588	-0.1421	-0.1539
10.9	-0.4351[1]	-0.3729	-0.1465	-0.1582
11.7	-0.4106[1]	-0.3935	-0.1528	-0.1643
12.5	-0.3899[1]	-0.4118	-0.1583	-0.1695
13.3	-0.3721[1]	-0.4284	-0.1632	-0.1742
14.1	-0.3566[1]	-0.4434	-0.1675	-0.1783
14.9	-0.3431[1]	-0.4570	-0.1713	-0.1819

$\hat{\alpha}_3 = 0.35$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.1	-1.4414	0.2630	0.7342[2]	0.6297
2.2	-1.2523	0.2667	0.2556[1]	0.5599
2.3	-1.0934	0.2674	0.4168[1]	0.4979
2.4	-0.9548	0.2653	0.5590[1]	0.4415
2.5	-0.8324	0.2604	0.6797[1]	0.3897
2.6	-0.7245	0.2527	0.7745[1]	0.3423
2.7	-0.6307	0.2424	0.8394[1]	0.2992
2.8	-0.5504	0.2297	0.8727[1]	0.2606
2.9	-0.4830	0.2153	0.8761[1]	0.2263
3.0	-0.4271	0.1996	0.8543[1]	0.1961
3.1	-0.3809	0.1832	0.8131[1]	0.1695
3.2	-0.3428	0.1665	0.7581[1]	0.1462
3.3	-0.3112	0.1498	0.6944[1]	0.1255
3.4	-0.2847	0.1333	0.6254[1]	0.1072
3.5	-0.2623	0.1171	0.5540[1]	0.9077[1]
3.6	-0.2432	0.1014	0.4819[1]	0.7602[1]
3.7	-0.2268	0.8619[1]	0.4104[1]	0.6267[1]
3.8	-0.2126	0.7144[1]	0.3403[1]	0.5053[1]
3.9	-0.2001	0.5718[1]	0.2722[1]	0.3943[1]
4.0	-0.1892	0.4341[1]	0.2062[1]	0.2925[1]
4.1	-0.1794	0.3011[1]	0.1427[1]	0.1986[1]
4.2	-0.1707	0.1727[1]	0.8158[2]	0.1116[1]
4.3	-0.1629	0.4870[2]	0.2292[2]	0.3090[2]
4.4	-0.1559	-0.7105[2]	-0.3332[2]	-0.4431[2]
4.5	-0.1495	-0.1868[1]	-0.8725[2]	-0.1146[1]
4.6	-0.1437	-0.2986[1]	-0.1389[1]	-0.1804[1]
4.7	-0.1383	-0.4067[1]	-0.1884[1]	-0.2423[1]
4.8	-0.1334	-0.5113[1]	-0.2358[1]	-0.3004[1]
4.9	-0.1289	-0.6125[1]	-0.2813[1]	-0.3553[1]
5.0	-0.1247	-0.7106[1]	-0.3249[1]	-0.4072[1]
5.2	-0.1173	-0.8977[1]	-0.4070[1]	-0.5029[1]
5.4	-0.1108	-0.1074	-0.4827[1]	-0.5893[1]
5.6	-0.1051	-0.1240	-0.5528[1]	-0.6677[1]
5.8	-0.1001	-0.1396	-0.6178[1]	-0.7392[1]
6.0	-0.9563[1]	-0.1545	-0.6782[1]	-0.8048[1]
6.2	-0.9162[1]	-0.1685	-0.7345[1]	-0.8652[1]
6.4	-0.8801[1]	-0.1818	-0.7871[1]	-0.9210[1]
6.6	-0.8473[1]	-0.1945	-0.8364[1]	-0.9727[1]
6.8	-0.8175[1]	-0.2066	-0.8826[1]	-0.1021
7.0	-0.7902[1]	-0.2181	-0.9261[1]	-0.1066
7.3	-0.7534[1]	-0.2344	-0.9866[1]	-0.1128
7.6	-0.7207[1]	-0.2497	-0.1042	-0.1184
7.9	-0.6916[1]	-0.2640	-0.1093	-0.1236
8.2	-0.6654[1]	-0.2774	-0.1141	-0.1283
8.5	-0.6417[1]	-0.2901	-0.1185	-0.1327
9.0	-0.6068[1]	-0.3097	-0.1251	-0.1393
9.5	-0.5768[1]	-0.3276	-0.1310	-0.1451
10.0	-0.5506[1]	-0.3440	-0.1364	-0.1503
10.5	-0.5276[1]	-0.3592	-0.1412	-0.1550
11.0	-0.5072[1]	-0.3732	-0.1456	-0.1592
11.8	-0.4789[1]	-0.3935	-0.1518	-0.1652
12.6	-0.4549[1]	-0.4117	-0.1573	-0.1704
13.4	-0.4343[1]	-0.4282	-0.1621	-0.1750
14.2	-0.4164[1]	-0.4430	-0.1664	-0.1790
15.0	-0.4007[1]	-0.4566	-0.1703	-0.1827

$\hat{\alpha}_3 = 0.40$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.2	-1.3537	0.2583	0.1295^{1}	0.5683
2.3	-1.1858	0.2595	0.2904^{1}	0.5058
2.4	-1.0429	0.2580	0.4305^{1}	0.4500
2.5	-0.9180	0.2539	0.5504^{1}	0.3991
2.6	-0.8081	0.2473	0.6481^{1}	0.3527
2.7	-0.7115	0.2384	0.7207^{1}	0.3103
2.8	-0.6277	0.2272	0.7666^{1}	0.2720
2.9	-0.5557	0.2143	0.7861^{1}	0.2376
3.0	-0.4947	0.2000	0.7817^{1}	0.2069
3.1	-0.4433	0.1848	0.7574^{1}	0.1797
3.2	-0.4001	0.1690	0.7177^{1}	0.1555
3.3	-0.3638	0.1530	0.6671^{1}	0.1341
3.4	-0.3331	0.1371	0.6091^{1}	0.1149
3.5	-0.3070	0.1214	0.5467^{1}	0.9782^{1}
3.6	-0.2846	0.1060	0.4818^{1}	0.8242^{1}
3.7	-0.2654	0.9099^{1}	0.4162^{1}	0.6849^{1}
3.8	-0.2486	0.7639^{1}	0.3508^{1}	0.5583^{1}
3.9	-0.2339	0.6223^{1}	0.2864^{1}	0.4427^{1}
4.0	-0.2210	0.4852^{1}	0.2235^{1}	0.3368^{1}
4.1	-0.2095	0.3525^{1}	0.1623^{1}	0.2391^{1}
4.2	-0.1992	0.2242^{1}	0.1031^{1}	0.1489^{1}
4.3	-0.1900	0.1002^{1}	0.4597^{2}	0.6521^{2}
4.4	-0.1816	-0.1984^{2}	-0.9089^{3}	-0.1269^{2}
4.5	-0.1741	-0.1358^{1}	-0.6204^{2}	-0.8534^{2}
4.6	-0.1672	-0.2480^{1}	-0.1130^{1}	-0.1533^{1}
4.7	-0.1610	-0.3566^{1}	-0.1619^{1}	-0.2171^{1}
4.8	-0.1552	-0.4617^{1}	-0.2089^{1}	-0.2771^{1}
4.9	-0.1499	-0.5634^{1}	-0.2541^{1}	-0.3336^{1}
5.0	-0.1450	-0.6620^{1}	-0.2975^{1}	-0.3870^{1}
5.1	-0.1404	-0.7576^{1}	-0.3392^{1}	-0.4375^{1}
5.3	-0.1322	-0.9401^{1}	-0.4179^{1}	-0.5307^{1}
5.5	-0.1251	-0.1112	-0.4908^{1}	-0.6150^{1}
5.7	-0.1189	-0.1274	-0.5585^{1}	-0.6916^{1}
5.9	-0.1133	-0.1428	-0.6214^{1}	-0.7616^{1}
6.1	-0.1084	-0.1573	-0.6801^{1}	-0.8258^{1}
6.3	-0.1039	-0.1711	-0.7349^{1}	-0.8850^{1}
6.5	-0.9989^{1}	-0.1842	-0.7861^{1}	-0.9397^{1}
6.7	-0.9624^{1}	-0.1967	-0.8342^{1}	-0.9905^{1}
6.9	-0.9290^{1}	-0.2085	-0.8794^{1}	-0.1038
7.1	-0.8985^{1}	-0.2199	-0.9220^{1}	-0.1082
7.4	-0.8573^{1}	-0.2359	-0.9814^{1}	-0.1143
7.7	-0.8207^{1}	-0.2510	-0.1036	-0.1198
8.0	-0.7879^{1}	-0.2651	-0.1086	-0.1249
8.3	-0.7584^{1}	-0.2783	-0.1133	-0.1296
8.6	-0.7317^{1}	-0.2909	-0.1176	-0.1339
9.1	-0.6925^{1}	-0.3102	-0.1242	-0.1404
9.6	-0.6586^{1}	-0.3279	-0.1301	-0.1461
10.1	-0.6290^{1}	-0.3442	-0.1354	-0.1513
10.6	-0.6029^{1}	-0.3592	-0.1402	-0.1559
11.1	-0.5798^{1}	-0.3730	-0.1445	-0.1601
11.9	-0.5477^{1}	-0.3932	-0.1507	-0.1660
12.7	-0.5205^{1}	-0.4113	-0.1562	-0.1712
13.5	-0.4970^{1}	-0.4276	-0.1610	-0.1757
14.3	-0.4767^{1}	-0.4424	-0.1653	-0.1797
15.1	-0.4588^{1}	-0.4559	-0.1691	-0.1833

$\hat{\alpha}_3 = 0.45$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.3	-1.2899	0.2519	0.1629^{1}	0.5173
2.4	-1.1383	0.2511	0.3051^{1}	0.4608
2.5	-1.0083	0.2479	0.4266^{1}	0.4101
2.6	-0.8946	0.2423	0.5277^{1}	0.3641
2.7	-0.7947	0.2346	0.6070^{1}	0.3221
2.8	-0.7071	0.2248	0.6633^{1}	0.2840
2.9	-0.6308	0.2133	0.6963^{1}	0.2495
3.0	-0.5650	0.2004	0.7072^{1}	0.2184
3.1	-0.5086	0.1863	0.6986^{1}	0.1906
3.2	-0.4605	0.1716	0.6737^{1}	0.1657
3.3	-0.4196	0.1565	0.6364^{1}	0.1435
3.4	-0.3846	0.1412	0.5898^{1}	0.1236
3.5	-0.3546	0.1260	0.5370^{1}	0.1058
3.6	-0.3288	0.1110	0.4801^{1}	0.8969^{1}
3.7	-0.3065	0.9623^{1}	0.4209^{1}	0.7514^{1}
3.8	-0.2870	0.8183^{1}	0.3608^{1}	0.6192^{1}
3.9	-0.2699	0.6781^{1}	0.3007^{1}	0.4986^{1}
4.0	-0.2548	0.5419^{1}	0.2412^{1}	0.3880^{1}
4.1	-0.2414	0.4098^{1}	0.1828^{1}	0.2864^{1}
4.2	-0.2294	0.2818^{1}	0.1258^{1}	0.1925^{1}
4.3	-0.2186	0.1577^{1}	0.7045^{2}	0.1055^{1}
4.4	-0.2090	0.3763^{2}	0.1680^{2}	0.2470^{2}
4.5	-0.2001	-0.7867^{2}	-0.3508^{2}	-0.5068^{2}
4.6	-0.1921	-0.1912^{1}	-0.8511^{2}	-0.1211^{1}
4.7	-0.1848	-0.3002^{1}	-0.1334^{1}	-0.1871^{1}
4.8	-0.1781	-0.4058^{1}	-0.1799^{1}	-0.2491^{1}
4.9	-0.1719	-0.5081^{1}	-0.2247^{1}	-0.3075^{1}
5.0	-0.1662	-0.6072^{1}	-0.2678^{1}	-0.3625^{1}
5.1	-0.1609	-0.7034^{1}	-0.3094^{1}	-0.4145^{1}
5.2	-0.1560	-0.7966^{1}	-0.3494^{1}	-0.4638^{1}
5.4	-0.1472	-0.9750^{1}	-0.4251^{1}	-0.5549^{1}
5.6	-0.1394	-0.1143	-0.4955^{1}	-0.6373^{1}
5.8	-0.1326	-0.1302	-0.5610^{1}	-0.7124^{1}
6.0	-0.1265	-0.1453	-0.6221^{1}	-0.7810^{1}
6.2	-0.1211	-0.1595	-0.6791^{1}	-0.8440^{1}
6.4	-0.1162	-0.1731	-0.7325^{1}	-0.9021^{1}
6.6	-0.1118	-0.1859	-0.7826^{1}	-0.9559^{1}
6.8	-0.1078	-0.1982	-0.8297^{1}	-0.1006
7.0	-0.1041	-0.2099	-0.8740^{1}	-0.1052
7.2	-0.1007	-0.2211	-0.9157^{1}	-0.1096
7.5	-0.9618^{1}	-0.2369	-0.9741^{1}	-0.1156
7.8	-0.9212^{1}	-0.2517	-0.1028	-0.1211
8.1	-0.8849^{1}	-0.2657	-0.1077	-0.1261
8.4	-0.8522^{1}	-0.2788	-0.1124	-0.1307
8.7	-0.8225^{1}	-0.2912	-0.1166	-0.1349
9.2	-0.7788^{1}	-0.3103	-0.1231	-0.1413
9.7	-0.7411^{1}	-0.3278	-0.1290	-0.1470
10.2	-0.7080^{1}	-0.3439	-0.1342	-0.1521
10.7	-0.6789^{1}	-0.3588	-0.1390	-0.1567
11.2	-0.6531^{1}	-0.3726	-0.1433	-0.1608
12.0	-0.6172^{1}	-0.3926	-0.1495	-0.1667
12.8	-0.5867^{1}	-0.4106	-0.1550	-0.1718
13.6	-0.5605^{1}	-0.4267	-0.1598	-0.1763
14.4	-0.5376^{1}	-0.4414	-0.1640	-0.1803
15.2	-0.5176^{1}	-0.4548	-0.1679	-0.1839

$\hat{\alpha}_3 = 0.50$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.3	-1.4124	0.2443	0.2873^{2}	0.5334
2.4	-1.2450	0.2445	0.1784^{1}	0.4748
2.5	-1.1060	0.2422	0.3045^{1}	0.4233
2.6	-0.9864	0.2376	0.4102^{1}	0.3770
2.7	-0.8818	0.2310	0.4961^{1}	0.3350
2.8	-0.7898	0.2225	0.5614^{1}	0.2969
2.9	-0.7091	0.2123	0.6060^{1}	0.2621
3.0	-0.6387	0.2007	0.6304^{1}	0.2307
3.1	-0.5775	0.1878	0.6361^{1}	0.2023
3.2	-0.5246	0.1742	0.6254^{1}	0.1768
3.3	-0.4790	0.1600	0.6013^{1}	0.1539
3.4	-0.4396	0.1454	0.5665^{1}	0.1332
3.5	-0.4057	0.1308	0.5239^{1}	0.1146
3.6	-0.3762	0.1163	0.4756^{1}	0.9786^{1}
3.7	-0.3506	0.1019	0.4237^{1}	0.8266^{1}
3.8	-0.3282	0.8771^{1}	0.3695^{1}	0.6885^{1}
3.9	-0.3084	0.7387^{1}	0.3142^{1}	0.5624^{1}
4.0	-0.2910	0.6038^{1}	0.2587^{1}	0.4469^{1}
4.1	-0.2755	0.4726^{1}	0.2035^{1}	0.3409^{1}
4.2	-0.2617	0.3450^{1}	0.1491^{1}	0.2429^{1}
4.3	-0.2492	0.2212^{1}	0.9582^{2}	0.1523^{1}
4.4	-0.2380	0.1011^{1}	0.4385^{2}	0.6817^{2}
4.5	-0.2278	-0.1539^{2}	-0.6683^{3}	-0.1018^{2}
4.6	-0.2186	-0.1283^{1}	-0.5570^{2}	-0.8335^{2}
4.7	-0.2101	-0.2377^{1}	-0.1032^{1}	-0.1518^{1}
4.8	-0.2023	-0.3438^{1}	-0.1491^{1}	-0.2161^{1}
4.9	-0.1952	-0.4466^{1}	-0.1934^{1}	-0.2765^{1}
5.0	-0.1886	-0.5463^{1}	-0.2362^{1}	-0.3335^{1}
5.1	-0.1825	-0.6431^{1}	-0.2775^{1}	-0.3872^{1}
5.2	-0.1769	-0.7370^{1}	-0.3174^{1}	-0.4381^{1}
5.4	-0.1667	-0.9167^{1}	-0.3931^{1}	-0.5320^{1}
5.6	-0.1578	-0.1086	-0.4635^{1}	-0.6169^{1}
5.8	-0.1500	-0.1246	-0.5293^{1}	-0.6941^{1}
6.0	-0.1430	-0.1398	-0.5907^{1}	-0.7645^{1}
6.2	-0.1368	-0.1542	-0.6482^{1}	-0.8291^{1}
6.4	-0.1312	-0.1679	-0.7020^{1}	-0.8886^{1}
6.6	-0.1261	-0.1809	-0.7526^{1}	-0.9437^{1}
6.8	-0.1216	-0.1932	-0.8001^{1}	-0.9947^{1}
7.0	-0.1174	-0.2050	-0.8449^{1}	-0.1042
7.2	-0.1135	-0.2163	-0.8872^{1}	-0.1087
7.5	-0.1083	-0.2323	-0.9462^{1}	-0.1148
7.8	-0.1037	-0.2472	-0.1001	-0.1204
8.1	-0.9957^{1}	-0.2613	-0.1051	-0.1255
8.4	-0.9585^{1}	-0.2745	-0.1098	-0.1301
8.7	-0.9248^{1}	-0.2870	-0.1141	-0.1345
9.2	-0.8752^{1}	-0.3063	-0.1207	-0.1410
9.7	-0.8324^{1}	-0.3240	-0.1266	-0.1467
10.2	-0.7950^{1}	-0.3402	-0.1319	-0.1519
10.7	-0.7620^{1}	-0.3552	-0.1368	-0.1565
11.2	-0.7328^{1}	-0.3691	-0.1412	-0.1607
12.0	-0.6922^{1}	-0.3892	-0.1475	-0.1666
12.8	-0.6578^{1}	-0.4073	-0.1530	-0.1718
13.6	-0.6282^{1}	-0.4236	-0.1579	-0.1763
14.4	-0.6024^{1}	-0.4384	-0.1622	-0.1803
15.2	-0.5798^{1}	-0.4519	-0.1661	-0.1839

$\hat{\alpha}_3 = 0.55$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.4	-1.3697	0.2379	0.4465^{2}	0.4931
2.5	-1.2153	0.2366	0.1798^{1}	0.4393
2.6	-1.0862	0.2331	0.2925^{1}	0.3920
2.7	-0.9748	0.2276	0.3854^{1}	0.3495
2.8	-0.8774	0.2202	0.4593^{1}	0.3109
2.9	-0.7918	0.2113	0.5143^{1}	0.2758
3.0	-0.7165	0.2009	0.5508^{1}	0.2440
3.1	-0.6504	0.1893	0.5696^{1}	0.2150
3.2	-0.5927	0.1767	0.5723^{1}	0.1889
3.3	-0.5424	0.1634	0.5611^{1}	0.1652
3.4	-0.4986	0.1497	0.5383^{1}	0.1438
3.5	-0.4605	0.1358	0.5063^{1}	0.1245
3.6	-0.4272	0.1217	0.4673^{1}	0.1070
3.7	-0.3981	0.1078	0.4233^{1}	0.9111^{1}
3.8	-0.3725	0.9394^{1}	0.3757^{1}	0.7666^{1}
3.9	-0.3499	0.8034^{1}	0.3260^{1}	0.6347^{1}
4.0	-0.3299	0.6702^{1}	0.2750^{1}	0.5139^{1}
4.1	-0.3121	0.5400^{1}	0.2236^{1}	0.4028^{1}
4.2	-0.2963	0.4134^{1}	0.1723^{1}	0.3006^{1}
4.3	-0.2820	0.2900^{1}	0.1215^{1}	0.2059^{1}
4.4	-0.2691	0.1700^{1}	0.7150^{2}	0.1181^{1}
4.5	-0.2574	0.5354^{2}	0.2258^{2}	0.3644^{2}
4.6	-0.2468	-0.5954^{2}	-0.2515^{2}	-0.3975^{2}
4.7	-0.2371	-0.1693^{1}	-0.7160^{2}	-0.1110^{1}
4.8	-0.2282	-0.2758^{1}	-0.1167^{1}	-0.1778^{1}
4.9	-0.2200	-0.3792^{1}	-0.1604^{1}	-0.2405^{1}
5.0	-0.2124	-0.4795^{1}	-0.2028^{1}	-0.2996^{1}
5.1	-0.2055	-0.5769^{1}	-0.2438^{1}	-0.3553^{1}
5.2	-0.1990	-0.6714^{1}	-0.2835^{1}	-0.4080^{1}
5.3	-0.1930	-0.7632^{1}	-0.3218^{1}	-0.4579^{1}
5.5	-0.1821	-0.9392^{1}	-0.3948^{1}	-0.5501^{1}
5.7	-0.1726	-0.1105	-0.4631^{1}	-0.6335^{1}
5.9	-0.1642	-0.1263	-0.5269^{1}	-0.7094^{1}
6.1	-0.1567	-0.1412	-0.5867^{1}	-0.7787^{1}
6.3	-0.1500	-0.1553	-0.6428^{1}	-0.8424^{1}
6.5	-0.1440	-0.1688	-0.6954^{1}	-0.9011^{1}
6.7	-0.1385	-0.1816	-0.7449^{1}	-0.9554^{1}
6.9	-0.1335	-0.1938	-0.7916^{1}	-0.1006
7.1	-0.1290	-0.2054	-0.8356^{1}	-0.1053
7.3	-0.1248	-0.2165	-0.8771^{1}	-0.1097
7.6	-0.1192	-0.2323	-0.9353^{1}	-0.1157
7.9	-0.1141	-0.2471	-0.9891^{1}	-0.1212
8.2	-0.1096	-0.2610	-0.1039	-0.1263
8.5	-0.1056	-0.2741	-0.1085	-0.1309
8.8	-0.1019	-0.2865	-0.1128	-0.1352
9.3	-0.9647^{1}	-0.3056	-0.1194	-0.1416
9.8	-0.9178^{1}	-0.3231	-0.1252	-0.1474
10.3	-0.8768^{1}	-0.3392	-0.1305	-0.1525
10.8	-0.8407^{1}	-0.3541	-0.1354	-0.1571
11.3	-0.8086^{1}	-0.3679	-0.1398	-0.1612
12.1	-0.7641^{1}	-0.3879	-0.1460	-0.1671
12.9	-0.7263^{1}	-0.4059	-0.1515	-0.1722
13.7	-0.6938^{1}	-0.4221	-0.1564	-0.1768
14.5	-0.6654^{1}	-0.4369	-0.1608	-0.1808
15.3	-0.6406^{1}	-0.4503	-0.1647	-0.1843

$\hat{\alpha}_3 = 0.60$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.5	-1.3428	0.2310	0.4715^{2}	0.4596
2.6	-1.1981	0.2286	0.1706^{1}	0.4098
2.7	-1.0766	0.2242	0.2723^{1}	0.3659
2.8	-0.9718	0.2180	0.3551^{1}	0.3265
2.9	-0.8801	0.2102	0.4200^{1}	0.2908
3.0	-0.7993	0.2010	0.4675^{1}	0.2583
3.1	-0.7281	0.1905	0.4984^{1}	0.2288
3.2	-0.6654	0.1791	0.5137^{1}	0.2020
3.3	-0.6103	0.1668	0.5151^{1}	0.1776
3.4	-0.5619	0.1540	0.5042^{1}	0.1554
3.5	-0.5195	0.1408	0.4833^{1}	0.1354
3.6	-0.4822	0.1274	0.4541^{1}	0.1171
3.7	-0.4493	0.1139	0.4187^{1}	0.1005
3.8	-0.4204	0.1004	0.3785^{1}	0.8541^{1}
3.9	-0.3948	0.8714^{1}	0.3350^{1}	0.7160^{1}
4.0	-0.3720	0.7404^{1}	0.2893^{1}	0.5894^{1}
4.1	-0.3517	0.6119^{1}	0.2422^{1}	0.4731^{1}
4.2	-0.3336	0.4863^{1}	0.1945^{1}	0.3660^{1}
4.3	-0.3173	0.3637^{1}	0.1467^{1}	0.2669^{1}
4.4	-0.3026	0.2442^{1}	0.9911^{2}	0.1750^{1}
4.5	-0.2892	0.1278^{1}	0.5215^{2}	0.8965^{2}
4.6	-0.2770	0.1491^{2}	0.6111^{3}	0.1025^{2}
4.7	-0.2659	-0.9531^{2}	-0.3916^{2}	-0.6425^{2}
4.8	-0.2558	-0.2022^{1}	-0.8328^{2}	-0.1339^{1}
4.9	-0.2464	-0.3061^{1}	-0.1262^{1}	-0.1992^{1}
5.0	-0.2378	-0.4069^{1}	-0.1680^{1}	-0.2606^{1}
5.1	-0.2299	-0.5048^{1}	-0.2085^{1}	-0.3186^{1}
5.2	-0.2225	-0.6001^{1}	-0.2479^{1}	-0.3733^{1}
5.3	-0.2157	-0.6926^{1}	-0.2861^{1}	-0.4250^{1}
5.4	-0.2093	-0.7825^{1}	-0.3230^{1}	-0.4740^{1}
5.6	-0.1977	-0.9549^{1}	-0.3936^{1}	-0.5647^{1}
5.8	-0.1876	-0.1118	-0.4597^{1}	-0.6469^{1}
6.0	-0.1786	-0.1273	-0.5218^{1}	-0.7217^{1}
6.2	-0.1706	-0.1419	-0.5801^{1}	-0.7902^{1}
6.4	-0.1634	-0.1559	-0.6349^{1}	-0.8531^{1}
6.6	-0.1569	-0.1691	-0.6865^{1}	-0.9111^{1}
6.8	-0.1510	-0.1817	-0.7351^{1}	-0.9648^{1}
7.0	-0.1457	-0.1938	-0.7809^{1}	-0.1015
7.2	-0.1408	-0.2053	-0.8242^{1}	-0.1061
7.4	-0.1363	-0.2163	-0.8652^{1}	-0.1105
7.7	-0.1302	-0.2319	-0.9226^{1}	-0.1165
8.0	-0.1247	-0.2465	-0.9758^{1}	-0.1219
8.3	-0.1198	-0.2603	-0.1025	-0.1269
8.6	-0.1154	-0.2732	-0.1071	-0.1315
8.9	-0.1115	-0.2855	-0.1114	-0.1358
9.4	-0.1056	-0.3045	-0.1179	-0.1422
9.9	-0.1005	-0.3219	-0.1237	-0.1479
10.4	-0.9601^{1}	-0.3379	-0.1290	-0.1529
10.9	-0.9208^{1}	-0.3526	-0.1338	-0.1575
11.4	-0.8859^{1}	-0.3663	-0.1382	-0.1616
12.2	-0.8373^{1}	-0.3863	-0.1445	-0.1675
13.0	-0.7961^{1}	-0.4042	-0.1500	-0.1726
13.8	-0.7605^{1}	-0.4204	-0.1549	-0.1771
14.6	-0.7296^{1}	-0.4350	-0.1592	-0.1811
15.4	-0.7024^{1}	-0.4484	-0.1631	-0.1846

$\hat{\alpha}_3 = 0.65$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.6	-1.3292	0.2240	0.3908^{2}	0.4318
2.7	-1.1915	0.2208	0.1533^{1}	0.3853
2.8	-1.0759	0.2157	0.2465^{1}	0.3443
2.9	-0.9761	0.2090	0.3215^{1}	0.3075
3.0	-0.8887	0.2010	0.3797^{1}	0.2742
3.1	-0.8117	0.1916	0.4220^{1}	0.2438
3.2	-0.7436	0.1813	0.4493^{1}	0.2162
3.3	-0.6834	0.1700	0.4627^{1}	0.1911
3.4	-0.6302	0.1581	0.4637^{1}	0.1682
3.5	-0.5832	0.1457	0.4539^{1}	0.1473
3.6	-0.5417	0.1330	0.4350^{1}	0.1283
3.7	-0.5049	0.1201	0.4088^{1}	0.1110
3.8	-0.4723	0.1071	0.3767^{1}	0.9516^{1}
3.9	-0.4434	0.9417^{1}	0.3402^{1}	0.8068^{1}
4.0	-0.4177	0.8136^{1}	0.3004^{1}	0.6740^{1}
4.1	-0.3946	0.6873^{1}	0.2584^{1}	0.5519^{1}
4.2	-0.3740	0.5632^{1}	0.2148^{1}	0.4395^{1}
4.3	-0.3555	0.4418^{1}	0.1706^{1}	0.3356^{1}
4.4	-0.3387	0.3229^{1}	0.1259^{1}	0.2393^{1}
4.5	-0.3235	0.2070^{1}	0.8137^{2}	0.1499^{1}
4.6	-0.3097	0.9398^{2}	0.3719^{2}	0.6659^{2}
4.7	-0.2970	-0.1607^{2}	-0.6394^{3}	-0.1116^{2}
4.8	-0.2855	-0.1232^{1}	-0.4921^{2}	-0.8391^{2}
4.9	-0.2749	-0.2274^{1}	-0.9114^{2}	-0.1521^{1}
5.0	-0.2651	-0.3287^{1}	-0.1321^{1}	-0.2162^{1}
5.1	-0.2560	-0.4273^{1}	-0.1721^{1}	-0.2766^{1}
5.2	-0.2477	-0.5231^{1}	-0.2110^{1}	-0.3335^{1}
5.3	-0.2399	-0.6163^{1}	-0.2488^{1}	-0.3873^{1}
5.4	-0.2327	-0.7069^{1}	-0.2855^{1}	-0.4383^{1}
5.5	-0.2259	-0.7950^{1}	-0.3212^{1}	-0.4866^{1}
5.7	-0.2137	-0.9643^{1}	-0.3895^{1}	-0.5761^{1}
5.9	-0.2029	-0.1125	-0.4537^{1}	-0.6572^{1}
6.1	-0.1933	-0.1277	-0.5142^{1}	-0.7312^{1}
6.3	-0.1848	-0.1421	-0.5711^{1}	-0.7989^{1}
6.5	-0.1771	-0.1558	-0.6248^{1}	-0.8612^{1}
6.7	-0.1701	-0.1689	-0.6754^{1}	-0.9187^{1}
6.9	-0.1638	-0.1814	-0.7231^{1}	-0.9720^{1}
7.1	-0.1581	-0.1933	-0.7682^{1}	-0.1021
7.3	-0.1528	-0.2046	-0.8109^{1}	-0.1068
7.5	-0.1480	-0.2155	-0.8513^{1}	-0.1111
7.8	-0.1414	-0.2309	-0.9081^{1}	-0.1170
8.1	-0.1355	-0.2454	-0.9607^{1}	-0.1225
8.4	-0.1303	-0.2591	-0.1010	-0.1274
8.7	-0.1255	-0.2720	-0.1055	-0.1320
9.0	-0.1212	-0.2841	-0.1098	-0.1362
9.5	-0.1149	-0.3030	-0.1162	-0.1426
10.0	-0.1093	-0.3202	-0.1221	-0.1482
10.5	-0.1045	-0.3362	-0.1273	-0.1533
11.0	-0.1003	-0.3508	-0.1321	-0.1578
11.5	-0.9646^{1}	-0.3645	-0.1365	-0.1620
12.3	-0.9120^{1}	-0.3843	-0.1428	-0.1678
13.1	-0.8672^{1}	-0.4022	-0.1483	-0.1729
13.9	-0.8286^{1}	-0.4183	-0.1532	-0.1774
14.7	-0.7950^{1}	-0.4329	-0.1576	-0.1813
15.5	-0.7655^{1}	-0.4462	-0.1615	-0.1849

$\hat{\alpha}_3 = 0.70$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.7	-1.3272	0.2170	0.2251^{2}	0.4090
2.8	-1.1943	0.2132	0.1298^{1}	0.3651
2.9	-1.0829	0.2077	0.2165^{1}	0.3264
3.0	-0.9867	0.2008	0.2858^{1}	0.2918
3.1	-0.9025	0.1926	0.3393^{1}	0.2604
3.2	-0.8283	0.1833	0.3781^{1}	0.2319
3.3	-0.7625	0.1731	0.4033^{1}	0.2059
3.4	-0.7041	0.1621	0.4159^{1}	0.1821
3.5	-0.6523	0.1505	0.4173^{1}	0.1604
3.6	-0.6062	0.1386	0.4091^{1}	0.1406
3.7	-0.5653	0.1263	0.3925^{1}	0.1225
3.8	-0.5288	0.1138	0.3691^{1}	0.1060
3.9	-0.4963	0.1014	0.3403^{1}	0.9077^{1}
4.0	-0.4673	0.8889^{1}	0.3072^{1}	0.7683^{1}
4.1	-0.4413	0.7654^{1}	0.2710^{1}	0.6401^{1}
4.2	-0.4179	0.6434^{1}	0.2324^{1}	0.5219^{1}
4.3	-0.3969	0.5234^{1}	0.1922^{1}	0.4126^{1}
4.4	-0.3779	0.4058^{1}	0.1511^{1}	0.3116^{1}
4.5	-0.3607	0.2905^{1}	0.1094^{1}	0.2177^{1}
4.6	-0.3450	0.1779^{1}	0.6767^{2}	0.1303^{1}
4.7	-0.3306	0.6799^{2}	0.2607^{2}	0.4872^{2}
4.8	-0.3175	-0.3918^{2}	-0.1512^{2}	-0.2751^{2}
4.9	-0.3055	-0.1436^{1}	-0.5574^{2}	-0.9894^{2}
5.0	-0.2944	-0.2453^{1}	-0.9565^{2}	-0.1660^{1}
5.1	-0.2842	-0.3443^{1}	-0.1348^{1}	-0.2291^{1}
5.2	-0.2747	-0.4407^{1}	-0.1730^{1}	-0.2885^{1}
5.3	-0.2659	-0.5345^{1}	-0.2103^{1}	-0.3447^{1}
5.4	-0.2577	-0.6258^{1}	-0.2467^{1}	-0.3978^{1}
5.5	-0.2501	-0.7147^{1}	-0.2822^{1}	-0.4481^{1}
5.6	-0.2429	-0.8012^{1}	-0.3167^{1}	-0.4958^{1}
5.8	-0.2300	-0.9675^{1}	-0.3828^{1}	-0.5843^{1}
6.0	-0.2186	-0.1125	-0.4453^{1}	-0.6647^{1}
6.2	-0.2084	-0.1275	-0.5043^{1}	-0.7380^{1}
6.4	-0.1993	-0.1417	-0.5599^{1}	-0.8051^{1}
6.6	-0.1911	-0.1553	-0.6125^{1}	-0.8669^{1}
6.8	-0.1837	-0.1682	-0.6622^{1}	-0.9240^{1}
7.0	-0.1769	-0.1805	-0.7091^{1}	-0.9769^{1}
7.2	-0.1708	-0.1923	-0.7536^{1}	-0.1026
7.4	-0.1651	-0.2035	-0.7957^{1}	-0.1072
7.6	-0.1599	-0.2143	-0.8357^{1}	-0.1115
7.9	-0.1529	-0.2296	-0.8919^{1}	-0.1174
8.2	-0.1466	-0.2440	-0.9440^{1}	-0.1228
8.5	-0.1409	-0.2575	-0.9925^{1}	-0.1278
8.8	-0.1358	-0.2703	-0.1038	-0.1323
9.1	-0.1312	-0.2824	-0.1080	-0.1365
9.6	-0.1243	-0.3011	-0.1145	-0.1429
10.1	-0.1184	-0.3183	-0.1203	-0.1485
10.6	-0.1132	-0.3341	-0.1255	-0.1535
11.1	-0.1086	-0.3487	-0.1303	-0.1581
11.6	-0.1045	-0.3623	-0.1347	-0.1622
12.4	-0.9883^{1}	-0.3821	-0.1410	-0.1680
13.2	-0.9399^{1}	-0.3999	-0.1466	-0.1731
14.0	-0.8981^{1}	-0.4159	-0.1515	-0.1775
14.8	-0.8618^{1}	-0.4305	-0.1559	-0.1815
15.6	-0.8299^{1}	-0.4438	-0.1598	-0.1850

$\hat{\alpha}_3 = 0.75$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
2.9	-1.2057	0.2061	0.1012^{1}	0.3487
3.0	-1.0968	0.2003	0.1835^{1}	0.3119
3.1	-1.0031	0.1932	0.2489^{1}	0.2790
3.2	-0.9212	0.1850	0.2993^{1}	0.2492
3.3	-0.8488	0.1758	0.3360^{1}	0.2221
3.4	-0.7846	0.1658	0.3602^{1}	0.1974
3.5	-0.7275	0.1552	0.3729^{1}	0.1748
3.6	-0.6765	0.1440	0.3753^{1}	0.1542
3.7	-0.6310	0.1324	0.3689^{1}	0.1352
3.8	-0.5904	0.1206	0.3548^{1}	0.1179
3.9	-0.5541	0.1086	0.3343^{1}	0.1019
4.0	-0.5215	0.9654^{1}	0.3087^{1}	0.8730^{1}
4.1	-0.4922	0.8453^{1}	0.2789^{1}	0.7381^{1}
4.2	-0.4659	0.7260^{1}	0.2460^{1}	0.6137^{1}
4.3	-0.4421	0.6080^{1}	0.2106^{1}	0.4988^{1}
4.4	-0.4206	0.4918^{1}	0.1736^{1}	0.3923^{1}
4.5	-0.4011	0.3778^{1}	0.1355^{1}	0.2936^{1}
4.6	-0.3833	0.2659^{1}	0.9663^{2}	0.2016^{1}
4.7	-0.3671	0.1564^{1}	0.5749^{2}	0.1159^{1}
4.8	-0.3523	0.4939^{2}	0.1833^{2}	0.3582^{2}
4.9	-0.3386	-0.5509^{2}	-0.2061^{2}	-0.3916^{2}
5.0	-0.3261	-0.1570^{1}	-0.5915^{2}	-0.1095^{1}
5.1	-0.3145	-0.2564^{1}	-0.9715^{2}	-0.1756^{1}
5.2	-0.3038	-0.3533^{1}	-0.1345^{1}	-0.2379^{1}
5.3	-0.2939	-0.4476^{1}	-0.1711^{1}	-0.2966^{1}
5.4	-0.2846	-0.5396^{1}	-0.2070^{1}	-0.3521^{1}
5.5	-0.2760	-0.6291^{1}	-0.2420^{1}	-0.4046^{1}
5.6	-0.2680	-0.7164^{1}	-0.2762^{1}	-0.4544^{1}
5.7	-0.2605	-0.8014^{1}	-0.3096^{1}	-0.5017^{1}
5.8	-0.2534	-0.8842^{1}	-0.3421^{1}	-0.5467^{1}
6.0	-0.2405	-0.1044	-0.4046^{1}	-0.6303^{1}
6.2	-0.2291	-0.1195	-0.4638^{1}	-0.7064^{1}
6.4	-0.2189	-0.1339	-0.5198^{1}	-0.7761^{1}
6.6	-0.2097	-0.1476	-0.5728^{1}	-0.8401^{1}
6.8	-0.2014	-0.1607	-0.6229^{1}	-0.8992^{1}
7.0	-0.1939	-0.1732	-0.6705^{1}	-0.9538^{1}
7.2	-0.1870	-0.1851	-0.7155^{1}	-0.1005
7.4	-0.1807	-0.1964	-0.7582^{1}	-0.1052
7.6	-0.1750	-0.2073	-0.7988^{1}	-0.1096
7.8	-0.1696	-0.2178	-0.8373^{1}	-0.1137
8.1	-0.1624	-0.2326	-0.8917^{1}	-0.1195
8.4	-0.1559	-0.2466	-0.9422^{1}	-0.1247
8.7	-0.1500	-0.2598	-0.9893^{1}	-0.1295
9.0	-0.1447	-0.2723	-0.1033	-0.1339
9.3	-0.1399	-0.2841	-0.1074	-0.1380
9.8	-0.1327	-0.3024	-0.1138	-0.1442
10.3	-0.1265	-0.3192	-0.1194	-0.1497
10.8	-0.1211	-0.3347	-0.1246	-0.1546
11.3	-0.1162	-0.3490	-0.1293	-0.1590
11.8	-0.1119	-0.3624	-0.1337	-0.1631
12.6	-0.1059	-0.3818	-0.1399	-0.1688
13.4	-0.1008	-0.3993	-0.1453	-0.1738
14.2	-0.9641^{1}	-0.4152	-0.1502	-0.1782
15.0	-0.9256^{1}	-0.4295	-0.1546	-0.1821
15.8	-0.8917^{1}	-0.4427	-0.1585	-0.1856

$\hat{\alpha}_3 = 0.80$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.0	-1.2251	0.1995	0.6849^{2}	0.3356
3.1	-1.1174	0.1936	0.1481^{1}	0.3002
3.2	-1.0250	0.1864	0.2111^{1}	0.2687
3.3	-0.9444	0.1783	0.2597^{1}	0.2402
3.4	-0.8732	0.1693	0.2955^{1}	0.2143
3.5	-0.8099	0.1595	0.3195^{1}	0.1907
3.6	-0.7535	0.1492	0.3330^{1}	0.1691
3.7	-0.7030	0.1383	0.3369^{1}	0.1492
3.8	-0.6578	0.1272	0.3325^{1}	0.1310
3.9	-0.6172	0.1158	0.3210^{1}	0.1143
4.0	-0.5807	0.1042	0.3035^{1}	0.9887^{1}
4.1	-0.5479	0.9260^{1}	0.2810^{1}	0.8468^{1}
4.2	-0.5183	0.8100^{1}	0.2544^{1}	0.7157^{1}
4.3	-0.4915	0.6947^{1}	0.2246^{1}	0.5945^{1}
4.4	-0.4672	0.5806^{1}	0.1924^{1}	0.4822^{1}
4.5	-0.4452	0.4679^{1}	0.1584^{1}	0.3780^{1}
4.6	-0.4252	0.3573^{1}	0.1231^{1}	0.2812^{1}
4.7	-0.4069	0.2485^{1}	0.8698^{2}	0.1909^{1}
4.8	-0.3901	0.1420^{1}	0.5035^{2}	0.1066^{1}
4.9	-0.3747	0.3770^{2}	0.1352^{2}	0.2770^{2}
5.0	-0.3605	-0.6425^{2}	-0.2326^{2}	-0.4627^{2}
5.1	-0.3474	-0.1638^{1}	-0.5981^{2}	-0.1157^{1}
5.2	-0.3353	-0.2610^{1}	-0.9599^{2}	-0.1811^{1}
5.3	-0.3241	-0.3558^{1}	-0.1317^{1}	-0.2427^{1}
5.4	-0.3137	-0.4483^{1}	-0.1667^{1}	-0.3009^{1}
5.5	-0.3040	-0.5385^{1}	-0.2012^{1}	-0.3559^{1}
5.6	-0.2949	-0.6265^{1}	-0.2349^{1}	-0.4080^{1}
5.7	-0.2865	-0.7122^{1}	-0.2680^{1}	-0.4574^{1}
5.8	-0.2785	-0.7958^{1}	-0.3003^{1}	-0.5044^{1}
5.9	-0.2711	-0.8773^{1}	-0.3318^{1}	-0.5491^{1}
6.1	-0.2575	-0.1034	-0.3925^{1}	-0.6322^{1}
6.3	-0.2454	-0.1184	-0.4502^{1}	-0.7080^{1}
6.5	-0.2346	-0.1326	-0.5050^{1}	-0.7773^{1}
6.7	-0.2248	-0.1461	-0.5570^{1}	-0.8411^{1}
6.9	-0.2160	-0.1591	-0.6063^{1}	-0.9000^{1}
7.1	-0.2080	-0.1714	-0.6531^{1}	-0.9545^{1}
7.3	-0.2007	-0.1832	-0.6975^{1}	-0.1005
7.5	-0.1940	-0.1945	-0.7398^{1}	-0.1052
7.7	-0.1878	-0.2053	-0.7799^{1}	-0.1096
7.9	-0.1821	-0.2156	-0.8181^{1}	-0.1138
8.2	-0.1744	-0.2304	-0.8720^{1}	-0.1195
8.5	-0.1674	-0.2443	-0.9222^{1}	-0.1247
8.8	-0.1611	-0.2574	-0.9691^{1}	-0.1295
9.1	-0.1555	-0.2698	-0.1013	-0.1339
9.4	-0.1503	-0.2815	-0.1054	-0.1380
9.9	-0.1426	-0.2997	-0.1117	-0.1442
10.4	-0.1360	-0.3165	-0.1174	-0.1497
10.9	-0.1301	-0.3319	-0.1226	-0.1546
11.4	-0.1250	-0.3462	-0.1273	-0.1591
11.9	-0.1203	-0.3595	-0.1316	-0.1631
12.7	-0.1139	-0.3790	-0.1379	-0.1688
13.5	-0.1084	-0.3964	-0.1434	-0.1738
14.3	-0.1037	-0.4122	-0.1483	-0.1782
15.1	-0.9953^{1}	-0.4266	-0.1527	-0.1821
15.9	-0.9589^{1}	-0.4398	-0.1566	-0.1856

$\hat{\alpha}_3 = 0.85$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.1	-1.2524	0.1935	0.3194^{2}	0.3254
3.2	-1.1444	0.1875	0.1104^{1}	0.2912
3.3	-1.0523	0.1804	0.1724^{1}	0.2607
3.4	-0.9721	0.1724	0.2205^{1}	0.2332
3.5	-0.9013	0.1635	0.2563^{1}	0.2083
3.6	-0.8384	0.1541	0.2809^{1}	0.1855
3.7	-0.7822	0.1440	0.2956^{1}	0.1647
3.8	-0.7318	0.1336	0.3014^{1}	0.1455
3.9	-0.6866	0.1228	0.2994^{1}	0.1279
4.0	-0.6458	0.1118	0.2905^{1}	0.1117
4.1	-0.6090	0.1007	0.2759^{1}	0.9668^{1}
4.2	-0.5758	0.8946^{1}	0.2565^{1}	0.8286^{1}
4.3	-0.5457	0.7825^{1}	0.2330^{1}	0.7006^{1}
4.4	-0.5184	0.6710^{1}	0.2064^{1}	0.5821^{1}
4.5	-0.4935	0.5605^{1}	0.1772^{1}	0.4720^{1}
4.6	-0.4709	0.4512^{1}	0.1461^{1}	0.3696^{1}
4.7	-0.4503	0.3438^{1}	0.1136^{1}	0.2744^{1}
4.8	-0.4313	0.2380^{1}	0.8002^{2}	0.1854^{1}
4.9	-0.4140	0.1342^{1}	0.4581^{2}	0.1022^{1}
5.0	-0.3979	0.3243^{2}	0.1122^{2}	0.2417^{2}
5.1	-0.3832	-0.6705^{2}	-0.2346^{2}	-0.4896^{2}
5.2	-0.3696	-0.1645^{1}	-0.5808^{2}	-0.1178^{1}
5.3	-0.3569	-0.2596^{1}	-0.9250^{2}	-0.1826^{1}
5.4	-0.3452	-0.3525^{1}	-0.1265^{1}	-0.2437^{1}
5.5	-0.3343	-0.4432^{1}	-0.1601^{1}	-0.3015^{1}
5.6	-0.3241	-0.5317^{1}	-0.1932^{1}	-0.3562^{1}
5.7	-0.3145	-0.6181^{1}	-0.2257^{1}	-0.4080^{1}
5.8	-0.3056	-0.7024^{1}	-0.2576^{1}	-0.4572^{1}
5.9	-0.2973	-0.7847^{1}	-0.2888^{1}	-0.5039^{1}
6.0	-0.2894	-0.8649^{1}	-0.3194^{1}	-0.5484^{1}
6.2	-0.2751	-0.1020	-0.3784^{1}	-0.6313^{1}
6.4	-0.2623	-0.1167	-0.4348^{1}	-0.7068^{1}
6.6	-0.2508	-0.1308	-0.4884^{1}	-0.7761^{1}
6.8	-0.2404	-0.1442	-0.5394^{1}	-0.8398^{1}
7.0	-0.2311	-0.1570	-0.5879^{1}	-0.8986^{1}
7.2	-0.2226	-0.1692	-0.6340^{1}	-0.9531^{1}
7.4	-0.2148	-0.1809	-0.6779^{1}	-0.1004
7.6	-0.2076	-0.1921	-0.7196^{1}	-0.1051
7.8	-0.2011	-0.2028	-0.7594^{1}	-0.1095
8.0	-0.1950	-0.2131	-0.7973^{1}	-0.1136
8.3	-0.1867	-0.2277	-0.8508^{1}	-0.1193
8.6	-0.1793	-0.2415	-0.9008^{1}	-0.1246
8.9	-0.1726	-0.2546	-0.9475^{1}	-0.1294
9.2	-0.1666	-0.2669	-0.9912^{1}	-0.1338
9.5	-0.1610	-0.2786	-0.1032	-0.1379
10.0	-0.1529	-0.2967	-0.1095	-0.1441
10.5	-0.1457	-0.3134	-0.1152	-0.1496
11.0	-0.1395	-0.3289	-0.1204	-0.1545
11.5	-0.1339	-0.3431	-0.1252	-0.1590
12.0	-0.1290	-0.3564	-0.1295	-0.1630
12.8	-0.1221	-0.3758	-0.1358	-0.1687
13.6	-0.1162	-0.3933	-0.1413	-0.1737
14.4	-0.1111	-0.4091	-0.1462	-0.1781
15.2	-0.1067	-0.4234	-0.1507	-0.1820
16.0	-0.1028	-0.4366	-0.1546	-0.1855

$\hat{\alpha}_3 = 0.90$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.3	-1.1782	0.1821	0.7038^2	0.2846
3.4	-1.0851	0.1751	0.1328^1	0.2548
3.5	-1.0043	0.1672	0.1813^1	0.2281
3.6	-0.9333	0.1586	0.2179^1	0.2039
3.7	-0.8702	0.1494	0.2438^1	0.1818
3.8	-0.8138	0.1397	0.2602^1	0.1615
3.9	-0.7632	0.1296	0.2681^1	0.1429
4.0	-0.7175	0.1192	0.2686^1	0.1258
4.1	-0.6764	0.1086	0.2626^1	0.1099
4.2	-0.6391	0.9788^1	0.2510^1	0.9533^1
4.3	-0.6053	0.8706^1	0.2346^1	0.8180^1
4.4	-0.5746	0.7622^1	0.2142^1	0.6926^1
4.5	-0.5466	0.6543^1	0.1906^1	0.5761^1
4.6	-0.5211	0.5472^1	0.1644^1	0.4678^1
4.7	-0.4978	0.4412^1	0.1361^1	0.3669^1
4.8	-0.4765	0.3367^1	0.1063^1	0.2729^1
4.9	-0.4569	0.2338^1	0.7531^2	0.1849^1
5.0	-0.4389	0.1327^1	0.4348^2	0.1025^1
5.1	-0.4223	0.3339^2	0.1111^2	0.2526^2
5.2	-0.4069	-0.6388^2	-0.2154^2	-0.4735^2
5.3	-0.3927	-0.1591^1	-0.5428^2	-0.1157^1
5.4	-0.3794	-0.2523^1	-0.8694^2	-0.1801^1
5.5	-0.3671	-0.3435^1	-0.1194^1	-0.2410^1
5.6	-0.3557	-0.4325^1	-0.1515^1	-0.2985^1
5.7	-0.3450	-0.5194^1	-0.1833^1	-0.3530^1
5.8	-0.3350	-0.6044^1	-0.2145^1	-0.4046^1
5.9	-0.3256	-0.6847^1	-0.2453^1	-0.4537^1
6.0	-0.3168	-0.7684^1	-0.2755^1	-0.5003^1
6.1	-0.3085	-0.8475^1	-0.3051^1	-0.5448^1
6.2	-0.3007	-0.9247^1	-0.3341^1	-0.5871^1
6.4	-0.2863	-0.1074	-0.3904^1	-0.6661^1
6.6	-0.2735	-0.1216	-0.4441^1	-0.7385^1
6.8	-0.2620	-0.1352	-0.4953^1	-0.8049^1
7.0	-0.2516	-0.1481	-0.5442^1	-0.8661^1
7.2	-0.2421	-0.1605	-0.5908^1	-0.9227^1
7.4	-0.2334	-0.1724	-0.6352^1	-0.9753^1
7.6	-0.2255	-0.1837	-0.6775^1	-0.1024
7.8	-0.2182	-0.1946	-0.7179^1	-0.1070
8.0	-0.2115	-0.2051	-0.7563^1	-0.1113
8.2	-0.2053	-0.2151	-0.7931^1	-0.1153
8.5	-0.1968	-0.2294	-0.8451^1	-0.1208
8.8	-0.1892	-0.2428	-0.8937^1	-0.1259
9.1	-0.1823	-0.2556	-0.9393^1	-0.1306
9.4	-0.1760	-0.2676	-0.9820^1	-0.1349
9.7	-0.1703	-0.2791	-0.1022	-0.1389
10.2	-0.1618	-0.2969	-0.1084	-0.1450
10.7	-0.1544	-0.3133	-0.1140	-0.1504
11.2	-0.1478	-0.3284	-0.1191	-0.1552
11.7	-0.1421	-0.3425	-0.1238	-0.1596
12.2	-0.1369	-0.3555	-0.1281	-0.1636
13.0	-0.1297	-0.3747	-0.1343	-0.1692
13.8	-0.1235	-0.3919	-0.1398	-0.1742
14.6	-0.1181	-0.4075	-0.1447	-0.1786
15.4	-0.1135	-0.4217	-0.1491	-0.1824
16.2	-0.1094	-0.4347	-0.1530	-0.1858

$\hat{\alpha}_3 = 0.95$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.4	-1.2193	0.1773	0.2760^2	0.2803
3.5	-1.1237	0.1704	0.9165^2	0.2509
3.6	-1.0415	0.1628	0.1416^1	0.2247
3.7	-0.9695	0.1544	0.1798^1	0.2010
3.8	-0.9056	0.1455	0.2075^1	0.1794
3.9	-0.8485	0.1362	0.2261^1	0.1596
4.0	-0.7972	0.1264	0.2365^1	0.1414
4.1	-0.7509	0.1164	0.2397^1	0.1246
4.2	-0.7091	0.1062	0.2366^1	0.1091
4.3	-0.6711	0.9579^1	0.2280^1	0.9477^1
4.4	-0.6366	0.8533^1	0.2147^1	0.8148^1
4.5	-0.6051	0.7486^1	0.1974^1	0.6913^1
4.6	-0.5764	0.6441^1	0.1768^1	0.5765^1
4.7	-0.5502	0.5402^1	0.1536^1	0.4695^1
4.8	-0.5261	0.4373^1	0.1281^1	0.3697^1
4.9	-0.5041	0.3358^1	0.1009^1	0.2766^1
5.0	-0.4838	0.2355^1	0.7243^2	0.1894^1
5.1	-0.4650	0.1369^1	0.4294^2	0.1076^1
5.2	-0.4477	0.3999^2	0.1276^2	0.3075^2
5.3	-0.4317	-0.5511^2	-0.1786^2	-0.4151^2
5.4	-0.4168	-0.1483^1	-0.4873^2	-0.1096^1
5.5	-0.4029	-0.2397^1	-0.7967^2	-0.1738^1
5.6	-0.3900	-0.3292^1	-0.1106^1	-0.2346^1
5.7	-0.3780	-0.4165^1	-0.1412^1	-0.2919^1
5.8	-0.3668	-0.5020^1	-0.1716^1	-0.3463^1
5.9	-0.3562	-0.5856^1	-0.2017^1	-0.3980^1
6.0	-0.3464	-0.6673^1	-0.2313^1	-0.4470^1
6.1	-0.3371	-0.7471^1	-0.2605^1	-0.4937^1
6.2	-0.3283	-0.8251^1	-0.2892^1	-0.5381^1
6.3	-0.3201	-0.9014^1	-0.3174^1	-0.5805^1
6.5	-0.3049	-0.1049	-0.3721^1	-0.6597^1
6.7	-0.3737^1	-0.7977	-0.2443	-0.2609
6.9	-0.2792	-0.1324	-0.4749^1	-0.7987^1
7.1	-0.2681	-0.1452	-0.5229^1	-0.8601^1
7.3	-0.2580	-0.1575	-0.5688^1	-0.9170^1
7.5	-0.2488	-0.1693	-0.6126^1	-0.9697^1
7.7	-0.2404	-0.1806	-0.6545^1	-0.1019
7.9	-0.2327	-0.1914	-0.6944^1	-0.1065
8.1	-0.2255	-0.2017	-0.7326^1	-0.1108
8.3	-0.2189	-0.2117	-0.7691^1	-0.1148
8.6	-0.2099	-0.2259	-0.8209^1	-0.1204
8.9	-0.2017	-0.2394	-0.8694^1	-0.1255
9.2	-0.1944	-0.2520	-0.9148^1	-0.1302
9.5	-0.1877	-0.2641	-0.9575^1	-0.1345
9.8	-0.1816	-0.2755	-0.9977^1	-0.1386
10.3	-0.1725	-0.2932	-0.1060	-0.1446
10.8	-0.1646	-0.3096	-0.1116	-0.1501
11.3	-0.1577	-0.3247	-0.1167	-0.1549
11.8	-0.1515	-0.3388	-0.1214	-0.1593
12.3	-0.1460	-0.3518	-0.1257	-0.1633
13.1	-0.1383	-0.3710	-0.1320	-0.1690
13.9	-0.1317	-0.3882	-0.1375	-0.1739
14.7	-0.1260	-0.4038	-0.1424	-0.1783
15.5	-0.1210	-0.4180	-0.1469	-0.1822
16.3	-0.1166	-0.4311	-0.1509	-0.1856

$\hat{\alpha}_3 = 1.00$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.6	-1.1690	0.1664	0.4829^{2}	0.2490
3.7	-1.0840	0.1590	0.1009^{1}	0.2230
3.8	-1.0101	0.1509	0.1414^{1}	0.1996
3.9	-0.9448	0.1423	0.1715^{1}	0.1783
4.0	-0.8865	0.1333	0.1927^{1}	0.1588
4.1	-0.8342	0.1239	0.2058^{1}	0.1409
4.2	-0.7869	0.1142	0.2119^{1}	0.1244
4.3	-0.7441	0.1044	0.2119^{1}	0.1091
4.4	-0.7053	0.9434^{1}	0.2064^{1}	0.9500^{1}
4.5	-0.6699	0.8424^{1}	0.1962^{1}	0.8187^{1}
4.6	-0.6376	0.7411^{1}	0.1820^{1}	0.6967^{1}
4.7	-0.6080	0.6399^{1}	0.1645^{1}	0.5830^{1}
4.8	-0.5809	0.5392^{1}	0.1442^{1}	0.4770^{1}
4.9	-0.5560	0.4392^{1}	0.1215^{1}	0.3780^{1}
5.0	-0.5331	0.3404^{1}	0.9698^{2}	0.2855^{1}
5.1	-0.5120	0.2428^{1}	0.7098^{2}	0.1987^{1}
5.2	-0.4925	0.1466^{1}	0.4383^{2}	0.1172^{1}
5.3	-0.4744	0.5192^{2}	0.1584^{2}	0.4061^{2}
5.4	-0.4577	-0.4110^{2}	-0.1276^{2}	-0.3149^{2}
5.5	-0.4421	-0.1324^{1}	-0.4176^{2}	-0.9946^{2}
5.6	-0.4276	-0.2219^{1}	-0.7097^{2}	-0.1636^{1}
5.7	-0.4141	-0.3096^{1}	-0.1002^{1}	-0.2243^{1}
5.8	-0.4014	-0.3956^{1}	-0.1295^{1}	-0.2818^{1}
5.9	-0.3896	-0.4797^{1}	-0.1586^{1}	-0.3363^{1}
6.0	-0.3785	-0.5619^{1}	-0.1874^{1}	-0.3880^{1}
6.1	-0.3681	-0.6424^{1}	-0.2159^{1}	-0.4372^{1}
6.2	-0.3583	-0.7212^{1}	-0.2441^{1}	-0.4840^{1}
6.3	-0.3491	-0.7982^{1}	-0.2718^{1}	-0.5286^{1}
6.4	-0.3404	-0.8734^{1}	-0.2992^{1}	-0.5711^{1}
6.5	-0.3322	-0.9470^{1}	-0.3260^{1}	-0.6117^{1}
6.7	-0.3170	-0.1089	-0.3783^{1}	-0.6877^{1}
6.9	-0.3034	-0.1226	-0.4285^{1}	-0.7574^{1}
7.1	-0.2910	-0.1356	-0.4768^{1}	-0.8217^{1}
7.3	-0.2799	-0.1480	-0.5230^{1}	-0.8810^{1}
7.5	-0.2697	-0.1600	-0.5672^{1}	-0.9360^{1}
7.7	-0.2603	-0.1714	-0.6095^{1}	-0.9872^{1}
7.9	-0.2517	-0.1824	-0.6500^{1}	-0.1035
8.1	-0.2438	-0.1929	-0.6888^{1}	-0.1080
8.3	-0.2365	-0.2030	-0.7258^{1}	-0.1121
8.5	-0.2297	-0.2128	-0.7613^{1}	-0.1161
8.8	-0.2205	-0.2267	-0.8118^{1}	-0.1215
9.1	-0.2121	-0.2398	-0.8591^{1}	-0.1265
9.4	-0.2045	-0.2522	-0.9035^{1}	-0.1311
9.7	-0.1976	-0.2640	-0.9454^{1}	-0.1354
10.0	-0.1913	-0.2752	-0.9848^{1}	-0.1393
10.5	-0.1819	-0.2927	-0.1046	-0.1453
11.0	-0.1737	-0.3087	-0.1101	-0.1506
11.5	-0.1665	-0.3236	-0.1152	-0.1554
12.0	-0.1600	-0.3375	-0.1198	-0.1598
12.5	-0.1543	-0.3504	-0.1241	-0.1637
13.3	-0.1462	-0.3693	-0.1303	-0.1693
14.1	-0.1393	-0.3863	-0.1358	-0.1742
14.9	-0.1334	-0.4018	-0.1407	-0.1785
15.7	-0.1281	-0.4158	-0.1451	-0.1824
16.5	-0.1235	-0.4288	-0.1491	-0.1858

$\hat{\alpha}_3 = 1.05$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.7	-1.2220	0.1631	0.1786^{3}	0.2491
3.8	-1.1327	0.1559	0.5815^{2}	0.2230
3.9	-1.0559	0.1481	0.1019^{1}	0.1997
4.0	-0.9883	0.1398	0.1351^{1}	0.1785
4.1	-0.9283	0.1311	0.1593^{1}	0.1592
4.2	-0.8744	0.1220	0.1755^{1}	0.1415
4.3	-0.8259	0.1127	0.1847^{1}	0.1251
4.4	-0.7819	0.1032	0.1878^{1}	0.1100
4.5	-0.7419	0.9349^{1}	0.1856^{1}	0.9599^{1}
4.6	-0.7054	0.8374^{1}	0.1786^{1}	0.8297^{1}
4.7	-0.6720	0.7394^{1}	0.1676^{1}	0.7086^{1}
4.8	-0.6415	0.6413^{1}	0.1531^{1}	0.5957^{1}
4.9	-0.6134	0.5436^{1}	0.1357^{1}	0.4902^{1}
5.0	-0.5875	0.4465^{1}	0.1159^{1}	0.3916^{1}
5.1	-0.5637	0.3502^{1}	0.9400^{2}	0.2992^{1}
5.2	-0.5418	0.2552^{1}	0.7055^{2}	0.2128^{1}
5.3	-0.5214	0.1613^{1}	0.4577^{2}	0.1314^{1}
5.4	-0.5026	0.6882^{2}	0.1998^{2}	0.5483^{2}
5.5	-0.4850	-0.2218^{2}	-0.6571^{3}	-0.1730^{2}
5.6	-0.4687	-0.1116^{1}	-0.3367^{2}	-0.8536^{2}
5.7	-0.4535	-0.1994^{1}	-0.6113^{2}	-0.1497^{1}
5.8	-0.4393	-0.2855^{1}	-0.8880^{2}	-0.2105^{1}
5.9	-0.4260	-0.3700^{1}	-0.1166^{1}	-0.2682^{1}
6.0	-0.4135	-0.4528^{1}	-0.1443^{1}	-0.3229^{1}
6.1	-0.4019	-0.5337^{1}	-0.1719^{1}	-0.3748^{1}
6.2	-0.3909	-0.6131^{1}	-0.1993^{1}	-0.4242^{1}
6.3	-0.3806	-0.6908^{1}	-0.2264^{1}	-0.4712^{1}
6.4	-0.3708	-0.7668^{1}	-0.2532^{1}	-0.5160^{1}
6.5	-0.3616	-0.8411^{1}	-0.2796^{1}	-0.5588^{1}
6.6	-0.3529	-0.9139^{1}	-0.3057^{1}	-0.5996^{1}
6.8	-0.3369	-0.1055	-0.3566^{1}	-0.6761^{1}
7.0	-0.3224	-0.1190	-0.4057^{1}	-0.7463^{1}
7.2	-0.3094	-0.1319	-0.4530^{1}	-0.8110^{1}
7.4	-0.2975	-0.1443	-0.4985^{1}	-0.8707^{1}
7.6	-0.2867	-0.1561	-0.5421^{1}	-0.9262^{1}
7.8	-0.2768	-0.1675	-0.5839^{1}	-0.9777^{1}
8.0	-0.2677	-0.1784	-0.6240^{1}	-0.1026
8.2	-0.2593	-0.1889	-0.6624^{1}	-0.1071
8.4	-0.2515	-0.1990	-0.6992^{1}	-0.1113
8.6	-0.2443	-0.2087	-0.7346^{1}	-0.1152
8.9	-0.2344	-0.2225	-0.7848^{1}	-0.1207
9.2	-0.2255	-0.2356	-0.8320^{1}	-0.1258
9.5	-0.2174	-0.2480	-0.8765^{1}	-0.1304
9.8	-0.2101	-0.2598	-0.9183^{1}	-0.1347
10.1	-0.2034	-0.2710	-0.9578^{1}	-0.1387
10.6	-0.1934	-0.2884	-0.1019	-0.1447
11.1	-0.1846	-0.3045	-0.1074	-0.1501
11.6	-0.1769	-0.3194	-0.1125	-0.1549
12.1	-0.1701	-0.3332	-0.1172	-0.1593
12.6	-0.1639	-0.3461	-0.1215	-0.1632
13.4	-0.1553	-0.3651	-0.1278	-0.1689
14.2	-0.1480	-0.3821	-0.1333	-0.1738
15.0	-0.1416	-0.3976	-0.1383	-0.1782
15.8	-0.1361	-0.4117	-0.1428	-0.1820
16.6	-0.1312	-0.4247	-0.1468	-0.1855

$\hat{\alpha}_3 = 1.10$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
3.9	-1.1889	0.1534	0.1235^{2}	0.2248
4.0	-1.1077	0.1458	0.6040^{2}	0.2013
4.1	-1.0369	0.1378	0.9740^{2}	0.1800
4.2	-0.9744	0.1294	0.1251^{1}	0.1607
4.3	-0.9185	0.1207	0.1448^{1}	0.1430
4.4	-0.8682	0.1117	0.1574^{1}	0.1267
4.5	-0.8226	0.1025	0.1639^{1}	0.1117
4.6	-0.7812	0.9320^{1}	0.1650^{1}	0.9774^{1}
4.7	-0.7434	0.8377^{1}	0.1613^{1}	0.8478^{1}
4.8	-0.7088	0.7428^{1}	0.1535^{1}	0.7270^{1}
4.9	-0.6771	0.6478^{1}	0.1422^{1}	0.6144^{1}
5.0	-0.6479	0.5530^{1}	0.1278^{1}	0.5091^{1}
5.1	-0.6210	0.4586^{1}	0.1108^{1}	0.4106^{1}
5.2	-0.5962	0.3650^{1}	0.9166^{2}	0.3182^{1}
5.3	-0.5733	0.2724^{1}	0.7076^{2}	0.2317^{1}
5.4	-0.5520	0.1808^{1}	0.4839^{2}	0.1502^{1}
5.5	-0.5323	0.9038^{2}	0.2484^{2}	0.7342^{2}
5.6	-0.5136	0.7603^{6}	0.2140^{10}	0.6045^{10}
5.7	-0.4968	-0.8629^{2}	-0.2479^{2}	-0.6726^{2}
5.8	-0.4808	-0.1724^{1}	-0.5046^{2}	-0.1319^{1}
5.9	-0.4659	-0.2570^{1}	-0.7648^{2}	-0.1930^{1}
6.0	-0.4519	-0.3401^{1}	-0.1027^{1}	-0.2510^{1}
6.1	-0.4388	-0.4214^{1}	-0.1290^{1}	-0.3060^{1}
6.2	-0.4264	-0.5013^{1}	-0.1553^{1}	-0.3583^{1}
6.3	-0.4148	-0.5796^{1}	-0.1816^{1}	-0.4081^{1}
6.4	-0.4039	-0.6561^{1}	-0.2076^{1}	-0.4553^{1}
6.5	-0.3936	-0.7312^{1}	-0.2335^{1}	-0.5005^{1}
6.6	-0.3839	-0.8047^{1}	-0.2590^{1}	-0.5436^{1}
6.7	-0.3747	-0.8767^{1}	-0.2843^{1}	-0.5848^{1}
6.8	-0.3660	-0.9471^{1}	-0.3092^{1}	-0.6242^{1}
7.0	-0.3499	-0.1084	-0.3580^{1}	-0.6980^{1}
7.2	-0.3353	-0.1215	-0.4051^{1}	-0.7660^{1}
7.4	-0.3221	-0.1340	-0.4506^{1}	-0.8287^{1}
7.6	-0.3101	-0.1460	-0.4944^{1}	-0.8867^{1}
7.8	-0.2991	-0.1576	-0.5366^{1}	-0.9407^{1}
8.0	-0.2890	-0.1687	-0.5771^{1}	-0.9909^{1}
8.2	-0.2797	-0.1793	-0.6159^{1}	-0.1038
8.4	-0.2711	-0.1896	-0.6533^{1}	-0.1082
8.6	-0.2632	-0.1994	-0.6891^{1}	-0.1123
8.8	-0.2558	-0.2089	-0.0724^{1}	-0.1162
9.1	-0.2456	-0.2224	-0.7726^{1}	-0.1216
9.4	-0.2365	-0.2353	-0.8188^{1}	-0.1265
9.7	-0.2282	-0.2475	-0.8624^{1}	-0.1311
10.0	-0.2206	-0.2590	-0.9035^{1}	-0.1353
10.3	-0.2136	-0.2700	-0.9423^{1}	-0.1392
10.8	-0.2033	-0.2872	-0.1002	-0.1451
11.3	-0.1942	-0.3030	-0.1057	-0.1504
11.8	-0.1862	-0.3177	-0.1108	-0.1552
12.3	-0.1791	-0.3314	-0.1154	-0.1595
12.8	-0.1727	-0.3441	-0.1197	-0.1635
13.6	-0.1637	-0.3628	-0.1259	-0.1691
14.4	-0.1561	-0.3797	-0.1314	-0.1739
15.2	-0.1494	-0.3951	-0.1364	-0.1783
16.0	-0.1436	-0.4091	-0.1408	-0.1821
16.8	-0.1384	-0.4219	-0.1449	-0.1855

$\hat{\alpha}_3 = 1.15$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
4.1	-1.1669	0.1442	0.1565^{2}	0.2045
4.2	-1.0915	0.1365	0.5746^{2}	0.1829
4.3	-1.0255	0.1284	0.8937^{2}	0.1635
4.4	-0.9669	0.1199	0.1130^{1}	0.1457
4.5	-0.9143	0.1113	0.1293^{1}	0.1293
4.6	-0.8668	0.1024	0.1395^{1}	0.1143
4.7	-0.8237	0.9341^{1}	0.1440^{1}	0.1003
4.8	-0.7843	0.8430^{1}	0.1438^{1}	0.8730^{1}
4.9	-0.7483	0.7512^{1}	0.1393^{1}	0.7521^{1}
5.0	-0.7152	0.6591^{1}	0.1312^{1}	0.6393^{1}
5.1	-0.6848	0.5671^{1}	0.1199^{1}	0.5338^{1}
5.2	-0.6567	0.4754^{1}	0.1058^{1}	0.4351^{1}
5.3	-0.6307	0.3843^{1}	0.8948^{2}	0.3424^{1}
5.4	-0.6068	0.2940^{1}	0.7120^{2}	0.2555^{1}
5.5	-0.5845	0.2046^{1}	0.5129^{2}	0.1736^{1}
5.6	-0.5637	0.1163^{1}	0.3005^{2}	0.9643^{2}
5.7	-0.5445	0.2912^{2}	0.7733^{3}	0.2364^{2}
5.8	-0.5265	-0.5674^{2}	-0.1544^{2}	-0.4513^{2}
5.9	-0.5096	-0.1412^{1}	-0.3927^{2}	-0.1102^{1}
6.0	-0.4939	-0.2244^{1}	-0.6361^{2}	-0.1718^{1}
6.1	-0.4792	-0.3059^{1}	-0.8824^{2}	-0.2302^{1}
6.2	-0.4653	-0.3861^{1}	-0.1131^{1}	-0.2857^{1}
6.3	-0.4523	-0.4647^{1}	-0.1381^{1}	-0.3384^{1}
6.4	-0.4401	-0.5419^{1}	-0.1631^{1}	-0.3886^{1}
6.5	-0.4286	-0.6175^{1}	-0.1881^{1}	-0.4364^{1}
6.6	-0.4177	-0.6917^{1}	-0.2129^{1}	-0.4820^{1}
6.7	-0.4074	-0.7643^{1}	-0.2375^{1}	-0.5255^{1}
6.8	-0.3976	-0.8355^{1}	-0.2619^{1}	-0.5671^{1}
6.9	-0.3884	-0.9053^{1}	-0.2861^{1}	-0.6069^{1}
7.0	-0.3796	-0.9736^{1}	-0.3100^{1}	-0.6450^{1}
7.2	-0.3634	-0.1106	-0.3567^{1}	-0.7166^{1}
7.4	-0.3487	-0.1233	-0.4020^{1}	-0.7825^{1}
7.6	-0.3353	-0.1355	-0.4459^{1}	-0.8435^{1}
7.8	-0.3231	-0.1473	-0.4882^{1}	-0.9001^{1}
8.0	-0.3119	-0.1585	-0.5290^{1}	-0.9527^{1}
8.2	-0.3017	-0.1693	-0.5682^{1}	-0.1002
8.4	-0.2922	-0.1797	-0.6060^{1}	-0.1048
8.6	-0.2834	-0.1898	-0.6423^{1}	-0.1091
8.8	-0.2752	-0.1994	-0.6772^{1}	-0.1131
9.0	-0.2676	-0.2087	-0.7108^{1}	-0.1169
9.3	-0.2572	-0.2220	-0.7588^{1}	-0.1222
9.6	-0.2478	-0.2346	-0.8041^{1}	-0.1271
9.9	-0.2392	-0.2465	-0.8468^{1}	-0.1316
10.2	-0.2314	-0.2579	-0.8872^{1}	-0.1357
10.5	-0.2242	-0.2687	-0.9254^{1}	-0.1396
11.0	-0.2135	-0.2856	-0.9847^{1}	-0.1454
11.5	-0.2041	-0.3012	-0.1039	-0.1507
12.0	-0.1958	-0.3157	-0.1089	-0.1554
12.5	-0.1883	-0.3292	-0.1135	-0.1597
13.0	-0.1817	-0.3419	-0.1177	-0.1636
13.8	-0.1723	-0.3604	-0.1239	-0.1691
14.6	-0.1643	-0.3771	-0.1294	-0.1740
15.4	-0.1574	-0.3923	-0.1344	-0.1783
16.2	-0.1513	-0.4062	-0.1388	-0.1821
17.0	-0.1459	-0.4190	-0.1429	-0.1855

$\hat{\alpha}_3 = 1.20$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
4.3	-1.1536	0.1356	0.1393^{2}	0.1874
4.4	-1.0828	0.1278	0.5100^{2}	0.1675
4.5	-1.0205	0.1197	0.7912^{2}	0.1494
4.6	-0.9650	0.1114	0.9969^{2}	0.1329
4.7	-0.9150	0.1028	0.1137^{1}	0.1177
4.8	-0.8697	0.9410^{1}	0.1221^{1}	0.1036
4.9	-0.8284	0.8529^{1}	0.1255^{1}	0.9057^{1}
5.0	-0.7907	0.7641^{1}	0.1245^{1}	0.7841^{1}
5.1	-0.7560	0.6749^{1}	0.1197^{1}	0.6706^{1}
5.2	-0.7241	0.5856^{1}	0.1115^{1}	0.5645^{1}
5.3	-0.6947	0.4964^{1}	0.1005^{1}	0.4651^{1}
5.4	-0.6675	0.4077^{1}	0.8705^{2}	0.3718^{1}
5.5	-0.6422	0.3196^{1}	0.7145^{2}	0.2841^{1}
5.6	-0.6189	0.2325^{1}	0.5411^{2}	0.2017^{1}
5.7	-0.5971	0.1462^{1}	0.3525^{2}	0.1239^{1}
5.8	-0.5768	0.6087^{2}	0.1515^{2}	0.5050^{2}
5.9	-0.5579	-0.2326^{2}	-0.5954^{3}	-0.1890^{2}
6.0	-0.5402	-0.1061^{1}	-0.2786^{2}	-0.8456^{2}
6.1	-0.5236	-0.1878^{1}	-0.5041^{2}	-0.1468^{1}
6.2	-0.5080	-0.2681^{1}	-0.7344^{2}	-0.2058^{1}
6.3	-0.4934	-0.3469^{1}	-0.9680^{2}	-0.2619^{1}
6.4	-0.4797	-0.4244^{1}	-0.1204^{1}	-0.3152^{1}
6.5	-0.4667	-0.5005^{1}	-0.1441^{1}	-0.3659^{1}
6.6	-0.4545	-0.5752^{1}	-0.1679^{1}	-0.4143^{1}
6.7	-0.4430	-0.6484^{1}	-0.1917^{1}	-0.4604^{1}
6.8	-0.4321	-0.7203^{1}	-0.2153^{1}	-0.5045^{1}
6.9	-0.4218	-0.7907^{1}	-0.2389^{1}	-0.5466^{1}
7.0	-0.4120	-0.8598^{1}	-0.2622^{1}	-0.5869^{1}
7.1	-0.4027	-0.9275^{1}	-0.2853^{1}	-0.6255^{1}
7.2	-0.3939	-0.9939^{1}	-0.3082^{1}	-0.6625^{1}
7.4	-0.3775	-0.1123	-0.3531^{1}	-0.7320^{1}
7.6	-0.3626	-0.1247	-0.3967^{1}	-0.7962^{1}
7.8	-0.3490	-0.1365	-0.4390^{1}	-0.8557^{1}
8.0	-0.3366	-0.1480	-0.4799^{1}	-0.9109^{1}
8.2	-0.3252	-0.1590	-0.5194^{1}	-0.9624^{1}
8.4	-0.3147	-0.1695	-0.5575^{1}	-0.1010
8.6	-0.3050	-0.1797	-0.5942^{1}	-0.1055
8.8	-0.2960	-0.1895	-0.6296^{1}	-0.1098
9.0	-0.2877	-0.1990	-0.6637^{1}	-0.1137
9.2	-0.2799	-0.2081	-0.6965^{1}	-0.1175
9.5	-0.2692	-0.2211	-0.7435^{1}	-0.1227
9.8	-0.2594	-0.2335	-0.7879^{1}	-0.1275
10.1	-0.2506	-0.2452	-0.8299^{1}	-0.1319
10.4	-0.2425	-0.2564	-0.8697^{1}	-0.1360
10.7	-0.2351	-0.2671	-0.9073^{1}	-0.1398
11.2	-0.2238	-0.2840	-0.9670^{1}	-0.1457
11.7	-0.2140	-0.2995	-0.1021	-0.1509
12.2	-0.2054	-0.3138	-0.1070	-0.1556
12.7	-0.1977	-0.3271	-0.1116	-0.1598
13.2	-0.1908	-0.3396	-0.1158	-0.1637
14.0	-0.1811	-0.3579	-0.1219	-0.1692
14.8	-0.1727	-0.3745	-0.1274	-0.1740
15.6	-0.1655	-0.3895	-0.1323	-0.1783
16.4	-0.1591	-0.4033	-0.1368	-0.1821
17.2	-0.1535	-0.4160	-0.1408	-0.1855

$\hat{\alpha}_3 = 1.25$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
4.5	-1.1479	0.1277	0.8589^{3}	0.1730
4.6	-1.0805	0.1199	0.4219^{2}	0.1545
4.7	-1.0211	0.1119	0.6755^{2}	0.1376
4.8	-0.9679	0.1036	0.8599^{2}	0.1221
4.9	-0.9199	0.9524^{1}	0.9844^{2}	0.1079
5.0	-0.8763	0.8672^{1}	0.1057^{1}	0.9463^{1}
5.1	-0.8365	0.7812^{1}	0.1084^{1}	0.8233^{1}
5.2	-0.8000	0.6947^{1}	0.1071^{1}	0.7086^{1}
5.3	-0.7664	0.6080^{1}	0.1023^{1}	0.6013^{1}
5.4	-0.7355	0.5214^{1}	0.9440^{2}	0.5009^{1}
5.5	-0.7068	0.4351^{1}	0.8391^{2}	0.4066^{1}
5.6	-0.6802	0.3492^{1}	0.7114^{2}	0.3180^{1}
5.7	-0.6556	0.2642^{1}	0.5643^{2}	0.2347^{1}
5.8	-0.6327	0.1798^{1}	0.4005^{2}	0.1560^{1}
5.9	-0.6113	0.9635^{2}	0.2227^{2}	0.8178^{2}
6.0	-0.5913	-0.1391^{2}	-0.3323^{3}	-0.1156^{2}
6.1	-0.5726	-0.6741^{2}	-0.1658^{2}	-0.5490^{2}
6.2	-0.5550	-0.1476^{1}	-0.3728^{2}	-0.1179^{1}
6.3	-0.5386	-0.2265^{1}	-0.5858^{2}	-0.1777^{1}
6.4	-0.5231	-0.3041^{1}	-0.8036^{2}	-0.2345^{1}
6.5	-0.5086	-0.3805^{1}	-0.1025^{1}	-0.2885^{1}
6.6	-0.4949	-0.4556^{1}	-0.1249^{1}	-0.3400^{1}
6.7	-0.4820	-0.5293^{1}	-0.1474^{1}	-0.3890^{1}
6.8	-0.4697	-0.6017^{1}	-0.1701^{1}	-0.4358^{1}
6.9	-0.4582	-0.6728^{1}	-0.1927^{1}	-0.4805^{1}
7.0	-0.4472	-0.7425^{1}	-0.2153^{1}	-0.5232^{1}
7.1	-0.4368	-0.8109^{1}	-0.2377^{1}	-0.5640^{1}
7.2	-0.4270	-0.8780^{1}	-0.2601^{1}	-0.6032^{1}
7.3	-0.4176	-0.9438^{1}	-0.2822^{1}	-0.6407^{1}
7.4	-0.4087	-0.1008	-0.3041^{1}	-0.6767^{1}
7.6	-0.3921	-0.1134	-0.3473^{1}	-0.7445^{1}
7.8	-0.3770	-0.1254	-0.3893^{1}	-0.8072^{1}
8.0	-0.3633	-0.1370	-0.4302^{1}	-0.8654^{1}
8.2	-0.3506	-0.1482	-0.4697^{1}	-0.9195^{1}
8.4	-0.3390	-0.1590	-0.5081^{1}	-0.9699^{1}
8.6	-0.3282	-0.1693	-0.5451^{1}	-0.1017
8.8	-0.3183	-0.1793	-0.5808^{1}	-0.1061
9.0	-0.3091	-0.1889	-0.6154^{1}	-0.1103
9.2	-0.3005	-0.1981	-0.6486^{1}	-0.1142
9.4	-0.2925	-0.2071	-0.6807^{1}	-0.1179
9.7	-0.2815	-0.2199	-0.7268^{1}	-0.1230
10.0	-0.2714	-0.2320	-0.7703^{1}	-0.1277
10.3	-0.2623	-0.2436	-0.8117^{1}	-0.1321
10.6	-0.2540	-0.2546	-0.8508^{1}	-0.1362
10.9	-0.2463	-0.2651	-0.8880^{1}	-0.1400
11.4	-0.2348	-0.2816	-0.9458^{1}	-0.1457
11.9	-0.2246	-0.2968	-0.9990^{1}	-0.1509
12.4	-0.2157	-0.3110	-0.1048	-0.1555
12.9	-0.2076	-0.3242	-0.1093	-0.1597
13.4	-0.2004	-0.3365	-0.1136	-0.1636
14.2	-0.1903	-0.3547	-0.1197	-0.1691
15.0	-0.1816	-0.3712	-0.1252	-0.1739
15.8	-0.1740	-0.3862	-0.1301	-0.1781
16.6	-0.1673	-0.3999	-0.1345	-0.1819
17.4	-0.1614	-0.4125	-0.1386	-0.1853

$\hat{\alpha}_3 = 1.30$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
4.7	-1.1487	0.1206	0.6892^{4}	0.1610
4.8	-1.0839	0.1129	0.3178^{2}	0.1435
4.9	-1.0265	0.1049	0.5525^{2}	0.1276
5.0	-0.9751	0.9679^{1}	0.7227^{2}	0.1130
5.1	-0.9286	0.8855^{1}	0.8373^{2}	0.9954^{1}
5.2	-0.8863	0.8022^{1}	0.9037^{2}	0.8702^{1}
5.3	-0.8476	0.7184^{1}	0.9278^{2}	0.7536^{1}
5.4	-0.8121	0.6342^{1}	0.9149^{2}	0.6447^{1}
5.5	-0.7793	0.5501^{1}	0.8696^{2}	0.5428^{1}
5.6	-0.7490	0.4660^{1}	0.7960^{2}	0.4472^{1}
5.7	-0.7209	0.3824^{1}	0.6977^{2}	0.3573^{1}
5.8	-0.6948	0.2992^{1}	0.5782^{2}	0.2726^{1}
5.9	-0.6706	0.2169^{1}	0.4405^{2}	0.1930^{1}
6.0	-0.6480	0.1353^{1}	0.2870^{2}	0.1176^{1}
6.1	-0.6268	0.5450^{2}	0.1202^{2}	0.4638^{2}
6.2	-0.6071	-0.2528^{2}	-0.5774^{3}	-0.2108^{2}
6.3	-0.5885	-0.1040^{1}	-0.2450^{2}	-0.8504^{2}
6.4	-0.5711	-0.1816^{1}	-0.4399^{2}	-0.1457^{1}
6.5	-0.5547	-0.2581^{1}	-0.6411^{2}	-0.2034^{1}
6.6	-0.5393	-0.3333^{1}	-0.8469^{2}	-0.2583^{1}
6.7	-0.5247	-0.4074^{1}	-0.1057^{1}	-0.3106^{1}
6.8	-0.5110	-0.4802^{1}	-0.1269^{1}	-0.3604^{1}
6.9	-0.4980	-0.5517^{1}	-0.1483^{1}	-0.4080^{1}
7.0	-0.4858	-0.6219^{1}	-0.1698^{1}	-0.4533^{1}
7.1	-0.4741	-0.6910^{1}	-0.1914^{1}	-0.4968^{1}
7.2	-0.4631	-0.7587^{1}	-0.2129^{1}	-0.5383^{1}
7.3	-0.4527	-0.8252^{1}	-0.2344^{1}	-0.5781^{1}
7.4	-0.4427	-0.8905^{1}	-0.2558^{1}	-0.6162^{1}
7.5	-0.4332	-0.9546^{1}	-0.2770^{1}	-0.6528^{1}
7.6	-0.4242	-0.1017	-0.2980^{1}	-0.6880^{1}
7.8	-0.4074	-0.1140	-0.3395^{1}	-0.7543^{1}
8.0	-0.3921	-0.1257	-0.3801^{1}	-0.8156^{1}
8.2	-0.3780	-0.1371	-0.4195^{1}	-0.8727^{1}
8.4	-0.3651	-0.1480	-0.4579^{1}	-0.9258^{1}
8.6	-0.3532	-0.1585	-0.4951^{1}	-0.9753^{1}
8.8	-0.3422	-0.1686	-0.5311^{1}	-0.1022
9.0	-0.3321	-0.1784	-0.5659^{1}	-0.1065
9.2	-0.3226	-0.1878	-0.5996^{1}	-0.1106
9.4	-0.3138	-0.1969	-0.6321^{1}	-0.1145
9.6	-0.3056	-0.2057	-0.6636^{1}	-0.1181
9.9	-0.2942	-0.2183	-0.7087^{1}	-0.1232
10.2	-0.2839	-0.2303	-0.7515^{1}	-0.1279
10.5	-0.2744	-0.2417	-0.7922^{1}	-0.1322
10.8	-0.2658	-0.2525	-0.8308^{1}	-0.1362
11.1	-0.2578	-0.2629	-0.8675^{1}	-0.1400
11.6	-0.2459	-0.2791	-0.9247^{1}	-0.1457
12.1	-0.2354	-0.2942	-0.9774^{1}	-0.1508
12.6	-0.2261	-0.3082	-0.1026	-0.1554
13.1	-0.2177	-0.3213	-0.1071	-0.1596
13.6	-0.2102	-0.3335	-0.1113	-0.1634
14.4	-0.1996	-0.3516	-0.1174	-0.1689
15.2	-0.1905	-0.3679	-0.1229	-0.1737
16.0	-0.1826	-0.3828	-0.1278	-0.1779
16.8	-0.1757	-0.3965	-0.1323	-0.1817
17.6	-0.1695	-0.4090	-0.1363	-0.1851

$\hat{\alpha}_3 = 1.35$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
5.0	-1.0925	0.1066	0.2023^{2}	0.1344
5.1	-1.0365	0.9874^{1}	0.4254^{2}	0.1193
5.2	-0.9863	0.9076^{1}	0.5874^{2}	0.1054
5.3	-0.9409	0.8269^{1}	0.6972^{2}	0.9257^{1}
5.4	-0.8995	0.7456^{1}	0.7615^{2}	0.8064^{1}
5.5	-0.8616	0.6640^{1}	0.7863^{2}	0.6953^{1}
5.6	-0.8267	0.5821^{1}	0.7763^{2}	0.5913^{1}
5.7	-0.7945	0.5004^{1}	0.7359^{2}	0.4938^{1}
5.8	-0.7647	0.4188^{1}	0.6689^{2}	0.4023^{1}
5.9	-0.7370	0.3377^{1}	0.5786^{2}	0.3161^{1}
6.0	-0.7113	0.2573^{1}	0.4683^{2}	0.2350^{1}
6.1	-0.6873	0.1774^{1}	0.3404^{2}	0.1583^{1}
6.2	-0.6648	0.9825^{2}	0.1976^{2}	0.8577^{2}
6.3	-0.6438	0.1993^{2}	0.4177^{3}	0.1704^{2}
6.4	-0.6242	-0.5370^{2}	-0.1246^{2}	-0.4801^{2}
6.5	-0.6056	-0.1337^{1}	-0.3006^{2}	-0.1099^{1}
6.6	-0.5883	-0.2088^{1}	-0.4834^{2}	-0.1685^{1}
6.7	-0.5719	-0.2830^{1}	-0.6728^{2}	-0.2244^{1}
6.8	-0.5565	-0.3560^{1}	-0.8671^{2}	-0.2776^{1}
6.9	-0.5419	-0.4278^{1}	-0.1065^{1}	-0.3284^{1}
6.9	-0.5419	-0.4279^{1}	-0.1066^{1}	-0.3284^{1}
7.0	-0.5280	-0.4987^{1}	-0.1267^{1}	-0.3769^{1}
7.1	-0.5151	-0.5680^{1}	-0.1470^{1}	-0.4230^{1}
7.2	-0.5027	-0.6363^{1}	-0.1674^{1}	-0.4672^{1}
7.3	-0.4910	-0.7034^{1}	-0.1880^{1}	-0.5096^{1}
7.4	-0.4798	-0.7693^{1}	-0.2085^{1}	-0.5501^{1}
7.5	-0.4692	-0.8341^{1}	-0.2291^{1}	-0.5890^{1}
7.6	-0.4592	-0.8977^{1}	-0.2495^{1}	-0.6263^{1}
7.7	-0.4496	-0.9602^{1}	-0.2699^{1}	-0.6621^{1}
7.8	-0.4404	-0.1021	-0.2901^{1}	-0.6965^{1}
7.9	-0.4317	-0.1082	-0.3101^{1}	-0.7296^{1}
8.1	-0.4154	-0.1199	-0.3497^{1}	-0.7921^{1}
8.3	-0.4004	-0.1312	-0.3883^{1}	-0.8502^{1}
8.5	-0.3867	-0.1420	-0.4260^{1}	-0.9043^{1}
8.7	-0.3740	-0.1525	-0.4626^{1}	-0.9548^{1}
8.9	-0.3623	-0.1627	-0.4982^{1}	-0.1002
9.1	-0.3515	-0.1724	-0.5327^{1}	-0.1046
9.3	-0.3414	-0.1818	-0.5661^{1}	-0.1088
9.5	-0.3320	-0.1909	-0.5985^{1}	-0.1127
9.7	-0.3232	-0.1997	-0.6298^{1}	-0.1164
9.9	-0.3150	-0.2082	-0.6601^{1}	-0.1199
10.2	-0.3037	-0.2204	-0.7036^{1}	-0.1248
10.5	-0.2933	-0.2320	-0.7450^{1}	-0.1293
10.8	-0.2839	-0.2430	-0.7844^{1}	-0.1335
11.1	-0.2752	-0.2536	-0.8219^{1}	-0.1374
11.4	-0.2672	-0.2637	-0.8576^{1}	-0.1411
11.9	-0.2551	-0.2795	-0.9133^{1}	-0.1466
12.4	-0.2445	-0.2941	-0.9648^{1}	-0.1516
12.9	-0.2350	-0.3078	-0.1012	-0.1561
13.4	-0.2265	-0.3206	-0.1057	-0.1602
13.9	-0.2188	-0.3326	-0.1098	-0.1639
14.7	-0.2080	-0.3503	-0.1158	-0.1693
15.5	-0.1987	-0.3664	-0.1212	-0.1740
16.3	-0.1906	-0.3810	-0.1261	-0.1782
17.1	-0.1834	-0.3945	-0.1305	-0.1819

$\hat{\alpha}_3 = 1.40$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
5.2	-1.1058	0.1011	0.7869^{3}	0.1268
5.3	-1.0507	0.9335^{1}	0.2960^{2}	0.1124
5.4	-1.0012	0.8553^{1}	0.4548^{2}	0.9909^{1}
5.5	-0.9565	0.7764^{1}	0.5638^{2}	0.8680^{1}
5.6	-0.9157	0.6970^{1}	0.6297^{2}	0.7537^{1}
5.7	-0.8783	0.6175^{1}	0.6579^{2}	0.6470^{1}
5.8	-0.8438	0.5379^{1}	0.6531^{2}	0.5471^{1}
5.9	-0.8119	0.4584^{1}	0.6194^{2}	0.4534^{1}
6.0	-0.7824	0.3792^{1}	0.5603^{2}	0.3653^{1}
6.1	-0.7550	0.3006^{1}	0.4791^{2}	0.2824^{1}
6.2	-0.7294	0.2225^{1}	0.3785^{2}	0.2041^{1}
6.3	-0.7056	0.1450^{1}	0.2611^{2}	0.1300^{1}
6.4	-0.6832	0.6823^{2}	0.1292^{2}	0.5987^{2}
6.5	-0.6622	-0.7667^{3}	-0.1518^{3}	-0.6591^{3}
6.6	-0.6426	-0.8266^{2}	-0.1702^{2}	-0.6968^{2}
6.7	-0.6241	-0.1567^{1}	-0.3343^{2}	-0.1296^{1}
6.8	-0.6067	-0.2297^{1}	-0.5060^{2}	-0.1867^{1}
6.9	-0.5902	-0.3017^{1}	-0.6840^{2}	-0.2410^{1}
7.0	-0.5747	-0.3726^{1}	-0.8671^{2}	-0.2928^{1}
7.1	-0.5600	-0.4424^{1}	-0.1054^{1}	-0.3422^{1}
7.2	-0.5461	-0.5112^{1}	-0.1245^{1}	-0.3894^{1}
7.3	-0.5330	-0.5789^{1}	-0.1438^{1}	-0.4346^{1}
7.4	-0.5205	-0.6452^{1}	-0.1632^{1}	-0.4778^{1}
7.5	-0.5086	-0.7106^{1}	-0.1827^{1}	-0.5191^{1}
7.6	-0.4974	-0.7748^{1}	-0.2023^{1}	-0.5588^{1}
7.7	-0.4866	-0.8380^{1}	-0.2220^{1}	-0.5969^{1}
7.8	-0.4764	-0.9000^{1}	-0.2415^{1}	-0.6335^{1}
7.9	-0.4667	-0.9609^{1}	-0.2610^{1}	-0.6686^{1}
8.0	-0.4574	-0.1021	-0.2804^{1}	-0.7023^{1}
8.1	-0.4485	-0.1080	-0.2997^{1}	-0.7349^{1}
8.3	-0.4318	-0.1194	-0.3378^{1}	-0.7963^{1}
8.5	-0.4165	-0.1305	-0.3751^{1}	-0.8535^{1}
8.7	-0.4025	-0.1411	-0.4116^{1}	-0.9068^{1}
8.9	-0.3895	-0.1514	-0.4472^{1}	-0.9566^{1}
9.1	-0.3775	-0.1613	-0.4818^{1}	-0.1003
9.3	-0.3664	-0.1709	-0.5154^{1}	-0.1047
9.5	-0.3560	-0.1801	-0.5481^{1}	-0.1088
9.7	-0.3464	-0.1891	-0.5797^{1}	-0.1127
9.9	-0.3373	-0.1977	-0.6104^{1}	-0.1164
10.1	-0.3289	-0.2061	-0.6401^{1}	-0.1198
10.4	-0.3172	-0.2181	-0.6829^{1}	-0.1247
10.7	-0.3065	-0.2295	-0.7236^{1}	-0.1292
11.0	-0.2967	-0.2404	-0.7625^{1}	-0.1334
11.3	-0.2877	-0.2509	-0.7995^{1}	-0.1372
11.6	-0.2794	-0.2608	-0.8347^{1}	-0.1409
12.1	-0.2669	-0.2764	-0.8899^{1}	-0.1464
12.6	-0.2558	-0.2910	-0.9410^{1}	-0.1513
13.1	-0.2460	-0.3046	-0.9884^{1}	-0.1558
13.6	-0.2371	-0.3173	-0.1032	-0.1599
14.1	-0.2291	-0.3292	-0.1073	-0.1637
14.9	-0.2179	-0.3467	-0.1133	-0.1690
15.7	-0.2082	-0.3627	-0.1187	-0.1737
16.5	-0.1997	-0.3773	-0.1236	-0.1779
17.3	-0.1922	-0.3907	-0.1281	-0.1816
18.1	-0.1855	-0.4030	-0.1321	-0.1850

$\hat{\alpha}_3 = 1.45$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
5.5	-1.0691	0.8871^{1}	0.1642^{2}	0.1068
5.6	-1.0199	0.8105^{1}	0.3245^{2}	0.9396^{1}
5.7	-0.9754	0.7333^{1}	0.4364^{2}	0.8210^{1}
5.8	-0.9348	0.6559^{1}	0.5068^{2}	0.7108^{1}
5.9	-0.8976	0.5784^{1}	0.5408^{2}	0.6078^{1}
6.0	-0.8634	0.5009^{1}	0.5432^{2}	0.5114^{1}
6.1	-0.8317	0.4236^{1}	0.5177^{2}	0.4208^{1}
6.2	-0.8022	0.3467^{1}	0.4677^{2}	0.3356^{1}
6.3	-0.7749	0.2703^{1}	0.3964^{2}	0.2553^{1}
6.4	-0.7494	0.1944^{1}	0.3063^{2}	0.1794^{1}
6.5	-0.7255	0.1192^{1}	0.1998^{2}	0.1076^{1}
6.6	-0.7031	0.4469^{2}	0.7911^{3}	0.3948^{2}
6.7	-0.6821	-0.2901^{2}	-0.5389^{3}	-0.2521^{2}
6.8	-0.6624	-0.1018^{1}	-0.1975^{2}	-0.8649^{2}
6.9	-0.6438	-0.1738^{1}	-0.3501^{2}	-0.1449^{1}
7.0	-0.6263	-0.2447^{1}	-0.5103^{2}	-0.2005^{1}
7.1	-0.6097	-0.3147^{1}	-0.6770^{2}	-0.2535^{1}
7.2	-0.5940	-0.3837^{1}	-0.8492^{2}	-0.3041^{1}
7.3	-0.5793	-0.4516^{1}	-0.1025^{1}	-0.3523^{1}
7.4	-0.5652	-0.5185^{1}	-0.1205^{1}	-0.3985^{1}
7.5	-0.5519	-0.5843^{1}	-0.1388^{1}	-0.4427^{1}
7.6	-0.5393	-0.6491^{1}	-0.1572^{1}	-0.4851^{1}
7.7	-0.5272	-0.7128^{1}	-0.1758^{1}	-0.5256^{1}
7.8	-0.5158	-0.7755^{1}	-0.1945^{1}	-0.5646^{1}
7.9	-0.5049	-0.8371^{1}	-0.2133^{1}	-0.6020^{1}
8.0	-0.4945	-0.8977^{1}	-0.2320^{1}	-0.6379^{1}
8.1	-0.4845	-0.9572^{1}	-0.2507^{1}	-0.6724^{1}
8.2	-0.4751	-0.1016	-0.2693^{1}	-0.7057^{1}
8.3	-0.4660	-0.1073	-0.2878^{1}	-0.7377^{1}
8.4	-0.4573	-0.1130	-0.3062^{1}	-0.7685^{1}
8.6	-0.4410	-0.1240	-0.3426^{1}	-0.8270^{1}
8.8	-0.4260	-0.1346	-0.3784^{1}	-0.8815^{1}
9.0	-0.4122	-0.1449	-0.4133^{1}	-0.9324^{1}
9.2	-0.3994	-0.1548	-0.4474^{1}	-0.9801^{1}
9.4	-0.3875	-0.1644	-0.4806^{1}	-0.1025
9.6	-0.3765	-0.1736	-0.5129^{1}	-0.1067
9.8	-0.3662	-0.1826	-0.5444^{1}	-0.1107
10.0	-0.3566	-0.1912	-0.5749^{1}	-0.1144
10.2	-0.3476	-0.1996	-0.6044^{1}	-0.1179
10.4	-0.3391	-0.2077	-0.6331^{1}	-0.1213
10.7	-0.3274	-0.2193	-0.6746^{1}	-0.1260
11.0	-0.3166	-0.2305	-0.7141^{1}	-0.1303
11.3	-0.3068	-0.2411	-0.7518^{1}	-0.1344
11.6	-0.2977	-0.2512	-0.7878^{1}	-0.1382
11.9	-0.2893	-0.2609	-0.8222^{1}	-0.1417
12.4	-0.2764	-0.2765	-0.8771^{1}	-0.1472
12.9	-0.2652	-0.2907	-0.9271^{1}	-0.1520
13.4	-0.2552	-0.3039	-0.9735^{1}	-0.1564
13.9	-0.2462	-0.3163	-0.1017	-0.1604
14.4	-0.2381	-0.3280	-0.1057	-0.1641
15.2	-0.2266	-0.3452	-0.1116	-0.1693
16.0	-0.2166	-0.3609	-0.1169	-0.1740
16.8	-0.2079	-0.3753	-0.1218	-0.1781
17.6	-0.2002	-0.3885	-0.1262	-0.1818
18.4	-0.1934	-0.4006	-0.1302	-0.1851

$\hat{\alpha}_3 = 1.50$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
5.7	-1.0915	0.8480^{1}	0.2943^{3}	0.1024
5.8	-1.0422	0.7728^{1}	0.1952^{2}	0.8991^{1}
5.9	-0.9976	0.6974^{1}	0.3133^{2}	0.7840^{1}
6.0	-0.9570	0.6219^{1}	0.3908^{2}	0.6770^{1}
6.1	-0.9197	0.5463^{1}	0.4329^{2}	0.5770^{1}
6.2	-0.8854	0.4708^{1}	0.4442^{2}	0.4834^{1}
6.3	-0.8536	0.3956^{1}	0.4283^{2}	0.3954^{1}
6.4	-0.8242	0.3209^{1}	0.3885^{2}	0.3127^{1}
6.5	-0.7968	0.2465^{1}	0.3279^{2}	0.2345^{1}
6.6	-0.7711	0.1728^{1}	0.2489^{2}	0.1606^{1}
6.7	-0.7471	0.9962^{2}	0.1538^{2}	0.9059^{2}
6.8	-0.7246	0.2719^{2}	0.4457^{3}	0.2422^{2}
6.9	-0.7035	-0.4446^{2}	-0.7684^{3}	-0.3881^{2}
7.0	-0.6836	-0.1153^{1}	-0.2089^{2}	-0.9875^{2}
7.1	-0.6648	-0.1852^{1}	-0.3500^{2}	-0.1558^{1}
7.2	-0.6471	-0.2543^{1}	-0.4989^{2}	-0.2102^{1}
7.3	-0.6304	-0.3224^{1}	-0.6545^{2}	-0.2621^{1}
7.4	-0.6146	-0.3896^{1}	-0.8157^{2}	-0.3116^{1}
7.5	-0.5995	-0.4559^{1}	-0.9818^{2}	-0.3591^{1}
7.6	-0.5853	-0.5212^{1}	-0.1152^{1}	-0.4045^{1}
7.7	-0.5718	-0.5851^{1}	-0.1324^{1}	-0.4477^{1}
7.8	-0.5590	-0.6483^{1}	-0.1498^{1}	-0.4893^{1}
7.9	-0.5468	-0.7105^{1}	-0.1675^{1}	-0.5292^{1}
8.0	-0.5351	-0.7716^{1}	-0.1853^{1}	-0.5676^{1}
8.1	-0.5240	-0.8318^{1}	-0.2032^{1}	-0.6044^{1}
8.2	-0.5134	-0.8910^{1}	-0.2210^{1}	-0.6398^{1}
8.3	-0.5033	-0.9492^{1}	-0.2389^{1}	-0.6738^{1}
8.4	-0.4936	-0.1006	-0.2568^{1}	-0.7066^{1}
8.5	-0.4843	-0.1063	-0.2746^{1}	-0.7382^{1}
8.6	-0.4754	-0.1118	-0.2923^{1}	-0.7687^{1}
8.8	-0.4588	-0.1226	-0.3274^{1}	-0.8265^{1}
9.0	-0.4434	-0.1330	-0.3620^{1}	-0.8804^{1}
9.2	-0.4292	-0.1431	-0.3959^{1}	-0.9308^{1}
9.4	-0.4160	-0.1528	-0.4290^{1}	-0.9781^{1}
9.6	-0.4038	-0.1622	-0.4613^{1}	-0.1022
9.8	-0.3925	-0.1713	-0.4929^{1}	-0.1064
10.0	-0.3818	-0.1802	-0.5236^{1}	-0.1104
10.2	-0.3719	-0.1887	-0.5535^{1}	-0.1141
10.4	-0.3626	-0.1969	-0.5825^{1}	-0.1176
10.6	-0.3539	-0.2049	-0.6107^{1}	-0.1209
10.9	-0.3417	-0.2164	-0.6514^{1}	-0.1256
11.2	-0.3306	-0.2274	-0.6904^{1}	-0.1299
11.5	-0.3204	-0.2379	-0.7276^{1}	-0.1340
11.8	-0.3110	-0.2479	-0.7632^{1}	-0.1377
12.1	-0.3023	-0.2575	-0.7973^{1}	-0.1413
12.6	-0.2892	-0.2727	-0.8508^{1}	-0.1466
13.1	-0.2775	-0.2868	-0.9005^{1}	-0.1515
13.6	-0.2671	-0.2999	-0.9467^{1}	-0.1559
14.1	-0.2577	-0.3123	-0.9899^{1}	-0.1599
14.6	-0.2492	-0.3239	-0.1030	-0.1636
15.4	-0.2372	-0.3411	-0.1089	-0.1688
16.2	-0.2268	-0.3567	-0.1143	-0.1735
17.0	-0.2177	-0.3710	-0.1191	-0.1776
17.8	-0.2096	-0.3842	-0.1235	-0.1813
18.6	-0.2025	-0.3963	-0.1276	-0.1847

$\hat{\alpha}_3 = 1.55$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
6.0	-1.0683	0.7421	0.6443^{3}	$.8687^{1}$
6.1	-1.0232	0.6683^{1}	0.1921^{2}	0.7562^{1}
6.2	-0.9822	0.5945^{1}	0.2795^{2}	0.6516^{1}
6.3	-0.9446	0.5207^{1}	0.3318^{2}	0.5540^{1}
6.4	-0.9100	0.4472^{1}	0.3536^{2}	0.4626^{1}
6.5	-0.8780	0.3740^{1}	0.3485^{2}	0.3768^{1}
6.6	-0.8483	0.3011^{1}	0.3201^{2}	0.2959^{1}
6.7	-0.8206	0.2288^{1}	0.2710^{2}	0.2195^{1}
6.8	-0.7947	0.1570^{1}	0.2037^{2}	0.1472^{1}
6.9	-0.7705	0.8580^{2}	0.1203^{2}	0.7874^{2}
7.0	-0.7478	0.1532^{2}	0.2299^{3}	0.1377^{2}
7.1	-0.7264	-0.5442^{2}	-0.8663^{3}	-0.4797^{2}
7.2	-0.7063	-0.1234^{1}	-0.2070^{2}	-0.1067^{1}
7.3	-0.6873	-0.1915^{1}	-0.3366^{2}	-0.1627^{1}
7.4	-0.6694	-0.2587^{1}	-0.4742^{2}	-0.2161^{1}
7.5	-0.6524	-0.3251^{1}	-0.6186^{2}	-0.2670^{1}
7.6	-0.6363	-0.3906^{1}	-0.7689^{2}	-0.3157^{1}
7.7	-0.6211	-0.4551^{1}	-0.9242^{2}	-0.3623^{1}
7.8	-0.6066	-0.5187^{1}	-0.1084^{1}	-0.4069^{1}
7.9	-0.5929	-0.5814^{1}	-0.1246^{1}	-0.4497^{1}
8.0	-0.5798	-0.6431^{1}	-0.1412^{1}	-0.4907^{1}
8.1	-0.5673	-0.7038^{1}	-0.1579^{1}	-0.5300^{1}
8.2	-0.5554	-0.7636^{1}	-0.1748^{1}	-0.5678^{1}
8.3	-0.5441	-0.8224^{1}	-0.1918^{1}	-0.6042^{1}
8.4	-0.5333	-0.8803^{1}	-0.2089^{1}	-0.6391^{1}
8.5	-0.5229	-0.9373^{1}	-0.2260^{1}	-0.6728^{1}
8.6	-0.5130	-0.9933^{1}	-0.2431^{1}	-0.7052^{1}
8.7	-0.5035	-0.1048	-0.2602^{1}	-0.7365^{1}
8.8	-0.4944	-0.1103	-0.2773^{1}	-0.7667^{1}
8.9	-0.4857	-0.1156	-0.2942^{1}	-0.7958^{1}
9.1	-0.4692	-0.1260	-0.3278^{1}	-0.8511^{1}
9.3	-0.4541	-0.1361	-0.3610^{1}	-0.9028^{1}
9.5	-0.4400	-0.1458	-0.3935^{1}	-0.9512^{1}
9.7	-0.4270	-0.1552	-0.4253^{1}	-0.9967^{1}
9.9	-0.4148	-0.1643	-0.4565^{1}	-0.1039
10.1	-0.4035	-0.1732	-0.4869^{1}	-0.1080
10.3	-0.3929	-0.1817	-0.5165^{1}	-0.1118
10.5	-0.3830	-0.1900	-0.5454^{1}	-0.1154
10.7	-0.3737	-0.1980	-0.5734^{1}	-0.1188
10.9	-0.3649	-0.2057	-0.6008^{1}	-0.1220
11.2	-0.3527	-0.2169	-0.6403^{1}	-0.1266
11.5	-0.3415	-0.2276	-0.6782^{1}	-0.1308
11.8	-0.3312	-0.2379	-0.7144^{1}	-0.1347
12.1	-0.3217	-0.2477	-0.7491^{1}	-0.1384
12.4	-0.3129	-0.2570	-0.7824^{1}	-0.1419
12.9	-0.2996	-0.2718	-0.8347^{1}	-0.1471
13.4	-0.2877	-0.2856	-0.8834^{1}	-0.1519
13.9	-0.2771	-0.2985	-0.9289^{1}	-0.1562
14.4	-0.2675	-0.3107	-0.9713^{1}	-0.1601
14.9	-0.2588	-0.3221	-0.1011	-0.1638
15.7	-0.2465	-0.3389	-0.1069	-0.1690
16.5	-0.2359	-0.3543	-0.1122	-0.1736
17.3	-0.2265	-0.3685	-0.1170	-0.1776
18.1	-0.2182	-0.3815	-0.1214	-0.1813
18.9	-0.2109	-0.3935	-0.1254	-0.1846

$\hat{\alpha}_3 = 1.60$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
6.3	-1.0524	0.6456^{1}	0.7024^{3}	0.7371^{1}
6.4	-1.0106	0.5734^{1}	0.1700^{2}	0.6343^{1}
6.5	-0.9725	0.5014^{1}	0.2344^{2}	0.5385^{1}
6.6	-0.9373	0.4296^{1}	0.2684^{2}	0.4487^{1}
6.7	-0.9048	0.3582^{1}	0.2758^{2}	0.3644^{1}
6.8	-0.8747	0.2872^{1}	0.2597^{2}	0.2850^{1}
6.9	-0.8466	0.2167^{1}	0.2229^{2}	0.2100^{1}
7.0	-0.8203	0.1467^{1}	0.1680^{2}	0.1390^{1}
7.1	-0.7958	0.7738^{2}	0.9694^{3}	0.7177^{2}
7.2	-0.7727	0.8695^{3}	0.1177^{3}	0.7902^{3}
7.3	-0.7510	-0.5924^{2}	-0.8576^{3}	-0.5279^{2}
7.4	-0.7305	-0.1265^{1}	-0.1943^{2}	-0.1106^{1}
7.5	-0.7113	-0.1928^{1}	-0.3121^{2}	-0.1657^{1}
7.6	-0.6930	-0.2584^{1}	-0.4383^{2}	-0.2182^{1}
7.7	-0.6757	-0.3232^{1}	-0.5716^{2}	-0.2685^{1}
7.8	-0.6594	-0.3870^{1}	-0.7111^{2}	-0.3165^{1}
7.9	-0.6438	-0.4500^{1}	-0.8558^{2}	-0.3624^{1}
8.0	-0.6291	-0.5121^{1}	-0.1005^{1}	-0.4065^{1}
8.1	-0.6150	-0.5733^{1}	-0.1158^{1}	-0.4487^{1}
8.2	-0.6016	-0.6338^{1}	-0.1314^{1}	-0.4893^{1}
8.3	-0.5889	-0.6931^{1}	-0.1472^{1}	-0.5282^{1}
8.4	-0.5768	-0.7516^{1}	-0.1632^{1}	-0.5655^{1}
8.5	-0.5652	-0.8092^{1}	-0.1794^{1}	-0.6015^{1}
8.6	-0.5541	-0.8659^{1}	-0.1957^{1}	-0.6361^{1}
8.7	-0.5435	-0.9217^{1}	-0.2120^{1}	-0.6695^{1}
8.8	-0.5333	-0.9766^{1}	-0.2284^{1}	-0.7016^{1}
8.9	-0.5236	-0.1031	-0.2448^{1}	-0.7326^{1}
9.0	-0.5142	-0.1084	-0.2611^{1}	-0.7625^{1}
9.1	-0.5053	-0.1136	-0.2775^{1}	-0.7914^{1}
9.2	-0.4967	-0.1188	-0.2937^{1}	-0.8193^{1}
9.4	-0.4804	-0.1288	-0.3260^{1}	-0.8724^{1}
9.6	-0.4654	-0.1386	-0.3578^{1}	-0.9222^{1}
9.8	-0.4515	-0.1480	-0.3890^{1}	-0.9689^{1}
10.0	-0.4385	-0.1571	-0.4197^{1}	-0.1013
10.2	-0.4264	-0.1659	-0.4497^{1}	-0.1054
10.4	-0.4150	-0.1745	-0.4790^{1}	-0.1093
10.6	-0.4044	-0.1828	-0.5077^{1}	-0.1130
10.8	-0.3945	-0.1908	-0.5356^{1}	-0.1165
11.0	-0.3852	-0.1986	-0.5628^{1}	-0.1198
11.2	-0.3763	-0.2061	-0.5893^{1}	-0.1230
11.5	-0.3641	-0.2171	-0.6277^{1}	-0.1274
11.8	-0.3528	-0.2275	-0.6645^{1}	-0.1315
12.1	-0.3424	-0.2375	-0.6999^{1}	-0.1354
12.4	-0.3327	-0.2470	-0.7338^{1}	-0.1390
12.7	-0.3238	-0.2562	-0.7663^{1}	-0.1423
13.2	-0.3103	-0.2707	-0.8175^{1}	-0.1475
13.7	-0.2982	-0.2842	-0.8653^{1}	-0.1522
14.2	-0.2874	-0.2969	-0.9100^{1}	-0.1564
14.7	-0.2776	-0.3088	-0.9518^{1}	-0.1603
15.2	-0.2687	-0.3200	-0.9910^{1}	-0.1639
16.0	-0.2561	-0.3366	-0.1049	-0.1690
16.8	-0.2452	-0.3518	-0.1101	-0.1736
17.6	-0.2356	-0.3658	-0.1149	-0.1776
18.4	-0.2270	-0.3786	-0.1192	-0.1812
19.2	-0.2195	-0.3905	-0.1232	-0.1845

$\hat{\alpha}_3 = 1.65$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
6.6	-1.0426	0.5584^{1}	0.5891^{3}	0.6247^{1}
6.7	-1.0034	0.4879^{1}	0.1379^{2}	0.5300^{1}
6.8	-0.9675	0.4178^{1}	0.1860^{2}	0.4414^{1}
6.9	-0.9343	0.3480^{1}	0.2070^{2}	0.3582^{1}
7.0	-0.9035	0.2787^{1}	0.2044^{2}	0.2798^{1}
7.1	-0.8748	0.2099^{1}	0.1809^{2}	0.2058^{1}
7.2	-0.8480	0.1417^{1}	0.1391^{2}	0.1358^{1}
7.3	-0.8230	0.7403^{2}	0.8092^{3}	0.6949^{2}
7.4	-0.7995	0.7037^{3}	0.842563	0.6472^{3}
7.5	-0.7773	-0.5924^{2}	-0.7669^{3}	-0.5340^{2}
7.6	-0.7565	-0.1248^{1}	-0.1730^{2}	-0.1105^{1}
7.7	-0.7368	-0.1896^{1}	-0.2790^{2}	-0.1649^{1}
7.8	-0.7182	-0.2536^{1}	-0.3936^{2}	-0.2169^{1}
7.9	-0.7005	-0.3168^{1}	-0.5157^{2}	-0.2665^{1}
8.0	-0.6838	-0.3792^{1}	-0.6443^{2}	-0.3140^{1}
8.1	-0.6679	-0.4408^{1}	-0.7784^{2}	-0.3595^{1}
8.2	-0.6528	-0.5015^{1}	-0.9173^{2}	-0.4030^{1}
8.3	-0.6384	-0.5614^{1}	-0.1060^{1}	-0.4449^{1}
8.4	-0.6248	-0.6204^{1}	-0.1207^{1}	-0.4850^{1}
8.5	-0.6116	-0.6787^{1}	-0.1356^{1}	-0.5237^{1}
8.6	-0.5992	-0.7359^{1}	-0.1508^{1}	-0.5607^{1}
8.7	-0.5873	-0.7923^{1}	-0.1661^{1}	-0.5964^{1}
8.8	-0.5759	-0.8479^{1}	-0.1816^{1}	-0.6308^{1}
8.9	-0.5650	-0.9026^{1}	-0.1971^{1}	-0.6639^{1}
9.0	-0.5546	-0.9565^{1}	-0.2128^{1}	-0.6958^{1}
9.1	-0.5446	-0.1009	-0.2284^{1}	-0.7266^{1}
9.2	-0.5350	-0.1062	-0.2441^{1}	-0.7563^{1}
9.3	-0.5258	-0.1113	-0.2598^{1}	-0.7851^{1}
9.4	-0.5169	-0.1164	-0.2755^{1}	-0.8129^{1}
9.5	-0.5084	-0.1214	-0.2910^{1}	-0.8397^{1}
9.7	-0.4923	-0.1311	-0.3220^{1}	-0.8909^{1}
9.9	-0.4774	-0.1405	-0.3526^{1}	-0.9389^{1}
10.1	-0.4635	-0.1496	-0.3827^{1}	-0.9840^{1}
10.3	-0.4505	-0.1585	-0.4122^{1}	-0.1027
10.5	-0.4384	-0.1671	-0.4412^{1}	-0.1067
10.7	-0.4271	-0.1754	-0.4695^{1}	-0.1105
10.9	-0.4165	-0.1834	-0.4972^{1}	-0.1140
11.1	-0.4065	-0.1912	-0.5243^{1}	-0.1174
11.3	-0.3971	-0.1988	-0.5506^{1}	-0.1207
11.5	-0.3882	-0.2062	-0.5764^{1}	-0.1237
11.8	-0.3758	-0.2168	-0.6137^{1}	-0.1281
12.1	-0.3644	-0.2270	-0.6496^{1}	-0.1321
12.4	-0.3539	-0.2368	-0.6841^{1}	-0.1359
12.7	-0.3441	-0.2461	-0.7172^{1}	-0.1394
13.0	-0.3351	-0.2551	-0.7490^{1}	-0.1427
13.5	-0.3213	-0.2693	-0.7992^{1}	-0.1478
14.0	-0.3090	-0.2825	-0.8462^{1}	-0.1524
14.5	-0.2980	-0.2950	-0.8902^{1}	-0.1566
15.0	-0.2880	-0.3067	-0.9314^{1}	-0.1604
15.5	-0.2789	-0.3177	-0.9701^{1}	-0.1639
16.3	-0.2660	-0.3341	-0.1027	-0.1690
17.1	-0.2548	-0.3491	-0.1079	-0.1735
17.9	-0.2449	-0.3629	-0.1127	-0.1775
18.7	-0.2361	-0.3756	-0.1170	-0.1811
19.5	-0.2283	-0.3873	-0.1210	-0.1844

$\hat{\alpha}_3 = 1.70$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
6.9	-1.0378	0.4800^{1}	0.3862^{3}	0.5286^{1}
7.0	-1.0007	0.4114^{1}	0.1027^{2}	0.4405^{1}
7.1	-0.9666	0.3432^{1}	0.1391^{2}	0.3580^{1}
7.2	-0.9349	0.2754^{1}	0.1513^{2}	0.2802^{1}
7.3	-0.9055	0.2082^{1}	0.1421^{2}	0.2069^{1}
7.4	-0.8780	0.1415^{1}	0.1142^{2}	0.1375^{1}
7.5	-0.8523	0.7547^{2}	0.6962^{3}	0.7179^{2}
7.6	-0.8282	0.1002^{2}	0.1033^{3}	0.9338^{3}
7.7	-0.8055	-0.5470^{2}	-0.6191^{3}	-0.5000^{2}
7.8	-0.7842	-0.1187^{1}	-0.1457^{2}	-0.1066^{1}
7.9	-0.7640	-0.1821^{1}	-0.2396^{2}	-0.1605^{1}
8.0	-0.7449	-0.2446^{1}	-0.3423^{2}	-0.2120^{1}
8.1	-0.7268	-0.3064^{1}	-0.4530^{2}	-0.2612^{1}
8.2	-0.7096	-0.3677^{1}	-0.5710^{2}	-0.3085^{1}
8.3	-0.6934	-0.4277^{1}	-0.6939^{2}	-0.3535^{1}
8.4	-0.6779	-0.4871^{1}	-0.8225^{2}	-0.3967^{1}
8.5	-0.6632	-0.5457^{1}	-0.9556^{2}	-0.4383^{1}
8.6	-0.6491	-0.6035^{1}	-0.1092^{1}	-0.4781^{1}
8.7	-0.6357	-0.6604^{1}	-0.1232^{1}	-0.5165^{1}
8.8	-0.6228	-0.7168^{1}	-0.1376^{1}	-0.5535^{1}
8.9	-0.6106	-0.7720^{1}	-0.1520^{1}	-0.5889^{1}
9.0	-0.5989	-0.8265^{1}	-0.1667^{1}	-0.6231^{1}
9.1	-0.5877	-0.8801^{1}	-0.1815^{1}	-0.6560^{1}
9.2	-0.5769	-0.9330^{1}	-0.1964^{1}	-0.6878^{1}
9.3	-0.5666	-0.9851^{1}	-0.2113^{1}	-0.7185^{1}
9.4	-0.5567	-0.1036	-0.2264^{1}	-0.7481^{1}
9.5	-0.5472	-0.1087	-0.2414^{1}	-0.7768^{1}
9.6	-0.5381	-0.1137	-0.2564^{1}	-0.8045^{1}
9.7	-0.5293	-0.1186	-0.2714^{1}	-0.8313^{1}
9.8	-0.5208	-0.1234	-0.2864^{1}	-0.8572^{1}
10.0	-0.5048	-0.1328	-0.3161^{1}	-0.9066^{1}
10.2	-0.4900	-0.1420	-0.3455^{1}	-0.9531^{1}
10.4	-0.4761	-0.1508	-0.3745^{1}	-0.9968^{1}
10.6	-0.4632	-0.1594	-0.4030^{1}	-0.1038
10.8	-0.4510	-0.1678	-0.4310^{1}	-0.1077
11.0	-0.4397	-0.1758	-0.4584^{1}	-0.1114
11.2	-0.4290	-0.1837	-0.4853^{1}	-0.1149
11.4	-0.4189	-0.1913	-0.5115^{1}	-0.1182
11.6	-0.4095	-0.1987	-0.5371^{1}	-0.1214
11.8	-0.4005	-0.2059	-0.5621^{1}	-0.1244
12.1	-0.3880	-0.2163	-0.5985^{1}	-0.1286
12.4	-0.3765	-0.2262	-0.6335^{1}	-0.1325
12.7	-0.3658	-0.2358	-0.6672^{1}	-0.1362
13.0	-0.3559	-0.2449	-0.6995^{1}	-0.1397
13.3	-0.3467	-0.2537	-0.7307^{1}	-0.1429
13.8	-0.3327	-0.2676	-0.7800^{1}	-0.1479
14.3	-0.3202	-0.2807	-0.8262^{1}	-0.1525
14.8	-0.3089	-0.2929	-0.8695^{1}	-0.1566
15.3	-0.2987	-0.3044	-0.9102^{1}	-0.1604
15.8	-0.2894	-0.3153	-0.9484^{1}	-0.1639
16.6	-0.2761	-0.3314	-0.1005	-0.1689
17.4	-0.2646	-0.3462	-0.1056	-0.1734
18.2	-0.2544	-0.3598	-0.1104	-0.1774
19.0	-0.2454	-0.3724	-0.1147	-0.1809
19.8	-0.2373	-0.3840	-0.1186	-0.1842

$\hat{\alpha}_3 = 1.75$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
7.2	-1.0375	0.4103^{1}	0.1488^{3}	0.4462^{1}
7.3	-1.0021	0.3434^{1}	0.6850^{3}	0.3637^{1}
7.4	-0.9693	0.2771^{1}	0.9698^{3}	0.2862^{1}
7.5	-0.9389	0.2113^{1}	0.1034^{2}	0.2131^{1}
7.6	-0.9105	0.1461^{1}	0.9041^{3}	0.1441^{1}
7.7	-0.8840	0.8145^{2}	0.6025^{3}	0.7864^{2}
7.8	-0.8604	0.1993^{2}	0.1688^{3}	0.1887^{2}
7.9	-0.8358	-0.4595^{2}	-0.4393^{3}	-0.4254^{2}
8.0	-0.8138	-0.1085^{1}	-0.1147^{2}	-0.9880^{2}
8.1	-0.7931	-0.1704^{1}	-0.1960^{2}	-0.1524^{1}
8.2	-0.7734	-0.2317^{1}	-0.2867^{2}	-0.2037^{1}
8.3	-0.7548	-0.2921^{1}	-0.3856^{2}	-0.2527^{1}
8.4	-0.7372	-0.3519^{1}	-0.4918^{2}	-0.2996^{1}
8.5	-0.7204	-0.4108^{1}	-0.6043^{2}	-0.3445^{1}
8.6	-0.7045	-0.4690^{1}	-0.7225^{2}	-0.3876^{1}
8.7	-0.6893	-0.5264^{1}	-0.8454^{2}	-0.4289^{1}
8.8	-0.6748	-0.5831^{1}	-0.9725^{2}	-0.4686^{1}
8.9	-0.6610	-0.6389^{1}	-0.1103^{1}	-0.5068^{1}
9.0	-0.6478	-0.6940^{1}	-0.1237^{1}	-0.5436^{1}
9.1	-0.6351	-0.7486^{1}	-0.1374^{1}	-0.5791^{1}
9.2	-0.6231	-0.8018^{1}	-0.1512^{1}	-0.6131^{1}
9.3	-0.6115	-0.8546^{1}	-0.1652^{1}	-0.6460^{1}
9.4	-0.6004	-0.9066^{1}	-0.1794^{1}	-0.6777^{1}
9.5	-0.5898	-0.9578^{1}	-0.1936^{1}	-0.7083^{1}
9.6	-0.5796	-0.1008	-0.2080^{1}	-0.7379^{1}
9.7	-0.5697	-0.1058	-0.2223^{1}	-0.7665^{1}
9.8	-0.5603	-0.1107	-0.2368^{1}	-0.7942^{1}
9.9	-0.5512	-0.1155	-0.2512^{1}	-0.8210^{1}
10.0	-0.5425	-0.1203	-0.2656^{1}	-0.8469^{1}
10.1	-0.5340	-0.1249	-0.2799^{1}	-0.8720^{1}
10.3	-0.5181	-0.1341	-0.3085^{1}	-0.9199^{1}
10.5	-0.5032	-0.1430	-0.3369^{1}	-0.9650^{1}
10.7	-0.4893	-0.1516	-0.3648^{1}	-0.1008
10.9	-0.4764	-0.1600	-0.3923^{1}	-0.1048
11.1	-0.4642	-0.1681	-0.4194^{1}	-0.1086
11.3	-0.4528	-0.1759	-0.4459^{1}	-0.1122
11.5	-0.4420	-0.1836	-0.4719^{1}	-0.1156
11.7	-0.4319	-0.1910	-0.4974^{1}	-0.1188
11.9	-0.4223	-0.1982	-0.5223^{1}	-0.1219
12.1	-0.4133	-0.2052	-0.5466^{1}	-0.1248
12.4	-0.4007	-0.2154	-0.5820^{1}	-0.1290
12.7	-0.3890	-0.2251	-0.6162^{1}	-0.1329
13.0	-0.3781	-0.2345	-0.6491^{1}	-0.1365
13.3	-0.3681	-0.2435	-0.6808^{1}	-0.1399
13.6	-0.3587	-0.2521	-0.7113^{1}	-0.1431
14.1	-0.3444	-0.2657	-0.7597^{1}	-0.1480
14.6	-0.3316	-0.2785	-0.8052^{1}	-0.1525
15.1	-0.3201	-0.2906	-0.8479^{1}	-0.1566
15.6	-0.3097	-0.3019	-0.8880^{1}	-0.1603
16.1	-0.3001	-0.3126	-0.9259^{1}	-0.1638
16.9	-0.2866	-0.3286	-0.9819^{1}	-0.1688
17.7	-0.2747	-0.3432	-0.1033	-0.1732
18.5	-0.2642	-0.3567	-0.1080	-0.1772
19.3	-0.2549	-0.3691	-0.1123	-0.1807
20.1	-0.2466	-0.3806	-0.1162	-0.1839

$\hat{\alpha}_3 = 1.80$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
7.6	-1.0070	0.2835^{1}	0.3780^{3}	0.2979^{1}
7.7	-0.9753	0.2190^{1}	0.6135^{3}	0.2246^{1}
7.8	-0.9458	0.1551^{1}	0.6456^{3}	0.1555^{1}
7.9	-0.9183	0.9177^{2}	0.4984^{3}	0.9006^{2}
8.0	-0.8925	0.2910^{2}	0.1925^{3}	0.2798^{2}
8.1	-0.8684	-0.3293^{2}	-0.2542^{3}	-0.3104^{2}
8.2	-0.8434	-0.1004^{1}	-0.8891^{3}	-0.9268^{2}
8.3	-0.8221	-0.1610^{1}	-0.1582^{2}	-0.1460^{1}
8.4	-0.8019	-0.2209^{1}	-0.2372^{2}	-0.1969^{1}
8.5	-0.7846	-0.2742^{1}	-0.3157^{2}	-0.2408^{1}
8.6	-0.7664	-0.3327^{1}	-0.4103^{2}	-0.2876^{1}
8.7	-0.7491	-0.3905^{1}	-0.5116^{2}	-0.3325^{1}
8.8	-0.7326	-0.4476^{1}	-0.6191^{2}	-0.3755^{1}
8.9	-0.7169	-0.5039^{1}	-0.7317^{2}	-0.4168^{1}
9.0	-0.7020	-0.5595^{1}	-0.8489^{2}	-0.4565^{1}
9.1	-0.6877	-0.6143^{1}	-0.9701^{2}	-0.4947^{1}
9.2	-0.6741	-0.6684^{1}	-0.1095^{1}	-0.5314^{1}
9.3	-0.6610	-0.7217^{1}	-0.1222^{1}	-0.5668^{1}
9.4	-0.6485	-0.7743^{1}	-0.1353^{1}	-0.6009^{1}
9.5	-0.6366	-0.8261^{1}	-0.1485^{1}	-0.6337^{1}
9.6	-0.6251	-0.8772^{1}	-0.1619^{1}	-0.6655^{1}
9.7	-0.6141	-0.9276^{1}	-0.1754^{1}	-0.6961^{1}
9.8	-0.6035	-0.9773^{1}	-0.1891^{1}	-0.7257^{1}
9.9	-0.5934	-0.1026	-0.2028^{1}	-0.7544^{1}
10.0	-0.5836	-0.1074	-0.2166^{1}	-0.7820^{1}
10.1	-0.5742	-0.1122	-0.2304^{1}	-0.8088^{1}
10.2	-0.5651	-0.1169	-0.2442^{1}	-0.8348^{1}
10.3	-0.5564	-0.1215	-0.2580^{1}	-0.8599^{1}
10.4	-0.5480	-0.1260	-0.2719^{1}	-0.8843^{1}
10.5	-0.5399	-0.1305	-0.2856^{1}	-0.9080^{1}
10.7	-0.5244	-0.1393	-0.3130^{1}	-0.9532^{1}
10.9	-0.5101	-0.1478	-0.3402^{1}	-0.9958^{1}
11.1	-0.4966	-0.1561	-0.3669^{1}	-0.1036
11.3	-0.4840	-0.1641	-0.3933^{1}	-0.1074
11.5	-0.4721	-0.1719	-0.4193^{1}	-0.1110
11.7	-0.4609	-0.1794	-0.4447^{1}	-0.1145
11.9	-0.4504	-0.1868	-0.4697^{1}	-0.1177
12.1	-0.4405	-0.1940	-0.4942^{1}	-0.1208
12.3	-0.4311	-0.2009	-0.5181^{1}	-0.1238
12.5	-0.4222	-0.2077	-0.5416^{1}	-0.1266
12.8	-0.4097	-0.2175	-0.5757^{1}	-0.1306
13.1	-0.3981	-0.2269	-0.6087^{1}	-0.1343
13.4	-0.3874	-0.2359	-0.6405^{1}	-0.1378
13.7	-0.3774	-0.2446	-0.6712^{1}	-0.1411
14.0	-0.3680	-0.2530	-0.7008^{1}	-0.1442
14.5	-0.3538	-0.2662	-0.7477^{1}	-0.1489
15.0	-0.3410	-0.2786	-0.7920^{1}	-0.1533
15.5	-0.3295	-0.2904	-0.8336^{1}	-0.1573
16.0	-0.3190	-0.3014	-0.8728^{1}	-0.1609
16.5	-0.3094	-0.3119	-0.9098^{1}	-0.1643
17.3	-0.2957	-0.3274	-0.9647^{1}	-0.1691
18.1	-0.2837	-0.3417	-0.1015	-0.1735
18.9	-0.2731	-0.3549	-0.1061	-0.1774
19.7	-0.2636	-0.3671	-0.1103	-0.1809
20.5	-0.2552	-0.3785	-0.1143	-0.1840

$\hat{\alpha}_3 = 1.85$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
7.9	-1.0151	0.2312^{1}	0.1210^{3}	0.2416^{1}
8.0	-0.9842	0.1684^{1}	0.3311^{3}	0.1720^{1}
8.1	-0.9555	0.1063^{1}	0.3515^{3}	0.1062^{1}
8.2	-0.9286	0.4480^{2}	0.2041^{3}	0.4384^{2}
8.3	-0.9034	-0.1603^{2}	-0.9175^{4}	-0.1538^{2}
8.4	-0.8798	-0.7621^{2}	-0.5195^{3}	-0.7173^{2}
8.5	-0.8574	-0.1357^{1}	-0.1065^{2}	-0.1254^{1}
8.6	-0.8364	-0.1945^{1}	-0.1714^{2}	-0.1767^{1}
8.7	-0.8164	-0.2527^{1}	-0.2456^{2}	-0.2257^{1}
8.8	-0.7975	-0.3101^{1}	-0.3281^{2}	-0.2726^{1}
8.9	-0.7796	-0.3668^{1}	-0.4179^{2}	-0.3175^{1}
9.0	-0.7625	-0.4228^{1}	-0.5142^{2}	-0.3606^{1}
9.1	-0.7462	-0.4782^{1}	-0.6163^{2}	-0.4020^{1}
9.2	-0.7307	-0.5327^{1}	-0.7233^{2}	-0.4417^{1}
9.3	-0.7160	-0.5866^{1}	-0.8347^{2}	-0.4799^{1}
9.4	-0.7018	-0.6397^{1}	-0.9501^{2}	-0.5167^{1}
9.5	-0.6883	-0.6922^{1}	-0.1069^{1}	-0.5522^{1}
9.6	-0.6754	-0.7439^{1}	-0.1190^{1}	-0.5863^{1}
9.7	-0.6630	-0.7949^{1}	-0.1314^{1}	-0.6193^{1}
9.8	-0.6511	-0.8452^{1}	-0.1441^{1}	-0.6511^{1}
9.9	-0.6397	-0.8948^{1}	-0.1569^{1}	-0.6818^{1}
10.0	-0.6287	-0.9437^{1}	-0.1698^{1}	-0.7115^{1}
10.1	-0.6182	-0.9919^{1}	-0.1828^{1}	-0.7402^{1}
10.2	-0.6080	-0.1039	-0.1960^{1}	-0.7680^{1}
10.3	-0.5983	-0.1086	-0.2092^{1}	-0.7949^{1}
10.4	-0.5889	-0.1132	-0.2224^{1}	-0.8209^{1}
10.5	-0.5798	-0.1178	-0.2357^{1}	-0.8462^{1}
10.6	-0.5711	-0.1223	-0.2490^{1}	-0.8706^{1}
10.7	-0.5627	-0.1267	-0.2623^{1}	-0.8944^{1}
10.8	-0.5545	-0.1311	-0.2755^{1}	-0.9174^{1}
11.0	-0.5391	-0.1396	-0.3019^{1}	-0.9615^{1}
11.2	-0.5246	-0.1479	-0.3280^{1}	-0.1003
11.4	-0.5111	-0.1559	-0.3539^{1}	-0.1042
11.6	-0.4983	-0.1638	-0.3794^{1}	-0.1080
11.8	-0.4864	-0.1714	-0.4046^{1}	-0.1115
12.0	-0.4751	-0.1788	-0.4293^{1}	-0.1149
12.2	-0.4644	-0.1860	-0.4535^{1}	-0.1181
12.4	-0.4544	-0.1929	-0.4773^{1}	-0.1211
12.6	-0.4449	-0.1997	-0.5007^{1}	-0.1240
12.8	-0.4358	-0.2064	-0.5235^{1}	-0.1268
13.1	-0.4228	-0.2163	-0.5579^{1}	-0.1308
13.4	-0.4110	-0.2255	-0.5901^{1}	-0.1345
13.7	-0.4001	-0.2343	-0.6212^{1}	-0.1379
14.0	-0.3900	-0.2429	-0.6513^{1}	-0.1412
14.4	-0.3789	-0.2537	-0.6878^{1}	-0.1451
14.8	-0.3663	-0.2638	-0.7256^{1}	-0.1488
15.3	-0.3532	-0.2761	-0.7692^{1}	-0.1531
15.8	-0.3413	-0.2876	-0.8102^{1}	-0.1571
16.3	-0.3306	-0.2985	-0.8490^{1}	-0.1607
16.8	-0.3207	-0.3088	-0.8856^{1}	-0.1640
17.6	-0.3067	-0.3242	-0.9401^{1}	-0.1689
18.4	-0.2943	-0.3384	-0.9901^{1}	-0.1732
19.2	-0.2834	-0.3514	-0.1036	-0.1771
20.0	-0.2737	-0.3635	-0.1078	-0.1806
20.8	-0.2649	-0.3748	-0.1117	-0.1837

$\hat{\alpha}_3 = 1.90$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
8.3	-0.9961	0.1249^{1}	0.1247^{3}	0.1273^{1}
8.4	-0.9678	0.6448^{2}	0.1501^{3}	0.6431^{2}
8.5	-0.9414	0.4676^{3}	0.1668^{4}	0.4571^{3}
8.6	-0.9166	-0.5444^{2}	-0.2575^{3}	-0.5220^{2}
8.7	-0.8933	-0.1129^{1}	-0.6567^{3}	-0.1063^{1}
8.8	-0.8713	-0.1707^{1}	-0.1167^{2}	-0.1579^{1}
8.9	-0.8505	-0.2278^{1}	-0.1777^{2}	-0.2071^{1}
9.0	-0.8308	-0.2842^{1}	-0.2475^{2}	-0.2543^{1}
9.1	-0.8121	-0.3399^{1}	-0.3253^{2}	-0.2994^{1}
9.2	-0.7943	-0.3950^{1}	-0.4100^{2}	-0.3427^{1}
9.3	-0.7774	-0.4493^{1}	-0.5010^{2}	-0.3843^{1}
9.4	-0.7613	-0.5030^{1}	-0.5976^{2}	-0.4243^{1}
9.5	-0.7459	-0.5560^{1}	-0.6989^{2}	-0.4626^{1}
9.6	-0.7312	-0.6082^{1}	-0.8046^{2}	-0.4996^{1}
9.7	-0.7172	-0.6598^{1}	-0.9141^{2}	-0.5352^{1}
9.8	-0.7037	-0.7107^{1}	-0.1027^{1}	-0.5695^{1}
9.9	-0.6908	-0.7610^{1}	-0.1143^{1}	-0.6026^{1}
10.0	-0.6785	-0.8105^{1}	-0.1261^{1}	-0.6346^{1}
10.1	-0.6666	-0.8593^{1}	-0.1381^{1}	-0.6655^{1}
10.2	-0.6552	-0.9075^{1}	-0.1503^{1}	-0.6953^{1}
10.3	-0.6443	-0.9551^{1}	-0.1627^{1}	-0.7241^{1}
10.4	-0.6338	-0.1002	-0.1751^{1}	-0.7521^{1}
10.5	-0.6236	-0.1048	-0.1877^{1}	-0.7791^{1}
10.6	-0.6138	-0.1094	-0.2003^{1}	-0.8053^{1}
10.7	-0.6044	-0.1139	-0.2130^{1}	-0.8307^{1}
10.8	-0.5953	-0.1183	-0.2258^{1}	-0.8553^{1}
10.9	-0.5866	-0.1227	-0.2385^{1}	-0.8791^{1}
11.0	-0.5781	-0.1270	-0.2513^{1}	-0.9023^{1}
11.1	-0.5699	-0.1312	-0.2640^{1}	-0.9248^{1}
11.2	-0.5620	-0.1354	-0.2767^{1}	-0.9466^{1}
11.4	-0.5470	-0.1436	-0.3021^{1}	-0.9885^{1}
11.6	-0.5329	-0.1516	-0.3272^{1}	-0.1028
11.8	-0.5196	-0.1593	-0.3521^{1}	-0.1066
12.0	-0.5072	-0.1669	-0.3766^{1}	-0.1101
12.2	-0.4954	-0.1742	-0.4008^{1}	-0.1135
12.4	-0.4843	-0.1813	-0.4245^{1}	-0.1167
12.6	-0.4739	-0.1883	-0.4479^{1}	-0.1198
12.8	-0.4639	-0.1950	-0.4708^{1}	-0.1227
13.0	-0.4545	-0.2016	-0.4933^{1}	-0.1255
13.2	-0.4456	-0.2080	-0.5154^{1}	-0.1282
13.5	-0.4326	-0.2176	-0.5487^{1}	-0.1321
13.8	-0.4210	-0.2265	-0.5798^{1}	-0.1356
14.1	-0.4101	-0.2351	-0.6100^{1}	-0.1390
14.4	-0.4000	-0.2434	-0.6392^{1}	-0.1421
14.7	-0.3905	-0.2513	-0.6674^{1}	-0.1451
15.2	-0.3763	-0.2637	-0.7114^{1}	-0.1495
15.7	-0.3632	-0.2757	-0.7539^{1}	-0.1537
16.2	-0.3513	-0.2869	-0.7940^{1}	-0.1576
16.7	-0.3404	-0.2976	-0.8319^{1}	-0.1611
17.2	-0.3305	-0.3077	-0.8678^{1}	-0.1644
18.0	-0.3163	-0.3227	-0.9213^{1}	-0.1691
18.8	-0.3038	-0.3366	-0.9705^{1}	-0.1734
19.6	-0.2927	-0.3494	-0.1016	-0.1772
20.4	-0.2828	-0.3613	-0.1057	-0.1806
21.2	-0.2739	-0.3724	-0.1096	-0.1837

$\hat{\alpha}_3 = 1.95$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
8.7	-0.9827	0.2917^{2}	0.3576^{4}	0.2911^{2}
8.8	-0.9771	0.4111^{2}	0.5806^{5}	0.4076^{3}
8.9	-0.9320	-0.8658^{2}	-0.3138^{3}	-0.8313^{2}
9.0	-0.9088	-0.1434^{1}	-0.6761^{3}	-0.1353^{1}
9.1	-0.8870	-0.1996^{1}	-0.1145^{2}	-0.1850^{1}
9.2	-0.8664	-0.2551^{1}	-0.1711^{2}	-0.2326^{1}
9.3	-0.8468	-0.3100^{1}	-0.2360^{2}	-0.2782^{1}
9.4	-0.8282	-0.3641^{1}	-0.3087^{2}	-0.3218^{1}
9.5	-0.8106	-0.4176^{1}	-0.3881^{2}	-0.3637^{1}
9.6	-0.7937	-0.4704^{1}	-0.4736^{2}	-0.4040^{1}
9.7	-0.7777	-0.5226^{1}	-0.5645^{2}	-0.4427^{1}
9.8	-0.7624	-0.5741^{1}	-0.6602^{2}	-0.4799^{1}
9.9	-0.7477	-0.6249^{1}	-0.7601^{2}	-0.5158^{1}
10.0	-0.7337	-0.6750^{1}	-0.8637^{2}	-0.5503^{1}
10.1	-0.7203	-0.7245^{1}	-0.9707^{2}	-0.5837^{1}
10.2	-0.7074	-0.7733^{1}	-0.1081^{1}	-0.6159^{1}
10.3	-0.6951	-0.8215^{1}	-0.1193^{1}	-0.6470^{1}
10.4	-0.6832	-0.8690^{1}	-0.1307^{1}	-0.6770^{1}
10.5	-0.6718	-0.9159^{1}	-0.1424^{1}	-0.7061^{1}
10.6	-0.6608	-0.9622^{1}	-0.1541^{1}	-0.7342^{1}
10.7	-0.6503	-0.1008	-0.1661^{1}	-0.7614^{1}
10.8	-0.6401	-0.1053	-0.1781^{1}	-0.7878^{1}
10.9	-0.6303	-0.1097	-0.1902^{1}	-0.8134^{1}
11.0	-0.6208	-0.1141	-0.2023^{1}	-0.8381^{1}
11.1	-0.6117	-0.1184	-0.2146^{1}	-0.8622^{1}
11.2	-0.6029	-0.1227	-0.2268^{1}	-0.8855^{1}
11.3	-0.5943	-0.1269	-0.2391^{1}	-0.9082^{1}
11.4	-0.5861	-0.1310	-0.2513^{1}	-0.9302^{1}
11.5	-0.5781	-0.1351	-0.2636^{1}	-0.9516^{1}
11.6	-0.5704	-0.1391	-0.2758^{1}	-0.9725^{1}
11.8	-0.5557	-0.1470	-0.3001^{1}	-0.1012
12.0	-0.5419	-0.1547	-0.3243^{1}	-0.1050
12.2	-0.5289	-0.1622	-0.3482^{1}	-0.1086
12.4	-0.5167	-0.1694	-0.3718^{1}	-0.1120
12.6	-0.5051	-0.1765	-0.3951^{1}	-0.1153
12.8	-0.4942	-0.1834	-0.4180^{1}	-0.1184
13.0	-0.4838	-0.1901	-0.4405^{1}	-0.1213
13.2	-0.4740	-0.1966	-0.4627^{1}	-0.1241
13.4	-0.4647	-0.2030	-0.4844^{1}	-0.1268
13.6	-0.4559	-0.2092	-0.5057^{1}	-0.1294
13.9	-0.4430	-0.2185	-0.5379^{1}	-0.1332
14.2	-0.4314	-0.2272	-0.5681^{1}	-0.1366
14.5	-0.4205	-0.2355	-0.5974^{1}	-0.1398
14.8	-0.4104	-0.2435	-0.6257^{1}	-0.1429
15.1	-0.4010	-0.2513	-0.6532^{1}	-0.1458
15.6	-0.3867	-0.2634	-0.6961^{1}	-0.1501
16.1	-0.3735	-0.2750	-0.7375^{1}	-0.1542
16.6	-0.3615	-0.2860	-0.7768^{1}	-0.1580
17.1	-0.3506	-0.2964	-0.8139^{1}	-0.1614
17.6	-0.3406	-0.3063	-0.8491^{1}	-0.1646
18.4	-0.3262	-0.3210	-0.9017^{1}	-0.1693
19.2	-0.3135	-0.3346	-0.9501^{1}	-0.1735
20.0	-0.3023	-0.3472	-0.9947^{1}	-0.1772
20.8	-0.2922	-0.3589	-0.1036	-0.1806
21.6	-0.2831	-0.3698	-0.1074	-0.1837

$\hat{\alpha}_3 = 2.00$

$\hat{\alpha}_4$	$\lambda_1(0,1)$	$\lambda_2(0,1)$	λ_3	λ_4
9.0	0.9998	0.9275^{4}	0.1184^{7}	0.9277^{4}
9.1	-0.9740	-0.5675^{2}	-0.7075^{4}	-0.5576^{2}
9.2	-0.9495	-0.1128^{1}	-0.2724^{3}	-0.1086^{1}
9.3	-0.9264	-0.1682^{1}	-0.5904^{3}	-0.1592^{1}
9.4	-0.9047	-0.2229^{1}	-0.1012^{2}	-0.2074^{1}
9.5	-0.8841	-0.2769^{1}	-0.1527^{2}	-0.2535^{1}
9.6	-0.8646	-0.3303^{1}	-0.2125^{2}	-0.2977^{1}
9.7	-0.8460	-0.3830^{1}	-0.2798^{2}	-0.3401^{1}
9.8	-0.8284	-0.4351^{1}	-0.3537^{2}	-0.3808^{1}
9.9	-0.8116	-0.4865^{1}	-0.4335^{2}	-0.4199^{1}
10.0	-0.7955	-0.5372^{1}	-0.5187^{2}	-0.4576^{1}
10.1	-0.7802	-0.5873^{1}	-0.6086^{2}	-0.4938^{1}
10.2	-0.7655	-0.6368^{1}	-0.7027^{2}	-0.5287^{1}
10.3	-0.7515	-0.6856^{1}	-0.8005^{2}	-0.5624^{1}
10.4	-0.7380	-0.7337^{1}	-0.9016^{2}	-0.5949^{1}
10.5	-0.7251	-0.7813^{1}	-0.1006^{1}	-0.6263^{1}
10.6	-0.7127	-0.8282^{1}	-0.1112^{1}	-0.6566^{1}
10.7	-0.7008	-0.8745^{1}	-0.1221^{1}	-0.6859^{1}
10.8	-0.6894	-0.9202^{1}	-0.1332^{1}	-0.7143^{1}
10.9	-0.6783	-0.9652^{1}	-0.1444^{1}	-0.7418^{1}
11.0	-0.6677	-0.1010	-0.1558^{1}	-0.7684^{1}
11.1	-0.6575	-0.1054	-0.1672^{1}	-0.7942^{1}
11.2	-0.6476	-0.1097	-0.1788^{1}	-0.8193^{1}
11.3	-0.6381	-0.1140	-0.1905^{1}	-0.8436^{1}
11.4	-0.6289	-0.1182	-0.2022^{1}	-0.8671^{1}
11.5	-0.6200	-0.1223	-0.2139^{1}	-0.8900^{1}
11.6	-0.6114	-0.1264	-0.2257^{1}	-0.9122^{1}
11.7	-0.6030	-0.1305	-0.2375^{1}	-0.9339^{1}
11.8	-0.5950	-0.1345	-0.2493^{1}	-0.9549^{1}
11.9	-0.5872	-0.1384	-0.2611^{1}	-0.9753^{1}
12.1	-0.5723	-0.1461	-0.2845^{1}	-0.1015
12.3	-0.5584	-0.1536	-0.3079^{1}	-0.1052
12.5	-0.5452	-0.1609	-0.3310^{1}	-0.1087
12.7	-0.5328	-0.1680	-0.3539^{1}	-0.1121
12.9	-0.5210	-0.1750	-0.3765^{1}	-0.1153
13.1	-0.5099	-0.1817	-0.3988^{1}	-0.1183
13.3	-0.4994	-0.1883	-0.4207^{1}	-0.1212
13.5	-0.4894	-0.1947	-0.4423^{1}	-0.1240
13.7	-0.4799	-0.2010	-0.4635^{1}	-0.1267
13.9	-0.4709	-0.2071	-0.4843^{1}	-0.1292
14.2	-0.4578	-0.2162	-0.5159^{1}	-0.1330
14.5	-0.4459	-0.2247	-0.5455^{1}	-0.1364
14.8	-0.4349	-0.2329	-0.5742^{1}	-0.1396
15.1	-0.4245	-0.2408	-0.6021^{1}	-0.1426
15.4	-0.4148	-0.2485	-0.6290^{1}	-0.1454
15.9	-0.4002	-0.2604	-0.6713^{1}	-0.1498
16.4	-0.3867	-0.2719	-0.7123^{1}	-0.1538
16.9	-0.3744	-0.2827	-0.7511^{1}	-0.1576
17.4	-0.3632	-0.2930	-0.7879^{1}	-0.1610
17.9	-0.3529	-0.3028	-0.8228^{1}	-0.1642
18.7	-0.3381	-0.3174	-0.8751^{1}	-0.1689
19.5	-0.3250	-0.3309	-0.9233^{1}	-0.1730
20.3	-0.3134	-0.3434	-0.9678^{1}	-0.1768
21.1	-0.3030	-0.3550	-0.1009	-0.1802
21.9	-0.2937	-0.3658	-0.1047	-0.1833

Appendix C

Table C–1 for GBD Fits: Method of Moments

For specified α_3 and α_4, Table C–1 gives the β_3 and β_4 values for GBD fits in the region designated in Figure 4.3–5.

The entries of Table C–1 are from "The Extended Generalized Lambda Distribution (EGLD) System for Fitting Distributions to Data with Moments, II: Tables" by E.J. Dudewicz and Z.A. Karian, *American Journal of Mathematical and Management Sciences*, Vol. 16 3 & 4 (1996), pp. 309–331, copyright ©1996 by American Sciences Press, Inc., 20 Cross Road, Syracuse, New York 13224-2104. Reprinted with permission.

