

Afshin Izadian

Fundamentals of Modern Electric Circuit Analysis and Filter Synthesis

A Transfer Function Approach



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Preface

The electric circuits are perhaps the fundamental step toward understanding complex electrical engineering concepts. A strong knowledge of circuit analysis leads to advanced techniques that are moving the industry forward. However, the traditional circuits as known by many of the educators have changed. For instance, most of the circuits are now being combined in Integrated Circuits and are built on chips. Strong processors are now accomplishing most of the operations in software and sometimes exceed performances that were previously just expected from hardware implementations. Therefore, although the principles of the circuit analysis are still required, their implementation is drastically changing. Some examples include programmable logic controller (PLCs), MP3 players, video games, modern video processors, vehicle's electronic control units (ECU), and other industrial microcontrollers. Their data processing power, filtration of noise, and operations are all accomplished in the code they run. Not much of the hardware implementation is expected except for the microprocessor itself and its supporting circuits to run in hardware.

In this book, the knowledge of circuit analysis and the design of hardware-based filters and operational amplifiers are provided to lay out a rich background. However, the approach in the second half of the book is more toward the mathematical analysis of circuits, their transfer functions, closed-loop operations, control actions, and filter applications. Numerous solved problems and end-of-chapter unsolved problems are provided to establish a strong background on circuits and emphasize the importance of the modern circuit analysis. The analysis of circuit in the frequency domain is limited to the Laplace transform as most of the signal processing courses offer their own basics of Fourier transform. Three-phase circuits are also expected to be covered in the basic power systems courses where the analysis of a three-phase circuit is needed.

Book Organization

The book has 12 chapters that cover from basics of electric circuits to the advanced Laplace and transfer function-based analysis. The book is designed to cover a two-semester circuit course, as most electrical engineering or technology programs do. The first part covers an introduction to circuits, components, voltage and current laws, sources and waveforms, first order and second order circuits, and sinusoidal steady-state analysis. The second part covers topics including mutual inductance, Laplace transform, application of Laplace transform in circuits, transfer functions, passive filters, operational amplifiers, active filters, and two-port networks. In some engineering schools, operational amplifiers can be covered in the first part, in a lower-level circuits course.

Indianapolis, IN, USA

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Chapter 1

Introduction to Electric Circuits



Introduction

Electric circuits are a method to demonstrate how an electric device receives power and operates. For instance, a flashlight has batteries to provide power to a light bulb, wires to bring the power from batteries to the bulb, and a switch to control the on-off action. The operation of this flashlight can be modeled by its electric circuit. Of course, it has mechanical components to hold the batteries, wires, the switch, and the bulb in place and make the entire unit waterproof. The wires and the switch have electric insulators to protect the flow of electric current in the wires. Nevertheless, these components have no electrical importance when it comes to the distribution of the power. Figure 1.1 shows a flashlight with its internal components and its equivalent circuit.

Electric circuits and their theories can provide models for electrical components such as flashlights, power distribution systems, electric motors, generators, transformers, integrated circuits, transistors, cell phones, computers, and many more (Figs. 1.2 and 1.3).

Some of the non-electrical components and devices can also use electric circuit equivalents in their model and operations. Examples are fuel cells, batteries, heat transfer circuits, electromechanical systems, sensors, biological systems, etc.

Solar cells convert the energy of the photons in sunrays into electric current. The solar cells can be modeled in multiple ways, one of which is in form of an electric circuit that explains the operation of the cell and specific characteristics observed when the cell is in use. A solar cell and its electric equivalent are shown in Fig. 1.4.

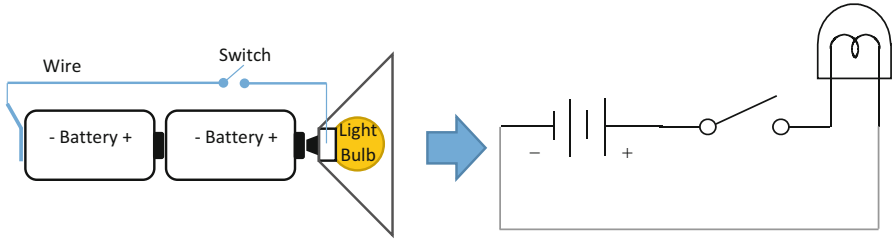


Fig. 1.1 A flashlight and its equivalent circuit. The battery, switch, and bulb are shown in circuit schematics. There is no need to model the non-electric parts of the circuit unless they participate an active role in the outcome

Fig. 1.2 An electric motor is shown with the electric circuits in the winding of the fixed part “stator” and the circuit of the rotating part “rotor.” The picture is used with permission from GE. Online at <http://www.directindustry.com/prod/ge-motors/product-22335-1411313.html>

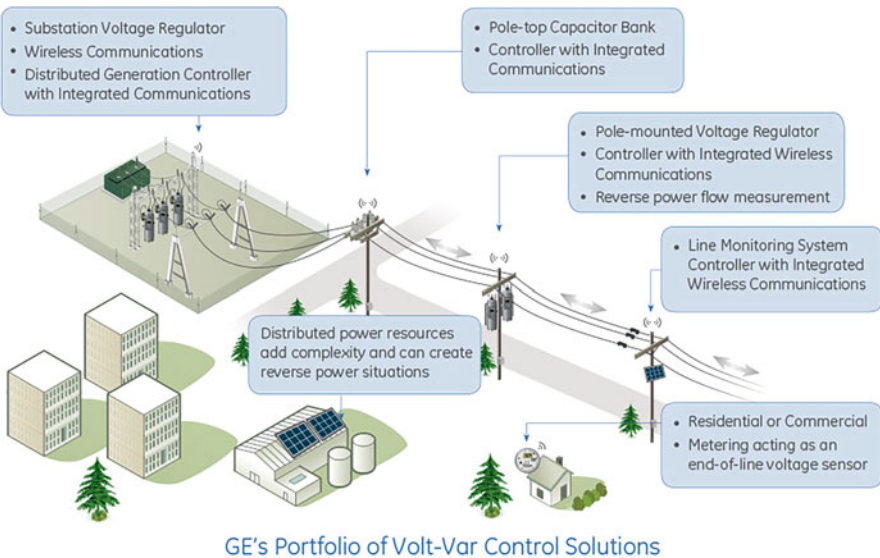
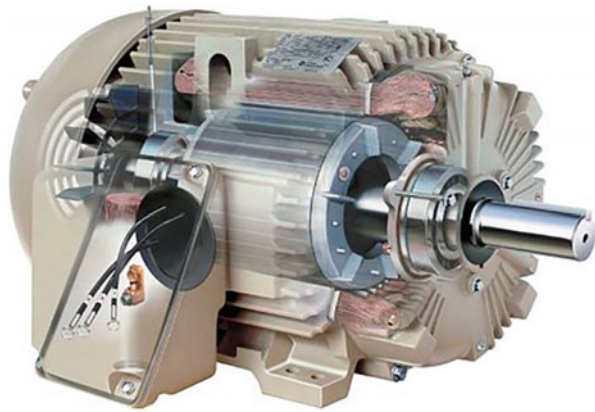


Fig. 1.3 Electric power distribution GE concept. Used with permission from GE. Can be found online at https://www.gegridsolutions.com/HVMV_Equipment/catalog/Voltage_Regulators.htm

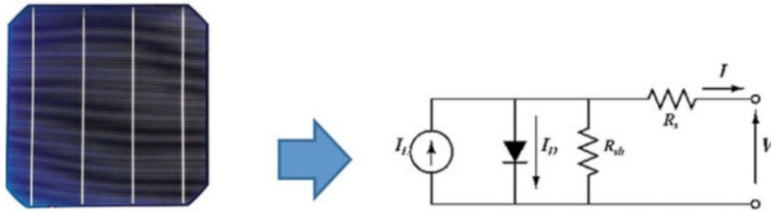


Fig. 1.4 A single solar cell and electric circuit equivalent of the cell. A single diode with series and parallel resistance demonstrates various operations of the cell

Electric Circuit Topologies

Consider a complicated system with many components and a complex network of wires to connect them. These components and their connecting wires might fit into a 3D space, meaning that there are wires that come out of the surface to connect components that themselves might be crossing other elements. Figure 1.4 shows a 3D circuit. However, the type of circuit studied in this book must fit only in a 2D plane, and no wire or element should pass any other wire or element.

An unacceptable example of a circuit that has components and wire cross each other, and its acceptable version is shown in Figs. 1.5 and 1.6.

Hinged Circuits

If an electric circuit is such that it splits into two halves such that the connecting point does not have any current passing in any direction toward two parts, the circuit is called electrically hinged. This makes the two parts of the circuit essentially independent unless there is a magnetic coupling or other forms of couplings that influences them (Fig. 1.7).

Hinged circuits may also occur as a result of voltage and current source configurations in the circuit. This is when a voltage source feeds a current source in series. This redundancy may cause a hinged circuit (Fig. 1.8).

Measurement Units

Throughout the world, the quantities are measured in various units. For instance, the speed in the USA is measured in miles per hour or MPH, but in the European countries, it is measured in kilometer per hour KMH. Generally, there are “Imperial Units” of measurement or BU and “International Standard of Units” or SI.

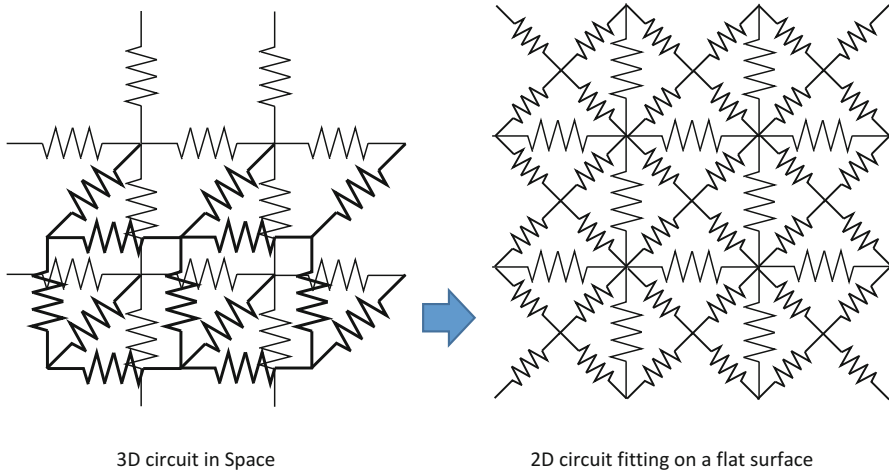


Fig. 1.5 A circuit build in 3D and an acceptable flat version of the circuit. Resistors and wire connectors do not pass over each other

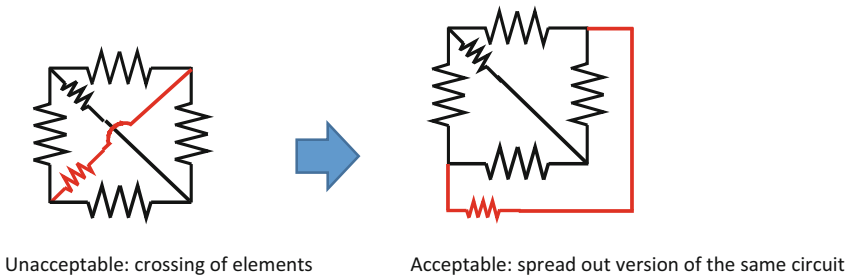
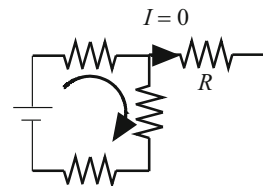


Fig. 1.6 A 2D circuit that has two elements or the wires crossing should be presented in a topology with no wires/elements crossing. A separate path on the surface can be found to prevent crossing

Fig. 1.7 Figure of a hinged circuit. There is no current passing through the resistor R , and the circuit is hinged at the node that connects to the resistor R



The quantities in these measurement units refer to different values. In this book the SI unit is used to measure the quantities as follows:

- Length (**L**) measured in meter (**m**)
- Mass (**M**) measured in kilograms (**kg**)

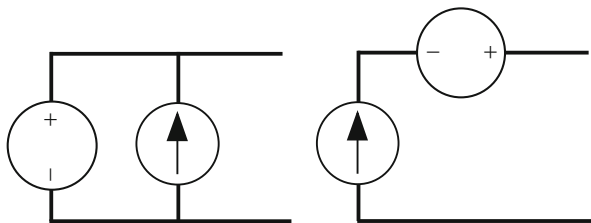


Fig. 1.8 The current source in the left circuit is redundant, and the circuit is hinged. The voltage source in the right circuit is redundant, and the circuit is hinged. These circuits are discussed in more details in the later chapters

- Time (**T**) measured in second (**s**)
- Electric current (**I**) measured in ampere (**A**)
- Electric charge (**q**) measured in Coulomb (**C**)

The quantities to be measured might vary from a very small to a very large number. The scales are useful to present these quantities. A list of these numbers is as follows:

Scales and Units

$$1 \text{ Zetta} = 10^{21} = 1e21$$

$$1 \text{ Exa} = 10^{18} = 1e18$$

$$1 \text{ Peta} = 10^{15} = 1e15$$

$$1 \text{ Tera} = 10^{12} = 1e12$$

$$1 \text{ Giga} = 10^9 = 1e9$$

$$1 \text{ Mega} = 10^6 = 1e6$$

$$1 \text{ Kilo} = 10^3 = 1e3$$

$$1 \text{ Hecto} = 10^2 = 1e2$$

$$1 \text{ Deca} = 10 = 1e1$$

$$1 \text{ Deci} = \frac{1}{10^1} = 1e-1$$

$$1 \text{ Centi} = \frac{1}{10^2} = 1e-2$$

$$1 \text{ Milli} = \frac{1}{10^3} = 1e-3$$

$$1 \text{ Micro} = \frac{1}{10^6} = 1e-6$$

$$1 \text{ Nano} = \frac{1}{10^9} = 1e-9$$

$$1 \text{ Pico} = \frac{1}{10^{12}} = 1e-12$$

$$1 \text{ Femto} = \frac{1}{10^{15}} = 1e-15$$

$$1 \text{ Atto} = \frac{1}{10^{18}} = 1e-18$$

$$1 \text{ Zepto} = \frac{1}{10^{21}} = 1e-21$$

Example 1.1 The current passing through a wire is $1.2e - 3$ A. Show the current in mA.

Solution The current is $1.2e - 3$ and $1e - 3$ equals 1 mA. Therefore, the current $1.2e - 3$ A = 1.2 mA.

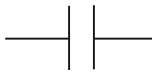
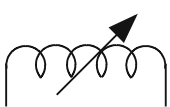
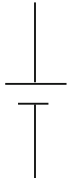
Example 1.2 Capacitance of a capacitor is 21 nF. Show the capacitance in PF.

Solution Each nF equals 1000 PF. Therefore, 21×1000 PF = 21,000 PF.

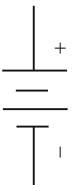
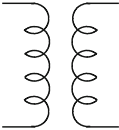
Example 1.3 Power generation of a power plant is 31 GW. Show the power generation in MW.

Solution Each GW equals 1000 MW. Therefore, 31 GW becomes 31×1000 MW or 31,000 MW.

Most Common Electric Circuit Symbols

Symbol	Circuit rep	Symbol letter	Unit of measurement
	Resistor	R	Ohm, Ω
	Variable resistor	R	Ohm, Ω
	Capacitor	C	Farad, F
	Variable capacitor	C	Farad, F
	Inductor	L	Henry, H
	Variable inductor	L	Henry, H
	Ground, Earth	—	
	Single-cell battery	E	Volt, V

(continued)

Symbol	Circuit rep	Symbol letter	Unit of measurement
	Multi-cell battery	E	Volt, V
	Voltage source	V	Volt, V
	Current source	I	Current, A
	Transformers or mutual inductance	–	–

Chapter 2

Component Voltage and Current Laws



Introduction

Electric circuit analysis is the collection of methods and tools to determine the voltages and currents as well as the power consumption and generation in electric circuits and components. The relations among the electric elements depend on the elements of the circuit and their configuration or topology. That is how the circuit elements are connected together.

In this chapter, the circuit elements are introduced, and the laws that determine the relation of voltages and currents in various elements are studied.

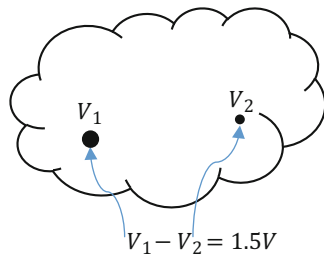
Definition of Voltage

Consider different electric charges on two points of an object. The voltage is defined as the difference of the charges between these two points. Considering potential energy at these two points, any difference in the potential energy applies forces on electrons to be displaced. The difference in these potential energies is known as voltage, and the displacement of electrons results in electric current flow. For instance, chemical reactions can generate different potentials on anode and cathode of a battery. The difference in these potential energies can be 1.3 or 1.5 V that leads to a 1.3 or 1.5 V batteries (Fig. 2.1).

Definition: Electric potential or voltage which is measured in units of volts is precisely defined as:

$$1 \text{ Volt} = \frac{1 \text{ Joule of potential energy}}{1 \text{ Coulomb of electric charge}}$$

Fig. 2.1 The potentials V_1 and V_2 are shown on an object. In this example, the difference between these potential energies results in a voltage equivalent to 1.5 V



This means that 1 V of electric potential energy is generated if it requires the equivalent of 1 J of work to move 1 Coulomb of electrons away from an oppositely charged place. 1 Coulomb of electron equals 6.24×10^{18} electrons.

Definition of Current

The difference in potential energy between two points in a circuit creates a voltage across the points. This means that a potential energy exists that can force the electrons to move. This movement is from the higher potential to the lower potential points. Providing a path through a conductor allows for electrons to pass. The current is the number of electrons passing through a cross section of the conductor per unit of time, 1 s. Precisely, the current of 1 ampere $i = 1$ A equals the charge of $q = 1$ C coulomb passing a conductor in $t = 1$ s.

$$i = \frac{dq}{dt} \quad (\text{A})$$

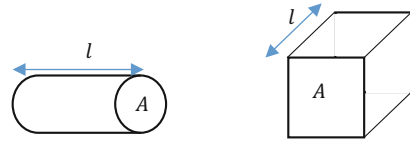
The current density J is defined as the amount of current i per cross-section area of the conductor A and is measured in A/m^2 as:

$$J = \frac{i}{A} \quad \left(\frac{\text{A}}{\text{m}^2} \right)$$

Resistor

A resistor is an element of electric circuits that limits the flow of current. A resistor which is shown by symbol R limits the current with several ways. Consider a wire made with a material that has specific resistance ρ against electric current.

Fig. 2.2 Pieces of conductors with length l and cross section A



The physical dimensions of this wire such as length l and cross-section area A influence the amount of passing electrons and consequently its resistance. If the length of the wire is long, the resistance increases. If the cross-section area is large, the resistance is less as it provides more room for the electrons to pass. Therefore, the resistance of that piece of material which is measured in ohms Ω (also shown as R (Ω)) can be calculated as (Fig. 2.2):

$$R = \rho \frac{l}{A} \quad (\Omega)$$

The units of these elements are listed as:

$$R(\Omega), \rho(\Omega \cdot \text{m}), l(\text{m}), A(\text{m}^2)$$

Example 2.1 A piece of material with specific resistance $\rho = 3000e - 6 \Omega \cdot \text{m}$ has a length of $l = 15 \text{ cm}$ and cross section of $A = 1 \text{ mm}^2$. Find the resistance of the material.

Solution

$$R = \frac{\rho l}{A} = 3000e - 6 \frac{15 \times 1e - 2(\text{cm} \rightarrow \text{m})}{1 \times 1e - 6(\text{mm}^2 \rightarrow \text{m}^2)} = 450 \Omega$$

Example 2.2 A material is formed into round cross-section wire. If the radius is cut in half and the length is increased by 50%, how much does the resistance change?

Solution The length is increased by 50% means $l_2 = 1.5l_1$. The area is $A = \pi r^2$. This results in $\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2}$ since $r_2 = \frac{1}{2}r_1$, then $\frac{A_1}{A_2} = 4$ or $A_2 = \frac{1}{4}A_1$. The material of the conductor has not changed; hence, $\rho_1 = \rho_2$.

Therefore,

$$\frac{R_1}{R_2} = \frac{\rho_1 l_1 A_2}{\rho_2 l_2 A_1} = \frac{l_1}{1.5l_1} \frac{\frac{1}{4}A_1}{A_1} = \frac{1}{4 \times 1.5} = \frac{1}{6}$$

$$R_2 = 6R_1$$

This means that the resistance is six times higher.

Conductors, Insulators, and Semiconductors

Specific resistance value depends on the type of material. A low specific resistance value makes materials conductors, and high specific resistance makes other materials insulators. Examples of conductors are aluminum ($\rho = 2.8e - 8 \Omega \cdot \text{m}$), brass ($\rho = 6e - 8 \Omega \cdot \text{m}$), and iron ($\rho = 9.8e - 8 \Omega \cdot \text{m}$). Examples of insulators are amber ($\rho = 5e14 \Omega \cdot \text{m}$), rubber ($\rho = 1e16 \Omega \cdot \text{m}$), and glass ($\rho = 1e11 \Omega \cdot \text{m}$).

There is a third class of material “semiconductors” in which their specific resistance is not fixed, rather it is influenced by external stimulators such as electric charge accumulation, light excitation, etc.

Effect of Temperature on Resistance

In *conductors*, the value of resistance is increased linearly by increasing the temperature. As the material’s temperature changes, its specific resistance changes which changes the resistance of the material. Each material has a different slope of change in specific resistance known as temperature coefficient α . Consider the resistance of a material at 0°C known as R_0 . The amount of resistance at temperature t or R_t is:

$$R_t = R_0(1 + \alpha t)$$

In *insulators* and *electrolytes*, the value of resistance decreases linearly by increasing the temperature. These materials show a negative value temperature coefficient.

If the temperature is decreased, the resistance value will decrease in conductors and increase in insulators and electrolytes. At very low temperatures, the amount of resistance is saturated to a very low value at which conductors become superconductors.

A list of material with their specific resistances and temperature coefficients is provided in the following table. This includes conductors and insulators.

Material	$\rho \times 10^{-8} (\Omega \cdot \text{m})$	α
Aluminum	2.8	40.3
Brass	6 – 8	20
Carbon	3000 – 7000	–5
Copper	1.72	39.3
Gold	2.44	36.5
Iron	9.8	65
Silver	1.64	38
Amber	5e14	
Glass	1e10 – 1e12	
Mica	1e15	
Rubber	1e16	

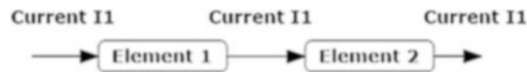


Fig. 2.3 Two circuit elements in series connection. The same current passes through the elements

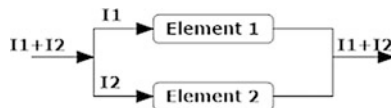


Fig. 2.4 Two circuit elements in parallel connection. The current is shared between the elements, but they share the same voltage across their terminals

Conductance

The ability of a material to pass electric current is known as conductance shown by symbol G and measured by various units as siemens, MHO, or Ω^{-1} . There is a reciprocal relation between the resistance and conductance. They relate as follows:

$$G = \frac{1}{R} \quad (\Omega^{-1})$$

Series Connection of Circuit Elements

Elements are connected in series if the entire current exiting from one element enters another element connected to the same node. The connection is still in series if more than one element is connected to the same node if the simplification of all elements constitutes two groups of elements connected in tandem (Fig. 2.3).

Parallel Connection of Circuit Elements

Elements are connected in parallel if they share similar nodes at both ends. Elements in parallel connection share the same voltage across themselves. The parallel connection provides a path to share the current (Fig. 2.4).

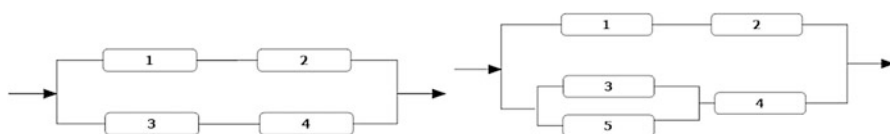


Fig. 2.5 A combination of series and parallel connections. In the left circuit, elements 1 and 2 are in series, 3 and 4 in series, and the entire 1–2 and 3–4 in parallel. It may read $(1+2)\parallel(3+4)$. In the circuit to the right, the elements 1 and 2 are in series, and the parallel combination of 3 and 5 is in series with 4. The 1–2 is in parallel to the series of 4 and 3 and 5 combined. It may read $(1+2)\parallel((3\parallel 5)+4)$

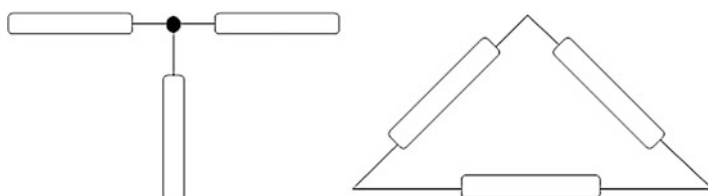


Fig. 2.6 These elements are not series nor parallel. They form a star and delta connection

Mixed Connection of Circuit Elements

Series and parallel connections may be combined together. In this case, special attention should be paid to identify the elements that are connected in tandem and the ones that share similar ending nodes. This may be a group of elements being in parallel connected in series to another group (Figs. 2.5 and 2.6).

There might be a connection that does not fit into series or parallel forms. The solution to these circuits is through Kirchhoff voltage law (KVL) or Kirchhoff current law (KCL).

Mesh

Consider an electric circuit that contains path(s) to carry current from the source or elements to other elements. These paths are also called branches. A closed path that starts and ends at the same element and contains no branches inside is called a mesh (Fig. 2.7).

Node

A node is a point of a circuit that connects at least two elements together. The elements connected to the node take a share of current entering the node but have the same voltage (potential) at the shared node (Fig. 2.8).

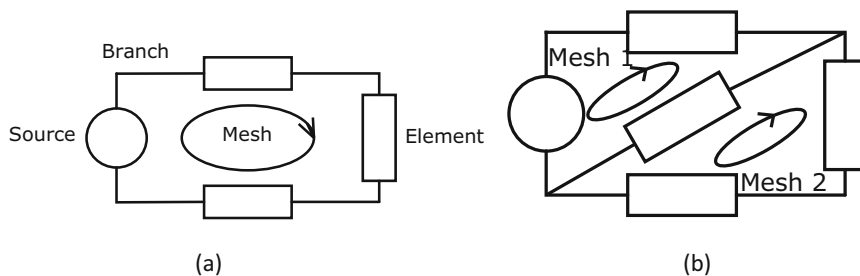


Fig. 2.7 A mesh is formed by the connection of circuit elements in a closed circuit that forms a loop. (a) Shows a single loop circuit (mesh), and (b) shows a multi-loop circuit (mesh 1 and mesh 2). Note that the circuit is planar and elements do not cross each other to connect nodes together. Elements might be shared in two or more loops, but each element is considered once in each loop

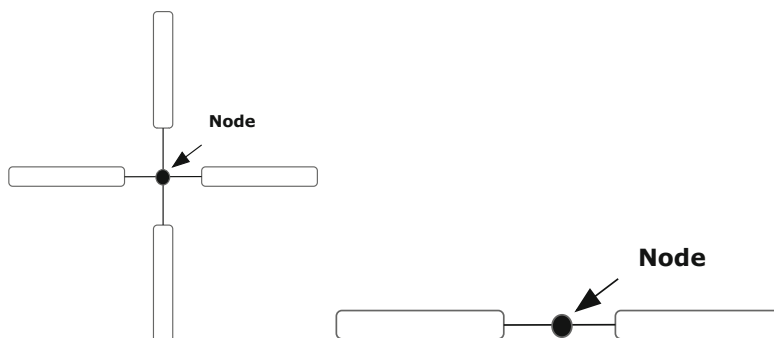


Fig. 2.8 Node is formed when at least two elements are connected together. The voltages of the nodes are important as they determine how much current passes through each element. The more potential difference in nodes, the more current passes through the elements

Ohm's Law

Consider an object with total resistance or R (Ω) (read R ohms). This object, when connected to a voltage source at voltage V (Volts) (e.g., battery) between its ports, passes a certain amount of current I (A). The current passing through the element is directly proportional to the applied voltage. The ratio of the voltage increase and the current increase is always constant, and it equals the resistance of the element. The Ohm's law equation is expressed as:

$$\frac{V}{I} = R = \text{constant}$$

This equation also shows that the voltage drop across a resistor equals the resistance times of the current passing through the resistor. It should be noted that the positive polarity of voltage drop is always at the terminal that receives the current. If the direction of current changes, the polarity of voltage drop also changes.

Example 2.3 The current through a $10\ \Omega$ resistor is 1.5 A . Find the voltage drop measured across the resistor. What is the voltage if the probes of the voltmeter are reverse connected to the terminals of the resistor.

Solution $V = RI$ with $R = 10$ and $I = 1.5$ leads to the voltage as follows:

$$V = 10 \times 1.5 = 15\text{ V}$$

If the probes are reverse connected (meaning that the positive probe is connected to the negative polarity and the negative probe is connected to the positive polarity), the voltmeter measures the voltage drop from the negative terminal with respect to the positive terminal. It can also be interpreted as a reverse direction of the current. Therefore, the voltage drop is measured as:

$$V = 10 \times (-1.5) = -15\text{ V}$$

Example 2.4 A $1\text{ k}\Omega$ resistor shows a voltage drop of 2 V . Find the current passing the resistor.

Solution

$$V = RI \rightarrow I = \frac{V}{R} = \frac{2}{1e3} = 2\text{ mA}$$

Example 2.5 A resistor is required to drain 10 A current from a 200 V node to the ground. Find a suitable resistor value.

Solution

$$V = RI \rightarrow R = \frac{V}{I} = \frac{200}{10} = 20\ \Omega$$

Kirchhoff Voltage Law (KVL)

Consider a circuit with multiple loops and nodes. KVL theory indicates that the summation of voltage drops across elements in a loop is zero. This involves all types of elements and is true in all meshes that either stand alone or share some branches with their adjacent loops. In case multiple meshes exist in a circuit, the summation of voltage drops in all individual meshes is independent and must be zero.

Example 2.6 Consider the circuit shown in Fig. 2.9. Write the KVL and find the current of the loop.

Solution The KVL suggests considering a direction for the current and following the flow of current writing the voltage drop of elements around the loop while paying attention to the polarity of the voltage drop across the element. The algebraic summation of these voltages in a closed loop must be zero. Now, the loop has a current I with the direction shown. Starting from any element, let us start with the negative terminal of the source, and following the direction of the current, the voltage drops across elements are:

Fig. 2.9 The circuit of Example 2.6

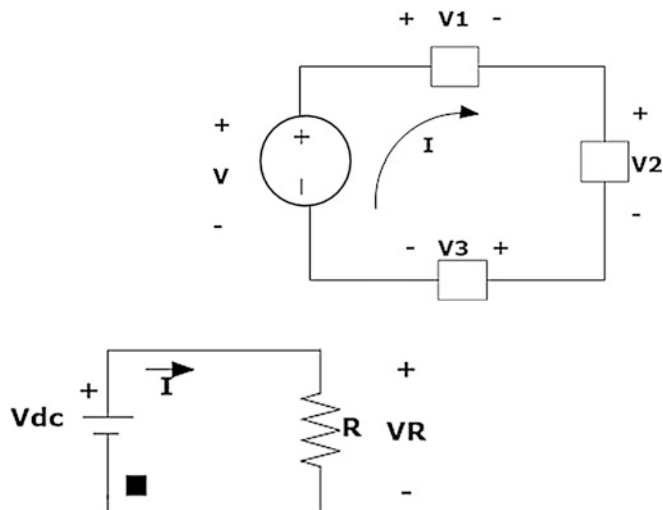


Fig. 2.10 In this example, starting from (■) sign and rotating clockwise, the first element is the voltage source. Following the current direction, it is observed that the voltage of the source is measured as $-V_{dc}$. This negative sign shows that the voltage is measured from the negative terminal with respect to its positive terminal, hence a negative value. Following the current direction in the loop, the voltage drop across the resistor is $+V_R$. The current should be calculated

Across the voltage source: $-V$ (negative, because the current enters the negative terminal of the source)

Across the element 1: $+V_1$ (positive, because the current entering any passive element generates a positive (+) polarity voltage drop)

Across the element 2: $+V_2$

Across the element 3: $+V_3$

Adding these voltages in KVL results in:

$$-V + V_1 + V_2 + V_3 = 0$$

Example 2.7 Consider the circuit of Fig. 2.10 with a voltage source V_{dc} (battery) and a resistor load R connected in series. The source forces a current in the circuit through the resistor. The resistor prevents the current passing and according to Ohms law the current I flows in the circuit. The source and the resistor form a loop that according to KVL can be analyzed to calculate the current I . Start with any arbitrary point in the loop, and calculate the voltage drop of elements, one by one. Write the KVL in this loop.