$0.000 \leq \hat{\alpha}_3 \leq 0.075$

	$\hat{\alpha}_3 = 0.000$		$\hat{\alpha}_3 = 0.025$		$\hat{\alpha}_3 = 0.050$		$\hat{\alpha}_3 = 0.075$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.05	-0.9615	-0.9615	-0.9625	-0.9616	-0.9644	-0.9626	-0.9673	-0.9647
1.08	-0.9375	-0.9375	-0.9388	-0.9372	-0.9411	-0.9381	-0.9443	-0.9400
1.11	-0.9127	-0.9127	-0.9143	-0.9121	-0.9170	-0.9127	-0.9206	-0.9144
1.14	-0.8871	-0.8871	-0.8891	-0.8862	-0.8921	-0.8866	-0.8962	-0.8881
1.17	-0.8607	-0.8607	-0.8630	-0.8595	-0.8664	-0.8595	-0.8709	-0.8608
1.20	-0.8333	-0.8333	-0.8360	-0.8318	-0.8399	-0.8316	-0.8449	-0.8327
1.23	-0.8051	-0.8051	-0.8082	-0.8033	-0.8125	-0.8027	-0.8179	-0.8036
1.26	-0.7759	-0.7759	-0.7794	-0.7737	-0.7841	-0.7729	-0.7901	-0.7735
1.29	-0.7456	-0.7456	-0.7495	-0.7431	-0.7548	-0.7419	-0.7613	-0.7423
1.32	-0.7143	-0.7143	-0.7187	-0.7113	-0.7244	-0.7099	-0.7314	-0.7100
1.35	-0.6818	-0.6818	-0.6867	-0.6785	-0.6929	-0.6767	-0.7005	-0.6765
1.38	-0.6481	-0.6481	-0.6535	-0.6444	-0.6603	-0.6422	-0.6685	-0.6418
1.41	-0.6132	-0.6132	-0.6190	-0.6090	-0.6264	-0.6065	-0.6353	-0.6057
1.44	-0.5769	-0.5769	-0.5833	-0.5723	-0.5913	-0.5694	-0.6009	-0.5683
1.47	-0.5392	-0.5392	-0.5462	-0.5341	-0.5548	-0.5308	-0.5651	-0.5294
1.50	-0.5000	-0.5000	-0.5076	-0.4943	-0.5169	-0.4906	-0.5279	-0.4889
1.53	-0.4592	-0.4592	-0.4674	-0.4530	-0.4774	-0.4488	-0.4892	-0.4467
1.56	-0.4167	-0.4167	-0.4255	-0.4099	-0.4363	-0.4052	-0.4490	-0.4028
1.59	-0.3723	-0.3723	-0.3819	-0.3649	-0.3935	-0.3598	-0.4071	-0.3570
1.62	-0.3261	-0.3261	-0.3364	-0.3180	-0.3489	-0.3124	-0.3634	-0.3091
1.65	-0.2778	-0.2778	-0.2889	-0.2690	-0.3023	-0.2628	-0.3178	-0.2592
1.68	-0.2273	-0.2273	-0.2392	-0.2178	-0.2536	-0.2110	-0.2702	-0.2069
1.71	-0.1744	-0.1744	-0.1873	-0.1642	-0.2027	-0.1567	-0.2204	-0.1522
1.74	-0.1190	-0.1190	-0.1329	-0.1080	-0.1494	-0.0999	-0.1683	-0.0948
1.77	-0.0610	-0.0610	-0.0758	-0.0491	-0.0935	-0.0402	-0.1137	-0.0346
1.80	-0.0000	-0.0000	-0.0160	0.0128	-0.0349	0.0224	-0.0565	0.0286
1.83	0.0641	0.0641	0.0470	0.0779	0.0267	0.0883	0.0036	0.0951
1.86	0.1316	0.1316	0.1132	0.1465	0.0915	0.1577	0.0667	0.1651
1.89	0.2027	0.2027	0.1829	0.2188	0.1597	0.2309	0.1332	0.2390
1.92	0.2778	0.2778	0.2565	0.2951	0.2316	0.3082	0.2033	0.3170
1.95	0.3571	0.3571	0.3343	0.3757	0.3076	0.3899	0.2772	0.3995
1.98	0.4412	0.4412	0.4167	0.4612	0.3880	0.4765	0.3554	0.4869
2.01	0.5303	0.5303	0.5039	0.5519	0.4731	0.5684	0.4382	0.5797
2.04	0.6250	0.6250	0.5966	0.6482	0.5635	0.6661	0.5260	0.6783
2.07	0.7258	0.7258	0.6952	0.7509	0.6596	0.7701	0.6193	0.7833
2.10	0.8333	0.8333	0.8004	0.8604	0.7620	0.8812	0.7186	0.8955
2.13	0.9483	0.9483	0.9127	0.9775	0.8713	0.9999	0.8245	1.0154
2.16	1.0714	1.0714	1.0330	1.1030	0.9883	1.1273	0.9378	1.1440
2.19	1.2037	1.2037	1.1621	1.2379	1.1137	1.2641	1.0591	1.2822
2.22	1.3462	1.3462	1.3011	1.3832	1.2487	1.4116	1.1895	1.4311
2.25	1.5000	1.5000	1.4511	1.5402	1.3941	1.5710	1.3300	1.5921
2.28	1.6667	1.6667	1.6134	1.7103	1.5515	1.7438	1.4817	1.7667
2.31	1.8478	1.8478	1.7898	1.8954	1.7222	1.9318	1.6462	1.9565
2.34	2.0455	2.0455	1.9820	2.0974	1.9082	2.1370	1.8250	2.1638
2.37	2.2619	2.2619	2.1924	2.3187	2.1114	2.3620	2.0202	2.3910
2.40	2.5000	2.5000	2.4236	2.5624	2.3344	2.6097	2.2341	2.6411
2.43	2.7632	2.7632	2.6788	2.8318	2.5804	2.8837	2.4696	2.9178
2.46	3.0556	3.0556	2.9622	3.1314	2.8530	3.1884	2.7301	3.2254
2.49	3.3824	3.3824	3.2784	3.4665	3.1568	3.5293	3.0199	3.5696
2.52	3.7500	3.7500	3.6338	3.8438	3.4976	3.9132	3.3443	3.9570
2.55	4.1667	4.1667	4.0360	4.2717	3.8826	4.3488	3.7098	4.3966
2.58	4.6429	4.6429	4.4949	4.7612	4.3209	4.8473	4.1248	4.8993
2.61	5.1923	5.1923	5.0235	5.3266	4.8245	5.4231	4.6003	5.4797
2.64	5.8333	5.8333	5.6391	5.9869	5.4094	6.0958	5.1505	6.1574
2.67	6.5909	6.5909	6.3649	6.7682	6.0969	6.8918	5.7947	6.9587

$0.100 \leq \hat{\alpha}_3 \leq 0.175$

	$\hat{\alpha}_3 = 0.100$		$\hat{\alpha}_3 = 0.125$		$\hat{\alpha}_3 = 0.150$		$\hat{\alpha}_3 = 0.175$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.05	-0.9710	-0.9679	-0.9755	-0.9722	-0.9808	-0.9777	-0.9867	-0.9842
1.08	-0.9484	-0.9430	-0.9534	-0.9472	-0.9592	-0.9525	-0.9656	-0.9590
1.11	-0.9252	-0.9173	-0.9306	-0.9214	-0.9369	-0.9267	-0.9439	-0.9331
1.14	-0.9012	-0.8908	-0.9071	-0.8947	-0.9139	-0.8999	-0.9214	-0.9064
1.17	-0.8765	-0.8634	-0.8829	-0.8672	-0.8902	-0.8724	-0.8983	-0.8788
1.20	-0.8509	-0.8351	-0.8579	-0.8388	-0.8658	-0.8439	-0.8745	-0.8503
1.23	-0.8245	-0.8058	-0.8320	-0.8094	-0.8405	-0.8144	-0.8498	-0.8208
1.26	-0.7972	-0.7755	-0.8053	-0.7790	-0.8144	-0.7839	-0.8244	-0.7903
1.29	-0.7689	-0.7441	-0.7777	-0.7475	-0.7874	-0.7524	-0.7981	-0.7588
1.32	-0.7397	-0.7116	-0.7491	-0.7149	-0.7595	-0.7197	-0.7709	-0.7261
1.35	-0.7094	-0.6780	-0.7195	-0.6811	-0.7306	-0.6858	-0.7428	-0.6923
1.38	-0.6781	-0.6430	-0.6888	-0.6460	-0.7007	-0.6507	-0.7137	-0.6572
1.41	-0.6456	-0.6068	-0.6571	-0.6096	-0.6698	-0.6143	-0.6835	-0.6208
1.44	-0.6118	-0.5691	-0.6242	-0.5718	-0.6377	-0.5764	-0.6523	-0.5830
1.47	-0.5768	-0.5300	-0.5900	-0.5325	-0.6044	-0.5371	-0.6199	-0.5437
1.50	-0.5405	-0.4892	-0.5545	-0.4917	-0.5698	-0.4962	-0.5864	-0.5028
1.53	-0.5027	-0.4468	-0.5176	-0.4491	-0.5340	-0.4536	-0.5515	-0.4603
1.56	-0.4634	-0.4026	-0.4793	-0.4048	-0.4967	-0.4092	-0.5153	-0.4159
1.59	-0.4224	-0.3565	-0.4394	-0.3585	-0.4579	-0.3629	-0.4776	-0.3698
1.62	-0.3798	-0.3084	-0.3978	-0.3102	-0.4175	-0.3146	-0.4385	-0.3216
1.65	-0.3353	-0.2582	-0.3545	-0.2598	-0.3754	-0.2642	-0.3977	-0.2712
1.68	-0.2888	-0.2056	-0.3093	-0.2071	-0.3315	-0.2114	-0.3552	-0.2186
1.71	-0.2403	-0.1505	-0.2621	-0.1519	-0.2857	-0.1562	-0.3109	-0.1635
1.74	-0.1895	-0.0928	-0.2128	-0.0940	-0.2379	-0.0983	-0.2646	-0.1058
1.77	-0.1364	-0.0323	-0.1612	-0.0333	-0.1879	-0.0376	-0.2163	-0.0453
1.80	-0.0807	0.0313	-0.1071	0.0305	-0.1355	0.0261	-0.1657	0.0182
1.83	-0.0222	0.0982	-0.0504	0.0975	-0.0807	0.0931	-0.1128	0.0850
1.86	0.0392	0.1686	0.0091	0.1682	-0.0231	0.1637	-0.0573	0.1553
1.89	0.1038	0.2429	0.0717	0.2426	0.0373	0.2381	0.0009	0.2294
1.92	0.1718	0.3214	0.1375	0.3212	0.1009	0.3166	0.0621	0.3076
1.95	0.2435	0.4043	0.2070	0.4044	0.1678	0.3997	0.1265	0.3902
1.98	0.3193	0.4922	0.2802	0.4925	0.2384	0.4876	0.1943	0.4777
2.01	0.3995	0.5855	0.3576	0.5859	0.3129	0.5809	0.2658	0.5704
2.04	0.4845	0.6847	0.4396	0.6853	0.3917	0.6800	0.3414	0.6689
2.07	0.5747	0.7903	0.5266	0.7910	0.4752	0.7855	0.4213	0.7737
2.10	0.6707	0.9031	0.6189	0.9039	0.5639	0.8980	0.5061	0.8854
2.13	0.7730	1.0236	0.7173	1.0246	0.6581	1.0182	0.5961	1.0048
2.16	0.8822	1.1529	0.8222	1.1539	0.7585	1.1471	0.6918	1.1325
2.19	0.9991	1.2918	0.9344	1.2929	0.8657	1.2854	0.7939	1.2696
2.22	1.1246	1.4415	1.0546	1.4425	0.9804	1.4344	0.9029	1.4171
2.25	1.2595	1.6032	1.1837	1.6042	1.1034	1.5951	1.0197	1.5763
2.28	1.4051	1.7785	1.3228	1.7794	1.2358	1.7692	1.1451	1.7484
2.31	1.5627	1.9692	1.4731	1.9697	1.3785	1.9583	1.2801	1.9352
2.34	1.7338	2.1773	1.6360	2.1774	1.5329	2.1644	1.4258	2.1386
2.37	1.9203	2.4053	1.8132	2.4049	1.7005	2.3899	1.5836	2.3609
2.40	2.1243	2.6562	2.0067	2.6550	1.8830	2.6377	1.7551	2.6048
2.43	2.3484	2.9337	2.2187	2.9313	2.0827	2.9111	1.9422	2.8737
2.46	2.5958	3.2420	2.4523	3.2382	2.3020	3.2143	2.1471	3.1713
2.49	2.8704	3.5868	2.7109	3.5809	2.5440	3.5525	2.3725	3.5027
2.52	3.1769	3.9746	2.9986	3.9660	2.8126	3.9320	2.6217	3.8738
2.55	3.5213	4.4142	3.3209	4.4019	3.1122	4.3606	2.8988	4.2921
2.58	3.9112	4.9164	3.6844	4.8991	3.4489	4.8486	3.2088	4.7671
2.61	4.3561	5.4957	4.0976	5.4715	3.8299	5.4090	3.5579	5.3109
2.64	4.8690	6.1710	4.5716	6.1374	4.2648	6.0590	3.9543	5.9395
2.67	5.4667	6.9681	5.1212	6.9213	4.7660	6.8216	4.4084	6.6739

$0.200 \leq \hat{\alpha}_3 \leq 0.275$

	$\hat{\alpha}_3 = 0.200$		$\hat{\alpha}_3 = 0.225$		$\hat{\alpha}_3 = 0.250$		$\hat{\alpha}_3 = 0.275$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.10	-0.9587	-0.9495	-0.9667	-0.9583	-0.9753	-0.9683	-0.9843	-0.9794
1.13	-0.9370	-0.9231	-0.9457	-0.9320	-0.9548	-0.9420	-0.9645	-0.9533
1.16	-0.9147	-0.8958	-0.9240	-0.9048	-0.9338	-0.9150	-0.9441	-0.9264
1.19	-0.8917	-0.8676	-0.9016	-0.8767	-0.9121	-0.8870	-0.9230	-0.8986
1.22	-0.8680	-0.8385	-0.8785	-0.8477	-0.8897	-0.8582	-0.9013	-0.8700
1.25	-0.8435	-0.8085	-0.8547	-0.8177	-0.8665	-0.8284	-0.8789	-0.8404
1.28	-0.8182	-0.7773	-0.8301	-0.7867	-0.8427	-0.7976	-0.8558	-0.8098
1.31	-0.7920	-0.7451	-0.8047	-0.7546	-0.8181	-0.7657	-0.8320	-0.7782
1.34	-0.7650	-0.7118	-0.7785	-0.7214	-0.7926	-0.7327	-0.8074	-0.7454
1.37	-0.7370	-0.6772	-0.7513	-0.6870	-0.7664	-0.6985	-0.7819	-0.7115
1.40	-0.7081	-0.6413	-0.7233	-0.6513	-0.7392	-0.6630	-0.7557	-0.6764
1.43	-0.6781	-0.6041	-0.6942	-0.6143	-0.7111	-0.6263	-0.7285	-0.6400
1.46	-0.6471	-0.5654	-0.6642	-0.5758	-0.6820	-0.5881	-0.7004	-0.6022
1.49	-0.6149	-0.5252	-0.6330	-0.5359	-0.6519	-0.5484	-0.6714	-0.5629
1.52	-0.5815	-0.4834	-0.6007	-0.4943	-0.6207	-0.5072	-0.6413	-0.5221
1.55	-0.5469	-0.4399	-0.5672	-0.4510	-0.5883	-0.4643	-0.6101	-0.4797
1.58	-0.5109	-0.3945	-0.5324	-0.4060	-0.5547	-0.4197	-0.5777	-0.4355
1.61	-0.4735	-0.3472	-0.4963	-0.3590	-0.5199	-0.3731	-0.5442	-0.3895
1.64	-0.4346	-0.2978	-0.4587	-0.3100	-0.4837	-0.3246	-0.5094	-0.3415
1.67	-0.3940	-0.2462	-0.4196	-0.2588	-0.4461	-0.2739	-0.4732	-0.2914
1.70	-0.3518	-0.1923	-0.3789	-0.2052	-0.4069	-0.2209	-0.4356	-0.2391
1.73	-0.3078	-0.1358	-0.3365	-0.1492	-0.3662	-0.1655	-0.3965	-0.1845
1.76	-0.2619	-0.0766	-0.2923	-0.0905	-0.3237	-0.1074	-0.3558	-0.1272
1.79	-0.2139	-0.0145	-0.2461	-0.0289	-0.2794	-0.0466	-0.3134	-0.0673
1.82	-0.1637	0.0508	-0.1979	0.0357	-0.2331	0.0172	-0.2691	-0.0044
1.85	-0.1111	0.1194	-0.1475	0.1036	-0.1848	0.0843	-0.2229	0.0616
1.88	-0.0561	0.1916	-0.0946	0.1751	-0.1343	0.1549	-0.1746	0.1311
1.91	0.0017	0.2678	-0.0393	0.2505	-0.0813	0.2292	-0.1241	0.2041
1.94	0.0624	0.3483	0.0188	0.3300	-0.0259	0.3076	-0.0713	0.2811
1.97	0.1263	0.4333	0.0798	0.4141	0.0323	0.3903	-0.0159	0.3624
2.00	0.1935	0.5234	0.1441	0.5030	0.0935	0.4779	0.0423	0.4483
2.03	0.2645	0.6190	0.2117	0.5974	0.1579	0.5707	0.1034	0.5393
2.06	0.3394	0.7206	0.2831	0.6975	0.2257	0.6691	0.1676	0.6357
2.09	0.4187	0.8288	0.3585	0.8041	0.2972	0.7738	0.2354	0.7381
2.12	0.5027	0.9441	0.4383	0.9177	0.3728	0.8852	0.3068	0.8470
2.15	0.5919	1.0675	0.5228	1.0391	0.4528	1.0041	0.3823	0.9631
2.18	0.6867	1.1996	0.6127	1.1690	0.5376	1.1313	0.4622	1.0872
2.21	0.7878	1.3416	0.7082	1.3084	0.6277	1.2676	0.5469	1.2199
2.24	0.8958	1.4944	0.8102	1.4583	0.7236	1.4140	0.6369	1.3624
2.27	1.0115	1.6594	0.9191	1.6199	0.8258	1.5717	0.7326	1.5156
2.30	1.1356	1.8381	1.0357	1.7948	0.9351	1.7420	0.8348	1.6808
2.33	1.2692	2.0322	1.1609	1.9844	1.0522	1.9265	0.9439	1.8594
2.36	1.4134	2.2437	1.2958	2.1909	1.1780	2.1270	1.0609	2.0532
2.39	1.5695	2.4753	1.4415	2.4164	1.3134	2.3456	1.1865	2.2641
2.42	1.7391	2.7296	1.5993	2.6637	1.4598	2.5848	1.3218	2.4943
2.45	1.9241	3.0103	1.7708	2.9361	1.6184	2.8477	1.4680	2.7468
2.48	2.1267	3.3216	1.9581	3.2375	1.7909	3.1378	1.6265	3.0247
2.51	2.3495	3.6687	2.1633	3.5727	1.9793	3.4597	1.7989	3.3321
2.54	2.5958	4.0581	2.3893	3.9478	2.1858	3.8187	1.9872	3.6738
2.57	2.8695	4.4979	2.6393	4.3699	2.4134	4.2214	2.1938	4.0558
2.60	3.1756	4.9983	2.9176	4.8487	2.6654	4.6763	2.4214	4.4855
2.63	3.5204	5.5726	3.2293	5.3958	2.9462	5.1940	2.6735	4.9723
2.66	3.9117	6.2383	3.5808	6.0270	3.2609	5.7881	2.9545	5.5279
2.69	4.3598	7.0186	3.9807	6.7626	3.6164	6.4765	3.2696	6.1680
2.72	4.8784	7.9452	4.4396	7.6306	4.0211	7.2832	3.6258	6.9127

$0.300 \leq \hat{\alpha}_3 \leq 0.375$

	$\hat{\alpha}_3 = 0.300$		$\hat{\alpha}_3 = 0.325$		$\hat{\alpha}_3 = 0.350$		$\hat{\alpha}_3 = 0.375$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.15	-0.9614	-0.9479	-0.9722	-0.9615	-0.9832	-0.9762	-0.9944	-0.9919
1.18	-0.9412	-0.9207	-0.9527	-0.9346	-0.9644	-0.9495	-0.9763	-0.9656
1.21	-0.9204	-0.8926	-0.9326	-0.9067	-0.9450	-0.9220	-0.9576	-0.9384
1.24	-0.8990	-0.8636	-0.9119	-0.8780	-0.9250	-0.8936	-0.9383	-0.9104
1.27	-0.8769	-0.8336	-0.8905	-0.8484	-0.9044	-0.8644	-0.9185	-0.8815
1.30	-0.8541	-0.8026	-0.8685	-0.8177	-0.8832	-0.8341	-0.8981	-0.8517
1.33	-0.8305	-0.7706	-0.8458	-0.7861	-0.8613	-0.8029	-0.8770	-0.8209
1.36	-0.8063	-0.7374	-0.8224	-0.7533	-0.8388	-0.7705	-0.8553	-0.7891
1.39	-0.7812	-0.7031	-0.7982	-0.7194	-0.8155	-0.7371	-0.8330	-0.7562
1.42	-0.7553	-0.6675	-0.7733	-0.6842	-0.7915	-0.7025	-0.8099	-0.7221
1.45	-0.7285	-0.6306	-0.7475	-0.6478	-0.7667	-0.6666	-0.7861	-0.6868
1.48	-0.7008	-0.5922	-0.7208	-0.6100	-0.7411	-0.6294	-0.7615	-0.6503
1.51	-0.6721	-0.5525	-0.6933	-0.5708	-0.7146	-0.5908	-0.7361	-0.6124
1.54	-0.6425	-0.5111	-0.6647	-0.5301	-0.6873	-0.5508	-0.7099	-0.5731
1.57	-0.6118	-0.4681	-0.6352	-0.4877	-0.6590	-0.5091	-0.6828	-0.5322
1.60	-0.5799	-0.4233	-0.6047	-0.4437	-0.6297	-0.4658	-0.6547	-0.4898
1.63	-0.5469	-0.3767	-0.5730	-0.3978	-0.5993	-0.4208	-0.6257	-0.4456
1.66	-0.5126	-0.3281	-0.5402	-0.3499	-0.5679	-0.3739	-0.5957	-0.3997
1.69	-0.4770	-0.2773	-0.5061	-0.3001	-0.5353	-0.3249	-0.5646	-0.3518
1.72	-0.4400	-0.2243	-0.4707	-0.2480	-0.5015	-0.2739	-0.5323	-0.3019
1.75	-0.4016	-0.1688	-0.4339	-0.1935	-0.4664	-0.2206	-0.4989	-0.2498
1.78	-0.3616	-0.1108	-0.3957	-0.1366	-0.4300	-0.1649	-0.4642	-0.1954
1.81	-0.3199	-0.0500	-0.3559	-0.0770	-0.3921	-0.1066	-0.4281	-0.1385
1.84	-0.2764	0.0138	-0.3145	-0.0145	-0.3527	-0.0455	-0.3907	-0.0789
1.87	-0.2310	0.0808	-0.2713	0.0511	-0.3116	0.0186	-0.3517	-0.0165
1.90	-0.1837	0.1513	-0.2263	0.1200	-0.2688	0.0858	-0.3111	0.0489
1.93	-0.1341	0.2254	-0.1792	0.1925	-0.2242	0.1564	-0.2689	0.1176
1.96	-0.0823	0.3036	-0.1300	0.2688	-0.1776	0.2308	-0.2248	0.1898
1.99	-0.0280	0.3861	-0.0786	0.3493	-0.1290	0.3091	-0.1788	0.2659
2.02	0.0290	0.4733	-0.0247	0.4343	-0.0780	0.3917	-0.1308	0.3460
2.05	0.0888	0.5656	0.0318	0.5242	-0.0247	0.4790	-0.0806	0.4305
2.08	0.1517	0.6635	0.0912	0.6194	0.0312	0.5713	-0.0280	0.5198
2.11	0.2179	0.7675	0.1535	0.7204	0.0898	0.6692	0.0271	0.6144
2.14	0.2877	0.8781	0.2192	0.8278	0.1515	0.7731	0.0849	0.7146
2.17	0.3614	0.9961	0.2884	0.9421	0.2164	0.8835	0.1457	0.8210
2.20	0.4394	1.1221	0.3615	1.0640	0.2847	1.0012	0.2095	0.9342
2.23	0.5220	1.2570	0.4387	1.1944	0.3569	1.1267	0.2768	1.0548
2.26	0.6097	1.4017	0.5206	1.3340	0.4331	1.2610	0.3478	1.1836
2.29	0.7030	1.5574	0.6074	1.4840	0.5139	1.4050	0.4228	1.3214
2.32	0.8023	1.7253	0.6997	1.6454	0.5995	1.5597	0.5021	1.4692
2.35	0.9084	1.9069	0.7981	1.8197	0.6905	1.7263	0.5862	1.6280
2.38	1.0220	2.1039	0.9030	2.0083	0.7874	1.9062	0.6755	1.7991
2.41	1.1438	2.3182	1.0154	2.2130	0.8908	2.1012	0.7705	1.9841
2.44	1.2750	2.5524	1.1359	2.4361	1.0013	2.3130	0.8719	2.1845
2.47	1.4165	2.8091	1.2655	2.6801	1.1199	2.5439	0.9802	2.4024
2.50	1.5697	3.0917	1.4053	2.9478	1.2473	2.7966	1.0962	2.6400
2.53	1.7362	3.4043	1.5566	3.2430	1.3847	3.0742	1.2209	2.9003
2.56	1.9177	3.7519	1.7210	3.5699	1.5333	3.3805	1.3552	3.1864
2.59	2.1165	4.1404	1.9001	3.9339	1.6946	3.7202	1.5003	3.5022
2.62	2.3351	4.5775	2.0962	4.3414	1.8702	4.0987	1.6576	3.8527
2.65	2.5769	5.0725	2.3118	4.8007	2.0624	4.5231	1.8288	4.2436
2.68	2.8458	5.6377	2.5501	5.3219	2.2735	5.0020	2.0158	4.6822
2.71	3.1467	6.2886	2.8149	5.9182	2.5066	5.5462	2.2211	5.1776
2.74	3.4859	7.0458	3.1111	6.6066	2.7654	6.1699	2.4474	5.7410
2.77	3.8712	7.9370	3.4448	7.4096	3.0545	6.8911	2.6984	6.3874

$0.400 \leq \hat{\alpha}_3 \leq 0.475$

	$\hat{\alpha}_3 = 0.400$		$\hat{\alpha}_3 = 0.425$		$\hat{\alpha}_3 = 0.450$		$\hat{\alpha}_3 = 0.475$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.25	-0.9455	-0.9188	-0.9592	-0.9378	-0.9729	-0.9577	-0.9865	-0.9784
1.28	-0.9262	-0.8901	-0.9407	-0.9095	-0.9552	-0.9299	-0.9695	-0.9512
1.31	-0.9064	-0.8604	-0.9217	-0.8804	-0.9369	-0.9013	-0.9520	-0.9232
1.34	-0.8860	-0.8298	-0.9021	-0.8503	-0.9181	-0.8718	-0.9340	-0.8943
1.37	-0.8649	-0.7982	-0.8819	-0.8193	-0.8988	-0.8414	-0.9155	-0.8644
1.40	-0.8432	-0.7655	-0.8611	-0.7872	-0.8788	-0.8099	-0.8965	-0.8337
1.43	-0.8208	-0.7316	-0.8396	-0.7540	-0.8583	-0.7774	-0.8768	-0.8019
1.46	-0.7978	-0.6966	-0.8175	-0.7197	-0.8372	-0.7439	-0.8566	-0.7691
1.49	-0.7740	-0.6603	-0.7947	-0.6841	-0.8154	-0.7091	-0.8358	-0.7351
1.52	-0.7494	-0.6227	-0.7712	-0.6473	-0.7929	-0.6731	-0.8143	-0.7000
1.55	-0.7240	-0.5837	-0.7470	-0.6091	-0.7697	-0.6358	-0.7922	-0.6637
1.58	-0.6978	-0.5432	-0.7219	-0.5695	-0.7458	-0.5972	-0.7694	-0.6260
1.61	-0.6707	-0.5011	-0.6960	-0.5284	-0.7211	-0.5571	-0.7459	-0.5870
1.64	-0.6426	-0.4574	-0.6693	-0.4857	-0.6956	-0.5155	-0.7216	-0.5464
1.67	-0.6136	-0.4118	-0.6416	-0.4413	-0.6693	-0.4722	-0.6966	-0.5044
1.70	-0.5836	-0.3644	-0.6130	-0.3951	-0.6421	-0.4273	-0.6707	-0.4607
1.73	-0.5525	-0.3150	-0.5835	-0.3470	-0.6140	-0.3805	-0.6440	-0.4153
1.76	-0.5203	-0.2635	-0.5528	-0.2968	-0.5849	-0.3317	-0.6164	-0.3680
1.79	-0.4869	-0.2097	-0.5211	-0.2445	-0.5548	-0.2809	-0.5878	-0.3187
1.82	-0.4522	-0.1534	-0.4882	-0.1898	-0.5236	-0.2279	-0.5583	-0.2674
1.85	-0.4162	-0.0946	-0.4540	-0.1327	-0.4912	-0.1726	-0.5277	-0.2139
1.88	-0.3788	-0.0330	-0.4186	-0.0730	-0.4577	-0.1147	-0.4960	-0.1580
1.91	-0.3398	0.0316	-0.3818	-0.0104	-0.4229	-0.0542	-0.4631	-0.0996
1.94	-0.2993	0.0993	-0.3435	0.0552	-0.3868	0.0091	-0.4291	-0.0384
1.97	-0.2571	0.1705	-0.3037	0.1240	-0.3492	0.0756	-0.3937	0.0256
2.00	-0.2131	0.2454	-0.2622	0.1963	-0.3102	0.1453	-0.3570	0.0927
2.03	-0.1672	0.3242	-0.2190	0.2724	-0.2696	0.2186	-0.3188	0.1631
2.06	-0.1192	0.4074	-0.1739	0.3525	-0.2273	0.2956	-0.2791	0.2371
2.09	-0.0691	0.4952	-0.1269	0.4370	-0.1832	0.3768	-0.2378	0.3149
2.12	-0.0166	0.5880	-0.0777	0.5263	-0.1371	0.4624	-0.1947	0.3968
2.15	0.0384	0.6863	-0.0263	0.6206	-0.0891	0.5528	-0.1498	0.4832
2.18	0.0961	0.7907	0.0276	0.7206	-0.0388	0.6484	-0.1029	0.5745
2.21	0.1567	0.9015	0.0841	0.8267	0.0138	0.7496	-0.0540	0.6710
2.24	0.2205	1.0195	0.1434	0.9394	0.0689	0.8570	-0.0027	0.7732
2.27	0.2876	1.1454	0.2057	1.0594	0.1267	0.9712	0.0509	0.8816
2.30	0.3584	1.2799	0.2713	1.1874	0.1875	1.0927	0.1071	0.9967
2.33	0.4332	1.4239	0.3404	1.3241	0.2513	1.2223	0.1661	1.1193
2.36	0.5124	1.5785	0.4134	1.4707	0.3186	1.3609	0.2281	1.2501
2.39	0.5963	1.7449	0.4905	1.6279	0.3895	1.5092	0.2933	1.3898
2.42	0.6853	1.9244	0.5722	1.7972	0.4645	1.6685	0.3620	1.5393
2.45	0.7801	2.1186	0.6589	1.9798	0.5437	1.8399	0.4345	1.6999
2.48	0.8812	2.3294	0.7511	2.1774	0.6278	2.0247	0.5112	1.8725
2.51	0.9892	2.5588	0.8492	2.3918	0.7170	2.2247	0.5923	2.0588
2.54	1.1049	2.8094	0.9540	2.6253	0.8119	2.4417	0.6784	2.2602
2.57	1.2291	3.0843	1.0662	2.8803	0.9132	2.6780	0.7698	2.4786
2.60	1.3630	3.3870	1.1865	3.1600	1.0214	2.9360	0.8672	2.7163
2.63	1.5077	3.7218	1.3159	3.4681	1.1373	3.2189	0.9712	2.9759
2.66	1.6645	4.0941	1.4556	3.8089	1.2618	3.5304	1.0824	3.2603
2.69	1.8351	4.5103	1.6068	4.1877	1.3960	3.8749	1.2017	3.5733
2.72	2.0216	4.9784	1.7710	4.6113	1.5410	4.2577	1.3299	3.9192
2.75	2.2261	5.5085	1.9502	5.0876	1.6982	4.6855	1.4683	4.3034
2.78	2.4518	6.1135	2.1464	5.6270	1.8694	5.1664	1.6180	4.7323
2.81	2.7019	6.8097	2.3623	6.2424	2.0564	5.7106	1.7807	5.2140
2.84	2.9810	7.6190	2.6012	6.9506	2.2618	6.3312	1.9580	5.7587
2.87	3.2944	8.5704	2.8670	7.7736	2.4884	7.0448	2.1521	6.3789

$0.500 \leq \hat{\alpha}_3 \leq 0.575$

	$\hat{\alpha}_3 = 0.500$		$\hat{\alpha}_3 = 0.525$		$\hat{\alpha}_3 = 0.550$		$\hat{\alpha}_3 = 0.575$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.35	-0.9439	-0.9079	-0.9597	-0.9322	-0.9751	-0.9571	-0.9902	-0.9827
1.38	-0.9261	-0.8785	-0.9426	-0.9034	-0.9588	-0.9291	-0.9747	-0.9553
1.41	-0.9077	-0.8481	-0.9250	-0.8737	-0.9421	-0.9001	-0.9587	-0.9271
1.44	-0.8887	-0.8167	-0.9070	-0.8432	-0.9248	-0.8703	-0.9423	-0.8981
1.47	-0.8692	-0.7843	-0.8884	-0.8116	-0.9071	-0.8396	-0.9254	-0.8682
1.50	-0.8492	-0.7508	-0.8692	-0.7790	-0.8889	-0.8078	-0.9081	-0.8373
1.53	-0.8285	-0.7162	-0.8495	-0.7453	-0.8702	-0.7751	-0.8903	-0.8055
1.56	-0.8072	-0.6804	-0.8293	-0.7104	-0.8509	-0.7412	-0.8719	-0.7726
1.59	-0.7853	-0.6433	-0.8084	-0.6744	-0.8310	-0.7062	-0.8530	-0.7387
1.62	-0.7626	-0.6049	-0.7869	-0.6371	-0.8106	-0.6700	-0.8336	-0.7036
1.65	-0.7393	-0.5650	-0.7647	-0.5984	-0.7895	-0.6325	-0.8136	-0.6673
1.68	-0.7152	-0.5237	-0.7419	-0.5583	-0.7678	-0.5937	-0.7931	-0.6298
1.71	-0.6904	-0.4808	-0.7183	-0.5167	-0.7455	-0.5535	-0.7719	-0.5908
1.74	-0.6648	-0.4361	-0.6940	-0.4735	-0.7224	-0.5117	-0.7501	-0.5505
1.77	-0.6383	-0.3897	-0.6689	-0.4286	-0.6987	-0.4684	-0.7276	-0.5087
1.80	-0.6109	-0.3414	-0.6430	-0.3820	-0.6742	-0.4233	-0.7044	-0.4653
1.83	-0.5827	-0.2911	-0.6162	-0.3334	-0.6489	-0.3765	-0.6805	-0.4202
1.86	-0.5534	-0.2387	-0.5886	-0.2829	-0.6228	-0.3278	-0.6559	-0.3734
1.89	-0.5231	-0.1840	-0.5600	-0.2301	-0.5958	-0.2771	-0.6304	-0.3246
1.92	-0.4917	-0.1268	-0.5304	-0.1751	-0.5679	-0.2242	-0.6042	-0.2739
1.95	-0.4592	-0.0671	-0.4998	-0.1177	-0.5391	-0.1691	-0.5771	-0.2210
1.98	-0.4255	-0.0046	-0.4681	-0.0577	-0.5093	-0.1115	-0.5490	-0.1659
2.01	-0.3906	0.0609	-0.4353	0.0051	-0.4785	-0.0514	-0.5201	-0.1083
2.04	-0.3543	0.1296	-0.4012	0.0709	-0.4465	0.0115	-0.4901	-0.0482
2.07	-0.3166	0.2016	-0.3659	0.1398	-0.4134	0.0774	-0.4591	0.0146
2.10	-0.2774	0.2773	-0.3292	0.2121	-0.3791	0.1464	-0.4270	0.0804
2.13	-0.2365	0.3570	-0.2910	0.2881	-0.3434	0.2188	-0.3937	0.1493
2.16	-0.1940	0.4409	-0.2514	0.3681	-0.3064	0.2948	-0.3591	0.2216
2.19	-0.1497	0.5295	-0.2101	0.4523	-0.2680	0.3748	-0.3233	0.2975
2.22	-0.1035	0.6230	-0.1671	0.5410	-0.2280	0.4590	-0.2861	0.3772
2.25	-0.0552	0.7219	-0.1223	0.6348	-0.1864	0.5477	-0.2475	0.4612
2.28	-0.0047	0.8267	-0.0755	0.7339	-0.1430	0.6414	-0.2073	0.5496
2.31	0.0481	0.9378	-0.0267	0.8389	-0.0978	0.7404	-0.1654	0.6429
2.34	0.1034	1.0560	0.0244	0.9502	-0.0506	0.8452	-0.1218	0.7415
2.37	0.1615	1.1818	0.0779	1.0686	-0.0013	0.9564	-0.0764	0.8457
2.40	0.2224	1.3160	0.1339	1.1945	0.0502	1.0744	-0.0290	0.9563
2.43	0.2865	1.4595	0.1927	1.3287	0.1041	1.1999	0.0206	1.0735
2.46	0.3541	1.6132	0.2545	1.4722	0.1607	1.3337	0.0724	1.1982
2.49	0.4253	1.7781	0.3195	1.6257	0.2200	1.4765	0.1266	1.3310
2.52	0.5005	1.9557	0.3879	1.7905	0.2823	1.6293	0.1835	1.4726
2.55	0.5801	2.1472	0.4601	1.9677	0.3479	1.7931	0.2431	1.6241
2.58	0.6644	2.3544	0.5364	2.1588	0.4170	1.9692	0.3058	1.7863
2.61	0.7540	2.5793	0.6172	2.3653	0.4899	2.1589	0.3717	1.9605
2.64	0.8493	2.8241	0.7028	2.5893	0.5670	2.3639	0.4412	2.1480
2.67	0.9510	3.0916	0.7937	2.8330	0.6486	2.5859	0.5146	2.3504
2.70	1.0596	3.3848	0.8906	3.0989	0.7351	2.8271	0.5920	2.5693
2.73	1.1760	3.7077	0.9939	3.3902	0.8271	3.0900	0.6741	2.8069
2.76	1.3011	4.0647	1.1044	3.7105	0.9250	3.3777	0.7611	3.0656
2.79	1.4359	4.4615	1.2229	4.0643	1.0295	3.6937	0.8536	3.3482
2.82	1.5816	4.9049	1.3503	4.4570	1.1413	4.0421	0.9522	3.6581
2.85	1.7397	5.4032	1.4877	4.8950	1.2613	4.4282	1.0573	3.9993
2.88	1.9117	5.9670	1.6363	5.3865	1.3903	4.8581	1.1699	4.3766
2.91	2.0999	6.6098	1.7977	5.9416	1.5295	5.3395	1.2906	4.7958
2.94	2.3066	7.3486	1.9736	6.5730	1.6802	5.8819	1.4205	5.2641
2.97	2.5348	8.2059	2.1661	7.2969	1.8439	6.4972	1.5607	5.7904

$0.600 \leq \hat{\alpha}_3 \leq 0.675$

	$\hat{\alpha}_3 = 0.600$		$\hat{\alpha}_3 = 0.625$		$\hat{\alpha}_3 = 0.650$		$\hat{\alpha}_3 = 0.675$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.50	-0.9268	-0.8674	-0.9449	-0.8978	-0.9624	-0.9287	-0.9793	-0.9598
1.53	-0.9098	-0.8365	-0.9288	-0.8679	-0.9471	-0.8996	-0.9648	-0.9316
1.56	-0.8924	-0.8046	-0.9122	-0.8370	-0.9314	-0.8697	-0.9499	-0.9026
1.59	-0.8745	-0.7717	-0.8952	-0.8051	-0.9153	-0.8389	-0.9346	-0.8728
1.62	-0.8560	-0.7377	-0.8777	-0.7723	-0.8987	-0.8071	-0.9189	-0.8421
1.65	-0.8371	-0.7026	-0.8598	-0.7383	-0.8817	-0.7743	-0.9028	-0.8104
1.68	-0.8176	-0.6663	-0.8413	-0.7032	-0.8642	-0.7404	-0.8862	-0.7778
1.71	-0.7975	-0.6287	-0.8223	-0.6670	-0.8462	-0.7055	-0.8692	-0.7441
1.74	-0.7768	-0.5898	-0.8027	-0.6295	-0.8277	-0.6693	-0.8517	-0.7093
1.77	-0.7556	-0.5495	-0.7826	-0.5906	-0.8087	-0.6319	-0.8338	-0.6733
1.80	-0.7337	-0.5077	-0.7619	-0.5504	-0.7891	-0.5933	-0.8153	-0.6361
1.83	-0.7111	-0.4643	-0.7406	-0.5087	-0.7690	-0.5532	-0.7963	-0.5977
1.86	-0.6879	-0.4193	-0.7187	-0.4655	-0.7483	-0.5117	-0.7768	-0.5579
1.89	-0.6639	-0.3725	-0.6961	-0.4206	-0.7270	-0.4687	-0.7568	-0.5166
1.92	-0.6391	-0.3238	-0.6728	-0.3739	-0.7051	-0.4240	-0.7361	-0.4739
1.95	-0.6136	-0.2732	-0.6488	-0.3254	-0.6825	-0.3776	-0.7149	-0.4295
1.98	-0.5873	-0.2204	-0.6240	-0.2750	-0.6593	-0.3294	-0.6930	-0.3835
2.01	-0.5601	-0.1654	-0.5985	-0.2225	-0.6353	-0.2793	-0.6705	-0.3357
2.04	-0.5320	-0.1080	-0.5721	-0.1677	-0.6106	-0.2271	-0.6473	-0.2860
2.07	-0.5029	-0.0481	-0.5449	-0.1106	-0.5850	-0.1728	-0.6234	-0.2343
2.10	-0.4729	0.0145	-0.5168	-0.0511	-0.5587	-0.1161	-0.5987	-0.1804
2.13	-0.4417	0.0800	-0.4877	0.0112	-0.5315	-0.0570	-0.5733	-0.1243
2.16	-0.4095	0.1487	-0.4576	0.0763	-0.5035	0.0047	-0.5471	-0.0658
2.19	-0.3761	0.2206	-0.4265	0.1444	-0.4744	0.0692	-0.5200	-0.0048
2.22	-0.3415	0.2961	-0.3943	0.2158	-0.4444	0.1367	-0.4921	0.0590
2.25	-0.3056	0.3754	-0.3609	0.2907	-0.4134	0.2074	-0.4632	0.1257
2.28	-0.2683	0.4588	-0.3263	0.3694	-0.3813	0.2815	-0.4333	0.1955
2.31	-0.2296	0.5467	-0.2904	0.4520	-0.3480	0.3593	-0.4024	0.2686
2.34	-0.1893	0.6393	-0.2531	0.5390	-0.3135	0.4410	-0.3705	0.3453
2.37	-0.1474	0.7371	-0.2144	0.6307	-0.2777	0.5269	-0.3374	0.4259
2.40	-0.1037	0.8405	-0.1741	0.7275	-0.2405	0.6174	-0.3030	0.5105
2.43	-0.0581	0.9500	-0.1322	0.8297	-0.2019	0.7128	-0.2675	0.5996
2.46	-0.0106	1.0662	-0.0886	0.9379	-0.1618	0.8136	-0.2305	0.6934
2.49	0.0390	1.1895	-0.0431	1.0525	-0.1201	0.9201	-0.1922	0.7924
2.52	0.0909	1.3208	0.0043	1.1742	-0.0766	1.0329	-0.1524	0.8969
2.55	0.1453	1.4608	0.0539	1.3036	-0.0314	1.1525	-0.1109	1.0075
2.58	0.2022	1.6103	0.1057	1.4413	0.0159	1.2795	-0.0678	1.1247
2.61	0.2619	1.7703	0.1600	1.5884	0.0652	1.4147	-0.0229	1.2491
2.64	0.3247	1.9419	0.2168	1.7456	0.1167	1.5587	0.0240	1.3812
2.67	0.3908	2.1265	0.2764	1.9140	0.1707	1.7126	0.0729	1.5219
2.70	0.4603	2.3254	0.3390	2.0948	0.2272	1.8772	0.1240	1.6720
2.73	0.5338	2.5403	0.4049	2.2895	0.2864	2.0538	0.1775	1.8324
2.76	0.6114	2.7733	0.4743	2.4996	0.3487	2.2436	0.2334	2.0041
2.79	0.6935	3.0265	0.5475	2.7270	0.4141	2.4481	0.2921	2.1885
2.82	0.7806	3.3027	0.6248	2.9737	0.4830	2.6691	0.3537	2.3868
2.85	0.8733	3.6050	0.7067	3.2423	0.5557	2.9084	0.4185	2.6007
2.88	0.9719	3.9372	0.7935	3.5358	0.6325	3.1685	0.4866	2.8319
2.91	1.0772	4.3038	0.8858	3.8576	0.7137	3.4521	0.5585	3.0827
2.94	1.1898	4.7102	0.9840	4.2119	0.7998	3.7624	0.6343	3.3556
2.97	1.3107	5.1630	1.0889	4.6037	0.8913	4.1031	0.7145	3.6534
3.00	1.4407	5.6704	1.2010	5.0390	0.9886	4.4789	0.7995	3.9796
3.03	1.5810	6.2425	1.3213	5.5253	1.0924	4.8952	0.8898	4.3384
3.06	1.7330	6.8920	1.4507	6.0716	1.2035	5.3586	0.9857	4.7346
3.09	1.8981	7.6352	1.5902	6.6895	1.3225	5.8775	1.0880	5.1742
3.12	2.0784	8.4931	1.7414	7.3935	1.4504	6.4620	1.1974	5.6645

$0.700 \leq \hat{\alpha}_3 \leq 0.775$

	$\hat{\alpha}_3 = 0.700$		$\hat{\alpha}_3 = 0.725$		$\hat{\alpha}_3 = 0.750$		$\hat{\alpha}_3 = 0.775$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.63	-0.9333	-0.8672	-0.9522	-0.9027	-0.9703	-0.9382	-0.9876	-0.9736
1.66	-0.9179	-0.8363	-0.9377	-0.8729	-0.9566	-0.9095	-0.9746	-0.9459
1.69	-0.9021	-0.8045	-0.9227	-0.8423	-0.9425	-0.8799	-0.9613	-0.9174
1.72	-0.8859	-0.7717	-0.9074	-0.8107	-0.9280	-0.8495	-0.9476	-0.8882
1.75	-0.8692	-0.7378	-0.8917	-0.7781	-0.9131	-0.8182	-0.9336	-0.8580
1.78	-0.8521	-0.7029	-0.8755	-0.7445	-0.8979	-0.7859	-0.9192	-0.8270
1.81	-0.8345	-0.6668	-0.8589	-0.7099	-0.8822	-0.7527	-0.9045	-0.7951
1.84	-0.8165	-0.6295	-0.8419	-0.6741	-0.8662	-0.7184	-0.8893	-0.7622
1.87	-0.7979	-0.5908	-0.8244	-0.6371	-0.8497	-0.6829	-0.8738	-0.7283
1.90	-0.7788	-0.5509	-0.8064	-0.5988	-0.8328	-0.6463	-0.8579	-0.6933
1.93	-0.7592	-0.5095	-0.7879	-0.5593	-0.8154	-0.6085	-0.8415	-0.6571
1.96	-0.7390	-0.4666	-0.7690	-0.5183	-0.7975	-0.5694	-0.8247	-0.6198
1.99	-0.7183	-0.4221	-0.7495	-0.4759	-0.7792	-0.5289	-0.8075	-0.5812
2.02	-0.6969	-0.3760	-0.7294	-0.4319	-0.7604	-0.4870	-0.7898	-0.5413
2.05	-0.6749	-0.3280	-0.7088	-0.3863	-0.7410	-0.4436	-0.7717	-0.5000
2.08	-0.6523	-0.2782	-0.6876	-0.3389	-0.7212	-0.3986	-0.7530	-0.4572
2.11	-0.6290	-0.2264	-0.6658	-0.2897	-0.7007	-0.3519	-0.7339	-0.4129
2.14	-0.6050	-0.1724	-0.6433	-0.2386	-0.6797	-0.3034	-0.7142	-0.3669
2.17	-0.5802	-0.1162	-0.6202	-0.1854	-0.6581	-0.2530	-0.6940	-0.3192
2.20	-0.5547	-0.0577	-0.5964	-0.1300	-0.6358	-0.2007	-0.6732	-0.2697
2.23	-0.5284	0.0035	-0.5718	-0.0723	-0.6129	-0.1462	-0.6518	-0.2182
2.26	-0.5012	0.0673	-0.5465	-0.0121	-0.5893	-0.0894	-0.6298	-0.1647
2.29	-0.4731	0.1340	-0.5204	0.0507	-0.5650	-0.0303	-0.6072	-0.1090
2.32	-0.4441	0.2038	-0.4934	0.1163	-0.5400	0.0314	-0.5839	-0.0510
2.35	-0.4142	0.2770	-0.4656	0.1850	-0.5142	0.0958	-0.5599	0.0094
2.38	-0.3831	0.3536	-0.4369	0.2568	-0.4875	0.1630	-0.5351	0.0724
2.41	-0.3511	0.4341	-0.4072	0.3320	-0.4600	0.2334	-0.5097	0.1383
2.44	-0.3178	0.5186	-0.3765	0.4108	-0.4317	0.3070	-0.4834	0.2070
2.47	-0.2834	0.6075	-0.3448	0.4936	-0.4023	0.3841	-0.4563	0.2789
2.50	-0.2477	0.7012	-0.3119	0.5807	-0.3720	0.4651	-0.4283	0.3542
2.53	-0.2107	0.7999	-0.2779	0.6722	-0.3407	0.5500	-0.3994	0.4332
2.56	-0.1722	0.9041	-0.2426	0.7687	-0.3083	0.6393	-0.3696	0.5159
2.59	-0.1322	1.0143	-0.2060	0.8704	-0.2747	0.7333	-0.3387	0.6028
2.62	-0.0907	1.1310	-0.1680	0.9779	-0.2399	0.8324	-0.3068	0.6942
2.65	-0.0474	1.2548	-0.1286	1.0915	-0.2039	0.9369	-0.2738	0.7904
2.68	-0.0024	1.3862	-0.0876	1.2119	-0.1665	1.0472	-0.2396	0.8918
2.71	0.0446	1.5261	-0.0449	1.3396	-0.1276	1.1640	-0.2042	0.9987
2.74	0.0937	1.6752	-0.0005	1.4753	-0.0873	1.2877	-0.1674	1.1117
2.77	0.1449	1.8344	0.0459	1.6197	-0.0453	1.4190	-0.1293	1.2313
2.80	0.1985	2.0047	0.0942	1.7737	-0.0016	1.5585	-0.0897	1.3580
2.83	0.2546	2.1874	0.1446	1.9382	0.0439	1.7071	-0.0486	1.4924
2.86	0.3134	2.3837	0.1973	2.1144	0.0913	1.8655	-0.0058	1.6353
2.89	0.3752	2.5952	0.2525	2.3033	0.1408	2.0349	0.0388	1.7875
2.92	0.4401	2.8237	0.3103	2.5065	0.1925	2.2162	0.0852	1.9499
2.95	0.5084	3.0713	0.3710	2.7256	0.2465	2.4108	0.1336	2.1234
2.98	0.5803	3.3402	0.4346	2.9623	0.3032	2.6201	0.1841	2.3093
3.01	0.6563	3.6333	0.5016	3.2190	0.3625	2.8459	0.2369	2.5088
3.04	0.7367	3.9539	0.5722	3.4979	0.4248	3.0900	0.2922	2.7234
3.07	0.8219	4.3060	0.6466	3.8023	0.4902	3.3547	0.3500	2.9550
3.10	0.9122	4.6941	0.7253	4.1353	0.5591	3.6426	0.4107	3.2054
3.13	1.0084	5.1239	0.8086	4.5014	0.6317	3.9567	0.4744	3.4769
3.16	1.1109	5.6022	0.8969	4.9052	0.7084	4.3008	0.5414	3.7724
3.19	1.2203	6.1375	0.9907	5.3529	0.7894	4.6790	0.6120	4.0948
3.22	1.3376	6.7400	1.0906	5.8517	0.8753	5.0966	0.6864	4.4481
3.25	1.4636	7.4230	1.1972	6.4104	0.9665	5.5599	0.7649	4.8366

$0.800 \leq \hat{\alpha}_3 \leq 0.875$

	$\hat{\alpha}_3 = 0.800$		$\hat{\alpha}_3 = 0.825$		$\hat{\alpha}_3 = 0.850$		$\hat{\alpha}_3 = 0.875$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4	β_3	β_4
1.80	-0.9303	-0.8474	-0.9502	-0.8886	-0.9690	-0.9292	-0.9868	-0.9692
1.83	-0.9162	-0.8162	-0.9369	-0.8586	-0.9565	-0.9004	-0.9751	-0.9416
1.86	-0.9017	-0.7840	-0.9232	-0.8278	-0.9436	-0.8709	-0.9630	-0.9133
1.89	-0.8868	-0.7509	-0.9092	-0.7960	-0.9305	-0.8405	-0.9506	-0.8842
1.92	-0.8716	-0.7167	-0.8949	-0.7633	-0.9170	-0.8092	-0.9379	-0.8543
1.95	-0.8559	-0.6814	-0.8802	-0.7296	-0.9031	-0.7770	-0.9249	-0.8235
1.98	-0.8399	-0.6450	-0.8651	-0.6949	-0.8890	-0.7438	-0.9116	-0.7918
2.01	-0.8234	-0.6074	-0.8496	-0.6590	-0.8744	-0.7097	-0.8979	-0.7593
2.04	-0.8065	-0.5685	-0.8337	-0.6220	-0.8595	-0.6744	-0.8839	-0.7257
2.07	-0.7892	-0.5283	-0.8175	-0.5838	-0.8443	-0.6380	-0.8696	-0.6911
2.10	-0.7714	-0.4867	-0.8008	-0.5443	-0.8286	-0.6005	-0.8549	-0.6554
2.13	-0.7531	-0.4436	-0.7837	-0.5034	-0.8126	-0.5617	-0.8399	-0.6186
2.16	-0.7343	-0.3990	-0.7661	-0.4611	-0.7961	-0.5216	-0.8245	-0.5805
2.19	-0.7151	-0.3527	-0.7481	-0.4173	-0.7792	-0.4802	-0.8087	-0.5413
2.22	-0.6953	-0.3047	-0.7296	-0.3719	-0.7619	-0.4373	-0.7925	-0.5007
2.25	-0.6749	-0.2549	-0.7106	-0.3249	-0.7442	-0.3928	-0.7759	-0.4587
2.28	-0.6540	-0.2031	-0.6911	-0.2761	-0.7260	-0.3468	-0.7588	-0.4153
2.31	-0.6325	-0.1493	-0.6710	-0.2254	-0.7073	-0.2991	-0.7414	-0.3703
2.34	-0.6104	-0.0933	-0.6504	-0.1728	-0.6881	-0.2496	-0.7235	-0.3237
2.37	-0.5877	-0.0350	-0.6292	-0.1181	-0.6683	-0.1982	-0.7051	-0.2754
2.40	-0.5642	0.0258	-0.6075	-0.0611	-0.6481	-0.1448	-0.6862	-0.2253
2.43	-0.5401	0.0891	-0.5851	-0.0018	-0.6273	-0.0893	-0.6669	-0.1733
2.46	-0.5153	0.1553	-0.5621	0.0600	-0.6059	-0.0315	-0.6470	-0.1192
2.49	-0.4897	0.2244	-0.5383	0.1244	-0.5839	0.0286	-0.6266	-0.0630
2.52	-0.4633	0.2966	-0.5139	0.1916	-0.5613	0.0913	-0.6056	-0.0046
2.55	-0.4361	0.3722	-0.4888	0.2619	-0.5381	0.1566	-0.5841	0.0562
2.58	-0.4080	0.4514	-0.4629	0.3353	-0.5141	0.2248	-0.5620	0.1196
2.61	-0.3790	0.5345	-0.4362	0.4122	-0.4895	0.2961	-0.5392	0.1857
2.64	-0.3490	0.6217	-0.4086	0.4928	-0.4641	0.3706	-0.5157	0.2547
2.67	-0.3180	0.7134	-0.3801	0.5772	-0.4379	0.4485	-0.4916	0.3268
2.70	-0.2860	0.8098	-0.3508	0.6659	-0.4109	0.5302	-0.4668	0.4021
2.73	-0.2528	0.9114	-0.3204	0.7591	-0.3831	0.6158	-0.4412	0.4809
2.76	-0.2185	1.0185	-0.2891	0.8571	-0.3544	0.7057	-0.4149	0.5634
2.79	-0.1830	1.1317	-0.2567	0.9604	-0.3247	0.8001	-0.3877	0.6500
2.82	-0.1461	1.2514	-0.2231	1.0693	-0.2941	0.8995	-0.3597	0.7408
2.85	-0.1078	1.3781	-0.1884	1.1844	-0.2625	1.0041	-0.3308	0.8362
2.88	-0.0681	1.5126	-0.1523	1.3060	-0.2297	1.1145	-0.3010	0.9366
2.91	-0.0268	1.6554	-0.1150	1.4348	-0.1959	1.2310	-0.2701	1.0423
2.94	0.0161	1.8074	-0.0763	1.5715	-0.1608	1.3542	-0.2383	1.1537
2.97	0.0608	1.9695	-0.0360	1.7167	-0.1244	1.4847	-0.2053	1.2714
3.00	0.1073	2.1427	0.0058	1.8712	-0.0868	1.6231	-0.1713	1.3957
3.03	0.1558	2.3280	0.0492	2.0359	-0.0477	1.7701	-0.1360	1.5274
3.06	0.2065	2.5268	0.0945	2.2118	-0.0070	1.9265	-0.0994	1.6670
3.09	0.2595	2.7405	0.1417	2.4001	0.0352	2.0931	-0.0615	1.8152
3.12	0.3148	2.9709	0.1909	2.6021	0.0790	2.2711	-0.0221	1.9729
3.15	0.3728	3.2198	0.2422	2.8192	0.1247	2.4616	0.0187	2.1409
3.18	0.4337	3.4894	0.2959	3.0531	0.1723	2.6658	0.0612	2.3202
3.21	0.4975	3.7825	0.3520	3.3058	0.2220	2.8853	0.1053	2.5120
3.24	0.5646	4.1020	0.4108	3.5796	0.2738	3.1217	0.1513	2.7176
3.27	0.6353	4.4517	0.4725	3.8771	0.3280	3.3770	0.1992	2.9385
3.30	0.7098	4.8356	0.5373	4.2014	0.3847	3.6536	0.2491	3.1763
3.33	0.7885	5.2591	0.6054	4.5561	0.4441	3.9539	0.3013	3.4330
3.36	0.8718	5.7282	0.6771	4.9456	0.5064	4.2812	0.3558	3.7108
3.39	0.9601	6.2504	0.7527	5.3750	0.5718	4.6390	0.4128	4.0125
3.42	1.0539	6.8350	0.8327	5.8506	0.6406	5.0317	0.4725	4.3409

$0.900 \leq \hat{\alpha}_3 \leq 0.950$

	$\hat{\alpha}_3 = 0.900$		$\hat{\alpha}_3 = 0.925$		$\hat{\alpha}_3 = 0.950$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
1.94	-0.9496	-0.8790	-0.9687	-0.9233	-0.9867	-0.9667
1.98	-0.9329	-0.8389	-0.9531	-0.8849	-0.9721	-0.9300
2.02	-0.9158	-0.7972	-0.9370	-0.8452	-0.9569	-0.8920
2.06	-0.8981	-0.7539	-0.9204	-0.8039	-0.9414	-0.8527
2.10	-0.8798	-0.7089	-0.9032	-0.7610	-0.9253	-0.8118
2.14	-0.8609	-0.6620	-0.8855	-0.7165	-0.9087	-0.7695
2.18	-0.8414	-0.6132	-0.8672	-0.6701	-0.8916	-0.7255
2.22	-0.8213	-0.5622	-0.8484	-0.6219	-0.8739	-0.6797
2.26	-0.8005	-0.5090	-0.8289	-0.5715	-0.8557	-0.6321
2.30	-0.7790	-0.4534	-0.8088	-0.5190	-0.8369	-0.5825
2.34	-0.7568	-0.3952	-0.7881	-0.4642	-0.8175	-0.5307
2.38	-0.7338	-0.3343	-0.7667	-0.4068	-0.7975	-0.4767
2.42	-0.7101	-0.2704	-0.7446	-0.3468	-0.7769	-0.4203
2.46	-0.6856	-0.2033	-0.7217	-0.2839	-0.7556	-0.3613
2.50	-0.6602	-0.1328	-0.6981	-0.2180	-0.7336	-0.2995
2.54	-0.6339	-0.0587	-0.6736	-0.1487	-0.7109	-0.2347
2.58	-0.6066	0.0195	-0.6484	-0.0759	-0.6874	-0.1667
2.62	-0.5784	0.1019	-0.6222	0.0008	-0.6631	-0.0954
2.66	-0.5491	0.1890	-0.5951	0.0816	-0.6380	-0.0203
2.70	-0.5187	0.2811	-0.5671	0.1668	-0.6120	0.0588
2.74	-0.4872	0.3787	-0.5380	0.2569	-0.5852	0.1421
2.78	-0.4544	0.4823	-0.5078	0.3523	-0.5573	0.2301
2.82	-0.4204	0.5924	-0.4765	0.4534	-0.5285	0.3231
2.86	-0.3849	0.7096	-0.4440	0.5607	-0.4986	0.4216
2.90	-0.3480	0.8347	-0.4101	0.6749	-0.4676	0.5260
2.94	-0.3095	0.9683	-0.3750	0.7965	-0.4354	0.6370
2.98	-0.2693	1.1114	-0.3384	0.9263	-0.4019	0.7550
3.02	-0.2274	1.2650	-0.3002	1.0650	-0.3671	0.8807
3.06	-0.1836	1.4303	-0.2605	1.2137	-0.3309	1.0150
3.10	-0.1377	1.6085	-0.2190	1.3735	-0.2933	1.1587
3.14	-0.0896	1.8013	-0.1756	1.5454	-0.2540	1.3127
3.18	-0.0392	2.0103	-0.1303	1.7309	-0.2130	1.4782
3.22	0.0137	2.2376	-0.0828	1.9317	-0.1702	1.6564
3.26	0.0694	2.4856	-0.0330	2.1496	-0.1255	1.8489
3.30	0.1280	2.7573	0.0193	2.3868	-0.0787	2.0574
3.34	0.1898	3.0559	0.0742	2.6459	-0.0297	2.2837
3.38	0.2552	3.3856	0.1320	2.9299	0.0217	2.5302
3.42	0.3243	3.7512	0.1929	3.2425	0.0757	2.7998
3.46	0.3976	4.1587	0.2572	3.5879	0.1325	3.0954
3.50	0.4755	4.6154	0.3252	3.9715	0.1924	3.4211
3.54	0.5585	5.1304	0.3973	4.3997	0.2555	3.7814
3.58	0.6470	5.7152	0.4738	4.8802	0.3222	4.1819
3.62	0.7417	6.3843	0.5552	5.4231	0.3927	4.6294
3.66	0.8433	7.1567	0.6420	6.0407	0.4676	5.1324
3.70	0.9527	8.0570	0.7348	6.7489	0.5471	5.7014
3.74	1.0707	9.1188	0.8341	7.5684	0.6319	6.3498
3.78	1.1986	10.3876	0.9410	8.5264	0.7223	7.0947
3.82	1.3376	11.9282	1.0561	9.6600	0.8191	7.9584
3.86	1.4895	13.8345	1.1807	11.0200	0.9230	8.9707
3.90	1.6561	16.2486	1.3160	12.6791	1.0348	10.1718
3.94	1.8399	19.3964	1.4634	14.7440	1.1556	11.6178
3.98	2.0439	23.6582	1.6250	17.3781	1.2865	13.3891
4.02	2.2718	29.7282	1.8028	20.8449	1.4289	15.6048
4.06	2.5283	39.0200	1.9996	25.5976	1.5846	18.4495
4.10	2.8197	54.9194	2.2190	32.4863	1.7555	22.2248

$0.975 \leq \hat{\alpha}_3 \leq 1.025$

	$\hat{\alpha}_3 = 0.975$		$\hat{\alpha}_3 = 1.000$		$\hat{\alpha}_3 = 1.025$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
2.10	-0.9460	-0.8613	-0.9655	-0.9095	-0.9837	-0.9564
2.14	-0.9305	-0.8210	-0.9509	-0.8711	-0.9701	-0.9198
2.18	-0.9144	-0.7792	-0.9359	-0.8313	-0.9560	-0.8820
2.22	-0.8979	-0.7358	-0.9204	-0.7901	-0.9415	-0.8428
2.26	-0.8808	-0.6907	-0.9044	-0.7473	-0.9266	-0.8022
2.30	-0.8633	-0.6437	-0.8880	-0.7029	-0.9112	-0.7601
2.34	-0.8451	-0.5949	-0.8711	-0.6567	-0.8954	-0.7164
2.38	-0.8265	-0.5439	-0.8536	-0.6087	-0.8790	-0.6710
2.42	-0.8072	-0.4908	-0.8356	-0.5586	-0.8622	-0.6238
2.46	-0.7873	-0.4354	-0.8170	-0.5065	-0.8449	-0.5747
2.50	-0.7668	-0.3774	-0.7979	-0.4521	-0.8270	-0.5235
2.54	-0.7457	-0.3168	-0.7782	-0.3953	-0.8087	-0.4702
2.58	-0.7238	-0.2533	-0.7579	-0.3359	-0.7897	-0.4146
2.62	-0.7013	-0.1868	-0.7369	-0.2737	-0.7702	-0.3565
2.66	-0.6780	-0.1169	-0.7153	-0.2086	-0.7501	-0.2957
2.70	-0.6539	-0.0435	-0.6930	-0.1404	-0.7294	-0.2322
2.74	-0.6291	0.0337	-0.6699	-0.0687	-0.7080	-0.1656
2.78	-0.6034	0.1150	-0.6461	0.0066	-0.6859	-0.0957
2.82	-0.5767	0.2008	-0.6215	0.0858	-0.6631	-0.0224
2.86	-0.5492	0.2914	-0.5961	0.1693	-0.6396	0.0547
2.90	-0.5207	0.3871	-0.5698	0.2573	-0.6154	0.1358
2.94	-0.4911	0.4886	-0.5426	0.3503	-0.5903	0.2212
2.98	-0.4604	0.5962	-0.5145	0.4487	-0.5644	0.3114
3.02	-0.4286	0.7105	-0.4853	0.5529	-0.5376	0.4066
3.06	-0.3956	0.8321	-0.4551	0.6634	-0.5098	0.5073
3.10	-0.3613	0.9618	-0.4237	0.7808	-0.4811	0.6141
3.14	-0.3256	1.1003	-0.3912	0.9059	-0.4514	0.7273
3.18	-0.2884	1.2486	-0.3574	1.0392	-0.4205	0.8477
3.22	-0.2497	1.4076	-0.3223	1.1816	-0.3885	0.9758
3.26	-0.2094	1.5785	-0.2857	1.3341	-0.3554	1.1124
3.30	-0.1673	1.7626	-0.2477	1.4977	-0.3209	1.2584
3.34	-0.1233	1.9616	-0.2081	1.6736	-0.2851	1.4148
3.38	-0.0774	2.1771	-0.1668	1.8632	-0.2478	1.5825
3.42	-0.0292	2.4113	-0.1237	2.0681	-0.2090	1.7629
3.46	0.0212	2.6666	-0.0786	2.2901	-0.1685	1.9574
3.50	0.0742	2.9457	-0.0315	2.5315	-0.1264	2.1676
3.54	0.1298	3.2522	0.0179	2.7946	-0.0823	2.3954
3.58	0.1884	3.5900	0.0697	3.0825	-0.0363	2.6431
3.62	0.2501	3.9639	0.1240	3.3987	0.0119	2.9133
3.66	0.3152	4.3800	0.1811	3.7474	0.0623	3.2089
3.70	0.3841	4.8453	0.2413	4.1337	0.1152	3.5337
3.74	0.4571	5.3689	0.3047	4.5637	0.1708	3.8920
3.78	0.5345	5.9618	0.3717	5.0450	0.2293	4.2890
3.82	0.6169	6.6384	0.4426	5.5868	0.2908	4.7311
3.86	0.7047	7.4167	0.5177	6.2010	0.3558	5.2261
3.90	0.7986	8.3208	0.5975	6.9025	0.4244	5.7836
3.94	0.8992	9.3825	0.6825	7.7103	0.4971	6.4160
3.98	1.0073	10.6451	0.7732	8.6499	0.5741	7.1385
4.02	1.1238	12.1695	0.8702	9.7548	0.6560	7.9713
4.06	1.2499	14.0430	0.9742	11.0712	0.7433	8.9405
4.10	1.3867	16.3966	1.0862	12.6638	0.8365	10.0815
4.14	1.5360	19.4345	1.2070	14.6263	0.9362	11.4423
4.18	1.6994	23.4950	1.3379	17.0996	1.0433	13.0908
4.22	1.8794	29.1798	1.4802	20.3055	1.1587	15.1258
4.26	2.0787	37.6727	1.6357	24.6143	1.2833	17.6959

$1.050 \leq \hat{\alpha}_3 \leq 1.100$

	$\hat{\alpha}_3 = 1.050$		$\hat{\alpha}_3 = 1.075$		$\hat{\alpha}_3 = 1.100$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
2.24	-0.9544	-0.8747	-0.9733	-0.9252	-0.9909	-0.9741
2.28	-0.9402	-0.8355	-0.9600	-0.8879	-0.9786	-0.9386
2.32	-0.9256	-0.7948	-0.9464	-0.8493	-0.9658	-0.9019
2.36	-0.9105	-0.7527	-0.9323	-0.8094	-0.9526	-0.8641
2.40	-0.8950	-0.7090	-0.9178	-0.7680	-0.9391	-0.8249
2.44	-0.8791	-0.6635	-0.9029	-0.7251	-0.9252	-0.7843
2.48	-0.8627	-0.6163	-0.8876	-0.6806	-0.9109	-0.7423
2.52	-0.8458	-0.5672	-0.8718	-0.6343	-0.8961	-0.6987
2.56	-0.8284	-0.5161	-0.8556	-0.5863	-0.8810	-0.6534
2.60	-0.8105	-0.4628	-0.8388	-0.5363	-0.8654	-0.6064
2.64	-0.7920	-0.4073	-0.8217	-0.4842	-0.8493	-0.5576
2.68	-0.7730	-0.3492	-0.8040	-0.4300	-0.8328	-0.5068
2.72	-0.7534	-0.2886	-0.7857	-0.3733	-0.8159	-0.4538
2.76	-0.7332	-0.2252	-0.7670	-0.3143	-0.7984	-0.3987
2.80	-0.7125	-0.1587	-0.7477	-0.2525	-0.7805	-0.3411
2.84	-0.6910	-0.0891	-0.7278	-0.1879	-0.7620	-0.2811
2.88	-0.6689	-0.0161	-0.7073	-0.1202	-0.7430	-0.2183
2.92	-0.6461	0.0607	-0.6862	-0.0493	-0.7234	-0.1526
2.96	-0.6226	0.1414	-0.6644	0.0251	-0.7032	-0.0838
3.00	-0.5983	0.2264	-0.6420	0.1033	-0.6825	-0.0117
3.04	-0.5732	0.3160	-0.6188	0.1856	-0.6611	0.0639
3.08	-0.5473	0.4106	-0.5950	0.2721	-0.6390	0.1434
3.12	-0.5204	0.5107	-0.5703	0.3634	-0.6163	0.2269
3.16	-0.4927	0.6166	-0.5448	0.4598	-0.5929	0.3149
3.20	-0.4640	0.7288	-0.5185	0.5617	-0.5687	0.4077
3.24	-0.4343	0.8481	-0.4913	0.6696	-0.5438	0.5057
3.28	-0.4035	0.9749	-0.4632	0.7840	-0.5180	0.6092
3.32	-0.3715	1.1101	-0.4341	0.9054	-0.4913	0.7188
3.36	-0.3384	1.2544	-0.4039	1.0346	-0.4638	0.8350
3.40	-0.3040	1.4087	-0.3727	1.1723	-0.4354	0.9583
3.44	-0.2683	1.5741	-0.3403	1.3193	-0.4059	1.0895
3.48	-0.2311	1.7518	-0.3067	1.4764	-0.3754	1.2293
3.52	-0.1924	1.9432	-0.2718	1.6449	-0.3438	1.3785
3.56	-0.1521	2.1498	-0.2355	1.8259	-0.3110	1.5381
3.60	-0.1100	2.3734	-0.1978	2.0208	-0.2770	1.7091
3.64	-0.0662	2.6161	-0.1585	2.2311	-0.2417	1.8928
3.68	-0.0203	2.8804	-0.1176	2.4588	-0.2050	2.0905
3.72	0.0276	3.1692	-0.0750	2.7059	-0.1669	2.3039
3.76	0.0778	3.4858	-0.0305	2.9749	-0.1272	2.5348
3.80	0.1304	3.8344	0.0160	3.2689	-0.0858	2.7853
3.84	0.1857	4.2199	0.0647	3.5911	-0.0427	3.0581
3.88	0.2437	4.6480	0.1156	3.9459	0.0023	3.3560
3.92	0.3049	5.1261	0.1691	4.3381	0.0494	3.6825
3.96	0.3694	5.6631	0.2251	4.7736	0.0986	4.0418
4.00	0.4375	6.2700	0.2841	5.2600	0.1501	4.4389
4.04	0.5095	6.9610	0.3462	5.8061	0.2041	4.8797
4.08	0.5859	7.7540	0.4117	6.4233	0.2609	5.3718
4.12	0.6670	8.6726	0.4809	7.1258	0.3205	5.9241
4.16	0.7534	9.7479	0.5541	7.9320	0.3833	6.5480
4.20	0.8455	11.0222	0.6317	8.8656	0.4496	7.2577
4.24	0.9441	12.5544	0.7142	9.9584	0.5196	8.0717
4.28	1.0499	14.4284	0.8021	11.2531	0.5936	9.0138
4.32	1.1637	16.7689	0.8958	12.8094	0.6722	10.1157
4.36	1.2866	19.7687	0.9962	14.7126	0.7557	11.4201
4.40	1.4197	23.7429	1.1039	17.0892	0.8446	12.9867

$1.125 \leq \hat{\alpha}_3 \leq 1.175$

	$\hat{\alpha}_3 = 1.125$		$\hat{\alpha}_3 = 1.150$		$\hat{\alpha}_3 = 1.175$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
2.44	-0.9460	-0.8413	-0.9653	-0.8961	-0.9833	-0.9490
2.48	-0.9326	-0.8015	-0.9528	-0.8585	-0.9716	-0.9133
2.52	-0.9188	-0.7603	-0.9400	-0.8196	-0.9596	-0.8764
2.56	-0.9046	-0.7177	-0.9267	-0.7793	-0.9473	-0.8384
2.60	-0.8901	-0.6735	-0.9132	-0.7376	-0.9346	-0.7990
2.64	-0.8751	-0.6276	-0.8992	-0.6944	-0.9216	-0.7583
2.68	-0.8598	-0.5799	-0.8848	-0.6496	-0.9082	-0.7161
2.72	-0.8439	-0.5303	-0.8701	-0.6031	-0.8945	-0.6725
2.76	-0.8277	-0.4788	-0.8550	-0.5548	-0.8803	-0.6272
2.80	-0.8110	-0.4251	-0.8394	-0.5046	-0.8658	-0.5802
2.84	-0.7938	-0.3691	-0.8234	-0.4524	-0.8509	-0.5313
2.88	-0.7761	-0.3107	-0.8070	-0.3980	-0.8356	-0.4806
2.92	-0.7579	-0.2498	-0.7901	-0.3414	-0.8199	-0.4278
2.96	-0.7393	-0.1861	-0.7727	-0.2823	-0.8038	-0.3728
3.00	-0.7200	-0.1195	-0.7549	-0.2206	-0.7872	-0.3156
3.04	-0.7002	-0.0497	-0.7365	-0.1561	-0.7702	-0.2558
3.08	-0.6798	0.0234	-0.7176	-0.0887	-0.7527	-0.1935
3.12	-0.6589	0.1001	-0.6982	-0.0181	-0.7347	-0.1284
3.16	-0.6372	0.1807	-0.6782	0.0559	-0.7162	-0.0603
3.20	-0.6150	0.2654	-0.6577	0.1335	-0.6972	0.0110
3.24	-0.5920	0.3546	-0.6365	0.2150	-0.6776	0.0857
3.28	-0.5683	0.4486	-0.6147	0.3007	-0.6575	0.1641
3.32	-0.5439	0.5479	-0.5923	0.3909	-0.6368	0.2463
3.36	-0.5187	0.6528	-0.5692	0.4860	-0.6155	0.3328
3.40	-0.4927	0.7638	-0.5453	0.5863	-0.5936	0.4238
3.44	-0.4658	0.8815	-0.5207	0.6924	-0.5711	0.5197
3.48	-0.4380	1.0065	-0.4953	0.8045	-0.5478	0.6209
3.52	-0.4093	1.1393	-0.4691	0.9234	-0.5238	0.7277
3.56	-0.3796	1.2808	-0.4420	1.0496	-0.4991	0.8408
3.60	-0.3488	1.4318	-0.4141	1.1837	-0.4736	0.9605
3.64	-0.3169	1.5933	-0.3852	1.3266	-0.4473	1.0876
3.68	-0.2839	1.7663	-0.3552	1.4789	-0.4201	1.2226
3.72	-0.2496	1.9520	-0.3243	1.6418	-0.3920	1.3663
3.76	-0.2140	2.1519	-0.2922	1.8162	-0.3630	1.5196
3.80	-0.1770	2.3676	-0.2589	2.0034	-0.3329	1.6833
3.84	-0.1385	2.6009	-0.2244	2.2048	-0.3019	1.8586
3.88	-0.0985	2.8539	-0.1886	2.4219	-0.2697	2.0466
3.92	-0.0568	3.1293	-0.1515	2.6567	-0.2363	2.2488
3.96	-0.0133	3.4299	-0.1128	2.9113	-0.2017	2.4667
4.00	0.0321	3.7592	-0.0726	3.1882	-0.1658	2.7022
4.04	0.0795	4.1215	-0.0307	3.4903	-0.1286	2.9573
4.08	0.1291	4.5216	0.0130	3.8210	-0.0898	3.2346
4.12	0.1810	4.9655	0.0586	4.1845	-0.0495	3.5369
4.16	0.2354	5.4607	0.1062	4.5857	-0.0075	3.8676
4.20	0.2926	6.0161	0.1560	5.0306	0.0362	4.2307
4.24	0.3527	6.6430	0.2082	5.5263	0.0819	4.6312
4.28	0.4160	7.3556	0.2628	6.0818	0.1295	5.0748
4.32	0.4827	8.1721	0.3202	6.7080	0.1793	5.5685
4.36	0.5532	9.1160	0.3805	7.4191	0.2315	6.1210
4.40	0.6278	10.2185	0.4440	8.2327	0.2862	6.7432
4.44	0.7070	11.5221	0.5110	9.1718	0.3435	7.4484
4.48	0.7911	13.0850	0.5817	10.2670	0.4038	8.2540
4.52	0.8806	14.9906	0.6565	11.5591	0.4673	9.1822
4.56	0.9762	17.3618	0.7358	13.1049	0.5341	10.2622
4.60	1.0785	20.3874	0.8201	14.9845	0.6048	11.5333

$1.200 \leq \hat{\alpha}_3 \leq 1.250$

	$\hat{\alpha}_3 = 1.200$		$\hat{\alpha}_3 = 1.225$		$\hat{\alpha}_3 = 1.250$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
2.60	-0.9546	-0.8579	-0.9732	-0.9144	-0.9904	-0.9686
2.64	-0.9425	-0.8194	-0.9618	-0.8780	-0.9798	-0.9342
2.68	-0.9300	-0.7797	-0.9502	-0.8405	-0.9690	-0.8987
2.72	-0.9171	-0.7386	-0.9382	-0.8017	-0.9578	-0.8621
2.76	-0.9040	-0.6960	-0.9259	-0.7617	-0.9463	-0.8243
2.80	-0.8904	-0.6519	-0.9133	-0.7202	-0.9346	-0.7853
2.84	-0.8766	-0.6062	-0.9004	-0.6773	-0.9225	-0.7450
2.88	-0.8623	-0.5588	-0.8871	-0.6329	-0.9101	-0.7033
2.92	-0.8477	-0.5095	-0.8735	-0.5868	-0.8974	-0.6601
2.96	-0.8326	-0.4583	-0.8595	-0.5390	-0.8844	-0.6153
3.00	-0.8172	-0.4050	-0.8451	-0.4893	-0.8710	-0.5690
3.04	-0.8014	-0.3496	-0.8304	-0.4377	-0.8573	-0.5209
3.08	-0.7851	-0.2918	-0.8153	-0.3841	-0.8433	-0.4709
3.12	-0.7684	-0.2316	-0.7998	-0.3283	-0.8288	-0.4191
3.16	-0.7513	-0.1687	-0.7839	-0.2701	-0.8140	-0.3651
3.20	-0.7337	-0.1030	-0.7675	-0.2095	-0.7989	-0.3090
3.24	-0.7156	-0.0344	-0.7508	-0.1462	-0.7833	-0.2505
3.28	-0.6970	0.0375	-0.7336	-0.0801	-0.7674	-0.1896
3.32	-0.6779	0.1127	-0.7159	-0.0111	-0.7510	-0.1261
3.36	-0.6583	0.1916	-0.6977	0.0612	-0.7342	-0.0598
3.40	-0.6381	0.2745	-0.6791	0.1368	-0.7170	0.0096
3.44	-0.6173	0.3615	-0.6600	0.2161	-0.6993	0.0821
3.48	-0.5960	0.4531	-0.6403	0.2994	-0.6811	0.1580
3.52	-0.5740	0.5496	-0.6200	0.3868	-0.6624	0.2375
3.56	-0.5513	0.6513	-0.5992	0.4787	-0.6433	0.3209
3.60	-0.5280	0.7588	-0.5778	0.5756	-0.6236	0.4085
3.64	-0.5040	0.8724	-0.5558	0.6776	-0.6034	0.5007
3.68	-0.4792	0.9927	-0.5331	0.7854	-0.5826	0.5976
3.72	-0.4536	1.1203	-0.5098	0.8993	-0.5612	0.6998
3.76	-0.4272	1.2558	-0.4858	1.0198	-0.5392	0.8075
3.80	-0.4000	1.4000	-0.4610	1.1476	-0.5166	0.9214
3.84	-0.3719	1.5537	-0.4354	1.2832	-0.4932	1.0419
3.88	-0.3428	1.7178	-0.4090	1.4275	-0.4692	1.1695
3.92	-0.3127	1.8934	-0.3818	1.5812	-0.4445	1.3049
3.96	-0.2816	2.0816	-0.3537	1.7452	-0.4190	1.4488
4.00	-0.2495	2.2839	-0.3247	1.9205	-0.3927	1.6020
4.04	-0.2161	2.5018	-0.2947	2.1084	-0.3656	1.7654
4.08	-0.1815	2.7371	-0.2637	2.3101	-0.3376	1.9400
4.12	-0.1456	2.9918	-0.2315	2.5272	-0.3087	2.1269
4.16	-0.1084	3.2684	-0.1983	2.7614	-0.2788	2.3274
4.20	-0.0697	3.5697	-0.1638	3.0148	-0.2479	2.5430
4.24	-0.0294	3.8990	-0.1280	3.2897	-0.2160	2.7754
4.28	0.0125	4.2602	-0.0909	3.5888	-0.1829	3.0266
4.32	0.0562	4.6581	-0.0523	3.9154	-0.1486	3.2988
4.36	0.1017	5.0983	-0.0123	4.2732	-0.1131	3.5946
4.40	0.1493	5.5876	0.0295	4.6668	-0.0762	3.9172
4.44	0.1990	6.1344	0.0729	5.1017	-0.0379	4.2703
4.48	0.2509	6.7491	0.1182	5.5843	0.0019	4.6580
4.52	0.3054	7.4446	0.1654	6.1227	0.0433	5.0857
4.56	0.3625	8.2375	0.2148	6.7268	0.0864	5.5596
4.60	0.4226	9.1489	0.2665	7.4090	0.1312	6.0872
4.64	0.4857	10.2066	0.3205	8.1848	0.1781	6.6781
4.68	0.5522	11.4478	0.3772	9.0744	0.2270	7.3437
4.72	0.6224	12.9230	0.4367	10.1036	0.2781	8.0987
4.76	0.6966	14.7034	0.4993	11.3072	0.3315	8.9618

$1.275 \leq \hat{\alpha}_3 \leq 1.325$

	$\hat{\alpha}_3 = 1.275$		$\hat{\alpha}_3 = 1.300$		$\hat{\alpha}_3 = 1.325$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
2.80	-0.9543	-0.8474	-0.9726	-0.9068	-0.9895	-0.9635
2.84	-0.9431	-0.8094	-0.9621	-0.8709	-0.9797	-0.9297
2.88	-0.9315	-0.7702	-0.9514	-0.8340	-0.9697	-0.8948
2.92	-0.9197	-0.7297	-0.9403	-0.7958	-0.9594	-0.8588
2.96	-0.9075	-0.6877	-0.9290	-0.7564	-0.9489	-0.8218
3.00	-0.8951	-0.6443	-0.9174	-0.7157	-0.9381	-0.7835
3.04	-0.8823	-0.5994	-0.9055	-0.6737	-0.9270	-0.7440
3.08	-0.8692	-0.5528	-0.8933	-0.6301	-0.9156	-0.7033
3.12	-0.8558	-0.5045	-0.8808	-0.5850	-0.9040	-0.6611
3.16	-0.8420	-0.4544	-0.8680	-0.5383	-0.8921	-0.6175
3.20	-0.8279	-0.4023	-0.8549	-0.4899	-0.8799	-0.5723
3.24	-0.8135	-0.3481	-0.8414	-0.4396	-0.8673	-0.5255
3.28	-0.7987	-0.2918	-0.8277	-0.3874	-0.8545	-0.4770
3.32	-0.7835	-0.2332	-0.8135	-0.3332	-0.8414	-0.4267
3.36	-0.7679	-0.1721	-0.7991	-0.2768	-0.8280	-0.3745
3.40	-0.7519	-0.1084	-0.7843	-0.2181	-0.8142	-0.3203
3.44	-0.7356	-0.0419	-0.7691	-0.1569	-0.8001	-0.2639
3.48	-0.7188	0.0275	-0.7535	-0.0932	-0.7857	-0.2052
3.52	-0.7015	0.1001	-0.7376	-0.0267	-0.7709	-0.1441
3.56	-0.6838	0.1761	-0.7212	0.0427	-0.7557	-0.0804
3.60	-0.6657	0.2557	-0.7045	0.1153	-0.7402	-0.0140
3.64	-0.6471	0.3392	-0.6873	0.1912	-0.7243	0.0553
3.68	-0.6279	0.4268	-0.6696	0.2707	-0.7080	0.1276
3.72	-0.6083	0.5188	-0.6515	0.3540	-0.6913	0.2033
3.76	-0.5881	0.6157	-0.6330	0.4414	-0.6742	0.2826
3.80	-0.5674	0.7177	-0.6139	0.5332	-0.6567	0.3655
3.84	-0.5460	0.8252	-0.5944	0.6298	-0.6387	0.4526
3.88	-0.5241	0.9388	-0.5743	0.7314	-0.6202	0.5439
3.92	-0.5016	1.0589	-0.5536	0.8385	-0.6013	0.6399
3.96	-0.4784	1.1860	-0.5324	0.9515	-0.5818	0.7409
4.00	-0.4545	1.3208	-0.5106	1.0709	-0.5619	0.8473
4.04	-0.4298	1.4640	-0.4882	1.1972	-0.5414	0.9595
4.08	-0.4045	1.6164	-0.4651	1.3311	-0.5203	1.0779
4.12	-0.3783	1.7787	-0.4414	1.4732	-0.4987	1.2032
4.16	-0.3514	1.9520	-0.4169	1.6242	-0.4764	1.3358
4.20	-0.3236	2.1373	-0.3918	1.7850	-0.4536	1.4764
4.24	-0.2948	2.3360	-0.3658	1.9566	-0.4300	1.6258
4.28	-0.2652	2.5495	-0.3391	2.1399	-0.4058	1.7847
4.32	-0.2345	2.7793	-0.3115	2.3362	-0.3809	1.9540
4.36	-0.2028	3.0275	-0.2830	2.5469	-0.3552	2.1348
4.40	-0.1699	3.2960	-0.2536	2.7735	-0.3287	2.3282
4.44	-0.1360	3.5876	-0.2233	3.0178	-0.3015	2.5356
4.48	-0.1007	3.9051	-0.1919	3.2819	-0.2733	2.7583
4.52	-0.0642	4.2520	-0.1594	3.5683	-0.2443	2.9981
4.56	-0.0263	4.6325	-0.1258	3.8797	-0.2143	3.2571
4.60	0.0131	5.0514	-0.0910	4.2193	-0.1833	3.5374
4.64	0.0540	5.5147	-0.0549	4.5912	-0.1513	3.8417
4.68	0.0966	6.0295	-0.0175	4.9999	-0.1181	4.1732
4.72	0.1410	6.6046	0.0214	5.4510	-0.0838	4.5354
4.76	0.1872	7.2509	0.0617	5.9512	-0.0483	4.9327
4.80	0.2354	7.9819	0.1037	6.5086	-0.0114	5.3703
4.84	0.2858	8.8150	0.1474	7.1332	0.0268	5.8543
4.88	0.3385	9.7723	0.1929	7.8377	0.0665	6.3923
4.92	0.3937	10.8830	0.2403	8.6377	0.1077	6.9935
4.96	0.4516	12.1859	0.2899	9.5536	0.1506	7.6694

$1.350 \leq \hat{\alpha}_3 \leq 1.400$

	$\hat{\alpha}_3 = 1.350$		$\hat{\alpha}_3 = 1.375$		$\hat{\alpha}_3 = 1.400$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
3.00	-0.9572	-0.8480	-0.9750	-0.9093	-0.9913	-0.9679
3.04	-0.9469	-0.8109	-0.9653	-0.8744	-0.9824	-0.9349
3.08	-0.9363	-0.7726	-0.9555	-0.8384	-0.9732	-0.9009
3.12	-0.9255	-0.7330	-0.9454	-0.8012	-0.9638	-0.8660
3.16	-0.9144	-0.6922	-0.9351	-0.7630	-0.9542	-0.8300
3.20	-0.9030	-0.6500	-0.9245	-0.7234	-0.9443	-0.7929
3.24	-0.8914	-0.6064	-0.9136	-0.6826	-0.9342	-0.7547
3.28	-0.8794	-0.5612	-0.9025	-0.6405	-0.9239	-0.7152
3.32	-0.8672	-0.5145	-0.8911	-0.5969	-0.9133	-0.6745
3.36	-0.8547	-0.4660	-0.8795	-0.5518	-0.9024	-0.6325
3.40	-0.8419	-0.4157	-0.8676	-0.5051	-0.8913	-0.5890
3.44	-0.8288	-0.3636	-0.8554	-0.4567	-0.8800	-0.5440
3.48	-0.8154	-0.3094	-0.8429	-0.4066	-0.8684	-0.4975
3.52	-0.8017	-0.2531	-0.8302	-0.3546	-0.8565	-0.4493
3.56	-0.7876	-0.1945	-0.8171	-0.3006	-0.8444	-0.3993
3.60	-0.7732	-0.1336	-0.8038	-0.2445	-0.8320	-0.3475
3.64	-0.7585	-0.0701	-0.7901	-0.1861	-0.8193	-0.2937
3.68	-0.7434	-0.0040	-0.7761	-0.1255	-0.8063	-0.2379
3.72	-0.7280	0.0651	-0.7618	-0.0623	-0.7930	-0.1799
3.76	-0.7122	0.1371	-0.7472	0.0035	-0.7795	-0.1196
3.80	-0.6960	0.2125	-0.7322	0.0722	-0.7656	-0.0568
3.84	-0.6794	0.2913	-0.7169	0.1438	-0.7514	0.0085
3.88	-0.6624	0.3738	-0.7011	0.2186	-0.7369	0.0767
3.92	-0.6449	0.4602	-0.6851	0.2969	-0.7220	0.1477
3.96	-0.6271	0.5510	-0.6686	0.3787	-0.7068	0.2219
4.00	-0.6087	0.6462	-0.6517	0.4645	-0.6912	0.2995
4.04	-0.5899	0.7464	-0.6344	0.5544	-0.6753	0.3805
4.08	-0.5707	0.8518	-0.6167	0.6488	-0.6590	0.4654
4.12	-0.5509	0.9630	-0.5986	0.7479	-0.6423	0.5544
4.16	-0.5306	1.0802	-0.5800	0.8522	-0.6252	0.6476
4.20	-0.5097	1.2041	-0.5609	0.9620	-0.6076	0.7456
4.24	-0.4883	1.3351	-0.5413	1.0778	-0.5897	0.8485
4.28	-0.4663	1.4740	-0.5212	1.2001	-0.5713	0.9568
4.32	-0.4436	1.6214	-0.5006	1.3293	-0.5524	1.0710
4.36	-0.4204	1.7780	-0.4794	1.4661	-0.5331	1.1913
4.40	-0.3964	1.9447	-0.4577	1.6111	-0.5133	1.3185
4.44	-0.3718	2.1225	-0.4353	1.7651	-0.4929	1.4529
4.48	-0.3464	2.3125	-0.4124	1.9288	-0.4720	1.5953
4.52	-0.3203	2.5160	-0.3887	2.1033	-0.4506	1.7464
4.56	-0.2934	2.7343	-0.3645	2.2895	-0.4286	1.9069
4.60	-0.2656	2.9691	-0.3395	2.4886	-0.4060	2.0776
4.64	-0.2370	3.2221	-0.3137	2.7020	-0.3827	2.2597
4.68	-0.2075	3.4957	-0.2872	2.9312	-0.3588	2.4541
4.72	-0.1769	3.7922	-0.2599	3.1779	-0.3343	2.6621
4.76	-0.1454	4.1146	-0.2318	3.4441	-0.3090	2.8852
4.80	-0.1128	4.4662	-0.2027	3.7321	-0.2829	3.1250
4.84	-0.0791	4.8511	-0.1728	4.0447	-0.2561	3.3834
4.88	-0.0442	5.2740	-0.1418	4.3850	-0.2285	3.6625
4.92	-0.0080	5.7407	-0.1098	4.7567	-0.2000	3.9647
4.96	0.0295	6.2581	-0.0767	5.1642	-0.1706	4.2931
5.00	0.0684	6.8345	-0.0425	5.6126	-0.1403	4.6510
5.04	0.1088	7.4804	-0.0071	6.1084	-0.1090	5.0423
5.08	0.1508	8.2087	0.0296	6.6591	-0.0767	5.4720
5.12	0.1945	9.0357	0.0676	7.2742	-0.0432	5.9457
5.16	0.2399	9.9823	0.1071	7.9650	-0.0086	6.4702

$1.425 \leq \hat{\alpha}_3 \leq 1.475$

	$\hat{\alpha}_3 = 1.425$		$\hat{\alpha}_3 = 1.450$		$\hat{\alpha}_3 = 1.475$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
3.20	-0.9626	-0.8588	-0.9796	-0.9214	-0.9951	-0.9810
3.24	-0.9532	-0.8229	-0.9708	-0.8876	-0.9870	-0.9491
3.28	-0.9436	-0.7859	-0.9619	-0.8528	-0.9787	-0.9163
3.32	-0.9337	-0.7478	-0.9527	-0.8171	-0.9702	-0.8827
3.36	-0.9237	-0.7085	-0.9433	-0.7802	-0.9614	-0.8481
3.40	-0.9133	-0.6679	-0.9337	-0.7422	-0.9525	-0.8124
3.44	-0.9028	-0.6259	-0.9238	-0.7031	-0.9433	-0.7758
3.48	-0.8920	-0.5826	-0.9138	-0.6626	-0.9340	-0.7380
3.52	-0.8809	-0.5378	-0.9035	-0.6209	-0.9244	-0.6990
3.56	-0.8696	-0.4915	-0.8930	-0.5778	-0.9146	-0.6589
3.60	-0.8581	-0.4435	-0.8823	-0.5333	-0.9046	-0.6174
3.64	-0.8463	-0.3939	-0.8713	-0.4873	-0.8944	-0.5746
3.68	-0.8342	-0.3424	-0.8601	-0.4396	-0.8839	-0.5304
3.72	-0.8219	-0.2889	-0.8486	-0.3902	-0.8733	-0.4846
3.76	-0.8093	-0.2335	-0.8369	-0.3391	-0.8624	-0.4374
3.80	-0.7964	-0.1759	-0.8249	-0.2861	-0.8512	-0.3884
3.84	-0.7832	-0.1160	-0.8127	-0.2311	-0.8399	-0.3377
3.88	-0.7698	-0.0538	-0.8002	-0.1740	-0.8282	-0.2852
3.92	-0.7560	0.0110	-0.7874	-0.1147	-0.8164	-0.2307
3.96	-0.7419	0.0786	-0.7744	-0.0530	-0.8043	-0.1742
4.00	-0.7275	0.1490	-0.7610	0.0112	-0.7919	-0.1155
4.04	-0.7128	0.2224	-0.7474	0.0780	-0.7793	-0.0545
4.08	-0.6978	0.2991	-0.7335	0.1476	-0.7664	0.0090
4.12	-0.6824	0.3793	-0.7192	0.2202	-0.7532	0.0750
4.16	-0.6666	0.4631	-0.7047	0.2959	-0.7397	0.1437
4.20	-0.6505	0.5509	-0.6898	0.3750	-0.7260	0.2153
4.24	-0.6340	0.6430	-0.6746	0.4577	-0.7119	0.2900
4.28	-0.6171	0.7395	-0.6590	0.5443	-0.6975	0.3679
4.32	-0.5998	0.8410	-0.6431	0.6349	-0.6829	0.4494
4.36	-0.5821	0.9476	-0.6268	0.7300	-0.6679	0.5346
4.40	-0.5639	1.0598	-0.6102	0.8297	-0.6525	0.6237
4.44	-0.5453	1.1781	-0.5931	0.9345	-0.6369	0.7171
4.48	-0.5263	1.3030	-0.5757	1.0447	-0.6208	0.8150
4.52	-0.5067	1.4348	-0.5578	1.1607	-0.6044	0.9177
4.56	-0.4867	1.5744	-0.5395	1.2830	-0.5876	1.0257
4.60	-0.4661	1.7222	-0.5207	1.4121	-0.5705	1.1393
4.64	-0.4450	1.8791	-0.5015	1.5486	-0.5529	1.2590
4.68	-0.4234	2.0459	-0.4818	1.6930	-0.5349	1.3851
4.72	-0.4012	2.2235	-0.4616	1.8461	-0.5165	1.5183
4.76	-0.3783	2.4128	-0.4409	2.0086	-0.4976	1.6591
4.80	-0.3549	2.6152	-0.4197	2.1814	-0.4783	1.8082
4.84	-0.3307	2.8319	-0.3979	2.3655	-0.4585	1.9663
4.88	-0.3059	3.0644	-0.3755	2.5619	-0.4382	2.1341
4.92	-0.2804	3.3146	-0.3525	2.7719	-0.4174	2.3127
4.96	-0.2542	3.5842	-0.3289	2.9969	-0.3960	2.5029
5.00	-0.2271	3.8757	-0.3046	3.2385	-0.3741	2.7060
5.04	-0.1992	4.1918	-0.2796	3.4985	-0.3516	2.9232
5.08	-0.1705	4.5354	-0.2539	3.7789	-0.3285	3.1561
5.12	-0.1409	4.9104	-0.2275	4.0824	-0.3048	3.4062
5.16	-0.1103	5.3209	-0.2003	4.4116	-0.2804	3.6755
5.20	-0.0787	5.7721	-0.1722	4.7699	-0.2553	3.9662
5.24	-0.0461	6.2702	-0.1433	5.1611	-0.2296	4.2809
5.28	-0.0124	6.8227	-0.1136	5.5900	-0.2030	4.6225
5.32	0.0225	7.4386	-0.0828	6.0618	-0.1757	4.9945
5.36	0.0585	8.1292	-0.0511	6.5834	-0.1476	5.4012

$1.500 \leq \hat{\alpha}_3 \leq 1.550$

	$\hat{\alpha}_3 = 1.500$		$\hat{\alpha}_3 = 1.525$		$\hat{\alpha}_3 = 1.550$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
3.45	-0.9591	-0.8357	-0.9759	-0.9013	-0.9912	-0.9636
3.50	-0.9482	-0.7910	-0.9657	-0.8593	-0.9818	-0.9240
3.55	-0.9369	-0.7446	-0.9551	-0.8158	-0.9720	-0.8831
3.60	-0.9253	-0.6964	-0.9443	-0.7707	-0.9619	-0.8408
3.65	-0.9133	-0.6463	-0.9332	-0.7239	-0.9516	-0.7970
3.70	-0.9011	-0.5943	-0.9218	-0.6754	-0.9410	-0.7517
3.75	-0.8885	-0.5401	-0.9101	-0.6250	-0.9301	-0.7046
3.80	-0.8756	-0.4837	-0.8981	-0.5726	-0.9189	-0.6558
3.85	-0.8623	-0.4248	-0.8857	-0.5180	-0.9074	-0.6051
3.90	-0.8487	-0.3634	-0.8731	-0.4612	-0.8956	-0.5524
3.95	-0.8347	-0.2993	-0.8601	-0.4020	-0.8835	-0.4976
4.00	-0.8204	-0.2322	-0.8468	-0.3402	-0.8711	-0.4405
4.05	-0.8057	-0.1620	-0.8331	-0.2757	-0.8584	-0.3810
4.10	-0.7906	-0.0885	-0.8191	-0.2083	-0.8453	-0.3189
4.15	-0.7751	-0.0114	-0.8047	-0.1377	-0.8320	-0.2541
4.20	-0.7593	0.0696	-0.7900	-0.0638	-0.8183	-0.1864
4.25	-0.7430	0.1547	-0.7748	0.0137	-0.8042	-0.1156
4.30	-0.7262	0.2443	-0.7593	0.0950	-0.7899	-0.0414
4.35	-0.7090	0.3387	-0.7434	0.1805	-0.7751	0.0363
4.40	-0.6914	0.4382	-0.7271	0.2704	-0.7600	0.1179
4.45	-0.6733	0.5434	-0.7104	0.3651	-0.7445	0.2036
4.50	-0.6547	0.6547	-0.6932	0.4650	-0.7286	0.2936
4.55	-0.6355	0.7725	-0.6755	0.5704	-0.7123	0.3885
4.60	-0.6159	0.8976	-0.6574	0.6819	-0.6956	0.4885
4.65	-0.5957	1.0304	-0.6388	0.8000	-0.6784	0.5940
4.70	-0.5749	1.1719	-0.6198	0.9252	-0.6608	0.7055
4.75	-0.5535	1.3227	-0.6001	1.0582	-0.6428	0.8235
4.80	-0.5315	1.4839	-0.5800	1.1996	-0.6242	0.9486
4.85	-0.5089	1.6564	-0.5593	1.3504	-0.6052	1.0813
4.90	-0.4855	1.8415	-0.5380	1.5114	-0.5857	1.2225
4.95	-0.4615	2.0405	-0.5160	1.6836	-0.5656	1.3729
5.00	-0.4368	2.2549	-0.4935	1.8682	-0.5450	1.5333
5.05	-0.4112	2.4867	-0.4703	2.0666	-0.5238	1.7048
5.10	-0.3849	2.7378	-0.4463	2.2802	-0.5020	1.8885
5.15	-0.3577	3.0107	-0.4217	2.5109	-0.4796	2.0858
5.20	-0.3296	3.3083	-0.3963	2.7606	-0.4565	2.2980
5.25	-0.3006	3.6339	-0.3701	3.0318	-0.4328	2.5270
5.30	-0.2706	3.9915	-0.3431	3.3271	-0.4083	2.7746
5.35	-0.2395	4.3859	-0.3152	3.6498	-0.3831	3.0432
5.40	-0.2074	4.8228	-0.2864	4.0039	-0.3572	3.3354
5.45	-0.1741	5.3092	-0.2566	4.3939	-0.3304	3.6543
5.50	-0.1396	5.8539	-0.2259	4.8253	-0.3027	4.0038
5.55	-0.1038	6.4674	-0.1940	5.3049	-0.2742	4.3881
5.60	-0.0666	7.1634	-0.1610	5.8409	-0.2447	4.8126
5.65	-0.0280	7.9590	-0.1268	6.4436	-0.2143	5.2837
5.70	0.0122	8.8767	-0.0914	7.1259	-0.1828	5.8093
5.75	0.0541	9.9459	-0.0546	7.9039	-0.1502	6.3991
5.80	0.0977	11.2067	-0.0164	8.7990	-0.1164	7.0651
5.85	0.1431	12.7140	0.0233	9.8386	-0.0814	7.8227
5.90	0.1906	14.5462	0.0646	11.0602	-0.0451	8.6917
5.95	0.2403	16.8185	0.1077	12.5145	-0.0074	9.6977
6.00	0.2923	19.7077	0.1525	14.2736	0.0317	10.8751
6.05	0.3467	23.4994	0.1993	16.4422	0.0724	12.2707
6.10	0.4039	28.6870	0.2482	19.1790	0.1147	13.9498
6.15	0.4640	36.2027	0.2994	22.7367	0.1588	16.0065

$1.575 \leq \hat{\alpha}_3 \leq 1.625$

	$\hat{\alpha}_3 = 1.575$		$\hat{\alpha}_3 = 1.600$		$\hat{\alpha}_3 = 1.625$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
3.68	-0.9627	-0.8407	-0.9787	-0.9074	-0.9934	-0.9706
3.73	-0.9526	-0.7973	-0.9693	-0.8667	-0.9847	-0.9323
3.78	-0.9423	-0.7523	-0.9597	-0.8246	-0.9758	-0.8928
3.83	-0.9317	-0.7057	-0.9499	-0.7810	-0.9666	-0.8519
3.88	-0.9208	-0.6574	-0.9397	-0.7359	-0.9572	-0.8097
3.93	-0.9097	-0.6073	-0.9294	-0.6892	-0.9475	-0.7661
3.98	-0.8982	-0.5552	-0.9187	-0.6407	-0.9376	-0.7208
4.03	-0.8865	-0.5010	-0.9078	-0.5904	-0.9274	-0.6740
4.08	-0.8745	-0.4446	-0.8966	-0.5382	-0.9170	-0.6254
4.13	-0.8622	-0.3859	-0.8851	-0.4839	-0.9063	-0.5751
4.18	-0.8496	-0.3247	-0.8734	-0.4274	-0.8954	-0.5227
4.23	-0.8367	-0.2608	-0.8614	-0.3685	-0.8842	-0.4684
4.28	-0.8234	-0.1941	-0.8490	-0.3072	-0.8727	-0.4118
4.33	-0.8098	-0.1244	-0.8364	-0.2433	-0.8610	-0.3529
4.38	-0.7959	-0.0515	-0.8235	-0.1765	-0.8489	-0.2916
4.43	-0.7817	0.0249	-0.8103	-0.1067	-0.8366	-0.2277
4.48	-0.7671	0.1050	-0.7967	-0.0338	-0.8240	-0.1610
4.53	-0.7522	0.1890	-0.7828	0.0426	-0.8112	-0.0913
4.58	-0.7368	0.2773	-0.7686	0.1226	-0.7980	-0.0184
4.63	-0.7211	0.3702	-0.7541	0.2066	-0.7845	0.0578
4.68	-0.7051	0.4680	-0.7392	0.2948	-0.7706	0.1377
4.73	-0.6886	0.5712	-0.7239	0.3875	-0.7565	0.2214
4.78	-0.6717	0.6800	-0.7083	0.4850	-0.7420	0.3093
4.83	-0.6543	0.7951	-0.6923	0.5878	-0.7272	0.4016
4.88	-0.6365	0.9170	-0.6759	0.6963	-0.7120	0.4988
4.93	-0.6183	1.0461	-0.6591	0.8109	-0.6965	0.6011
4.98	-0.5996	1.1833	-0.6419	0.9322	-0.6806	0.7090
5.03	-0.5804	1.3292	-0.6242	1.0607	-0.6643	0.8229
5.08	-0.5606	1.4846	-0.6061	1.1970	-0.6476	0.9434
5.13	-0.5404	1.6506	-0.5875	1.3419	-0.6305	1.0709
5.18	-0.5196	1.8280	-0.5684	1.4961	-0.6130	1.2061
5.23	-0.4982	2.0182	-0.5489	1.6607	-0.5951	1.3497
5.28	-0.4762	2.2224	-0.5288	1.8365	-0.5766	1.5025
5.33	-0.4536	2.4423	-0.5082	2.0247	-0.5578	1.6653
5.38	-0.4303	2.6797	-0.4870	2.2267	-0.5384	1.8390
5.43	-0.4064	2.9365	-0.4652	2.4439	-0.5185	2.0249
5.48	-0.3817	3.2153	-0.4428	2.6781	-0.4981	2.2242
5.53	-0.3563	3.5187	-0.4198	2.9313	-0.4772	2.4383
5.58	-0.3301	3.8503	-0.3961	3.2057	-0.4557	2.6688
5.63	-0.3032	4.2138	-0.3718	3.5040	-0.4336	2.9177
5.68	-0.2753	4.6140	-0.3467	3.8295	-0.4109	3.1871
5.73	-0.2466	5.0564	-0.3209	4.1857	-0.3875	3.4797
5.78	-0.2169	5.5480	-0.2942	4.5773	-0.3635	3.7982
5.83	-0.1862	6.0971	-0.2668	5.0094	-0.3388	4.1464
5.88	-0.1545	6.7142	-0.2385	5.4885	-0.3133	4.5284
5.93	-0.1217	7.4121	-0.2092	6.0224	-0.2871	4.9491
5.98	-0.0877	8.2075	-0.1790	6.6209	-0.2601	5.4146
6.03	-0.0525	9.1217	-0.1478	7.2960	-0.2322	5.9321
6.08	-0.0160	10.1828	-0.1156	8.0629	-0.2035	6.5107
6.13	0.0219	11.4282	-0.0822	8.9413	-0.1738	7.1614
6.18	0.0612	12.9093	-0.0476	9.9567	-0.1432	7.8983
6.23	0.1021	14.6987	-0.0118	11.1429	-0.1115	8.7392
6.28	0.1447	16.9014	0.0254	12.5461	-0.0788	9.7071
6.33	0.1890	19.6767	0.0639	14.2302	-0.0450	10.8325
6.38	0.2352	23.2772	0.1039	16.2874	-0.0099	12.1562

$1.650 \leq \hat{\alpha}_3 \leq 1.700$

	$\hat{\alpha}_3 = 1.650$		$\hat{\alpha}_3 = 1.675$		$\hat{\alpha}_3 = 1.700$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
3.95	-0.9606	-0.8216	-0.9763	-0.8908	-0.9906	-0.9562
4.00	-0.9513	-0.7788	-0.9676	-0.8507	-0.9826	-0.9185
4.05	-0.9417	-0.7344	-0.9587	-0.8092	-0.9743	-0.8796
4.10	-0.9319	-0.6885	-0.9496	-0.7663	-0.9658	-0.8394
4.15	-0.9219	-0.6410	-0.9402	-0.7220	-0.9571	-0.7980
4.20	-0.9116	-0.5917	-0.9306	-0.6761	-0.9482	-0.7551
4.25	-0.9010	-0.5405	-0.9208	-0.6286	-0.9391	-0.7109
4.30	-0.8903	-0.4874	-0.9108	-0.5793	-0.9297	-0.6650
4.35	-0.8792	-0.4322	-0.9005	-0.5282	-0.9201	-0.6176
4.40	-0.8679	-0.3748	-0.8899	-0.4751	-0.9103	-0.5684
4.45	-0.8563	-0.3150	-0.8792	-0.4200	-0.9003	-0.5174
4.50	-0.8445	-0.2527	-0.8681	-0.3627	-0.8900	-0.4645
4.55	-0.8324	-0.1878	-0.8569	-0.3031	-0.8795	-0.4096
4.60	-0.8200	-0.1201	-0.8454	-0.2410	-0.8687	-0.3525
4.65	-0.8074	-0.0493	-0.8336	-0.1763	-0.8578	-0.2931
4.70	-0.7944	0.0246	-0.8215	-0.1088	-0.8466	-0.2312
4.75	-0.7812	0.1020	-0.8092	-0.0384	-0.8351	-0.1668
4.80	-0.7676	0.1830	-0.7966	0.0352	-0.8234	-0.0997
4.85	-0.7537	0.2680	-0.7837	0.1122	-0.8114	-0.0296
4.90	-0.7396	0.3572	-0.7706	0.1928	-0.7992	0.0435
4.95	-0.7250	0.4509	-0.7571	0.2772	-0.7867	0.1200
5.00	-0.7102	0.5494	-0.7434	0.3658	-0.7739	0.2000
5.05	-0.6950	0.6532	-0.7293	0.4588	-0.7608	0.2838
5.10	-0.6794	0.7626	-0.7149	0.5566	-0.7475	0.3717
5.15	-0.6635	0.8781	-0.7002	0.6595	-0.7339	0.4638
5.20	-0.6472	1.0002	-0.6851	0.7679	-0.7199	0.5607
5.25	-0.6305	1.1295	-0.6697	0.8823	-0.7057	0.6625
5.30	-0.6134	1.2666	-0.6540	1.0031	-0.6911	0.7697
5.35	-0.5959	1.4121	-0.6378	1.1309	-0.6762	0.8828
5.40	-0.5780	1.5669	-0.6213	1.2662	-0.6610	1.0021
5.45	-0.5596	1.7317	-0.6044	1.4098	-0.6454	1.1282
5.50	-0.5407	1.9078	-0.5871	1.5624	-0.6295	1.2617
5.55	-0.5214	2.0960	-0.5694	1.7249	-0.6132	1.4031
5.60	-0.5016	2.2977	-0.5513	1.8981	-0.5965	1.5533
5.65	-0.4812	2.5144	-0.5327	2.0831	-0.5795	1.7130
5.70	-0.4604	2.7476	-0.5136	2.2812	-0.5620	1.8831
5.75	-0.4389	2.9994	-0.4940	2.4936	-0.5441	2.0646
5.80	-0.4169	3.2718	-0.4740	2.7221	-0.5258	2.2587
5.85	-0.3943	3.5675	-0.4534	2.9683	-0.5070	2.4666
5.90	-0.3710	3.8895	-0.4323	3.2343	-0.4878	2.6899
5.95	-0.3471	4.2413	-0.4107	3.5227	-0.4681	2.9302
6.00	-0.3225	4.6270	-0.3884	3.8361	-0.4478	3.1894
6.05	-0.2972	5.0517	-0.3656	4.1778	-0.4271	3.4699
6.10	-0.2711	5.5214	-0.3421	4.5518	-0.4058	3.7743
6.15	-0.2443	6.0435	-0.3179	4.9628	-0.3840	4.1055
6.20	-0.2166	6.6268	-0.2931	5.4161	-0.3616	4.4673
6.25	-0.1881	7.2825	-0.2676	5.9187	-0.3386	4.8639
6.30	-0.1586	8.0247	-0.2412	6.4786	-0.3149	5.3004
6.35	-0.1283	8.8710	-0.2142	7.1060	-0.2906	5.7830
6.40	-0.0969	9.8446	-0.1862	7.8136	-0.2656	6.3190
6.45	-0.0644	10.9757	-0.1574	8.6173	-0.2398	6.9178
6.50	-0.0309	12.3050	-0.1277	9.5375	-0.2134	7.5906
6.60	0.0398	15.8052	-0.0654	11.8436	-0.1580	9.2192
6.71	0.1236	21.8563	0.0080	15.4729	-0.0931	11.6291
6.83	0.2234	34.5155	0.0947	21.8805	-0.0169	15.4823

$1.725 \leq \hat{\alpha}_3 \leq 1.775$

	$\hat{\alpha}_3 = 1.725$		$\hat{\alpha}_3 = 1.750$		$\hat{\alpha}_3 = 1.775$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
4.22	-0.9610	-0.8129	-0.9761	-0.8838	-0.9900	-0.9506
4.27	-0.9524	-0.7710	-0.9681	-0.8446	-0.9826	-0.9138
4.32	-0.9436	-0.7277	-0.9600	-0.8041	-0.9750	-0.8758
4.37	-0.9346	-0.6829	-0.9516	-0.7623	-0.9672	-0.8367
4.42	-0.9254	-0.6366	-0.9430	-0.7191	-0.9592	-0.7964
4.47	-0.9159	-0.5886	-0.9342	-0.6745	-0.9510	-0.7547
4.52	-0.9063	-0.5389	-0.9252	-0.6283	-0.9426	-0.7118
4.57	-0.8964	-0.4873	-0.9160	-0.5805	-0.9340	-0.6673
4.62	-0.8863	-0.4338	-0.9065	-0.5310	-0.9252	-0.6214
4.67	-0.8760	-0.3783	-0.8969	-0.4797	-0.9163	-0.5739
4.72	-0.8654	-0.3206	-0.8871	-0.4265	-0.9071	-0.5246
4.77	-0.8547	-0.2605	-0.8771	-0.3713	-0.8977	-0.4736
4.82	-0.8437	-0.1981	-0.8668	-0.3139	-0.8882	-0.4208
4.87	-0.8324	-0.1330	-0.8563	-0.2543	-0.8784	-0.3659
4.92	-0.8209	-0.0652	-0.8456	-0.1922	-0.8684	-0.3090
4.97	-0.8092	0.0056	-0.8347	-0.1277	-0.8583	-0.2498
5.02	-0.7972	0.0795	-0.8236	-0.0604	-0.8479	-0.1883
5.07	-0.7850	0.1567	-0.8122	0.0098	-0.8373	-0.1243
5.12	-0.7725	0.2375	-0.8006	0.0830	-0.8264	-0.0576
5.17	-0.7597	0.3221	-0.7887	0.1595	-0.8154	0.0119
5.22	-0.7467	0.4107	-0.7766	0.2394	-0.8041	0.0843
5.27	-0.7334	0.5037	-0.7642	0.3231	-0.7926	0.1600
5.32	-0.7198	0.6014	-0.7516	0.4107	-0.7809	0.2390
5.37	-0.7059	0.7041	-0.7387	0.5026	-0.7689	0.3217
5.42	-0.6917	0.8122	-0.7255	0.5990	-0.7567	0.4082
5.47	-0.6772	0.9261	-0.7121	0.7003	-0.7442	0.4988
5.52	-0.6623	1.0463	-0.6984	0.8069	-0.7315	0.5938
5.57	-0.6472	1.1733	-0.6844	0.9191	-0.7185	0.6936
5.62	-0.6317	1.3077	-0.6700	1.0374	-0.7053	0.7985
5.67	-0.6158	1.4501	-0.6554	1.1622	-0.6918	0.9088
5.72	-0.5996	1.6012	-0.6405	1.2943	-0.6779	1.0250
5.77	-0.5830	1.7618	-0.6252	1.4340	-0.6638	1.1476
5.82	-0.5661	1.9329	-0.6096	1.5821	-0.6495	1.2771
5.87	-0.5487	2.1153	-0.5937	1.7394	-0.6347	1.4140
5.92	-0.5309	2.3102	-0.5774	1.9067	-0.6197	1.5590
5.97	-0.5127	2.5190	-0.5607	2.0849	-0.6044	1.7128
6.02	-0.4941	2.7431	-0.5436	2.2751	-0.5887	1.8761
6.07	-0.4750	2.9841	-0.5262	2.4786	-0.5727	2.0500
6.12	-0.4555	3.2440	-0.5083	2.6966	-0.5563	2.2352
6.17	-0.4354	3.5251	-0.4900	2.9308	-0.5396	2.4331
6.22	-0.4149	3.8298	-0.4713	3.1829	-0.5225	2.6449
6.27	-0.3938	4.1613	-0.4521	3.4550	-0.5050	2.8720
6.32	-0.3722	4.5231	-0.4325	3.7495	-0.4871	3.1161
6.37	-0.3500	4.9193	-0.4124	4.0692	-0.4687	3.3791
6.42	-0.3273	5.3551	-0.3918	4.4173	-0.4500	3.6632
6.47	-0.3039	5.8364	-0.3706	4.7978	-0.4308	3.9709
6.52	-0.2798	6.3706	-0.3490	5.2151	-0.4111	4.3054
6.57	-0.2551	6.9666	-0.3267	5.6747	-0.3909	4.6699
6.62	-0.2297	7.6354	-0.3039	6.1832	-0.3703	5.0688
6.67	-0.2036	8.3911	-0.2804	6.7486	-0.3491	5.5069
6.72	-0.1767	9.2511	-0.2563	7.3808	-0.3274	5.9901
6.82	-0.1205	11.3823	-0.2061	8.8976	-0.2823	7.1219
6.92	-0.0607	14.3149	-0.1531	10.8766	-0.2348	8.5435
7.04	0.0161	19.6946	-0.0851	14.2160	-0.1743	10.8112
7.16	0.0993	29.3837	-0.0121	19.4357	-0.1096	14.0621

$1.800 \leq \hat{\alpha}_3 \leq 1.850$

	$\hat{\alpha}_3 = 1.800$		$\hat{\alpha}_3 = 1.825$		$\hat{\alpha}_3 = 1.850$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
4.48	-0.9649	-0.8221	-0.9793	-0.8933	-0.9924	-0.9604
4.54	-0.9555	-0.7734	-0.9705	-0.8478	-0.9843	-0.9176
4.60	-0.9458	-0.7229	-0.9615	-0.8006	-0.9759	-0.8734
4.66	-0.9358	-0.6704	-0.9522	-0.7516	-0.9672	-0.8276
4.72	-0.9256	-0.6158	-0.9426	-0.7008	-0.9584	-0.7802
4.78	-0.9151	-0.5589	-0.9329	-0.6479	-0.9492	-0.7310
4.84	-0.9043	-0.4997	-0.9228	-0.5930	-0.9399	-0.6799
4.90	-0.8933	-0.4378	-0.9125	-0.5358	-0.9303	-0.6268
4.96	-0.8819	-0.3732	-0.9019	-0.4761	-0.9204	-0.5715
5.02	-0.8703	-0.3057	-0.8911	-0.4139	-0.9103	-0.5140
5.08	-0.8585	-0.2351	-0.8800	-0.3490	-0.9000	-0.4541
5.14	-0.8463	-0.1611	-0.8687	-0.2811	-0.8893	-0.3917
5.20	-0.8338	-0.0835	-0.8570	-0.2101	-0.8785	-0.3264
5.26	-0.8210	-0.0020	-0.8451	-0.1358	-0.8674	-0.2583
5.32	-0.8079	0.0835	-0.8329	-0.0579	-0.8560	-0.1871
5.38	-0.7946	0.1736	-0.8205	0.0239	-0.8444	-0.1125
5.44	-0.7808	0.2684	-0.8077	0.1098	-0.8325	-0.0344
5.50	-0.7668	0.3685	-0.7946	0.2001	-0.8203	0.0476
5.56	-0.7524	0.4742	-0.7812	0.2953	-0.8078	0.1337
5.62	-0.7377	0.5859	-0.7676	0.3956	-0.7951	0.2242
5.68	-0.7226	0.7043	-0.7535	0.5015	-0.7821	0.3195
5.74	-0.7072	0.8298	-0.7392	0.6134	-0.7687	0.4199
5.80	-0.6913	0.9632	-0.7245	0.7319	-0.7551	0.5258
5.86	-0.6751	1.1051	-0.7095	0.8576	-0.7411	0.6378
5.92	-0.6585	1.2565	-0.6941	0.9910	-0.7269	0.7562
5.98	-0.6415	1.4181	-0.6784	1.1329	-0.7123	0.8818
6.04	-0.6241	1.5911	-0.6623	1.2841	-0.6973	1.0150
6.10	-0.6062	1.7766	-0.6458	1.4455	-0.6821	1.1566
6.16	-0.5878	1.9761	-0.6288	1.6181	-0.6664	1.3073
6.22	-0.5690	2.1910	-0.6115	1.8031	-0.6504	1.4682
6.28	-0.5497	2.4231	-0.5938	2.0019	-0.6340	1.6401
6.34	-0.5299	2.6746	-0.5756	2.2159	-0.6172	1.8242
6.40	-0.5096	2.9479	-0.5569	2.4468	-0.6001	2.0218
6.46	-0.4887	3.2458	-0.5377	2.6969	-0.5825	2.2344
6.52	-0.4672	3.5717	-0.5181	2.9682	-0.5644	2.4637
6.58	-0.4452	3.9295	-0.4980	3.2638	-0.5459	2.7117
6.64	-0.4225	4.3241	-0.4773	3.5867	-0.5270	2.9806
6.70	-0.3992	4.7613	-0.4560	3.9410	-0.5076	3.2730
6.76	-0.3752	5.2479	-0.4342	4.3311	-0.4876	3.5923
6.82	-0.3505	5.7928	-0.4118	4.7626	-0.4672	3.9420
6.88	-0.3251	6.4067	-0.3888	5.2423	-0.4462	4.3266
6.94	-0.2989	7.1032	-0.3651	5.7785	-0.4247	4.7514
7.00	-0.2719	7.8998	-0.3407	6.3815	-0.4026	5.2228
7.06	-0.2440	8.8190	-0.3156	7.0642	-0.3798	5.7489
7.12	-0.2153	9.8910	-0.2898	7.8431	-0.3565	6.3392
7.18	-0.1856	11.1562	-0.2632	8.7396	-0.3325	7.0061
7.24	-0.1550	12.6711	-0.2357	9.7818	-0.3077	7.7651
7.30	-0.1233	14.5162	-0.2074	11.0076	-0.2823	8.6362
7.36	-0.0905	16.8105	-0.1783	12.4692	-0.2561	9.6455
7.42	-0.0566	19.7384	-0.1481	14.2404	-0.2292	10.8281
7.48	-0.0215	23.6004	-0.1170	16.4295	-0.2014	12.2320
7.54	0.0150	28.9225	-0.0848	19.2020	-0.1727	13.9243
7.60	0.0528	36.7165	-0.0515	22.8234	-0.1431	16.0026
7.66	0.0920	49.2080	-0.0170	27.7491	-0.1126	18.6139
7.72	0.1328	72.4385	0.0187	34.8307	-0.0811	21.9904

$1.875 \leq \hat{\alpha}_3 \leq 1.925$

	$\hat{\alpha}_3 = 1.875$		$\hat{\alpha}_3 = 1.900$		$\hat{\alpha}_3 = 1.925$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
4.75	-0.9686	-0.8318	-0.9822	-0.9032	-0.9946	-0.9704
4.81	-0.9600	-0.7850	-0.9742	-0.8594	-0.9872	-0.9293
4.87	-0.9512	-0.7365	-0.9660	-0.8140	-0.9796	-0.8867
4.93	-0.9421	-0.6861	-0.9575	-0.7670	-0.9717	-0.8428
4.99	-0.9328	-0.6338	-0.9488	-0.7183	-0.9636	-0.7973
5.05	-0.9232	-0.5794	-0.9399	-0.6678	-0.9553	-0.7502
5.11	-0.9135	-0.5229	-0.9308	-0.6153	-0.9468	-0.7014
5.17	-0.9034	-0.4640	-0.9214	-0.5608	-0.9381	-0.6508
5.23	-0.8932	-0.4026	-0.9119	-0.5041	-0.9291	-0.5982
5.29	-0.8827	-0.3386	-0.9021	-0.4451	-0.9200	-0.5436
5.35	-0.8720	-0.2718	-0.8920	-0.3836	-0.9106	-0.4869
5.41	-0.8610	-0.2020	-0.8818	-0.3195	-0.9010	-0.4278
5.47	-0.8497	-0.1290	-0.8713	-0.2526	-0.8912	-0.3663
5.53	-0.8382	-0.0525	-0.8606	-0.1828	-0.8812	-0.3021
5.59	-0.8265	0.0276	-0.8496	-0.1097	-0.8710	-0.2352
5.65	-0.8145	0.1117	-0.8384	-0.0332	-0.8605	-0.1654
5.71	-0.8022	0.2000	-0.8269	0.0469	-0.8498	-0.0924
5.77	-0.7897	0.2928	-0.8152	0.1309	-0.8389	-0.0160
5.83	-0.7768	0.3906	-0.8033	0.2191	-0.8277	0.0640
5.89	-0.7637	0.4936	-0.7911	0.3119	-0.8163	0.1479
5.95	-0.7503	0.6023	-0.7786	0.4094	-0.8047	0.2359
6.01	-0.7366	0.7172	-0.7658	0.5122	-0.7928	0.3283
6.07	-0.7226	0.8389	-0.7528	0.6206	-0.7807	0.4255
6.13	-0.7082	0.9677	-0.7395	0.7351	-0.7683	0.5279
6.19	-0.6936	1.1045	-0.7259	0.8562	-0.7556	0.6358
6.25	-0.6786	1.2500	-0.7119	0.9845	-0.7427	0.7497
6.31	-0.6632	1.4049	-0.6977	1.1205	-0.7295	0.8701
6.37	-0.6475	1.5702	-0.6832	1.2651	-0.7160	0.9976
6.43	-0.6315	1.7469	-0.6683	1.4190	-0.7022	1.1327
6.49	-0.6151	1.9363	-0.6531	1.5830	-0.6881	1.2761
6.55	-0.5983	2.1395	-0.6376	1.7583	-0.6737	1.4287
6.61	-0.5810	2.3582	-0.6217	1.9458	-0.6590	1.5912
6.67	-0.5634	2.5942	-0.6055	2.1470	-0.6440	1.7647
6.73	-0.5454	2.8495	-0.5889	2.3634	-0.6287	1.9502
6.79	-0.5269	3.1264	-0.5719	2.5965	-0.6130	2.1489
6.85	-0.5079	3.4278	-0.5545	2.8484	-0.5969	2.3624
6.91	-0.4885	3.7569	-0.5366	3.1214	-0.5805	2.5923
6.97	-0.4686	4.1176	-0.5184	3.4180	-0.5637	2.8404
7.03	-0.4482	4.5145	-0.4997	3.7416	-0.5466	3.1088
7.09	-0.4272	4.9532	-0.4805	4.0956	-0.5290	3.4003
7.15	-0.4057	5.4405	-0.4609	4.4846	-0.5110	3.7176
7.21	-0.3836	5.9845	-0.4408	4.9138	-0.4926	4.0644
7.27	-0.3609	6.5957	-0.4202	5.3896	-0.4738	4.4447
7.33	-0.3376	7.2869	-0.3990	5.9198	-0.4545	4.8635
7.39	-0.3137	8.0745	-0.3773	6.5139	-0.4347	5.3269
7.45	-0.2890	8.9797	-0.3550	7.1840	-0.4144	5.8421
7.51	-0.2637	10.0304	-0.3321	7.9454	-0.3937	6.4181
7.57	-0.2376	11.2638	-0.3086	8.8175	-0.3723	7.0660
7.63	-0.2108	12.7314	-0.2845	9.8259	-0.3505	7.7998
7.69	-0.1831	14.5055	-0.2596	11.0044	-0.3280	8.6375
7.75	-0.1546	16.6919	-0.2341	12.3995	-0.3050	9.6023
7.83	-0.1151	20.5380	-0.1988	14.7116	-0.2733	11.1407
7.97	-0.0420	32.1944	-0.1337	20.7854	-0.2150	14.8576
8.11	0.-370	63.9400	-0.0639	32.4901	-0.1527	20.9276
8.25	0.1227	475.8773	0.0114	64.1704	-0.0859	32.5601

$1.950 \leq \hat{\alpha}_3 \leq 2.000$

	$\hat{\alpha}_3 = 1.950$		$\hat{\alpha}_3 = 1.975$		$\hat{\alpha}_3 = 2.000$	
$\hat{\alpha}_4$	β_3	β_4	β_3	β_4	β_3	β_4
5.08	-0.9655	-0.8047	-0.9788	-0.8785	-0.9910	-0.9477
5.15	-0.9562	-0.7505	-0.9701	-0.8278	-0.9829	-0.9002
5.22	-0.9465	-0.6940	-0.9611	-0.7750	-0.9745	-0.8509
5.29	-0.9365	-0.6350	-0.9518	-0.7201	-0.9659	-0.7996
5.36	-0.9263	-0.5735	-0.9423	-0.6630	-0.9570	-0.7463
5.43	-0.9158	-0.5092	-0.9325	-0.6033	-0.9478	-0.6908
5.50	-0.9050	-0.4418	-0.9224	-0.5410	-0.9384	-0.6330
5.57	-0.8940	-0.3713	-0.9121	-0.4759	-0.9288	-0.5727
5.64	-0.8826	-0.2973	-0.9014	-0.4078	-0.9189	-0.5097
5.71	-0.8710	-0.2196	-0.8906	-0.3364	-0.9087	-0.4439
5.78	-0.8591	-0.1379	-0.8794	-0.2615	-0.8983	-0.3750
5.85	-0.8468	-0.0520	-0.8680	-0.1829	-0.8876	-0.3029
5.92	-0.8343	0.0386	-0.8563	-0.1003	-0.8766	-0.2273
5.99	-0.8215	0.1342	-0.8443	-0.0134	-0.8654	-0.1479
6.06	-0.8083	0.2353	-0.8320	0.0782	-0.8540	-0.0644
6.13	-0.7949	0.3422	-0.8195	0.1749	-0.8422	0.0234
6.20	-0.7811	0.4556	-0.8066	0.2771	-0.8302	0.1159
6.27	-0.7670	0.5759	-0.7934	0.3851	-0.8179	0.2135
6.34	-0.7525	0.7038	-0.7799	0.4997	-0.8053	0.3166
6.41	-0.7377	0.8401	-0.7661	0.6212	-0.7924	0.4256
6.48	-0.7225	0.9856	-0.7520	0.7504	-0.7792	0.5411
6.55	-0.7069	1.1411	-0.7375	0.8880	-0.7657	0.6637
6.62	-0.6910	1.3077	-0.7227	1.0348	-0.7520	0.7940
6.69	-0.6747	1.4866	-0.7076	1.1917	-0.7378	0.9327
6.76	-0.6579	1.6792	-0.6920	1.3598	-0.7234	1.0806
6.83	-0.6408	1.8870	-0.6761	1.5402	-0.7086	1.2386
6.90	-0.6232	2.1118	-0.6598	1.7343	-0.6935	1.4078
6.97	-0.6052	2.3557	-0.6432	1.9437	-0.6780	1.5894
7.04	-0.5867	2.6213	-0.6261	2.1702	-0.6622	1.7847
7.11	-0.5677	2.9113	-0.6086	2.4158	-0.6460	1.9952
7.18	-0.5483	3.2292	-0.5906	2.6830	-0.6294	2.2228
7.25	-0.5283	3.5790	-0.5722	2.9747	-0.6124	2.4695
7.32	-0.5078	3.9658	-0.5533	3.2943	-0.5950	2.7378
7.39	-0.4867	4.3954	-0.5340	3.6459	-0.5771	3.0306
7.46	-0.4651	4.8751	-0.5141	4.0342	-0.5589	3.3511
7.53	-0.4428	5.4141	-0.4938	4.4654	-0.5401	3.7034
7.60	-0.4199	6.0235	-0.4728	4.9465	-0.5209	4.0924
7.67	-0.3964	6.7180	-0.4513	5.4865	-0.5012	4.5238
7.74	-0.3722	7.5161	-0.4293	6.0968	-0.4810	5.0048
7.81	-0.3472	8.4423	-0.4066	6.7915	-0.4603	5.5443
7.88	-0.3215	9.5294	-0.3833	7.5889	-0.4390	6.1533
7.95	-0.2951	10.8224	-0.3593	8.5133	-0.4171	6.8457
8.02	-0.2678	12.3848	-0.3346	9.5967	-0.3947	7.6396
8.09	-0.2396	14.3089	-0.3091	10.8833	-0.3716	8.5584
8.16	-0.2106	16.7347	-0.2830	12.4349	-0.3479	9.6336
8.23	-0.1806	19.8851	-0.2560	14.3415	-0.3235	10.9079
8.30	-0.1496	24.1372	-0.2282	16.7387	-0.2984	12.4412
8.37	-0.1175	30.1849	-0.1995	19.8408	-0.2725	14.3201
8.44	-0.0843	39.4587	-0.1699	24.0086	-0.2459	16.6745
8.51	-0.0499	55.4564	-0.1393	29.8991	-0.2185	19.7083
8.58	-0.0143	89.5900	-0.1076	38.8481	-0.1902	23.7616
8.65	0.0226	212.7215	-0.0749	54.0532	-0.1610	29.4467
8.72			-0.0411	85.5496	-0.1309	37.9880
8.79			-0.0061	189.4380	-0.0998	52.2426
8.86					-0.0676	80.7819

Appendix D

Tables D–1 through D–5 for GLD Fits: Method of Percentiles

For specified ρ_3 and ρ_4, Tables D–1 through D–5 give the λ_3 and λ_4 values for GLD fits in the regions designated by "T1," "T2," "T3," "T4," and "T5," respectively, in Figure 5.3–1.

The entries for Tables D–1, D–2, and D–5 are from "Fitting the Generalized Lambda Distribution (GLD) System by a Method of Percentiles, II: Tables" by E.J. Dudewicz and Z.A. Karian, *American Journal of Mathematical and Management Sciences*, Vol. 19, 1 & 2 (1999), pp. 1–73, copyright ©1999 by American Sciences Press, Inc., 20 Cross Road, Syracuse, New York 13224-2104. Reprinted with permission.

Table D–1: $0.005 \leq \hat{\rho}_4 \leq 0.008$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.005$		$\hat{\rho}_4 = 0.006$		$\hat{\rho}_4 = 0.007$		$\hat{\rho}_4 = 0.008$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	28.9513	66.0812	27.9517	65.0816	27.1063	64.2362	26.3739	63.5039
.04	28.8455	59.3966	27.8459	58.3970	27.0005	57.5516	26.2681	56.8192
.06	28.7427	55.4454	27.7431	54.4458	26.8977	53.6004	26.1653	52.8681
.08	28.6432	52.6155	27.6436	51.6159	26.7982	50.7705	26.0658	50.0381
.10	28.5472	50.4016	27.5476	49.4020	26.7022	48.5566	25.9698	47.8242
.12	28.4549	48.5788	27.4553	47.5792	26.6099	46.7339	25.8776	46.0015
.14	28.3666	47.0274	27.3670	46.0278	26.5216	45.1824	25.7892	44.4501
.16	28.2824	45.6758	27.2828	44.6762	26.4374	43.8309	25.7050	43.0985
.18	28.2025	44.4780	27.2029	43.4784	26.3575	42.6330	25.6251	41.9007
.20	28.1271	43.4026	27.1275	42.4030	26.2821	41.5576	25.5497	40.8252
.22	28.0562	42.4272	27.0566	41.4276	26.2113	40.5822	25.4789	39.8498
.24	27.9902	41.5353	26.9906	40.5357	26.1452	39.6903	25.4128	38.9579
.26	27.9290	40.7144	26.9294	39.7148	26.0840	38.8694	25.3517	38.1371
.28	27.8729	39.9549	26.8733	38.9553	26.0279	38.1099	25.2955	37.3775
.30	27.8219	39.2490	26.8223	38.2495	25.9769	37.4041	25.2445	36.6717
.32	27.7761	38.5907	26.7765	37.5911	25.9311	36.7457	25.1987	36.0133
.34	27.7355	37.9748	26.7359	36.9752	25.8905	36.1298	25.1582	35.3974
.36	27.7003	37.3971	26.7007	36.3975	25.8554	35.5521	25.1230	34.8197
.38	27.6706	36.8541	26.6710	35.8545	25.8256	35.0091	25.0932	34.2768
.40	27.6462	36.3429	26.6466	35.3433	25.8012	34.4979	25.0688	33.7656
.42	27.6273	35.8609	26.6277	34.8613	25.7823	34.0159	25.0499	33.2835
.44	27.6138	35.4059	26.6142	34.4063	25.7688	33.5609	25.0364	32.8285
.46	27.6057	34.9759	26.6061	33.9763	25.7607	33.1309	25.0283	32.3985
.48	27.6030	34.5693	26.6034	33.5697	25.7580	32.7243	25.0256	31.9919
.50	27.6057	34.1845	26.6061	33.1849	25.7607	32.3395	25.0283	31.6071
.52	27.6136	33.8202	26.6140	32.8206	25.7686	31.9752	25.0362	31.2428
.54	27.6268	33.4751	26.6272	32.4755	25.7818	31.6301	25.0494	30.8978
.56	27.6450	33.1482	26.6454	32.1486	25.8000	31.3032	25.0677	30.5709
.58	27.6683	32.8384	26.6687	31.8388	25.8233	30.9934	25.0909	30.2611
.60	27.6965	32.5449	26.6969	31.5453	25.8515	30.6999	25.1191	29.9675
.62	27.7295	32.2666	26.7299	31.2670	25.8845	30.4216	25.1521	29.6893
.64	27.7671	32.0029	26.7675	31.0033	25.9221	30.1579	25.1897	29.4256
.66	27.8092	31.7530	26.8096	30.7534	25.9642	29.9080	25.2319	29.1756
.68	27.8557	31.5161	26.8561	30.5165	26.0107	29.6712	25.2784	28.9388
.70	27.9064	31.2917	26.9068	30.2921	26.0614	29.4467	25.3291	28.7144
.72	27.9612	31.0791	26.9616	30.0795	26.1162	29.2341	25.3838	28.5017
.74	28.0198	30.8777	27.0202	29.8781	26.1748	29.0327	25.4425	28.3003
.76	28.0822	30.6869	27.0826	29.6873	26.2372	28.8420	25.5048	28.1096
.78	28.1481	30.5063	27.1485	29.5067	26.3031	28.6613	25.5708	27.9290
.80	28.2174	30.3353	27.2178	29.3357	26.3724	28.4903	25.6400	27.7580
.82	28.2899	30.1735	27.2903	29.1739	26.4449	28.3285	25.7126	27.5961
.84	28.3655	30.0203	27.3659	29.0207	26.5205	28.1753	25.7881	27.4429
.86	28.4439	29.8754	27.4443	28.8758	26.5989	28.0304	25.8665	27.2980
.88	28.5250	29.7383	27.5254	28.7387	26.6800	27.8933	25.9477	27.1610
.90	28.6087	29.6087	27.6091	28.6091	26.7637	27.7637	26.0313	27.0313
.92	28.6948	29.4862	27.6952	28.4866	26.8498	27.6412	26.1174	26.9088
.94	28.7831	29.3703	27.7835	28.3707	26.9381	27.5253	26.2057	26.7930
.96	28.8734	29.2609	27.8738	28.2613	27.0284	27.4159	26.2960	26.6835
.98	28.9657	29.1574	27.9661	28.1578	27.1207	27.3125	26.3883	26.5801
1.00	29.0598	29.0598	28.0602	28.0602	27.2148	27.2148	26.4824	26.4824

Table D–1: $0.009 \leq \hat{\rho}_4 \leq 0.012$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.009$		$\hat{\rho}_4 = 0.010$		$\hat{\rho}_4 = 0.011$		$\hat{\rho}_4 = 0.012$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	25.7279	62.8579	25.1500	62.2801	24.6273	61.7574	24.1500	61.2802
.04	25.6221	56.1732	25.0442	55.5954	24.5215	55.0727	24.0442	54.5955
.06	25.5193	52.2221	24.9414	51.6442	24.4187	51.1215	23.9414	50.6443
.08	25.4198	49.3921	24.8419	48.8142	24.3192	48.2915	23.8419	47.8143
.10	25.3238	47.1782	24.7459	46.6003	24.2232	46.0776	23.7459	45.6004
.12	25.2316	45.3555	24.6537	44.7776	24.1309	44.2549	23.6537	43.7777
.14	25.1432	43.8041	24.5654	43.2262	24.0426	42.7035	23.5654	42.2262
.16	25.0590	42.4525	24.4811	41.8746	23.9584	41.3519	23.4811	40.8747
.18	24.9791	41.2547	24.4012	40.6768	23.8785	40.1541	23.4012	39.6768
.20	24.9037	40.1792	24.3258	39.6014	23.8030	39.0786	23.3258	38.6014
.22	24.8329	39.2038	24.2550	38.6259	23.7322	38.1032	23.2550	37.6260
.24	24.7668	38.3119	24.1889	37.7340	23.6662	37.2113	23.1889	36.7341
.26	24.7057	37.4911	24.1278	36.9132	23.6050	36.3904	23.1278	35.9132
.28	24.6495	36.7315	24.0716	36.1537	23.5489	35.6309	23.0716	35.1537
.30	24.5985	36.0257	24.0206	35.4478	23.4979	34.9251	23.0206	34.4478
.32	24.5527	35.3673	23.9748	34.7895	23.4520	34.2667	22.9748	33.7895
.34	24.5122	34.7514	23.9343	34.1735	23.4115	33.6508	22.9343	33.1735
.36	24.4770	34.1737	23.8991	33.5958	23.3763	33.0731	22.8991	32.5959
.38	24.4472	33.6308	23.8693	33.0529	23.3465	32.5301	22.8693	32.0529
.40	24.4228	33.1195	23.8449	32.5417	23.3222	32.0189	22.8449	31.5417
.42	24.4039	32.6375	23.8260	32.0597	23.3032	31.5369	22.8260	31.0597
.44	24.3904	32.1825	23.8125	31.6046	23.2897	31.0819	22.8125	30.6047
.46	24.3823	31.7525	23.8044	31.1747	23.2817	30.6519	22.8044	30.1747
.48	24.3796	31.3459	23.8017	30.7680	23.2790	30.2453	22.8017	29.7681
.50	24.3823	30.9611	23.8044	30.3832	23.2816	29.8605	22.8044	29.3833
.52	24.3902	30.5968	23.8123	30.0189	23.2896	29.4962	22.8123	29.0190
.54	24.4034	30.2518	23.8255	29.6739	23.3027	29.1511	22.8255	28.6739
.56	24.4216	29.9248	23.8438	29.3470	23.3210	28.8242	22.8438	28.3470
.58	24.4449	29.6151	23.8670	29.0372	23.3443	28.5144	22.8670	28.0372
.60	24.4731	29.3215	23.8952	28.7436	23.3725	28.2209	22.8952	27.7436
.62	24.5061	29.0433	23.9282	28.4654	23.4055	27.9426	22.9282	27.4654
.64	24.5437	28.7795	23.9658	28.2017	23.4431	27.6789	22.9658	27.2017
.66	24.5859	28.5296	24.0080	27.9517	23.4852	27.4290	23.0080	26.9518
.68	24.6324	28.2928	24.0545	27.7149	23.5317	27.1921	23.0545	26.7149
.70	24.6831	28.0684	24.1052	27.4905	23.5824	26.9677	23.1052	26.4905
.72	24.7378	27.8557	24.1599	27.2779	23.6372	26.7551	23.1599	26.2779
.74	24.7965	27.6543	24.2186	27.0764	23.6958	26.5537	23.2186	26.0765
.76	24.8588	27.4636	24.2809	26.8857	23.7582	26.3629	23.2809	25.8857
.78	24.9247	27.2829	24.3469	26.7051	23.8241	26.1823	23.3469	25.7051
.80	24.9940	27.1119	24.4162	26.5341	23.8934	26.0113	23.4162	25.5341
.82	25.0665	26.9501	24.4887	26.3722	23.9659	25.8495	23.4887	25.3722
.84	25.1421	26.7969	24.5642	26.2190	24.0415	25.6963	23.5642	25.2191
.86	25.2205	26.6520	24.6426	26.0741	24.1199	25.5514	23.6426	25.0741
.88	25.3016	26.5149	24.7238	25.9371	24.2010	25.4143	23.7238	24.9371
.90	25.3853	26.3853	24.8074	25.8074	24.2847	25.2847	23.8074	24.8075
.92	25.4714	26.2628	24.8935	25.6849	24.3707	25.1621	23.8935	24.6849
.94	25.5597	26.1469	24.9818	25.5691	24.4590	25.0463	23.9818	24.5691
.96	25.6500	26.0375	25.0722	25.4596	24.5494	24.9368	24.0722	24.4596
.98	25.7423	25.9341	25.1644	25.3562	24.6417	24.8334	24.1644	24.3562
1.00	25.8364	25.8364	25.2585	25.2585	24.7357	24.7357	24.2585	24.2585

Table D–1: $0.0135 \leq \hat{\rho}_4 \leq 0.0180$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.0135$		$\hat{\rho}_4 = 0.0150$		$\hat{\rho}_4 = 0.0165$		$\hat{\rho}_4 = 0.0180$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	23.5038	60.6342	22.9260	60.0566	22.4033	59.5341	21.9261	59.0572
.04	23.3980	53.9493	22.8202	53.3717	22.2975	52.8491	21.8203	52.3721
.06	23.2952	49.9981	22.7175	49.4204	22.1948	48.8978	21.7175	48.4207
.08	23.1957	47.1681	22.6180	46.5904	22.0953	46.0678	21.6180	45.5907
.10	23.0997	44.9542	22.5220	44.3765	21.9993	43.8539	21.5220	43.3767
.12	23.0074	43.1314	22.4297	42.5538	21.9070	42.0311	21.4298	41.5540
.14	22.9191	41.5800	22.3414	41.0023	21.8187	40.4797	21.3415	40.0025
.16	22.8349	40.2284	22.2572	39.6507	21.7345	39.1281	21.2572	38.6509
.18	22.7550	39.0306	22.1773	38.4529	21.6546	37.9303	21.1773	37.4531
.20	22.6796	37.9552	22.1018	37.3775	21.5791	36.8548	21.1019	36.3777
.22	22.6087	36.9798	22.0310	36.4021	21.5083	35.8794	21.0311	35.4022
.24	22.5427	36.0879	21.9650	35.5102	21.4423	34.9875	20.9650	34.5103
.26	22.4815	35.2670	21.9038	34.6893	21.3811	34.1666	20.9039	33.6895
.28	22.4254	34.5075	21.8477	33.9298	21.3250	33.4071	20.8477	32.9300
.30	22.3744	33.8016	21.7966	33.2239	21.2739	32.7013	20.7967	32.2241
.32	22.3286	33.1433	21.7508	32.5656	21.2281	32.0429	20.7509	31.5657
.34	22.2880	32.5273	21.7103	31.9496	21.1876	31.4270	20.7104	30.9498
.36	22.2528	31.9496	21.6751	31.3719	21.1524	30.8493	20.6752	30.3721
.38	22.2230	31.4067	21.6453	30.8290	21.1226	30.3063	20.6454	29.8291
.40	22.1987	30.8955	21.6209	30.3178	21.0982	29.7951	20.6210	29.3179
.42	22.1797	30.4135	21.6020	29.8358	21.0793	29.3131	20.6021	28.8359
.44	22.1663	29.9584	21.5885	29.3807	21.0658	28.8581	20.5886	28.3809
.46	22.1582	29.5285	21.5804	28.9507	21.0577	28.4281	20.5805	27.9509
.48	22.1555	29.1218	21.5778	28.5441	21.0551	28.0214	20.5778	27.5443
.50	22.1582	28.7370	21.5804	28.1593	21.0577	27.6366	20.5805	27.1595
.52	22.1661	28.3727	21.5884	27.7950	21.0657	27.2723	20.5884	26.7951
.54	22.1792	28.0277	21.6015	27.4499	21.0788	26.9273	20.6016	26.4501
.56	22.1975	27.7007	21.6198	27.1230	21.0971	26.6004	20.6198	26.1232
.58	22.2208	27.3910	21.6431	26.8133	21.1204	26.2906	20.6431	25.8134
.60	22.2490	27.0974	21.6712	26.5197	21.1485	25.9970	20.6713	25.5198
.62	22.2820	26.8191	21.7042	26.2414	21.1815	25.7187	20.7043	25.2416
.64	22.3196	26.5554	21.7418	25.9777	21.2191	25.4550	20.7419	24.9778
.66	22.3617	26.3055	21.7840	25.7278	21.2613	25.2051	20.7840	24.7279
.68	22.4082	26.0687	21.8305	25.4909	21.3078	24.9683	20.8305	24.4911
.70	22.4589	25.8442	21.8812	25.2665	21.3585	24.7438	20.8813	24.2666
.72	22.5137	25.6316	21.9359	25.0539	21.4132	24.5312	20.9360	24.0540
.74	22.5723	25.4302	21.9946	24.8525	21.4719	24.3298	20.9947	23.8526
.76	22.6347	25.2395	22.0569	24.6617	21.5342	24.1390	21.0570	23.6618
.78	22.7006	25.0588	22.1229	24.4811	21.6002	23.9584	21.1229	23.4812
.80	22.7699	24.8878	22.1922	24.3101	21.6695	23.7874	21.1922	23.3102
.82	22.8424	24.7260	22.2647	24.1482	21.7420	23.6255	21.2647	23.1483
.84	22.9180	24.5728	22.3402	23.9951	21.8175	23.4724	21.3403	22.9952
.86	22.9964	24.4279	22.4186	23.8502	21.8959	23.3275	21.4187	22.8502
.88	23.0775	24.2908	22.4998	23.7131	21.9771	23.1904	21.4998	22.7132
.90	23.1612	24.1612	22.5835	23.5835	22.0607	23.0608	21.5835	22.5836
.92	23.2472	24.0386	22.6695	23.4609	22.1468	22.9382	21.6696	22.4610
.94	23.3355	23.9228	22.7578	23.3451	22.2351	22.8224	21.7579	22.3452
.96	23.4259	23.8134	22.8482	23.2356	22.3255	22.7129	21.8482	22.2357
.98	23.5182	23.7099	22.9404	23.1322	22.4177	22.6095	21.9405	22.1323
1.00	23.6122	23.6122	23.0345	23.0345	22.5118	22.5118	22.0346	22.0346

Table D–1: $0.0195 \leq \hat{\rho}_4 \leq 0.0240$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.0195$		$\hat{\rho}_4 = 0.0210$		$\hat{\rho}_4 = 0.0225$		$\hat{\rho}_4 = 0.0240$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	21.4871	58.6186	21.0807	58.2125	20.7022	57.8346	20.3483	57.4812
.04	21.3813	51.9332	20.9749	51.5270	20.5965	51.1489	20.2425	50.7952
.06	21.2785	47.9819	20.8721	47.5756	20.4937	47.1974	20.1397	46.8436
.08	21.1790	45.1518	20.7726	44.7455	20.3942	44.3672	20.0402	44.0134
.10	21.0830	42.9378	20.6766	42.5315	20.2982	42.1532	19.9442	41.7994
.12	20.9908	41.1151	20.5843	40.7087	20.2059	40.3304	19.8519	39.9766
.14	20.9024	39.5636	20.4960	39.1573	20.1176	38.7790	19.7636	38.4251
.16	20.8182	38.2120	20.4118	37.8056	20.0334	37.4273	19.6794	37.0735
.18	20.7383	37.0142	20.3319	36.6078	19.9535	36.2295	19.5995	35.8757
.20	20.6629	35.9387	20.2564	35.5324	19.8780	35.1540	19.5241	34.8002
.22	20.5921	34.9633	20.1856	34.5569	19.8072	34.1786	19.4532	33.8247
.24	20.5260	34.0714	20.1196	33.6650	19.7412	33.2867	19.3872	32.9328
.26	20.4649	33.2505	20.0584	32.8441	19.6800	32.4658	19.3260	32.1120
.28	20.4087	32.4910	20.0023	32.0846	19.6239	31.7063	19.2699	31.3524
.30	20.3577	31.7851	19.9513	31.3788	19.5728	31.0004	19.2189	30.6466
.32	20.3119	31.1268	19.9054	30.7204	19.5270	30.3421	19.1731	29.9882
.34	20.2714	30.5108	19.8649	30.1044	19.4865	29.7261	19.1325	29.3722
.36	20.2362	29.9331	19.8297	29.5267	19.4513	29.1484	19.0973	28.7945
.38	20.2064	29.3902	19.7999	28.9838	19.4215	28.6054	19.0675	28.2516
.40	20.1820	28.8790	19.7756	28.4726	19.3971	28.0942	19.0432	27.7403
.42	20.1631	28.3969	19.7566	27.9905	19.3782	27.6122	19.0242	27.2583
.44	20.1496	27.9419	19.7431	27.5355	19.3647	27.1572	19.0107	26.8033
.46	20.1415	27.5119	19.7350	27.1055	19.3566	26.7272	19.0027	26.3733
.48	20.1388	27.1053	19.7324	26.6989	19.3539	26.3205	19.0000	25.9666
.50	20.1415	26.7205	19.7350	26.3141	19.3566	25.9357	19.0026	25.5818
.52	20.1494	26.3562	19.7430	25.9498	19.3645	25.5714	19.0106	25.2175
.54	20.1626	26.0111	19.7561	25.6047	19.3777	25.2263	19.0237	24.8724
.56	20.1808	25.6842	19.7744	25.2778	19.3959	24.8994	19.0420	24.5455
.58	20.2041	25.3744	19.7976	24.9680	19.4192	24.5896	19.0652	24.2357
.60	20.2323	25.0808	19.8258	24.6744	19.4474	24.2960	19.0934	23.9421
.62	20.2653	24.8026	19.8588	24.3961	19.4804	24.0178	19.1264	23.6638
.64	20.3029	24.5389	19.8964	24.1324	19.5180	23.7541	19.1640	23.4001
.66	20.3450	24.2889	19.9386	23.8825	19.5601	23.5041	19.2061	23.1502
.68	20.3915	24.0521	19.9851	23.6456	19.6066	23.2673	19.2526	22.9133
.70	20.4422	23.8276	20.0358	23.4212	19.6573	23.0428	19.3034	22.6889
.72	20.4970	23.6150	20.0905	23.2086	19.7121	22.8302	19.3581	22.4762
.74	20.5556	23.4136	20.1492	23.0072	19.7707	22.6288	19.4167	22.2748
.76	20.6180	23.2228	20.2115	22.8164	19.8331	22.4380	19.4791	22.0840
.78	20.6839	23.0422	20.2774	22.6358	19.8990	22.2574	19.5450	21.9034
.80	20.7532	22.8712	20.3467	22.4647	19.9683	22.0864	19.6143	21.7324
.82	20.8257	22.7093	20.4192	22.3029	20.0408	21.9245	19.6868	21.5705
.84	20.9013	22.5562	20.4948	22.1497	20.1164	21.7713	19.7624	21.4173
.86	20.9797	22.4112	20.5732	22.0048	20.1948	21.6264	19.8408	21.2724
.88	21.0608	22.2742	20.6544	21.8677	20.2759	21.4893	19.9219	21.1353
.90	21.1445	22.1445	20.7380	21.7381	20.3596	21.3597	20.0056	21.0057
.92	21.2306	22.0220	20.8241	21.6155	20.4457	21.2371	20.0917	20.8832
.94	21.3189	21.9062	20.9124	21.4997	20.5340	21.1213	20.1800	20.7673
.96	21.4092	21.7967	21.0028	21.3902	20.6243	21.0118	20.2704	20.6578
.98	21.5015	21.6933	21.0950	21.2868	20.7166	20.9084	20.3626	20.5544
1.00	21.5956	21.5956	21.1891	21.1891	20.8107	20.8107	20.4567	20.4567

Table D–1: $0.026 \leq \hat{\rho}_4 \leq 0.032$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.026$		$\hat{\rho}_4 = 0.028$		$\hat{\rho}_4 = 0.030$		$\hat{\rho}_4 = 0.032$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	19.9093	57.0431	19.5028	56.6378	19.1244	56.2606	18.7704	55.9081
.04	19.8035	50.3567	19.3970	49.9508	19.0185	49.5730	18.6646	49.2199
.06	19.7007	46.4050	19.2942	45.9989	18.9157	45.6209	18.5618	45.2676
.08	19.6012	43.5747	19.1947	43.1686	18.8162	42.7905	18.4623	42.4371
.10	19.5052	41.3606	19.0987	40.9545	18.7202	40.5763	18.3663	40.2228
.12	19.4129	39.5378	19.0065	39.1316	18.6280	38.7534	18.2741	38.3999
.14	19.3246	37.9863	18.9182	37.5801	18.5397	37.2019	18.1858	36.8483
.16	19.2404	36.6347	18.8339	36.2285	18.4555	35.8502	18.1016	35.4966
.18	19.1605	35.4368	18.7540	35.0306	18.3756	34.6524	18.0217	34.2987
.20	19.0851	34.3614	18.6786	33.9551	18.3001	33.5769	17.9462	33.2232
.22	19.0142	33.3859	18.6078	32.9796	18.2293	32.6014	17.8754	32.2478
.24	18.9482	32.4940	18.5417	32.0877	18.1632	31.7095	17.8093	31.3558
.26	18.8870	31.6731	18.4806	31.2668	18.1021	30.8886	17.7482	30.5349
.28	18.8309	30.9136	18.4244	30.5073	18.0459	30.1290	17.6920	29.7754
.30	18.7799	30.2077	18.3734	29.8014	17.9949	29.4231	17.6410	29.0695
.32	18.7340	29.5493	18.3276	29.1430	17.9491	28.7647	17.5952	28.4111
.34	18.6935	28.9334	18.2871	28.5271	17.9086	28.1488	17.5547	27.7951
.36	18.6583	28.3557	18.2519	27.9494	17.8734	27.5711	17.5195	27.2174
.38	18.6285	27.8127	18.2221	27.4064	17.8436	27.0281	17.4897	26.6744
.40	18.6042	27.3015	18.1977	26.8952	17.8192	26.5168	17.4653	26.1631
.42	18.5852	26.8194	18.1788	26.4131	17.8002	26.0348	17.4463	25.6811
.44	18.5717	26.3644	18.1652	25.9581	17.7867	25.5797	17.4328	25.2260
.46	18.5636	25.9344	18.1572	25.5281	17.7787	25.1497	17.4247	24.7960
.48	18.5609	25.5277	18.1545	25.1214	17.7760	24.7431	17.4220	24.3893
.50	18.5636	25.1429	18.1571	24.7366	17.7786	24.3582	17.4247	24.0045
.52	18.5715	24.7786	18.1651	24.3723	17.7865	23.9939	17.4326	23.6402
.54	18.5847	24.4335	18.1782	24.0272	17.7997	23.6488	17.4458	23.2950
.56	18.6029	24.1066	18.1965	23.7002	17.8179	23.3219	17.4640	22.9681
.58	18.6262	23.7968	18.2197	23.3904	17.8412	23.0120	17.4873	22.6583
.60	18.6544	23.5032	18.2479	23.0968	17.8694	22.7184	17.5155	22.3646
.62	18.6874	23.2249	18.2809	22.8185	17.9024	22.4401	17.5484	22.0864
.64	18.7250	22.9612	18.3185	22.5548	17.9400	22.1764	17.5860	21.8226
.66	18.7671	22.7112	18.3606	22.3048	17.9821	21.9264	17.6282	21.5726
.68	18.8136	22.4744	18.4071	22.0680	18.0286	21.6895	17.6747	21.3357
.70	18.8643	22.2499	18.4578	21.8435	18.0793	21.4651	17.7254	21.1113
.72	18.9191	22.0373	18.5126	21.6309	18.1341	21.2524	17.7801	20.8986
.74	18.9777	21.8358	18.5712	21.4294	18.1927	21.0510	17.8388	20.6971
.76	19.0401	21.6451	18.6336	21.2387	18.2551	20.8602	17.9011	20.5064
.78	19.1060	21.4644	18.6995	21.0580	18.3210	20.6795	17.9670	20.3257
.80	19.1753	21.2934	18.7688	20.8870	18.3903	20.5085	18.0363	20.1546
.82	19.2478	21.1315	18.8413	20.7251	18.4628	20.3466	18.1088	19.9928
.84	19.3234	20.9783	18.9169	20.5719	18.5383	20.1934	18.1844	19.8396
.86	19.4018	20.8334	18.9953	20.4270	18.6168	20.0485	18.2628	19.6946
.88	19.4829	20.6963	19.0764	20.2899	18.6979	19.9114	18.3440	19.5575
.90	19.5666	20.5667	19.1601	20.1603	18.7816	19.7818	18.4277	19.4279
.92	19.6527	20.4441	19.2462	20.0377	18.8677	19.6592	18.5137	19.3053
.94	19.7410	20.3283	19.3345	19.9218	18.9560	19.5433	18.6020	19.1894
.96	19.8313	20.2188	19.4249	19.8124	19.0463	19.4339	18.6924	19.0799
.98	19.9236	20.1154	19.5172	19.7089	19.1386	19.3304	18.7847	18.9765
1.00	20.0177	20.0177	19.6112	19.6112	19.2327	19.2327	18.8788	18.8788