Therefore, combining all elements in this KVL results in:

$$\sum \Delta V = 0$$

$$-V_{dc} + V_R = 0$$

The actual value of V_R according to Ohm's law can be obtained as $V_R = RI$. Replacing in the KVL equation results in:

$$-V_{dc} + RI = 0$$

Solving for I yields:

$$RI = V_{dc}$$

$$I = \frac{V_{dc}}{R}$$

Note 2.1 Voltage drop across passive elements always shows positive polarity at the entry terminal. Passive elements are R , L , C . Therefore, the voltage drop of the resistor is $+V_R$.

Kirchhoff Current Law (KCL)

Consider a node in a circuit that connects two or more elements (node A in Fig. 2.11, for instance). Some of these elements feed the current into the node and some of them drain the current out of the node. KCL theory indicates that the summation of all currents entering or leaving a node must be zero. In node n the balance of currents can be written as:

$$\sum I_n = 0$$

Note 2.2 Currents may enter or leave a node. *Consider all current entering a node as a negative value and all currents leaving a node as a positive value.*

Note 2.3 Current sources force the current in or out of the node at a fixed value.

Note 2.4 If passive elements are connected to a node, they always drain the current out of the node.

Note 2.5 These rules apply to elements connected to each node regardless of all other considerations at the other nodes. For instance, the direction of current at both sides of a resistor is always inward to the terminals.

Example 2.8 Consider circuit of Fig. 2.11. Write KCL for the node A. KCL indicates that, the summation of all currents entering and leaving a node must be zero. Therefore:

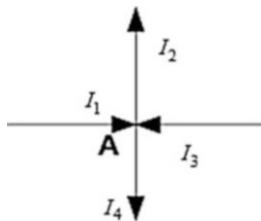
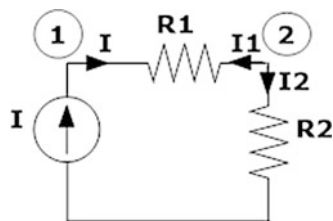


Fig. 2.11 Node A connects four elements together. Element 1 directs the current I_1 into the node, hence it carries a negative value in KCL. Element 2 directs the current I_2 out of the node hence it carries a positive value in KCL. Element 3 directs the current I_3 into the node hence it carries a negative value in KCL, and current I_4 directs the current out of the node hence it carries a positive value in KCL

Fig. 2.12 The circuit of Example 2.9. A current source forces the current into the node ①, and node ② is connected to two passive elements



$$\sum I_A = 0$$

Since I_1 and I_4 are entering the node, they will be written with a negative sign, and since I_2 and I_3 are leaving the node, they are considered positive values. Therefore, KCL:

$$-I_1 + I_2 + I_3 - I_4 = 0$$

Example 2.9 In the circuit shown in Fig. 2.12, find the voltages V_1 and V_2 once by using KCL and once by using KVL.

Solution 1 (Using KCL) The circuit has two nodes which are labeled 1 and 2. Each node has a voltage V_1 and V_2 . These voltages determine the current directions. However, as these nodes are connected to either current source or passive elements, their voltage needs to be determined.

Since there are two nodes, two KCLs need to be written.

KCL ①: The current source forces the current I in the node, hence, showing a negative value. The resistor R_1 is a passive element that drains the current out of node ①. Hence, the KCL in this node is written as:

$$-I + I_1 = 0$$

The current I_1 can be obtained from the Ohm's law for resistors. The current is the voltage drop across the resistor divided by the resistance. As the current direction is from V_1 to V_2 , $I_1 = \frac{V_1 - V_2}{R_1}$.

The KCL can now be completed as:

$$-I + \frac{V_1 - V_2}{R_1} = 0 \quad (2.1)$$

KCL ②: This node is connected to two resistors R_1 and R_2 . Hence, the currents I_1 and I_2 exit the node. The KCL becomes:

$$I_1 + I_2 = 0$$

The currents can be obtained considering their direction and voltages of the nodes as follows:

$I_1 = \frac{V_2 - V_1}{R_1}$ because the current is leaving node ②, and hence V_2 is considered larger than V_1 .

$I_2 = \frac{V_2}{R_2}$ because the voltage drop across the resistor R_2 is $(V_2 - 0)$.

Replacing the values, the KCL ② can be written as:

$$\frac{V_2 - V_1}{R_1} + \frac{V_2}{R_2} = 0. \quad (2.2)$$

Considering Eqs. (2.1) and (2.2), there is a set of two equations with two unknowns, V_1 and V_2 . Solving for V_1 and V_2 yields:

Simplifying the equations as:

$$\begin{cases} -I + \frac{V_1 - V_2}{R_1} = 0 \\ \frac{V_2 - V_1}{R_1} + \frac{V_2}{R_2} = 0 \end{cases}$$

$$\begin{cases} \frac{V_1}{R_1} - \frac{V_2}{R_1} = I \\ \frac{-V_1}{R_1} + V_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = 0 \end{cases}$$

Adding two equations eliminates V_1 and results in:

$$\frac{V_2}{R_2} = I$$

or

$$V_2 = R_2 I$$

The voltage V_1 becomes:

$$V_1 = (R_1 + R_2)I$$

Solution 2 (Using KVL) This problem can be solved using KVL. The existence of the current source imposes the same current I through R_1 and R_2 , as a known value. Therefore, the purpose of KVL is already served. The voltage drop across each element can be obtained from the Ohm's law.

Therefore, knowing the current I :

$$V_2 = R_2 I$$

Since V_1 is measured from the node ① to the ground, the total resistance from this node to ground must be considered in voltage calculations. Hence, V_1 becomes:

$$V_1 = (R_1 + R_2)I$$

Equivalent of Resistors in Series

The equivalent resistance of several resistors in series is the summation of those elements. Equivalent resistance is the amount of one resistor that can be replaced with the complex that is being considered to follow the same Ohm's law as the individual elements. The equivalent of n resistors in series can be obtained as follows (Fig. 2.13):

$$R_{eq} = \sum_{i=1}^n R_i$$

Example 2.10 Two resistors of $R_1 = 1 \text{ k}\Omega$ and $R_2 = 5 \text{ k}\Omega$ are connected in series to a 110 V source (as shown in Fig. 2.14). Find the current passing through the circuit, and find the voltage drop across each resistor.

Fig. 2.13 Resistors are connected in series. The equivalent becomes a summation of all resistors

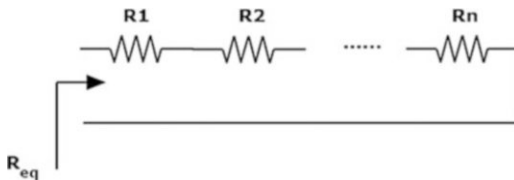


Fig. 2.14 The circuit of Example 2.10

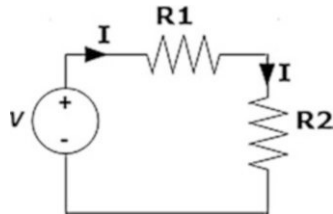
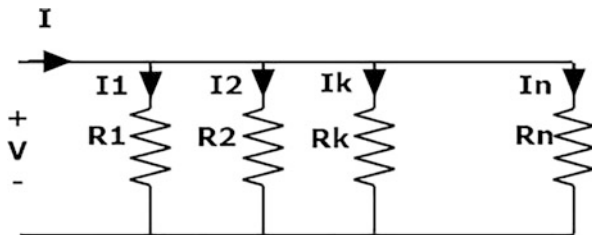


Fig. 2.15 The connection of several resistors in parallel. They all share the same voltage across the complex, but the current is shared inversely proportional to their resistance. Higher resistance takes lower current



Solution The equivalent resistor is $R_{eq} = R_1 + R_2 = 1\text{ k} + 5\text{ k} = 6\text{ k}\Omega$.

The current passing through the circuit becomes:

$$I = \frac{V}{R_{eq}} = \frac{110}{6\text{ k}} = 18.3\text{ mA}$$

The current passes each resistor and results in a voltage drop across each element proportional to its resistance. Therefore,

$$V_1 = IR_1 = 18.3e-3 \times 1e3 = 18.3\text{ V}$$

$$V_2 = IR_2 = 18.3e-3 \times 5e3 = 91.7\text{ V}$$

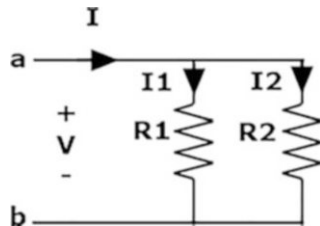
Equivalent of Resistors in Parallel

Resistors connected in parallel share the current proportional to their resistance values. Consider the following circuit with n resistors in parallel. The voltage across the circuit is V . Therefore, the current of the resistor k is $I_k = \frac{V}{R_k}$ for $k = 1, \dots, n$. The current drawn from the source can be calculated using KCL as follows (Fig. 2.15):

$$I = \sum_{k=1}^n I_k$$

$$I = \sum_{k=1}^n \left(\frac{V}{R_k} \right) = \left(\sum_{k=1}^n \frac{1}{R_k} \right) V = \frac{1}{R_{eq}} V$$

Fig. 2.16 The circuit of Example 2.11



$$R_{eq} = \frac{1}{\sum_{k=1}^n \frac{1}{R_k}}$$

Example 2.11 For a circuit connecting R_1 and R_2 in parallel (circuit of Fig. 2.16), find the equivalent resistance from ports a and b.

Solution

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

This can be simplified as:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Example 2.12 In the circuit of Fig. 2.16, the value of $R_1 = 10 \, \Omega$ and $R_2 = 15 \, \Omega$ which are connected to a 50 V source. Find the equivalent resistance of the circuit, the current drawn from the source, and the current in each resistor.

Solution The equivalent resistance is the parallel of R_1 and R_2 as:

$$R_{eq} = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{eq} = 10 \parallel 15 = \frac{10 \times 15}{10 + 15} = 6 \, \Omega$$

Therefore, the current I drawn from the source is:

$$I = \frac{50}{6} = 8.33 \, \text{A}$$

This current is shared between the resistors as follows:

$$I_1 = \frac{V}{R_1} = \frac{50}{10} = 5 \, \text{A}$$



Fig. 2.17 The figure of Example 2.13

$$I_2 = \frac{V}{R_1} = \frac{50}{15} = 3.33 \text{ A}$$

Example 2.13 Simplify the circuits A and B shown in Fig 2.17 and find the equivalent resistance.

Solution Circuit A. The same current passes through R_1 and R_2 ; therefore, they are connected in series. R_3 and R_4 are connected in series. However, the equivalent of series $R_1 + R_2$ is connected in parallel to the series connection of $R_3 + R_4$. That reads, $(R_1 + R_2) \parallel (R_3 + R_4)$. The equivalent resistance of circuit A becomes:

$$R_{eq} = (R_1 + R_2) \parallel (R_3 + R_4) = \frac{(R_1 + R_2)(R_3 + R_4)}{(R_1 + R_2) + (R_3 + R_4)}$$

Circuit B. In this circuit R_1 and R_2 are in series and in parallel to the other branch of $R_3 \parallel R_5$ in series to R_4 . Therefore,

$$R_{eq} = (R_1 + R_2) \parallel ((R_3 \parallel R_5) + R_4)$$

$$R_{eq} = \frac{(R_1 + R_2)((R_3 \parallel R_5) + R_4)}{(R_1 + R_2) + ((R_3 \parallel R_5) + R_4)}$$

Considering:

$$R_3 \parallel R_5 = \frac{R_3 R_5}{R_3 + R_5}$$

$$R_{eq} = \frac{(R_1 + R_2) \left(\left(\frac{R_3 R_5}{R_3 + R_5} \right) + R_4 \right)}{(R_1 + R_2) + \left(\left(\frac{R_3 R_5}{R_3 + R_5} \right) + R_4 \right)}$$

Power and Energy in Resistors

Resistors are passive elements and cannot store electric energy. However, they can consume power and generate heat. The power loss through a resistor P is measured in watts (W) and is directly proportional to the resistance and the square of the current as follows:

$$P = RI^2 = \frac{V^2}{R}$$

Example 2.14 A $100\ \Omega$ resistor passes a current of 2.2 A. Find the voltage drop across the resistor and the amount of power loss in the resistor.

Solution

$$V = RI \rightarrow V = 100 \times 2.2 = 220\text{ V}$$

The power of the resistor is:

$$P = RI^2 = 100 \times 2.2^2 = 484\text{ W.}$$

Example 2.15 An electric heater operates at a voltage of 110 V. If it takes 15 A to operate, find its resistance and power rating.

Solution

$$R = \frac{V}{I} = \frac{110}{15} = 7.33\ \Omega$$

$$P = VI = 110 \times 15 = 1650\text{ W.}$$

Example 2.16 Find a proper resistor (resistance and power rating) that can pass 1.5 A current at a voltage drop of 100 V.

Solution The resistance can be:

$$R = \frac{V}{I} = \frac{100}{1.5} = 66.6\ \Omega.$$

However, passing this current through the resistor generates power loss. The resistor has to be sized properly to be able to dissipate the heat. The power loss is:

$$P = VI = 100 \times 1.5 = 15\text{ W}$$

Therefore, a 15-W resistor is needed.

Definition of a Short Circuit

Part of an electric circuit can be called a “short circuit” if the total resistance connecting two points of that section becomes very small and ideally zero. For instance, if an electric switch is used to turn an electric bulb on and off, when the switch is closed, it fully conducts the current without any resistance (or small resistance), and in a sense, it shorts that part of the circuit. Short circuits happen

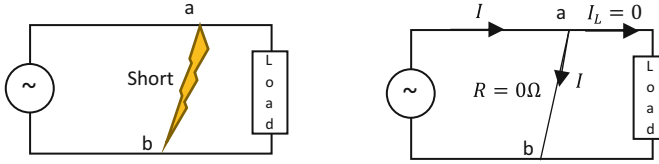


Fig. 2.18 Creation of a short circuit between points a and b

because of many reasons; some of it is intentional, e.g., a switch, and some of it is unwanted, e.g., a fault in the circuit. When two points of a circuit are shorted together, they are forced to become equipotential. That may cause a current to flow into the short circuit section. In fact, most of the short circuits created in existing circuits are due to a fault and a significant current passes the short circuit part. The short circuit analysis is to find the amount of current passing through the short circuit segment. Figure 2.18 shows the schematic of a short circuit.

What Is an Inductor?

Consider a straight line of wire. When a current passes through this wire, it builds a magnetic field around the wire. However, the magnetic field can be increased if more wires are grouped together so that their fields are added. One way to increase this field is to wrap the wire such that it forms a cylindrical shape. The wire can be wrapped around a toroidal core to form a toroidal inductor, or it can be circular but on a flat surface to form a winding Fig. 2.19. Any of these shapes form an inductor, where its inductance directly depends on the square of the number of turns (N^2) as follows:

$$L \propto N^2$$

Example 2.17 An inductor has inductance of $L = 1$ mH. If the number of turns in this inductor is increased by 40%, what would be the new inductance value?

Solution The value of inductance is directly proportional to the N^2 and for the new condition $N_{\text{new}} = 1.4N$. Therefore,

$$\frac{L_{\text{new}}}{L} = \left(\frac{N_{\text{new}}}{N} \right)^2$$

$$\frac{L_{\text{new}}}{L} = \left(\frac{1.4N}{N} \right)^2 = 1.4^2 = 1.96$$

The inductance is increased by 196%.

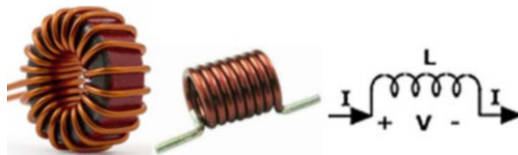


Fig. 2.19 The voltage drop across the terminals of an inductor depends on the time variation of the current passing through the inductor. Once the current variations are zero, the inductor shows zero volt drop and its equivalent to a short circuit

The voltage drop across an inductor is directly proportional to the amount of current variation over time. The slope of this dependency is the inductance L measured in henrys (H). The Ohm's law for an inductor L is expressed as:

$$v = L \frac{di}{dt}$$

Considering an initial current I_0 in the inductor, the instantaneous current becomes:

$$i = \frac{1}{L} \int v dt + I_0$$

It can be interpreted that the time variation of inductor current induces a voltage across the element. Therefore, if there is no current variation across the inductor, the voltage generated across the inductor falls down to zero. A zero voltage-induced value indicates a short circuit.

Example 2.18 A 1 mH inductor experiences a 10 A current change in 2 ms. Find the voltage induced at the terminals of the inductor.

Solution The voltage induced at the terminal of an inductor is:

$$v = L \frac{di}{dt} = L \frac{\Delta i}{\Delta t}$$

Therefore,

$$v = L \frac{\Delta i}{\Delta t} = 1e-3 \frac{10}{2e-3} = 5V.$$

Example 2.19 A 1 mH inductor experiences a 10 A current change in 2 μ s. Find the voltage induced at the terminals of the inductor.

Solution The voltage induced at the terminal of an inductor is:

$$v = L \frac{di}{dt} = L \frac{\Delta i}{\Delta t}$$

Therefore,

$$v = L \frac{\Delta i}{\Delta t} = 1e - 3 \frac{10}{2e - 6} = 5000V.$$

Energy and Power of an Inductor

Inductors store energy in the space around the coil wires. The amount of stored energy W (J) (joules) in an inductor L (H) when a current I (A) is passing through can be calculated by:

$$W = \frac{1}{2} LI^2 \text{ (J)}$$

The power of an inductor is the capability of discharging the stored energy over time. That is:

$$P = \frac{W}{t} \text{ (Watts)}$$

Example 2.20 Find the energy stored in a 1 H inductor that passes 10 A current.

Solution The amount of stored energy is:

$$W = \frac{1}{2} LI^2 = \frac{1}{2} \times 1 \times 10^2 = 50 \text{ (J)}.$$

Example 2.21 An inductor is used as an energy storage unit. To feed 100 W of energy in 2 min at the current discharge of 5 A, find the inductance.

Solution The inductor must store $W = P \times t$ joules of energy. To be able to discharge 100 Watts in $2 \times 60 = 120$ seconds it needs:

$$W = 100 \times 120 = 12 \text{ kJ}.$$

The amount of inductance needed is:

$$W = \frac{1}{2} LI^2$$

$$L = \frac{2W}{I^2} = \frac{2 \times 12,000}{5^2} = 960 \text{ H.}$$

The inductor in this example is large because the discharge rate is small. At higher discharge rates, the amount of inductance can be lower.

Equivalent of Inductors in Series

Equivalent inductances in series (Fig. 2.20) are the summation of the inductances. As the inductors share the same current, KVL determines that the summation of all voltages must add up to the source value. This leads to:

$$V = \sum_{k=1}^n V_k = \sum_{k=1}^n \left(L_k \frac{dI}{dt} \right).$$

Series connection results in the same current through all inductors.

$$V = \left(\sum_{k=1}^n L_k \right) \frac{dI}{dt} \triangleq L_{\text{eq}} \frac{dI}{dt}$$

$$L_{\text{eq}} = \sum_{k=1}^n L_k$$

Equivalent of Inductors in Parallel

Inductors connected in parallel share the current proportional to the integral of voltage and the value of inductance. Consider the following circuit with n inductors in parallel. The voltage across the circuit is V . Therefore, the current of each inductors is $I_k = \frac{1}{L_k} \int V$ for $k = 1, \dots, n$. The current drawn from the source can be calculated using KCL, as follows:

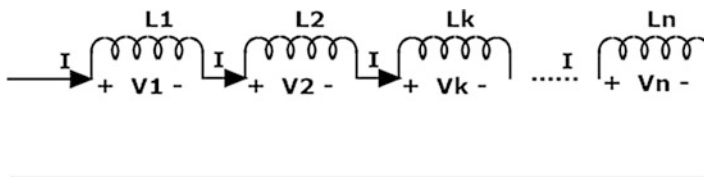


Fig. 2.20 The connection of inductors in series

$$I = \sum_{k=1}^n I_k$$

$$I = \sum_{k=1}^n \left(\frac{1}{L_k} \int V \right) = \left(\sum_{k=1}^n \frac{1}{L_k} \right) \int V$$

$$L_{\text{eq}} = \frac{1}{\sum_{k=1}^n \frac{1}{L_k}}$$

Example 2.22 For a circuit connecting L_1 and L_2 in parallel, find the equivalent inductance from ports a and b (Fig. 2.21).

$$L_{\text{eq}} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2}}$$

This can be simplified as:

$$L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2}$$

Example 2.23 Find the equivalent inductance and current of each inductor in the circuit of Fig. 2.21 when the voltage $v = 2 \sin 10t$ and $L_1 = 15 \text{ H}$, $L_2 = 20 \text{ H}$.

Solution The two inductors are connected in parallel. Therefore, the equivalent inductance is:

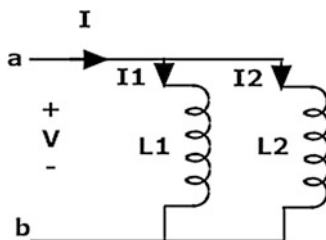
$$L_{\text{eq}} = 15 \text{ m} \parallel 20 \text{ m} = \frac{15 \times 20}{15 + 20} = 8.57 \text{ H}$$

The current of equivalent inductance is:

$$v = L_{\text{eq}} \frac{di}{dt}$$

Therefore, the current becomes:

Fig. 2.21 Circuit of Example 2.22



$$i = \frac{1}{L_{eq}} \int v dt = \frac{1}{8.57} \int 2 \sin 10t \, dt$$

$$i = \frac{1}{8.57} \frac{-2}{10} \cos 10t = -0.023 \cos 10t \, \text{A}$$

$$i_1 = \frac{1}{L_1} \int 2 \sin 10t \, dt = \frac{1}{15} \frac{-2}{10} \cos 10t = -0.013 \cos 10t \, \text{A}$$

$$i_2 = \frac{1}{L_2} \int 2 \sin 10t \, dt = \frac{1}{20} \frac{-2}{10} \cos 10t = -0.01 \cos 10t \, \text{A}$$

What Is a Capacitor?

Consider two conductive plates facing each other and form an overlap area A (m^2) in a close distance d (m). The shape of the blades and shape of the distance are not important as long as they maintain the same area and a constant distance. When the effective area between the plates is filled with a dielectric material with permittivity ϵ , the collection of the plates and the dielectric forms a capacitor in which its capacitance C is measured in Farads F as follows:

$$C = \epsilon \frac{A}{d}$$

Example 2.24 The capacitance of a capacitor with a relative dielectric constant of 40, and plates in a distance of 1 mm with an area of $A = 200 \, \text{mm}^2$ is 70.1 pF.

Ohm's Law and Capacitors

The amount of voltage drop across a capacitor is directly proportional to the integral of the current passing through the capacitor. The slope of this dependency is the inverse of capacitance $\frac{1}{C}$. Considering an initial voltage V_0 in the capacitor, the instantaneous voltage becomes:

$$v = \frac{1}{C} \int di + V_0$$

Therefore, the current passing through the inductor becomes:

$$i = C \frac{dv}{dt}$$

It can be interpreted that the voltage across a capacitor depends on the integral of the current over time. Another word, the current of a capacitor depends on the

instantaneous time variation of the voltage. The current of a capacitor exists as a result of its terminal voltage variations. If the voltage has no variation over time, like a DC source, the current of the capacitor reaches zero after it is being fully charged. The fully charged capacitor under DC current shows open-circuit behavior.

Example 2.25 A 1 μF capacitor experiences a current change with slope 10 A/s. Find the voltage drop across the terminals of the capacitor.

Solution The current is increasing with slope 10 A/s resulting in a linear equation of $i(t) = 10t$. Therefore,

$$v = \frac{1}{C} \int i dt$$

$$v = \frac{1}{1\mu} \int 10t dt = 1e6 \frac{10}{2} t^2 = 5e6 t^2 \text{ V}$$

Example 2.26 A 5 μF capacitor is connected to a voltage of $v(t) = 110\sqrt{2} \sin 377t$. Find the current passing through the capacitor.

Solution The current of the capacitor is obtained by:

$$i = C \frac{dv}{dt} = 5e - 6 \frac{d}{dt} (110\sqrt{2} \sin 377t) = -5e$$

$$-6 \times 110\sqrt{2} \times 377 \cos 377t = -0.293 \cos 377t$$

Energy and Power of a Capacitor

Capacitors store energy in form of electric charges on the plates interfaced by a dielectric material. The amount of stored energy W (J) depends on the applied voltage V and the capacitance, expressed as follows:

$$W = \frac{1}{2} CV^2 \text{ (J)}$$

Example 2.27 An ultra-capacitor is used to store energy in an electric vehicle. The amount of energy needed is 1 MJ and it is delivered at a 400 V system. Find the size of the capacitor needed.

Solution The amount of energy and the operating voltage are given. Therefore,

$$1e6 = \frac{1}{2} C 400^2$$

$$C = \frac{2 \times 1e6}{400^2} = 12.5 \text{ F}$$

Example 2.28 A capacitor is needed to smooth out the output voltage of a 500 W power supply when rectifying the 60 Hz waveforms. Find the amount of capacitor needed at full load in a half-wave rectifier when operating at 12 V.

Solution The amount of time that is needed to deliver the power is half cycle as $\frac{1}{2} \times \frac{1}{60} = \frac{1}{120}$ s. The amount of energy needed is:

$$W = P \times t$$

$$W = 500 \times \frac{1}{120} = 4.16 \text{ J}$$

Therefore,

$$C = \frac{2W}{V^2} = \frac{2 \times 4.16}{12^2} = 0.0577 \text{ F}$$

Or:

$$C = 57.77 \text{ mF}$$

Note A half-wave rectifier generates a voltage waveform that only selects the positive peaks of a sinusoidal waveform. Figure 2.22 shows the waveform and the time that the capacitor needs to feed the load (Fig. 2.23).

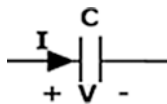
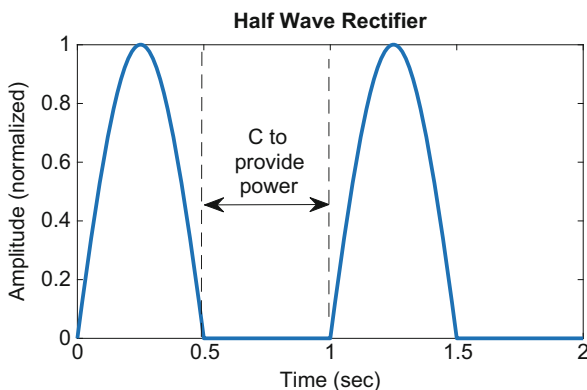


Fig. 2.22 The current passing a capacitor depends on the time variation of voltage across its terminals. Once the capacitor is fully charged, the current reaches zero and it becomes an open circuit

Fig. 2.23 The figure of half-wave rectifier



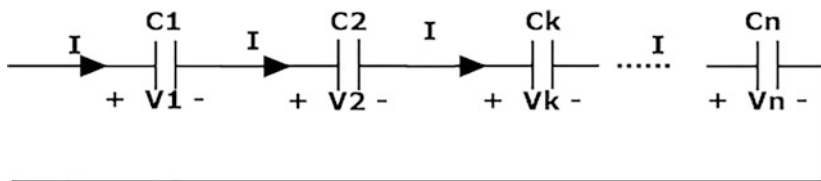


Fig. 2.24 The connection of capacitors in series

Equivalent of Capacitors in Series

The series connection of capacitors suggests similar current passing through each capacitor. As such, the voltages around a loop are added to hold KVL (Fig. 2.24).

Therefore, the voltage V across the series network can be obtained by:

$$V = \sum_{k=1}^n V_k$$

The voltage of the capacitor k can be obtained from the Ohm's law as:

$$V_k = \frac{1}{C_k} \int I \text{ for } k = 1, \dots, n$$

Sharing the same current, the KVL can be rewritten as:

$$V = \sum_{k=1}^n \left(\frac{1}{C_k} \int I \right) = \left(\sum_{k=1}^n \frac{1}{C_k} \right) \int I \triangleq \frac{1}{C_{\text{eq}}} \int I$$

Therefore,

$$\frac{1}{C_{\text{eq}}} = \left(\sum_{k=1}^n \frac{1}{C_k} \right)$$

Or

$$C_{\text{eq}} = \frac{1}{\sum_{k=1}^n \frac{1}{C_k}}$$

Example 2.29 Find the series connection of two capacitors C_1 and C_2 .

Solution

$$C_{eq} = C_1 \parallel C_2 = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$$

Equivalent of Capacitors in Parallel

The equivalent of capacitors in parallel shares the same voltage across the complex. The voltage imposes current to pass through each capacitor k as:

$$I_k = C_k \frac{dV}{dt} \text{ for } k = 1, \dots, n$$

Total current drawn from the source is shown in Fig. 2.26.

$$I = \sum_{k=1}^n I_k = \sum_{k=1}^n \left(C_k \frac{dV}{dt} \right)$$

As the voltage for all capacitors is the same, the KCL can be rewritten as:

$$I = \left(\sum_{k=1}^n C_k \right) \frac{dV}{dt} \triangleq C_{eq} \frac{dV}{dt}$$

Parallel connection of capacitors results in equivalent capacitance as follows:

Fig. 2.25 The circuit of Example 2.27

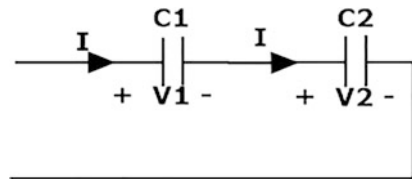
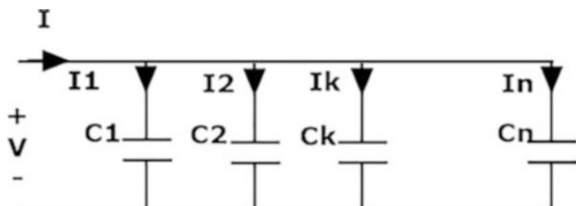


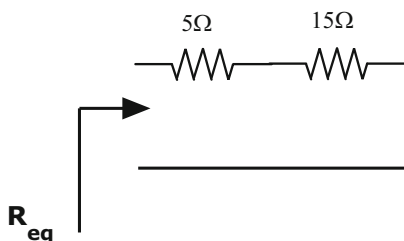
Fig. 2.26 The connection of capacitors in parallel



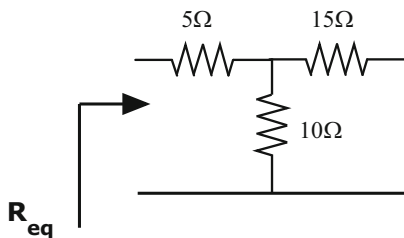
$$C_{eq} = \sum_{k=1}^n C_k$$

Problems

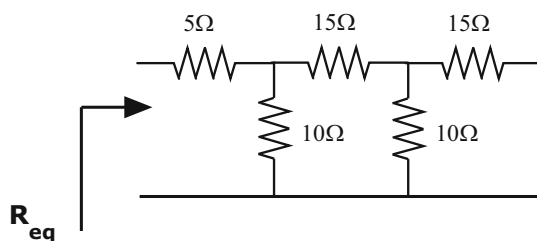
- 2.1 Find the number of electrons that needs to pass a section of a wire in unit of time to create a current of 2.2 A.
- 2.2 Find the current density of a wire with cross section of 25 mm^2 that passes 1.2 A current.
- 2.3 Find the resistance of a 100 m wire made of copper, with cross section of 25 mm^2 .
- 2.4 What is the resistance of a 100 g copper when it is shaped as a wire with cross section of 4 mm^2 ?
- 2.5 Find the equivalent resistance of the following circuit.



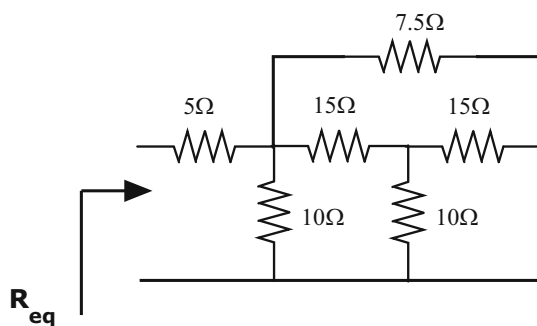
- 2.6 Find the equivalent resistance of the following circuit.



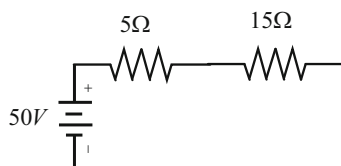
2.7 Find the equivalent resistance of the following circuit.