Table D–1: $0.034 \leq \hat{\rho}_4 \leq 0.040$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.034$		$\hat{\rho}_4 = 0.036$		$\hat{\rho}_4 = 0.038$		$\hat{\rho}_4 = 0.040$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	18.4379	55.5773	18.1245	55.2658	17.8279	54.9714	17.5466	54.6926
.04	18.3321	48.8884	18.0187	48.5759	17.7222	48.2806	17.4408	48.0006
.06	18.2294	44.9358	17.9159	44.6231	17.6194	44.3274	17.3381	44.0471
.08	18.1299	42.1051	17.8164	41.7923	17.5199	41.4964	17.2386	41.2159
.10	18.0339	39.8908	17.7204	39.5779	17.4239	39.2820	17.1426	39.0013
.12	17.9416	38.0678	17.6281	37.7548	17.3316	37.4589	17.0503	37.1782
.14	17.8533	36.5162	17.5398	36.2032	17.2433	35.9072	16.9620	35.6264
.16	17.7691	35.1645	17.4556	34.8515	17.1591	34.5554	16.8778	34.2746
.18	17.6892	33.9666	17.3757	33.6535	17.0792	33.3575	16.7979	33.0766
.20	17.6137	32.8911	17.3003	32.5780	17.0038	32.2819	16.7225	32.0011
.22	17.5429	31.9156	17.2295	31.6025	16.9330	31.3064	16.6517	31.0255
.24	17.4769	31.0237	17.1634	30.7105	16.8669	30.4144	16.5856	30.1335
.26	17.4157	30.2027	17.1023	29.8896	16.8057	29.5935	16.5244	29.3126
.28	17.3596	29.4432	17.0461	29.1300	16.7496	28.8339	16.4683	28.5530
.30	17.3085	28.7373	16.9951	28.4241	16.6986	28.1280	16.4173	27.8471
.32	17.2627	28.0789	16.9492	27.7657	16.6527	27.4695	16.3714	27.1886
.34	17.2222	27.4629	16.9087	27.1497	16.6122	26.8536	16.3309	26.5726
.36	17.1870	26.8852	16.8735	26.5720	16.5770	26.2758	16.2957	25.9949
.38	17.1572	26.3422	16.8437	26.0290	16.5472	25.7328	16.2659	25.4518
.40	17.1328	25.8309	16.8193	25.5177	16.5228	25.2215	16.2415	24.9406
.42	17.1139	25.3489	16.8004	25.0357	16.5038	24.7394	16.2225	24.4585
.44	17.1003	24.8938	16.7869	24.5806	16.4903	24.2843	16.2090	24.0034
.46	17.0923	24.4638	16.7788	24.1505	16.4822	23.8543	16.2009	23.5733
.48	17.0896	24.0571	16.7761	23.7438	16.4795	23.4476	16.1982	23.1666
.50	17.0922	23.6722	16.7787	23.3590	16.4821	23.0627	16.2008	22.7817
.52	17.1001	23.3079	16.7866	22.9946	16.4901	22.6983	16.2087	22.4173
.54	17.1133	22.9628	16.7997	22.6495	16.5032	22.3532	16.2218	22.0721
.56	17.1315	22.6358	16.8180	22.3225	16.5214	22.0262	16.2401	21.7451
.58	17.1548	22.3259	16.8413	22.0127	16.5447	21.7163	16.2633	21.4352
.60	17.1830	22.0323	16.8694	21.7190	16.5729	21.4227	16.2915	21.1416
.62	17.2159	21.7540	16.9024	21.4407	16.6058	21.1443	16.3245	20.8632
.64	17.2535	21.4902	16.9400	21.1769	16.6434	20.8805	16.3621	20.5994
.66	17.2956	21.2403	16.9821	20.9269	16.6856	20.6305	16.4042	20.3494
.68	17.3421	21.0034	17.0286	20.6900	16.7320	20.3936	16.4507	20.1124
.70	17.3928	20.7789	17.0793	20.4655	16.7827	20.1691	16.5014	19.8879
.72	17.4476	20.5662	17.1341	20.2528	16.8375	19.9564	16.5561	19.6752
.74	17.5062	20.3647	17.1927	20.0514	16.8961	19.7549	16.6148	19.4737
.76	17.5686	20.1739	17.2551	19.8605	16.9585	19.5641	16.6771	19.2829
.78	17.6345	19.9933	17.3210	19.6799	17.0244	19.3834	16.7430	19.1022
.80	17.7038	19.8222	17.3903	19.5088	17.0937	19.2123	16.8123	18.9311
.82	17.7763	19.6603	17.4628	19.3469	17.1662	19.0504	16.8849	18.7692
.84	17.8519	19.5071	17.5384	19.1937	17.2418	18.8972	16.9604	18.6159
.86	17.9303	19.3622	17.6168	19.0487	17.3202	18.7522	17.0389	18.4709
.88	18.0115	19.2251	17.6979	18.9116	17.4014	18.6151	17.1200	18.3338
.90	18.0952	19.0954	17.7816	18.7819	17.4851	18.4854	17.2037	18.2041
.92	18.1812	18.9728	17.8677	18.6593	17.5712	18.3628	17.2898	18.0815
.94	18.2695	18.8569	17.9560	18.5435	17.6595	18.2469	17.3781	17.9656
.96	18.3599	18.7475	18.0464	18.4340	17.7499	18.1374	17.4685	17.8561
.98	18.4522	18.6440	18.1387	18.3305	17.8422	18.0340	17.5608	17.7527
1.00	18.5463	18.5463	18.2328	18.2328	17.9363	17.9363	17.6549	17.6549

Table D–1: $0.0425 \leq \hat{\rho}_4 \leq 0.0500$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.0425$		$\hat{\rho}_4 = 0.0450$		$\hat{\rho}_4 = 0.0475$		$\hat{\rho}_4 = 0.0500$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	17.2142	54.3636	16.9007	54.0541	16.6042	53.7621	16.3229	53.4858
.04	17.1084	47.6700	16.7949	47.3587	16.4984	47.0647	16.2171	46.7861
.06	17.0056	43.7160	16.6921	43.4040	16.3957	43.1093	16.1144	42.8300
.08	16.9061	40.8846	16.5927	40.5724	16.2962	40.2773	16.0149	39.9976
.10	16.8101	38.6698	16.4966	38.3574	16.2002	38.0622	15.9189	37.7823
.12	16.7179	36.8465	16.4044	36.5341	16.1079	36.2387	15.8267	35.9586
.14	16.6295	35.2948	16.3160	34.9821	16.0196	34.6867	15.7384	34.4066
.16	16.5453	33.9429	16.2318	33.6302	15.9354	33.3347	15.6542	33.0545
.18	16.4654	32.7449	16.1520	32.4322	15.8555	32.1366	15.5743	31.8564
.20	16.3900	31.6693	16.0765	31.3565	15.7801	31.0609	15.4988	30.7806
.22	16.3192	30.6937	16.0057	30.3810	15.7093	30.0853	15.4280	29.8050
.24	16.2531	29.8017	15.9396	29.4889	15.6432	29.1933	15.3619	28.9129
.26	16.1920	28.9807	15.8784	28.6679	15.5820	28.3722	15.3008	28.0919
.28	16.1358	28.2211	15.8223	27.9082	15.5259	27.6126	15.2446	27.3322
.30	16.0848	27.5152	15.7712	27.2023	15.4748	26.9066	15.1936	26.6262
.32	16.0389	26.8567	15.7254	26.5438	15.4290	26.2481	15.1477	25.9676
.34	15.9984	26.2407	15.6849	25.9278	15.3884	25.6321	15.1072	25.3516
.36	15.9632	25.6629	15.6496	25.3500	15.3532	25.0542	15.0719	24.7737
.38	15.9334	25.1199	15.6198	24.8069	15.3234	24.5111	15.0421	24.2306
.40	15.9090	24.6086	15.5954	24.2956	15.2990	23.9998	15.0177	23.7193
.42	15.8900	24.1265	15.5764	23.8135	15.2800	23.5177	14.9987	23.2371
.44	15.8765	23.6713	15.5629	23.3583	15.2665	23.0625	14.9852	22.7819
.46	15.8684	23.2413	15.5548	22.9282	15.2583	22.6324	14.9770	22.3517
.48	15.8657	22.8345	15.5521	22.5214	15.2556	22.2256	14.9743	21.9449
.50	15.8683	22.4496	15.5547	22.1365	15.2582	21.8406	14.9769	21.5599
.52	15.8762	22.0852	15.5626	21.7721	15.2661	21.4761	14.9848	21.1954
.54	15.8893	21.7400	15.5757	21.4269	15.2792	21.1309	14.9979	20.8502
.56	15.9075	21.4130	15.5939	21.0998	15.2974	20.8038	15.0161	20.5230
.58	15.9308	21.1031	15.6172	20.7899	15.3207	20.4939	15.0393	20.2131
.60	15.9590	20.8094	15.6453	20.4962	15.3488	20.2001	15.0674	19.9193
.62	15.9919	20.5310	15.6783	20.2178	15.3817	19.9217	15.1004	19.6408
.64	16.0295	20.2672	15.7159	19.9539	15.4193	19.6578	15.1379	19.3769
.66	16.0716	20.0171	15.7580	19.7038	15.4614	19.4077	15.1800	19.1268
.68	16.1181	19.7802	15.8044	19.4669	15.5079	19.1707	15.2265	18.8897
.70	16.1688	19.5556	15.8552	19.2423	15.5586	18.9461	15.2772	18.6651
.72	16.2235	19.3429	15.9099	19.0295	15.6133	18.7333	15.3319	18.4523
.74	16.2822	19.1414	15.9685	18.8280	15.6720	18.5318	15.3906	18.2507
.76	16.3445	18.9505	16.0309	18.6371	15.7344	18.3409	15.4529	18.0598
.78	16.4105	18.7698	16.0968	18.4564	15.8003	18.1601	15.5189	17.8790
.80	16.4798	18.5987	16.1661	18.2853	15.8696	17.9889	15.5882	17.7078
.82	16.5523	18.4367	16.2386	18.1233	15.9421	17.8270	15.6607	17.5458
.84	16.6279	18.2835	16.3142	17.9700	16.0177	17.6737	15.7363	17.3925
.86	16.7063	18.1385	16.3927	17.8250	16.0961	17.5286	15.8147	17.2474
.88	16.7875	18.0014	16.4738	17.6878	16.1773	17.3915	15.8959	17.1102
.90	16.8712	17.8717	16.5575	17.5581	16.2610	17.2617	15.9796	16.9805
.92	16.9573	17.7490	16.6436	17.4355	16.3471	17.1391	16.0658	16.8578
.94	17.0456	17.6331	16.7320	17.3196	16.4355	17.0232	16.1541	16.7419
.96	17.1360	17.5236	16.8224	17.2101	16.5259	16.9136	16.2445	16.6323
.98	17.2283	17.4202	16.9147	17.1066	16.6182	16.8101	16.3369	16.5288
1.00	17.3224	17.3224	17.0089	17.0089	16.7123	16.7123	16.4310	16.4310

Table D–1: $0.053 \leq \hat{\rho}_4 \leq 0.062$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.053$		$\hat{\rho}_4 = 0.056$		$\hat{\rho}_4 = 0.059$		$\hat{\rho}_4 = 0.062$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	16.0034	53.1731	15.7015	52.8790	15.4154	52.6016	15.1434	52.3395
.04	15.8976	46.4703	15.5957	46.1726	15.3096	45.8912	15.0377	45.6246
.06	15.7949	42.5132	15.4930	42.2143	15.2069	41.9316	14.9350	41.6635
.08	15.6954	39.6803	15.3935	39.3808	15.1074	39.0975	14.8355	38.8287
.10	15.5994	37.4646	15.2975	37.1649	15.0114	36.8811	14.7395	36.6119
.12	15.5072	35.6408	15.2053	35.3408	14.9192	35.0568	14.6473	34.7873
.14	15.4188	34.0886	15.1170	33.7885	14.8309	33.5043	14.5590	33.2346
.16	15.3347	32.7364	15.0328	32.4362	14.7467	32.1520	14.4748	31.8821
.18	15.2548	31.5382	14.9529	31.2379	14.6668	30.9535	14.3949	30.6836
.20	15.1793	30.4624	14.8774	30.1621	14.5913	29.8776	14.3195	29.6076
.22	15.1085	29.4867	14.8066	29.1863	14.5205	28.9018	14.2486	28.6317
.24	15.0424	28.5946	14.7406	28.2941	14.4545	28.0096	14.1826	27.7394
.26	14.9813	27.7735	14.6794	27.4730	14.3933	27.1884	14.1214	26.9182
.28	14.9251	27.0138	14.6232	26.7132	14.3371	26.4286	14.0652	26.1583
.30	14.8740	26.3078	14.5722	26.0072	14.2860	25.7225	14.0141	25.4522
.32	14.8282	25.6492	14.5263	25.3486	14.2402	25.0639	13.9683	24.7935
.34	14.7876	25.0331	14.4857	24.7324	14.1996	24.4477	13.9277	24.1773
.36	14.7524	24.4553	14.4505	24.1545	14.1643	23.8697	13.8924	23.5993
.38	14.7225	23.9121	14.4206	23.6114	14.1345	23.3265	13.8625	23.0560
.40	14.6981	23.4007	14.3962	23.0999	14.1100	22.8150	13.8381	22.5445
.42	14.6791	22.9185	14.3772	22.6177	14.0910	22.3327	13.8190	22.0622
.44	14.6656	22.4633	14.3636	22.1624	14.0774	21.8774	13.8054	21.6068
.46	14.6574	22.0331	14.3555	21.7322	14.0692	21.4471	13.7972	21.1764
.48	14.6547	21.6262	14.3527	21.3253	14.0665	21.0402	13.7944	20.7694
.50	14.6573	21.2412	14.3553	20.9402	14.0690	20.6551	13.7970	20.3843
.52	14.6651	20.8766	14.3631	20.5756	14.0769	20.2904	13.8048	20.0196
.54	14.6782	20.5313	14.3762	20.2302	14.0899	19.9450	13.8178	19.6741
.56	14.6964	20.2042	14.3944	19.9030	14.1081	19.6177	13.8360	19.3468
.58	14.7196	19.8941	14.4176	19.5930	14.1313	19.3076	13.8592	19.0366
.60	14.7478	19.6003	14.4457	19.2991	14.1594	19.0137	13.8873	18.7426
.62	14.7807	19.3218	14.4786	19.0205	14.1923	18.7351	13.9201	18.4639
.64	14.8183	19.0579	14.5162	18.7565	14.2298	18.4710	13.9577	18.1998
.66	14.8603	18.8077	14.5583	18.5063	14.2719	18.2207	13.9997	17.9495
.68	14.9068	18.5706	14.6047	18.2692	14.3184	17.9835	14.0462	17.7122
.70	14.9575	18.3459	14.6554	18.0445	14.3690	17.7588	14.0968	17.4874
.72	15.0122	18.1331	14.7101	17.8316	14.4238	17.5458	14.1516	17.2744
.74	15.0709	17.9315	14.7688	17.6299	14.4824	17.3441	14.2102	17.0726
.76	15.1332	17.7405	14.8311	17.4389	14.5447	17.1530	14.2726	16.8815
.78	15.1991	17.5596	14.8970	17.2580	14.6107	16.9721	14.3385	16.7005
.80	15.2685	17.3885	14.9664	17.0867	14.6800	16.8008	14.4078	16.5292
.82	15.3410	17.2264	15.0389	16.9247	14.7525	16.6387	14.4804	16.3670
.84	15.4166	17.0731	15.1145	16.7713	14.8281	16.4853	14.5560	16.2135
.86	15.4950	16.9280	15.1930	16.6262	14.9066	16.3401	14.6345	16.0683
.88	15.5762	16.7907	15.2742	16.4889	14.9878	16.2028	14.7157	15.9310
.90	15.6600	16.6610	15.3579	16.3591	15.0716	16.0730	14.7995	15.8011
.92	15.7461	16.5383	15.4440	16.2364	15.1577	15.9503	14.8856	15.6783
.94	15.8345	16.4223	15.5324	16.1204	15.2461	15.8342	14.9740	15.5623
.96	15.9249	16.3127	15.6229	16.0108	15.3366	15.7246	15.0645	15.4526
.98	16.0173	16.2092	15.7153	15.9073	15.4290	15.6210	15.1570	15.3490
1.00	16.1114	16.1114	15.8095	15.8095	15.5232	15.5232	15.2512	15.2512

Table D–1: $0.0655 \leq \hat{\rho}_4 \leq 0.0760$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.0655$		$\hat{\rho}_4 = 0.0690$		$\hat{\rho}_4 = 0.0725$		$\hat{\rho}_4 = 0.0760$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	14.8424	52.0513	14.5570	51.7804	14.2858	51.5255	14.0274	51.2852
.04	14.7366	45.3305	14.4513	45.0530	14.1801	44.7905	13.9217	44.5420
.06	14.6339	41.3675	14.3486	41.0878	14.0774	40.8229	13.8190	40.5716
.08	14.5344	38.5317	14.2491	38.2509	13.9779	37.9849	13.7196	37.7322
.10	14.4385	36.3144	14.1532	36.0330	13.8820	35.7662	13.6236	35.5128
.12	14.3463	34.4895	14.0609	34.2077	13.7898	33.9404	13.5314	33.6865
.14	14.2580	32.9365	13.9726	32.6544	13.7015	32.3869	13.4432	32.1326
.16	14.1738	31.5838	13.8885	31.3015	13.6173	31.0337	13.3590	30.7792
.18	14.0939	30.3851	13.8086	30.1027	13.5374	29.8347	13.2791	29.5800
.20	14.0184	29.3090	13.7331	29.0264	13.4620	28.7584	13.2037	28.5035
.22	13.9476	28.3330	13.6623	28.0504	13.3911	27.7822	13.1328	27.5271
.24	13.8815	27.4407	13.5962	27.1579	13.3250	26.8896	13.0667	26.6345
.26	13.8204	26.6194	13.5350	26.3365	13.2639	26.0682	13.0055	25.8129
.28	13.7642	25.8595	13.4788	25.5766	13.2076	25.3081	12.9493	25.0528
.30	13.7131	25.1533	13.4277	24.8703	13.1565	24.6017	12.8982	24.3464
.32	13.6672	24.4945	13.3819	24.2115	13.1106	23.9429	12.8523	23.6874
.34	13.6266	23.8782	13.3412	23.5951	13.0700	23.3265	12.8116	23.0709
.36	13.5913	23.3002	13.3059	23.0170	13.0347	22.7483	12.7763	22.4926
.38	13.5614	22.7569	13.2760	22.4736	13.0047	22.2048	12.7463	21.9491
.40	13.5369	22.2453	13.2515	21.9620	12.9802	21.6930	12.7218	21.4372
.42	13.5179	21.7629	13.2324	21.4795	12.9611	21.2105	12.7026	20.9546
.44	13.5043	21.3074	13.2188	21.0240	12.9474	20.7549	12.6889	20.4989
.46	13.4961	20.8770	13.2106	20.5935	12.9392	20.3243	12.6806	20.0682
.48	13.4932	20.4699	13.2077	20.1863	12.9363	19.9170	12.6777	19.6609
.50	13.4957	20.0847	13.2102	19.8010	12.9388	19.5316	12.6802	19.2753
.52	13.5035	19.7199	13.2180	19.4361	12.9465	19.1666	12.6879	18.9102
.54	13.5166	19.3744	13.2310	19.0905	12.9595	18.8209	12.7008	18.5644
.56	13.5347	19.0469	13.2491	18.7630	12.9775	18.4933	12.7188	18.2367
.58	13.5578	18.7367	13.2722	18.4526	13.0006	18.1829	12.7419	17.9261
.60	13.5859	18.4426	13.3002	18.1584	13.0287	17.8886	12.7699	17.6317
.62	13.6188	18.1639	13.3331	17.8796	13.0615	17.6096	12.8027	17.3527
.64	13.6563	17.8996	13.3706	17.6153	13.0990	17.3452	12.8402	17.0882
.66	13.6984	17.6492	13.4126	17.3648	13.1410	17.0946	12.8822	16.8374
.68	13.7448	17.4119	13.4590	17.1274	13.1874	16.8571	12.9286	16.5998
.70	13.7954	17.1870	13.5097	16.9024	13.2380	16.6320	12.9792	16.3746
.72	13.8502	16.9739	13.5644	16.6892	13.2927	16.4188	13.0339	16.1612
.74	13.9088	16.7721	13.6230	16.4873	13.3514	16.2167	13.0925	15.9591
.76	13.9711	16.5809	13.6854	16.2960	13.4137	16.0254	13.1549	15.7677
.78	14.0371	16.3998	13.7513	16.1149	13.4797	15.8442	13.2208	15.5864
.80	14.1064	16.2284	13.8207	15.9434	13.5490	15.6726	13.2901	15.4147
.82	14.1790	16.0662	13.8932	15.7811	13.6216	15.5102	13.3627	15.2522
.84	14.2546	15.9127	13.9689	15.6275	13.6972	15.3566	13.4384	15.0985
.86	14.3331	15.7674	14.0474	15.4822	13.7758	15.2112	13.5170	14.9530
.88	14.4143	15.6300	14.1286	15.3448	13.8570	15.0737	13.5983	14.8155
.90	14.4981	15.5001	14.2124	15.2148	13.9409	14.9437	13.6821	14.6854
.92	14.5843	15.3773	14.2987	15.0919	14.0271	14.8207	13.7684	14.5624
.94	14.6727	15.2612	14.3871	14.9758	14.1156	14.7045	13.8569	14.4461
.96	14.7633	15.1515	14.4777	14.8661	14.2062	14.5947	13.9476	14.3363
.98	14.8557	15.0479	14.5702	14.7624	14.2987	14.4910	14.0401	14.2325
1.00	14.9500	14.9500	14.6645	14.6645	14.3931	14.3931	14.1345	14.1345

Table D–1: $0.080 \leq \hat{\rho}_4 \leq 0.092$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.080$		$\hat{\rho}_4 = 0.084$		$\hat{\rho}_4 = 0.088$		$\hat{\rho}_4 = 0.092$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	13.7464	51.0273	13.4790	50.7860	13.2242	50.5601	12.9808	50.3487
.04	13.6407	44.2734	13.3734	44.0201	13.1185	43.7808	12.8751	43.5546
.06	13.5380	40.2995	13.2707	40.0423	13.0159	39.7987	12.7725	39.5677
.08	13.4385	37.4585	13.1713	37.1993	12.9165	36.9536	12.6731	36.7202
.10	13.3426	35.2380	13.0754	34.9778	12.8206	34.7308	12.5772	34.4960
.12	13.2504	33.4111	12.9832	33.1501	12.7284	32.9023	12.4850	32.6667
.14	13.1621	31.8568	12.8949	31.5953	12.6402	31.3469	12.3968	31.1106
.16	13.0780	30.5030	12.8107	30.2412	12.5560	29.9924	12.3126	29.7557
.18	12.9981	29.3036	12.7309	29.0415	12.4761	28.7924	12.2328	28.5553
.20	12.9227	28.2268	12.6554	27.9645	12.4007	27.7152	12.1573	27.4778
.22	12.8518	27.2504	12.5846	26.9878	12.3298	26.7383	12.0865	26.5008
.24	12.7857	26.3576	12.5185	26.0949	12.2637	25.8452	12.0203	25.6074
.26	12.7245	25.5359	12.4573	25.2731	12.2025	25.0232	11.9591	24.7853
.28	12.6683	24.7756	12.4010	24.5127	12.1462	24.2627	11.9028	24.0246
.30	12.6172	24.0691	12.3499	23.8060	12.0951	23.5559	11.8516	23.3177
.32	12.5712	23.4100	12.3039	23.1469	12.0491	22.8966	11.8056	22.6582
.34	12.5305	22.7934	12.2632	22.5301	12.0083	22.2798	11.7648	22.0412
.36	12.4952	22.2150	12.2278	21.9516	11.9729	21.7011	11.7294	21.4625
.38	12.4652	21.6714	12.1978	21.4078	11.9429	21.1572	11.6993	20.9184
.40	12.4406	21.1594	12.1732	20.8958	11.9182	20.6450	11.6746	20.4060
.42	12.4214	20.6767	12.1539	20.4129	11.8989	20.1620	11.6553	19.9228
.44	12.4077	20.2208	12.1402	19.9569	11.8851	19.7059	11.6414	19.4665
.46	12.3993	19.7900	12.1318	19.5260	11.8767	19.2748	11.6329	19.0353
.48	12.3964	19.3826	12.1288	19.1184	11.8736	18.8670	11.6298	18.6273
.50	12.3988	18.9969	12.1311	18.7325	11.8759	18.4810	11.6320	18.2411
.52	12.4065	18.6317	12.1388	18.3672	11.8835	18.1154	11.6395	17.8754
.54	12.4193	18.2857	12.1516	18.0210	11.8963	17.7691	11.6523	17.5289
.56	12.4374	17.9578	12.1696	17.6930	11.9142	17.4409	11.6701	17.2005
.58	12.4604	17.6471	12.1926	17.3821	11.9372	17.1299	11.6930	16.8892
.60	12.4884	17.3526	12.2205	17.0874	11.9650	16.8350	11.7209	16.5942
.62	12.5211	17.0734	12.2532	16.8081	11.9977	16.5555	11.7535	16.3144
.64	12.5586	16.8087	12.2906	16.5432	12.0351	16.2905	11.7909	16.0492
.66	12.6005	16.5579	12.3326	16.2922	12.0770	16.0392	11.8327	15.7978
.68	12.6469	16.3201	12.3789	16.0543	12.1233	15.8011	11.8790	15.5595
.70	12.6975	16.0948	12.4295	15.8288	12.1739	15.5755	11.9296	15.3337
.72	12.7522	15.8813	12.4842	15.6151	12.2286	15.3617	11.9842	15.1197
.74	12.8108	15.6790	12.5428	15.4127	12.2872	15.1591	12.0428	14.9169
.76	12.8732	15.4874	12.6051	15.2210	12.3495	14.9672	12.1052	14.7249
.78	12.9391	15.3060	12.6711	15.0395	12.4155	14.7855	12.1711	14.5430
.80	13.0085	15.1342	12.7404	14.8676	12.4848	14.6135	12.2405	14.3708
.82	13.0811	14.9716	12.8131	14.7049	12.5575	14.4506	12.3132	14.2078
.84	13.1567	14.8178	12.8888	14.5509	12.6332	14.2965	12.3889	14.0535
.86	13.2353	14.6722	12.9674	14.4052	12.7119	14.1507	12.4676	13.9076
.88	13.3167	14.5346	13.0487	14.2674	12.7933	14.0128	12.5490	13.7696
.90	13.4006	14.4044	13.1327	14.1372	12.8772	13.8824	12.6331	13.6391
.92	13.4869	14.2813	13.2190	14.0140	12.9637	13.7592	12.7195	13.5157
.94	13.5754	14.1650	13.3077	13.8976	13.0523	13.6427	12.8083	13.3991
.96	13.6661	14.0550	13.3984	13.7876	13.1431	13.5326	12.8991	13.2889
.98	13.7587	13.9512	13.4911	13.6837	13.2358	13.4286	12.9919	13.1848
1.00	13.8532	13.8532	13.5855	13.5855	13.3304	13.3304	13.0866	13.0866

Table D–1: $0.0965 \leq \hat{\rho}_4 \leq 0.1100$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.0965$		$\hat{\rho}_4 = 0.1010$		$\hat{\rho}_4 = 0.1055$		$\hat{\rho}_4 = 0.1100$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	12.7193	50.1273	12.4698	49.9225	12.2312	49.7336	12.0027	49.5598
.04	12.6136	43.3146	12.3642	43.0890	12.1257	42.8769	11.8972	42.6776
.06	12.5110	39.3217	12.2616	39.0894	12.0231	38.8699	11.7947	38.6623
.08	12.4117	36.4713	12.1622	36.2357	11.9238	36.0126	11.6954	35.8010
.10	12.3158	34.2454	12.0664	34.0080	11.8280	33.7827	11.5996	33.5688
.12	12.2237	32.4149	11.9743	32.1762	11.7359	31.9496	11.5075	31.7342
.14	12.1354	30.8581	11.8861	30.6186	11.6477	30.3910	11.4193	30.1746
.16	12.0513	29.5026	11.8019	29.2624	11.5635	29.0341	11.3352	28.8170
.18	11.9714	28.3018	11.7221	28.0611	11.4837	27.8324	11.2554	27.6146
.20	11.8960	27.2240	11.6466	26.9830	11.4082	26.7537	11.1799	26.5355
.22	11.8251	26.2466	11.5758	26.0053	11.3374	25.7757	11.1091	25.5571
.24	11.7590	25.3531	11.5097	25.1114	11.2712	24.8816	11.0429	24.6627
.26	11.6977	24.5307	11.4484	24.2889	11.2099	24.0588	10.9816	23.8396
.28	11.6414	23.7698	11.3920	23.5278	11.1536	23.2974	10.9252	23.0780
.30	11.5902	23.0627	11.3408	22.8205	11.1023	22.5899	10.8739	22.3702
.32	11.5442	22.4031	11.2947	22.1606	11.0562	21.9299	10.8277	21.7099
.34	11.5034	21.7859	11.2538	21.5433	11.0153	21.3123	10.7867	21.0921
.36	11.4679	21.2070	11.2183	20.9641	10.9796	20.7329	10.7511	20.5125
.38	11.4377	20.6627	11.1881	20.4197	10.9494	20.1882	10.7207	19.9676
.40	11.4129	20.1502	11.1632	19.9069	10.9245	19.6753	10.6957	19.4543
.42	11.3936	19.6668	11.1438	19.4233	10.9050	19.1914	10.6762	18.9702
.44	11.3796	19.2103	11.1298	18.9666	10.8909	18.7344	10.6620	18.5130
.46	11.3711	18.7788	11.1212	18.5349	10.8822	18.3024	10.6532	18.0807
.48	11.3679	18.3706	11.1179	18.1264	10.8788	17.8937	10.6497	17.6717
.50	11.3701	17.9842	11.1200	17.7398	10.8808	17.5068	10.6517	17.2844
.52	11.3775	17.6183	11.1274	17.3736	10.8881	17.1403	10.6588	16.9176
.54	11.3902	17.2715	11.1400	17.0265	10.9006	16.7930	10.6713	16.5700
.56	11.4080	16.9429	11.1577	16.6976	10.9183	16.4638	10.6888	16.2405
.58	11.4308	16.6314	11.1805	16.3858	10.9409	16.1517	10.7114	15.9280
.60	11.4586	16.3360	11.2082	16.0902	10.9686	15.8558	10.7390	15.6318
.62	11.4912	16.0560	11.2407	15.8100	11.0010	15.5752	10.7713	15.3509
.64	11.5285	15.7906	11.2779	15.5442	11.0382	15.3092	10.8084	15.0845
.66	11.5703	15.5389	11.3197	15.2923	11.0799	15.0569	10.8501	14.8319
.68	11.6165	15.3004	11.3659	15.0535	11.1261	14.8178	10.8962	14.5925
.70	11.6671	15.0743	11.4164	14.8272	11.1766	14.5912	10.9467	14.3655
.72	11.7217	14.8601	11.4710	14.6127	11.2311	14.3764	11.0012	14.1504
.74	11.7803	14.6571	11.5296	14.4094	11.2897	14.1729	11.0597	13.9466
.76	11.8426	14.4648	11.5919	14.2169	11.3520	13.9801	11.1221	13.7536
.78	11.9086	14.2827	11.6579	14.0346	11.4180	13.7975	11.1881	13.5707
.80	11.9780	14.1103	11.7273	13.8620	11.4875	13.6247	11.2575	13.3976
.82	12.0507	13.9472	11.8000	13.6986	11.5602	13.4610	11.3303	13.2337
.84	12.1265	13.7927	11.8759	13.5440	11.6361	13.3062	11.4062	13.0786
.86	12.2052	13.6466	11.9546	13.3977	11.7149	13.1597	11.4851	12.9318
.88	12.2867	13.5084	12.0362	13.2593	11.7965	13.0211	11.5668	12.7931
.90	12.3708	13.3778	12.1203	13.1285	11.8808	12.8901	11.6511	12.6618
.92	12.4573	13.2543	12.2069	13.0048	11.9674	12.7663	11.7379	12.5378
.94	12.5461	13.1375	12.2958	12.8879	12.0564	12.6492	11.8270	12.4206
.96	12.6370	13.0272	12.3869	12.7775	12.1475	12.5386	11.9182	12.3098
.98	12.7299	12.9230	12.4798	12.6732	12.2406	12.4342	12.0114	12.2052
1.00	12.8246	12.8246	12.5747	12.5746	12.3355	12.3355	12.1065	12.1065

Table D–1: $0.115 \leq \hat{\rho}_4 \leq 0.130$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.115$		$\hat{\rho}_4 = 0.120$		$\hat{\rho}_4 = 0.125$		$\hat{\rho}_4 = 0.130$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	11.7596	49.3841	11.5270	49.2261	11.3040	49.0857	11.0898	48.9624
.04	11.6541	42.4703	11.4215	42.2771	11.1986	42.0976	10.9844	41.9311
.06	11.5516	38.4449	11.3191	38.2406	11.0962	38.0487	10.8821	37.8686
.08	11.4524	35.5787	11.2199	35.3690	10.9970	35.1711	10.7830	34.9845
.10	11.3566	33.3436	11.1242	33.1307	10.9013	32.9294	10.6873	32.7389
.12	11.2646	31.5072	11.0321	31.2923	10.8093	31.0886	10.5954	30.8957
.14	11.1764	29.9463	10.9440	29.7299	10.7212	29.5247	10.5073	29.3301
.16	11.0923	28.5877	10.8599	28.3703	10.6371	28.1640	10.4232	27.9682
.18	11.0125	27.3846	10.7801	27.1664	10.5573	26.9593	10.3434	26.7625
.20	10.9371	26.3050	10.7047	26.0862	10.4819	25.8783	10.2680	25.6808
.22	10.8662	25.3261	10.6338	25.1068	10.4110	24.8984	10.1971	24.7002
.24	10.8000	24.4313	10.5676	24.2116	10.3448	24.0027	10.1309	23.8040
.26	10.7387	23.6078	10.5062	23.3877	10.2834	23.1784	10.0694	22.9793
.28	10.6822	22.8460	10.4498	22.6255	10.2269	22.4158	10.0129	22.2162
.30	10.6309	22.1379	10.3983	21.9171	10.1754	21.7070	9.9613	21.5071
.32	10.5846	21.4773	10.3520	21.2562	10.1290	21.0458	9.9149	20.8455
.34	10.5436	20.8592	10.3109	20.6378	10.0879	20.4270	9.8736	20.2263
.36	10.5079	20.2793	10.2751	20.0575	10.0519	19.8465	9.8376	19.6454
.38	10.4774	19.7341	10.2446	19.5120	10.0213	19.3005	9.8069	19.0990
.40	10.4524	19.2205	10.2194	18.9981	9.9960	18.7863	9.7815	18.5844
.42	10.4327	18.7360	10.1996	18.5133	9.9761	18.3011	9.7614	18.0987
.44	10.4184	18.2785	10.1852	18.0553	9.9615	17.8427	9.7467	17.6400
.46	10.4095	17.8459	10.1762	17.6223	9.9524	17.4093	9.7374	17.2061
.48	10.4059	17.4365	10.1725	17.2126	9.9486	16.9991	9.7335	16.7954
.50	10.4077	17.0489	10.1742	16.8245	9.9501	16.6106	9.7348	16.4064
.52	10.4148	16.6817	10.1811	16.4569	9.9569	16.2425	9.7415	16.0378
.54	10.4271	16.3336	10.1933	16.1084	9.9689	15.8936	9.7534	15.6884
.56	10.4445	16.0037	10.2106	15.7781	9.9861	15.5627	9.7704	15.3570
.58	10.4670	15.6909	10.2330	15.4648	10.0084	15.2490	9.7925	15.0427
.60	10.4945	15.3942	10.2603	15.1677	10.0356	14.9513	9.8196	14.7445
.62	10.5268	15.1129	10.2925	14.8859	10.0677	14.6691	9.8516	14.4617
.64	10.5638	14.8461	10.3294	14.6187	10.1045	14.4013	9.8883	14.1934
.66	10.6054	14.5931	10.3710	14.3652	10.1460	14.1474	9.9296	13.9390
.68	10.6515	14.3533	10.4170	14.1250	10.1919	13.9067	9.9755	13.6977
.70	10.7018	14.1259	10.4673	13.8972	10.2422	13.6784	10.0257	13.4689
.72	10.7564	13.9105	10.5218	13.6813	10.2966	13.4620	10.0801	13.2520
.74	10.8149	13.7063	10.5803	13.4767	10.3551	13.2570	10.1385	13.0465
.76	10.8772	13.5128	10.6426	13.2828	10.4174	13.0627	10.2009	12.8517
.78	10.9432	13.3296	10.7086	13.0992	10.4834	12.8787	10.2669	12.6672
.80	11.0127	13.1562	10.7781	12.9254	10.5530	12.7044	10.3365	12.4926
.82	11.0855	12.9920	10.8510	12.7608	10.6259	12.5395	10.4095	12.3272
.84	11.1615	12.8366	10.9270	12.6051	10.7020	12.3834	10.4857	12.1707
.86	11.2405	12.6895	11.0061	12.4578	10.7811	12.2357	10.5649	12.0226
.88	11.3222	12.5505	11.0879	12.3184	10.8631	12.0961	10.6469	11.8826
.90	11.4066	12.4190	11.1725	12.1867	10.9477	11.9640	10.7317	11.7502
.92	11.4935	12.2947	11.2595	12.0622	11.0349	11.8392	10.8190	11.6251
.94	11.5827	12.1773	11.3488	11.9445	11.1244	11.7212	10.9087	11.5069
.96	11.6741	12.0664	11.4403	11.8333	11.2160	11.6098	11.0005	11.3952
.98	11.7675	11.9616	11.5338	11.7283	11.3098	11.5046	11.0944	11.2897
1.00	11.8627	11.8627	11.6292	11.6292	11.4053	11.4053	11.1902	11.1902

Table D–1: $0.136 \leq \hat{\rho}_4 \leq 0.154$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.136$		$\hat{\rho}_4 = 0.142$		$\hat{\rho}_4 = 0.148$		$\hat{\rho}_4 = 0.154$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	10.8436	48.8373	10.6082	48.7371	10.3827	48.6621	10.1664	48.6129
.04	10.7383	41.7481	10.5030	41.5827	10.2776	41.4347	10.0613	41.3036
.06	10.6360	37.6675	10.4007	37.4820	10.1754	37.3115	9.9592	37.1556
.08	10.5369	34.7746	10.3017	34.5794	10.0765	34.3982	9.8604	34.2304
.10	10.4414	32.5240	10.2062	32.3232	9.9810	32.1358	9.7650	31.9612
.12	10.3494	30.6775	10.1143	30.4731	9.8892	30.2817	9.6732	30.1027
.14	10.2614	29.1097	10.0263	28.9027	9.8012	28.7085	9.5853	28.5264
.16	10.1774	27.7461	9.9423	27.5372	9.7173	27.3411	9.5014	27.1568
.18	10.0976	26.5391	9.8625	26.3289	9.6375	26.1312	9.4217	25.9453
.20	10.0222	25.4564	9.7871	25.2452	9.5621	25.0462	9.3463	24.8590
.22	9.9512	24.4751	9.7162	24.2629	9.4911	24.0630	9.2753	23.8747
.24	9.8849	23.5782	9.6499	23.3652	9.4248	23.1645	9.2089	22.9752
.26	9.8235	22.7528	9.5884	22.5393	9.3632	22.3378	9.1473	22.1478
.28	9.7669	21.9893	9.5317	21.7751	9.3065	21.5729	9.0905	21.3822
.30	9.7152	21.2796	9.4800	21.0648	9.2547	20.8621	9.0386	20.6706
.32	9.6687	20.6175	9.4334	20.4022	9.2079	20.1988	8.9917	20.0067
.34	9.6273	19.9978	9.3918	19.7820	9.1663	19.5780	8.9499	19.3853
.36	9.5912	19.4164	9.3556	19.2000	9.1299	18.9954	8.9133	18.8020
.38	9.5603	18.8695	9.3245	18.6526	9.0987	18.4474	8.8819	18.2533
.40	9.5347	18.3543	9.2988	18.1368	9.0727	17.9310	8.8558	17.7362
.42	9.5145	17.8681	9.2784	17.6500	9.0521	17.4435	8.8349	17.2480
.44	9.4997	17.4088	9.2633	17.1900	9.0368	16.9828	8.8194	16.7866
.46	9.4902	16.9743	9.2536	16.7549	9.0269	16.5470	8.8092	16.3500
.48	9.4860	16.5630	9.2492	16.3429	9.0223	16.1343	8.8044	15.9364
.50	9.4872	16.1734	9.2502	15.9526	9.0230	15.7432	8.8048	15.5445
.52	9.4936	15.8041	9.2564	15.5826	9.0290	15.3724	8.8105	15.1728
.54	9.5053	15.4540	9.2679	15.2317	9.0402	15.0207	8.8215	14.8202
.56	9.5222	15.1219	9.2845	14.8988	9.0566	14.6870	8.8376	14.4856
.58	9.5441	14.8069	9.3062	14.5830	9.0781	14.3703	8.8588	14.1680
.60	9.5710	14.5080	9.3329	14.2834	9.1046	14.0698	8.8851	13.8666
.62	9.6028	14.2245	9.3645	13.9991	9.1360	13.7846	8.9163	13.5804
.64	9.6393	13.9555	9.4009	13.7293	9.1722	13.5140	8.9523	13.3089
.66	9.6806	13.7004	9.4420	13.4733	9.2131	13.2572	8.9930	13.0511
.68	9.7263	13.4584	9.4876	13.2306	9.2586	13.0135	9.0384	12.8065
.70	9.7764	13.2289	9.5377	13.0003	9.3085	12.7825	9.0882	12.5745
.72	9.8307	13.0113	9.5919	12.7820	9.3627	12.5633	9.1423	12.3545
.74	9.8892	12.8051	9.6503	12.5751	9.4210	12.3556	9.2006	12.1459
.76	9.9515	12.6097	9.7126	12.3790	9.4833	12.1587	9.2628	11.9481
.78	10.0176	12.4246	9.7787	12.1932	9.5494	11.9722	9.3290	11.7608
.80	10.0872	12.2493	9.8484	12.0173	9.6192	11.7956	9.3988	11.5834
.82	10.1603	12.0834	9.9215	11.8507	9.6924	11.6283	9.4721	11.4154
.84	10.2365	11.9264	9.9979	11.6931	9.7689	11.4700	9.5487	11.2564
.86	10.3159	11.7778	10.0774	11.5440	9.8485	11.3203	9.6285	11.1060
.88	10.3981	11.6373	10.1598	11.4030	9.9311	11.1787	9.7113	10.9637
.90	10.4830	11.5045	10.2449	11.2697	10.0165	11.0448	9.7968	10.8293
.92	10.5705	11.3790	10.3326	11.1437	10.1044	10.9183	9.8851	10.7022
.94	10.6604	11.2604	10.4227	11.0246	10.1948	10.7988	9.9757	10.5822
.96	10.7525	11.1483	10.5151	10.9122	10.2875	10.6860	10.0688	10.4688
.98	10.8467	11.0426	10.6095	10.8061	10.3822	10.5794	10.1639	10.3619
1.00	10.9427	10.9427	10.7059	10.7059	10.4789	10.4789	10.2609	10.2609

Table D–1: $0.161 \leq \hat{\rho}_4 \leq 0.182$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.161$		$\hat{\rho}_4 = 0.168$		$\hat{\rho}_4 = 0.175$		$\hat{\rho}_4 = 0.182$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	9.9246	48.5893	9.6935	48.6037	9.4721	48.6584	9.2597	48.7564
.04	9.8197	41.1721	9.5887	41.0635	9.3675	40.9779	9.1552	40.9158
.06	9.7177	36.9917	9.4869	36.8468	9.2658	36.7206	9.0537	36.6130
.08	9.6190	34.0512	9.3882	33.8891	9.1672	33.7438	8.9553	33.6150
.10	9.5237	31.7730	9.2930	31.6011	9.0722	31.4449	8.8603	31.3040
.12	9.4320	29.9089	9.2014	29.7306	8.9807	29.5674	8.7689	29.4187
.14	9.3442	28.3287	9.1137	28.1461	8.8930	27.9781	8.6813	27.8241
.16	9.2603	26.9562	9.0298	26.7705	8.8092	26.5989	8.5976	26.4410
.18	9.1806	25.7425	8.9502	25.5544	8.7295	25.3801	8.5180	25.2194
.20	9.1052	24.6545	8.8748	24.4645	8.6541	24.2882	8.4426	24.1251
.22	9.0342	23.6689	8.8038	23.4773	8.5831	23.2993	8.3716	23.1344
.24	8.9678	22.7682	8.7373	22.5754	8.5166	22.3960	8.3050	22.2295
.26	8.9061	21.9397	8.6755	21.7457	8.4547	21.5651	8.2430	21.3973
.28	8.8492	21.1732	8.6185	20.9782	8.3976	20.7965	8.1858	20.6275
.30	8.7971	20.4608	8.5663	20.2648	8.3453	20.0821	8.1333	19.9119
.32	8.7501	19.7961	8.5191	19.5992	8.2979	19.4154	8.0857	19.2442
.34	8.7081	19.1738	8.4770	18.9760	8.2555	18.7913	8.0431	18.6190
.36	8.6713	18.5897	8.4399	18.3910	8.2182	18.2053	8.0055	18.0319
.38	8.6397	18.0402	8.4080	17.8405	8.1860	17.6538	7.9730	17.4793
.40	8.6133	17.5221	8.3813	17.3216	8.1590	17.1338	7.9456	16.9582
.42	8.5921	17.0330	8.3599	16.8315	8.1372	16.6426	7.9235	16.4659
.44	8.5763	16.5707	8.3437	16.3681	8.1207	16.1781	7.9066	16.0001
.46	8.5658	16.1331	8.3328	15.9294	8.1094	15.7382	7.8949	15.5590
.48	8.5606	15.7185	8.3273	15.5137	8.1035	15.3213	7.8885	15.1407
.50	8.5607	15.3255	8.3270	15.1195	8.1028	14.9258	7.8873	14.7439
.52	8.5660	14.9527	8.3320	14.7455	8.1073	14.5505	7.8914	14.3671
.54	8.5767	14.5989	8.3422	14.3905	8.1171	14.1942	7.9008	14.0093
.56	8.5924	14.2632	8.3576	14.0535	8.1322	13.8557	7.9154	13.6694
.58	8.6133	13.9444	8.3782	13.7334	8.1523	13.5342	7.9351	13.3463
.60	8.6393	13.6418	8.4038	13.4294	8.1776	13.2288	7.9599	13.0393
.62	8.6702	13.3544	8.4344	13.1407	8.2078	12.9386	7.9898	12.7475
.64	8.7060	13.0816	8.4699	12.8666	8.2430	12.6630	8.0246	12.4703
.66	8.7465	12.8226	8.5101	12.6063	8.2830	12.4012	8.0643	12.2069
.68	8.7917	12.5768	8.5551	12.3592	8.3277	12.1526	8.1087	11.9567
.70	8.8413	12.3437	8.6045	12.1246	8.3769	11.9167	8.1578	11.7192
.72	8.8953	12.1225	8.6584	11.9022	8.4307	11.6928	8.2113	11.4937
.74	8.9535	11.9127	8.7165	11.6911	8.4887	11.4804	8.2693	11.2798
.76	9.0158	11.7139	8.7788	11.4911	8.5509	11.2790	8.3314	11.0769
.78	9.0819	11.5255	8.8449	11.3015	8.6171	11.0881	8.3976	10.8846
.80	9.1518	11.3470	8.9149	11.1219	8.6871	10.9072	8.4677	10.7023
.82	9.2252	11.1780	8.9884	10.9519	8.7608	10.7360	8.5416	10.5297
.84	9.3020	11.0181	9.0654	10.7909	8.8380	10.5739	8.6190	10.3664
.86	9.3820	10.8668	9.1457	10.6386	8.9185	10.4205	8.6998	10.2118
.88	9.4650	10.7237	9.2290	10.4946	9.0021	10.2755	8.7837	10.0657
.90	9.5509	10.5885	9.3152	10.3585	9.0887	10.1384	8.8707	9.9275
.92	9.6395	10.4607	9.4041	10.2299	9.1781	10.0089	8.9605	9.7970
.94	9.7305	10.3400	9.4956	10.1085	9.2700	9.8866	9.0530	9.6738
.96	9.8239	10.2261	9.5895	9.9938	9.3644	9.7712	9.1479	9.5575
.98	9.9195	10.1185	9.6855	9.8856	9.4610	9.6623	9.2452	9.4479
1.00	10.0171	10.0171	9.7836	9.7836	9.5597	9.5597	9.3445	9.3445

Table D–1: $0.190 \leq \hat{\rho}_4 \leq 0.214$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.190$		$\hat{\rho}_4 = 0.198$		$\hat{\rho}_4 = 0.206$		$\hat{\rho}_4 = 0.214$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	9.0272	48.9259	8.8047	49.1635	8.5916	49.4781	8.3870	49.8814
.04	8.9229	40.8741	8.7006	40.8647	8.4877	40.8892	8.2834	40.9492
.06	8.8215	36.5128	8.5994	36.4371	8.3867	36.3863	8.1826	36.3610
.08	8.7233	33.4878	8.5014	33.3817	8.2889	33.2968	8.0850	33.2332
.10	8.6285	31.1613	8.4067	31.0380	8.1943	30.9339	7.9906	30.8488
.12	8.5372	29.2662	8.3156	29.1319	8.1034	29.0156	7.8998	28.9171
.14	8.4497	27.6648	8.2282	27.5230	8.0161	27.3983	7.8127	27.2906
.16	8.3661	26.2769	8.1447	26.1297	7.9326	25.9990	7.7293	25.8847
.18	8.2865	25.0516	8.0651	24.9003	7.8532	24.7651	7.6499	24.6457
.20	8.2111	23.9545	7.9898	23.8000	7.7778	23.6613	7.5746	23.5380
.22	8.1400	22.9614	7.9187	22.8043	7.7067	22.6628	7.5034	22.5364
.24	8.0734	22.0545	7.8520	21.8953	7.6399	21.7514	7.4366	21.6225
.26	8.0113	21.2206	7.7898	21.0596	7.5776	20.9136	7.3741	20.7824
.28	7.9539	20.4493	7.7322	20.2865	7.5198	20.1388	7.3162	20.0057
.30	7.9012	19.7323	7.6793	19.5681	7.4667	19.4186	7.2628	19.2837
.32	7.8534	19.0633	7.6312	18.8975	7.4183	18.7465	7.2141	18.6099
.34	7.8105	18.4367	7.5880	18.2696	7.3748	18.1170	7.1702	17.9787
.36	7.7726	17.8483	7.5497	17.6798	7.3361	17.5257	7.1311	17.3857
.38	7.7397	17.2944	7.5164	17.1244	7.3023	16.9687	7.0969	16.8271
.40	7.7119	16.7719	7.4881	16.6004	7.2736	16.4431	7.0676	16.2997
.42	7.6893	16.2781	7.4650	16.1051	7.2499	15.9461	7.0434	15.8010
.44	7.6719	15.8109	7.4471	15.6362	7.2314	15.4755	7.0242	15.3285
.46	7.6597	15.3682	7.4343	15.1918	7.2180	15.0293	7.0101	14.8803
.48	7.6527	14.9483	7.4267	14.7701	7.2098	14.6057	7.0012	14.4547
.50	7.6510	14.5497	7.4244	14.3697	7.2068	14.2033	6.9975	14.0502
.52	7.6545	14.1712	7.4273	13.9893	7.2090	13.8208	6.9990	13.6655
.54	7.6633	13.8116	7.4355	13.6276	7.2165	13.4570	7.0057	13.2994
.56	7.6773	13.4697	7.4489	13.2837	7.2292	13.1109	7.0176	12.9508
.58	7.6965	13.1447	7.4674	12.9566	7.2471	12.7815	7.0348	12.6190
.60	7.7208	12.8357	7.4912	12.6455	7.2702	12.4680	7.0571	12.3029
.62	7.7502	12.5420	7.5200	12.3495	7.2984	12.1697	7.0847	12.0020
.64	7.7845	12.2627	7.5539	12.0681	7.3317	11.8858	7.1174	11.7155
.66	7.8238	11.9973	7.5927	11.8004	7.3700	11.6157	7.1551	11.4428
.68	7.8679	11.7451	7.6364	11.5460	7.4133	11.3589	7.1979	11.1833
.70	7.9167	11.5056	7.6848	11.3043	7.4614	11.1147	7.2456	10.9364
.72	7.9700	11.2781	7.7380	11.0747	7.5142	10.8827	7.2981	10.7017
.74	8.0279	11.0623	7.7956	10.8567	7.5717	10.6623	7.3553	10.4788
.76	8.0899	10.8575	7.8576	10.6498	7.6336	10.4531	7.4172	10.2670
.78	8.1562	10.6634	7.9239	10.4536	7.6999	10.2547	7.4834	10.0661
.80	8.2264	10.4794	7.9942	10.2676	7.7703	10.0666	7.5540	9.8755
.82	8.3004	10.3051	8.0684	10.0915	7.8448	9.8883	7.6287	9.6949
.84	8.3781	10.1401	8.1464	9.9247	7.9230	9.7195	7.7073	9.5239
.86	8.4592	9.9840	8.2279	9.7669	8.0050	9.5598	7.7897	9.3621
.88	8.5436	9.8364	8.3128	9.6177	8.0904	9.4088	7.8757	9.2091
.90	8.6311	9.6969	8.4009	9.4767	8.1791	9.2661	7.9651	9.0646
.92	8.7215	9.5651	8.4919	9.3435	8.2709	9.1314	8.0576	8.9281
.94	8.8147	9.4408	8.5858	9.2178	8.3655	9.0042	8.1532	8.7993
.96	8.9103	9.3234	8.6822	9.0993	8.4629	8.8843	8.2515	8.6779
.98	9.0083	9.2128	8.7811	8.9875	8.5627	8.7713	8.3525	8.5635
1.00	9.1085	9.1085	8.8822	8.8822	8.6649	8.6649	8.4558	8.4558

Table D–1: $0.223 \leq \hat{\rho}_4 \leq 0.250$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.223$		$\hat{\rho}_4 = 0.232$		$\hat{\rho}_4 = 0.241$		$\hat{\rho}_4 = 0.250$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	8.1664	50.4604	7.9552	51.2028	7.7528	52.1529	7.5585	53.3831
.04	8.0631	41.0621	7.8522	41.2271	7.6501	41.4492	7.4561	41.7349
.06	7.9626	36.3640	7.7520	36.4018	7.5502	36.4761	7.3566	36.5894
.08	7.8652	33.1877	7.6549	33.1703	7.4534	33.1818	7.2601	33.2235
.10	7.7711	30.7763	7.5610	30.7284	7.3598	30.7057	7.1668	30.7087
.12	7.6805	28.8275	7.4706	28.7606	7.2696	28.7164	7.0768	28.6955
.14	7.5935	27.1895	7.3838	27.1096	7.1830	27.0510	6.9904	27.0140
.16	7.5103	25.7752	7.3007	25.6861	7.1000	25.6172	6.9076	25.5687
.18	7.4309	24.5301	7.2215	24.4340	7.0209	24.3575	6.8286	24.3005
.20	7.3556	23.4176	7.1462	23.3162	6.9456	23.2337	6.7534	23.1701
.22	7.2844	22.4121	7.0750	22.3064	6.8744	22.2192	6.6821	22.1504
.24	7.2175	21.4950	7.0079	21.3857	6.8073	21.2946	6.6149	21.2216
.26	7.1549	20.6522	6.9452	20.5400	6.7443	20.4456	6.5518	20.3689
.28	7.0967	19.8730	6.8868	19.7581	6.6857	19.6608	6.4929	19.5811
.30	7.0431	19.1488	6.8328	19.0315	6.6314	18.9317	6.4383	18.8491
.32	6.9940	18.4729	6.7834	18.3534	6.5816	18.2511	6.3880	18.1660
.34	6.9496	17.8397	6.7386	17.7181	6.5363	17.6135	6.3422	17.5259
.36	6.9100	17.2447	6.6984	17.1209	6.4955	17.0141	6.3009	16.9241
.38	6.8752	16.6841	6.6630	16.5582	6.4595	16.4491	6.2641	16.3567
.40	6.8453	16.1547	6.6324	16.0266	6.4282	15.9152	6.2320	15.8204
.42	6.8204	15.6538	6.6067	15.5235	6.4016	15.4097	6.2046	15.3123
.44	6.8004	15.1791	6.5859	15.0464	6.3800	14.9301	6.1821	14.8302
.46	6.7856	14.7286	6.5702	14.5934	6.3633	14.4745	6.1643	14.3718
.48	6.7758	14.3006	6.5595	14.1628	6.3516	14.0411	6.1516	13.9356
.50	6.7712	13.8935	6.5539	13.7530	6.3449	13.6284	6.1438	13.5198
.52	6.7717	13.5061	6.5534	13.3627	6.3434	13.2351	6.1411	13.1233
.54	6.7775	13.1372	6.5582	12.9908	6.3471	12.8600	6.1435	12.7449
.56	6.7885	12.7858	6.5682	12.6362	6.3559	12.5021	6.1512	12.3835
.58	6.8048	12.4509	6.5834	12.2981	6.3701	12.1606	6.1641	12.0382
.60	6.8263	12.1318	6.6039	11.9757	6.3895	11.8345	6.1823	11.7083
.62	6.8530	11.8277	6.6297	11.6681	6.4142	11.5232	6.2058	11.3931
.64	6.8849	11.5379	6.6607	11.3749	6.4442	11.2262	6.2348	11.0919
.66	6.9219	11.2619	6.6970	11.0953	6.4795	10.9427	6.2691	10.8042
.68	6.9641	10.9991	6.7384	10.8289	6.5202	10.6724	6.3087	10.5296
.70	7.0112	10.7490	6.7850	10.5751	6.5660	10.4146	6.3538	10.2675
.72	7.0633	10.5111	6.8366	10.3336	6.6171	10.1691	6.4042	10.0175
.74	7.1203	10.2848	6.8931	10.1037	6.6732	9.9352	6.4599	9.7793
.76	7.1819	10.0699	6.9546	9.8852	6.7344	9.7128	6.5207	9.5526
.78	7.2482	9.8658	7.0208	9.6777	6.8005	9.5015	6.5867	9.3369
.80	7.3189	9.6723	7.0916	9.4808	6.8714	9.3007	6.6577	9.1320
.82	7.3938	9.4888	7.1668	9.2940	6.9469	9.1103	6.7335	8.9375
.84	7.4729	9.3150	7.2463	9.1170	7.0269	8.9298	6.8140	8.7530
.86	7.5558	9.1505	7.3299	8.9496	7.1112	8.7590	6.8990	8.5784
.88	7.6425	8.9950	7.4174	8.7912	7.1996	8.5974	6.9883	8.4132
.90	7.7328	8.8480	7.5086	8.6416	7.2918	8.4448	7.0817	8.2572
.92	7.8264	8.7093	7.6033	8.5004	7.3878	8.3007	7.1790	8.1099
.94	7.9230	8.5785	7.7012	8.3672	7.4871	8.1649	7.2800	7.9711
.96	8.0226	8.4551	7.8023	8.2417	7.5897	8.0370	7.3843	7.8405
.98	8.1249	8.3390	7.9061	8.1236	7.6953	7.9167	7.4918	7.7176
1.00	8.2297	8.2297	8.0125	8.0125	7.8036	7.8036	7.6022	7.6022

Table D–1: $0.26 \leq \hat{\rho}_4 \leq 0.29$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.26$		$\hat{\rho}_4 = 0.27$		$\hat{\rho}_4 = 0.28$		$\hat{\rho}_4 = 0.29$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	7.3514	55.2232	7.1532	57.8791	6.9630	62.1808	6.7805	71.9138
.04	7.2495	42.1368	7.0517	42.6421	6.8621	43.2708	6.6801	44.0506
.06	7.1504	36.7643	6.9530	36.9955	6.7638	37.2891	6.5823	37.6529
.08	7.0543	33.3069	6.8573	33.4318	6.6685	33.6009	6.4875	33.8181
.10	6.9613	30.7433	6.7647	30.8122	6.5763	30.9172	6.3957	31.0604
.12	6.8717	28.7001	6.6754	28.7350	6.4873	28.8012	6.3071	28.9004
.14	6.7855	26.9986	6.5894	27.0108	6.4017	27.0514	6.2217	27.1215
.16	6.7029	25.5390	6.5070	25.5352	6.3195	25.5578	6.1398	25.6077
.18	6.6239	24.2602	6.4282	24.2445	6.2409	24.2539	6.0613	24.2890
.20	6.5488	23.1217	6.3532	23.0970	6.1659	23.0964	5.9864	23.1202
.22	6.4775	22.0956	6.2819	22.0638	6.0946	22.0552	5.9151	22.0702
.24	6.4102	21.1616	6.2144	21.1240	6.0271	21.1090	5.8475	21.1170
.26	6.3469	20.3046	6.1510	20.2622	5.9634	20.2419	5.7836	20.2440
.28	6.2877	19.5129	6.0915	19.4664	5.9036	19.4416	5.7236	19.4389
.30	6.2327	18.7776	6.0361	18.7275	5.8478	18.6988	5.6674	18.6917
.32	6.1820	18.0914	5.9848	18.0379	5.7960	18.0057	5.6151	17.9949
.34	6.1356	17.4484	5.9378	17.3918	5.7484	17.3563	5.5668	17.3420
.36	6.0935	16.8438	5.8951	16.7843	5.7049	16.7456	5.5225	16.7281
.38	6.0560	16.2736	5.8567	16.2112	5.6656	16.1695	5.4824	16.1487
.40	6.0230	15.7345	5.8227	15.6691	5.6307	15.6243	5.4464	15.6004
.42	5.9946	15.2235	5.7933	15.1551	5.6001	15.1072	5.4147	15.0802
.44	5.9709	14.7383	5.7684	14.6667	5.5740	14.6157	5.3873	14.5854
.46	5.9520	14.2768	5.7482	14.2020	5.5525	14.1475	5.3643	14.1138
.48	5.9379	13.8372	5.7328	13.7589	5.5356	13.7009	5.3459	13.6637
.50	5.9288	13.4180	5.7222	13.3360	5.5235	13.2743	5.3321	13.2332
.52	5.9247	13.0177	5.7166	12.9319	5.5162	12.8662	5.3231	12.8211
.54	5.9257	12.6353	5.7160	12.5454	5.5139	12.4755	5.3190	12.4260
.56	5.9318	12.2698	5.7205	12.1755	5.5166	12.1011	5.3198	12.0470
.58	5.9432	11.9202	5.7303	11.8213	5.5246	11.7421	5.3258	11.6831
.60	5.9600	11.5857	5.7453	11.4821	5.5379	11.3978	5.3371	11.3335
.62	5.9821	11.2657	5.7658	11.1570	5.5566	11.0674	5.3538	10.9976
.64	6.0096	10.9597	5.7918	10.8457	5.5808	10.7505	5.3761	10.6748
.66	6.0426	10.6669	5.8233	10.5475	5.6107	10.4464	5.4040	10.3645
.68	6.0811	10.3871	5.8605	10.2620	5.6462	10.1549	5.4378	10.0665
.70	6.1251	10.1197	5.9033	9.9888	5.6876	9.8754	5.4776	9.7803
.72	6.1747	9.8644	5.9518	9.7277	5.7349	9.6078	5.5234	9.5057
.74	6.2296	9.6208	6.0059	9.4782	5.7880	9.3518	5.5753	9.2425
.76	6.2901	9.3887	6.0658	9.2401	5.8471	9.1071	5.6334	8.9905
.78	6.3558	9.1677	6.1312	9.0131	5.9121	8.8735	5.6978	8.7495
.80	6.4268	8.9576	6.2022	8.7971	5.9830	8.6509	5.7684	8.5195
.82	6.5029	8.7579	6.2786	8.5917	6.0596	8.4389	5.8452	8.3002
.84	6.5840	8.5686	6.3603	8.3967	6.1419	8.2375	5.9280	8.0915
.86	6.6699	8.3892	6.4471	8.2118	6.2297	8.0465	6.0168	7.8934
.88	6.7604	8.2194	6.5388	8.0369	6.3228	7.8656	6.1115	7.7057
.90	6.8552	8.0591	6.6353	7.8716	6.4211	7.6946	6.2117	7.5282
.92	6.9543	7.9078	6.7362	7.7157	6.5242	7.5334	6.3172	7.3607
.94	7.0572	7.7652	6.8414	7.5688	6.6318	7.3815	6.4278	7.2031
.96	7.1638	7.6311	6.9505	7.4307	6.7438	7.2387	6.5430	7.0550
.98	7.2737	7.5050	7.0633	7.3010	6.8598	7.1048	6.6627	6.9162
1.00	7.3868	7.3868	7.1793	7.1793	6.9794	6.9794	6.7863	6.7863

Table D–1: $0.301 \leq \hat{\rho}_4 \leq 0.334$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.301$		$\hat{\rho}_4 = 0.312$		$\hat{\rho}_4 = 0.323$		$\hat{\rho}_4 = 0.334$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	6.4882	45.1317	6.3045	46.5288	6.1283	48.3901	5.9594	51.0040
.06	6.3910	38.1458	6.2079	38.7527	6.0324	39.4976	5.8641	40.4142
.08	6.2967	34.1179	6.1141	34.4888	5.9392	34.9405	5.7716	35.4857
.10	6.2054	31.2653	6.0233	31.5239	5.8489	31.8419	5.6818	32.2257
.12	6.1172	29.0494	5.9356	29.2433	5.7617	29.4854	5.5951	29.7800
.14	6.0322	27.2343	5.8510	27.3864	5.6775	27.5803	5.5113	27.8189
.16	5.9506	25.6952	5.7696	25.8184	5.5965	25.9793	5.4307	26.1800
.18	5.8723	24.3581	5.6916	24.4604	5.5187	24.5975	5.3532	24.7710
.20	5.7975	23.1754	5.6169	23.2620	5.4442	23.3812	5.2789	23.5346
.22	5.7262	22.1146	5.5457	22.1891	5.3730	22.2946	5.2078	22.4325
.24	5.6585	21.1528	5.4779	21.2176	5.3052	21.3123	5.1399	21.4381
.26	5.5945	20.2728	5.4137	20.3297	5.2408	20.4157	5.0754	20.5318
.28	5.5341	19.4617	5.3530	19.5122	5.1798	19.5910	5.0141	19.6991
.30	5.4774	18.7095	5.2959	18.7544	5.1223	18.8272	4.9562	18.9288
.32	5.4246	18.0081	5.2425	18.0483	5.0683	18.1159	4.9016	18.2118
.34	5.3756	17.3513	5.1927	17.3871	5.0178	17.4502	4.8505	17.5412
.36	5.3304	16.7335	5.1467	16.7654	4.9710	16.8244	4.8027	16.9111
.38	5.2893	16.1506	5.1045	16.1788	4.9278	16.2339	4.7585	16.3167
.40	5.2521	15.5988	5.0662	15.6234	4.8882	15.6749	4.7178	15.7540
.42	5.2191	15.0750	5.0318	15.0960	4.8525	15.1440	4.6807	15.2196
.44	5.1902	14.5766	5.0015	14.5940	4.8206	14.6384	4.6472	14.7106
.46	5.1657	14.1013	4.9752	14.1150	4.7926	14.1558	4.6175	14.2245
.48	5.1455	13.6471	4.9532	13.6570	4.7687	13.6940	4.5917	13.7591
.50	5.1298	13.2125	4.9355	13.2182	4.7490	13.2513	4.5698	13.3127
.52	5.1187	12.7959	4.9223	12.7972	4.7335	12.8260	4.5520	12.8834
.54	5.1124	12.3961	4.9137	12.3926	4.7225	12.4168	4.5385	12.4699
.56	5.1110	12.0119	4.9099	12.0034	4.7161	12.0226	4.5294	12.0710
.58	5.1147	11.6425	4.9110	11.6284	4.7145	11.6423	4.5249	11.6856
.60	5.1236	11.2870	4.9173	11.2670	4.7179	11.2750	4.5253	11.3126
.62	5.1378	10.9448	4.9289	10.9183	4.7265	10.9199	4.5307	10.9514
.64	5.1577	10.6152	4.9460	10.5818	4.7407	10.5764	4.5415	10.6011
.66	5.1833	10.2978	4.9689	10.2570	4.7605	10.2441	4.5580	10.2613
.68	5.2148	9.9921	4.9978	9.9434	4.7864	9.9223	4.5805	9.9313
.70	5.2524	9.6980	5.0329	9.6408	4.8186	9.6110	4.6093	9.6110
.72	5.2963	9.4151	5.0745	9.3490	4.8574	9.3098	4.6448	9.3001
.74	5.3466	9.1433	5.1227	9.0679	4.9031	9.0186	4.6874	8.9984
.76	5.4034	8.8824	5.1778	8.7972	4.9560	8.7374	4.7375	8.7058
.78	5.4668	8.6324	5.2399	8.5372	5.0164	8.4662	4.7955	8.4225
.80	5.5369	8.3932	5.3092	8.2876	5.0844	8.2051	4.8618	8.1487
.82	5.6136	8.1647	5.3857	8.0487	5.1604	7.9543	4.9366	7.8846
.84	5.6971	7.9469	5.4695	7.8204	5.2443	7.7140	5.0203	7.6306
.86	5.7870	7.7399	5.5606	7.6030	5.3364	7.4845	5.1131	7.3871
.88	5.8834	7.5434	5.6588	7.3963	5.4365	7.2659	5.2150	7.1544
.90	5.9860	7.3576	5.7640	7.2006	5.5445	7.0585	5.3261	6.9330
.92	6.0945	7.1822	5.8760	7.0158	5.6603	6.8624	5.4461	6.7234
.94	6.2087	7.0170	5.9943	6.8418	5.7834	6.6778	5.5748	6.5259
.96	6.3282	6.8620	6.1186	6.6785	5.9135	6.5046	5.7116	6.3406
.98	6.4525	6.7169	6.2485	6.5258	6.0500	6.3428	5.8560	6.1677
1.00	6.5813	6.5813	6.3835	6.3835	6.1923	6.1923	6.0072	6.0072

Table D–1: $0.346 \leq \hat{\rho}_4 \leq 0.382$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.346$		$\hat{\rho}_4 = 0.358$		$\hat{\rho}_4 = 0.370$		$\hat{\rho}_4 = 0.382$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	5.7828	55.5853	5.6137	66.5241				
.06	5.6882	41.6700	5.5199	43.2973	5.3587	45.4912	5.2042	48.6694
.08	5.5964	36.2068	5.4288	37.0896	5.2684	38.1788	5.1147	39.5434
.10	5.5073	32.7298	5.3404	33.3372	5.1807	34.0678	5.0278	34.9489
.12	5.4211	30.1677	5.2549	30.6332	5.0958	31.1878	4.9435	31.8466
.14	5.3379	28.1348	5.1722	28.5147	5.0137	28.9659	4.8620	29.4982
.16	5.2577	26.4478	5.0924	26.7708	4.9344	27.1547	4.7832	27.6064
.18	5.1805	25.0046	5.0156	25.2878	4.8580	25.6249	4.7072	26.0212
.20	5.1064	23.7430	4.9418	23.9969	4.7844	24.3000	4.6340	24.6565
.22	5.0354	22.6217	4.8709	22.8536	4.7138	23.1311	4.5636	23.4578
.24	4.9676	21.6122	4.8031	21.8270	4.6460	22.0848	4.4959	22.3888
.26	4.9029	20.6940	4.7383	20.8953	4.5812	21.1377	4.4310	21.4239
.28	4.8414	19.8517	4.6765	20.0421	4.5192	20.2722	4.3689	20.5443
.30	4.7830	19.0734	4.6178	19.2549	4.4601	19.4750	4.3096	19.7359
.32	4.7279	18.3498	4.5622	18.5240	4.4040	18.7360	4.2530	18.9878
.34	4.6760	17.6735	4.5096	17.8415	4.3508	18.0468	4.1992	18.2912
.36	4.6274	17.0385	4.4601	17.2012	4.3004	17.4009	4.1481	17.6392
.38	4.5820	16.4397	4.4137	16.5979	4.2530	16.7928	4.0998	17.0261
.40	4.5400	15.8730	4.3704	16.0271	4.2086	16.2179	4.0542	16.4470
.42	4.5014	15.3348	4.3304	15.4853	4.1672	15.6725	4.0114	15.8981
.44	4.4663	14.8222	4.2936	14.9692	4.1287	15.1533	3.9715	15.3758
.46	4.4347	14.3325	4.2601	14.4763	4.0934	14.6574	3.9344	14.8773
.48	4.4067	13.8635	4.2300	14.0041	4.0612	14.1824	3.9002	14.4000
.50	4.3825	13.4133	4.2034	13.5506	4.0323	13.7261	3.8690	13.9416
.52	4.3621	12.9801	4.1804	13.1139	4.0067	13.2867	3.8409	13.5001
.54	4.3457	12.5624	4.1612	12.6926	3.9846	12.8624	3.8159	13.0738
.56	4.3336	12.1588	4.1458	12.2851	3.9661	12.4518	3.7943	12.6611
.58	4.3258	11.7683	4.1346	11.8902	3.9514	12.0535	3.7761	12.2605
.60	4.3226	11.3897	4.1277	11.5067	3.9407	11.6662	3.7616	11.8706
.62	4.3243	11.0222	4.1255	11.1337	3.9343	11.2888	3.7510	11.4903
.64	4.3312	10.6649	4.1281	10.7703	3.9325	10.9204	3.7445	11.1183
.66	4.3436	10.3173	4.1360	10.4156	3.9355	10.5599	3.7426	10.7538
.68	4.3618	9.9787	4.1496	10.0691	3.9440	10.2067	3.7456	10.3955
.70	4.3864	9.6488	4.1693	9.7301	3.9583	9.8598	3.7540	10.0427
.72	4.4177	9.3271	4.1956	9.3982	3.9789	9.5188	3.7683	9.6944
.74	4.4562	9.0136	4.2292	9.0731	4.0066	9.1831	3.7894	9.3500
.76	4.5025	8.7082	4.2706	8.7547	4.0422	8.8523	3.8179	9.0086
.78	4.5570	8.4109	4.3207	8.4428	4.0864	8.5262	3.8549	8.6698
.80	4.6205	8.1219	4.3801	8.1377	4.1404	8.2047	3.9017	8.3331
.82	4.6933	7.8416	4.4497	7.8397	4.2051	7.8881	3.9595	7.9983
.84	4.7760	7.5704	4.5302	7.5493	4.2818	7.5767	4.0300	7.6655
.86	4.8690	7.3091	4.6225	7.2673	4.3717	7.2714	4.1150	7.3351
.88	4.9726	7.0581	4.7272	6.9946	4.4761	6.9733	4.2167	7.0081
.90	5.0871	6.8185	4.8448	6.7325	4.5960	6.6839	4.3370	6.6861
.92	5.2122	6.5907	4.9755	6.4822	4.7323	6.4052	4.4780	6.3717
.94	5.3479	6.3757	5.1193	6.2450	4.8854	6.1392	4.6411	6.0681
.96	5.4936	6.1740	5.2758	6.0220	5.0550	5.8883	4.8268	5.7791
.98	5.6486	5.9859	5.4440	5.8144	5.2402	5.6544	5.0343	5.5090
1.00	5.8118	5.8118	5.6227	5.6227	5.4393	5.4393	5.2613	5.2613

Table D–1: $0.394 \leq \hat{\rho}_4 \leq 0.430$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.394$		$\hat{\rho}_4 = 0.406$		$\hat{\rho}_4 = 0.418$		$\hat{\rho}_4 = 0.430$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.06	5.0560	54.0416	4.9138	70.4214				
.08	4.9673	41.2989	4.8259	43.6607	4.6901	47.1054	4.5596	53.1079
.10	4.8812	36.0202	4.7405	37.3425	4.6056	39.0148	4.4760	41.2144
.12	4.7976	32.6299	4.6577	33.5666	4.5235	34.6979	4.3947	36.0876
.14	4.7167	30.1244	4.5775	30.8616	4.4440	31.7329	4.3159	32.7713
.16	4.6385	28.1347	4.4998	28.7512	4.3669	29.4712	4.2394	30.3151
.18	4.5629	26.4834	4.4248	27.0200	4.2924	27.6419	4.1655	28.3636
.20	4.4901	25.0715	4.3523	25.5519	4.2204	26.1060	4.0939	26.7445
.22	4.4199	23.8380	4.2824	24.2772	4.1508	24.7821	4.0247	25.3613
.24	4.3524	22.7426	4.2151	23.1507	4.0837	23.6189	3.9579	24.1541
.26	4.2875	21.7571	4.1503	22.1414	4.0191	22.5814	3.8934	23.0833
.28	4.2253	20.8615	4.0881	21.2270	3.9568	21.6452	3.8313	22.1212
.30	4.1658	20.0402	4.0283	20.3910	3.8970	20.7920	3.7714	21.2479
.32	4.1088	19.2818	3.9711	19.6208	3.8395	20.0083	3.7137	20.4483
.34	4.0544	18.5769	3.9163	18.9066	3.7843	19.2833	3.6582	19.7108
.36	4.0027	17.9182	3.8639	18.2404	3.7314	18.6086	3.6049	19.0264
.38	3.9535	17.2997	3.8139	17.6159	3.6808	17.9775	3.5537	18.3877
.40	3.9069	16.7163	3.7664	17.0279	3.6324	17.3844	3.5046	17.7888
.42	3.8629	16.1638	3.7213	16.4718	3.5862	16.8244	3.4575	17.2246
.44	3.8215	15.6387	3.6785	15.9439	3.5423	16.2938	3.4126	16.6911
.46	3.7827	15.1379	3.6382	15.4411	3.5006	15.7892	3.3696	16.1847
.48	3.7467	14.6587	3.6004	14.9605	3.4612	15.3076	3.3287	15.7024
.50	3.7133	14.1988	3.5650	14.4998	3.4240	14.8465	3.2899	15.2414
.52	3.6827	13.7561	3.5322	14.0566	3.3890	14.4036	3.2531	14.7994
.54	3.6551	13.3287	3.5019	13.6291	3.3564	13.9769	3.2183	14.3744
.56	3.6304	12.9150	3.4743	13.2155	3.3261	13.5646	3.1856	13.9644
.58	3.6088	12.5133	3.4495	12.8142	3.2983	13.1650	3.1550	13.5678
.60	3.5905	12.1224	3.4276	12.4237	3.2730	12.7765	3.1267	13.1829
.62	3.5757	11.7407	3.4088	12.0425	3.2503	12.3977	3.1005	12.8083
.64	3.5646	11.3672	3.3932	11.6694	3.2305	12.0272	3.0768	12.4425
.66	3.5576	11.0005	3.3811	11.3030	3.2136	11.6636	3.0556	12.0841
.68	3.5550	10.6395	3.3729	10.9421	3.2000	11.3056	3.0370	11.7319
.70	3.5572	10.2832	3.3688	10.5853	3.1899	10.9518	3.0213	11.3843
.72	3.5648	9.9304	3.3695	10.2315	3.1838	10.6007	3.0087	11.0400
.74	3.5784	9.5801	3.3754	9.8792	3.1820	10.2510	2.9996	10.6974
.76	3.5990	9.2313	3.3874	9.5271	3.1852	9.9010	2.9945	10.3550
.78	3.6275	8.8829	3.4064	9.1739	3.1942	9.5490	2.9938	10.0108
.80	3.6653	8.5339	3.4336	8.8178	3.2101	9.1929	2.9985	9.6627
.82	3.7138	8.1836	3.4708	8.4574	3.2344	8.8306	3.0096	9.3083
.84	3.7752	7.8311	3.5200	8.0908	3.2689	8.4593	3.0285	8.9442
.86	3.8519	7.4762	3.5840	7.7162	3.3164	8.0756	3.0574	8.5666
.88	3.9469	7.1189	3.6668	7.3319	3.3809	7.6757	3.0995	8.1700
.90	4.0637	6.7602	3.7735	6.9368	3.4683	7.2547	3.1599	7.7470
.92	4.2064	6.4023	3.9108	6.5305	3.5878	6.8069	3.2470	7.2863
.94	4.3787	6.0492	4.0872	6.1153	3.7536	6.3264	3.3766	6.7703
.96	4.5834	5.7072	4.3115	5.6983	3.9875	5.8108	3.5817	6.1696
.98	4.8210	5.3841	4.5899	5.2931	4.3172	5.2721	3.9399	5.4389
1.00	5.0881	5.0881	4.9194	4.9194	4.7547	4.7548	4.5939	4.5939

Table D–1: $0.442 \leq \hat{\rho}_4 \leq 0.478$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.442$		$\hat{\rho}_4 = 0.454$		$\hat{\rho}_4 = 0.466$		$\hat{\rho}_4 = 0.478$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.08	4.4343	79.8432						
.10	4.3515	44.3122	4.2317	49.3212	4.1164	61.9369		
.12	4.2710	37.8406	4.1520	40.1475	4.0376	43.4154	3.9274	48.8048
.14	4.1928	34.0258	4.0746	35.5738	3.9610	37.5463	3.8517	40.1963
.16	4.1171	31.3119	3.9996	32.5038	3.8867	33.9555	3.7781	35.7737
.18	4.0437	29.2044	3.9268	30.1916	3.8145	31.3642	3.7066	32.7818
.20	3.9726	27.4819	3.8563	28.3375	3.7446	29.3378	3.6372	30.5218
.22	3.9039	26.0260	3.7880	26.7907	3.6767	27.6755	3.5699	28.7079
.24	3.8374	24.7655	3.7218	25.4647	3.6110	26.2674	3.5047	27.1948
.26	3.7731	23.6545	3.6579	24.3049	3.5474	25.0473	3.4414	25.8985
.28	3.7111	22.6617	3.5960	23.2749	3.4858	23.9716	3.3801	24.7659
.30	3.6512	21.7644	3.5362	22.3488	3.4261	23.0104	3.3206	23.7613
.32	3.5934	20.9460	3.4784	21.5080	3.3683	22.1424	3.2630	22.8597
.34	3.5377	20.1939	3.4226	20.7383	3.3125	21.3514	3.2071	22.0426
.36	3.4841	19.4980	3.3687	20.0287	3.2584	20.6252	3.1530	21.2961
.38	3.4324	18.8504	3.3166	19.3706	3.2061	19.9544	3.1006	20.6095
.40	3.3827	18.2448	3.2665	18.7571	3.1556	19.3311	3.0498	19.9742
.42	3.3349	17.6759	3.2181	18.1823	3.1067	18.7493	3.0006	19.3835
.44	3.2890	17.1391	3.1714	17.6417	3.0595	18.2038	2.9529	18.8318
.46	3.2450	16.6308	3.1265	17.1311	3.0138	17.6903	2.9067	18.3144
.48	3.2029	16.1477	3.0833	16.6473	2.9697	17.2053	2.8619	17.8275
.50	3.1626	15.6872	3.0418	16.1872	2.9272	16.7457	2.8185	17.3677
.52	3.1241	15.2467	3.0019	15.7484	2.8861	16.3087	2.7765	16.9323
.54	3.0874	14.8240	2.9636	15.3287	2.8465	15.8921	2.7358	16.5188
.56	3.0526	14.4173	2.9269	14.9260	2.8083	15.4937	2.6964	16.1250
.58	3.0196	14.0248	2.8919	14.5385	2.7716	15.1119	2.6583	15.7492
.60	2.9885	13.6449	2.8585	14.1646	2.7362	14.7449	2.6213	15.3896
.62	2.9594	13.2760	2.8267	13.8029	2.7022	14.3913	2.5856	15.0447
.64	2.9322	12.9168	2.7965	13.4519	2.6696	14.0497	2.5511	14.7133
.66	2.9070	12.5659	2.7681	13.1103	2.6384	13.7188	2.5177	14.3940
.68	2.8840	12.2220	2.7413	12.7768	2.6086	13.3975	2.4854	14.0858
.70	2.8633	11.8837	2.7164	12.4503	2.5802	13.0845	2.4543	13.7876
.72	2.8451	11.5497	2.6933	12.1294	2.5532	12.7789	2.4243	13.4984
.74	2.8295	11.2185	2.6722	11.8130	2.5277	12.4794	2.3954	13.2173
.76	2.8169	10.8886	2.6533	11.4995	2.5037	12.1851	2.3676	12.9434
.78	2.8076	10.5583	2.6367	11.1877	2.4814	11.8947	2.3410	12.6758
.80	2.8021	10.2256	2.6227	10.8759	2.4608	11.6070	2.3155	12.4136
.82	2.8011	9.8881	2.6116	10.5622	2.4420	11.3208	2.2913	12.1560
.84	2.8055	9.5430	2.6040	10.2444	2.4253	11.0344	2.2682	11.9019
.86	2.8166	9.1865	2.6004	9.9197	2.4109	10.7462	2.2465	11.6505
.88	2.8364	8.8137	2.6019	9.5845	2.3992	10.4541	2.2261	11.4005
.90	2.8681	8.4171	2.6098	9.2338	2.3907	10.1552	2.2073	11.1507
.92	2.9167	7.9856	2.6263	8.8605	2.3860	9.8460	2.1901	10.8996
.94	2.9923	7.4999	2.6553	8.4527	2.3862	9.5215	2.1748	10.6453
.96	3.1170	6.9211	2.7042	7.9896	2.3932	9.1740	2.1618	10.3856
.98	3.3615	6.1418	2.7910	7.4240	2.4100	8.7907	2.1514	10.1172
1.00	3.3635	5.2636	2.9874	6.5859	2.4430	8.3468	2.1444	9.8357

Table D–1: $0.490 \leq \hat{\rho}_4 \leq 0.526$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.490$		$\hat{\rho}_4 = 0.502$		$\hat{\rho}_4 = 0.514$		$\hat{\rho}_4 = 0.526$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.12	3.8214	64.2211						
.14	3.7464	44.1155	3.6450	51.3837				
.16	3.6736	38.1536	3.5729	41.5147	3.4759	47.0716	3.3824	63.7499
.18	3.6028	34.5411	3.5029	36.8151	3.4066	39.9625	3.3138	44.9480
.20	3.5341	31.9479	3.4348	33.7114	3.3392	35.9819	3.2471	39.1087
.22	3.4673	29.9280	3.3686	31.3973	3.2737	33.2157	3.1823	35.5637
.24	3.4025	28.2765	3.3044	29.5562	3.2100	31.1011	3.1191	33.0233
.26	3.3397	26.8819	3.2420	28.0307	3.1481	29.3943	3.0578	31.0502
.28	3.2787	25.6769	3.1814	26.7312	3.0879	27.9671	2.9981	29.4427
.30	3.2195	24.6177	3.1225	25.6014	3.0294	26.7438	2.9400	28.0907
.32	3.1621	23.6740	3.0653	24.6041	2.9725	25.6761	2.8835	26.9280
.34	3.1063	22.8242	3.0098	23.7128	2.9172	24.7310	2.8285	25.9108
.36	3.0522	22.0523	2.9558	22.9087	2.8634	23.8850	2.7749	25.0094
.38	2.9997	21.3460	2.9033	22.1771	2.8111	23.1211	2.7228	24.2024
.40	2.9488	20.6956	2.8523	21.5074	2.7602	22.4260	2.6720	23.4739
.42	2.8993	20.0935	2.8028	20.8906	2.7106	21.7898	2.6225	22.8117
.44	2.8513	19.5336	2.7546	20.3198	2.6623	21.2044	2.5743	22.2064
.46	2.8047	19.0108	2.7077	19.7894	2.6153	20.6633	2.5272	21.6505
.48	2.7594	18.5208	2.6620	19.2946	2.5694	20.1614	2.4813	21.1380
.50	2.7154	18.0601	2.6176	18.8317	2.5248	19.6943	2.4365	20.6639
.52	2.6727	17.6256	2.5744	18.3972	2.4812	19.2582	2.3927	20.2241
.54	2.6312	17.2149	2.5323	17.9885	2.4386	18.8503	2.3500	19.8152
.56	2.5908	16.8256	2.4912	17.6030	2.3971	18.4679	2.3081	19.4343
.58	2.5516	16.4557	2.4512	17.2389	2.3566	18.1087	2.2673	19.0789
.60	2.5135	16.1037	2.4122	16.8942	2.3169	17.7708	2.2273	18.7470
.62	2.4764	15.7679	2.3741	16.5674	2.2782	17.4526	2.1881	18.4367
.64	2.4404	15.4470	2.3370	16.2571	2.2403	17.1527	2.1497	18.1466
.66	2.4053	15.1398	2.3007	15.9621	2.2032	16.8697	2.1121	17.8752
.68	2.3712	14.8452	2.2653	15.6814	2.1668	16.6026	2.0751	17.6215
.70	2.3380	14.5623	2.2306	15.4139	2.1312	16.3505	2.0389	17.3843
.72	2.3058	14.2902	2.1968	15.1589	2.0963	16.1124	2.0033	17.1629
.74	2.2744	14.0280	2.1637	14.9156	2.0620	15.8877	1.9684	16.9565
.76	2.2439	13.7749	2.1313	14.6833	2.0284	15.6757	1.9340	16.7644
.78	2.2142	13.5304	2.0995	14.4614	1.9953	15.4758	1.9002	16.5860
.80	2.1854	13.2937	2.0684	14.2494	1.9628	15.2877	1.8668	16.4209
.82	2.1573	13.0642	2.0379	14.0469	1.9308	15.1108	1.8340	16.2687
.84	2.1301	12.8414	2.0080	13.8534	1.8993	14.9448	1.8017	16.1290
.86	2.1036	12.6246	1.9787	13.6685	1.8683	14.7895	1.7697	16.0016
.88	2.0779	12.4133	1.9498	13.4920	1.8377	14.6446	1.7382	15.8861
.90	2.0530	12.2069	1.9215	13.3237	1.8075	14.5100	1.7071	15.7825
.92	2.0289	12.0049	1.8936	13.1631	1.7776	14.3855	1.6763	15.6907
.94	2.0055	11.8067	1.8661	13.0103	1.7481	14.2709	1.6459	15.6104
.96	1.9830	11.6118	1.8390	12.8651	1.7189	14.1664	1.6158	15.5416
.98	1.9613	11.4193	1.8123	12.7274	1.6900	14.0720	1.5859	15.4844
1.00	1.9405	11.2286	1.7859	12.5972	1.6613	13.9875	1.5564	15.4387

Table D–1: $0.538 \leq \hat{\rho}_4 \leq 0.574$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.538$		$\hat{\rho}_4 = 0.550$		$\hat{\rho}_4 = 0.562$		$\hat{\rho}_4 = 0.574$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.18	3.2243	56.8961						
.20	3.1584	44.0215	3.0726	55.4584				
.22	3.0942	38.8200	3.0091	44.0299	2.9270	57.3016		
.24	3.0316	35.5298	2.9472	39.0740	2.8658	45.0216	2.7871	66.3575
.26	2.9708	33.1312	2.8870	35.8939	2.8062	39.9409	2.7282	47.4182
.28	2.9116	31.2523	2.8284	33.5636	2.7481	36.7209	2.6707	41.6362
.30	2.8540	29.7140	2.7713	31.7340	2.6915	34.3782	2.6147	38.1597
.32	2.7979	28.4169	2.7156	30.2357	2.6364	32.5487	2.5600	35.6918
.34	2.7432	27.2999	2.6614	28.9730	2.5826	31.0571	2.5067	33.7943
.36	2.6900	26.3226	2.6085	27.8870	2.5301	29.8056	2.4547	32.2646
.38	2.6381	25.4570	2.5569	26.9387	2.4790	28.7337	2.4040	30.9928
.40	2.5876	24.6829	2.5067	26.1006	2.4290	27.8016	2.3544	29.9124
.42	2.5382	23.9853	2.4576	25.3533	2.3802	26.9817	2.3060	28.9800
.44	2.4901	23.3526	2.4097	24.6820	2.3326	26.2541	2.2587	28.1658
.46	2.4432	22.7759	2.3629	24.0754	2.2860	25.6038	2.2124	27.4486
.48	2.3973	22.2479	2.3171	23.5248	2.2405	25.0195	2.1672	26.8124
.50	2.3525	21.7629	2.2724	23.0231	2.1960	24.4923	2.1229	26.2455
.52	2.3087	21.3161	2.2287	22.5647	2.1524	24.0151	2.0796	25.7384
.54	2.2658	20.9036	2.1859	22.1449	2.1098	23.5824	2.0372	25.2838
.56	2.2239	20.5221	2.1440	21.7599	2.0680	23.1894	1.9956	24.8758
.58	2.1829	20.1690	2.1029	21.4065	2.0271	22.8323	1.9549	24.5097
.60	2.1427	19.8417	2.0627	21.0820	1.9869	22.5080	1.9149	24.1814
.62	2.1033	19.5384	2.0233	20.7842	1.9476	22.2137	1.8758	23.8877
.64	2.0646	19.2573	1.9846	20.5111	1.9089	21.9472	1.8373	23.6259
.66	2.0267	18.9969	1.9466	20.2610	1.8710	21.7066	1.7995	23.3937
.68	1.9895	18.7561	1.9093	20.0326	1.8338	21.4902	1.7625	23.1892
.70	1.9530	18.5336	1.8726	19.8246	1.7972	21.2967	1.7260	23.0107
.72	1.9170	18.3286	1.8366	19.6360	1.7612	21.1249	1.6902	22.8568
.74	1.8817	18.1402	1.8011	19.4659	1.7258	20.9737	1.6550	22.7265
.76	1.8469	17.9678	1.7662	19.3134	1.6910	20.8423	1.6204	22.6186
.78	1.8127	17.8107	1.7319	19.1780	1.6567	20.7300	1.5863	22.5323
.80	1.7790	17.6684	1.6981	19.0590	1.6230	20.6360	1.5528	22.4669
.82	1.7458	17.5405	1.6648	18.9560	1.5897	20.5598	1.5198	22.4219
.84	1.7130	17.4266	1.6319	18.8685	1.5570	20.5011	1.4873	22.3968
.86	1.6807	17.3264	1.5995	18.7963	1.5248	20.4593	1.4553	22.3910
.88	1.6488	17.2396	1.5676	18.7389	1.4930	20.4342	1.4237	22.4044
.90	1.6174	17.1660	1.5361	18.6961	1.4616	20.4254	1.3927	22.4367
.92	1.5862	17.1055	1.5050	18.6678	1.4307	20.4329	1.3621	22.4877
.94	1.5555	17.0579	1.4743	18.6537	1.4002	20.4563	1.3319	22.5574
.96	1.5251	17.0230	1.4440	18.6538	1.3702	20.4957	1.3022	22.6456
.98	1.4951	17.0009	1.4140	18.6678	1.3405	20.5508	1.2729	22.7525
1.00	1.4654	16.9915	1.3844	18.6959	1.3112	20.6218	1.2440	22.8781

Table D–1: $0.586 \leq \hat{\rho}_4 \leq 0.622$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.586$		$\hat{\rho}_4 = 0.598$		$\hat{\rho}_4 = 0.610$		$\hat{\rho}_4 = 0.622$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.28	2.5959	52.8597						
.30	2.5405	44.7330						
.32	2.4864	40.5450	2.4152	51.3449				
.34	2.4336	37.7435	2.3629	44.8007				
.36	2.3821	35.6598	2.3119	41.1053	2.2441	55.7224		
.38	2.3318	34.0174	2.2621	38.5612	2.1948	47.7863		
.40	2.2826	32.6749	2.2134	36.6469	2.1466	43.7192		
.42	2.2346	31.5503	2.1658	35.1318	2.0995	41.0277	2.0353	60.2810
.44	2.1877	30.5915	2.1193	33.8936	2.0534	39.0527	1.9897	51.3813
.46	2.1418	29.7640	2.0738	32.8599	2.0083	37.5208	1.9451	47.1513
.48	2.0969	29.0430	2.0293	31.9837	1.9642	36.2915	1.9014	44.4470
.50	2.0529	28.4108	1.9857	31.2338	1.9211	35.2839	1.8587	42.5175
.52	2.0099	27.8539	1.9431	30.5873	1.8788	34.4465	1.8168	41.0617
.54	1.9678	27.3622	1.9013	30.0279	1.8374	33.7450	1.7758	39.9293
.56	1.9265	26.9273	1.8603	29.5432	1.7968	33.1555	1.7356	39.0347
.58	1.8860	26.5431	1.8202	29.1237	1.7570	32.6604	1.6962	38.3246
.60	1.8464	26.2042	1.7808	28.7618	1.7180	32.2471	1.6576	37.7641
.62	1.8074	25.9065	1.7422	28.4516	1.6797	31.9055	1.6197	37.3292
.64	1.7692	25.6465	1.7043	28.1884	1.6422	31.6282	1.5825	37.0031
.66	1.7317	25.4214	1.6671	27.9684	1.6053	31.4094	1.5460	36.7740
.68	1.6949	25.2288	1.6306	27.7884	1.5691	31.2445	1.5102	36.6336
.70	1.6587	25.0667	1.5947	27.6460	1.5336	31.1301	1.4750	36.5762
.72	1.6232	24.9334	1.5595	27.5390	1.4987	31.0636	1.4404	36.5985
.74	1.5882	24.8276	1.5248	27.4659	1.4644	31.0431	1.4065	36.6989
.76	1.5538	24.7482	1.4908	27.4254	1.4307	31.0673	1.3731	36.8779
.78	1.5201	24.6941	1.4573	27.4166	1.3975	31.1355	1.3403	37.1375
.80	1.4868	24.6647	1.4244	27.4386	1.3650	31.2473	1.3081	37.4817
.82	1.4541	24.6594	1.3920	27.4910	1.3329	31.4032	1.2764	37.9170
.84	1.4219	24.6776	1.3601	27.5736	1.3014	31.6040	1.2452	38.4527
.86	1.3902	24.7192	1.3288	27.6864	1.2704	31.8510	1.2146	39.1019
.88	1.3590	24.7837	1.2979	27.8295	1.2399	32.1462	1.1844	39.8834
.90	1.3283	24.8712	1.2676	28.0034	1.2099	32.4923	1.1547	40.8236
.92	1.2980	24.9818	1.2376	28.2088	1.1803	32.8928	1.1255	41.9624
.94	1.2682	25.1154	1.2082	28.4466	1.1513	33.3524	1.0967	43.3598
.96	1.2388	25.2724	1.1792	28.7179	1.1226	33.8767	1.0684	45.1157
.98	1.2099	25.4531	1.1507	29.0242	1.0944	34.4737	1.0405	47.4101
1.00	1.1814	25.6581	1.1225	29.3676	1.0666	35.1528	1.0131	50.6210

Table D–2: $0.0965 \leq \hat{\rho}_4 \leq 0.2500$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0965$		$\hat{\rho}_4 = 0.1010$		$\hat{\rho}_4 = 0.1055$		$\hat{\rho}_4 = 0.1100$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	14.9985	.007375	14.5980	.007678	14.2241	.007968	13.8734	.008246
.04	19.4018	.009919	18.4847	.010926	17.6989	.011868	17.0114	.012758

	$\hat{\rho}_4 = 0.115$		$\hat{\rho}_4 = 0.120$		$\hat{\rho}_4 = 0.125$		$\hat{\rho}_4 = 0.130$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	13.5076	.008540	13.1639	.008819	12.8396	.009083	12.5326	.009331
.04	16.3361	.013694	15.7342	.014583	15.1912	.054308	14.6963	.016241
.06	24.1625	.009595	21.8283	.012275	20.2013	.014576	18.9505	.016633

	$\hat{\rho}_4 = 0.136$		$\hat{\rho}_4 = 0.142$		$\hat{\rho}_4 = 0.148$		$\hat{\rho}_4 = 0.154$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	12.1846	.009607	11.8561	.009859	11.5451	.010086	11.2497	.010285
.04	14.1549	.017169	13.6610	.018052	13.2067	.018891	12.7860	.019690
.06	17.7511	.001888	16.7680	.020937	15.9343	.022854	15.2103	.024654
.08			28.4134	.008739	23.2586	.015052	20.6849	.019749

	$\hat{\rho}_4 = 0.161$		$\hat{\rho}_4 = 0.168$		$\hat{\rho}_4 = 0.175$		$\hat{\rho}_4 = 0.182$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	10.9227	.010480	10.6126	.010632	10.3177	.010738	10.0365	.010794
.04	12.3312	.020570	11.9095	.021395	11.5163	.022164	11.1478	.022874
.06	14.4700	.026630	13.8166	.028490	13.2313	.030247	12.7009	.031908
.08	18.7108	.024323	17.2687	.028317	16.1306	.031920	15.1896	.035233
.10					25.5898	.015803	21.2151	.025055

	$\hat{\rho}_4 = 0.190$		$\hat{\rho}_4 = 0.198$		$\hat{\rho}_4 = 0.206$		$\hat{\rho}_4 = 0.214$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	9.73016	.010791	9.43839	.010709	9.15975	.010541	8.89307	.010278
.04	10.7530	.023611	10.3826	.024262	10.0336	.024820	9.70358	.025278
.06	12.1496	.033696	11.6464	.035368	11.1834	.036925	10.7542	.038363
.08	14.2809	.038738	13.5007	.041997	12.8164	.045043	12.2064	.047898
.10	18.5972	.033008	16.8527	.039657	15.5401	.045513	14.4858	.050812
.12					23.8252	.024628	19.6285	.038262

	$\hat{\rho}_4 = 0.223$		$\hat{\rho}_4 = 0.232$		$\hat{\rho}_4 = 0.241$		$\hat{\rho}_4 = 0.250$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	8.60607	.009858	8.33168	.009294	8.06884	.008571	7.81665	.007675
.04	9.35238	.025663	9.02009	.025896	8.70467	.025960	8.40445	.025838
.06	10.3058	.039833	9.88868	.041134	9.49843	.042254	9.13161	.043174
.08	11.5903	.050895	11.0338	.053670	10.5260	.056222	10.0585	.058543
.10	13.5025	.056267	12.6691	.061285	11.9448	.065925	11.3035	.070222
.12	17.1282	.049722	15.4479	.059261	14.1772	.067624	13.1528	.075148
.14			25.7910	.025339	19.3559	.049624	16.6760	.065567

Table D–2: $0.260 \leq \hat{\rho}_4 \leq 0.382$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.260$		$\hat{\rho}_4 = 0.270$		$\hat{\rho}_4 = 0.280$		$\hat{\rho}_4 = 0.290$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	7.54804	.006456	7.29078	.004982	7.04415	.003231	6.80757	.001182
.04	8.08698	.025462	7.78492	.024805	7.49694	.023839	7.22196	.022533
.06	8.74814	.043940	8.38704	.044405	8.04579	.044535	7.72234	.044292
.08	9.57880	.060834	9.13454	.062795	8.72054	.064394	8.33276	.065592
.10	10.6667	.074609	10.0934	.078590	9.57138	.082147	9.09181	.085253
.12	12.2052	.082738	11.3984	.089645	10.6946	.095933	10.0694	.101628
.14	14.7797	.079769	13.4016	.091871	12.3147	.102544	11.4147	.112106
.16	23.7947	.038842	17.7305	.072665	15.1968	.094167	13.5468	.111299
.18							19.0640	.077452

	$\hat{\rho}_4 = 0.301$		$\hat{\rho}_4 = 0.312$		$\hat{\rho}_4 = 0.323$		$\hat{\rho}_4 = 0.334$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	6.93341	.020667	6.65856	.018312	6.39667	.015432	6.14717	.011993
.06	7.38512	.043547	7.06565	.042247	6.76250	.040333	6.47458	.037750
.08	7.93267	.066390	7.55704	.066579	7.20317	.066085	6.86896	.064829
.10	8.60525	.088099	8.15503	.090282	7.73594	.091721	7.34393	.092322
.12	9.45282	.107200	8.89575	.112007	8.38716	.115984	7.91886	.119042
.14	10.5737	.121534	9.84557	.129907	9.20235	.137241	8.62528	.143499
.16	12.2057	.127238	11.1483	.141044	10.2720	.153145	9.52160	.163727
.18	15.2413	.113826	13.2200	.139106	11.8174	.159456	10.7345	.176652
.20			19.1473	.092960	14.6904	.144014	12.6286	.175609
.22							17.9296	.125390

	$\hat{\rho}_4 = 0.346$		$\hat{\rho}_4 = 0.358$		$\hat{\rho}_4 = 0.370$		$\hat{\rho}_4 = 0.382$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	5.88870	.007574	5.64421	.002441				
.06	6.17689	.034107	5.89558	.029549	5.63019	.024036	5.38050	.017556
.08	6.52489	.062495	6.20069	.059057	5.89523	.054424	5.60781	.048525
.10	6.94351	.091902	6.56849	.090222	6.21652	.087137	5.88596	.082509
.12	7.44679	.121201	7.00939	.121972	6.60215	.121170	6.22173	.118599
.14	8.05593	.149002	7.53770	.152975	7.06182	.155225	6.62186	.155525
.16	8.80771	.173598	8.17658	.181671	7.60992	.187810	7.09504	.191808
.18	9.77462	.192656	8.96861	.206155	8.27148	.217252	7.65554	.225873
.20	11.1288	.202121	10.0031	.223662	9.09496	.241468	8.32982	.255994
.22	13.4798	.190940	11.5146	.228712	10.1879	.257200	9.17124	.279957
.24			14.6527	.200377	11.8673	.256937	10.3043	.293986
.26					18.1400	.171123	12.1431	.287898

Table D–2: $0.394 \le \hat{\rho}_4 \le 0.478$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.394$		$\hat{\rho}_4 = 0.406$		$\hat{\rho}_4 = 0.418$		$\hat{\rho}_4 = 0.430$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.06	5.14648	.010131	4.92823	.001824				
.08	5.33809	.041326	5.08603	.032838	4.85183	.023135	4.63580	.012363
.10	5.57574	.076223	5.28529	.068202	5.01456	.058438	4.76385	.047009
.12	5.86568	.114061	5.53240	.107377	5.22100	.098411	4.93132	.087108
.14	6.21303	.153620	5.83187	.149242	5.47599	.142127	5.14401	.132052
.16	6.62284	.193401	6.18666	.192274	5.78168	.188080	5.40450	.180453
.18	7.10256	.231829	6.59992	.234832	6.13864	.234516	5.71217	.230448
.20	7.66600	.267329	7.07776	.275346	6.54810	.279762	6.06521	.280169
.22	8.34012	.298201	7.63302	.312336	7.01474	.322364	6.46318	.328039
.24	9.18074	.322169	8.29169	.344205	7.54983	.361017	6.90924	.372811
.26	10.3230	.335029	9.10812	.368726	8.17661	.394307	7.41250	.413412
.28	12.2680	.323744	10.2158	.381544	8.94332	.420203	7.99185	.448685
.30			12.1391	.367957	9.96654	.434642	8.68558	.476942
.32							9.58155	.494863
.34							10.9695	.492530

	$\hat{\rho}_4 = 0.442$		$\hat{\rho}_4 = 0.454$		$\hat{\rho}_4 = 0.466$		$\hat{\rho}_4 = 0.478$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.08	4.43824	.000738						
.10	4.53373	.034109	4.32479	.020049	4.13740	.005237		
.12	4.66379	.073542	4.41934	.057960	4.19908	.040807	4.00382	.022699
.14	4.83556	.118887	4.55121	.102672	4.29242	.837115	4.06108	.062638
.16	5.05303	.169048	4.72655	.153615	4.42575	.134118	4.15282	.110907
.18	5.31586	.222142	4.94674	.209097	4.60353	.190882	4.28702	.167313
.20	5.62059	.276049	5.20817	.266777	4.82387	.251655	4.46571	.229992
.22	5.96339	.328927	5.50486	.324421	5.07996	.313740	4.68340	.295932
.24	6.34245	.379408	5.83149	.380354	5.36386	.374949	4.93044	.362224
.26	6.75945	.426506	6.18525	.433487	5.66938	.433878	5.19766	.426864
.28	7.22091	.469421	6.56646	.483123	5.99307	.489751	5.47855	.488778
.30	7.74053	.507297	6.97905	.528742	6.33418	.542156	5.76940	.547503
.32	8.34498	.538890	7.43162	.569804	6.69440	.590839	6.06870	.602891
.34	9.09082	.561863	7.94021	.605526	7.07815	.635550	6.37657	.654918
.36	10.1322	.570281	8.53574	.634516	7.49348	.675916	6.69455	.703581
.38			9.28746	.653763	7.95452	.711277	7.02565	.748823
.40					8.48787	.740354	7.37490	.790473
.42					9.15249	.760307	7.75048	.828159
.44					10.1394	.762243	8.16661	.861145
.46							8.65100	.887904
.48							9.27138	.904692
.50							10.3282	.895569

Table D–2: $0.490 \leq \hat{\rho}_4 \leq 0.526$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.490$		$\hat{\rho}_4 = 0.502$		$\hat{\rho}_4 = 0.514$		$\hat{\rho}_4 = 0.526$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.12	3.83361	.004340						
.14	3.85879	.040387	3.68590	.018038				
.16	3.91090	.084877	3.70284	.057466	3.52951	.030338	3.38879	.004896
.18	4.00037	.138753	3.74867	.106479	3.53682	.072733	3.36622	.040089
.20	4.13450	.201343	3.83474	.166034	3.57456	.125918	3.36202	.084587
.22	4.31234	.269941	3.96750	.234895	3.65533	.190993	3.38885	.141080
.24	4.52468	.340901	4.14266	.309397	3.78501	.266158	3.46121	.211152
.26	4.75964	.411239	4.34743	.385286	3.95590	.346644	3.58550	.292636
.28	5.00749	.479237	4.56831	.459600	4.15174	.427520	3.75098	.379423
.30	5.26187	.544167	4.79551	.530952	4.35773	.505874	3.93761	.465623
.32	5.51912	.605858	5.02312	.598958	4.56439	.580663	4.12927	.548289
.34	5.77736	.664390	5.24781	.663709	4.76644	.651831	4.31680	.626716
.36	6.03580	.719936	5.46768	.725482	4.96105	.719710	4.49571	.701175
.38	6.29436	.772682	5.68162	.784604	5.14661	.784744	4.66378	.772231
.40	6.55346	.822790	5.88888	.841400	5.32207	.847385	4.81986	.840484
.42	6.81396	.870387	6.08885	.896180	5.48662	.908061	4.96325	.906494
.44	7.07720	.915544	6.28093	.949233	5.63951	.967171	5.09339	.970762
.46	7.34517	.958265	6.46445	1.00084	5.77993	1.02509	5.20978	1.03374
.48	7.62083	.998462	6.63856	1.05126	5.90697	1.08218	5.31188	1.09586
.50	7.90886	1.03590	6.80217	1.10079	6.01962	1.13880	5.39915	1.15749
.52	8.21722	1.07010	6.95386	1.14971	6.11670	1.19533	5.47099	1.21903
.54	8.56118	1.10001	7.09179	1.19838	6.19692	1.25214	5.52684	1.28084
.56	8.97677	1.12315	7.21358	1.24717	6.25888	1.30965	5.56612	1.34328
.58	9.60174	1.13013	7.31614	1.29658	6.30112	1.36830	5.58837	1.40673
.60			7.39559	1.34720	6.32221	1.42855	5.59323	1.47156
.62			7.44719	1.39977	6.32085	1.49091	5.58050	1.53812
.64			7.46549	1.45521	6.29605	1.55591	5.55025	1.60679
.66			7.44484	1.51460	6.24723	1.62409	5.50278	1.67791
.68			7.38038	1.57914	6.17441	1.69599	5.43873	1.75182
.70			7.26943	1.65003	6.07832	1.77213	5.35903	1.82882
.72			7.11278	1.72832	5.96044	1.85298	5.26495	1.90917
.74			6.91505	1.81476	5.82295	1.93892	5.15802	1.99309
.76			6.68388	1.90979	5.66859	2.03028	5.03997	2.08075
.78			6.42828	2.01356	5.50051	2.12729	4.91270	2.17222
.80			6.15717	2.12606	5.32202	2.23005	4.77817	2.26756
.82			5.87838	2.24719	5.13642	2.33861	4.63835	2.36672
.84	7.03552	2.13737	5.59829	2.37678	4.94681	2.45288	4.49510	2.46959
.86	6.38593	2.33102	5.32185	2.51463	4.75599	2.57273	4.35016	2.57600
.88	5.87348	2.52119	5.05272	2.66049	4.56637	2.69790	4.20510	2.68570
.90	5.43394	2.71637	4.79349	2.81407	4.37996	2.82808	4.06127	2.79842
.92	5.04513	2.91944	4.54588	2.97505	4.19834	2.96291	3.91982	2.91379
.94	4.69622	3.13158	4.31093	3.14307	4.02272	3.10194	3.78166	3.03143
.96	4.38057	3.35402	4.08912	3.31771	3.85396	3.24468	3.64755	3.15090
.98	4.09369	3.58721	3.88054	3.49852	3.69264	3.39060	3.51801	3.27174
1.00	3.83170	3.83170	3.68500	3.68500	3.53910	3.53910	3.39345	3.39345

Table D–2: $0.538 \leq \hat{\rho}_4 \leq 0.574$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.538$		$\hat{\rho}_4 = 0.550$		$\hat{\rho}_4 = 0.562$		$\hat{\rho}_4 = 0.574$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.18	3.23328	.010462						
.20	3.19873	.045969	3.07799	.012406				
.22	3.18073	.091087	3.03099	.046730	2.92725	.010178		
.24	3.19250	.149191	2.99706	.090281	2.86847	.041649	2.78571	.003807
.26	3.25123	.223004	2.98855	.147025	2.81803	.080997	2.71751	.031020
.28	3.36657	.310720	3.02506	.221630	2.78508	.132063	2.65306	.063947
.30	3.52595	.404569	3.12429	.314715	2.78758	.201153	2.59712	.105215
.32	3.70446	.496937	3.27660	.417098	2.85390	.294908	2.55962	.159906
.34	3.88375	.584543	3.44978	.517498	2.99341	.407609	2.56418	.238368
.36	4.05484	.667015	3.62048	.611397	3.16634	.520313	2.65642	.353873
.38	4.21400	.744991	3.77918	.698752	3.33530	.623650	2.83262	.488804
.40	4.35978	.819296	3.92295	.780709	3.48781	.717768	3.01472	.610728
.42	4.49160	.890698	4.05123	.858453	3.62186	.804761	3.17262	.716684
.44	4.60923	.959861	4.16416	.932976	3.73812	.886475	3.30561	.811446
.46	4.71254	1.02735	4.26205	1.00508	3.83767	.964291	3.41688	.898608
.48	4.80140	1.09367	4.34525	1.07540	3.92156	1.03923	3.50930	.980461
.50	4.87570	1.15925	4.41408	1.14448	3.99066	1.11206	3.58499	1.05851
.52	4.93533	1.22448	4.46882	1.21277	4.04573	1.18339	3.64558	1.13378
.54	4.98019	1.28973	4.50976	1.28066	4.08745	1.25369	3.69236	1.20704
.56	5.01025	1.35534	4.53721	1.34849	4.11642	1.32337	3.72640	1.27882
.58	5.02554	1.42163	4.55152	1.41656	4.13325	1.39274	3.74861	1.34957
.60	5.02619	1.48891	4.55308	1.48515	4.13856	1.46207	3.75985	1.41962
.62	5.01247	1.55745	4.54239	1.55449	4.13297	1.53161	3.76091	1.48922
.64	4.98478	1.62754	4.52002	1.62480	4.11716	1.60153	3.75256	1.55859
.66	4.94371	1.69940	4.48663	1.69625	4.09182	1.67198	3.73555	1.62789
.68	4.89000	1.77326	4.44298	1.76901	4.05771	1.74310	3.71066	1.69722
.70	4.82455	1.84932	4.38990	1.84318	4.01560	1.81496	3.67862	1.76668
.72	4.74843	1.92771	4.32831	1.91887	3.96631	1.88762	3.64019	1.83631
.74	4.66282	2.00854	4.25919	1.99610	3.91067	1.96110	3.59612	1.90612
.76	4.56901	2.09190	4.18353	2.07491	3.84951	2.03539	3.54713	1.97611
.78	4.46834	2.17778	4.10238	2.15527	3.78367	2.11045	3.49392	2.04623
.80	4.36216	2.26617	4.01674	2.23711	3.71396	2.18620	3.43719	2.11643
.82	4.25181	2.35697	3.92762	2.32033	3.64116	2.26256	3.37758	2.18661
.84	4.13857	2.45005	3.83595	2.40479	3.56601	2.33939	3.31570	2.25668
.86	4.02364	2.54524	3.74263	2.49034	3.48920	2.41655	3.25211	2.32651
.88	3.90811	2.64232	3.64845	2.57677	3.41136	2.49388	3.18733	2.39597
.90	3.79295	2.74101	3.55415	2.66385	3.33307	2.57118	3.12183	2.46491
.92	3.67898	2.84102	3.46035	2.75135	3.25481	2.64826	3.05603	2.53318
.94	3.56692	2.94202	3.36762	2.83899	3.17705	2.72491	2.99029	2.60063
.96	3.45734	3.04366	3.27640	2.92650	3.10015	2.80092	2.92493	2.66709
.98	3.35070	3.14557	3.18708	3.01359	3.02443	2.87608	2.86023	2.73240
1.00	3.24736	3.24736	3.09997	3.09997	2.95016	2.95016	2.79641	2.79641

Table D–2: $0.586 \leq \hat{\rho}_4 \leq 0.622$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.586$		$\hat{\rho}_4 = 0.598$		$\hat{\rho}_4 = 0.610$		$\hat{\rho}_4 = 0.622$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.28	2.58357	.016265						
.30	2.51056	.041966						
.32	2.44008	.072213	2.39377	.018533				
.34	2.37504	.109053	2.31856	.039593				
.36	2.32155	.156827	2.24445	.062899				
.38	2.29739	.227101	2.17225	.089033				
.40	2.37404	.356359	2.10308	.119078				
.42	2.60480	.536847	2.03907	.155386				
.44	2.80566	.676842	1.98626	.204788				
.46	2.95769	.788904	2.00065	.321329				
.48	3.07646	.886201	2.49982	.689537				
.50	3.17066	.974588	2.67832	.824975				
.52	3.24526	1.05708	2.79756	.930794				
.54	3.30339	1.13545	2.88479	1.02317	2.17967	.723125		
.56	3.34725	1.21082	2.94991	1.10763	2.43072	.911825		
.58	3.37845	1.28396	2.99796	1.18682	2.54082	1.02242		
.60	3.39833	1.35542	3.03202	1.26225	2.61264	1.11488		
.62	3.40797	1.42559	3.05425	1.33486	2.66181	1.19788		
.64	3.40837	1.49478	3.06622	1.40526	2.69484	1.27483	2.10969	.988449
.66	3.40039	1.56320	3.06923	1.47390	2.71536	1.34748	2.23420	1.11870
.68	3.38488	1.63101	3.06435	1.54108	2.72570	1.41685	2.29480	1.21200
.70	3.36261	1.69831	3.05252	1.60699	2.72753	1.48358	2.33032	1.29212
.72	3.33432	1.76518	3.03457	1.67179	2.72215	1.54809	2.35036	1.36463
.74	3.30072	1.83165	3.01125	1.73558	2.71060	1.61066	2.35934	1.43187
.76	3.26249	1.89773	2.98325	1.79839	2.69375	1.67149	2.35983	1.49510
.78	3.22025	1.96340	2.95120	1.86026	2.67235	1.73070	2.35354	1.55506
.80	3.17463	2.02861	2.91568	1.92118	2.64707	1.78837	2.34170	1.61219
.82	3.12618	2.09330	2.87724	1.98112	2.61848	1.84454	2.32527	1.66681
.84	3.07544	2.15740	2.83635	2.04004	2.58709	1.89923	2.30502	1.71912
.86	3.02289	2.22082	2.79348	2.09789	2.55335	1.95243	2.28156	1.76925
.88	2.96897	2.28345	2.74901	2.15460	2.51769	2.00413	2.25542	1.81728
.90	2.91409	2.34519	2.70333	2.21010	2.48046	2.05430	2.22706	1.86328
.92	2.85861	2.40592	2.65675	2.26432	2.44198	2.10290	2.19684	1.90728
.94	2.80285	2.46553	2.60957	2.31718	2.40254	2.14989	2.16511	1.94929
.96	2.74709	2.52391	2.56204	2.36860	2.36238	2.19523	2.13214	1.98933
.98	2.69157	2.58093	2.51439	2.41850	2.32174	2.23888	2.09818	2.02738
1.00	2.63649	2.63649	2.46680	2.46680	2.28080	2.28080	2.06344	2.06344

Table D–3: $0.502 \leq \hat{\rho}_4 \leq 0.526$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.502$		$\hat{\rho}_4 = 0.514$		$\hat{\rho}_4 = 0.526$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.38	.00013	.01106				
.40	.00039	.01130	.00019	.07795		
.42	.00067	.01153	.00181	.07966		
.44	.00097	.01176	.00353	.08134	.00154	.14887
.46	.00127	.01197	.00535	.08296	.00431	.15202
.48	.00160	.01217	.00728	.08451	.00725	.15511
.50	.00193	.01236	.00930	.08600	.01037	.15811
.52	.00228	.01252	.01143	.08741	.01366	.16102
.54	.00264	.01268	.01366	.08872	.01714	.16383
.56	.00301	.01281	.01599	.08994	.02082	.16650
.58	.00339	.01293	.01842	.09106	.02469	.16904
.60	.00378	.01303	.02095	.09205	.02877	.17142
.62	.00418	.01311	.02357	.09292	.03306	.17362
.64	.00458	.01316	.02628	.09364	.03756	.17561
.66	.00498	.01320	.02907	.09422	.04229	.17738
.68	.00539	.01321	.03194	.09465	.04725	.17890
.70	.00579	.01320	.03487	.09490	.05244	.18014
.72	.00620	.01317	.03786	.09499	.05786	.18107
.74	.00660	.01312	.04090	.09489	.06351	.18166
.76	.00699	.01305	.04398	.09461	.06939	.18187
.78	.00739	.01296	.04707	.09414	.07549	.18167
.80	.00777	.01285	.05017	.09348	.08180	.18103
.82	.00814	.01272	.05325	.09263	.08829	.17991
.84	.00850	.01258	.05631	.09159	.09494	.17829
.86	.00886	.01241	.05932	.09037	.10171	.17614
.88	.00919	.01223	.06227	.08898	.10854	.17345
.90	.00952	.01204	.06513	.08742	.11538	.17022
.92	.00983	.01184	.06790	.08572	.12216	.16646
.94	.01013	.01162	.07056	.08388	.12879	.16219
.96	.01041	.01140	.07309	.08193	.13519	.15749
.98	.01068	.01117	.07548	.07987	.14129	.15240
1.00	.01093	.01093	.07773	.07773	.14701	.14701

Table D–3: $0.538 \leq \hat{\rho}_4 \leq 0.574$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.538$		$\hat{\rho}_4 = 0.550$		$\hat{\rho}_4 = 0.562$		$\hat{\rho}_4 = 0.574$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.48	.00301	.22336						
.50	.00668	.22795						
.52	.01057	.23246	.00381	.30117				
.54	.01468	.23689	.00808	.30721				
.56	.01904	.24122	.01260	.31319	.00328	.38199		
.58	.02365	.24542	.01737	.31910	.00785	.38950		
.60	.02853	.24948	.02242	.32492	.01266	.39697	.00097	.46541
.62	.03370	.25338	.02777	.33063	.01773	.40439	.00553	.47436
.64	.03918	.25710	.03344	.33621	.02307	.41177	.01031	.48331
.66	.04499	.26059	.03945	.34165	.02871	.41907	.01532	.49227
.68	.05115	.26383	.04584	.34691	.03467	.42630	.02057	.50121
.70	.05768	.26679	.05264	.35196	.04099	.43342	.02607	.51014
.72	.06463	.26940	.05991	.35677	.04769	.44042	.03186	.51904
.74	.07203	.27163	.06768	.36128	.05483	.44727	.03795	.52790
.76	.07991	.27340	.07604	.36544	.06245	.45393	.04437	.53672
.78	.08831	.27465	.08505	.36918	.07062	.46038	.05117	.54547
.80	.09729	.27527	.09481	.37240	.07942	.46655	.05837	.55415
.82	.10689	.27518	.10546	.37498	.08895	.47239	.06603	.56272
.84	.11715	.27424	.11716	.37677	.09935	.47780	.07423	.57116
.86	.12813	.27230	.13012	.37753	.11081	.48266	.08303	.57943
.88	.13984	.26921	.14463	.37696	.12359	.48681	.09255	.58749
.90	.15228	.26478	.16112	.37460	.13808	.48998	.10294	.59528
.92	.16538	.25884	.18016	.36973	.15488	.49176	.11439	.60269
.94	.17899	.25126	.20255	.36125	.17506	.49140	.12721	.60959
.96	.19281	.24203	.22934	.34737	.20081	.48733	.14183	.61578
.98	.20642	.23127	.26111	.32571	.23811	.47506	.15905	.62087
1.00	.21933	.21933	.29544	.29544	.37634	.37634	.46346	.46346

Table D–3: $0.586 \leq \hat{\rho}_4 \leq 0.622$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.586$		$\hat{\rho}_4 = 0.598$		$\hat{\rho}_4 = 0.610$		$\hat{\rho}_4 = 0.622$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.54					2.03531	.63682		
.56					1.77814	.51340		
.58					1.65593	.47138		
.60					1.56598	.45088		
.62					1.49268	.44293		
.64					1.42962	.44349		
.66					1.37344	.45007		
.68					1.32222	.46094	1.58799	.73803
.70					1.27479	.47491	1.51059	.72949
.72					1.23040	.49112	1.44543	.72855
.74					1.18857	.50897	1.38847	.73267
.76					1.14896	.52805	1.33755	.74044
.78					1.11133	.54805	1.29132	.75098
.80			.96798	.36556	1.07548	.56876	1.24890	.76368
.82			.93249	.40578	1.04128	.59000	1.20967	.77811
.84			.89864	.44062	1.00859	.61166	1.17318	.79396
.86			.86610	.47263	.97733	.63364	1.13909	.81097
.88			.83472	.50286	.94739	.65587	1.10713	.82897
.90	.73415	.35942	.80440	.53185	.91871	.67829	1.07710	.84779
.92	.69452	.41179	.77507	.55993	.89122	.70086	1.04883	.86732
.94	.65865	.45379	.74666	.58730	.86485	.72354	1.02218	.88746
.96	.62454	.49119	.71909	.61408	.83954	.74629	.99703	.90814
.98	.59146	.52592	.69232	.64039	.81525	.76910	.97329	.92929
1.00	.55894	.55894	.66628	.66628	.79194	.79194	.95087	.95087

	$\hat{\rho}_4 = 0.634$	
$\hat{\rho}_3$	λ_3	λ_4
.82	1.57391	1.10986
.84	1.49927	1.09761
.86	1.44174	1.09742
.88	1.39373	1.10368
.90	1.35223	1.11420
.92	1.31565	1.12786
.94	1.28306	1.14399
.96	1.25382	1.16220
.98	1.22752	1.18221
1.00	1.20385	1.20385

Table D–4: $0.586 \leq \hat{\rho}_4 \leq 0.622$ and $\lambda_3,\ \lambda_4 > 0$

	$\hat{\rho}_4 = 0.586$		$\hat{\rho}_4 = 0.598$		$\hat{\rho}_4 = 0.610$		$\hat{\rho}_4 = 0.622$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.66	.00096	.56123						
.68	.00544	.57161						
.70	.01009	.58203						
.72	.01492	.59248						
.74	.01995	.60297	.00231	.67288				
.76	.02518	.61348	.00651	.68471				
.78	.03063	.62402	.01084	.69660				
.80	.03632	.63459	.01529	.70857				
.82	.04226	.64517	.01987	.72060				
.84	.04848	.65577	.02458	.73271	.00309	.80327		
.86	.05501	.66638	.02943	.74488	.00674	.81653		
.88	.06188	.67699	.03444	.75714	.01044	.82989		
.90	.06914	.68759	.03959	.76947	.01419	.84334		
.92	.07682	.69817	.04492	.78187	.01800	.85688		
.94	.08500	.70871	.05042	.79436	.02185	.87053		
.96	.09376	.71920	.05611	.80693	.02575	.88427	.00061	.95405
.98	.10321	.72960	.06201	.81958	.02970	.89812	.00336	.96862
1.00	.11349	.73988	.06813	.83232	.03369	.91208	.00610	.98327

Table D–5: $0.005 \leq \hat{\rho}_4 \leq 0.008$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.005$		$\hat{\rho}_4 = 0.006$		$\hat{\rho}_4 = 0.007$		$\hat{\rho}_4 = 0.008$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-4.1567	-5.8568	-3.9567	-5.6573	-3.7873	-5.4884	-3.6403	-5.3419
.04	-4.4853	-5.8837	-4.2855	-5.6842	-4.1164	-5.5153	-3.9698	-5.3689
.06	-4.6782	-5.9004	-4.4786	-5.7008	-4.3096	-5.5320	-4.1631	-5.3856
.08	-4.8145	-5.9116	-4.6149	-5.7121	-4.4460	-5.5433	-4.2996	-5.3970
.10	-4.9193	-5.9194	-4.7197	-5.7200	-4.5509	-5.5512	-4.4045	-5.4048
.12	-5.0039	-5.9249	-4.8044	-5.7254	-4.6355	-5.5566	-4.4892	-5.4103
.14	-5.0745	-5.9285	-4.8750	-5.7290	-4.7062	-5.5603	-4.5598	-5.4139
.16	-5.1348	-5.9307	-4.9353	-5.7313	-4.7665	-5.5625	-4.6202	-5.4162
.18	-5.1871	-5.9319	-4.9876	-5.7325	-4.8189	-5.5637	-4.6725	-5.4174
.20	-5.2332	-5.9322	-5.0337	-5.7328	-4.8649	-5.5640	-4.7186	-5.4177
.22	-5.2741	-5.9318	-5.0747	-5.7324	-4.9059	-5.5636	-4.7596	-5.4173
.24	-5.3109	-5.9308	-5.1115	-5.7313	-4.9427	-5.5626	-4.7964	-5.4163
.26	-5.3441	-5.9292	-5.1447	-5.7298	-4.9760	-5.5611	-4.8297	-5.4148
.28	-5.3744	-5.9273	-5.1750	-5.7279	-5.0062	-5.5591	-4.8599	-5.4129
.30	-5.4020	-5.9249	-5.2026	-5.7255	-5.0339	-5.5568	-4.8876	-5.4106
.32	-5.4274	-5.9223	-5.2280	-5.7229	-5.0593	-5.5542	-4.9130	-5.4079
.34	-5.4509	-5.9194	-5.2515	-5.7200	-5.0827	-5.5513	-4.9365	-5.4050
.36	-5.4726	-5.9163	-5.2732	-5.7169	-5.1044	-5.5482	-4.9582	-5.4019
.38	-5.4927	-5.9130	-5.2933	-5.7136	-5.1246	-5.5449	-4.9783	-5.3986
.40	-5.5115	-5.9095	-5.3121	-5.7101	-5.1434	-5.5414	-4.9971	-5.3951
.42	-5.5291	-5.9058	-5.3297	-5.7064	-5.1609	-5.5377	-5.0147	-5.3915
.44	-5.5455	-5.9020	-5.3461	-5.7027	-5.1774	-5.5339	-5.0311	-5.3877
.46	-5.5609	-5.8982	-5.3615	-5.6988	-5.1928	-5.5301	-5.0465	-5.3838
.48	-5.5754	-5.8942	-5.3760	-5.6948	-5.2073	-5.5261	-5.0610	-5.3798
.50	-5.5891	-5.8901	-5.3897	-5.6907	-5.2209	-5.5220	-5.0747	-5.3758
.52	-5.6019	-5.8860	-5.4026	-5.6866	-5.2338	-5.5179	-5.0876	-5.3716
.54	-5.6141	-5.8818	-5.4147	-5.6824	-5.2460	-5.5137	-5.0998	-5.3674
.56	-5.6257	-5.8775	-5.4263	-5.6781	-5.2576	-5.5094	-5.1113	-5.3632
.58	-5.6366	-5.8732	-5.4372	-5.6738	-5.2685	-5.5051	-5.1223	-5.3589
.60	-5.6470	-5.8689	-5.4476	-5.6695	-5.2789	-5.5008	-5.1327	-5.3545
.62	-5.6569	-5.8645	-5.4575	-5.6651	-5.2888	-5.4964	-5.1425	-5.3502
.64	-5.6662	-5.8601	-5.4669	-5.6607	-5.2982	-5.4920	-5.1519	-5.3458
.66	-5.6752	-5.8557	-5.4758	-5.6563	-5.3071	-5.4876	-5.1609	-5.3414
.68	-5.6837	-5.8512	-5.4843	-5.6518	-5.3156	-5.4831	-5.1694	-5.3369
.70	-5.6919	-5.8468	-5.4925	-5.6474	-5.3238	-5.4787	-5.1775	-5.3325
.72	-5.6996	-5.8423	-5.5003	-5.6429	-5.3316	-5.4742	-5.1853	-5.3280
.74	-5.7071	-5.8378	-5.5077	-5.6385	-5.3390	-5.4698	-5.1928	-5.3235
.76	-5.7142	-5.8334	-5.5148	-5.6340	-5.3461	-5.4653	-5.1999	-5.3191
.78	-5.7210	-5.8289	-5.5216	-5.6295	-5.3529	-5.4608	-5.2067	-5.3146
.80	-5.7275	-5.8244	-5.5282	-5.6251	-5.3595	-5.4564	-5.2132	-5.3102
.82	-5.7338	-5.8200	-5.5344	-5.6206	-5.3657	-5.4519	-5.2195	-5.3057
.84	-5.7398	-5.8155	-5.5404	-5.6162	-5.3717	-5.4475	-5.2255	-5.3012
.86	-5.7456	-5.8111	-5.5462	-5.6117	-5.3775	-5.4430	-5.2313	-5.2968
.88	-5.7511	-5.8067	-5.5518	-5.6073	-5.3831	-5.4386	-5.2368	-5.2924
.90	-5.7565	-5.8022	-5.5571	-5.6029	-5.3884	-5.4342	-5.2422	-5.2879
.92	-5.7616	-5.7978	-5.5622	-5.5984	-5.3935	-5.4297	-5.2473	-5.2835
.94	-5.7665	-5.7934	-5.5672	-5.5940	-5.3985	-5.4253	-5.2522	-5.2791
.96	-5.7713	-5.7890	-5.5719	-5.5897	-5.4032	-5.4210	-5.2570	-5.2747
.98	-5.7759	-5.7847	-5.5765	-5.5853	-5.4078	-5.4166	-5.2616	-5.2704
1.00	-5.7803	-5.7803	-5.5809	-5.5809	-5.4122	-5.4122	-5.2660	-5.2660

Table D–5: $0.009 \leq \hat{\rho}_4 \leq 0.012$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.009$		$\hat{\rho}_4 = 0.010$		$\hat{\rho}_4 = 0.011$		$\hat{\rho}_4 = 0.012$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-3.5104	-5.2126	-3.3940	-5.0969	-3.2885	-4.9921	-3.1920	-4.8963
.04	-3.8403	-5.2397	-3.7243	-5.1240	-3.6193	-5.0192	-3.5233	-4.9235
.06	-4.0337	-5.2564	-3.9179	-5.1408	-3.8130	-5.0361	-3.7171	-4.9404
.08	-4.1703	-5.2678	-4.0545	-5.1521	-3.9497	-5.0475	-3.8539	-4.9518
.10	-4.2752	-5.2757	-4.1595	-5.1600	-4.0547	-5.0554	-3.9590	-4.9598
.12	-4.3600	-5.2811	-4.2443	-5.1655	-4.1396	-5.0609	-4.0439	-4.9653
.14	-4.4306	-5.2848	-4.3150	-5.1692	-4.2103	-5.0646	-4.1146	-4.9690
.16	-4.4910	-5.2871	-4.3753	-5.1715	-4.2707	-5.0669	-4.1750	-4.9713
.18	-4.5434	-5.2883	-4.4278	-5.1727	-4.3231	-5.0681	-4.2275	-4.9725
.20	-4.5895	-5.2886	-4.4739	-5.1731	-4.3692	-5.0685	-4.2736	-4.9729
.22	-4.6305	-5.2882	-4.5149	-5.1727	-4.4103	-5.0681	-4.3147	-4.9725
.24	-4.6673	-5.2872	-4.5517	-5.1717	-4.4471	-5.0671	-4.3515	-4.9715
.26	-4.7006	-5.2857	-4.5850	-5.1702	-4.4804	-5.0656	-4.3848	-4.9700
.28	-4.7308	-5.2838	-4.6152	-5.1682	-4.5106	-5.0636	-4.4151	-4.9681
.30	-4.7585	-5.2815	-4.6429	-5.1659	-4.5383	-5.0613	-4.4427	-4.9658
.32	-4.7839	-5.2789	-4.6683	-5.1633	-4.5637	-5.0587	-4.4682	-4.9632
.34	-4.8074	-5.2760	-4.6918	-5.1604	-4.5872	-5.0559	-4.4917	-4.9603
.36	-4.8291	-5.2729	-4.7135	-5.1573	-4.6089	-5.0527	-4.5134	-4.9572
.38	-4.8492	-5.2695	-4.7337	-5.1540	-4.6291	-5.0494	-4.5336	-4.9539
.40	-4.8680	-5.2660	-4.7525	-5.1505	-4.6479	-5.0460	-4.5524	-4.9504
.42	-4.8856	-5.2624	-4.7701	-5.1469	-4.6655	-5.0423	-4.5699	-4.9468
.44	-4.9020	-5.2586	-4.7865	-5.1431	-4.6819	-5.0386	-4.5864	-4.9430
.46	-4.9175	-5.2548	-4.8019	-5.1392	-4.6974	-5.0347	-4.6018	-4.9392
.48	-4.9320	-5.2508	-4.8164	-5.1353	-4.7119	-5.0307	-4.6163	-4.9352
.50	-4.9456	-5.2467	-4.8301	-5.1312	-4.7255	-5.0266	-4.6300	-4.9311
.52	-4.9585	-5.2426	-4.8430	-5.1271	-4.7384	-5.0225	-4.6429	-4.9270
.54	-4.9707	-5.2384	-4.8552	-5.1229	-4.7506	-5.0183	-4.6551	-4.9228
.56	-4.9823	-5.2341	-4.8668	-5.1186	-4.7622	-5.0141	-4.6667	-4.9185
.58	-4.9932	-5.2298	-4.8777	-5.1143	-4.7731	-5.0098	-4.6776	-4.9142
.60	-5.0036	-5.2255	-4.8881	-5.1100	-4.7835	-5.0054	-4.6880	-4.9099
.62	-5.0135	-5.2211	-4.8980	-5.1056	-4.7934	-5.0011	-4.6979	-4.9055
.64	-5.0229	-5.2167	-4.9074	-5.1012	-4.8028	-4.9967	-4.7073	-4.9011
.66	-5.0318	-5.2123	-4.9163	-5.0968	-4.8117	-4.9922	-4.7162	-4.8967
.68	-5.0403	-5.2079	-4.9248	-5.0924	-4.8203	-4.9878	-4.7248	-4.8923
.70	-5.0485	-5.2034	-4.9330	-5.0879	-4.8284	-4.9834	-4.7329	-4.8879
.72	-5.0563	-5.1990	-4.9408	-5.0835	-4.8362	-4.9789	-4.7407	-4.8834
.74	-5.0637	-5.1945	-4.9482	-5.0790	-4.8436	-4.9744	-4.7481	-4.8789
.76	-5.0708	-5.1900	-4.9553	-5.0745	-4.8508	-4.9700	-4.7553	-4.8745
.78	-5.0776	-5.1856	-4.9621	-5.0701	-4.8576	-4.9655	-4.7621	-4.8700
.80	-5.0842	-5.1811	-4.9687	-5.0656	-4.8641	-4.9610	-4.7686	-4.8655
.82	-5.0904	-5.1766	-4.9749	-5.0611	-4.8704	-4.9566	-4.7749	-4.8611
.84	-5.0965	-5.1722	-4.9809	-5.0567	-4.8764	-4.9521	-4.7809	-4.8566
.86	-5.1022	-5.1677	-4.9867	-5.0522	-4.8822	-4.9477	-4.7867	-4.8522
.88	-5.1078	-5.1633	-4.9923	-5.0478	-4.8877	-4.9433	-4.7922	-4.8477
.90	-5.1131	-5.1589	-4.9976	-5.0434	-4.8931	-4.9388	-4.7976	-4.8433
.92	-5.1183	-5.1545	-5.0027	-5.0390	-4.8982	-4.9344	-4.8027	-4.8389
.94	-5.1232	-5.1501	-5.0077	-5.0346	-4.9031	-4.9300	-4.8076	-4.8345
.96	-5.1280	-5.1457	-5.0124	-5.0302	-4.9079	-4.9256	-4.8124	-4.8301
.98	-5.1325	-5.1413	-5.0170	-5.0258	-4.9125	-4.9213	-4.8170	-4.8258
1.00	-5.1370	-5.1370	-5.0215	-5.0215	-4.9169	-4.9169	-4.8214	-4.8214

Table D–5: $0.0135 \leq \hat{\rho}_4 \leq 0.0180$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0135$		$\hat{\rho}_4 = 0.0150$		$\hat{\rho}_4 = 0.0165$		$\hat{\rho}_4 = 0.0180$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-3.0609	-4.7665	-2.9434	-4.6503	-2.8367	-4.5450	-2.7388	-4.4487
.04	-3.3930	-4.7938	-3.2763	-4.6777	-3.1705	-4.5725	-3.0737	-4.4763
.06	-3.5872	-4.8107	-3.4708	-4.6947	-3.3652	-4.5895	-3.2687	-4.4934
.08	-3.7241	-4.8222	-3.6078	-4.7062	-3.5025	-4.6011	-3.4061	-4.5050
.10	-3.8293	-4.8302	-3.7131	-4.7142	-3.6078	-4.6091	-3.5116	-4.5131
.12	-3.9142	-4.8357	-3.7980	-4.7197	-3.6929	-4.6147	-3.5967	-4.5187
.14	-3.9850	-4.8394	-3.8689	-4.7235	-3.7638	-4.6185	-3.6676	-4.5225
.16	-4.0454	-4.8418	-3.9294	-4.7258	-3.8243	-4.6208	-3.7282	-4.5249
.18	-4.0979	-4.8430	-3.9819	-4.7271	-3.8768	-4.6221	-3.7807	-4.5262
.20	-4.1440	-4.8434	-4.0281	-4.7275	-3.9230	-4.6225	-3.8270	-4.5265
.22	-4.1851	-4.8430	-4.0691	-4.7271	-3.9641	-4.6221	-3.8681	-4.5262
.24	-4.2219	-4.8420	-4.1060	-4.7261	-4.0010	-4.6212	-3.9050	-4.5253
.26	-4.2553	-4.8405	-4.1393	-4.7247	-4.0343	-4.6197	-3.9383	-4.5238
.28	-4.2855	-4.8386	-4.1696	-4.7227	-4.0646	-4.6178	-3.9687	-4.5219
.30	-4.3132	-4.8363	-4.1973	-4.7205	-4.0924	-4.6155	-3.9964	-4.5196
.32	-4.3387	-4.8337	-4.2228	-4.7179	-4.1178	-4.6129	-4.0219	-4.5171
.34	-4.3622	-4.8309	-4.2463	-4.7150	-4.1413	-4.6101	-4.0454	-4.5142
.36	-4.3839	-4.8278	-4.2680	-4.7119	-4.1631	-4.6070	-4.0672	-4.5111
.38	-4.4041	-4.8245	-4.2882	-4.7086	-4.1833	-4.6037	-4.0874	-4.5078
.40	-4.4229	-4.8210	-4.3070	-4.7051	-4.2021	-4.6002	-4.1062	-4.5044
.42	-4.4405	-4.8174	-4.3246	-4.7015	-4.2197	-4.5966	-4.1238	-4.5007
.44	-4.4569	-4.8136	-4.3411	-4.6978	-4.2361	-4.5929	-4.1403	-4.4970
.46	-4.4724	-4.8097	-4.3565	-4.6939	-4.2516	-4.5890	-4.1557	-4.4931
.48	-4.4869	-4.8058	-4.3710	-4.6899	-4.2661	-4.5850	-4.1702	-4.4892
.50	-4.5006	-4.8017	-4.3847	-4.6859	-4.2798	-4.5810	-4.1839	-4.4851
.52	-4.5135	-4.7976	-4.3976	-4.6817	-4.2927	-4.5768	-4.1969	-4.4810
.54	-4.5257	-4.7934	-4.4098	-4.6775	-4.3049	-4.5727	-4.2091	-4.4768
.56	-4.5372	-4.7891	-4.4214	-4.6733	-4.3165	-4.5684	-4.2206	-4.4726
.58	-4.5482	-4.7848	-4.4323	-4.6690	-4.3274	-4.5641	-4.2316	-4.4683
.60	-4.5586	-4.7805	-4.4427	-4.6647	-4.3378	-4.5598	-4.2420	-4.4639
.62	-4.5685	-4.7761	-4.4526	-4.6603	-4.3477	-4.5554	-4.2519	-4.4596
.64	-4.5779	-4.7717	-4.4620	-4.6559	-4.3571	-4.5510	-4.2613	-4.4552
.66	-4.5868	-4.7673	-4.4710	-4.6515	-4.3661	-4.5466	-4.2702	-4.4508
.68	-4.5953	-4.7629	-4.4795	-4.6471	-4.3746	-4.5422	-4.2788	-4.4463
.70	-4.6035	-4.7584	-4.4877	-4.6426	-4.3828	-4.5377	-4.2869	-4.4419
.72	-4.6113	-4.7540	-4.4955	-4.6382	-4.3906	-4.5333	-4.2947	-4.4374
.74	-4.6187	-4.7495	-4.5029	-4.6337	-4.3980	-4.5288	-4.3022	-4.4330
.76	-4.6258	-4.7451	-4.5100	-4.6292	-4.4051	-4.5244	-4.3093	-4.4285
.78	-4.6327	-4.7406	-4.5168	-4.6248	-4.4120	-4.5199	-4.3161	-4.4241
.80	-4.6392	-4.7361	-4.5234	-4.6203	-4.4185	-4.5154	-4.3226	-4.4196
.82	-4.6455	-4.7317	-4.5296	-4.6159	-4.4248	-4.5110	-4.3289	-4.4151
.84	-4.6515	-4.7272	-4.5357	-4.6114	-4.4308	-4.5065	-4.3349	-4.4107
.86	-4.6573	-4.7228	-4.5414	-4.6070	-4.4366	-4.5021	-4.3407	-4.4062
.88	-4.6628	-4.7183	-4.5470	-4.6025	-4.4421	-4.4977	-4.3463	-4.4018
.90	-4.6681	-4.7139	-4.5523	-4.5981	-4.4475	-4.4932	-4.3516	-4.3974
.92	-4.6733	-4.7095	-4.5575	-4.5937	-4.4526	-4.4888	-4.3567	-4.3930
.94	-4.6782	-4.7051	-4.5624	-4.5893	-4.4575	-4.4844	-4.3617	-4.3886
.96	-4.6830	-4.7007	-4.5672	-4.5849	-4.4623	-4.4800	-4.3665	-4.3842
.98	-4.6876	-4.6963	-4.5718	-4.5805	-4.4669	-4.4757	-4.3710	-4.3798
1.00	-4.6920	-4.6920	-4.5762	-4.5762	-4.4713	-4.4713	-4.3755	-4.3755

Table D–5: $0.0195 \leq \hat{\rho}_4 \leq 0.0240$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0195$		$\hat{\rho}_4 = 0.0210$		$\hat{\rho}_4 = 0.0225$		$\hat{\rho}_4 = 0.0240$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-2.6485	-4.3600	-2.5645	-4.2777	-2.4860	-4.2010	-2.4122	-4.1291
.04	-2.9844	-4.3877	-2.9015	-4.3056	-2.8242	-4.2290	-2.7516	-4.1572
.06	-3.1798	-4.4049	-3.0973	-4.3228	-3.0203	-4.2463	-2.9481	-4.1746
.08	-3.3173	-4.4165	-3.2350	-4.3345	-3.1582	-4.2580	-3.0862	-4.1864
.10	-3.4229	-4.4246	-3.3407	-4.3426	-3.2640	-4.2662	-3.1922	-4.1946
.12	-3.5081	-4.4302	-3.4259	-4.3483	-3.3493	-4.2719	-3.2776	-4.2003
.14	-3.5791	-4.4341	-3.4970	-4.3521	-3.4204	-4.2757	-3.3487	-4.2042
.16	-3.6397	-4.4365	-3.5576	-4.3545	-3.4811	-4.2782	-3.4095	-4.2067
.18	-3.6923	-4.4378	-3.6102	-4.3559	-3.5338	-4.2795	-3.4622	-4.2080
.20	-3.7385	-4.4382	-3.6565	-4.3563	-3.5801	-4.2799	-3.5085	-4.2085
.22	-3.7797	-4.4379	-3.6977	-4.3560	-3.6213	-4.2796	-3.5497	-4.2082
.24	-3.8166	-4.4369	-3.7346	-4.3550	-3.6583	-4.2787	-3.5867	-4.2073
.26	-3.8500	-4.4355	-3.7680	-4.3536	-3.6917	-4.2773	-3.6201	-4.2058
.28	-3.8803	-4.4336	-3.7984	-4.3517	-3.7220	-4.2754	-3.6505	-4.2040
.30	-3.9080	-4.4313	-3.8261	-4.3495	-3.7498	-4.2732	-3.6783	-4.2017
.32	-3.9335	-4.4287	-3.8516	-4.3469	-3.7753	-4.2706	-3.7038	-4.1992
.34	-3.9570	-4.4259	-3.8752	-4.3441	-3.7988	-4.2678	-3.7274	-4.1964
.36	-3.9788	-4.4228	-3.8969	-4.3410	-3.8206	-4.2647	-3.7492	-4.1933
.38	-3.9990	-4.4195	-3.9172	-4.3377	-3.8409	-4.2614	-3.7694	-4.1900
.40	-4.0179	-4.4161	-3.9360	-4.3342	-3.8597	-4.2580	-3.7883	-4.1866
.42	-4.0355	-4.4125	-3.9536	-4.3306	-3.8773	-4.2544	-3.8059	-4.1830
.44	-4.0519	-4.4087	-3.9701	-4.3269	-3.8938	-4.2506	-3.8224	-4.1792
.46	-4.0674	-4.4049	-3.9856	-4.3230	-3.9093	-4.2468	-3.8379	-4.1754
.48	-4.0819	-4.4009	-4.0001	-4.3191	-3.9238	-4.2428	-3.8524	-4.1714
.50	-4.0956	-4.3968	-4.0138	-4.3150	-3.9375	-4.2388	-3.8661	-4.1674
.52	-4.1086	-4.3927	-4.0267	-4.3109	-3.9505	-4.2347	-3.8791	-4.1633
.54	-4.1208	-4.3885	-4.0390	-4.3067	-3.9627	-4.2305	-3.8913	-4.1591
.56	-4.1323	-4.3843	-4.0505	-4.3025	-3.9743	-4.2263	-3.9029	-4.1549
.58	-4.1433	-4.3800	-4.0615	-4.2982	-3.9852	-4.2220	-3.9138	-4.1506
.60	-4.1537	-4.3757	-4.0719	-4.2939	-3.9956	-4.2176	-3.9242	-4.1463
.62	-4.1636	-4.3713	-4.0818	-4.2895	-4.0055	-4.2133	-3.9341	-4.1419
.64	-4.1730	-4.3669	-4.0912	-4.2851	-4.0149	-4.2089	-3.9435	-4.1375
.66	-4.1820	-4.3625	-4.1002	-4.2807	-4.0239	-4.2045	-3.9525	-4.1331
.68	-4.1905	-4.3581	-4.1087	-4.2763	-4.0325	-4.2001	-3.9611	-4.1287
.70	-4.1987	-4.3536	-4.1169	-4.2718	-4.0406	-4.1956	-3.9692	-4.1242
.72	-4.2064	-4.3492	-4.1246	-4.2674	-4.0484	-4.1912	-3.9770	-4.1198
.74	-4.2139	-4.3447	-4.1321	-4.2629	-4.0559	-4.1867	-3.9845	-4.1153
.76	-4.2210	-4.3403	-4.1392	-4.2585	-4.0630	-4.1823	-3.9916	-4.1109
.78	-4.2278	-4.3358	-4.1460	-4.2540	-4.0698	-4.1778	-3.9984	-4.1064
.80	-4.2344	-4.3313	-4.1526	-4.2496	-4.0764	-4.1733	-4.0050	-4.1020
.82	-4.2407	-4.3269	-4.1589	-4.2451	-4.0826	-4.1689	-4.0112	-4.0975
.84	-4.2467	-4.3224	-4.1649	-4.2406	-4.0886	-4.1644	-4.0173	-4.0930
.86	-4.2525	-4.3180	-4.1707	-4.2362	-4.0944	-4.1600	-4.0230	-4.0886
.88	-4.2580	-4.3136	-4.1762	-4.2318	-4.1000	-4.1555	-4.0286	-4.0842
.90	-4.2634	-4.3091	-4.1816	-4.2273	-4.1053	-4.1511	-4.0339	-4.0797
.92	-4.2685	-4.3047	-4.1867	-4.2229	-4.1105	-4.1467	-4.0391	-4.0753
.94	-4.2734	-4.3003	-4.1916	-4.2185	-4.1154	-4.1423	-4.0440	-4.0709
.96	-4.2782	-4.2959	-4.1964	-4.2141	-4.1202	-4.1379	-4.0488	-4.0665
.98	-4.2828	-4.2916	-4.2010	-4.2098	-4.1248	-4.1336	-4.0534	-4.0622
1.00	-4.2872	-4.2872	-4.2054	-4.2054	-4.1292	-4.1292	-4.0578	-4.0578

Table D–5: $0.026 \leq \hat{\rho}_4 \leq 0.032$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.026$		$\hat{\rho}_4 = 0.028$		$\hat{\rho}_4 = 0.030$		$\hat{\rho}_4 = 0.032$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-2.3201	-4.0398	-2.2344	-3.9569	-2.1540	-3.8795	-2.0782	-3.8069
.04	-2.6613	-4.0681	-2.5774	-3.9853	-2.4990	-3.9082	-2.4254	-3.8359
.06	-2.8584	-4.0855	-2.7751	-4.0029	-2.6973	-3.9259	-2.6243	-3.8537
.08	-2.9968	-4.0974	-2.9137	-4.0148	-2.8363	-3.9378	-2.7636	-3.8657
.10	-3.1029	-4.1056	-3.0200	-4.0232	-2.9427	-3.9462	-2.8702	-3.8741
.12	-3.1884	-4.1114	-3.1056	-4.0289	-3.0285	-3.9521	-2.9561	-3.8800
.14	-3.2597	-4.1153	-3.1770	-4.0329	-3.0999	-3.9560	-3.0276	-3.8840
.16	-3.3204	-4.1178	-3.2379	-4.0354	-3.1608	-3.9586	-3.0886	-3.8866
.18	-3.3732	-4.1192	-3.2907	-4.0368	-3.2137	-3.9600	-3.1415	-3.8881
.20	-3.4196	-4.1197	-3.3371	-4.0373	-3.2602	-3.9605	-3.1881	-3.8886
.22	-3.4608	-4.1194	-3.3784	-4.0371	-3.3015	-3.9603	-3.2294	-3.8884
.24	-3.4978	-4.1185	-3.4154	-4.0362	-3.3386	-3.9594	-3.2665	-3.8876
.26	-3.5313	-4.1171	-3.4489	-4.0348	-3.3720	-3.9581	-3.3001	-3.8862
.28	-3.5617	-4.1152	-3.4793	-4.0330	-3.4025	-3.9562	-3.3305	-3.8844
.30	-3.5895	-4.1130	-3.5072	-4.0308	-3.4303	-3.9540	-3.3584	-3.8822
.32	-3.6150	-4.1105	-3.5327	-4.0282	-3.4559	-3.9515	-3.3840	-3.8797
.34	-3.6386	-4.1077	-3.5563	-4.0254	-3.4795	-3.9487	-3.4076	-3.8769
.36	-3.6604	-4.1046	-3.5781	-4.0224	-3.5013	-3.9457	-3.4295	-3.8738
.38	-3.6807	-4.1013	-3.5984	-4.0191	-3.5216	-3.9424	-3.4497	-3.8706
.40	-3.6995	-4.0979	-3.6173	-4.0157	-3.5405	-3.9390	-3.4686	-3.8672
.42	-3.7172	-4.0943	-3.6349	-4.0121	-3.5582	-3.9354	-3.4863	-3.8636
.44	-3.7337	-4.0906	-3.6514	-4.0084	-3.5747	-3.9317	-3.5028	-3.8599
.46	-3.7491	-4.0867	-3.6669	-4.0045	-3.5902	-3.9279	-3.5183	-3.8561
.48	-3.7637	-4.0828	-3.6815	-4.0006	-3.6048	-3.9239	-3.5329	-3.8521
.50	-3.7774	-4.0787	-3.6952	-3.9965	-3.6185	-3.9199	-3.5467	-3.8481
.52	-3.7904	-4.0746	-3.7081	-3.9924	-3.6314	-3.9158	-3.5596	-3.8440
.54	-3.8026	-4.0705	-3.7204	-3.9883	-3.6437	-3.9116	-3.5719	-3.8398
.56	-3.8142	-4.0662	-3.7320	-3.9840	-3.6553	-3.9074	-3.5835	-3.8356
.58	-3.8252	-4.0619	-3.7429	-3.9797	-3.6663	-3.9031	-3.5945	-3.8313
.60	-3.8356	-4.0576	-3.7534	-3.9754	-3.6767	-3.8988	-3.6049	-3.8270
.62	-3.8455	-4.0533	-3.7633	-3.9711	-3.6866	-3.8944	-3.6148	-3.8227
.64	-3.8549	-4.0489	-3.7727	-3.9667	-3.6960	-3.8901	-3.6242	-3.8183
.66	-3.8639	-4.0445	-3.7817	-3.9623	-3.7050	-3.8857	-3.6332	-3.8139
.68	-3.8724	-4.0400	-3.7902	-3.9579	-3.7136	-3.8812	-3.6418	-3.8095
.70	-3.8806	-4.0356	-3.7984	-3.9534	-3.7217	-3.8768	-3.6499	-3.8050
.72	-3.8884	-4.0312	-3.8062	-3.9490	-3.7295	-3.8723	-3.6577	-3.8006
.74	-3.8958	-4.0267	-3.8136	-3.9445	-3.7370	-3.8679	-3.6652	-3.7961
.76	-3.9030	-4.0222	-3.8208	-3.9401	-3.7441	-3.8634	-3.6723	-3.7917
.78	-3.9098	-4.0178	-3.8276	-3.9356	-3.7509	-3.8590	-3.6792	-3.7872
.80	-3.9163	-4.0133	-3.8341	-3.9311	-3.7575	-3.8545	-3.6857	-3.7827
.82	-3.9226	-4.0089	-3.8404	-3.9267	-3.7638	-3.8501	-3.6920	-3.7783
.84	-3.9286	-4.0044	-3.8464	-3.9222	-3.7698	-3.8456	-3.6980	-3.7738
.86	-3.9344	-4.0000	-3.8522	-3.9178	-3.7756	-3.8412	-3.7038	-3.7694
.88	-3.9400	-3.9955	-3.8578	-3.9134	-3.7812	-3.8367	-3.7094	-3.7650
.90	-3.9453	-3.9911	-3.8631	-3.9089	-3.7865	-3.8323	-3.7147	-3.7605
.92	-3.9505	-3.9867	-3.8683	-3.9045	-3.7916	-3.8279	-3.7199	-3.7561
.94	-3.9554	-3.9823	-3.8732	-3.9001	-3.7966	-3.8235	-3.7248	-3.7517
.96	-3.9602	-3.9779	-3.8780	-3.8957	-3.8014	-3.8191	-3.7296	-3.7473
.98	-3.9648	-3.9735	-3.8826	-3.8914	-3.8059	-3.8147	-3.7342	-3.7430
1.00	-3.9692	-3.9692	-3.8870	-3.8870	-3.8104	-3.8104	-3.7386	-3.7386