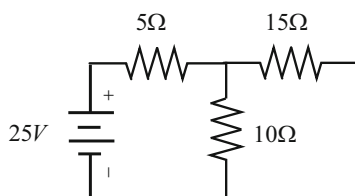
2.8 Find the equivalent resistance of the following circuit.



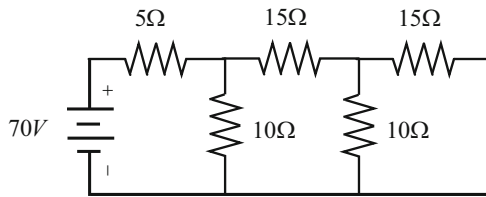
2.9 Find the current and voltage drop of all resistors in the following circuit.



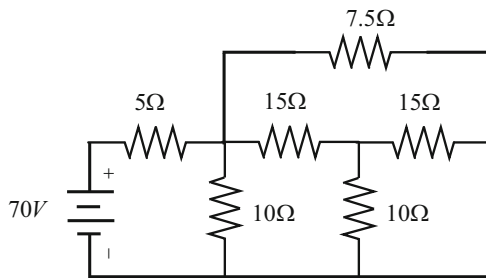
2.10 Find the current and voltage drop of all resistors in the following circuit.



2.11 Find the current and voltage drop of all resistors in the following circuit.



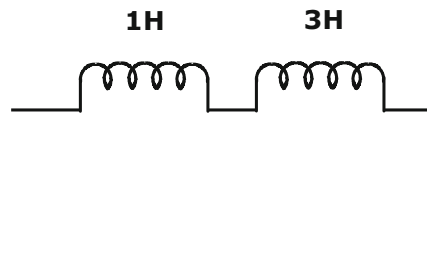
2.12 Find the current and voltage drop of all resistors in the following circuit.



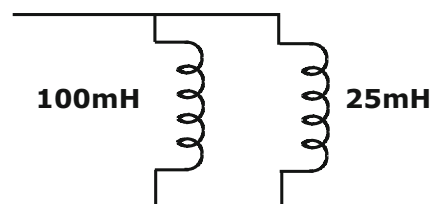
2.13 In the circuits of problems 2.9–2.12, find the power consumption in each resistor. How much power is drawn from the source?

2.14 An inductor shows a voltage of 150 V when its current varies 2.5 A in 2 ms. Find the inductance.

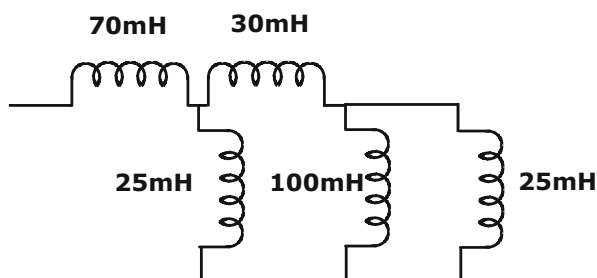
2.15 Find the equivalent inductance of the following circuit.



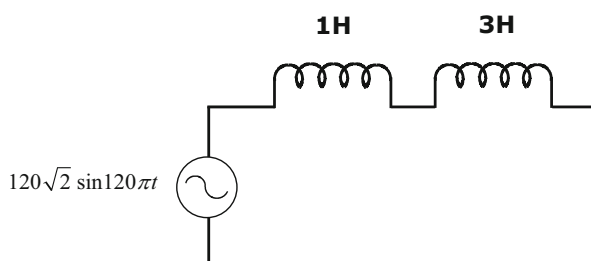
2.16 Find the equivalent inductance of the following circuit.



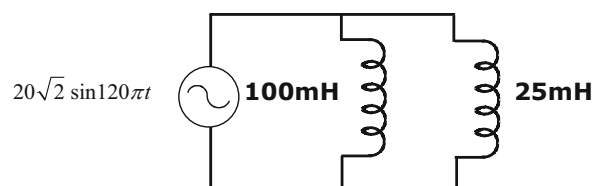
2.17 Find the equivalent inductance of the following circuit.



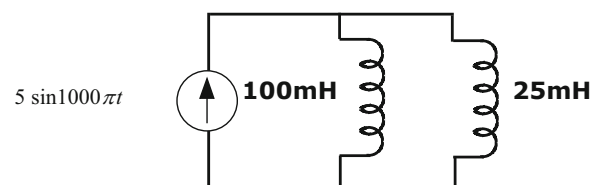
2.18 Find the current and voltage of all inductors in the following circuit.



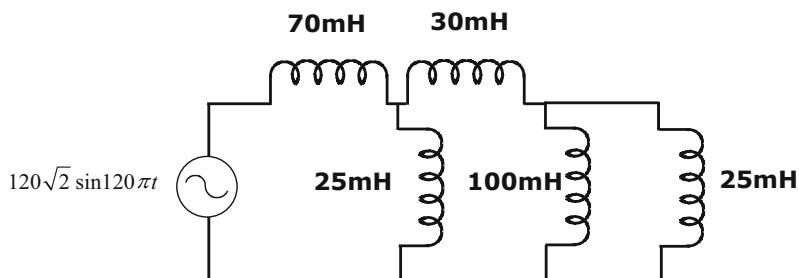
2.19 Find the current and voltage of all inductors in the following circuit.



2.20 Find the current and voltage of all inductors in the following circuit.

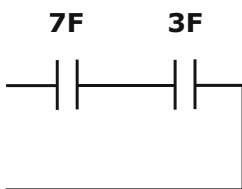


2.21 Find the current and voltage of all inductors in the following circuit.

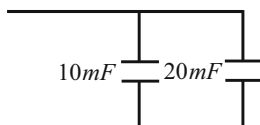


2.22 Find the energy stored in the circuits of problems 2.19–2.22.

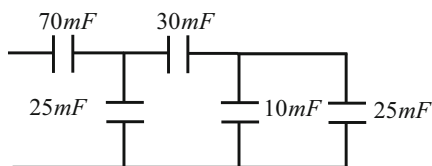
2.23 Find the equivalent capacitance of the following circuit.



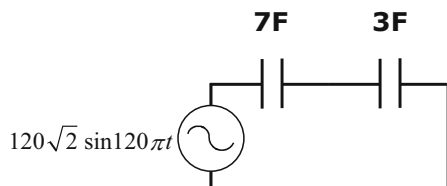
2.24 Find the equivalent capacitance of the following circuit.



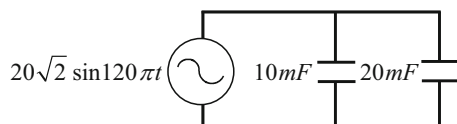
2.25 Find the equivalent capacitance of the following circuit.



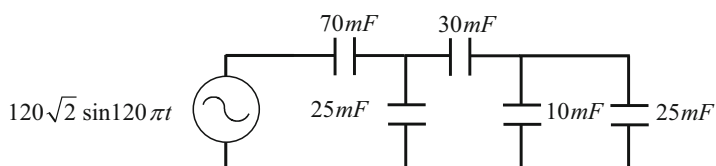
2.26 Find the current and voltage of each capacitor in the following circuit.



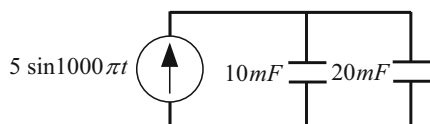
2.27 Find the current and voltage of each capacitor in the following circuit.



2.28 Find the current and voltage of each capacitor in the following circuits.



2.29 Find the current and voltage of each capacitor in the following circuits.



2.30 Calculate the energy stored in the circuits of problems 2.26–2.29.

Chapter 3

Waveform and Source Analyses



Introduction

Electric circuits consist of several components that form a certain topology. The drivers of the circuit can be voltage sources and/or current source. They force the current pass through the circuit by generating voltage drops across elements. A circuit performs certain tasks and has a certain output as well. The sources which are considered as input to the circuit can generate various waveforms. These waveforms excite the circuit and cause different effects.

This chapter is to introduce the common waveforms that might be seen from the sources or as a result of the excitation throughout the circuit. One important aspect of the waveform analysis is their expression in mathematical terms. This allows the programming of the circuit in computer codes and obtaining their results without building them. Later in the chapter, the types of sources as independent and dependent are discussed, and their effects in the circuit are studied.

Waveform Analysis

Impulse Function $f(t) = \delta(t)$

The impulse function, Dirac delta or chronicle impulse, $\delta(t)$, has value only at a single time, where the argument of the function is zero. The impulse function is zero elsewhere. As the function is shown in Fig. 3.1, its mathematical expression is as follows:

Fig. 3.1 Impulse at time $t = 0$

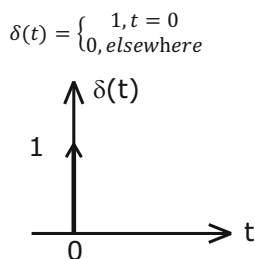
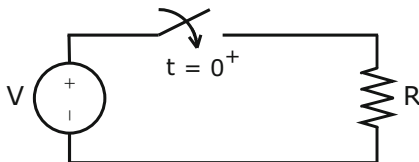


Fig. 3.2 The voltage source and the switch closing at time $t = 0^+$ resemble a step function that is applied to the resistor



$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{elsewhere} \end{cases}$$

Note 3.1 The $\int_0^\infty f(t)\delta(t-a)dt$ has value only at $t-a=0$ or $t=a$. The value of the integral is therefore $f(a)$. For instance, $f(t) = 2e^{-t}+1$ and $\delta(t-1)$ yields $\int_0^\infty (2e^{-t}+1)\delta(t-1)dt = 2e^{-1}+1 = 1.758$.

Unit Step Function $f(t) = u(t)$

Consider an on-off switch that is used to connect a source to a circuit and acts at a certain time. When the switch is on, the source is connected to the circuit, and the current flows, and when the switch is turned off, the source is disconnected, and the current flow stops. To show this even mathematically, a unit step can be used. The switching event changes the circuit topology. In simplest form, it connects or disconnects a source to a circuit and activates part of the circuit.

As the Fig. 3.2 shows, the voltage at times less than zero (means before switching action) is zero, and the voltage applied across the resistor jumps to a certain voltage a moment after the time of switching $t = 0^+$. The source amplitude determines the amplitude of the function, and the switching mechanism makes it a unit step.

As Fig. 3.3 shows the sketch of the unit step function in time domain, the amplitude of this function is 1, and it starts at time $t = 0^+$.

Mathematical expression of the unit step function is identified as:

Fig. 3.3 Step function of amplitude 1 at time $t = 0^+$. The time means that the switching event occurs in a moment after time $t = 0$

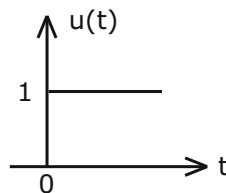
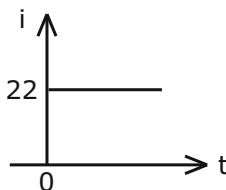


Fig. 3.4 The current in the form of a step function with amplitude 22 A and starting at time $t = 0^+$



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Example 3.1 If the unit step's amplitude changes, it will be expressed as a coefficient. For instance, the amplitude of current as a unit step function can be $i(t) = 22 u(t)$. It means that the current started at time $t = 0^+$ with an amplitude of 22 A. The current was zero at $t < 0$ (Fig. 3.4).

The step function can also start at any arbitrary time. A shift of time in the function affects all terms with time element t . A shift to the right of time axis by a seconds means a delay or lagging effect in the function, which institutes a transform of all $t \rightarrow t - a$, and a shift to the left of time axis by a seconds means a leading effect in the function, which institutes a transform of all $t \rightarrow t + a$.

For instance, as Fig. 3.5 demonstrates, the shift of 7 s to the right transforms the function $V(t) = 3u(t)$ to $V(t) = 3u(t - 7)$, and a shift of the same time to the left makes the function transforms to $V(t) = 3u(t + 7)$.

Note 3.2 In fact, the start time of any function can be found by making the argument of the signal to zero and solving for time. For instance, the signal $V(t) = 3u(t - 7)$ has the time argument of $(t - 7)$. Making $t - 7 = 0$ and solving for the time yields $t = 7$, as the starting time of the signal. The same procedure can be used to find the start time of any function.

Example 3.2 Sketch the following functions in time domain: (a) $f(t) = 15u(t)$, (b) $f(t) = 15u(t - 10)$, (c) $f(t) = 15u(3t - 18)$.

Solution All functions are unit steps with amplitude 15. The only difference is in their start time. Function *a* has start time at $t = 0$, function *b* has start time obtained from $t - 10 = 0$, $t = 10$ s. Function *c* has start time where its argument is zero, or $3t - 18 = 0$, which yields $3t = 18$, or $t = 6$ s. Figure 3.6 shows the functions.

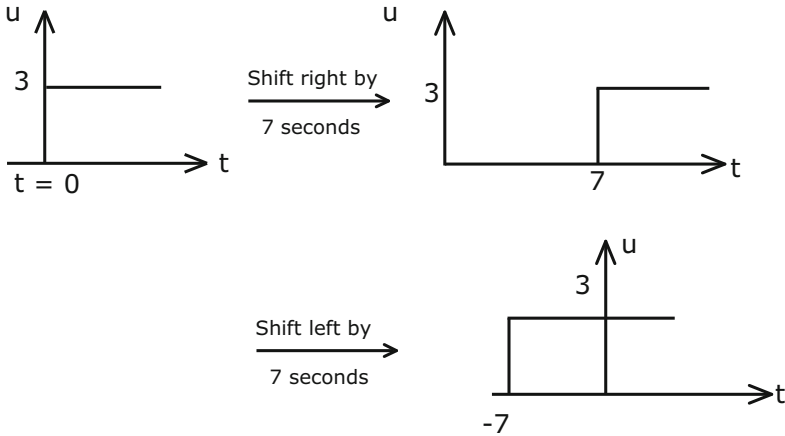


Fig. 3.5 A shift of step function to the right (delay in switching) and shift left (early switching)

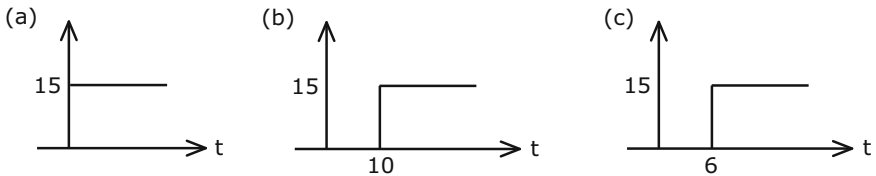


Fig. 3.6 Waveforms of Example 3.1

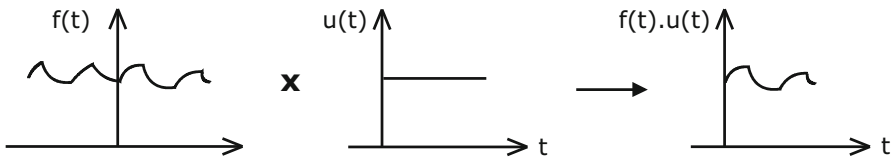


Fig. 3.7 Product of a function $f(t)$ by a unit step function. The amplitude does not change; however, only positive part of the waveform $f(t)$, where the unit step exists, is selected

Note 3.3 Step function has the ability to eliminate a part of a function. Production of the function $f(t)$ by the unit step $u(t)$ guarantees that the waveform is forced to zero before the unit step starts. A shift in time, when required, must appear in both functions as $f(t - a)$ and $u(t - a)$. Figure 3.7 shows the results of an arbitrary function $f(t)$ (that exists for both time positive and negative) both before and after it is multiplied by the unit step function $u(t)$.

Note 3.4 The actual value of a step function is its amplitude. When any function multiplied by a step function, this only indicates that the function has a value when the step indicates. Therefore, $\int_0^\infty f(t - a)u(t - a)dt$ means that the borders of integral are set for $t \geq a$. The integral can be rewritten as $\int_a^\infty f(t - a)dt$.