Table D–5: $0.034 \leq \hat{\rho}_4 \leq 0.040$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.034$		$\hat{\rho}_4 = 0.036$		$\hat{\rho}_4 = 0.038$		$\hat{\rho}_4 = 0.040$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-2.0065	-3.7386	-1.9383	-3.6740	-1.8733	-3.6128	-1.8111	-3.5545
.04	-2.3559	-3.7678	-2.2901	-3.7034	-2.2276	-3.6424	-2.1680	-3.5844
.06	-2.5555	-3.7857	-2.4904	-3.7215	-2.4286	-3.6606	-2.3698	-3.6027
.08	-2.6951	-3.7978	-2.6304	-3.7337	-2.5690	-3.6729	-2.5106	-3.6151
.10	-2.8020	-3.8063	-2.7374	-3.7422	-2.6762	-3.6815	-2.6180	-3.6237
.12	-2.8880	-3.8122	-2.8236	-3.7482	-2.7625	-3.6875	-2.7045	-3.6298
.14	-2.9596	-3.8163	-2.8953	-3.7523	-2.8344	-3.6916	-2.7764	-3.6340
.16	-3.0207	-3.8189	-2.9565	-3.7549	-2.8956	-3.6943	-2.8377	-3.6367
.18	-3.0736	-3.8204	-3.0095	-3.7564	-2.9487	-3.6958	-2.8909	-3.6382
.20	-3.1202	-3.8209	-3.0561	-3.7570	-2.9954	-3.6964	-2.9376	-3.6389
.22	-3.1616	-3.8207	-3.0976	-3.7568	-3.0369	-3.6963	-2.9791	-3.6387
.24	-3.1988	-3.8199	-3.1347	-3.7560	-3.0741	-3.6955	-3.0164	-3.6379
.26	-3.2323	-3.8186	-3.1683	-3.7547	-3.1077	-3.6942	-3.0500	-3.6366
.28	-3.2628	-3.8168	-3.1988	-3.7529	-3.1382	-3.6924	-3.0806	-3.6349
.30	-3.2907	-3.8146	-3.2267	-3.7507	-3.1661	-3.6902	-3.1086	-3.6327
.32	-3.3163	-3.8121	-3.2524	-3.7482	-3.1918	-3.6877	-3.1342	-3.6303
.34	-3.3399	-3.8093	-3.2760	-3.7455	-3.2155	-3.6850	-3.1579	-3.6275
.36	-3.3618	-3.8063	-3.2979	-3.7424	-3.2374	-3.6820	-3.1798	-3.6245
.38	-3.3821	-3.8030	-3.3182	-3.7392	-3.2577	-3.6788	-3.2002	-3.6213
.40	-3.4010	-3.7996	-3.3371	-3.7358	-3.2766	-3.6754	-3.2191	-3.6179
.42	-3.4187	-3.7960	-3.3548	-3.7322	-3.2943	-3.6718	-3.2368	-3.6144
.44	-3.4352	-3.7923	-3.3714	-3.7285	-3.3109	-3.6681	-3.2534	-3.6107
.46	-3.4507	-3.7885	-3.3869	-3.7247	-3.3264	-3.6643	-3.2689	-3.6068
.48	-3.4653	-3.7846	-3.4015	-3.7208	-3.3410	-3.6604	-3.2835	-3.6029
.50	-3.4791	-3.7806	-3.4152	-3.7168	-3.3548	-3.6563	-3.2973	-3.5989
.52	-3.4920	-3.7765	-3.4282	-3.7127	-3.3678	-3.6523	-3.3103	-3.5948
.54	-3.5043	-3.7723	-3.4405	-3.7085	-3.3800	-3.6481	-3.3226	-3.5907
.56	-3.5159	-3.7681	-3.4521	-3.7043	-3.3916	-3.6439	-3.3342	-3.5865
.58	-3.5269	-3.7638	-3.4631	-3.7000	-3.4026	-3.6396	-3.3452	-3.5822
.60	-3.5373	-3.7595	-3.4735	-3.6957	-3.4131	-3.6353	-3.3556	-3.5779
.62	-3.5472	-3.7551	-3.4834	-3.6914	-3.4230	-3.6310	-3.3656	-3.5736
.64	-3.5567	-3.7508	-3.4929	-3.6870	-3.4324	-3.6266	-3.3750	-3.5692
.66	-3.5657	-3.7464	-3.5019	-3.6826	-3.4414	-3.6222	-3.3840	-3.5648
.68	-3.5742	-3.7419	-3.5104	-3.6782	-3.4500	-3.6178	-3.3926	-3.5604
.70	-3.5824	-3.7375	-3.5186	-3.6737	-3.4582	-3.6133	-3.4008	-3.5559
.72	-3.5902	-3.7331	-3.5264	-3.6693	-3.4660	-3.6089	-3.4086	-3.5515
.74	-3.5977	-3.7286	-3.5339	-3.6648	-3.4735	-3.6044	-3.4161	-3.5471
.76	-3.6048	-3.7241	-3.5410	-3.6604	-3.4806	-3.6000	-3.4232	-3.5426
.78	-3.6116	-3.7197	-3.5479	-3.6559	-3.4875	-3.5955	-3.4301	-3.5381
.80	-3.6182	-3.7152	-3.5544	-3.6515	-3.4940	-3.5911	-3.4366	-3.5337
.82	-3.6245	-3.7108	-3.5607	-3.6470	-3.5003	-3.5866	-3.4429	-3.5292
.84	-3.6305	-3.7063	-3.5667	-3.6426	-3.5063	-3.5822	-3.4489	-3.5248
.86	-3.6363	-3.7019	-3.5725	-3.6381	-3.5121	-3.5777	-3.4547	-3.5203
.88	-3.6419	-3.6974	-3.5781	-3.6337	-3.5177	-3.5733	-3.4603	-3.5159
.90	-3.6472	-3.6930	-3.5834	-3.6293	-3.5230	-3.5689	-3.4656	-3.5115
.92	-3.6524	-3.6886	-3.5886	-3.6249	-3.5282	-3.5645	-3.4708	-3.5071
.94	-3.6573	-3.6842	-3.5935	-3.6205	-3.5331	-3.5601	-3.4757	-3.5027
.96	-3.6621	-3.6798	-3.5983	-3.6161	-3.5379	-3.5557	-3.4805	-3.4983
.98	-3.6667	-3.6754	-3.6029	-3.6117	-3.5425	-3.5513	-3.4851	-3.4939
1.00	-3.6711	-3.6711	-3.6073	-3.6073	-3.5469	-3.5469	-3.4895	-3.4895

Table D–5: $0.0425 \leq \hat{\rho}_4 \leq 0.0500$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0425$		$\hat{\rho}_4 = 0.0450$		$\hat{\rho}_4 = 0.0475$		$\hat{\rho}_4 = 0.0500$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-1.7368	-3.4854	-1.6660	-3.4201	-1.5981	-3.3580	-1.5329	-3.2989
.04	-2.0972	-3.5157	-2.0301	-3.4507	-1.9662	-3.3890	-1.9051	-3.3303
.06	-2.3001	-3.5341	-2.2340	-3.4693	-2.1712	-3.4078	-2.1113	-3.3492
.08	-2.4413	-3.5466	-2.3757	-3.4819	-2.3134	-3.4205	-2.2541	-3.3621
.10	-2.5490	-3.5553	-2.4837	-3.4907	-2.4218	-3.4294	-2.3628	-3.3711
.12	-2.6357	-3.5615	-2.5706	-3.4969	-2.5089	-3.4356	-2.4501	-3.3774
.14	-2.7078	-3.5657	-2.6428	-3.5011	-2.5812	-3.4400	-2.5226	-3.3818
.16	-2.7692	-3.5684	-2.7043	-3.5039	-2.6429	-3.4428	-2.5843	-3.3847
.18	-2.8224	-3.5700	-2.7577	-3.5056	-2.6963	-3.4445	-2.6378	-3.3864
.20	-2.8692	-3.5707	-2.8045	-3.5063	-2.7432	-3.4452	-2.6849	-3.3871
.22	-2.9108	-3.5706	-2.8462	-3.5062	-2.7849	-3.4451	-2.7266	-3.3871
.24	-2.9481	-3.5698	-2.8835	-3.5054	-2.8223	-3.4444	-2.7640	-3.3864
.26	-2.9817	-3.5685	-2.9172	-3.5042	-2.8560	-3.4432	-2.7978	-3.3852
.28	-3.0124	-3.5668	-2.9479	-3.5024	-2.8867	-3.4415	-2.8286	-3.3835
.30	-3.0403	-3.5647	-2.9759	-3.5003	-2.9148	-3.4394	-2.8566	-3.3814
.32	-3.0660	-3.5622	-3.0016	-3.4979	-2.9405	-3.4369	-2.8824	-3.3790
.34	-3.0897	-3.5595	-3.0253	-3.4952	-2.9643	-3.4342	-2.9062	-3.3763
.36	-3.1117	-3.5565	-3.0473	-3.4922	-2.9862	-3.4313	-2.9282	-3.3733
.38	-3.1320	-3.5533	-3.0677	-3.4890	-3.0066	-3.4281	-2.9486	-3.3702
.40	-3.1510	-3.5499	-3.0866	-3.4856	-3.0256	-3.4247	-2.9676	-3.3668
.42	-3.1687	-3.5463	-3.1044	-3.4821	-3.0434	-3.4212	-2.9854	-3.3633
.44	-3.1853	-3.5427	-3.1210	-3.4784	-3.0600	-3.4175	-3.0020	-3.3596
.46	-3.2009	-3.5389	-3.1365	-3.4746	-3.0756	-3.4137	-3.0176	-3.3558
.48	-3.2155	-3.5349	-3.1512	-3.4707	-3.0902	-3.4098	-3.0323	-3.3519
.50	-3.2293	-3.5309	-3.1650	-3.4667	-3.1040	-3.4058	-3.0461	-3.3480
.52	-3.2423	-3.5269	-3.1780	-3.4626	-3.1170	-3.4018	-3.0591	-3.3439
.54	-3.2545	-3.5227	-3.1903	-3.4585	-3.1293	-3.3976	-3.0714	-3.3398
.56	-3.2662	-3.5185	-3.2019	-3.4543	-3.1410	-3.3934	-3.0831	-3.3356
.58	-3.2772	-3.5142	-3.2129	-3.4500	-3.1520	-3.3892	-3.0941	-3.3313
.60	-3.2876	-3.5099	-3.2234	-3.4457	-3.1625	-3.3849	-3.1046	-3.3270
.62	-3.2976	-3.5056	-3.2333	-3.4414	-3.1724	-3.3805	-3.1145	-3.3227
.64	-3.3070	-3.5012	-3.2428	-3.4370	-3.1819	-3.3762	-3.1240	-3.3183
.66	-3.3160	-3.4968	-3.2518	-3.4326	-3.1909	-3.3718	-3.1330	-3.3139
.68	-3.3246	-3.4924	-3.2604	-3.4282	-3.1995	-3.3674	-3.1416	-3.3095
.70	-3.3328	-3.4880	-3.2685	-3.4238	-3.2077	-3.3629	-3.1498	-3.3051
.72	-3.3406	-3.4836	-3.2764	-3.4193	-3.2155	-3.3585	-3.1576	-3.3007
.74	-3.3481	-3.4791	-3.2838	-3.4149	-3.2230	-3.3541	-3.1651	-3.2962
.76	-3.3552	-3.4746	-3.2910	-3.4104	-3.2301	-3.3496	-3.1723	-3.2918
.78	-3.3621	-3.4702	-3.2979	-3.4060	-3.2370	-3.3452	-3.1791	-3.2873
.80	-3.3686	-3.4657	-3.3044	-3.4015	-3.2436	-3.3407	-3.1857	-3.2829
.82	-3.3749	-3.4613	-3.3107	-3.3971	-3.2499	-3.3362	-3.1920	-3.2784
.84	-3.3810	-3.4568	-3.3167	-3.3926	-3.2559	-3.3318	-3.1981	-3.2740
.86	-3.3868	-3.4524	-3.3225	-3.3882	-3.2617	-3.3274	-3.2039	-3.2695
.88	-3.3923	-3.4480	-3.3281	-3.3837	-3.2673	-3.3229	-3.2094	-3.2651
.90	-3.3977	-3.4435	-3.3335	-3.3793	-3.2726	-3.3185	-3.2148	-3.2607
.92	-3.4028	-3.4391	-3.3386	-3.3749	-3.2778	-3.3141	-3.2199	-3.2562
.94	-3.4078	-3.4347	-3.3436	-3.3705	-3.2827	-3.3097	-3.2249	-3.2518
.96	-3.4126	-3.4303	-3.3484	-3.3661	-3.2875	-3.3053	-3.2297	-3.2474
.98	-3.4172	-3.4259	-3.3529	-3.3617	-3.2921	-3.3009	-3.2343	-3.2431
1.00	-3.4216	-3.4216	-3.3574	-3.3574	-3.2965	-3.2965	-3.2387	-3.2387

Table D–5: $0.053 \leq \hat{\rho}_4 \leq 0.062$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.053$		$\hat{\rho}_4 = 0.056$		$\hat{\rho}_4 = 0.059$		$\hat{\rho}_4 = 0.062$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-1.4577	-3.2315	-1.3854	-3.1675	-1.3155	-3.1065	-1.2478	-3.0483
.04	-1.8353	-3.2633	-1.7687	-3.1998	-1.7051	-3.1394	-1.6441	-3.0818
.06	-2.0429	-3.2825	-1.9779	-3.2193	-1.9159	-3.1591	-1.8567	-3.1017
.08	-2.1864	-3.2955	-2.1221	-3.2324	-2.0609	-3.1724	-2.0024	-3.1152
.10	-2.2955	-3.3046	-2.2317	-3.2416	-2.1709	-3.1817	-2.1128	-3.1246
.12	-2.3831	-3.3110	-2.3195	-3.2481	-2.2590	-3.1883	-2.2012	-3.1313
.14	-2.4558	-3.3155	-2.3924	-3.2526	-2.3321	-3.1929	-2.2746	-3.1360
.16	-2.5177	-3.3184	-2.4544	-3.2556	-2.3943	-3.1960	-2.3369	-3.1391
.18	-2.5713	-3.3202	-2.5082	-3.2574	-2.4481	-3.1978	-2.3909	-3.1410
.20	-2.6184	-3.3210	-2.5554	-3.2583	-2.4954	-3.1987	-2.4383	-3.1419
.22	-2.6602	-3.3210	-2.5973	-3.2583	-2.5374	-3.1988	-2.4803	-3.1420
.24	-2.6977	-3.3203	-2.6348	-3.2577	-2.5750	-3.1982	-2.5180	-3.1414
.26	-2.7316	-3.3191	-2.6687	-3.2565	-2.6090	-3.1970	-2.5520	-3.1403
.28	-2.7623	-3.3174	-2.6995	-3.2549	-2.6399	-3.1954	-2.5830	-3.1387
.30	-2.7904	-3.3154	-2.7277	-3.2528	-2.6681	-3.1934	-2.6112	-3.1367
.32	-2.8163	-3.3130	-2.7535	-3.2504	-2.6940	-3.1910	-2.6371	-3.1344
.34	-2.8401	-3.3103	-2.7774	-3.2478	-2.7178	-3.1884	-2.6610	-3.1318
.36	-2.8621	-3.3074	-2.7994	-3.2449	-2.7399	-3.1855	-2.6831	-3.1289
.38	-2.8825	-3.3042	-2.8199	-3.2417	-2.7604	-3.1823	-2.7037	-3.1257
.40	-2.9016	-3.3009	-2.8390	-3.2384	-2.7795	-3.1790	-2.7228	-3.1224
.42	-2.9194	-3.2973	-2.8568	-3.2349	-2.7973	-3.1755	-2.7406	-3.1190
.44	-2.9360	-3.2937	-2.8734	-3.2312	-2.8140	-3.1719	-2.7573	-3.1153
.46	-2.9516	-3.2899	-2.8891	-3.2275	-2.8296	-3.1681	-2.7730	-3.1116
.48	-2.9663	-3.2860	-2.9038	-3.2236	-2.8443	-3.1643	-2.7877	-3.1077
.50	-2.9801	-3.2821	-2.9176	-3.2196	-2.8582	-3.1603	-2.8016	-3.1038
.52	-2.9931	-3.2780	-2.9306	-3.2156	-2.8712	-3.1562	-2.8146	-3.0997
.54	-3.0055	-3.2739	-2.9430	-3.2114	-2.8836	-3.1521	-2.8270	-3.0956
.56	-3.0171	-3.2697	-2.9546	-3.2073	-2.8953	-3.1479	-2.8387	-3.0914
.58	-3.0282	-3.2654	-2.9657	-3.2030	-2.9063	-3.1437	-2.8497	-3.0872
.60	-3.0386	-3.2611	-2.9762	-3.1987	-2.9168	-3.1394	-2.8603	-3.0829
.62	-3.0486	-3.2568	-2.9862	-3.1944	-2.9268	-3.1351	-2.8702	-3.0786
.64	-3.0581	-3.2525	-2.9956	-3.1901	-2.9363	-3.1308	-2.8797	-3.0743
.66	-3.0671	-3.2481	-3.0047	-3.1857	-2.9453	-3.1264	-2.8888	-3.0699
.68	-3.0757	-3.2437	-3.0133	-3.1813	-2.9539	-3.1220	-2.8974	-3.0655
.70	-3.0839	-3.2392	-3.0215	-3.1768	-2.9622	-3.1176	-2.9056	-3.0611
.72	-3.0917	-3.2348	-3.0293	-3.1724	-2.9700	-3.1131	-2.9135	-3.0566
.74	-3.0992	-3.2304	-3.0368	-3.1680	-2.9775	-3.1087	-2.9210	-3.0522
.76	-3.1064	-3.2259	-3.0440	-3.1635	-2.9847	-3.1042	-2.9282	-3.0478
.78	-3.1133	-3.2215	-3.0508	-3.1591	-2.9915	-3.0998	-2.9350	-3.0433
.80	-3.1198	-3.2170	-3.0574	-3.1546	-2.9981	-3.0953	-2.9416	-3.0389
.82	-3.1261	-3.2126	-3.0637	-3.1502	-3.0044	-3.0909	-2.9479	-3.0344
.84	-3.1322	-3.2081	-3.0698	-3.1457	-3.0105	-3.0864	-2.9540	-3.0300
.86	-3.1380	-3.2037	-3.0756	-3.1413	-3.0163	-3.0820	-2.9598	-3.0255
.88	-3.1436	-3.1992	-3.0812	-3.1368	-3.0219	-3.0776	-2.9654	-3.0211
.90	-3.1489	-3.1948	-3.0865	-3.1324	-3.0272	-3.0731	-2.9707	-3.0166
.92	-3.1541	-3.1904	-3.0917	-3.1280	-3.0324	-3.0687	-2.9759	-3.0122
.94	-3.1590	-3.1860	-3.0966	-3.1236	-3.0374	-3.0643	-2.9809	-3.0078
.96	-3.1638	-3.1816	-3.1014	-3.1192	-3.0421	-3.0599	-2.9856	-3.0034
.98	-3.1684	-3.1772	-3.1060	-3.1148	-3.0467	-3.0555	-2.9902	-2.9991
1.00	-3.1728	-3.1728	-3.1104	-3.1104	-3.0512	-3.0512	-2.9947	-2.9947

Table D–5: $0.0655 \leq \hat{\rho}_4 \leq 0.0760$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0655$		$\hat{\rho}_4 = 0.0690$		$\hat{\rho}_4 = 0.0725$		$\hat{\rho}_4 = 0.0760$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-1.1711	-2.9834	-1.0964	-2.9215	-1.0233	-2.8623	-.9515	-2.8055
.04	-1.5759	-3.0176	-1.5105	-2.9565	-1.4476	-2.8981	-1.3870	-2.8422
.06	-1.7905	-3.0379	-1.7274	-2.9771	-1.6669	-2.9191	-1.6087	-2.8635
.08	-1.9372	-3.0516	-1.8751	-2.9910	-1.8156	-2.9332	-1.7586	-2.8778
.10	-2.0482	-3.0611	-1.9866	-3.0007	-1.9278	-2.9430	-1.8713	-2.8879
.12	-2.1370	-3.0679	-2.0758	-3.0076	-2.0173	-2.9501	-1.9613	-2.8950
.14	-2.2106	-3.0727	-2.1496	-3.0125	-2.0914	-2.9550	-2.0357	-2.9000
.16	-2.2731	-3.0759	-2.2124	-3.0157	-2.1544	-2.9583	-2.0988	-2.9035
.18	-2.3272	-3.0778	-2.2666	-3.0177	-2.2088	-2.9604	-2.1534	-2.9056
.20	-2.3747	-3.0788	-2.3143	-3.0188	-2.2565	-2.9615	-2.2013	-2.9067
.22	-2.4169	-3.0789	-2.3565	-3.0190	-2.2989	-2.9617	-2.2438	-2.9070
.24	-2.4547	-3.0784	-2.3944	-3.0185	-2.3369	-2.9613	-2.2818	-2.9066
.26	-2.4888	-3.0773	-2.4286	-3.0174	-2.3711	-2.9603	-2.3161	-2.9056
.28	-2.5197	-3.0758	-2.4596	-3.0159	-2.4022	-2.9588	-2.3473	-2.9042
.30	-2.5480	-3.0738	-2.4879	-3.0139	-2.4306	-2.9569	-2.3758	-2.9023
.32	-2.5740	-3.0715	-2.5140	-3.0117	-2.4567	-2.9546	-2.4019	-2.9000
.34	-2.5979	-3.0689	-2.5379	-3.0091	-2.4807	-2.9520	-2.4259	-2.8975
.36	-2.6201	-3.0660	-2.5601	-3.0062	-2.5029	-2.9492	-2.4482	-2.8947
.38	-2.6406	-3.0629	-2.5807	-3.0031	-2.5235	-2.9461	-2.4689	-2.8916
.40	-2.6598	-3.0596	-2.5999	-2.9998	-2.5427	-2.9429	-2.4881	-2.8884
.42	-2.6777	-3.0561	-2.6178	-2.9964	-2.5606	-2.9394	-2.5060	-2.8850
.44	-2.6944	-3.0525	-2.6345	-2.9928	-2.5774	-2.9358	-2.5228	-2.8814
.46	-2.7101	-3.0488	-2.6502	-2.9891	-2.5931	-2.9321	-2.5386	-2.8777
.48	-2.7248	-3.0449	-2.6650	-2.9852	-2.6079	-2.9283	-2.5534	-2.8739
.50	-2.7387	-3.0410	-2.6789	-2.9813	-2.6218	-2.9244	-2.5673	-2.8699
.52	-2.7518	-3.0370	-2.6920	-2.9773	-2.6349	-2.9203	-2.5804	-2.8659
.54	-2.7641	-3.0328	-2.7044	-2.9732	-2.6473	-2.9162	-2.5928	-2.8618
.56	-2.7758	-3.0287	-2.7161	-2.9690	-2.6591	-2.9121	-2.6046	-2.8577
.58	-2.7869	-3.0245	-2.7272	-2.9648	-2.6702	-2.9079	-2.6157	-2.8535
.60	-2.7974	-3.0202	-2.7377	-2.9605	-2.6807	-2.9036	-2.6263	-2.8492
.62	-2.8074	-3.0159	-2.7477	-2.9562	-2.6907	-2.8993	-2.6363	-2.8449
.64	-2.8169	-3.0115	-2.7572	-2.9519	-2.7003	-2.8950	-2.6458	-2.8406
.66	-2.8260	-3.0072	-2.7663	-2.9475	-2.7093	-2.8906	-2.6549	-2.8362
.68	-2.8346	-3.0028	-2.7749	-2.9431	-2.7180	-2.8862	-2.6635	-2.8319
.70	-2.8428	-2.9983	-2.7831	-2.9387	-2.7262	-2.8818	-2.6718	-2.8275
.72	-2.8507	-2.9939	-2.7910	-2.9343	-2.7341	-2.8774	-2.6797	-2.8230
.74	-2.8582	-2.9895	-2.7985	-2.9298	-2.7416	-2.8730	-2.6872	-2.8186
.76	-2.8654	-2.9850	-2.8057	-2.9254	-2.7488	-2.8685	-2.6944	-2.8142
.78	-2.8723	-2.9806	-2.8126	-2.9209	-2.7557	-2.8641	-2.7013	-2.8097
.80	-2.8789	-2.9761	-2.8192	-2.9165	-2.7623	-2.8596	-2.7079	-2.8053
.82	-2.8852	-2.9717	-2.8255	-2.9120	-2.7686	-2.8552	-2.7142	-2.8008
.84	-2.8912	-2.9672	-2.8316	-2.9076	-2.7747	-2.8507	-2.7203	-2.7964
.86	-2.8970	-2.9628	-2.8374	-2.9032	-2.7805	-2.8463	-2.7261	-2.7919
.88	-2.9026	-2.9584	-2.8430	-2.8987	-2.7861	-2.8418	-2.7317	-2.7875
.90	-2.9080	-2.9539	-2.8483	-2.8943	-2.7915	-2.8374	-2.7371	-2.7831
.92	-2.9132	-2.9495	-2.8535	-2.8899	-2.7966	-2.8330	-2.7423	-2.7786
.94	-2.9181	-2.9451	-2.8585	-2.8855	-2.8016	-2.8286	-2.7472	-2.7742
.96	-2.9229	-2.9407	-2.8633	-2.8811	-2.8064	-2.8242	-2.7520	-2.7698
.98	-2.9275	-2.9363	-2.8679	-2.8767	-2.8110	-2.8198	-2.7566	-2.7655
1.00	-2.9320	-2.9320	-2.8723	-2.8723	-2.8154	-2.8154	-2.7611	-2.7611

Table D–5: $0.080 \leq \hat{\rho}_4 \leq 0.092$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.080$		$\hat{\rho}_4 = 0.084$		$\hat{\rho}_4 = 0.088$		$\hat{\rho}_4 = 0.092$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-.8704	-2.7431	-.7897	-2.6833	-.7090	-2.6257	-.6273	-2.5700
.04	-1.3201	-2.7810	-1.2556	-2.7224	-1.1931	-2.6661	-1.1324	-2.6120
.06	-1.5449	-2.8027	-1.4836	-2.7446	-1.4245	-2.6888	-1.3675	-2.6352
.08	-1.6961	-2.8173	-1.6361	-2.7595	-1.5785	-2.7040	-1.5230	-2.6507
.10	-1.8096	-2.8276	-1.7504	-2.7699	-1.6936	-2.7146	-1.6389	-2.6615
.12	-1.9000	-2.8348	-1.8413	-2.7773	-1.7850	-2.7222	-1.7309	-2.6693
.14	-1.9747	-2.8400	-1.9164	-2.7826	-1.8605	-2.7276	-1.8067	-2.6748
.16	-2.0381	-2.8435	-1.9801	-2.7862	-1.9244	-2.7313	-1.8709	-2.6786
.18	-2.0929	-2.8457	-2.0350	-2.7885	-1.9796	-2.7337	-1.9263	-2.6811
.20	-2.1409	-2.8469	-2.0832	-2.7897	-2.0279	-2.7350	-1.9748	-2.6825
.22	-2.1836	-2.8472	-2.1260	-2.7902	-2.0708	-2.7355	-2.0178	-2.6830
.24	-2.2217	-2.8469	-2.1642	-2.7899	-2.1092	-2.7352	-2.0563	-2.6828
.26	-2.2561	-2.8460	-2.1987	-2.7890	-2.1437	-2.7344	-2.0910	-2.6820
.28	-2.2873	-2.8445	-2.2300	-2.7876	-2.1751	-2.7330	-2.1224	-2.6807
.30	-2.3159	-2.8427	-2.2586	-2.7857	-2.2038	-2.7312	-2.1512	-2.6789
.32	-2.3420	-2.8405	-2.2848	-2.7836	-2.2301	-2.7291	-2.1775	-2.6768
.34	-2.3661	-2.8379	-2.3090	-2.7811	-2.2543	-2.7266	-2.2018	-2.6744
.36	-2.3884	-2.8351	-2.3314	-2.7783	-2.2767	-2.7239	-2.2242	-2.6716
.38	-2.4091	-2.8321	-2.3521	-2.7753	-2.2975	-2.7209	-2.2451	-2.6687
.40	-2.4284	-2.8289	-2.3714	-2.7721	-2.3168	-2.7177	-2.2644	-2.6655
.42	-2.4464	-2.8255	-2.3894	-2.7687	-2.3349	-2.7143	-2.2825	-2.6622
.44	-2.4632	-2.8219	-2.4063	-2.7652	-2.3517	-2.7108	-2.2994	-2.6587
.46	-2.4790	-2.8182	-2.4221	-2.7615	-2.3676	-2.7071	-2.3153	-2.6550
.48	-2.4938	-2.8144	-2.4369	-2.7577	-2.3824	-2.7034	-2.3302	-2.6512
.50	-2.5077	-2.8105	-2.4509	-2.7538	-2.3964	-2.6995	-2.3442	-2.6474
.52	-2.5209	-2.8065	-2.4640	-2.7498	-2.4096	-2.6955	-2.3574	-2.6434
.54	-2.5333	-2.8024	-2.4765	-2.7457	-2.4221	-2.6914	-2.3699	-2.6394
.56	-2.5451	-2.7983	-2.4883	-2.7416	-2.4339	-2.6873	-2.3817	-2.6352
.58	-2.5562	-2.7941	-2.4994	-2.7374	-2.4451	-2.6831	-2.3929	-2.6311
.60	-2.5668	-2.7899	-2.5100	-2.7332	-2.4556	-2.6789	-2.4035	-2.6268
.62	-2.5768	-2.7856	-2.5201	-2.7289	-2.4657	-2.6746	-2.4136	-2.6226
.64	-2.5864	-2.7812	-2.5296	-2.7246	-2.4753	-2.6703	-2.4231	-2.6182
.66	-2.5955	-2.7769	-2.5387	-2.7202	-2.4844	-2.6659	-2.4323	-2.6139
.68	-2.6041	-2.7725	-2.5474	-2.7158	-2.4931	-2.6616	-2.4409	-2.6095
.70	-2.6124	-2.7681	-2.5556	-2.7114	-2.5013	-2.6572	-2.4492	-2.6051
.72	-2.6203	-2.7637	-2.5635	-2.7070	-2.5092	-2.6528	-2.4571	-2.6007
.74	-2.6278	-2.7592	-2.5711	-2.7026	-2.5168	-2.6483	-2.4647	-2.5963
.76	-2.6350	-2.7548	-2.5783	-2.6981	-2.5240	-2.6439	-2.4719	-2.5919
.78	-2.6419	-2.7504	-2.5852	-2.6937	-2.5309	-2.6395	-2.4788	-2.5874
.80	-2.6485	-2.7459	-2.5918	-2.6893	-2.5375	-2.6350	-2.4855	-2.5830
.82	-2.6548	-2.7415	-2.5981	-2.6848	-2.5439	-2.6306	-2.4918	-2.5785
.84	-2.6609	-2.7370	-2.6042	-2.6804	-2.5500	-2.6261	-2.4979	-2.5741
.86	-2.6667	-2.7326	-2.6101	-2.6759	-2.5558	-2.6217	-2.5037	-2.5697
.88	-2.6723	-2.7281	-2.6157	-2.6715	-2.5614	-2.6172	-2.5094	-2.5652
.90	-2.6777	-2.7237	-2.6210	-2.6671	-2.5668	-2.6128	-2.5147	-2.5608
.92	-2.6829	-2.7193	-2.6262	-2.6626	-2.5720	-2.6084	-2.5199	-2.5564
.94	-2.6879	-2.7149	-2.6312	-2.6582	-2.5770	-2.6040	-2.5249	-2.5520
.96	-2.6927	-2.7105	-2.6360	-2.6538	-2.5818	-2.5996	-2.5297	-2.5476
.98	-2.6973	-2.7061	-2.6406	-2.6494	-2.5864	-2.5952	-2.5343	-2.5432
1.00	-2.7017	-2.7017	-2.6451	-2.6451	-2.5908	-2.5908	-2.5388	-2.5388

Table D–5: $0.0965 \leq \hat{\rho}_4 \leq 0.1100$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.0965$		$\hat{\rho}_4 = 0.1010$		$\hat{\rho}_4 = 0.1055$		$\hat{\rho}_4 = 0.1100$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-.5333	-2.5095	-.4354	-2.4509	-.3310	-2.3940	-.2158	-2.3383
.04	-1.0660	-2.5534	-1.0013	-2.4970	-.9381	-2.4426	-.8762	-2.3900
.06	-1.3055	-2.5773	-1.2456	-2.5216	-1.1875	-2.4679	-1.1311	-2.4160
.08	-1.4628	-2.5931	-1.4048	-2.5377	-1.3488	-2.4845	-1.2945	-2.4330
.10	-1.5797	-2.6042	-1.5227	-2.5491	-1.4678	-2.4961	-1.4146	-2.4450
.12	-1.6723	-2.6121	-1.6160	-2.5572	-1.5617	-2.5044	-1.5092	-2.4535
.14	-1.7485	-2.6178	-1.6927	-2.5630	-1.6388	-2.5104	-1.5869	-2.4596
.16	-1.8130	-2.6217	-1.7575	-2.5671	-1.7040	-2.5145	-1.6524	-2.4639
.18	-1.8687	-2.6243	-1.8134	-2.5697	-1.7602	-2.5173	-1.7088	-2.4668
.20	-1.9174	-2.6257	-1.8623	-2.5713	-1.8093	-2.5190	-1.7582	-2.4685
.22	-1.9606	-2.6263	-1.9056	-2.5720	-1.8528	-2.5197	-1.8018	-2.4694
.24	-1.9992	-2.6262	-1.9444	-2.5719	-1.8917	-2.5197	-1.8408	-2.4694
.26	-2.0340	-2.6255	-1.9793	-2.5712	-1.9267	-2.5191	-1.8760	-2.4689
.28	-2.0656	-2.6242	-2.0110	-2.5700	-1.9585	-2.5179	-1.9079	-2.4678
.30	-2.0944	-2.6225	-2.0399	-2.5683	-1.9874	-2.5163	-1.9369	-2.4662
.32	-2.1208	-2.6204	-2.0663	-2.5663	-2.0140	-2.5143	-1.9636	-2.4642
.34	-2.1451	-2.6180	-2.0907	-2.5639	-2.0385	-2.5120	-1.9881	-2.4619
.36	-2.1676	-2.6153	-2.1133	-2.5613	-2.0611	-2.5093	-2.0108	-2.4593
.38	-2.1885	-2.6124	-2.1342	-2.5584	-2.0820	-2.5064	-2.0318	-2.4565
.40	-2.2079	-2.6092	-2.1537	-2.5552	-2.1016	-2.5034	-2.0513	-2.4534
.42	-2.2260	-2.6059	-2.1718	-2.5519	-2.1198	-2.5001	-2.0696	-2.4501
.44	-2.2430	-2.6024	-2.1888	-2.5485	-2.1368	-2.4966	-2.0866	-2.4467
.46	-2.2588	-2.5988	-2.2047	-2.5448	-2.1527	-2.4930	-2.1026	-2.4431
.48	-2.2738	-2.5950	-2.2197	-2.5411	-2.1677	-2.4893	-2.1176	-2.4394
.50	-2.2878	-2.5912	-2.2338	-2.5373	-2.1818	-2.4855	-2.1318	-2.4356
.52	-2.3011	-2.5872	-2.2470	-2.5333	-2.1951	-2.4815	-2.1451	-2.4317
.54	-2.3136	-2.5832	-2.2595	-2.5293	-2.2076	-2.4775	-2.1577	-2.4277
.56	-2.3254	-2.5791	-2.2714	-2.5252	-2.2195	-2.4734	-2.1695	-2.4236
.58	-2.3366	-2.5749	-2.2826	-2.5210	-2.2308	-2.4693	-2.1808	-2.4195
.60	-2.3472	-2.5707	-2.2933	-2.5168	-2.2414	-2.4651	-2.1915	-2.4153
.62	-2.3573	-2.5664	-2.3034	-2.5126	-2.2515	-2.4608	-2.2016	-2.4110
.64	-2.3669	-2.5621	-2.3130	-2.5083	-2.2612	-2.4565	-2.2113	-2.4067
.66	-2.3760	-2.5578	-2.3221	-2.5039	-2.2703	-2.4522	-2.2204	-2.4024
.68	-2.3847	-2.5534	-2.3308	-2.4996	-2.2790	-2.4479	-2.2292	-2.3981
.70	-2.3930	-2.5490	-2.3391	-2.4952	-2.2874	-2.4435	-2.2375	-2.3937
.72	-2.4009	-2.5446	-2.3471	-2.4908	-2.2953	-2.4391	-2.2454	-2.3893
.74	-2.4085	-2.5402	-2.3546	-2.4864	-2.3029	-2.4347	-2.2530	-2.3849
.76	-2.4157	-2.5357	-2.3619	-2.4819	-2.3101	-2.4302	-2.2603	-2.3805
.78	-2.4227	-2.5313	-2.3688	-2.4775	-2.3171	-2.4258	-2.2672	-2.3760
.80	-2.4293	-2.5269	-2.3754	-2.4731	-2.3237	-2.4214	-2.2739	-2.3716
.82	-2.4357	-2.5224	-2.3818	-2.4686	-2.3301	-2.4169	-2.2803	-2.3671
.84	-2.4417	-2.5180	-2.3879	-2.4642	-2.3362	-2.4125	-2.2864	-2.3627
.86	-2.4476	-2.5135	-2.3938	-2.4597	-2.3420	-2.4080	-2.2922	-2.3583
.88	-2.4532	-2.5091	-2.3994	-2.4553	-2.3477	-2.4036	-2.2979	-2.3538
.90	-2.4586	-2.5047	-2.4048	-2.4509	-2.3531	-2.3992	-2.3033	-2.3494
.92	-2.4638	-2.5002	-2.4100	-2.4464	-2.3583	-2.3947	-2.3085	-2.3450
.94	-2.4688	-2.4958	-2.4150	-2.4420	-2.3633	-2.3903	-2.3135	-2.3406
.96	-2.4736	-2.4914	-2.4198	-2.4376	-2.3681	-2.3859	-2.3183	-2.3361
.98	-2.4782	-2.4870	-2.4244	-2.4332	-2.3727	-2.3815	-2.3229	-2.3317
1.00	-2.4827	-2.4827	-2.4288	-2.4288	-2.3771	-2.3771	-2.3274	-2.3274

Table D–5: $0.115 \leq \hat{\rho}_4 \leq 0.130$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.115$		$\hat{\rho}_4 = 0.120$		$\hat{\rho}_4 = 0.125$		$\hat{\rho}_4 = 0.130$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.02	-.0648	-2.2773						
.04	-.8086	-2.3335	-.7420	-2.2787	-.6761	-2.2256	-.6108	-2.1740
.06	-1.0703	-2.3604	-1.0110	-2.3067	-.9533	-2.2547	-.8969	-2.2042
.08	-1.2361	-2.3779	-1.1795	-2.3247	-1.1246	-2.2733	-1.0712	-2.2234
.10	-1.3576	-2.3902	-1.3023	-2.3373	-1.2489	-2.2862	-1.1969	-2.2367
.12	-1.4530	-2.3989	-1.3986	-2.3463	-1.3460	-2.2955	-1.2949	-2.2463
.14	-1.5312	-2.4053	-1.4774	-2.3528	-1.4253	-2.3022	-1.3749	-2.2532
.16	-1.5971	-2.4097	-1.5437	-2.3575	-1.4921	-2.3070	-1.4421	-2.2582
.18	-1.6538	-2.4127	-1.6008	-2.3606	-1.5495	-2.3103	-1.4998	-2.2616
.20	-1.7034	-2.4146	-1.6506	-2.3626	-1.5996	-2.3123	-1.5502	-2.2638
.22	-1.7473	-2.4155	-1.6946	-2.3636	-1.6438	-2.3134	-1.5947	-2.2650
.24	-1.7864	-2.4156	-1.7340	-2.3638	-1.6834	-2.3138	-1.6344	-2.2654
.26	-1.8217	-2.4151	-1.7694	-2.3633	-1.7189	-2.3134	-1.6701	-2.2651
.28	-1.8537	-2.4141	-1.8015	-2.3624	-1.7511	-2.3125	-1.7024	-2.2642
.30	-1.8829	-2.4125	-1.8308	-2.3609	-1.7805	-2.3110	-1.7319	-2.2629
.32	-1.9096	-2.4106	-1.8576	-2.3590	-1.8074	-2.3092	-1.7589	-2.2611
.34	-1.9342	-2.4084	-1.8823	-2.3568	-1.8322	-2.3070	-1.7838	-2.2590
.36	-1.9570	-2.4058	-1.9051	-2.3543	-1.8551	-2.3046	-1.8067	-2.2565
.38	-1.9780	-2.4030	-1.9262	-2.3515	-1.8763	-2.3018	-1.8280	-2.2538
.40	-1.9976	-2.3999	-1.9459	-2.3485	-1.8960	-2.2988	-1.8478	-2.2509
.42	-2.0159	-2.3967	-1.9642	-2.3453	-1.9144	-2.2957	-1.8662	-2.2477
.44	-2.0330	-2.3933	-1.9814	-2.3419	-1.9316	-2.2923	-1.8834	-2.2444
.46	-2.0490	-2.3897	-1.9974	-2.3384	-1.9477	-2.2888	-1.8996	-2.2409
.48	-2.0641	-2.3861	-2.0125	-2.3347	-1.9628	-2.2851	-1.9147	-2.2373
.50	-2.0783	-2.3823	-2.0267	-2.3309	-1.9770	-2.2814	-1.9290	-2.2335
.52	-2.0916	-2.3784	-2.0401	-2.3270	-1.9904	-2.2775	-1.9424	-2.2297
.54	-2.1042	-2.3744	-2.0527	-2.3230	-2.0031	-2.2735	-1.9551	-2.2257
.56	-2.1161	-2.3703	-2.0647	-2.3190	-2.0150	-2.2695	-1.9671	-2.2217
.58	-2.1274	-2.3662	-2.0760	-2.3149	-2.0264	-2.2654	-1.9784	-2.2176
.60	-2.1381	-2.3620	-2.0867	-2.3107	-2.0371	-2.2612	-1.9892	-2.2135
.62	-2.1482	-2.3578	-2.0969	-2.3065	-2.0473	-2.2570	-1.9994	-2.2092
.64	-2.1579	-2.3535	-2.1065	-2.3022	-2.0570	-2.2528	-2.0091	-2.2050
.66	-2.1671	-2.3492	-2.1157	-2.2979	-2.0662	-2.2485	-2.0183	-2.2007
.68	-2.1758	-2.3448	-2.1245	-2.2936	-2.0750	-2.2441	-2.0271	-2.1964
.70	-2.1842	-2.3404	-2.1328	-2.2892	-2.0833	-2.2398	-2.0355	-2.1920
.72	-2.1921	-2.3361	-2.1408	-2.2848	-2.0913	-2.2354	-2.0435	-2.1876
.74	-2.1997	-2.3317	-2.1484	-2.2804	-2.0989	-2.2310	-2.0511	-2.1833
.76	-2.2070	-2.3272	-2.1557	-2.2760	-2.1062	-2.2266	-2.0584	-2.1788
.78	-2.2140	-2.3228	-2.1627	-2.2716	-2.1132	-2.2222	-2.0654	-2.1744
.80	-2.2206	-2.3184	-2.1693	-2.2671	-2.1199	-2.2177	-2.0721	-2.1700
.82	-2.2270	-2.3139	-2.1757	-2.2627	-2.1263	-2.2133	-2.0785	-2.1656
.84	-2.2331	-2.3095	-2.1818	-2.2583	-2.1324	-2.2088	-2.0846	-2.1611
.86	-2.2390	-2.3050	-2.1877	-2.2538	-2.1383	-2.2044	-2.0905	-2.1567
.88	-2.2446	-2.3006	-2.1934	-2.2494	-2.1439	-2.2000	-2.0962	-2.1522
.90	-2.2500	-2.2962	-2.1988	-2.2449	-2.1493	-2.1955	-2.1016	-2.1478
.92	-2.2552	-2.2918	-2.2040	-2.2405	-2.1546	-2.1911	-2.1068	-2.1434
.94	-2.2602	-2.2873	-2.2090	-2.2361	-2.1596	-2.1867	-2.1118	-2.1390
.96	-2.2650	-2.2829	-2.2138	-2.2317	-2.1644	-2.1823	-2.1166	-2.1346
.98	-2.2697	-2.2785	-2.2184	-2.2273	-2.1690	-2.1779	-2.1213	-2.1301
1.00	-2.2741	-2.2741	-2.2229	-2.2229	-2.1735	-2.1735	-2.1258	-2.1258

Table D–5: $0.136 \leq \hat{\rho}_4 \leq 0.154$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.136$		$\hat{\rho}_4 = 0.142$		$\hat{\rho}_4 = 0.148$		$\hat{\rho}_4 = 0.154$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	-.5325	-2.1138	-.4539	-2.0551	-.3744	-1.9979	-.2930	-1.9419
.06	-.8307	-2.1455	-.7659	-2.0886	-.7024	-2.0334	-.6399	-1.9797
.08	-1.0088	-2.1655	-.9482	-2.1094	-.8892	-2.0551	-.8315	-2.0023
.10	-1.1365	-2.1793	-1.0779	-2.1238	-1.0210	-2.0700	-.9656	-2.0178
.12	-1.2356	-2.1892	-1.1782	-2.1341	-1.1226	-2.0807	-1.0685	-2.0289
.14	-1.3164	-2.1964	-1.2598	-2.1415	-1.2049	-2.0884	-1.1517	-2.0370
.16	-1.3841	-2.2016	-1.3281	-2.1469	-1.2738	-2.0941	-1.2212	-2.0428
.18	-1.4423	-2.2051	-1.3866	-2.1507	-1.3328	-2.0980	-1.2806	-2.0470
.20	-1.4929	-2.2075	-1.4376	-2.1532	-1.3842	-2.1007	-1.3323	-2.0498
.22	-1.5377	-2.2088	-1.4827	-2.1546	-1.4294	-2.1023	-1.3779	-2.0515
.24	-1.5776	-2.2093	-1.5228	-2.1552	-1.4698	-2.1030	-1.4185	-2.0524
.26	-1.6135	-2.2091	-1.5589	-2.1551	-1.5061	-2.1030	-1.4549	-2.0525
.28	-1.6460	-2.2083	-1.5915	-2.1544	-1.5389	-2.1024	-1.4879	-2.0520
.30	-1.6756	-2.2070	-1.6213	-2.1532	-1.5688	-2.1013	-1.5180	-2.0509
.32	-1.7027	-2.2053	-1.6485	-2.1516	-1.5961	-2.0997	-1.5454	-2.0494
.34	-1.7276	-2.2033	-1.6736	-2.1496	-1.6213	-2.0977	-1.5707	-2.0475
.36	-1.7507	-2.2009	-1.6967	-2.1472	-1.6445	-2.0954	-1.5940	-2.0453
.38	-1.7720	-2.1982	-1.7181	-2.1446	-1.6660	-2.0929	-1.6156	-2.0428
.40	-1.7919	-2.1953	-1.7380	-2.1418	-1.6860	-2.0900	-1.6357	-2.0400
.42	-1.8104	-2.1922	-1.7566	-2.1387	-1.7046	-2.0870	-1.6544	-2.0370
.44	-1.8277	-2.1889	-1.7739	-2.1354	-1.7220	-2.0838	-1.6718	-2.0338
.46	-1.8438	-2.1854	-1.7902	-2.1320	-1.7383	-2.0804	-1.6882	-2.0305
.48	-1.8590	-2.1818	-1.8054	-2.1284	-1.7536	-2.0768	-1.7035	-2.0269
.50	-1.8733	-2.1781	-1.8198	-2.1247	-1.7680	-2.0732	-1.7179	-2.0233
.52	-1.8868	-2.1743	-1.8333	-2.1209	-1.7815	-2.0694	-1.7315	-2.0195
.54	-1.8995	-2.1703	-1.8460	-2.1170	-1.7943	-2.0655	-1.7443	-2.0157
.56	-1.9115	-2.1663	-1.8581	-2.1130	-1.8064	-2.0615	-1.7564	-2.0117
.58	-1.9229	-2.1623	-1.8695	-2.1090	-1.8178	-2.0575	-1.7679	-2.0077
.60	-1.9337	-2.1581	-1.8803	-2.1048	-1.8287	-2.0534	-1.7788	-2.0036
.62	-1.9439	-2.1539	-1.8905	-2.1006	-1.8390	-2.0492	-1.7891	-1.9994
.64	-1.9537	-2.1497	-1.9003	-2.0964	-1.8487	-2.0450	-1.7989	-1.9952
.66	-1.9629	-2.1454	-1.9095	-2.0921	-1.8580	-2.0407	-1.8082	-1.9910
.68	-1.9717	-2.1411	-1.9184	-2.0878	-1.8669	-2.0364	-1.8170	-1.9867
.70	-1.9801	-2.1367	-1.9268	-2.0835	-1.8753	-2.0321	-1.8255	-1.9824
.72	-1.9881	-2.1324	-1.9348	-2.0791	-1.8833	-2.0277	-1.8335	-1.9780
.74	-1.9958	-2.1280	-1.9425	-2.0748	-1.8910	-2.0234	-1.8412	-1.9737
.76	-2.0031	-2.1236	-1.9498	-2.0704	-1.8983	-2.0190	-1.8486	-1.9693
.78	-2.0101	-2.1192	-1.9568	-2.0659	-1.9054	-2.0146	-1.8556	-1.9649
.80	-2.0168	-2.1147	-1.9635	-2.0615	-1.9121	-2.0101	-1.8623	-1.9605
.82	-2.0232	-2.1103	-1.9699	-2.0571	-1.9185	-2.0057	-1.8688	-1.9560
.84	-2.0293	-2.1059	-1.9761	-2.0527	-1.9247	-2.0013	-1.8749	-1.9516
.86	-2.0352	-2.1014	-1.9820	-2.0482	-1.9306	-1.9968	-1.8808	-1.9472
.88	-2.0409	-2.0970	-1.9876	-2.0438	-1.9362	-1.9924	-1.8865	-1.9427
.90	-2.0463	-2.0926	-1.9931	-2.0393	-1.9417	-1.9880	-1.8920	-1.9383
.92	-2.0515	-2.0881	-1.9983	-2.0349	-1.9469	-1.9835	-1.8972	-1.9339
.94	-2.0566	-2.0837	-2.0033	-2.0305	-1.9519	-1.9791	-1.9022	-1.9294
.96	-2.0614	-2.0793	-2.0082	-2.0261	-1.9568	-1.9747	-1.9071	-1.9250
.98	-2.0660	-2.0749	-2.0128	-2.0217	-1.9614	-1.9703	-1.9117	-1.9206
1.00	-2.0705	-2.0705	-2.0173	-2.0173	-1.9659	-1.9659	-1.9162	-1.9162

Table D–5: $0.161 \leq \hat{\rho}_4 \leq 0.182$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.161$		$\hat{\rho}_4 = 0.168$		$\hat{\rho}_4 = 0.175$		$\hat{\rho}_4 = 0.182$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.04	-.1943	-1.8777	-.0889	-1.8143				
.06	-.5681	-1.9186	-.4971	-1.8591	-.4268	-1.8009	-.3567	-1.7439
.08	-.7659	-1.9425	-.7018	-1.8843	-.6389	-1.8277	-.5773	-1.7723
.10	-.9027	-1.9587	-.8415	-1.9013	-.7818	-1.8455	-.7236	-1.7912
.12	-1.0072	-1.9703	-.9476	-1.9135	-.8897	-1.8583	-.8333	-1.8045
.14	-1.0914	-1.9788	-1.0330	-1.9224	-.9762	-1.8676	-.9209	-1.8143
.16	-1.1616	-1.9849	-1.1040	-1.9289	-1.0480	-1.8744	-.9935	-1.8215
.18	-1.2216	-1.9893	-1.1645	-1.9335	-1.1091	-1.8794	-1.0552	-1.8267
.20	-1.2738	-1.9924	-1.2171	-1.9368	-1.1621	-1.8828	-1.1087	-1.8304
.22	-1.3197	-1.9943	-1.2633	-1.9389	-1.2087	-1.8851	-1.1557	-1.8329
.24	-1.3605	-1.9953	-1.3045	-1.9400	-1.2502	-1.8864	-1.1974	-1.8344
.26	-1.3972	-1.9955	-1.3414	-1.9404	-1.2873	-1.8870	-1.2348	-1.8351
.28	-1.4304	-1.9951	-1.3748	-1.9401	-1.3209	-1.8868	-1.2686	-1.8350
.30	-1.4606	-1.9941	-1.4052	-1.9392	-1.3515	-1.8860	-1.2993	-1.8344
.32	-1.4882	-1.9927	-1.4329	-1.9379	-1.3794	-1.8848	-1.3274	-1.8333
.34	-1.5136	-1.9909	-1.4585	-1.9362	-1.4050	-1.8832	-1.3532	-1.8317
.36	-1.5370	-1.9888	-1.4820	-1.9341	-1.4287	-1.8812	-1.3770	-1.8298
.38	-1.5587	-1.9863	-1.5038	-1.9317	-1.4506	-1.8788	-1.3990	-1.8275
.40	-1.5789	-1.9836	-1.5240	-1.9290	-1.4709	-1.8762	-1.4194	-1.8250
.42	-1.5976	-1.9806	-1.5429	-1.9261	-1.4898	-1.8734	-1.4384	-1.8222
.44	-1.6152	-1.9775	-1.5605	-1.9230	-1.5075	-1.8703	-1.4561	-1.8192
.46	-1.6316	-1.9741	-1.5769	-1.9198	-1.5240	-1.8671	-1.4727	-1.8160
.48	-1.6470	-1.9707	-1.5924	-1.9163	-1.5396	-1.8637	-1.4883	-1.8126
.50	-1.6615	-1.9671	-1.6069	-1.9127	-1.5541	-1.8601	-1.5029	-1.8091
.52	-1.6751	-1.9633	-1.6206	-1.9090	-1.5679	-1.8565	-1.5167	-1.8055
.54	-1.6880	-1.9595	-1.6335	-1.9052	-1.5808	-1.8527	-1.5297	-1.8018
.56	-1.7001	-1.9556	-1.6457	-1.9013	-1.5930	-1.8488	-1.5420	-1.7979
.58	-1.7116	-1.9516	-1.6572	-1.8973	-1.6046	-1.8449	-1.5536	-1.7940
.60	-1.7225	-1.9475	-1.6682	-1.8933	-1.6156	-1.8408	-1.5646	-1.7900
.62	-1.7328	-1.9433	-1.6785	-1.8892	-1.6260	-1.8367	-1.5750	-1.7859
.64	-1.7427	-1.9392	-1.6884	-1.8850	-1.6358	-1.8326	-1.5849	-1.7817
.66	-1.7520	-1.9349	-1.6977	-1.8808	-1.6452	-1.8284	-1.5943	-1.7776
.68	-1.7609	-1.9306	-1.7066	-1.8765	-1.6541	-1.8241	-1.6033	-1.7733
.70	-1.7693	-1.9263	-1.7151	-1.8722	-1.6626	-1.8198	-1.6118	-1.7690
.72	-1.7774	-1.9220	-1.7232	-1.8679	-1.6708	-1.8155	-1.6199	-1.7647
.74	-1.7851	-1.9176	-1.7309	-1.8635	-1.6785	-1.8112	-1.6277	-1.7604
.76	-1.7925	-1.9132	-1.7383	-1.8591	-1.6859	-1.8068	-1.6351	-1.7560
.78	-1.7995	-1.9089	-1.7454	-1.8548	-1.6930	-1.8024	-1.6421	-1.7517
.80	-1.8063	-1.9044	-1.7521	-1.8504	-1.6997	-1.7980	-1.6489	-1.7473
.82	-1.8127	-1.9000	-1.7586	-1.8459	-1.7062	-1.7936	-1.6554	-1.7429
.84	-1.8189	-1.8956	-1.7648	-1.8415	-1.7124	-1.7892	-1.6616	-1.7384
.86	-1.8248	-1.8912	-1.7707	-1.8371	-1.7183	-1.7848	-1.6676	-1.7340
.88	-1.8305	-1.8867	-1.7764	-1.8327	-1.7240	-1.7803	-1.6733	-1.7296
.90	-1.8359	-1.8823	-1.7818	-1.8282	-1.7295	-1.7759	-1.6787	-1.7252
.92	-1.8412	-1.8779	-1.7871	-1.8238	-1.7347	-1.7715	-1.6840	-1.7207
.94	-1.8462	-1.8734	-1.7921	-1.8194	-1.7398	-1.7670	-1.6891	-1.7163
.96	-1.8511	-1.8690	-1.7970	-1.8149	-1.7446	-1.7626	-1.6939	-1.7119
.98	-1.8557	-1.8646	-1.8016	-1.8105	-1.7493	-1.7582	-1.6986	-1.7075
1.00	-1.8602	-1.8602	-1.8061	-1.8061	-1.7538	-1.7538	-1.7031	-1.7031

Table D–5: $0.190 \leq \hat{\rho}_4 \leq 0.214$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.190$		$\hat{\rho}_4 = 0.198$		$\hat{\rho}_4 = 0.206$		$\hat{\rho}_4 = 0.214$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.06	-.2767	-1.6799	-.1962	-1.6169	-.1148	-1.5545	-.0319	-1.4925
.08	-.5082	-1.7105	-.4403	-1.6501	-.3734	-1.5907	-.3075	-1.5322
.10	-.6586	-1.7305	-.5950	-1.6713	-.5329	-1.6134	-.4721	-1.5565
.12	-.7705	-1.7447	-.7092	-1.6863	-.6495	-1.6293	-.5911	-1.5735
.14	-.8594	-1.7550	-.7996	-1.6973	-.7413	-1.6409	-.6844	-1.5858
.16	-.9330	-1.7627	-.8741	-1.7054	-.8168	-1.6495	-.7609	-1.5949
.18	-.9954	-1.7682	-.9372	-1.7114	-.8807	-1.6559	-.8255	-1.6017
.20	-1.0494	-1.7722	-.9918	-1.7157	-.9358	-1.6605	-.8812	-1.6067
.22	-1.0968	-1.7750	-1.0396	-1.7186	-.9840	-1.6638	-.9299	-1.6103
.24	-1.1389	-1.7767	-1.0821	-1.7206	-1.0268	-1.6659	-.9730	-1.6126
.26	-1.1766	-1.7775	-1.1201	-1.7216	-1.0651	-1.6672	-1.0116	-1.6141
.28	-1.2106	-1.7776	-1.1543	-1.7219	-1.0997	-1.6676	-1.0464	-1.6147
.30	-1.2415	-1.7771	-1.1855	-1.7215	-1.1310	-1.6674	-1.0780	-1.6147
.32	-1.2698	-1.7761	-1.2139	-1.7206	-1.1596	-1.6667	-1.1068	-1.6141
.34	-1.2957	-1.7747	-1.2400	-1.7193	-1.1859	-1.6655	-1.1332	-1.6130
.36	-1.3196	-1.7728	-1.2641	-1.7176	-1.2101	-1.6638	-1.1575	-1.6115
.38	-1.3418	-1.7707	-1.2863	-1.7155	-1.2324	-1.6618	-1.1800	-1.6096
.40	-1.3623	-1.7682	-1.3069	-1.7131	-1.2532	-1.6595	-1.2009	-1.6074
.42	-1.3814	-1.7655	-1.3261	-1.7104	-1.2725	-1.6570	-1.2202	-1.6049
.44	-1.3992	-1.7625	-1.3441	-1.7076	-1.2905	-1.6542	-1.2383	-1.6021
.46	-1.4159	-1.7594	-1.3608	-1.7045	-1.3073	-1.6511	-1.2553	-1.5992
.48	-1.4315	-1.7561	-1.3765	-1.7012	-1.3231	-1.6479	-1.2711	-1.5961
.50	-1.4462	-1.7526	-1.3913	-1.6978	-1.3379	-1.6446	-1.2860	-1.5928
.52	-1.4601	-1.7490	-1.4052	-1.6943	-1.3519	-1.6411	-1.3000	-1.5893
.54	-1.4731	-1.7453	-1.4183	-1.6906	-1.3650	-1.6374	-1.3132	-1.5857
.56	-1.4854	-1.7415	-1.4306	-1.6868	-1.3774	-1.6337	-1.3256	-1.5820
.58	-1.4971	-1.7376	-1.4423	-1.6830	-1.3891	-1.6299	-1.3374	-1.5782
.60	-1.5081	-1.7336	-1.4534	-1.6790	-1.4002	-1.6259	-1.3486	-1.5743
.62	-1.5186	-1.7296	-1.4639	-1.6750	-1.4108	-1.6219	-1.3591	-1.5703
.64	-1.5285	-1.7254	-1.4738	-1.6709	-1.4208	-1.6179	-1.3691	-1.5663
.66	-1.5379	-1.7213	-1.4833	-1.6667	-1.4303	-1.6137	-1.3787	-1.5622
.68	-1.5469	-1.7171	-1.4923	-1.6625	-1.4393	-1.6096	-1.3877	-1.5580
.70	-1.5555	-1.7128	-1.5009	-1.6583	-1.4479	-1.6053	-1.3963	-1.5538
.72	-1.5636	-1.7085	-1.5090	-1.6540	-1.4561	-1.6011	-1.4045	-1.5496
.74	-1.5714	-1.7042	-1.5168	-1.6497	-1.4639	-1.5968	-1.4124	-1.5453
.76	-1.5788	-1.6998	-1.5243	-1.6454	-1.4713	-1.5925	-1.4198	-1.5410
.78	-1.5859	-1.6955	-1.5314	-1.6410	-1.4785	-1.5881	-1.4270	-1.5367
.80	-1.5927	-1.6911	-1.5382	-1.6366	-1.4853	-1.5837	-1.4338	-1.5323
.82	-1.5992	-1.6867	-1.5447	-1.6322	-1.4918	-1.5794	-1.4403	-1.5279
.84	-1.6054	-1.6823	-1.5509	-1.6278	-1.4980	-1.5750	-1.4466	-1.5235
.86	-1.6113	-1.6778	-1.5569	-1.6234	-1.5040	-1.5705	-1.4526	-1.5191
.88	-1.6171	-1.6734	-1.5626	-1.6190	-1.5097	-1.5661	-1.4583	-1.5147
.90	-1.6225	-1.6690	-1.5681	-1.6146	-1.5152	-1.5617	-1.4638	-1.5103
.92	-1.6278	-1.6646	-1.5734	-1.6101	-1.5205	-1.5573	-1.4691	-1.5059
.94	-1.6329	-1.6601	-1.5784	-1.6057	-1.5256	-1.5529	-1.4742	-1.5015
.96	-1.6377	-1.6557	-1.5833	-1.6013	-1.5304	-1.5484	-1.4790	-1.4970
.98	-1.6424	-1.6513	-1.5880	-1.5969	-1.5351	-1.5440	-1.4837	-1.4926
1.00	-1.6469	-1.6469	-1.5925	-1.5925	-1.5396	-1.5396	-1.4882	-1.4882

Table D–5: $0.223 \leq \hat{\rho}_4 \leq 0.250$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.223$		$\hat{\rho}_4 = 0.232$		$\hat{\rho}_4 = 0.241$		$\hat{\rho}_4 = 0.250$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.08	-.2346	-1.4674	-.1628	-1.4032	-.0925	-1.3395	-.0241	-1.2761
.10	-.4051	-1.4937	-.3397	-1.4318	-.2759	-1.3707	-.2139	-1.3102
.12	-.5270	-1.5118	-.4645	-1.4513	-.4036	-1.3917	-.3442	-1.3328
.14	-.6220	-1.5250	-.5612	-1.4654	-.5019	-1.4068	-.4441	-1.3491
.16	-.6997	-1.5348	-.6400	-1.4759	-.5818	-1.4181	-.5250	-1.3612
.18	-.7651	-1.5421	-.7062	-1.4838	-.6488	-1.4266	-.5928	-1.3703
.20	-.8214	-1.5475	-.7631	-1.4897	-.7063	-1.4329	-.6509	-1.3772
.22	-.8706	-1.5514	-.8128	-1.4939	-.7565	-1.4376	-.7015	-1.3823
.24	-.9141	-1.5541	-.8567	-1.4970	-.8008	-1.4410	-.7462	-1.3860
.26	-.9530	-1.5558	-.8960	-1.4989	-.8404	-1.4432	-.7861	-1.3886
.28	-.9881	-1.5567	-.9313	-1.5000	-.8760	-1.4446	-.8220	-1.3902
.30	-1.0199	-1.5568	-.9634	-1.5004	-.9083	-1.4452	-.8544	-1.3910
.32	-1.0489	-1.5564	-.9926	-1.5001	-.9377	-1.4451	-.8841	-1.3912
.34	-1.0755	-1.5555	-1.0194	-1.4993	-.9646	-1.4445	-.9112	-1.3908
.36	-1.1000	-1.5541	-1.0440	-1.4981	-.9894	-1.4434	-.9361	-1.3898
.38	-1.1226	-1.5523	-1.0668	-1.4965	-1.0123	-1.4419	-.9591	-1.3885
.40	-1.1436	-1.5502	-1.0879	-1.4945	-1.0335	-1.4400	-.9805	-1.3867
.42	-1.1631	-1.5478	-1.1075	-1.4922	-1.0533	-1.4378	-1.0003	-1.3847
.44	-1.1813	-1.5452	-1.1258	-1.4896	-1.0716	-1.4354	-1.0188	-1.3823
.46	-1.1983	-1.5423	-1.1429	-1.4868	-1.0888	-1.4327	-1.0360	-1.3797
.48	-1.2142	-1.5392	-1.1589	-1.4838	-1.1049	-1.4298	-1.0522	-1.3769
.50	-1.2292	-1.5360	-1.1739	-1.4807	-1.1200	-1.4267	-1.0674	-1.3739
.52	-1.2432	-1.5326	-1.1880	-1.4773	-1.1342	-1.4234	-1.0816	-1.3707
.54	-1.2565	-1.5290	-1.2013	-1.4739	-1.1475	-1.4200	-1.0950	-1.3673
.56	-1.2690	-1.5254	-1.2139	-1.4702	-1.1602	-1.4164	-1.1077	-1.3638
.58	-1.2808	-1.5216	-1.2257	-1.4665	-1.1721	-1.4128	-1.1197	-1.3602
.60	-1.2920	-1.5178	-1.2370	-1.4627	-1.1833	-1.4090	-1.1310	-1.3565
.62	-1.3026	-1.5138	-1.2476	-1.4588	-1.1940	-1.4052	-1.1417	-1.3527
.64	-1.3127	-1.5098	-1.2577	-1.4548	-1.2041	-1.4012	-1.1518	-1.3488
.66	-1.3222	-1.5058	-1.2673	-1.4508	-1.2137	-1.3972	-1.1615	-1.3448
.68	-1.3313	-1.5016	-1.2764	-1.4467	-1.2229	-1.3931	-1.1706	-1.3408
.70	-1.3399	-1.4974	-1.2851	-1.4425	-1.2316	-1.3890	-1.1793	-1.3367
.72	-1.3482	-1.4932	-1.2933	-1.4383	-1.2398	-1.3848	-1.1876	-1.3325
.74	-1.3560	-1.4890	-1.3012	-1.4341	-1.2477	-1.3806	-1.1955	-1.3283
.76	-1.3635	-1.4847	-1.3087	-1.4298	-1.2553	-1.3764	-1.2031	-1.3241
.78	-1.3707	-1.4803	-1.3159	-1.4255	-1.2624	-1.3721	-1.2103	-1.3198
.80	-1.3775	-1.4760	-1.3227	-1.4212	-1.2693	-1.3678	-1.2172	-1.3155
.82	-1.3840	-1.4716	-1.3293	-1.4168	-1.2759	-1.3634	-1.2237	-1.3112
.84	-1.3903	-1.4673	-1.3355	-1.4125	-1.2821	-1.3591	-1.2300	-1.3069
.86	-1.3963	-1.4629	-1.3415	-1.4081	-1.2882	-1.3547	-1.2360	-1.3025
.88	-1.4020	-1.4585	-1.3473	-1.4037	-1.2939	-1.3503	-1.2418	-1.2981
.90	-1.4075	-1.4540	-1.3528	-1.3993	-1.2994	-1.3459	-1.2473	-1.2938
.92	-1.4128	-1.4496	-1.3581	-1.3949	-1.3047	-1.3415	-1.2526	-1.2894
.94	-1.4179	-1.4452	-1.3632	-1.3905	-1.3098	-1.3371	-1.2577	-1.2850
.96	-1.4228	-1.4408	-1.3680	-1.3861	-1.3147	-1.3327	-1.2626	-1.2806
.98	-1.4275	-1.4364	-1.3727	-1.3816	-1.3194	-1.3283	-1.2672	-1.2761
1.00	-1.4320	-1.4320	-1.3772	-1.3772	-1.3239	-1.3239	-1.2717	-1.2717

Table D–5: $0.26 \leq \hat{\rho}_4 \leq 0.29$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.26$		$\hat{\rho}_4 = 0.27$		$\hat{\rho}_4 = 0.28$		$\hat{\rho}_4 = 0.29$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.10	-.1474	-1.2434	-.0839	-1.1771	-.0245	-1.1110		
.12	-.2803	-1.2681	-.2188	-1.2040	-.1600	-1.1402	-.1047	-1.0767
.14	-.3817	-1.2858	-.3214	-1.2231	-.2632	-1.1609	-.2076	-1.0991
.16	-.4636	-1.2989	-.4041	-1.2373	-.3464	-1.1763	-.2908	-1.1157
.18	-.5322	-1.3088	-.4732	-1.2480	-.4160	-1.1879	-.3605	-1.1283
.20	-.5908	-1.3163	-.5323	-1.2562	-.4755	-1.1969	-.4202	-1.1381
.22	-.6419	-1.3219	-.5838	-1.2624	-.5273	-1.2037	-.4722	-1.1455
.24	-.6870	-1.3261	-.6293	-1.2670	-.5730	-1.2088	-.5181	-1.1512
.26	-.7272	-1.3290	-.6698	-1.2704	-.6137	-1.2126	-.5590	-1.1555
.28	-.7633	-1.3309	-.7062	-1.2727	-.6503	-1.2152	-.5957	-1.1585
.30	-.7961	-1.3320	-.7391	-1.2741	-.6834	-1.2170	-.6290	-1.1606
.32	-.8259	-1.3324	-.7691	-1.2747	-.7136	-1.2179	-.6593	-1.1619
.34	-.8532	-1.3322	-.7966	-1.2747	-.7412	-1.2182	-.6871	-1.1624
.36	-.8783	-1.3315	-.8218	-1.2742	-.7666	-1.2179	-.7125	-1.1624
.38	-.9015	-1.3303	-.8451	-1.2732	-.7900	-1.2171	-.7361	-1.1618
.40	-.9229	-1.3287	-.8667	-1.2718	-.8117	-1.2159	-.7578	-1.1607
.42	-.9429	-1.3268	-.8867	-1.2700	-.8318	-1.2143	-.7780	-1.1593
.44	-.9614	-1.3246	-.9054	-1.2679	-.8506	-1.2123	-.7969	-1.1575
.46	-.9788	-1.3221	-.9228	-1.2656	-.8681	-1.2101	-.8145	-1.1554
.48	-.9950	-1.3194	-.9392	-1.2630	-.8845	-1.2076	-.8309	-1.1531
.50	-1.0103	-1.3164	-.9545	-1.2601	-.8998	-1.2049	-.8463	-1.1505
.52	-1.0246	-1.3133	-.9688	-1.2571	-.9143	-1.2019	-.8608	-1.1477
.54	-1.0381	-1.3100	-.9824	-1.2539	-.9279	-1.1989	-.8744	-1.1447
.56	-1.0508	-1.3066	-.9951	-1.2506	-.9407	-1.1956	-.8873	-1.1415
.58	-1.0628	-1.3031	-1.0072	-1.2471	-.9527	-1.1922	-.8994	-1.1382
.60	-1.0741	-1.2994	-1.0186	-1.2435	-.9642	-1.1887	-.9108	-1.1347
.62	-1.0849	-1.2957	-1.0293	-1.2398	-.9750	-1.1850	-.9217	-1.1312
.64	-1.0951	-1.2918	-1.0396	-1.2360	-.9852	-1.1813	-.9319	-1.1275
.66	-1.1047	-1.2879	-1.0493	-1.2321	-.9949	-1.1775	-.9417	-1.1237
.68	-1.1139	-1.2839	-1.0585	-1.2282	-1.0042	-1.1736	-.9509	-1.1199
.70	-1.1226	-1.2798	-1.0672	-1.2242	-1.0130	-1.1696	-.9597	-1.1160
.72	-1.1310	-1.2757	-1.0756	-1.2201	-1.0213	-1.1656	-.9681	-1.1120
.74	-1.1389	-1.2715	-1.0835	-1.2160	-1.0293	-1.1615	-.9761	-1.1079
.76	-1.1464	-1.2673	-1.0911	-1.2118	-1.0368	-1.1573	-.9836	-1.1038
.78	-1.1537	-1.2631	-1.0983	-1.2076	-1.0441	-1.1531	-.9909	-1.0997
.80	-1.1605	-1.2588	-1.1052	-1.2033	-1.0510	-1.1489	-.9978	-1.0955
.82	-1.1671	-1.2545	-1.1118	-1.1990	-1.0576	-1.1447	-1.0044	-1.0913
.84	-1.1734	-1.2502	-1.1181	-1.1947	-1.0639	-1.1404	-1.0107	-1.0870
.86	-1.1794	-1.2459	-1.1241	-1.1904	-1.0699	-1.1361	-1.0167	-1.0827
.88	-1.1852	-1.2415	-1.1299	-1.1861	-1.0757	-1.1318	-1.0225	-1.0784
.90	-1.1907	-1.2371	-1.1354	-1.1817	-1.0812	-1.1274	-1.0280	-1.0741
.92	-1.1960	-1.2327	-1.1407	-1.1773	-1.0865	-1.1231	-1.0333	-1.0698
.94	-1.2011	-1.2283	-1.1458	-1.1730	-1.0915	-1.1187	-1.0383	-1.0654
.96	-1.2060	-1.2239	-1.1506	-1.1686	-1.0964	-1.1143	-1.0432	-1.0611
.98	-1.2106	-1.2195	-1.1553	-1.1642	-1.1011	-1.1099	-1.0478	-1.0567
1.00	-1.2151	-1.2151	-1.1598	-1.1598	-1.1056	-1.1056	-1.0523	-1.0523

Table D–5: $0.301 \leq \hat{\rho}_4 \leq 0.334$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.301$		$\hat{\rho}_4 = 0.312$		$\hat{\rho}_4 = 0.323$		$\hat{\rho}_4 = 0.334$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.12	-.0486	-1.0071						
.14	-.1499	-1.0314	-.0965	-.9638	-.0486	-.8965	-.0070	-.8295
.16	-.2324	-1.0494	-.1773	-.9833	-.1263	-.9174	-.0800	-.8515
.18	-.3018	-1.0632	-.2458	-.9983	-.1930	-.9336	-.1439	-.8688
.20	-.3614	-1.0739	-.3049	-1.0100	-.2510	-.9462	-.2001	-.8825
.22	-.4134	-1.0821	-.3566	-1.0190	-.3021	-.9562	-.2501	-.8933
.24	-.4593	-1.0884	-.4024	-1.0261	-.3474	-.9639	-.2947	-.9019
.26	-.5003	-1.0932	-.4433	-1.0315	-.3881	-.9700	-.3348	-.9087
.28	-.5372	-1.0968	-.4802	-1.0356	-.4248	-.9746	-.3711	-.9139
.30	-.5706	-1.0993	-.5136	-1.0385	-.4581	-.9781	-.4041	-.9179
.32	-.6010	-1.1009	-.5440	-1.0405	-.4884	-.9806	-.4343	-.9208
.34	-.6288	-1.1018	-.5718	-1.0418	-.5162	-.9822	-.4620	-.9229
.36	-.6544	-1.1020	-.5975	-1.0423	-.5418	-.9831	-.4875	-.9241
.38	-.6780	-1.1017	-.6211	-1.0423	-.5655	-.9833	-.5110	-.9247
.40	-.6998	-1.1009	-.6430	-1.0417	-.5874	-.9831	-.5329	-.9248
.42	-.7201	-1.0997	-.6633	-1.0407	-.6077	-.9823	-.5532	-.9243
.44	-.7390	-1.0981	-.6823	-1.0394	-.6266	-.9812	-.5721	-.9234
.46	-.7566	-1.0962	-.6999	-1.0376	-.6443	-.9797	-.5897	-.9221
.48	-.7731	-1.0940	-.7165	-1.0356	-.6609	-.9778	-.6062	-.9205
.50	-.7886	-1.0915	-.7320	-1.0333	-.6764	-.9757	-.6217	-.9185
.52	-.8031	-1.0888	-.7465	-1.0308	-.6909	-.9733	-.6363	-.9163
.54	-.8168	-1.0860	-.7602	-1.0280	-.7046	-.9707	-.6500	-.9139
.56	-.8297	-1.0829	-.7731	-1.0251	-.7175	-.9679	-.6629	-.9112
.58	-.8418	-1.0797	-.7853	-1.0220	-.7297	-.9649	-.6750	-.9084
.60	-.8533	-1.0763	-.7968	-1.0187	-.7412	-.9618	-.6865	-.9054
.62	-.8641	-1.0728	-.8076	-1.0153	-.7521	-.9585	-.6974	-.9022
.64	-.8744	-1.0693	-.8179	-1.0118	-.7624	-.9551	-.7077	-.8989
.66	-.8842	-1.0656	-.8277	-1.0082	-.7722	-.9516	-.7174	-.8955
.68	-.8935	-1.0618	-.8370	-1.0045	-.7814	-.9479	-.7267	-.8920
.70	-.9023	-1.0579	-.8458	-1.0007	-.7902	-.9442	-.7354	-.8883
.72	-.9106	-1.0540	-.8542	-.9968	-.7986	-.9404	-.7438	-.8846
.74	-.9186	-1.0500	-.8621	-.9929	-.8065	-.9366	-.7517	-.8808
.76	-.9262	-1.0459	-.8697	-.9889	-.8141	-.9326	-.7593	-.8769
.78	-.9334	-1.0418	-.8769	-.9849	-.8213	-.9286	-.7665	-.8730
.80	-.9403	-1.0377	-.8838	-.9808	-.8282	-.9246	-.7734	-.8690
.82	-.9469	-1.0335	-.8904	-.9766	-.8348	-.9205	-.7799	-.8650
.84	-.9532	-1.0293	-.8967	-.9724	-.8411	-.9164	-.7862	-.8609
.86	-.9592	-1.0250	-.9027	-.9682	-.8471	-.9122	-.7921	-.8568
.88	-.9650	-1.0208	-.9085	-.9640	-.8528	-.9080	-.7978	-.8526
.90	-.9705	-1.0165	-.9140	-.9597	-.8583	-.9038	-.8033	-.8484
.92	-.9758	-1.0121	-.9192	-.9554	-.8635	-.8995	-.8085	-.8442
.94	-.9808	-1.0078	-.9243	-.9511	-.8685	-.8952	-.8135	-.8400
.96	-.9857	-1.0035	-.9291	-.9468	-.8733	-.8909	-.8183	-.8358
.98	-.9903	-.9991	-.9337	-.9425	-.8779	-.8866	-.8228	-.8315
1.00	-.9948	-.9948	-.9381	-.9381	-.8823	-.8823	-.8272	-.8272

Table D–5: $0.346 \leq \hat{\rho}_4 \leq 0.382$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.346$		$\hat{\rho}_4 = 0.358$		$\hat{\rho}_4 = 0.370$		$\hat{\rho}_4 = 0.382$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.16	-.0360	-.7799						
.18	-.0954	-.7982	-.0533	-.7277	-.0184	-.6575		
.20	-.1488	-.8130	-.1025	-.7433	-.0624	-.6736	-.0291	-.6042
.22	-.1967	-.8247	-.1476	-.7560	-.1036	-.6870	-.0655	-.6179
.24	-.2400	-.8342	-.1888	-.7662	-.1419	-.6980	-.1002	-.6294
.26	-.2791	-.8417	-.2265	-.7745	-.1775	-.7070	-.1328	-.6391
.28	-.3147	-.8476	-.2610	-.7812	-.2104	-.7144	-.1635	-.6471
.30	-.3472	-.8522	-.2926	-.7864	-.2408	-.7203	-.1922	-.6536
.32	-.3770	-.8557	-.3218	-.7905	-.2689	-.7250	-.2190	-.6589
.34	-.4044	-.8583	-.3487	-.7936	-.2951	-.7286	-.2440	-.6631
.36	-.4296	-.8600	-.3735	-.7958	-.3193	-.7313	-.2673	-.6663
.38	-.4530	-.8610	-.3966	-.7972	-.3419	-.7332	-.2891	-.6688
.40	-.4747	-.8614	-.4180	-.7980	-.3629	-.7344	-.3095	-.6704
.42	-.4949	-.8612	-.4380	-.7982	-.3825	-.7350	-.3286	-.6715
.44	-.5137	-.8606	-.4566	-.7979	-.4008	-.7351	-.3465	-.6719
.46	-.5313	-.8596	-.4740	-.7972	-.4180	-.7347	-.3634	-.6719
.48	-.5477	-.8582	-.4904	-.7961	-.4341	-.7339	-.3791	-.6715
.50	-.5632	-.8565	-.5057	-.7947	-.4493	-.7328	-.3940	-.6706
.52	-.5777	-.8545	-.5201	-.7929	-.4635	-.7313	-.4080	-.6694
.54	-.5913	-.8523	-.5336	-.7909	-.4769	-.7295	-.4212	-.6679
.56	-.6042	-.8498	-.5464	-.7886	-.4895	-.7274	-.4336	-.6661
.58	-.6163	-.8471	-.5584	-.7861	-.5015	-.7252	-.4453	-.6640
.60	-.6277	-.8443	-.5698	-.7834	-.5127	-.7227	-.4565	-.6618
.62	-.6386	-.8412	-.5806	-.7806	-.5234	-.7200	-.4670	-.6593
.64	-.6488	-.8381	-.5908	-.7775	-.5335	-.7171	-.4769	-.6566
.66	-.6585	-.8348	-.6004	-.7744	-.5431	-.7141	-.4864	-.6538
.68	-.6678	-.8314	-.6096	-.7711	-.5521	-.7110	-.4953	-.6509
.70	-.6765	-.8278	-.6183	-.7677	-.5608	-.7077	-.5038	-.6478
.72	-.6848	-.8242	-.6266	-.7642	-.5689	-.7044	-.5119	-.6445
.74	-.6927	-.8205	-.6344	-.7606	-.5767	-.7009	-.5196	-.6412
.76	-.7003	-.8167	-.6419	-.7569	-.5841	-.6973	-.5269	-.6378
.78	-.7074	-.8129	-.6490	-.7532	-.5912	-.6937	-.5338	-.6343
.80	-.7142	-.8090	-.6558	-.7493	-.5979	-.6900	-.5404	-.6307
.82	-.7208	-.8050	-.6623	-.7455	-.6043	-.6862	-.5467	-.6270
.84	-.7270	-.8010	-.6684	-.7415	-.6104	-.6823	-.5527	-.6233
.86	-.7329	-.7969	-.6743	-.7375	-.6162	-.6785	-.5585	-.6195
.88	-.7386	-.7928	-.6799	-.7335	-.6217	-.6745	-.5639	-.6157
.90	-.7440	-.7887	-.6853	-.7295	-.6270	-.6705	-.5691	-.6118
.92	-.7492	-.7846	-.6904	-.7254	-.6321	-.6665	-.5741	-.6079
.94	-.7541	-.7804	-.6953	-.7213	-.6369	-.6625	-.5789	-.6039
.96	-.7588	-.7762	-.7000	-.7171	-.6416	-.6584	-.5834	-.5999
.98	-.7634	-.7720	-.7045	-.7129	-.6460	-.6543	-.5877	-.5959
1.00	-.7677	-.7677	-.7088	-.7088	-.6502	-.6502	-.5919	-.5919

Table D–5: $0.394 \leq \hat{\rho}_4 \leq 0.430$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.394$		$\hat{\rho}_4 = 0.406$		$\hat{\rho}_4 = 0.418$		$\hat{\rho}_4 = 0.430$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.20	-.0032	-.5353						
.22	-.0341	-.5489	-.0099	-.4805				
.24	-.0643	-.5607	-.0350	-.4921	-.0127	-.4240		
.26	-.0934	-.5708	-.0599	-.5023	-.0330	-.4339	-.0130	-.3659
.28	-.1212	-.5793	-.0842	-.5111	-.0533	-.4426	-.0290	-.3742
.30	-.1475	-.5864	-.1076	-.5185	-.0732	-.4502	-.0452	-.3816
.32	-.1724	-.5922	-.1301	-.5248	-.0928	-.4568	-.0613	-.3882
.34	-.1959	-.5970	-.1516	-.5301	-.1117	-.4623	-.0772	-.3938
.36	-.2180	-.6007	-.1720	-.5343	-.1300	-.4670	-.0929	-.3987
.38	-.2388	-.6037	-.1914	-.5377	-.1475	-.4708	-.1081	-.4029
.40	-.2583	-.6058	-.2097	-.5404	-.1643	-.4739	-.1229	-.4063
.42	-.2767	-.6073	-.2270	-.5423	-.1803	-.4763	-.1371	-.4091
.44	-.2940	-.6082	-.2435	-.5436	-.1955	-.4781	-.1508	-.4112
.46	-.3102	-.6086	-.2590	-.5444	-.2100	-.4793	-.1639	-.4128
.48	-.3256	-.6085	-.2736	-.5447	-.2238	-.4800	-.1765	-.4140
.50	-.3400	-.6080	-.2875	-.5446	-.2368	-.4802	-.1885	-.4146
.52	-.3536	-.6071	-.3006	-.5441	-.2493	-.4801	-.2000	-.4149
.54	-.3665	-.6059	-.3130	-.5432	-.2610	-.4796	-.2109	-.4147
.56	-.3786	-.6044	-.3248	-.5420	-.2722	-.4787	-.2213	-.4142
.58	-.3901	-.6026	-.3359	-.5405	-.2829	-.4776	-.2313	-.4134
.60	-.4010	-.6005	-.3464	-.5388	-.2929	-.4761	-.2407	-.4124
.62	-.4113	-.5983	-.3564	-.5368	-.3025	-.4744	-.2498	-.4110
.64	-.4211	-.5959	-.3659	-.5346	-.3116	-.4726	-.2583	-.4095
.66	-.4303	-.5933	-.3749	-.5322	-.3203	-.4705	-.2665	-.4077
.68	-.4391	-.5905	-.3835	-.5297	-.3285	-.4682	-.2743	-.4057
.70	-.4474	-.5876	-.3916	-.5270	-.3363	-.4657	-.2817	-.4035
.72	-.4554	-.5845	-.3993	-.5242	-.3437	-.4631	-.2887	-.4012
.74	-.4629	-.5814	-.4066	-.5212	-.3508	-.4604	-.2954	-.3988
.76	-.4700	-.5781	-.4136	-.5181	-.3575	-.4576	-.3018	-.3962
.78	-.4769	-.5748	-.4202	-.5149	-.3639	-.4546	-.3079	-.3935
.80	-.4833	-.5713	-.4266	-.5117	-.3700	-.4515	-.3137	-.3906
.82	-.4895	-.5678	-.4326	-.5083	-.3758	-.4484	-.3192	-.3877
.84	-.4954	-.5642	-.4383	-.5049	-.3813	-.4451	-.3245	-.3847
.86	-.5010	-.5605	-.4438	-.5014	-.3866	-.4418	-.3295	-.3816
.88	-.5064	-.5568	-.4490	-.4978	-.3916	-.4384	-.3343	-.3785
.90	-.5115	-.5531	-.4539	-.4942	-.3964	-.4350	-.3388	-.3753
.92	-.5163	-.5493	-.4587	-.4905	-.4010	-.4315	-.3431	-.3720
.94	-.5210	-.5454	-.4632	-.4868	-.4053	-.4280	-.3472	-.3687
.96	-.5254	-.5415	-.4675	-.4831	-.4094	-.4244	-.3511	-.3653
.98	-.5297	-.5376	-.4716	-.4793	-.4134	-.4208	-.3549	-.3619
1.00	-.5337	-.5337	-.4755	-.4755	-.4171	-.4171	-.3584	-.3584

Table D–5: $0.442 \leq \hat{\rho}_4 \leq 0.478$ and $\lambda_3,\ \lambda_4 < 0$

	$\hat{\rho}_4 = 0.442$		$\hat{\rho}_4 = 0.454$		$\hat{\rho}_4 = 0.466$		$\hat{\rho}_4 = 0.478$	
$\hat{\rho}_3$	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4	λ_3	λ_4
.28	-.0114	-.3064	-.0003	-.2397				
.30	-.0237	-.3132	-.0089	-.2455	-.0002	-.1789	.0033	-.1140
.32	-.0363	-.3194	-.0179	-.2509	-.0060	-.1832	.0001	-.1168
.34	-.0489	-.3249	-.0272	-.2558	-.0121	-.1872	-.0032	-.1196
.36	-.0615	-.3297	-.0366	-.2603	-.0184	-.1909	-.0068	-.1222
.38	-.0740	-.3339	-.0461	-.2643	-.0250	-.1943	-.0106	-.1247
.40	-.0863	-.3375	-.0557	-.2678	-.0316	-.1974	-.0145	-.1270
.42	-.0984	-.3406	-.0652	-.2708	-.0384	-.2002	-.0186	-.1291
.44	-.1102	-.3430	-.0746	-.2734	-.0452	-.2026	-.0227	-.1310
.46	-.1215	-.3450	-.0839	-.2756	-.0520	-.2047	-.0269	-.1327
.48	-.1326	-.3465	-.0930	-.2773	-.0588	-.2065	-.0312	-.1342
.50	-.1432	-.3475	-.1018	-.2786	-.0655	-.2079	-.0355	-.1355
.52	-.1534	-.3481	-.1104	-.2796	-.0721	-.2090	-.0398	-.1366
.54	-.1632	-.3484	-.1188	-.2802	-.0786	-.2099	-.0441	-.1374
.56	-.1726	-.3483	-.1268	-.2804	-.0850	-.2104	-.0484	-.1381
.58	-.1816	-.3478	-.1346	-.2804	-.0912	-.2107	-.0526	-.1386
.60	-.1903	-.3471	-.1421	-.2801	-.0972	-.2107	-.0568	-.1388
.62	-.1985	-.3462	-.1493	-.2795	-.1031	-.2105	-.0609	-.1389
.64	-.2064	-.3449	-.1563	-.2786	-.1087	-.2101	-.0649	-.1388
.66	-.2139	-.3435	-.1629	-.2776	-.1142	-.2095	-.0688	-.1386
.68	-.2211	-.3419	-.1693	-.2764	-.1194	-.2086	-.0726	-.1382
.70	-.2279	-.3401	-.1754	-.2749	-.1245	-.2076	-.0763	-.1376
.72	-.2345	-.3381	-.1812	-.2733	-.1294	-.2065	-.0799	-.1369
.74	-.2407	-.3359	-.1868	-.2716	-.1341	-.2051	-.0833	-.1360
.76	-.2466	-.3337	-.1921	-.2696	-.1385	-.2037	-.0866	-.1351
.78	-.2523	-.3313	-.1972	-.2676	-.1428	-.2021	-.0898	-.1340
.80	-.2577	-.3287	-.2020	-.2655	-.1469	-.2003	-.0929	-.1328
.82	-.2628	-.3261	-.2066	-.2632	-.1508	-.1985	-.0959	-.1315
.84	-.2677	-.3234	-.2110	-.2608	-.1546	-.1966	-.0987	-.1302
.86	-.2724	-.3206	-.2152	-.2584	-.1582	-.1946	-.1014	-.1287
.88	-.2768	-.3177	-.2192	-.2559	-.1616	-.1925	-.1040	-.1272
.90	-.2810	-.3148	-.2230	-.2533	-.1648	-.1904	-.1064	-.1256
.92	-.2850	-.3118	-.2266	-.2506	-.1679	-.1881	-.1087	-.1240
.94	-.2889	-.3087	-.2301	-.2479	-.1708	-.1859	-.1110	-.1223
.96	-.2925	-.3056	-.2334	-.2451	-.1736	-.1835	-.1131	-.1205
.98	-.2960	-.3025	-.2365	-.2423	-.1762	-.1811	-.1151	-.1188
1.00	-.2993	-.2993	-.2394	-.2394	-.1787	-.1787	-.1169	-.1169

Appendix E

Tables E–1 through E–5 for GLD Fits: Method of *L*-Moments

For specified τ_3 and τ_4, Tables E–1 through E–5 give the λ_3 and λ_4 values, along with the values of A and B that were discussed in Chapter 6, for GLD fits. Tables E–1 through E–5 provide fits, respectively, for the regions T1 through T5 that are shown in Figure 6.3–6.

Table E–1: $0 \le \hat{\tau}_3 \le 0.075$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.000,\quad 0.00 \le \hat{\tau}_4 \le 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.01	0.3215	0.0000	0.8398	0.8398
0.02	0.3084	0.0000	0.7242	0.7242
0.03	0.2942	0.0000	0.6314	0.6314
0.04	0.2789	0.0000	0.5529	0.5529
0.05	0.2627	0.0000	0.4845	0.4845
0.06	0.2455	0.0000	0.4236	0.4236
0.07	0.2274	0.0000	0.3686	0.3686
0.08	0.2083	0.0000	0.3183	0.3183
0.09	0.1882	0.0000	0.2720	0.2720
0.10	0.1672	0.0000	0.2290	0.2290
0.11	0.1451	0.0000	0.1888	0.1888
0.12	0.1220	0.0000	0.1511	0.1511
0.13	0.0979	0.0000	0.1155	0.1155
0.14	0.0727	0.0000	0.0818	0.0818
0.15	0.0464	0.0000	0.0499	0.0499
0.16	0.0189	0.0000	0.0195	0.0195

$\hat{\tau}_3 = 0.050,\quad 0.02 \le \hat{\tau}_4 \le 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.02	0.1616	0.4256	0.0094	0.7695
0.03	0.2082	-0.3967	0.9498	0.0994
0.04	0.2283	-0.3155	0.8542	0.1698
0.05	0.1686	0.3267	0.0521	0.6031
0.06	0.1693	0.2885	0.0681	0.5438
0.07	0.1684	0.2478	0.0842	0.4825
0.08	0.1651	0.2053	0.0991	0.4194
0.09	0.1585	0.1631	0.1102	0.3558
0.10	0.1477	0.1242	0.1148	0.2940
0.11	0.1329	0.0913	0.1111	0.2364
0.12	0.1144	0.0651	0.0996	0.1844
0.13	0.0933	0.0448	0.0826	0.1378
0.14	0.0700	0.0289	0.0620	0.0957
0.15	0.0450	0.0163	0.0395	0.0574
0.16	0.0185	0.0060	0.0159	0.0221

$\hat{\tau}_3 = 0.025,\quad 0.01 \le \hat{\tau}_4 \le 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.01	0.1654	0.4617	0.0059	0.8782
0.02	0.1869	-0.4424	0.9490	0.0466
0.03	0.1955	-0.3995	0.8793	0.0734
0.04	0.2057	-0.3464	0.8022	0.1096
0.05	0.2202	-0.2689	0.7081	0.1705
0.06	0.1839	0.2748	0.0977	0.5719
0.07	0.1847	0.2243	0.1223	0.4998
0.08	0.1822	0.1702	0.1460	0.4237
0.09	0.1742	0.1189	0.1615	0.3475
0.10	0.1600	0.0793	0.1613	0.2790
0.11	0.1413	0.0529	0.1471	0.2212
0.12	0.1199	0.0356	0.1250	0.1720
0.13	0.0966	0.0237	0.0993	0.1288
0.14	0.0720	0.0150	0.0723	0.0899
0.15	0.0460	0.0084	0.0449	0.0541
0.16	0.0188	0.0030	0.0178	0.0209

$\hat{\tau}_3 = 0.075,\quad 0.03 \le \hat{\tau}_4 \le 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.03	0.1557	0.3915	0.0107	0.6725
0.04	0.1570	0.3614	0.0224	0.6215
0.05	0.1576	0.3293	0.0345	0.5692
0.06	0.1571	0.2954	0.0469	0.5157
0.07	0.1553	0.2598	0.0590	0.4610
0.08	0.1515	0.2230	0.0700	0.4053
0.09	0.1453	0.1858	0.0788	0.3492
0.10	0.1361	0.1497	0.0838	0.2937
0.11	0.1234	0.1163	0.0836	0.2400
0.12	0.1075	0.0869	0.0779	0.1893
0.13	0.0885	0.0619	0.0670	0.1424
0.14	0.0671	0.0409	0.0519	0.0992
0.15	0.0434	0.0234	0.0339	0.0596
0.16	0.0179	0.0086	0.0140	0.0229

Table E–1: $0.1 \leq \hat{\tau}_3 \leq 0.175$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.100,\quad 0.03 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.03	0.1476	0.3873	0.0006	0.6336
0.04	0.1480	0.3592	0.0104	0.5861
0.05	0.1478	0.3297	0.0205	0.5379
0.06	0.1465	0.2986	0.0306	0.4888
0.07	0.1441	0.2663	0.0403	0.4389
0.08	0.1400	0.2329	0.0492	0.3885
0.09	0.1340	0.1989	0.0565	0.3376
0.10	0.1255	0.1652	0.0613	0.2870
0.11	0.1144	0.1327	0.0628	0.2373
0.12	0.1002	0.1024	0.0601	0.1893
0.13	0.0832	0.0750	0.0533	0.1436
0.14	0.0635	0.0507	0.0425	0.1007
0.15	0.0414	0.0295	0.0285	0.0607
0.16	0.0172	0.0110	0.0120	0.0234

$\hat{\tau}_3 = 0.150,\quad 0.06 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.06	0.1286	0.2988	0.0068	0.4399
0.07	0.1254	0.2704	0.0140	0.3970
0.08	0.1210	0.2412	0.0206	0.3539
0.09	0.1152	0.2113	0.0263	0.3105
0.10	0.1077	0.1809	0.0308	0.2671
0.11	0.0982	0.1506	0.0335	0.2239
0.12	0.0865	0.1207	0.0339	0.1813
0.13	0.0724	0.0918	0.0317	0.1396
0.14	0.0558	0.0643	0.0267	0.0993
0.15	0.0368	0.0386	0.0188	0.0605
0.16	0.0155	0.0148	0.0083	0.0236

$\hat{\tau}_3 = 0.175,\quad 0.04 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.04	0.1397	0.3562	0.0003	0.5539
0.05	0.1389	0.3284	0.0090	0.5090
0.06	0.1371	0.2995	0.0175	0.4635
0.07	0.1342	0.2694	0.0257	0.4175
0.08	0.1300	0.2384	0.0333	0.3710
0.09	0.1241	0.2068	0.0397	0.3243
0.10	0.1161	0.1749	0.0443	0.2776
0.11	0.1059	0.1435	0.0465	0.2314
0.12	0.0932	0.1133	0.0457	0.1862
0.13	0.0777	0.0847	0.0416	0.1424
0.14	0.0597	0.0585	0.0341	0.1006
0.15	0.0392	0.0346	0.0234	0.0610
0.16	0.0164	0.0131	0.0101	0.0236

$\hat{\tau}_3 = 0.175,\quad 0.07 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.07	0.1174	0.2700	0.0042	0.3777
0.08	0.1130	0.2421	0.0102	0.3374
0.09	0.1073	0.2135	0.0156	0.2968
0.10	0.1001	0.1844	0.0199	0.2562
0.11	0.0911	0.1550	0.0229	0.2156
0.12	0.0803	0.1256	0.0242	0.1755
0.13	0.0673	0.0967	0.0234	0.1359
0.14	0.0521	0.0686	0.0203	0.0971
0.15	0.0345	0.0417	0.0147	0.0595
0.16	0.0146	0.0162	0.0067	0.0233

Table E–1: $0.2 \leq \hat{\tau}_3 \leq 0.3$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.200,\quad 0.08 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.08	0.1057	0.2417	0.0017	0.3216
0.09	0.1000	0.2142	0.0067	0.2835
0.10	0.0931	0.1861	0.0110	0.2453
0.11	0.0847	0.1575	0.0142	0.2071
0.12	0.0746	0.1288	0.0161	0.1691
0.13	0.0625	0.1000	0.0164	0.1315
0.14	0.0484	0.0717	0.0148	0.0944
0.15	0.0322	0.0440	0.0111	0.0581
0.16	0.0136	0.0172	0.0052	0.0229

$\hat{\tau}_3 = 0.275,\quad 0.13 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.13	0.0502	0.1036	0.0013	0.1172
0.14	0.0389	0.0755	0.0027	0.0849
0.15	0.0258	0.0473	0.0028	0.0528
0.16	0.0110	0.0189	0.0016	0.0210

$\hat{\tau}_3 = 0.225,\quad 0.10 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.10	0.0868	0.1865	0.0036	0.2347
0.11	0.0788	0.1587	0.0070	0.1986
0.12	0.0693	0.1305	0.0094	0.1626
0.13	0.0581	0.1021	0.0105	0.1268
0.14	0.0450	0.0737	0.0101	0.0914
0.15	0.0299	0.0456	0.0079	0.0565
0.16	0.0127	0.0180	0.0038	0.0223

$\hat{\tau}_3 = 0.300,\quad 0.15 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.15	0.0240	0.0475	0.0008	0.0508
0.16	0.0102	0.0191	0.0008	0.0202

$\hat{\tau}_3 = 0.250,\quad 0.11 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.11	0.0734	0.1589	0.0010	0.1902
0.12	0.0645	0.1312	0.0038	0.1560
0.13	0.0540	0.1032	0.0056	0.1220
0.14	0.0418	0.0749	0.0061	0.0882
0.15	0.0278	0.0467	0.0052	0.0547
0.16	0.0118	0.0185	0.0027	0.0217

Table E–2: $0.025 \leq \hat{\tau}_3 \leq 0.3$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.025,\quad 0.00 \leq \hat{\tau}_4 \leq 0.01$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1720	-0.5113	1.0761	0.0070
0.01	0.1791	-0.4791	1.0142	0.0250

$\hat{\tau}_3 = 0.175,\quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1744	-0.6343	1.8195	0.0111
0.01	0.1897	-0.5960	1.7536	0.0426
0.02	0.2114	-0.5418	1.6870	0.0942

$\hat{\tau}_3 = 0.025,\quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1766	-0.5243	1.1629	0.0136
0.01	0.1855	-0.4895	1.0972	0.0349
0.02	0.1956	-0.4484	1.0271	0.0619

$\hat{\tau}_3 = 0.200,\quad 0.00 \leq \hat{\tau}_4 \leq 0.03$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1667	-0.6667	2.0000	0.0000
0.01	0.1811	-0.6322	1.9394	0.0284
0.02	0.2006	-0.5857	1.8801	0.0719
0.03	0.2400	-0.4938	1.8562	0.1850

$\hat{\tau}_3 = 0.075,\quad 0.00 \leq \hat{\tau}_4 \leq 0.03$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1802	-0.5394	1.2623	0.0189
0.01	0.1910	-0.5020	1.1933	0.0440
0.02	0.2042	-0.4555	1.1184	0.0781
0.03	0.2236	-0.3880	1.0318	0.1361

$\hat{\tau}_3 = 0.225,\quad 0.01 \leq \hat{\tau}_4 \leq 0.03$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.01	0.1700	-0.6706	2.1389	0.0109
0.02	0.1868	-0.6323	2.0864	0.0457
0.03	0.2118	-0.5754	2.0489	0.1070

$\hat{\tau}_3 = 0.100,\quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1823	-0.5574	1.3763	0.0222
0.01	0.1950	-0.5177	1.3051	0.0510
0.02	0.2117	-0.4651	1.2269	0.0939

$\hat{\tau}_3 = 0.250,\quad 0.02 \leq \hat{\tau}_4 \leq 0.04$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.02	0.1716	-0.6780	2.3023	0.0196
0.03	0.1903	-0.6368	2.2653	0.0604
0.04	0.2203	-0.5709	2.2748	0.1412

$\hat{\tau}_3 = 0.125,\quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1823	-0.5791	1.5067	0.0225
0.01	0.1966	-0.5380	1.4351	0.0541
0.02	0.2168	-0.4803	1.3569	0.1055

$\hat{\tau}_3 = 0.275,\quad 0.03 \leq \hat{\tau}_4 \leq 0.05$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.03	0.1706	-0.6898	2.4906	0.0243
0.04	0.1900	-0.6483	2.4752	0.0683
0.05	0.2202	-0.5835	2.5382	0.1545

$\hat{\tau}_3 = 0.150,\quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.1797	-0.6048	1.6545	0.0188
0.01	0.1949	-0.5641	1.5847	0.0515
0.02	0.2171	-0.5054	1.5110	0.1067

$\hat{\tau}_3 = 0.300,\quad 0.04 \leq \hat{\tau}_4 \leq 0.06$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.04	0.1668	-0.7064	2.7005	0.0239
0.05	0.1854	-0.6678	2.7074	0.0666
0.06	0.2114	-0.6127	2.7977	0.1415

Table E–2: $0.325 \leq \hat{\tau}_3 \leq 0.45$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.325,\quad 0.05 \leq \hat{\tau}_4 \leq 0.08$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.05	0.1603	-0.7271	2.9258	0.0185
0.06	0.1769	-0.6934	2.9508	0.0565
0.07	0.1982	-0.6492	3.0472	0.1157
0.08	0.2227	-0.5958	3.3257	0.2093

$\hat{\tau}_3 = 0.350,\quad 0.06 \leq \hat{\tau}_4 \leq 0.10$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.06	0.1518	-0.7505	3.1599	0.0092
0.07	0.1660	-0.7221	3.1979	0.0413
0.08	0.1832	-0.6872	3.2922	0.0868
0.09	0.2024	-0.6465	3.5023	0.1512
0.10	0.2188	-0.6088	3.8823	0.2290

$\hat{\tau}_3 = 0.375,\quad 0.8 \leq \hat{\tau}_4 \leq 0.12$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.08	0.1540	-0.7514	3.4442	0.0242
0.09	0.1679	-0.7238	3.5352	0.0589
0.10	0.1831	-0.6925	3.7036	0.1048
0.11	0.1977	-0.6609	3.9893	0.1611
0.12	0.2084	-0.6348	4.4002	0.2196

$\hat{\tau}_3 = 0.400,\quad 0.09 \leq \hat{\tau}_4 \leq 0.15$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.09	0.1418	-0.7796	3.6862	0.0070
0.10	0.1530	-0.7578	3.7750	0.0339
0.11	0.1652	-0.7335	3.9181	0.0675
0.12	0.1774	-0.7079	4.1416	0.1081
0.13	0.1881	-0.6842	4.4650	0.1532
0.14	0.1956	-0.6652	4.8865	0.1975
0.15	0.1998	-0.6519	5.3913	0.2370

$\hat{\tau}_3 = 0.425,\quad 0.11 \leq \hat{\tau}_4 \leq 0.19$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.11	0.1391	-0.7885	4.0088	0.0120
0.12	0.1489	-0.7694	4.1367	0.0373
0.13	0.1589	-0.7491	4.3214	0.0672
0.14	0.1684	-0.7290	4.5800	0.1010
0.15	0.1763	-0.7110	4.9225	0.1365
0.16	0.1819	-0.6966	5.3487	0.1709
0.17	0.1850	-0.6862	5.8536	0.2018
0.18	0.1860	-0.6797	6.4355	0.2282
0.19	0.1851	-0.6768	7.1000	0.2497

$\hat{\tau}_3 = 0.450,\quad 0.13 \leq \hat{\tau}_4 \leq 0.26$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.13	0.1343	-0.8006	4.3525	0.0127
0.14	0.1424	-0.7846	4.5131	0.0353
0.15	0.1505	-0.7682	4.7280	0.0607
0.16	0.1578	-0.7525	5.0086	0.0883
0.17	0.1638	-0.7385	5.3620	0.1164
0.18	0.1680	-0.7272	5.7903	0.1435
0.19	0.1704	-0.7189	6.2942	0.1682
0.20	0.1711	-0.7137	6.8762	0.1896
0.21	0.1703	-0.7112	7.5444	0.2074
0.22	0.1682	-0.7113	8.3130	0.2214
0.23	0.1648	-0.7139	9.2054	0.2317
0.24	0.1603	-0.7188	10.257	0.2382
0.25	0.1546	-0.7261	11.522	0.2408
0.26	0.1478	-0.7359	13.089	0.2393

Table E–2: $0.475 \leq \hat{\tau}_3 \leq 0.55$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.475,\quad 0.15 \leq \hat{\tau}_4 \leq 0.28$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.15	0.1280	-0.8147	4.7058	0.0101
0.16	0.1346	-0.8016	4.8920	0.0296
0.17	0.1410	-0.7885	5.1286	0.0508
0.18	0.1466	-0.7761	5.4239	0.0732
0.19	0.1512	-0.7652	5.7838	0.0957
0.20	0.1545	-0.7563	6.2126	0.1174
0.21	0.1564	-0.7496	6.7137	0.1374
0.22	0.1569	-0.7452	7.2925	0.1550
0.23	0.1562	-0.7431	7.9580	0.1700
0.24	0.1542	-0.7432	8.7249	0.1820
0.25	0.1512	-0.7454	9.6151	0.1910
0.26	0.1471	-0.7497	10.661	0.1970
0.27	0.1420	-0.7560	11.913	0.1998
0.28	0.1358	-0.7646	13.447	0.1993

$\hat{\tau}_3 = 0.525,\quad 0.20 \leq \hat{\tau}_4 \leq 0.32$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.20	0.1178	-0.8361	5.6294	0.0132
0.21	0.1217	-0.8278	5.8934	0.0279
0.22	0.1251	-0.8200	6.2057	0.0430
0.23	0.1279	-0.8131	6.5720	0.0580
0.24	0.1299	-0.8073	6.9974	0.0726
0.25	0.1310	-0.8029	7.4881	0.0862
0.26	0.1312	-0.7998	8.0514	0.0985
0.27	0.1306	-0.7983	8.6969	0.1094
0.28	0.1291	-0.7982	9.4382	0.1186
0.29	0.1267	-0.7996	10.294	0.1259
0.30	0.1236	-0.8026	11.290	0.1313
0.31	0.1196	-0.8070	12.464	0.1347
0.32	0.1148	-0.8130	13.869	0.1360

$\hat{\tau}_3 = 0.500,\quad 0.17 \leq \hat{\tau}_4 \leq 0.30$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.17	0.1209	-0.8298	5.0597	0.0052
0.18	0.1262	-0.8191	5.2658	0.0218
0.19	0.1312	-0.8086	5.5181	0.0395
0.20	0.1356	-0.7988	5.8234	0.0578
0.21	0.1392	-0.7902	6.1875	0.0760
0.22	0.1417	-0.7830	6.6152	0.0937
0.23	0.1432	-0.7776	7.1119	0.1101
0.24	0.1436	-0.7739	7.6842	0.1247
0.25	0.1429	-0.7721	8.3419	0.1374
0.26	0.1411	-0.7722	9.0990	0.1479
0.27	0.1385	-0.7740	9.9760	0.1559
0.28	0.1349	-0.7776	11.003	0.1616
0.29	0.1303	-0.7830	12.222	0.1647
0.30	0.1248	-0.7902	13.698	0.1651

$\hat{\tau}_3 = 0.550,\quad 0.22 \leq \hat{\tau}_4 \leq 0.34$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.22	0.1096	-0.8522	5.9797	0.0045
0.23	0.1127	-0.8455	6.2523	0.0169
0.24	0.1154	-0.8392	6.5694	0.0295
0.25	0.1175	-0.8337	6.9363	0.0420
0.26	0.1191	-0.8290	7.3584	0.0541
0.27	0.1199	-0.8254	7.8418	0.0656
0.28	0.1200	-0.8228	8.3940	0.0761
0.29	0.1194	-0.8215	9.0245	0.0855
0.30	0.1181	-0.8213	9.7457	0.0937
0.31	0.1161	-0.8224	10.574	0.1004
0.32	0.1134	-0.8247	11.533	0.1056
0.33	0.1099	-0.8283	12.654	0.1092
0.34	0.1058	-0.8332	13.981	0.1111

Table E–2: $0.575 \leq \hat{\tau}_3 \leq 0.75$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.575, \quad 0.25 \leq \hat{\tau}_4 \leq 0.36$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.25	0.1044	-0.8616	6.5935	0.0067
0.26	0.1065	-0.8566	6.9136	0.0174
0.27	0.1081	-0.8521	7.2798	0.0279
0.28	0.1093	-0.8483	7.6975	0.0382
0.29	0.1099	-0.8452	8.1727	0.0479
0.30	0.1100	-0.8431	8.7125	0.0570
0.31	0.1094	-0.8419	9.3259	0.0653
0.32	0.1083	-0.8417	10.024	0.0725
0.33	0.1065	-0.8424	10.823	0.0787
0.34	0.1041	-0.8442	11.741	0.0837
0.35	0.1012	-0.8471	12.805	0.0875
0.36	0.0976	-0.8510	14.053	0.0899

$\hat{\tau}_3 = 0.600, \quad 0.28 \leq \hat{\tau}_4 \leq 0.38$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.28	0.0984	-0.8720	7.2378	0.0066
0.29	0.0997	-0.8683	7.6023	0.0156
0.30	0.1006	-0.8652	8.0147	0.0245
0.31	0.1010	-0.8627	8.4808	0.0329
0.32	0.1010	-0.8609	9.0074	0.0408
0.33	0.1005	-0.8598	9.6026	0.0481
0.34	0.0995	-0.8595	10.277	0.0547
0.35	0.0979	-0.8600	11.044	0.0604
0.36	0.0959	-0.8614	11.920	0.0652
0.37	0.0934	-0.8636	12.928	0.0690
0.38	0.0903	-0.8668	14.098	0.0718

$\hat{\tau}_3 = 0.625, \quad 0.31 \leq \hat{\tau}_4 \leq 0.40$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.31	0.0921	-0.8827	7.9040	0.0050
0.32	0.0928	-0.8801	8.3105	0.0127
0.33	0.0931	-0.8780	8.7669	0.0200
0.34	0.0930	-0.8764	9.2797	0.0270
0.35	0.0925	-0.8754	9.8564	0.0335
0.36	0.0916	-0.8751	10.506	0.0395
0.37	0.0903	-0.8754	11.241	0.0448
0.38	0.0886	-0.8764	12.075	0.0494
0.39	0.0864	-0.8782	13.029	0.0532
0.40	0.0838	-0.8806	14.126	0.0563

$\hat{\tau}_3 = 0.650, \quad 0.34 \leq \hat{\tau}_4 \leq 0.42$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.34	0.0859	-0.8931	8.5854	0.0025
0.35	0.0861	-0.8913	9.0319	0.0091
0.36	0.0860	-0.8900	9.5307	0.0153
0.37	0.0855	-0.8891	10.089	0.0212
0.38	0.0847	-0.8887	10.714	0.0266
0.39	0.0836	-0.8889	11.418	0.0315
0.40	0.0821	-0.8896	12.212	0.0359
0.41	0.0802	-0.8910	13.113	0.0397
0.42	0.0780	-0.8929	14.142	0.0429

$\hat{\tau}_3 = 0.675, \quad 0.38 \leq \hat{\tau}_4 \leq 0.44$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.38	0.0797	-0.9019	9.7620	0.0052
0.39	0.0793	-0.9011	10.302	0.0106
0.40	0.0786	-0.9007	10.904	0.0156
0.41	0.0776	-0.9007	11.577	0.0202
0.42	0.0763	-0.9012	12.333	0.0244
0.43	0.0747	-0.9022	13.185	0.0281
0.44	0.0728	-0.9037	14.151	0.0313

$\hat{\tau}_3 = 0.700, \quad 0.41 \leq \hat{\tau}_4 \leq 0.46$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.41	0.0738	-0.9116	10.497	0.0014
0.42	0.0731	-0.9112	11.076	0.0061
0.43	0.0723	-0.9111	11.721	0.0104
0.44	0.0711	-0.9114	12.441	0.0144
0.45	0.0697	-0.9121	13.248	0.0180
0.46	0.0681	-0.9132	14.156	0.0212

$\hat{\tau}_3 = 0.725, \quad 0.45 \leq \hat{\tau}_4 \leq 0.48$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.45	0.0675	-0.9203	11.851	0.0019
0.46	0.0665	-0.9204	12.538	0.0057
0.47	0.0653	-0.9209	13.303	0.0092
0.48	0.0639	-0.9218	14.159	0.0124

$\hat{\tau}_3 = 0.750, \quad 0.49 \leq \hat{\tau}_4 \leq 0.50$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.49	0.0614	-0.9288	13.351	0.0016
0.50	0.0601	-0.9294	14.160	0.0047

Table E–3: $0.025 \leq \hat{\tau}_3 \leq 0.25$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.025, \quad 0.00 \leq \hat{\tau}_4 \leq 0.00$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3330	-0.0497	1.1135	0.9127

$\hat{\tau}_3 = 0.050, \quad 0.00 \leq \hat{\tau}_4 \leq 0.01$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3323	-0.0974	1.2595	0.8520
0.01	0.3193	-0.1119	1.0680	0.6795

$\hat{\tau}_3 = 0.075, \quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3317	-0.1405	1.4594	0.8278
0.01	0.3173	-0.1633	1.2148	0.6266
0.02	0.2999	-0.1913	1.0743	0.4851

$\hat{\tau}_3 = 0.100, \quad 0.00 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3333	-0.1667	2.0000	1.0000
0.01	0.3157	-0.2083	1.3979	0.5992
0.02	0.2950	-0.2475	1.2227	0.4338

$\hat{\tau}_3 = 0.125, \quad 0.01 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.01	0.3164	-0.2422	1.6593	0.6176
0.02	0.2917	-0.2940	1.3983	0.4066

$\hat{\tau}_3 = 0.150, \quad 0.02 \leq \hat{\tau}_4 \leq 0.02$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.02	0.2932	-0.3233	1.6318	0.4219

$\hat{\tau}_3 = 0.175, \quad 0.02 \leq \hat{\tau}_4 \leq 0.03$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.02	0.3117	-0.2999	2.3943	0.6820
0.03	0.3111	-0.2770	3.2260	0.9469

$\hat{\tau}_3 = 0.200, \quad 0.03 \leq \hat{\tau}_4 \leq 0.05$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.03	0.3038	-0.3361	2.7263	0.6544
0.04	0.3031	-0.3179	3.4565	0.8439
0.05	0.2980	-0.3153	4.0161	0.9429

$\hat{\tau}_3 = 0.225, \quad 0.04 \leq \hat{\tau}_4 \leq 0.14$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.04	0.2930	-0.3807	2.9023	0.5698
0.05	0.2942	-0.3601	3.6268	0.7353
0.06	0.2901	-0.3562	4.2146	0.8250
0.07	0.2846	-0.3575	4.7948	0.8867
0.08	0.2784	-0.3620	5.4090	0.9304
0.09	0.2717	-0.3691	6.0898	0.9601
0.10	0.2644	-0.3789	6.8767	0.9770
0.11	0.2563	-0.3915	7.8299	0.9812
0.12	0.2472	-0.4079	9.0602	0.9710
0.13	0.2363	-0.4303	10.821	0.9420
0.14	0.2207	-0.4658	14.020	0.8784

$\hat{\tau}_3 = 0.250, \quad 0.05 \leq \hat{\tau}_4 \leq 0.15$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.05	0.2794	-0.4304	3.0097	0.4710
0.06	0.2841	-0.4042	3.7367	0.6252
0.07	0.2813	-0.3978	4.3480	0.7102
0.08	0.2766	-0.3977	4.9553	0.7680
0.09	0.2708	-0.4014	5.6004	0.8088
0.10	0.2644	-0.4079	6.3171	0.8362
0.11	0.2572	-0.4172	7.1464	0.8519
0.12	0.2492	-0.4296	8.1510	0.8559
0.13	0.2401	-0.4457	9.4441	0.8470
0.14	0.2290	-0.4675	11.278	0.8217
0.15	0.2137	-0.5010	14.483	0.7681

Table E–3: $0.275 \le \hat{\tau}_3 \le 0.425$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.275,\quad 0.06 \le \hat{\tau}_4 \le 0.16$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.06	0.2626	-0.4847	3.0890	0.3713
0.07	0.2719	-0.4509	3.7924	0.5162
0.08	0.2712	-0.4407	4.4172	0.5993
0.09	0.2676	-0.4384	5.0411	0.6558
0.10	0.2626	-0.4406	5.7035	0.6956
0.11	0.2567	-0.4460	6.4376	0.7227
0.12	0.2499	-0.4544	7.2833	0.7388
0.13	0.2421	-0.4657	8.2996	0.7444
0.14	0.2333	-0.4807	9.5896	0.7388
0.15	0.2226	-0.5007	11.368	0.7195
0.16	0.2085	-0.5298	14.237	0.6794

$\hat{\tau}_3 = 0.300,\quad 0.07 \le \hat{\tau}_4 \le 0.17$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.07	0.2432	-0.5410	3.1864	0.2822
0.08	0.2571	-0.5010	3.8121	0.4109
0.09	0.2593	-0.4856	4.4316	0.4932
0.10	0.2574	-0.4802	5.0586	0.5497
0.11	0.2535	-0.4802	5.7240	0.5900
0.12	0.2483	-0.4839	6.4578	0.6180
0.13	0.2421	-0.4908	7.2960	0.6358
0.14	0.2349	-0.5006	8.2904	0.6441
0.15	0.2266	-0.5138	9.5265	0.6426
0.16	0.2167	-0.5313	11.168	0.6301
0.17	0.2042	-0.5555	13.605	0.6026

$\hat{\tau}_3 = 0.325,\quad 0.09 \le \hat{\tau}_4 \le 0.18$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.09	0.2392	-0.5543	3.8293	0.3133
0.10	0.2450	-0.5333	4.4117	0.3927
0.11	0.2454	-0.5238	5.0238	0.4497
0.12	0.2430	-0.5208	5.6770	0.4912
0.13	0.2389	-0.5222	6.3944	0.5210
0.14	0.2336	-0.5270	7.2061	0.5411
0.15	0.2272	-0.5349	8.1546	0.5526
0.16	0.2196	-0.5458	9.3066	0.5555
0.17	0.2107	-0.5605	10.779	0.5494
0.18	0.1998	-0.5802	12.814	0.5324

$\hat{\tau}_3 = 0.350,\quad 0.11 \le \hat{\tau}_4 \le 0.20$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.11	0.2279	-0.5835	4.3894	0.3003
0.12	0.2311	-0.5695	4.9617	0.3564
0.13	0.2307	-0.5630	5.5855	0.3989
0.14	0.2281	-0.5615	6.2718	0.4306
0.15	0.2240	-0.5637	7.0425	0.4533
0.16	0.2186	-0.5692	7.9303	0.4681
0.17	0.2121	-0.5777	8.9851	0.4753
0.18	0.2043	-0.5894	10.288	0.4750
0.19	0.1949	-0.6050	11.988	0.4664
0.20	0.1833	-0.6260	14.401	0.4473

$\hat{\tau}_3 = 0.375,\quad 0.13 \le \hat{\tau}_4 \le 0.21$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.13	0.2143	-0.6172	4.9052	0.2715
0.14	0.2163	-0.6069	5.4793	0.3137
0.15	0.2156	-0.6020	6.1195	0.3468
0.16	0.2130	-0.6014	6.8376	0.3718
0.17	0.2089	-0.6041	7.6563	0.3895
0.18	0.2036	-0.6099	8.6111	0.4007
0.19	0.1971	-0.6186	9.7576	0.4053
0.20	0.1892	-0.6305	11.188	0.4033
0.21	0.1797	-0.6464	13.072	0.3938

$\hat{\tau}_3 = 0.400,\quad 0.16 \le \hat{\tau}_4 \le 0.23$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.16	0.2011	-0.6439	5.9700	0.2700
0.17	0.2003	-0.6402	6.6249	0.2964
0.18	0.1978	-0.6400	7.3688	0.3166
0.19	0.1939	-0.6429	8.2254	0.3310
0.20	0.1887	-0.6487	9.2318	0.3398
0.21	0.1823	-0.6572	10.446	0.3430
0.22	0.1746	-0.6689	11.966	0.3405
0.23	0.1652	-0.6842	13.967	0.3315

$\hat{\tau}_3 = 0.425,\quad 0.20 \le \hat{\tau}_4 \le 0.25$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.20	0.1828	-0.6768	7.8612	0.2663
0.21	0.1791	-0.6796	8.7427	0.2783
0.22	0.1742	-0.6851	9.7816	0.2857
0.23	0.1681	-0.6931	11.036	0.2883
0.24	0.1608	-0.7040	12.600	0.2861

Table E–4: $0 \leq \hat{\tau}_3 \leq 0.025$ and λ_3, $\lambda_4 > 0$

$\hat{\tau}_3 = 0.000$, $0.00 \leq \hat{\tau}_4 \leq 0.53$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3333	0.0000	2.0000	2.0000
0.01	0.3254	0.0000	2.2612	2.2612
0.02	0.3181	0.0000	2.4799	2.4799
0.03	0.3113	0.0000	2.6779	2.6779
0.04	0.3048	0.0000	2.8638	2.8638
0.05	0.2985	0.0000	3.0418	3.0418
0.06	0.2925	0.0000	3.2147	3.2147
0.07	0.2867	0.0000	3.3841	3.3841
0.08	0.2811	0.0000	3.5512	3.5512
0.09	0.2757	0.0000	3.7170	3.7170
0.10	0.2704	0.0000	3.8821	3.8821
0.11	0.2652	0.0000	4.0472	4.0472
0.12	0.2602	0.0000	4.2126	4.2126
0.13	0.2552	0.0000	4.3788	4.3788
0.14	0.2504	0.0000	4.5461	4.5461
0.15	0.2457	0.0000	4.7148	4.7148
0.16	0.2411	0.0000	4.8853	4.8853
0.17	0.2366	0.0000	5.0577	5.0577
0.18	0.2322	0.0000	5.2324	5.2324
0.19	0.2278	0.0000	5.4096	5.4096
0.20	0.2235	0.0000	5.5895	5.5895
0.21	0.2193	0.0000	5.7723	5.7723
0.22	0.2152	0.0000	5.9582	5.9582
0.23	0.2111	0.0000	6.1476	6.1476
0.24	0.2071	0.0000	6.3405	6.3405
0.25	0.2032	0.0000	6.5373	6.5373
0.26	0.1993	0.0000	6.7381	6.7381
0.27	0.1955	0.0000	6.9433	6.9433
0.28	0.1917	0.0000	7.1530	7.1530
0.29	0.1880	0.0000	7.3674	7.3674
0.30	0.1843	0.0000	7.5870	7.5870
0.31	0.1807	0.0000	7.8119	7.8119
0.32	0.1771	0.0000	8.0423	8.0423
0.33	0.1736	0.0000	8.2787	8.2787
0.34	0.1701	0.0000	8.5214	8.5214
0.35	0.1667	0.0000	8.7705	8.7705
0.36	0.1633	0.0000	9.0266	9.0266
0.37	0.1599	0.0000	9.2899	9.2899
0.38	0.1566	0.0000	9.5609	9.5609
0.39	0.1533	0.0000	9.8399	9.8399
0.40	0.1501	0.0000	10.1275	10.1275
0.41	0.1469	0.0000	10.4239	10.4239
0.42	0.1437	0.0000	10.7299	10.7299
0.43	0.1406	0.0000	11.0458	11.0458
0.44	0.1375	0.0000	11.3722	11.3722
0.45	0.1344	0.0000	11.7097	11.7097
0.46	0.1314	0.0000	12.0591	12.0591
0.47	0.1284	0.0000	12.4209	12.4209
0.48	0.1254	0.0000	12.7959	12.7959
0.49	0.1224	0.0000	13.1849	13.1849
0.50	0.1195	0.0000	13.5887	13.5887
0.51	0.1166	0.0000	14.0084	14.0084
0.52	0.1137	0.0000	14.4449	14.4449
0.53	0.1109	0.0000	14.8992	14.8992

$\hat{\tau}_3 = 0.025$, $0.00 \leq \hat{\tau}_4 \leq 0.51$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3334	-0.0388	2.1549	1.8109
0.01	0.3255	-0.0389	2.4547	2.0454
0.02	0.3182	-0.0396	2.7099	2.2351
0.03	0.3113	-0.0406	2.9462	2.4016
0.04	0.3048	-0.0419	3.1741	2.5527
0.05	0.2985	-0.0436	3.3995	2.6920
0.06	0.2924	-0.0456	3.6268	2.8212
0.07	0.2866	-0.0480	3.8601	2.9408
0.08	0.2809	-0.0509	4.1035	3.0504
0.09	0.2753	-0.0546	4.3624	3.1487
0.10	0.2698	-0.0590	4.6441	3.2337
0.11	0.2643	-0.0647	4.9601	3.3016
0.12	0.2589	-0.0721	5.3299	3.3460
0.13	0.2533	-0.0823	5.7921	3.3558
0.14	0.2197	-0.2478	14.6707	2.2087
0.15	0.2390	-0.1299	7.7677	3.0995
0.17	0.2081	0.1969	2.8495	14.8996
0.18	0.2089	0.1766	3.0713	13.4875
0.19	0.2092	0.1562	3.3201	12.2817
0.20	0.2090	0.1361	3.5958	11.2763
0.21	0.2082	0.1171	3.8942	10.4691
0.22	0.2068	0.0999	4.2073	9.8538
0.23	0.2047	0.0850	4.5251	9.4139
0.24	0.2022	0.0725	4.8391	9.1227
0.25	0.1993	0.0622	5.1444	8.9495
0.26	0.1962	0.0539	5.4398	8.8665
0.27	0.1930	0.0472	5.7261	8.8513
0.28	0.1896	0.0416	6.0050	8.8875
0.29	0.1863	0.0370	6.2784	8.9632
0.30	0.1828	0.0332	6.5483	9.0702
0.31	0.1794	0.0299	6.8163	9.2026
0.32	0.1760	0.0271	7.0838	9.3562
0.33	0.1726	0.0247	7.3521	9.5281
0.34	0.1693	0.0227	7.6224	9.7161
0.35	0.1659	0.0208	7.8955	9.9187
0.36	0.1626	0.0192	8.1723	10.1349
0.37	0.1593	0.0178	8.4536	10.3639
0.38	0.1560	0.0165	8.7401	10.6052
0.39	0.1528	0.0153	9.0326	10.8586
0.40	0.1496	0.0143	9.3316	11.1240
0.41	0.1464	0.0134	9.6379	11.4014
0.42	0.1433	0.0125	9.9521	11.6909
0.43	0.1402	0.0117	10.2748	11.9927
0.44	0.1371	0.0110	10.6068	12.3072
0.45	0.1341	0.0103	10.9487	12.6347
0.46	0.1311	0.0097	11.3013	12.9757
0.47	0.1281	0.0092	11.6653	13.3308
0.48	0.1251	0.0087	12.0415	13.7006
0.49	0.1222	0.0082	12.4308	14.0858
0.50	0.1193	0.0077	12.8340	14.4871
0.51	0.1164	0.0073	13.2521	14.9055

Table E–4: $0.05 \leq \hat{\tau}_3 \leq 0.075$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.050, \quad 0.00 \leq \hat{\tau}_4 \leq 0.48$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3337	-0.0779	2.2567	1.5979
0.01	0.3258	-0.0777	2.6097	1.8190
0.02	0.3184	-0.0788	2.9080	1.9878
0.03	0.3115	-0.0807	3.1877	2.1304
0.04	0.3049	-0.0831	3.4627	2.2550
0.05	0.2984	-0.0863	3.7415	2.3651
0.06	0.2922	-0.0901	4.0311	2.4620
0.07	0.2862	-0.0947	4.3390	2.5457
0.08	0.2802	-0.1004	4.6747	2.6151
0.09	0.2742	-0.1074	5.0515	2.6682
0.10	0.2682	-0.1161	5.4913	2.7010
0.11	0.2619	-0.1276	6.0338	2.7070
0.12	0.2551	-0.1436	6.7686	2.6726
0.13	0.2466	-0.1699	8.0020	2.5591
0.20	0.1996	0.1611	3.4257	14.4200
0.21	0.1989	0.1468	3.6415	13.5755
0.22	0.1978	0.1333	3.8697	12.8741
0.23	0.1963	0.1206	4.1085	12.3031
0.24	0.1945	0.1089	4.3558	11.8495
0.25	0.1925	0.0983	4.6093	11.5000
0.26	0.1902	0.0887	4.8671	11.2412
0.27	0.1876	0.0803	5.1278	11.0604
0.28	0.1849	0.0728	5.3904	10.9457
0.29	0.1821	0.0662	5.6542	10.8871
0.30	0.1791	0.0603	5.9193	10.8759
0.31	0.1761	0.0552	6.1856	10.9050
0.32	0.1731	0.0506	6.4536	10.9685
0.33	0.1700	0.0466	6.7236	11.0620
0.34	0.1669	0.0430	6.9963	11.1816
0.35	0.1638	0.0397	7.2722	11.3246
0.36	0.1606	0.0369	7.5517	11.4887
0.37	0.1575	0.0343	7.8357	11.6721
0.38	0.1544	0.0319	8.1246	11.8735
0.39	0.1513	0.0298	8.4190	12.0919
0.40	0.1483	0.0278	8.7197	12.3266
0.41	0.1452	0.0261	9.0271	12.5772
0.42	0.1421	0.0245	9.3419	12.8432
0.43	0.1391	0.0230	9.6648	13.1246
0.44	0.1361	0.0216	9.9964	13.4213
0.45	0.1331	0.0203	10.3375	13.7336
0.46	0.1302	0.0192	10.6887	14.0616
0.47	0.1273	0.0181	11.0507	14.4058
0.48	0.1244	0.0171	11.4245	14.7665

$\hat{\tau}_3 = 0.075, \quad 0.00 \leq \hat{\tau}_4 \leq 0.44$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3341	-0.1181	2.2704	1.3591
0.01	0.3262	-0.1165	2.7052	1.5879
0.02	0.3188	-0.1176	3.0541	1.7454
0.03	0.3118	-0.1199	3.3806	1.8720
0.04	0.3050	-0.1232	3.7046	1.9782
0.05	0.2984	-0.1275	4.0383	2.0680
0.06	0.2920	-0.1327	4.3923	2.1430
0.07	0.2856	-0.1391	4.7784	2.2031
0.08	0.2793	-0.1470	5.2131	2.2470
0.09	0.2728	-0.1568	5.7218	2.2723
0.10	0.2660	-0.1694	6.3504	2.2739
0.11	0.2585	-0.1865	7.1993	2.2421
0.12	0.2490	-0.2133	8.5920	2.1492
0.24	0.1874	0.1285	4.1789	14.4777
0.25	0.1856	0.1188	4.3943	14.0281
0.26	0.1837	0.1098	4.6167	13.6637
0.27	0.1816	0.1015	4.8455	13.3755
0.28	0.1793	0.0938	5.0798	13.1548
0.29	0.1769	0.0868	5.3193	12.9942
0.30	0.1744	0.0803	5.5634	12.8866
0.31	0.1718	0.0745	5.8121	12.8259
0.32	0.1691	0.0691	6.0651	12.8068
0.33	0.1663	0.0642	6.3224	12.8248
0.34	0.1635	0.0598	6.5843	12.8761
0.35	0.1607	0.0557	6.8508	12.9572
0.36	0.1578	0.0520	7.1224	13.0656
0.37	0.1549	0.0486	7.3992	13.1990
0.38	0.1520	0.0455	7.6817	13.3557
0.39	0.1491	0.0427	7.9703	13.5340
0.40	0.1461	0.0401	8.2655	13.7330
0.41	0.1432	0.0376	8.5678	13.9517
0.42	0.1403	0.0354	8.8776	14.1894
0.43	0.1374	0.0333	9.1956	14.4457
0.44	0.1345	0.0314	9.5224	14.7205

Table E–4: $0.1 \leq \hat{\tau}_3 \leq 0.2$ and $\lambda_3,\ \lambda_4 > 0$

$\hat{\tau}_3 = 0.1, \quad 0.00 \leq \hat{\tau}_4 \leq 0.37$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.00	0.3333	-0.1667	2.0000	1.0000
0.01	0.3265	-0.1560	2.7138	1.3513
0.02	0.3192	-0.1561	3.1263	1.5099
0.03	0.3121	-0.1583	3.5022	1.6287
0.04	0.3053	-0.1619	3.8742	1.7241
0.05	0.2985	-0.1668	4.2596	1.8016
0.06	0.2919	-0.1730	4.6728	1.8633
0.07	0.2852	-0.1806	5.1304	1.9096
0.08	0.2785	-0.1900	5.6557	1.9394
0.09	0.2715	-0.2017	6.2868	1.9503
0.10	0.2639	-0.2170	7.0974	1.9370
0.11	0.2550	-0.2384	8.2703	1.8875
0.12	0.2418	-0.2778	10.6834	1.7517
0.31	0.1670	0.0887	5.6003	14.9264
0.32	0.1646	0.0831	5.8360	14.8349
0.33	0.1621	0.0779	6.0775	14.7848
0.34	0.1596	0.0731	6.3248	14.7724
0.35	0.1570	0.0686	6.5780	14.7946
0.36	0.1543	0.0645	6.8372	14.8485
0.37	0.1517	0.0606	7.1028	14.9318

$\hat{\tau}_3 = 0.125, \quad 0.01 \leq \hat{\tau}_4 \leq 0.12$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.01	0.3261	-0.1987	2.5819	1.0925
0.02	0.3194	-0.1951	3.1002	1.2778
0.03	0.3125	-0.1962	3.5317	1.3987
0.04	0.3056	-0.1995	3.9497	1.4905
0.05	0.2988	-0.2044	4.3800	1.5623
0.06	0.2920	-0.2108	4.8414	1.6177
0.07	0.2851	-0.2189	5.3544	1.6577
0.08	0.2781	-0.2289	5.9468	1.6818
0.09	0.2706	-0.2415	6.6646	1.6881
0.10	0.2625	-0.2579	7.5975	1.6720
0.11	0.2528	-0.2810	8.9742	1.6226
0.12	0.2373	-0.3251	12.0178	1.4882

$\hat{\tau}_3 = 0.150, \quad 0.02 \leq \hat{\tau}_4 \leq 0.12$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.02	0.3187	-0.2372	2.9372	1.0360
0.03	0.3124	-0.2348	3.4496	1.1759
0.04	0.3058	-0.2366	3.9152	1.2722
0.05	0.2990	-0.2408	4.3836	1.3446
0.06	0.2922	-0.2467	4.8801	1.3992
0.07	0.2852	-0.2546	5.4280	1.4383
0.08	0.2781	-0.2644	6.0568	1.4623
0.09	0.2705	-0.2768	6.8126	1.4703
0.10	0.2621	-0.2928	7.7814	1.4592
0.11	0.2523	-0.3146	9.1665	1.4214
0.12	0.2383	-0.3511	11.7807	1.3294

$\hat{\tau}_3 = 0.175, \quad 0.04 \leq \hat{\tau}_4 \leq 0.12$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.04	0.3054	-0.2749	3.7581	1.0614
0.05	0.2990	-0.2771	4.2634	1.1411
0.06	0.2924	-0.2817	4.7849	1.1997
0.07	0.2855	-0.2885	5.3499	1.2421
0.08	0.2784	-0.2974	5.9874	1.2700
0.09	0.2708	-0.3087	6.7386	1.2833
0.10	0.2626	-0.3232	7.6733	1.2807
0.11	0.2532	-0.3422	8.9368	1.2580
0.12	0.2413	-0.3701	10.9491	1.2034

$\hat{\tau}_3 = 0.200, \quad 0.06 \leq \hat{\tau}_4 \leq 0.13$				
$\hat{\tau}_4$	A	B	λ_3	λ_4
0.06	0.2920	-0.3173	4.5612	1.0115
0.07	0.2855	-0.3222	5.1330	1.0609
0.08	0.2787	-0.3294	5.7616	1.0953
0.09	0.2714	-0.3389	6.4821	1.1160
0.10	0.2635	-0.3512	7.3470	1.1230
0.11	0.2548	-0.3671	8.4504	1.1146
0.12	0.2444	-0.3885	10.0033	1.0860
0.13	0.2302	-0.4217	12.7223	1.0218

Table E–5: $\hat{\tau}_3 = 0$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.000, \quad 0.18 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.18	0.0395	0.0000	0.0373	0.0373
0.19	0.0705	0.0000	0.0639	0.0639
0.20	0.1028	0.0000	0.0895	0.0895
0.21	0.1365	0.0000	0.1140	0.1140
0.22	0.1715	0.0000	0.1377	0.1377
0.23	0.2080	0.0000	0.1606	0.1606
0.24	0.2459	0.0000	0.1826	0.1826
0.25	0.2853	0.0000	0.2040	0.2040
0.26	0.3263	0.0000	0.2246	0.2246
0.27	0.3690	0.0000	0.2446	0.2446
0.28	0.4134	0.0000	0.2641	0.2641
0.29	0.4596	0.0000	0.2829	0.2829
0.30	0.5076	0.0000	0.3013	0.3013
0.31	0.5576	0.0000	0.3191	0.3191
0.32	0.6096	0.0000	0.3365	0.3365
0.33	0.6637	0.0000	0.3534	0.3534
0.34	0.7200	0.0000	0.3698	0.3698
0.35	0.7787	0.0000	0.3859	0.3859
0.36	0.8397	0.0000	0.4016	0.4016
0.37	0.9033	0.0000	0.4169	0.4169
0.38	0.9695	0.0000	0.4319	0.4319
0.39	1.0385	0.0000	0.4465	0.4465
0.40	1.1104	0.0000	0.4608	0.4608
0.41	1.1854	0.0000	0.4748	0.4748
0.42	1.2636	0.0000	0.4885	0.4885
0.43	1.3452	0.0000	0.5019	0.5019
0.44	1.4304	0.0000	0.5150	0.5150
0.45	1.5193	0.0000	0.5279	0.5279
0.46	1.6123	0.0000	0.5405	0.5405
0.47	1.7094	0.0000	0.5529	0.5529
0.48	1.8110	0.0000	0.5651	0.5651
0.49	1.9173	0.0000	0.5770	0.5770
0.50	2.0286	0.0000	0.5887	0.5887
0.51	2.1452	0.0000	0.6002	0.6002
0.52	2.2674	0.0000	0.6115	0.6115
0.53	2.3957	0.0000	0.6226	0.6226
0.54	2.5303	0.0000	0.6335	0.6335
0.55	2.6717	0.0000	0.6443	0.6443
0.56	2.8205	0.0000	0.6548	0.6548
0.57	2.9770	0.0000	0.6652	0.6652
0.58	3.1418	0.0000	0.6754	0.6754

$\hat{\tau}_3 = 0.000, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.3157	0.0000	0.6855	0.6855
0.60	3.4991	0.0000	0.6954	0.6954
0.61	3.6930	0.0000	0.7051	0.7051
0.62	3.8980	0.0000	0.7147	0.7147
0.63	4.1152	0.0000	0.7242	0.7242
0.64	4.3455	0.0000	0.7335	0.7335
0.65	4.5901	0.0000	0.7426	0.7426
0.66	4.8502	0.0000	0.7517	0.7517
0.67	5.1272	0.0000	0.7606	0.7606
0.68	5.4227	0.0000	0.7694	0.7694
0.69	5.7385	0.0000	0.7781	0.7781
0.70	6.0767	0.0000	0.7866	0.7866
0.71	6.4396	0.0000	0.7951	0.7951
0.72	6.8298	0.0000	0.8034	0.8034
0.73	7.2503	0.0000	0.8116	0.8116
0.74	7.7047	0.0000	0.8197	0.8197
0.75	8.1971	0.0000	0.8277	0.8277
0.76	8.7321	0.0000	0.8356	0.8356
0.77	9.3154	0.0000	0.8434	0.8434
0.78	9.9535	0.0000	0.8511	0.8511
0.79	10.6543	0.0000	0.8587	0.8587
0.80	11.4272	0.0000	0.8663	0.8663
0.81	12.2836	0.0000	0.8737	0.8737
0.82	13.2375	0.0000	0.8810	0.8810
0.83	14.3059	0.0000	0.8883	0.8883
0.84	15.5104	0.0000	0.8955	0.8955
0.85	16.8783	0.0000	0.9025	0.9025
0.86	18.4445	0.0000	0.9096	0.9096
0.87	20.2548	0.0000	0.9165	0.9165
0.88	22.3704	0.0000	0.9233	0.9233
0.89	24.8744	0.0000	0.9301	0.9301
0.90	27.8834	0.0000	0.9368	0.9368
0.91	31.5657	0.0000	0.9434	0.9434
0.92	36.1739	0.0000	0.9500	0.9500
0.93	42.1048	0.0000	0.9565	0.9565
0.94	50.0197	0.0000	0.9629	0.9629
0.95	61.1092	0.0000	0.9692	0.9692
0.96	77.7541	0.0000	0.9755	0.9755
0.97	105.510	0.0000	0.9817	0.9817
0.98	161.044	0.0000	0.9879	0.9879
0.99	327.689	0.0000	0.9940	0.9940

Table E–5: $\hat{\tau}_3 = 0.025$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.025$, $0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0096	0.0014	0.0088	0.0102
0.18	0.0393	0.0052	0.0347	0.0395
0.19	0.0702	0.0085	0.0599	0.0674
0.20	0.1025	0.0114	0.0844	0.0939
0.21	0.1361	0.0141	0.1081	0.1192
0.22	0.1710	0.0166	0.1312	0.1435
0.23	0.2074	0.0189	0.1535	0.1668
0.24	0.2453	0.0211	0.1752	0.1893
0.25	0.2847	0.0232	0.1962	0.2109
0.26	0.3257	0.0252	0.2166	0.2318
0.27	0.3684	0.0271	0.2365	0.2520
0.28	0.4127	0.0290	0.2558	0.2715
0.29	0.4589	0.0309	0.2746	0.2905
0.30	0.5069	0.0328	0.2929	0.3089
0.31	0.5568	0.0346	0.3107	0.3267
0.32	0.6088	0.0365	0.3280	0.3441
0.33	0.6629	0.0383	0.3450	0.3610
0.34	0.7192	0.0402	0.3615	0.3775
0.35	0.7778	0.0421	0.3776	0.3935
0.36	0.8388	0.0440	0.3934	0.4091
0.37	0.9024	0.0459	0.4088	0.4244
0.38	0.9686	0.0479	0.4238	0.4393
0.39	1.0376	0.0499	0.4385	0.4538
0.40	1.1095	0.0519	0.4529	0.4680
0.41	1.1844	0.0540	0.4670	0.4819
0.42	1.2626	0.0562	0.4808	0.4956
0.43	1.3442	0.0584	0.4944	0.5089
0.44	1.4294	0.0607	0.5076	0.5219
0.45	1.5183	0.0631	0.5206	0.5347
0.46	1.6112	0.0655	0.5334	0.5472
0.47	1.7083	0.0681	0.5459	0.5595
0.48	1.8099	0.0707	0.5581	0.5715
0.49	1.9162	0.0734	0.5702	0.5833
0.50	2.0274	0.0762	0.5820	0.5949
0.51	2.1440	0.0792	0.5937	0.6063
0.52	2.2662	0.0822	0.6051	0.6175
0.53	2.3944	0.0854	0.6163	0.6285
0.54	2.5290	0.0888	0.6274	0.6393
0.55	2.6705	0.0923	0.6382	0.6499
0.56	2.8192	0.0959	0.6489	0.6603
0.57	2.9756	0.0998	0.6594	0.6706
0.58	3.1405	0.1038	0.6698	0.6807

$\hat{\tau}_3 = 0.025$, $0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.3143	0.1080	0.6799	0.6906
0.60	3.4977	0.1125	0.6900	0.7004
0.61	3.6916	0.1172	0.6998	0.7100
0.62	3.8966	0.1221	0.7096	0.7195
0.63	4.1137	0.1273	0.7191	0.7288
0.64	4.3440	0.1329	0.7286	0.7380
0.65	4.5885	0.1388	0.7379	0.7471
0.66	4.8486	0.1450	0.7471	0.7560
0.67	5.1255	0.1517	0.7561	0.7648
0.68	5.4210	0.1588	0.7650	0.7735
0.69	5.7368	0.1664	0.7738	0.7821
0.70	6.0750	0.1745	0.7825	0.7905
0.71	6.4378	0.1833	0.7911	0.7988
0.72	6.8280	0.1927	0.7996	0.8070
0.73	7.2485	0.2028	0.8079	0.8151
0.74	7.7029	0.2137	0.8161	0.8231
0.75	8.1952	0.2256	0.8243	0.8310
0.76	8.7302	0.2385	0.8323	0.8388
0.77	9.3134	0.2526	0.8402	0.8464
0.78	9.9515	0.2681	0.8481	0.8540
0.79	10.6522	0.2851	0.8558	0.8615
0.80	11.4251	0.3038	0.8635	0.8689
0.81	12.2815	0.3247	0.8710	0.8762
0.82	13.2352	0.3479	0.8785	0.8834
0.83	14.3036	0.3740	0.8859	0.8906
0.84	15.5081	0.4034	0.8932	0.8976
0.85	16.8759	0.4369	0.9004	0.9046
0.86	18.4420	0.4753	0.9076	0.9114
0.87	20.2523	0.5198	0.9146	0.9182
0.88	22.3678	0.5719	0.9216	0.9250
0.89	24.8717	0.6337	0.9285	0.9316
0.90	27.8807	0.7080	0.9353	0.9382
0.91	31.5629	0.7992	0.9421	0.9447
0.92	36.1710	0.9135	0.9488	0.9511
0.93	42.1018	1.0607	0.9554	0.9574
0.94	50.0167	1.2576	0.9620	0.9637
0.95	61.1061	1.5337	0.9685	0.9699
0.96	77.7510	1.9487	0.9749	0.9761
0.97	105.507	2.6415	0.9813	0.9822
0.98	161.040	4.0286	0.9876	0.9882
0.99	327.685	8.1935	0.9938	0.9941

Table E–5: $\hat{\tau}_3 = 0.05$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.050$, $0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0095	0.0027	0.0080	0.0107
0.18	0.0388	0.0102	0.0319	0.0414
0.19	0.0694	0.0168	0.0555	0.0703
0.20	0.1014	0.0227	0.0788	0.0977
0.21	0.1348	0.0280	0.1016	0.1237
0.22	0.1696	0.0330	0.1239	0.1485
0.23	0.2059	0.0376	0.1457	0.1723
0.24	0.2436	0.0420	0.1670	0.1951
0.25	0.2829	0.0461	0.1877	0.2170
0.26	0.3238	0.0502	0.2079	0.2381
0.27	0.3664	0.0541	0.2275	0.2585
0.28	0.4107	0.0579	0.2467	0.2782
0.29	0.4567	0.0617	0.2654	0.2973
0.30	0.5047	0.0654	0.2837	0.3157
0.31	0.5546	0.0691	0.3015	0.3336
0.32	0.6065	0.0728	0.3189	0.3510
0.33	0.6605	0.0765	0.3358	0.3679
0.34	0.7167	0.0802	0.3524	0.3844
0.35	0.7753	0.0840	0.3686	0.4004
0.36	0.8362	0.0878	0.3844	0.4160
0.37	0.8997	0.0917	0.3999	0.4312
0.38	0.9659	0.0956	0.4151	0.4460
0.39	1.0348	0.0996	0.4299	0.4605
0.40	1.1066	0.1037	0.4444	0.4747
0.41	1.1815	0.1080	0.4586	0.4885
0.42	1.2596	0.1123	0.4726	0.5021
0.43	1.3412	0.1167	0.4862	0.5153
0.44	1.4263	0.1213	0.4996	0.5282
0.45	1.5151	0.1261	0.5127	0.5409
0.46	1.6080	0.1309	0.5256	0.5534
0.47	1.7050	0.1360	0.5383	0.5656
0.48	1.8065	0.1412	0.5507	0.5775
0.49	1.9128	0.1467	0.5629	0.5892
0.50	2.0240	0.1523	0.5749	0.6007
0.51	2.1405	0.1582	0.5866	0.6120
0.52	2.2626	0.1643	0.5982	0.6231
0.53	2.3908	0.1707	0.6096	0.6340
0.54	2.5253	0.1774	0.6208	0.6447
0.55	2.6666	0.1844	0.6318	0.6552
0.56	2.8153	0.1917	0.6426	0.6655
0.57	2.9717	0.1993	0.6532	0.6757
0.58	3.1364	0.2074	0.6637	0.6856

$\hat{\tau}_3 = 0.050$, $0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.3102	0.2158	0.6740	0.6955
0.60	3.4935	0.2247	0.6842	0.7051
0.61	3.6873	0.2341	0.6942	0.7147
0.62	3.8922	0.2440	0.7041	0.7240
0.63	4.1093	0.2545	0.7138	0.7332
0.64	4.3394	0.2656	0.7234	0.7423
0.65	4.5839	0.2774	0.7329	0.7513
0.66	4.8438	0.2899	0.7422	0.7601
0.67	5.1207	0.3032	0.7514	0.7688
0.68	5.4161	0.3174	0.7604	0.7774
0.69	5.7318	0.3326	0.7694	0.7858
0.70	6.0698	0.3489	0.7782	0.7941
0.71	6.4326	0.3663	0.7869	0.8023
0.72	6.8226	0.3851	0.7955	0.8104
0.73	7.2430	0.4053	0.8040	0.8184
0.74	7.6972	0.4272	0.8123	0.8263
0.75	8.1894	0.4510	0.8206	0.8340
0.76	8.7243	0.4768	0.8288	0.8417
0.77	9.3074	0.5050	0.8369	0.8493
0.78	9.9453	0.5359	0.8448	0.8568
0.79	10.6460	0.5699	0.8527	0.8641
0.80	11.4187	0.6074	0.8605	0.8714
0.81	12.2749	0.6491	0.8682	0.8786
0.82	13.2285	0.6955	0.8758	0.8857
0.83	14.2967	0.7477	0.8834	0.8927
0.84	15.5010	0.8065	0.8908	0.8996
0.85	16.8687	0.8735	0.8982	0.9065
0.86	18.4346	0.9504	0.9054	0.9132
0.87	20.2448	1.0393	0.9126	0.9199
0.88	22.3601	1.1435	0.9198	0.9265
0.89	24.8638	1.2670	0.9268	0.9330
0.90	27.8725	1.4157	0.9338	0.9395
0.91	31.5546	1.5980	0.9407	0.9459
0.92	36.1625	1.8265	0.9476	0.9522
0.93	42.0930	2.1211	0.9543	0.9584
0.94	50.0076	2.5147	0.9611	0.9645
0.95	61.0968	3.0670	0.9677	0.9706
0.96	77.7414	3.8970	0.9743	0.9766
0.97	105.497	5.2825	0.9808	0.9826
0.98	161.030	8.0567	0.9873	0.9884
0.99	327.675	16.3864	0.9937	0.9943

Table E–5: $\hat{\tau}_3 = 0.075$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.075,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0092	0.0040	0.0071	0.0111
0.18	0.0379	0.0150	0.0288	0.0428
0.19	0.0681	0.0248	0.0508	0.0726
0.20	0.0997	0.0336	0.0727	0.1007
0.21	0.1328	0.0416	0.0945	0.1274
0.22	0.1673	0.0490	0.1160	0.1527
0.23	0.2033	0.0559	0.1372	0.1769
0.24	0.2408	0.0625	0.1580	0.2001
0.25	0.2800	0.0688	0.1783	0.2223
0.26	0.3207	0.0748	0.1983	0.2437
0.27	0.3631	0.0807	0.2178	0.2642
0.28	0.4072	0.0865	0.2368	0.2841
0.29	0.4532	0.0921	0.2555	0.3032
0.30	0.5010	0.0977	0.2737	0.3218
0.31	0.5508	0.1033	0.2915	0.3398
0.32	0.6026	0.1088	0.3089	0.3572
0.33	0.6565	0.1144	0.3259	0.3741
0.34	0.7126	0.1200	0.3425	0.3906
0.35	0.7710	0.1256	0.3588	0.4066
0.36	0.8319	0.1314	0.3747	0.4222
0.37	0.8953	0.1372	0.3903	0.4374
0.38	0.9613	0.1431	0.4056	0.4522
0.39	1.0301	0.1491	0.4206	0.4666
0.40	1.1019	0.1553	0.4352	0.4808
0.41	1.1767	0.1616	0.4496	0.4945
0.42	1.2547	0.1681	0.4636	0.5080
0.43	1.3361	0.1748	0.4774	0.5212
0.44	1.4211	0.1817	0.4910	0.5341
0.45	1.5099	0.1888	0.5043	0.5467
0.46	1.6026	0.1961	0.5173	0.5590
0.47	1.6996	0.2037	0.5301	0.5711
0.48	1.8010	0.2115	0.5427	0.5830
0.49	1.9071	0.2197	0.5550	0.5946
0.50	2.0182	0.2282	0.5672	0.6061
0.51	2.1346	0.2370	0.5791	0.6173
0.52	2.2566	0.2461	0.5908	0.6283
0.53	2.3846	0.2557	0.6023	0.6390
0.54	2.5190	0.2657	0.6137	0.6496
0.55	2.6602	0.2762	0.6248	0.6601
0.56	2.8087	0.2871	0.6358	0.6703
0.57	2.9650	0.2986	0.6466	0.6804
0.58	3.1297	0.3107	0.6573	0.6903

$\hat{\tau}_3 = 0.075,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.3032	0.3234	0.6677	0.7000
0.60	3.4865	0.3367	0.6781	0.7096
0.61	3.6801	0.3508	0.6882	0.7190
0.62	3.8849	0.3656	0.6983	0.7282
0.63	4.1018	0.3813	0.7081	0.7374
0.64	4.3318	0.3980	0.7179	0.7464
0.65	4.5761	0.4156	0.7275	0.7552
0.66	4.8359	0.4344	0.7369	0.7639
0.67	5.1126	0.4543	0.7463	0.7725
0.68	5.4078	0.4756	0.7555	0.7810
0.69	5.7233	0.4984	0.7646	0.7893
0.70	6.0612	0.5228	0.7736	0.7975
0.71	6.4238	0.5489	0.7824	0.8056
0.72	6.8136	0.5771	0.7912	0.8136
0.73	7.2338	0.6074	0.7998	0.8215
0.74	7.6878	0.6403	0.8083	0.8293
0.75	8.1798	0.6759	0.8167	0.8369
0.76	8.7145	0.7146	0.8251	0.8445
0.77	9.2974	0.7569	0.8333	0.8520
0.78	9.9351	0.8032	0.8414	0.8593
0.79	10.6355	0.8542	0.8494	0.8666
0.80	11.4080	0.9104	0.8574	0.8738
0.81	12.2639	0.9729	0.8652	0.8809
0.82	13.2173	1.0425	0.8730	0.8878
0.83	14.2852	1.1207	0.8807	0.8947
0.84	15.4893	1.2090	0.8883	0.9016
0.85	16.8566	1.3095	0.8958	0.9083
0.86	18.4223	1.4247	0.9032	0.9149
0.87	20.2321	1.5582	0.9106	0.9215
0.88	22.3471	1.7144	0.9179	0.9280
0.89	24.8505	1.8996	0.9251	0.9344
0.90	27.8589	2.1226	0.9322	0.9407
0.91	31.5406	2.3960	0.9393	0.9470
0.92	36.1481	2.7388	0.9463	0.9532
0.93	42.0783	3.1806	0.9532	0.9593
0.94	49.9925	3.7710	0.9601	0.9653
0.95	61.0813	4.5995	0.9669	0.9712
0.96	77.7255	5.8444	0.9736	0.9771
0.97	105.481	7.9225	0.9803	0.9830
0.98	161.014	12.0838	0.9869	0.9887
0.99	327.657	24.5783	0.9935	0.9944

Table E–5: $\hat{\tau}_3 = 0.1$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.100, \quad 0.17 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0089	0.0052	0.0062	0.0113
0.18	0.0368	0.0195	0.0256	0.0438
0.19	0.0662	0.0323	0.0458	0.0743
0.20	0.0973	0.0440	0.0663	0.1031
0.21	0.1299	0.0546	0.0869	0.1303
0.22	0.1640	0.0645	0.1076	0.1561
0.23	0.1997	0.0738	0.1281	0.1807
0.24	0.2370	0.0825	0.1483	0.2042
0.25	0.2758	0.0909	0.1683	0.2267
0.26	0.3163	0.0990	0.1879	0.2483
0.27	0.3585	0.1069	0.2072	0.2691
0.28	0.4024	0.1145	0.2261	0.2891
0.29	0.4482	0.1221	0.2447	0.3084
0.30	0.4958	0.1296	0.2628	0.3271
0.31	0.5454	0.1370	0.2806	0.3452
0.32	0.5971	0.1444	0.2981	0.3627
0.33	0.6508	0.1518	0.3151	0.3797
0.34	0.7068	0.1593	0.3319	0.3961
0.35	0.7651	0.1669	0.3482	0.4122
0.36	0.8258	0.1745	0.3643	0.4278
0.37	0.8890	0.1823	0.3800	0.4430
0.38	0.9549	0.1902	0.3954	0.4577
0.39	1.0236	0.1982	0.4105	0.4722
0.40	1.0952	0.2064	0.4253	0.4862
0.41	1.1699	0.2149	0.4398	0.5000
0.42	1.2477	0.2235	0.4540	0.5134
0.43	1.3290	0.2324	0.4680	0.5265
0.44	1.4139	0.2416	0.4817	0.5394
0.45	1.5025	0.2510	0.4952	0.5519
0.46	1.5951	0.2608	0.5084	0.5642
0.47	1.6919	0.2709	0.5213	0.5763
0.48	1.7931	0.2814	0.5341	0.5881
0.49	1.8991	0.2922	0.5466	0.5996
0.50	2.0100	0.3035	0.5589	0.6110
0.51	2.1262	0.3153	0.5710	0.6221
0.52	2.2481	0.3275	0.5829	0.6330
0.53	2.3760	0.3403	0.5946	0.6437
0.54	2.5102	0.3536	0.6061	0.6543
0.55	2.6513	0.3675	0.6174	0.6646
0.56	2.7996	0.3821	0.6286	0.6748
0.57	2.9557	0.3974	0.6396	0.6847
0.58	3.1201	0.4135	0.6504	0.6945

$\hat{\tau}_3 = 0.100, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2935	0.4304	0.6610	0.7042
0.60	3.4766	0.4481	0.6715	0.7137
0.61	3.6700	0.4669	0.6819	0.7230
0.62	3.8746	0.4867	0.6920	0.7322
0.63	4.0913	0.5076	0.7021	0.7412
0.64	4.3211	0.5298	0.7120	0.7501
0.65	4.5651	0.5533	0.7218	0.7589
0.66	4.8247	0.5783	0.7314	0.7675
0.67	5.1012	0.6049	0.7409	0.7760
0.68	5.3962	0.6332	0.7503	0.7844
0.69	5.7114	0.6636	0.7595	0.7926
0.70	6.0491	0.6960	0.7687	0.8007
0.71	6.4114	0.7309	0.7777	0.8087
0.72	6.8009	0.7684	0.7866	0.8166
0.73	7.2209	0.8089	0.7954	0.8244
0.74	7.6746	0.8526	0.8041	0.8321
0.75	8.1663	0.9000	0.8126	0.8397
0.76	8.7007	0.9517	0.8211	0.8471
0.77	9.2833	1.0080	0.8295	0.8545
0.78	9.9207	1.0697	0.8378	0.8618
0.79	10.6207	1.1376	0.8460	0.8689
0.80	11.3929	1.2126	0.8541	0.8760
0.81	12.2485	1.2958	0.8621	0.8830
0.82	13.2015	1.3887	0.8700	0.8899
0.83	14.2690	1.4929	0.8778	0.8967
0.84	15.4727	1.6106	0.8856	0.9034
0.85	16.8396	1.7445	0.8933	0.9100
0.86	18.4049	1.8981	0.9009	0.9166
0.87	20.2143	2.0759	0.9084	0.9230
0.88	22.3289	2.2842	0.9158	0.9294
0.89	24.8318	2.5311	0.9232	0.9357
0.90	27.8397	2.8284	0.9305	0.9419
0.91	31.5209	3.1929	0.9377	0.9480
0.92	36.1279	3.6498	0.9449	0.9541
0.93	42.0576	4.2388	0.9520	0.9601
0.94	49.9712	5.0260	0.9590	0.9660
0.95	61.0594	6.1305	0.9660	0.9719
0.96	77.7030	7.7904	0.9729	0.9776
0.97	105.458	10.5611	0.9798	0.9833
0.98	160.990	16.1094	0.9866	0.9890
0.99	327.633	32.7686	0.9933	0.9945

Table E–5: $\hat{\tau}_3 = 0.125$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.125,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0085	0.0062	0.0054	0.0115
0.18	0.0354	0.0236	0.0224	0.0444
0.19	0.0640	0.0393	0.0406	0.0755
0.20	0.0943	0.0537	0.0595	0.1047
0.21	0.1263	0.0669	0.0789	0.1324
0.22	0.1600	0.0792	0.0986	0.1587
0.23	0.1952	0.0908	0.1183	0.1837
0.24	0.2320	0.1018	0.1380	0.2075
0.25	0.2705	0.1124	0.1575	0.2303
0.26	0.3107	0.1225	0.1768	0.2522
0.27	0.3526	0.1324	0.1958	0.2732
0.28	0.3963	0.1420	0.2146	0.2934
0.29	0.4418	0.1515	0.2330	0.3129
0.30	0.4892	0.1608	0.2511	0.3316
0.31	0.5386	0.1702	0.2689	0.3498
0.32	0.5900	0.1794	0.2864	0.3674
0.33	0.6436	0.1887	0.3036	0.3845
0.34	0.6993	0.1981	0.3204	0.4010
0.35	0.7574	0.2076	0.3368	0.4171
0.36	0.8179	0.2171	0.3530	0.4327
0.37	0.8810	0.2268	0.3689	0.4479
0.38	0.9467	0.2367	0.3844	0.4627
0.39	1.0152	0.2467	0.3997	0.4771
0.40	1.0866	0.2570	0.4146	0.4912
0.41	1.1611	0.2676	0.4293	0.5049
0.42	1.2388	0.2784	0.4437	0.5183
0.43	1.3199	0.2895	0.4579	0.5314
0.44	1.4045	0.3010	0.4717	0.5442
0.45	1.4929	0.3128	0.4854	0.5567
0.46	1.5853	0.3250	0.4988	0.5690
0.47	1.6819	0.3376	0.5119	0.5810
0.48	1.7830	0.3507	0.5249	0.5927
0.49	1.8887	0.3642	0.5376	0.6042
0.50	1.9995	0.3783	0.5501	0.6155
0.51	2.1155	0.3930	0.5624	0.6266
0.52	2.2372	0.4083	0.5744	0.6374
0.53	2.3648	0.4242	0.5863	0.6481
0.54	2.4988	0.4408	0.5980	0.6585
0.55	2.6396	0.4582	0.6095	0.6688
0.56	2.7877	0.4765	0.6209	0.6789
0.57	2.9436	0.4956	0.6320	0.6888
0.58	3.1078	0.5156	0.6430	0.6985

$\hat{\tau}_3 = 0.125,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2810	0.5367	0.6539	0.7081
0.60	3.4638	0.5589	0.6645	0.7175
0.61	3.6569	0.5823	0.6751	0.7268
0.62	3.8612	0.6070	0.6854	0.7359
0.63	4.0777	0.6331	0.6956	0.7448
0.64	4.3072	0.6608	0.7057	0.7536
0.65	4.5510	0.6902	0.7157	0.7623
0.66	4.8103	0.7214	0.7255	0.7709
0.67	5.0864	0.7546	0.7351	0.7793
0.68	5.3811	0.7900	0.7447	0.7875
0.69	5.6961	0.8279	0.7541	0.7957
0.70	6.0334	0.8685	0.7634	0.8038
0.71	6.3953	0.9120	0.7726	0.8117
0.72	6.7846	0.9588	0.7817	0.8195
0.73	7.2042	1.0094	0.7907	0.8272
0.74	7.6576	1.0640	0.7995	0.8348
0.75	8.1489	1.1232	0.8083	0.8422
0.76	8.6828	1.1877	0.8169	0.8496
0.77	9.2650	1.2581	0.8255	0.8569
0.78	9.9020	1.3352	0.8339	0.8641
0.79	10.6017	1.4200	0.8423	0.8711
0.80	11.3734	1.5137	0.8505	0.8781
0.81	12.2285	1.6177	0.8587	0.8850
0.82	13.1810	1.7337	0.8668	0.8918
0.83	14.2481	1.8638	0.8748	0.8985
0.84	15.4512	2.0109	0.8827	0.9051
0.85	16.8177	2.1782	0.8906	0.9116
0.86	18.3824	2.3701	0.8984	0.9181
0.87	20.1912	2.5923	0.9060	0.9244
0.88	22.3052	2.8526	0.9137	0.9307
0.89	24.8075	3.1612	0.9212	0.9369
0.90	27.8148	3.5327	0.9287	0.9430
0.91	31.4954	3.9882	0.9361	0.9491
0.92	36.1017	4.5592	0.9434	0.9550
0.93	42.0306	5.2953	0.9507	0.9609
0.94	49.9436	6.2793	0.9579	0.9667
0.95	61.0310	7.6598	0.9651	0.9724
0.96	77.6738	9.7344	0.9722	0.9781
0.97	105.427	13.1978	0.9792	0.9837
0.98	160.959	20.1330	0.9862	0.9892
0.99	327.601	40.9569	0.9931	0.9946

Table E–5: $\hat{\tau}_3 = 0.15$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.150, \quad 0.17 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0081	0.0071	0.0045	0.0115
0.18	0.0338	0.0271	0.0192	0.0446
0.19	0.0614	0.0456	0.0354	0.0760
0.20	0.0909	0.0626	0.0526	0.1056
0.21	0.1221	0.0784	0.0706	0.1337
0.22	0.1551	0.0931	0.0892	0.1604
0.23	0.1897	0.1070	0.1080	0.1857
0.24	0.2261	0.1203	0.1270	0.2099
0.25	0.2641	0.1330	0.1460	0.2330
0.26	0.3039	0.1452	0.1649	0.2552
0.27	0.3455	0.1571	0.1836	0.2764
0.28	0.3888	0.1687	0.2022	0.2968
0.29	0.4340	0.1801	0.2205	0.3165
0.30	0.4811	0.1914	0.2386	0.3354
0.31	0.5302	0.2026	0.2564	0.3537
0.32	0.5814	0.2137	0.2739	0.3714
0.33	0.6346	0.2249	0.2911	0.3886
0.34	0.6902	0.2362	0.3080	0.4052
0.35	0.7480	0.2475	0.3246	0.4213
0.36	0.8083	0.2590	0.3409	0.4370
0.37	0.8711	0.2707	0.3569	0.4522
0.38	0.9366	0.2825	0.3726	0.4671
0.39	1.0049	0.2946	0.3880	0.4815
0.40	1.0760	0.3070	0.4032	0.4956
0.41	1.1503	0.3196	0.4180	0.5093
0.42	1.2277	0.3326	0.4326	0.5227
0.43	1.3086	0.3459	0.4470	0.5358
0.44	1.3930	0.3596	0.4610	0.5486
0.45	1.4812	0.3738	0.4749	0.5610
0.46	1.5734	0.3884	0.4885	0.5733
0.47	1.6697	0.4036	0.5018	0.5852
0.48	1.7705	0.4193	0.5150	0.5969
0.49	1.8760	0.4355	0.5279	0.6084
0.50	1.9865	0.4524	0.5406	0.6196
0.51	2.1023	0.4700	0.5531	0.6307
0.52	2.2237	0.4883	0.5654	0.6415
0.53	2.3511	0.5074	0.5775	0.6521
0.54	2.4848	0.5273	0.5894	0.6625
0.55	2.6254	0.5482	0.6011	0.6727
0.56	2.7732	0.5700	0.6126	0.6827
0.57	2.9288	0.5929	0.6240	0.6925
0.58	3.0927	0.6169	0.6352	0.7022

$\hat{\tau}_3 = 0.150, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2655	0.6422	0.6462	0.7117
0.60	3.4480	0.6688	0.6571	0.7211
0.61	3.6408	0.6968	0.6678	0.7302
0.62	3.8449	0.7265	0.6784	0.7393
0.63	4.0610	0.7578	0.6888	0.7482
0.64	4.2902	0.7909	0.6990	0.7569
0.65	4.5336	0.8261	0.7092	0.7655
0.66	4.7925	0.8635	0.7192	0.7740
0.67	5.0683	0.9033	0.7290	0.7823
0.68	5.3626	0.9458	0.7388	0.7905
0.69	5.6772	0.9912	0.7484	0.7986
0.70	6.0141	1.0398	0.7579	0.8066
0.71	6.3756	1.0920	0.7673	0.8144
0.72	6.7644	1.1481	0.7765	0.8221
0.73	7.1836	1.2087	0.7857	0.8298
0.74	7.6365	1.2742	0.7947	0.8373
0.75	8.1274	1.3452	0.8036	0.8447
0.76	8.6608	1.4225	0.8125	0.8520
0.77	9.2425	1.5069	0.8212	0.8591
0.78	9.8790	1.5993	0.8298	0.8662
0.79	10.5781	1.7010	0.8384	0.8732
0.80	11.3493	1.8133	0.8468	0.8801
0.81	12.2039	1.9380	0.8552	0.8869
0.82	13.1558	2.0771	0.8634	0.8936
0.83	14.2223	2.2332	0.8716	0.9002
0.84	15.4248	2.4096	0.8797	0.9068
0.85	16.7906	2.6102	0.8878	0.9132
0.86	18.3546	2.8404	0.8957	0.9195
0.87	20.1628	3.1070	0.9036	0.9258
0.88	22.2760	3.4191	0.9114	0.9320
0.89	24.7776	3.7893	0.9191	0.9381
0.90	27.7841	4.2350	0.9268	0.9441
0.91	31.4638	4.7815	0.9343	0.9500
0.92	36.0693	5.4665	0.9419	0.9559
0.93	41.9974	6.3497	0.9493	0.9617
0.94	49.9094	7.5303	0.9567	0.9674
0.95	60.9959	9.1867	0.9641	0.9730
0.96	77.6377	11.6761	0.9714	0.9785
0.97	105.390	15.8319	0.9786	0.9840
0.98	160.921	24.1540	0.9858	0.9894
0.99	327.562	49.1424	0.9929	0.9947

Table E–5: $\hat{\tau}_3 = 0.175$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.175,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0076	0.0078	0.0037	0.0114
0.18	0.0320	0.0302	0.0161	0.0445
0.19	0.0585	0.0511	0.0302	0.0759
0.20	0.0869	0.0705	0.0456	0.1058
0.21	0.1173	0.0888	0.0621	0.1342
0.22	0.1495	0.1059	0.0794	0.1612
0.23	0.1834	0.1222	0.0973	0.1869
0.24	0.2192	0.1376	0.1154	0.2114
0.25	0.2567	0.1525	0.1338	0.2349
0.26	0.2960	0.1668	0.1522	0.2573
0.27	0.3370	0.1807	0.1707	0.2788
0.28	0.3800	0.1943	0.1890	0.2994
0.29	0.4248	0.2077	0.2072	0.3193
0.30	0.4715	0.2209	0.2251	0.3384
0.31	0.5203	0.2341	0.2429	0.3569
0.32	0.5711	0.2471	0.2604	0.3747
0.33	0.6241	0.2602	0.2777	0.3920
0.34	0.6793	0.2734	0.2947	0.4087
0.35	0.7369	0.2867	0.3114	0.4249
0.36	0.7969	0.3001	0.3278	0.4407
0.37	0.8594	0.3137	0.3440	0.4559
0.38	0.9246	0.3275	0.3599	0.4708
0.39	0.9926	0.3416	0.3755	0.4853
0.40	1.0635	0.3561	0.3908	0.4994
0.41	1.1375	0.3708	0.4059	0.5131
0.42	1.2146	0.3859	0.4207	0.5266
0.43	1.2952	0.4015	0.4353	0.5396
0.44	1.3794	0.4175	0.4496	0.5524
0.45	1.4673	0.4340	0.4636	0.5649
0.46	1.5592	0.4511	0.4774	0.5771
0.47	1.6552	0.4687	0.4910	0.5891
0.48	1.7557	0.4870	0.5044	0.6007
0.49	1.8610	0.5059	0.5175	0.6122
0.50	1.9711	0.5256	0.5304	0.6234
0.51	2.0866	0.5461	0.5432	0.6344
0.52	2.2077	0.5674	0.5557	0.6451
0.53	2.3348	0.5896	0.5680	0.6557
0.54	2.4682	0.6129	0.5801	0.6660
0.55	2.6084	0.6372	0.5920	0.6762
0.56	2.7559	0.6626	0.6038	0.6862
0.57	2.9111	0.6892	0.6154	0.6960
0.58	3.0747	0.7172	0.6268	0.7056

$\hat{\tau}_3 = 0.175,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2472	0.7467	0.6380	0.7150
0.60	3.4293	0.7777	0.6491	0.7243
0.61	3.6217	0.8103	0.6600	0.7335
0.62	3.8253	0.8448	0.6708	0.7424
0.63	4.0410	0.8813	0.6814	0.7513
0.64	4.2698	0.9199	0.6919	0.7599
0.65	4.5128	0.9609	0.7023	0.7685
0.66	4.7713	1.0045	0.7125	0.7769
0.67	5.0467	1.0508	0.7225	0.7852
0.68	5.3405	1.1003	0.7325	0.7933
0.69	5.6547	1.1532	0.7423	0.8013
0.70	5.9911	1.2098	0.7520	0.8092
0.71	6.3521	1.2706	0.7615	0.8170
0.72	6.7404	1.3360	0.7710	0.8246
0.73	7.1590	1.4066	0.7803	0.8322
0.74	7.6114	1.4829	0.7896	0.8396
0.75	8.1017	1.5657	0.7987	0.8469
0.76	8.6346	1.6558	0.8077	0.8542
0.77	9.2157	1.7541	0.8166	0.8613
0.78	9.8516	1.8618	0.8255	0.8683
0.79	10.5501	1.9803	0.8342	0.8752
0.80	11.3206	2.1113	0.8428	0.8820
0.81	12.1745	2.2566	0.8514	0.8887
0.82	13.1257	2.4187	0.8599	0.8953
0.83	14.1914	2.6007	0.8682	0.9019
0.84	15.3932	2.8063	0.8765	0.9083
0.85	16.7582	3.0402	0.8848	0.9147
0.86	18.3215	3.3086	0.8929	0.9209
0.87	20.1288	3.6195	0.9010	0.9271
0.88	22.2411	3.9835	0.9089	0.9332
0.89	24.7418	4.4152	0.9169	0.9392
0.90	27.7474	4.9350	0.9247	0.9451
0.91	31.4261	5.5723	0.9325	0.9510
0.92	36.0306	6.3713	0.9402	0.9567
0.93	41.9576	7.4015	0.9479	0.9624
0.94	49.8685	8.7785	0.9555	0.9680
0.95	60.9538	10.7108	0.9630	0.9735
0.96	77.5945	13.6148	0.9705	0.9790
0.97	105.346	18.4630	0.9780	0.9843
0.98	160.875	28.1718	0.9854	0.9896
0.99	327.515	57.3246	0.9927	0.9949

Table E–5: $\hat{\tau}_3 = 0.2$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.200, \quad 0.17 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0072	0.0084	0.0029	0.0112
0.18	0.0302	0.0327	0.0131	0.0440
0.19	0.0554	0.0558	0.0251	0.0754
0.20	0.0827	0.0775	0.0387	0.1054
0.21	0.1120	0.0981	0.0536	0.1340
0.22	0.1432	0.1175	0.0695	0.1612
0.23	0.1764	0.1361	0.0862	0.1872
0.24	0.2114	0.1538	0.1034	0.2121
0.25	0.2482	0.1708	0.1211	0.2358
0.26	0.2869	0.1872	0.1389	0.2585
0.27	0.3274	0.2032	0.1569	0.2803
0.28	0.3698	0.2189	0.1749	0.3011
0.29	0.4142	0.2342	0.1929	0.3212
0.30	0.4605	0.2494	0.2108	0.3406
0.31	0.5088	0.2645	0.2285	0.3592
0.32	0.5593	0.2795	0.2460	0.3772
0.33	0.6119	0.2945	0.2633	0.3946
0.34	0.6668	0.3096	0.2804	0.4115
0.35	0.7240	0.3248	0.2972	0.4278
0.36	0.7836	0.3402	0.3138	0.4437
0.37	0.8458	0.3557	0.3301	0.4590
0.38	0.9107	0.3716	0.3462	0.4740
0.39	0.9783	0.3877	0.3620	0.4885
0.40	1.0489	0.4042	0.3776	0.5027
0.41	1.1226	0.4210	0.3929	0.5165
0.42	1.1994	0.4383	0.4079	0.5299
0.43	1.2797	0.4561	0.4227	0.5430
0.44	1.3635	0.4744	0.4372	0.5558
0.45	1.4511	0.4932	0.4515	0.5683
0.46	1.5426	0.5127	0.4656	0.5805
0.47	1.6384	0.5328	0.4794	0.5924
0.48	1.7385	0.5537	0.4930	0.6041
0.49	1.8434	0.5753	0.5064	0.6156
0.50	1.9532	0.5977	0.5196	0.6267
0.51	2.0684	0.6211	0.5325	0.6377
0.52	2.1891	0.6454	0.5453	0.6485
0.53	2.3158	0.6708	0.5578	0.6590
0.54	2.4488	0.6973	0.5702	0.6693
0.55	2.5886	0.7250	0.5824	0.6794
0.56	2.7357	0.7540	0.5944	0.6894
0.57	2.8906	0.7844	0.6062	0.6991
0.58	3.0537	0.8164	0.6178	0.7087

$\hat{\tau}_3 = 0.200, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2258	0.8499	0.6293	0.7181
0.60	3.4075	0.8853	0.6406	0.7274
0.61	3.5995	0.9225	0.6518	0.7364
0.62	3.8026	0.9619	0.6628	0.7454
0.63	4.0179	1.0035	0.6736	0.7541
0.64	4.2462	1.0476	0.6843	0.7628
0.65	4.4887	1.0943	0.6949	0.7712
0.66	4.7467	1.1440	0.7053	0.7796
0.67	5.0215	1.1969	0.7156	0.7878
0.68	5.3148	1.2534	0.7257	0.7959
0.69	5.6284	1.3137	0.7358	0.8038
0.70	5.9643	1.3783	0.7457	0.8117
0.71	6.3247	1.4477	0.7554	0.8194
0.72	6.7124	1.5223	0.7651	0.8270
0.73	7.1304	1.6028	0.7747	0.8345
0.74	7.5822	1.6899	0.7841	0.8418
0.75	8.0718	1.7844	0.7934	0.8491
0.76	8.6040	1.8872	0.8027	0.8562
0.77	9.1844	1.9994	0.8118	0.8633
0.78	9.8196	2.1224	0.8208	0.8702
0.79	10.5173	2.2577	0.8298	0.8770
0.80	11.2870	2.4071	0.8386	0.8838
0.81	12.1401	2.5731	0.8474	0.8904
0.82	13.0905	2.7582	0.8561	0.8970
0.83	14.1554	2.9660	0.8646	0.9034
0.84	15.3563	3.2007	0.8731	0.9098
0.85	16.7204	3.4679	0.8816	0.9160
0.86	18.2827	3.7744	0.8899	0.9222
0.87	20.0890	4.1294	0.8982	0.9283
0.88	22.2003	4.5452	0.9064	0.9343
0.89	24.6999	5.0383	0.9145	0.9402
0.90	27.7043	5.6321	0.9226	0.9461
0.91	31.3819	6.3601	0.9306	0.9518
0.92	35.9852	7.2730	0.9385	0.9575
0.93	41.9110	8.4501	0.9464	0.9631
0.94	49.8206	10.0235	0.9542	0.9686
0.95	60.9045	12.2314	0.9619	0.9740
0.96	77.5437	15.5500	0.9696	0.9794
0.97	105.2947	21.0904	0.9773	0.9847
0.98	160.821	32.1858	0.9849	0.9898
0.99	327.459	65.5029	0.9925	0.9950

Table E–5: $\hat{\tau}_3 = 0.225$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.225,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0067	0.0089	0.0023	0.0110
0.18	0.0284	0.0348	0.0103	0.0433
0.19	0.0522	0.0597	0.0203	0.0744
0.20	0.0782	0.0834	0.0320	0.1043
0.21	0.1063	0.1061	0.0451	0.1330
0.22	0.1365	0.1278	0.0595	0.1604
0.23	0.1686	0.1486	0.0749	0.1867
0.24	0.2027	0.1685	0.0911	0.2118
0.25	0.2388	0.1877	0.1078	0.2358
0.26	0.2767	0.2063	0.1250	0.2588
0.27	0.3166	0.2244	0.1425	0.2808
0.28	0.3584	0.2421	0.1601	0.3020
0.29	0.4022	0.2594	0.1778	0.3224
0.30	0.4480	0.2766	0.1955	0.3419
0.31	0.4959	0.2936	0.2131	0.3608
0.32	0.5459	0.3106	0.2306	0.3790
0.33	0.5980	0.3276	0.2479	0.3966
0.34	0.6525	0.3446	0.2651	0.4136
0.35	0.7093	0.3617	0.2820	0.4301
0.36	0.7685	0.3790	0.2988	0.4460
0.37	0.8303	0.3966	0.3153	0.4615
0.38	0.8948	0.4144	0.3315	0.4765
0.39	0.9621	0.4326	0.3475	0.4912
0.40	1.0323	0.4511	0.3633	0.5054
0.41	1.1056	0.4701	0.3788	0.5192
0.42	1.1821	0.4895	0.3941	0.5327
0.43	1.2619	0.5095	0.4092	0.5459
0.44	1.3454	0.5301	0.4240	0.5587
0.45	1.4326	0.5513	0.4385	0.5712
0.46	1.5238	0.5731	0.4528	0.5835
0.47	1.6191	0.5957	0.4669	0.5954
0.48	1.7189	0.6192	0.4808	0.6071
0.49	1.8233	0.6434	0.4944	0.6185
0.50	1.9328	0.6687	0.5079	0.6297
0.51	2.0475	0.6949	0.5211	0.6407
0.52	2.1678	0.7222	0.5341	0.6514
0.53	2.2940	0.7507	0.5469	0.6619
0.54	2.4267	0.7805	0.5596	0.6722
0.55	2.5660	0.8116	0.5720	0.6823
0.56	2.7127	0.8441	0.5842	0.6923
0.57	2.8670	0.8783	0.5963	0.7020
0.58	3.0297	0.9141	0.6082	0.7115

$\hat{\tau}_3 = 0.225,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.2013	0.9518	0.6199	0.7209
0.60	3.3825	0.9915	0.6315	0.7301
0.61	3.5740	1.0333	0.6429	0.7391
0.62	3.7766	1.0774	0.6541	0.7480
0.63	3.9913	1.1242	0.6652	0.7568
0.64	4.2191	1.1736	0.6762	0.7654
0.65	4.4610	1.2261	0.6870	0.7738
0.66	4.7184	1.2819	0.6976	0.7821
0.67	4.9927	1.3413	0.7081	0.7903
0.68	5.2854	1.4047	0.7185	0.7983
0.69	5.5983	1.4724	0.7288	0.8062
0.70	5.9335	1.5450	0.7389	0.8140
0.71	6.2933	1.6229	0.7489	0.8216
0.72	6.6803	1.7067	0.7588	0.8292
0.73	7.0976	1.7971	0.7686	0.8366
0.74	7.5486	1.8949	0.7783	0.8439
0.75	8.0375	2.0010	0.7878	0.8511
0.76	8.5689	2.1165	0.7973	0.8582
0.77	9.1485	2.2426	0.8067	0.8651
0.78	9.7828	2.3807	0.8159	0.8720
0.79	10.4797	2.5327	0.8251	0.8788
0.80	11.2485	2.7006	0.8341	0.8854
0.81	12.1007	2.8871	0.8431	0.8920
0.82	13.0501	3.0951	0.8520	0.8985
0.83	14.1140	3.3286	0.8608	0.9049
0.84	15.3138	3.5925	0.8695	0.9112
0.85	16.6768	3.8927	0.8782	0.9174
0.86	18.2380	4.2373	0.8867	0.9235
0.87	20.0432	4.6364	0.8952	0.9295
0.88	22.1533	5.1038	0.9036	0.9354
0.89	24.6517	5.6582	0.9120	0.9412
0.90	27.6548	6.3259	0.9203	0.9470
0.91	31.3310	7.1446	0.9285	0.9527
0.92	35.9329	8.1712	0.9366	0.9582
0.93	41.8572	9.4950	0.9447	0.9638
0.94	49.7653	11.2647	0.9528	0.9692
0.95	60.8476	13.7481	0.9608	0.9745
0.96	77.4852	17.4811	0.9687	0.9798
0.97	105.233	23.7135	0.9766	0.9849
0.98	160.759	36.1953	0.9844	0.9900
0.99	327.395	73.6765	0.9922	0.9951

Table E–5: $\hat{\tau}_3 = 0.25$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.250, \quad 0.17 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0063	0.0092	0.0016	0.0107
0.18	0.0265	0.0363	0.0078	0.0423
0.19	0.0489	0.0627	0.0158	0.0730
0.20	0.0736	0.0883	0.0255	0.1027
0.21	0.1004	0.1129	0.0368	0.1313
0.22	0.1293	0.1367	0.0496	0.1588
0.23	0.1603	0.1596	0.0635	0.1853
0.24	0.1934	0.1816	0.0785	0.2106
0.25	0.2285	0.2030	0.0942	0.2349
0.26	0.2656	0.2237	0.1106	0.2582
0.27	0.3047	0.2439	0.1274	0.2805
0.28	0.3458	0.2637	0.1445	0.3020
0.29	0.3889	0.2831	0.1618	0.3226
0.30	0.4341	0.3023	0.1793	0.3424
0.31	0.4814	0.3213	0.1967	0.3615
0.32	0.5308	0.3403	0.2142	0.3800
0.33	0.5825	0.3592	0.2315	0.3977
0.34	0.6365	0.3782	0.2487	0.4149
0.35	0.6928	0.3973	0.2658	0.4316
0.36	0.7516	0.4166	0.2826	0.4477
0.37	0.8129	0.4361	0.2993	0.4633
0.38	0.8770	0.4560	0.3158	0.4785
0.39	0.9438	0.4762	0.3320	0.4932
0.40	1.0136	0.4968	0.3480	0.5075
0.41	1.0864	0.5179	0.3638	0.5214
0.42	1.1625	0.5395	0.3793	0.5350
0.43	1.2419	0.5616	0.3946	0.5482
0.44	1.3249	0.5845	0.4097	0.5611
0.45	1.4117	0.6080	0.4245	0.5737
0.46	1.5024	0.6322	0.4391	0.5860
0.47	1.5973	0.6573	0.4535	0.5979
0.48	1.6967	0.6833	0.4677	0.6097
0.49	1.8007	0.7102	0.4816	0.6211
0.50	1.9097	0.7382	0.4953	0.6323
0.51	2.0239	0.7673	0.5088	0.6433
0.52	2.1437	0.7976	0.5221	0.6540
0.53	2.2695	0.8292	0.5352	0.6645
0.54	2.4016	0.8621	0.5482	0.6748
0.55	2.5405	0.8966	0.5609	0.6849
0.56	2.6866	0.9327	0.5734	0.6948
0.57	2.8405	0.9705	0.5857	0.7045
0.58	3.0026	1.0103	0.5979	0.7141

$\hat{\tau}_3 = 0.250, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.1736	1.0520	0.6099	0.7234
0.60	3.3542	1.0960	0.6217	0.7326
0.61	3.5452	1.1423	0.6334	0.7416
0.62	3.7472	1.1913	0.6449	0.7505
0.63	3.9613	1.2431	0.6563	0.7592
0.64	4.1884	1.2979	0.6675	0.7677
0.65	4.4297	1.3561	0.6785	0.7761
0.66	4.6865	1.4179	0.6894	0.7844
0.67	4.9600	1.4838	0.7002	0.7925
0.68	5.2520	1.5540	0.7108	0.8005
0.69	5.5642	1.6291	0.7214	0.8084
0.70	5.8987	1.7096	0.7317	0.8161
0.71	6.2577	1.7959	0.7420	0.8237
0.72	6.6439	1.8889	0.7521	0.8312
0.73	7.0604	1.9891	0.7622	0.8385
0.74	7.5106	2.0976	0.7721	0.8458
0.75	7.9986	2.2153	0.7819	0.8529
0.76	8.5291	2.3433	0.7916	0.8600
0.77	9.1078	2.4832	0.8012	0.8669
0.78	9.7411	2.6364	0.8106	0.8737
0.79	10.4370	2.8050	0.8200	0.8804
0.80	11.2048	2.9914	0.8293	0.8870
0.81	12.0559	3.1982	0.8385	0.8935
0.82	13.0042	3.4291	0.8477	0.8999
0.83	14.0669	3.6882	0.8567	0.9062
0.84	15.2656	3.9810	0.8657	0.9125
0.85	16.6274	4.3143	0.8745	0.9186
0.86	18.1873	4.6968	0.8833	0.9246
0.87	19.9911	5.1399	0.8920	0.9306
0.88	22.0998	5.6588	0.9007	0.9364
0.89	24.5967	6.2744	0.9093	0.9422
0.90	27.5984	7.0158	0.9178	0.9479
0.91	31.2731	7.9250	0.9263	0.9535
0.92	35.8733	9.0652	0.9347	0.9590
0.93	41.7959	10.5356	0.9430	0.9644
0.94	49.7022	12.5014	0.9513	0.9697
0.95	60.7827	15.2602	0.9595	0.9750
0.96	77.4183	19.4073	0.9677	0.9801
0.97	105.164	26.3316	0.9758	0.9852
0.98	160.688	40.1995	0.9839	0.9902
0.99	327.322	81.8446	0.9920	0.9952

Table E–5: $\hat{\tau}_3 = 0.275$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.275,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0058	0.0094	0.0011	0.0104
0.18	0.0247	0.0374	0.0055	0.0412
0.19	0.0457	0.0651	0.0116	0.0713
0.20	0.0689	0.0921	0.0194	0.1006
0.21	0.0943	0.1185	0.0289	0.1290
0.22	0.1218	0.1441	0.0399	0.1565
0.23	0.1516	0.1690	0.0522	0.1830
0.24	0.1835	0.1931	0.0658	0.2086
0.25	0.2175	0.2166	0.0803	0.2331
0.26	0.2536	0.2394	0.0957	0.2566
0.27	0.2918	0.2617	0.1117	0.2793
0.28	0.3320	0.2836	0.1282	0.3010
0.29	0.3744	0.3051	0.1451	0.3219
0.30	0.4188	0.3263	0.1622	0.3420
0.31	0.4654	0.3473	0.1794	0.3614
0.32	0.5143	0.3683	0.1967	0.3801
0.33	0.5653	0.3892	0.2140	0.3981
0.34	0.6187	0.4102	0.2312	0.4155
0.35	0.6745	0.4313	0.2483	0.4324
0.36	0.7327	0.4526	0.2653	0.4486
0.37	0.7936	0.4741	0.2822	0.4644
0.38	0.8571	0.4960	0.2988	0.4797
0.39	0.9234	0.5182	0.3153	0.4946
0.40	0.9927	0.5409	0.3315	0.5091
0.41	1.0650	0.5641	0.3476	0.5231
0.42	1.1406	0.5879	0.3634	0.5368
0.43	1.2196	0.6123	0.3790	0.5501
0.44	1.3021	0.6373	0.3943	0.5630
0.45	1.3884	0.6632	0.4095	0.5757
0.46	1.4786	0.6898	0.4244	0.5880
0.47	1.5730	0.7174	0.4391	0.6000
0.48	1.6719	0.7459	0.4535	0.6118
0.49	1.7754	0.7755	0.4678	0.6233
0.50	1.8838	0.8062	0.4818	0.6345
0.51	1.9975	0.8381	0.4956	0.6455
0.52	2.1168	0.8713	0.5093	0.6562
0.53	2.2421	0.9060	0.5227	0.6668
0.54	2.3736	0.9421	0.5359	0.6771
0.55	2.5119	0.9799	0.5489	0.6872
0.56	2.6575	1.0195	0.5617	0.6971
0.57	2.8107	1.0610	0.5744	0.7068
0.58	2.9722	1.1046	0.5869	0.7163

$\hat{\tau}_3 = 0.275,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.1427	1.1504	0.5991	0.7257
0.60	3.3226	1.1987	0.6113	0.7348
0.61	3.5129	1.2495	0.6232	0.7438
0.62	3.7143	1.3032	0.6350	0.7527
0.63	3.9277	1.3600	0.6467	0.7613
0.64	4.1541	1.4201	0.6581	0.7699
0.65	4.3947	1.4840	0.6695	0.7782
0.66	4.6506	1.5518	0.6807	0.7865
0.67	4.9234	1.6240	0.6917	0.7946
0.68	5.2146	1.7011	0.7026	0.8025
0.69	5.5260	1.7835	0.7134	0.8103
0.70	5.8596	1.8718	0.7240	0.8180
0.71	6.2178	1.9666	0.7346	0.8256
0.72	6.6031	2.0685	0.7450	0.8330
0.73	7.0186	2.1786	0.7552	0.8404
0.74	7.4679	2.2976	0.7654	0.8476
0.75	7.9549	2.4268	0.7755	0.8547
0.76	8.4844	2.5674	0.7854	0.8616
0.77	9.0620	2.7209	0.7953	0.8685
0.78	9.6942	2.8892	0.8050	0.8753
0.79	10.3890	3.0743	0.8147	0.8819
0.80	11.1556	3.2789	0.8242	0.8885
0.81	12.0055	3.5061	0.8337	0.8949
0.82	12.9526	3.7597	0.8430	0.9013
0.83	14.0140	4.0444	0.8523	0.9075
0.84	15.2113	4.3661	0.8615	0.9137
0.85	16.5716	4.7322	0.8706	0.9198
0.86	18.1301	5.1525	0.8797	0.9257
0.87	19.9324	5.6394	0.8887	0.9316
0.88	22.0395	6.2097	0.8976	0.9374
0.89	24.5348	6.8864	0.9064	0.9431
0.90	27.5347	7.7013	0.9152	0.9487
0.91	31.2076	8.7009	0.9239	0.9542
0.92	35.8059	9.9545	0.9325	0.9596
0.93	41.7266	11.5712	0.9411	0.9650
0.94	49.6308	13.7329	0.9497	0.9702
0.95	60.7092	16.7669	0.9582	0.9754
0.96	77.3425	21.3280	0.9666	0.9805
0.97	105.086	28.9438	0.9750	0.9855
0.98	160.607	44.1978	0.9834	0.9904
0.99	327.239	90.0064	0.9917	0.9953

Table E–5: $\hat{\tau}_3 = 0.3$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.300, \quad 0.17 \leq \hat{\tau}_4 \leq 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0054	0.0095	0.0006	0.0101
0.18	0.0229	0.0382	0.0034	0.0400
0.19	0.0425	0.0667	0.0078	0.0694
0.20	0.0642	0.0949	0.0138	0.0982
0.21	0.0881	0.1227	0.0214	0.1263
0.22	0.1143	0.1500	0.0305	0.1536
0.23	0.1426	0.1767	0.0412	0.1801
0.24	0.1732	0.2027	0.0532	0.2057
0.25	0.2059	0.2282	0.0664	0.2304
0.26	0.2408	0.2531	0.0806	0.2542
0.27	0.2779	0.2775	0.0956	0.2771
0.28	0.3172	0.3015	0.1113	0.2991
0.29	0.3586	0.3251	0.1275	0.3203
0.30	0.4022	0.3484	0.1442	0.3407
0.31	0.4480	0.3715	0.1611	0.3604
0.32	0.4961	0.3945	0.1781	0.3793
0.33	0.5464	0.4174	0.1953	0.3976
0.34	0.5992	0.4404	0.2125	0.4153
0.35	0.6543	0.4635	0.2297	0.4324
0.36	0.7119	0.4868	0.2468	0.4489
0.37	0.7722	0.5104	0.2638	0.4649
0.38	0.8351	0.5343	0.2806	0.4803
0.39	0.9009	0.5586	0.2973	0.4954
0.40	0.9696	0.5834	0.3138	0.5100
0.41	1.0414	0.6087	0.3301	0.5241
0.42	1.1164	0.6346	0.3462	0.5379
0.43	1.1949	0.6612	0.3621	0.5513
0.44	1.2768	0.6885	0.3778	0.5644
0.45	1.3626	0.7167	0.3932	0.5771
0.46	1.4522	0.7457	0.4085	0.5895
0.47	1.5461	0.7757	0.4235	0.6016
0.48	1.6444	0.8067	0.4383	0.6135
0.49	1.7473	0.8389	0.4529	0.6250
0.50	1.8552	0.8723	0.4673	0.6363
0.51	1.9683	0.9071	0.4814	0.6473
0.52	2.0870	0.9432	0.4954	0.6581
0.53	2.2116	0.9809	0.5091	0.6687
0.54	2.3425	1.0202	0.5227	0.6790
0.55	2.4802	1.0614	0.5360	0.6891
0.56	2.6251	1.1044	0.5492	0.6990
0.57	2.7777	1.1496	0.5622	0.7088
0.58	2.9385	1.1970	0.5750	0.7183

$\hat{\tau}_3 = 0.300, \quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.1082	1.2468	0.5876	0.7276
0.60	3.2875	1.2992	0.6000	0.7368
0.61	3.4770	1.3545	0.6123	0.7458
0.62	3.6776	1.4129	0.6244	0.7546
0.63	3.8903	1.4747	0.6363	0.7633
0.64	4.1159	1.5401	0.6481	0.7718
0.65	4.3557	1.6095	0.6597	0.7801
0.66	4.6108	1.6833	0.6712	0.7884
0.67	4.8827	1.7619	0.6826	0.7964
0.68	5.1730	1.8457	0.6938	0.8044
0.69	5.4835	1.9353	0.7048	0.8121
0.70	5.8162	2.0314	0.7158	0.8198
0.71	6.1733	2.1345	0.7266	0.8273
0.72	6.5577	2.2454	0.7373	0.8348
0.73	6.9721	2.3652	0.7478	0.8420
0.74	7.4203	2.4947	0.7583	0.8492
0.75	7.9062	2.6353	0.7686	0.8563
0.76	8.4346	2.7883	0.7788	0.8632
0.77	9.0110	2.9554	0.7890	0.8700
0.78	9.6420	3.1386	0.7990	0.8767
0.79	10.3354	3.3402	0.8089	0.8834
0.80	11.1007	3.5630	0.8187	0.8899
0.81	11.9492	3.8104	0.8284	0.8963
0.82	12.8949	4.0865	0.8381	0.9026
0.83	13.9548	4.3966	0.8476	0.9088
0.84	15.1506	4.7470	0.8571	0.9149
0.85	16.5093	5.1459	0.8665	0.9209
0.86	18.0661	5.6038	0.8758	0.9268
0.87	19.8666	6.1344	0.8851	0.9326
0.88	21.9720	6.7560	0.8942	0.9383
0.89	24.4654	7.4935	0.9033	0.9439
0.90	27.4633	8.3818	0.9124	0.9495
0.91	31.1341	9.4716	0.9214	0.9549
0.92	35.7303	10.8383	0.9303	0.9603
0.93	41.6487	12.6012	0.9391	0.9656
0.94	49.5507	14.9586	0.9480	0.9707
0.95	60.6265	18.2675	0.9567	0.9758
0.96	77.2573	23.2422	0.9655	0.9808
0.97	104.998	31.5494	0.9742	0.9858
0.98	160.517	48.1890	0.9828	0.9906
0.99	327.145	98.1610	0.9914	0.9953

Table E–5: $\hat{\tau}_3 = 0.325$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.325,\quad 0.17 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.17	0.0050	0.0096	0.0002	0.0097
0.18	0.0213	0.0386	0.0016	0.0386
0.19	0.0395	0.0677	0.0044	0.0672
0.20	0.0597	0.0968	0.0086	0.0954
0.21	0.0821	0.1257	0.0144	0.1230
0.22	0.1067	0.1544	0.0217	0.1501
0.23	0.1335	0.1827	0.0306	0.1764
0.24	0.1625	0.2105	0.0409	0.2020
0.25	0.1939	0.2379	0.0525	0.2268
0.26	0.2275	0.2648	0.0653	0.2508
0.27	0.2633	0.2912	0.0792	0.2739
0.28	0.3014	0.3172	0.0939	0.2962
0.29	0.3418	0.3429	0.1094	0.3177
0.30	0.3844	0.3683	0.1254	0.3385
0.31	0.4292	0.3935	0.1418	0.3584
0.32	0.4764	0.4185	0.1585	0.3777
0.33	0.5260	0.4435	0.1755	0.3963
0.34	0.5779	0.4686	0.1926	0.4142
0.35	0.6323	0.4937	0.2097	0.4315
0.36	0.6892	0.5191	0.2269	0.4483
0.37	0.7488	0.5447	0.2440	0.4645
0.38	0.8111	0.5707	0.2610	0.4802
0.39	0.8762	0.5970	0.2779	0.4954
0.40	0.9443	0.6239	0.2947	0.5102
0.41	1.0154	0.6513	0.3113	0.5246
0.42	1.0898	0.6794	0.3277	0.5385
0.43	1.1676	0.7082	0.3439	0.5520
0.44	1.2490	0.7378	0.3599	0.5652
0.45	1.3341	0.7682	0.3757	0.5781
0.46	1.4232	0.7996	0.3913	0.5906
0.47	1.5164	0.8321	0.4067	0.6028
0.48	1.6140	0.8656	0.4219	0.6147
0.49	1.7163	0.9004	0.4368	0.6263
0.50	1.8236	0.9365	0.4516	0.6377
0.51	1.9360	0.9740	0.4661	0.6488
0.52	2.0540	1.0131	0.4804	0.6596
0.53	2.1780	1.0538	0.4945	0.6702
0.54	2.3082	1.0963	0.5084	0.6806
0.55	2.4452	1.1407	0.5222	0.6907
0.56	2.5894	1.1872	0.5357	0.7007
0.57	2.7412	1.2359	0.5490	0.7104
0.58	2.9013	1.2870	0.5621	0.7200

$\hat{\tau}_3 = 0.325,\quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.0702	1.3408	0.5751	0.7293
0.60	3.2487	1.3974	0.5879	0.7385
0.61	3.4374	1.4571	0.6005	0.7475
0.62	3.6372	1.5202	0.6129	0.7563
0.63	3.8490	1.5868	0.6252	0.7650
0.64	4.0737	1.6575	0.6373	0.7735
0.65	4.3126	1.7324	0.6493	0.7818
0.66	4.5668	1.8121	0.6611	0.7900
0.67	4.8377	1.8969	0.6728	0.7981
0.68	5.1270	1.9874	0.6843	0.8060
0.69	5.4365	2.0842	0.6957	0.8138
0.70	5.7681	2.1879	0.7069	0.8214
0.71	6.1242	2.2993	0.7180	0.8289
0.72	6.5073	2.4192	0.7290	0.8363
0.73	6.9207	2.5485	0.7399	0.8436
0.74	7.3676	2.6885	0.7506	0.8507
0.75	7.8522	2.8404	0.7613	0.8577
0.76	8.3793	3.0057	0.7718	0.8646
0.77	8.9544	3.1863	0.7822	0.8714
0.78	9.5840	3.3843	0.7925	0.8781
0.79	10.2760	3.6022	0.8027	0.8847
0.80	11.0398	3.8430	0.8128	0.8911
0.81	11.8868	4.1105	0.8228	0.8975
0.82	12.8308	4.4091	0.8328	0.9037
0.83	13.8891	4.7444	0.8426	0.9099
0.84	15.0831	5.1234	0.8524	0.9159
0.85	16.4400	5.5549	0.8620	0.9219
0.86	17.9949	6.0502	0.8716	0.9278
0.87	19.7934	6.6243	0.8812	0.9335
0.88	21.8967	7.2969	0.8906	0.9392
0.89	24.3880	8.0951	0.9000	0.9447
0.90	27.3837	9.0566	0.9094	0.9502
0.91	31.0522	10.2363	0.9186	0.9556
0.92	35.6459	11.7160	0.9278	0.9609
0.93	41.5618	13.6248	0.9370	0.9661
0.94	49.4610	16.1775	0.9461	0.9712
0.95	60.5341	19.7611	0.9552	0.9762
0.96	77.1619	25.1492	0.9642	0.9812
0.97	104.900	34.1474	0.9732	0.9860
0.98	160.415	52.1723	0.9822	0.9908
0.99	327.040	106.307	0.9911	0.9954

Table E–5: $\hat{\tau}_3 = 0.35$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.350, \quad 0.18 \le \hat{\tau}_4 \le 0.58$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.18	0.0197	0.0387	0.0000	0.0373
0.19	0.0366	0.0682	0.0014	0.0650
0.20	0.0554	0.0979	0.0040	0.0925
0.21	0.0762	0.1276	0.0080	0.1195
0.22	0.0992	0.1574	0.0135	0.1461
0.23	0.1244	0.1870	0.0206	0.1722
0.24	0.1518	0.2164	0.0291	0.1976
0.25	0.1816	0.2455	0.0390	0.2224
0.26	0.2137	0.2742	0.0502	0.2465
0.27	0.2481	0.3026	0.0627	0.2698
0.28	0.2848	0.3306	0.0762	0.2924
0.29	0.3239	0.3584	0.0907	0.3142
0.30	0.3653	0.3858	0.1058	0.3352
0.31	0.4091	0.4131	0.1216	0.3555
0.32	0.4553	0.4403	0.1379	0.3751
0.33	0.5039	0.4673	0.1545	0.3940
0.34	0.5549	0.4945	0.1714	0.4122
0.35	0.6084	0.5217	0.1885	0.4299
0.36	0.6645	0.5491	0.2056	0.4469
0.37	0.7233	0.5768	0.2228	0.4634
0.38	0.7848	0.6048	0.2400	0.4793
0.39	0.8492	0.6333	0.2571	0.4948
0.40	0.9166	0.6623	0.2741	0.5098
0.41	0.9870	0.6919	0.2910	0.5243
0.42	1.0608	0.7221	0.3077	0.5384
0.43	1.1379	0.7531	0.3242	0.5522
0.44	1.2185	0.7849	0.3406	0.5655
0.45	1.3030	0.8177	0.3568	0.5785
0.46	1.3913	0.8514	0.3728	0.5911
0.47	1.4839	0.8863	0.3885	0.6034
0.48	1.5808	0.9223	0.4041	0.6154
0.49	1.6824	0.9597	0.4194	0.6272
0.50	1.7889	0.9985	0.4346	0.6386
0.51	1.9006	1.0387	0.4495	0.6498
0.52	2.0179	1.0806	0.4642	0.6607
0.53	2.1411	1.1243	0.4787	0.6713
0.54	2.2706	1.1699	0.4930	0.6818
0.55	2.4068	1.2176	0.5071	0.6920
0.56	2.5501	1.2674	0.5210	0.7020
0.57	2.7012	1.3197	0.5348	0.7117
0.58	2.8604	1.3746	0.5483	0.7213

$\hat{\tau}_3 = 0.350, \quad 0.59 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	3.0285	1.4323	0.5616	0.7307
0.60	3.2061	1.4930	0.5748	0.7399
0.61	3.3939	1.5571	0.5878	0.7489
0.62	3.5927	1.6247	0.6006	0.7578
0.63	3.8035	1.6962	0.6132	0.7664
0.64	4.0273	1.7720	0.6257	0.7750
0.65	4.2651	1.8524	0.6380	0.7833
0.66	4.5183	1.9379	0.6502	0.7915
0.67	4.7882	2.0289	0.6622	0.7996
0.68	5.0764	2.1260	0.6740	0.8075
0.69	5.3847	2.2299	0.6858	0.8152
0.70	5.7151	2.3412	0.6973	0.8229
0.71	6.0700	2.4607	0.7088	0.8304
0.72	6.4519	2.5894	0.7201	0.8377
0.73	6.8639	2.7282	0.7313	0.8450
0.74	7.3095	2.8785	0.7424	0.8521
0.75	7.7927	3.0416	0.7533	0.8591
0.76	8.3183	3.2191	0.7642	0.8659
0.77	8.8919	3.4131	0.7749	0.8727
0.78	9.5200	3.6257	0.7855	0.8794
0.79	10.2104	3.8598	0.7961	0.8859
0.80	10.9725	4.1185	0.8065	0.8923
0.81	11.8177	4.4059	0.8168	0.8986
0.82	12.7599	4.7268	0.8271	0.9048
0.83	13.8163	5.0872	0.8372	0.9110
0.84	15.0083	5.4946	0.8473	0.9170
0.85	16.3632	5.9585	0.8573	0.9229
0.86	17.9160	6.4911	0.8672	0.9287
0.87	19.7123	7.1085	0.8770	0.9344
0.88	21.8133	7.8319	0.8868	0.9400
0.89	24.3021	8.6904	0.8965	0.9455
0.90	27.2952	9.7249	0.9061	0.9509
0.91	30.9611	10.9942	0.9157	0.9563
0.92	35.5520	12.5866	0.9252	0.9615
0.93	41.4650	14.6410	0.9347	0.9666
0.94	49.3611	17.3889	0.9442	0.9717
0.95	60.4310	21.2467	0.9535	0.9766
0.96	77.0554	27.0479	0.9629	0.9815
0.97	104.790	36.7367	0.9722	0.9862
0.98	160.301	56.1465	0.9815	0.9909
0.99	326.922	114.444	0.9908	0.9955

Table E–5: $\hat{\tau}_3 = 0.375$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.375,\quad 0.21 \le \hat{\tau}_4 \le 0.60$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.21	0.0706	0.1286	0.0023	0.1157
0.22	0.0920	0.1591	0.0061	0.1418
0.23	0.1155	0.1898	0.0112	0.1675
0.24	0.1412	0.2204	0.0179	0.1927
0.25	0.1693	0.2510	0.0260	0.2173
0.26	0.1997	0.2814	0.0355	0.2414
0.27	0.2325	0.3116	0.0464	0.2648
0.28	0.2677	0.3415	0.0584	0.2876
0.29	0.3053	0.3713	0.0716	0.3096
0.30	0.3453	0.4008	0.0858	0.3309
0.31	0.3878	0.4301	0.1007	0.3515
0.32	0.4328	0.4594	0.1163	0.3715
0.33	0.4802	0.4885	0.1325	0.3907
0.34	0.5302	0.5178	0.1490	0.4093
0.35	0.5827	0.5471	0.1659	0.4272
0.36	0.6379	0.5766	0.1829	0.4446
0.37	0.6957	0.6064	0.2001	0.4614
0.38	0.7564	0.6365	0.2174	0.4776
0.39	0.8200	0.6671	0.2347	0.4933
0.40	0.8865	0.6982	0.2519	0.5086
0.41	0.9562	0.7300	0.2690	0.5234
0.42	1.0291	0.7624	0.2860	0.5377
0.43	1.1054	0.7956	0.3029	0.5516
0.44	1.1853	0.8297	0.3197	0.5651
0.45	1.2690	0.8647	0.3362	0.5783
0.46	1.3566	0.9008	0.3526	0.5911
0.47	1.4484	0.9381	0.3688	0.6035
0.48	1.5445	0.9766	0.3848	0.6157
0.49	1.6453	1.0165	0.4006	0.6275
0.50	1.7510	1.0579	0.4161	0.6391
0.51	1.8620	1.1009	0.4315	0.6503
0.52	1.9784	1.1457	0.4466	0.6613
0.53	2.1008	1.1923	0.4616	0.6721
0.54	2.2294	1.2410	0.4763	0.6826
0.55	2.3647	1.2918	0.4909	0.6929
0.56	2.5072	1.3450	0.5052	0.7029
0.57	2.6573	1.4008	0.5193	0.7128
0.58	2.8157	1.4594	0.5333	0.7224
0.59	2.9828	1.5209	0.5470	0.7318
0.60	3.1594	1.5857	0.5606	0.7410

$\hat{\tau}_3 = 0.375,\quad 0.61 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.61	3.3462	1.6540	0.5740	0.7501
0.62	3.5440	1.7262	0.5872	0.7590
0.63	3.7538	1.8025	0.6002	0.7677
0.64	3.9765	1.8833	0.6131	0.7762
0.65	4.2132	1.9691	0.6258	0.7846
0.66	4.4652	2.0603	0.6384	0.7928
0.67	4.7339	2.1574	0.6507	0.8008
0.68	5.0208	2.2611	0.6630	0.8087
0.69	5.3279	2.3719	0.6751	0.8165
0.70	5.6570	2.4907	0.6870	0.8241
0.71	6.0105	2.6183	0.6988	0.8316
0.72	6.3910	2.7556	0.7105	0.8390
0.73	6.8015	2.9039	0.7221	0.8462
0.74	7.2456	3.0643	0.7335	0.8533
0.75	7.7273	3.2385	0.7448	0.8603
0.76	8.2513	3.4281	0.7560	0.8671
0.77	8.8232	3.6353	0.7671	0.8739
0.78	9.4495	3.8624	0.7780	0.8805
0.79	10.1381	4.1125	0.7889	0.8870
0.80	10.8983	4.3890	0.7997	0.8934
0.81	11.7416	4.6961	0.8103	0.8997
0.82	12.6817	5.0391	0.8209	0.9059
0.83	13.7360	5.4244	0.8314	0.9119
0.84	14.9258	5.8600	0.8418	0.9179
0.85	16.2784	6.3560	0.8521	0.9238
0.86	17.8288	6.9257	0.8623	0.9295
0.87	19.6226	7.5861	0.8725	0.9352
0.88	21.7209	8.3601	0.8826	0.9408
0.89	24.2070	9.2788	0.8927	0.9462
0.90	27.1973	10.3859	0.9026	0.9516
0.91	30.8601	11.7445	0.9126	0.9569
0.92	35.4479	13.4493	0.9224	0.9620
0.93	41.3575	15.6490	0.9322	0.9671
0.94	49.2502	18.5915	0.9420	0.9721
0.95	60.3164	22.7233	0.9518	0.9770
0.96	76.9369	28.9370	0.9615	0.9818
0.97	104.667	39.3160	0.9711	0.9865
0.98	160.174	60.1103	0.9808	0.9911
0.99	326.791	122.570	0.9904	0.9956

Table E–5: $\hat{\tau}_3 = 0.4$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.400, \quad 0.23 \leq \hat{\tau}_4 \leq 0.61$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.23	0.1069	0.1911	0.0028	0.1624
0.24	0.1309	0.2227	0.0075	0.1872
0.25	0.1571	0.2545	0.0137	0.2116
0.26	0.1857	0.2863	0.0214	0.2356
0.27	0.2167	0.3181	0.0304	0.2590
0.28	0.2501	0.3498	0.0409	0.2818
0.29	0.2861	0.3814	0.0525	0.3040
0.30	0.3245	0.4129	0.0654	0.3256
0.31	0.3655	0.4443	0.0792	0.3465
0.32	0.4090	0.4756	0.0939	0.3668
0.33	0.4552	0.5069	0.1094	0.3864
0.34	0.5039	0.5382	0.1254	0.4053
0.35	0.5552	0.5697	0.1419	0.4236
0.36	0.6093	0.6013	0.1587	0.4413
0.37	0.6661	0.6332	0.1758	0.4585
0.38	0.7258	0.6655	0.1931	0.4750
0.39	0.7883	0.6982	0.2105	0.4910
0.40	0.8540	0.7315	0.2279	0.5066
0.41	0.9227	0.7654	0.2453	0.5216
0.42	0.9948	0.8000	0.2626	0.5362
0.43	1.0702	0.8354	0.2798	0.5504
0.44	1.1493	0.8717	0.2969	0.5641
0.45	1.2321	0.9091	0.3139	0.5775
0.46	1.3188	0.9475	0.3307	0.5904
0.47	1.4097	0.9872	0.3473	0.6031
0.48	1.5050	1.0282	0.3638	0.6154
0.49	1.6049	1.0706	0.3800	0.6274
0.50	1.7098	1.1146	0.3961	0.6390
0.51	1.8198	1.1604	0.4119	0.6504
0.52	1.9354	1.2079	0.4275	0.6616
0.53	2.0568	1.2574	0.4430	0.6724
0.54	2.1845	1.3091	0.4582	0.6830
0.55	2.3189	1.3631	0.4732	0.6934
0.56	2.4604	1.4197	0.4880	0.7035
0.57	2.6095	1.4789	0.5026	0.7134
0.58	2.7669	1.5411	0.5170	0.7231
0.59	2.9329	1.6064	0.5312	0.7326
0.60	3.1085	1.6752	0.5452	0.7419
0.61	3.2942	1.7477	0.5591	0.7510

$\hat{\tau}_3 = 0.400, \quad 0.62 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.62	3.4909	1.8243	0.5727	0.7599
0.63	3.6994	1.9053	0.5862	0.7686
0.64	3.9209	1.9911	0.5995	0.7772
0.65	4.1564	2.0822	0.6126	0.7856
0.66	4.4071	2.1790	0.6256	0.7938
0.67	4.6745	2.2822	0.6384	0.8019
0.68	4.9601	2.3923	0.6510	0.8098
0.69	5.2657	2.5100	0.6635	0.8176
0.70	5.5934	2.6361	0.6759	0.8252
0.71	5.9454	2.7716	0.6881	0.8327
0.72	6.3243	2.9175	0.7001	0.8401
0.73	6.7333	3.0750	0.7121	0.8473
0.74	7.1757	3.2455	0.7239	0.8544
0.75	7.6556	3.4306	0.7356	0.8613
0.76	8.1778	3.6321	0.7471	0.8682
0.77	8.7478	3.8523	0.7586	0.8749
0.78	9.3722	4.0938	0.7699	0.8815
0.79	10.0587	4.3597	0.7811	0.8880
0.80	10.8169	4.6538	0.7923	0.8944
0.81	11.6579	4.9804	0.8033	0.9006
0.82	12.5958	5.3453	0.8142	0.9068
0.83	13.6477	5.7553	0.8251	0.9128
0.84	14.8350	6.2188	0.8358	0.9188
0.85	16.1850	6.7468	0.8465	0.9246
0.86	17.7326	7.3533	0.8571	0.9303
0.87	19.5236	8.0564	0.8677	0.9360
0.88	21.6190	8.8806	0.8781	0.9415
0.89	24.1019	9.8592	0.8885	0.9469
0.90	27.0889	11.0386	0.8989	0.9522
0.91	30.7483	12.4862	0.9091	0.9574
0.92	35.3325	14.3030	0.9194	0.9626
0.93	41.2383	16.6475	0.9296	0.9676
0.94	49.1270	19.7843	0.9397	0.9725
0.95	60.1890	24.1895	0.9498	0.9773
0.96	76.8052	30.8154	0.9599	0.9821
0.97	104.531	41.8841	0.9700	0.9867
0.98	160.033	64.0621	0.9800	0.9912
0.99	326.644	130.683	0.9900	0.9957

Table E–5: $\hat{\tau}_3 = 0.425$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.425,\quad 0.25 \le \hat{\tau}_4 \le 0.62$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.25	0.1452	0.2561	0.0024	0.2054
0.26	0.1719	0.2890	0.0081	0.2291
0.27	0.2009	0.3221	0.0152	0.2524
0.28	0.2325	0.3554	0.0238	0.2752
0.29	0.2665	0.3887	0.0337	0.2975
0.30	0.3031	0.4220	0.0449	0.3193
0.31	0.3424	0.4553	0.0574	0.3404
0.32	0.3843	0.4886	0.0709	0.3610
0.33	0.4288	0.5220	0.0854	0.3809
0.34	0.4761	0.5555	0.1006	0.4002
0.35	0.5260	0.5891	0.1165	0.4189
0.36	0.5788	0.6229	0.1330	0.4370
0.37	0.6344	0.6569	0.1499	0.4545
0.38	0.6929	0.6914	0.1670	0.4714
0.39	0.7543	0.7263	0.1844	0.4878
0.40	0.8189	0.7617	0.2019	0.5037
0.41	0.8866	0.7978	0.2195	0.5190
0.42	0.9576	0.8346	0.2371	0.5339
0.43	1.0321	0.8722	0.2547	0.5483
0.44	1.1102	0.9108	0.2722	0.5623
0.45	1.1920	0.9505	0.2896	0.5759
0.46	1.2778	0.9913	0.3068	0.5891
0.47	1.3678	1.0333	0.3239	0.6020
0.48	1.4621	1.0768	0.3409	0.6145
0.49	1.5611	1.1217	0.3576	0.6266
0.50	1.6650	1.1683	0.3742	0.6385
0.51	1.7740	1.2167	0.3905	0.6500
0.52	1.8886	1.2670	0.4067	0.6613
0.53	2.0091	1.3194	0.4226	0.6723
0.54	2.1357	1.3741	0.4384	0.6830
0.55	2.2691	1.4312	0.4539	0.6935
0.56	2.4095	1.4910	0.4692	0.7037
0.57	2.5575	1.5536	0.4843	0.7137
0.58	2.7137	1.6193	0.4992	0.7235
0.59	2.8787	1.6884	0.5139	0.7330
0.60	3.0530	1.7611	0.5285	0.7424
0.61	3.2375	1.8377	0.5428	0.7515
0.62	3.4329	1.9187	0.5569	0.7605

$\hat{\tau}_3 = 0.425,\quad 0.63 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.63	3.6402	2.0043	0.5709	0.7693
0.64	3.8604	2.0950	0.5846	0.7779
0.65	4.0945	2.1913	0.5982	0.7863
0.66	4.3438	2.2937	0.6116	0.7946
0.67	4.6097	2.4027	0.6249	0.8027
0.68	4.8938	2.5191	0.6380	0.8107
0.69	5.1979	2.6435	0.6509	0.8185
0.70	5.5239	2.7770	0.6637	0.8261
0.71	5.8742	2.9203	0.6764	0.8336
0.72	6.2515	3.0746	0.6889	0.8410
0.73	6.6586	3.2412	0.7012	0.8482
0.74	7.0992	3.4216	0.7134	0.8553
0.75	7.5772	3.6174	0.7255	0.8623
0.76	8.0974	3.8307	0.7375	0.8691
0.77	8.6653	4.0637	0.7494	0.8758
0.78	9.2875	4.3194	0.7611	0.8824
0.79	9.9718	4.6009	0.7727	0.8889
0.80	10.7276	4.9123	0.7843	0.8953
0.81	11.5662	5.2583	0.7957	0.9015
0.82	12.5015	5.6448	0.8070	0.9076
0.83	13.5507	6.0791	0.8183	0.9137
0.84	14.7352	6.5704	0.8294	0.9196
0.85	16.0823	7.1300	0.8405	0.9254
0.86	17.6268	7.7729	0.8515	0.9311
0.87	19.4146	8.5185	0.8624	0.9367
0.88	21.5066	9.3926	0.8733	0.9421
0.89	23.9860	10.4306	0.8840	0.9475
0.90	26.9693	11.6820	0.8948	0.9528
0.91	30.6248	13.2182	0.9055	0.9580
0.92	35.2049	15.1464	0.9161	0.9630
0.93	41.1064	17.6354	0.9267	0.9680
0.94	48.9906	20.9659	0.9372	0.9729
0.95	60.0478	25.6440	0.9477	0.9777
0.96	76.6589	32.6814	0.9582	0.9823
0.97	104.379	44.4391	0.9687	0.9869
0.98	159.876	68.0004	0.9791	0.9914
0.99	326.481	138.782	0.9896	0.9957

Table E–5: $\hat{\tau}_3 = 0.45$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.450, \quad 0.27 \leq \hat{\tau}_4 \leq 0.63$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.27	0.1855	0.3238	0.0009	0.2452
0.28	0.2150	0.3583	0.0075	0.2678
0.29	0.2469	0.3931	0.0154	0.2901
0.30	0.2815	0.4280	0.0248	0.3120
0.31	0.3187	0.4631	0.0355	0.3333
0.32	0.3587	0.4983	0.0476	0.3541
0.33	0.4014	0.5337	0.0607	0.3744
0.34	0.4469	0.5692	0.0749	0.3940
0.35	0.4953	0.6049	0.0900	0.4131
0.36	0.5465	0.6409	0.1058	0.4316
0.37	0.6006	0.6771	0.1222	0.4495
0.38	0.6578	0.7138	0.1391	0.4668
0.39	0.7179	0.7508	0.1564	0.4835
0.40	0.7812	0.7885	0.1739	0.4998
0.41	0.8478	0.8268	0.1917	0.5155
0.42	0.9176	0.8658	0.2095	0.5307
0.43	0.9910	0.9057	0.2274	0.5455
0.44	1.0680	0.9466	0.2452	0.5598
0.45	1.1487	0.9885	0.2630	0.5736
0.46	1.2335	1.0316	0.2808	0.5871
0.47	1.3224	1.0761	0.2983	0.6002
0.48	1.4157	1.1220	0.3158	0.6129
0.49	1.5136	1.1694	0.3331	0.6253
0.50	1.6164	1.2186	0.3502	0.6373
0.51	1.7244	1.2696	0.3671	0.6491
0.52	1.8379	1.3226	0.3838	0.6605
0.53	1.9572	1.3779	0.4003	0.6716
0.54	2.0827	1.4355	0.4166	0.6825
0.55	2.2149	1.4956	0.4327	0.6931
0.56	2.3542	1.5586	0.4486	0.7035
0.57	2.5010	1.6245	0.4643	0.7136
0.58	2.6560	1.6937	0.4798	0.7235
0.59	2.8196	1.7665	0.4951	0.7331
0.60	2.9927	1.8430	0.5101	0.7426
0.61	3.1759	1.9237	0.5250	0.7518
0.62	3.3699	2.0089	0.5396	0.7608
0.63	3.5758	2.0991	0.5541	0.7697

$\hat{\tau}_3 = 0.450, \quad 0.64 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.64	3.7945	2.1946	0.5684	0.7784
0.65	4.0271	2.2960	0.5825	0.7868
0.66	4.2749	2.4037	0.5965	0.7952
0.67	4.5392	2.5185	0.6102	0.8033
0.68	4.8216	2.6411	0.6238	0.8113
0.69	5.1240	2.7722	0.6372	0.8191
0.70	5.4482	2.9127	0.6505	0.8268
0.71	5.7967	3.0637	0.6636	0.8343
0.72	6.1720	3.2262	0.6766	0.8417
0.73	6.5772	3.4018	0.6894	0.8490
0.74	7.0157	3.5918	0.7021	0.8561
0.75	7.4915	3.7982	0.7146	0.8631
0.76	8.0095	4.0231	0.7270	0.8699
0.77	8.5751	4.2688	0.7393	0.8766
0.78	9.1948	4.5384	0.7515	0.8832
0.79	9.8766	4.8353	0.7636	0.8897
0.80	10.6297	5.1637	0.7756	0.8960
0.81	11.4656	5.5288	0.7874	0.9023
0.82	12.3980	5.9367	0.7992	0.9084
0.83	13.4442	6.3951	0.8108	0.9144
0.84	14.6256	6.9137	0.8224	0.9203
0.85	15.9694	7.5047	0.8339	0.9261
0.86	17.5105	8.1837	0.8453	0.9317
0.87	19.2947	8.9713	0.8567	0.9373
0.88	21.3829	9.8949	0.8680	0.9428
0.89	23.8583	10.9920	0.8792	0.9481
0.90	26.8374	12.3148	0.8903	0.9533
0.91	30.4884	13.9391	0.9014	0.9585
0.92	35.0639	15.9783	0.9125	0.9635
0.93	40.9604	18.6111	0.9235	0.9684
0.94	48.8394	22.1348	0.9345	0.9733
0.95	59.8911	27.0851	0.9454	0.9780
0.96	76.4963	34.5334	0.9564	0.9826
0.97	104.210	46.9794	0.9673	0.9871
0.98	159.700	71.9230	0.9782	0.9915
0.99	326.299	146.864	0.9891	0.9958

Table E–5: $\hat{\tau}_3 = 0.475$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.475,\ \ 0.30 \le \hat{\tau}_4 \le 0.64$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.30	0.2599	0.4309	0.0054	0.3038
0.31	0.2949	0.4675	0.0141	0.3252
0.32	0.3327	0.5045	0.0243	0.3461
0.33	0.3733	0.5417	0.0358	0.3666
0.34	0.4168	0.5792	0.0485	0.3866
0.35	0.4632	0.6170	0.0624	0.4060
0.36	0.5126	0.6550	0.0773	0.4249
0.37	0.5650	0.6935	0.0930	0.4432
0.38	0.6205	0.7323	0.1094	0.4609
0.39	0.6791	0.7716	0.1263	0.4781
0.40	0.7410	0.8114	0.1437	0.4948
0.41	0.8061	0.8520	0.1614	0.5109
0.42	0.8747	0.8933	0.1794	0.5265
0.43	0.9468	0.9354	0.1975	0.5416
0.44	1.0225	0.9786	0.2157	0.5563
0.45	1.1020	1.0228	0.2339	0.5705
0.46	1.1856	1.0683	0.2521	0.5843
0.47	1.2733	1.1151	0.2702	0.5976
0.48	1.3654	1.1634	0.2882	0.6106
0.49	1.4621	1.2133	0.3061	0.6233
0.50	1.5638	1.2651	0.3238	0.6356
0.51	1.6706	1.3187	0.3413	0.6475
0.52	1.7829	1.3744	0.3587	0.6591
0.53	1.9010	1.4324	0.3758	0.6705
0.54	2.0253	1.4929	0.3928	0.6815
0.55	2.1562	1.5561	0.4095	0.6923
0.56	2.2942	1.6222	0.4260	0.7028
0.57	2.4397	1.6914	0.4423	0.7130
0.58	2.5933	1.7640	0.4584	0.7230
0.59	2.7556	1.8403	0.4743	0.7328
0.60	2.9272	1.9206	0.4900	0.7424
0.61	3.1089	2.0052	0.5054	0.7517
0.62	3.3015	2.0946	0.5207	0.7608
0.63	3.5058	2.1892	0.5358	0.7698
0.64	3.7229	2.2894	0.5506	0.7785

$\hat{\tau}_3 = 0.475,\ \ 0.65 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.65	3.9539	2.3957	0.5653	0.7871
0.66	4.1999	2.5088	0.5798	0.7954
0.67	4.4625	2.6292	0.5941	0.8037
0.68	4.7431	2.7578	0.6083	0.8117
0.69	5.0435	2.8954	0.6222	0.8196
0.70	5.3658	3.0428	0.6360	0.8273
0.71	5.7122	3.2012	0.6497	0.8349
0.72	6.0854	3.3719	0.6631	0.8423
0.73	6.4884	3.5562	0.6765	0.8496
0.74	6.9246	3.7558	0.6897	0.8567
0.75	7.3980	3.9725	0.7027	0.8637
0.76	7.9135	4.2086	0.7156	0.8706
0.77	8.4765	4.4668	0.7284	0.8773
0.78	9.0936	4.7500	0.7411	0.8839
0.79	9.7725	5.0621	0.7536	0.8904
0.80	10.5227	5.4073	0.7661	0.8967
0.81	11.3555	5.7911	0.7784	0.9029
0.82	12.2847	6.2200	0.7906	0.9091
0.83	13.3275	6.7022	0.8027	0.9151
0.84	14.5054	7.2478	0.8148	0.9209
0.85	15.8454	7.8697	0.8267	0.9267
0.86	17.3826	8.5844	0.8386	0.9323
0.87	19.1627	9.4137	0.8504	0.9379
0.88	21.2466	10.3863	0.8622	0.9433
0.89	23.7175	11.5419	0.8738	0.9486
0.90	26.6918	12.9356	0.8855	0.9538
0.91	30.3378	14.6474	0.8970	0.9589
0.92	34.9079	16.7971	0.9086	0.9639
0.93	40.7988	19.5730	0.9201	0.9688
0.94	48.6718	23.2891	0.9315	0.9736
0.95	59.7171	28.5109	0.9429	0.9783
0.96	76.3156	36.3692	0.9544	0.9828
0.97	104.023	49.5026	0.9658	0.9873
0.98	159.505	75.8276	0.9772	0.9916
0.99	326.095	154.927	0.9886	0.9959

Table E–5: $\hat{\tau}_3 = 0.5$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.500, \quad 0.32 \leq \hat{\tau}_4 \leq 0.65$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.32	0.3066	0.5070	0.0016	0.3371
0.33	0.3448	0.5459	0.0110	0.3577
0.34	0.3859	0.5851	0.0219	0.3779
0.35	0.4300	0.6248	0.0342	0.3977
0.36	0.4773	0.6649	0.0477	0.4169
0.37	0.5276	0.7055	0.0623	0.4356
0.38	0.5812	0.7464	0.0778	0.4538
0.39	0.6380	0.7880	0.0941	0.4714
0.40	0.6981	0.8301	0.1112	0.4885
0.41	0.7617	0.8729	0.1287	0.5051
0.42	0.8287	0.9164	0.1467	0.5211
0.43	0.8992	0.9609	0.1649	0.5367
0.44	0.9735	1.0064	0.1834	0.5518
0.45	1.0517	1.0529	0.2020	0.5663
0.46	1.1339	1.1007	0.2206	0.5805
0.47	1.2203	1.1499	0.2392	0.5942
0.48	1.3110	1.2007	0.2578	0.6075
0.49	1.4065	1.2531	0.2763	0.6205
0.50	1.5068	1.3073	0.2946	0.6330
0.51	1.6123	1.3635	0.3128	0.6452
0.52	1.7233	1.4219	0.3309	0.6571
0.53	1.8400	1.4827	0.3487	0.6687
0.54	1.9630	1.5460	0.3663	0.6799
0.55	2.0925	1.6121	0.3838	0.6909
0.56	2.2291	1.6812	0.4010	0.7016
0.57	2.3732	1.7536	0.4180	0.7120
0.58	2.5253	1.8295	0.4348	0.7221
0.59	2.6861	1.9093	0.4514	0.7321
0.60	2.8562	1.9933	0.4678	0.7417
0.61	3.0363	2.0818	0.4839	0.7512
0.62	3.2272	2.1753	0.4998	0.7605
0.63	3.4298	2.2741	0.5155	0.7695
0.64	3.6451	2.3788	0.5311	0.7783
0.65	3.8743	2.4900	0.5464	0.7870

$\hat{\tau}_3 = 0.500, \quad 0.66 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.66	4.1184	2.6082	0.5615	0.7954
0.67	4.3790	2.7342	0.5764	0.8037
0.68	4.6576	2.8686	0.5912	0.8118
0.69	4.9560	3.0124	0.6057	0.8198
0.70	5.2761	3.1667	0.6201	0.8276
0.71	5.6203	3.3324	0.6343	0.8352
0.72	5.9911	3.5109	0.6484	0.8426
0.73	6.3916	3.7037	0.6623	0.8500
0.74	6.8253	3.9126	0.6760	0.8571
0.75	7.2961	4.1394	0.6896	0.8642
0.76	7.8088	4.3866	0.7031	0.8710
0.77	8.3689	4.6569	0.7164	0.8778
0.78	8.9830	4.9535	0.7296	0.8844
0.79	9.6588	5.2804	0.7427	0.8909
0.80	10.4056	5.6421	0.7556	0.8973
0.81	11.2350	6.0443	0.7685	0.9035
0.82	12.1606	6.4939	0.7812	0.9096
0.83	13.1996	6.9994	0.7939	0.9156
0.84	14.3734	7.5716	0.8064	0.9215
0.85	15.7093	8.2240	0.8189	0.9272
0.86	17.2421	8.9739	0.8313	0.9329
0.87	19.0175	9.8442	0.8436	0.9384
0.88	21.0965	10.8654	0.8558	0.9438
0.89	23.5622	12.0789	0.8680	0.9491
0.90	26.5311	13.5429	0.8801	0.9543
0.91	30.1713	15.3415	0.8922	0.9594
0.92	34.7353	17.6008	0.9043	0.9643
0.93	40.6197	20.5191	0.9163	0.9692
0.94	48.4858	24.4269	0.9282	0.9739
0.95	59.5238	29.9192	0.9402	0.9786
0.96	76.1146	38.1865	0.9522	0.9831
0.97	103.813	52.0062	0.9641	0.9875
0.98	159.287	79.7115	0.9761	0.9918
0.99	325.867	162.969	0.9880	0.9959

Table E–5: $\hat{\tau}_3 = 0.525$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.525,\quad 0.35 \le \hat{\tau}_4 \le 0.67$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.35	0.3963	0.6283	0.0058	0.3881
0.36	0.4410	0.6702	0.0174	0.4076
0.37	0.4889	0.7127	0.0304	0.4267
0.38	0.5402	0.7558	0.0447	0.4452
0.39	0.5948	0.7995	0.0600	0.4633
0.40	0.6528	0.8439	0.0763	0.4809
0.41	0.7144	0.8889	0.0934	0.4980
0.42	0.7795	0.9348	0.1111	0.5145
0.43	0.8484	0.9816	0.1293	0.5305
0.44	0.9210	1.0294	0.1479	0.5460
0.45	0.9975	1.0783	0.1667	0.5611
0.46	1.0782	1.1285	0.1857	0.5757
0.47	1.1630	1.1801	0.2049	0.5898
0.48	1.2523	1.2332	0.2240	0.6035
0.49	1.3463	1.2881	0.2431	0.6168
0.50	1.4452	1.3448	0.2622	0.6297
0.51	1.5492	1.4036	0.2811	0.6422
0.52	1.6587	1.4646	0.2999	0.6543
0.53	1.7740	1.5281	0.3185	0.6662
0.54	1.8955	1.5942	0.3370	0.6777
0.55	2.0235	1.6631	0.3552	0.6889
0.56	2.1585	1.7352	0.3732	0.6998
0.57	2.3010	1.8107	0.3910	0.7104
0.58	2.4515	1.8899	0.4086	0.7207
0.59	2.6107	1.9731	0.4259	0.7308
0.60	2.7790	2.0606	0.4431	0.7407
0.61	2.9574	2.1529	0.4600	0.7503
0.62	3.1465	2.2503	0.4767	0.7597
0.63	3.3473	2.3533	0.4931	0.7688
0.64	3.5607	2.4624	0.5094	0.7778
0.65	3.7878	2.5783	0.5254	0.7866
0.66	4.0299	2.7014	0.5412	0.7951
0.67	4.2884	2.8327	0.5568	0.8035

$\hat{\tau}_3 = 0.525,\quad 0.68 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.68	4.5647	2.9728	0.5723	0.8117
0.69	4.8608	3.1228	0.5875	0.8197
0.70	5.1785	3.2835	0.6025	0.8276
0.71	5.5202	3.4563	0.6174	0.8353
0.72	5.8884	3.6425	0.6321	0.8428
0.73	6.2862	3.8436	0.6466	0.8502
0.74	6.7170	4.0614	0.6610	0.8574
0.75	7.1849	4.2981	0.6752	0.8644
0.76	7.6945	4.5561	0.6893	0.8714
0.77	8.2514	4.8382	0.7032	0.8782
0.78	8.8621	5.1479	0.7170	0.8848
0.79	9.5344	5.4892	0.7306	0.8913
0.80	10.2775	5.8669	0.7442	0.8977
0.81	11.1030	6.2872	0.7576	0.9040
0.82	12.0245	6.7570	0.7709	0.9101
0.83	13.0592	7.2855	0.7841	0.9161
0.84	14.2286	7.8838	0.7972	0.9220
0.85	15.5597	8.5660	0.8102	0.9277
0.86	17.0875	9.3506	0.8232	0.9334
0.87	18.8576	10.2615	0.8361	0.9389
0.88	20.9311	11.3304	0.8489	0.9443
0.89	23.3909	12.6012	0.8616	0.9496
0.90	26.3535	14.1348	0.8743	0.9547
0.91	29.9871	16.0194	0.8869	0.9598
0.92	34.5441	18.3874	0.8995	0.9647
0.93	40.4210	21.4471	0.9121	0.9695
0.94	48.2791	25.5455	0.9247	0.9742
0.95	59.3087	31.3073	0.9372	0.9788
0.96	75.8904	39.9824	0.9497	0.9833
0.97	103.579	54.4871	0.9623	0.9877
0.98	159.042	83.5712	0.9748	0.9919
0.99	325.612	170.984	0.9874	0.9960

Table E–5: $\hat{\tau}_3 = 0.55$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.550, \quad 0.38 \leq \hat{\tau}_4 \leq 0.68$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.38	0.4977	0.7600	0.0105	0.4352
0.39	0.5497	0.8057	0.0242	0.4537
0.40	0.6052	0.8522	0.0393	0.4718
0.41	0.6644	0.8995	0.0554	0.4893
0.42	0.7273	0.9477	0.0725	0.5064
0.43	0.7941	0.9968	0.0903	0.5229
0.44	0.8647	1.0469	0.1088	0.5390
0.45	0.9394	1.0982	0.1278	0.5545
0.46	1.0182	1.1508	0.1471	0.5696
0.47	1.1013	1.2048	0.1667	0.5842
0.48	1.1889	1.2604	0.1864	0.5983
0.49	1.2812	1.3177	0.2061	0.6120
0.50	1.3785	1.3769	0.2259	0.6253
0.51	1.4809	1.4383	0.2456	0.6382
0.52	1.5887	1.5019	0.2652	0.6507
0.53	1.7024	1.5680	0.2847	0.6629
0.54	1.8222	1.6368	0.3040	0.6747
0.55	1.9486	1.7086	0.3232	0.6861
0.56	2.0819	1.7837	0.3421	0.6973
0.57	2.2227	1.8622	0.3608	0.7082
0.58	2.3715	1.9445	0.3793	0.7187
0.59	2.5288	2.0310	0.3975	0.7290
0.60	2.6953	2.1219	0.4155	0.7391
0.61	2.8717	2.2178	0.4333	0.7489
0.62	3.0588	2.3190	0.4508	0.7584
0.63	3.2576	2.4260	0.4681	0.7678
0.64	3.4688	2.5394	0.4852	0.7769
0.65	3.6938	2.6598	0.5020	0.7858
0.66	3.9336	2.7878	0.5186	0.7945
0.67	4.1897	2.9242	0.5351	0.8030
0.68	4.4636	3.0698	0.5512	0.8113

$\hat{\tau}_3 = 0.550, \quad 0.69 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.69	4.7571	3.2256	0.5672	0.8194
0.70	5.0722	3.3927	0.5830	0.8273
0.71	5.4111	3.5723	0.5986	0.8351
0.72	5.7765	3.7658	0.6140	0.8427
0.73	6.1713	3.9749	0.6293	0.8501
0.74	6.5990	4.2015	0.6443	0.8574
0.75	7.0635	4.4477	0.6592	0.8645
0.76	7.5697	4.7161	0.6740	0.8715
0.77	8.1230	5.0096	0.6885	0.8784
0.78	8.7299	5.3320	0.7030	0.8850
0.79	9.3983	5.6873	0.7173	0.8916
0.80	10.1373	6.0808	0.7315	0.8980
0.81	10.9584	6.5185	0.7455	0.9043
0.82	11.8753	7.0081	0.7595	0.9104
0.83	12.9052	7.5589	0.7733	0.9165
0.84	14.0695	8.1827	0.7870	0.9224
0.85	15.3952	8.8943	0.8007	0.9281
0.86	16.9173	9.7129	0.8142	0.9338
0.87	18.6814	10.6635	0.8277	0.9393
0.88	20.7485	11.7796	0.8411	0.9447
0.89	23.2015	13.1068	0.8545	0.9499
0.90	26.1570	14.7090	0.8678	0.9551
0.91	29.7830	16.6786	0.8811	0.9601
0.92	34.3318	19.1544	0.8943	0.9651
0.93	40.2001	22.3544	0.9075	0.9698
0.94	48.0490	26.6422	0.9207	0.9745
0.95	59.0687	32.6721	0.9338	0.9791
0.96	75.6398	41.7535	0.9470	0.9835
0.97	103.318	56.9417	0.9602	0.9878
0.98	158.768	87.4028	0.9734	0.9920
0.99	325.325	178.970	0.9867	0.9961

Table E–5: $\hat{\tau}_3 = 0.575$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.575,\ \ 0.40 \le \hat{\tau}_4 \le 0.69$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.40	0.5558	0.8546	0.0005	0.4609
0.41	0.6121	0.9040	0.0150	0.4790
0.42	0.6722	0.9544	0.0309	0.4966
0.43	0.7364	1.0058	0.0480	0.5137
0.44	0.8046	1.0583	0.0660	0.5303
0.45	0.8770	1.1120	0.0847	0.5464
0.46	0.9537	1.1670	0.1041	0.5620
0.47	1.0347	1.2234	0.1240	0.5772
0.48	1.1204	1.2814	0.1442	0.5918
0.49	1.2108	1.3413	0.1646	0.6060
0.50	1.3062	1.4030	0.1851	0.6198
0.51	1.4067	1.4669	0.2056	0.6331
0.52	1.5128	1.5330	0.2261	0.6460
0.53	1.6247	1.6018	0.2466	0.6586
0.54	1.7427	1.6733	0.2668	0.6708
0.55	1.8672	1.7479	0.2870	0.6826
0.56	1.9987	1.8257	0.3069	0.6941
0.57	2.1376	1.9072	0.3267	0.7052
0.58	2.2844	1.9926	0.3462	0.7161
0.59	2.4397	2.0822	0.3654	0.7266
0.60	2.6042	2.1765	0.3844	0.7369
0.61	2.7785	2.2759	0.4032	0.7469
0.62	2.9635	2.3807	0.4217	0.7567
0.63	3.1600	2.4916	0.4400	0.7662
0.64	3.3689	2.6091	0.4580	0.7755
0.65	3.5915	2.7338	0.4758	0.7846
0.66	3.8288	2.8664	0.4933	0.7934
0.67	4.0823	3.0077	0.5106	0.8020
0.68	4.3536	3.1586	0.5277	0.8105
0.69	4.6442	3.3200	0.5446	0.8187

$\hat{\tau}_3 = 0.575,\ \ 0.70 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.70	4.9564	3.4932	0.5612	0.8268
0.71	5.2922	3.6793	0.5776	0.8346
0.72	5.6544	3.8799	0.5939	0.8423
0.73	6.0459	4.0967	0.6099	0.8499
0.74	6.4701	4.3316	0.6257	0.8572
0.75	6.9310	4.5870	0.6414	0.8644
0.76	7.4334	4.8654	0.6569	0.8715
0.77	7.9826	5.1700	0.6722	0.8784
0.78	8.5853	5.5046	0.6874	0.8851
0.79	9.2492	5.8735	0.7025	0.8917
0.80	9.9836	6.2821	0.7174	0.8982
0.81	10.7997	6.7368	0.7321	0.9045
0.82	11.7115	7.2456	0.7468	0.9107
0.83	12.7359	7.8181	0.7613	0.9167
0.84	13.8944	8.4668	0.7757	0.9226
0.85	15.2140	9.2070	0.7901	0.9284
0.86	16.7296	10.0588	0.8043	0.9341
0.87	18.4868	11.0483	0.8185	0.9396
0.88	20.5466	12.2106	0.8326	0.9450
0.89	22.9919	13.5932	0.8466	0.9503
0.90	25.9391	15.2630	0.8606	0.9554
0.91	29.5562	17.3165	0.8745	0.9605
0.92	34.0957	19.8988	0.8884	0.9654
0.93	39.9539	23.2378	0.9023	0.9701
0.94	47.7920	27.7134	0.9162	0.9748
0.95	58.8002	34.0098	0.9301	0.9793
0.96	75.3588	43.4958	0.9440	0.9837
0.97	103.023	59.3653	0.9579	0.9880
0.98	158.459	91.2014	0.9719	0.9921
0.99	325.000	186.920	0.9859	0.9961

Table E–5: $\hat{\tau}_3 = 0.6$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.600, \quad 0.43 \leq \hat{\tau}_4 \leq 0.71$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.43	0.6756	1.0077	0.0023	0.5025
0.44	0.7408	1.0625	0.0192	0.5197
0.45	0.8104	1.1186	0.0372	0.5364
0.46	0.8844	1.1760	0.0563	0.5527
0.47	0.9630	1.2349	0.0761	0.5685
0.48	1.0463	1.2954	0.0966	0.5837
0.49	1.1345	1.3577	0.1175	0.5985
0.50	1.2277	1.4220	0.1387	0.6129
0.51	1.3262	1.4884	0.1601	0.6267
0.52	1.4302	1.5572	0.1815	0.6402
0.53	1.5401	1.6285	0.2030	0.6532
0.54	1.6561	1.7027	0.2244	0.6658
0.55	1.7786	1.7800	0.2457	0.6780
0.56	1.9080	1.8607	0.2668	0.6899
0.57	2.0448	1.9450	0.2877	0.7014
0.58	2.1895	2.0333	0.3084	0.7126
0.59	2.3427	2.1261	0.3289	0.7235
0.60	2.5049	2.2236	0.3491	0.7340
0.61	2.6770	2.3262	0.3690	0.7443
0.62	2.8596	2.4346	0.3887	0.7543
0.63	3.0536	2.5492	0.4081	0.7641
0.64	3.2601	2.6706	0.4272	0.7736
0.65	3.4800	2.7994	0.4461	0.7828
0.66	3.7146	2.9364	0.4647	0.7919
0.67	3.9652	3.0823	0.4831	0.8007
0.68	4.2335	3.2382	0.5012	0.8093
0.69	4.5211	3.4050	0.5190	0.8177
0.70	4.8300	3.5839	0.5366	0.8258
0.71	5.1625	3.7763	0.5540	0.8338

$\hat{\tau}_3 = 0.600, \quad 0.72 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.72	5.5211	3.9837	0.5712	0.8417
0.73	5.9088	4.2078	0.5881	0.8493
0.74	6.3291	4.4507	0.6049	0.8568
0.75	6.7860	4.7147	0.6214	0.8641
0.76	7.2841	5.0027	0.6378	0.8712
0.77	7.8289	5.3179	0.6540	0.8782
0.78	8.4268	5.6642	0.6700	0.8850
0.79	9.0857	6.0461	0.6858	0.8916
0.80	9.8148	6.4693	0.7015	0.8982
0.81	10.6254	6.9404	0.7171	0.9045
0.82	11.5313	7.4676	0.7325	0.9108
0.83	12.5495	8.0612	0.7479	0.9169
0.84	13.7014	8.7339	0.7631	0.9228
0.85	15.0140	9.5018	0.7782	0.9286
0.86	16.5222	10.3859	0.7932	0.9343
0.87	18.2716	11.4134	0.8081	0.9398
0.88	20.3229	12.6208	0.8230	0.9453
0.89	22.7592	14.0576	0.8378	0.9505
0.90	25.6968	15.7937	0.8525	0.9557
0.91	29.3038	17.9298	0.8672	0.9607
0.92	33.8323	20.6170	0.8819	0.9656
0.93	39.6787	24.0932	0.8966	0.9704
0.94	47.5042	28.7549	0.9113	0.9750
0.95	58.4986	35.3158	0.9259	0.9795
0.96	75.0426	45.2041	0.9406	0.9839
0.97	102.691	61.7527	0.9554	0.9881
0.98	158.110	94.9610	0.9702	0.9922
0.99	324.632	194.828	0.9850	0.9962

Table E–5: $\hat{\tau}_3 = 0.625$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.625,\ \ 0.46 \le \hat{\tau}_4 \le 0.72$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.46	0.8104	1.1767	0.0031	0.5412
0.47	0.8858	1.2381	0.0224	0.5577
0.48	0.9662	1.3011	0.0427	0.5737
0.49	1.0517	1.3660	0.0638	0.5892
0.50	1.1424	1.4328	0.0855	0.6042
0.51	1.2384	1.5018	0.1077	0.6187
0.52	1.3401	1.5732	0.1302	0.6327
0.53	1.4476	1.6472	0.1528	0.6463
0.54	1.5614	1.7240	0.1754	0.6595
0.55	1.6816	1.8040	0.1980	0.6722
0.56	1.8088	1.8873	0.2205	0.6846
0.57	1.9434	1.9745	0.2428	0.6965
0.58	2.0858	2.0657	0.2649	0.7081
0.59	2.2366	2.1614	0.2868	0.7194
0.60	2.3964	2.2620	0.3084	0.7303
0.61	2.5660	2.3679	0.3298	0.7409
0.62	2.7460	2.4796	0.3508	0.7512
0.63	2.9374	2.5977	0.3715	0.7613
0.64	3.1410	2.7227	0.3920	0.7710
0.65	3.3581	2.8555	0.4122	0.7805
0.66	3.5897	2.9966	0.4320	0.7898
0.67	3.8372	3.1470	0.4516	0.7988
0.68	4.1022	3.3076	0.4709	0.8076
0.69	4.3864	3.4794	0.4899	0.8162
0.70	4.6917	3.6638	0.5087	0.8245
0.71	5.0205	3.8620	0.5272	0.8327
0.72	5.3752	4.0757	0.5454	0.8406

$\hat{\tau}_3 = 0.625,\ \ 0.73 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.73	5.7588	4.3067	0.5634	0.8484
0.74	6.1748	4.5572	0.5812	0.8560
0.75	6.6271	4.8295	0.5988	0.8634
0.76	7.1204	5.1266	0.6161	0.8707
0.77	7.6601	5.4518	0.6333	0.8777
0.78	8.2527	5.8092	0.6503	0.8846
0.79	8.9060	6.2035	0.6671	0.8914
0.80	9.6291	6.6405	0.6837	0.8980
0.81	10.4334	7.1272	0.7002	0.9044
0.82	11.3326	7.6721	0.7165	0.9107
0.83	12.3438	8.2858	0.7327	0.9169
0.84	13.4881	8.9816	0.7488	0.9229
0.85	14.7928	9.7763	0.7648	0.9287
0.86	16.2924	10.6916	0.7807	0.9344
0.87	18.0327	11.7559	0.7964	0.9400
0.88	20.0743	13.0070	0.8122	0.9454
0.89	22.5003	14.4967	0.8278	0.9508
0.90	25.4267	16.2975	0.8434	0.9559
0.91	29.0217	18.5144	0.8590	0.9609
0.92	33.5373	21.3047	0.8746	0.9658
0.93	39.3699	24.9161	0.8901	0.9706
0.94	47.1804	29.7615	0.9057	0.9752
0.95	58.1587	36.5844	0.9212	0.9797
0.96	74.6851	46.8723	0.9368	0.9841
0.97	102.314	64.0969	0.9525	0.9883
0.98	157.713	98.6739	0.9682	0.9923
0.99	324.212	202.686	0.9841	0.9962

Table E–5: $\hat{\tau}_3 = 0.65$ and λ_3, $\lambda_4 < 0$

$\hat{\tau}_3 = 0.650, \quad 0.49 \leq \hat{\tau}_4 \leq 0.75$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.49	0.9619	1.3644	0.0023	0.5775
0.50	1.0493	1.4338	0.0241	0.5933
0.51	1.1424	1.5055	0.0468	0.6086
0.52	1.2413	1.5795	0.0702	0.6233
0.53	1.3462	1.6561	0.0940	0.6377
0.54	1.4573	1.7356	0.1180	0.6515
0.55	1.5751	1.8183	0.1421	0.6649
0.56	1.6998	1.9044	0.1662	0.6778
0.57	1.8318	1.9943	0.1902	0.6903
0.58	1.9717	2.0884	0.2140	0.7024
0.59	2.1200	2.1869	0.2376	0.7142
0.60	2.2772	2.2905	0.2609	0.7255
0.61	2.4440	2.3994	0.2839	0.7366
0.62	2.6213	2.5143	0.3066	0.7473
0.63	2.8097	2.6358	0.3290	0.7576
0.64	3.0104	2.7643	0.3511	0.7677
0.65	3.2243	2.9007	0.3728	0.7775
0.66	3.4527	3.0457	0.3942	0.7871
0.67	3.6968	3.2002	0.4152	0.7963
0.68	3.9582	3.3652	0.4360	0.8054
0.69	4.2386	3.5418	0.4564	0.8142
0.70	4.5400	3.7312	0.4765	0.8227
0.71	4.8646	3.9349	0.4963	0.8311
0.72	5.2150	4.1545	0.5159	0.8392
0.73	5.5940	4.3920	0.5351	0.8472
0.74	6.0052	4.6494	0.5541	0.8549
0.75	6.4524	4.9294	0.5729	0.8625

$\hat{\tau}_3 = 0.650, \quad 0.76 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.76	6.9404	5.2350	0.5914	0.8698
0.77	7.4744	5.5696	0.6097	0.8770
0.78	8.0610	5.9374	0.6278	0.8840
0.79	8.7079	6.3434	0.6457	0.8909
0.80	9.4243	6.7934	0.6634	0.8976
0.81	10.2215	7.2949	0.6809	0.9041
0.82	11.1131	7.8565	0.6983	0.9105
0.83	12.1161	8.4894	0.7155	0.9167
0.84	13.2519	9.2072	0.7326	0.9228
0.85	14.5473	10.0274	0.7496	0.9287
0.86	16.0371	10.9725	0.7664	0.9344
0.87	17.7669	12.0721	0.7832	0.9401
0.88	19.7972	13.3654	0.7999	0.9455
0.89	22.2110	14.9062	0.8165	0.9509
0.90	25.1244	16.7699	0.8331	0.9561
0.91	28.7053	19.0655	0.8497	0.9611
0.92	33.2057	21.9566	0.8662	0.9660
0.93	39.0219	25.7007	0.8827	0.9708
0.94	46.8145	30.7270	0.8993	0.9754
0.95	57.7734	37.8088	0.9159	0.9799
0.96	74.2786	48.4928	0.9325	0.9842
0.97	101.885	66.3893	0.9492	0.9884
0.98	157.258	102.332	0.9660	0.9924
0.99	323.729	210.483	0.9829	0.9963

Table E–5: $0.675 \le \hat{\tau}_3 \le 0.7$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.675, \quad 0.53 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.53	1.2340	1.6534	0.0239	0.6266
0.54	1.3422	1.7357	0.0493	0.6413
0.55	1.4570	1.8211	0.0751	0.6554
0.56	1.5789	1.9100	0.1011	0.6691
0.57	1.7082	2.0027	0.1271	0.6824
0.58	1.8454	2.0995	0.1530	0.6951
0.59	1.9908	2.2010	0.1788	0.7075
0.60	2.1452	2.3074	0.2043	0.7194
0.61	2.3092	2.4193	0.2294	0.7310
0.62	2.4834	2.5372	0.2542	0.7421
0.63	2.6688	2.6618	0.2787	0.7530
0.64	2.8662	2.7936	0.3027	0.7635
0.65	3.0768	2.9335	0.3264	0.7737
0.66	3.3016	3.0821	0.3497	0.7835
0.67	3.5420	3.2404	0.3726	0.7931
0.68	3.7995	3.4094	0.3951	0.8025
0.69	4.0758	3.5903	0.4172	0.8115
0.70	4.3729	3.7844	0.4390	0.8204
0.71	4.6929	3.9931	0.4604	0.8290
0.72	5.0385	4.2182	0.4815	0.8373
0.73	5.4125	4.4615	0.5023	0.8455
0.74	5.8183	4.7254	0.5228	0.8534
0.75	6.2598	5.0125	0.5430	0.8611
0.76	6.7417	5.3259	0.5629	0.8687
0.77	7.2694	5.6691	0.5825	0.8760
0.78	7.8492	6.0465	0.6019	0.8832
0.79	8.4889	6.4632	0.6211	0.8901
0.80	9.1977	6.9254	0.6400	0.8969
0.81	9.9867	7.4406	0.6588	0.9036
0.82	10.8696	8.0178	0.6774	0.9100
0.83	11.8633	8.6686	0.6958	0.9164
0.84	12.9892	9.4070	0.7140	0.9225
0.85	14.2739	10.2513	0.7321	0.9285
0.86	15.7524	11.2248	0.7501	0.9343
0.87	17.4698	12.3579	0.7680	0.9400
0.88	19.4870	13.6914	0.7859	0.9455
0.89	21.8865	15.2812	0.8036	0.9509
0.90	24.7846	17.2055	0.8213	0.9561
0.91	28.3488	19.5772	0.8390	0.9612
0.92	32.8311	22.5662	0.8566	0.9661
0.93	38.6276	26.4398	0.8743	0.9709
0.94	46.3988	31.6435	0.8920	0.9755
0.95	57.3342	38.9800	0.9097	0.9800
0.96	73.8137	50.0556	0.9275	0.9843
0.97	101.392	68.6191	0.9454	0.9885
0.98	156.733	105.919	0.9635	0.9925
0.99	323.171	218.205	0.9816	0.9963

$\hat{\tau}_3 = 0.700, \quad 0.56 \le \hat{\tau}_4 \le 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.56	1.4432	1.9013	0.0208	0.6578
0.57	1.5695	1.9969	0.0494	0.6720
0.58	1.7037	2.0967	0.0780	0.6857
0.59	1.8462	2.2010	0.1066	0.6988
0.60	1.9976	2.3103	0.1349	0.7115
0.61	2.1585	2.4251	0.1629	0.7237
0.62	2.3296	2.5460	0.1905	0.7355
0.63	2.5117	2.6736	0.2176	0.7469
0.64	2.7057	2.8085	0.2443	0.7580
0.65	2.9127	2.9516	0.2706	0.7686
0.66	3.1337	3.1035	0.2963	0.7790
0.67	3.3701	3.2653	0.3215	0.7890
0.68	3.6234	3.4381	0.3463	0.7987
0.69	3.8953	3.6229	0.3707	0.8081
0.70	4.1877	3.8212	0.3945	0.8173
0.71	4.5027	4.0344	0.4180	0.8262
0.72	4.8430	4.2643	0.4410	0.8348
0.73	5.2113	4.5130	0.4637	0.8432
0.74	5.6112	4.7826	0.4860	0.8514
0.75	6.0464	5.0761	0.5079	0.8593
0.76	6.5215	5.3965	0.5295	0.8670
0.77	7.0420	5.7475	0.5507	0.8746
0.78	7.6142	6.1336	0.5717	0.8819
0.79	8.2458	6.5600	0.5924	0.8891
0.80	8.9458	7.0332	0.6129	0.8960
0.81	9.7255	7.5608	0.6331	0.9028
0.82	10.5984	8.1523	0.6531	0.9094
0.83	11.5815	8.8195	0.6729	0.9158
0.84	12.6958	9.5771	0.6925	0.9221
0.85	13.9682	10.4437	0.7119	0.9281
0.86	15.4334	11.4434	0.7313	0.9341
0.87	17.1365	12.6079	0.7505	0.9398
0.88	19.1381	13.9794	0.7696	0.9454
0.89	21.5208	15.6156	0.7886	0.9509
0.90	24.4005	17.5974	0.8076	0.9561
0.91	27.9449	20.0420	0.8266	0.9613
0.92	32.4055	23.1253	0.8455	0.9662
0.93	38.1782	27.1242	0.8645	0.9710
0.94	45.9233	32.5008	0.8835	0.9757
0.95	56.8300	40.0870	0.9026	0.9801
0.96	73.2779	51.5483	0.9217	0.9845
0.97	100.821	70.7722	0.9410	0.9886
0.98	156.124	109.423	0.9605	0.9926
0.99	322.517	225.833	0.9801	0.9964

Table E–5: $0.725 \leq \hat{\tau}_3 \leq 0.75$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.725,\quad 0.59 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.59	1.6810	2.1832	0.0140	0.6872
0.60	1.8294	2.2956	0.0463	0.7010
0.61	1.9873	2.4135	0.0783	0.7142
0.62	2.1553	2.5374	0.1099	0.7268
0.63	2.3341	2.6679	0.1409	0.7390
0.64	2.5247	2.8059	0.1713	0.7508
0.65	2.7280	2.9520	0.2011	0.7621
0.66	2.9451	3.1071	0.2302	0.7730
0.67	3.1774	3.2721	0.2587	0.7836
0.68	3.4262	3.4482	0.2866	0.7938
0.69	3.6934	3.6365	0.3138	0.8037
0.70	3.9807	3.8385	0.3405	0.8133
0.71	4.2903	4.0557	0.3666	0.8225
0.72	4.6248	4.2899	0.3922	0.8315
0.73	4.9870	4.5432	0.4172	0.8402
0.74	5.3802	4.8180	0.4418	0.8487
0.75	5.8084	5.1169	0.4659	0.8569
0.76	6.2760	5.4434	0.4896	0.8649
0.77	6.7884	5.8012	0.5129	0.8727
0.78	7.3520	6.1949	0.5359	0.8802
0.79	7.9743	6.6299	0.5585	0.8876
0.80	8.6644	7.1127	0.5807	0.8947
0.81	9.4334	7.6514	0.6027	0.9017
0.82	10.2949	8.2556	0.6244	0.9084
0.83	11.2655	8.9374	0.6459	0.9150
0.84	12.3665	9.7121	0.6672	0.9214
0.85	13.6246	10.5988	0.6882	0.9276
0.86	15.0741	11.6225	0.7091	0.9336
0.87	16.7602	12.8158	0.7299	0.9395
0.88	18.7434	14.2223	0.7506	0.9451
0.89	21.1061	15.9015	0.7711	0.9507
0.90	23.9639	17.9372	0.7916	0.9560
0.91	27.4843	20.4506	0.8121	0.9612
0.92	31.9185	23.6235	0.8325	0.9662
0.93	37.6623	27.7426	0.8530	0.9711
0.94	45.3753	33.2860	0.8736	0.9757
0.95	56.2466	41.1152	0.8942	0.9802
0.96	72.6550	52.9545	0.9149	0.9845
0.97	100.154	72.8298	0.9359	0.9887
0.98	155.407	112.822	0.9570	0.9926
0.99	321.746	233.345	0.9783	0.9964

$\hat{\tau}_3 = 0.750,\quad 0.62 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.62	1.9519	2.5057	0.0016	0.7149
0.63	2.1280	2.6396	0.0389	0.7283
0.64	2.3156	2.7808	0.0751	0.7411
0.65	2.5156	2.9300	0.1104	0.7533
0.66	2.7291	3.0882	0.1448	0.7651
0.67	2.9574	3.2564	0.1781	0.7764
0.68	3.2019	3.4356	0.2105	0.7873
0.69	3.4643	3.6272	0.2420	0.7978
0.70	3.7464	3.8325	0.2726	0.8080
0.71	4.0504	4.0531	0.3024	0.8177
0.72	4.3787	4.2910	0.3315	0.8272
0.73	4.7343	4.5482	0.3598	0.8363
0.74	5.1203	4.8272	0.3875	0.8452
0.75	5.5407	5.1308	0.4146	0.8538
0.76	6.0000	5.4623	0.4411	0.8621
0.77	6.5034	5.8258	0.4670	0.8702
0.78	7.0573	6.2258	0.4925	0.8780
0.79	7.6691	6.6679	0.5175	0.8856
0.80	8.3479	7.1588	0.5421	0.8930
0.81	9.1046	7.7068	0.5663	0.9001
0.82	9.9528	8.3217	0.5901	0.9071
0.83	10.9092	9.0161	0.6137	0.9138
0.84	11.9946	9.8055	0.6369	0.9204
0.85	13.2357	10.7098	0.6600	0.9267
0.86	14.6668	11.7545	0.6828	0.9329
0.87	16.3328	12.9732	0.7054	0.9389
0.88	18.2941	14.4110	0.7279	0.9447
0.89	20.6326	16.1292	0.7503	0.9503
0.90	23.4640	18.2141	0.7726	0.9558
0.91	26.9551	20.7909	0.7948	0.9610
0.92	31.3571	24.0475	0.8171	0.9661
0.93	37.0650	28.2799	0.8394	0.9710
0.94	44.7382	33.9823	0.8617	0.9757
0.95	55.5649	42.0453	0.8842	0.9803
0.96	71.9233	54.2522	0.9068	0.9846
0.97	99.3659	74.7668	0.9297	0.9888
0.98	154.556	116.086	0.9528	0.9927
0.99	320.822	240.705	0.9762	0.9965

Table E–5: $0.775 \leq \hat{\tau}_3 \leq 0.8$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.775,\ \ 0.66 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.66	2.4729	3.0392	0.0251	0.7540
0.67	2.6986	3.2109	0.0671	0.7665
0.68	2.9398	3.3935	0.1072	0.7784
0.69	3.1981	3.5882	0.1457	0.7899
0.70	3.4755	3.7967	0.1827	0.8008
0.71	3.7741	4.0205	0.2184	0.8113
0.72	4.0964	4.2615	0.2527	0.8214
0.73	4.4451	4.5220	0.2860	0.8312
0.74	4.8237	4.8044	0.3182	0.8405
0.75	5.2358	5.1117	0.3495	0.8496
0.76	5.6860	5.4472	0.3799	0.8584
0.77	6.1795	5.8150	0.4095	0.8668
0.78	6.7226	6.2199	0.4384	0.8750
0.79	7.3226	6.6675	0.4666	0.8829
0.80	7.9885	7.1648	0.4943	0.8906
0.81	8.7312	7.7200	0.5214	0.8981
0.82	9.5642	8.3434	0.5480	0.9053
0.83	10.5038	9.0477	0.5742	0.9123
0.84	11.5710	9.8490	0.6001	0.9190
0.85	12.7922	10.7675	0.6256	0.9256
0.86	14.2014	11.8295	0.6508	0.9319
0.87	15.8434	13.0696	0.6757	0.9381
0.88	17.7782	14.5340	0.7005	0.9440
0.89	20.0875	16.2858	0.7251	0.9498
0.90	22.8865	18.4140	0.7496	0.9554
0.91	26.3417	21.0473	0.7740	0.9607
0.92	30.7036	24.3796	0.7984	0.9659
0.93	36.3667	28.7162	0.8229	0.9709
0.94	43.9895	34.5670	0.8474	0.9757
0.95	54.7593	42.8516	0.8721	0.9803
0.96	71.0533	55.4116	0.8970	0.9846
0.97	98.4225	76.5485	0.9222	0.9888
0.98	153.528	119.174	0.9477	0.9928
0.99	319.697	247.865	0.9736	0.9965

$\hat{\tau}_3 = 0.800,\ \ 0.69 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.69	2.8728	3.5071	0.0018	0.7782
0.70	3.1488	3.7197	0.0518	0.7905
0.71	3.4444	3.9471	0.0987	0.8022
0.72	3.7622	4.1915	0.1427	0.8134
0.73	4.1054	4.4551	0.1845	0.8240
0.74	4.4770	4.7405	0.2243	0.8342
0.75	4.8812	5.0507	0.2624	0.8439
0.76	5.3221	5.3892	0.2989	0.8533
0.77	5.8052	5.7602	0.3341	0.8623
0.78	6.3366	6.1684	0.3681	0.8710
0.79	6.9236	6.6198	0.4011	0.8794
0.80	7.5752	7.1212	0.4331	0.8875
0.81	8.3020	7.6812	0.4643	0.8953
0.82	9.1174	8.3104	0.4948	0.9028
0.83	10.0377	9.0216	0.5246	0.9101
0.84	11.0835	9.8311	0.5538	0.9172
0.85	12.2811	10.7599	0.5825	0.9240
0.86	13.6642	11.8347	0.6109	0.9306
0.87	15.2774	13.0909	0.6388	0.9369
0.88	17.1800	14.5759	0.6665	0.9431
0.89	19.4536	16.3545	0.6939	0.9490
0.90	22.2126	18.5180	0.7211	0.9548
0.91	25.6229	21.1987	0.7483	0.9603
0.92	29.9344	24.5961	0.7754	0.9656
0.93	35.5404	29.0245	0.8025	0.9707
0.94	43.0984	35.0091	0.8298	0.9755
0.95	53.7943	43.4981	0.8572	0.9802
0.96	70.0034	56.3909	0.8849	0.9846
0.97	97.2746	78.1256	0.9129	0.9888
0.98	152.266	122.028	0.9414	0.9928
0.99	318.302	254.755	0.9704	0.9965

Table E–5: $0.825 \leq \hat{\tau}_3 \leq 0.85$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.825,\quad 0.73 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.73	3.6818	4.3281	0.0252	0.8132
0.74	4.0519	4.6179	0.0820	0.8248
0.75	4.4515	4.9316	0.1340	0.8357
0.76	4.8856	5.2731	0.1824	0.8461
0.77	5.3596	5.6465	0.2277	0.8559
0.78	5.8798	6.0570	0.2706	0.8654
0.79	6.4536	6.5104	0.3114	0.8744
0.80	7.0899	7.0140	0.3504	0.8831
0.81	7.7993	7.5763	0.3880	0.8914
0.82	8.5950	8.2081	0.4242	0.8994
0.83	9.4932	8.9225	0.4593	0.9072
0.84	10.5142	9.7361	0.4934	0.9146
0.85	11.6839	10.6702	0.5268	0.9218
0.86	13.0359	11.7520	0.5594	0.9287
0.87	14.6140	13.0177	0.5914	0.9353
0.88	16.4774	14.5155	0.6229	0.9418
0.89	18.7067	16.3119	0.6541	0.9479
0.90	21.4158	18.5002	0.6850	0.9539
0.91	24.7694	21.2161	0.7156	0.9596
0.92	29.0163	24.6641	0.7462	0.9650
0.93	34.5485	29.1670	0.7768	0.9703
0.94	42.0215	35.2650	0.8074	0.9753
0.95	52.6190	43.9337	0.8383	0.9800
0.96	68.7134	57.1293	0.8695	0.9846
0.97	95.8500	79.4253	0.9011	0.9888
0.98	150.683	124.560	0.9333	0.9928
0.99	316.529	261.267	0.9662	0.9966

$\hat{\tau}_3 = 0.850,\quad 0.77 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.77	4.7919	5.4427	0.0517	0.8461
0.78	5.3089	5.8574	0.1161	0.8569
0.79	5.8744	6.3134	0.1741	0.8671
0.80	6.4981	6.8183	0.2274	0.8767
0.81	7.1911	7.3811	0.2769	0.8859
0.82	7.9666	8.0127	0.3234	0.8946
0.83	8.8408	8.7266	0.3676	0.9030
0.84	9.8339	9.5395	0.4098	0.9110
0.85	10.9714	10.4730	0.4503	0.9186
0.86	12.2865	11.5549	0.4895	0.9260
0.87	13.8224	12.8216	0.5276	0.9331
0.88	15.6376	14.3224	0.5648	0.9398
0.89	17.8118	16.1247	0.6012	0.9463
0.90	20.4577	18.3237	0.6371	0.9526
0.91	23.7386	21.0579	0.6725	0.9585
0.92	27.9015	24.5362	0.7078	0.9642
0.93	33.3360	29.0892	0.7429	0.9697
0.94	40.6948	35.2708	0.7781	0.9749
0.95	51.1578	44.0826	0.8135	0.9798
0.96	67.0925	57.5356	0.8492	0.9844
0.97	94.0378	80.3360	0.8855	0.9888
0.98	148.639	126.630	0.9226	0.9928
0.99	314.202	267.225	0.9607	0.9966

Table E–5: $0.875 \leq \hat{\tau}_3 \leq 0.975$ and $\lambda_3,\ \lambda_4 < 0$

$\hat{\tau}_3 = 0.875,\quad 0.81 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.81	6.3962	7.0413	0.0820	0.8770
0.82	7.1628	7.6753	0.1558	0.8871
0.83	8.0189	8.3880	0.2211	0.8966
0.84	8.9859	9.1972	0.2803	0.9055
0.85	10.0898	10.1249	0.3351	0.9140
0.86	11.3636	11.1992	0.3864	0.9220
0.87	12.8503	12.4572	0.4350	0.9297
0.88	14.6071	13.9484	0.4815	0.9370
0.89	16.7128	15.7412	0.5264	0.9440
0.90	19.2783	17.9319	0.5700	0.9506
0.91	22.4648	20.6609	0.6126	0.9570
0.92	26.5163	24.1407	0.6546	0.9630
0.93	31.8187	28.7081	0.6962	0.9688
0.94	39.0198	34.9285	0.7377	0.9742
0.95	49.2929	43.8269	0.7793	0.9793
0.96	64.9966	57.4653	0.8214	0.9841
0.97	91.6577	80.6770	0.8641	0.9886
0.98	145.905	128.008	0.9079	0.9928
0.99	311.020	272.327	0.9530	0.9966

$\hat{\tau}_3 = 0.900,\quad 0.84 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.84	7.7963	8.5888	0.0124	0.8958
0.85	8.8988	9.5224	0.1179	0.9062
0.86	10.1460	10.5908	0.2042	0.9156
0.87	11.5870	11.8344	0.2791	0.9244
0.88	13.2808	13.3044	0.3463	0.9326
0.89	15.3058	15.0699	0.4082	0.9403
0.90	17.7711	17.2279	0.4663	0.9476
0.91	20.8351	19.9198	0.5217	0.9546
0.92	24.7371	23.3597	0.5750	0.9611
0.93	29.8569	27.8878	0.6270	0.9673
0.94	36.8336	34.0776	0.6783	0.9731
0.95	46.8280	42.9717	0.7293	0.9786
0.96	62.1817	56.6745	0.7806	0.9837
0.97	88.3965	80.1337	0.8327	0.9884
0.98	142.064	128.277	0.8862	0.9927
0.99	306.412	276.001	0.9417	0.9966

$\hat{\tau}_3 = 0.925,\quad 0.88 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.88	11.3148	12.1265	0.0302	0.9240
0.89	13.3099	13.8824	0.1640	0.9337
0.90	15.6842	15.9967	0.2684	0.9424
0.91	18.6070	18.6187	0.3572	0.9504
0.92	22.3162	21.9647	0.4366	0.9579
0.93	27.1833	26.3750	0.5101	0.9648
0.94	33.8322	32.4239	0.5799	0.9713
0.95	43.4001	41.1596	0.6476	0.9773
0.96	58.1919	54.7096	0.7145	0.9828
0.97	83.6523	78.1043	0.7818	0.9879
0.98	136.282	126.598	0.8509	0.9925
0.99	299.157	277.022	0.9230	0.9966

$\hat{\tau}_3 = 0.950,\quad 0.92 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.92	18.3863	19.2156	0.0482	0.9507
0.93	23.0497	23.4970	0.2328	0.9597
0.94	29.2707	29.2703	0.3677	0.9676
0.95	38.1826	37.5943	0.4813	0.9747
0.96	52.0178	50.5787	0.5846	0.9811
0.97	76.0744	73.2447	0.6837	0.9869
0.98	126.578	120.989	0.7830	0.9920
0.99	286.065	272.195	0.8867	0.9964

$\hat{\tau}_3 = 0.975,\quad 0.96 \leq \hat{\tau}_4 \leq 0.99$				
$\hat{\tau}_4$	$-A$	$-B$	$-\lambda_3$	$-\lambda_4$
0.96	39.6673	40.5128	0.0662	0.9760
0.97	61.2841	61.2712	0.3778	0.9841
0.98	106.565	105.095	0.5892	0.9907
0.99	255.322	249.694	0.7841	0.9961

Appendix F

Table F–1 for Kappa Distribution Fits: Method of L-Moments

For specified τ_3 and τ_4, the table that follows gives the h and k values, along with the values of A and B that were discussed in Chapter 17 (see Section 17.2), for kappa distribution fits through the use of L-moments.

$$-0.975 \leq \tau_3 \leq -0.800$$

$\hat{\tau}_3 = -0.975, \quad 0.95 \leq \hat{\tau}_4 \leq 0.95$				
$\hat{\tau}_4$	A	B	h	k
0.95	37.41125	-37.6233	-0.3853	2.3908

$\hat{\tau}_3 = -0.950, \quad 0.89 \leq \hat{\tau}_4 \leq 0.91$				
$\hat{\tau}_4$	A	B	h	k
0.89	26.72951	-27.2864	-0.0538	4.5729
0.90	16.98128	-17.1575	-0.3220	2.5435
0.91	15.72812	-15.7406	-0.5794	1.5924

$\hat{\tau}_3 = -0.925, \quad 0.83 \leq \hat{\tau}_4 \leq 0.87$				
$\hat{\tau}_4$	A	B	h	k
0.83	13.72800	-14.1831	0.2870	8.5206
0.84	11.53917	-11.8196	-0.0460	4.1342
0.85	10.05634	-10.1934	-0.2503	2.7315
0.86	9.656103	-9.6907	-0.4282	1.9577
0.87	10.24782	-10.2077	-0.6421	1.3993

$\hat{\tau}_3 = -0.900, \quad 0.77 \leq \hat{\tau}_4 \leq 0.84$				
$\hat{\tau}_4$	A	B	h	k
0.77	0.143028	-0.1073	0.8014	23.0005
0.78	5.577399	-5.7554	0.2633	6.9823
0.79	6.455508	-6.6131	0.0090	4.1789
0.80	6.510171	-6.6048	-0.1692	2.9652
0.81	6.575946	-6.6076	-0.3194	2.2485
0.82	6.837989	-6.8131	-0.4698	1.7416
0.83	7.455962	-7.3845	-0.6521	1.3328
0.84	8.788012	-8.6924	-0.9319	0.9624

$\hat{\tau}_3 = -0.875, \quad 0.72 \leq \hat{\tau}_4 \leq 0.80$				
$\hat{\tau}_4$	A	B	h	k
0.72	0.305401	-0.2607	0.7025	15.1263
0.73	2.689331	-2.7411	0.3158	6.8538
0.74	3.835212	-3.9073	0.0894	4.4369
0.75	4.324528	-4.3743	-0.0774	3.2592
0.76	4.618015	-4.6322	-0.2165	2.5399
0.77	4.883730	-4.8590	-0.3450	2.0362
0.78	5.218744	-5.1561	-0.4779	1.6458
0.79	5.720139	-5.6242	-0.6347	1.3166
0.80	6.557120	-6.4399	-0.8520	1.0158

$\hat{\tau}_3 = -0.850, \quad 0.67 \leq \hat{\tau}_4 \leq 0.76$				
$\hat{\tau}_4$	A	B	h	k
0.67	0.115750	-0.0556	0.7530	14.2656
0.68	1.249724	-1.2243	0.4087	7.2950
0.69	2.234356	-2.2381	0.1907	4.8743
0.70	2.834143	-2.8378	0.0268	3.6355
0.71	3.230909	-3.2185	-0.1090	2.8689
0.72	3.537320	-3.5011	-0.2299	2.3362
0.73	3.820195	-3.7566	-0.3453	1.9335
0.74	4.131890	-4.0404	-0.4649	1.6074
0.75	4.528253	-4.4109	-0.6013	1.3268
0.76	5.094368	-4.9578	-0.7761	1.0705

$\hat{\tau}_3 = -0.825, \quad 0.62 \leq \hat{\tau}_4 \leq 0.73$				
$\hat{\tau}_4$	A	B	h	k
0.62	0.019478	0.0425	0.8769	15.6288
0.63	0.483704	-0.4131	0.5354	8.2304
0.64	1.192527	-1.1419	0.3133	5.5256
0.65	1.763908	-1.7229	0.1458	4.1286
0.66	2.190120	-2.1460	0.0082	3.2672
0.67	2.522720	-2.4666	-0.1118	2.6748
0.68	2.803941	-2.7305	-0.2219	2.2355
0.69	3.066451	-2.9725	-0.3281	1.8896
0.70	3.340002	-3.2243	-0.4372	1.6029
0.71	3.657408	-3.5207	-0.5576	1.3539
0.72	4.064272	-3.9100	-0.7026	1.1274
0.73	4.647539	-4.4851	-0.8984	0.9101

$\hat{\tau}_3 = -0.800, \quad 0.57 \leq \hat{\tau}_4 \leq 0.70$				
$\hat{\tau}_4$	A	B	h	k
0.57	0.000794	0.0511	1.0660	19.2234
0.58	0.125418	-0.0389	0.6989	9.8333
0.59	0.530132	-0.4422	0.4610	6.4862
0.60	0.989987	-0.9098	0.2833	4.7962
0.61	1.391131	-1.3139	0.1390	3.7743
0.62	1.725026	-1.6441	0.0154	3.0842
0.63	2.008094	-1.9179	-0.0953	2.5819
0.64	2.258799	-2.1552	-0.1981	2.1952
0.65	2.494426	-2.3749	-0.2976	1.8835
0.66	2.732391	-2.5954	-0.3984	1.6219
0.67	2.992408	-2.8378	-0.5065	1.3938
0.68	3.300285	-3.1296	-0.6301	1.1876
0.69	3.697749	-3.5157	-0.7838	0.9934
0.70	4.275837	-4.0948	-1.0000	0.8000

$$-0.775 \leq \tau_3 \leq -0.700$$

$\hat{\tau}_3 = -0.775, \quad 0.52 \leq \hat{\tau}_4 \leq 0.66$				
$\hat{\tau}_4$	A	B	h	k
0.52	0.000001	0.0370	1.3363	27.0511
0.53	0.013702	0.0638	0.9091	12.6010
0.54	0.164687	-0.0621	0.6407	7.9556
0.55	0.459254	-0.3517	0.4445	5.7407
0.56	0.786650	-0.6802	0.2883	4.4518
0.57	1.092497	-0.9856	0.1566	3.6064
0.58	1.365225	-1.2543	0.0411	3.0055
0.59	1.608607	-1.4901	-0.0637	2.5531
0.60	1.830676	-1.7015	-0.1616	2.1967
0.61	2.040663	-1.8986	-0.2561	1.9055
0.62	2.248794	-2.0924	-0.3506	1.6595
0.63	2.467032	-2.2956	-0.4493	1.4449
0.64	2.710572	-2.5245	-0.5576	1.2521
0.65	3.001421	-2.8030	-0.6842	1.0731
0.66	3.378411	-3.1733	-0.8447	0.9006

$\hat{\tau}_3 = -0.725, \quad 0.44 \leq \hat{\tau}_4 \leq 0.60$				
$\hat{\tau}_4$	A	B	h	k
0.44	0.000583	0.0666	1.1558	14.8631
0.45	0.022524	0.0789	0.8776	9.4926
0.46	0.111784	0.0139	0.6706	6.8560
0.47	0.265039	-0.1259	0.5050	5.3100
0.48	0.450683	-0.3042	0.3659	4.2971
0.49	0.644718	-0.4930	0.2449	3.5815
0.50	0.835107	-0.6781	0.1365	3.0473
0.51	1.017446	-0.8540	0.0371	2.6314
0.52	1.191352	-1.0201	-0.0558	2.2968
0.53	1.358536	-1.1780	-0.1445	2.0198
0.54	1.521999	-1.3310	-0.2312	1.7849
0.55	1.685868	-1.4835	-0.3181	1.5810
0.56	1.855570	-1.6414	-0.4078	1.4001
0.57	2.038348	-1.8124	-0.5040	1.2359
0.58	2.244448	-2.0078	-0.6117	1.0835
0.59	2.490070	-2.2453	-0.7392	0.9380
0.60	2.805672	-2.5591	-0.9022	0.7941

$\hat{\tau}_3 = -0.750, \quad 0.48 \leq \hat{\tau}_4 \leq 0.63$				
$\hat{\tau}_4$	A	B	h	k
0.48	0.000181	0.0558	1.1872	17.8657
0.49	0.024188	0.0684	0.8647	10.3735
0.50	0.148352	-0.0325	0.6379	7.1611
0.51	0.361322	-0.2361	0.4622	5.4082
0.52	0.605875	-0.4772	0.3174	4.3082
0.53	0.847896	-0.7164	0.1928	3.5521
0.54	1.075032	-0.9391	0.0820	2.9979
0.55	1.285551	-1.1429	-0.0192	2.5719
0.56	1.482128	-1.3306	-0.1139	2.2317
0.57	1.669349	-1.5071	-0.2049	1.9514
0.58	1.853038	-1.6787	-0.2948	1.7138
0.59	2.040373	-1.8530	-0.3865	1.5070
0.60	2.240478	-2.0400	-0.4839	1.3224
0.61	2.465753	-2.2529	-0.5924	1.1530
0.62	2.735240	-2.5127	-0.7204	0.9932
0.63	3.084136	-2.8580	-0.8843	0.8366

$\hat{\tau}_3 = -0.700, \quad 0.39 \leq \hat{\tau}_4 \leq 0.57$				
$\hat{\tau}_4$	A	B	h	k
0.39	0.000001	0.0398	1.5604	25.0988
0.40	0.000539	0.0715	1.1888	13.8653
0.41	0.014933	0.0900	0.9296	9.2721
0.42	0.073278	0.0579	0.7310	6.8548
0.43	0.180673	-0.0320	0.5693	5.3812
0.44	0.320431	-0.1606	0.4321	4.3927
0.45	0.475324	-0.3076	0.3118	3.6836
0.46	0.634260	-0.4599	0.2039	3.1490
0.47	0.791521	-0.6104	0.1050	2.7302
0.48	0.944859	-0.7563	0.0127	2.3919
0.49	1.094085	-0.8971	-0.0750	2.1116
0.50	1.240325	-1.0341	-0.1598	1.8741
0.51	1.385686	-1.1693	-0.2436	1.6686
0.52	1.533241	-1.3061	-0.3282	1.4875
0.53	1.687204	-1.4490	-0.4161	1.3247
0.54	1.853376	-1.6044	-0.5107	1.1754
0.55	2.040138	-1.7815	-0.6165	1.0357
0.56	2.260857	-1.9951	-0.7412	0.9015
0.57	2.540514	-2.2733	-0.8993	0.7684

$$-0.675 \le \tau_3 \le -0.600$$

$\hat{\tau}_3 = -0.675, \quad 0.36 \le \hat{\tau}_4 \le 0.54$				
$\hat{\tau}_4$	A	B	h	k
0.36	0.000236	0.0714	1.2693	13.9496
0.37	0.007380	0.0962	1.0146	9.5122
0.38	0.041270	0.0905	0.8165	7.1006
0.39	0.112326	0.0408	0.6540	5.6044
0.40	0.214166	-0.0460	0.5154	4.5904
0.41	0.335219	-0.1561	0.3938	3.8586
0.42	0.465886	-0.2780	0.2846	3.3050
0.43	0.599971	-0.4043	0.1848	2.8707
0.44	0.734068	-0.5307	0.0918	2.5199
0.45	0.866662	-0.6552	0.0041	2.2295
0.46	0.997652	-0.7775	-0.0802	1.9840
0.47	1.127670	-0.8982	-0.1623	1.7726
0.48	1.258259	-1.0190	-0.2438	1.5873
0.49	1.391736	-1.1421	-0.3265	1.4223
0.50	1.531386	-1.2714	-0.4126	1.2727
0.51	1.681817	-1.4117	-0.5049	1.1346
0.52	1.849743	-1.5705	-0.6078	1.0047
0.53	2.045819	-1.7597	-0.7279	0.8798
0.54	2.289483	-2.0015	-0.8778	0.7559

$\hat{\tau}_3 = -0.625, \quad 0.28 \le \hat{\tau}_4 \le 0.49$				
$\hat{\tau}_4$	A	B	h	k
0.28	0.000003	0.0588	1.5642	16.9911
0.29	0.000527	0.0870	1.2845	11.4111
0.30	0.006530	0.1106	1.0672	8.4165
0.31	0.027949	0.1166	0.8895	6.5815
0.32	0.070778	0.0969	0.7387	5.3524
0.33	0.134072	0.0521	0.6072	4.4750
0.34	0.213286	-0.0124	0.4900	3.8181
0.35	0.303405	-0.0904	0.3836	3.3078
0.36	0.400322	-0.1769	0.2856	2.8994
0.37	0.501125	-0.2682	0.1941	2.5646
0.38	0.603964	-0.3620	0.1076	2.2845
0.39	0.707837	-0.4568	0.0249	2.0459
0.40	0.812401	-0.5523	-0.0552	1.8396
0.41	0.917867	-0.6484	-0.1338	1.6586
0.42	1.024954	-0.7458	-0.2120	1.4976
0.43	1.134921	-0.8460	-0.2912	1.3525
0.44	1.249673	-0.9509	-0.3732	1.2198
0.45	1.371959	-1.0636	-0.4601	1.0969
0.46	1.505786	-1.1886	-0.5550	0.9811
0.47	1.657294	-1.3328	-0.6627	0.8700
0.48	1.836843	-1.5080	-0.7910	0.7610
0.49	2.064950	-1.7381	-0.9560	0.6501

$\hat{\tau}_3 = -0.650, \quad 0.32 \le \hat{\tau}_4 \le 0.51$				
$\hat{\tau}_4$	A	B	h	k
0.32	0.000048	0.0669	1.3933	14.9317
0.33	0.002559	0.0950	1.1319	10.1913
0.34	0.019028	0.1080	0.9277	7.6007
0.35	0.061505	0.0903	0.7599	5.9912
0.36	0.131190	0.0398	0.6169	4.9012
0.37	0.221909	-0.0363	0.4916	4.1158
0.38	0.326212	-0.1292	0.3795	3.5232
0.39	0.438118	-0.2316	0.2771	3.0595
0.40	0.553604	-0.3384	0.1824	2.6860
0.41	0.670292	-0.4466	0.0934	2.3780
0.42	0.787021	-0.5548	0.0086	2.1188
0.43	0.903545	-0.6624	-0.0732	1.8966
0.44	1.020284	-0.7699	-0.1533	1.7032
0.45	1.138312	-0.8782	-0.2330	1.5321
0.46	1.259361	-0.9893	-0.3139	1.3787
0.47	1.385960	-1.1058	-0.3980	1.2388
0.48	1.521707	-1.2318	-0.4877	1.1093
0.49	1.671855	-1.3731	-0.5869	0.9873
0.50	1.844612	-1.5389	-0.7011	0.8700
0.51	2.054430	-1.7457	-0.8406	0.7541

$\hat{\tau}_3 = -0.600, \quad 0.25 \le \hat{\tau}_4 \le 0.46$				
$\hat{\tau}_4$	A	B	h	k
0.25	0.000045	0.0743	1.4800	13.4401
0.26	0.001393	0.1015	1.2406	9.6868
0.27	0.009501	0.1219	1.0470	7.4562
0.28	0.031501	0.1261	0.8842	5.9954
0.29	0.070740	0.1094	0.7433	4.9708
0.30	0.126207	0.0726	0.6186	4.2144
0.31	0.194715	0.0194	0.5062	3.6337
0.32	0.272721	-0.0456	0.4034	3.1738
0.33	0.357194	-0.1187	0.3080	2.8000
0.34	0.445860	-0.1971	0.2185	2.4899
0.35	0.537173	-0.2787	0.1335	2.2278
0.36	0.630203	-0.3623	0.0521	2.0028
0.37	0.724535	-0.4472	-0.0269	1.8070
0.38	0.820189	-0.5333	-0.1044	1.6344
0.39	0.917603	-0.6211	-0.1814	1.4802
0.40	1.017653	-0.7114	-0.2591	1.3410
0.41	1.121726	-0.8057	-0.3390	1.2136
0.42	1.231870	-0.9063	-0.4229	1.0955
0.43	1.351063	-1.0165	-0.5135	0.9845
0.44	1.483783	-1.1413	-0.6144	0.8785
0.45	1.637267	-1.2892	-0.7318	0.7751
0.46	1.824770	-1.4754	-0.8768	0.6716

$$-0.575 \leq \tau_3 \leq -0.550$$

$\hat{\tau}_3 = -0.575, \quad 0.21 \leq \hat{\tau}_4 \leq 0.44$				
$\hat{\tau}_4$	A	B	h	k
0.21	0.000001	0.0592	1.7318	16.8894
0.22	0.000129	0.0853	1.4573	11.6972
0.23	0.001992	0.1114	1.2395	8.7699
0.24	0.010312	0.1309	1.0589	6.9230
0.25	0.030280	0.1367	0.9044	5.6630
0.26	0.064279	0.1253	0.7690	4.7530
0.27	0.111728	0.0971	0.6481	4.0666
0.28	0.170448	0.0546	0.5383	3.5307
0.29	0.237857	0.0012	0.4372	3.1008
0.30	0.311613	-0.0602	0.3431	2.7479
0.31	0.389848	-0.1273	0.2546	2.4526
0.32	0.471207	-0.1982	0.1704	2.2014
0.33	0.554799	-0.2718	0.0896	1.9848
0.34	0.640136	-0.3474	0.0113	1.7954
0.35	0.727086	-0.4246	-0.0654	1.6279
0.36	0.815860	-0.5036	-0.1415	1.4780
0.37	0.907018	-0.5849	-0.2179	1.3425
0.38	1.001528	-0.6696	-0.2958	1.2186
0.39	1.100861	-0.7593	-0.3770	1.1040
0.40	1.207176	-0.8564	-0.4633	0.9965
0.41	1.323667	-0.9644	-0.5579	0.8943
0.42	1.455300	-1.0892	-0.6652	0.7955
0.43	1.610544	-1.2405	-0.7929	0.6978
0.44	1.806272	-1.4383	-0.9564	0.5978

$\hat{\tau}_3 = -0.550, \quad 0.18 \leq \hat{\tau}_4 \leq 0.41$				
$\hat{\tau}_4$	A	B	h	k
0.18	0.000002	0.0663	1.7334	15.0768
0.19	0.000174	0.0921	1.4782	10.8347
0.20	0.001997	0.1177	1.2712	8.3100
0.21	0.009214	0.1379	1.0970	6.6589
0.22	0.025858	0.1470	0.9465	5.5040
0.23	0.054036	0.1419	0.8135	4.6545
0.24	0.093683	0.1224	0.6940	4.0049
0.25	0.143397	0.0901	0.5851	3.4926
0.26	0.201288	0.0473	0.4845	3.0781
0.27	0.265502	-0.0034	0.3907	2.7357
0.28	0.334464	-0.0603	0.3022	2.4477
0.29	0.406953	-0.1215	0.2181	2.2019
0.30	0.482097	-0.1860	0.1374	1.9891
0.31	0.559350	-0.2529	0.0593	1.8027
0.32	0.638454	-0.3219	-0.0171	1.6376
0.33	0.719435	-0.3929	-0.0926	1.4899
0.34	0.802598	-0.4661	-0.1679	1.3563
0.35	0.888572	-0.5421	-0.2443	1.2343
0.36	0.978368	-0.6221	-0.3229	1.1216
0.37	1.073501	-0.7077	-0.4054	1.0165
0.38	1.176202	-0.8014	-0.4943	0.9170
0.39	1.289834	-0.9071	-0.5927	0.8216
0.40	1.419798	-1.0310	-0.7060	0.7284
0.41	1.575768	-1.1847	-0.8437	0.6350

$$-0.525 \le \tau_3 \le -0.500$$

$\hat{\tau}_3 = -0.525, \quad 0.15 \le \hat{\tau}_4 \le 0.39$				
$\hat{\tau}_4$	A	B	h	k
0.15	0.000002	0.0697	1.7825	14.3509
0.16	0.000144	0.0949	1.5348	10.5158
0.17	0.001538	0.1204	1.3315	8.1638
0.18	0.007008	0.1421	1.1590	6.5946
0.19	0.019850	0.1553	1.0090	5.4815
0.20	0.042150	0.1568	0.8759	4.6543
0.21	0.074343	0.1459	0.7560	4.0170
0.22	0.115672	0.1233	0.6464	3.5114
0.23	0.164803	0.0907	0.5451	3.1006
0.24	0.220275	0.0499	0.4505	2.7602
0.25	0.280746	0.0027	0.3614	2.4732
0.26	0.345104	-0.0494	0.2767	2.2276
0.27	0.412498	-0.1053	0.1955	2.0149
0.28	0.482329	-0.1642	0.1170	1.8284
0.29	0.554244	-0.2255	0.0406	1.6632
0.30	0.628113	-0.2890	-0.0346	1.5154
0.31	0.704044	-0.3546	-0.1093	1.3820
0.32	0.782390	-0.4228	-0.1845	1.2603
0.33	0.863804	-0.4941	-0.2610	1.1484
0.34	0.949303	-0.5698	-0.3403	1.0443
0.35	1.040404	-0.6514	-0.4242	0.9464
0.36	1.139356	-0.7415	-0.5152	0.8533
0.37	1.249607	-0.8442	-0.6168	0.7633
0.38	1.376844	-0.9661	-0.7352	0.6746
0.39	1.531700	-1.1200	-0.8817	0.5848

$\hat{\tau}_3 = -0.500, \quad 0.12 \le \hat{\tau}_4 \le 0.37$				
$\hat{\tau}_4$	A	B	h	k
0.12	0.000001	0.0695	1.8735	14.3828
0.13	0.000079	0.0941	1.6247	10.6170
0.14	0.000924	0.1195	1.4193	8.2782
0.15	0.004534	0.1429	1.2444	6.7050
0.16	0.013618	0.1601	1.0919	5.5828
0.17	0.030291	0.1682	0.9565	4.7459
0.18	0.055446	0.1656	0.8343	4.0993
0.19	0.088899	0.1524	0.7226	3.5856
0.20	0.129811	0.1296	0.6194	3.1677
0.21	0.177076	0.0985	0.5231	2.8211
0.22	0.229567	0.0606	0.4325	2.5289
0.23	0.286281	0.0172	0.3465	2.2789
0.24	0.346390	-0.0304	0.2642	2.0624
0.25	0.409263	-0.0815	0.1849	1.8727
0.26	0.474467	-0.1355	0.1079	1.7049
0.27	0.541754	-0.1918	0.0324	1.5550
0.28	0.611069	-0.2504	-0.0420	1.4199
0.29	0.682555	-0.3114	-0.1164	1.2971
0.30	0.756581	-0.3750	-0.1914	1.1844
0.31	0.833787	-0.4420	-0.2682	1.0802
0.32	0.915165	-0.5134	-0.3482	0.9827
0.33	1.002188	-0.5909	-0.4331	0.8906
0.34	1.097056	-0.6770	-0.5255	0.8025
0.35	1.203187	-0.7758	-0.6295	0.7170
0.36	1.326326	-0.8940	-0.7514	0.6322
0.37	1.477547	-1.0450	-0.9040	0.5455

$$-0.475 \leq \tau_3 \leq -0.450$$

$\hat{\tau}_3 = -0.475, \quad 0.09 \leq \hat{\tau}_4 \leq 0.35$				
$\hat{\tau}_4$	A	B	h	k
0.09	0.000000	0.0662	2.0066	15.1116
0.10	0.000028	0.0899	1.7486	11.1243
0.11	0.000419	0.1150	1.5356	8.6526
0.12	0.002428	0.1396	1.3544	6.9934
0.13	0.008172	0.1604	1.1964	5.8124
0.14	0.019719	0.1745	1.0563	4.9334
0.15	0.038343	0.1798	0.9300	4.2558
0.16	0.064388	0.1756	0.8148	3.7184
0.17	0.097494	0.1623	0.7084	3.2822
0.18	0.136910	0.1407	0.6094	2.9210
0.19	0.181740	0.1120	0.5163	2.6171
0.20	0.231099	0.0774	0.4282	2.3576
0.21	0.284204	0.0380	0.3442	2.1333
0.22	0.340409	-0.0055	0.2634	1.9372
0.23	0.399221	-0.0521	0.1853	1.7641
0.24	0.460301	-0.1016	0.1091	1.6098
0.25	0.523468	-0.1535	0.0343	1.4711
0.26	0.588698	-0.2077	-0.0398	1.3454
0.27	0.656145	-0.2643	-0.1140	1.2306
0.28	0.726164	-0.3236	-0.1890	1.1248
0.29	0.799360	-0.3864	-0.2660	1.0265
0.30	0.876663	-0.4536	-0.3463	0.9342
0.31	0.959462	-0.5267	-0.4317	0.8468
0.32	1.049837	-0.6083	-0.5249	0.7629
0.33	1.151043	-0.7021	-0.6299	0.6811
0.34	1.268608	-0.8147	-0.7533	0.5998
0.35	1.413336	-0.9594	-0.9085	0.5163

$\hat{\tau}_3 = -0.450, \quad 0.07 \leq \hat{\tau}_4 \leq 0.33$				
$\hat{\tau}_4$	A	B	h	k
0.07	0.000005	0.0825	1.9106	12.1132
0.08	0.000131	0.1070	1.6838	9.3337
0.09	0.001014	0.1320	1.4917	7.4907
0.10	0.004116	0.1556	1.3249	6.1916
0.11	0.011287	0.1747	1.1774	5.2325
0.12	0.024046	0.1871	1.0450	4.4980
0.13	0.043205	0.1914	0.9244	3.9188
0.14	0.068886	0.1874	0.8135	3.4510
0.15	0.100720	0.1752	0.7105	3.0655
0.16	0.138076	0.1558	0.6140	2.7424
0.17	0.180225	0.1300	0.5229	2.4675
0.18	0.226453	0.0989	0.4363	2.2307
0.19	0.276125	0.0633	0.3534	2.0244
0.20	0.328713	0.0239	0.2735	1.8428
0.21	0.383812	-0.0186	0.1960	1.6815
0.22	0.441141	-0.0639	0.1203	1.5371
0.23	0.500551	-0.1116	0.0458	1.4066
0.24	0.562029	-0.1617	-0.0282	1.2880
0.25	0.625718	-0.2142	-0.1022	1.1792
0.26	0.691940	-0.2695	-0.1773	1.0786
0.27	0.761242	-0.3281	-0.2543	0.9849
0.28	0.834470	-0.3910	-0.3346	0.8969
0.29	0.912886	-0.4595	-0.4200	0.8132
0.30	0.998386	-0.5360	-0.5131	0.7328
0.31	1.093944	-0.6238	-0.6177	0.6543
0.32	1.204597	-0.7291	-0.7405	0.5762
0.33	1.340163	-0.8640	-0.8942	0.4960

$$-0.425 \le \tau_3 \le -0.400$$

$\hat{\tau}_3 = -0.425, \quad 0.04 \le \hat{\tau}_4 \le 0.31$				
$\hat{\tau}_4$	A	B	h	k
0.04	0.000000	0.0726	2.1182	13.7799
0.05	0.000023	0.0958	1.8698	10.4316
0.06	0.000296	0.1206	1.6611	8.2644
0.07	0.001621	0.1454	1.4813	6.7647
0.08	0.005408	0.1683	1.3232	5.6734
0.09	0.013199	0.1866	1.1820	4.8476
0.10	0.026160	0.1987	1.0540	4.2028
0.11	0.044875	0.2034	0.9368	3.6864
0.12	0.069397	0.2006	0.8284	3.2639
0.13	0.099411	0.1905	0.7273	2.9121
0.14	0.134396	0.1738	0.6321	2.6145
0.15	0.173755	0.1512	0.5420	2.3594
0.16	0.216897	0.1237	0.4561	2.1383
0.17	0.263290	0.0918	0.3737	1.9445
0.18	0.312485	0.0564	0.2942	1.7732
0.19	0.364132	0.0178	0.2168	1.6203
0.20	0.417983	-0.0235	0.1412	1.4830
0.21	0.473903	-0.0673	0.0667	1.3585
0.22	0.531872	-0.1135	-0.0072	1.2450
0.23	0.592012	-0.1621	-0.0813	1.1407
0.24	0.654593	-0.2133	-0.1563	1.0441
0.25	0.720092	-0.2678	-0.2331	0.9540
0.26	0.789248	-0.3263	-0.3132	0.8692
0.27	0.863170	-0.3900	-0.3980	0.7886
0.28	0.943523	-0.4609	-0.4901	0.7111
0.29	1.032903	-0.5420	-0.5931	0.6356
0.30	1.135666	-0.6388	-0.7131	0.5605
0.31	1.260125	-0.7613	-0.8615	0.4837

$\hat{\tau}_3 = -0.400, \quad 0.02 \le \hat{\tau}_4 \le 0.30$				
$\hat{\tau}_4$	A	B	h	k
0.02	0.000002	0.0822	2.1032	12.1703
0.03	0.000049	0.1058	1.8702	9.4443
0.04	0.000440	0.1307	1.6716	7.6127
0.05	0.001996	0.1555	1.4986	6.3099
0.06	0.005986	0.1781	1.3451	5.3419
0.07	0.013710	0.1966	1.2071	4.5973
0.08	0.026129	0.2093	1.0814	4.0083
0.09	0.043732	0.2154	0.9657	3.5315
0.10	0.066578	0.2146	0.8584	3.1380
0.11	0.094424	0.2071	0.7579	2.8079
0.12	0.126851	0.1935	0.6631	2.5270
0.13	0.163364	0.1745	0.5732	2.2850
0.14	0.203466	0.1506	0.4873	2.0743
0.15	0.246696	0.1226	0.4048	1.8889
0.16	0.292662	0.0909	0.3251	1.7245
0.17	0.341046	0.0562	0.2476	1.5774
0.18	0.391621	0.0187	0.1718	1.4449
0.19	0.444250	-0.0213	0.0971	1.3246
0.20	0.498900	-0.0636	0.0230	1.2147
0.21	0.555651	-0.1084	-0.0511	1.1136
0.22	0.614726	-0.1558	-0.1260	1.0199
0.23	0.676517	-0.2061	-0.2026	0.9325
0.24	0.741645	-0.2601	-0.2821	0.8502
0.25	0.811048	-0.3189	-0.3659	0.7720
0.26	0.886133	-0.3840	-0.4564	0.6970
0.27	0.969075	-0.4581	-0.5567	0.6240
0.28	1.063456	-0.5454	-0.6721	0.5518
0.29	1.175878	-0.6542	-0.8125	0.4783
0.30	1.321306	-0.8033	-1.0000	0.4000

$$-0.375 \leq \tau_3 \leq -0.350$$

$\hat{\tau}_3 = -0.375, \quad 0.00 \leq \hat{\tau}_4 \leq 0.28$				
$\hat{\tau}_4$	A	B	h	k
0.00	0.000003	0.0886	2.1307	11.2850
0.01	0.000065	0.1124	1.9049	8.8866
0.02	0.000490	0.1374	1.7108	7.2379
0.03	0.002034	0.1623	1.5405	6.0450
0.04	0.005808	0.1853	1.3887	5.1469
0.05	0.012942	0.2046	1.2515	4.4489
0.06	0.024292	0.2187	1.1261	3.8922
0.07	0.040322	0.2268	1.0105	3.4386
0.08	0.061139	0.2286	0.9029	3.0621
0.09	0.086578	0.2243	0.8020	2.7448
0.10	0.116314	0.2141	0.7068	2.4738
0.11	0.149936	0.1987	0.6163	2.2395
0.12	0.187021	0.1786	0.5299	2.0350
0.13	0.227165	0.1544	0.4468	1.8546
0.14	0.270012	0.1267	0.3665	1.6943
0.15	0.315268	0.0957	0.2884	1.5507
0.16	0.362711	0.0620	0.2121	1.4212
0.17	0.412199	0.0258	0.1370	1.3035
0.18	0.463675	-0.0129	0.0626	1.1959
0.19	0.517181	-0.0539	-0.0117	1.0969
0.20	0.572879	-0.0974	-0.0865	1.0051
0.21	0.631076	-0.1437	-0.1628	0.9195
0.22	0.692272	-0.1933	-0.2415	0.8391
0.23	0.757229	-0.2471	-0.3240	0.7628
0.24	0.827092	-0.3064	-0.4123	0.6897
0.25	0.903614	-0.3733	-0.5092	0.6189
0.26	0.989612	-0.4511	-0.6191	0.5491
0.27	1.090063	-0.5461	-0.7498	0.4789
0.28	1.215371	-0.6713	-0.9177	0.4054

$\hat{\tau}_3 = -0.350, \quad -0.02 \leq \hat{\tau}_4 \leq 0.26$				
$\hat{\tau}_4$	A	B	h	k
-0.02	0.000003	0.0918	2.1958	10.8870
-0.01	0.000060	0.1157	1.9711	8.6379
0.00	0.000433	0.1407	1.7770	7.0725
0.01	0.001766	0.1659	1.6060	5.9294
0.02	0.005019	0.1896	1.4532	5.0628
0.03	0.011200	0.2101	1.3148	4.3856
0.04	0.021117	0.2262	1.1882	3.8430
0.05	0.035256	0.2369	1.0711	3.3994
0.06	0.053790	0.2418	0.9622	3.0302
0.07	0.076642	0.2409	0.8599	2.7183
0.08	0.103572	0.2345	0.7634	2.4513
0.09	0.134252	0.2230	0.6717	2.2203
0.10	0.168316	0.2068	0.5840	2.0182
0.11	0.205406	0.1866	0.4998	1.8399
0.12	0.245196	0.1627	0.4185	1.6814
0.13	0.287407	0.1356	0.3395	1.5392
0.14	0.331818	0.1057	0.2624	1.4110
0.15	0.378274	0.0731	0.1866	1.2944
0.16	0.426693	0.0382	0.1117	1.1879
0.17	0.477078	0.0009	0.0372	1.0899
0.18	0.529528	-0.0388	-0.0377	0.9992
0.19	0.584268	-0.0811	-0.1137	0.9147
0.20	0.641678	-0.1264	-0.1916	0.8353
0.21	0.702354	-0.1754	-0.2727	0.7603
0.22	0.767191	-0.2290	-0.3586	0.6887
0.23	0.837550	-0.2888	-0.4517	0.6196
0.24	0.915566	-0.3576	-0.5556	0.5520
0.25	1.004846	-0.4395	-0.6762	0.4846
0.26	1.112327	-0.5435	-0.8251	0.4154

$$-0.325 \leq \tau_3 \leq -0.300$$

$\hat{\tau}_3 = -0.325, \quad -0.04 \leq \hat{\tau}_4 \leq 0.25$				
$\hat{\tau}_4$	A	B	h	k
-0.04	0.000002	0.0919	2.2978	10.8816
-0.03	0.000041	0.1156	2.0687	8.6483
-0.02	0.000308	0.1406	1.8704	7.0883
-0.01	0.001313	0.1661	1.6956	5.9463
0.00	0.003864	0.1906	1.5393	5.0789
0.01	0.008873	0.2128	1.3977	4.4001
0.02	0.017127	0.2312	1.2681	3.8559
0.03	0.029159	0.2448	1.1483	3.4105
0.04	0.045224	0.2531	1.0368	3.0398
0.05	0.065340	0.2560	0.9322	2.7265
0.06	0.089358	0.2536	0.8335	2.4583
0.07	0.117023	0.2462	0.7398	2.2262
0.08	0.148028	0.2342	0.6504	2.0233
0.09	0.182057	0.2180	0.5645	1.8443
0.10	0.218807	0.1982	0.4817	1.6851
0.11	0.258011	0.1750	0.4014	1.5425
0.12	0.299446	0.1489	0.3231	1.4138
0.13	0.342943	0.1201	0.2463	1.2971
0.14	0.388394	0.0888	0.1707	1.1904
0.15	0.435762	0.0552	0.0956	1.0924
0.16	0.485091	0.0193	0.0205	1.0018
0.17	0.536524	-0.0191	-0.0553	0.9176
0.18	0.590334	-0.0602	-0.1325	0.8387
0.19	0.646960	-0.1044	-0.2122	0.7644
0.20	0.707078	-0.1526	-0.2958	0.6937
0.21	0.771704	-0.2059	-0.3852	0.6258
0.22	0.842408	-0.2661	-0.4831	0.5599
0.23	0.921728	-0.3365	-0.5942	0.4948
0.24	1.014187	-0.4225	-0.7262	0.4292
0.25	1.129363	-0.5365	-0.8961	0.3601