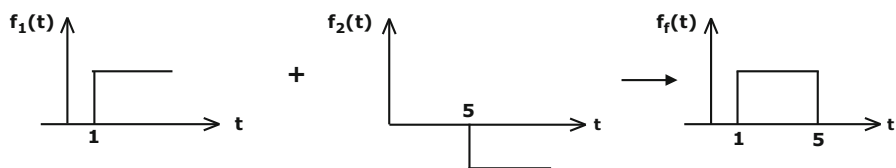


Fig. 3.8 Summation of two unit steps and formation of a pulse. Functions $f_1(t)$ and $f_2(t)$ have opposite amplitudes but start at different times. Therefore, when adding them together, they cancel each other when they both exist, e.g., for $t > 5$ and for the time both have not started yet, e.g., $t < 1$

Note 3.5 When two step functions, for instance, $f_1(t) = u(t - 1)$ and $f_2(t) = -u(t - 5)$, are being added together, the result is obtained by adding point-to-point values of both signals. Therefore, considering the functions, the summation result has no value until $t = 1$ when the function f_1 starts. Within the window of $1 < t < 5$, only $f_1(t)$ has a value. Therefore, the summation has a function with amplitude 1 for the time $1 < t < 5$. When the function $f_2(t)$ starts at time $t = 5$, the result summation of both amplitudes becomes $+1 - 1 = 0$. Therefore, the amplitude of the summation becomes zero for any time $t > 5$. This yields a pulse function between $1 < t < 5$ with amplitude 1 as shown in Fig. 3.8.

Ramp Function $f(t) = r(t)$

If the value of the function linearly (proportionally) increases by time, the function is called a ramp. The slope of this line, k , determines the rate of increment. The function is written in mathematical form as:

$$r(t) = ktu(t).$$

The ramp function can also be expressed as:

$$r(t) = \begin{cases} kt, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

The ramp function has the value zero for time negative and increases by a factor k every second. The slope is defined as the ratio of the value gained over the time it took to gain the value. Figure 3.9 demonstrates the ramp function and its slope calculation.

Note 3.6 The slope is determined by the amount of amplitude gain over the time it took to reach the gain.

Note 3.7 A shift in the ramp function shifts the waveform's starting time but keeps the slope intact.

Fig. 3.9 A ramp function with slope k . The slope is the ratio of the gain in the amplitude over time it takes.

$$k = \frac{\text{Gain}}{\text{Time}}$$

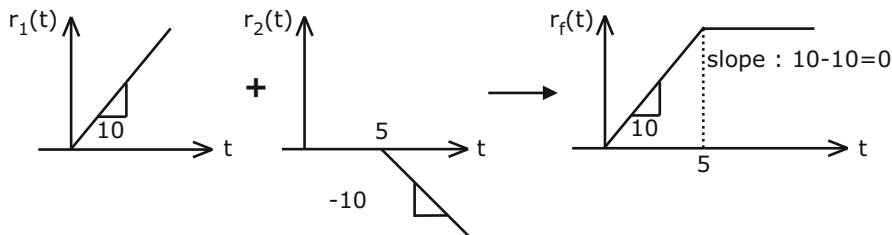
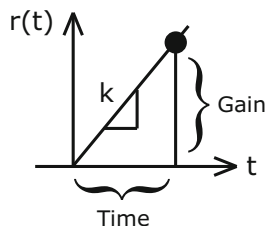


Fig. 3.10 Ramp functions of Example 3.3. When ramp functions are added together, their slopes are algebraically added. It is important to consider the slope changes when any of the functions experience a slope change. For instance, the points of $t = 0$ and $t = 5$ are needed to be observed for the slope changes because $r_1(t)$ and $r_2(t)$ start at these times

Example 3.3 Considering functions r_1 and r_2 shown in Fig. 3.10, sketch the summation of the functions and express the result in mathematical terms.

Solution Function $r_1(t) = 10tu(t)$ is shown in Fig. 3.10. The function reaches amplitude 10 in 1 s and amplitude 20 in 2 s. It reaches 50 in 5 s, and it continues to gain amplitude at the same rate 10 units/s. Function $r_2(t)$ starts at time $t = 5$ s. It has the slope -10 , which means in each second it drops by -10 . Therefore, in 1 s after it start (or $t = 5 + 1 = 6$ s), it reaches -10 and continues to drop -10 each second. Mathematical expression of this function is $r_2(t) = -10(t - 5)u(t - 5)$, with slope -10 and start time 5.

Now, to add $r_1(t)$ and $r_2(t)$ point by point, the result starts from $t = 0$. The slope is 10 in time $0 < t < 5$, and it shows a change of slope at $t = 5$ to $(+10 - 10 = 0)$. Since the effect of two signals has been considered and there is no change in them for $t > 5$, the result summation function continues with slope 0 onward.

Example 3.4 Consider the previous example, and add a third function $r_3(t) = -10(t - 7)u(t - 7)$ to the functions $r_1(t)$ and $r_2(t)$. Sketch the summation result.

Solution Function $r_3(t)$ starts at $t = 7$, which means no change in the summation of $r_1(t)$ and $r_2(t)$ until $t = 7$. From $5 \leq t < 7$, the slope of $r_1(t) + r_2(t)$ is zero. At time $= 7$, $r_3(t)$ with slope -10 is added to the signals. The slope is therefore changed to $0 - 10 = -10$. The function of $r_1(t) + r_2(t) + r_3(t)$ is shown in Fig. 3.11.

Note 3.8 Adding ramp functions together changes their slope at the starting time the functions.

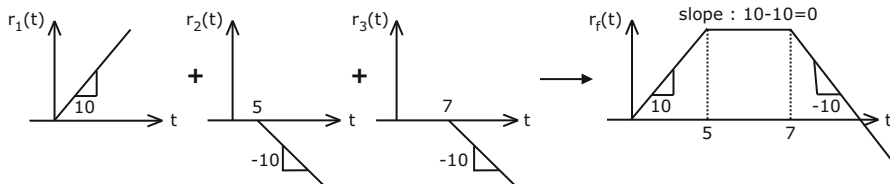


Fig. 3.11 Ramp functions of Example 3.4

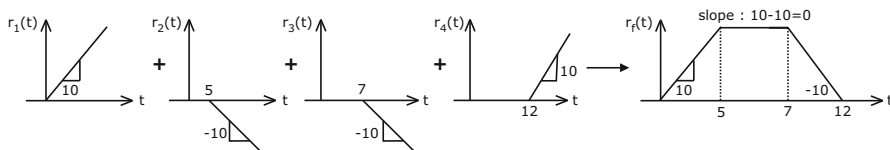
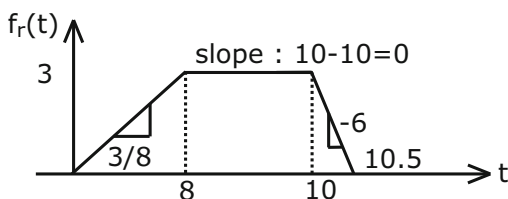


Fig. 3.12 Ramp functions of Example 3.5

Fig. 3.13 Ramp function of Example 3.6



Example 3.5 Note that the function $r_1(t) + r_2(t) + r_3(t)$ continues with slope -10 even after it reaches zero at time $t > 12$. Suggest a fourth function $r_4(t)$ to stop the function from proceeding to negative values and make its slope zero at $t = 12$ and onward.

Solution The functions $r_1(t) + r_2(t) + r_3(t)$ result in slope -10 at $t > 12$, and to reach slope zero, a function $r_4(t)$ with slope $+10$ must be added. However, the function must be started at the time $t = 12$. Therefore, if a shift of 12 s is applied, the function needs to be added is obtained as $r_4(t) = +10(t - 12)u(t - 12)$. The summation is shown in Fig. 3.12.

Note 3.9 Note that the term of unit step function $u(t - 12)$ presented in the function $r_4(t) = 10(t - 12)u(t - 12)$ guarantees that the value of $r_4(t)$ remains zero for all $t < 12$.

Example 3.6 Write the mathematical expression of the signal shown in Fig. 3.13.

Solution The function starts at time $t = 0$ with slope $3/8$. This suggests existence of a signal $r_1(t) = \frac{3}{8}tu(t)$. The function continues with this slope until time $t = 8$. At this point the slope has reached zero. This requires a $-\frac{3}{8}$ slope change to the existing signal of $r_1(t)$ but starting at time $t = 8$. Therefore, $r_2(t) = -\frac{3}{8}(t - 8)u(t - 8)$ must be added to the function $r_1(t)$. The summation of these two signals continues with slope zero until the time $t = 10$. At this point the slope needs to reach -6 .

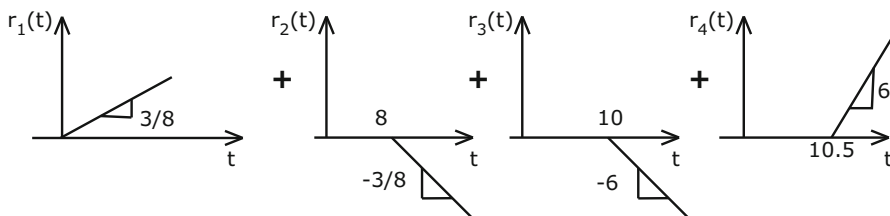


Fig. 3.14 Split of ramp functions in Example 3.6

This requires a $0 - 6 = -6$ slope change introduced by a third function at $t = 10$ or $r_3(t) = -6(t - 10)u(t - 10)$. With slope -6 , amplitude of the resultant signal reaches zero in 0.5 s. Summation of these three signals leaves the slope -6 for time $10 \leq t < 10.5$.

After this time, or $t \geq 10.5$, the amplitude and slope should remain 0. Therefore, another slope change of $+6$ must be added to the signals starting at $t = 10.5$. Therefore, the fourth signal is $r_4(t) = +6(t - 10.5)u(t - 10.5)$ (Fig. 3.14).

Example 3.7 Sketch the function $f(t) = 5(t - 10)u(t - 7)$.

Solution The argument of the unit step function and the time shift of the ramp are different. In this case, an operation is needed to make the ramp format as $k(t - a)u(t - a)$. It is critical to have the same amount of time shift for the ramp and its unit step function.

In this example, 10 is split to a 7 and a 3 as $f(t) = 5(t - 7 - 3)u(t - 7)$. Then, an expansion results in:

$$f(t) = 5(t - 7)u(t - 7) + 5(-3)u(t - 7)$$

The function becomes a combination of a ramp $f_1(t) = 5(t - 7)u(t - 7)$ and a unit step function $f_2(t) = 5(-3)u(t - 7)$. The ramp has slope of 5 and a start time of 7. The step function has an amplitude of $-15 = 5(-3)$ and a start time of 7.

The waveform is shown in Fig. 3.15.

Power Function $f(t) = \frac{A t^n}{n!} u(t)$

For a given natural number n , the function has an amplitude of $\frac{A}{n!}$ and a time factor of t^n . For $n = 1$, the function becomes a ramp $f(t) = A t u(t)$, shown in Fig. 3.16. For $n = 2$ the function becomes a parabolic that exists in positive time (the effect of $u(t)$), as $f(t) = \frac{A t^2}{2!} u(t)$, shown in Fig. 3.17.

Note 3.10 Derivative of a ramp function $r(t) = k t u(t)$ is a unit step function with amplitude k . Likewise, the integral of a step function $k u(t)$ is a ramp function

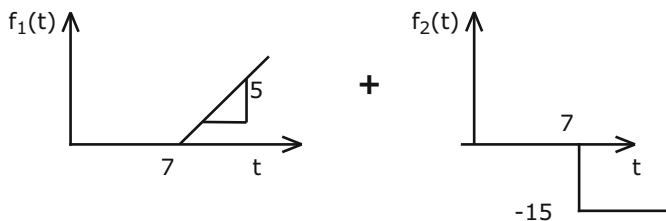


Fig. 3.15 Figure of function in Example 3.7

Fig. 3.16 Power function becomes a ramp function when $n = 1$. Coefficient A becomes the slope

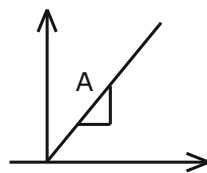


Fig. 3.17 Power function becomes a parabolic function when $n = 2$

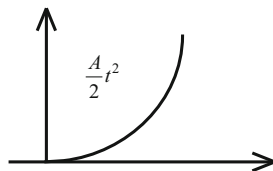
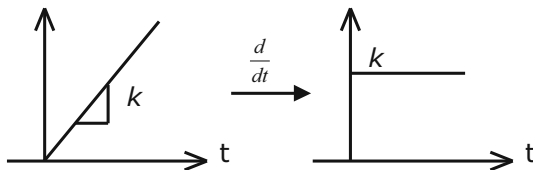


Fig. 3.18 Derivative of a ramp function with slope k becomes a unit step with amplitude k



$r(t) = ktu(t)$. Figure 3.17 shows the derivative and integral functions of the ramp and unit step (Fig. 3.18).

$$\frac{d}{dt}r(t) = \frac{d}{dt}ktu(t) = ku(t)$$

$$\int ku(t)dt = ktu(t)$$

Note 3.11 Derivative of a step function $ku(t)$ is an impulse with amplitude $k\delta(t)$. Likewise, the integral of $k\delta(t)$ is $ku(t)$ (Fig. 3.19).

Fig. 3.19 Derivative of a unit step function with amplitude k becomes an impulse with amplitude k

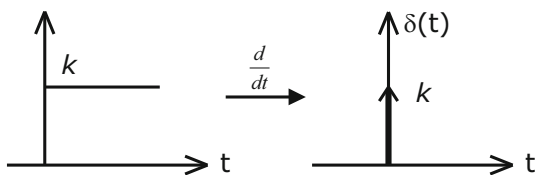
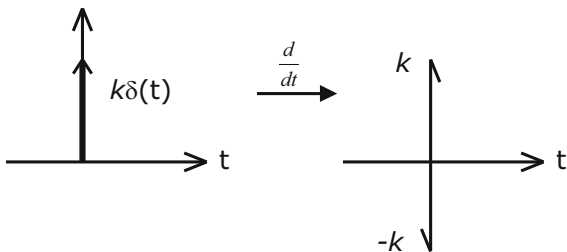


Fig. 3.20 Derivative of an impulse with amplitude k becomes a function with amplitudes split to $-k$ in negative time $t = 0^-$ and $+k$ in positive time $t = 0^+$



$$\frac{d}{dt}ku(t) = k\delta(t)$$

$$\int k\delta(t)dt = ku(t)$$

Note 3.12 Derivative of an impulse function $k\delta(t)$ is $k\dot{\delta}(t)$. Likewise, the integral of $k\dot{\delta}(t)$ is $k\delta(t)$.

$$\frac{d}{dt}k\delta(t) = k\dot{\delta}(t)$$

$$\int k\dot{\delta}(t)dt = k\delta(t)$$

Exponential Function $f(t) = Ae^{\alpha t}u(t)$

This function shows an exponentially steady raise or decay. The decay factor α determines the rate of change, where a positive value shows a raise and a negative value shows a decay in the amplitude. The functions for $\alpha > 0$ and $\alpha < 0$ are shown in Fig. 3.20. The value of function $f(t) = Ae^{\alpha t}u(t)$ at $t = 0$ is $f(t = 0) = A$. The value of $1/\alpha$ shows the time constant of the signal.

The decay factor determines how fast the signal reaches zero. For instance, $\alpha = -1$ has slower decay rate than $\alpha = -2$. Figure 3.21 shows the effect of decay factor on the shape of signal.

For $\alpha > 0$, the waveform exponentially increases. Figure 3.22 shows the waveforms $\alpha = +1, -1$ and $\alpha = +2, -2$.

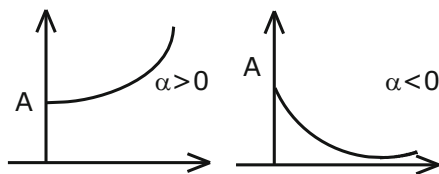


Fig. 3.21 Exponential function with positive damping factor $\alpha > 0$. The amplitude increases by time. Exponential function with negative damping factor $\alpha < 0$. The amplitude decreases by time

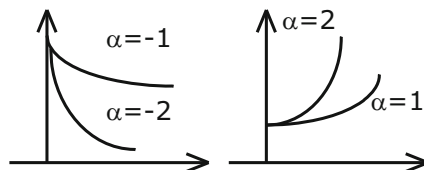


Fig. 3.22 As damping rate becomes more negative, the rate of amplitude decrease accelerates and the function reaches zero faster. As damping rate becomes more positive, the rate of amplitude increase accelerates and the function reaches infinite faster

Note 3.13 Derivative of an exponential function repeats itself and generates coefficients as follows:

$$\frac{d}{dt}(Ae^{\alpha t}) = \alpha(Ae^{\alpha t})$$

$$\int Ae^{\alpha t} dt = A \frac{1}{\alpha} e^{\alpha t}$$

Example 3.8 Do the math.