$\hat{\tau}_3 = -0.3000, \quad -0.06 \leq \hat{\tau}_4 \leq 0.24$				
$\hat{\tau}_4$	A	B	h	k
-0.06	0.000001	0.0888	2.4386	11.2577
-0.05	0.000019	0.1121	2.1995	8.9143
-0.04	0.000173	0.1370	1.9928	7.2846
-0.03	0.000822	0.1627	1.8110	6.0960
-0.02	0.002616	0.1881	1.6485	5.1959
-0.01	0.006367	0.2119	1.5015	4.4936
0.00	0.012847	0.2327	1.3672	3.9317
0.01	0.022647	0.2495	1.2432	3.4730
0.02	0.036115	0.2615	1.1279	3.0917
0.03	0.053374	0.2685	1.0199	2.7702
0.04	0.074371	0.2704	0.9182	2.4954
0.05	0.098930	0.2673	0.8217	2.2579
0.06	0.126806	0.2597	0.7297	2.0506
0.07	0.157724	0.2479	0.6416	1.8680
0.08	0.191406	0.2322	0.5567	1.7058
0.09	0.227595	0.2132	0.4745	1.5607
0.10	0.266067	0.1910	0.3947	1.4301
0.11	0.306639	0.1661	0.3166	1.3116
0.12	0.349179	0.1386	0.2398	1.2036
0.13	0.393612	0.1087	0.1639	1.1046
0.14	0.439931	0.0765	0.0883	1.0132
0.15	0.488212	0.0420	0.0125	0.9284
0.16	0.538620	0.0050	-0.0642	0.8493
0.17	0.591469	-0.0348	-0.1428	0.7749
0.18	0.647237	-0.0779	-0.2242	0.7045
0.19	0.706661	-0.1251	-0.3102	0.6373
0.20	0.770856	-0.1778	-0.4027	0.5725
0.21	0.841572	-0.2381	-0.5052	0.5092
0.22	0.921727	-0.3095	-0.6230	0.4463
0.23	1.016807	-0.3990	-0.7663	0.3819
0.24	1.139679	-0.5231	-0.9591	0.3126

$$-0.275 \leq \tau_3 \leq -0.2500$$

$\hat{\tau}_3 = -0.275, \quad -0.08 \leq \hat{\tau}_4 \leq 0.22$					$\hat{\tau}_3 = -0.250, \quad -0.09 \leq \hat{\tau}_4 \leq 0.21$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.08	0.000001	0.0828	2.6237	12.0797	-0.09	0.000001	0.0959	2.5813	10.4254
-0.07	0.000006	0.1055	2.3679	9.4747	-0.08	0.000020	0.1197	2.3417	8.3531
-0.06	0.000072	0.1300	2.1480	7.6865	-0.07	0.000158	0.1450	2.1331	6.8836
-0.05	0.000416	0.1557	1.9552	6.3955	-0.06	0.000710	0.1713	1.9485	5.7957
-0.04	0.001513	0.1817	1.7835	5.4261	-0.05	0.002200	0.1977	1.7829	4.9623
-0.03	0.004049	0.2070	1.6287	4.6747	-0.04	0.005296	0.2230	1.6325	4.3060
-0.02	0.008767	0.2302	1.4876	4.0772	-0.03	0.010658	0.2459	1.4946	3.7771
-0.01	0.016305	0.2502	1.3577	3.5917	-0.02	0.018826	0.2656	1.3671	3.3426
0.00	0.027112	0.2660	1.2372	3.1901	-0.01	0.030157	0.2812	1.2484	2.9798
0.01	0.041425	0.2772	1.1246	2.8526	0.00	0.044825	0.2923	1.1370	2.6724
0.02	0.059298	0.2837	1.0187	2.5652	0.01	0.062853	0.2989	1.0320	2.4089
0.03	0.080646	0.2853	0.9185	2.3175	0.02	0.084146	0.3009	0.9324	2.1805
0.04	0.105291	0.2823	0.8232	2.1020	0.03	0.108537	0.2986	0.8374	1.9806
0.05	0.133008	0.2751	0.7320	1.9126	0.04	0.135818	0.2922	0.7464	1.8042
0.06	0.163549	0.2639	0.6445	1.7448	0.05	0.165767	0.2821	0.6588	1.6473
0.07	0.196669	0.2492	0.5600	1.5951	0.06	0.198165	0.2685	0.5741	1.5067
0.08	0.232145	0.2312	0.4780	1.4605	0.07	0.232813	0.2518	0.4918	1.3800
0.09	0.269784	0.2102	0.3981	1.3388	0.08	0.269540	0.2323	0.4115	1.2651
0.10	0.309431	0.1866	0.3198	1.2281	0.09	0.308211	0.2102	0.3328	1.1602
0.11	0.350980	0.1606	0.2427	1.1269	0.10	0.348734	0.1856	0.2551	1.0640
0.12	0.394377	0.1321	0.1663	1.0337	0.11	0.391068	0.1588	0.1780	0.9754
0.13	0.439633	0.1014	0.0901	0.9474	0.12	0.435234	0.1296	0.1011	0.8932
0.14	0.486842	0.0684	0.0135	0.8672	0.13	0.481330	0.0982	0.0236	0.8165
0.15	0.536186	0.0329	-0.0643	0.7921	0.14	0.529545	0.0643	-0.0551	0.7446
0.16	0.587996	-0.0055	-0.1441	0.7214	0.15	0.580209	0.0275	-0.1361	0.6768
0.17	0.642774	-0.0472	-0.2271	0.6542	0.16	0.633828	-0.0126	-0.2204	0.6123
0.18	0.701291	-0.0932	-0.3151	0.5899	0.17	0.691178	-0.0570	-0.3100	0.5504
0.19	0.764727	-0.1449	-0.4104	0.5277	0.18	0.753456	-0.1071	-0.4073	0.4903
0.20	0.834957	-0.2045	-0.5167	0.4666	0.19	0.822585	-0.1653	-0.5162	0.4312
0.21	0.915198	-0.2760	-0.6402	0.4055	0.20	0.901927	-0.2357	-0.6438	0.3719
0.22	1.011780	-0.3675	-0.7934	0.3424	0.21	0.998325	-0.3272	-0.8039	0.3101

$$-0.225 \leq \tau_3 \leq -0.200$$

$\hat{\tau}_3 = -0.225, \quad -0.11 \leq \hat{\tau}_4 \leq 0.201$				
$\hat{\tau}_4$	A	B	h	k
-0.11	0.000001	0.0836	2.8514	11.9626
-0.10	0.000003	0.1064	2.5832	9.4003
-0.09	0.000040	0.1309	2.3523	7.6332
-0.08	0.000248	0.1570	2.1496	6.3533
-0.07	0.000960	0.1838	1.9691	5.3900
-0.06	0.002709	0.2106	1.8062	4.6422
-0.05	0.006120	0.2361	1.6576	4.0469
-0.04	0.011791	0.2592	1.5210	3.5628
-0.03	0.020192	0.2791	1.3942	3.1621
-0.02	0.031631	0.2951	1.2758	2.8253
-0.01	0.046251	0.3068	1.1646	2.5385
0.00	0.064064	0.3141	1.0594	2.2914
0.01	0.084980	0.3170	0.9595	2.0764
0.02	0.108846	0.3158	0.8641	1.8875
0.03	0.135472	0.3107	0.7726	1.7203
0.04	0.164659	0.3019	0.6844	1.5711
0.05	0.196203	0.2898	0.5991	1.4372
0.06	0.229928	0.2746	0.5161	1.3161
0.07	0.265674	0.2567	0.4351	1.2061
0.08	0.303319	0.2362	0.3555	1.1056
0.09	0.342779	0.2133	0.2771	1.0133
0.10	0.384018	0.1882	0.1992	0.9280
0.11	0.427057	0.1607	0.1214	0.8488
0.12	0.471989	0.1310	0.0431	0.7750
0.13	0.518999	0.0989	-0.0365	0.7056
0.14	0.568400	0.0639	-0.1184	0.6401
0.15	0.620680	0.0256	-0.2037	0.5777
0.16	0.676589	-0.0168	-0.2942	0.5179
0.17	0.737288	-0.0648	-0.3926	0.4597
0.18	0.804657	-0.1207	-0.5029	0.4025
0.19	0.882004	-0.1886	-0.6321	0.3448
0.20	0.976160	-0.2774	-0.7951	0.2847

$\hat{\tau}_3 = -0.200, \quad -0.12 \leq \hat{\tau}_4 \leq 0.20$				
$\hat{\tau}_4$	A	B	h	k
-0.12	0.000001	0.0904	2.8871	11.0560
-0.11	0.000005	0.1139	2.6240	8.7817
-0.10	0.000055	0.1390	2.3960	7.1868
-0.09	0.000303	0.1657	2.1949	6.0166
-0.08	0.001089	0.1931	2.0150	5.1269
-0.07	0.002926	0.2203	1.8523	4.4305
-0.06	0.006401	0.2464	1.7034	3.8724
-0.05	0.012065	0.2702	1.5662	3.4161
-0.04	0.020351	0.2908	1.4387	3.0366
-0.03	0.031541	0.3077	1.3194	2.7164
-0.02	0.045774	0.3204	1.2072	2.4427
-0.01	0.063063	0.3290	1.1010	2.2063
0.00	0.083333	0.3333	1.0000	2.0000
0.01	0.106447	0.3336	0.9035	1.8184
0.02	0.132232	0.3301	0.8109	1.6573
0.03	0.160505	0.3230	0.7216	1.5134
0.04	0.191082	0.3126	0.6352	1.3840
0.05	0.223793	0.2992	0.5511	1.2668
0.06	0.258492	0.2831	0.4690	1.1603
0.07	0.295061	0.2644	0.3884	1.0628
0.08	0.333418	0.2433	0.3090	0.9732
0.09	0.373524	0.2199	0.2301	0.8904
0.10	0.415396	0.1943	0.1514	0.8135
0.11	0.459113	0.1664	0.0722	0.7417
0.12	0.504842	0.1361	-0.0082	0.6742
0.13	0.552871	0.1032	-0.0907	0.6105
0.14	0.603646	0.0670	-0.1766	0.5499
0.15	0.657862	0.0269	-0.2675	0.4917
0.16	0.716597	-0.0185	-0.3661	0.4353
0.17	0.781591	-0.0713	-0.4762	0.3797
0.18	0.855904	-0.1354	-0.6047	0.3239
0.19	0.945797	-0.2187	-0.7658	0.2657
0.20	1.068959	-0.3448	-1.0000	0.2000

$$-0.175 \leq \tau_3 \leq -0.150$$

$\hat{\tau}_3 = -0.175, \quad -0.13 \leq \hat{\tau}_4 \leq 0.19$					$\hat{\tau}_3 = -0.150, \quad -0.14 \leq \hat{\tau}_4 \leq 0.18$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.13	0.000001	0.0943	2.9662	10.6019	-0.14	0.000001	0.0951	3.0892	10.5198
-0.12	0.000005	0.1182	2.7017	8.4617	-0.13	0.000004	0.1191	2.8169	8.3940
-0.11	0.000057	0.1438	2.4717	6.9488	-0.12	0.000046	0.1450	2.5802	6.8904
-0.10	0.000303	0.1709	2.2684	5.8318	-0.11	0.000251	0.1725	2.3710	5.7801
-0.09	0.001065	0.1989	2.0862	4.9784	-0.10	0.000903	0.2011	2.1836	4.9315
-0.08	0.002823	0.2268	1.9211	4.3078	-0.09	0.002441	0.2298	2.0137	4.2648
-0.07	0.006129	0.2537	1.7700	3.7686	-0.08	0.005384	0.2577	1.8583	3.7287
-0.06	0.011506	0.2785	1.6305	3.3266	-0.07	0.010239	0.2838	1.7149	3.2892
-0.05	0.019372	0.3003	1.5008	2.9582	-0.06	0.017425	0.3072	1.5816	2.9230
-0.04	0.030009	0.3186	1.3795	2.6468	-0.05	0.027239	0.3274	1.4569	2.6134
-0.03	0.043560	0.3329	1.2652	2.3802	-0.04	0.039849	0.3437	1.3396	2.3485
-0.02	0.060058	0.3432	1.1571	2.1496	-0.03	0.055311	0.3562	1.2286	2.1194
-0.01	0.079444	0.3494	1.0542	1.9482	-0.02	0.073591	0.3646	1.1230	1.9192
0.00	0.101600	0.3516	0.9560	1.7707	-0.01	0.094595	0.3691	1.0223	1.7430
0.01	0.126373	0.3500	0.8617	1.6131	0.00	0.118186	0.3698	0.9256	1.5865
0.02	0.153593	0.3449	0.7707	1.4723	0.01	0.144210	0.3670	0.8325	1.4467
0.03	0.183088	0.3365	0.6828	1.3455	0.02	0.172504	0.3609	0.7425	1.3209
0.04	0.214697	0.3251	0.5972	1.2307	0.03	0.202915	0.3517	0.6551	1.2070
0.05	0.248279	0.3109	0.5137	1.1263	0.04	0.235300	0.3397	0.5698	1.1035
0.06	0.283715	0.2941	0.4319	1.0307	0.05	0.269542	0.3251	0.4863	1.0088
0.07	0.320923	0.2750	0.3512	0.9429	0.06	0.305551	0.3080	0.4041	0.9217
0.08	0.359858	0.2535	0.2712	0.8617	0.07	0.343270	0.2887	0.3228	0.8413
0.09	0.400521	0.2299	0.1914	0.7863	0.08	0.382687	0.2671	0.2419	0.7668
0.10	0.442976	0.2040	0.1114	0.7159	0.09	0.423841	0.2433	0.1609	0.6972
0.11	0.487366	0.1758	0.0302	0.6498	0.10	0.466841	0.2173	0.0790	0.6320
0.12	0.533930	0.1450	-0.0528	0.5875	0.11	0.511887	0.1887	-0.0044	0.5706
0.13	0.583070	0.1111	-0.1389	0.5282	0.12	0.559308	0.1574	-0.0906	0.5123
0.14	0.635396	0.0736	-0.2297	0.4714	0.13	0.609620	0.1227	-0.1808	0.4565
0.15	0.691862	0.0312	-0.3276	0.4164	0.14	0.663630	0.0837	-0.2774	0.4027
0.16	0.754002	-0.0178	-0.4362	0.3624	0.15	0.722622	0.0388	-0.3835	0.3501
0.17	0.824472	-0.0769	-0.5618	0.3084	0.16	0.788775	-0.0147	-0.5044	0.2977
0.18	0.908555	-0.1529	-0.7168	0.2525	0.17	0.866241	-0.0821	-0.6504	0.2442
0.19	1.020213	-0.2641	-0.9346	0.1905	0.18	0.964995	-0.1766	-0.8463	0.1863

$$-0.125 \leq \tau_3 \leq -0.100$$

$\hat{\tau}_3 = -0.125, \quad -0.15 \leq \hat{\tau}_4 \leq 0.17$					$\hat{\tau}_3 = -0.100, \quad -0.15 \leq \hat{\tau}_4 \leq 0.17$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.15	0.000001	0.0926	3.2600	10.8024	-0.15	0.000001	0.1106	3.1768	9.0375
-0.14	0.000002	0.1166	2.9732	8.5739	-0.14	0.000012	0.1365	2.9102	7.3257
-0.13	0.000028	0.1426	2.7247	7.0087	-0.13	0.000090	0.1642	2.6762	6.0836
-0.12	0.000170	0.1703	2.5055	5.8593	-0.12	0.000402	0.1935	2.4678	5.1478
-0.11	0.000658	0.1993	2.3095	4.9848	-0.11	0.001261	0.2236	2.2799	4.4210
-0.10	0.001878	0.2288	2.1322	4.3003	-0.10	0.003104	0.2539	2.1089	3.8423
-0.09	0.004313	0.2579	1.9703	3.7517	-0.09	0.006414	0.2832	1.9516	3.3718
-0.08	0.008459	0.2856	1.8211	3.3031	-0.08	0.011643	0.3108	1.8061	2.9826
-0.07	0.014749	0.3110	1.6826	2.9302	-0.07	0.019153	0.3356	1.6704	2.6556
-0.06	0.023509	0.3333	1.5532	2.6157	-0.06	0.029194	0.3572	1.5431	2.3774
-0.05	0.034941	0.3521	1.4315	2.3471	-0.05	0.041904	0.3752	1.4231	2.1379
-0.04	0.049137	0.3671	1.3166	2.1151	-0.04	0.057326	0.3894	1.3093	1.9297
-0.03	0.066096	0.3782	1.2075	1.9128	-0.03	0.075425	0.3997	1.2010	1.7470
-0.02	0.085750	0.3854	1.1035	1.7349	-0.02	0.096111	0.4062	1.0974	1.5855
-0.01	0.107984	0.3888	1.0038	1.5771	-0.01	0.119265	0.4091	0.9980	1.4417
0.00	0.132658	0.3887	0.9079	1.4363	0.00	0.144743	0.4085	0.9021	1.3127
0.01	0.159621	0.3851	0.8152	1.3098	0.01	0.172402	0.4047	0.8094	1.1963
0.02	0.188721	0.3785	0.7254	1.1955	0.02	0.202100	0.3979	0.7192	1.0908
0.03	0.219818	0.3689	0.6380	1.0916	0.03	0.233711	0.3882	0.6312	0.9946
0.04	0.252791	0.3566	0.5525	0.9967	0.04	0.267125	0.3760	0.5451	0.9064
0.05	0.287540	0.3419	0.4685	0.9095	0.05	0.302261	0.3614	0.4602	0.8252
0.06	0.323996	0.3248	0.3856	0.8292	0.06	0.339065	0.3445	0.3763	0.7502
0.07	0.362128	0.3055	0.3033	0.7547	0.07	0.377527	0.3253	0.2928	0.6804
0.08	0.401950	0.2839	0.2212	0.6854	0.08	0.417682	0.3040	0.2091	0.6153
0.09	0.443535	0.2602	0.1385	0.6205	0.09	0.459632	0.2804	0.1247	0.5542
0.10	0.487034	0.2341	0.0546	0.5595	0.10	0.503565	0.2544	0.0386	0.4965
0.11	0.532704	0.2054	-0.0314	0.5017	0.11	0.549791	0.2257	-0.0503	0.4418
0.12	0.580959	0.1736	-0.1210	0.4467	0.12	0.598810	0.1936	-0.1434	0.3894
0.13	0.632449	0.1381	-0.2159	0.3937	0.13	0.651404	0.1574	-0.2430	0.3388
0.14	0.688201	0.0975	-0.3188	0.3423	0.14	0.708844	0.1153	-0.3526	0.2892
0.15	0.749914	0.0498	-0.4341	0.2915	0.15	0.773327	0.0648	-0.4781	0.2398
0.16	0.820672	-0.0089	-0.5698	0.2402	0.16	0.849178	0.0002	-0.6310	0.1890
0.17	0.907160	-0.0874	-0.7431	0.1861	0.17	0.947496	-0.0932	-0.8414	0.1333

$$-0.075 \leq \tau_3 \leq -0.050$$

$\hat{\tau}_3 = -0.075, \quad -0.16 \leq \hat{\tau}_4 \leq 0.17$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000001	0.1012	3.4381	9.8766
-0.15	0.0000034	0.1266	3.1450	7.8968
-0.14	0.0000351	0.1540	2.8898	6.4885
-0.13	0.0001946	0.1833	2.6641	5.4440
-0.12	0.0007113	0.2139	2.4617	4.6431
-0.11	0.0019532	0.2451	2.2782	4.0121
-0.10	0.0043747	0.2761	2.1103	3.5037
-0.09	0.0084390	0.3058	1.9554	3.0862
-0.08	0.0145511	0.3334	1.8114	2.7379
-0.07	0.0230182	0.3581	1.6768	2.4432
-0.06	0.0340379	0.3795	1.5502	2.1907
-0.05	0.0477055	0.3972	1.4306	1.9723
-0.04	0.0640314	0.4110	1.3169	1.7814
-0.03	0.0829614	0.4211	1.2085	1.6132
-0.02	0.1043972	0.4274	1.1046	1.4639
-0.01	0.1282132	0.4302	1.0047	1.3304
0.00	0.1542715	0.4296	0.9083	1.2104
0.01	0.1824324	0.4258	0.8148	1.1018
0.02	0.2125637	0.4192	0.7238	1.0030
0.03	0.2445476	0.4098	0.6349	0.9127
0.04	0.2782866	0.3979	0.5477	0.8298
0.05	0.3137093	0.3836	0.4617	0.7534
0.06	0.3507767	0.3670	0.3764	0.6825
0.07	0.3894904	0.3483	0.2915	0.6165
0.08	0.4299043	0.3273	0.2062	0.5548
0.09	0.4721411	0.3041	0.1198	0.4967
0.10	0.5164217	0.2783	0.0314	0.4418
0.11	0.5630982	0.2497	-0.0602	0.3895
0.12	0.6127470	0.2176	-0.1568	0.3394
0.13	0.6662781	0.1809	-0.2611	0.2907
0.14	0.7252086	0.1377	-0.3773	0.2427
0.15	0.7922886	0.0845	-0.5130	0.1944
0.16	0.8734346	0.0137	-0.6849	0.1437
0.17	0.9876201	-0.1004	-0.9469	0.0848

$\hat{\tau}_3 = -0.050, \quad -0.16 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000005	0.1132	3.4420	8.8334
-0.15	0.0000086	0.1400	3.1569	7.1428
-0.14	0.0000681	0.1688	2.9069	5.9183
-0.13	0.0003163	0.1994	2.6846	4.9972
-0.12	0.0010267	0.2313	2.4844	4.2826
-0.11	0.0025950	0.2636	2.3023	3.7143
-0.10	0.0054764	0.2954	2.1350	3.2528
-0.09	0.0101114	0.3256	1.9804	2.8713
-0.08	0.0168689	0.3536	1.8363	2.5511
-0.07	0.0260176	0.3786	1.7013	2.2788
-0.06	0.0377211	0.4002	1.5741	2.0446
-0.05	0.0520483	0.4181	1.4537	1.8411
-0.04	0.0689911	0.4322	1.3392	1.6627
-0.03	0.0884848	0.4424	1.2298	1.5051
-0.02	0.1104261	0.4490	1.1249	1.3648
-0.01	0.1346891	0.4521	1.0240	1.2391
0.00	0.1611385	0.4519	0.9263	1.1257
0.01	0.1896393	0.4486	0.8316	1.0230
0.02	0.2200654	0.4424	0.7394	0.9294
0.03	0.2523054	0.4335	0.6492	0.8437
0.04	0.2862690	0.4221	0.5606	0.7649
0.05	0.3218921	0.4084	0.4732	0.6921
0.06	0.3591436	0.3924	0.3864	0.6245
0.07	0.3980343	0.3742	0.2998	0.5615
0.08	0.4386285	0.3537	0.2128	0.5025
0.09	0.4810638	0.3310	0.1244	0.4469
0.10	0.5255817	0.3058	0.0339	0.3943
0.11	0.5725673	0.2775	-0.0603	0.3441
0.12	0.6226541	0.2456	-0.1601	0.2958
0.13	0.6768569	0.2088	-0.2686	0.2487
0.14	0.7369108	0.1649	-0.3908	0.2021
0.15	0.8061036	0.1097	-0.5361	0.1548
0.16	0.8921686	0.0331	-0.7273	0.1041

$$-0.025 \leq \tau_3 \leq 0.000$$

$\hat{\tau}_3 = -0.025, \quad -0.16 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000010	0.1223	3.4931	8.1797
-0.15	0.0000145	0.1502	3.2091	6.6545
-0.14	0.0000989	0.1803	2.9592	5.5383
-0.13	0.0004169	0.2122	2.7362	4.6917
-0.12	0.0012663	0.2453	2.5350	4.0305
-0.11	0.0030545	0.2787	2.3515	3.5017
-0.10	0.0062321	0.3115	2.1827	3.0702
-0.09	0.0112223	0.3427	2.0264	2.7122
-0.08	0.0183717	0.3715	1.8806	2.4106
-0.07	0.0279264	0.3972	1.7439	2.1534
-0.06	0.0400312	0.4195	1.6149	1.9316
-0.05	0.0547411	0.4381	1.4928	1.7385
-0.04	0.0720394	0.4528	1.3765	1.5689
-0.03	0.0918563	0.4638	1.2654	1.4187
-0.02	0.1140870	0.4711	1.1587	1.2848
-0.01	0.1386065	0.4749	1.0560	1.1646
0.00	0.1652816	0.4754	0.9567	1.0562
0.01	0.1939804	0.4729	0.8603	0.9578
0.02	0.2245801	0.4674	0.7664	0.8681
0.03	0.2569728	0.4593	0.6745	0.7858
0.04	0.2910717	0.4486	0.5843	0.7101
0.05	0.3268159	0.4356	0.4951	0.6401
0.06	0.3641777	0.4204	0.4067	0.5751
0.07	0.4031711	0.4029	0.3183	0.5144
0.08	0.4438646	0.3832	0.2295	0.4576
0.09	0.4864012	0.3612	0.1392	0.4040
0.10	0.5310310	0.3367	0.0466	0.3532
0.11	0.5781568	0.3091	-0.0499	0.3047
0.12	0.6284413	0.2777	-0.1524	0.2579
0.13	0.6829650	0.2413	-0.2643	0.2123
0.14	0.7436116	0.1974	-0.3912	0.1671
0.15	0.8140797	0.1414	-0.5442	0.1207
0.16	0.9037454	0.0607	-0.7519	0.0703

$\hat{\tau}_3 = 0.000, \quad -0.16 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000013	0.1280	3.5903	7.8146
-0.15	0.0000176	0.1570	3.3012	6.3690
-0.14	0.0001138	0.1882	3.0466	5.3069
-0.13	0.0004619	0.2212	2.8192	4.4988
-0.12	0.0013676	0.2555	2.6138	3.8661
-0.11	0.0032405	0.2902	2.4264	3.3590
-0.10	0.0065271	0.3242	2.2539	2.9444
-0.09	0.0116430	0.3566	2.0941	2.5999
-0.08	0.0189260	0.3867	1.9451	2.3093
-0.07	0.0286145	0.4136	1.8052	2.0613
-0.06	0.0408478	0.4372	1.6733	1.8471
-0.05	0.0556773	0.4569	1.5484	1.6605
-0.04	0.0730847	0.4729	1.4294	1.4964
-0.03	0.0930001	0.4850	1.3157	1.3511
-0.02	0.1153195	0.4935	1.2066	1.2215
-0.01	0.1399189	0.4984	1.1016	1.1051
0.00	0.1666666	0.5000	1.0000	1.0000
0.01	0.1954320	0.4985	0.9014	0.9046
0.02	0.2260930	0.4941	0.8054	0.8176
0.03	0.2585424	0.4869	0.7115	0.7378
0.04	0.2926932	0.4773	0.6192	0.6643
0.05	0.3284840	0.4652	0.5281	0.5964
0.06	0.3658856	0.4509	0.4378	0.5333
0.07	0.4049098	0.4344	0.3476	0.4744
0.08	0.4456222	0.4157	0.2568	0.4192
0.09	0.4881618	0.3946	0.1648	0.3671
0.10	0.5327736	0.3710	0.0703	0.3178
0.11	0.5798585	0.3443	-0.0281	0.2708
0.12	0.6300765	0.3139	-0.1327	0.2254
0.13	0.6845187	0.2784	-0.2470	0.1811
0.14	0.7451124	0.2355	-0.3768	0.1371
0.15	0.8157184	0.1800	-0.5344	0.0919
0.16	0.9065603	0.0985	-0.7519	0.0423

$$0.025 \leq \tau_3 \leq 0.050$$

$\hat{\tau}_3 = 0.025, \quad -0.16 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000012	0.1300	3.7360	7.6941
-0.15	0.0000160	0.1598	3.4356	6.2576
-0.14	0.0001056	0.1919	3.1713	5.2045
-0.13	0.0004357	0.2260	2.9355	4.4044
-0.12	0.0013049	0.2615	2.7226	3.7789
-0.11	0.0031178	0.2976	2.5286	3.2781
-0.10	0.0063197	0.3331	2.3502	2.8691
-0.09	0.0113286	0.3671	2.1850	2.5294
-0.08	0.0184871	0.3989	2.0310	2.2432
-0.07	0.0280401	0.4276	1.8866	1.9990
-0.06	0.0401332	0.4528	1.7505	1.7883
-0.05	0.0548241	0.4743	1.6216	1.6047
-0.04	0.0720993	0.4920	1.4990	1.4435
-0.03	0.0918931	0.5058	1.3819	1.3007
-0.02	0.1141042	0.5158	1.2696	1.1733
-0.01	0.1386105	0.5223	1.1615	1.0591
0.00	0.1652805	0.5253	1.0571	0.9559
0.01	0.1939833	0.5252	0.9558	0.8623
0.02	0.2245955	0.5221	0.8571	0.7770
0.03	0.2570076	0.5163	0.7608	0.6987
0.04	0.2911292	0.5079	0.6662	0.6267
0.05	0.3268945	0.4970	0.5729	0.5602
0.06	0.3642685	0.4839	0.4804	0.4984
0.07	0.4032557	0.4685	0.3882	0.4408
0.08	0.4439118	0.4509	0.2956	0.3868
0.09	0.4863626	0.4310	0.2017	0.3359
0.10	0.5308350	0.4085	0.1056	0.2878
0.11	0.5777517	0.3830	0.0055	0.2418
0.12	0.6276015	0.3540	-0.1003	0.1976
0.13	0.6815616	0.3201	-0.2157	0.1545
0.14	0.7414346	0.2789	-0.3466	0.1117
0.15	0.8109290	0.2258	-0.5048	0.0679
0.16	0.8998603	0.1475	-0.7224	0.0199

$\hat{\tau}_3 = 0.050, \quad -0.16 \leq \hat{\tau}_4 \leq 0.16$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000007	0.1280	3.9361	7.8151
-0.15	0.0000106	0.1583	3.6170	6.3169
-0.14	0.0000785	0.1911	3.3374	5.2276
-0.13	0.0003479	0.2261	3.0886	4.4056
-0.12	0.0010953	0.2628	2.8647	3.7664
-0.11	0.0027119	0.3003	2.6611	3.2569
-0.10	0.0056430	0.3376	2.4742	2.8423
-0.09	0.0103180	0.3736	2.3014	2.4991
-0.08	0.0170973	0.4074	2.1406	2.2107
-0.07	0.0262460	0.4384	1.9900	1.9653
-0.06	0.0379286	0.4659	1.8483	1.7539
-0.05	0.0522187	0.4897	1.7142	1.5702
-0.04	0.0691153	0.5096	1.5869	1.4090
-0.03	0.0885613	0.5256	1.4654	1.2665
-0.02	0.1104608	0.5377	1.3490	1.1396
-0.01	0.1346940	0.5461	1.2370	1.0258
0.00	0.1611298	0.5510	1.1290	0.9233
0.01	0.1896355	0.5527	1.0244	0.8303
0.02	0.2200842	0.5513	0.9227	0.7457
0.03	0.2523616	0.5470	0.8234	0.6682
0.04	0.2863710	0.5401	0.7261	0.5969
0.05	0.3220388	0.5307	0.6302	0.5311
0.06	0.3593205	0.5190	0.5354	0.4701
0.07	0.3982081	0.5049	0.4410	0.4132
0.08	0.4387415	0.4887	0.3464	0.3600
0.09	0.4810255	0.4701	0.2507	0.3099
0.10	0.5252572	0.4490	0.1531	0.2626
0.11	0.5717747	0.4251	0.0519	0.2176
0.12	0.6211293	0.3978	-0.0547	0.1743
0.13	0.6742665	0.3658	-0.1702	0.1323
0.14	0.7328418	0.3273	-0.3001	0.0907
0.15	0.8001316	0.2779	-0.4551	0.0484
0.16	0.8844343	0.2067	-0.6635	0.0027

$$0.075 \le \tau_3 \le 0.100$$

$\hat{\tau}_3 = 0.075, \quad -0.16 \le \hat{\tau}_4 \le 0.17$				
$\hat{\tau}_4$	A	B	h	k
-0.16	0.0000002	0.1217	4.2007	8.2162
-0.15	0.0000049	0.1522	3.8538	6.5686
-0.14	0.0000448	0.1854	3.5518	5.3893
-0.13	0.0002290	0.2212	3.2846	4.5103
-0.12	0.0007940	0.2589	3.0451	3.8335
-0.11	0.0021046	0.2978	2.8281	3.2984
-0.10	0.0046020	0.3369	2.6296	2.8661
-0.09	0.0087324	0.3752	2.4466	2.5103
-0.08	0.0148859	0.4116	2.2768	2.2127
-0.07	0.0233616	0.4453	2.1181	1.9606
-0.06	0.0343569	0.4756	1.9691	1.7443
-0.05	0.0479727	0.5023	1.8284	1.5569
-0.04	0.0642295	0.5250	1.6950	1.3930
-0.03	0.0830847	0.5437	1.5680	1.2484
-0.02	0.1044518	0.5584	1.4465	1.1200
-0.01	0.1282152	0.5693	1.3298	1.0052
0.00	0.1542444	0.5766	1.2175	0.9019
0.01	0.1824040	0.5805	1.1088	0.8085
0.02	0.2125624	0.5811	1.0033	0.7235
0.03	0.2445980	0.5788	0.9006	0.6459
0.04	0.2784055	0.5737	0.8000	0.5746
0.05	0.3139003	0.5660	0.7012	0.5089
0.06	0.3510249	0.5559	0.6036	0.4480
0.07	0.3897551	0.5434	0.5068	0.3914
0.08	0.4301103	0.5287	0.4100	0.3385
0.09	0.4721675	0.5117	0.3125	0.2889
0.10	0.5160859	0.4923	0.2134	0.2421
0.11	0.5621459	0.4701	0.1112	0.1976
0.12	0.6109342	0.4447	0.0040	0.1550
0.13	0.6628985	0.4153	-0.1104	0.1139
0.14	0.7197828	0.3799	-0.2378	0.0735
0.15	0.7841514	0.3353	-0.3869	0.0327
0.16	0.8623154	0.2732	-0.5798	-0.0104
0.17	0.9811157	0.1567	-0.9118	-0.0642

$\hat{\tau}_3 = 0.100, \quad -0.15 \le \hat{\tau}_4 \le 0.17$				
$\hat{\tau}_4$	A	B	h	k
-0.15	0.0000013	0.1414	4.1591	7.0695
-0.14	0.0000181	0.1747	3.8251	5.7239
-0.13	0.0001176	0.2107	3.5320	4.7403
-0.12	0.0004788	0.2491	3.2711	3.9947
-0.11	0.0014191	0.2893	3.0359	3.4127
-0.10	0.0033624	0.3303	2.8220	2.9474
-0.09	0.0067697	0.3710	2.6255	2.5677
-0.08	0.0120690	0.4102	2.4439	2.2527
-0.07	0.0196076	0.4471	2.2747	1.9875
-0.06	0.0296310	0.4808	2.1163	1.7613
-0.05	0.0422822	0.5110	1.9673	1.5663
-0.04	0.0576145	0.5371	1.8262	1.3965
-0.03	0.0756091	0.5592	1.6922	1.2474
-0.02	0.0961945	0.5772	1.5644	1.1154
-0.01	0.1192631	0.5912	1.4420	0.9977
0.00	0.1446864	0.6013	1.3243	0.8922
0.01	0.1723266	0.6079	1.2107	0.7970
0.02	0.2020463	0.6110	1.1007	0.7106
0.03	0.2337154	0.6110	0.9938	0.6319
0.04	0.2672173	0.6081	0.8894	0.5598
0.05	0.3024543	0.6025	0.7871	0.4935
0.06	0.3393529	0.5943	0.6864	0.4322
0.07	0.3778696	0.5837	0.5867	0.3754
0.08	0.4179995	0.5707	0.4874	0.3224
0.09	0.4597883	0.5555	0.3878	0.2728
0.10	0.5033514	0.5379	0.2871	0.2261
0.11	0.5489060	0.5177	0.1840	0.1819
0.12	0.5968282	0.4946	0.0770	0.1398
0.13	0.6477475	0.4678	-0.0366	0.0993
0.14	0.7027727	0.4360	-0.1606	0.0598
0.15	0.7639557	0.3966	-0.3024	0.0205
0.16	0.8357459	0.3438	-0.4778	-0.0202
0.17	0.9322672	0.2586	-0.7381	-0.0666

$$0.125 \leq \tau_3 \leq 0.150$$

$\hat{\tau}_3 = 0.125, \quad -0.15 \leq \hat{\tau}_4 \leq 0.17$					$\hat{\tau}_3 = 0.150, \quad -0.14 \leq \hat{\tau}_4 \leq 0.18$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.15	0.0000002	0.1260	4.5537	7.9392	-0.14	0.0000005	0.1376	4.6233	7.2658
-0.14	0.0000044	0.1587	4.1737	6.3020	-0.13	0.0000088	0.1726	4.2411	5.7937
-0.13	0.0000424	0.1945	3.8441	5.1397	-0.12	0.0000720	0.2108	3.9087	4.7394
-0.12	0.0002242	0.2331	3.5535	4.2787	-0.11	0.0003380	0.2520	3.6151	3.9531
-0.11	0.0007931	0.2742	3.2938	3.6190	-0.10	0.0011016	0.2957	3.3521	3.3473
-0.10	0.0021284	0.3168	3.0591	3.0996	-0.09	0.0027875	0.3407	3.1141	2.8680
-0.09	0.0046907	0.3598	2.8448	2.6812	-0.08	0.0058815	0.3859	2.8966	2.4805
-0.08	0.0089460	0.4021	2.6477	2.3378	-0.07	0.0108515	0.4300	2.6962	2.1613
-0.07	0.0153009	0.4425	2.4650	2.0513	-0.06	0.0180886	0.4720	2.5103	1.8942
-0.06	0.0240660	0.4802	2.2947	1.8090	-0.05	0.0278771	0.5109	2.3368	1.6677
-0.05	0.0354444	0.5144	2.1349	1.6015	-0.04	0.0403906	0.5460	2.1740	1.4734
-0.04	0.0495385	0.5447	1.9842	1.4219	-0.03	0.0557027	0.5770	2.0204	1.3049
-0.03	0.0663663	0.5709	1.8416	1.2651	-0.02	0.0738060	0.6037	1.8749	1.1574
-0.02	0.0858806	0.5928	1.7059	1.1270	-0.01	0.0946323	0.6260	1.7364	1.0274
-0.01	0.1079884	0.6105	1.5763	1.0044	0.00	0.1180720	0.6440	1.6041	0.9118
0.00	0.1325667	0.6242	1.4521	0.8950	0.01	0.1439901	0.6578	1.4771	0.8085
0.01	0.1594774	0.6340	1.3325	0.7965	0.02	0.1722397	0.6678	1.3549	0.7154
0.02	0.1885769	0.6403	1.2170	0.7076	0.03	0.2026718	0.6741	1.2368	0.6313
0.03	0.2197259	0.6431	1.1050	0.6268	0.04	0.2351440	0.6769	1.1223	0.5547
0.04	0.2527951	0.6428	0.9961	0.5530	0.05	0.2695263	0.6767	1.0107	0.4847
0.05	0.2876714	0.6396	0.8896	0.4853	0.06	0.3057068	0.6735	0.9016	0.4205
0.06	0.3242631	0.6337	0.7851	0.4229	0.07	0.3435971	0.6675	0.7944	0.3613
0.07	0.3625050	0.6252	0.6820	0.3652	0.08	0.3831377	0.6590	0.6886	0.3064
0.08	0.4023658	0.6143	0.5798	0.3116	0.09	0.4243060	0.6481	0.5836	0.2554
0.09	0.4438574	0.6011	0.4778	0.2615	0.10	0.4671271	0.6348	0.4786	0.2077
0.10	0.4870497	0.5856	0.3751	0.2146	0.11	0.5116915	0.6192	0.3727	0.1630
0.11	0.5320953	0.5676	0.2709	0.1703	0.12	0.5581843	0.6009	0.2647	0.1207
0.12	0.5792709	0.5468	0.1636	0.1284	0.13	0.6069376	0.5797	0.1531	0.0806
0.13	0.6290581	0.5228	0.0512	0.0882	0.14	0.6585494	0.5550	0.0353	0.0420
0.14	0.6822614	0.4946	-0.0694	0.0494	0.15	0.7139832	0.5256	-0.0925	0.0046
0.15	0.7403846	0.4603	-0.2037	0.0112	0.16	0.7752238	0.4890	-0.2378	-0.0325
0.16	0.8064060	0.4162	-0.3628	-0.0275	0.17	0.8464880	0.4397	-0.4164	-0.0707
0.17	0.8879585	0.3521	-0.5747	-0.0689	0.18	0.9413811	0.3593	-0.6809	-0.1141

$$0.175 \le \tau_3 \le 0.200$$

$\hat{\tau}_3 = 0.175, \quad -0.13 \le \hat{\tau}_4 \le 0.19$				
$\hat{\tau}_4$	A	B	h	k
-0.13	0.0000007	0.1454	4.7558	6.8769
-0.12	0.0000124	0.1824	4.3624	5.4821
-0.11	0.0000941	0.2229	4.0202	4.4818
-0.10	0.0004206	0.2665	3.7180	3.7348
-0.09	0.0013229	0.3127	3.4473	3.1586
-0.08	0.0032593	0.3603	3.2023	2.7023
-0.07	0.0067377	0.4079	2.9784	2.3330
-0.06	0.0122337	0.4543	2.7722	2.0287
-0.05	0.0201336	0.4983	2.5809	1.7738
-0.04	0.0307089	0.5389	2.4024	1.5576
-0.03	0.0441161	0.5756	2.2349	1.3719
-0.02	0.0604106	0.6079	2.0769	1.2108
-0.01	0.0795676	0.6357	1.9272	1.0699
0.00	0.1015035	0.6589	1.7847	0.9455
0.01	0.1260953	0.6777	1.6486	0.8349
0.02	0.1531972	0.6922	1.5181	0.7359
0.03	0.1826536	0.7027	1.3924	0.6468
0.04	0.2143089	0.7095	1.2710	0.5662
0.05	0.2480159	0.7127	1.1532	0.4929
0.06	0.2836414	0.7128	1.0385	0.4258
0.07	0.3210721	0.7099	0.9263	0.3642
0.08	0.3602195	0.7042	0.8160	0.3074
0.09	0.4010263	0.6960	0.7072	0.2548
0.10	0.4434742	0.6852	0.5990	0.2059
0.11	0.4875967	0.6721	0.4907	0.1601
0.12	0.5334985	0.6564	0.3814	0.1172
0.13	0.5813888	0.6382	0.2697	0.0766
0.14	0.6316358	0.6169	0.1539	0.0380
0.15	0.6849481	0.5918	0.0309	0.0008
0.16	0.7424825	0.5614	-0.1039	-0.0354
0.17	0.8066483	0.5228	-0.2598	-0.0715
0.18	0.8831962	0.4683	-0.4593	-0.1095
0.19	0.9976464	0.3628	-0.8045	-0.1565

$\hat{\tau}_3 = 0.200, \quad -0.12 \le \hat{\tau}_4 \le 0.20$				
$\hat{\tau}_4$	A	B	h	k
-0.12	0.0000007	0.1486	4.9569	6.7304
-0.11	0.0000125	0.1873	4.5422	5.3390
-0.10	0.0000972	0.2298	4.1827	4.3468
-0.09	0.0004400	0.2758	3.8658	3.6090
-0.08	0.0013914	0.3245	3.5825	3.0417
-0.07	0.0034354	0.3748	3.3265	2.5938
-0.06	0.0071046	0.4252	3.0929	2.2320
-0.05	0.0128932	0.4743	2.8781	1.9344
-0.04	0.0211975	0.5209	2.6790	1.6856
-0.03	0.0322913	0.5639	2.4933	1.4747
-0.02	0.0463274	0.6028	2.3193	1.2938
-0.01	0.0633531	0.6371	2.1553	1.1371
0.00	0.0833333	0.6667	2.0000	1.0000
0.01	0.1061732	0.6914	1.8523	0.8791
0.02	0.1317388	0.7116	1.7114	0.7717
0.03	0.1598737	0.7273	1.5762	0.6757
0.04	0.1904127	0.7388	1.4462	0.5893
0.05	0.2231918	0.7465	1.3207	0.5112
0.06	0.2580563	0.7505	1.1989	0.4401
0.07	0.2948671	0.7513	1.0804	0.3751
0.08	0.3335058	0.7490	0.9646	0.3155
0.09	0.3738803	0.7438	0.8508	0.2605
0.10	0.4159309	0.7361	0.7385	0.2097
0.11	0.4596390	0.7257	0.6269	0.1623
0.12	0.5050398	0.7130	0.5152	0.1181
0.13	0.5522429	0.6977	0.4024	0.0766
0.14	0.6014595	0.6796	0.2870	0.0373
0.15	0.6531498	0.6585	0.1670	0.0000*
0.16	0.7079738	0.6333	0.0394	-0.0360
0.17	0.7673204	0.6025	-0.1015	-0.0711
0.18	0.8339474	0.5626	-0.2666	-0.1063
0.19	0.9151295	0.5039	-0.4852	-0.1439
0.20	1.0689593	0.3448	-1.0000	-0.2000

* Actually, -3.776×10^{-5}

$$0.225 \leq \tau_3 \leq 0.250$$

$\hat{\tau}_3 = 0.225, \quad -0.11 \leq \hat{\tau}_4 \leq 0.20$					$\hat{\tau}_3 = 0.250, \quad -0.10 \leq \hat{\tau}_4 \leq 0.21$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.11	0.0000004	0.1465	5.2381	6.8280	-0.10	0.0000001	0.1385	5.6190	7.2204
-0.10	0.0000089	0.1865	4.7899	5.3623	-0.09	0.0000042	0.1793	5.1206	5.5775
-0.09	0.0000789	0.2307	4.4037	4.3310	-0.08	0.0000480	0.2247	4.6954	4.4481
-0.08	0.0003853	0.2788	4.0648	3.5721	-0.07	0.0002748	0.2745	4.3252	3.6315
-0.07	0.0012793	0.3300	3.7632	2.9934	-0.06	0.0010104	0.3281	3.9978	3.0174
-0.06	0.0032619	0.3832	3.4915	2.5393	-0.05	0.0027576	0.3842	3.7045	2.5411
-0.05	0.0068950	0.4367	3.2443	2.1747	-0.04	0.0061098	0.4413	3.4388	2.1619
-0.04	0.0127056	0.4891	3.0175	1.8760	-0.03	0.0116424	0.4975	3.1958	1.8537
-0.03	0.0211185	0.5390	2.8078	1.6272	-0.02	0.0198304	0.5514	2.9720	1.5987
-0.02	0.0324278	0.5852	2.6126	1.4171	-0.01	0.0310103	0.6017	2.7644	1.3845
-0.01	0.0467971	0.6271	2.4299	1.2374	0.00	0.0453755	0.6475	2.5705	1.2021
0.00	0.0642772	0.6642	2.2580	1.0820	0.01	0.0629940	0.6882	2.3886	1.0451
0.01	0.0848301	0.6962	2.0956	0.9464	0.02	0.0838341	0.7237	2.2170	0.9086
0.02	0.1083540	0.7233	1.9413	0.8271	0.03	0.1077923	0.7538	2.0543	0.7889
0.03	0.1347055	0.7454	1.7942	0.7213	0.04	0.1347177	0.7788	1.8996	0.6830
0.04	0.1637173	0.7629	1.6533	0.6268	0.05	0.1644322	0.7988	1.7517	0.5886
0.05	0.1952132	0.7760	1.5180	0.5419	0.06	0.1967465	0.8141	1.6098	0.5041
0.06	0.2290183	0.7851	1.3874	0.4652	0.07	0.2314721	0.8250	1.4731	0.4279
0.07	0.2649675	0.7903	1.2609	0.3956	0.08	0.2684299	0.8320	1.3410	0.3588
0.08	0.3029121	0.7922	1.1380	0.3320	0.09	0.3074575	0.8353	1.2128	0.2959
0.09	0.3427249	0.7908	1.0179	0.2737	0.10	0.3484145	0.8353	1.0877	0.2383
0.10	0.3843058	0.7865	0.9001	0.2200	0.11	0.3911882	0.8321	0.9652	0.1853
0.11	0.4275880	0.7794	0.7838	0.1703	0.12	0.4356991	0.8262	0.8446	0.1364
0.12	0.4725463	0.7698	0.6685	0.1241	0.13	0.4819089	0.8175	0.7251	0.0910
0.13	0.5192099	0.7576	0.5531	0.0811	0.14	0.5298275	0.8062	0.6058	0.0487
0.14	0.5676745	0.7429	0.4367	0.0406	0.15	0.5795876	0.7923	0.4856	0.0090
0.15	0.6182164	0.7254	0.3177	0.0025	0.16	0.6313479	0.7756	0.3631	-0.0283
0.16	0.6711847	0.7047	0.1941	-0.0338	0.17	0.6855168	0.7557	0.2362	-0.0637
0.17	0.7273470	0.6799	0.0624	-0.0687	0.18	0.7428107	0.7318	0.1014	-0.0977
0.18	0.7881148	0.6494	-0.0831	-0.1028	0.19	0.8046033	0.7023	-0.0471	-0.1309
0.19	0.8564770	0.6093	-0.2547	-0.1371	0.20	0.8738359	0.6632	-0.2217	-0.1641
0.20	0.9406956	0.5485	-0.4866	-0.1740	0.21	0.9587430	0.6037	-0.4572	-0.1998

$$0.275 \le \tau_3 \le 0.300$$

$\hat{\tau}_3 = 0.275, \quad -0.08 \le \hat{\tau}_4 \le 0.22$				
$\hat{\tau}_4$	A	B	h	k
-0.08	0.0000011	0.1652	5.5592	6.0527
-0.07	0.0000197	0.2111	5.0770	4.7356
-0.06	0.0001500	0.2620	4.6623	3.8093
-0.05	0.0006587	0.3175	4.2989	3.1275
-0.04	0.0020215	0.3764	3.9758	2.6075
-0.03	0.0048544	0.4371	3.6851	2.1993
-0.02	0.0097930	0.4977	3.4208	1.8713
-0.01	0.0173877	0.5564	3.1784	1.6025
0.00	0.0280450	0.6117	2.9545	1.3785
0.01	0.0420133	0.6624	2.7463	1.1891
0.02	0.0593971	0.7080	2.5515	1.0271
0.03	0.0801846	0.7479	2.3683	0.8869
0.04	0.1042782	0.7822	2.1952	0.7646
0.05	0.1315233	0.8108	2.0310	0.6568
0.06	0.1617312	0.8341	1.8744	0.5612
0.07	0.1946979	0.8524	1.7245	0.4758
0.08	0.2302180	0.8660	1.5805	0.3991
0.09	0.2680946	0.8752	1.4416	0.3297
0.10	0.3081470	0.8805	1.3070	0.2666
0.11	0.3502167	0.8823	1.1761	0.2091
0.12	0.3941730	0.8807	1.0482	0.1563
0.13	0.4399179	0.8761	0.9225	0.1076
0.14	0.4873913	0.8687	0.7983	0.0626
0.15	0.5365092	0.8586	0.6750	0.0208
0.16	0.5874862	0.8459	0.5509	-0.0184
0.17	0.6405985	0.8303	0.4246	-0.0552
0.18	0.6959159	0.8116	0.2946	-0.0901
0.19	0.7541889	0.7890	0.1574	-0.1234
0.20	0.8165744	0.7611	0.0077	-0.1556
0.21	0.8859826	0.7244	-0.1665	-0.1878
0.22	0.9692983	0.6695	-0.3955	-0.2218

$\hat{\tau}_3 = 0.300, \quad -0.07 \le \hat{\tau}_4 \le 0.24$				
$\hat{\tau}_4$	A	B	h	k
-0.07	0.0000001	0.1441	6.1466	6.9409
-0.06	0.0000044	0.1895	5.5793	5.2756
-0.05	0.0000553	0.2406	5.0999	4.1527
-0.04	0.0003283	0.2972	4.6855	3.3523
-0.03	0.0012224	0.3583	4.3213	2.7569
-0.02	0.0033432	0.4225	3.9964	2.2989
-0.01	0.0073850	0.4878	3.7034	1.9368
0.00	0.0139961	0.5519	3.4365	1.6441
0.01	0.0236880	0.6131	3.1913	1.4029
0.02	0.0367987	0.6700	2.9644	1.2011
0.03	0.0534980	0.7216	2.7532	1.0299
0.04	0.0738145	0.7672	2.5553	0.8828
0.05	0.0976702	0.8068	2.3690	0.7553
0.06	0.1249134	0.8404	2.1929	0.6436
0.07	0.1553473	0.8680	2.0255	0.5450
0.08	0.1887524	0.8902	1.8658	0.4574
0.09	0.2249031	0.9072	1.7128	0.3789
0.10	0.2635803	0.9195	1.5656	0.3083
0.11	0.3045799	0.9275	1.4235	0.2443
0.12	0.3477201	0.9315	1.2856	0.1861
0.13	0.3928459	0.9320	1.1512	0.1329
0.14	0.4398342	0.9293	1.0197	0.0840
0.15	0.4885853	0.9235	0.8902	0.0389
0.16	0.5392135	0.9149	0.7616	-0.0030
0.17	0.5914372	0.9036	0.6332	-0.0420
0.18	0.6457005	0.8894	0.5034	-0.0786
0.19	0.7021481	0.8722	0.3703	-0.1130
0.20	0.7613312	0.8514	0.2312	-0.1458
0.21	0.8242843	0.8257	0.0811	-0.1774
0.22	0.8931850	0.7923	-0.0896	-0.2085
0.23	0.9735475	0.7440	-0.3054	-0.2407
0.24	1.0936201	0.6413	-0.6945	-0.2809

$$0.325 \leq \tau_3 \leq 0.350$$

$\hat{\tau}_3 = 0.325, \quad -0.05 \leq \hat{\tau}_4 \leq 0.25$				
$\hat{\tau}_4$	A	B	h	k
-0.05	0.0000003	0.1600	6.2544	6.2483
-0.04	0.0000107	0.2101	5.6768	4.7598
-0.03	0.0001076	0.2663	5.1878	3.7492
-0.02	0.0005552	0.3285	4.7646	3.0248
-0.01	0.0018769	0.3955	4.3922	2.4836
0.00	0.0047880	0.4651	4.0599	2.0656
0.01	0.0100421	0.5351	3.7598	1.7341
0.02	0.0182920	0.6031	3.4863	1.4654
0.03	0.0300121	0.6673	3.2350	1.2435
0.04	0.0454811	0.7263	3.0024	1.0574
0.05	0.0648021	0.7792	2.7856	0.8992
0.06	0.0879396	0.8257	2.5826	0.7631
0.07	0.1147590	0.8655	2.3914	0.6449
0.08	0.1450616	0.8989	2.2105	0.5413
0.09	0.1786137	0.9262	2.0386	0.4497
0.10	0.2151675	0.9477	1.8744	0.3682
0.11	0.2544769	0.9640	1.7172	0.2951
0.12	0.2963090	0.9754	1.5658	0.2293
0.13	0.3404520	0.9825	1.4195	0.1696
0.14	0.3867203	0.9857	1.2774	0.1153
0.15	0.4349574	0.9853	1.1388	0.0655
0.16	0.4849371	0.9816	1.0032	0.0199
0.17	0.5368603	0.9749	0.8690	-0.0224
0.18	0.5906610	0.9654	0.7352	-0.0617
0.19	0.6462393	0.9530	0.6012	-0.0983
0.20	0.7038869	0.9376	0.4648	-0.1327
0.21	0.7640224	0.9188	0.3238	-0.1652
0.22	0.8274481	0.8957	0.1739	-0.1963
0.23	0.8957321	0.8662	0.0077	-0.2266
0.24	0.9729885	0.8251	-0.1928	-0.2571
0.25	1.0729942	0.7534	-0.4913	-0.2914

$\hat{\tau}_3 = 0.350, \quad -0.03 \leq \hat{\tau}_4 \leq 0.26$				
$\hat{\tau}_4$	A	B	h	k
-0.03	0.0000006	0.1708	6.4606	5.8535
-0.02	0.0000169	0.2252	5.8556	4.4399
-0.01	0.0001572	0.2865	5.3444	3.4826
0.00	0.0007646	0.3545	4.9029	2.7977
0.01	0.0024728	0.4273	4.5148	2.2864
0.02	0.0060968	0.5027	4.1689	1.8919
0.03	0.0124492	0.5780	3.8568	1.5792
0.04	0.0221952	0.6506	3.5726	1.3258
0.05	0.0357855	0.7185	3.3117	1.1166
0.06	0.0534532	0.7805	3.0702	0.9411
0.07	0.0752471	0.8356	2.8454	0.7919
0.08	0.1010767	0.8836	2.6350	0.6636
0.09	0.1307569	0.9244	2.4368	0.5522
0.10	0.1640458	0.9584	2.2495	0.4545
0.11	0.2006734	0.9859	2.0714	0.3681
0.12	0.2403634	1.0073	1.9016	0.2912
0.13	0.2828477	1.0234	1.7387	0.2223
0.14	0.3278773	1.0344	1.5820	0.1602
0.15	0.3752285	1.0410	1.4306	0.1040
0.16	0.4247006	1.0437	1.2835	0.0527
0.17	0.4748337	1.0428	1.1434	0.0069
0.18	0.5294836	1.0384	0.9988	-0.0374
0.19	0.5846899	1.0311	0.8592	-0.0773
0.20	0.6417089	1.0208	0.7201	-0.1144
0.21	0.7006923	1.0076	0.5798	-0.1490
0.22	0.7619304	0.9911	0.4363	-0.1816
0.23	0.8259983	0.9708	0.2863	-0.2125
0.24	0.8940486	0.9452	0.1239	-0.2422
0.25	0.9686470	0.9110	-0.0626	-0.2715
0.26	1.0571176	0.8585	-0.3072	-0.3023

$$0.375 \leq \tau_3 \leq 0.400$$

$\hat{\tau}_3 = 0.375, \quad -0.01 \leq \hat{\tau}_4 \leq 0.28$					$\hat{\tau}_3 = 0.400, \quad 0.01 \leq \hat{\tau}_4 \leq 0.30$				
$\hat{\tau}_4$	A	B	h	k	$\hat{\tau}_4$	A	B	h	k
-0.01	0.0000006	0.1749	6.7800	5.7167	0.01	0.0000003	0.1708	7.2408	5.8547
0.00	0.0000184	0.2331	6.1272	4.2880	0.02	0.0000132	0.2321	6.5123	4.3070
0.01	0.0001776	0.2993	5.5791	3.3330	0.03	0.0001534	0.3026	5.9078	3.2982
0.02	0.0008762	0.3729	5.1080	2.6559	0.04	0.0008325	0.3816	5.3926	2.5961
0.03	0.0028419	0.4520	4.6954	2.1541	0.05	0.0028484	0.4672	4.9446	2.0828
0.04	0.0069910	0.5338	4.3289	1.7690	0.06	0.0072281	0.5562	4.5486	1.6930
0.05	0.0142117	0.6152	3.9991	1.4650	0.07	0.0149685	0.6448	4.1941	1.3880
0.06	0.0252051	0.6934	3.6995	1.2194	0.08	0.0268457	0.7299	3.8732	1.1433
0.07	0.0404204	0.7662	3.4248	1.0173	0.09	0.0433415	0.8090	3.5801	0.9430
0.08	0.0600634	0.8322	3.1712	0.8481	0.10	0.0646582	0.8805	3.3102	0.7762
0.09	0.0841406	0.8905	2.9354	0.7046	0.11	0.0907742	0.9436	3.0599	0.6353
0.10	0.1125138	0.9411	2.7150	0.5814	0.12	0.1215097	0.9979	2.8264	0.5147
0.11	0.1449506	0.9838	2.5077	0.4746	0.13	0.1565855	1.0437	2.6074	0.4103
0.12	0.1811655	1.0191	2.3119	0.3810	0.14	0.1956695	1.0814	2.4009	0.3192
0.13	0.2208505	1.0476	2.1261	0.2983	0.15	0.2384107	1.1116	2.2052	0.2390
0.14	0.2636972	1.0696	1.9489	0.2248	0.16	0.2844624	1.1350	2.0189	0.1678
0.15	0.3094115	1.0860	1.7792	0.1590	0.17	0.3334964	1.1523	1.8407	0.1041
0.16	0.3577226	1.0972	1.6160	0.0998	0.18	0.3851966	1.1641	1.6696	0.0469
0.17	0.4083750	1.1037	1.4583	0.0461	0.19	0.4394005	1.1712	1.5043	-0.0049
0.18	0.4543360	1.1061	1.3245	0.0033	0.20	0.4957173	1.1739	1.3441	-0.0519
0.19	0.5160499	1.1050	1.1555	-0.0475	0.21	0.5541626	1.1727	1.1876	-0.0950
0.20	0.5728222	1.1004	1.0085	-0.0886	0.22	0.6145274	1.1681	1.0339	-0.1345
0.21	0.6314336	1.0926	0.8630	-0.1266	0.23	0.6767896	1.1603	0.8818	-0.1710
0.22	0.6919440	1.0819	0.7177	-0.1619	0.24	0.7410102	1.1493	0.7296	-0.2048
0.23	0.7545241	1.0680	0.5706	-0.1949	0.25	0.8073953	1.1351	0.5753	-0.2365
0.24	0.8195481	1.0506	0.4193	-0.2259	0.26	0.8763986	1.1171	0.4159	-0.2663
0.25	0.8877724	1.0289	0.2596	-0.2555	0.27	0.9489608	1.0943	0.2461	-0.2946
0.26	0.9607882	1.0008	0.0835	-0.2841	0.28	1.0271667	1.0638	0.0557	-0.3222
0.27	1.0425167	0.9612	-0.1274	-0.3127	0.29	1.1167832	1.0181	-0.1833	-0.3504
0.28	1.1479170	0.8892	-0.4446	-0.3447	0.30	1.2574950	0.8995	-0.6684	-0.3871

$$0.425 \leq \tau_3 \leq 0.450$$

$\hat{\tau}_3 = 0.425, \quad 0.03 \leq \hat{\tau}_4 \leq 0.31$				
$\hat{\tau}_4$	A	B	h	k
0.03	0.0000001	0.1572	7.8931	6.3625
0.04	0.0000056	0.2204	7.0468	4.5369
0.05	0.0000951	0.2940	6.3572	3.3965
0.06	0.0006341	0.3779	5.7775	2.6265
0.07	0.0024433	0.4700	5.2788	2.0762
0.08	0.0066653	0.5669	4.8417	1.6656
0.09	0.0144525	0.6641	4.4531	1.3485
0.10	0.0267152	0.7579	4.1034	1.0970
0.11	0.0440194	0.8453	3.7855	0.8930
0.12	0.0665993	0.9242	3.4941	0.7244
0.13	0.0944249	0.9936	3.2249	0.5828
0.14	0.1272821	1.0534	2.9746	0.4623
0.15	0.1648442	1.1037	2.7406	0.3586
0.16	0.2067269	1.1450	2.5205	0.2684
0.17	0.2525275	1.1781	2.3124	0.1892
0.18	0.3018508	1.2038	2.1149	0.1192
0.19	0.3543144	1.2228	1.9264	0.0568
0.20	0.4095982	1.2359	1.7459	0.0009
0.21	0.4674336	1.2438	1.5716	-0.0497
0.22	0.5276052	1.2473	1.4027	-0.0955
0.23	0.5898606	1.2466	1.2383	-0.1373
0.24	0.6541000	1.2424	1.0770	-0.1756
0.25	0.7202831	1.2348	0.9174	-0.2109
0.26	0.7884747	1.2239	0.7578	-0.2437
0.27	0.8589033	1.2096	0.5959	-0.2742
0.28	0.9320831	1.1914	0.4282	-0.3029
0.29	1.0091102	1.1678	0.2487	-0.3303
0.30	1.0925380	1.1356	0.0444	-0.3570
0.31	1.1902868	1.0837	-0.2237	-0.3848

$\hat{\tau}_3 = 0.450, \quad 0.06 \leq \hat{\tau}_4 \leq 0.33$				
$\hat{\tau}_4$	A	B	h	k
0.06	0.0000010	0.1966	7.7926	5.0864
0.07	0.0000363	0.2717	6.9716	3.6789
0.08	0.0003570	0.3589	6.2960	2.7734
0.09	0.0017126	0.4569	5.7239	2.1485
0.10	0.0053313	0.5621	5.2288	1.6944
0.11	0.0125698	0.6694	4.7930	1.3510
0.12	0.0245584	0.7739	4.4040	1.0831
0.13	0.0420275	0.8718	4.0530	0.8686
0.14	0.0653007	0.9606	3.7331	0.6933
0.15	0.0943723	1.0389	3.4391	0.5475
0.16	0.1290073	1.1063	3.1671	0.4244
0.17	0.1688311	1.1630	2.9138	0.3191
0.18	0.2133993	1.2096	2.6765	0.2281
0.19	0.2622452	1.2469	2.4531	0.1487
0.20	0.3149079	1.2759	2.2415	0.0787
0.21	0.3706423	1.2974	2.0414	0.0170
0.22	0.4299863	1.3127	1.8481	-0.0388
0.23	0.4918373	1.3223	1.6629	-0.0887
0.24	0.5560411	1.3270	1.4842	-0.1338
0.25	0.6224364	1.3273	1.3104	-0.1749
0.26	0.6908825	1.3238	1.1403	-0.2124
0.27	0.7613203	1.3168	0.9723	-0.2469
0.28	0.8338046	1.3064	0.8047	-0.2788
0.29	0.9085661	1.2925	0.6348	-0.3085
0.30	0.9861450	1.2744	0.4590	-0.3363
0.31	1.0677269	1.2508	0.2703	-0.3628
0.32	1.1561865	1.2176	0.0541	-0.3887
0.33	1.2612678	1.1613	-0.2387	-0.4159

$$0.475 \le \tau_3 \le 0.500$$

$\hat{\tau}_3 = 0.475, \quad 0.09 \le \hat{\tau}_4 \le 0.35$				
$\hat{\tau}_4$	A	B	h	k
0.09	0.0000060	0.2343	7.8291	4.2684
0.10	0.0001243	0.3223	7.0038	3.0970
0.11	0.0008898	0.4242	6.3217	2.3320
0.12	0.0034930	0.5371	5.7422	1.7980
0.13	0.0095045	0.6553	5.2395	1.4065
0.14	0.0203810	0.7727	4.7961	1.1082
0.15	0.0371440	0.8842	4.3998	0.8740
0.16	0.0603066	0.9860	4.0417	0.6856
0.17	0.0899488	1.0763	3.7150	0.5310
0.18	0.1258425	1.1542	3.4146	0.4018
0.19	0.1675707	1.2198	3.1364	0.2925
0.20	0.2146198	1.2738	2.8771	0.1987
0.21	0.2664432	1.3172	2.6340	0.1174
0.22	0.3224737	1.3511	2.4050	0.0463
0.23	0.3822972	1.3767	2.1877	-0.0165
0.24	0.4453974	1.3949	1.9810	-0.0722
0.25	0.5114370	1.4069	1.7829	-0.1222
0.26	0.5800419	1.4135	1.5922	-0.1672
0.27	0.6509662	1.4153	1.4074	-0.2079
0.28	0.7240291	1.4131	1.2271	-0.2451
0.29	0.7991391	1.4071	1.0496	-0.2791
0.30	0.8763233	1.3976	0.8730	-0.3104
0.31	0.9557887	1.3844	0.6947	-0.3395
0.32	1.0380571	1.3671	0.5107	-0.3667
0.33	1.1243077	1.3441	0.3140	-0.3925
0.34	1.2174685	1.3114	0.0892	-0.4176
0.35	1.3278472	1.2544	-0.2168	-0.4438

$\hat{\tau}_3 = 0.500, \quad 0.11 \le \hat{\tau}_4 \le 0.37$				
$\hat{\tau}_4$	A	B	h	k
0.11	0.0000002	0.1821	9.0807	5.4908
0.12	0.0000176	0.2671	8.0020	3.7420
0.13	0.0002771	0.3691	7.1442	2.6989
0.14	0.0016665	0.4868	6.4357	2.0157
0.15	0.0058263	0.6153	5.8341	1.5376
0.16	0.0146150	0.7471	5.3125	1.1862
0.17	0.0295287	0.8752	4.8527	0.9180
0.18	0.0514462	0.9942	4.4419	0.7070
0.19	0.0806552	1.1005	4.0708	0.5370
0.20	0.1169995	1.1929	3.7325	0.3972
0.21	0.1600413	1.2710	3.4214	0.2804
0.22	0.2091934	1.3356	3.1334	0.1813
0.23	0.2638090	1.3876	2.8651	0.0962
0.24	0.3230251	1.4284	2.6145	0.0227
0.25	0.3868249	1.4596	2.3768	-0.0421
0.26	0.4542890	1.4824	2.1517	-0.0993
0.27	0.5248971	1.4979	1.9374	-0.1501
0.28	0.5983193	1.5072	1.7320	-0.1956
0.29	0.6742411	1.5113	1.5339	-0.2366
0.30	0.7524261	1.5108	1.3414	-0.2738
0.31	0.8327326	1.5064	1.1528	-0.3077
0.32	0.9151396	1.4982	0.9661	-0.3388
0.33	0.9998024	1.4864	0.7786	-0.3675
0.34	1.0871770	1.4704	0.5863	-0.3943
0.35	1.1783339	1.4488	0.3825	-0.4195
0.36	1.2759494	1.4180	0.1525	-0.4439
0.37	1.3892261	1.3648	-0.1520	-0.4689

$$0.525 \leq \tau_3 \leq 0.550$$

$\hat{\tau}_3 = 0.525, \quad 0.14 \leq \hat{\tau}_4 \leq 0.39$				
$\hat{\tau}_4$	A	B	h	k
0.14	0.0000003	0.1954	9.4969	5.1181
0.15	0.0000305	0.2914	8.3272	3.4298
0.16	0.0004437	0.4077	7.4047	2.4375
0.17	0.0024999	0.5423	6.6474	1.7936
0.18	0.0083002	0.6878	6.0073	1.3458
0.19	0.0199926	0.8350	5.4541	1.0181
0.20	0.0391130	0.9754	4.9679	0.7686
0.21	0.0664005	1.1033	4.5346	0.5729
0.22	0.1019215	1.2156	4.1440	0.4155
0.23	0.1452864	1.3113	3.7885	0.2862
0.24	0.1958483	1.3908	3.4622	0.1782
0.25	0.2528420	1.4553	3.1605	0.0867
0.26	0.3155015	1.5063	2.8798	0.0083
0.27	0.3830215	1.5454	2.6171	-0.0597
0.28	0.4549305	1.5746	2.3692	-0.1194
0.29	0.5304714	1.5951	2.1344	-0.1720
0.30	0.6091811	1.6084	1.9107	-0.2188
0.31	0.6906551	1.6157	1.6960	-0.2607
0.32	0.7745782	1.6178	1.4887	-0.2984
0.33	0.8607367	1.6155	1.2868	-0.3327
0.34	0.9490386	1.6093	1.0882	-0.3640
0.35	1.0395604	1.5992	0.8901	-0.3927
0.36	1.1326499	1.5851	0.6890	-0.4192
0.37	1.2291867	1.5658	0.4784	-0.4441
0.38	1.3313686	1.5381	0.2457	-0.4678
0.39	1.4462446	1.4920	-0.0471	-0.4916

$\hat{\tau}_3 = 0.550, \quad 0.17 \leq \hat{\tau}_4 \leq 0.42$				
$\hat{\tau}_4$	A	B	h	k
0.17	0.0000002	0.1963	10.1702	5.0949
0.18	0.0000321	0.3022	8.8414	3.3067
0.19	0.0005269	0.4329	7.8117	2.2932
0.20	0.0030802	0.5854	6.9764	1.6507
0.21	0.0103000	0.7502	6.2766	1.2110
0.22	0.0247091	0.9154	5.6761	0.8930
0.23	0.0479513	1.0711	5.1512	0.6530
0.24	0.0806607	1.2107	4.6856	0.4659
0.25	0.1226896	1.3313	4.2675	0.3162
0.26	0.1734066	1.4324	3.8883	0.1938
0.27	0.2319335	1.5149	3.5413	0.0920
0.28	0.2972984	1.5806	3.2214	0.0059
0.29	0.3685941	1.6316	2.9245	-0.0677
0.30	0.4450913	1.6701	2.6466	-0.1314
0.31	0.5258930	1.6979	2.3853	-0.1871
0.32	0.6103898	1.7170	2.1380	-0.2362
0.33	0.6980505	1.7287	1.9024	-0.2799
0.34	0.7884483	1.7344	1.6764	-0.3189
0.35	0.8812680	1.7351	1.4580	-0.3540
0.36	0.9763205	1.7315	1.2448	-0.3858
0.37	1.0735762	1.7238	1.0341	-0.4149
0.38	1.1732436	1.7122	0.8225	-0.4416
0.39	1.2759628	1.6957	0.6045	-0.4663
0.40	1.3833620	1.6722	0.3695	-0.4897
0.41	1.5002388	1.6346	0.0909	-0.5124
0.42	1.6583284	1.5339	-0.4068	-0.5387

$$0.575 \leq \tau_3 \leq 0.600$$

$\hat{\tau}_3 = 0.575, \quad 0.21 \leq \hat{\tau}_4 \leq 0.44$				
$\hat{\tau}_4$	A	B	h	k
0.21	0.0000184	0.2942	9.6152	3.3973
0.22	0.0004489	0.4376	8.4125	2.2699
0.23	0.0030893	0.6088	7.4567	1.5850
0.24	0.0111286	0.7960	6.6677	1.1297
0.25	0.0276651	0.9836	5.9982	0.8070
0.26	0.0546148	1.1591	5.4182	0.5673
0.27	0.0925847	1.3147	4.9074	0.3827
0.28	0.1412243	1.4472	4.4516	0.2363
0.29	0.1996336	1.5566	4.0403	0.1176
0.30	0.2661918	1.6439	3.6682	0.0200
0.31	0.3412151	1.7132	3.3215	-0.0632
0.32	0.4220891	1.7657	3.0031	-0.1335
0.33	0.5083234	1.8045	2.7064	-0.1942
0.34	0.5990586	1.8320	2.4281	-0.2470
0.35	0.6935732	1.8502	2.1652	-0.2934
0.36	0.7912801	1.8611	1.9151	-0.3345
0.37	0.8917238	1.8659	1.6755	-0.3712
0.38	0.9945847	1.8657	1.4437	-0.4041
0.39	1.0996985	1.8612	1.2172	-0.4340
0.40	1.2071092	1.8526	0.9924	-0.4612
0.41	1.3172001	1.8396	0.7647	-0.4862
0.42	1.4310508	1.8207	0.5259	-0.5094
0.43	1.5516576	1.7915	0.2579	-0.5316
0.44	1.6912352	1.7332	-0.1075	-0.5542

$\hat{\tau}_3 = 0.600, \quad 0.24 \leq \hat{\tau}_4 \leq 0.46$				
$\hat{\tau}_4$	A	B	h	k
0.24	0.0000040	0.2623	10.7844	3.8125
0.25	0.0002402	0.4142	9.2934	2.4047
0.26	0.0023865	0.6030	8.1475	1.6100
0.27	0.0102116	0.8156	7.2233	1.1064
0.28	0.0277376	1.0316	6.4525	0.7613
0.29	0.0574874	1.2338	5.7936	0.5112
0.30	0.1002090	1.4119	5.2195	0.3222
0.31	0.1553802	1.5618	4.7118	0.1745
0.32	0.2217573	1.6838	4.2573	0.0561
0.33	0.2976346	1.7799	3.8474	-0.0406
0.34	0.3825314	1.8547	3.4698	-0.1218
0.35	0.4740059	1.9104	3.1238	-0.1903
0.36	0.5712478	1.9509	2.8028	-0.2489
0.37	0.6732506	1.9791	2.5029	-0.2997
0.38	0.7791845	1.9975	2.2205	-0.3442
0.39	0.8883864	2.0082	1.9526	-0.3835
0.40	1.0003539	2.0127	1.6964	-0.4184
0.41	1.1147488	2.0122	1.4488	-0.4496
0.42	1.2314252	2.0074	1.2066	-0.4779
0.43	1.3505079	1.9984	0.9654	-0.5035
0.44	1.4726022	1.9845	0.7190	-0.5271
0.45	1.5994319	1.9631	0.4551	-0.5491
0.46	1.7365797	1.9254	0.1409	-0.5703

$$0.625 \leq \tau_3 \leq 0.650$$

$\hat{\tau}_3 = 0.625, \quad 0.27 \leq \hat{\tau}_4 \leq 0.49$				
$\hat{\tau}_4$	A	B	h	k
0.27	0.0000001	0.2035	12.6419	4.9134
0.28	0.0000557	0.3555	10.6220	2.8098
0.29	0.0012256	0.5558	9.1562	1.7682
0.30	0.0074409	0.7947	8.0175	1.1593
0.31	0.0240801	1.0463	7.0927	0.7640
0.32	0.0549293	1.2852	6.3176	0.4883
0.33	0.1012663	1.4955	5.6528	0.2858
0.34	0.1625024	1.6713	5.0723	0.1310
0.35	0.2348453	1.8091	4.5715	0.0120
0.36	0.3231205	1.9227	4.0966	-0.0895
0.37	0.4187008	2.0062	3.6786	-0.1707
0.38	0.5221321	2.0677	3.2965	-0.2387
0.39	0.6319503	2.1118	2.9442	-0.2965
0.40	0.7469375	2.1420	2.6167	-0.3463
0.41	0.8661068	2.1615	2.3098	-0.3895
0.42	0.9886825	2.1729	2.0199	-0.4275
0.43	1.1140860	2.1779	1.7435	-0.4610
0.44	1.2419389	2.1779	1.4770	-0.4910
0.45	1.3720982	2.1736	1.2164	-0.5180
0.46	1.5047661	2.1649	0.9564	-0.5424
0.47	1.6408059	2.1506	0.6886	-0.5648
0.48	1.7828315	2.1266	0.3951	-0.5857
0.49	1.9414424	2.0757	0.0135	-0.6064

$\hat{\tau}_3 = 0.650, \quad 0.31 \leq \hat{\tau}_4 \leq 0.52$				
$\hat{\tau}_4$	A	B	h	k
0.31	0.0000017	0.2598	12.7734	3.8491
0.32	0.0002808	0.4565	10.6985	2.1795
0.33	0.0036507	0.7141	9.1885	1.3365
0.34	0.0167351	1.0063	8.0136	0.8368
0.35	0.0457283	1.2948	7.0587	0.5092
0.36	0.0933785	1.5525	6.2585	0.2788
0.37	0.1593884	1.7675	5.5723	0.1085
0.38	0.2419021	1.9387	4.9731	-0.0223
0.39	0.3382531	2.0701	4.4428	-0.1258
0.40	0.4460855	2.1682	3.9674	-0.2097
0.41	0.5631354	2.2394	3.5368	-0.2790
0.42	0.6875250	2.2896	3.1430	-0.3373
0.43	0.8177205	2.3237	2.7798	-0.3869
0.44	0.9524993	2.3457	2.4417	-0.4297
0.45	1.0909137	2.3587	2.1242	-0.4669
0.46	1.2322643	2.3651	1.8229	-0.4997
0.47	1.3760966	2.3665	1.5336	-0.5287
0.48	1.5222405	2.3635	1.2514	-0.5547
0.49	1.6709517	2.3561	0.9699	-0.5781
0.50	1.8233528	2.3425	0.6782	-0.5996
0.51	1.9831570	2.3164	0.3512	-0.6196
0.52	2.1716324	2.2423	-0.1379	-0.6402

$$0.675 \le \tau_3 \le 0.750$$

$\hat{\tau}_3 = 0.675, \quad 0.35 \le \hat{\tau}_4 \le 0.54$				
$\hat{\tau}_4$	A	B	h	k
0.35	0.0000072	0.3067	13.2917	3.2604
0.36	0.0007572	0.5560	11.0295	1.7761
0.37	0.0076655	0.8799	9.4009	1.0359
0.38	0.0302062	1.2302	8.1430	0.5997
0.39	0.0747654	1.5568	7.1262	0.3144
0.40	0.1425473	1.8328	6.2778	0.1140
0.41	0.2315014	2.0516	5.5531	-0.0341
0.42	0.3382827	2.2177	4.9228	-0.1478
0.43	0.4595059	2.3396	4.3662	-0.2378
0.44	0.5921139	2.4266	3.8686	-0.3108
0.45	0.7335735	2.4868	3.4187	-0.3711
0.46	0.8818527	2.5274	3.0078	-0.4217
0.47	1.0353606	2.5536	2.6288	-0.4649
0.48	1.1928808	2.5697	2.2757	-0.5021
0.49	1.3535208	2.5786	1.9432	-0.5346
0.50	1.5166925	2.5824	1.6258	-0.5631
0.51	1.6821476	2.5820	1.3179	-0.5884
0.52	1.8501414	2.5772	1.0116	-0.6111
0.53	2.0219899	2.5657	0.6938	-0.6318
0.54	2.2025128	2.5391	0.3314	-0.6510

$\hat{\tau}_3 = 0.725, \quad 0.43 \le \hat{\tau}_4 \le 0.60$				
$\hat{\tau}_4$	A	B	h	k
0.43	0.0000039	0.3121	16.1487	3.2036
0.44	0.0013228	0.6741	12.7937	1.4498
0.45	0.0157775	1.1690	10.5662	0.7039
0.46	0.0614515	1.6785	8.9278	0.2993
0.47	0.1454175	2.1113	7.6477	0.0476
0.48	0.2645562	2.4410	6.6068	-0.1235
0.49	0.4120876	2.6766	5.7359	-0.2470
0.50	0.5811843	2.8380	4.9910	-0.3402
0.51	0.7661619	2.9451	4.3425	-0.4131
0.52	0.9626254	3.0146	3.7695	-0.4716
0.53	1.1672872	3.0590	3.2562	-0.5195
0.54	1.3777308	3.0874	2.7905	-0.5594
0.55	1.5922144	3.1061	2.3622	-0.5933
0.56	1.8095405	3.1192	1.9627	-0.6223
0.57	2.0290160	3.1287	1.5830	-0.6475
0.58	2.2505774	3.1350	1.2127	-0.6697
0.59	2.4753922	3.1354	0.8351	-0.6894
0.60	2.7088926	3.1190	0.4082	-0.7074

$\hat{\tau}_3 = 0.700, \quad 0.39 \le \hat{\tau}_4 \le 0.57$				
$\hat{\tau}_4$	A	B	h	k
0.39	0.0000104	0.3303	14.3164	3.0268
0.40	0.0012714	0.6374	11.6793	1.5366
0.41	0.0123888	1.0389	9.8325	0.8299
0.42	0.0462460	1.4582	8.4307	0.4245
0.43	0.1092558	1.8298	7.3115	0.1636
0.44	0.2006866	2.1278	6.3867	-0.0177
0.45	0.3164130	2.3524	5.6029	-0.1507
0.46	0.4515782	2.5147	4.9255	-0.2523
0.47	0.6017600	2.6282	4.3308	-0.3323
0.48	0.7633128	2.7054	3.8014	-0.3970
0.49	0.9333689	2.7567	3.3245	-0.4503
0.50	1.1097282	2.7898	2.8901	-0.4951
0.51	1.2907364	2.8107	2.4898	-0.5331
0.52	1.4751866	2.8235	2.1166	-0.5658
0.53	1.6622643	2.8311	1.7638	-0.5943
0.54	1.8515623	2.8347	1.4243	-0.6193
0.55	2.0432434	2.8343	1.0891	-0.6416
0.56	2.2386645	2.8273	0.7430	-0.6616
0.57	2.4433566	2.8032	0.3465	-0.6802

$\hat{\tau}_3 = 0.750, \quad 0.47 \le \hat{\tau}_4 \le 0.63$				
$\hat{\tau}_4$	A	B	h	k
0.47	0.0000001	0.2348	19.5944	4.2589
0.48	0.0006702	0.6273	14.6999	1.5728
0.49	0.0150977	1.2302	11.7716	0.6643
0.50	0.0703708	1.8691	9.7334	0.2215
0.51	0.1768754	2.3957	8.1964	-0.0379
0.52	0.3285339	2.7765	6.9779	-0.2074
0.53	0.5146890	3.0330	5.9780	-0.3266
0.54	0.7256283	3.1984	5.1362	-0.4148
0.55	0.9538535	3.3021	4.4126	-0.4828
0.56	1.1939103	3.3664	3.7798	-0.5366
0.57	1.4419200	3.4070	3.2177	-0.5803
0.58	1.6951556	3.4342	2.7108	-0.6165
0.59	1.9517408	3.4549	2.2462	-0.6470
0.60	2.2104864	3.4726	1.8123	-0.6730
0.61	2.4709147	3.4885	1.3967	-0.6954
0.62	2.7337323	3.5003	0.9816	-0.7151
0.63	3.0033732	3.4980	0.5269	-0.7327

$$0.775 \le \tau_3 \le 0.875$$

$\hat{\tau}_3 = 0.775, \quad 0.51 \le \hat{\tau}_4 \le 0.67$				
$\hat{\tau}_4$	A	B	h	k
0.51	0.0000000	0.1004	27.7719	9.9632
0.52	0.0000575	0.4574	18.2642	2.1826
0.53	0.0088669	1.1474	13.8298	0.7597
0.54	0.0657217	1.9794	11.0503	0.2015
0.55	0.1933818	2.6650	9.0760	-0.0914
0.56	0.3830115	3.1363	7.5716	-0.2708
0.57	0.6169608	3.4322	6.3759	-0.3913
0.58	0.8812709	3.6101	5.3909	-0.4779
0.59	1.1652798	3.7146	4.5600	-0.5431
0.60	1.4618643	3.7771	3.8443	-0.5938
0.61	1.7662347	3.8176	3.2162	-0.6344
0.62	2.0751836	3.8488	2.6547	-0.6676
0.63	2.3866153	3.8773	2.1426	-0.6953
0.64	2.6993464	3.9062	1.6640	-0.7187
0.65	3.0133311	3.9343	1.1992	-0.7388
0.66	3.3313945	3.9533	0.7126	-0.7564
0.67	3.6800166	3.9066	0.0149	-0.7730

$\hat{\tau}_3 = 0.825, \quad 0.62 \le \hat{\tau}_4 \le 0.73$				
$\hat{\tau}_4$	A	B	h	k
0.62	0.0081126	1.3023	18.0915	0.6582
0.63	0.1170944	2.8048	13.2199	-0.0275
0.64	0.3794023	3.8455	10.1972	-0.3155
0.65	0.7490593	4.3992	8.0827	-0.4728
0.66	1.1801779	4.6679	6.4968	-0.5716
0.67	1.6445258	4.7981	5.2503	-0.6394
0.68	2.1260574	4.8722	4.2348	-0.6887
0.69	2.6155618	4.9323	3.3819	-0.7261
0.70	3.0626328	4.9916	2.7078	-0.7530
0.71	3.5987757	5.0762	1.9850	-0.7790
0.72	4.0882433	5.1640	1.3663	-0.7983
0.73	4.5796220	5.2438	0.7221	-0.8147

$\hat{\tau}_3 = 0.800, \quad 0.57 \le \hat{\tau}_4 \le 0.70$				
$\hat{\tau}_4$	A	B	h	k
0.57	0.0013997	0.8082	17.7983	1.2007
0.58	0.0416004	1.8845	13.3536	0.2824
0.59	0.1793004	2.8626	10.5398	-0.1017
0.60	0.4099327	3.5099	8.5374	-0.3105
0.61	0.7046625	3.8857	7.0167	-0.4411
0.62	1.0394897	4.0917	5.8092	-0.5304
0.63	1.3984486	4.2035	4.8184	-0.5952
0.64	1.7714752	4.2686	3.9836	-0.6443
0.65	2.1522450	4.3148	3.2638	-0.6827
0.66	2.5367538	4.3572	2.6287	-0.7137
0.67	2.9225250	4.4029	2.0540	-0.7391
0.68	3.3083705	4.4530	1.5158	-0.7603
0.69	3.6952259	4.5015	0.9810	-0.7784
0.70	4.0943235	4.5187	0.3485	-0.7943

$\hat{\tau}_3 = 0.850, \quad 0.67 \le \hat{\tau}_4 \le 0.77$				
$\hat{\tau}_4$	A	B	h	k
0.67	0.0216348	1.8760	19.3738	0.3465
0.68	0.2460028	3.8862	13.4780	-0.2412
0.69	0.6914941	4.9362	10.0230	-0.4739
0.70	1.2559312	5.3691	7.6905	-0.5980
0.71	1.8761122	5.5425	5.9860	-0.6749
0.72	2.5209017	5.6344	4.6704	-0.7271
0.73	3.1747010	5.7211	3.6104	-0.7649
0.74	3.8291786	5.8309	2.7219	-0.7934
0.75	4.4796787	5.9689	1.9434	-0.8156
0.76	5.1246543	6.1241	1.2120	-0.8334
0.77	5.7778524	6.2418	0.3575	-0.8483

$\hat{\tau}_3 = 0.875, \quad 0.72 \le \hat{\tau}_4 \le 0.80$				
$\hat{\tau}_4$	A	B	h	k
0.72	0.0296175	2.3125	22.5402	0.2118
0.73	0.4241025	5.1481	14.3310	-0.3920
0.74	1.1520144	6.2007	10.0502	-0.5960
0.75	2.0134594	6.5059	7.3459	-0.6979
0.76	2.9223134	6.6289	5.4530	-0.7588
0.77	3.8443036	6.7604	4.0321	-0.7992
0.78	4.7640903	6.9521	2.9021	-0.8280
0.79	5.6736386	7.2038	1.9465	-0.8494
0.80	6.5701534	7.4856	1.0500	-0.8659

$$0.900 \leq \tau_3 \leq 0.975$$

$\hat{\tau}_3 = 0.900, \quad 0.77 \leq \hat{\tau}_4 \leq 0.84$				
$\hat{\tau}_4$	A	B	h	k
0.77	0.0090326	1.8450	31.3980	0.4141
0.78	0.6070567	6.6547	16.3917	-0.4918
0.79	1.8171111	7.7877	10.3941	-0.6923
0.80	3.1915197	7.9884	7.0388	-0.7796
0.81	4.5999853	8.1814	4.8518	-0.8283
0.82	6.0027635	8.5273	3.2750	-0.8593
0.83	7.3825468	9.0110	2.0285	-0.8806
0.84	8.7314723	9.5525	0.8738	-0.8961

$\hat{\tau}_3 = 0.925, \quad 0.83 \leq \hat{\tau}_4 \leq 0.88$				
$\hat{\tau}_4$	A	B	h	k
0.83	0.5977733	8.2894	22.2029	-0.5166
0.84	2.7593659	10.0289	11.4094	-0.7683
0.85	5.1920040	10.2285	6.7617	-0.8479
0.86	7.6299913	10.8535	4.0952	-0.8867
0.87	10.0172025	11.8717	2.2739	-0.9095
0.88	12.3238489	13.0344	0.6708	-0.9244

$\hat{\tau}_3 = 0.950, \quad 0.89 \leq \hat{\tau}_4 \leq 0.92$				
$\hat{\tau}_4$	A	B	h	k
0.89	3.8736851	13.8393	14.6753	-0.8222
0.90	9.2276517	14.4486	6.5090	-0.9060
0.91	14.4877828	16.9393	2.9652	-0.9359
0.92	19.4933321	20.0581	0.4008	-0.9510

$\hat{\tau}_3 = 0.975, \quad 0.94 \leq \hat{\tau}_4 \leq 0.95$				
$\hat{\tau}_4$	A	B	h	k
0.94	1.0347608	21.8259	55.5494	-0.6743
0.95	21.3733573	26.7324	6.2765	-0.9561

Appendix G

Table G–1 for Kappa Distribution Fits: Method of Percentiles

For specified ρ_3 and ρ_4, the following table gives the h and k values, along with the values of A and B that were discussed in Chapter 17 (see Section 17.3), kappa distribution for fits through the use of percentiles.

$$0.05 \le \hat{\rho}_3 \le 0.10$$

$\hat{\rho}_3 = 0.05, \quad 0.26 \le \hat{\rho}_4 \le 0.63$				
$\hat{\rho}_4$	A	B	h	k
0.26	7.8105850	2.6310	3.2718	-0.9910
0.27	6.0180570	2.5230	3.4748	-0.8627
0.28	4.5716972	2.3814	3.6638	-0.7290
0.29	3.4273094	2.2184	3.8411	-0.5902
0.30	2.5374263	2.0444	4.0081	-0.4466
0.31	1.8562321	1.8676	4.1663	-0.2985
0.32	1.3422820	1.6942	4.3165	-0.1459
0.33	0.9597355	1.5286	4.4598	0.0110
0.34	0.6786359	1.3737	4.5967	0.1723
0.35	0.4746176	1.2313	4.7279	0.3378
0.36	0.3283101	1.1021	4.8540	0.5077
0.37	0.2246160	0.9862	4.9754	0.6818
0.38	0.1519737	0.8830	5.0924	0.8604
0.39	0.1016714	0.7918	5.2055	1.0435
0.40	0.0672417	0.7116	5.3150	1.2312
0.41	0.0439514	0.6412	5.4212	1.4236
0.42	0.0283833	0.5796	5.5243	1.6209
0.43	0.0181030	0.5258	5.6245	1.8233
0.44	0.0113986	0.4787	5.7221	2.0309
0.45	0.0070821	0.4374	5.8173	2.2440
0.46	0.0043397	0.4011	5.9102	2.4628
0.47	0.0026211	0.3692	6.0010	2.6875
0.48	0.0015594	0.3410	6.0899	2.9185
0.49	0.0009132	0.3159	6.1769	3.1560
0.50	0.0005261	0.2935	6.2623	3.4004
0.51	0.0002978	0.2735	6.3461	3.6521
0.52	0.0001655	0.2555	6.4285	3.9113
0.53	0.0000903	0.2392	6.5095	4.1787
0.54	0.0000482	0.2244	6.5894	4.4545
0.55	0.0000252	0.2110	6.6680	4.7394
0.56	0.0000129	0.1986	6.7457	5.0338
0.57	0.0000064	0.1873	6.8224	5.3384
0.58	0.0000031	0.1769	6.8982	5.6538
0.59	0.0000015	0.1672	6.9733	5.9806
0.60	0.0000007	0.1582	7.0477	6.3197
0.61	0.0000003	0.1499	7.1215	6.6719
0.62	0.0000001	0.1421	7.1947	7.0380
0.63	0.0000001	0.1348	7.2675	7.4190

$\hat{\rho}_3 = 0.10, \quad 0.30 \le \hat{\rho}_4 \le 0.74$				
$\hat{\rho}_4$	A	B	h	k
0.30	8.2358575	1.3154	1.5243	-0.9669
0.31	7.0764268	1.4197	1.7754	-0.8851
0.32	6.0067283	1.4850	1.9948	-0.7981
0.33	5.0424353	1.5190	2.1927	-0.7065
0.34	4.1889831	1.5274	2.3748	-0.6105
0.35	3.4454039	1.5144	2.5446	-0.5103
0.36	2.8065794	1.4839	2.7043	-0.4060
0.37	2.2647743	1.4395	2.8556	-0.2978
0.38	1.8107527	1.3844	2.9996	-0.1857
0.39	1.4346064	1.3214	3.1373	-0.0697
0.40	1.1263622	1.2530	3.2693	0.0501
0.41	0.8764126	1.1816	3.3964	0.1737
0.42	0.6758033	1.1089	3.5188	0.3012
0.43	0.5164069	1.0365	3.6372	0.4325
0.44	0.3910084	0.9656	3.7517	0.5677
0.45	0.2933242	0.8972	3.8628	0.7069
0.46	0.2179752	0.8319	3.9707	0.8501
0.47	0.1604263	0.7703	4.0756	0.9975
0.48	0.1169093	0.7126	4.1778	1.1491
0.49	0.0843345	0.6589	4.2774	1.3052
0.50	0.0602011	0.6093	4.3747	1.4658
0.51	0.0425095	0.5636	4.4699	1.6312
0.52	0.0296806	0.5218	4.5629	1.8015
0.53	0.0204816	0.4835	4.6542	1.9769
0.54	0.0139616	0.4486	4.7436	2.1576
0.55	0.0093960	0.4168	4.8314	2.3441
0.56	0.0062390	0.3879	4.9178	2.5364
0.57	0.0040846	0.3615	5.0027	2.7350
0.58	0.0026346	0.3375	5.0864	2.9401
0.59	0.0016728	0.3156	5.1689	3.1522
0.60	0.0010446	0.2956	5.2504	3.3717
0.61	0.0006408	0.2772	5.3308	3.5990
0.62	0.0003858	0.2604	5.4104	3.8347
0.63	0.0002277	0.2449	5.4892	4.0792
0.64	0.0001315	0.2306	5.5673	4.3333
0.65	0.0000743	0.2174	5.6448	4.5975
0.66	0.0000409	0.2052	5.7217	4.8726
0.67	0.0000220	0.1938	5.7983	5.1595
0.68	0.0000114	0.1832	5.8745	5.4589
0.69	0.0000058	0.1732	5.9505	5.7721
0.70	0.0000028	0.1639	6.0263	6.1000
0.71	0.0000013	0.1552	6.1020	6.4438
0.72	0.0000006	0.1469	6.1779	6.8052
0.73	0.0000003	0.1392	6.2538	7.1855
0.74	0.0000001	0.1318	6.3301	7.5865

$$0.15 \le \hat{\rho}_3 \le 0.20$$

$\hat{\rho}_3 = 0.15, \quad 0.33 \le \hat{\rho}_4 \le 0.81$				
$\hat{\rho}_4$	A	B	h	k
0.33	7.6530388	0.5510	0.2417	-0.8816
0.34	6.7623342	0.8080	0.7686	-0.8129
0.35	5.9434983	0.9441	1.0734	-0.7419
0.36	5.1842300	1.0357	1.3130	-0.6674
0.37	4.4875046	1.0985	1.5182	-0.5896
0.38	3.8550710	1.1395	1.7014	-0.5084
0.39	3.2869017	1.1627	1.8691	-0.4238
0.40	2.7814739	1.1711	2.0250	-0.3359
0.41	2.3361070	1.1669	2.1717	-0.2446
0.42	1.9472675	1.1521	2.3109	-0.1500
0.43	1.6108335	1.1284	2.4437	-0.0519
0.44	1.3223200	1.0976	2.5710	0.0496
0.45	1.0770703	1.0610	2.6936	0.1545
0.46	0.8704130	1.0199	2.8119	0.2631
0.47	0.6977888	0.9757	2.9266	0.3752
0.48	0.5548478	0.9293	3.0378	0.4910
0.49	0.4375210	0.8818	3.1459	0.6105
0.50	0.3420674	0.8339	3.2512	0.7339
0.51	0.2651021	0.7863	3.3540	0.8613
0.52	0.2036062	0.7397	3.4543	0.9927
0.53	0.1549241	0.6945	3.5524	1.1284
0.54	0.1167496	0.6510	3.6485	1.2685
0.55	0.0871049	0.6094	3.7427	1.4131
0.56	0.0643137	0.5701	3.8351	1.5625
0.57	0.0469724	0.5329	3.9259	1.7168
0.58	0.0339190	0.4981	4.0152	1.8764
0.59	0.0242027	0.4656	4.1032	2.0414
0.60	0.0170544	0.4353	4.1898	2.2122
0.61	0.0118595	0.4071	4.2752	2.3890
0.62	0.0081325	0.3810	4.3596	2.5722
0.63	0.0054948	0.3568	4.4430	2.7623
0.64	0.0036546	0.3345	4.5255	2.9596
0.65	0.0023903	0.3138	4.6073	3.1646
0.66	0.0015356	0.2946	4.6883	3.3779
0.67	0.0009678	0.2769	4.7687	3.6000
0.68	0.0005975	0.2604	4.8487	3.8317
0.69	0.0003608	0.2452	4.9282	4.0735
0.70	0.0002127	0.2310	5.0074	4.3264
0.71	0.0001222	0.2177	5.0864	4.5912
0.72	0.0000683	0.2053	5.1653	4.8689
0.73	0.0000370	0.1937	5.2442	5.1608
0.74	0.0000194	0.1829	5.3232	5.4681
0.75	0.0000098	0.1726	5.4024	5.7923
0.76	0.0000047	0.1630	5.4820	6.1351
0.77	0.0000022	0.1539	5.5621	6.4984
0.78	0.0000010	0.1453	5.6429	6.8843
0.79	0.0000004	0.1371	5.7244	7.2956
0.80	0.0000002	0.1293	5.8069	7.7351
0.81	0.0000001	0.1219	5.8906	8.2063

$\hat{\rho}_3 = 0.20, \quad 0.37 \le \hat{\rho}_4 \le 0.85$				
$\hat{\rho}_4$	A	B	h	k
0.37	6.1128933	0.4841	0.1447	-0.7085
0.38	5.3593021	0.6785	0.5896	-0.6371
0.39	4.7134311	0.7886	0.8667	-0.5667
0.40	4.1307568	0.8643	1.0867	-0.4946
0.41	3.6024169	0.9178	1.2757	-0.4201
0.42	3.1245539	0.9546	1.4447	-0.3431
0.43	2.6944609	0.9780	1.5995	-0.2635
0.44	2.3096360	0.9900	1.7436	-0.1810
0.45	1.9675066	0.9924	1.8794	-0.0956
0.46	1.6653627	0.9866	2.0085	-0.0072
0.47	1.4003661	0.9737	2.1319	0.0842
0.48	1.1695884	0.9549	2.2506	0.1788
0.49	0.9700597	0.9311	2.3651	0.2767
0.50	0.7988192	0.9032	2.4761	0.3779
0.51	0.6529621	0.8720	2.5839	0.4826
0.52	0.5296814	0.8385	2.6888	0.5908
0.53	0.4263030	0.8031	2.7911	0.7028
0.54	0.3403135	0.7666	2.8912	0.8185
0.55	0.2693808	0.7296	2.9891	0.9383
0.56	0.2113675	0.6924	3.0851	1.0622
0.57	0.1643379	0.6556	3.1794	1.1903
0.58	0.1265592	0.6195	3.2720	1.3230
0.59	0.0964979	0.5844	3.3631	1.4604
0.60	0.0728118	0.5505	3.4528	1.6027
0.61	0.0543397	0.5180	3.5413	1.7502
0.62	0.0400876	0.4869	3.6286	1.9032
0.63	0.0292147	0.4575	3.7149	2.0620
0.64	0.0210175	0.4296	3.8002	2.2269
0.65	0.0149143	0.4034	3.8847	2.3982
0.66	0.0104301	0.3787	3.9683	2.5765
0.67	0.0071814	0.3556	4.0514	2.7622
0.68	0.0048628	0.3340	4.1338	2.9558
0.69	0.0032345	0.3138	4.2158	3.1579
0.70	0.0021103	0.2950	4.2974	3.3692
0.71	0.0013485	0.2774	4.3787	3.5903
0.72	0.0008425	0.2609	4.4598	3.8221
0.73	0.0005137	0.2455	4.5408	4.0655
0.74	0.0003049	0.2311	4.6219	4.3215
0.75	0.0001758	0.2176	4.7031	4.5915
0.76	0.0000981	0.2050	4.7846	4.8766
0.77	0.0000529	0.1931	4.8665	5.1785
0.78	0.0000274	0.1818	4.9490	5.4989
0.79	0.0000136	0.1712	5.0322	5.8400
0.80	0.0000064	0.1612	5.1163	6.2041
0.81	0.0000029	0.1516	5.2015	6.5942
0.82	0.0000012	0.1426	5.2880	7.0134
0.83	0.0000005	0.1339	5.3761	7.4659
0.84	0.0000002	0.1257	5.4660	7.9564
0.85	0.0000001	0.1178	5.5582	8.4907

$$0.25 \le \hat{\rho}_3 \le 0.30$$

$\hat{\rho}_3 = 0.25, \quad 0.40 \le \hat{\rho}_4 \le 0.87$					$\hat{\rho}_3 = 0.30, \quad 0.42 \le \hat{\rho}_4 \le 0.89$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.40	5.2997191	0.4226	0.0324	-0.5833	0.42	5.1030786	0.2728	-0.3155	-0.5160
0.41	4.6268617	0.5901	0.4423	-0.5101	0.43	4.3688586	0.4874	0.2209	-0.4345
0.42	4.0713640	0.6876	0.7045	-0.4400	0.44	3.8391438	0.5906	0.5006	-0.3638
0.43	3.5788384	0.7555	0.9131	-0.3692	0.45	3.3817467	0.6609	0.7146	-0.2938
0.44	3.1359404	0.8043	1.0924	-0.2967	0.46	2.9748198	0.7119	0.8950	-0.2230
0.45	2.7366108	0.8391	1.2526	-0.2223	0.47	2.6094509	0.7491	1.0544	-0.1506
0.46	2.3770707	0.8626	1.3993	-0.1455	0.48	2.2806368	0.7756	1.1993	-0.0765
0.47	2.0544154	0.8766	1.5359	-0.0664	0.49	1.9849439	0.7934	1.3335	-0.0002
0.48	1.7661047	0.8825	1.6646	0.0153	0.50	1.7196880	0.8036	1.4594	0.0784
0.49	1.5097559	0.8815	1.7869	0.0998	0.51	1.4825423	0.8074	1.5788	0.1595
0.50	1.2830569	0.8744	1.9041	0.1871	0.52	1.2714045	0.8055	1.6929	0.2431
0.51	1.0837339	0.8622	2.0168	0.2773	0.53	1.0842962	0.7988	1.8026	0.3295
0.52	0.9095456	0.8455	2.1258	0.3706	0.54	0.9193252	0.7878	1.9085	0.4188
0.53	0.7582907	0.8251	2.2315	0.4672	0.55	0.7746670	0.7732	2.0113	0.5112
0.54	0.6278207	0.8015	2.3344	0.5671	0.56	0.6485584	0.7554	2.1113	0.6068
0.55	0.5160563	0.7754	2.4348	0.6704	0.57	0.5392976	0.7350	2.2089	0.7058
0.56	0.4210027	0.7474	2.5330	0.7775	0.58	0.4452487	0.7124	2.3044	0.8083
0.57	0.3407646	0.7179	2.6291	0.8883	0.59	0.3648475	0.6881	2.3981	0.9145
0.58	0.2735585	0.6874	2.7235	1.0031	0.60	0.2966082	0.6624	2.4902	1.0247
0.59	0.2177229	0.6563	2.8163	1.1222	0.61	0.2391297	0.6358	2.5808	1.1390
0.60	0.1717251	0.6251	2.9075	1.2455	0.62	0.1911008	0.6086	2.6701	1.2577
0.61	0.1341653	0.5939	2.9975	1.3735	0.63	0.1513049	0.5811	2.7582	1.3811
0.62	0.1037781	0.5632	3.0863	1.5064	0.64	0.1186228	0.5535	2.8454	1.5093
0.63	0.0794313	0.5331	3.1739	1.6443	0.65	0.0920340	0.5262	2.9316	1.6427
0.64	0.0601223	0.5038	3.2606	1.7877	0.66	0.0706167	0.4992	3.0171	1.7816
0.65	0.0449725	0.4756	3.3464	1.9368	0.67	0.0535463	0.4729	3.1019	1.9264
0.66	0.0332205	0.4484	3.4315	2.0921	0.68	0.0400928	0.4473	3.1861	2.0774
0.67	0.0242134	0.4225	3.5158	2.2538	0.69	0.0296161	0.4226	3.2698	2.2352
0.68	0.0173980	0.3977	3.5996	2.4225	0.70	0.0215616	0.3988	3.3532	2.4002
0.69	0.0123111	0.3743	3.6829	2.5986	0.71	0.0154539	0.3760	3.4362	2.5730
0.70	0.0085694	0.3521	3.7658	2.7827	0.72	0.0108906	0.3543	3.5191	2.7542
0.71	0.0058601	0.3312	3.8484	2.9755	0.73	0.0075355	0.3336	3.6020	2.9445
0.72	0.0039313	0.3114	3.9308	3.1776	0.74	0.0051110	0.3139	3.6848	3.1447
0.73	0.0025830	0.2929	4.0131	3.3897	0.75	0.0033919	0.2953	3.7678	3.3556
0.74	0.0016592	0.2754	4.0954	3.6129	0.76	0.0021980	0.2777	3.8511	3.5784
0.75	0.0010397	0.2590	4.1779	3.8481	0.77	0.0013874	0.2611	3.9347	3.8142
0.76	0.0006340	0.2436	4.2606	4.0965	0.78	0.0008506	0.2454	4.0189	4.0644
0.77	0.0003752	0.2291	4.3437	4.3593	0.79	0.0005049	0.2305	4.1038	4.3305
0.78	0.0002148	0.2154	4.4273	4.6382	0.80	0.0002891	0.2165	4.1896	4.6144
0.79	0.0001186	0.2025	4.5116	4.9350	0.81	0.0001590	0.2032	4.2764	4.9182
0.80	0.0000628	0.1904	4.5968	5.2516	0.82	0.0000835	0.1906	4.3645	5.2446
0.81	0.0000317	0.1788	4.6831	5.5905	0.83	0.0000416	0.1786	4.4542	5.5966
0.82	0.0000152	0.1679	4.7706	5.9547	0.84	0.0000195	0.1673	4.5456	5.9779
0.83	0.0000069	0.1575	4.8597	6.3475	0.85	0.0000086	0.1564	4.6393	6.3930
0.84	0.0000029	0.1476	4.9505	6.7730	0.86	0.0000035	0.1460	4.7355	6.8473
0.85	0.0000011	0.1382	5.0436	7.2363	0.87	0.0000013	0.1361	4.8347	7.3478
0.86	0.0000004	0.1291	5.1392	7.7436	0.88	0.0000004	0.1265	4.9375	7.9032
0.87	0.0000001	0.1204	5.2378	8.3024	0.89	0.0000001	0.1173	5.0446	8.5248

$0.35 \le \hat{\rho}_3 \le 0.40$

$\hat{\rho}_3 = 0.35, \quad 0.44 \le \hat{\rho}_4 \le 0.90$					$\hat{\rho}_3 = 0.40, \quad 0.46 \le \hat{\rho}_4 \le 0.91$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.44	4.6099116	0.2993	-0.2408	-0.4202	0.46	4.0289839	0.3666	-0.0554	-0.3089
0.45	3.9810079	0.4604	0.1779	-0.3421	0.47	3.5233516	0.4732	0.2403	-0.2357
0.46	3.5002820	0.5508	0.4345	-0.2713	0.48	3.1051269	0.5439	0.4572	-0.1656
0.47	3.0861495	0.6139	0.6358	-0.2015	0.49	2.7400689	0.5953	0.6364	-0.0957
0.48	2.7189941	0.6602	0.8071	-0.1311	0.50	2.4153806	0.6337	0.7929	-0.0251
0.49	2.3901060	0.6945	0.9592	-0.0594	0.51	2.1243632	0.6623	0.9339	0.0468
0.50	2.0944445	0.7193	1.0979	0.0140	0.52	1.8627802	0.6829	1.0637	0.1204
0.51	1.8285592	0.7363	1.2265	0.0893	0.53	1.6275885	0.6968	1.1850	0.1959
0.52	1.5897953	0.7467	1.3475	0.1667	0.54	1.4163930	0.7051	1.2995	0.2735
0.53	1.3759371	0.7513	1.4623	0.2465	0.55	1.2271821	0.7084	1.4087	0.3533
0.54	1.1850298	0.7509	1.5721	0.3288	0.56	1.0581862	0.7073	1.5135	0.4357
0.55	1.0152856	0.7461	1.6777	0.4137	0.57	0.9077984	0.7024	1.6146	0.5207
0.56	0.8650313	0.7375	1.7799	0.5014	0.58	0.7745283	0.6941	1.7127	0.6086
0.57	0.7326803	0.7255	1.8791	0.5922	0.59	0.6569746	0.6829	1.8081	0.6995
0.58	0.6167173	0.7106	1.9758	0.6861	0.60	0.5538088	0.6691	1.9012	0.7936
0.59	0.5156911	0.6932	2.0703	0.7834	0.61	0.4637665	0.6531	1.9925	0.8912
0.60	0.4282122	0.6737	2.1629	0.8842	0.62	0.3856421	0.6352	2.0821	0.9924
0.61	0.3529538	0.6525	2.2538	0.9888	0.63	0.3182874	0.6157	2.1703	1.0974
0.62	0.2886533	0.6300	2.3433	1.0973	0.64	0.2606117	0.5950	2.2572	1.2066
0.63	0.2341158	0.6064	2.4315	1.2100	0.65	0.2115824	0.5733	2.3431	1.3201
0.64	0.1882172	0.5820	2.5187	1.3272	0.66	0.1702265	0.5508	2.4281	1.4383
0.65	0.1499067	0.5572	2.6049	1.4492	0.67	0.1356329	0.5279	2.5124	1.5616
0.66	0.1182094	0.5322	2.6902	1.5762	0.68	0.1069531	0.5047	2.5961	1.6902
0.67	0.0922279	0.5071	2.7749	1.7086	0.69	0.0834029	0.4815	2.6793	1.8245
0.68	0.0711429	0.4823	2.8590	1.8468	0.70	0.0642630	0.4584	2.7621	1.9650
0.69	0.0542133	0.4578	2.9426	1.9912	0.71	0.0488792	0.4355	2.8446	2.1122
0.70	0.0407747	0.4339	3.0258	2.1422	0.72	0.0366617	0.4131	2.9270	2.2665
0.71	0.0302374	0.4106	3.1088	2.3003	0.73	0.0270839	0.3912	3.0094	2.4287
0.72	0.0220838	0.3881	3.1916	2.4662	0.74	0.0196805	0.3699	3.0919	2.5994
0.73	0.0158644	0.3663	3.2744	2.6404	0.75	0.0140451	0.3493	3.1745	2.7793
0.74	0.0111933	0.3455	3.3572	2.8237	0.76	0.0098271	0.3295	3.2575	2.9694
0.75	0.0077440	0.3255	3.4402	3.0169	0.77	0.0067277	0.3104	3.3410	3.1707
0.76	0.0052435	0.3064	3.5235	3.2210	0.78	0.0044963	0.2922	3.4250	3.3842
0.77	0.0034673	0.2883	3.6072	3.4371	0.79	0.0029256	0.2748	3.5098	3.6115
0.78	0.0022335	0.2711	3.6915	3.6663	0.80	0.0018476	0.2581	3.5956	3.8540
0.79	0.0013974	0.2547	3.7766	3.9102	0.81	0.0011282	0.2423	3.6825	4.1137
0.80	0.0008464	0.2391	3.8625	4.1704	0.82	0.0006633	0.2272	3.7707	4.3926
0.81	0.0004942	0.2244	3.9495	4.4489	0.83	0.0003735	0.2128	3.8605	4.6935
0.82	0.0002769	0.2104	4.0378	4.7481	0.84	0.0002001	0.1991	3.9522	5.0196
0.83	0.0001481	0.1971	4.1277	5.0708	0.85	0.0001012	0.1860	4.0462	5.3746
0.84	0.0000750	0.1844	4.2194	5.4203	0.86	0.0000479	0.1735	4.1427	5.7634
0.85	0.0000357	0.1724	4.3133	5.8008	0.87	0.0000209	0.1615	4.2424	6.1920
0.86	0.0000158	0.1608	4.4098	6.2174	0.88	0.0000083	0.1500	4.3458	6.6679
0.87	0.0000064	0.1498	4.5093	6.6764	0.89	0.0000030	0.1389	4.4536	7.2010
0.88	0.0000024	0.1392	4.6125	7.1858	0.90	0.0000009	0.1281	4.5667	7.8043
0.89	0.0000008	0.1289	4.7201	7.7561	0.91	0.0000002	0.1177	4.6864	8.4953
0.90	0.0000002	0.1190	4.8329	8.4011					

$0.45 \leq \hat{\rho}_3 \leq 0.50$

$\hat{\rho}_3 = 0.45, \quad 0.47 \leq \hat{\rho}_4 \leq 0.92$					$\hat{\rho}_3 = 0.50, \quad 0.48 \leq \hat{\rho}_4 \leq 0.93$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.47	3.9797896	0.3135	-0.1865	-0.2647	0.48	3.8360945	0.2985	-0.2192	-0.2093
0.48	3.4659907	0.4259	0.1252	-0.1894	0.49	3.3418693	0.4037	0.0774	-0.1339
0.49	3.0521018	0.4985	0.3469	-0.1187	0.50	2.9429927	0.4730	0.2923	-0.0629
0.50	2.6946707	0.5512	0.5279	-0.0488	0.51	2.5992400	0.5236	0.4688	0.0072
0.51	2.3785064	0.5907	0.6847	0.0215	0.52	2.2957624	0.5618	0.6221	0.0775
0.52	2.0959333	0.6204	0.8253	0.0928	0.53	2.0248968	0.5907	0.7598	0.1488
0.53	1.8422421	0.6423	0.9543	0.1656	0.54	1.7819049	0.6122	0.8861	0.2215
0.54	1.6141483	0.6577	1.0744	0.2402	0.55	1.5634847	0.6277	1.0039	0.2958
0.55	1.4091422	0.6676	1.1877	0.3167	0.56	1.3671284	0.6379	1.1149	0.3720
0.56	1.2251732	0.6728	1.2955	0.3953	0.57	1.1908067	0.6435	1.2206	0.4503
0.57	1.0604817	0.6737	1.3988	0.4764	0.58	1.0327969	0.6452	1.3219	0.5309
0.58	0.9135034	0.6709	1.4984	0.5600	0.59	0.8915851	0.6433	1.4195	0.6141
0.59	0.7828128	0.6648	1.5949	0.6464	0.60	0.7658052	0.6383	1.5141	0.6999
0.60	0.6670873	0.6558	1.6888	0.7357	0.61	0.6542020	0.6306	1.6062	0.7887
0.61	0.5650866	0.6443	1.7804	0.8282	0.62	0.5556065	0.6203	1.6962	0.8806
0.62	0.4756379	0.6305	1.8701	0.9240	0.63	0.4689206	0.6080	1.7843	0.9758
0.63	0.3976287	0.6149	1.9583	1.0234	0.64	0.3931067	0.5938	1.8709	1.0747
0.64	0.3300015	0.5976	2.0450	1.1266	0.65	0.3271816	0.5780	1.9562	1.1773
0.65	0.2717516	0.5790	2.1306	1.2339	0.66	0.2702125	0.5608	2.0404	1.2841
0.66	0.2219263	0.5593	2.2152	1.3456	0.67	0.2213149	0.5425	2.1237	1.3954
0.67	0.1796248	0.5387	2.2990	1.4620	0.68	0.1796517	0.5233	2.2063	1.5114
0.68	0.1439994	0.5175	2.3822	1.5834	0.69	0.1444331	0.5035	2.2883	1.6325
0.69	0.1142557	0.4959	2.4648	1.7101	0.70	0.1149171	0.4831	2.3699	1.7591
0.70	0.0896542	0.4741	2.5470	1.8428	0.71	0.0904098	0.4625	2.4512	1.8917
0.71	0.0695109	0.4522	2.6290	1.9816	0.72	0.0702662	0.4417	2.5323	2.0308
0.72	0.0531976	0.4305	2.7108	2.1273	0.73	0.0538907	0.4209	2.6135	2.1769
0.73	0.0401423	0.4090	2.7926	2.2804	0.74	0.0407375	0.4003	2.6947	2.3306
0.74	0.0298288	0.3878	2.8745	2.4414	0.75	0.0303108	0.3799	2.7762	2.4928
0.75	0.0217957	0.3672	2.9566	2.6113	0.76	0.0221637	0.3600	2.8580	2.6641
0.76	0.0156350	0.3471	3.0390	2.7907	0.77	0.0158986	0.3405	2.9404	2.8455
0.77	0.0109901	0.3277	3.1220	2.9807	0.78	0.0111647	0.3215	3.0234	3.0381
0.78	0.0075535	0.3089	3.2056	3.1824	0.79	0.0076572	0.3031	3.1073	3.2432
0.79	0.0050633	0.2908	3.2900	3.3970	0.80	0.0051145	0.2854	3.1921	3.4621
0.80	0.0033006	0.2735	3.3754	3.6262	0.81	0.0033161	0.2683	3.2782	3.6966
0.81	0.0020850	0.2568	3.4619	3.8715	0.82	0.0020788	0.2518	3.3657	3.9487
0.82	0.0012712	0.2409	3.5498	4.1352	0.83	0.0012541	0.2361	3.4549	4.2209
0.83	0.0007443	0.2257	3.6394	4.4198	0.84	0.0007239	0.2209	3.5461	4.5161
0.84	0.0004160	0.2112	3.7309	4.7282	0.85	0.0003971	0.2064	3.6397	4.8378
0.85	0.0002204	0.1973	3.8247	5.0642	0.86	0.0002052	0.1925	3.7360	5.1904
0.86	0.0001096	0.1840	3.9212	5.4323	0.87	0.0000988	0.1792	3.8355	5.5797
0.87	0.0000506	0.1712	4.0209	5.8383	0.88	0.0000437	0.1663	3.9389	6.0125
0.88	0.0000214	0.1590	4.1243	6.2895	0.89	0.0000174	0.1539	4.0469	6.4982
0.89	0.0000081	0.1472	4.2322	6.7952	0.90	0.0000061	0.1419	4.1605	7.0490
0.90	0.0000027	0.1357	4.3456	7.3680	0.91	0.0000018	0.1302	4.2810	7.6815
0.91	0.0000008	0.1246	4.4657	8.0250	0.92	0.0000004	0.1188	4.4099	8.4193
0.92	0.0000002	0.1138	4.5941	8.7899	0.93	0.0000001	0.1076	4.5497	9.2965

$$0.55 \le \hat{\rho}_3 \le 0.60$$

$\hat{\rho}_3 = 0.55, \quad 0.48 \le \hat{\rho}_4 \le 0.93$					$\hat{\rho}_3 = 0.60, \quad 0.49 \le \hat{\rho}_4 \le 0.93$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.48	4.3189330	0.1246	-0.6768	-0.2367	0.49	3.9205342	0.2075	-0.4563	-0.1547
0.49	3.6245733	0.3085	-0.1847	-0.1448	0.50	3.3767491	0.3302	-0.1155	-0.0737
0.50	3.1684518	0.3994	0.0794	-0.0705	0.51	2.9637142	0.4058	0.1139	-0.0007
0.51	2.7928840	0.4619	0.2799	0.0005	0.52	2.6156413	0.4601	0.2970	0.0702
0.52	2.4676766	0.5082	0.4475	0.0708	0.53	2.3118979	0.5011	0.4537	0.1408
0.53	2.1802084	0.5435	0.5945	0.1414	0.54	2.0425787	0.5325	0.5929	0.2119
0.54	1.9235787	0.5703	0.7273	0.2130	0.55	1.8018584	0.5564	0.7197	0.2840
0.55	1.6933798	0.5903	0.8496	0.2859	0.56	1.5858456	0.5742	0.8373	0.3575
0.56	1.4864832	0.6047	0.9640	0.3605	0.57	1.3916971	0.5869	0.9476	0.4328
0.57	1.3004908	0.6142	1.0720	0.4369	0.58	1.2171934	0.5951	1.0522	0.5099
0.58	1.1334552	0.6195	1.1750	0.5154	0.59	1.0605117	0.5994	1.1522	0.5892
0.59	0.9837250	0.6211	1.2739	0.5963	0.60	0.9200969	0.6003	1.2484	0.6708
0.60	0.8498524	0.6193	1.3693	0.6797	0.61	0.7945818	0.5981	1.3414	0.7550
0.61	0.7305368	0.6146	1.4619	0.7658	0.62	0.6827376	0.5932	1.4318	0.8420
0.62	0.6245882	0.6074	1.5520	0.8548	0.63	0.5834407	0.5859	1.5200	0.9320
0.63	0.5309032	0.5978	1.6402	0.9471	0.64	0.4956510	0.5765	1.6063	1.0252
0.64	0.4484491	0.5862	1.7266	1.0427	0.65	0.4183966	0.5651	1.6911	1.1219
0.65	0.3762534	0.5728	1.8116	1.1419	0.66	0.3507637	0.5521	1.7745	1.2224
0.66	0.3133965	0.5578	1.8955	1.2451	0.67	0.2918897	0.5377	1.8570	1.3270
0.67	0.2590074	0.5416	1.9783	1.3525	0.68	0.2409587	0.5220	1.9385	1.4359
0.68	0.2122609	0.5243	2.0604	1.4644	0.69	0.1971981	0.5053	2.0194	1.5495
0.69	0.1723761	0.5061	2.1418	1.5812	0.70	0.1598771	0.4878	2.0998	1.6683
0.70	0.1386156	0.4872	2.2228	1.7033	0.71	0.1283057	0.4696	2.1798	1.7925
0.71	0.1102854	0.4678	2.3034	1.8312	0.72	0.1018343	0.4509	2.2597	1.9228
0.72	0.0867355	0.4481	2.3839	1.9652	0.73	0.0798538	0.4319	2.3395	2.0597
0.73	0.0673599	0.4282	2.4644	2.1061	0.74	0.0617959	0.4127	2.4194	2.2037
0.74	0.0515973	0.4082	2.5449	2.2543	0.75	0.0471336	0.3935	2.4995	2.3555
0.75	0.0389316	0.3884	2.6257	2.4105	0.76	0.0353812	0.3743	2.5800	2.5160
0.76	0.0288919	0.3688	2.7069	2.5757	0.77	0.0260950	0.3554	2.6611	2.6860
0.77	0.0210523	0.3495	2.7886	2.7506	0.78	0.0188731	0.3367	2.7429	2.8666
0.78	0.0150317	0.3306	2.8711	2.9363	0.79	0.0133553	0.3184	2.8256	3.0588
0.79	0.0104930	0.3121	2.9543	3.1340	0.80	0.0092223	0.3005	2.9094	3.2643
0.80	0.0071416	0.2942	3.0386	3.3452	0.81	0.0061952	0.2831	2.9944	3.4844
0.81	0.0047241	0.2769	3.1242	3.5716	0.82	0.0040335	0.2662	3.0809	3.7214
0.82	0.0030256	0.2601	3.2112	3.8150	0.83	0.0025339	0.2498	3.1693	3.9774
0.83	0.0018676	0.2440	3.3000	4.0778	0.84	0.0015276	0.2340	3.2598	4.2553
0.84	0.0011050	0.2285	3.3909	4.3631	0.85	0.0008780	0.2188	3.3527	4.5586
0.85	0.0006224	0.2135	3.4841	4.6741	0.86	0.0004771	0.2041	3.4485	4.8917
0.86	0.0003310	0.1992	3.5802	5.0154	0.87	0.0002425	0.1900	3.5477	5.2598
0.87	0.0001644	0.1853	3.6795	5.3923	0.88	0.0001138	0.1763	3.6510	5.6701
0.88	0.0000752	0.1720	3.7829	5.8119	0.89	0.0000484	0.1631	3.7592	6.1316
0.89	0.0000312	0.1591	3.8910	6.2832	0.90	0.0000183	0.1502	3.8732	6.6563
0.90	0.0000114	0.1467	4.0048	6.8184	0.91	0.0000059	0.1377	3.9944	7.2610
0.91	0.0000036	0.1345	4.1256	7.4340	0.92	0.0000016	0.1255	4.1247	7.9692
0.92	0.0000009	0.1226	4.2551	8.1534	0.93	0.0000003	0.1134	4.2664	8.8156
0.93	0.0000002	0.1110	4.3958	9.0109					

$$0.65 \le \hat{\rho}_3 \le 0.70$$

$\hat{\rho}_3 = 0.65$, $0.49 \le \hat{\rho}_4 \le 0.94$					$\hat{\rho}_3 = 0.70$, $0.50 \le \hat{\rho}_4 \le 0.94$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.49	4.2863569	0.0811	-0.7893	-0.1696	0.50	3.7544226	0.2042	-0.4604	-0.0714
0.50	3.5715083	0.2651	-0.2951	-0.0738	0.51	3.2456769	0.3110	-0.1575	0.0087
0.51	3.1142593	0.3554	-0.0315	0.0023	0.52	2.8496697	0.3804	0.0579	0.0824
0.52	2.7423728	0.4175	0.1678	0.0744	0.53	2.5149903	0.4311	0.2328	0.1540
0.53	2.4227828	0.4637	0.3340	0.1453	0.54	2.2231085	0.4698	0.3836	0.2252
0.54	2.1416775	0.4990	0.4793	0.2164	0.55	1.9646318	0.4996	0.5183	0.2969
0.55	1.8915405	0.5261	0.6103	0.2881	0.56	1.7338797	0.5225	0.6413	0.3696
0.56	1.6676122	0.5468	0.7307	0.3611	0.57	1.5270032	0.5397	0.7554	0.4436
0.57	1.4665555	0.5620	0.8432	0.4355	0.58	1.3411745	0.5521	0.8627	0.5193
0.58	1.2858505	0.5725	0.9492	0.5117	0.59	1.1741868	0.5604	0.9646	0.5968
0.59	1.1234851	0.5791	1.0502	0.5899	0.60	1.0242359	0.5651	1.0619	0.6764
0.60	0.9777818	0.5821	1.1471	0.6704	0.61	0.8897925	0.5665	1.1556	0.7584
0.61	0.8472949	0.5819	1.2405	0.7533	0.62	0.7695233	0.5651	1.2463	0.8430
0.62	0.7307456	0.5790	1.3310	0.8388	0.63	0.6622395	0.5612	1.3344	0.9303
0.63	0.6269806	0.5735	1.4192	0.9272	0.64	0.5668635	0.5550	1.4205	1.0206
0.64	0.5349434	0.5658	1.5054	1.0187	0.65	0.4824048	0.5467	1.5048	1.1142
0.65	0.4536554	0.5562	1.5899	1.1136	0.66	0.4079443	0.5366	1.5876	1.2114
0.66	0.3822024	0.5448	1.6731	1.2122	0.67	0.3426224	0.5250	1.6692	1.3124
0.67	0.3197252	0.5319	1.7551	1.3146	0.68	0.2856314	0.5119	1.7498	1.4175
0.68	0.2654135	0.5176	1.8362	1.4213	0.69	0.2362093	0.4977	1.8297	1.5271
0.69	0.2185012	0.5023	1.9165	1.5326	0.70	0.1936360	0.4824	1.9090	1.6415
0.70	0.1782635	0.4859	1.9963	1.6488	0.71	0.1572306	0.4663	1.9879	1.7613
0.71	0.1440152	0.4688	2.0758	1.7705	0.72	0.1263500	0.4495	2.0666	1.8868
0.72	0.1151095	0.4512	2.1550	1.8980	0.73	0.1003876	0.4321	2.1452	2.0186
0.73	0.0909380	0.4330	2.2342	2.0319	0.74	0.0787727	0.4144	2.2239	2.1573
0.74	0.0709298	0.4146	2.3135	2.1728	0.75	0.0609710	0.3964	2.3028	2.3036
0.75	0.0545527	0.3960	2.3930	2.3214	0.76	0.0464844	0.3783	2.3822	2.4582
0.76	0.0413127	0.3773	2.4729	2.4785	0.77	0.0348514	0.3601	2.4621	2.6220
0.77	0.0307551	0.3588	2.5534	2.6448	0.78	0.0256473	0.3421	2.5428	2.7960
0.78	0.0224641	0.3404	2.6346	2.8216	0.79	0.0184851	0.3243	2.6244	2.9815
0.79	0.0160634	0.3223	2.7167	3.0098	0.80	0.0130149	0.3067	2.7071	3.1797
0.80	0.0112159	0.3045	2.8000	3.2110	0.81	0.0089243	0.2894	2.7912	3.3924
0.81	0.0076234	0.2871	2.8845	3.4268	0.82	0.0059381	0.2726	2.8768	3.6215
0.82	0.0050256	0.2702	2.9706	3.6591	0.83	0.0038173	0.2562	2.9644	3.8694
0.83	0.0031993	0.2538	3.0585	3.9102	0.84	0.0023582	0.2402	3.0542	4.1388
0.84	0.0019561	0.2379	3.1486	4.1830	0.85	0.0013908	0.2247	3.1466	4.4333
0.85	0.0011413	0.2225	3.2413	4.4809	0.86	0.0007767	0.2097	3.2420	4.7572
0.86	0.0006301	0.2076	3.3369	4.8083	0.87	0.0004064	0.1952	3.3410	5.1160
0.87	0.0003258	0.1932	3.4360	5.1706	0.88	0.0001966	0.1811	3.4443	5.5169
0.88	0.0001557	0.1793	3.5392	5.5748	0.89	0.0000864	0.1675	3.5528	5.9691
0.89	0.0000676	0.1658	3.6475	6.0301	0.90	0.0000337	0.1542	3.6675	6.4850
0.90	0.0000260	0.1527	3.7618	6.5486	0.91	0.0000113	0.1412	3.7898	7.0819
0.91	0.0000086	0.1399	3.8836	7.1474	0.92	0.0000031	0.1285	3.9217	7.7845
0.92	0.0000023	0.1274	4.0146	7.8502	0.93	0.0000006	0.1159	4.0659	8.6294
0.93	0.0000005	0.1150	4.1575	8.6928	0.94	0.0000001	0.1034	4.2263	9.6743
0.94	0.0000001	0.1028	4.3160	9.7307					

$0.75 \leq \hat{\rho}_3 \leq 0.80$

$\hat{\rho}_3 = 0.75, \quad 0.50 \leq \hat{\rho}_4 \leq 0.94$					$\hat{\rho}_3 = 0.80, \quad 0.50 \leq \hat{\rho}_4 \leq 0.94$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.50	3.9250765	0.1480	-0.6110	-0.0669	0.50	4.0803595	0.0977	-0.7446	-0.0601
0.51	3.3581594	0.2726	-0.2648	0.0182	0.51	3.4513749	0.2405	-0.3537	0.0303
0.52	2.9387514	0.3487	-0.0345	0.0935	0.52	3.0105568	0.3222	-0.1107	0.1073
0.53	2.5901889	0.4033	0.1481	0.1659	0.53	2.6497836	0.3798	0.0782	0.1806
0.54	2.2887117	0.4446	0.3036	0.2376	0.54	2.3400645	0.4233	0.2374	0.2528
0.55	2.0230011	0.4765	0.4413	0.3094	0.55	2.0682645	0.4568	0.3774	0.3249
0.56	1.7864651	0.5012	0.5663	0.3821	0.56	1.8269531	0.4828	0.5039	0.3977
0.57	1.5747571	0.5200	0.6819	0.4559	0.57	1.6113307	0.5028	0.6205	0.4716
0.58	1.3847527	0.5339	0.7902	0.5313	0.58	1.4180028	0.5177	0.7294	0.5468
0.59	1.2140571	0.5436	0.8926	0.6084	0.59	1.2444035	0.5284	0.8322	0.6238
0.60	1.0607433	0.5495	0.9903	0.6876	0.60	1.0884924	0.5354	0.9302	0.7028
0.61	0.9231996	0.5522	1.0842	0.7690	0.61	0.9485814	0.5390	1.0241	0.7839
0.62	0.8000373	0.5520	1.1749	0.8529	0.62	0.8232295	0.5398	1.1148	0.8675
0.63	0.6900312	0.5493	1.2630	0.9395	0.63	0.7111758	0.5380	1.2027	0.9537
0.64	0.5920796	0.5442	1.3488	1.0291	0.64	0.6112949	0.5338	1.2883	1.0428
0.65	0.5051780	0.5370	1.4328	1.1218	0.65	0.5225671	0.5275	1.3720	1.1350
0.66	0.4283993	0.5281	1.5153	1.2181	0.66	0.4440562	0.5194	1.4541	1.2307
0.67	0.3608812	0.5175	1.5965	1.3180	0.67	0.3748952	0.5097	1.5350	1.3300
0.68	0.3018159	0.5054	1.6767	1.4221	0.68	0.3142750	0.4985	1.6148	1.4334
0.69	0.2504436	0.4921	1.7562	1.5305	0.69	0.2614366	0.4859	1.6938	1.5412
0.70	0.2060476	0.4777	1.8350	1.6438	0.70	0.2156651	0.4723	1.7721	1.6537
0.71	0.1679506	0.4625	1.9134	1.7622	0.71	0.1762863	0.4578	1.8500	1.7713
0.72	0.1355126	0.4464	1.9915	1.8864	0.72	0.1426629	0.4424	1.9277	1.8947
0.73	0.1081296	0.4298	2.0696	2.0168	0.73	0.1141933	0.4264	2.0053	2.0242
0.74	0.0852323	0.4127	2.1478	2.1540	0.74	0.0903101	0.4099	2.0830	2.1604
0.75	0.0662862	0.3953	2.2262	2.2986	0.75	0.0704794	0.3930	2.1609	2.3042
0.76	0.0507911	0.3776	2.3050	2.4516	0.76	0.0542009	0.3758	2.2393	2.4561
0.77	0.0382820	0.3600	2.3844	2.6137	0.77	0.0410076	0.3585	2.3182	2.6172
0.78	0.0283289	0.3423	2.4646	2.7860	0.78	0.0304662	0.3412	2.3980	2.7885
0.79	0.0205371	0.3247	2.5457	2.9696	0.79	0.0221777	0.3240	2.4787	2.9711
0.80	0.0145483	0.3074	2.6280	3.1660	0.80	0.0157774	0.3069	2.5606	3.1665
0.81	0.0100397	0.2903	2.7117	3.3768	0.81	0.0109358	0.2900	2.6439	3.3764
0.82	0.0067249	0.2736	2.7970	3.6039	0.82	0.0073581	0.2734	2.7289	3.6027
0.83	0.0043532	0.2572	2.8842	3.8498	0.83	0.0047848	0.2572	2.8159	3.8478
0.84	0.0027086	0.2413	2.9737	4.1174	0.84	0.0029908	0.2414	2.9052	4.1147
0.85	0.0016093	0.2258	3.0659	4.4101	0.85	0.0017850	0.2259	2.9972	4.4070
0.86	0.0009055	0.2108	3.1612	4.7323	0.86	0.0010087	0.2109	3.0925	4.7292
0.87	0.0004775	0.1962	3.2603	5.0898	0.87	0.0005340	0.1963	3.1916	5.0870
0.88	0.0002327	0.1820	3.3637	5.4896	0.88	0.0002612	0.1821	3.2952	5.4877
0.89	0.0001030	0.1683	3.4724	5.9413	0.89	0.0001158	0.1682	3.4042	5.9413
0.90	0.0000404	0.1548	3.5875	6.4577	0.90	0.0000455	0.1548	3.5199	6.4608
0.91	0.0000136	0.1417	3.7105	7.0565	0.91	0.0000152	0.1415	3.6437	7.0647
0.92	0.0000037	0.1288	3.8434	7.7631	0.92	0.0000041	0.1285	3.7777	7.7793
0.93	0.0000008	0.1161	3.9891	8.6158	0.93	0.0000008	0.1157	3.9249	8.6447
0.94	0.0000001	0.1034	4.1516	9.6749	0.94	0.0000001	0.1028	4.0898	9.7243

$0.85 \le \hat{\rho}_3 \le 0.90$

$\hat{\rho}_3 = 0.85, \quad 0.50 \le \hat{\rho}_4 \le 0.94$					$\hat{\rho}_3 = 0.90, \quad 0.50 \le \hat{\rho}_4 \le 0.94$				
$\hat{\rho}_4$	A	B	h	k	$\hat{\rho}_4$	A	B	h	k
0.50	4.2136540	0.0553	-0.8561	-0.0507	0.50	4.3150957	0.0238	-0.9383	-0.0379
0.51	3.5249445	0.2147	-0.4245	0.0449	0.51	3.5788783	0.1950	-0.4782	0.0619
0.52	3.0659791	0.3006	-0.1720	0.1235	0.52	3.1060160	0.2835	-0.2198	0.1418
0.53	2.6950565	0.3604	0.0214	0.1977	0.53	2.7272494	0.3447	-0.0237	0.2167
0.54	2.3785763	0.4053	0.1831	0.2704	0.54	2.4055485	0.3905	0.1395	0.2900
0.55	2.1018526	0.4400	0.3247	0.3429	0.55	2.1250569	0.4259	0.2819	0.3630
0.56	1.8567431	0.4669	0.4523	0.4159	0.56	1.8770844	0.4534	0.4100	0.4363
0.57	1.6380626	0.4878	0.5695	0.4900	0.57	1.6561426	0.4748	0.5274	0.5106
0.58	1.4421853	0.5035	0.6788	0.5653	0.58	1.4584207	0.4910	0.6368	0.5862
0.59	1.2663975	0.5149	0.7818	0.6424	0.59	1.2810841	0.5030	0.7398	0.6634
0.60	1.1085599	0.5226	0.8798	0.7213	0.60	1.1219117	0.5112	0.8377	0.7424
0.61	0.9669166	0.5270	0.9736	0.8023	0.61	0.9790904	0.5161	0.9313	0.8236
0.62	0.8399797	0.5285	1.0641	0.8858	0.62	0.8510919	0.5181	1.0216	0.9071
0.63	0.7264555	0.5274	1.1518	0.9718	0.63	0.7365943	0.5175	1.1090	0.9932
0.64	0.6251965	0.5239	1.2372	1.0607	0.64	0.6344299	0.5145	1.1940	1.0821
0.65	0.5351679	0.5183	1.3206	1.1527	0.65	0.5435503	0.5095	1.2771	1.1741
0.66	0.4554242	0.5109	1.4023	1.2481	0.66	0.4630013	0.5026	1.3585	1.2695
0.67	0.3850933	0.5019	1.4828	1.3472	0.67	0.3919054	0.4941	1.4386	1.3686
0.68	0.3233636	0.4913	1.5622	1.4503	0.68	0.3294483	0.4841	1.5176	1.4716
0.69	0.2694760	0.4794	1.6407	1.5577	0.69	0.2748702	0.4728	1.5957	1.5789
0.70	0.2227168	0.4664	1.7187	1.6698	0.70	0.2274580	0.4603	1.6732	1.6910
0.71	0.1824137	0.4525	1.7962	1.7871	0.71	0.1865407	0.4468	1.7503	1.8083
0.72	0.1479318	0.4377	1.8734	1.9100	0.72	0.1514856	0.4325	1.8271	1.9311
0.73	0.1186718	0.4222	1.9505	2.0391	0.73	0.1216952	0.4175	1.9038	2.0602
0.74	0.0940683	0.4062	2.0278	2.1749	0.74	0.0966062	0.4019	1.9807	2.1960
0.75	0.0735887	0.3898	2.1053	2.3182	0.75	0.0756874	0.3859	2.0578	2.3394
0.76	0.0567330	0.3730	2.1832	2.4698	0.76	0.0584397	0.3696	2.1353	2.4910
0.77	0.0430336	0.3561	2.2617	2.6305	0.77	0.0443959	0.3530	2.2135	2.6519
0.78	0.0320557	0.3392	2.3411	2.8014	0.78	0.0331205	0.3364	2.2925	2.8230
0.79	0.0233974	0.3222	2.4214	2.9838	0.79	0.0242101	0.3197	2.3726	3.0057
0.80	0.0166903	0.3054	2.5030	3.1790	0.80	0.0172940	0.3031	2.4538	3.2013
0.81	0.0115998	0.2887	2.5860	3.3887	0.81	0.0120345	0.2867	2.5366	3.4116
0.82	0.0078256	0.2723	2.6707	3.6150	0.82	0.0081276	0.2705	2.6212	3.6388
0.83	0.0051017	0.2563	2.7575	3.8604	0.83	0.0053028	0.2546	2.7078	3.8852
0.84	0.0031964	0.2405	2.8467	4.1278	0.84	0.0033238	0.2390	2.7969	4.1540
0.85	0.0019115	0.2252	2.9387	4.4208	0.85	0.0019875	0.2237	2.8889	4.4490
0.86	0.0010819	0.2102	3.0340	4.7443	0.86	0.0011239	0.2088	2.9843	4.7749
0.87	0.0005732	0.1956	3.1332	5.1039	0.87	0.0005944	0.1943	3.0837	5.1378
0.88	0.0002803	0.1814	3.2371	5.5074	0.88	0.0002896	0.1802	3.1879	5.5456
0.89	0.0001241	0.1676	3.3465	5.9649	0.89	0.0001276	0.1664	3.2979	6.0087
0.90	0.0000485	0.1541	3.4628	6.4899	0.90	0.0000494	0.1528	3.4149	6.5415
0.91	0.0000161	0.1408	3.5875	7.1017	0.91	0.0000162	0.1396	3.5406	7.1638
0.92	0.0000043	0.1277	3.7228	7.8279	0.92	0.0000043	0.1265	3.6773	7.9048
0.93	0.0000009	0.1148	3.8718	8.7105	0.93	0.0000008	0.1135	3.8281	8.8088
0.94	0.0000001	0.1019	4.0391	9.8166	0.94	0.0000001	0.1005	3.9981	9.9471

$$0.95 \le \hat{\rho}_3 \le 1.00$$

$\hat{\rho}_3 = 0.95, \quad 0.51 \le \hat{\rho}_4 \le 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.51	3.6138583	0.1810	-0.5161	0.0811
0.52	3.1318349	0.2705	-0.2556	0.1619
0.53	2.7476016	0.3323	-0.0586	0.2376
0.54	2.4222046	0.3785	0.1050	0.3114
0.55	2.1390573	0.4141	0.2476	0.3848
0.56	1.8891012	0.4420	0.3757	0.4585
0.57	1.6666333	0.4636	0.4931	0.5332
0.58	1.4677062	0.4802	0.6023	0.6091
0.59	1.2893933	0.4924	0.7050	0.6866
0.60	1.1294095	0.5009	0.8027	0.7659
0.61	0.9858960	0.5061	0.8961	0.8473
0.62	0.8572915	0.5085	0.9860	0.9310
0.63	0.7422504	0.5083	1.0730	1.0173
0.64	0.6395880	0.5057	1.1577	1.1064
0.65	0.5482442	0.5011	1.2404	1.1986
0.66	0.4672568	0.4946	1.3214	1.2942
0.67	0.3957434	0.4865	1.4011	1.3934
0.68	0.3328870	0.4769	1.4797	1.4967
0.69	0.2779267	0.4660	1.5575	1.6042
0.70	0.2301499	0.4540	1.6346	1.7165
0.71	0.1888866	0.4409	1.7113	1.8340
0.72	0.1535056	0.4270	1.7877	1.9572
0.73	0.1234115	0.4124	1.8641	2.0865
0.74	0.0980424	0.3972	1.9405	2.2227
0.75	0.0768691	0.3815	2.0173	2.3665
0.76	0.0593936	0.3655	2.0945	2.5186
0.77	0.0451494	0.3493	2.1724	2.6801
0.78	0.0337011	0.3329	2.2511	2.8519
0.79	0.0246449	0.3166	2.3309	3.0354
0.80	0.0176090	0.3002	2.4119	3.2320
0.81	0.0122540	0.2840	2.4945	3.4436
0.82	0.0082734	0.2680	2.5789	3.6723
0.83	0.0053942	0.2523	2.6655	3.9206
0.84	0.0033768	0.2368	2.7546	4.1916
0.85	0.0020152	0.2217	2.8466	4.4894
0.86	0.0011362	0.2069	2.9422	4.8189
0.87	0.0005983	0.1925	3.0419	5.1862
0.88	0.0002898	0.1784	3.1465	5.5997
0.89	0.0001265	0.1647	3.2571	6.0703
0.90	0.0000484	0.1512	3.3749	6.6127
0.91	0.0000156	0.1380	3.5017	7.2481
0.92	0.0000040	0.1249	3.6399	8.0069
0.93	0.0000007	0.1119	3.7928	8.9361

$\hat{\rho}_3 = 1.00, \quad 0.51 \le \hat{\rho}_4 \le 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.51	3.6312845	0.1720	-0.5400	0.1022
0.52	3.1447629	0.2611	-0.2810	0.1837
0.53	2.7573491	0.3227	-0.0847	0.2599
0.54	2.4297008	0.3688	0.0784	0.3342
0.55	2.1449376	0.4044	0.2206	0.4081
0.56	1.8938096	0.4323	0.3483	0.4823
0.57	1.6704859	0.4540	0.4653	0.5573
0.58	1.4709301	0.4707	0.5742	0.6336
0.59	1.2921521	0.4830	0.6766	0.7115
0.60	1.1318200	0.4917	0.7738	0.7912
0.61	0.9880407	0.4971	0.8668	0.8729
0.62	0.8592277	0.4997	0.9563	0.9570
0.63	0.7440162	0.4997	1.0430	1.0437
0.64	0.6412082	0.4974	1.1273	1.1331
0.65	0.5497335	0.4931	1.2096	1.2257
0.66	0.4686234	0.4869	1.2902	1.3217
0.67	0.3969911	0.4791	1.3695	1.4213
0.68	0.3340173	0.4699	1.4477	1.5249
0.69	0.2789401	0.4593	1.5251	1.6329
0.70	0.2310469	0.4476	1.6018	1.7457
0.71	0.1896687	0.4349	1.6781	1.8636
0.72	0.1541757	0.4213	1.7542	1.9873
0.73	0.1239740	0.4070	1.8302	2.1173
0.74	0.0985036	0.3921	1.9064	2.2542
0.75	0.0772367	0.3768	1.9828	2.3987
0.76	0.0596770	0.3611	2.0597	2.5517
0.77	0.0453590	0.3452	2.1373	2.7141
0.78	0.0338480	0.3291	2.2158	2.8871
0.79	0.0247405	0.3129	2.2954	3.0719
0.80	0.0176643	0.2968	2.3762	3.2700
0.81	0.0122793	0.2809	2.4587	3.4834
0.82	0.0082782	0.2651	2.5430	3.7142
0.83	0.0053863	0.2495	2.6296	3.9650
0.84	0.0033627	0.2342	2.7187	4.2391
0.85	0.0019995	0.2192	2.8109	4.5406
0.86	0.0011219	0.2046	2.9067	4.8746
0.87	0.0005869	0.1903	3.0067	5.2475
0.88	0.0002818	0.1763	3.1118	5.6680
0.89	0.0001216	0.1626	3.2230	6.1475
0.90	0.0000458	0.1492	3.3418	6.7015
0.91	0.0000145	0.1360	3.4698	7.3521
0.92	0.0000036	0.1230	3.6095	8.1316
0.93	0.0000006	0.1100	3.7646	9.0898

$1.05 \le \hat{\rho}_3 \le 1.10$

$\hat{\rho}_3 = 1.05, \quad 0.50 \le \hat{\rho}_4 \le 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.3792855	0.0042	-0.9892	0.0242
0.51	3.6331060	0.1673	-0.5522	0.1250
0.52	3.1462297	0.2547	-0.2974	0.2068
0.53	2.7577084	0.3155	-0.1031	0.2835
0.54	2.4291253	0.3612	0.0587	0.3583
0.55	2.1436922	0.3965	0.2000	0.4327
0.56	1.8921279	0.4242	0.3269	0.5074
0.57	1.6685524	0.4458	0.4433	0.5829
0.58	1.4688840	0.4624	0.5516	0.6596
0.59	1.2900951	0.4747	0.6534	0.7379
0.60	1.1298237	0.4834	0.7502	0.8181
0.61	0.9861532	0.4889	0.8427	0.9003
0.62	0.8574785	0.4916	0.9318	0.9848
0.63	0.7424213	0.4918	1.0180	1.0720
0.64	0.6397727	0.4896	1.1019	1.1619
0.65	0.5484551	0.4855	1.1838	1.2550
0.66	0.4674944	0.4796	1.2640	1.3515
0.67	0.3960002	0.4720	1.3429	1.4516
0.68	0.3331514	0.4630	1.4208	1.5558
0.69	0.2781852	0.4526	1.4978	1.6644
0.70	0.2303891	0.4412	1.5741	1.7779
0.71	0.1890948	0.4288	1.6501	1.8966
0.72	0.1536738	0.4155	1.7258	2.0210
0.73	0.1235337	0.4015	1.8015	2.1519
0.74	0.0981162	0.3869	1.8774	2.2897
0.75	0.0768952	0.3718	1.9535	2.4353
0.76	0.0593759	0.3564	2.0302	2.5894
0.77	0.0450942	0.3407	2.1076	2.7532
0.78	0.0336164	0.3248	2.1858	2.9276
0.79	0.0245398	0.3089	2.2652	3.1141
0.80	0.0174928	0.2931	2.3460	3.3143
0.81	0.0121355	0.2773	2.4284	3.5300
0.82	0.0081604	0.2617	2.5127	3.7634
0.83	0.0052926	0.2463	2.5992	4.0174
0.84	0.0032908	0.2312	2.6885	4.2952
0.85	0.0019466	0.2164	2.7808	4.6012
0.86	0.0010851	0.2019	2.8769	4.9406
0.87	0.0005629	0.1877	2.9773	5.3202
0.88	0.0002673	0.1738	3.0830	5.7489
0.89	0.0001137	0.1602	3.1950	6.2387
0.90	0.0000420	0.1469	3.3147	6.8059
0.91	0.0000129	0.1338	3.4440	7.4739
0.92	0.0000031	0.1208	3.5855	8.2768
0.93	0.0000005	0.1079	3.7428	9.2676

$\hat{\rho}_3 = 1.10, \quad 0.50 \le \hat{\rho}_4 \le 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.3408255	0.0143	-0.9628	0.0505
0.51	3.6215464	0.1660	-0.5547	0.1492
0.52	3.1376914	0.2509	-0.3063	0.2311
0.53	2.7498509	0.3105	-0.1151	0.3082
0.54	2.4214924	0.3554	0.0449	0.3835
0.55	2.1362303	0.3902	0.1848	0.4584
0.56	1.8848852	0.4175	0.3107	0.5335
0.57	1.6615957	0.4388	0.4262	0.6096
0.58	1.4622729	0.4552	0.5338	0.6868
0.59	1.2838747	0.4674	0.6350	0.7657
0.60	1.1240241	0.4760	0.7311	0.8463
0.61	0.9807906	0.4815	0.8231	0.9291
0.62	0.8525575	0.4842	0.9117	1.0142
0.63	0.7379367	0.4843	0.9975	1.1019
0.64	0.6357121	0.4823	1.0809	1.1925
0.65	0.5448005	0.4783	1.1623	1.2861
0.66	0.4642238	0.4725	1.2422	1.3833
0.67	0.3930891	0.4650	1.3207	1.4841
0.68	0.3305736	0.4562	1.3981	1.5891
0.69	0.2759141	0.4461	1.4748	1.6985
0.70	0.2283982	0.4348	1.5508	1.8127
0.71	0.1873585	0.4226	1.6265	1.9323
0.72	0.1521675	0.4095	1.7019	2.0578
0.73	0.1222343	0.3958	1.7773	2.1897
0.74	0.0970022	0.3814	1.8529	2.3287
0.75	0.0759469	0.3666	1.9288	2.4756
0.76	0.0585750	0.3514	2.0052	2.6312
0.77	0.0444241	0.3359	2.0824	2.7966
0.78	0.0330621	0.3203	2.1605	2.9729
0.79	0.0240873	0.3047	2.2398	3.1615
0.80	0.0171293	0.2890	2.3204	3.3641
0.81	0.0118491	0.2734	2.4028	3.5825
0.82	0.0079399	0.2580	2.4871	3.8191
0.83	0.0051276	0.2428	2.5737	4.0768
0.84	0.0031714	0.2279	2.6631	4.3590
0.85	0.0018639	0.2132	2.7557	4.6702
0.86	0.0010305	0.1989	2.8521	5.0158
0.87	0.0005291	0.1848	2.9530	5.4030
0.88	0.0002480	0.1711	3.0593	5.8411
0.89	0.0001037	0.1576	3.1721	6.3425
0.90	0.0000375	0.1444	3.2929	6.9247
0.91	0.0000112	0.1314	3.4236	7.6120
0.92	0.0000026	0.1185	3.5668	8.4409
0.93	0.0000004	0.1056	3.7265	9.4676

$1.15 \leq \hat{\rho}_3 \leq 1.20$

$\hat{\rho}_3 = 1.15, \quad 0.50 \leq \hat{\rho}_4 \leq 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.2875307	0.0277	-0.9279	0.0783
0.51	3.5988430	0.1674	-0.5498	0.1745
0.52	3.1205612	0.2492	-0.3091	0.2565
0.53	2.7348790	0.3071	-0.1217	0.3339
0.54	2.4077374	0.3510	0.0359	0.4096
0.55	2.1233782	0.3851	0.1742	0.4850
0.56	1.8728274	0.4119	0.2989	0.5607
0.57	1.6502972	0.4328	0.4134	0.6372
0.58	1.4517235	0.4489	0.5200	0.7151
0.59	1.2740696	0.4609	0.6205	0.7945
0.60	1.1149558	0.4693	0.7160	0.8758
0.61	0.9724465	0.4747	0.8074	0.9591
0.62	0.8449191	0.4773	0.8955	1.0449
0.63	0.7309801	0.4774	0.9808	1.1332
0.64	0.6294088	0.4754	1.0637	1.2245
0.65	0.5391184	0.4714	1.1447	1.3189
0.66	0.4591281	0.4657	1.2241	1.4168
0.67	0.3885432	0.4583	1.3022	1.5185
0.68	0.3265401	0.4496	1.3793	1.6243
0.69	0.2723552	0.4396	1.4556	1.7346
0.70	0.2252765	0.4285	1.5313	1.8499
0.71	0.1846374	0.4165	1.6067	1.9706
0.72	0.1498115	0.4036	1.6818	2.0972
0.73	0.1202093	0.3900	1.7570	2.2304
0.74	0.0952758	0.3759	1.8323	2.3709
0.75	0.0744882	0.3612	1.9080	2.5193
0.76	0.0573551	0.3462	1.9842	2.6766
0.77	0.0434156	0.3310	2.0612	2.8439
0.78	0.0322394	0.3156	2.1392	3.0224
0.79	0.0234265	0.3001	2.2184	3.2134
0.80	0.0166080	0.2847	2.2990	3.4187
0.81	0.0114465	0.2693	2.3814	3.6403
0.82	0.0076367	0.2541	2.4657	3.8806
0.83	0.0049059	0.2391	2.5525	4.1425
0.84	0.0030151	0.2243	2.6421	4.4297
0.85	0.0017583	0.2098	2.7350	4.7467
0.86	0.0009629	0.1957	2.8318	5.0994
0.87	0.0004885	0.1818	2.9332	5.4950
0.88	0.0002256	0.1681	3.0402	5.9435
0.89	0.0000925	0.1548	3.1539	6.4579
0.90	0.0000326	0.1417	3.2758	7.0565
0.91	0.0000094	0.1288	3.4079	7.7651
0.92	0.0000021	0.1160	3.5530	8.6224
0.93	0.0000003	0.1032	3.7151	9.6884

$\hat{\rho}_3 = 1.20, \quad 0.50 \leq \hat{\rho}_4 \leq 0.93$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.2249823	0.0427	-0.8887	0.1068
0.51	3.5670663	0.1709	-0.5390	0.2008
0.52	3.0961571	0.2491	-0.3067	0.2827
0.53	2.7138090	0.3052	-0.1237	0.3604
0.54	2.3887126	0.3479	0.0312	0.4365
0.55	2.1058815	0.3811	0.1676	0.5124
0.56	1.8566229	0.4072	0.2908	0.5886
0.57	1.6352646	0.4277	0.4041	0.6658
0.58	1.4377928	0.4434	0.5099	0.7442
0.59	1.2611922	0.4550	0.6095	0.8243
0.60	1.1030915	0.4632	0.7043	0.9062
0.61	0.9615570	0.4684	0.7951	0.9903
0.62	0.8349653	0.4709	0.8826	1.0767
0.63	0.7219211	0.4709	0.9673	1.1658
0.64	0.6212016	0.4689	1.0498	1.2579
0.65	0.5317182	0.4649	1.1304	1.3531
0.66	0.4524886	0.4591	1.2094	1.4519
0.67	0.3826173	0.4518	1.2871	1.5545
0.68	0.3212801	0.4432	1.3638	1.6613
0.69	0.2677137	0.4333	1.4398	1.7727
0.70	0.2212064	0.4223	1.5152	1.8891
0.71	0.1810924	0.4104	1.5902	2.0110
0.72	0.1467466	0.3976	1.6651	2.1390
0.73	0.1175809	0.3842	1.7400	2.2737
0.74	0.0930419	0.3702	1.8151	2.4157
0.75	0.0726085	0.3558	1.8906	2.5659
0.76	0.0557912	0.3410	1.9667	2.7252
0.77	0.0421309	0.3259	2.0436	2.8947
0.78	0.0311993	0.3107	2.1215	3.0756
0.79	0.0225985	0.2955	2.2006	3.2694
0.80	0.0159615	0.2802	2.2812	3.4778
0.81	0.0109529	0.2650	2.3636	3.7029
0.82	0.0072697	0.2500	2.4481	3.9473
0.83	0.0046416	0.2352	2.5350	4.2139
0.84	0.0028317	0.2206	2.6248	4.5066
0.85	0.0016367	0.2063	2.7181	4.8301
0.86	0.0008866	0.1923	2.8153	5.1905
0.87	0.0004438	0.1785	2.9173	5.5955
0.88	0.0002015	0.1651	3.0250	6.0553
0.89	0.0000809	0.1518	3.1396	6.5839
0.90	0.0000277	0.1389	3.2627	7.2003
0.91	0.0000077	0.1261	3.3963	7.9321
0.92	0.0000016	0.1134	3.5434	8.8201
0.93	0.0000002	0.1007	3.7080	9.9285

$1.25 \leq \hat{\rho}_3 \leq 1.30$

$\hat{\rho}_3 = 1.25, \quad 0.50 \leq \hat{\rho}_4 \leq 0.92$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.1566683	0.0581	-0.8477	0.1359
0.51	3.5280292	0.1758	-0.5239	0.2279
0.52	3.0656710	0.2504	-0.3004	0.3095
0.53	2.6875615	0.3045	-0.1219	0.3875
0.54	2.3651859	0.3458	0.0301	0.4641
0.55	2.0844090	0.3781	0.1643	0.5405
0.56	1.8368684	0.4034	0.2860	0.6173
0.57	1.6170382	0.4233	0.3981	0.6951
0.58	1.4209744	0.4385	0.5028	0.7742
0.59	1.2456961	0.4498	0.6016	0.8549
0.60	1.0888488	0.4577	0.6956	0.9375
0.61	0.9485069	0.4626	0.7857	1.0223
0.62	0.8230507	0.4649	0.8726	1.1096
0.63	0.7110859	0.4648	0.9568	1.1995
0.64	0.6113900	0.4627	1.0388	1.2924
0.65	0.5228738	0.4586	1.1189	1.3886
0.66	0.4445550	0.4528	1.1975	1.4883
0.67	0.3755377	0.4455	1.2748	1.5920
0.68	0.3149979	0.4369	1.3512	1.6999
0.69	0.2621725	0.4271	1.4269	1.8124
0.70	0.2163508	0.4162	1.5020	1.9301
0.71	0.1768677	0.4043	1.5767	2.0534
0.72	0.1430995	0.3917	1.6514	2.1829
0.73	0.1144594	0.3784	1.7260	2.3192
0.74	0.0903956	0.3646	1.8010	2.4630
0.75	0.0703889	0.3503	1.8763	2.6152
0.76	0.0539516	0.3356	1.9523	2.7766
0.77	0.0406269	0.3208	2.0291	2.9485
0.78	0.0299885	0.3057	2.1069	3.1321
0.79	0.0216408	0.2907	2.1860	3.3290
0.80	0.0152194	0.2756	2.2666	3.5408
0.81	0.0103914	0.2606	2.3491	3.7698
0.82	0.0068565	0.2458	2.4337	4.0186
0.83	0.0043474	0.2311	2.5208	4.2904
0.84	0.0026303	0.2167	2.6109	4.5892
0.85	0.0015051	0.2026	2.7045	4.9198
0.86	0.0008055	0.1887	2.8023	5.2885
0.87	0.0003972	0.1751	2.9049	5.7036
0.88	0.0001770	0.1618	3.0134	6.1758
0.89	0.0000694	0.1488	3.1290	6.7197
0.90	0.0000231	0.1359	3.2533	7.3554
0.91	0.0000062	0.1233	3.3884	8.1120
0.92	0.0000012	0.1107	3.5374	9.0331