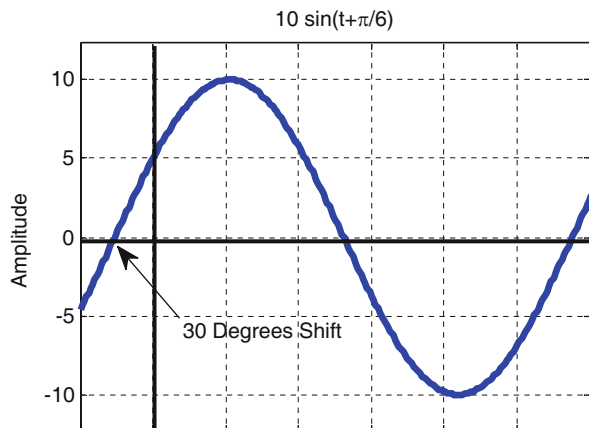
- (a) $\int 10e^{-3t} dt = \frac{10}{-3} e^{-3t}$
 (b) $\frac{d}{dt}(5e^{-7t}) = 5(-7)e^{-7t}$
 (c) $\int \left(\frac{7}{8} e^{\frac{3}{16}t}\right) dt = \frac{7}{8} \frac{1}{\frac{3}{16}} e^{\frac{3}{16}t} = \frac{7}{8} \frac{16}{3} e^{\frac{3}{16}t} = \frac{14}{3} e^{\frac{3}{16}t}$

Sinusoidal Function $f(t) = A \sin(\omega t + \varphi)$

The sinusoidal function is periodical and repeats every 2π degrees. Maximum amplitude reaches A at frequency ω rad/s. The frequency in Hertz is obtained by $f = \frac{\omega}{2\pi}$. The signal can lead or lag to cross the origin by φ degrees. Therefore, the angle φ is called a phase shift.

Note 3.14 A lead signal has a positive phase shift, and a lag signal has a negative phase shift (Fig. 3.23).

Fig. 3.23 A sinusoidal waveform with amplitude $A = 10$ (peak) and phase shift of angle $\varphi = -\frac{\pi}{6}$ or -30° can be mathematically expressed as $A \sin(\omega t - 30)$



Note 3.15 Derivative of a sinusoidal function is as follows:

$$\frac{d}{dt} A \sin(\omega t + \varphi) = A\omega \cos(\omega t + \varphi)$$

$$\int A \sin(\omega t + \varphi) dt = -\frac{A}{\omega} \cos(\omega t + \varphi)$$

Note 3.16 Several information are presented in a sinusoidal function including the amplitude, phase shift, and frequency. When a circuit is excited by a sinusoidal waveform, the frequency remains constant throughout the circuit. In every element of the circuit whether current or voltage, the frequency is the frequency of the source. However, the amplitude and phase of the voltages and currents are influenced by the circuit topology and its element values. Therefore, a sinusoidal function when representing values of a circuit can be presented by conveying their amplitudes and phase information. A phasor equivalent of a sinusoidal conveys this information and is presented as follows:

$$A \sin(\omega t + \varphi) \equiv A \angle \varphi$$

Polar to Cartesian (Rectangle) Conversion

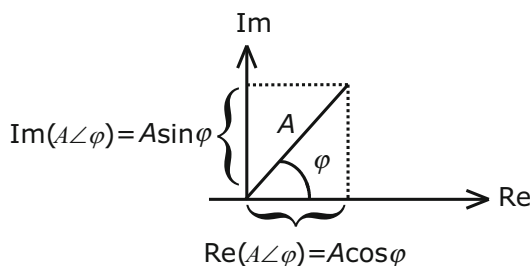
Phasor is a polar representation of the function. The polar coordinates can be converted to rectangle as follows. Consider a function $f(t)$ as:

$$f(t) = A \angle \varphi$$

Rectangle presentation of this phasor, shown in Fig. 3.24, is obtained by projecting the vector on the real and imaginary axes as follows:

$$A \angle \varphi = A \cos(\varphi) + jA \sin(\varphi) = \text{Re}(f(t)) + j\text{Im}(f(t)) = p + jq. \quad (3.1)$$

Fig. 3.24 Polar representation of a sinusoidal waveform (amplitude and phase) can be converted by projecting the vector onto the real and imaginary axes to obtain the rectangular representation



Real part of this function $p = \text{Re}(A \angle \varphi) = A \cos(\varphi)$ and the imaginary part of this function is $q = \text{Im}(A \angle \varphi) = A \sin(\varphi)$.

Cartesian (Rectangle) to Polar Conversion

Any complex conjugate number $p + jq$ can be converted to polar coordinates as follows:

$$A \angle \varphi = p + jq = \sqrt{p^2 + q^2} \angle \tan^{-1} \frac{q}{p}$$

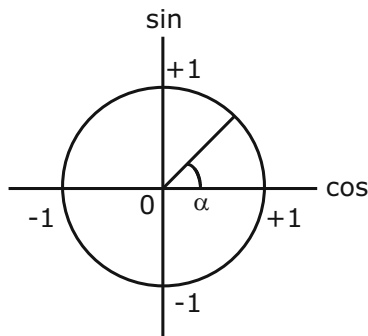
Example 3.9 Find polar transform of the following numbers:

- (a) $2 + j2$ Answer: $\sqrt{2^2 + 2^2} \angle \tan^{-1} \frac{2}{2} = 2\sqrt{2} \angle 45$
- (b) $2 - j2$ Answer: $\sqrt{2^2 + (-2)^2} \angle \tan^{-1} \frac{-2}{2} = 2\sqrt{2} \angle -45$
- (c) $1 + j0.866$ Answer: $\sqrt{1^2 + 0.866^2} \angle \tan^{-1} \frac{0.866}{1} = 1.3228 \angle 40.91$
- (d) $0 + j1$ Answer: $\sqrt{0^2 + 1^2} \angle \tan^{-1} \frac{1}{0} = 1 \angle 90$
- (e) $-j1$ Answer: $\sqrt{0^2 + (-1)^2} \angle \tan^{-1} \frac{-1}{0} = 1 \angle -90$

Example 3.10 Find the rectangle transform of the following numbers:

- (a) $10 \angle 30$ Answer: $10 \cos(30) + j10 \sin(30) = 8.66 + j5$
- (b) $10 \angle 0$ Answer: $10 \cos(0) + j10 \sin(0) = 10 + j0$
- (c) $10 \angle 90$ Answer: $10 \cos(90) + j10 \sin(90) = 0 + j10$
- (d) $10 \angle -90$ Answer: $10 \cos(-90) + j10 \sin(-90) = 0 - j10$

Fig. 3.25 Unit circle, sin, and cos axes are shown



Example 3.11 Find the value for j , j^2 , j^3 .

Solution Considering $j = \sqrt{-1}$, then $j^2 = \sqrt{-1}\sqrt{-1} = -1$, and $j^3 = jj^2 = j(-1) = -j$.

Example 3.12 The signal $10 \sin(377t + 40)$ has the peak amplitude of 10 and angular frequency of 377 rad/s, with phase angle 40° . This can also be presented in phasor as $10 \angle 40$.

Note 3.17 Sine and cosine functions can be converted to each other in mathematical operations. Following is the list of these conversions obtained from the unit circle as shown in Fig. 3.25.

$$\cos(\omega t) = \sin(\omega t + 90)$$

$$-\cos(\omega t) = \sin(\omega t - 90)$$

$$\sin(\omega t) = \cos(\omega t - 90)$$

$$-\sin(\omega t) = \cos(\omega t + 90)$$

Note 3.18 To add or subtract several sine and cosine functions, (1) all of them must have the same frequency, and (2) all must be converted to either sine or cosine. Otherwise, a phase shift of $\pm 90^\circ$ will be lost in the calculations.

Example 3.13 Show the following functions in sine form and phasor:

$$f(t) = 10 \sin(377t + 30)$$

Solution The functions is already in sine form; therefore the phasor presentation is $10 \angle 30$.

Example 3.14 The function needs to be converted to a sine function.

$$f(t) = 5 \cos(377t + 60)$$

Solution The phase shift according to Note 3.17 must be considered:

$$\cos(\omega t) = \sin(\omega t + 90)$$

Therefore, $f(t) = 5 \sin(377t + 60 + 90)$. The 90° phase shift is added to convert the cosine to sine. The phasor representation is $5 \angle (60 + 90)$ or $5 \angle 150$.

Mathematical Operation of Polar and Complex Numbers

It is recommended to add complex numbers in Cartesian coordinate and to multiply numbers in polar coordinate. It is possible to add numbers in polar coordinate and multiply numbers in Cartesian coordinates; however it requires longer mathematical operations.

Adding Complex Numbers

Add real parts together, and add imaginary parts together. It becomes:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

Product of Complex Numbers

Expansion production results in:

$$(a + jb)(c + jd) = (ac + jad + jbc + jjbd)$$

$jj = j^2 = -1$, which leads to:

$$(ac + jad + jbc - bd)$$

Now, collecting real parts and imaginary parts separately results in:

$$(ac - bd) + j(ad + jbc)$$

Product of Polar Numbers

Amplitudes are multiplied, and angles are summed to result in:

$$(A \angle \varphi)(B \angle \theta) = (AB) \angle (\varphi + \theta)$$

Division of Polar Numbers

Amplitudes are divided, and angles are subtracted as:

$$\frac{(A \angle \varphi)}{(B \angle \theta)} = \left(\frac{A}{B}\right) \angle (\varphi - \theta)$$

The angle is always the angle of numerator φ minus the angle of denominator θ .

Summation of Polar Numbers

$$\begin{aligned} (A \angle \varphi) + (B \angle \theta) &\rightarrow (A \cos \varphi + jA \sin \varphi) + (B \cos \theta + jB \sin \theta) \\ &= (A \cos \varphi + B \cos \theta) + j(A \sin \varphi + B \sin \theta) \end{aligned}$$

Both numbers must be converted to rectangle and then be added as real and imaginary numbers.

Note 3.19 Mathematical operation of a mix of polar and rectangle suggests a transform from one form to another to ease the mathematical operations. It is more suitable to do division and product in polar form and summation in rectangle form. However, as explained earlier, it is possible to do all mathematical operations in one form.

Example 3.15 Do the following mathematical operations in a suitable form (either polar or rectangle):

$(1 + j20)(1 - j20).$	Answer: 401
$\frac{1+j20}{1-j1}.$	Answer: $-9.5 + j10.5$
$\frac{2-j5}{3+j7}.$	Answer: $-0.5 - j0.5$
$(10 \angle -90) + (10 \angle -30).$	Answer: $8.66 - j15$
$\frac{10 \angle -60}{5 \angle 30}.$	Answer: $-j2$

Note 3.20 Conjugate of a complex number is shown by a star and is obtained as follows:

$$(a + jb)^* = a - jb$$

$$(A \angle \varphi)^* = A \angle -\varphi$$

Example 3.16 Solve $\frac{1+j2}{4-j3}$.

Solution To simplify the function, the complex number of the denominator should be converted to a real number. That is possible by multiplying and dividing the entire function by the complex conjugate of the denominator as follows:

$$\begin{aligned} \frac{1+j2}{4-j3} &= \frac{1+j2}{4-j3} \times \frac{(4-j3)^*}{(4-j3)^*} = \frac{1+j2}{4-j3} \times \frac{4+j3}{4+j3} = \frac{(4+j3)(1+j2)}{(4^2+3^2)} \\ &= \frac{4 \times 1 + 4 \times j2 + j3 \times 1 + j3 \times j2}{16+9} \\ \frac{4+j8+j3-6}{25} &= \frac{(4-6) + j(8+6)}{25} = \frac{-2+j11}{25} \end{aligned}$$

Note 3.21 $(a + jb)(a - jb) = a^2 + b^2$. For instance, with $a = 4$, $b = 3$, the argument becomes:

$$(4 - j3)(4 + j3) = 16 + 9 = 25$$

Summation of Sinusoidal Functions

Functions with similar frequencies can be added using phasor, as follows:

$$f(t) = M \sin(\omega t + \alpha) + N \sin(\omega t + \beta)$$

Note 3.22 Check the frequencies to be the same.

Note 3.23 Check the form of functions to be the same. If they are not the same, both must be converted to sin or cos.

Converting the functions to phasor obtains:

$$f(t) = M \angle \alpha + N \angle \beta$$

The rectangle form of this function is obtained by converting each term individually to a complex number as follows:

$$f(t) = (M \cos \alpha + jM \sin \alpha) + (N \cos \beta + jN \sin \beta)$$

Adding the real parts and imaginary parts separately results in:

$$f(t) = (M \cos \alpha + N \cos \beta) + j(M \sin \alpha + N \sin \beta)$$

Considering

$$p \equiv M \cos \alpha + N \cos \beta$$

$$q \equiv M \sin \alpha + N \sin \beta$$

The function becomes:

$$f(t) = p + jq$$

The rectangle to polar conversion can be accomplished as follows:

In polar form, there is a need for an amplitude and a phase. The amplitude can be obtained by:

$$A = \sqrt{p^2 + q^2}$$

The phase can be obtained by:

$$\varphi = \tan^{-1} \frac{\text{Im}(f(t))}{\text{Re}(f(t))}$$

$$\varphi = \tan^{-1} \frac{q}{p}.$$

Example 3.17 Find the result of $e(t) = 10 \sin(50t + 30) + 20 \cos(50t + 20)$.

Solution Since the frequencies are the same, the values can be added. Otherwise, they have to be added by calculating point-by-point values of each sine function.

However, the functions are sine and cosine. There is a need to convert one to another, i.e., both have to be sine or both have to be cosine. Let us convert both functions to sine. Using the transform, both in sine function can be written as:

$$e(t) = 10 \sin(50t + 30) + 20 \sin(50t + 20 + 90)$$

Phasor presentation results in:

$$e(t) = 10 \angle 30 + 20 \angle 110$$

Adding two polar numbers is better done when both are converted to Cartesian or rectangle. This results in:

$$e(t) = 10 \cos 30 + j10 \sin 30 + 20 \cos 110 + j20 \sin 110$$

$$e(t) = 10 \times 0.866 + j10 \times 0.5 + 20 \times -0.342 + j20 \times 0.939$$

$$e(t) = 8.66 + j5 - 6.84 + j18.78$$

$$e(t) = 1.82 - j13.78$$

Converting to polar results in:

$$e(t) = \sqrt{1.82^2 + (-13.78)^2} \angle \tan^{-1} \frac{-13.78}{1.82}$$

$$e(t) = 13.899 \angle -82.47$$

In time series function, considering the original frequency, the summation becomes:

$$e(t) = 13.899 \sin(50t - 82.47)$$

Example 3.18 Simplify these functions.

$$f(t) = 20 \sin(100t + 10) + 150 \cos(100t - 26) - 120 \cos(100t + 30)$$

$$f(t) = 20 \cos(377t) - 50 \sin(377t - 30) + 20 \cos(150t) - 50 \sin(150t - 30)$$

Damped Sinusoidal Function

A negatively damped exponential function shows a decay trajectory, which if multiplied by a sinusoidal function, results in a decaying sinusoidal waveform. Figure 3.26 shows the function $f_1(t) = e^{-\alpha t}u(t)$ and $f_2(t) = \sin(\omega t)$. The product of these functions is sinusoidal wrapped around a decaying factor as $f_3(t) = e^{-\alpha t} \sin(\omega t)u(t)$.

Note 3.24 Derivative of a function consisting the product of two functions $f_1 f_2$ can be found as:

$$\frac{d}{dt}(f_1 f_2) = \left(\frac{d}{dt}f_1\right)f_2 + f_1\left(\frac{d}{dt}f_2\right)$$

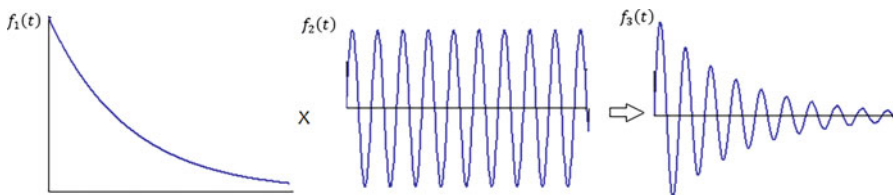


Fig. 3.26 Product of a sinusoidal and exponentially damping function results in a damped sinusoidal

Therefore,

$$\frac{d}{dt}(e^{-\alpha t} \sin(\omega t)u(t)) = \left(\frac{d}{dt}e^{-\alpha t}\right) \sin(\omega t) + e^{-\alpha t} \left(\frac{d}{dt}\sin(\omega t)\right)$$

Considering the definitions done in this chapter yields:

$$(-\alpha e^{-\alpha t}) \sin(\omega t) + e^{-\alpha t} (\omega \cos(\omega t))$$

Example 3.19 Sketch

$$f(t) = 5e^{-t} \sin(10t)u(t).$$

Solution The amplitude of this signal at $t = 0$ is $f(t = 0) = 5e^{-0} \sin(0)$ or $f(0) = 0$. The $u(t)$ function shows that the function starts at time $t \geq 0$. This is a decaying sinusoidal because the power of exponential argument is negative, i.e., negative damping. As time increases, the function value decreases and reaches zero when $t \rightarrow \infty$. In this decaying function, there is a sinusoidal that is bounded by the exponential. The amplitude drops as the exponential function continues to drop, limiting the amplitude of the sinusoidal but not its frequency. The sinusoidal function has a frequency of 10 rad/s. The function oscillates every $\frac{10}{2\pi}$ s. Figure 3.27 shows the time variation of the signal $f(t)$.