$\hat{\rho}_3 = 1.30, \quad 0.50 \leq \hat{\rho}_4 \leq 0.92$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.0847791	0.0736	-0.8064	0.1653
0.51	3.4832627	0.1818	-0.5056	0.2555
0.52	3.0301566	0.2527	-0.2907	0.3370
0.53	2.6569581	0.3046	-0.1170	0.4153
0.54	2.3378437	0.3445	0.0318	0.4923
0.55	2.0595567	0.3758	0.1639	0.5692
0.56	1.8140942	0.4003	0.2839	0.6466
0.57	1.5960972	0.4195	0.3947	0.7250
0.58	1.4017053	0.4342	0.4983	0.8048
0.59	1.2279819	0.4451	0.5962	0.8862
0.60	1.0725963	0.4526	0.6894	0.9696
0.61	0.9336360	0.4573	0.7789	1.0552
0.62	0.8094881	0.4593	0.8651	1.1433
0.63	0.6987623	0.4591	0.9488	1.2341
0.64	0.6002378	0.4567	1.0303	1.3280
0.65	0.5128268	0.4526	1.1099	1.4251
0.66	0.4355474	0.4467	1.1881	1.5259
0.67	0.3675042	0.4394	1.2651	1.6307
0.68	0.3078739	0.4308	1.3412	1.7398
0.69	0.2558940	0.4210	1.4165	1.8537
0.70	0.2108546	0.4101	1.4913	1.9727
0.71	0.1720918	0.3983	1.5658	2.0975
0.72	0.1389830	0.3858	1.6402	2.2287
0.73	0.1109431	0.3726	1.7147	2.3667
0.74	0.0874219	0.3589	1.7895	2.5125
0.75	0.0679020	0.3447	1.8647	2.6668
0.76	0.0518979	0.3303	1.9405	2.8306
0.77	0.0389548	0.3155	2.0172	3.0052
0.78	0.0286490	0.3007	2.0950	3.1917
0.79	0.0205875	0.2858	2.1741	3.3918
0.80	0.0144086	0.2709	2.2548	3.6074
0.81	0.0097827	0.2561	2.3374	3.8406
0.82	0.0064126	0.2414	2.4222	4.0943
0.83	0.0040346	0.2270	2.5096	4.3717
0.84	0.0024187	0.2128	2.6000	4.6769
0.85	0.0013689	0.1988	2.6940	5.0152
0.86	0.0007229	0.1851	2.7923	5.3930
0.87	0.0003507	0.1717	2.8956	5.8190
0.88	0.0001531	0.1586	3.0049	6.3044
0.89	0.0000585	0.1456	3.1215	6.8646
0.90	0.0000188	0.1330	3.2471	7.5209
0.91	0.0000048	0.1204	3.3838	8.3041
0.92	0.0000009	0.1080	3.5348	9.2603

$1.35 \leq \hat{\rho}_3 \leq 1.40$

$\hat{\rho}_3 = 1.35, \quad 0.50 \leq \hat{\rho}_4 \leq 0.92$				
$\hat{\rho}_4$	A	B	h	k
0.50	4.0107279	0.0888	-0.7654	0.1949
0.51	3.4340313	0.1885	-0.4849	0.2836
0.52	2.9905291	0.2557	-0.2785	0.3650
0.53	2.6227237	0.3055	-0.1096	0.4435
0.54	2.3072940	0.3439	0.0361	0.5210
0.55	2.0318534	0.3741	0.1659	0.5984
0.56	1.7887705	0.3978	0.2842	0.6764
0.57	1.5728656	0.4163	0.3936	0.7555
0.58	1.3803717	0.4304	0.4961	0.8360
0.59	1.2084033	0.4408	0.5930	0.9182
0.60	1.0546591	0.4480	0.6855	1.0024
0.61	0.9172433	0.4523	0.7742	1.0888
0.62	0.7945529	0.4541	0.8599	1.1778
0.63	0.6852034	0.4536	0.9430	1.2696
0.64	0.5879777	0.4511	1.0240	1.3644
0.65	0.5017901	0.4468	1.1032	1.4626
0.66	0.4256600	0.4409	1.1810	1.5646
0.67	0.3586934	0.4335	1.2576	1.6705
0.68	0.3000677	0.4248	1.3333	1.7809
0.69	0.2490215	0.4150	1.4084	1.8962
0.70	0.2048459	0.4041	1.4829	2.0168
0.71	0.1668783	0.3924	1.5572	2.1432
0.72	0.1344973	0.3800	1.6314	2.2761
0.73	0.1071195	0.3669	1.7057	2.4161
0.74	0.0841963	0.3532	1.7803	2.5639
0.75	0.0652125	0.3392	1.8554	2.7206
0.76	0.0496844	0.3248	1.9312	2.8870
0.77	0.0371600	0.3103	2.0078	3.0643
0.78	0.0272180	0.2956	2.0856	3.2540
0.79	0.0194684	0.2808	2.1648	3.4577
0.80	0.0135528	0.2661	2.2455	3.6773
0.81	0.0091449	0.2515	2.3282	3.9151
0.82	0.0059515	0.2370	2.4132	4.1739
0.83	0.0037130	0.2228	2.5009	4.4573
0.84	0.0022037	0.2087	2.5917	4.7695
0.85	0.0012324	0.1949	2.6861	5.1159
0.86	0.0006415	0.1814	2.7850	5.5034
0.87	0.0003058	0.1682	2.8890	5.9410
0.88	0.0001306	0.1552	2.9991	6.4405
0.89	0.0000486	0.1425	3.1168	7.0181
0.90	0.0000151	0.1299	3.2437	7.6963
0.91	0.0000037	0.1175	3.3820	8.5077
0.92	0.0000006	0.1053	3.5350	9.5010

$\hat{\rho}_3 = 1.40, \quad 0.50 \leq \hat{\rho}_4 \leq 0.92$				
$\hat{\rho}_4$	A	B	h	k
0.50	3.9354561	0.1034	-0.7252	0.2246
0.51	3.3813660	0.1956	-0.4627	0.3121
0.52	2.9475745	0.2593	-0.2644	0.3933
0.53	2.5854922	0.3069	-0.1002	0.4721
0.54	2.2740725	0.3439	0.0424	0.5501
0.55	2.0017658	0.3729	0.1700	0.6281
0.56	1.7613124	0.3957	0.2865	0.7067
0.57	1.5477176	0.4135	0.3945	0.7865
0.58	1.3573139	0.4270	0.4959	0.8677
0.59	1.1872721	0.4369	0.5919	0.9507
0.60	1.0353235	0.4437	0.6835	1.0357
0.61	0.8995921	0.4476	0.7715	1.1231
0.62	0.7784873	0.4491	0.8566	1.2130
0.63	0.6706320	0.4484	0.9391	1.3058
0.64	0.5748140	0.4457	1.0196	1.4017
0.65	0.4899506	0.4412	1.0984	1.5010
0.66	0.4150635	0.4351	1.1758	1.6042
0.67	0.3492601	0.4277	1.2521	1.7114
0.68	0.2917194	0.4189	1.3275	1.8232
0.69	0.2416810	0.4091	1.4022	1.9400
0.70	0.1984373	0.3983	1.4765	2.0621
0.71	0.1613271	0.3866	1.5506	2.1903
0.72	0.1297303	0.3742	1.6246	2.3250
0.73	0.1030652	0.3611	1.6987	2.4671
0.74	0.0807852	0.3476	1.7732	2.6172
0.75	0.0623768	0.3337	1.8482	2.7763
0.76	0.0473589	0.3194	1.9239	2.9454
0.77	0.0352821	0.3050	2.0006	3.1258
0.78	0.0257279	0.2905	2.0784	3.3189
0.79	0.0183095	0.2759	2.1576	3.5263
0.80	0.0126721	0.2613	2.2385	3.7502
0.81	0.0084935	0.2469	2.3214	3.9928
0.82	0.0054846	0.2326	2.4066	4.2572
0.83	0.0033906	0.2185	2.4945	4.5470
0.84	0.0019908	0.2046	2.5857	4.8666
0.85	0.0010991	0.1910	2.6806	5.2217
0.86	0.0005633	0.1777	2.7801	5.6194
0.87	0.0002635	0.1647	2.8848	6.0693
0.88	0.0001100	0.1518	2.9958	6.5837
0.89	0.0000397	0.1393	3.1146	7.1797
0.90	0.0000119	0.1269	3.2428	7.8810
0.91	0.0000028	0.1147	3.3827	8.7220
0.92	0.0000005	0.1025	3.5378	9.7544

$1.45 \leq \hat{\rho}_3 \leq 1.50$

$\hat{\rho}_3 = 1.45, \quad 0.50 \leq \hat{\rho}_4 \leq 0.91$				
$\hat{\rho}_4$	A	B	h	k
0.50	3.8596112	0.1175	-0.6861	0.2543
0.51	3.3261016	0.2030	-0.4394	0.3408
0.52	2.9019620	0.2633	-0.2486	0.4220
0.53	2.5458142	0.3088	-0.0890	0.5011
0.54	2.2386484	0.3442	0.0504	0.5795
0.55	1.9697044	0.3721	0.1757	0.6581
0.56	1.7320855	0.3940	0.2905	0.7374
0.57	1.5209832	0.4110	0.3971	0.8179
0.58	1.3328322	0.4239	0.4973	0.8999
0.59	1.1648625	0.4333	0.5924	0.9837
0.60	1.0148411	0.4396	0.6832	1.0696
0.61	0.8809140	0.4432	0.7705	1.1579
0.62	0.7615042	0.4444	0.8549	1.2488
0.63	0.6552438	0.4434	0.9370	1.3426
0.64	0.5609261	0.4404	1.0170	1.4397
0.65	0.4774725	0.4358	1.0953	1.5402
0.66	0.4039074	0.4296	1.1723	1.6446
0.67	0.3393403	0.4220	1.2483	1.7532
0.68	0.2829518	0.4132	1.3234	1.8665
0.69	0.2339829	0.4033	1.3978	1.9848
0.70	0.1917274	0.3925	1.4719	2.1087
0.71	0.1555255	0.3808	1.5458	2.2387
0.72	0.1247589	0.3684	1.6196	2.3754
0.73	0.0988473	0.3554	1.6936	2.5196
0.74	0.0772461	0.3420	1.7680	2.6721
0.75	0.0594442	0.3282	1.8429	2.8338
0.76	0.0449628	0.3140	1.9186	3.0059
0.77	0.0333553	0.2997	1.9952	3.1895
0.78	0.0242064	0.2853	2.0731	3.3861
0.79	0.0171329	0.2709	2.1524	3.5976
0.80	0.0117839	0.2565	2.2334	3.8259
0.81	0.0078414	0.2423	2.3165	4.0737
0.82	0.0050214	0.2281	2.4019	4.3440
0.83	0.0030740	0.2142	2.4902	4.6405
0.84	0.0017842	0.2006	2.5818	4.9679
0.85	0.0009716	0.1871	2.6772	5.3321
0.86	0.0004898	0.1740	2.7773	5.7407
0.87	0.0002246	0.1611	2.8827	6.2035
0.88	0.0000914	0.1485	2.9947	6.7336
0.89	0.0000320	0.1361	3.1146	7.3489
0.90	0.0000092	0.1238	3.2441	8.0745
0.91	0.0000020	0.1118	3.3857	8.9465

$\hat{\rho}_3 = 1.50, \quad 0.49 \leq \hat{\rho}_4 \leq 0.91$				
$\hat{\rho}_4$	A	B	h	k
0.49	4.5062944	0.0019	-0.9952	0.1857
0.50	3.7836508	0.1310	-0.6481	0.2841
0.51	3.2689125	0.2105	-0.4154	0.3698
0.52	2.8542582	0.2676	-0.2317	0.4510
0.53	2.5041658	0.3110	-0.0765	0.5304
0.54	2.2014314	0.3449	0.0599	0.6093
0.55	1.9360290	0.3716	0.1829	0.6885
0.56	1.7014111	0.3926	0.2959	0.7685
0.57	1.4929523	0.4089	0.4012	0.8497
0.58	1.3071902	0.4212	0.5003	0.9325
0.59	1.1414158	0.4300	0.5944	1.0172
0.60	0.9934333	0.4358	0.6844	1.1040
0.61	0.8614122	0.4390	0.7710	1.1933
0.62	0.7437905	0.4398	0.8548	1.2852
0.63	0.6392102	0.4386	0.9363	1.3801
0.64	0.5464713	0.4354	1.0158	1.4783
0.65	0.4644996	0.4306	1.0938	1.5801
0.66	0.3923229	0.4242	1.1704	1.6858
0.67	0.3290530	0.4165	1.2460	1.7959
0.68	0.2738722	0.4076	1.3208	1.9107
0.69	0.2260235	0.3976	1.3950	2.0306
0.70	0.1848022	0.3868	1.4689	2.1563
0.71	0.1495498	0.3751	1.5425	2.2882
0.72	0.1196497	0.3627	1.6162	2.4271
0.73	0.0945238	0.3498	1.6901	2.5736
0.74	0.0736291	0.3364	1.7644	2.7286
0.75	0.0564571	0.3227	1.8392	2.8930
0.76	0.0425316	0.3087	1.9149	3.0681
0.77	0.0314089	0.2945	1.9916	3.2551
0.78	0.0226773	0.2802	2.0695	3.4555
0.79	0.0159573	0.2660	2.1489	3.6712
0.80	0.0109023	0.2517	2.2301	3.9043
0.81	0.0071993	0.2376	2.3133	4.1575
0.82	0.0045694	0.2237	2.3991	4.4339
0.83	0.0027684	0.2100	2.4877	4.7375
0.84	0.0015873	0.1965	2.5797	5.0731
0.85	0.0008519	0.1832	2.6757	5.4469
0.86	0.0004220	0.1703	2.7764	5.8669
0.87	0.0001894	0.1576	2.8826	6.3432
0.88	0.0000751	0.1451	2.9955	6.8898
0.89	0.0000255	0.1329	3.1165	7.5253
0.90	0.0000070	0.1208	3.2473	8.2762
0.91	0.0000015	0.1089	3.3906	9.1806

$1.55 \leq \hat{\rho}_3 \leq 1.60$

$\hat{\rho}_3 = 1.55, \quad 0.49 \leq \hat{\rho}_4 \leq 0.91$				
$\hat{\rho}_4$	A	B	h	k
0.49	4.3809208	0.0291	-0.9241	0.2187
0.50	3.7079064	0.1438	-0.6114	0.3138
0.51	3.2103435	0.2179	-0.3909	0.3989
0.52	2.8049415	0.2720	-0.2140	0.4801
0.53	2.4609568	0.3134	-0.0630	0.5599
0.54	2.1627777	0.3458	0.0705	0.6393
0.55	1.9010547	0.3714	0.1913	0.7191
0.56	1.6695712	0.3915	0.3026	0.7998
0.57	1.4638797	0.4070	0.4065	0.8818
0.58	1.2806200	0.4186	0.5045	0.9655
0.59	1.1171441	0.4269	0.5976	1.0510
0.60	0.9712946	0.4322	0.6868	1.1388
0.61	0.8412652	0.4350	0.7727	1.2291
0.62	0.7255100	0.4355	0.8560	1.3221
0.63	0.6226816	0.4339	0.9369	1.4181
0.64	0.5315872	0.4305	1.0160	1.5176
0.65	0.4511577	0.4254	1.0935	1.6206
0.66	0.3804244	0.4189	1.1698	1.7278
0.67	0.3185017	0.4111	1.2451	1.8393
0.68	0.2645743	0.4021	1.3196	1.9557
0.69	0.2178868	0.3921	1.3936	2.0774
0.70	0.1777362	0.3812	1.4672	2.2049
0.71	0.1434657	0.3695	1.5407	2.3388
0.72	0.1144607	0.3571	1.6143	2.4799
0.73	0.0901446	0.3442	1.6880	2.6288
0.74	0.0699771	0.3309	1.7622	2.7864
0.75	0.0534517	0.3173	1.8371	2.9538
0.76	0.0400953	0.3034	1.9127	3.1320
0.77	0.0294674	0.2893	1.9894	3.3226
0.78	0.0211600	0.2752	2.0674	3.5269
0.79	0.0147978	0.2610	2.1470	3.7470
0.80	0.0100390	0.2470	2.2283	3.9851
0.81	0.0065756	0.2330	2.3118	4.2440
0.82	0.0041344	0.2193	2.3979	4.5268
0.83	0.0024775	0.2057	2.4869	4.8378
0.84	0.0014023	0.1924	2.5793	5.1820
0.85	0.0007412	0.1794	2.6759	5.5659
0.86	0.0003605	0.1666	2.7772	5.9977
0.87	0.0001583	0.1541	2.8842	6.4882
0.88	0.0000611	0.1418	2.9980	7.0519
0.89	0.0000200	0.1297	3.1201	7.7085
0.90	0.0000053	0.1178	3.2523	8.4857
0.91	0.0000010	0.1061	3.3972	9.4239

$\hat{\rho}_3 = 1.60, \quad 0.49 \leq \hat{\rho}_4 \leq 0.91$				
$\hat{\rho}_4$	A	B	h	k
0.49	4.2660398	0.0527	-0.8617	0.2508
0.50	3.6326225	0.1560	-0.5758	0.3435
0.51	3.1508354	0.2252	-0.3663	0.4281
0.52	2.7544147	0.2765	-0.1955	0.5095
0.53	2.4165393	0.3159	-0.0486	0.5896
0.54	2.1229968	0.3469	0.0820	0.6696
0.55	1.8650562	0.3714	0.2007	0.7500
0.56	1.6368124	0.3905	0.3104	0.8314
0.57	1.4339888	0.4052	0.4129	0.9142
0.58	1.2533248	0.4162	0.5098	0.9987
0.59	1.0922333	0.4239	0.6020	1.0852
0.60	0.9485952	0.4288	0.6904	1.1740
0.61	0.8206293	0.4312	0.7756	1.2653
0.62	0.7068062	0.4313	0.8583	1.3594
0.63	0.6057895	0.4294	0.9387	1.4567
0.64	0.5163942	0.4257	1.0173	1.5573
0.65	0.4375565	0.4205	1.0945	1.6618
0.66	0.3683115	0.4138	1.1704	1.7703
0.67	0.3077768	0.4058	1.2454	1.8834
0.68	0.2551392	0.3967	1.3196	2.0015
0.69	0.2096453	0.3866	1.3934	2.1250
0.70	0.1705939	0.3756	1.4668	2.2544
0.71	0.1373302	0.3639	1.5402	2.3905
0.72	0.1092413	0.3516	1.6136	2.5338
0.73	0.0857527	0.3387	1.6873	2.6852
0.74	0.0663263	0.3255	1.7614	2.8456
0.75	0.0504585	0.3119	1.8362	3.0160
0.76	0.0376792	0.2981	1.9119	3.1976
0.77	0.0275514	0.2841	1.9886	3.3918
0.78	0.0196709	0.2701	2.0667	3.6003
0.79	0.0136671	0.2562	2.1464	3.8250
0.80	0.0092031	0.2422	2.2280	4.0683
0.81	0.0059768	0.2285	2.3117	4.3330
0.82	0.0037210	0.2149	2.3981	4.6226
0.83	0.0022042	0.2015	2.4875	4.9413
0.84	0.0012309	0.1884	2.5804	5.2945
0.85	0.0006402	0.1755	2.6775	5.6888
0.86	0.0003055	0.1629	2.7795	6.1329
0.87	0.0001310	0.1506	2.8873	6.6382
0.88	0.0000491	0.1385	3.0021	7.2197
0.89	0.0000155	0.1266	3.1253	7.8982
0.90	0.0000039	0.1149	3.2588	8.7027
0.91	0.0000007	0.1034	3.4052	9.6757

$1.65 \le \hat{\rho}_3 \le 1.70$

$\hat{\rho}_3 = 1.65, \quad 0.49 \le \hat{\rho}_4 \le 0.91$				
$\hat{\rho}_4$	A	B	h	k
0.49	4.1586799	0.0736	-0.8055	0.2823
0.50	3.5579820	0.1676	-0.5414	0.3732
0.51	3.0907456	0.2324	-0.3417	0.4574
0.52	2.7030168	0.2810	-0.1767	0.5389
0.53	2.3712152	0.3186	-0.0335	0.6195
0.54	2.0823565	0.3482	0.0944	0.7000
0.55	1.8282731	0.3715	0.2110	0.7811
0.56	1.6033506	0.3897	0.3190	0.8633
0.57	1.4034752	0.4036	0.4202	0.9469
0.58	1.2254829	0.4140	0.5160	1.0323
0.59	1.0668463	0.4211	0.6073	1.1197
0.60	0.9254846	0.4255	0.6949	1.2095
0.61	0.7996415	0.4275	0.7795	1.3019
0.62	0.6878046	0.4272	0.8616	1.3971
0.63	0.5886488	0.4250	0.9415	1.4956
0.64	0.5009971	0.4211	1.0197	1.5976
0.65	0.4237914	0.4156	1.0964	1.7035
0.66	0.3560710	0.4087	1.1720	1.8135
0.67	0.2969564	0.4006	1.2467	1.9282
0.68	0.2456371	0.3914	1.3207	2.0480
0.69	0.2013617	0.3812	1.3943	2.1733
0.70	0.1634309	0.3702	1.4675	2.3048
0.71	0.1311919	0.3584	1.5407	2.4431
0.72	0.1040337	0.3461	1.6140	2.5888
0.73	0.0813841	0.3333	1.6876	2.7428
0.74	0.0627076	0.3201	1.7617	2.9060
0.75	0.0475032	0.3066	1.8365	3.0795
0.76	0.0353042	0.2929	1.9123	3.2646
0.77	0.0256773	0.2790	1.9891	3.4626
0.78	0.0182228	0.2652	2.0673	3.6754
0.79	0.0125747	0.2513	2.1471	3.9049
0.80	0.0084018	0.2376	2.2289	4.1536
0.81	0.0054077	0.2240	2.3129	4.4245
0.82	0.0033320	0.2106	2.3996	4.7211
0.83	0.0019502	0.1974	2.4894	5.0478
0.84	0.0010737	0.1844	2.5828	5.4103
0.85	0.0005492	0.1717	2.6805	5.8155
0.86	0.0002568	0.1593	2.7832	6.2724
0.87	0.0001075	0.1472	2.8918	6.7929
0.88	0.0000391	0.1352	3.0075	7.3929
0.89	0.0000119	0.1235	3.1318	8.0940
0.90	0.0000029	0.1120	3.2666	8.9268
0.91	0.0000005	0.1006	3.4146	9.9356

$\hat{\rho}_3 = 1.70, \quad 0.49 \le \hat{\rho}_4 \le 0.90$				
$\hat{\rho}_4$	A	B	h	k
0.49	4.0570850	0.0924	-0.7542	0.3132
0.50	3.4841234	0.1785	-0.5081	0.4027
0.51	3.0303647	0.2394	-0.3173	0.4868
0.52	2.6510330	0.2854	-0.1575	0.5685
0.53	2.3252430	0.3212	-0.0180	0.6495
0.54	2.0410887	0.3494	0.1073	0.7306
0.55	1.7909137	0.3717	0.2219	0.8124
0.56	1.5693740	0.3890	0.3284	0.8954
0.57	1.3725100	0.4022	0.4283	0.9798
0.58	1.1972503	0.4118	0.5230	1.0661
0.59	1.0411258	0.4185	0.6135	1.1545
0.60	0.9020933	0.4224	0.7003	1.2453
0.61	0.7784214	0.4239	0.7843	1.3388
0.62	0.6686146	0.4233	0.8658	1.4353
0.63	0.5713595	0.4208	0.9452	1.5350
0.64	0.4854873	0.4166	1.0230	1.6383
0.65	0.4099456	0.4108	1.0993	1.7456
0.66	0.3437780	0.4038	1.1746	1.8572
0.67	0.2861083	0.3955	1.2490	1.9736
0.68	0.2361286	0.3861	1.3228	2.0952
0.69	0.1930897	0.3759	1.3961	2.2224
0.70	0.1562946	0.3648	1.4692	2.3560
0.71	0.1250921	0.3530	1.5423	2.4965
0.72	0.0988737	0.3407	1.6155	2.6447
0.73	0.0770694	0.3279	1.6890	2.8014
0.74	0.0591464	0.3147	1.7631	2.9676
0.75	0.0446068	0.3013	1.8379	3.1444
0.76	0.0329873	0.2877	1.9137	3.3330
0.77	0.0238587	0.2740	1.9906	3.5351
0.78	0.0168259	0.2602	2.0689	3.7523
0.79	0.0115282	0.2465	2.1490	3.9868
0.80	0.0076401	0.2329	2.2310	4.2411
0.81	0.0048717	0.2195	2.3153	4.5183
0.82	0.0029695	0.2063	2.4023	4.8221
0.83	0.0017163	0.1933	2.4925	5.1572
0.84	0.0009311	0.1805	2.5864	5.5292
0.85	0.0004681	0.1680	2.6847	5.9457
0.86	0.0002144	0.1558	2.7880	6.4159
0.87	0.0000875	0.1438	2.8974	6.9522
0.88	0.0000309	0.1321	3.0141	7.5712
0.89	0.0000091	0.1205	3.1395	8.2956
0.90	0.0000021	0.1092	3.2756	9.1574

$1.75 \leq \hat{\rho}_3 \leq 1.80$

$\hat{\rho}_3 = 1.75, \quad 0.49 \leq \hat{\rho}_4 \leq 0.90$				
$\hat{\rho}_4$	A	B	h	k
0.49	3.9601338	0.1094	-0.7068	0.3438
0.50	3.4111523	0.1889	-0.4759	0.4322
0.51	2.9699296	0.2461	-0.2930	0.5162
0.52	2.5987031	0.2898	-0.1382	0.5982
0.53	2.2788437	0.3239	-0.0020	0.6796
0.54	1.9993937	0.3508	0.1208	0.7614
0.55	1.7531592	0.3719	0.2335	0.8439
0.56	1.5350470	0.3883	0.3384	0.9276
0.57	1.3412426	0.4008	0.4371	1.0129
0.58	1.1687634	0.4098	0.5308	1.1002
0.59	1.0151965	0.4159	0.6204	1.1896
0.60	0.8785354	0.4193	0.7065	1.2814
0.61	0.7570735	0.4204	0.7898	1.3761
0.62	0.6493318	0.4194	0.8707	1.4737
0.63	0.5540089	0.4166	0.9497	1.5748
0.64	0.4699440	0.4121	1.0270	1.6795
0.65	0.3960908	0.4061	1.1031	1.7883
0.66	0.3314973	0.3989	1.1780	1.9015
0.67	0.2752907	0.3904	1.2522	2.0196
0.68	0.2266656	0.3810	1.3257	2.1430
0.69	0.1848754	0.3706	1.3989	2.2722
0.70	0.1492251	0.3595	1.4718	2.4080
0.71	0.1190659	0.3477	1.5447	2.5508
0.72	0.0937912	0.3354	1.6179	2.7016
0.73	0.0728339	0.3226	1.6914	2.8610
0.74	0.0556637	0.3095	1.7654	3.0303
0.75	0.0417863	0.2961	1.8403	3.2104
0.76	0.0307420	0.2826	1.9161	3.4028
0.77	0.0221059	0.2690	1.9931	3.6089
0.78	0.0154879	0.2554	2.0716	3.8307
0.79	0.0105330	0.2418	2.1518	4.0704
0.80	0.0069216	0.2284	2.2340	4.3305
0.81	0.0043709	0.2151	2.3187	4.6143
0.82	0.0026346	0.2021	2.4060	4.9256
0.83	0.0015030	0.1892	2.4967	5.2692
0.84	0.0008030	0.1767	2.5911	5.6512
0.85	0.0003964	0.1643	2.6899	6.0793
0.86	0.0001777	0.1523	2.7939	6.5631
0.87	0.0000707	0.1405	2.9041	7.1157
0.88	0.0000242	0.1290	3.0217	7.7543
0.89	0.0000068	0.1176	3.1482	8.5027
0.90	0.0000015	0.1064	3.2856	9.3943

$\hat{\rho}_3 = 1.80, \quad 0.48 \leq \hat{\rho}_4 \leq 0.90$				
$\hat{\rho}_4$	A	B	h	k
0.48	4.5769606	0.0128	-0.9668	0.2766
0.49	3.8670686	0.1251	-0.6627	0.3740
0.50	3.3391490	0.1987	-0.4448	0.4617
0.51	2.9096339	0.2526	-0.2690	0.5456
0.52	2.5462288	0.2940	-0.1187	0.6279
0.53	2.2322072	0.3265	0.0142	0.7099
0.54	1.9574443	0.3521	0.1346	0.7922
0.55	1.7151667	0.3722	0.2455	0.8755
0.56	1.5005124	0.3878	0.3489	0.9600
0.57	1.3098033	0.3995	0.4465	1.0462
0.58	1.1401411	0.4079	0.5391	1.1344
0.59	0.9891671	0.4133	0.6279	1.2248
0.60	0.8549104	0.4163	0.7133	1.3178
0.61	0.7356885	0.4170	0.7960	1.4136
0.62	0.6300392	0.4156	0.8764	1.5125
0.63	0.5366724	0.4125	0.9549	1.6149
0.64	0.4544359	0.4077	1.0318	1.7211
0.65	0.3822891	0.4015	1.1075	1.8314
0.66	0.3192848	0.3941	1.1822	1.9462
0.67	0.2645533	0.3855	1.2561	2.0660
0.68	0.2172924	0.3759	1.3294	2.1913
0.69	0.1767578	0.3655	1.4024	2.3227
0.70	0.1422566	0.3543	1.4752	2.4606
0.71	0.1131424	0.3425	1.5480	2.6059
0.72	0.0888111	0.3301	1.6211	2.7593
0.73	0.0686982	0.3174	1.6945	2.9216
0.74	0.0522766	0.3043	1.7686	3.0940
0.75	0.0390553	0.2910	1.8435	3.2776
0.76	0.0285788	0.2776	1.9194	3.4738
0.77	0.0204269	0.2641	1.9965	3.6842
0.78	0.0142146	0.2506	2.0751	3.9108
0.79	0.0095927	0.2372	2.1555	4.1557
0.80	0.0062487	0.2239	2.2380	4.4218
0.81	0.0039064	0.2108	2.3229	4.7123
0.82	0.0023274	0.1979	2.4107	5.0314
0.83	0.0013100	0.1853	2.5018	5.3839
0.84	0.0006889	0.1729	2.5967	5.7761
0.85	0.0003338	0.1608	2.6961	6.2161
0.86	0.0001463	0.1489	2.8008	6.7140
0.87	0.0000566	0.1373	2.9118	7.2833
0.88	0.0000187	0.1259	3.0303	7.9420
0.89	0.0000051	0.1147	3.1578	8.7150
0.90	0.0000010	0.1038	3.2964	9.6371

$1.85 \leq \hat{\rho}_3 \leq 1.90$

$\hat{\rho}_3 = 1.85, \quad 0.48 \leq \hat{\rho}_4 \leq 0.90$				
$\hat{\rho}_4$	A	B	h	k
0.48	4.4433765	0.0386	-0.8991	0.3090
0.49	3.7773537	0.1394	-0.6212	0.4039
0.50	3.2681750	0.2080	-0.4147	0.4910
0.51	2.8496359	0.2589	-0.2453	0.5750
0.52	2.4937798	0.2982	-0.0993	0.6577
0.53	2.1854942	0.3290	0.0306	0.7402
0.54	1.9153896	0.3534	0.1488	0.8232
0.55	1.6770724	0.3725	0.2579	0.9072
0.56	1.4658946	0.3872	0.3599	0.9926
0.57	1.2783056	0.3982	0.4563	1.0797
0.58	1.1114873	0.4059	0.5480	1.1688
0.59	0.9631325	0.4109	0.6359	1.2603
0.60	0.8313050	0.4134	0.7206	1.3544
0.61	0.7143457	0.4136	0.8027	1.4514
0.62	0.6108088	0.4119	0.8826	1.5516
0.63	0.5194156	0.4084	0.9607	1.6553
0.64	0.4390221	0.4034	1.0373	1.7630
0.65	0.3685940	0.3970	1.1126	1.8748
0.66	0.3071882	0.3894	1.1870	1.9914
0.67	0.2539389	0.3806	1.2606	2.1130
0.68	0.2080466	0.3709	1.3338	2.2403
0.69	0.1687695	0.3604	1.4066	2.3737
0.70	0.1354173	0.3491	1.4793	2.5139
0.71	0.1073458	0.3373	1.5520	2.6617
0.72	0.0839537	0.3249	1.6250	2.8178
0.73	0.0646792	0.3122	1.6984	2.9831
0.74	0.0489985	0.2992	1.7725	3.1587
0.75	0.0364245	0.2860	1.8474	3.3459
0.76	0.0265058	0.2726	1.9234	3.5460
0.77	0.0188273	0.2593	2.0007	3.7608
0.78	0.0130095	0.2459	2.0795	3.9922
0.79	0.0087098	0.2326	2.1601	4.2427
0.80	0.0056222	0.2195	2.2428	4.5149
0.81	0.0034784	0.2066	2.3281	4.8124
0.82	0.0020477	0.1939	2.4162	5.1394
0.83	0.0011366	0.1814	2.5077	5.5010
0.84	0.0005881	0.1692	2.6031	5.9038
0.85	0.0002795	0.1572	2.7031	6.3560
0.86	0.0001197	0.1456	2.8085	6.8683
0.87	0.0000451	0.1341	2.9203	7.4547
0.88	0.0000144	0.1229	3.0397	8.1341
0.89	0.0000037	0.1120	3.1682	8.9322
0.90	0.0000007	0.1012	3.3080	9.8855

$\hat{\rho}_3 = 1.90, \quad 0.48 \leq \hat{\rho}_4 \leq 0.89$				
$\hat{\rho}_4$	A	B	h	k
0.48	4.3208860	0.0610	-0.8390	0.3406
0.49	3.6905966	0.1527	-0.5821	0.4336
0.50	3.1982764	0.2167	-0.3855	0.5203
0.51	2.7900651	0.2648	-0.2220	0.6044
0.52	2.4414989	0.3021	-0.0799	0.6875
0.53	2.1388428	0.3315	0.0472	0.7706
0.54	1.8733577	0.3547	0.1631	0.8542
0.55	1.6389943	0.3728	0.2706	0.9390
0.56	1.4313015	0.3866	0.3713	1.0253
0.57	1.2468483	0.3969	0.4665	1.1133
0.58	1.0828920	0.4041	0.5573	1.2034
0.59	0.9371750	0.4085	0.6444	1.2960
0.60	0.8077942	0.4105	0.7285	1.3912
0.61	0.6931135	0.4103	0.8100	1.4894
0.62	0.5917028	0.4082	0.8894	1.5910
0.63	0.5022948	0.4045	0.9671	1.6961
0.64	0.4237537	0.3992	1.0433	1.8052
0.65	0.3550510	0.3925	1.1183	1.9187
0.66	0.2952484	0.3847	1.1924	2.0370
0.67	0.2434835	0.3758	1.2658	2.1605
0.68	0.1989598	0.3660	1.3388	2.2897
0.69	0.1609381	0.3553	1.4114	2.4253
0.70	0.1287307	0.3440	1.4840	2.5679
0.71	0.1016961	0.3322	1.5566	2.7182
0.72	0.0792354	0.3198	1.6296	2.8771
0.73	0.0607901	0.3071	1.7030	3.0454
0.74	0.0458399	0.2942	1.7771	3.2244
0.75	0.0339015	0.2810	1.8521	3.4152
0.76	0.0245286	0.2678	1.9282	3.6194
0.77	0.0173109	0.2545	2.0056	3.8387
0.78	0.0118751	0.2413	2.0845	4.0751
0.79	0.0078851	0.2282	2.1654	4.3312
0.80	0.0050424	0.2152	2.2484	4.6097
0.81	0.0030864	0.2024	2.3339	4.9144
0.82	0.0017945	0.1899	2.4224	5.2495
0.83	0.0009819	0.1776	2.5144	5.6205
0.84	0.0004996	0.1656	2.6103	6.0340
0.85	0.0002328	0.1538	2.7109	6.4988
0.86	0.0000974	0.1423	2.8169	7.0259
0.87	0.0000356	0.1311	2.9295	7.6298
0.88	0.0000110	0.1200	3.0498	8.3303
0.89	0.0000027	0.1092	3.1793	9.1540

$$1.95 \leq \hat{\rho}_3 \leq 2.00$$

$\hat{\rho}_3 = 1.95, \quad 0.48 \leq \hat{\rho}_4 \leq 0.89$				
$\hat{\rho}_4$	A	B	h	k
0.48	4.2068256	0.0810	-0.7847	0.3714
0.49	3.6065008	0.1650	-0.5451	0.4631
0.50	3.1294878	0.2250	-0.3572	0.5495
0.51	2.7310276	0.2705	-0.1991	0.6338
0.52	2.3895058	0.3059	-0.0607	0.7173
0.53	2.0923704	0.3339	0.0637	0.8010
0.54	1.8314590	0.3559	0.1777	0.8854
0.55	1.6010343	0.3730	0.2835	0.9709
0.56	1.3968267	0.3861	0.3829	1.0581
0.57	1.2155169	0.3957	0.4771	1.1470
0.58	1.0544335	0.4022	0.5670	1.2382
0.59	0.9113660	0.4061	0.6533	1.3318
0.60	0.7844433	0.4076	0.7368	1.4283
0.61	0.6720511	0.4071	0.8177	1.5277
0.62	0.5727749	0.4046	0.8967	1.6306
0.63	0.4853585	0.4006	0.9739	1.7372
0.64	0.4086740	0.3950	1.0497	1.8478
0.65	0.3416990	0.3881	1.1245	1.9629
0.66	0.2834997	0.3801	1.1983	2.0830
0.67	0.2332172	0.3711	1.2715	2.2084
0.68	0.1900582	0.3611	1.3443	2.3397
0.69	0.1532862	0.3504	1.4168	2.4775
0.70	0.1222161	0.3390	1.4893	2.6225
0.71	0.0962089	0.3271	1.5619	2.7754
0.72	0.0746690	0.3148	1.6348	2.9371
0.73	0.0570410	0.3021	1.7082	3.1085
0.74	0.0428083	0.2892	1.7823	3.2909
0.75	0.0314921	0.2761	1.8574	3.4855
0.76	0.0226508	0.2630	1.9335	3.6939
0.77	0.0158798	0.2498	2.0111	3.9178
0.78	0.0108122	0.2367	2.0902	4.1594
0.79	0.0071187	0.2238	2.1713	4.4211
0.80	0.0045086	0.2110	2.2546	4.7062
0.81	0.0027293	0.1983	2.3404	5.0182
0.82	0.0015669	0.1860	2.4293	5.3616
0.83	0.0008447	0.1739	2.5217	5.7422
0.84	0.0004224	0.1620	2.6181	6.1668
0.85	0.0001929	0.1504	2.7193	6.6444
0.86	0.0000788	0.1391	2.8260	7.1866
0.87	0.0000280	0.1281	2.9393	7.8084
0.88	0.0000084	0.1172	3.0605	8.5303
0.89	0.0000020	0.1066	3.1910	9.3802

$\hat{\rho}_3 = 2.00, \quad 0.48 \leq \hat{\rho}_4 \leq 0.89$				
$\hat{\rho}_4$	A	B	h	k
0.48	4.0994702	0.0990	-0.7350	0.4017
0.49	3.5248361	0.1765	-0.5099	0.4923
0.50	3.0618343	0.2328	-0.3297	0.5787
0.51	2.6726103	0.2759	-0.1765	0.6632
0.52	2.3379014	0.3096	-0.0416	0.7472
0.53	2.0461772	0.3361	0.0803	0.8315
0.54	1.7897881	0.3570	0.1924	0.9166
0.55	1.5632801	0.3732	0.2966	1.0029
0.56	1.3625510	0.3855	0.3948	1.0909
0.57	1.1843855	0.3944	0.4879	1.1809
0.58	1.0261795	0.4003	0.5770	1.2731
0.59	0.8857674	0.4037	0.6626	1.3679
0.60	0.7613084	0.4048	0.7454	1.4655
0.61	0.6512095	0.4038	0.8258	1.5662
0.62	0.5540712	0.4011	0.9043	1.6704
0.63	0.4686481	0.3967	0.9812	1.7785
0.64	0.3938202	0.3909	1.0567	1.8907
0.65	0.3285709	0.3838	1.1311	2.0075
0.66	0.2719710	0.3756	1.2047	2.1293
0.67	0.2231654	0.3664	1.2777	2.2567
0.68	0.1813636	0.3563	1.3503	2.3901
0.69	0.1458321	0.3455	1.4227	2.5302
0.70	0.1158885	0.3341	1.4951	2.6776
0.71	0.0908969	0.3222	1.5676	2.8332
0.72	0.0702644	0.3099	1.6405	2.9978
0.73	0.0534395	0.2972	1.7139	3.1725
0.74	0.0399093	0.2843	1.7881	3.3583
0.75	0.0291998	0.2714	1.8632	3.5568
0.76	0.0208745	0.2583	1.9395	3.7694
0.77	0.0145349	0.2453	2.0172	3.9980
0.78	0.0098206	0.2323	2.0965	4.2449
0.79	0.0064098	0.2195	2.1778	4.5126
0.80	0.0040196	0.2068	2.2614	4.8042
0.81	0.0024058	0.1944	2.3476	5.1237
0.82	0.0013632	0.1822	2.4368	5.4757
0.83	0.0007239	0.1702	2.5296	5.8660
0.84	0.0003557	0.1586	2.6266	6.3019
0.85	0.0001590	0.1472	2.7283	6.7927
0.86	0.0000634	0.1360	2.8357	7.3502
0.87	0.0000219	0.1251	2.9497	7.9903
0.88	0.0000063	0.1145	3.0717	8.7340
0.89	0.0000014	0.1041	3.2032	9.6104

Appendix H

Table H–1 for Johnson System Fits in the S_U Region: Method of Moments

For specified α_3 and α_4, the table that follows gives the values of δ and $-\gamma$ for Johnson system fits from the S_U region (see Section 16.3). To assist with computations the table also gives values for μ_Y and σ_Y, as described in Section 16.3.

$0.00 \leq \hat{\alpha}_3 \leq 0.10$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.00$				$\hat{\alpha}_3 = 0.05$				$\hat{\alpha}_3 = 0.10$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
3.1	0.000	0.157	6.441	0.000	0.108	0.155	6.548	0.701	0.234	0.150	6.904	1.584
3.2	0.000	0.221	4.634	0.000	0.076	0.220	4.671	0.348	0.158	0.216	4.787	0.737
3.3	0.000	0.269	3.846	0.000	0.062	0.268	3.866	0.233	0.128	0.265	3.927	0.484
3.4	0.000	0.309	3.384	0.000	0.054	0.308	3.396	0.176	0.110	0.306	3.435	0.363
3.5	0.000	0.344	3.072	0.000	0.049	0.343	3.081	0.142	0.099	0.341	3.108	0.291
3.6	0.000	0.374	2.845	0.000	0.045	0.374	2.852	0.120	0.090	0.372	2.872	0.244
3.7	0.000	0.402	2.670	0.000	0.042	0.402	2.676	0.104	0.084	0.400	2.692	0.211
3.8	0.000	0.428	2.531	0.000	0.039	0.427	2.535	0.092	0.079	0.426	2.548	0.186
3.9	0.000	0.452	2.417	0.000	0.037	0.451	2.420	0.082	0.074	0.450	2.431	0.166
4.0	0.000	0.474	2.321	0.000	0.035	0.474	2.324	0.075	0.071	0.472	2.333	0.151
4.1	0.000	0.495	2.240	0.000	0.034	0.495	2.242	0.068	0.068	0.493	2.250	0.138
4.2	0.000	0.515	2.169	0.000	0.032	0.514	2.171	0.063	0.065	0.513	2.178	0.128
4.3	0.000	0.533	2.107	0.000	0.031	0.533	2.109	0.059	0.063	0.532	2.115	0.119
4.4	0.000	0.551	2.052	0.000	0.030	0.551	2.054	0.055	0.061	0.550	2.060	0.111
4.5	0.000	0.568	2.004	0.000	0.029	0.568	2.005	0.052	0.059	0.567	2.010	0.105
4.6	0.000	0.584	1.960	0.000	0.029	0.584	1.961	0.049	0.057	0.583	1.966	0.099
4.7	0.000	0.600	1.920	0.000	0.028	0.600	1.921	0.047	0.056	0.599	1.925	0.094
4.8	0.000	0.615	1.884	0.000	0.027	0.615	1.885	0.044	0.054	0.614	1.889	0.089
4.9	0.000	0.630	1.851	0.000	0.026	0.629	1.852	0.042	0.053	0.628	1.855	0.085
5.0	0.000	0.644	1.820	0.000	0.026	0.643	1.822	0.041	0.052	0.642	1.825	0.082
5.1	0.000	0.657	1.792	0.000	0.025	0.657	1.793	0.039	0.051	0.656	1.796	0.078
5.2	0.000	0.670	1.766	0.000	0.025	0.670	1.767	0.037	0.050	0.669	1.770	0.075
5.3	0.000	0.683	1.742	0.000	0.024	0.683	1.743	0.036	0.049	0.682	1.746	0.073
5.4	0.000	0.695	1.720	0.000	0.024	0.695	1.721	0.035	0.048	0.694	1.723	0.070
5.5	0.000	0.707	1.699	0.000	0.024	0.707	1.699	0.034	0.047	0.706	1.702	0.068
5.6	0.000	0.719	1.679	0.000	0.023	0.718	1.680	0.033	0.047	0.718	1.682	0.066
5.7	0.000	0.730	1.660	0.000	0.023	0.730	1.661	0.032	0.046	0.729	1.663	0.064
5.8	0.000	0.741	1.643	0.000	0.022	0.741	1.643	0.031	0.045	0.740	1.645	0.062
5.9	0.000	0.752	1.626	0.000	0.022	0.752	1.627	0.030	0.044	0.751	1.628	0.060
6.0	0.000	0.762	1.610	0.000	0.022	0.762	1.611	0.029	0.044	0.761	1.613	0.058
6.1	0.000	0.773	1.596	0.000	0.022	0.772	1.596	0.028	0.043	0.772	1.598	0.057
6.2	0.000	0.783	1.581	0.000	0.021	0.782	1.582	0.028	0.043	0.782	1.583	0.055
6.3	0.000	0.792	1.568	0.000	0.021	0.792	1.568	0.027	0.042	0.792	1.570	0.054
6.4	0.000	0.802	1.555	0.000	0.021	0.802	1.556	0.026	0.042	0.801	1.557	0.053
6.5	0.000	0.811	1.543	0.000	0.021	0.811	1.543	0.026	0.041	0.811	1.545	0.052
6.6	0.000	0.821	1.531	0.000	0.020	0.820	1.532	0.025	0.041	0.820	1.533	0.050
6.7	0.000	0.830	1.520	0.000	0.020	0.829	1.520	0.025	0.040	0.829	1.522	0.049
6.8	0.000	0.838	1.509	0.000	0.020	0.838	1.510	0.024	0.040	0.838	1.511	0.048
6.9	0.000	0.847	1.499	0.000	0.020	0.847	1.499	0.024	0.039	0.846	1.501	0.047
7.0	0.000	0.856	1.489	0.000	0.019	0.855	1.490	0.023	0.039	0.855	1.491	0.046
7.1	0.000	0.864	1.480	0.000	0.019	0.864	1.480	0.023	0.039	0.863	1.481	0.046
7.2	0.000	0.872	1.471	0.000	0.019	0.872	1.471	0.022	0.038	0.871	1.472	0.045
7.3	0.000	0.880	1.462	0.000	0.019	0.880	1.462	0.022	0.038	0.880	1.463	0.044
7.4	0.000	0.888	1.453	0.000	0.019	0.888	1.454	0.022	0.038	0.887	1.455	0.043
7.5	0.000	0.896	1.445	0.000	0.019	0.896	1.445	0.021	0.037	0.895	1.446	0.042
7.6	0.000	0.904	1.437	0.000	0.018	0.903	1.438	0.021	0.037	0.903	1.438	0.042
7.7	0.000	0.911	1.430	0.000	0.018	0.911	1.430	0.020	0.037	0.911	1.431	0.041
7.8	0.000	0.919	1.422	0.000	0.018	0.918	1.423	0.020	0.036	0.918	1.423	0.040
7.9	0.000	0.926	1.415	0.000	0.018	0.926	1.415	0.020	0.036	0.925	1.416	0.040
8.0	0.000	0.933	1.408	0.000	0.018	0.933	1.408	0.020	0.036	0.933	1.409	0.039
8.1	0.000	0.940	1.401	0.000	0.018	0.940	1.402	0.019	0.036	0.940	1.402	0.039
8.2	0.000	0.947	1.395	0.000	0.018	0.947	1.395	0.019	0.035	0.947	1.396	0.038
8.3	0.000	0.954	1.389	0.000	0.017	0.954	1.389	0.019	0.035	0.954	1.390	0.038
8.4	0.000	0.961	1.383	0.000	0.017	0.961	1.383	0.018	0.035	0.960	1.383	0.037
8.5	0.000	0.968	1.377	0.000	0.017	0.968	1.377	0.018	0.034	0.967	1.378	0.037
8.6	0.000	0.974	1.371	0.000	0.017	0.974	1.371	0.018	0.034	0.974	1.372	0.036
8.7	0.000	0.981	1.365	0.000	0.017	0.981	1.365	0.018	0.034	0.980	1.366	0.036
8.8	0.000	0.987	1.360	0.000	0.017	0.987	1.360	0.018	0.034	0.987	1.361	0.035
8.9	0.000	0.994	1.354	0.000	0.017	0.994	1.355	0.017	0.034	0.993	1.355	0.035
9.0	0.000	1.000	1.349	0.000	0.017	1.000	1.349	0.017	0.033	0.999	1.350	0.034
9.1	0.000	1.006	1.344	0.000	0.017	1.006	1.344	0.017	0.033	1.006	1.345	0.034
9.2	0.000	1.012	1.339	0.000	0.016	1.012	1.339	0.017	0.033	1.012	1.340	0.033
9.3	0.000	1.018	1.334	0.000	0.016	1.018	1.335	0.016	0.033	1.018	1.335	0.033
9.4	0.000	1.024	1.330	0.000	0.016	1.024	1.330	0.016	0.033	1.024	1.330	0.033
9.5	0.000	1.030	1.325	0.000	0.016	1.030	1.325	0.016	0.032	1.030	1.326	0.032
9.6	0.000	1.036	1.321	0.000	0.016	1.036	1.321	0.016	0.032	1.036	1.321	0.032
9.7	0.000	1.042	1.316	0.000	0.016	1.042	1.317	0.016	0.032	1.041	1.317	0.032
9.8	0.000	1.048	1.312	0.000	0.016	1.048	1.312	0.016	0.032	1.047	1.313	0.031
9.9	0.000	1.053	1.308	0.000	0.016	1.053	1.308	0.015	0.032	1.053	1.309	0.031
10.0	0.000	1.059	1.304	0.000	0.016	1.059	1.304	0.015	0.032	1.058	1.305	0.031
10.1	0.000	1.064	1.300	0.000	0.016	1.064	1.300	0.015	0.031	1.064	1.301	0.030
10.2	0.000	1.070	1.296	0.000	0.016	1.070	1.296	0.015	0.031	1.069	1.297	0.030
10.3	0.000	1.075	1.292	0.000	0.016	1.075	1.292	0.015	0.031	1.075	1.293	0.030
10.4	0.000	1.081	1.288	0.000	0.015	1.081	1.289	0.015	0.031	1.080	1.289	0.029
10.5	0.000	1.086	1.285	0.000	0.015	1.086	1.285	0.015	0.031	1.086	1.285	0.029
10.6	0.000	1.091	1.281	0.000	0.015	1.091	1.281	0.014	0.031	1.091	1.282	0.029
10.7	0.000	1.096	1.278	0.000	0.015	1.096	1.278	0.014	0.030	1.096	1.278	0.029
10.8	0.000	1.102	1.274	0.000	0.015	1.102	1.274	0.014	0.030	1.101	1.275	0.028
10.9	0.000	1.107	1.271	0.000	0.015	1.107	1.271	0.014	0.030	1.106	1.271	0.028
11.0	0.000	1.112	1.268	0.000	0.015	1.112	1.268	0.014	0.030	1.111	1.268	0.028

$0.15 \leq \hat{\alpha}_3 \leq 0.25$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.15$				$\hat{\alpha}_3 = 0.20$				$\hat{\alpha}_3 = 0.25$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
3.1	0.411	0.142	7.662	3.039	0.789	0.137	9.351	6.740				
3.2	0.253	0.210	5.004	1.228	0.376	0.202	5.368	1.939	0.565	0.194	5.992	3.188
3.3	0.200	0.260	4.037	0.776	0.283	0.254	4.209	1.145	0.390	0.246	4.472	1.662
3.4	0.170	0.302	3.504	0.571	0.238	0.296	3.608	0.818	0.317	0.289	3.759	1.133
3.5	0.152	0.337	3.156	0.454	0.209	0.332	3.228	0.640	0.275	0.326	3.329	0.864
3.6	0.138	0.369	2.908	0.378	0.189	0.364	2.960	0.527	0.246	0.358	3.034	0.701
3.7	0.128	0.397	2.720	0.324	0.175	0.393	2.760	0.449	0.225	0.388	2.816	0.592
3.8	0.120	0.423	2.571	0.285	0.163	0.419	2.604	0.393	0.210	0.414	2.648	0.513
3.9	0.113	0.447	2.450	0.255	0.154	0.444	2.477	0.350	0.197	0.439	2.514	0.454
4.0	0.108	0.470	2.349	0.230	0.146	0.466	2.372	0.315	0.186	0.462	2.403	0.408
4.1	0.103	0.491	2.264	0.211	0.139	0.488	2.283	0.288	0.177	0.484	2.309	0.371
4.2	0.099	0.511	2.190	0.195	0.133	0.508	2.207	0.265	0.170	0.504	2.230	0.341
4.3	0.095	0.530	2.126	0.181	0.128	0.527	2.141	0.246	0.163	0.523	2.160	0.316
4.4	0.092	0.548	2.069	0.169	0.124	0.545	2.082	0.230	0.157	0.542	2.100	0.294
4.5	0.089	0.565	2.018	0.159	0.120	0.562	2.030	0.216	0.152	0.559	2.046	0.276
4.6	0.087	0.581	1.973	0.150	0.117	0.579	1.984	0.203	0.148	0.576	1.998	0.260
4.7	0.084	0.597	1.932	0.142	0.113	0.595	1.942	0.193	0.143	0.592	1.955	0.245
4.8	0.082	0.612	1.895	0.135	0.111	0.610	1.904	0.183	0.140	0.607	1.916	0.233
4.9	0.080	0.627	1.861	0.129	0.108	0.625	1.869	0.174	0.136	0.622	1.880	0.222
5.0	0.078	0.641	1.830	0.124	0.105	0.639	1.837	0.167	0.133	0.636	1.847	0.212
5.1	0.077	0.654	1.801	0.118	0.103	0.652	1.808	0.160	0.130	0.650	1.817	0.203
5.2	0.075	0.668	1.775	0.114	0.101	0.666	1.781	0.154	0.128	0.663	1.790	0.195
5.3	0.074	0.680	1.750	0.110	0.099	0.678	1.756	0.148	0.125	0.676	1.764	0.187
5.4	0.073	0.693	1.727	0.106	0.097	0.691	1.732	0.143	0.123	0.688	1.740	0.181
5.5	0.071	0.705	1.705	0.102	0.096	0.703	1.711	0.138	0.120	0.701	1.718	0.174
5.6	0.070	0.716	1.685	0.099	0.094	0.715	1.690	0.133	0.118	0.712	1.697	0.169
5.7	0.069	0.728	1.666	0.096	0.092	0.726	1.671	0.129	0.116	0.724	1.677	0.163
5.8	0.068	0.739	1.648	0.093	0.091	0.737	1.653	0.125	0.115	0.735	1.659	0.158
5.9	0.067	0.750	1.631	0.090	0.090	0.748	1.636	0.122	0.113	0.746	1.641	0.154
6.0	0.066	0.760	1.615	0.088	0.088	0.759	1.619	0.118	0.111	0.757	1.625	0.149
6.1	0.065	0.771	1.600	0.086	0.087	0.769	1.604	0.115	0.110	0.767	1.609	0.145
6.2	0.064	0.781	1.586	0.084	0.086	0.779	1.590	0.112	0.108	0.777	1.594	0.142
6.3	0.063	0.790	1.572	0.081	0.085	0.789	1.576	0.109	0.107	0.787	1.580	0.138
6.4	0.063	0.800	1.559	0.080	0.084	0.799	1.563	0.107	0.106	0.797	1.567	0.135
6.5	0.062	0.810	1.547	0.078	0.083	0.808	1.550	0.104	0.104	0.806	1.554	0.132
6.6	0.061	0.819	1.535	0.076	0.082	0.817	1.538	0.102	0.103	0.816	1.542	0.129
6.7	0.061	0.828	1.524	0.074	0.081	0.826	1.527	0.100	0.102	0.825	1.530	0.126
6.8	0.060	0.837	1.513	0.073	0.080	0.835	1.516	0.098	0.101	0.834	1.519	0.123
6.9	0.059	0.845	1.502	0.071	0.079	0.844	1.505	0.096	0.100	0.842	1.509	0.121
7.0	0.059	0.854	1.492	0.070	0.079	0.853	1.495	0.094	0.099	0.851	1.498	0.118
7.1	0.058	0.862	1.483	0.069	0.078	0.861	1.485	0.092	0.098	0.859	1.489	0.116
7.2	0.058	0.871	1.474	0.067	0.077	0.869	1.476	0.090	0.097	0.868	1.479	0.114
7.3	0.057	0.879	1.465	0.066	0.076	0.877	1.467	0.089	0.096	0.876	1.470	0.112
7.4	0.057	0.887	1.456	0.065	0.076	0.885	1.458	0.087	0.095	0.884	1.461	0.110
7.5	0.056	0.894	1.448	0.064	0.075	0.893	1.450	0.086	0.094	0.892	1.453	0.108
7.6	0.056	0.902	1.440	0.063	0.074	0.901	1.442	0.084	0.093	0.899	1.445	0.106
7.7	0.055	0.910	1.432	0.062	0.074	0.909	1.434	0.083	0.093	0.907	1.437	0.104
7.8	0.055	0.917	1.425	0.061	0.073	0.916	1.427	0.082	0.092	0.915	1.429	0.103
7.9	0.054	0.925	1.418	0.060	0.072	0.923	1.419	0.080	0.091	0.922	1.422	0.101
8.0	0.054	0.932	1.411	0.059	0.072	0.931	1.412	0.079	0.090	0.929	1.415	0.099
8.1	0.053	0.939	1.404	0.058	0.071	0.938	1.406	0.078	0.090	0.936	1.408	0.098
8.2	0.053	0.946	1.397	0.057	0.071	0.945	1.399	0.077	0.089	0.943	1.401	0.096
8.3	0.053	0.953	1.391	0.056	0.070	0.952	1.392	0.076	0.088	0.950	1.395	0.095
8.4	0.052	0.960	1.385	0.056	0.070	0.959	1.386	0.075	0.088	0.957	1.388	0.094
8.5	0.052	0.966	1.379	0.055	0.069	0.965	1.380	0.074	0.087	0.964	1.382	0.092
8.6	0.051	0.973	1.373	0.054	0.069	0.972	1.374	0.073	0.086	0.971	1.376	0.091
8.7	0.051	0.980	1.367	0.054	0.068	0.979	1.369	0.072	0.086	0.977	1.371	0.090
8.8	0.051	0.986	1.362	0.053	0.068	0.985	1.363	0.071	0.085	0.984	1.365	0.089
8.9	0.050	0.992	1.356	0.052	0.067	0.991	1.358	0.070	0.085	0.990	1.360	0.088
9.0	0.050	0.999	1.351	0.052	0.067	0.998	1.352	0.069	0.084	0.997	1.354	0.087
9.1	0.050	1.005	1.346	0.051	0.067	1.004	1.347	0.068	0.084	1.003	1.349	0.086
9.2	0.050	1.011	1.341	0.050	0.066	1.010	1.342	0.067	0.083	1.009	1.344	0.085
9.3	0.049	1.017	1.336	0.050	0.066	1.016	1.337	0.067	0.083	1.015	1.339	0.084
9.4	0.049	1.023	1.331	0.049	0.065	1.022	1.333	0.066	0.082	1.021	1.334	0.083
9.5	0.049	1.029	1.327	0.049	0.065	1.028	1.328	0.065	0.082	1.027	1.330	0.082
9.6	0.048	1.035	1.322	0.048	0.065	1.034	1.324	0.064	0.081	1.033	1.325	0.081
9.7	0.048	1.041	1.318	0.048	0.064	1.040	1.319	0.064	0.081	1.039	1.321	0.080
9.8	0.048	1.047	1.314	0.047	0.064	1.046	1.315	0.063	0.080	1.045	1.316	0.079
9.9	0.048	1.052	1.309	0.047	0.064	1.051	1.311	0.062	0.080	1.050	1.312	0.078
10.0	0.047	1.058	1.305	0.046	0.063	1.057	1.306	0.062	0.079	1.056	1.308	0.077
10.1	0.047	1.063	1.301	0.046	0.063	1.062	1.302	0.061	0.079	1.061	1.304	0.077
10.2	0.047	1.069	1.297	0.045	0.063	1.068	1.298	0.060	0.078	1.067	1.300	0.076
10.3	0.047	1.074	1.294	0.045	0.062	1.073	1.295	0.060	0.078	1.072	1.296	0.075
10.4	0.046	1.080	1.290	0.044	0.062	1.079	1.291	0.059	0.078	1.078	1.292	0.074
10.5	0.046	1.085	1.286	0.044	0.062	1.084	1.287	0.059	0.077	1.083	1.288	0.074
10.6	0.046	1.090	1.282	0.043	0.061	1.089	1.283	0.058	0.077	1.088	1.285	0.073
10.7	0.046	1.095	1.279	0.043	0.061	1.095	1.280	0.058	0.077	1.094	1.281	0.072
10.8	0.046	1.101	1.275	0.043	0.061	1.100	1.276	0.057	0.076	1.099	1.278	0.072
10.9	0.045	1.106	1.272	0.042	0.060	1.105	1.273	0.057	0.076	1.104	1.274	0.071
11.0	0.045	1.111	1.269	0.042	0.060	1.110	1.270	0.056	0.075	1.109	1.271	0.070

$0.30 \leq \hat{\alpha}_3 \leq 0.40$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.30$				$\hat{\alpha}_3 = 0.35$				$\hat{\alpha}_3 = 0.40$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
3.2	1.014	0.199	7.197	6.367								
3.3	0.543	0.238	4.880	2.488	0.829	0.236	5.566	4.147	2.266	0.359	6.944	10.744
3.4	0.417	0.281	3.978	1.567	0.559	0.273	4.300	2.236	0.804	0.272	4.809	3.472
3.5	0.352	0.318	3.468	1.151	0.451	0.310	3.663	1.546	0.590	0.303	3.943	2.146
3.6	0.311	0.351	3.133	0.914	0.389	0.344	3.266	1.187	0.490	0.336	3.448	1.565
3.7	0.282	0.381	2.891	0.760	0.348	0.374	2.989	0.968	0.429	0.366	3.120	1.238
3.8	0.261	0.408	2.707	0.653	0.319	0.402	2.783	0.820	0.387	0.394	2.882	1.028
3.9	0.244	0.434	2.561	0.573	0.296	0.427	2.623	0.713	0.356	0.420	2.701	0.881
4.0	0.230	0.457	2.442	0.512	0.278	0.451	2.493	0.632	0.332	0.444	2.556	0.773
4.1	0.218	0.479	2.343	0.464	0.263	0.473	2.386	0.568	0.312	0.467	2.439	0.690
4.2	0.208	0.499	2.259	0.424	0.250	0.494	2.295	0.517	0.296	0.488	2.340	0.624
4.3	0.200	0.519	2.186	0.391	0.239	0.514	2.217	0.475	0.282	0.508	2.256	0.571
4.4	0.192	0.537	2.122	0.364	0.230	0.532	2.150	0.440	0.270	0.527	2.184	0.526
4.5	0.186	0.555	2.066	0.340	0.222	0.550	2.091	0.411	0.260	0.545	2.121	0.489
4.6	0.180	0.572	2.016	0.320	0.214	0.567	2.038	0.385	0.251	0.562	2.065	0.457
4.7	0.175	0.588	1.971	0.302	0.208	0.584	1.991	0.363	0.243	0.578	2.015	0.430
4.8	0.170	0.603	1.930	0.286	0.202	0.599	1.949	0.343	0.236	0.594	1.970	0.405
4.9	0.166	0.618	1.894	0.272	0.197	0.614	1.910	0.326	0.229	0.610	1.930	0.384
5.0	0.162	0.633	1.860	0.260	0.192	0.629	1.875	0.310	0.223	0.624	1.893	0.365
5.1	0.158	0.646	1.829	0.248	0.187	0.643	1.843	0.296	0.218	0.638	1.860	0.348
5.2	0.155	0.660	1.800	0.238	0.183	0.656	1.813	0.284	0.213	0.652	1.829	0.333
5.3	0.152	0.673	1.774	0.229	0.179	0.669	1.786	0.273	0.208	0.665	1.800	0.319
5.4	0.149	0.685	1.749	0.220	0.176	0.682	1.760	0.262	0.204	0.678	1.774	0.307
5.5	0.146	0.698	1.726	0.213	0.172	0.694	1.737	0.253	0.200	0.690	1.749	0.296
5.6	0.143	0.710	1.705	0.205	0.169	0.706	1.715	0.244	0.196	0.702	1.726	0.285
5.7	0.141	0.721	1.685	0.199	0.166	0.718	1.694	0.236	0.193	0.714	1.705	0.276
5.8	0.139	0.732	1.666	0.193	0.164	0.729	1.674	0.229	0.190	0.726	1.685	0.267
5.9	0.137	0.743	1.648	0.187	0.161	0.740	1.656	0.222	0.186	0.737	1.666	0.258
6.0	0.135	0.754	1.631	0.182	0.159	0.751	1.639	0.215	0.184	0.748	1.648	0.251
6.1	0.133	0.764	1.615	0.177	0.156	0.762	1.623	0.209	0.181	0.758	1.631	0.244
6.2	0.131	0.775	1.600	0.172	0.154	0.772	1.607	0.204	0.178	0.769	1.616	0.237
6.3	0.129	0.785	1.586	0.168	0.152	0.782	1.593	0.198	0.176	0.779	1.600	0.231
6.4	0.128	0.794	1.572	0.164	0.150	0.792	1.579	0.194	0.173	0.789	1.586	0.225
6.5	0.126	0.804	1.559	0.160	0.148	0.801	1.565	0.189	0.171	0.798	1.573	0.219
6.6	0.125	0.813	1.547	0.156	0.147	0.811	1.553	0.184	0.169	0.808	1.560	0.214
6.7	0.123	0.822	1.535	0.153	0.145	0.820	1.541	0.180	0.167	0.817	1.547	0.209
6.8	0.122	0.831	1.524	0.149	0.143	0.829	1.529	0.176	0.165	0.826	1.535	0.204
6.9	0.120	0.840	1.513	0.146	0.142	0.838	1.518	0.173	0.163	0.835	1.524	0.200
7.0	0.119	0.849	1.502	0.143	0.140	0.846	1.507	0.169	0.162	0.844	1.513	0.196
7.1	0.118	0.857	1.492	0.140	0.139	0.855	1.497	0.166	0.160	0.852	1.503	0.192
7.2	0.117	0.866	1.483	0.138	0.137	0.863	1.488	0.163	0.158	0.861	1.493	0.188
7.3	0.116	0.874	1.474	0.135	0.136	0.872	1.478	0.160	0.157	0.869	1.483	0.185
7.4	0.115	0.882	1.465	0.133	0.135	0.880	1.469	0.157	0.155	0.877	1.474	0.181
7.5	0.114	0.890	1.456	0.130	0.133	0.888	1.460	0.154	0.154	0.885	1.465	0.178
7.6	0.113	0.898	1.448	0.128	0.132	0.895	1.452	0.151	0.152	0.893	1.457	0.175
7.7	0.112	0.905	1.440	0.126	0.131	0.903	1.444	0.149	0.151	0.901	1.448	0.172
7.8	0.111	0.913	1.432	0.124	0.130	0.911	1.436	0.146	0.150	0.908	1.440	0.169
7.9	0.110	0.920	1.425	0.122	0.129	0.918	1.428	0.144	0.148	0.916	1.433	0.166
8.0	0.109	0.927	1.418	0.120	0.128	0.925	1.421	0.142	0.147	0.923	1.425	0.163
8.1	0.108	0.935	1.411	0.118	0.127	0.933	1.414	0.139	0.146	0.930	1.418	0.161
8.2	0.107	0.942	1.404	0.117	0.126	0.940	1.407	0.137	0.145	0.937	1.411	0.158
8.3	0.106	0.949	1.397	0.115	0.125	0.947	1.401	0.135	0.144	0.944	1.404	0.156
8.4	0.106	0.956	1.391	0.113	0.124	0.954	1.394	0.133	0.142	0.951	1.398	0.154
8.5	0.105	0.962	1.385	0.112	0.123	0.960	1.388	0.131	0.141	0.958	1.391	0.152
8.6	0.104	0.969	1.379	0.110	0.122	0.967	1.382	0.130	0.140	0.965	1.385	0.150
8.7	0.103	0.976	1.373	0.109	0.121	0.974	1.376	0.128	0.139	0.972	1.379	0.148
8.8	0.103	0.982	1.367	0.107	0.120	0.980	1.370	0.126	0.138	0.978	1.373	0.146
8.9	0.102	0.989	1.362	0.106	0.120	0.987	1.365	0.125	0.137	0.985	1.368	0.144
9.0	0.101	0.995	1.356	0.105	0.119	0.993	1.359	0.123	0.137	0.991	1.362	0.142
9.1	0.101	1.001	1.351	0.103	0.118	1.000	1.354	0.121	0.136	0.997	1.357	0.140
9.2	0.100	1.007	1.346	0.102	0.117	1.006	1.349	0.120	0.135	1.004	1.352	0.138
9.3	0.099	1.014	1.341	0.101	0.117	1.012	1.344	0.119	0.134	1.010	1.347	0.137
9.4	0.099	1.020	1.336	0.100	0.116	1.018	1.339	0.117	0.133	1.016	1.342	0.135
9.5	0.098	1.026	1.332	0.099	0.115	1.024	1.334	0.116	0.132	1.022	1.337	0.133
9.6	0.098	1.032	1.327	0.097	0.114	1.030	1.329	0.115	0.131	1.028	1.332	0.132
9.7	0.097	1.037	1.323	0.096	0.114	1.036	1.325	0.113	0.131	1.034	1.327	0.130
9.8	0.097	1.043	1.318	0.095	0.113	1.041	1.320	0.112	0.130	1.040	1.323	0.129
9.9	0.096	1.049	1.314	0.094	0.113	1.047	1.316	0.111	0.129	1.045	1.318	0.128
10.0	0.096	1.054	1.310	0.093	0.112	1.053	1.312	0.110	0.128	1.051	1.314	0.126
10.1	0.095	1.060	1.306	0.092	0.111	1.058	1.308	0.108	0.128	1.057	1.310	0.125
10.2	0.094	1.066	1.302	0.091	0.111	1.064	1.304	0.107	0.127	1.062	1.306	0.124
10.3	0.094	1.071	1.298	0.091	0.110	1.070	1.300	0.106	0.126	1.068	1.302	0.122
10.4	0.094	1.076	1.294	0.090	0.110	1.075	1.296	0.105	0.126	1.073	1.298	0.121
10.5	0.093	1.082	1.290	0.089	0.109	1.080	1.292	0.104	0.125	1.079	1.294	0.120
10.6	0.093	1.087	1.286	0.088	0.108	1.086	1.288	0.103	0.124	1.084	1.290	0.119
10.7	0.092	1.092	1.283	0.087	0.108	1.091	1.285	0.102	0.124	1.089	1.287	0.118
10.8	0.092	1.098	1.279	0.086	0.107	1.096	1.281	0.101	0.123	1.094	1.283	0.117
10.9	0.091	1.103	1.276	0.086	0.107	1.101	1.277	0.100	0.123	1.100	1.280	0.115
11.0	0.091	1.108	1.272	0.085	0.106	1.106	1.274	0.099	0.122	1.105	1.276	0.114
11.1	0.090	1.113	1.269	0.084	0.106	1.111	1.271	0.099	0.122	1.110	1.273	0.113

$$0.45 \leq \hat{\alpha}_3 \leq 0.55$$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.45$				$\hat{\alpha}_3 = 0.50$				$\hat{\alpha}_3 = 0.55$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
3.4	1.569	0.329	5.719	6.974								
3.5	0.826	0.303	4.369	3.218	1.489	0.357	5.090	5.970				
3.6	0.633	0.330	3.705	2.139	0.877	0.333	4.087	3.157	1.569	0.400	4.718	5.726
3.7	0.534	0.359	3.295	1.614	0.685	0.354	3.539	2.188	0.953	0.363	3.900	3.218
3.8	0.471	0.386	3.011	1.302	0.583	0.380	3.184	1.687	0.749	0.378	3.424	2.284
3.9	0.427	0.412	2.801	1.095	0.517	0.405	2.931	1.378	0.639	0.400	3.104	1.783
4.0	0.394	0.437	2.637	0.947	0.470	0.429	2.739	1.169	0.568	0.423	2.872	1.469
4.1	0.368	0.459	2.505	0.836	0.435	0.452	2.588	1.018	0.517	0.445	2.694	1.253
4.2	0.347	0.481	2.396	0.750	0.407	0.474	2.465	0.903	0.478	0.467	2.552	1.095
4.3	0.330	0.501	2.304	0.681	0.384	0.494	2.363	0.813	0.447	0.487	2.435	0.975
4.4	0.315	0.520	2.226	0.625	0.365	0.514	2.276	0.741	0.422	0.507	2.338	0.880
4.5	0.302	0.539	2.157	0.578	0.348	0.532	2.202	0.681	0.401	0.525	2.255	0.803
4.6	0.291	0.556	2.097	0.538	0.334	0.550	2.136	0.631	0.384	0.543	2.183	0.740
4.7	0.281	0.573	2.044	0.504	0.322	0.567	2.079	0.589	0.368	0.560	2.120	0.686
4.8	0.272	0.589	1.997	0.474	0.311	0.583	2.028	0.552	0.355	0.577	2.065	0.641
4.9	0.264	0.604	1.954	0.448	0.301	0.599	1.982	0.520	0.343	0.592	2.015	0.602
5.0	0.257	0.619	1.915	0.425	0.293	0.614	1.941	0.492	0.332	0.608	1.971	0.567
5.1	0.250	0.633	1.880	0.405	0.285	0.628	1.903	0.467	0.322	0.622	1.931	0.537
5.2	0.244	0.647	1.847	0.387	0.278	0.642	1.869	0.445	0.314	0.636	1.894	0.511
5.3	0.239	0.661	1.817	0.370	0.271	0.655	1.837	0.425	0.306	0.650	1.860	0.487
5.4	0.234	0.674	1.790	0.355	0.265	0.669	1.808	0.408	0.298	0.663	1.830	0.465
5.5	0.229	0.686	1.764	0.342	0.259	0.681	1.781	0.391	0.292	0.676	1.801	0.446
5.6	0.224	0.698	1.740	0.329	0.254	0.694	1.756	0.376	0.286	0.688	1.775	0.428
5.7	0.220	0.710	1.718	0.318	0.249	0.705	1.733	0.363	0.280	0.700	1.750	0.412
5.8	0.216	0.722	1.697	0.307	0.245	0.717	1.711	0.350	0.274	0.712	1.728	0.398
5.9	0.213	0.733	1.677	0.297	0.240	0.728	1.691	0.339	0.269	0.724	1.706	0.384
6.0	0.209	0.744	1.659	0.288	0.236	0.740	1.672	0.328	0.265	0.735	1.686	0.371
6.1	0.206	0.754	1.642	0.280	0.233	0.750	1.654	0.318	0.260	0.746	1.667	0.360
6.2	0.203	0.765	1.625	0.272	0.229	0.761	1.637	0.309	0.256	0.756	1.649	0.349
6.3	0.200	0.775	1.610	0.265	0.226	0.771	1.620	0.301	0.252	0.767	1.633	0.339
6.4	0.197	0.785	1.595	0.258	0.222	0.781	1.605	0.292	0.249	0.777	1.617	0.330
6.5	0.195	0.795	1.581	0.251	0.219	0.791	1.591	0.285	0.245	0.787	1.602	0.321
6.6	0.192	0.804	1.568	0.245	0.217	0.800	1.577	0.278	0.242	0.796	1.587	0.313
6.7	0.190	0.814	1.555	0.239	0.214	0.810	1.564	0.271	0.239	0.806	1.574	0.305
6.8	0.188	0.823	1.543	0.234	0.211	0.819	1.551	0.265	0.236	0.815	1.561	0.298
6.9	0.186	0.832	1.531	0.229	0.209	0.828	1.539	0.259	0.233	0.824	1.548	0.291
7.0	0.184	0.841	1.520	0.224	0.206	0.837	1.528	0.253	0.230	0.833	1.537	0.284
7.1	0.182	0.849	1.509	0.219	0.204	0.846	1.517	0.248	0.227	0.842	1.525	0.278
7.2	0.180	0.858	1.499	0.215	0.202	0.854	1.506	0.243	0.225	0.851	1.514	0.272
7.3	0.178	0.866	1.489	0.211	0.200	0.863	1.496	0.238	0.222	0.859	1.504	0.267
7.4	0.176	0.874	1.480	0.207	0.198	0.871	1.487	0.233	0.220	0.867	1.494	0.261
7.5	0.174	0.882	1.471	0.203	0.196	0.879	1.477	0.229	0.218	0.875	1.485	0.256
7.6	0.173	0.890	1.462	0.199	0.194	0.887	1.468	0.225	0.216	0.883	1.475	0.252
7.7	0.171	0.898	1.454	0.196	0.192	0.895	1.460	0.221	0.213	0.891	1.466	0.247
7.8	0.170	0.905	1.446	0.192	0.190	0.902	1.451	0.217	0.211	0.899	1.458	0.243
7.9	0.168	0.913	1.438	0.189	0.189	0.910	1.443	0.213	0.210	0.907	1.450	0.238
8.0	0.167	0.920	1.430	0.186	0.187	0.917	1.435	0.210	0.208	0.914	1.442	0.234
8.1	0.165	0.928	1.423	0.183	0.185	0.925	1.428	0.206	0.206	0.922	1.434	0.230
8.2	0.164	0.935	1.416	0.180	0.184	0.932	1.421	0.203	0.204	0.929	1.426	0.227
8.3	0.163	0.942	1.409	0.178	0.182	0.939	1.414	0.200	0.202	0.936	1.419	0.223
8.4	0.161	0.949	1.402	0.175	0.181	0.946	1.407	0.197	0.201	0.943	1.412	0.220
8.5	0.160	0.956	1.396	0.172	0.179	0.953	1.400	0.194	0.199	0.950	1.405	0.217
8.6	0.159	0.963	1.389	0.170	0.178	0.960	1.394	0.191	0.198	0.957	1.399	0.213
8.7	0.158	0.969	1.383	0.168	0.177	0.967	1.388	0.189	0.196	0.964	1.392	0.210
8.8	0.157	0.976	1.377	0.165	0.176	0.973	1.381	0.186	0.195	0.970	1.386	0.207
8.9	0.156	0.982	1.371	0.163	0.174	0.980	1.376	0.184	0.193	0.977	1.380	0.205
9.0	0.155	0.989	1.366	0.161	0.173	0.986	1.370	0.181	0.192	0.983	1.374	0.202
9.1	0.154	0.995	1.360	0.159	0.172	0.993	1.364	0.179	0.191	0.990	1.369	0.199
9.2	0.153	1.001	1.355	0.157	0.171	0.999	1.359	0.176	0.189	0.996	1.363	0.197
9.3	0.152	1.008	1.350	0.155	0.170	1.005	1.354	0.174	0.188	1.002	1.358	0.194
9.4	0.151	1.014	1.345	0.153	0.169	1.011	1.348	0.172	0.187	1.009	1.353	0.192
9.5	0.150	1.020	1.340	0.152	0.167	1.017	1.343	0.170	0.186	1.015	1.348	0.189
9.6	0.149	1.026	1.335	0.150	0.166	1.023	1.339	0.168	0.184	1.021	1.343	0.187
9.7	0.148	1.032	1.330	0.148	0.165	1.029	1.334	0.166	0.183	1.027	1.338	0.185
9.8	0.147	1.037	1.326	0.146	0.164	1.035	1.329	0.164	0.182	1.033	1.333	0.183
9.9	0.146	1.043	1.321	0.145	0.163	1.041	1.325	0.162	0.181	1.038	1.328	0.181
10.0	0.145	1.049	1.317	0.143	0.163	1.047	1.320	0.161	0.180	1.044	1.324	0.179
10.1	0.145	1.055	1.313	0.142	0.162	1.052	1.316	0.159	0.179	1.050	1.319	0.177
10.2	0.144	1.060	1.309	0.140	0.161	1.058	1.312	0.157	0.178	1.055	1.315	0.175
10.3	0.143	1.066	1.305	0.139	0.160	1.063	1.308	0.156	0.177	1.061	1.311	0.173
10.4	0.142	1.071	1.301	0.137	0.159	1.069	1.304	0.154	0.176	1.066	1.307	0.171
10.5	0.142	1.077	1.297	0.136	0.158	1.074	1.300	0.153	0.175	1.072	1.303	0.169
10.6	0.141	1.082	1.293	0.135	0.157	1.080	1.296	0.151	0.174	1.077	1.299	0.168
10.7	0.140	1.087	1.289	0.133	0.157	1.085	1.292	0.150	0.173	1.083	1.295	0.166
10.8	0.139	1.092	1.285	0.132	0.156	1.090	1.288	0.148	0.172	1.088	1.291	0.164
10.9	0.139	1.098	1.282	0.131	0.155	1.096	1.285	0.147	0.172	1.093	1.288	0.163
11.0	0.138	1.103	1.278	0.130	0.154	1.101	1.281	0.145	0.171	1.098	1.284	0.161
11.1	0.137	1.108	1.275	0.129	0.153	1.106	1.278	0.144	0.170	1.104	1.280	0.160
11.2	0.137	1.113	1.272	0.127	0.153	1.111	1.274	0.143	0.169	1.109	1.277	0.158
11.3	0.136	1.118	1.268	0.126	0.152	1.116	1.271	0.141	0.168	1.114	1.274	0.157

$0.60 \leq \hat{\alpha}_3 \leq 0.70$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.60$				$\hat{\alpha}_3 = 0.65$				$\hat{\alpha}_3 = 0.70$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
3.7	1.785	0.463	4.487	5.930								
3.8	1.057	0.396	3.776	3.383	2.251	0.575	4.348	6.638				
3.9	0.827	0.402	3.344	2.426	1.204	0.434	3.699	3.664	3.787	0.929	4.278	8.623
4.0	0.703	0.420	3.049	1.906	0.924	0.428	3.294	2.621	1.425	0.488	3.659	4.105
4.1	0.624	0.440	2.830	1.577	0.780	0.440	3.013	2.060	1.050	0.459	3.269	2.886
4.2	0.568	0.460	2.661	1.349	0.689	0.457	2.804	1.705	0.873	0.463	2.996	2.254
4.3	0.525	0.480	2.526	1.182	0.625	0.476	2.641	1.460	0.764	0.476	2.791	1.860
4.4	0.491	0.500	2.414	1.054	0.576	0.494	2.510	1.280	0.690	0.491	2.631	1.589
4.5	0.463	0.518	2.320	0.953	0.538	0.512	2.401	1.141	0.634	0.508	2.502	1.391
4.6	0.440	0.536	2.240	0.870	0.507	0.530	2.309	1.032	0.591	0.525	2.395	1.240
4.7	0.421	0.553	2.170	0.802	0.482	0.547	2.231	0.943	0.556	0.541	2.304	1.121
4.8	0.404	0.570	2.109	0.745	0.460	0.563	2.162	0.870	0.527	0.557	2.226	1.024
4.9	0.389	0.586	2.055	0.696	0.441	0.579	2.102	0.808	0.502	0.573	2.159	0.944
5.0	0.376	0.601	2.007	0.654	0.425	0.595	2.049	0.755	0.481	0.588	2.099	0.876
5.1	0.364	0.616	1.963	0.617	0.410	0.610	2.001	0.709	0.463	0.603	2.046	0.818
5.2	0.353	0.630	1.924	0.584	0.397	0.624	1.958	0.669	0.447	0.618	1.999	0.769
5.3	0.344	0.644	1.888	0.556	0.385	0.638	1.919	0.634	0.432	0.632	1.956	0.725
5.4	0.335	0.657	1.855	0.530	0.375	0.651	1.884	0.603	0.419	0.645	1.918	0.687
5.5	0.327	0.670	1.824	0.507	0.365	0.664	1.851	0.575	0.407	0.658	1.882	0.653
5.6	0.319	0.683	1.796	0.486	0.356	0.677	1.821	0.550	0.397	0.671	1.850	0.623
5.7	0.313	0.695	1.770	0.467	0.348	0.689	1.794	0.527	0.387	0.683	1.820	0.595
5.8	0.306	0.707	1.746	0.449	0.341	0.701	1.768	0.506	0.378	0.696	1.793	0.571
5.9	0.300	0.719	1.724	0.433	0.334	0.713	1.744	0.487	0.370	0.707	1.767	0.548
6.0	0.295	0.730	1.703	0.418	0.327	0.724	1.722	0.470	0.362	0.719	1.743	0.528
6.1	0.290	0.741	1.683	0.405	0.321	0.735	1.701	0.454	0.355	0.730	1.721	0.509
6.2	0.285	0.751	1.664	0.392	0.316	0.746	1.681	0.439	0.348	0.741	1.700	0.492
6.3	0.280	0.762	1.647	0.381	0.310	0.757	1.663	0.426	0.342	0.752	1.681	0.476
6.4	0.276	0.772	1.630	0.370	0.305	0.767	1.645	0.413	0.336	0.762	1.662	0.461
6.5	0.272	0.782	1.614	0.359	0.301	0.777	1.629	0.401	0.331	0.772	1.645	0.447
6.6	0.268	0.792	1.599	0.350	0.296	0.787	1.613	0.390	0.326	0.782	1.628	0.434
6.7	0.264	0.801	1.585	0.341	0.292	0.797	1.598	0.380	0.321	0.792	1.613	0.422
6.8	0.261	0.811	1.572	0.333	0.288	0.806	1.584	0.370	0.316	0.801	1.598	0.411
6.9	0.258	0.820	1.559	0.325	0.284	0.815	1.571	0.361	0.312	0.811	1.584	0.400
7.0	0.255	0.829	1.547	0.317	0.280	0.825	1.558	0.352	0.308	0.820	1.571	0.390
7.1	0.251	0.838	1.535	0.310	0.277	0.833	1.546	0.344	0.304	0.829	1.558	0.381
7.2	0.249	0.846	1.524	0.303	0.274	0.842	1.534	0.337	0.300	0.838	1.546	0.372
7.3	0.246	0.855	1.513	0.297	0.270	0.851	1.523	0.329	0.296	0.846	1.534	0.364
7.4	0.243	0.863	1.503	0.291	0.267	0.859	1.512	0.322	0.293	0.855	1.523	0.356
7.5	0.241	0.872	1.493	0.285	0.264	0.867	1.502	0.316	0.289	0.863	1.512	0.349
7.6	0.238	0.880	1.483	0.280	0.262	0.876	1.492	0.310	0.286	0.871	1.502	0.342
7.7	0.236	0.888	1.474	0.274	0.259	0.884	1.483	0.304	0.283	0.879	1.492	0.335
7.8	0.233	0.895	1.465	0.270	0.256	0.891	1.473	0.298	0.280	0.887	1.483	0.328
7.9	0.231	0.903	1.457	0.265	0.254	0.899	1.465	0.293	0.277	0.895	1.474	0.322
8.0	0.229	0.911	1.448	0.260	0.251	0.907	1.456	0.287	0.275	0.903	1.465	0.316
8.1	0.227	0.918	1.441	0.256	0.249	0.914	1.448	0.282	0.272	0.910	1.456	0.311
8.2	0.225	0.925	1.433	0.252	0.247	0.922	1.440	0.278	0.270	0.918	1.448	0.305
8.3	0.223	0.933	1.425	0.248	0.245	0.929	1.432	0.273	0.267	0.925	1.440	0.300
8.4	0.221	0.940	1.418	0.244	0.243	0.936	1.425	0.269	0.265	0.932	1.433	0.295
8.5	0.220	0.947	1.411	0.240	0.241	0.943	1.418	0.265	0.262	0.939	1.425	0.290
8.6	0.218	0.954	1.405	0.236	0.239	0.950	1.411	0.260	0.260	0.946	1.418	0.286
8.7	0.216	0.960	1.398	0.233	0.237	0.957	1.404	0.257	0.258	0.953	1.411	0.281
8.8	0.215	0.967	1.392	0.230	0.235	0.964	1.398	0.253	0.256	0.960	1.404	0.277
8.9	0.213	0.974	1.386	0.226	0.233	0.970	1.391	0.249	0.254	0.967	1.398	0.273
9.0	0.211	0.980	1.380	0.223	0.231	0.977	1.385	0.246	0.252	0.973	1.392	0.269
9.1	0.210	0.987	1.374	0.220	0.230	0.983	1.379	0.242	0.250	0.980	1.385	0.266
9.2	0.208	0.993	1.368	0.217	0.228	0.990	1.374	0.239	0.248	0.986	1.380	0.262
9.3	0.207	0.999	1.363	0.215	0.226	0.996	1.368	0.236	0.247	0.993	1.374	0.258
9.4	0.206	1.006	1.357	0.212	0.225	1.002	1.362	0.233	0.245	0.999	1.368	0.255
9.5	0.204	1.012	1.352	0.209	0.223	1.009	1.357	0.230	0.243	1.005	1.363	0.252
9.6	0.203	1.018	1.347	0.207	0.222	1.015	1.352	0.227	0.241	1.011	1.357	0.248
9.7	0.202	1.024	1.342	0.204	0.220	1.021	1.347	0.224	0.240	1.017	1.352	0.245
9.8	0.200	1.030	1.337	0.202	0.219	1.027	1.342	0.222	0.238	1.023	1.347	0.242
9.9	0.199	1.036	1.332	0.200	0.218	1.032	1.337	0.219	0.237	1.029	1.342	0.240
10.0	0.198	1.041	1.328	0.197	0.216	1.038	1.332	0.217	0.235	1.035	1.337	0.237
10.1	0.197	1.047	1.323	0.195	0.215	1.044	1.328	0.214	0.234	1.041	1.333	0.234
10.2	0.196	1.053	1.319	0.193	0.214	1.050	1.323	0.212	0.232	1.047	1.328	0.231
10.3	0.195	1.058	1.315	0.191	0.213	1.055	1.319	0.209	0.231	1.052	1.323	0.229
10.4	0.194	1.064	1.311	0.189	0.211	1.061	1.315	0.207	0.230	1.058	1.319	0.226
10.5	0.192	1.069	1.306	0.187	0.210	1.066	1.310	0.205	0.228	1.063	1.315	0.224
10.6	0.191	1.075	1.302	0.185	0.209	1.072	1.306	0.203	0.227	1.069	1.311	0.221
10.7	0.190	1.080	1.298	0.183	0.208	1.077	1.302	0.201	0.226	1.074	1.307	0.219
10.8	0.189	1.085	1.295	0.181	0.207	1.083	1.298	0.199	0.225	1.080	1.303	0.217
10.9	0.188	1.091	1.291	0.180	0.206	1.088	1.295	0.197	0.223	1.085	1.299	0.215
11.0	0.187	1.096	1.287	0.178	0.205	1.093	1.291	0.195	0.222	1.090	1.295	0.213
11.1	0.187	1.101	1.284	0.176	0.204	1.098	1.287	0.193	0.221	1.095	1.291	0.211
11.2	0.186	1.106	1.280	0.175	0.203	1.103	1.284	0.191	0.220	1.101	1.287	0.209
11.3	0.185	1.111	1.277	0.173	0.202	1.109	1.280	0.189	0.219	1.106	1.284	0.207
11.4	0.184	1.116	1.273	0.171	0.201	1.114	1.277	0.188	0.218	1.111	1.280	0.205
11.5	0.183	1.121	1.270	0.170	0.200	1.119	1.273	0.186	0.217	1.116	1.277	0.203
11.6	0.182	1.126	1.267	0.168	0.199	1.123	1.270	0.184	0.216	1.121	1.273	0.201

$0.75 \leq \hat{\alpha}_3 \leq 0.85$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.75$				$\hat{\alpha}_3 = 0.80$				$\hat{\alpha}_3 = 0.85$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
4.1	1.808	0.579	3.654	4.828								
4.2	1.225	0.501	3.266	3.253	2.763	0.815	3.681	6.281				
4.3	0.990	0.491	2.994	2.503	1.495	0.565	3.286	3.792				
4.4	0.856	0.497	2.791	2.050	1.149	0.527	3.010	2.835	2.011	0.693	3.329	4.689
4.5	0.766	0.508	2.631	1.744	0.971	0.522	2.803	2.290	1.384	0.582	3.042	3.301
4.6	0.700	0.522	2.502	1.522	0.857	0.528	2.642	1.931	1.124	0.557	2.828	2.603
4.7	0.650	0.537	2.395	1.353	0.778	0.539	2.511	1.676	0.973	0.554	2.662	2.166
4.8	0.609	0.553	2.305	1.221	0.717	0.552	2.403	1.485	0.871	0.559	2.528	1.864
4.9	0.576	0.568	2.227	1.113	0.670	0.565	2.312	1.335	0.797	0.568	2.418	1.641
5.0	0.548	0.583	2.160	1.025	0.632	0.579	2.233	1.215	0.740	0.579	2.325	1.469
5.1	0.525	0.597	2.100	0.951	0.599	0.593	2.165	1.117	0.695	0.591	2.245	1.332
5.2	0.504	0.612	2.048	0.888	0.572	0.607	2.105	1.034	0.657	0.604	2.176	1.221
5.3	0.486	0.625	2.000	0.833	0.549	0.620	2.052	0.964	0.625	0.616	2.115	1.128
5.4	0.470	0.639	1.958	0.786	0.528	0.633	2.005	0.904	0.598	0.629	2.061	1.050
5.5	0.455	0.652	1.919	0.744	0.510	0.646	1.962	0.851	0.575	0.642	2.012	0.982
5.6	0.442	0.665	1.884	0.707	0.494	0.659	1.923	0.805	0.554	0.654	1.969	0.924
5.7	0.430	0.677	1.851	0.673	0.479	0.672	1.887	0.765	0.536	0.666	1.929	0.873
5.8	0.420	0.690	1.821	0.644	0.466	0.684	1.855	0.728	0.519	0.678	1.893	0.828
5.9	0.410	0.701	1.794	0.617	0.454	0.696	1.825	0.696	0.504	0.690	1.860	0.788
6.0	0.400	0.713	1.768	0.592	0.443	0.707	1.797	0.666	0.491	0.702	1.830	0.752
6.1	0.392	0.724	1.744	0.570	0.433	0.719	1.771	0.640	0.478	0.713	1.802	0.720
6.2	0.384	0.735	1.722	0.550	0.423	0.730	1.747	0.615	0.467	0.724	1.776	0.690
6.3	0.377	0.746	1.701	0.531	0.415	0.740	1.725	0.593	0.457	0.735	1.752	0.664
6.4	0.370	0.756	1.682	0.513	0.407	0.751	1.704	0.572	0.447	0.745	1.729	0.639
6.5	0.364	0.767	1.663	0.497	0.399	0.761	1.684	0.553	0.438	0.756	1.708	0.617
6.6	0.358	0.777	1.646	0.482	0.392	0.771	1.666	0.536	0.430	0.766	1.688	0.596
6.7	0.352	0.787	1.629	0.468	0.385	0.781	1.648	0.520	0.422	0.776	1.669	0.577
6.8	0.346	0.796	1.614	0.455	0.379	0.791	1.632	0.505	0.415	0.785	1.652	0.559
6.9	0.341	0.806	1.599	0.443	0.373	0.800	1.616	0.490	0.408	0.795	1.635	0.543
7.0	0.337	0.815	1.585	0.432	0.368	0.810	1.601	0.477	0.401	0.804	1.619	0.527
7.1	0.332	0.824	1.572	0.421	0.362	0.819	1.587	0.465	0.395	0.814	1.604	0.513
7.2	0.328	0.833	1.559	0.411	0.357	0.828	1.573	0.453	0.389	0.823	1.590	0.499
7.3	0.323	0.841	1.547	0.401	0.353	0.836	1.561	0.442	0.384	0.831	1.576	0.487
7.4	0.320	0.850	1.535	0.392	0.348	0.845	1.548	0.432	0.378	0.840	1.563	0.475
7.5	0.316	0.858	1.524	0.384	0.344	0.854	1.537	0.422	0.373	0.849	1.551	0.464
7.6	0.312	0.867	1.513	0.376	0.340	0.862	1.525	0.413	0.369	0.857	1.539	0.453
7.7	0.309	0.875	1.503	0.368	0.336	0.870	1.515	0.404	0.364	0.865	1.528	0.443
7.8	0.305	0.883	1.493	0.361	0.332	0.878	1.504	0.396	0.360	0.873	1.517	0.433
7.9	0.302	0.891	1.483	0.354	0.328	0.886	1.494	0.388	0.356	0.881	1.507	0.424
8.0	0.299	0.898	1.474	0.347	0.325	0.894	1.485	0.380	0.352	0.889	1.497	0.416
8.1	0.296	0.906	1.466	0.341	0.321	0.901	1.476	0.373	0.348	0.897	1.487	0.408
8.2	0.293	0.913	1.457	0.335	0.318	0.909	1.467	0.366	0.344	0.904	1.478	0.400
8.3	0.290	0.921	1.449	0.329	0.315	0.916	1.458	0.359	0.341	0.912	1.469	0.392
8.4	0.288	0.928	1.441	0.323	0.312	0.924	1.450	0.353	0.337	0.919	1.460	0.385
8.5	0.285	0.935	1.433	0.318	0.309	0.931	1.442	0.347	0.334	0.926	1.452	0.378
8.6	0.283	0.942	1.426	0.313	0.306	0.938	1.435	0.341	0.331	0.934	1.444	0.372
8.7	0.280	0.949	1.419	0.308	0.304	0.945	1.427	0.336	0.328	0.941	1.437	0.366
8.8	0.278	0.956	1.412	0.303	0.301	0.952	1.420	0.330	0.325	0.948	1.429	0.360
8.9	0.276	0.963	1.405	0.299	0.298	0.959	1.413	0.325	0.322	0.954	1.422	0.354
9.0	0.274	0.969	1.399	0.294	0.296	0.965	1.406	0.320	0.319	0.961	1.415	0.348
9.1	0.271	0.976	1.392	0.290	0.294	0.972	1.400	0.316	0.317	0.968	1.408	0.343
9.2	0.269	0.983	1.386	0.286	0.291	0.979	1.393	0.311	0.314	0.975	1.401	0.338
9.3	0.267	0.989	1.380	0.282	0.289	0.985	1.387	0.307	0.312	0.981	1.395	0.333
9.4	0.265	0.995	1.374	0.278	0.287	0.991	1.381	0.302	0.309	0.987	1.389	0.328
9.5	0.264	1.002	1.369	0.274	0.285	0.998	1.375	0.298	0.307	0.994	1.383	0.324
9.6	0.262	1.008	1.363	0.271	0.283	1.004	1.370	0.294	0.305	1.000	1.377	0.319
9.7	0.260	1.014	1.358	0.267	0.281	1.010	1.364	0.291	0.302	1.006	1.371	0.315
9.8	0.258	1.020	1.353	0.264	0.279	1.016	1.359	0.287	0.300	1.012	1.366	0.311
9.9	0.256	1.026	1.348	0.261	0.277	1.022	1.354	0.283	0.298	1.018	1.360	0.307
10.0	0.255	1.032	1.343	0.258	0.275	1.028	1.349	0.280	0.296	1.024	1.355	0.303
10.1	0.253	1.038	1.338	0.255	0.273	1.034	1.344	0.276	0.294	1.030	1.350	0.299
10.2	0.252	1.043	1.333	0.252	0.271	1.040	1.339	0.273	0.292	1.036	1.345	0.296
10.3	0.250	1.049	1.329	0.249	0.270	1.045	1.334	0.270	0.290	1.042	1.340	0.292
10.4	0.249	1.055	1.324	0.246	0.268	1.051	1.329	0.267	0.288	1.047	1.335	0.289
10.5	0.247	1.060	1.320	0.243	0.266	1.057	1.325	0.264	0.287	1.053	1.331	0.285
10.6	0.246	1.066	1.315	0.241	0.265	1.062	1.321	0.261	0.285	1.059	1.326	0.282
10.7	0.244	1.071	1.311	0.238	0.263	1.068	1.316	0.258	0.283	1.064	1.322	0.279
10.8	0.243	1.077	1.307	0.236	0.262	1.073	1.312	0.255	0.281	1.070	1.318	0.276
10.9	0.242	1.082	1.303	0.233	0.260	1.079	1.308	0.253	0.280	1.075	1.313	0.273
11.0	0.240	1.087	1.299	0.231	0.259	1.084	1.304	0.250	0.278	1.080	1.309	0.270
11.1	0.239	1.092	1.295	0.229	0.258	1.089	1.300	0.248	0.277	1.086	1.305	0.267
11.2	0.238	1.098	1.292	0.226	0.256	1.094	1.296	0.245	0.275	1.091	1.301	0.265
11.3	0.237	1.103	1.288	0.224	0.255	1.100	1.292	0.243	0.274	1.096	1.297	0.262
11.4	0.235	1.108	1.284	0.222	0.253	1.105	1.289	0.240	0.272	1.101	1.294	0.259
11.5	0.234	1.113	1.281	0.220	0.252	1.110	1.285	0.238	0.271	1.106	1.290	0.257
11.6	0.233	1.118	1.277	0.218	0.251	1.115	1.282	0.236	0.269	1.111	1.286	0.254
11.7	0.232	1.123	1.274	0.216	0.250	1.120	1.278	0.234	0.268	1.116	1.283	0.252
11.8	0.231	1.128	1.271	0.214	0.248	1.125	1.275	0.232	0.267	1.121	1.279	0.250
11.9	0.230	1.132	1.267	0.212	0.247	1.129	1.271	0.229	0.265	1.126	1.276	0.247
12.0	0.229	1.137	1.264	0.210	0.246	1.134	1.268	0.227	0.264	1.131	1.272	0.245

$0.90 \le \hat{\alpha}_3 \le 1.00$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 0.90$				$\hat{\alpha}_3 = 0.95$				$\hat{\alpha}_3 = 1.00$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
4.5	3.938	1.223	3.400	6.928								
4.6	1.796	0.687	3.093	4.031								
4.7	1.347	0.610	2.868	3.037	2.917	1.001	3.167	5.527				
4.8	1.127	0.588	2.694	2.472	1.729	0.709	2.925	3.701				
4.9	0.990	0.584	2.555	2.099	1.352	0.643	2.739	2.895	2.687	0.986	3.001	4.990
5.0	0.895	0.588	2.440	1.831	1.150	0.621	2.592	2.406	1.737	0.746	2.799	3.540
5.1	0.824	0.596	2.344	1.629	1.019	0.615	2.471	2.071	1.385	0.680	2.641	2.831
5.2	0.768	0.605	2.262	1.470	0.927	0.617	2.371	1.824	1.189	0.655	2.512	2.385
5.3	0.723	0.616	2.191	1.342	0.856	0.623	2.285	1.635	1.060	0.647	2.405	2.072
5.4	0.685	0.627	2.128	1.236	0.801	0.631	2.211	1.484	0.966	0.646	2.315	1.838
5.5	0.654	0.639	2.073	1.147	0.755	0.640	2.146	1.361	0.895	0.650	2.237	1.656
5.6	0.626	0.651	2.023	1.071	0.717	0.650	2.089	1.259	0.838	0.656	2.170	1.510
5.7	0.602	0.662	1.979	1.005	0.685	0.661	2.038	1.172	0.791	0.664	2.110	1.390
5.8	0.581	0.674	1.939	0.948	0.657	0.672	1.992	1.098	0.752	0.673	2.057	1.289
5.9	0.563	0.686	1.902	0.898	0.632	0.682	1.951	1.033	0.719	0.682	2.009	1.203
6.0	0.546	0.697	1.868	0.854	0.611	0.693	1.913	0.976	0.690	0.692	1.967	1.129
6.1	0.531	0.708	1.837	0.814	0.591	0.704	1.879	0.926	0.665	0.702	1.928	1.065
6.2	0.517	0.719	1.809	0.778	0.574	0.715	1.847	0.882	0.642	0.712	1.892	1.008
6.3	0.504	0.730	1.782	0.746	0.558	0.725	1.818	0.842	0.622	0.722	1.860	0.958
6.4	0.492	0.740	1.758	0.716	0.544	0.735	1.791	0.806	0.604	0.732	1.830	0.913
6.5	0.481	0.750	1.735	0.689	0.531	0.746	1.766	0.773	0.587	0.742	1.802	0.873
6.6	0.471	0.761	1.713	0.665	0.519	0.756	1.742	0.744	0.572	0.752	1.776	0.836
6.7	0.462	0.770	1.693	0.642	0.507	0.766	1.721	0.716	0.559	0.761	1.752	0.803
6.8	0.453	0.780	1.674	0.621	0.497	0.775	1.700	0.691	0.546	0.771	1.729	0.773
6.9	0.445	0.790	1.656	0.601	0.487	0.785	1.681	0.668	0.534	0.780	1.708	0.745
7.0	0.438	0.799	1.639	0.583	0.478	0.794	1.663	0.647	0.523	0.789	1.689	0.720
7.1	0.431	0.808	1.623	0.567	0.470	0.803	1.645	0.627	0.513	0.799	1.670	0.696
7.2	0.424	0.817	1.608	0.551	0.462	0.812	1.629	0.609	0.504	0.808	1.653	0.674
7.3	0.417	0.826	1.594	0.536	0.454	0.821	1.614	0.592	0.495	0.816	1.636	0.654
7.4	0.411	0.835	1.580	0.522	0.447	0.830	1.599	0.575	0.486	0.825	1.620	0.635
7.5	0.406	0.844	1.567	0.509	0.440	0.839	1.585	0.560	0.479	0.834	1.605	0.618
7.6	0.400	0.852	1.555	0.497	0.434	0.847	1.572	0.546	0.471	0.842	1.591	0.601
7.7	0.395	0.860	1.543	0.486	0.428	0.855	1.559	0.533	0.464	0.850	1.578	0.585
7.8	0.390	0.868	1.531	0.475	0.422	0.863	1.547	0.520	0.457	0.859	1.565	0.571
7.9	0.385	0.876	1.520	0.464	0.417	0.872	1.536	0.508	0.451	0.867	1.552	0.557
8.0	0.381	0.884	1.510	0.455	0.412	0.879	1.524	0.497	0.445	0.875	1.541	0.544
8.1	0.376	0.892	1.500	0.445	0.407	0.887	1.514	0.486	0.439	0.882	1.529	0.532
8.2	0.372	0.900	1.490	0.436	0.402	0.895	1.504	0.476	0.434	0.890	1.519	0.520
8.3	0.368	0.907	1.481	0.428	0.397	0.902	1.494	0.467	0.429	0.898	1.508	0.509
8.4	0.364	0.915	1.472	0.420	0.393	0.910	1.484	0.457	0.424	0.905	1.498	0.499
8.5	0.360	0.922	1.463	0.412	0.389	0.917	1.475	0.449	0.419	0.913	1.489	0.489
8.6	0.357	0.929	1.455	0.405	0.385	0.924	1.467	0.440	0.414	0.920	1.479	0.479
8.7	0.354	0.936	1.447	0.398	0.381	0.932	1.458	0.432	0.410	0.927	1.471	0.470
8.8	0.350	0.943	1.439	0.391	0.377	0.939	1.450	0.425	0.406	0.934	1.462	0.462
8.9	0.347	0.950	1.431	0.385	0.373	0.946	1.442	0.418	0.401	0.941	1.454	0.453
9.0	0.344	0.957	1.424	0.378	0.370	0.952	1.434	0.411	0.398	0.948	1.446	0.445
9.1	0.341	0.964	1.417	0.372	0.367	0.959	1.427	0.404	0.394	0.955	1.438	0.438
9.2	0.338	0.970	1.410	0.367	0.363	0.966	1.420	0.397	0.390	0.961	1.431	0.431
9.3	0.335	0.977	1.404	0.361	0.360	0.972	1.413	0.391	0.386	0.968	1.423	0.424
9.4	0.333	0.983	1.397	0.356	0.357	0.979	1.406	0.385	0.383	0.975	1.416	0.417
9.5	0.330	0.990	1.391	0.351	0.354	0.985	1.400	0.379	0.380	0.981	1.410	0.410
9.6	0.327	0.996	1.385	0.346	0.351	0.992	1.394	0.374	0.377	0.987	1.403	0.404
9.7	0.325	1.002	1.379	0.341	0.349	0.998	1.387	0.369	0.373	0.994	1.397	0.398
9.8	0.322	1.008	1.373	0.336	0.346	1.004	1.381	0.363	0.370	1.000	1.390	0.392
9.9	0.320	1.014	1.368	0.332	0.343	1.010	1.376	0.358	0.368	1.006	1.384	0.387
10.0	0.318	1.020	1.362	0.328	0.341	1.016	1.370	0.354	0.365	1.012	1.379	0.382
10.1	0.316	1.026	1.357	0.323	0.338	1.022	1.365	0.349	0.362	1.018	1.373	0.376
10.2	0.313	1.032	1.352	0.319	0.336	1.028	1.359	0.345	0.359	1.024	1.367	0.371
10.3	0.311	1.038	1.347	0.315	0.333	1.034	1.354	0.340	0.357	1.030	1.362	0.367
10.4	0.309	1.044	1.342	0.312	0.331	1.040	1.349	0.336	0.354	1.036	1.357	0.362
10.5	0.307	1.049	1.337	0.308	0.329	1.045	1.344	0.332	0.352	1.041	1.352	0.357
10.6	0.305	1.055	1.332	0.304	0.327	1.051	1.339	0.328	0.349	1.047	1.347	0.353
10.7	0.303	1.061	1.328	0.301	0.325	1.057	1.334	0.324	0.347	1.053	1.342	0.349
10.8	0.302	1.066	1.323	0.298	0.323	1.062	1.330	0.320	0.345	1.058	1.337	0.345
10.9	0.300	1.071	1.319	0.294	0.321	1.068	1.325	0.317	0.342	1.064	1.332	0.341
11.0	0.298	1.077	1.315	0.291	0.319	1.073	1.321	0.313	0.340	1.069	1.328	0.337
11.1	0.296	1.082	1.311	0.288	0.317	1.078	1.317	0.310	0.338	1.075	1.323	0.333
11.2	0.295	1.087	1.307	0.285	0.315	1.084	1.313	0.306	0.336	1.080	1.319	0.329
11.3	0.293	1.093	1.303	0.282	0.313	1.089	1.308	0.303	0.334	1.085	1.315	0.326
11.4	0.291	1.098	1.299	0.279	0.311	1.094	1.304	0.300	0.332	1.090	1.311	0.322
11.5	0.290	1.103	1.295	0.276	0.310	1.099	1.301	0.297	0.330	1.096	1.307	0.319
11.6	0.288	1.108	1.291	0.274	0.308	1.104	1.297	0.294	0.328	1.101	1.303	0.315
11.7	0.287	1.113	1.288	0.271	0.306	1.109	1.293	0.291	0.327	1.106	1.299	0.312
11.8	0.285	1.118	1.284	0.268	0.305	1.114	1.289	0.288	0.325	1.111	1.295	0.309
11.9	0.284	1.123	1.280	0.266	0.303	1.119	1.286	0.286	0.323	1.116	1.291	0.306
12.0	0.282	1.128	1.277	0.264	0.301	1.124	1.282	0.283	0.321	1.121	1.288	0.303
12.1	0.281	1.133	1.274	0.261	0.300	1.129	1.279	0.280	0.320	1.126	1.284	0.300
12.2	0.280	1.137	1.270	0.259	0.298	1.134	1.275	0.278	0.318	1.130	1.281	0.297
12.3	0.278	1.142	1.267	0.257	0.297	1.139	1.272	0.275	0.316	1.135	1.277	0.295
12.4	0.277	1.147	1.264	0.254	0.296	1.144	1.269	0.273	0.315	1.140	1.274	0.292

$1.05 \leq \hat{\alpha}_3 \leq 1.15$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.05$				$\hat{\alpha}_3 = 1.10$				$\hat{\alpha}_3 = 1.15$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
5.1	2.710	1.039	2.878	4.795								
5.2	1.799	0.795	2.704	3.490								
5.3	1.445	0.722	2.564	2.826	2.924	1.151	2.785	4.828				
5.4	1.244	0.692	2.449	2.401	1.915	0.859	2.630	3.529				
5.5	1.111	0.680	2.353	2.098	1.533	0.770	2.505	2.872	3.421	1.363	2.714	5.100
5.6	1.014	0.676	2.270	1.871	1.317	0.732	2.400	2.450	2.101	0.946	2.574	3.656
5.7	0.940	0.678	2.199	1.692	1.175	0.715	2.311	2.149	1.654	0.830	2.458	2.968
5.8	0.881	0.682	2.136	1.547	1.072	0.708	2.234	1.921	1.411	0.779	2.361	2.532
5.9	0.832	0.688	2.080	1.428	0.993	0.706	2.167	1.741	1.253	0.754	2.278	2.223
6.0	0.791	0.696	2.030	1.327	0.930	0.709	2.108	1.595	1.140	0.742	2.206	1.989
6.1	0.756	0.704	1.986	1.241	0.878	0.713	2.056	1.474	1.054	0.737	2.143	1.805
6.2	0.726	0.713	1.945	1.167	0.835	0.719	2.009	1.372	0.986	0.737	2.087	1.655
6.3	0.700	0.722	1.908	1.102	0.798	0.726	1.966	1.285	0.930	0.739	2.037	1.531
6.4	0.676	0.731	1.875	1.044	0.766	0.733	1.928	1.209	0.883	0.743	1.992	1.426
6.5	0.655	0.740	1.843	0.993	0.737	0.741	1.893	1.143	0.844	0.748	1.951	1.336
6.6	0.636	0.749	1.815	0.948	0.712	0.749	1.860	1.084	0.809	0.754	1.914	1.258
6.7	0.619	0.758	1.788	0.907	0.690	0.758	1.830	1.032	0.779	0.761	1.880	1.190
6.8	0.603	0.768	1.763	0.869	0.670	0.766	1.803	0.986	0.752	0.768	1.849	1.130
6.9	0.588	0.777	1.740	0.836	0.652	0.775	1.777	0.943	0.728	0.776	1.820	1.076
7.0	0.575	0.786	1.719	0.805	0.635	0.783	1.753	0.905	0.707	0.783	1.793	1.027
7.1	0.563	0.795	1.698	0.776	0.620	0.792	1.731	0.870	0.687	0.791	1.768	0.984
7.2	0.551	0.803	1.679	0.750	0.606	0.800	1.710	0.839	0.669	0.799	1.745	0.944
7.3	0.540	0.812	1.661	0.726	0.593	0.809	1.690	0.809	0.653	0.807	1.723	0.908
7.4	0.531	0.821	1.644	0.703	0.580	0.817	1.671	0.782	0.638	0.815	1.702	0.875
7.5	0.521	0.829	1.628	0.682	0.569	0.826	1.654	0.757	0.624	0.823	1.683	0.845
7.6	0.512	0.838	1.613	0.663	0.559	0.834	1.637	0.734	0.611	0.831	1.665	0.817
7.7	0.504	0.846	1.598	0.645	0.549	0.842	1.622	0.713	0.599	0.839	1.648	0.791
7.8	0.496	0.854	1.584	0.628	0.539	0.850	1.607	0.692	0.588	0.847	1.632	0.767
7.9	0.489	0.862	1.571	0.612	0.531	0.858	1.593	0.674	0.578	0.855	1.616	0.744
8.0	0.482	0.870	1.559	0.597	0.522	0.866	1.579	0.656	0.568	0.862	1.602	0.724
8.1	0.475	0.878	1.547	0.582	0.514	0.874	1.566	0.639	0.558	0.870	1.588	0.704
8.2	0.469	0.886	1.535	0.569	0.507	0.881	1.554	0.624	0.550	0.878	1.575	0.686
8.3	0.463	0.893	1.524	0.556	0.500	0.889	1.542	0.609	0.541	0.885	1.562	0.668
8.4	0.457	0.901	1.514	0.544	0.493	0.896	1.531	0.595	0.533	0.892	1.550	0.652
8.5	0.451	0.908	1.504	0.533	0.487	0.904	1.520	0.582	0.526	0.900	1.538	0.637
8.6	0.446	0.915	1.494	0.522	0.481	0.911	1.510	0.569	0.519	0.907	1.527	0.622
8.7	0.441	0.922	1.484	0.512	0.475	0.918	1.500	0.557	0.512	0.914	1.517	0.608
8.8	0.436	0.930	1.475	0.502	0.469	0.925	1.490	0.546	0.506	0.921	1.507	0.595
8.9	0.432	0.937	1.467	0.492	0.464	0.932	1.481	0.535	0.499	0.928	1.497	0.583
9.0	0.427	0.943	1.458	0.483	0.459	0.939	1.472	0.525	0.493	0.935	1.487	0.571
9.1	0.423	0.950	1.450	0.475	0.454	0.946	1.463	0.515	0.488	0.942	1.478	0.560
9.2	0.419	0.957	1.442	0.467	0.449	0.953	1.455	0.506	0.482	0.948	1.469	0.549
9.3	0.415	0.964	1.435	0.459	0.445	0.959	1.447	0.497	0.477	0.955	1.461	0.539
9.4	0.411	0.970	1.427	0.451	0.440	0.966	1.440	0.488	0.472	0.962	1.453	0.529
9.5	0.407	0.977	1.420	0.444	0.436	0.972	1.432	0.480	0.467	0.968	1.445	0.520
9.6	0.403	0.983	1.413	0.437	0.432	0.979	1.425	0.472	0.463	0.975	1.437	0.511
9.7	0.400	0.989	1.407	0.430	0.428	0.985	1.418	0.465	0.458	0.981	1.430	0.503
9.8	0.396	0.996	1.400	0.424	0.424	0.991	1.411	0.458	0.454	0.987	1.423	0.494
9.9	0.393	1.002	1.394	0.418	0.421	0.998	1.404	0.451	0.450	0.993	1.416	0.487
10.0	0.390	1.008	1.388	0.412	0.417	1.004	1.398	0.444	0.446	1.000	1.409	0.479
10.1	0.387	1.014	1.382	0.406	0.413	1.010	1.392	0.437	0.442	1.006	1.403	0.472
10.2	0.384	1.020	1.376	0.400	0.410	1.016	1.386	0.431	0.438	1.012	1.396	0.465
10.3	0.381	1.026	1.371	0.395	0.407	1.022	1.380	0.425	0.434	1.017	1.390	0.458
10.4	0.378	1.032	1.365	0.390	0.404	1.027	1.374	0.419	0.431	1.023	1.384	0.451
10.5	0.376	1.037	1.360	0.384	0.401	1.033	1.369	0.414	0.427	1.029	1.378	0.445
10.6	0.373	1.043	1.355	0.380	0.398	1.039	1.363	0.408	0.424	1.035	1.373	0.439
10.7	0.370	1.049	1.349	0.375	0.395	1.045	1.358	0.403	0.421	1.041	1.367	0.433
10.8	0.368	1.054	1.345	0.370	0.392	1.050	1.353	0.398	0.418	1.046	1.362	0.427
10.9	0.365	1.060	1.340	0.366	0.389	1.056	1.348	0.393	0.415	1.052	1.357	0.422
11.0	0.363	1.065	1.335	0.362	0.387	1.061	1.343	0.388	0.412	1.057	1.352	0.416
11.1	0.361	1.071	1.330	0.357	0.384	1.067	1.338	0.383	0.409	1.063	1.347	0.411
11.2	0.358	1.076	1.326	0.353	0.381	1.072	1.334	0.379	0.406	1.068	1.342	0.406
11.3	0.356	1.081	1.322	0.349	0.379	1.077	1.329	0.375	0.403	1.074	1.337	0.401
11.4	0.354	1.087	1.317	0.345	0.377	1.083	1.325	0.370	0.400	1.079	1.332	0.397
11.5	0.352	1.092	1.313	0.342	0.374	1.088	1.320	0.366	0.398	1.084	1.328	0.392
11.6	0.350	1.097	1.309	0.338	0.372	1.093	1.316	0.362	0.395	1.089	1.324	0.388
11.7	0.348	1.102	1.305	0.335	0.370	1.098	1.312	0.358	0.393	1.094	1.319	0.383
11.8	0.346	1.107	1.301	0.331	0.368	1.103	1.308	0.354	0.390	1.100	1.315	0.379
11.9	0.344	1.112	1.297	0.328	0.365	1.108	1.304	0.351	0.388	1.105	1.311	0.375
12.0	0.342	1.117	1.294	0.325	0.363	1.113	1.300	0.347	0.386	1.110	1.307	0.371
12.1	0.340	1.122	1.290	0.321	0.361	1.118	1.296	0.344	0.384	1.115	1.303	0.367
12.2	0.338	1.127	1.286	0.318	0.359	1.123	1.292	0.340	0.381	1.119	1.299	0.364
12.3	0.336	1.132	1.283	0.315	0.357	1.128	1.289	0.337	0.379	1.124	1.295	0.360
12.4	0.335	1.136	1.279	0.312	0.355	1.133	1.285	0.334	0.377	1.129	1.292	0.356
12.5	0.333	1.141	1.276	0.309	0.354	1.138	1.282	0.330	0.375	1.134	1.288	0.353
12.6	0.331	1.146	1.272	0.307	0.352	1.142	1.278	0.327	0.373	1.139	1.284	0.349
12.7	0.330	1.151	1.269	0.304	0.350	1.147	1.275	0.324	0.371	1.143	1.281	0.346
12.8	0.328	1.155	1.266	0.301	0.348	1.152	1.272	0.321	0.369	1.148	1.278	0.343
12.9	0.326	1.160	1.263	0.299	0.346	1.156	1.268	0.319	0.367	1.153	1.274	0.340
13.0	0.325	1.164	1.260	0.296	0.345	1.161	1.265	0.316	0.365	1.157	1.271	0.337

$1.20 \leq \hat{\alpha}_3 \leq 1.30$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.20$				$\hat{\alpha}_3 = 1.25$				$\hat{\alpha}_3 = 1.30$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
5.7	4.706	1.877	2.661	5.812								
5.8	2.400	1.075	2.531	3.892								
5.9	1.825	0.907	2.423	3.122								
6.0	1.535	0.836	2.331	2.652	2.934	1.297	2.499	4.303				
6.1	1.352	0.800	2.252	2.324	2.076	1.014	2.396	3.354				
6.2	1.223	0.781	2.184	2.078	1.700	0.908	2.309	2.819	4.234	1.835	2.477	5.129
6.3	1.127	0.771	2.124	1.885	1.477	0.855	2.234	2.458	2.483	1.185	2.378	3.708
6.4	1.051	0.767	2.070	1.728	1.326	0.826	2.168	2.191	1.935	1.008	2.294	3.052
6.5	0.990	0.766	2.022	1.599	1.214	0.810	2.109	1.983	1.643	0.926	2.220	2.634
6.6	0.939	0.768	1.978	1.489	1.128	0.801	2.057	1.817	1.455	0.881	2.156	2.334
6.7	0.895	0.772	1.939	1.396	1.059	0.797	2.011	1.679	1.321	0.855	2.099	2.106
6.8	0.858	0.776	1.903	1.314	1.001	0.796	1.968	1.563	1.220	0.840	2.049	1.923
6.9	0.825	0.782	1.870	1.243	0.953	0.797	1.930	1.464	1.140	0.831	2.003	1.774
7.0	0.796	0.788	1.840	1.180	0.912	0.800	1.895	1.378	1.074	0.827	1.962	1.650
7.1	0.770	0.794	1.811	1.124	0.876	0.804	1.863	1.303	1.020	0.825	1.924	1.543
7.2	0.747	0.801	1.785	1.074	0.844	0.808	1.833	1.237	0.973	0.826	1.890	1.452
7.3	0.726	0.808	1.761	1.028	0.816	0.814	1.805	1.178	0.933	0.828	1.858	1.372
7.4	0.707	0.816	1.738	0.987	0.790	0.820	1.780	1.125	0.898	0.831	1.829	1.301
7.5	0.689	0.823	1.717	0.949	0.768	0.826	1.756	1.077	0.866	0.835	1.802	1.238
7.6	0.673	0.830	1.697	0.915	0.747	0.832	1.734	1.034	0.839	0.839	1.776	1.182
7.7	0.658	0.838	1.678	0.883	0.728	0.839	1.713	0.994	0.813	0.845	1.753	1.131
7.8	0.644	0.845	1.660	0.854	0.710	0.846	1.693	0.958	0.791	0.850	1.731	1.086
7.9	0.631	0.853	1.644	0.827	0.694	0.853	1.675	0.925	0.770	0.856	1.710	1.044
8.0	0.619	0.860	1.628	0.802	0.679	0.860	1.657	0.894	0.751	0.862	1.691	1.006
8.1	0.608	0.867	1.613	0.779	0.666	0.866	1.640	0.866	0.734	0.868	1.672	0.970
8.2	0.598	0.875	1.598	0.757	0.653	0.873	1.625	0.840	0.717	0.874	1.655	0.938
8.3	0.588	0.882	1.585	0.736	0.641	0.880	1.610	0.815	0.702	0.881	1.639	0.908
8.4	0.578	0.889	1.571	0.717	0.629	0.887	1.596	0.792	0.688	0.887	1.623	0.880
8.5	0.569	0.896	1.559	0.699	0.619	0.894	1.582	0.771	0.675	0.894	1.608	0.854
8.6	0.561	0.904	1.547	0.682	0.609	0.901	1.569	0.751	0.663	0.900	1.594	0.830
8.7	0.553	0.911	1.536	0.666	0.599	0.908	1.557	0.732	0.652	0.907	1.581	0.808
8.8	0.546	0.918	1.525	0.651	0.590	0.915	1.545	0.714	0.641	0.913	1.568	0.787
8.9	0.538	0.924	1.514	0.637	0.582	0.921	1.534	0.697	0.631	0.920	1.556	0.767
9.0	0.531	0.931	1.504	0.623	0.574	0.928	1.523	0.681	0.621	0.926	1.544	0.748
9.1	0.525	0.938	1.495	0.610	0.566	0.935	1.513	0.666	0.612	0.933	1.533	0.730
9.2	0.519	0.945	1.485	0.598	0.559	0.941	1.503	0.652	0.603	0.939	1.522	0.714
9.3	0.513	0.951	1.476	0.586	0.552	0.948	1.493	0.638	0.595	0.945	1.512	0.698
9.4	0.507	0.958	1.468	0.575	0.545	0.954	1.484	0.626	0.587	0.952	1.502	0.683
9.5	0.501	0.964	1.459	0.564	0.538	0.961	1.475	0.613	0.580	0.958	1.492	0.668
9.6	0.496	0.971	1.451	0.554	0.532	0.967	1.466	0.602	0.572	0.964	1.483	0.655
9.7	0.491	0.977	1.443	0.544	0.526	0.973	1.458	0.590	0.566	0.970	1.474	0.642
9.8	0.486	0.983	1.436	0.535	0.521	0.980	1.450	0.580	0.559	0.977	1.466	0.630
9.9	0.481	0.989	1.428	0.526	0.515	0.986	1.442	0.569	0.553	0.983	1.458	0.618
10.0	0.477	0.996	1.421	0.517	0.510	0.992	1.435	0.560	0.547	0.989	1.450	0.607
10.1	0.472	1.002	1.414	0.509	0.505	0.998	1.428	0.550	0.541	0.995	1.442	0.596
10.2	0.468	1.008	1.408	0.501	0.500	1.004	1.420	0.541	0.535	1.001	1.434	0.586
10.3	0.464	1.014	1.401	0.494	0.495	1.010	1.414	0.533	0.530	1.006	1.427	0.576
10.4	0.460	1.019	1.395	0.486	0.491	1.016	1.407	0.524	0.525	1.012	1.420	0.566
10.5	0.456	1.025	1.389	0.479	0.486	1.021	1.401	0.516	0.520	1.018	1.413	0.557
10.6	0.452	1.031	1.383	0.472	0.482	1.027	1.394	0.509	0.515	1.024	1.407	0.549
10.7	0.448	1.037	1.377	0.466	0.478	1.033	1.388	0.501	0.510	1.029	1.400	0.540
10.8	0.445	1.042	1.372	0.459	0.474	1.038	1.382	0.494	0.506	1.035	1.394	0.532
10.9	0.442	1.048	1.366	0.453	0.470	1.044	1.377	0.487	0.501	1.040	1.388	0.524
11.0	0.438	1.053	1.361	0.447	0.467	1.050	1.371	0.480	0.497	1.046	1.382	0.517
11.1	0.435	1.059	1.356	0.441	0.463	1.055	1.366	0.474	0.493	1.051	1.377	0.509
11.2	0.432	1.064	1.351	0.436	0.459	1.060	1.360	0.468	0.489	1.057	1.371	0.502
11.3	0.429	1.070	1.346	0.430	0.456	1.066	1.355	0.462	0.485	1.062	1.366	0.496
11.4	0.426	1.075	1.341	0.425	0.453	1.071	1.350	0.456	0.481	1.068	1.360	0.489
11.5	0.423	1.080	1.336	0.420	0.449	1.076	1.345	0.450	0.478	1.073	1.355	0.483
11.6	0.420	1.085	1.332	0.415	0.446	1.082	1.341	0.445	0.474	1.078	1.350	0.477
11.7	0.417	1.091	1.327	0.410	0.443	1.087	1.336	0.439	0.471	1.083	1.345	0.471
11.8	0.415	1.096	1.323	0.406	0.440	1.092	1.331	0.434	0.467	1.088	1.340	0.465
11.9	0.412	1.101	1.319	0.401	0.437	1.097	1.327	0.429	0.464	1.093	1.336	0.459
12.0	0.409	1.106	1.314	0.397	0.434	1.102	1.322	0.424	0.461	1.098	1.331	0.454
12.1	0.407	1.111	1.310	0.393	0.432	1.107	1.318	0.420	0.458	1.103	1.327	0.449
12.2	0.404	1.116	1.306	0.388	0.429	1.112	1.314	0.415	0.455	1.108	1.322	0.444
12.3	0.402	1.121	1.302	0.384	0.426	1.117	1.310	0.411	0.452	1.113	1.318	0.439
12.4	0.400	1.125	1.299	0.380	0.424	1.122	1.306	0.406	0.449	1.118	1.314	0.434
12.5	0.398	1.130	1.295	0.377	0.421	1.127	1.302	0.402	0.446	1.123	1.310	0.429
12.6	0.395	1.135	1.291	0.373	0.419	1.131	1.298	0.398	0.443	1.128	1.306	0.425
12.7	0.393	1.140	1.287	0.369	0.416	1.136	1.295	0.394	0.441	1.132	1.302	0.420
12.8	0.391	1.144	1.284	0.366	0.414	1.141	1.291	0.390	0.438	1.137	1.298	0.416
12.9	0.389	1.149	1.280	0.362	0.412	1.145	1.287	0.386	0.436	1.142	1.295	0.412
13.0	0.387	1.154	1.277	0.359	0.409	1.150	1.284	0.382	0.433	1.146	1.291	0.407
13.1	0.385	1.158	1.274	0.356	0.407	1.155	1.280	0.379	0.431	1.151	1.287	0.403
13.2	0.383	1.163	1.270	0.352	0.405	1.159	1.277	0.375	0.428	1.156	1.284	0.400
13.3	0.381	1.167	1.267	0.349	0.403	1.164	1.274	0.372	0.426	1.160	1.280	0.396
13.4	0.379	1.172	1.264	0.346	0.401	1.168	1.270	0.368	0.424	1.165	1.277	0.392
13.5	0.377	1.176	1.261	0.343	0.399	1.173	1.267	0.365	0.421	1.169	1.274	0.388
13.6	0.375	1.181	1.258	0.340	0.397	1.177	1.264	0.362	0.419	1.174	1.270	0.385

$1.35 \leq \hat{\alpha}_3 \leq 1.45$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.35$				$\hat{\alpha}_3 = 1.40$				$\hat{\alpha}_3 = 1.45$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
6.5	3.299	1.530	2.367	4.318								
6.6	2.302	1.163	2.284	3.390								
6.7	1.876	1.025	2.213	2.872	6.802	3.048	2.363	5.972				
6.8	1.626	0.954	2.150	2.520	2.997	1.462	2.281	3.939				
6.9	1.457	0.913	2.094	2.258	2.237	1.179	2.210	3.210				
7.0	1.333	0.888	2.044	2.054	1.866	1.056	2.148	2.766	5.311	2.489	2.284	5.202
7.1	1.237	0.872	1.999	1.889	1.637	0.989	2.092	2.453	2.914	1.475	2.213	3.750
7.2	1.161	0.862	1.958	1.752	1.477	0.948	2.043	2.216	2.242	1.218	2.150	3.116
7.3	1.097	0.857	1.921	1.636	1.358	0.922	1.998	2.028	1.893	1.098	2.095	2.713
7.4	1.044	0.855	1.887	1.536	1.265	0.905	1.958	1.874	1.671	1.030	2.045	2.424
7.5	0.998	0.854	1.856	1.450	1.190	0.895	1.921	1.745	1.514	0.987	2.000	2.201
7.6	0.958	0.855	1.827	1.374	1.127	0.888	1.887	1.635	1.396	0.960	1.960	2.023
7.7	0.923	0.858	1.800	1.306	1.074	0.885	1.856	1.540	1.303	0.941	1.923	1.876
7.8	0.892	0.861	1.775	1.246	1.028	0.883	1.827	1.457	1.227	0.929	1.889	1.752
7.9	0.864	0.865	1.751	1.192	0.988	0.883	1.800	1.384	1.163	0.921	1.858	1.646
8.0	0.839	0.869	1.730	1.143	0.953	0.885	1.775	1.318	1.109	0.915	1.829	1.553
8.1	0.816	0.874	1.709	1.098	0.921	0.887	1.752	1.260	1.063	0.913	1.802	1.472
8.2	0.795	0.879	1.690	1.058	0.893	0.890	1.730	1.207	1.022	0.912	1.777	1.401
8.3	0.776	0.884	1.671	1.020	0.868	0.894	1.709	1.159	0.986	0.912	1.754	1.337
8.4	0.759	0.890	1.654	0.986	0.844	0.898	1.690	1.115	0.954	0.913	1.732	1.279
8.5	0.742	0.896	1.638	0.954	0.823	0.902	1.672	1.075	0.925	0.915	1.711	1.227
8.6	0.727	0.902	1.622	0.925	0.804	0.907	1.655	1.038	0.899	0.918	1.692	1.179
8.7	0.713	0.908	1.608	0.897	0.785	0.912	1.639	1.004	0.875	0.922	1.674	1.136
8.8	0.699	0.914	1.594	0.872	0.769	0.917	1.623	0.973	0.853	0.925	1.657	1.096
8.9	0.687	0.920	1.580	0.848	0.753	0.922	1.608	0.943	0.833	0.929	1.640	1.060
9.0	0.675	0.926	1.568	0.825	0.739	0.928	1.594	0.916	0.815	0.934	1.625	1.026
9.1	0.664	0.932	1.556	0.804	0.725	0.933	1.581	0.891	0.797	0.938	1.610	0.994
9.2	0.654	0.938	1.544	0.784	0.712	0.939	1.568	0.867	0.781	0.943	1.596	0.965
9.3	0.644	0.944	1.533	0.766	0.700	0.945	1.556	0.844	0.766	0.948	1.583	0.938
9.4	0.635	0.950	1.522	0.748	0.689	0.950	1.545	0.823	0.752	0.953	1.570	0.912
9.5	0.626	0.956	1.512	0.731	0.678	0.956	1.533	0.803	0.739	0.958	1.558	0.888
9.6	0.617	0.962	1.502	0.715	0.668	0.962	1.523	0.785	0.727	0.964	1.546	0.866
9.7	0.609	0.968	1.492	0.700	0.658	0.968	1.512	0.767	0.715	0.969	1.535	0.844
9.8	0.601	0.974	1.483	0.686	0.649	0.973	1.503	0.750	0.704	0.974	1.524	0.824
9.9	0.594	0.980	1.474	0.672	0.640	0.979	1.493	0.734	0.693	0.979	1.514	0.806
10.0	0.587	0.986	1.466	0.659	0.632	0.985	1.484	0.719	0.683	0.985	1.504	0.788
10.1	0.580	0.992	1.458	0.647	0.624	0.990	1.475	0.705	0.674	0.990	1.495	0.771
10.2	0.574	0.998	1.450	0.635	0.617	0.996	1.467	0.691	0.665	0.996	1.485	0.755
10.3	0.568	1.004	1.442	0.624	0.609	1.002	1.458	0.678	0.656	1.001	1.477	0.739
10.4	0.562	1.009	1.435	0.613	0.602	1.007	1.450	0.665	0.648	1.006	1.468	0.725
10.5	0.556	1.015	1.427	0.603	0.596	1.013	1.443	0.653	0.640	1.012	1.460	0.711
10.6	0.550	1.021	1.420	0.593	0.589	1.018	1.435	0.642	0.633	1.017	1.452	0.697
10.7	0.545	1.026	1.414	0.583	0.583	1.024	1.428	0.631	0.625	1.022	1.444	0.685
10.8	0.540	1.032	1.407	0.574	0.577	1.029	1.421	0.620	0.619	1.028	1.437	0.673
10.9	0.535	1.037	1.401	0.565	0.571	1.035	1.414	0.610	0.612	1.033	1.429	0.661
11.0	0.530	1.043	1.394	0.557	0.566	1.040	1.408	0.601	0.605	1.038	1.422	0.650
11.1	0.525	1.048	1.388	0.548	0.561	1.045	1.401	0.591	0.599	1.043	1.416	0.639
11.2	0.521	1.054	1.383	0.540	0.555	1.051	1.395	0.582	0.593	1.049	1.409	0.629
11.3	0.516	1.059	1.377	0.533	0.550	1.056	1.389	0.574	0.588	1.054	1.403	0.619
11.4	0.512	1.064	1.371	0.525	0.546	1.061	1.383	0.565	0.582	1.059	1.396	0.609
11.5	0.508	1.069	1.366	0.518	0.541	1.066	1.378	0.557	0.577	1.064	1.390	0.600
11.6	0.504	1.075	1.361	0.511	0.536	1.072	1.372	0.549	0.572	1.069	1.384	0.591
11.7	0.500	1.080	1.355	0.505	0.532	1.077	1.367	0.542	0.567	1.074	1.379	0.583
11.8	0.496	1.085	1.350	0.498	0.528	1.082	1.361	0.535	0.562	1.079	1.373	0.575
11.9	0.493	1.090	1.346	0.492	0.524	1.087	1.356	0.528	0.557	1.084	1.368	0.567
12.0	0.489	1.095	1.341	0.486	0.520	1.092	1.351	0.521	0.552	1.089	1.362	0.559
12.1	0.486	1.100	1.336	0.480	0.516	1.097	1.346	0.514	0.548	1.094	1.357	0.552
12.2	0.482	1.105	1.332	0.474	0.512	1.102	1.342	0.508	0.544	1.099	1.352	0.545
12.3	0.479	1.110	1.327	0.469	0.508	1.107	1.337	0.502	0.540	1.104	1.347	0.538
12.4	0.476	1.115	1.323	0.464	0.505	1.111	1.332	0.496	0.536	1.109	1.343	0.531
12.5	0.473	1.119	1.319	0.458	0.501	1.116	1.328	0.490	0.532	1.113	1.338	0.524
12.6	0.470	1.124	1.314	0.453	0.498	1.121	1.324	0.484	0.528	1.118	1.333	0.518
12.7	0.467	1.129	1.310	0.448	0.494	1.126	1.319	0.479	0.524	1.123	1.329	0.512
12.8	0.464	1.134	1.306	0.444	0.491	1.130	1.315	0.474	0.520	1.128	1.325	0.506
12.9	0.461	1.138	1.303	0.439	0.488	1.135	1.311	0.468	0.517	1.132	1.320	0.500
13.0	0.458	1.143	1.299	0.434	0.485	1.140	1.307	0.463	0.513	1.137	1.316	0.495
13.1	0.456	1.148	1.295	0.430	0.482	1.144	1.303	0.458	0.510	1.141	1.312	0.489
13.2	0.453	1.152	1.291	0.426	0.479	1.149	1.299	0.454	0.507	1.146	1.308	0.484
13.3	0.450	1.157	1.288	0.421	0.476	1.153	1.296	0.449	0.504	1.150	1.304	0.479
13.4	0.448	1.161	1.284	0.417	0.473	1.158	1.292	0.444	0.500	1.155	1.300	0.474
13.5	0.445	1.166	1.281	0.413	0.471	1.162	1.288	0.440	0.497	1.159	1.297	0.469
13.6	0.443	1.170	1.277	0.409	0.468	1.167	1.285	0.436	0.494	1.164	1.293	0.464
13.7	0.441	1.174	1.274	0.406	0.465	1.171	1.281	0.432	0.492	1.168	1.289	0.459
13.8	0.438	1.179	1.271	0.402	0.463	1.176	1.278	0.427	0.489	1.172	1.286	0.455
13.9	0.436	1.183	1.268	0.398	0.460	1.180	1.275	0.423	0.486	1.177	1.282	0.450
14.0	0.434	1.187	1.264	0.395	0.458	1.184	1.271	0.420	0.483	1.181	1.279	0.446
14.1	0.432	1.192	1.261	0.391	0.455	1.188	1.268	0.416	0.481	1.185	1.276	0.442
14.2	0.429	1.196	1.258	0.388	0.453	1.193	1.265	0.412	0.478	1.190	1.272	0.438
14.3	0.427	1.200	1.255	0.385	0.451	1.197	1.262	0.408	0.475	1.194	1.269	0.434
14.4	0.425	1.204	1.252	0.381	0.448	1.201	1.259	0.405	0.473	1.198	1.266	0.430

$1.50 \le \hat{\alpha}_3 \le 1.60$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.50$				$\hat{\alpha}_3 = 1.55$				$\hat{\alpha}_3 = 1.60$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
7.3	5.126	2.482	2.220	4.968								
7.4	2.964	1.541	2.157	3.680								
7.5	2.302	1.278	2.101	3.085								
7.6	1.952	1.152	2.051	2.703	5.615	2.781	2.169	5.036				
7.7	1.727	1.078	2.006	2.425	3.134	1.659	2.112	3.704				
7.8	1.567	1.032	1.965	2.209	2.419	1.361	2.061	3.110				
7.9	1.446	1.001	1.928	2.036	2.044	1.219	2.015	2.729	7.396	3.716	2.127	5.507
8.0	1.350	0.979	1.894	1.892	1.806	1.136	1.974	2.453	3.465	1.854	2.075	3.828
8.1	1.272	0.965	1.862	1.771	1.637	1.083	1.936	2.239	2.604	1.477	2.028	3.189
8.2	1.207	0.955	1.833	1.667	1.509	1.047	1.901	2.066	2.178	1.304	1.985	2.792
8.3	1.151	0.948	1.806	1.576	1.408	1.022	1.869	1.923	1.912	1.205	1.946	2.508
8.4	1.103	0.943	1.781	1.496	1.326	1.004	1.840	1.802	1.727	1.142	1.911	2.289
8.5	1.061	0.941	1.757	1.425	1.258	0.991	1.812	1.698	1.588	1.099	1.878	2.114
8.6	1.024	0.940	1.735	1.361	1.199	0.982	1.787	1.607	1.479	1.069	1.848	1.968
8.7	0.990	0.940	1.715	1.304	1.149	0.976	1.763	1.527	1.391	1.047	1.821	1.845
8.8	0.960	0.942	1.695	1.252	1.105	0.972	1.741	1.455	1.318	1.031	1.795	1.739
8.9	0.933	0.943	1.677	1.204	1.066	0.970	1.720	1.391	1.255	1.019	1.770	1.647
9.0	0.909	0.946	1.660	1.161	1.031	0.968	1.700	1.334	1.202	1.011	1.748	1.565
9.1	0.886	0.949	1.643	1.121	1.000	0.968	1.682	1.281	1.155	1.005	1.727	1.493
9.2	0.865	0.952	1.628	1.084	0.971	0.969	1.664	1.233	1.113	1.001	1.707	1.428
9.3	0.846	0.956	1.613	1.050	0.945	0.971	1.648	1.190	1.076	0.998	1.688	1.369
9.4	0.828	0.960	1.599	1.018	0.922	0.973	1.632	1.149	1.043	0.997	1.670	1.316
9.5	0.811	0.964	1.586	0.989	0.900	0.976	1.617	1.112	1.013	0.996	1.653	1.267
9.6	0.796	0.968	1.573	0.962	0.880	0.979	1.603	1.078	0.986	0.997	1.637	1.223
9.7	0.781	0.973	1.560	0.936	0.861	0.982	1.589	1.046	0.961	0.998	1.622	1.182
9.8	0.768	0.978	1.549	0.912	0.844	0.985	1.576	1.016	0.938	1.000	1.608	1.144
9.9	0.755	0.982	1.537	0.889	0.828	0.989	1.564	0.988	0.917	1.002	1.594	1.109
10.0	0.743	0.987	1.527	0.867	0.812	0.993	1.552	0.962	0.897	1.004	1.581	1.076
10.1	0.731	0.992	1.516	0.847	0.798	0.997	1.541	0.937	0.879	1.007	1.568	1.046
10.2	0.720	0.997	1.506	0.828	0.785	1.002	1.530	0.914	0.862	1.010	1.556	1.017
10.3	0.710	1.002	1.497	0.810	0.772	1.006	1.519	0.892	0.846	1.014	1.545	0.990
10.4	0.700	1.007	1.488	0.793	0.760	1.010	1.509	0.872	0.831	1.017	1.534	0.965
10.5	0.691	1.012	1.479	0.776	0.748	1.015	1.500	0.852	0.817	1.021	1.523	0.942
10.6	0.682	1.017	1.470	0.761	0.738	1.020	1.490	0.834	0.803	1.025	1.513	0.919
10.7	0.673	1.022	1.462	0.746	0.727	1.024	1.482	0.816	0.791	1.029	1.504	0.898
10.8	0.665	1.027	1.454	0.732	0.718	1.029	1.473	0.800	0.779	1.033	1.494	0.878
10.9	0.657	1.032	1.446	0.718	0.708	1.034	1.465	0.784	0.767	1.037	1.485	0.859
11.0	0.650	1.038	1.439	0.705	0.699	1.038	1.456	0.768	0.756	1.042	1.476	0.842
11.1	0.642	1.043	1.431	0.693	0.691	1.043	1.449	0.754	0.746	1.046	1.468	0.824
11.2	0.635	1.048	1.424	0.681	0.683	1.048	1.441	0.740	0.736	1.050	1.460	0.808
11.3	0.629	1.053	1.417	0.670	0.675	1.053	1.434	0.727	0.727	1.055	1.452	0.793
11.4	0.622	1.058	1.411	0.659	0.667	1.057	1.427	0.714	0.718	1.059	1.444	0.778
11.5	0.616	1.063	1.404	0.648	0.660	1.062	1.420	0.702	0.710	1.064	1.437	0.764
11.6	0.610	1.068	1.398	0.638	0.653	1.067	1.413	0.691	0.701	1.068	1.430	0.750
11.7	0.604	1.072	1.392	0.629	0.646	1.072	1.407	0.680	0.693	1.073	1.423	0.737
11.8	0.599	1.077	1.386	0.619	0.640	1.076	1.400	0.669	0.686	1.077	1.416	0.725
11.9	0.594	1.082	1.380	0.610	0.634	1.081	1.394	0.658	0.679	1.082	1.409	0.713
12.0	0.588	1.087	1.375	0.601	0.628	1.086	1.388	0.649	0.672	1.086	1.403	0.701
12.1	0.583	1.092	1.369	0.593	0.622	1.091	1.383	0.639	0.665	1.091	1.397	0.690
12.2	0.578	1.097	1.364	0.585	0.616	1.095	1.377	0.630	0.658	1.095	1.391	0.680
12.3	0.574	1.102	1.359	0.577	0.611	1.100	1.371	0.621	0.652	1.100	1.385	0.670
12.4	0.569	1.106	1.354	0.570	0.606	1.105	1.366	0.612	0.646	1.104	1.379	0.660
12.5	0.565	1.111	1.349	0.562	0.601	1.109	1.361	0.604	0.640	1.108	1.374	0.650
12.6	0.560	1.116	1.344	0.555	0.596	1.114	1.356	0.596	0.634	1.113	1.368	0.641
12.7	0.556	1.120	1.339	0.548	0.591	1.118	1.351	0.588	0.629	1.117	1.363	0.632
12.8	0.552	1.125	1.335	0.542	0.586	1.123	1.346	0.581	0.624	1.122	1.358	0.624
12.9	0.548	1.130	1.330	0.535	0.582	1.128	1.341	0.573	0.618	1.126	1.353	0.615
13.0	0.544	1.134	1.326	0.529	0.577	1.132	1.337	0.566	0.613	1.131	1.348	0.607
13.1	0.540	1.139	1.322	0.523	0.573	1.136	1.332	0.559	0.609	1.135	1.344	0.600
13.2	0.537	1.143	1.318	0.517	0.569	1.141	1.328	0.553	0.604	1.139	1.339	0.592
13.3	0.533	1.148	1.313	0.511	0.565	1.145	1.323	0.546	0.599	1.144	1.334	0.585
13.4	0.530	1.152	1.309	0.505	0.561	1.150	1.319	0.540	0.595	1.148	1.330	0.578
13.5	0.526	1.156	1.306	0.500	0.557	1.154	1.315	0.534	0.590	1.152	1.326	0.571
13.6	0.523	1.161	1.302	0.495	0.553	1.158	1.311	0.528	0.586	1.157	1.321	0.564
13.7	0.520	1.165	1.298	0.489	0.550	1.163	1.307	0.522	0.582	1.161	1.317	0.558
13.8	0.516	1.170	1.294	0.484	0.546	1.167	1.303	0.517	0.578	1.165	1.313	0.552
13.9	0.513	1.174	1.291	0.480	0.543	1.171	1.300	0.511	0.574	1.169	1.309	0.546
14.0	0.510	1.178	1.287	0.475	0.539	1.176	1.296	0.506	0.570	1.174	1.305	0.540
14.1	0.507	1.182	1.284	0.470	0.536	1.180	1.292	0.501	0.567	1.178	1.301	0.534
14.2	0.504	1.187	1.280	0.466	0.533	1.184	1.289	0.496	0.563	1.182	1.298	0.528
14.3	0.502	1.191	1.277	0.461	0.530	1.188	1.285	0.491	0.559	1.186	1.294	0.523
14.4	0.499	1.195	1.274	0.457	0.526	1.192	1.282	0.486	0.556	1.190	1.290	0.517
14.5	0.496	1.199	1.270	0.453	0.523	1.196	1.278	0.481	0.553	1.194	1.287	0.512
14.6	0.493	1.203	1.267	0.449	0.520	1.201	1.275	0.477	0.549	1.198	1.283	0.507
14.7	0.491	1.207	1.264	0.444	0.518	1.205	1.272	0.472	0.546	1.202	1.280	0.502
14.8	0.488	1.211	1.261	0.441	0.515	1.209	1.269	0.468	0.543	1.206	1.277	0.497
14.9	0.486	1.215	1.258	0.437	0.512	1.213	1.265	0.464	0.540	1.210	1.273	0.493
15.0	0.483	1.219	1.255	0.433	0.509	1.217	1.262	0.460	0.537	1.214	1.270	0.488
15.1	0.481	1.223	1.252	0.429	0.507	1.221	1.259	0.455	0.534	1.218	1.267	0.484
15.2	0.479	1.227	1.249	0.426	0.504	1.225	1.256	0.452	0.531	1.222	1.264	0.479

$1.65 \leq \hat{\alpha}_3 \leq 1.75$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.65$				$\hat{\alpha}_3 = 1.70$				$\hat{\alpha}_3 = 1.75$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
8.3	4.092	2.199	2.044	4.090								
8.4	2.893	1.643	2.000	3.334								
8.5	2.367	1.416	1.960	2.898								
8.6	2.055	1.292	1.924	2.594	5.575	2.993	2.019	4.642				
8.7	1.843	1.213	1.891	2.363	3.372	1.909	1.978	3.576				
8.8	1.687	1.160	1.860	2.180	2.644	1.574	1.940	3.058				
8.9	1.566	1.122	1.831	2.029	2.250	1.404	1.905	2.717	25.515	13.618	1.999	7.612
9.0	1.469	1.095	1.805	1.901	1.994	1.302	1.874	2.465	4.312	2.423	1.960	4.000
9.1	1.388	1.075	1.780	1.792	1.812	1.234	1.844	2.268	3.080	1.817	1.923	3.300
9.2	1.321	1.060	1.757	1.697	1.673	1.186	1.817	2.107	2.528	1.562	1.890	2.890
9.3	1.262	1.049	1.735	1.613	1.563	1.151	1.791	1.972	2.198	1.418	1.860	2.602
9.4	1.212	1.040	1.715	1.538	1.473	1.125	1.768	1.857	1.973	1.327	1.831	2.382
9.5	1.167	1.034	1.696	1.471	1.397	1.106	1.745	1.757	1.807	1.264	1.805	2.206
9.6	1.128	1.030	1.677	1.411	1.333	1.091	1.724	1.670	1.678	1.218	1.781	2.060
9.7	1.092	1.027	1.660	1.356	1.277	1.079	1.705	1.592	1.574	1.184	1.758	1.937
9.8	1.060	1.025	1.644	1.306	1.229	1.071	1.686	1.522	1.488	1.159	1.736	1.830
9.9	1.031	1.024	1.629	1.261	1.185	1.064	1.669	1.460	1.416	1.139	1.716	1.737
10.0	1.004	1.024	1.614	1.219	1.147	1.059	1.652	1.403	1.354	1.124	1.697	1.655
10.1	0.980	1.025	1.600	1.180	1.112	1.056	1.636	1.351	1.299	1.112	1.679	1.582
10.2	0.957	1.026	1.587	1.144	1.081	1.054	1.621	1.304	1.252	1.102	1.662	1.516
10.3	0.936	1.028	1.574	1.110	1.052	1.052	1.607	1.260	1.209	1.095	1.646	1.456
10.4	0.917	1.030	1.562	1.079	1.026	1.052	1.594	1.220	1.171	1.090	1.630	1.402
10.5	0.899	1.032	1.550	1.049	1.001	1.052	1.581	1.182	1.136	1.086	1.616	1.352
10.6	0.882	1.035	1.539	1.022	0.979	1.053	1.568	1.148	1.105	1.083	1.602	1.307
10.7	0.866	1.038	1.528	0.996	0.958	1.054	1.556	1.115	1.077	1.081	1.588	1.265
10.8	0.851	1.041	1.518	0.972	0.939	1.056	1.545	1.085	1.050	1.080	1.576	1.226
10.9	0.837	1.045	1.508	0.949	0.921	1.058	1.534	1.056	1.026	1.079	1.564	1.189
11.0	0.823	1.048	1.499	0.927	0.904	1.060	1.524	1.030	1.004	1.080	1.552	1.156
11.1	0.811	1.052	1.489	0.907	0.888	1.062	1.513	1.005	0.983	1.080	1.541	1.124
11.2	0.799	1.055	1.480	0.887	0.873	1.065	1.504	0.981	0.963	1.081	1.530	1.094
11.3	0.787	1.059	1.472	0.869	0.859	1.068	1.494	0.959	0.945	1.083	1.520	1.067
11.4	0.777	1.063	1.464	0.851	0.845	1.071	1.485	0.937	0.928	1.085	1.510	1.041
11.5	0.766	1.067	1.456	0.834	0.832	1.074	1.477	0.917	0.912	1.087	1.500	1.016
11.6	0.756	1.071	1.448	0.819	0.820	1.078	1.468	0.898	0.897	1.089	1.491	0.993
11.7	0.747	1.075	1.440	0.803	0.809	1.081	1.460	0.880	0.882	1.091	1.482	0.971
11.8	0.738	1.079	1.433	0.789	0.798	1.085	1.452	0.863	0.869	1.094	1.474	0.950
11.9	0.729	1.084	1.426	0.775	0.788	1.088	1.445	0.846	0.856	1.097	1.465	0.930
12.0	0.721	1.088	1.419	0.761	0.778	1.092	1.437	0.830	0.843	1.100	1.457	0.911
12.1	0.713	1.092	1.413	0.749	0.768	1.096	1.430	0.815	0.832	1.103	1.450	0.893
12.2	0.705	1.096	1.406	0.736	0.759	1.100	1.423	0.801	0.821	1.106	1.442	0.876
12.3	0.698	1.101	1.400	0.725	0.750	1.104	1.417	0.787	0.810	1.109	1.435	0.860
12.4	0.691	1.105	1.394	0.713	0.742	1.107	1.410	0.774	0.800	1.113	1.428	0.844
12.5	0.684	1.109	1.388	0.702	0.734	1.111	1.404	0.761	0.790	1.116	1.421	0.829
12.6	0.677	1.113	1.382	0.692	0.726	1.115	1.398	0.749	0.781	1.120	1.415	0.815
12.7	0.671	1.118	1.377	0.682	0.718	1.119	1.392	0.737	0.772	1.123	1.408	0.801
12.8	0.665	1.122	1.371	0.672	0.711	1.123	1.386	0.726	0.763	1.127	1.402	0.788
12.9	0.659	1.126	1.366	0.662	0.704	1.127	1.380	0.715	0.755	1.131	1.396	0.775
13.0	0.653	1.130	1.361	0.653	0.697	1.131	1.375	0.705	0.747	1.134	1.390	0.763
13.1	0.648	1.135	1.356	0.644	0.691	1.135	1.369	0.695	0.740	1.138	1.384	0.751
13.2	0.642	1.139	1.351	0.636	0.685	1.139	1.364	0.685	0.732	1.142	1.379	0.740
13.3	0.637	1.143	1.346	0.628	0.679	1.143	1.359	0.675	0.725	1.146	1.373	0.729
13.4	0.632	1.147	1.341	0.620	0.673	1.147	1.354	0.666	0.718	1.149	1.368	0.719
13.5	0.627	1.151	1.337	0.612	0.667	1.152	1.349	0.657	0.712	1.153	1.363	0.708
13.6	0.622	1.156	1.332	0.604	0.661	1.156	1.344	0.649	0.705	1.157	1.358	0.699
13.7	0.617	1.160	1.328	0.597	0.656	1.160	1.340	0.641	0.699	1.161	1.353	0.689
13.8	0.613	1.164	1.324	0.590	0.651	1.164	1.335	0.633	0.693	1.165	1.348	0.680
13.9	0.608	1.168	1.320	0.583	0.646	1.168	1.331	0.625	0.687	1.168	1.343	0.671
14.0	0.604	1.172	1.315	0.577	0.641	1.172	1.326	0.617	0.682	1.172	1.339	0.663
14.1	0.600	1.176	1.311	0.570	0.636	1.176	1.322	0.610	0.676	1.176	1.334	0.654
14.2	0.596	1.180	1.307	0.564	0.632	1.180	1.318	0.603	0.671	1.180	1.330	0.646
14.3	0.592	1.184	1.304	0.558	0.627	1.183	1.314	0.596	0.666	1.184	1.325	0.639
14.4	0.588	1.188	1.300	0.552	0.623	1.187	1.310	0.589	0.661	1.187	1.321	0.631
14.5	0.584	1.192	1.296	0.546	0.618	1.191	1.306	0.583	0.656	1.191	1.317	0.624
14.6	0.580	1.196	1.293	0.540	0.614	1.195	1.302	0.577	0.651	1.195	1.313	0.617
14.7	0.577	1.200	1.289	0.535	0.610	1.199	1.299	0.570	0.646	1.199	1.309	0.610
14.8	0.573	1.204	1.286	0.529	0.606	1.203	1.295	0.564	0.642	1.203	1.305	0.603
14.9	0.570	1.208	1.282	0.524	0.602	1.207	1.291	0.559	0.637	1.206	1.301	0.596
15.0	0.566	1.212	1.279	0.519	0.598	1.211	1.288	0.553	0.633	1.210	1.298	0.590
15.1	0.563	1.216	1.275	0.514	0.595	1.215	1.284	0.547	0.629	1.214	1.294	0.584
15.2	0.560	1.220	1.272	0.509	0.591	1.218	1.281	0.542	0.625	1.218	1.291	0.578
15.3	0.557	1.224	1.269	0.505	0.587	1.222	1.278	0.537	0.621	1.221	1.287	0.572
15.4	0.554	1.228	1.266	0.500	0.584	1.226	1.274	0.532	0.617	1.225	1.284	0.566
15.5	0.551	1.231	1.263	0.495	0.581	1.230	1.271	0.527	0.613	1.229	1.280	0.560
15.6	0.548	1.235	1.260	0.491	0.577	1.234	1.268	0.522	0.609	1.232	1.277	0.555
15.7	0.545	1.239	1.257	0.487	0.574	1.237	1.265	0.517	0.606	1.236	1.274	0.550
15.8	0.542	1.243	1.254	0.482	0.571	1.241	1.262	0.512	0.602	1.240	1.271	0.544
15.9	0.539	1.247	1.251	0.478	0.568	1.245	1.259	0.508	0.599	1.243	1.267	0.539
16.0	0.536	1.250	1.248	0.474	0.565	1.248	1.256	0.503	0.595	1.247	1.264	0.534
16.1	0.534	1.254	1.246	0.470	0.562	1.252	1.253	0.499	0.592	1.251	1.261	0.530
16.2	0.531	1.258	1.243	0.467	0.559	1.256	1.250	0.495	0.589	1.254	1.258	0.525

$1.80 \leq \hat{\alpha}_3 \leq 1.90$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.80$				$\hat{\alpha}_3 = 1.85$				$\hat{\alpha}_3 = 1.90$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
9.3	7.544	4.191	1.945	5.033								
9.4	3.891	2.266	1.910	3.698								
9.5	2.962	1.804	1.878	3.143								
9.6	2.489	1.582	1.849	2.790								
9.7	2.191	1.450	1.821	2.533	6.261	3.588	1.900	4.556				
9.8	1.981	1.362	1.796	2.333	3.762	2.252	1.869	3.547				
9.9	1.824	1.301	1.772	2.171	2.944	1.838	1.841	3.059				
10.0	1.700	1.256	1.750	2.035	2.504	1.626	1.814	2.737				
10.1	1.599	1.221	1.729	1.919	2.218	1.496	1.789	2.499	6.112	3.586	1.863	4.412
10.2	1.515	1.195	1.709	1.818	2.014	1.408	1.766	2.311	3.808	2.328	1.835	3.494
10.3	1.443	1.175	1.690	1.730	1.859	1.346	1.744	2.157	3.006	1.911	1.809	3.032
10.4	1.382	1.158	1.673	1.652	1.736	1.299	1.723	2.027	2.565	1.693	1.784	2.723
10.5	1.328	1.146	1.656	1.581	1.635	1.263	1.704	1.916	2.277	1.557	1.761	2.493
10.6	1.280	1.135	1.640	1.518	1.551	1.235	1.686	1.819	2.070	1.465	1.740	2.311
10.7	1.238	1.127	1.625	1.460	1.480	1.213	1.668	1.734	1.912	1.398	1.720	2.161
10.8	1.200	1.121	1.611	1.408	1.418	1.196	1.652	1.657	1.787	1.348	1.700	2.035
10.9	1.165	1.116	1.597	1.360	1.363	1.182	1.636	1.589	1.684	1.310	1.682	1.926
11.0	1.134	1.113	1.584	1.315	1.315	1.170	1.622	1.527	1.598	1.279	1.665	1.831
11.1	1.105	1.110	1.572	1.274	1.272	1.161	1.608	1.471	1.525	1.255	1.649	1.747
11.2	1.078	1.108	1.560	1.236	1.233	1.154	1.594	1.420	1.461	1.236	1.634	1.672
11.3	1.054	1.107	1.548	1.201	1.198	1.148	1.581	1.372	1.405	1.220	1.619	1.605
11.4	1.031	1.107	1.537	1.167	1.166	1.144	1.569	1.329	1.356	1.207	1.605	1.544
11.5	1.010	1.107	1.527	1.136	1.137	1.140	1.557	1.288	1.312	1.197	1.592	1.488
11.6	0.990	1.107	1.517	1.107	1.110	1.138	1.546	1.251	1.272	1.188	1.579	1.437
11.7	0.971	1.108	1.507	1.080	1.085	1.136	1.535	1.216	1.236	1.181	1.567	1.390
11.8	0.954	1.109	1.498	1.054	1.061	1.135	1.525	1.183	1.203	1.176	1.556	1.347
11.9	0.938	1.111	1.489	1.030	1.040	1.134	1.515	1.152	1.172	1.171	1.544	1.307
12.0	0.922	1.113	1.480	1.007	1.019	1.134	1.505	1.123	1.145	1.168	1.534	1.269
12.1	0.908	1.115	1.471	0.985	1.000	1.134	1.496	1.096	1.119	1.165	1.523	1.234
12.2	0.894	1.117	1.463	0.964	0.983	1.135	1.487	1.070	1.095	1.163	1.514	1.202
12.3	0.881	1.120	1.455	0.945	0.966	1.136	1.478	1.046	1.073	1.162	1.504	1.171
12.4	0.868	1.122	1.448	0.926	0.950	1.137	1.470	1.023	1.052	1.161	1.495	1.142
12.5	0.856	1.125	1.440	0.908	0.935	1.139	1.462	1.002	1.032	1.161	1.486	1.115
12.6	0.845	1.128	1.433	0.891	0.921	1.140	1.454	0.981	1.014	1.161	1.477	1.090
12.7	0.834	1.131	1.426	0.875	0.907	1.142	1.446	0.961	0.997	1.161	1.469	1.065
12.8	0.824	1.134	1.420	0.859	0.895	1.145	1.439	0.943	0.980	1.162	1.461	1.042
12.9	0.814	1.137	1.413	0.844	0.882	1.147	1.432	0.925	0.965	1.163	1.453	1.021
13.0	0.804	1.140	1.407	0.830	0.871	1.149	1.425	0.908	0.950	1.164	1.446	1.000
13.1	0.795	1.143	1.400	0.816	0.860	1.152	1.418	0.891	0.936	1.166	1.438	0.980
13.2	0.786	1.146	1.394	0.803	0.849	1.155	1.412	0.876	0.923	1.167	1.431	0.961
13.3	0.778	1.150	1.389	0.790	0.839	1.157	1.406	0.861	0.910	1.169	1.425	0.943
13.4	0.770	1.153	1.383	0.778	0.829	1.160	1.400	0.846	0.898	1.171	1.418	0.926
13.5	0.762	1.157	1.377	0.766	0.820	1.163	1.394	0.833	0.887	1.174	1.411	0.910
13.6	0.755	1.160	1.372	0.755	0.811	1.166	1.388	0.819	0.876	1.176	1.405	0.894
13.7	0.747	1.164	1.367	0.744	0.802	1.169	1.382	0.807	0.865	1.178	1.399	0.879
13.8	0.740	1.167	1.362	0.734	0.794	1.172	1.377	0.794	0.855	1.181	1.393	0.864
13.9	0.733	1.171	1.357	0.723	0.786	1.175	1.371	0.783	0.846	1.183	1.387	0.851
14.0	0.727	1.174	1.352	0.714	0.778	1.179	1.366	0.771	0.836	1.186	1.382	0.837
14.1	0.720	1.178	1.347	0.704	0.770	1.182	1.361	0.760	0.827	1.189	1.376	0.824
14.2	0.714	1.182	1.342	0.695	0.763	1.185	1.356	0.750	0.819	1.192	1.371	0.812
14.3	0.708	1.185	1.338	0.686	0.756	1.188	1.351	0.739	0.810	1.194	1.366	0.800
14.4	0.702	1.189	1.333	0.677	0.749	1.192	1.346	0.729	0.802	1.197	1.361	0.789
14.5	0.697	1.192	1.329	0.669	0.743	1.195	1.342	0.720	0.795	1.200	1.356	0.777
14.6	0.691	1.196	1.325	0.661	0.736	1.199	1.337	0.711	0.787	1.203	1.351	0.767
14.7	0.686	1.200	1.320	0.653	0.730	1.202	1.333	0.702	0.780	1.206	1.346	0.756
14.8	0.681	1.203	1.316	0.645	0.724	1.205	1.328	0.693	0.773	1.209	1.342	0.746
14.9	0.676	1.207	1.312	0.638	0.718	1.209	1.324	0.684	0.766	1.213	1.337	0.737
15.0	0.671	1.210	1.308	0.631	0.713	1.212	1.320	0.676	0.760	1.216	1.333	0.727
15.1	0.666	1.214	1.305	0.624	0.707	1.216	1.316	0.668	0.753	1.219	1.328	0.718
15.2	0.661	1.218	1.301	0.617	0.702	1.219	1.312	0.661	0.747	1.222	1.324	0.709
15.3	0.657	1.221	1.297	0.610	0.697	1.222	1.308	0.653	0.741	1.225	1.320	0.701
15.4	0.653	1.225	1.294	0.604	0.692	1.226	1.304	0.646	0.735	1.228	1.316	0.692
15.5	0.648	1.228	1.290	0.598	0.687	1.229	1.301	0.639	0.729	1.232	1.312	0.684
15.6	0.644	1.232	1.287	0.592	0.682	1.233	1.297	0.632	0.724	1.235	1.308	0.676
15.7	0.640	1.236	1.283	0.586	0.677	1.236	1.293	0.625	0.719	1.238	1.304	0.669
15.8	0.636	1.239	1.280	0.580	0.673	1.240	1.290	0.619	0.713	1.241	1.301	0.661
15.9	0.632	1.243	1.277	0.574	0.668	1.243	1.286	0.612	0.708	1.245	1.297	0.654
16.0	0.628	1.246	1.273	0.569	0.664	1.247	1.283	0.606	0.703	1.248	1.293	0.647
16.1	0.624	1.250	1.270	0.563	0.660	1.250	1.280	0.600	0.699	1.251	1.290	0.640
16.2	0.621	1.253	1.267	0.558	0.656	1.253	1.276	0.594	0.694	1.255	1.286	0.634
16.3	0.617	1.257	1.264	0.553	0.652	1.257	1.273	0.588	0.689	1.258	1.283	0.627
16.4	0.614	1.260	1.261	0.548	0.648	1.260	1.270	0.583	0.685	1.261	1.280	0.621
16.5	0.610	1.264	1.258	0.543	0.644	1.264	1.267	0.577	0.680	1.264	1.276	0.615
16.6	0.607	1.267	1.255	0.538	0.640	1.267	1.264	0.572	0.676	1.268	1.273	0.609
16.7	0.604	1.271	1.252	0.533	0.636	1.270	1.261	0.567	0.672	1.271	1.270	0.603
16.8	0.600	1.274	1.250	0.529	0.633	1.274	1.258	0.561	0.668	1.274	1.267	0.597
16.9	0.597	1.278	1.247	0.524	0.629	1.277	1.255	0.556	0.664	1.277	1.264	0.592
17.0	0.594	1.281	1.244	0.520	0.626	1.280	1.252	0.552	0.660	1.281	1.261	0.586
17.1	0.591	1.284	1.241	0.516	0.622	1.284	1.249	0.547	0.656	1.284	1.258	0.581
17.2	0.588	1.288	1.239	0.511	0.619	1.287	1.247	0.542	0.652	1.287	1.255	0.576

$1.95 \leq \hat{\alpha}_3 \leq 2.05$

$\hat{\alpha}_4$	$\hat{\alpha}_3 = 1.95$				$\hat{\alpha}_3 = 2.00$				$\hat{\alpha}_3 = 2.05$			
	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$	μ_Y	σ_Y	δ	$-\gamma$
10.6	4.016	2.493	1.805	3.521								
10.7	3.145	2.026	1.781	3.054								
10.8	2.674	1.784	1.759	2.744								
10.9	2.369	1.635	1.737	2.514	8.559	5.173	1.804	4.856				
11.0	2.151	1.534	1.717	2.332	4.439	2.783	1.780	3.637				
11.1	1.985	1.460	1.699	2.183	3.381	2.196	1.758	3.126				
11.2	1.854	1.405	1.681	2.057	2.841	1.909	1.737	2.800				
11.3	1.746	1.363	1.664	1.949	2.501	1.735	1.717	2.562	31.448	19.150	1.782	7.098
11.4	1.656	1.329	1.648	1.854	2.261	1.618	1.698	2.375	5.275	3.327	1.759	3.882
11.5	1.580	1.302	1.633	1.770	2.081	1.535	1.681	2.223	3.765	2.456	1.738	3.262
11.6	1.513	1.280	1.618	1.695	1.939	1.472	1.664	2.094	3.089	2.081	1.718	2.898
11.7	1.455	1.262	1.604	1.628	1.824	1.423	1.648	1.984	2.685	1.866	1.699	2.640
11.8	1.403	1.247	1.591	1.567	1.727	1.385	1.633	1.888	2.409	1.725	1.682	2.442
11.9	1.357	1.235	1.579	1.511	1.646	1.354	1.618	1.803	2.205	1.625	1.665	2.281
12.0	1.316	1.225	1.567	1.460	1.575	1.329	1.604	1.727	2.047	1.551	1.649	2.148
12.1	1.278	1.217	1.555	1.413	1.513	1.308	1.591	1.659	1.919	1.494	1.634	2.033
12.2	1.244	1.210	1.544	1.370	1.459	1.291	1.579	1.598	1.814	1.449	1.619	1.934
12.3	1.213	1.204	1.533	1.330	1.410	1.276	1.567	1.541	1.725	1.413	1.606	1.847
12.4	1.184	1.200	1.523	1.292	1.366	1.265	1.555	1.490	1.648	1.383	1.593	1.769
12.5	1.157	1.196	1.513	1.257	1.327	1.255	1.544	1.442	1.582	1.359	1.580	1.699
12.6	1.132	1.193	1.504	1.224	1.291	1.246	1.534	1.398	1.523	1.339	1.568	1.636
12.7	1.109	1.191	1.494	1.194	1.258	1.239	1.523	1.357	1.471	1.322	1.556	1.578
12.8	1.087	1.189	1.486	1.165	1.227	1.233	1.513	1.319	1.424	1.308	1.545	1.525
12.9	1.067	1.188	1.477	1.137	1.199	1.229	1.504	1.284	1.382	1.296	1.535	1.476
13.0	1.048	1.188	1.469	1.111	1.173	1.225	1.495	1.251	1.343	1.285	1.525	1.432
13.1	1.030	1.187	1.461	1.087	1.149	1.222	1.486	1.219	1.308	1.277	1.515	1.390
13.2	1.013	1.188	1.453	1.064	1.126	1.219	1.478	1.190	1.276	1.270	1.505	1.351
13.3	0.997	1.188	1.446	1.042	1.105	1.217	1.469	1.162	1.246	1.263	1.496	1.315
13.4	0.982	1.189	1.438	1.021	1.085	1.216	1.461	1.136	1.218	1.258	1.487	1.281
13.5	0.967	1.190	1.431	1.001	1.066	1.215	1.454	1.111	1.193	1.254	1.479	1.249
13.6	0.953	1.191	1.425	0.982	1.048	1.214	1.446	1.088	1.169	1.250	1.470	1.219
13.7	0.940	1.192	1.418	0.964	1.031	1.214	1.439	1.066	1.146	1.248	1.463	1.191
13.8	0.928	1.194	1.412	0.946	1.015	1.214	1.432	1.044	1.125	1.245	1.455	1.164
13.9	0.916	1.196	1.405	0.930	1.000	1.215	1.425	1.024	1.105	1.244	1.447	1.139
14.0	0.905	1.198	1.399	0.914	0.986	1.215	1.419	1.005	1.087	1.242	1.440	1.115
14.1	0.894	1.200	1.393	0.899	0.972	1.216	1.412	0.986	1.069	1.241	1.433	1.092
14.2	0.883	1.202	1.388	0.884	0.959	1.217	1.406	0.969	1.052	1.241	1.426	1.070
14.3	0.873	1.204	1.382	0.870	0.947	1.218	1.400	0.952	1.036	1.240	1.420	1.050
14.4	0.863	1.206	1.376	0.856	0.935	1.220	1.394	0.936	1.021	1.240	1.413	1.030
14.5	0.854	1.209	1.371	0.843	0.924	1.221	1.388	0.920	1.007	1.241	1.407	1.011
14.6	0.845	1.211	1.366	0.831	0.913	1.223	1.383	0.905	0.993	1.241	1.401	0.993
14.7	0.837	1.214	1.361	0.819	0.902	1.225	1.377	0.891	0.980	1.242	1.395	0.976
14.8	0.828	1.216	1.356	0.807	0.892	1.227	1.372	0.877	0.968	1.243	1.389	0.959
14.9	0.820	1.219	1.351	0.796	0.882	1.229	1.367	0.864	0.956	1.244	1.384	0.944
15.0	0.812	1.222	1.346	0.785	0.873	1.231	1.362	0.851	0.944	1.245	1.378	0.928
15.1	0.805	1.225	1.342	0.774	0.864	1.233	1.357	0.839	0.933	1.247	1.373	0.914
15.2	0.798	1.227	1.337	0.764	0.856	1.236	1.352	0.827	0.923	1.248	1.368	0.900
15.3	0.791	1.230	1.333	0.754	0.847	1.238	1.347	0.816	0.913	1.250	1.363	0.886
15.4	0.784	1.233	1.329	0.745	0.839	1.241	1.342	0.804	0.903	1.252	1.358	0.873
15.5	0.777	1.236	1.324	0.736	0.831	1.243	1.338	0.794	0.894	1.254	1.353	0.861
15.6	0.771	1.239	1.320	0.727	0.824	1.246	1.334	0.783	0.884	1.256	1.348	0.848
15.7	0.765	1.242	1.316	0.718	0.816	1.248	1.329	0.773	0.876	1.258	1.344	0.837
15.8	0.758	1.245	1.312	0.709	0.809	1.251	1.325	0.764	0.867	1.260	1.339	0.825
15.9	0.753	1.248	1.308	0.701	0.802	1.254	1.321	0.754	0.859	1.262	1.335	0.814
16.0	0.747	1.251	1.305	0.693	0.796	1.256	1.317	0.745	0.851	1.264	1.330	0.804
16.1	0.741	1.254	1.301	0.686	0.789	1.259	1.313	0.736	0.844	1.267	1.326	0.794
16.2	0.736	1.257	1.297	0.678	0.783	1.262	1.309	0.728	0.836	1.269	1.322	0.784
16.3	0.731	1.260	1.294	0.671	0.777	1.265	1.305	0.719	0.829	1.271	1.318	0.774
16.4	0.726	1.263	1.290	0.664	0.771	1.267	1.302	0.711	0.822	1.274	1.314	0.765
16.5	0.721	1.266	1.287	0.657	0.765	1.270	1.298	0.703	0.815	1.276	1.310	0.755
16.6	0.716	1.269	1.283	0.650	0.760	1.273	1.294	0.695	0.809	1.279	1.306	0.747
16.7	0.711	1.273	1.280	0.643	0.754	1.276	1.291	0.688	0.802	1.281	1.303	0.738
16.8	0.706	1.276	1.277	0.637	0.749	1.279	1.287	0.681	0.796	1.284	1.299	0.730
16.9	0.702	1.279	1.274	0.631	0.744	1.282	1.284	0.674	0.790	1.287	1.295	0.722
17.0	0.697	1.282	1.270	0.624	0.738	1.285	1.281	0.667	0.784	1.289	1.292	0.714
17.1	0.693	1.285	1.267	0.619	0.733	1.288	1.277	0.660	0.778	1.292	1.288	0.706
17.2	0.689	1.288	1.264	0.613	0.729	1.291	1.274	0.653	0.773	1.295	1.285	0.699
17.3	0.685	1.291	1.261	0.607	0.724	1.293	1.271	0.647	0.768	1.297	1.282	0.691
17.4	0.681	1.294	1.258	0.601	0.719	1.296	1.268	0.641	0.762	1.300	1.278	0.684
17.5	0.677	1.297	1.256	0.596	0.715	1.299	1.265	0.635	0.757	1.303	1.275	0.678
17.6	0.673	1.301	1.253	0.591	0.710	1.302	1.262	0.629	0.752	1.306	1.272	0.671
17.7	0.669	1.304	1.250	0.586	0.706	1.305	1.259	0.623	0.747	1.308	1.269	0.664
17.8	0.665	1.307	1.247	0.580	0.702	1.308	1.256	0.617	0.742	1.311	1.266	0.658
17.9	0.662	1.310	1.245	0.576	0.698	1.311	1.253	0.612	0.738	1.314	1.263	0.652
18.0	0.658	1.313	1.242	0.571	0.694	1.314	1.251	0.606	0.733	1.317	1.260	0.646
18.1	0.655	1.316	1.239	0.566	0.690	1.317	1.248	0.601	0.729	1.320	1.257	0.640
18.2	0.651	1.319	1.237	0.561	0.686	1.320	1.245	0.596	0.724	1.322	1.254	0.634
18.3	0.648	1.322	1.234	0.557	0.682	1.323	1.243	0.591	0.720	1.325	1.252	0.628
18.4	0.645	1.325	1.232	0.552	0.679	1.326	1.240	0.586	0.716	1.328	1.249	0.623
18.5	0.641	1.328	1.229	0.548	0.675	1.329	1.237	0.581	0.712	1.331	1.246	0.617

Appendix I

Table I–1 Johnson System Fits in the S_B Region: Method of Moments

For specified α_3 and α_4, the table that follows gives the values of δ and γ for Johnson system fits from the S_B region (see Section 16.4). To assist with computations the table also gives values for μ_Y and σ_Y, as described in Section 16.4.

$$0.00 \leq \hat{\alpha}_3 \leq 0.05$$

$\hat{\alpha}_3 = 0,\ \ 1.00 < \hat{\alpha}_4 < 2.96$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.000	0.5000	0.5000	0.0000	0.0000
1.200	0.5000	0.4306	0.1691	0.0000
1.400	0.5000	0.3715	0.3226	0.0000
1.600	0.5000	0.3204	0.4780	0.0000
1.800	0.5000	0.2754	0.6464	0.0000
1.900	0.5000	0.2547	0.7393	0.0000
2.000	0.5000	0.2348	0.8403	0.0000
2.100	0.5000	0.2157	0.9521	0.0000
2.200	0.5000	0.1971	1.0785	0.0000
2.300	0.5000	0.1788	1.2247	0.0000
2.400	0.5000	0.1608	1.3985	0.0000
2.500	0.5000	0.1426	1.6132	0.0000
2.600	0.5000	0.1241	1.8926	0.0000
2.700	0.5000	0.1046	2.2865	0.0000
2.750	0.5000	0.0942	2.5594	0.0000
2.800	0.5000	0.0832	2.9221	0.0000
2.850	0.5000	0.0711	3.4434	0.0000
2.900	0.5000	0.0573	4.3035	0.0000
2.950	0.5000	0.0400	6.2091	0.0000
2.960	0.5000	0.0357	6.9620	0.0000

$\hat{\alpha}_3 = 0.05,\ \ 1.05 < \hat{\alpha}_4 < 2.96$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.050	0.4875	0.4824	0.0432	0.0314
1.200	0.4875	0.4313	0.1672	0.0326
1.400	0.4875	0.3721	0.3205	0.0361
1.600	0.4872	0.3210	0.4757	0.0413
1.800	0.4868	0.2760	0.6436	0.0487
2.000	0.4862	0.2353	0.8367	0.0595
2.200	0.4852	0.1976	1.0737	0.0755
2.300	0.4844	0.1794	1.2187	0.0873
2.400	0.4834	0.1613	1.3908	0.1030
2.500	0.4822	0.1432	1.6031	0.1245
2.550	0.4814	0.1341	1.7309	0.1392
2.600	0.4804	0.1247	1.8789	0.1574
2.650	0.4792	0.1152	2.0534	0.1804
2.700	0.4777	0.1053	2.2654	0.2114
2.750	0.4759	0.0950	2.5319	0.2537
2.800	0.4732	0.0840	2.8841	0.3183
2.850	0.4694	0.0720	3.3856	0.4232
2.875	0.4668	0.0655	3.7337	0.5054
2.900	0.4632	0.0585	4.1958	0.6282
2.920	0.4591	0.0522	4.7012	0.7798
2.940	0.4534	0.0454	5.4169	1.0219
2.960	0.4441	0.0374	6.5621	1.4822

$$0.10 \leq \hat{\alpha}_3 \leq 0.15$$

$\hat{\alpha}_3 = 0.1,\ 1.05 < \hat{\alpha}_4 < 2.96$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.050	0.4750	0.4846	0.0365	0.0627
1.200	0.4750	0.4334	0.1608	0.0652
1.400	0.4749	0.3740	0.3141	0.0718
1.600	0.4745	0.3228	0.4684	0.0820
1.800	0.4737	0.2776	0.6351	0.0967
2.000	0.4725	0.2369	0.8259	0.1176
2.200	0.4705	0.1992	1.0589	0.1489
2.300	0.4690	0.1810	1.2009	0.1716
2.400	0.4672	0.1630	1.3688	0.2014
2.500	0.4647	0.1450	1.5745	0.2431
2.550	0.4632	0.1359	1.6974	0.2704
2.600	0.4614	0.1266	1.8392	0.3044
2.650	0.4592	0.1171	2.0052	0.3476
2.700	0.4564	0.1074	2.2054	0.4048
2.750	0.4528	0.0972	2.4536	0.4828
2.800	0.4482	0.0864	2.7762	0.5956
2.850	0.4414	0.0748	3.2236	0.7762
2.875	0.4367	0.0685	3.5266	0.9148
2.900	0.4307	0.0617	3.9162	1.1098
2.925	0.4224	0.0543	4.4438	1.4077
2.950	0.4099	0.0459	5.2268	1.9212
2.960	0.4028	0.0422	5.6653	2.2471

$\hat{\alpha}_3 = 0.15,\ 1.20 < \hat{\alpha}_4 < 2.975$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.200	0.4626	0.4370	0.1503	0.0972
1.400	0.4625	0.3772	0.3034	0.1068
1.600	0.4619	0.3257	0.4565	0.1217
1.800	0.4608	0.2804	0.6211	0.1427
2.000	0.4590	0.2396	0.8084	0.1730
2.100	0.4577	0.2204	0.9154	0.1932
2.200	0.4561	0.2019	1.0353	0.2182
2.300	0.4541	0.1837	1.1725	0.2502
2.400	0.4515	0.1658	1.3335	0.2922
2.500	0.4481	0.1479	1.5291	0.3499
2.550	0.4459	0.1388	1.6448	0.3876
2.600	0.4434	0.1297	1.7772	0.4336
2.650	0.4404	0.1204	1.9308	0.4916
2.700	0.4367	0.1108	2.1134	0.5666
2.750	0.4320	0.1008	2.3364	0.6672
2.800	0.4259	0.0903	2.6192	0.8098
2.850	0.4174	0.0791	2.9972	1.0258
2.875	0.4119	0.0731	3.2424	1.1814
2.900	0.4049	0.0667	3.5464	1.3918
2.925	0.3958	0.0599	3.9369	1.6910
2.950	0.3833	0.0523	4.4682	2.1508
2.975	0.3646	0.0438	5.2533	2.9438

$$0.20 \leq \hat{\alpha}_3 \leq 0.25$$

$\hat{\alpha}_3 = 0.2,\ \ 1.10 < \hat{\alpha}_4 < 2.975$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.100	0.4502	0.4758	0.0536	0.1257
1.200	0.4503	0.4419	0.1358	0.1287
1.400	0.4501	0.3817	0.2887	0.1407
1.600	0.4494	0.3298	0.4403	0.1597
1.800	0.4481	0.2843	0.6020	0.1867
2.000	0.4458	0.2433	0.7847	0.2247
2.100	0.4442	0.2241	0.8883	0.2502
2.200	0.4422	0.2056	1.0038	0.2814
2.300	0.4397	0.1875	1.1347	0.3212
2.400	0.4365	0.1696	1.2872	0.3726
2.500	0.4323	0.1519	1.4699	0.4422
2.550	0.4298	0.1429	1.5771	0.4869
2.600	0.4267	0.1339	1.6982	0.5414
2.650	0.4232	0.1247	1.8369	0.6084
2.700	0.4189	0.1154	1.9992	0.6927
2.750	0.4136	0.1057	2.1936	0.8036
2.800	0.4068	0.0956	2.4334	0.9548
2.850	0.3978	0.0849	2.7416	1.1728
2.875	0.3922	0.0792	2.9342	1.3218
2.900	0.3853	0.0733	3.1624	1.5128
2.925	0.3768	0.0670	3.4408	1.7662
2.950	0.3659	0.0603	3.7946	2.1208
2.975	0.3513	0.0529	4.2620	2.6481

$\hat{\alpha}_3 = 0.25,\ \ 1.10 < \hat{\alpha}_4 < 3.025$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.100	0.4380	0.4826	0.0338	0.1563
1.400	0.4379	0.3874	0.2700	0.1733
1.600	0.4371	0.3351	0.4199	0.1957
1.800	0.4356	0.2892	0.5783	0.2276
2.000	0.4330	0.2481	0.7552	0.2724
2.100	0.4312	0.2289	0.8549	0.3017
2.200	0.4289	0.2103	0.9651	0.3379
2.300	0.4260	0.1922	1.0892	0.3834
2.400	0.4223	0.1745	1.2318	0.4414
2.500	0.4177	0.1569	1.4004	0.5184
2.550	0.4148	0.1481	1.4981	0.5672
2.600	0.4116	0.1392	1.6072	0.6254
2.650	0.4077	0.1302	1.7307	0.6958
2.700	0.4031	0.1211	1.8724	0.7834
2.750	0.3976	0.1117	2.0388	0.8946
2.800	0.3908	0.1020	2.2380	1.0406
2.850	0.3821	0.0918	2.4851	1.2406
2.900	0.3705	0.0810	2.8044	1.5318
2.925	0.3631	0.0753	3.0035	1.7322
2.950	0.3541	0.0692	3.2434	1.9932
2.975	0.3429	0.0627	3.5370	2.3441
3.000	0.3285	0.0557	3.9102	2.8383
3.025	0.3091	0.0480	4.4083	3.5886

$$0.30 \leq \hat{\alpha}_3 \leq 0.35$$

$\hat{\alpha}_3 = 0.30,\ 1.20 < \hat{\alpha}_4 < 3.075$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.200	0.4259	0.4562	0.0942	0.1896
1.400	0.4257	0.3945	0.2475	0.2046
1.600	0.4250	0.3416	0.3956	0.2295
1.800	0.4234	0.2952	0.5502	0.2652
2.000	0.4205	0.2539	0.7212	0.3152
2.200	0.4161	0.2160	0.9209	0.3872
2.300	0.4130	0.1979	1.0375	0.4362
2.400	0.4091	0.1803	1.1700	0.4981
2.500	0.4042	0.1629	1.3244	0.5788
2.600	0.3978	0.1454	1.5096	0.6878
2.650	0.3939	0.1367	1.6181	0.7577
2.700	0.3893	0.1277	1.7409	0.8431
2.750	0.3839	0.1187	1.8819	0.9484
2.800	0.3773	0.1093	2.0466	1.0823
2.850	0.3693	0.0997	2.2432	1.2567
2.900	0.3590	0.0896	2.4870	1.4964
2.925	0.3528	0.0843	2.6318	1.6512
2.950	0.3455	0.0788	2.7999	1.8428
2.975	0.3369	0.0731	2.9926	2.0798
3.000	0.3265	0.0670	3.2252	2.3878
3.025	0.3137	0.0605	3.5065	2.7968
3.050	0.2974	0.0535	3.8636	3.3742
3.075	0.2756	0.0457	4.3323	4.2389

$\hat{\alpha}_3 = 0.35,\ 1.20 < \hat{\alpha}_4 < 3.175$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.200	0.4138	0.4655	0.0669	0.2192
1.400	0.4137	0.4029	0.2213	0.2342
1.600	0.4131	0.3492	0.3677	0.2610
1.800	0.4114	0.3024	0.5184	0.2993
2.000	0.4085	0.2607	0.6830	0.3528
2.200	0.4039	0.2227	0.8724	0.4291
2.300	0.4007	0.2046	0.9813	0.4799
2.400	0.3968	0.1871	1.1037	0.5432
2.500	0.3918	0.1698	1.2440	0.6242
2.600	0.3855	0.1526	1.4091	0.7309
2.700	0.3772	0.1353	1.6101	0.8781
2.750	0.3720	0.1265	1.7292	0.9744
2.800	0.3660	0.1176	1.8651	1.0926
2.850	0.3587	0.1084	2.0228	1.2416
2.900	0.3498	0.0989	2.2114	1.4358
2.950	0.3387	0.0889	2.4396	1.6962
3.000	0.3242	0.0783	2.7294	2.0678
3.050	0.3043	0.0667	3.1182	2.6403
3.075	0.2912	0.0604	3.3672	3.0558
3.100	0.2748	0.0536	3.6768	3.6277
3.125	0.2536	0.0461	4.0716	4.4542
3.150	0.2247	0.0377	4.6048	5.7617
3.165	0.2012	0.0318	5.0362	7.0033
3.175	0.1819	0.0276	5.3851	8.1561

$$0.40 \leq \hat{\alpha}_3 \leq 0.45$$

$\hat{\alpha}_3 = 0.40,\ 1.20 < \hat{\alpha}_4 < 3.205$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.200	0.4020	0.4763	0.0349	0.2487
1.300	0.4020	0.4433	0.1159	0.2536
1.400	0.4019	0.4127	0.1913	0.2625
1.600	0.4013	0.3581	0.3364	0.2901
1.800	0.3998	0.3106	0.4834	0.3302
2.000	0.3969	0.2685	0.6416	0.3858
2.200	0.3923	0.2303	0.8206	0.4640
2.400	0.3852	0.1947	1.0348	0.5784
2.600	0.3742	0.1606	1.3089	0.7590
2.700	0.3664	0.1437	1.4831	0.8946
2.800	0.3562	0.1265	1.6970	1.0837
2.900	0.3421	0.1087	1.9731	1.3663
3.000	0.3215	0.0898	2.3528	1.8302
3.050	0.3071	0.0795	2.6114	2.1972
3.100	0.2878	0.0683	2.9446	2.7388
3.125	0.2755	0.0624	3.1528	3.1177
3.150	0.2606	0.0559	3.4044	3.6198
3.170	0.2461	0.0504	3.6466	4.1512
3.175	0.2420	0.0490	3.7136	4.3082
3.185	0.2332	0.0460	3.8584	4.6608
3.190	0.2285	0.0445	3.9358	4.8572
3.195	0.2234	0.0429	4.0202	5.0778
3.200	0.2180	0.0413	4.1072	5.3133
3.205	0.2124	0.0396	4.1995	5.5711

$\hat{\alpha}_3 = 0.45,\ 1.40 < \hat{\alpha}_4 < 3.33$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.400	0.3903	0.4239	0.1578	0.2896
1.600	0.3898	0.3682	0.3020	0.3168
1.800	0.3884	0.3200	0.4454	0.3577
2.000	0.3857	0.2773	0.5974	0.4142
2.200	0.3812	0.2388	0.7665	0.4924
2.400	0.3743	0.2032	0.9646	0.6046
2.600	0.3639	0.1694	1.2106	0.7752
2.700	0.3567	0.1528	1.3624	0.8982
2.800	0.3475	0.1361	1.5431	1.0632
2.900	0.3353	0.1191	1.7658	1.2956
3.000	0.3186	0.1013	2.0563	1.6476
3.100	0.2938	0.0821	2.4614	2.2398
3.150	0.2762	0.0715	2.7393	2.7192
3.200	0.2525	0.0600	3.1036	3.4472
3.225	0.2371	0.0537	3.3348	3.9741
3.250	0.2184	0.0469	3.6148	4.6860
3.260	0.2095	0.0439	3.7463	5.0511
3.270	0.1998	0.0409	3.8904	5.4750
3.280	0.1889	0.0377	4.0505	5.9790
3.290	0.1768	0.0343	4.2264	6.5769
3.300	0.1630	0.0308	4.4274	7.3192
3.310	0.1471	0.0269	4.6583	8.2629
3.320	0.1289	0.0228	4.9206	9.4777
3.330	0.1081	0.0185	5.2176	11.0882

$$0.50 \leq \hat{\alpha}_3 \leq 0.55$$

$\hat{\alpha}_3 = 0.50,\ 1.30 < \hat{\alpha}_4 < 3.355$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.300	0.3788	0.4683	0.0423	0.3097
1.400	0.3788	0.4364	0.1206	0.3159
1.600	0.3785	0.3796	0.2646	0.3417
1.800	0.3773	0.3305	0.4050	0.3822
2.000	0.3748	0.2872	0.5512	0.4384
2.200	0.3706	0.2483	0.7109	0.5155
2.400	0.3641	0.2126	0.8942	0.6233
2.600	0.3545	0.1790	1.1157	0.7824
2.700	0.3479	0.1626	1.2484	0.8930
2.800	0.3397	0.1463	1.4027	1.0364
2.900	0.3292	0.1298	1.5867	1.2295
3.000	0.3153	0.1130	1.8133	1.5017
3.100	0.2964	0.0953	2.1093	1.9164
3.150	0.2838	0.0860	2.2967	2.2169
3.200	0.2683	0.0761	2.5233	2.6212
3.250	0.2482	0.0654	2.8083	3.1993
3.280	0.2332	0.0585	3.0183	3.6800
3.300	0.2213	0.0536	3.1816	4.0881
3.320	0.2076	0.0485	3.3690	4.5973
3.330	0.2000	0.0458	3.4727	4.8994
3.340	0.1918	0.0430	3.5835	5.2400
3.345	0.1873	0.0416	3.6437	5.4325
3.350	0.1826	0.0401	3.7072	5.6414
3.355	0.1777	0.0386	3.7734	5.8656

$\hat{\alpha}_3 = 0.55,\ 1.40 < \hat{\alpha}_4 < 3.485$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.400	0.3674	0.4505	0.0791	0.3421
1.600	0.3673	0.3923	0.2244	0.3649
1.800	0.3664	0.3421	0.3624	0.4040
2.000	0.3642	0.2980	0.5032	0.4589
2.200	0.3604	0.2586	0.6546	0.5336
2.400	0.3544	0.2227	0.8245	0.6362
2.600	0.3456	0.1892	1.0245	0.7828
2.800	0.3324	0.1570	1.2742	1.0062
2.900	0.3234	0.1410	1.4281	1.1676
3.000	0.3119	0.1249	1.6119	1.3843
3.100	0.2968	0.1083	1.8373	1.6892
3.200	0.2762	0.0908	2.1300	2.1528
3.250	0.2628	0.0816	2.3141	2.4863
3.300	0.2461	0.0718	2.5363	2.9362
3.350	0.2250	0.0612	2.8118	3.5732
3.375	0.2120	0.0555	2.9782	4.0047
3.400	0.1969	0.0495	3.1695	4.5490
3.420	0.1828	0.0444	3.3464	5.1043
3.440	0.1664	0.0390	3.5508	5.8144
3.450	0.1571	0.0361	3.6658	6.2507
3.460	0.1470	0.0331	3.7906	6.7582
3.470	0.1358	0.0299	3.9275	7.3600
3.475	0.1297	0.0282	4.0018	7.7082
3.480	0.1237	0.0266	4.0748	8.0692
3.485	0.1171	0.0249	4.1556	8.4877

$0.60 \leq \hat{\alpha}_3 \leq 0.65$

$\hat{\alpha}_3 = 0.60,\ 1.40 < \hat{\alpha}_4 < 3.60$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.400	0.3563	0.4659	0.0330	0.3689
1.600	0.3563	0.4063	0.1812	0.3871
1.800	0.3557	0.3549	0.3176	0.4238
2.000	0.3538	0.3099	0.4539	0.4765
2.200	0.3505	0.2699	0.5975	0.5478
2.400	0.3451	0.2337	0.7557	0.6445
2.600	0.3372	0.2001	0.9374	0.7788
2.700	0.3319	0.1840	1.0413	0.8670
2.800	0.3256	0.1682	1.1567	0.9752
2.900	0.3178	0.1526	1.2880	1.1114
3.000	0.3081	0.1370	1.4392	1.2866
3.100	0.2959	0.1212	1.6180	1.5201
3.150	0.2886	0.1132	1.7217	1.6689
3.200	0.2802	0.1050	1.8374	1.8473
3.250	0.2703	0.0966	1.9683	2.0657
3.300	0.2589	0.0879	2.1181	2.3374
3.350	0.2451	0.0789	2.2921	2.6855
3.400	0.2283	0.0693	2.5006	3.1507
3.450	0.2073	0.0590	2.7553	3.7989
3.475	0.1946	0.0535	2.9071	4.2321
3.500	0.1799	0.0476	3.0808	4.7758
3.525	0.1629	0.0415	3.2782	5.4661
3.550	0.1426	0.0348	3.5137	6.4044
3.570	0.1231	0.0289	3.7373	7.4394
3.590	0.1005	0.0227	3.9917	8.8466
3.600	0.0873	0.0193	4.1408	9.8172

$\hat{\alpha}_3 = 0.65,\ 1.60 < \hat{\alpha}_4 < 3.72$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.600	0.3455	0.4217	0.1347	0.4090
1.800	0.3451	0.3689	0.2707	0.4418
2.000	0.3437	0.3228	0.4035	0.4914
2.200	0.3409	0.2821	0.5405	0.5588
2.400	0.3361	0.2454	0.6884	0.6491
2.600	0.3291	0.2116	0.8546	0.7717
2.800	0.3189	0.1800	1.0490	0.9447
3.000	0.3041	0.1494	1.2897	1.2046
3.200	0.2816	0.1188	1.6080	1.6328
3.300	0.2656	0.1030	1.8162	1.9694
3.400	0.2445	0.0864	2.0771	2.4620
3.500	0.2149	0.0684	2.4264	3.2585
3.550	0.1950	0.0586	2.6533	3.8746
3.600	0.1700	0.0479	2.9327	4.7618
3.620	0.1579	0.0433	3.0656	5.2419
3.640	0.1444	0.0384	3.2125	5.8250
3.650	0.1369	0.0359	3.2938	6.1734
3.660	0.1293	0.0334	3.3758	6.5482
3.670	0.1207	0.0307	3.4684	6.9971
3.680	0.1112	0.0278	3.5705	7.5295
3.690	0.1016	0.0250	3.6728	8.1142
3.700	0.0907	0.0219	3.7891	8.8430
3.705	0.0851	0.0204	3.8479	9.2456
3.710	0.0796	0.0189	3.9061	9.6710
3.715	0.0731	0.0172	3.9746	10.2018
3.720	0.0668	0.0155	4.0406	10.7629

$$0.70 \leq \hat{\alpha}_3 \leq 0.75$$

$\hat{\alpha}_3 = 0.70,\ 1.60 < \hat{\alpha}_4 < 3.825$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.600	0.3349	0.4385	0.0845	0.4314
1.800	0.3348	0.3842	0.2218	0.4589
2.000	0.3338	0.3369	0.3518	0.5045
2.200	0.3314	0.2953	0.4833	0.5674
2.400	0.3274	0.2580	0.6225	0.6510
2.600	0.3213	0.2239	0.7753	0.7626
2.800	0.3124	0.1922	0.9498	0.9156
3.000	0.2998	0.1621	1.1577	1.1348
3.100	0.2915	0.1473	1.2801	1.2836
3.200	0.2814	0.1324	1.4199	1.4722
3.300	0.2690	0.1175	1.5821	1.7180
3.400	0.2533	0.1022	1.7759	2.0516
3.500	0.2329	0.0863	2.0145	2.5284
3.550	0.2202	0.0778	2.1590	2.8550
3.600	0.2053	0.0691	2.3240	3.2690
3.650	0.1873	0.0598	2.5194	3.8187
3.700	0.1653	0.0499	2.7532	4.5786
3.725	0.1520	0.0445	2.8917	5.0883
3.750	0.1373	0.0389	3.0443	5.7129
3.775	0.1202	0.0330	3.2190	6.5229
3.790	0.1089	0.0292	3.3345	7.1262
3.800	0.1006	0.0266	3.4188	7.6070
3.810	0.0916	0.0238	3.5098	8.1698
3.815	0.0870	0.0225	3.5554	8.4733
3.820	0.0821	0.0211	3.6043	8.8144
3.825	0.0770	0.0196	3.6555	9.1928

$\hat{\alpha}_3 = 0.75,\ 1.80 < \hat{\alpha}_4 < 3.94$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.800	0.3245	0.4008	0.1703	0.4760
2.000	0.3239	0.3521	0.2990	0.5165
2.200	0.3221	0.3094	0.4262	0.5743
2.400	0.3188	0.2714	0.5578	0.6512
2.600	0.3136	0.2369	0.6996	0.7525
2.800	0.3060	0.2050	0.8577	0.8880
2.900	0.3011	0.1899	0.9452	0.9735
3.000	0.2953	0.1750	1.0407	1.0753
3.100	0.2884	0.1605	1.1453	1.1978
3.200	0.2801	0.1461	1.2613	1.3476
3.300	0.2702	0.1318	1.3925	1.5352
3.400	0.2580	0.1173	1.5440	1.7768
3.500	0.2430	0.1026	1.7214	2.0973
3.550	0.2340	0.0950	1.8236	2.3011
3.600	0.2239	0.0873	1.9362	2.5435
3.650	0.2122	0.0793	2.0633	2.8408
3.700	0.1988	0.0711	2.2060	3.2075
3.750	0.1829	0.0624	2.3713	3.6800
3.775	0.1738	0.0578	2.4639	3.9697
3.800	0.1639	0.0532	2.5648	4.3073
3.825	0.1529	0.0483	2.6760	4.7090
3.850	0.1408	0.0433	2.7955	5.1821
3.875	0.1271	0.0379	2.9315	5.7745
3.900	0.1118	0.0324	3.0800	6.5079
3.920	0.0980	0.0277	3.2135	7.2566
3.930	0.0905	0.0252	3.2853	7.7032
3.940	0.0825	0.0227	3.3630	8.2256

$$0.80 \le \hat{\alpha}_3 \le 0.85$$

$\hat{\alpha}_3 = 0.80,\ 1.70 < \hat{\alpha}_4 < 4.10$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.700	0.3143	0.4466	0.0454	0.4853
2.000	0.3141	0.3685	0.2449	0.5280
2.200	0.3129	0.3246	0.3688	0.5802
2.400	0.3103	0.2857	0.4945	0.6503
2.600	0.3060	0.2506	0.6270	0.7419
2.800	0.2995	0.2184	0.7715	0.8621
3.000	0.2905	0.1884	0.9347	1.0237
3.100	0.2847	0.1740	1.0255	1.1261
3.200	0.2779	0.1599	1.1247	1.2484
3.300	0.2698	0.1459	1.2344	1.3967
3.400	0.2602	0.1320	1.3568	1.5796
3.500	0.2486	0.1181	1.4960	1.8108
3.600	0.2345	0.1038	1.6579	2.1127
3.700	0.2169	0.0893	1.8489	2.5201
3.800	0.1944	0.0739	2.0824	3.1021
3.850	0.1807	0.0659	2.2205	3.4967
3.900	0.1646	0.0574	2.3791	4.0021
3.950	0.1456	0.0484	2.5637	4.6728
3.975	0.1347	0.0437	2.6673	5.0962
4.000	0.1228	0.0389	2.7794	5.5980
4.025	0.1094	0.0337	2.9054	6.2253
4.040	0.1007	0.0305	2.9865	6.6718
4.050	0.0946	0.0284	3.0421	7.0020
4.060	0.0881	0.0261	3.1022	7.3800
4.070	0.0812	0.0238	3.1664	7.8143
4.080	0.0739	0.0214	3.2333	8.3052
4.090	0.0664	0.0190	3.3016	8.8573
4.100	0.0587	0.0166	3.3718	9.4892

$\hat{\alpha}_3 = 0.85,\ 1.80 < \hat{\alpha}_4 < 4.26$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.800	0.3044	0.4381	0.0570	0.5144
2.100	0.3042	0.3628	0.2506	0.5607
2.300	0.3030	0.3203	0.3715	0.6152
2.500	0.3003	0.2825	0.4938	0.6877
2.700	0.2959	0.2483	0.6226	0.7816
2.900	0.2895	0.2171	0.7621	0.9036
3.100	0.2806	0.1878	0.9181	1.0655
3.300	0.2683	0.1601	1.0982	1.2872
3.500	0.2514	0.1331	1.3143	1.6066
3.700	0.2274	0.1060	1.5870	2.1019
3.800	0.2115	0.0921	1.7561	2.4672
3.900	0.1916	0.0777	1.9576	2.9696
4.000	0.1663	0.0624	2.2034	3.6991
4.050	0.1508	0.0543	2.3498	4.2067
4.100	0.1326	0.0457	2.5180	4.8715
4.125	0.1222	0.0412	2.6138	5.2958
4.150	0.1110	0.0365	2.7157	5.7922
4.175	0.0986	0.0317	2.8265	6.3938
4.190	0.0906	0.0287	2.8983	6.8245
4.200	0.0849	0.0266	2.9493	7.1528
4.210	0.0790	0.0245	3.0017	7.5133
4.220	0.0729	0.0224	3.0554	7.9107
4.230	0.0664	0.0202	3.1125	8.3667
4.240	0.0598	0.0180	3.1704	8.8752
4.245	0.0563	0.0168	3.2001	9.1571
4.250	0.0530	0.0157	3.2293	9.4499
4.255	0.0490	0.0145	3.2645	9.8212
4.260	0.0455	0.0134	3.2941	10.1608

$$0.90 \leq \hat{\alpha}_3 \leq 0.95$$

$\hat{\alpha}_3 = 0.90,\ 1.90 < \hat{\alpha}_4 < 4.40$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
1.900	0.2949	0.4311	0.0644	0.5429
2.200	0.2946	0.3583	0.2527	0.5919
2.400	0.2933	0.3170	0.3705	0.6479
2.600	0.2906	0.2803	0.4895	0.7219
2.800	0.2863	0.2470	0.6141	0.8169
3.000	0.2801	0.2165	0.7483	0.9393
3.200	0.2714	0.1881	0.8969	1.0993
3.400	0.2596	0.1611	1.0662	1.3149
3.500	0.2522	0.1480	1.1618	1.4528
3.600	0.2436	0.1349	1.2669	1.6189
3.700	0.2335	0.1220	1.3827	1.8209
3.800	0.2216	0.1089	1.5134	2.0734
3.900	0.2073	0.0957	1.6617	2.3954
4.000	0.1900	0.0821	1.8346	2.8226
4.050	0.1797	0.0751	1.9336	3.0951
4.100	0.1684	0.0679	2.0405	3.4161
4.150	0.1556	0.0605	2.1594	3.8078
4.200	0.1410	0.0528	2.2926	4.2966
4.225	0.1328	0.0487	2.3657	4.5904
4.250	0.1241	0.0446	2.4434	4.9254
4.275	0.1148	0.0405	2.5256	5.3087
4.300	0.1045	0.0360	2.6163	5.7698
4.320	0.0958	0.0325	2.6916	6.1898
4.340	0.0863	0.0287	2.7740	6.6937
4.350	0.0816	0.0269	2.8136	6.9576
4.360	0.0764	0.0250	2.8586	7.2717
4.370	0.0711	0.0230	2.9036	7.6081
4.380	0.0658	0.0211	2.9489	7.9719
4.390	0.0599	0.0191	2.9985	8.4001
4.400	0.0539	0.0170	3.0494	8.8826

$\hat{\alpha}_3 = 0.95,\ 2.20 < \hat{\alpha}_4 < 4.58$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.200	0.2855	0.3769	0.1929	0.5993
2.400	0.2848	0.3342	0.3091	0.6477
2.600	0.2829	0.2964	0.4235	0.7132
2.800	0.2795	0.2624	0.5411	0.7976
3.000	0.2744	0.2313	0.6653	0.9050
3.200	0.2673	0.2027	0.7994	1.0423
3.400	0.2576	0.1757	0.9489	1.2220
3.600	0.2448	0.1500	1.1196	1.4639
3.800	0.2275	0.1249	1.3214	1.8045
3.900	0.2166	0.1124	1.4387	2.0322
4.000	0.2039	0.0998	1.5696	2.3154
4.100	0.1887	0.0870	1.7191	2.6794
4.200	0.1704	0.0737	1.8922	3.1626
4.250	0.1598	0.0669	1.9890	3.4663
4.300	0.1479	0.0598	2.0971	3.8362
4.350	0.1346	0.0526	2.2146	4.2828
4.375	0.1273	0.0489	2.2784	4.5472
4.400	0.1194	0.0450	2.3469	4.8490
4.425	0.1111	0.0411	2.4178	5.1858
4.450	0.1022	0.0371	2.4933	5.5743
4.475	0.0927	0.0330	2.5733	6.0248
4.490	0.0864	0.0304	2.6270	6.3512
4.500	0.0822	0.0287	2.6618	6.5769
4.510	0.0779	0.0270	2.6980	6.8231
4.520	0.0732	0.0251	2.7369	7.1012
4.530	0.0688	0.0235	2.7728	7.3764
4.540	0.0641	0.0217	2.8118	7.6917
4.550	0.0588	0.0197	2.8560	8.0734
4.560	0.0536	0.0178	2.8983	8.4718
4.570	0.0484	0.0159	2.9411	8.9135
4.580	0.0428	0.0140	2.9865	9.4302

$1.00 \leq \hat{\alpha}_3 \leq 1.05$

$\hat{\alpha}_3 = 1.00,\ 2.10 < \hat{\alpha}_4 < 4.80$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.100	0.2764	0.4212	0.0681	0.5979
2.300	0.2764	0.3740	0.1900	0.6268
2.500	0.2758	0.3324	0.3032	0.6754
2.700	0.2740	0.2955	0.4147	0.7411
2.900	0.2708	0.2622	0.5283	0.8251
3.100	0.2659	0.2320	0.6472	0.9309
3.300	0.2591	0.2040	0.7754	1.0650
3.500	0.2500	0.1777	0.9164	1.2379
3.700	0.2380	0.1527	1.0749	1.4658
3.900	0.2222	0.1284	1.2592	1.7788
4.000	0.2124	0.1165	1.3636	1.9816
4.100	0.2011	0.1044	1.4797	2.2301
4.200	0.1879	0.0922	1.6098	2.5402
4.300	0.1722	0.0798	1.7564	2.9360
4.400	0.1535	0.0670	1.9258	3.4619
4.500	0.1307	0.0536	2.1244	4.1939
4.550	0.1174	0.0466	2.2379	4.6817
4.600	0.1025	0.0393	2.3621	5.2896
4.625	0.0942	0.0355	2.4308	5.6651
4.650	0.0854	0.0316	2.5036	6.0997
4.675	0.0761	0.0277	2.5789	6.5999
4.700	0.0662	0.0236	2.6595	7.2024
4.720	0.0577	0.0203	2.7271	7.7768
4.740	0.0484	0.0168	2.8020	8.5074
4.760	0.0390	0.0133	2.8764	9.3781
4.770	0.0339	0.0115	2.9162	9.9247
4.780	0.0287	0.0096	2.9575	10.5743
4.785	0.0261	0.0087	2.9779	10.9361
4.790	0.0230	0.0077	3.0023	11.4080
4.795	0.0206	0.0068	3.0214	11.8324
4.800	0.0178	0.0059	3.0435	12.3725

$\hat{\alpha}_3 = 1.05,\ 2.40 < \hat{\alpha}_4 < 4.98$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.400	0.2677	0.3721	0.1843	0.6528
2.600	0.2670	0.3316	0.2946	0.7010
2.800	0.2654	0.2954	0.4030	0.7662
3.000	0.2623	0.2629	0.5126	0.8490
3.200	0.2578	0.2333	0.6267	0.9524
3.400	0.2514	0.2058	0.7487	1.0823
3.600	0.2429	0.1803	0.8811	1.2469
3.800	0.2318	0.1559	1.0286	1.4606
4.000	0.2174	0.1324	1.1967	1.7462
4.200	0.1987	0.1093	1.3938	2.1449
4.400	0.1738	0.0861	1.6325	2.7355
4.500	0.1581	0.0742	1.7745	3.1531
4.600	0.1395	0.0619	1.9364	3.7034
4.700	0.1172	0.0490	2.1245	4.4655
4.750	0.1044	0.0423	2.2303	4.9680
4.775	0.0974	0.0389	2.2871	5.2652
4.800	0.0902	0.0354	2.3451	5.5927
4.820	0.0839	0.0326	2.3956	5.8976
4.840	0.0775	0.0297	2.4467	6.2304
4.860	0.0707	0.0267	2.5005	6.6106
4.880	0.0636	0.0237	2.5570	7.0475
4.890	0.0597	0.0221	2.5877	7.3035
4.900	0.0559	0.0206	2.6173	7.5658
4.910	0.0520	0.0190	2.6479	7.8553
4.920	0.0481	0.0175	2.6783	8.1634
4.930	0.0441	0.0159	2.7086	8.4964
4.940	0.0397	0.0142	2.7431	8.9036
4.950	0.0355	0.0126	2.7754	9.3289
4.960	0.0309	0.0109	2.8115	9.8543
4.970	0.0262	0.0092	2.8474	10.4577
4.980	0.0216	0.0075	2.8825	11.1540

$1.10 \leq \hat{\alpha}_3 \leq 1.15$

$\hat{\alpha}_3 = 1.10,\ 2.30 < \hat{\alpha}_4 < 5.20$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.300	0.2591	0.4160	0.0587	0.6498
2.600	0.2590	0.3507	0.2303	0.6988
2.900	0.2571	0.2961	0.3887	0.7886
3.200	0.2524	0.2494	0.5486	0.9171
3.500	0.2441	0.2083	0.7198	1.0947
3.700	0.2363	0.1833	0.8445	1.2508
3.800	0.2315	0.1713	0.9114	1.3441
3.900	0.2261	0.1596	0.9815	1.4498
4.100	0.2131	0.1368	1.1352	1.7097
4.300	0.1964	0.1146	1.3117	2.0614
4.500	0.1748	0.0925	1.5202	2.5610
4.600	0.1616	0.0813	1.6411	2.8990
4.700	0.1463	0.0699	1.7753	3.3239
4.800	0.1284	0.0582	1.9266	3.8769
4.850	0.1183	0.0522	2.0101	4.2227
4.900	0.1071	0.0460	2.1012	4.6381
4.950	0.0951	0.0396	2.1983	5.1347
5.000	0.0819	0.0332	2.3016	5.7401
5.025	0.0746	0.0298	2.3591	6.1174
5.050	0.0671	0.0264	2.4176	6.5402
5.075	0.0591	0.0229	2.4792	7.0361
5.100	0.0509	0.0194	2.5421	7.6121
5.120	0.0436	0.0164	2.5981	8.1970
5.130	0.0400	0.0150	2.6253	8.5169
5.140	0.0362	0.0135	2.6543	8.8856
5.150	0.0327	0.0121	2.6808	9.2577
5.160	0.0286	0.0105	2.7113	9.7292
5.170	0.0244	0.0089	2.7431	10.2875
5.180	0.0204	0.0074	2.7729	10.9058
5.190	0.0161	0.0058	2.8057	11.7185
5.200	0.0119	0.0043	2.8361	12.6939

$\hat{\alpha}_3 = 1.15,\ 2.40 < \hat{\alpha}_4 < 5.40$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.400	0.2508	0.4150	0.0495	0.6747
2.600	0.2509	0.3711	0.1647	0.7005
2.800	0.2505	0.3321	0.2702	0.7464
3.000	0.2491	0.2974	0.3724	0.8086
3.200	0.2466	0.2662	0.4745	0.8872
3.400	0.2427	0.2377	0.5793	0.9839
3.600	0.2372	0.2113	0.6895	1.1030
3.800	0.2300	0.1868	0.8067	1.2502
4.000	0.2208	0.1637	0.9343	1.4348
4.200	0.2091	0.1415	1.0756	1.6716
4.400	0.1942	0.1201	1.2347	1.9829
4.500	0.1855	0.1096	1.3223	2.1769
4.600	0.1755	0.0990	1.4175	2.4072
4.700	0.1643	0.0884	1.5210	2.6831
4.800	0.1514	0.0777	1.6351	3.0219
4.900	0.1368	0.0669	1.7605	3.4425
4.950	0.1286	0.0614	1.8283	3.6941
5.100	0.1001	0.0443	2.0592	4.7120
5.150	0.0891	0.0384	2.1460	5.1776
5.200	0.0769	0.0323	2.2402	5.7521
5.250	0.0638	0.0261	2.3409	6.4726
5.275	0.0568	0.0229	2.3934	6.9066
5.300	0.0493	0.0196	2.4500	7.4306
5.325	0.0414	0.0162	2.5096	8.0665
5.340	0.0371	0.0144	2.5409	8.4543
5.350	0.0333	0.0129	2.5698	8.8369
5.360	0.0300	0.0116	2.5937	9.1919
5.370	0.0266	0.0102	2.6196	9.6148
5.380	0.0230	0.0088	2.6454	10.0913
5.390	0.0194	0.0073	2.6720	10.6547
5.400	0.0155	0.0058	2.7009	11.3845

$1.20 \leq \hat{\alpha}_3 \leq 1.25$

$\hat{\alpha}_3 = 1.20,\ 2.50 < \hat{\alpha}_4 < 5.63$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.500	0.2428	0.4148	0.0376	0.6991
2.800	0.2428	0.3521	0.2037	0.7422
3.100	0.2414	0.2995	0.3539	0.8263
3.400	0.2376	0.2543	0.5028	0.9462
3.700	0.2307	0.2148	0.6579	1.1078
4.000	0.2202	0.1793	0.8268	1.3270
4.300	0.2053	0.1466	1.0171	1.6318
4.500	0.1921	0.1259	1.1609	1.9073
4.700	0.1758	0.1055	1.3238	2.2726
4.900	0.1553	0.0854	1.5111	2.7748
5.000	0.1431	0.0753	1.6170	3.1044
5.100	0.1293	0.0650	1.7331	3.5110
5.200	0.1136	0.0545	1.8615	4.0266
5.250	0.1049	0.0492	1.9307	4.3392
5.300	0.0956	0.0438	2.0038	4.7010
5.350	0.0856	0.0383	2.0813	5.1265
5.400	0.0748	0.0326	2.1637	5.6361
5.425	0.0692	0.0298	2.2060	5.9272
5.450	0.0631	0.0269	2.2515	6.2645
5.475	0.0570	0.0240	2.2971	6.6352
5.500	0.0505	0.0210	2.3458	7.0723
5.520	0.0452	0.0186	2.3850	7.4652
5.540	0.0397	0.0162	2.4257	7.9185
5.560	0.0338	0.0136	2.4685	8.4618
5.570	0.0309	0.0124	2.4904	8.7717
5.580	0.0278	0.0111	2.5125	9.1151
5.590	0.0248	0.0098	2.5347	9.4962
5.600	0.0217	0.0086	2.5571	9.9281
5.610	0.0186	0.0073	2.5792	10.4147
5.620	0.0154	0.0060	2.6025	11.0097
5.630	0.0122	0.0048	2.6250	11.7136

$\hat{\alpha}_3 = 1.25,\ 2.60 < \hat{\alpha}_4 < 5.87$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.600	0.2350	0.4156	0.0230	0.7231
2.900	0.2351	0.3540	0.1872	0.7617
3.200	0.2340	0.3021	0.3338	0.8419
3.500	0.2306	0.2576	0.4775	0.9566
3.800	0.2245	0.2188	0.6255	1.1098
4.100	0.2150	0.1839	0.7845	1.3143
4.400	0.2016	0.1519	0.9608	1.5925
4.600	0.1900	0.1318	1.0914	1.8372
4.800	0.1757	0.1122	1.2368	2.1525
5.000	0.1582	0.0930	1.4013	2.5719
5.100	0.1479	0.0834	1.4919	2.8358
5.200	0.1365	0.0738	1.5900	3.1523
5.300	0.1237	0.0641	1.6960	3.5352
5.400	0.1092	0.0542	1.8123	4.0134
5.450	0.1014	0.0493	1.8739	4.2963
5.500	0.0930	0.0442	1.9391	4.6224
5.550	0.0840	0.0391	2.0077	4.9994
5.600	0.0744	0.0338	2.0811	5.4485
5.625	0.0694	0.0312	2.1178	5.6967
5.650	0.0642	0.0285	2.1572	5.9804
5.675	0.0588	0.0259	2.1967	6.2884
5.700	0.0532	0.0231	2.2381	6.6400
5.725	0.0474	0.0204	2.2812	7.0432
5.750	0.0415	0.0176	2.3239	7.4927
5.775	0.0352	0.0148	2.3694	8.0387
5.800	0.0289	0.0120	2.4144	8.6774
5.820	0.0234	0.0096	2.4546	9.3561
5.840	0.0178	0.0073	2.4942	10.1929
5.850	0.0152	0.0062	2.5128	10.6756
5.860	0.0122	0.0049	2.5341	11.3259
5.870	0.0092	0.0037	2.5551	12.1357

$$1.30 \leq \hat{\alpha}_3 \leq 1.35$$

$\hat{\alpha}_3 = 1.30,\ 2.80 < \hat{\alpha}_4 < 6.11$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.800	0.2276	0.3956	0.0642	0.7519
3.100	0.2275	0.3385	0.2178	0.8015
3.400	0.2261	0.2899	0.3588	0.8886
3.700	0.2224	0.2481	0.4978	1.0088
4.000	0.2160	0.2112	0.6414	1.1673
4.300	0.2064	0.1780	0.7954	1.3764
4.600	0.1932	0.1476	0.9648	1.6569
4.900	0.1753	0.1189	1.1571	2.0471
5.100	0.1602	0.1004	1.3025	2.4010
5.200	0.1515	0.0913	1.3814	2.6175
5.300	0.1418	0.0822	1.4653	2.8692
5.400	0.1312	0.0731	1.5557	3.1677
5.500	0.1194	0.0639	1.6525	3.5241
5.600	0.1063	0.0547	1.7565	3.9568
5.700	0.0916	0.0453	1.8709	4.5048
5.750	0.0837	0.0405	1.9316	4.8344
5.800	0.0754	0.0358	1.9939	5.2082
5.850	0.0664	0.0309	2.0608	5.6558
5.900	0.0569	0.0259	2.1300	6.1845
5.950	0.0469	0.0209	2.2030	6.8386
5.975	0.0416	0.0183	2.2417	7.2380
6.000	0.0362	0.0158	2.2798	7.6830
6.025	0.0305	0.0132	2.3204	8.2240
6.050	0.0247	0.0106	2.3617	8.8780
6.070	0.0199	0.0084	2.3961	9.5397
6.080	0.0176	0.0074	2.4123	9.9065
6.090	0.0150	0.0063	2.4303	10.3658
6.095	0.0138	0.0058	2.4384	10.6010
6.100	0.0126	0.0053	2.4468	10.8642
6.105	0.0112	0.0047	2.4571	11.2078
6.110	0.0101	0.0042	2.4646	11.4983

$\hat{\alpha}_3 = 1.35,\ 2.90 < \hat{\alpha}_4 < 6.37$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
2.900	0.2203	0.3982	0.0447	0.7737
3.200	0.2204	0.3418	0.1969	0.8175
3.500	0.2193	0.2938	0.3345	0.8992
3.800	0.2161	0.2525	0.4684	1.0127
4.100	0.2105	0.2161	0.6058	1.1616
4.400	0.2020	0.1834	0.7505	1.3546
4.700	0.1902	0.1535	0.9077	1.6089
5.000	0.1746	0.1256	1.0828	1.9526
5.200	0.1615	0.1078	1.2122	2.2539
5.300	0.1540	0.0991	1.2822	2.4354
5.400	0.1458	0.0904	1.3558	2.6423
5.500	0.1368	0.0817	1.4339	2.8810
5.600	0.1271	0.0731	1.5159	3.1564
5.700	0.1163	0.0645	1.6042	3.4839
5.800	0.1044	0.0558	1.6983	3.8754
5.850	0.0980	0.0514	1.7479	4.1023
5.900	0.0915	0.0471	1.7984	4.3504
5.950	0.0844	0.0426	1.8525	4.6368
6.000	0.0770	0.0381	1.9080	4.9572
6.050	0.0692	0.0336	1.9659	5.3258
6.100	0.0610	0.0291	2.0255	5.7501
6.150	0.0524	0.0245	2.0884	6.2590
6.200	0.0434	0.0199	2.1531	6.8712
6.225	0.0385	0.0175	2.1878	7.2484
6.250	0.0337	0.0152	2.2221	7.6672
6.275	0.0287	0.0128	2.2575	8.1616
6.300	0.0237	0.0105	2.2921	8.7284
6.320	0.0193	0.0085	2.3230	9.3282
6.340	0.0150	0.0065	2.3533	10.0578
6.360	0.0107	0.0046	2.3831	11.0022
6.370	0.0084	0.0036	2.3984	11.6337

$1.40 \leq \hat{\alpha}_3 \leq 1.45$

$\hat{\alpha}_3 = 1.40,\ 3.00 < \hat{\alpha}_4 < 6.65$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.000	0.2133	0.4015	0.0226	0.7958
3.300	0.2134	0.3457	0.1744	0.8327
3.600	0.2126	0.2982	0.3091	0.9086
3.900	0.2100	0.2573	0.4386	1.0152
4.200	0.2051	0.2212	0.5701	1.1545
4.500	0.1977	0.1890	0.7068	1.3330
4.800	0.1873	0.1597	0.8533	1.5640
5.100	0.1735	0.1325	1.0132	1.8678
5.300	0.1621	0.1151	1.1307	2.1292
5.500	0.1487	0.0984	1.2575	2.4538
5.700	0.1328	0.0818	1.3979	2.8717
5.900	0.1141	0.0656	1.5523	3.4205
6.000	0.1034	0.0574	1.6373	3.7708
6.100	0.0917	0.0492	1.7275	4.1901
6.200	0.0791	0.0409	1.8233	4.7023
6.300	0.0652	0.0326	1.9253	5.3473
6.350	0.0579	0.0284	1.9790	5.7416
6.400	0.0501	0.0242	2.0352	6.2087
6.450	0.0419	0.0199	2.0932	6.7666
6.500	0.0334	0.0156	2.1531	7.4590
6.525	0.0290	0.0134	2.1839	7.8793
6.550	0.0245	0.0113	2.2152	8.3704
6.575	0.0198	0.0090	2.2483	8.9847
6.600	0.0153	0.0069	2.2795	9.7122
6.610	0.0132	0.0059	2.2940	10.1116
6.620	0.0113	0.0050	2.3071	10.5323
6.630	0.0093	0.0041	2.3208	11.0479
6.635	0.0084	0.0037	2.3267	11.3092
6.640	0.0073	0.0033	2.3341	11.6655
6.645	0.0064	0.0028	2.3404	12.0138
6.650	0.0052	0.0023	2.3489	12.5512

$\hat{\alpha}_3 = 1.45,\ 3.20 < \hat{\alpha}_4 < 6.925$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.200	0.2066	0.3859	0.0528	0.8222
3.500	0.2067	0.3336	0.1957	0.8669
3.800	0.2057	0.2888	0.3250	0.9471
4.100	0.2029	0.2500	0.4503	1.0566
4.400	0.1980	0.2158	0.5768	1.1972
4.700	0.1906	0.1850	0.7083	1.3756
5.000	0.1806	0.1569	0.8485	1.6037
5.200	0.1722	0.1393	0.9486	1.7925
5.400	0.1623	0.1225	1.0546	2.0188
5.600	0.1507	0.1062	1.1694	2.2970
5.800	0.1371	0.0904	1.2924	2.6400
5.900	0.1295	0.0826	1.3582	2.8447
6.000	0.1213	0.0748	1.4275	3.0787
6.100	0.1125	0.0671	1.5000	3.3460
6.200	0.1029	0.0594	1.5764	3.6566
6.300	0.0926	0.0518	1.6561	4.0184
6.400	0.0815	0.0441	1.7403	4.4501
6.450	0.0755	0.0402	1.7854	4.7058
6.500	0.0693	0.0363	1.8310	4.9870
6.550	0.0629	0.0324	1.8773	5.2984
6.600	0.0563	0.0285	1.9256	5.6570
6.650	0.0493	0.0246	1.9753	6.0698
6.700	0.0421	0.0207	2.0268	6.5578
6.750	0.0347	0.0168	2.0786	7.1361
6.775	0.0307	0.0147	2.1066	7.4946
6.800	0.0268	0.0127	2.1338	7.8872
6.825	0.0228	0.0108	2.1614	8.3435
6.850	0.0186	0.0087	2.1904	8.9071
6.875	0.0145	0.0067	2.2182	9.5735
6.900	0.0103	0.0047	2.2473	10.4865
6.925	0.0059	0.0027	2.2771	11.8952

$$1.50 \leq \hat{\alpha}_3 \leq 1.55$$

$\hat{\alpha}_3 = 1.50,\ 3.30 < \hat{\alpha}_4 < 7.20$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.300	0.2000	0.3905	0.0269	0.8426
3.600	0.2002	0.3387	0.1695	0.8792
3.900	0.1996	0.2941	0.2967	0.9526
4.200	0.1973	0.2556	0.4182	1.0546
4.500	0.1931	0.2216	0.5393	1.1851
4.800	0.1868	0.1912	0.6642	1.3495
5.100	0.1780	0.1635	0.7951	1.5561
5.400	0.1665	0.1379	0.9357	1.8197
5.600	0.1571	0.1217	1.0360	2.0376
5.800	0.1463	0.1062	1.1426	2.2992
6.000	0.1338	0.0911	1.2569	2.6196
6.200	0.1193	0.0763	1.3813	3.0220
6.400	0.1028	0.0618	1.5154	3.5351
6.500	0.0936	0.0546	1.5872	3.8510
6.600	0.0838	0.0474	1.6621	4.2191
6.700	0.0732	0.0401	1.7416	4.6613
6.800	0.0619	0.0329	1.8242	5.1951
6.850	0.0559	0.0293	1.8678	5.5152
6.900	0.0498	0.0257	1.9107	5.8658
6.950	0.0434	0.0221	1.9567	6.2849
7.000	0.0367	0.0184	2.0037	6.7779
7.050	0.0299	0.0148	2.0511	7.3679
7.075	0.0264	0.0130	2.0749	7.7114
7.100	0.0228	0.0111	2.1000	8.1206
7.125	0.0192	0.0093	2.1243	8.5789
7.150	0.0154	0.0074	2.1505	9.1648
7.170	0.0125	0.0060	2.1705	9.7132
7.180	0.0110	0.0052	2.1803	10.0293
7.185	0.0102	0.0048	2.1860	10.2280
7.190	0.0093	0.0044	2.1919	10.4481
7.200	0.0079	0.0037	2.2016	10.8685

$\hat{\alpha}_3 = 1.55,\ 3.50 < \hat{\alpha}_4 < 7.50$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.500	0.1938	0.3776	0.0500	0.8677
3.800	0.1939	0.3287	0.1853	0.9102
4.100	0.1932	0.2865	0.3073	0.9862
4.400	0.1909	0.2498	0.4246	1.0894
4.700	0.1868	0.2173	0.5416	1.2200
5.000	0.1806	0.1882	0.6615	1.3825
5.300	0.1721	0.1616	0.7865	1.5845
5.600	0.1613	0.1371	0.9197	1.8387
5.800	0.1525	0.1216	1.0141	2.0462
6.000	0.1425	0.1066	1.1143	2.2937
6.200	0.1309	0.0922	1.2205	2.5916
6.400	0.1178	0.0781	1.3343	2.9573
6.500	0.1106	0.0712	1.3936	3.1708
6.600	0.1029	0.0644	1.4560	3.4138
6.700	0.0947	0.0575	1.5209	3.6900
6.800	0.0860	0.0508	1.5876	4.0041
6.900	0.0767	0.0440	1.6581	4.3740
7.000	0.0669	0.0373	1.7306	4.8061
7.100	0.0564	0.0306	1.8062	5.3284
7.150	0.0509	0.0272	1.8463	5.6420
7.200	0.0453	0.0239	1.8856	5.9852
7.250	0.0394	0.0205	1.9273	6.3914
7.300	0.0335	0.0172	1.9685	6.8545
7.325	0.0303	0.0155	1.9906	7.1318
7.350	0.0272	0.0138	2.0118	7.4270
7.375	0.0242	0.0122	2.0329	7.7514
7.400	0.0209	0.0104	2.0555	8.1413
7.425	0.0176	0.0087	2.0778	8.5863
7.450	0.0144	0.0071	2.0993	9.0972
7.475	0.0109	0.0054	2.1232	9.7930
7.500	0.0077	0.0037	2.1452	10.6613

$1.60 \leq \hat{\alpha}_3 \leq 1.65$

$\hat{\alpha}_3 = 1.60,\ 3.60 < \hat{\alpha}_4 < 7.85$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.600	0.1877	0.3834	0.0203	0.8872
3.900	0.1878	0.3346	0.1565	0.9203
4.200	0.1874	0.2927	0.2767	0.9889
4.500	0.1857	0.2561	0.3908	1.0840
4.800	0.1823	0.2238	0.5033	1.2047
5.100	0.1770	0.1948	0.6173	1.3538
5.400	0.1697	0.1685	0.7351	1.5371
5.700	0.1602	0.1443	0.8588	1.7638
6.000	0.1484	0.1217	0.9910	2.0488
6.300	0.1339	0.1005	1.1330	2.4121
6.500	0.1227	0.0869	1.2342	2.7133
6.700	0.1101	0.0737	1.3420	3.0806
6.900	0.0959	0.0608	1.4563	3.5338
7.100	0.0800	0.0480	1.5791	4.1153
7.200	0.0713	0.0417	1.6442	4.4748
7.300	0.0622	0.0354	1.7113	4.8947
7.400	0.0526	0.0292	1.7803	5.3947
7.500	0.0424	0.0229	1.8528	6.0208
7.550	0.0372	0.0199	1.8889	6.3878
7.600	0.0318	0.0168	1.9266	6.8223
7.650	0.0263	0.0137	1.9645	7.3356
7.700	0.0205	0.0105	2.0044	7.9886
7.725	0.0176	0.0090	2.0242	8.3793
7.750	0.0148	0.0075	2.0431	8.8159
7.775	0.0118	0.0060	2.0631	9.3636
7.800	0.0089	0.0045	2.0831	10.0598
7.820	0.0063	0.0032	2.1003	10.8520
7.830	0.0053	0.0026	2.1074	11.2834
7.840	0.0040	0.0020	2.1160	11.9191
7.845	0.0033	0.0017	2.1204	12.3191
7.850	0.0028	0.0014	2.1241	12.7427

$\hat{\alpha}_3 = 1.65,\ 3.80 < \hat{\alpha}_4 < 8.125$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.800	0.1818	0.3726	0.0378	0.9105
4.100	0.1820	0.3264	0.1674	0.9480
4.400	0.1816	0.2864	0.2832	1.0180
4.700	0.1798	0.2515	0.3930	1.1132
5.000	0.1765	0.2205	0.5016	1.2327
5.300	0.1714	0.1927	0.6110	1.3787
5.600	0.1645	0.1674	0.7238	1.5569
5.800	0.1588	0.1516	0.8016	1.6972
6.000	0.1522	0.1366	0.8823	1.8585
6.200	0.1447	0.1224	0.9657	2.0440
6.400	0.1361	0.1086	1.0533	2.2610
6.600	0.1264	0.0954	1.1450	2.5154
6.800	0.1155	0.0826	1.2410	2.8165
6.900	0.1097	0.0763	1.2910	2.9890
7.000	0.1035	0.0701	1.3425	3.1797
7.100	0.0970	0.0640	1.3952	3.3895
7.200	0.0900	0.0579	1.4502	3.6268
7.300	0.0828	0.0519	1.5064	3.8910
7.400	0.0752	0.0460	1.5634	4.1862
7.500	0.0672	0.0401	1.6232	4.5291
7.600	0.0589	0.0343	1.6844	4.9253
7.700	0.0501	0.0285	1.7475	5.3953
7.750	0.0456	0.0256	1.7796	5.6646
7.800	0.0409	0.0227	1.8125	5.9675
7.850	0.0361	0.0198	1.8463	6.3134
7.900	0.0314	0.0170	1.8791	6.6932
7.950	0.0264	0.0141	1.9135	7.1507
8.000	0.0213	0.0113	1.9484	7.7022
8.025	0.0188	0.0099	1.9655	8.0196
8.050	0.0162	0.0085	1.9828	8.3816
8.075	0.0135	0.0070	2.0015	8.8333
8.100	0.0110	0.0057	2.0184	9.3263
8.125	0.0083	0.0043	2.0362	9.9726

$$1.70 \leq \hat{\alpha}_3 \leq 1.75$$

$\hat{\alpha}_3 = 1.70,\ 3.90 < \hat{\alpha}_4 < 8.51$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
3.900	0.1762	0.3793	0.0049	0.9300
4.300	0.1764	0.3193	0.1755	0.9743
4.700	0.1755	0.2695	0.3228	1.0741
5.100	0.1723	0.2274	0.4628	1.2154
5.400	0.1680	0.1997	0.5677	1.3494
5.700	0.1621	0.1745	0.6745	1.5113
6.000	0.1544	0.1515	0.7847	1.7065
6.300	0.1449	0.1301	0.9001	1.9445
6.500	0.1374	0.1167	0.9800	2.1319
6.700	0.1290	0.1037	1.0638	2.3499
6.900	0.1197	0.0912	1.1508	2.6029
7.100	0.1094	0.0791	1.2415	2.8998
7.300	0.0979	0.0673	1.3374	3.2566
7.500	0.0853	0.0559	1.4372	3.6866
7.700	0.0716	0.0447	1.5421	4.2213
7.800	0.0642	0.0392	1.5973	4.5458
7.900	0.0565	0.0337	1.6534	4.9165
8.000	0.0486	0.0284	1.7102	5.3452
8.100	0.0403	0.0230	1.7696	5.8679
8.150	0.0360	0.0203	1.7991	6.1670
8.200	0.0318	0.0177	1.8288	6.5029
8.250	0.0273	0.0151	1.8600	6.8995
8.300	0.0229	0.0125	1.8902	7.3491
8.325	0.0206	0.0112	1.9058	7.6119
8.350	0.0183	0.0099	1.9213	7.9006
8.375	0.0159	0.0085	1.9379	8.2445
8.400	0.0137	0.0073	1.9529	8.6059
8.425	0.0112	0.0060	1.9696	9.0707
8.446	0.0093	0.0050	1.9820	9.4893
8.465	0.0075	0.0039	1.9947	9.9982
8.480	0.0061	0.0032	2.0039	10.4558
8.490	0.0051	0.0027	2.0102	10.8271
8.495	0.0046	0.0024	2.0136	11.0539
8.500	0.0042	0.0022	2.0165	11.2706
8.505	0.0036	0.0019	2.0205	11.5935
8.510	0.0032	0.0017	2.0229	11.8336

$\hat{\alpha}_3 = 1.75,\ 4.10 < \hat{\alpha}_4 < 8.80$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
4.100	0.1708	0.3703	0.0176	0.9517
4.500	0.1710	0.3131	0.1809	0.9993
4.900	0.1701	0.2654	0.3233	1.0990
5.300	0.1670	0.2250	0.4584	1.2380
5.600	0.1630	0.1983	0.5592	1.3685
5.900	0.1574	0.1740	0.6613	1.5246
6.200	0.1503	0.1517	0.7665	1.7118
6.500	0.1415	0.1311	0.8754	1.9366
6.700	0.1346	0.1181	0.9511	2.1129
6.900	0.1269	0.1056	1.0292	2.3143
7.100	0.1184	0.0936	1.1098	2.5452
7.300	0.1090	0.0820	1.1942	2.8152
7.500	0.0987	0.0707	1.2817	3.1310
7.700	0.0875	0.0599	1.3725	3.5052
7.800	0.0815	0.0545	1.4198	3.7221
7.900	0.0753	0.0492	1.4678	3.9608
8.000	0.0688	0.0440	1.5169	4.2270
8.100	0.0621	0.0388	1.5668	4.5256
8.200	0.0551	0.0337	1.6183	4.8679
8.300	0.0479	0.0287	1.6701	5.2583
8.350	0.0442	0.0262	1.6963	5.4771
8.400	0.0404	0.0237	1.7233	5.7196
8.450	0.0366	0.0212	1.7506	5.9882
8.500	0.0328	0.0188	1.7771	6.2755
8.525	0.0307	0.0176	1.7915	6.4429
8.550	0.0288	0.0164	1.8043	6.6040
8.575	0.0268	0.0152	1.8186	6.7912
8.600	0.0248	0.0140	1.8324	6.9869
8.625	0.0228	0.0128	1.8458	7.1913
8.650	0.0208	0.0116	1.8598	7.4209
8.675	0.0186	0.0104	1.8747	7.6864
8.700	0.0167	0.0092	1.8878	7.9499
8.725	0.0146	0.0080	1.9018	8.2612
8.750	0.0125	0.0069	1.9160	8.6203
8.775	0.0104	0.0057	1.9302	9.0400
8.800	0.0083	0.0045	1.9444	9.5482

$1.80 \leq \hat{\alpha}_3 \leq 1.85$

$\hat{\alpha}_3 = 1.80,\ 4.30 < \hat{\alpha}_4 < 9.24$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
4.300	0.1655	0.3625	0.0270	0.9731
4.700	0.1657	0.3078	0.1842	1.0229
5.100	0.1649	0.2620	0.3219	1.1217
5.500	0.1620	0.2231	0.4522	1.2577
5.900	0.1567	0.1892	0.5816	1.4318
6.300	0.1488	0.1593	0.7133	1.6503
6.600	0.1412	0.1388	0.8157	1.8510
6.900	0.1320	0.1197	0.9216	2.0906
7.200	0.1212	0.1019	1.0321	2.3794
7.500	0.1088	0.0850	1.1476	2.7316
7.700	0.0995	0.0742	1.2287	3.0148
7.900	0.0894	0.0638	1.3122	3.3443
8.100	0.0785	0.0536	1.3988	3.7351
8.300	0.0669	0.0438	1.4878	4.2043
8.400	0.0607	0.0389	1.5340	4.4810
8.500	0.0543	0.0341	1.5809	4.7919
8.600	0.0478	0.0294	1.6283	5.1443
8.700	0.0411	0.0248	1.6762	5.5499
8.800	0.0342	0.0202	1.7251	6.0327
8.850	0.0306	0.0180	1.7502	6.3146
8.900	0.0270	0.0157	1.7748	6.6235
8.950	0.0235	0.0135	1.7992	6.9693
9.000	0.0198	0.0113	1.8245	7.3809
9.050	0.0161	0.0091	1.8501	7.8722
9.075	0.0143	0.0080	1.8622	8.1463
9.100	0.0124	0.0069	1.8754	8.4760
9.120	0.0110	0.0061	1.8847	8.7460
9.140	0.0095	0.0053	1.8948	9.0722
9.160	0.0080	0.0044	1.9049	9.4503
9.180	0.0063	0.0035	1.9159	9.9405
9.190	0.0056	0.0031	1.9207	10.1957
9.200	0.0048	0.0027	1.9261	10.5192
9.210	0.0041	0.0022	1.9313	10.8831
9.220	0.0034	0.0019	1.9354	11.2390
9.230	0.0026	0.0014	1.9408	11.7926
9.240	0.0019	0.0010	1.9459	12.4831

$\hat{\alpha}_3 = 1.85,\ 4.50 < \hat{\alpha}_4 < 9.60$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
4.500	0.1605	0.3557	0.0336	0.9943
4.900	0.1607	0.3033	0.1853	1.0450
5.300	0.1599	0.2593	0.3185	1.1423
5.700	0.1572	0.2217	0.4443	1.2747
6.100	0.1523	0.1889	0.5688	1.4430
6.400	0.1471	0.1668	0.6630	1.5954
6.700	0.1405	0.1465	0.7591	1.7748
7.000	0.1325	0.1277	0.8580	1.9866
7.300	0.1232	0.1100	0.9605	2.2384
7.600	0.1123	0.0934	1.0669	2.5399
7.800	0.1043	0.0829	1.1402	2.7752
8.000	0.0956	0.0727	1.2159	3.0459
8.200	0.0862	0.0628	1.2938	3.3590
8.400	0.0762	0.0532	1.3744	3.7269
8.600	0.0655	0.0440	1.4567	4.1622
8.700	0.0598	0.0394	1.4991	4.4155
8.800	0.0541	0.0349	1.5418	4.6951
8.900	0.0482	0.0305	1.5849	5.0087
9.000	0.0420	0.0262	1.6291	5.3700
9.050	0.0389	0.0240	1.6511	5.5686
9.100	0.0358	0.0219	1.6734	5.7848
9.150	0.0327	0.0198	1.6954	6.0169
9.200	0.0295	0.0177	1.7177	6.2728
9.250	0.0262	0.0156	1.7403	6.5585
9.300	0.0230	0.0136	1.7624	6.8730
9.350	0.0197	0.0115	1.7854	7.2413
9.375	0.0181	0.0106	1.7963	7.4380
9.400	0.0164	0.0095	1.8082	7.6699
9.425	0.0147	0.0085	1.8195	7.9117
9.450	0.0131	0.0075	1.8307	8.1796
9.475	0.0114	0.0066	1.8419	8.4776
9.500	0.0097	0.0055	1.8539	8.8418
9.525	0.0081	0.0046	1.8644	9.2253
9.550	0.0063	0.0036	1.8764	9.7443
9.575	0.0047	0.0027	1.8873	10.3619
9.600	0.0030	0.0017	1.8985	11.2598

$1.90 \leq \hat{\alpha}_3 \leq 1.95$

$\hat{\alpha}_3 = 1.90,\ 4.70 < \hat{\alpha}_4 < 9.99$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
4.700	0.1556	0.3499	0.0377	1.0149
5.200	0.1558	0.2883	0.2178	1.0860
5.700	0.1542	0.2382	0.3749	1.2212
6.100	0.1507	0.2043	0.4951	1.3662
6.500	0.1451	0.1744	0.6149	1.5462
6.900	0.1373	0.1478	0.7366	1.7672
7.300	0.1273	0.1237	0.8620	2.0393
7.700	0.1150	0.1016	0.9928	2.3783
8.000	0.1043	0.0862	1.0947	2.6889
8.300	0.0923	0.0716	1.2005	3.0642
8.500	0.0836	0.0623	1.2732	3.3599
8.700	0.0743	0.0532	1.3482	3.7043
8.900	0.0645	0.0444	1.4245	4.1066
9.100	0.0541	0.0359	1.5026	4.5894
9.200	0.0487	0.0318	1.5419	4.8677
9.300	0.0433	0.0278	1.5815	5.1797
9.400	0.0377	0.0238	1.6217	5.5350
9.500	0.0319	0.0198	1.6625	5.9495
9.550	0.0291	0.0179	1.6825	6.1785
9.600	0.0261	0.0159	1.7035	6.4412
9.650	0.0232	0.0141	1.7232	6.7158
9.700	0.0203	0.0122	1.7439	7.0363
9.750	0.0173	0.0103	1.7643	7.3974
9.775	0.0158	0.0094	1.7745	7.5999
9.800	0.0143	0.0084	1.7852	7.8311
9.825	0.0128	0.0075	1.7956	8.0792
9.850	0.0113	0.0066	1.8054	8.3404
9.875	0.0097	0.0057	1.8161	8.6608
9.900	0.0083	0.0048	1.8261	9.0111
9.925	0.0068	0.0039	1.8364	9.4333
9.950	0.0053	0.0031	1.8460	9.9267
9.960	0.0047	0.0027	1.8502	10.1792
9.970	0.0041	0.0024	1.8544	10.4621
9.980	0.0035	0.0020	1.8586	10.7937
9.985	0.0032	0.0018	1.8606	10.9745
9.990	0.0028	0.0016	1.8627	11.1810

$\hat{\alpha}_3 = 1.95,\ 4.90 < \hat{\alpha}_4 < 10.40$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
4.900	0.1510	0.3448	0.0397	1.0349
5.400	0.1512	0.2855	0.2142	1.1048
5.900	0.1497	0.2371	0.3665	1.2361
6.300	0.1465	0.2042	0.4824	1.3758
6.700	0.1414	0.1752	0.5976	1.5481
7.100	0.1342	0.1493	0.7140	1.7579
7.400	0.1276	0.1316	0.8029	1.9442
7.700	0.1197	0.1150	0.8940	2.1614
8.000	0.1108	0.0994	0.9873	2.4147
8.200	0.1042	0.0895	1.0515	2.6094
8.400	0.0971	0.0801	1.1164	2.8258
8.600	0.0895	0.0709	1.1833	3.0719
8.800	0.0815	0.0621	1.2508	3.3489
9.000	0.0729	0.0535	1.3203	3.6685
9.100	0.0684	0.0493	1.3556	3.8467
9.200	0.0639	0.0452	1.3910	4.0385
9.300	0.0592	0.0412	1.4267	4.2464
9.400	0.0544	0.0372	1.4629	4.4740
9.500	0.0495	0.0333	1.4990	4.7219
9.600	0.0446	0.0294	1.5357	4.9987
9.700	0.0395	0.0257	1.5721	5.3038
9.800	0.0344	0.0220	1.6089	5.6510
9.900	0.0292	0.0184	1.6461	6.0532
9.950	0.0265	0.0166	1.6647	6.2796
10.000	0.0239	0.0148	1.6830	6.5248
10.050	0.0212	0.0130	1.7021	6.8044
10.100	0.0185	0.0113	1.7205	7.1098
10.150	0.0158	0.0096	1.7391	7.4607
10.175	0.0145	0.0088	1.7484	7.6569
10.200	0.0132	0.0080	1.7572	7.8597
10.225	0.0118	0.0071	1.7668	8.1010
10.250	0.0104	0.0062	1.7761	8.3602
10.275	0.0091	0.0054	1.7852	8.6496
10.300	0.0077	0.0046	1.7945	8.9868
10.325	0.0064	0.0038	1.8033	9.3626
10.350	0.0050	0.0030	1.8129	9.8613
10.375	0.0037	0.0022	1.8216	10.4505
10.400	0.0024	0.0014	1.8309	11.3429

$2.00 \le \hat{\alpha}_3 \le 2.05$

$\hat{\alpha}_3 = 2.00,\ 5.10 < \hat{\alpha}_4 < 10.835$					$\hat{\alpha}_3 = 2.05,\ 5.30 < \hat{\alpha}_4 < 11.275$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ	$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
5.100	0.1465	0.3405	0.0395	1.0544	5.300	0.1421	0.3370	0.0373	1.0733
5.600	0.1467	0.2833	0.2093	1.1222	5.800	0.1424	0.2815	0.2028	1.1383
6.100	0.1454	0.2363	0.3569	1.2491	6.300	0.1413	0.2360	0.3462	1.2605
6.500	0.1425	0.2044	0.4690	1.3836	6.700	0.1387	0.2049	0.4546	1.3895
6.900	0.1379	0.1762	0.5796	1.5480	7.100	0.1345	0.1775	0.5610	1.5462
7.300	0.1313	0.1510	0.6907	1.7463	7.500	0.1286	0.1529	0.6675	1.7339
7.700	0.1230	0.1283	0.8037	1.9852	7.900	0.1210	0.1307	0.7752	1.9580
8.000	0.1155	0.1125	0.8906	2.1971	8.200	0.1142	0.1154	0.8572	2.1542
8.300	0.1071	0.0977	0.9793	2.4425	8.500	0.1065	0.1010	0.9411	2.3803
8.600	0.0976	0.0838	1.0701	2.7287	8.800	0.0979	0.0875	1.0261	2.6398
8.900	0.0872	0.0706	1.1634	3.0668	9.100	0.0885	0.0746	1.1133	2.9430
9.100	0.0797	0.0622	1.2271	3.3280	9.300	0.0817	0.0665	1.1725	3.1739
9.300	0.0718	0.0541	1.2912	3.6222	9.500	0.0747	0.0587	1.2318	3.4300
9.500	0.0635	0.0462	1.3571	3.9626	9.700	0.0672	0.0510	1.2926	3.7219
9.600	0.0593	0.0424	1.3897	4.1487	9.800	0.0634	0.0474	1.3227	3.8800
9.700	0.0549	0.0386	1.4233	4.3546	9.900	0.0595	0.0437	1.3538	4.0533
9.800	0.0505	0.0349	1.4567	4.5764	10.000	0.0555	0.0402	1.3843	4.2350
9.900	0.0460	0.0313	1.4902	4.8179	10.100	0.0515	0.0367	1.4152	4.4333
10.000	0.0414	0.0278	1.5235	5.0824	10.200	0.0474	0.0333	1.4459	4.6452
10.100	0.0367	0.0242	1.5578	5.3829	10.300	0.0432	0.0299	1.4768	4.8771
10.200	0.0320	0.0208	1.5918	5.7183	10.400	0.0390	0.0266	1.5083	5.1350
10.300	0.0272	0.0175	1.6256	6.1012	10.500	0.0347	0.0233	1.5393	5.4169
10.400	0.0224	0.0142	1.6594	6.5494	10.600	0.0304	0.0202	1.5699	5.7293
10.450	0.0201	0.0126	1.6757	6.7997	10.700	0.0261	0.0171	1.6010	6.0885
10.500	0.0176	0.0110	1.6932	7.0945	10.750	0.0239	0.0156	1.6164	6.2863
10.550	0.0152	0.0094	1.7097	7.4135	10.800	0.0217	0.0140	1.6323	6.5088
10.600	0.0128	0.0079	1.7265	7.7886	10.850	0.0195	0.0125	1.6475	6.7426
10.625	0.0115	0.0071	1.7350	8.0040	10.900	0.0173	0.0110	1.6630	7.0065
10.650	0.0103	0.0063	1.7432	8.2325	10.950	0.0152	0.0096	1.6778	7.2890
10.675	0.0092	0.0056	1.7512	8.4804	11.000	0.0129	0.0082	1.6933	7.6251
10.700	0.0079	0.0048	1.7595	8.7721	11.025	0.0119	0.0075	1.7008	7.8064
10.725	0.0067	0.0041	1.7679	9.1097	11.050	0.0107	0.0067	1.7087	8.0151
10.750	0.0055	0.0033	1.7761	9.5038	11.075	0.0096	0.0060	1.7163	8.2377
10.775	0.0043	0.0026	1.7844	9.9913	11.100	0.0085	0.0053	1.7238	8.4796
10.795	0.0033	0.0020	1.7909	10.4788	11.125	0.0075	0.0046	1.7312	8.7510
10.810	0.0026	0.0015	1.7962	10.9828	11.150	0.0064	0.0039	1.7387	9.0618
10.815	0.0023	0.0014	1.7979	11.1792	11.175	0.0053	0.0033	1.7458	9.4132
10.820	0.0021	0.0012	1.7997	11.4012	11.200	0.0042	0.0026	1.7533	9.8562
10.825	0.0018	0.0011	1.8012	11.6250	11.225	0.0031	0.0019	1.7609	10.4321
10.830	0.0016	0.0010	1.8025	11.8438	11.250	0.0021	0.0013	1.7682	11.2176
10.835	0.0014	0.0008	1.8044	12.1743	11.275	0.0010	0.0006	1.7757	12.6253

$2.10 \leq \hat{\alpha}_3 \leq 2.15$

$\hat{\alpha}_3 = 2.10,\ 5.50 < \hat{\alpha}_4 < 11.69$					$\hat{\alpha}_3 = 2.15,\ 5.70 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ	$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
5.500	0.1380	0.3339	0.0337	1.0916	5.700	0.1339	0.3315	0.0283	1.1095
6.000	0.1382	0.2803	0.1951	1.1532	6.200	0.1342	0.2794	0.1863	1.1671
6.500	0.1373	0.2360	0.3345	1.2703	6.700	0.1335	0.2362	0.3223	1.2790
7.000	0.1342	0.1987	0.4653	1.4287	7.200	0.1308	0.1998	0.4490	1.4303
7.400	0.1300	0.1726	0.5677	1.5848	7.500	0.1281	0.1805	0.5230	1.5390
7.800	0.1243	0.1493	0.6699	1.7700	7.800	0.1246	0.1627	0.5964	1.6621
8.200	0.1170	0.1282	0.7727	1.9887	8.100	0.1203	0.1463	0.6699	1.8012
8.600	0.1082	0.1088	0.8777	2.2490	8.400	0.1152	0.1309	0.7439	1.9582
8.900	0.1007	0.0954	0.9571	2.4749	8.700	0.1094	0.1166	0.8183	2.1350
9.200	0.0924	0.0827	1.0385	2.7356	9.000	0.1028	0.1033	0.8933	2.3343
9.500	0.0834	0.0707	1.1209	3.0362	9.300	0.0956	0.0907	0.9692	2.5603
9.800	0.0737	0.0594	1.2044	3.3852	9.600	0.0876	0.0787	1.0465	2.8190
10.000	0.0669	0.0522	1.2608	3.6524	9.800	0.0819	0.0712	1.0983	3.0115
10.200	0.0598	0.0452	1.3178	3.9541	10.000	0.0760	0.0639	1.1504	3.2223
10.400	0.0524	0.0385	1.3749	4.2956	10.200	0.0699	0.0569	1.2028	3.4549
10.600	0.0449	0.0320	1.4321	4.6883	10.400	0.0634	0.0501	1.2559	3.7148
10.700	0.0411	0.0289	1.4608	4.9102	10.600	0.0569	0.0436	1.3086	4.0027
10.800	0.0372	0.0258	1.4895	5.1518	10.700	0.0534	0.0404	1.3356	4.1634
10.900	0.0333	0.0228	1.5177	5.4134	10.800	0.0501	0.0373	1.3617	4.3290
11.000	0.0294	0.0199	1.5466	5.7105	10.900	0.0466	0.0343	1.3883	4.5090
11.100	0.0254	0.0170	1.5747	6.0395	11.000	0.0431	0.0313	1.4150	4.7027
11.150	0.0235	0.0156	1.5887	6.2193	11.100	0.0395	0.0283	1.4417	4.9124
11.200	0.0214	0.0141	1.6032	6.4215	11.200	0.0360	0.0254	1.4682	5.1380
11.250	0.0195	0.0128	1.6168	6.6274	11.300	0.0324	0.0227	1.4941	5.3810
11.300	0.0175	0.0114	1.6311	6.8648	11.350	0.0307	0.0213	1.5073	5.5128
11.350	0.0155	0.0100	1.6452	7.1227	11.400	0.0288	0.0199	1.5209	5.6565
11.400	0.0136	0.0087	1.6587	7.4028	11.450	0.0270	0.0185	1.5339	5.8032
11.450	0.0115	0.0074	1.6729	7.7373	11.500	0.0252	0.0172	1.5468	5.9570
11.475	0.0106	0.0067	1.6796	7.9164	11.550	0.0234	0.0158	1.5600	6.1244
11.500	0.0096	0.0061	1.6865	8.1157	11.600	0.0216	0.0145	1.5729	6.3005
11.525	0.0086	0.0055	1.6933	8.3307	11.650	0.0198	0.0133	1.5858	6.4898
11.550	0.0076	0.0048	1.7003	8.5778	11.700	0.0180	0.0120	1.5983	6.6902
11.575	0.0066	0.0042	1.7068	8.8398	11.750	0.0162	0.0107	1.6113	6.9169
11.600	0.0057	0.0036	1.7134	9.1417	11.800	0.0144	0.0095	1.6237	7.1570
11.620	0.0048	0.0030	1.7193	9.4469	11.850	0.0126	0.0082	1.6368	7.4387
11.640	0.0040	0.0025	1.7247	9.7817	11.875	0.0117	0.0076	1.6430	7.5874
11.650	0.0037	0.0023	1.7271	9.9592	11.900	0.0108	0.0070	1.6494	7.7497
11.660	0.0033	0.0021	1.7296	10.1570	11.925	0.0099	0.0065	1.6552	7.9132
11.670	0.0029	0.0018	1.7323	10.3892	11.950	0.0090	0.0059	1.6615	8.1003
11.680	0.0025	0.0016	1.7350	10.6541	11.975	0.0081	0.0053	1.6678	8.3049
11.690	0.0021	0.0013	1.7377	10.9621	12.000	0.0072	0.0047	1.6740	8.5305

$$2.20 \le \hat{\alpha}_3 \le 2.25$$

$\hat{\alpha}_3 = 2.20,\ 5.90 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
5.900	0.1300	0.3296	0.0213	1.1271
6.400	0.1303	0.2789	0.1763	1.1801
6.900	0.1298	0.2368	0.3088	1.2864
7.400	0.1275	0.2012	0.4319	1.4304
7.800	0.1241	0.1763	0.5273	1.5713
8.200	0.1194	0.1540	0.6214	1.7361
8.600	0.1135	0.1338	0.7153	1.9279
9.000	0.1063	0.1153	0.8100	2.1515
9.300	0.1001	0.1026	0.8812	2.3424
9.600	0.0933	0.0904	0.9535	2.5584
9.900	0.0859	0.0791	1.0263	2.8019
10.100	0.0806	0.0718	1.0752	2.9824
10.200	0.0779	0.0683	1.0997	3.0785
10.300	0.0752	0.0649	1.1240	3.1778
10.400	0.0724	0.0615	1.1488	3.2839
10.500	0.0695	0.0581	1.1736	3.3947
10.600	0.0666	0.0549	1.1980	3.5091
10.700	0.0636	0.0516	1.2230	3.6316
10.800	0.0606	0.0485	1.2477	3.7591
10.900	0.0575	0.0454	1.2722	3.8926
11.000	0.0544	0.0424	1.2973	4.0361
11.100	0.0513	0.0394	1.3217	4.1851
11.200	0.0481	0.0365	1.3466	4.3450
11.300	0.0449	0.0336	1.3713	4.5148
11.400	0.0417	0.0308	1.3959	4.6957
11.450	0.0401	0.0294	1.4086	4.7941
11.500	0.0385	0.0281	1.4206	4.8914
11.550	0.0368	0.0267	1.4327	4.9928
11.600	0.0352	0.0254	1.4451	5.1010
11.650	0.0336	0.0241	1.4571	5.2105
11.700	0.0319	0.0227	1.4696	5.3294
11.750	0.0303	0.0215	1.4813	5.4470
11.800	0.0286	0.0202	1.4938	5.5771
11.825	0.0278	0.0195	1.4997	5.6412
11.850	0.0270	0.0189	1.5059	5.7110
11.875	0.0261	0.0183	1.5118	5.7792
11.900	0.0253	0.0176	1.5178	5.8504
11.925	0.0245	0.0170	1.5237	5.9216
11.950	0.0237	0.0164	1.5298	5.9990
11.975	0.0229	0.0158	1.5356	6.0745
12.000	0.0220	0.0152	1.5416	6.1548

$\hat{\alpha}_3 = 2.25,\ 6.10 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
6.100	0.1263	0.3281	0.0129	1.1444
6.400	0.1265	0.2974	0.1082	1.1649
6.700	0.1266	0.2699	0.1927	1.2091
6.900	0.1265	0.2532	0.2448	1.2478
7.100	0.1262	0.2376	0.2950	1.2929
7.300	0.1257	0.2230	0.3437	1.3438
7.500	0.1248	0.2093	0.3913	1.4000
7.700	0.1236	0.1964	0.4377	1.4612
7.900	0.1221	0.1842	0.4837	1.5279
8.100	0.1203	0.1727	0.5293	1.6000
8.300	0.1183	0.1618	0.5748	1.6780
8.500	0.1159	0.1514	0.6199	1.7616
8.700	0.1132	0.1415	0.6650	1.8516
8.900	0.1102	0.1320	0.7099	1.9480
9.100	0.1069	0.1230	0.7552	2.0521
9.300	0.1034	0.1144	0.8003	2.1635
9.500	0.0996	0.1061	0.8456	2.2833
9.700	0.0956	0.0981	0.8910	2.4119
9.800	0.0934	0.0943	0.9137	2.4801
9.900	0.0913	0.0905	0.9367	2.5512
10.000	0.0890	0.0868	0.9595	2.6243
10.100	0.0867	0.0831	0.9825	2.7009
10.200	0.0844	0.0796	1.0055	2.7804
10.300	0.0820	0.0760	1.0288	2.8641
10.400	0.0795	0.0726	1.0517	2.9494
10.500	0.0770	0.0693	1.0746	3.0385
10.600	0.0745	0.0659	1.0979	3.1325
10.700	0.0719	0.0627	1.1209	3.2295
10.800	0.0692	0.0595	1.1443	3.3319
10.900	0.0665	0.0564	1.1674	3.4379
11.000	0.0638	0.0533	1.1908	3.5500
11.100	0.0610	0.0503	1.2138	3.6657
11.200	0.0582	0.0473	1.2370	3.7878
11.300	0.0554	0.0444	1.2600	3.9152
11.400	0.0524	0.0415	1.2836	4.0523
11.500	0.0496	0.0387	1.3066	4.1943
11.600	0.0466	0.0360	1.3297	4.3449
11.700	0.0437	0.0333	1.3523	4.5017
11.800	0.0407	0.0307	1.3753	4.6719
11.900	0.0377	0.0281	1.3980	4.8516
12.000	0.0347	0.0255	1.4211	5.0481

$$2.30 \leq \hat{\alpha}_3 \leq 2.35$$

$\hat{\alpha}_3 = 2.30,\ 6.30 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
6.300	0.1227	0.3270	0.0034	1.1616
6.700	0.1229	0.2878	0.1260	1.1902
7.100	0.1230	0.2539	0.2315	1.2561
7.500	0.1223	0.2243	0.3279	1.3469
7.800	0.1211	0.2045	0.3968	1.4286
8.100	0.1193	0.1863	0.4641	1.5218
8.400	0.1168	0.1696	0.5300	1.6259
8.700	0.1137	0.1541	0.5952	1.7419
8.900	0.1113	0.1444	0.6384	1.8263
9.100	0.1086	0.1351	0.6817	1.9168
9.300	0.1056	0.1263	0.7247	2.0131
9.500	0.1024	0.1178	0.7677	2.1165
9.700	0.0990	0.1096	0.8111	2.2277
9.900	0.0953	0.1019	0.8542	2.3459
10.100	0.0914	0.0943	0.8977	2.4735
10.300	0.0873	0.0872	0.9410	2.6099
10.400	0.0852	0.0837	0.9628	2.6819
10.500	0.0830	0.0802	0.9847	2.7571
10.600	0.0808	0.0768	1.0065	2.8348
10.700	0.0785	0.0736	1.0281	2.9145
10.800	0.0762	0.0703	1.0499	2.9980
10.900	0.0738	0.0671	1.0717	3.0847
11.000	0.0714	0.0640	1.0935	3.1752
11.100	0.0690	0.0609	1.1153	3.2689
11.200	0.0665	0.0579	1.1372	3.3672
11.300	0.0640	0.0550	1.1590	3.4691
11.400	0.0614	0.0521	1.1808	3.5751
11.500	0.0588	0.0492	1.2025	3.6860
11.550	0.0575	0.0478	1.2137	3.7450
11.600	0.0562	0.0464	1.2246	3.8038
11.650	0.0549	0.0450	1.2354	3.8638
11.700	0.0536	0.0437	1.2460	3.9243
11.750	0.0522	0.0423	1.2571	3.9884
11.800	0.0509	0.0410	1.2676	4.0514
11.850	0.0495	0.0396	1.2786	4.1187
11.900	0.0482	0.0383	1.2895	4.1869
11.925	0.0475	0.0377	1.2945	4.2196
11.950	0.0469	0.0371	1.2999	4.2548
11.975	0.0462	0.0364	1.3054	4.2909
12.000	0.0455	0.0358	1.3108	4.3270

$\hat{\alpha}_3 = 2.35,\ 6.60 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
6.600	0.1192	0.3162	0.0252	1.1800
6.800	0.1194	0.2971	0.0856	1.1923
7.000	0.1195	0.2793	0.1408	1.2150
7.200	0.1196	0.2628	0.1924	1.2457
7.400	0.1195	0.2473	0.2415	1.2829
7.600	0.1193	0.2327	0.2889	1.3260
7.800	0.1188	0.2192	0.3344	1.3740
8.000	0.1181	0.2064	0.3789	1.4269
8.200	0.1171	0.1943	0.4225	1.4844
8.400	0.1158	0.1829	0.4653	1.5464
8.600	0.1143	0.1721	0.5078	1.6134
8.800	0.1125	0.1618	0.5499	1.6850
9.000	0.1105	0.1520	0.5916	1.7617
9.200	0.1082	0.1427	0.6333	1.8437
9.400	0.1057	0.1338	0.6747	1.9311
9.600	0.1029	0.1253	0.7158	2.0240
9.700	0.1014	0.1212	0.7365	2.0729
9.800	0.0999	0.1171	0.7572	2.1237
9.900	0.0983	0.1132	0.7776	2.1752
10.000	0.0967	0.1093	0.7984	2.2294
10.100	0.0950	0.1055	0.8189	2.2849
10.200	0.0932	0.1018	0.8395	2.3423
10.300	0.0914	0.0982	0.8602	2.4018
10.400	0.0896	0.0946	0.8808	2.4632
10.500	0.0877	0.0911	0.9015	2.5270
10.600	0.0858	0.0877	0.9221	2.5923
10.700	0.0838	0.0843	0.9429	2.6606
10.800	0.0818	0.0810	0.9632	2.7301
10.900	0.0797	0.0778	0.9839	2.8028
11.000	0.0776	0.0746	1.0046	2.8783
11.100	0.0754	0.0715	1.0252	2.9560
11.200	0.0732	0.0684	1.0461	3.0379
11.300	0.0710	0.0654	1.0665	3.1207
11.400	0.0688	0.0624	1.0870	3.2075
11.500	0.0665	0.0595	1.1078	3.2983
11.600	0.0641	0.0567	1.1283	3.3919
11.700	0.0618	0.0539	1.1486	3.4884
11.800	0.0594	0.0512	1.1692	3.5903
11.900	0.0570	0.0485	1.1897	3.6959
12.000	0.0546	0.0458	1.2102	3.8065

$$2.40 \leq \hat{\alpha}_3 \leq 2.45$$

$\hat{\alpha}_3 = 2.40,\ 6.80 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
6.800	0.1159	0.3162	0.0127	1.1961
7.000	0.1160	0.2975	0.0725	1.2056
7.200	0.1161	0.2800	0.1271	1.2257
7.400	0.1162	0.2638	0.1781	1.2540
7.600	0.1162	0.2486	0.2264	1.2889
7.800	0.1160	0.2344	0.2726	1.3292
8.000	0.1157	0.2210	0.3173	1.3747
8.200	0.1151	0.2084	0.3607	1.4247
8.400	0.1142	0.1966	0.4031	1.4791
8.600	0.1132	0.1853	0.4448	1.5379
8.800	0.1118	0.1747	0.4858	1.6009
9.000	0.1103	0.1646	0.5264	1.6684
9.200	0.1085	0.1549	0.5669	1.7407
9.400	0.1064	0.1458	0.6068	1.8174
9.600	0.1041	0.1370	0.6466	1.8992
9.800	0.1016	0.1287	0.6862	1.9860
10.000	0.0989	0.1206	0.7258	2.0785
10.200	0.0960	0.1129	0.7654	2.1770
10.400	0.0929	0.1056	0.8046	2.2813
10.600	0.0896	0.0985	0.8438	2.3921
10.800	0.0861	0.0917	0.8833	2.5108
10.900	0.0844	0.0884	0.9028	2.5722
11.000	0.0825	0.0851	0.9223	2.6360
11.100	0.0806	0.0819	0.9420	2.7022
11.200	0.0787	0.0788	0.9617	2.7707
11.300	0.0767	0.0757	0.9814	2.8416
11.400	0.0748	0.0727	1.0008	2.9138
11.500	0.0727	0.0698	1.0202	2.9886
11.600	0.0707	0.0669	1.0397	3.0664
11.700	0.0686	0.0640	1.0592	3.1468
11.800	0.0665	0.0612	1.0788	3.2307
11.900	0.0643	0.0585	1.0981	3.3166
12.000	0.0622	0.0558	1.1175	3.4058

$\hat{\alpha}_3 = 2.45,\ 7.10 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
7.100	0.1127	0.3071	0.0297	1.2141
7.300	0.1128	0.2894	0.0864	1.2265
7.500	0.1129	0.2729	0.1383	1.2483
7.700	0.1130	0.2575	0.1871	1.2774
7.900	0.1130	0.2430	0.2335	1.3126
8.100	0.1128	0.2295	0.2781	1.3530
8.300	0.1124	0.2167	0.3210	1.3980
8.500	0.1119	0.2047	0.3629	1.4474
8.700	0.1110	0.1933	0.4038	1.5008
8.900	0.1100	0.1826	0.4440	1.5584
9.100	0.1088	0.1723	0.4838	1.6202
9.300	0.1073	0.1626	0.5229	1.6858
9.500	0.1056	0.1534	0.5617	1.7559
9.700	0.1036	0.1446	0.6001	1.8300
9.900	0.1015	0.1361	0.6383	1.9088
10.100	0.0992	0.1281	0.6763	1.9924
10.300	0.0966	0.1203	0.7142	2.0812
10.500	0.0939	0.1129	0.7519	2.1750
10.600	0.0925	0.1093	0.7710	2.2246
10.700	0.0910	0.1058	0.7896	2.2747
10.800	0.0895	0.1023	0.8084	2.3266
10.900	0.0879	0.0990	0.8271	2.3800
11.000	0.0863	0.0956	0.8461	2.4357
11.100	0.0847	0.0924	0.8646	2.4920
11.200	0.0830	0.0891	0.8836	2.5512
11.300	0.0813	0.0860	0.9021	2.6109
11.400	0.0796	0.0830	0.9206	2.6727
11.500	0.0778	0.0799	0.9393	2.7368
11.600	0.0760	0.0769	0.9581	2.8032
11.700	0.0741	0.0740	0.9766	2.8712
11.800	0.0723	0.0712	0.9950	2.9409
11.900	0.0704	0.0684	1.0135	3.0131
12.000	0.0684	0.0656	1.0322	3.0884

$$2.50 \leq \hat{\alpha}_3 \leq 2.55$$

$\hat{\alpha}_3 = 2.50,\ 7.30 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
7.300	0.1096	0.3078	0.0150	1.2293
7.500	0.1097	0.2904	0.0714	1.2386
7.700	0.1098	0.2742	0.1230	1.2576
7.900	0.1099	0.2590	0.1713	1.2842
8.100	0.1099	0.2448	0.2169	1.3167
8.300	0.1098	0.2315	0.2607	1.3545
8.500	0.1096	0.2189	0.3029	1.3969
8.700	0.1091	0.2071	0.3437	1.4433
8.900	0.1084	0.1959	0.3837	1.4938
9.100	0.1075	0.1852	0.4229	1.5482
9.300	0.1065	0.1752	0.4615	1.6064
9.500	0.1052	0.1656	0.4994	1.6682
9.700	0.1036	0.1565	0.5370	1.7342
9.900	0.1019	0.1478	0.5741	1.8039
10.100	0.1000	0.1395	0.6109	1.8776
10.300	0.0979	0.1315	0.6476	1.9558
10.450	0.0962	0.1257	0.6751	2.0177
10.600	0.0945	0.1202	0.7022	2.0817
10.750	0.0926	0.1148	0.7292	2.1483
10.900	0.0906	0.1095	0.7565	2.2183
11.000	0.0892	0.1061	0.7744	2.2662
11.100	0.0878	0.1028	0.7924	2.3157
11.200	0.0864	0.0995	0.8104	2.3667
11.300	0.0849	0.0963	0.8282	2.4188
11.400	0.0833	0.0931	0.8463	2.4729
11.500	0.0818	0.0901	0.8639	2.5276
11.600	0.0802	0.0870	0.8818	2.5845
11.700	0.0786	0.0840	0.8996	2.6431
11.800	0.0769	0.0811	0.9173	2.7030
11.900	0.0753	0.0783	0.9349	2.7644
12.000	0.0735	0.0754	0.9526	2.8280

$\hat{\alpha}_3 = 2.55,\ 7.60 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
7.600	0.1066	0.3001	0.0282	1.2465
7.800	0.1067	0.2836	0.0818	1.2579
8.000	0.1068	0.2682	0.1312	1.2781
8.200	0.1069	0.2537	0.1775	1.3051
8.400	0.1070	0.2401	0.2217	1.3378
8.600	0.1068	0.2273	0.2639	1.3753
8.800	0.1066	0.2153	0.3045	1.4170
9.000	0.1061	0.2040	0.3440	1.4626
9.200	0.1055	0.1932	0.3828	1.5122
9.400	0.1047	0.1830	0.4205	1.5651
9.600	0.1036	0.1733	0.4578	1.6218
9.800	0.1024	0.1641	0.4944	1.6819
10.000	0.1010	0.1553	0.5305	1.7454
10.200	0.0994	0.1469	0.5664	1.8131
10.400	0.0976	0.1388	0.6020	1.8844
10.600	0.0957	0.1312	0.6371	1.9594
10.700	0.0946	0.1274	0.6547	1.9985
10.800	0.0935	0.1238	0.6723	2.0389
10.900	0.0924	0.1202	0.6895	2.0798
11.000	0.0912	0.1167	0.7071	2.1224
11.100	0.0900	0.1133	0.7243	2.1657
11.200	0.0888	0.1099	0.7416	2.2102
11.300	0.0875	0.1066	0.7588	2.2559
11.400	0.0862	0.1034	0.7759	2.3027
11.500	0.0849	0.1002	0.7933	2.3514
11.600	0.0835	0.0971	0.8103	2.4005
11.700	0.0821	0.0941	0.8273	2.4512
11.800	0.0806	0.0911	0.8444	2.5035
11.900	0.0792	0.0881	0.8613	2.5568
12.000	0.0777	0.0852	0.8783	2.6118

$$2.60 \leq \hat{\alpha}_3 \leq 2.65$$

$\hat{\alpha}_3 = 2.60,\ 7.80 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
7.800	0.1037	0.3015	0.0114	1.2611
8.000	0.1038	0.2852	0.0653	1.2691
8.200	0.1039	0.2700	0.1142	1.2863
8.400	0.1040	0.2558	0.1602	1.3105
8.600	0.1041	0.2424	0.2038	1.3405
8.800	0.1040	0.2298	0.2454	1.3754
9.000	0.1039	0.2179	0.2854	1.4145
9.200	0.1035	0.2067	0.3242	1.4574
9.400	0.1030	0.1960	0.3620	1.5039
9.600	0.1023	0.1859	0.3989	1.5538
9.800	0.1014	0.1764	0.4351	1.6071
10.000	0.1004	0.1672	0.4708	1.6638
10.150	0.0995	0.1607	0.4973	1.7086
10.300	0.0985	0.1543	0.5234	1.7550
10.450	0.0974	0.1482	0.5495	1.8037
10.600	0.0962	0.1423	0.5752	1.8541
10.700	0.0953	0.1385	0.5922	1.8887
10.800	0.0944	0.1347	0.6093	1.9245
10.900	0.0935	0.1310	0.6261	1.9610
11.000	0.0925	0.1274	0.6429	1.9983
11.100	0.0915	0.1239	0.6597	2.0368
11.200	0.0905	0.1204	0.6765	2.0762
11.300	0.0894	0.1170	0.6932	2.1166
11.400	0.0883	0.1137	0.7100	2.1584
11.500	0.0871	0.1104	0.7265	2.2007
11.600	0.0859	0.1072	0.7430	2.2442
11.700	0.0847	0.1041	0.7594	2.2886
11.800	0.0835	0.1010	0.7759	2.3344
11.900	0.0822	0.0980	0.7924	2.3817
12.000	0.0809	0.0951	0.8086	2.4294

$\hat{\alpha}_3 = 2.65,\ 8.10 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
8.100	0.1009	0.2949	0.0213	1.2773
8.300	0.1010	0.2794	0.0725	1.2868
8.500	0.1011	0.2649	0.1198	1.3047
8.700	0.1012	0.2513	0.1642	1.3292
8.900	0.1013	0.2384	0.2063	1.3590
9.100	0.1013	0.2263	0.2466	1.3935
9.300	0.1011	0.2149	0.2852	1.4317
9.500	0.1008	0.2041	0.3228	1.4737
9.700	0.1003	0.1939	0.3593	1.5191
9.900	0.0997	0.1842	0.3950	1.5677
10.100	0.0988	0.1749	0.4302	1.6195
10.300	0.0979	0.1661	0.4646	1.6743
10.500	0.0967	0.1577	0.4986	1.7324
10.600	0.0961	0.1536	0.5155	1.7626
10.700	0.0954	0.1496	0.5322	1.7936
10.800	0.0947	0.1458	0.5488	1.8254
10.900	0.0939	0.1419	0.5654	1.8580
11.000	0.0931	0.1382	0.5818	1.8912
11.100	0.0923	0.1346	0.5982	1.9256
11.200	0.0914	0.1310	0.6145	1.9606
11.300	0.0905	0.1275	0.6308	1.9967
11.400	0.0896	0.1241	0.6471	2.0336
11.500	0.0886	0.1207	0.6631	2.0711
11.600	0.0876	0.1175	0.6791	2.1096
11.700	0.0866	0.1142	0.6952	2.1495
11.800	0.0855	0.1110	0.7112	2.1900
11.900	0.0844	0.1079	0.7271	2.2315
12.000	0.0833	0.1049	0.7429	2.2737

$$2.70 \leq \hat{\alpha}_3 \leq 2.75$$

$\hat{\alpha}_3 = 2.70,\ 8.30 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
8.300	0.0982	0.2968	0.0029	1.2917
8.500	0.0983	0.2815	0.0542	1.2977
8.700	0.0984	0.2672	0.1014	1.3124
8.900	0.0985	0.2537	0.1457	1.3339
9.100	0.0986	0.2410	0.1875	1.3610
9.300	0.0986	0.2291	0.2271	1.3927
9.500	0.0985	0.2178	0.2654	1.4284
9.700	0.0983	0.2071	0.3023	1.4677
9.900	0.0979	0.1969	0.3383	1.5104
10.100	0.0974	0.1873	0.3732	1.5560
10.300	0.0967	0.1781	0.4076	1.6048
10.500	0.0959	0.1694	0.4411	1.6563
10.600	0.0954	0.1652	0.4578	1.6833
10.700	0.0949	0.1610	0.4743	1.7110
10.800	0.0944	0.1570	0.4907	1.7393
10.900	0.0938	0.1531	0.5069	1.7684
11.000	0.0932	0.1492	0.5232	1.7985
11.100	0.0925	0.1455	0.5392	1.8290
11.200	0.0918	0.1418	0.5551	1.8602
11.300	0.0911	0.1382	0.5711	1.8925
11.400	0.0903	0.1346	0.5869	1.9253
11.500	0.0895	0.1312	0.6026	1.9589
11.600	0.0887	0.1278	0.6183	1.9935
11.700	0.0878	0.1245	0.6338	2.0285
11.800	0.0869	0.1212	0.6494	2.0647
11.900	0.0859	0.1180	0.6649	2.1016
12.000	0.0850	0.1149	0.6802	2.1390

$\hat{\alpha}_3 = 2.75,\ 8.60 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
8.600	0.0956	0.2913	0.0097	1.3071
8.800	0.0957	0.2766	0.0592	1.3140
9.000	0.0958	0.2629	0.1049	1.3291
9.200	0.0959	0.2499	0.1478	1.3507
9.400	0.0960	0.2377	0.1881	1.3774
9.600	0.0960	0.2262	0.2266	1.4085
9.800	0.0960	0.2153	0.2637	1.4434
10.000	0.0958	0.2050	0.2996	1.4817
10.200	0.0954	0.1952	0.3345	1.5232
10.400	0.0950	0.1859	0.3684	1.5676
10.600	0.0943	0.1770	0.4016	1.6147
10.700	0.0940	0.1727	0.4179	1.6393
10.800	0.0936	0.1685	0.4343	1.6649
10.900	0.0931	0.1645	0.4503	1.6908
11.000	0.0927	0.1605	0.4663	1.7175
11.100	0.0921	0.1565	0.4823	1.7450
11.200	0.0916	0.1527	0.4979	1.7729
11.300	0.0910	0.1490	0.5135	1.8017
11.400	0.0904	0.1454	0.5290	1.8311
11.500	0.0898	0.1418	0.5445	1.8613
11.600	0.0891	0.1383	0.5599	1.8922
11.700	0.0884	0.1348	0.5751	1.9235
11.800	0.0876	0.1315	0.5903	1.9558
11.900	0.0868	0.1282	0.6054	1.9887
12.000	0.0860	0.1249	0.6204	2.0224

$$2.80 \leq \hat{\alpha}_3 \leq 2.85$$

$\hat{\alpha}_3 = 2.80,\ 8.90 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
8.900	0.0931	0.2862	0.0152	1.3222
9.100	0.0932	0.2722	0.0629	1.3298
9.300	0.0933	0.2590	0.1070	1.3451
9.500	0.0934	0.2465	0.1485	1.3665
9.700	0.0935	0.2348	0.1876	1.3927
9.900	0.0936	0.2237	0.2252	1.4231
10.100	0.0935	0.2132	0.2611	1.4571
10.300	0.0933	0.2032	0.2960	1.4944
10.400	0.0932	0.1984	0.3130	1.5142
10.500	0.0930	0.1937	0.3298	1.5346
10.600	0.0928	0.1891	0.3465	1.5559
10.700	0.0926	0.1847	0.3629	1.5778
10.800	0.0923	0.1804	0.3789	1.6001
10.900	0.0920	0.1761	0.3950	1.6233
11.000	0.0917	0.1720	0.4109	1.6472
11.100	0.0913	0.1679	0.4266	1.6717
11.200	0.0909	0.1639	0.4423	1.6969
11.300	0.0905	0.1600	0.4578	1.7227
11.400	0.0900	0.1563	0.4731	1.7489
11.500	0.0895	0.1525	0.4884	1.7761
11.600	0.0890	0.1489	0.5034	1.8036
11.700	0.0884	0.1453	0.5186	1.8322
11.800	0.0878	0.1419	0.5334	1.8608
11.900	0.0872	0.1385	0.5483	1.8905
12.000	0.0865	0.1351	0.5631	1.9208

$\hat{\alpha}_3 = 2.85,\ 9.20 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
9.200	0.0907	0.2818	0.0190	1.3371
9.400	0.0908	0.2683	0.0652	1.3451
9.600	0.0909	0.2556	0.1080	1.3604
9.800	0.0910	0.2436	0.1481	1.3813
10.000	0.0911	0.2322	0.1864	1.4071
10.200	0.0912	0.2215	0.2227	1.4366
10.400	0.0911	0.2113	0.2578	1.4698
10.600	0.0910	0.2017	0.2916	1.5059
10.700	0.0908	0.1970	0.3081	1.5250
10.800	0.0907	0.1925	0.3244	1.5448
10.900	0.0905	0.1880	0.3407	1.5655
11.000	0.0903	0.1837	0.3566	1.5866
11.100	0.0900	0.1795	0.3723	1.6084
11.200	0.0898	0.1754	0.3879	1.6308
11.300	0.0895	0.1714	0.4032	1.6537
11.400	0.0891	0.1674	0.4186	1.6775
11.500	0.0888	0.1636	0.4338	1.7017
11.600	0.0884	0.1598	0.4486	1.7264
11.700	0.0879	0.1561	0.4636	1.7519
11.800	0.0875	0.1525	0.4783	1.7778
11.900	0.0870	0.1490	0.4931	1.8046
12.000	0.0865	0.1455	0.5077	1.8319

$$2.90 \leq \hat{\alpha}_3 \leq 2.95$$

$\hat{\alpha}_3 = 2.90,\ 9.50 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
9.500	0.0884	0.2778	0.0213	1.3517
9.700	0.0885	0.2648	0.0663	1.3599
9.900	0.0886	0.2525	0.1079	1.3749
10.100	0.0887	0.2410	0.1468	1.3954
10.300	0.0888	0.2300	0.1839	1.4203
10.500	0.0888	0.2196	0.2194	1.4492
10.600	0.0888	0.2146	0.2366	1.4648
10.700	0.0888	0.2097	0.2536	1.4813
10.800	0.0887	0.2050	0.2701	1.4984
10.900	0.0887	0.2004	0.2864	1.5162
11.000	0.0886	0.1959	0.3025	1.5348
11.100	0.0884	0.1915	0.3185	1.5541
11.200	0.0883	0.1872	0.3342	1.5739
11.300	0.0881	0.1830	0.3496	1.5943
11.400	0.0879	0.1789	0.3649	1.6153
11.500	0.0876	0.1749	0.3800	1.6369
11.600	0.0873	0.1710	0.3951	1.6592
11.700	0.0870	0.1671	0.4099	1.6819
11.800	0.0867	0.1634	0.4247	1.7054
11.900	0.0863	0.1597	0.4392	1.7292
12.000	0.0860	0.1561	0.4536	1.7537

$\hat{\alpha}_3 = 2.95,\ 9.80 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
9.800	0.0862	0.2742	0.0225	1.3660
10.000	0.0862	0.2617	0.0660	1.3741
10.200	0.0863	0.2499	0.1065	1.3888
10.400	0.0865	0.2386	0.1446	1.4087
10.600	0.0866	0.2280	0.1808	1.4329
10.800	0.0866	0.2180	0.2154	1.4608
10.900	0.0866	0.2131	0.2321	1.4759
11.000	0.0866	0.2084	0.2485	1.4917
11.100	0.0865	0.2038	0.2648	1.5084
11.200	0.0865	0.1993	0.2807	1.5257
11.300	0.0864	0.1949	0.2964	1.5436
11.400	0.0863	0.1907	0.3118	1.5621
11.500	0.0861	0.1865	0.3271	1.5813
11.600	0.0860	0.1824	0.3422	1.6010
11.700	0.0858	0.1784	0.3572	1.6214
11.800	0.0855	0.1745	0.3718	1.6421
11.900	0.0853	0.1707	0.3864	1.6635
12.000	0.0850	0.1670	0.4009	1.6856

$$3.00 \leq \hat{\alpha}_3 \leq 3.00$$

$\hat{\alpha}_3 = 3.00,\ \ 10.30 < \hat{\alpha}_4 < 12.00$				
$\hat{\alpha}_4$	μ_Y	σ_Y	δ	γ
10.300	0.0841	0.2589	0.0650	1.3878
10.500	0.0842	0.2475	0.1043	1.4020
10.700	0.0843	0.2366	0.1415	1.4212
10.900	0.0844	0.2264	0.1768	1.4445
11.100	0.0844	0.2166	0.2104	1.4714
11.200	0.0844	0.2119	0.2268	1.4860
11.300	0.0844	0.2073	0.2429	1.5014
11.400	0.0844	0.2028	0.2588	1.5174
11.500	0.0843	0.1985	0.2743	1.5341
11.600	0.0843	0.1942	0.2896	1.5514
11.700	0.0842	0.1901	0.3047	1.5693
11.800	0.0841	0.1860	0.3197	1.5878
11.900	0.0839	0.1820	0.3343	1.6068
12.000	0.0837	0.1781	0.3488	1.6264

Appendix J

Table J–1 for p-Values Associated with Kolmogorov-Smirnov Statistics

For a dataset of n observations let the Kolmogorov-Smirnov statistic associated with a fit of the data be designated by KS. The p-value of the Kolmogorov-Smirnov test for this fit can be computed from $\sqrt{n}KS$ by using the formula of Theorem 13.4.2 (p. 670) of Dudewicz and Mishra, *Modern Mathematical Statistics*, John Wiley & Sons (1988). The table that follows gives the p-values for specified $\sqrt{n}KS$.

$\sqrt{n}KS$	p-Value	$\sqrt{n}KS$	p-Value	$\sqrt{n}KS$	p-Value	$\sqrt{n}KS$	p-Value
0.374	0.999	0.676	0.750	0.828	0.500	1.019	0.250
0.417	0.995	0.680	0.745	0.831	0.495	1.024	0.245
0.441	0.990	0.683	0.740	0.834	0.490	1.029	0.240
0.457	0.985	0.686	0.735	0.837	0.485	1.034	0.235
0.470	0.980	0.689	0.730	0.840	0.480	1.040	0.230
0.481	0.975	0.692	0.725	0.844	0.475	1.045	0.225
0.490	0.970	0.695	0.720	0.847	0.470	1.050	0.220
0.498	0.965	0.698	0.715	0.850	0.465	1.056	0.215
0.506	0.960	0.701	0.710	0.854	0.460	1.061	0.210
0.513	0.955	0.704	0.705	0.857	0.455	1.067	0.205
0.520	0.950	0.707	0.700	0.860	0.450	1.073	0.200
0.526	0.945	0.710	0.695	0.864	0.445	1.079	0.195
0.532	0.940	0.713	0.690	0.867	0.440	1.085	0.190
0.537	0.935	0.716	0.685	0.870	0.435	1.091	0.185
0.543	0.930	0.719	0.680	0.874	0.430	1.097	0.180
0.548	0.925	0.722	0.675	0.877	0.425	1.104	0.175
0.553	0.920	0.725	0.670	0.881	0.420	1.110	0.170
0.558	0.915	0.728	0.665	0.884	0.415	1.117	0.165
0.562	0.910	0.731	0.660	0.888	0.410	1.124	0.160
0.567	0.905	0.733	0.655	0.891	0.405	1.131	0.155
0.571	0.900	0.736	0.650	0.895	0.400	1.138	0.150
0.575	0.895	0.739	0.645	0.898	0.395	1.145	0.145
0.580	0.890	0.742	0.640	0.902	0.390	1.153	0.140
0.584	0.885	0.745	0.635	0.906	0.385	1.161	0.135
0.588	0.880	0.748	0.630	0.909	0.380	1.169	0.130
0.592	0.875	0.751	0.625	0.913	0.375	1.177	0.125
0.596	0.870	0.754	0.620	0.917	0.370	1.186	0.120
0.599	0.865	0.757	0.615	0.921	0.365	1.195	0.115
0.603	0.860	0.760	0.610	0.924	0.360	1.204	0.110
0.607	0.855	0.763	0.605	0.928	0.355	1.214	0.105
0.611	0.850	0.766	0.600	0.932	0.350	1.224	0.100
0.614	0.845	0.769	0.595	0.936	0.345	1.234	0.095
0.618	0.840	0.772	0.590	0.940	0.340	1.245	0.090
0.621	0.835	0.775	0.585	0.944	0.335	1.257	0.085
0.625	0.830	0.778	0.580	0.948	0.330	1.269	0.080
0.628	0.825	0.781	0.575	0.952	0.325	1.281	0.075
0.632	0.820	0.784	0.570	0.956	0.320	1.295	0.070
0.635	0.815	0.787	0.565	0.960	0.315	1.309	0.065
0.638	0.810	0.790	0.560	0.965	0.310	1.324	0.060
0.642	0.805	0.793	0.555	0.969	0.305	1.340	0.055
0.645	0.800	0.796	0.550	0.973	0.300	1.358	0.050
0.648	0.795	0.799	0.545	0.977	0.295	1.377	0.045
0.651	0.790	0.803	0.540	0.982	0.290	1.399	0.040
0.655	0.785	0.806	0.535	0.986	0.285	1.422	0.035
0.658	0.780	0.809	0.530	0.991	0.280	1.449	0.030
0.661	0.775	0.812	0.525	0.995	0.275	1.480	0.025
0.664	0.770	0.815	0.520	1.000	0.270	1.517	0.020
0.667	0.765	0.818	0.515	1.005	0.265	1.564	0.015
0.670	0.760	0.821	0.510	1.009	0.260	1.628	0.010
0.673	0.755	0.824	0.505	1.014	0.255	1.731	0.005

Appendix K

Table K–1 Normal Distribution Probabilities

A random variable X is said to have the normal distribution $N(\mu, \sigma^2)$ if (for some $-\infty < \mu < +\infty\,, \quad \sigma^2 > 0$)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \qquad -\infty < x < +\infty\,.$$

The table on the following two pages gives for various values of y (when $\mu = 0$ and $\sigma^2 = 1$) the probability $P[X \le y] = \int_{-\infty}^{y} f_X(x)\,dx$. A table entry is either a number, a, or a number, a, and an exponent, b. If an exponent is not present, then a is the value of the integral to 4 significant digits. If an exponent, b, is present, then it signifies the number of 0s (in case $y < 0$) or the number of 9s (in case $y > 0$) that must be inserted immediately following the decimal point of a. For example, if $y = -3.26$, $a = .5771$ and $b = 3$; hence the value of the integral is .0005771. If $y = 4.52$, then $a = .6908$ and $b = 5$; hence the value of the integral is .999996908.

This table is from Appendix C of *Modern Statistical, Systems, and GPSS Simulation*, Second Edition, by Z.A. Karian and E.J. Dudewicz, CRC Press (1999), pp. 470–471. Reprinted with the permission of CRC Press.

The Normal Distribution (continues)

y	9	8	7	6	5	4	3	2	1	0
-4.90	$.3019^{6}$	$.3179^{6}$	$.3347^{6}$	$.3524^{6}$	$.3710^{6}$	$.3906^{6}$	$.4111^{6}$	$.4327^{6}$	$.4554^{6}$	$.4792^{6}$
-4.80	$.5042^{6}$	$.5304^{6}$	$.5580^{6}$	$.5869^{6}$	$.6173^{6}$	$.6492^{6}$	$.6826^{6}$	$.7178^{6}$	$.7546^{6}$	$.7933^{6}$
-4.70	$.8339^{6}$	$.8765^{6}$	$.9211^{6}$	$.9679^{6}$	$.1017^{5}$	$.1069^{5}$	$.1123^{5}$	$.1179^{5}$	$.1239^{5}$	$.1301^{5}$
-4.60	$.1366^{5}$	$.1434^{5}$	$.1506^{5}$	$.1581^{5}$	$.1660^{5}$	$.1742^{5}$	$.1828^{5}$	$.1919^{5}$	$.2013^{5}$	$.2112^{5}$
-4.50	$.2216^{5}$	$.2325^{5}$	$.2439^{5}$	$.2558^{5}$	$.2682^{5}$	$.2813^{5}$	$.2949^{5}$	$.3092^{5}$	$.3241^{5}$	$.3398^{5}$
-4.40	$.3561^{5}$	$.3732^{5}$	$.3911^{5}$	$.4098^{5}$	$.4294^{5}$	$.4498^{5}$	$.4712^{5}$	$.4935^{5}$	$.5169^{5}$	$.5413^{5}$
-4.30	$.5668^{5}$	$.5934^{5}$	$.6212^{5}$	$.6503^{5}$	$.6807^{5}$	$.7124^{5}$	$.7455^{5}$	$.7801^{5}$	$.8163^{5}$	$.8540^{5}$
-4.20	$.8934^{5}$	$.9345^{5}$	$.9774^{5}$	$.1022^{4}$	$.1069^{4}$	$.1118^{4}$	$.1168^{4}$	$.1222^{4}$	$.1277^{4}$	$.1335^{4}$
-4.10	$.1395^{4}$	$.1458^{4}$	$.1523^{4}$	$.1591^{4}$	$.1662^{4}$	$.1737^{4}$	$.1814^{4}$	$.1894^{4}$	$.1978^{4}$	$.2066^{4}$
-4.00	$.2157^{4}$	$.2252^{4}$	$.2351^{4}$	$.2454^{4}$	$.2561^{4}$	$.2673^{4}$	$.2789^{4}$	$.2910^{4}$	$.3036^{4}$	$.3167^{4}$
-3.90	$.3304^{4}$	$.3446^{4}$	$.3594^{4}$	$.3747^{4}$	$.3908^{4}$	$.4074^{4}$	$.4247^{4}$	$.4427^{4}$	$.4615^{4}$	$.4810^{4}$
-3.80	$.5012^{4}$	$.5223^{4}$	$.5442^{4}$	$.5669^{4}$	$.5906^{4}$	$.6152^{4}$	$.6407^{4}$	$.6673^{4}$	$.6948^{4}$	$.7235^{4}$
-3.70	$.7532^{4}$	$.7841^{4}$	$.8162^{4}$	$.8496^{4}$	$.8842^{4}$	$.9201^{4}$	$.9574^{4}$	$.9961^{4}$	$.1036^{3}$	$.1078^{3}$
-3.60	$.1121^{3}$	$.1166^{3}$	$.1213^{3}$	$.1261^{3}$	$.1311^{3}$	$.1363^{3}$	$.1417^{3}$	$.1473^{3}$	$.1531^{3}$	$.1591^{3}$
-3.50	$.1653^{3}$	$.1718^{3}$	$.1785^{3}$	$.1854^{3}$	$.1926^{3}$	$.2001^{3}$	$.2078^{3}$	$.2158^{3}$	$.2241^{3}$	$.2326^{3}$
-3.40	$.2415^{3}$	$.2507^{3}$	$.2602^{3}$	$.2701^{3}$	$.2803^{3}$	$.2909^{3}$	$.3018^{3}$	$.3131^{3}$	$.3248^{3}$	$.3369^{3}$
-3.30	$.3495^{3}$	$.3624^{3}$	$.3758^{3}$	$.3897^{3}$	$.4041^{3}$	$.4189^{3}$	$.4342^{3}$	$.4501^{3}$	$.4665^{3}$	$.4834^{3}$
-3.20	$.5009^{3}$	$.5190^{3}$	$.5377^{3}$	$.5571^{3}$	$.5770^{3}$	$.5976^{3}$	$.6190^{3}$	$.6410^{3}$	$.6637^{3}$	$.6871^{3}$
-3.10	$.7114^{3}$	$.7364^{3}$	$.7622^{3}$	$.7888^{3}$	$.8164^{3}$	$.8447^{3}$	$.8740^{3}$	$.9043^{3}$	$.9354^{3}$	$.9676^{3}$
-3.00	$.1001^{2}$	$.1035^{2}$	$.1070^{2}$	$.1107^{2}$	$.1144^{2}$	$.1183^{2}$	$.1223^{2}$	$.1264^{2}$	$.1306^{2}$	$.1350^{2}$
-2.90	$.1395^{2}$	$.1441^{2}$	$.1489^{2}$	$.1538^{2}$	$.1589^{2}$	$.1641^{2}$	$.1695^{2}$	$.1750^{2}$	$.1807^{2}$	$.1866^{2}$
-2.80	$.1926^{2}$	$.1988^{2}$	$.2052^{2}$	$.2118^{2}$	$.2186^{2}$	$.2256^{2}$	$.2327^{2}$	$.2401^{2}$	$.2477^{2}$	$.2555^{2}$
-2.70	$.2635^{2}$	$.2718^{2}$	$.2803^{2}$	$.2890^{2}$	$.2980^{2}$	$.3072^{2}$	$.3167^{2}$	$.3264^{2}$	$.3364^{2}$	$.3467^{2}$
-2.60	$.3573^{2}$	$.3681^{2}$	$.3793^{2}$	$.3907^{2}$	$.4025^{2}$	$.4145^{2}$	$.4269^{2}$	$.4396^{2}$	$.4527^{2}$	$.4661^{2}$
-2.50	$.4799^{2}$	$.4940^{2}$	$.5085^{2}$	$.5234^{2}$	$.5386^{2}$	$.5543^{2}$	$.5703^{2}$	$.5868^{2}$	$.6037^{2}$	$.6210^{2}$
-2.40	$.6387^{2}$	$.6569^{2}$	$.6756^{2}$	$.6947^{2}$	$.7143^{2}$	$.7344^{2}$	$.7549^{2}$	$.7760^{2}$	$.7976^{2}$	$.8198^{2}$
-2.30	$.8424^{2}$	$.8656^{2}$	$.8894^{2}$	$.9137^{2}$	$.9387^{2}$	$.9642^{2}$	$.9903^{2}$	$.1017^{1}$	$.1044^{1}$	$.1072^{1}$
-2.20	$.1101^{1}$	$.1130^{1}$	$.1160^{1}$	$.1191^{1}$	$.1222^{1}$	$.1255^{1}$	$.1287^{1}$	$.1321^{1}$	$.1355^{1}$	$.1390^{1}$
-2.10	$.1426^{1}$	$.1463^{1}$	$.1500^{1}$	$.1539^{1}$	$.1578^{1}$	$.1618^{1}$	$.1659^{1}$	$.1700^{1}$	$.1743^{1}$	$.1786^{1}$
-2.00	$.1831^{1}$	$.1876^{1}$	$.1923^{1}$	$.1970^{1}$	$.2018^{1}$	$.2068^{1}$	$.2118^{1}$	$.2169^{1}$	$.2222^{1}$	$.2275^{1}$
-1.90	$.2330^{1}$	$.2385^{1}$	$.2442^{1}$	$.2500^{1}$	$.2559^{1}$	$.2619^{1}$	$.2680^{1}$	$.2743^{1}$	$.2807^{1}$	$.2872^{1}$
-1.80	$.2938^{1}$	$.3005^{1}$	$.3074^{1}$	$.3144^{1}$	$.3216^{1}$	$.3288^{1}$	$.3362^{1}$	$.3438^{1}$	$.3515^{1}$	$.3593^{1}$
-1.70	$.3673^{1}$	$.3754^{1}$	$.3836^{1}$	$.3920^{1}$	$.4006^{1}$	$.4093^{1}$	$.4182^{1}$	$.4272^{1}$	$.4363^{1}$	$.4457^{1}$
-1.60	$.4551^{1}$	$.4648^{1}$	$.4746^{1}$	$.4846^{1}$	$.4947^{1}$	$.5050^{1}$	$.5155^{1}$	$.5262^{1}$	$.5370^{1}$	$.5480^{1}$
-1.50	$.5592^{1}$	$.5705^{1}$	$.5821^{1}$	$.5938^{1}$	$.6057^{1}$	$.6178^{1}$	$.6301^{1}$	$.6426^{1}$	$.6552^{1}$	$.6681^{1}$
-1.40	$.6811^{1}$	$.6944^{1}$	$.7078^{1}$	$.7215^{1}$	$.7353^{1}$	$.7493^{1}$	$.7636^{1}$	$.7780^{1}$	$.7927^{1}$	$.8076^{1}$
-1.30	$.8226^{1}$	$.8379^{1}$	$.8534^{1}$	$.8691^{1}$	$.8851^{1}$	$.9012^{1}$	$.9176^{1}$	$.9342^{1}$	$.9510^{1}$	$.9680^{1}$
-1.20	$.9853^{1}$	.1003	.1020	.1038	.1056	.1075	.1093	.1112	.1131	.1151
-1.10	.1170	.1190	.1210	.1230	.1251	.1271	.1292	.1314	.1335	.1357
-1.00	.1379	.1401	.1423	.1446	.1469	.1492	.1515	.1539	.1562	.1587
-.90	.1611	.1635	.1660	.1685	.1711	.1736	.1762	.1788	.1814	.1841
-.80	.1867	.1894	.1922	.1949	.1977	.2005	.2033	.2061	.2090	.2119
-.70	.2148	.2177	.2206	.2236	.2266	.2296	.2327	.2358	.2389	.2420
-.60	.2451	.2483	.2514	.2546	.2578	.2611	.2643	.2676	.2709	.2743
-.50	.2776	.2810	.2843	.2877	.2912	.2946	.2981	.3015	.3050	.3085
-.40	.3121	.3156	.3192	.3228	.3264	.3300	.3336	.3372	.3409	.3446
-.30	.3483	.3520	.3557	.3594	.3632	.3669	.3707	.3745	.3783	.3821
-.20	.3859	.3897	.3936	.3974	.4013	.4052	.4090	.4129	.4168	.4207
-.10	.4247	.4286	.4325	.4364	.4404	.4443	.4483	.4522	.4562	.4602
-0.00	.4641	.4681	.4721	.4761	.4801	.4840	.4880	.4920	.4960	.5000

The Normal Distribution (concluded)

y	0	1	2	3	4	5	6	7	8	9
.00	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.10	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.20	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.30	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.40	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.60	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.70	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.80	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.00	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.10	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.20	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	$.0148^{1}$
1.30	$.0320^{1}$	$.0490^{1}$	$.0658^{1}$	$.0824^{1}$	$.0988^{1}$	$.1149^{1}$	$.1309^{1}$	$.1466^{1}$	$.1621^{1}$	$.1774^{1}$
1.40	$.1924^{1}$	$.2073^{1}$	$.2220^{1}$	$.2364^{1}$	$.2507^{1}$	$.2647^{1}$	$.2785^{1}$	$.2922^{1}$	$.3056^{1}$	$.3189^{1}$
1.50	$.3319^{1}$	$.3448^{1}$	$.3574^{1}$	$.3699^{1}$	$.3822^{1}$	$.3943^{1}$	$.4062^{1}$	$.4179^{1}$	$.4295^{1}$	$.4408^{1}$
1.60	$.4520^{1}$	$.4630^{1}$	$.4738^{1}$	$.4845^{1}$	$.4950^{1}$	$.5053^{1}$	$.5154^{1}$	$.5254^{1}$	$.5352^{1}$	$.5449^{1}$
1.70	$.5543^{1}$	$.5637^{1}$	$.5728^{1}$	$.5818^{1}$	$.5907^{1}$	$.5994^{1}$	$.6080^{1}$	$.6164^{1}$	$.6246^{1}$	$.6327^{1}$
1.80	$.6407^{1}$	$.6485^{1}$	$.6562^{1}$	$.6638^{1}$	$.6712^{1}$	$.6784^{1}$	$.6856^{1}$	$.6926^{1}$	$.6995^{1}$	$.7062^{1}$
1.90	$.7128^{1}$	$.7193^{1}$	$.7257^{1}$	$.7320^{1}$	$.7381^{1}$	$.7441^{1}$	$.7500^{1}$	$.7558^{1}$	$.7615^{1}$	$.7670^{1}$
2.00	$.7725^{1}$	$.7778^{1}$	$.7831^{1}$	$.7882^{1}$	$.7932^{1}$	$.7982^{1}$	$.8030^{1}$	$.8077^{1}$	$.8124^{1}$	$.8169^{1}$
2.10	$.8214^{1}$	$.8257^{1}$	$.8300^{1}$	$.8341^{1}$	$.8382^{1}$	$.8422^{1}$	$.8461^{1}$	$.8500^{1}$	$.8537^{1}$	$.8574^{1}$
2.20	$.8610^{1}$	$.8645^{1}$	$.8679^{1}$	$.8713^{1}$	$.8745^{1}$	$.8778^{1}$	$.8809^{1}$	$.8840^{1}$	$.8870^{1}$	$.8899^{1}$
2.30	$.8928^{1}$	$.8956^{1}$	$.8983^{1}$	$.0097^{2}$	$.0358^{2}$	$.0613^{2}$	$.0863^{2}$	$.1106^{2}$	$.1344^{2}$	$.1576^{2}$
2.40	$.1802^{2}$	$.2024^{2}$	$.2240^{2}$	$.2451^{2}$	$.2656^{2}$	$.2857^{2}$	$.3053^{2}$	$.3244^{2}$	$.3431^{2}$	$.3613^{2}$
2.50	$.3790^{2}$	$.3963^{2}$	$.4132^{2}$	$.4297^{2}$	$.4457^{2}$	$.4614^{2}$	$.4766^{2}$	$.4915^{2}$	$.5060^{2}$	$.5201^{2}$
2.60	$.5339^{2}$	$.5473^{2}$	$.5604^{2}$	$.5731^{2}$	$.5855^{2}$	$.5975^{2}$	$.6093^{2}$	$.6207^{2}$	$.6319^{2}$	$.6427^{2}$
2.70	$.6533^{2}$	$.6636^{2}$	$.6736^{2}$	$.6833^{2}$	$.6928^{2}$	$.7020^{2}$	$.7110^{2}$	$.7197^{2}$	$.7282^{2}$	$.7365^{2}$
2.80	$.7445^{2}$	$.7523^{2}$	$.7599^{2}$	$.7673^{2}$	$.7744^{2}$	$.7814^{2}$	$.7882^{2}$	$.7948^{2}$	$.8012^{2}$	$.8074^{2}$
2.90	$.8134^{2}$	$.8193^{2}$	$.8250^{2}$	$.8305^{2}$	$.8359^{2}$	$.8411^{2}$	$.8462^{2}$	$.8511^{2}$	$.8559^{2}$	$.8605^{2}$
3.00	$.8650^{2}$	$.8694^{2}$	$.8736^{2}$	$.8777^{2}$	$.8817^{2}$	$.8856^{2}$	$.8893^{2}$	$.8930^{2}$	$.8965^{2}$	$.8999^{2}$
3.10	$.0324^{3}$	$.0646^{3}$	$.0957^{3}$	$.1260^{3}$	$.1553^{3}$	$.1836^{3}$	$.2112^{3}$	$.2378^{3}$	$.2636^{3}$	$.2886^{3}$
3.20	$.3129^{3}$	$.3363^{3}$	$.3590^{3}$	$.3810^{3}$	$.4024^{3}$	$.4230^{3}$	$.4429^{3}$	$.4623^{3}$	$.4810^{3}$	$.4991^{3}$
3.30	$.5166^{3}$	$.5335^{3}$	$.5499^{3}$	$.5658^{3}$	$.5811^{3}$	$.5959^{3}$	$.6103^{3}$	$.6242^{3}$	$.6376^{3}$	$.6505^{3}$
3.40	$.6631^{3}$	$.6752^{3}$	$.6869^{3}$	$.6982^{3}$	$.7091^{3}$	$.7197^{3}$	$.7299^{3}$	$.7398^{3}$	$.7493^{3}$	$.7585^{3}$
3.50	$.7674^{3}$	$.7759^{3}$	$.7842^{3}$	$.7922^{3}$	$.7999^{3}$	$.8074^{3}$	$.8146^{3}$	$.8215^{3}$	$.8282^{3}$	$.8347^{3}$
3.60	$.8409^{3}$	$.8469^{3}$	$.8527^{3}$	$.8583^{3}$	$.8637^{3}$	$.8689^{3}$	$.8739^{3}$	$.8787^{3}$	$.8834^{3}$	$.8879^{3}$
3.70	$.8922^{3}$	$.8964^{3}$	$.0039^{4}$	$.0426^{4}$	$.0799^{4}$	$.1158^{4}$	$.1504^{4}$	$.1838^{4}$	$.2159^{4}$	$.2468^{4}$
3.80	$.2765^{4}$	$.3052^{4}$	$.3327^{4}$	$.3593^{4}$	$.3848^{4}$	$.4094^{4}$	$.4331^{4}$	$.4558^{4}$	$.4777^{4}$	$.4988^{4}$
3.90	$.5190^{4}$	$.5385^{4}$	$.5573^{4}$	$.5753^{4}$	$.5926^{4}$	$.6092^{4}$	$.6253^{4}$	$.6406^{4}$	$.6554^{4}$	$.6696^{4}$
4.00	$.6833^{4}$	$.6964^{4}$	$.7090^{4}$	$.7211^{4}$	$.7327^{4}$	$.7439^{4}$	$.7546^{4}$	$.7649^{4}$	$.7748^{4}$	$.7843^{4}$
4.10	$.7934^{4}$	$.8022^{4}$	$.8106^{4}$	$.8186^{4}$	$.8263^{4}$	$.8338^{4}$	$.8409^{4}$	$.8477^{4}$	$.8542^{4}$	$.8605^{4}$
4.20	$.8665^{4}$	$.8723^{4}$	$.8778^{4}$	$.8832^{4}$	$.8882^{4}$	$.8931^{4}$	$.8978^{4}$	$.0226^{5}$	$.0655^{5}$	$.1066^{5}$
4.30	$.1460^{5}$	$.1837^{5}$	$.2199^{5}$	$.2545^{5}$	$.2876^{5}$	$.3193^{5}$	$.3497^{5}$	$.3788^{5}$	$.4066^{5}$	$.4333^{5}$
4.40	$.4588^{5}$	$.4832^{5}$	$.5065^{5}$	$.5288^{5}$	$.5502^{5}$	$.5707^{5}$	$.5902^{5}$	$.6089^{5}$	$.6268^{5}$	$.6439^{5}$
4.50	$.6602^{5}$	$.6759^{5}$	$.6908^{5}$	$.7051^{5}$	$.7187^{5}$	$.7318^{5}$	$.7442^{5}$	$.7561^{5}$	$.7675^{5}$	$.7784^{5}$
4.60	$.7888^{5}$	$.7987^{5}$	$.8081^{5}$	$.8172^{5}$	$.8258^{5}$	$.8340^{5}$	$.8419^{5}$	$.8494^{5}$	$.8566^{5}$	$.8634^{5}$
4.70	$.8699^{5}$	$.8761^{5}$	$.8821^{5}$	$.8877^{5}$	$.8931^{5}$	$.8983^{5}$	$.0321^{6}$	$.0789^{6}$	$.1235^{6}$	$.1661^{6}$
4.80	$.2067^{6}$	$.2454^{6}$	$.2822^{6}$	$.3174^{6}$	$.3508^{6}$	$.3827^{6}$	$.4131^{6}$	$.4420^{6}$	$.4696^{6}$	$.4958^{6}$
4.90	$.5208^{6}$	$.5446^{6}$	$.5673^{6}$	$.5889^{6}$	$.6094^{6}$	$.6290^{6}$	$.6476^{6}$	$.6653^{6}$	$.6821^{6}$	$.6981^{6}$