Example 3.20 Sketch

$$f(t) = 5e^{-0.5t} \sin(10t + 30)u(t).$$

Solution The function has initial value of $f(t = 0) = 5e^0 \sin(30) = 2.5$. A phase $+30^\circ$ shows a 30° shift of the signal to the left. The term $u(t)$ guarantees that the function appears starting at time positive. Therefore, the negative time is eliminated. If the entire waveform existed for time positive and negative, the first zero crossing

Fig. 3.27 Function of Example 3.16. Note the frequency is 10 rad/s

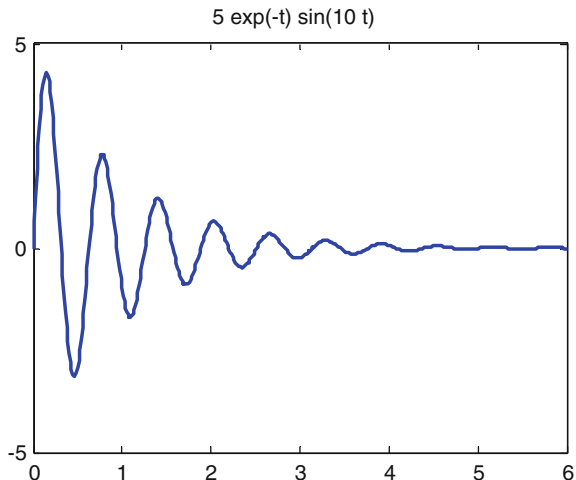
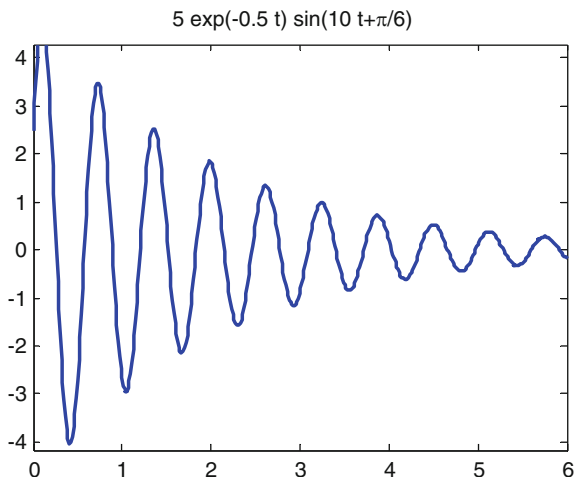


Fig. 3.28 Function of Example 3.18



would have occurred when the argument was zero at $10t + \frac{\pi}{6} = 0$ or $t = -\frac{\pi}{60}$ s. Note that in this calculation, the phase shift angle must be in radians. Figure 3.28 shows the waveform in positive time.

Average of a Signal

A periodic waveform at period T is presented by $f(t) = f(t+T)$. The average or DC value of this periodic signal can be obtained by $1/T$ times of its integral over a period, as follows:

$$f_{dc} = f_{ave} = \frac{1}{T} \int_0^T f(t) dt$$

Example 3.21 Find the DC value of the periodic waveform expressed in Fig. 3.29.

Solution The waveform shows a period of 4 s, because it repeats itself every 4 s. This time can also be measured from first zero crossing of the waveform to the next zero crossing or the time from one peak to the next peak.

According to Fig. 3.29, the function can be split in the time sections and written as $f(t) = \begin{cases} 10 u(t), & 0 < t < 2 \\ 0, & 2 < t < 4 \end{cases}$. Accordingly, the average of the signal can be found as:

$$f_{dc} = f_{ave} = \frac{1}{T} \int_0^T f(t) dt$$

Replacing with the mathematical expression in time intervals results in:

Fig. 3.29 Function of Example 3.21

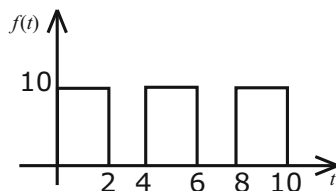
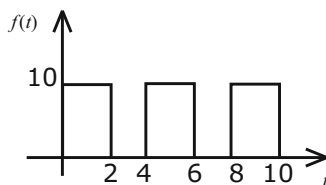


Fig. 3.30 Function of Example 3.22



$$\begin{aligned}
 f_{dc} = f_{ave} &= \frac{1}{4} \int_0^4 f(t) dt = \frac{1}{4} \left(\int_0^2 10u(t) dt + \int_2^4 0 dt \right) \\
 &= \frac{1}{4} \left(10t \Big|_0^2 + 0 \right) = \frac{1}{4} (10(2 - 0)) = 5
 \end{aligned}$$

Root Mean Square (RMS)

The root mean square of a periodic function $f(t) = f(t+T)$ at period T can be obtained as follows:

$$f_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t) dt}$$

Note 3.25 The function must be squared first and then the root of its average must be calculated.

Example 3.22 Find RMS of the function shown in Fig. 3.30.

To calculate the rms, period of the waveform and mathematical expression or the value of function in each time interval must be expressed. In this example, the period is $T = 4$ s, as the signal repeats itself every 4 s. The function has a value of 10 in time $0 < t < 2$ and is zero in time $2 < t < 4$, and this sequence is repeated. The rms can be obtained as follows:

$$f_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t) dt} = \sqrt{\frac{1}{4} \left(\int_0^2 (10)^2 dt + \int_2^4 0 dt \right)}$$

$$f_{\text{rms}} = \sqrt{\frac{1}{4} \left(100t \Big|_0^2 + 0 \right)} = \sqrt{\frac{1}{4} (100(2 - 0) + 0)} = \sqrt{50} = 7.07$$

Sources

Sources are generally divided into voltage source and current source. Each source can generate a voltage waveform across the terminals of the circuit, or it may force a specific current through the circuit. A voltage source tends to keep the voltage constant no matter how much current is fed to the terminals of the circuit. A current source tends to keep the current constant no matter how much voltage is required across the source. The value of the voltage in a voltage source or the value of current in a current source can be either a fixed number, or it may depend on other parameters of the circuit. For instance, a voltage source might be 12 V fixed, or it may depend on α times of a current that passes through a resistor, somewhere in the circuit, as αI .

The sources with fixed value are called independent sources, and the sources that their values depend on other parameters of the circuit are called dependent sources.

Independent Voltage Source

A voltage source generates a voltage value regardless of the current drawn from the source. A voltage source keeps the voltage at the terminal by allowing the current vary. $V(t)$ can have any of the waveforms discussed in this chapter. In reality, all sources have internal resistance that causes a voltage drop as the current passes through. The voltage drop from the desired value is called voltage regulation. A schematic of an ideal independent voltage source is shown in Fig. 3.31.

A unit step voltage source keeps the voltage at a steady value and ideally can provide infinite current without drop of voltage. The voltage measured at the terminal of the voltage source (or its V – I characteristics) is shown in Fig. 3.32. As the figure demonstrates, the voltage amplitude is fixed at any given current. A sinusoidal waveform is shown in Fig. 3.33, and a bipolar step function is shown in Fig. 3.34.

Independent Current Source

An independent current source generates a constant current value regardless of the voltage across the source. This source changes the voltage at the terminal to keep the current constant. $I(t)$ can force a desired amount of current through a circuit even if

Fig. 3.31 Schematic of a voltage source with positive and negative terminals and amplitude V

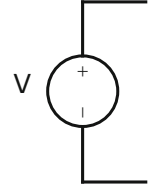


Fig. 3.32 An independent voltage source and its $V - I$ characteristics. The voltage amplitude is constant at any drawn current

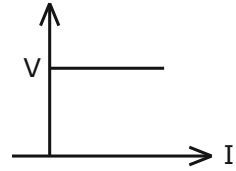


Fig. 3.33 A voltage source generating sinusoidal waveform

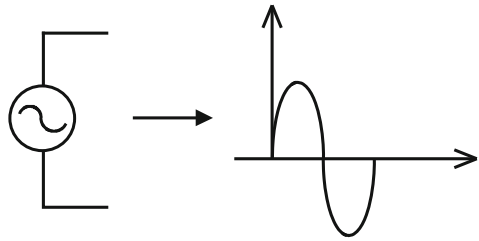
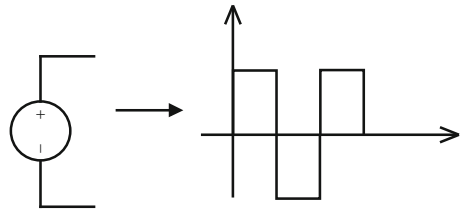


Fig. 3.34 A step function generator with bipolar amplitude. It means the voltage becomes positive and negative



the voltage virtually reaches infinite. A schematic of an independent current source is shown in Fig. 3.35. The $V - I$ characteristics are shown in Fig. 3.36. As the figure shows, the amount of current is fixed at any voltage. In reality, the amount of current will drop because of the internal resistances of the source.

Note 3.26 A voltage source keeps the voltage of the nodes constant by sourcing current to the branches connected to the terminals.

Note 3.27 A current source keeps the current of a branch constant by varying the voltage at terminals.

Note 3.28 A zero-volt (0 V) voltage source resembles a short circuit. Figure 3.37 shows an equivalent of a zero-volt voltage source.

Fig. 3.35 Schematic of an ideal independent current source

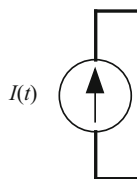


Fig. 3.36 $V - I$ characteristics of a current source. The current source can deliver the current at an arbitrary voltage level

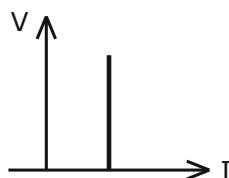


Fig. 3.37 Equivalent of a zero-volt voltage source

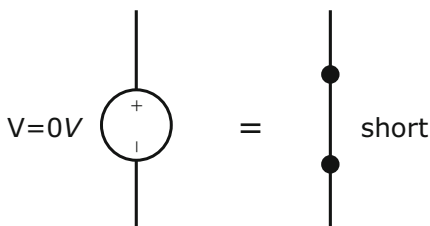
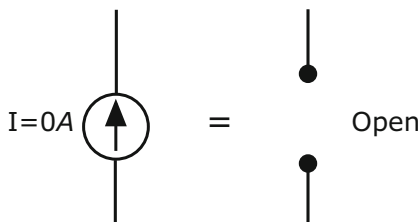


Fig. 3.38 Equivalent of a zero-amp current source



Note 3.29 A zero-amp (0 A) current source resembles an open circuit. Figure 3.38 shows an equivalent of a zero-amp current source.

Example 3.23 Find the currents I , I_1 , I_2 of this circuit, when $v(t) = 10 u(t)$ (Fig. 3.39).

Solution The voltage source generates a pulse at amplitude V . The voltage source is connected in parallel to two resistors. Each of these resistors takes a current proportional to its resistance. The current through R_1 is calculated to be $i_1(t) = \frac{V(t)}{R_1}$ or $i_1(t) = \frac{10u(t)}{R_1}$, and the current of R_2 is $i_2(t) = \frac{V(t)}{R_2}$ or $i_2(t) = \frac{10u(t)}{R_2}$. The total current taken from the source is the summation of $i_1(t)$ and $i_2(t)$. Therefore, $i(t) = i_1(t) + i_2(t) = \frac{V(t)}{R_{eq}}$ and $R_{eq} = R_1 \parallel R_2$. This reads R_1 in parallel to R_2 . Parallel of two resistors can be calculated as:

Fig. 3.39 Circuit of Example 3.22

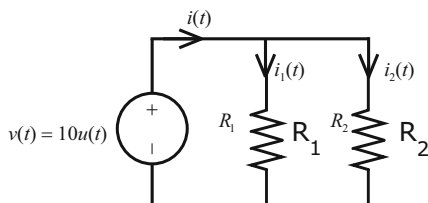
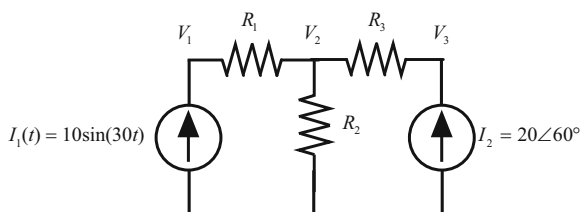


Fig. 3.40 Circuit of Example 3.24



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$I = \frac{V(t)}{\frac{R_1 R_2}{R_1 + R_2}} = V(t) \frac{R_1 + R_2}{R_1 R_2} = 10 \frac{R_1 + R_2}{R_1 R_2} u(t)$$

which results in total current of

Example 3.24 Find the voltages of V_1, V_2, V_3 , when $I_1(t) = 10 \sin 30t(\text{A})$ and $I_2(t) = 20 \angle 60^\circ (\text{A})$ (Fig. 3.40).

Solution The circuit has two current sources of $I_1(t) = 10 \sin 30t(\text{A})$ and $I_2(t) = 20 \angle 60^\circ (\text{A})$, which operate at the same frequency. The circuit has three nodes at unknown voltages of V_1, V_2, V_3 . The voltages can be obtained using KCL written for each node.

KCL ①. Summation of entering current (negative sign) and exiting currents (positive sign) must be zero.

$$-I_1 + \frac{V_1 - V_2}{R_1} = 0$$

KCL ②.

$$\frac{V_2 - V_1}{R_1} + \frac{V_2 - 0}{R_2} + \frac{V_2 - V_3}{R_3} = 0$$

KCL ③.

$$-I_2 + \frac{V_3 - V_2}{R_3} = 0.$$

This results in three equations and three unknowns. Note that the current source values are replaced by phasors.

Simplifying the equations results in:

$$\begin{cases} \frac{V_1}{R_1} - \frac{V_2}{R_1} = I_1 \\ \frac{-V_1}{R_1} + V_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) + \frac{-V_3}{R_3} = 0 \\ \frac{-V_2}{R_3} + \frac{V_3}{R_3} = I_2 \end{cases}$$

Adding all three equations results in:

$$\begin{aligned} \frac{V_2}{R_2} &= I_1 + I_2 \\ V_2 &= R_2(I_1 + I_2) \end{aligned}$$

Replacing in the second equation and solving for V_3 result in:

$$V_3 = R_3 I_2 + R_2(I_1 + I_2) = R_2 I_1 + (R_2 + R_3) I_2,$$

and using the first equation with the value of V_2 , V_1 becomes:

$$\begin{aligned} V_1 &= V_2 + R_1 I_1 \\ V_1 &= (R_1 + R_2) I_1 + R_2 I_2. \end{aligned}$$

Note 3.30 A mix of voltage and current source might exist in a circuit. It should be noted that a voltage source keeps the voltage of the terminal nodes constant and a current source keeps the current of a circuit branch constant.

Example 3.25 Find the voltage V_2 and the current I_1 , when the sources have phasor values of V and I (Fig. 3.41).

Solution As the circuit demonstrates, the voltage source keeps the voltage of node ① at constant V volts. Considering the voltage of node ② at V_2 , the current I_1 can be found.

KCL ②:

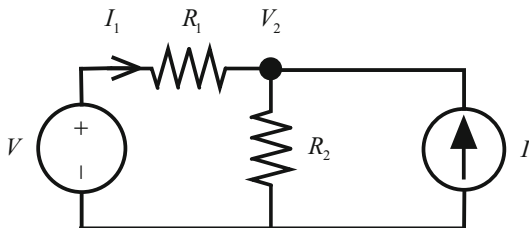
$$I_{R_1} + I_{R_2} - I = 0.$$

Note 3.31 The current of passive elements should always drain the node, hence positive. The current of the current source if forced to the node (in this circuit) is hence negative.

Therefore,

$$\frac{V_2 - V}{R_1} + \frac{V_2}{R_2} - I = 0$$

Fig. 3.41 Circuit of Example 3.25



Solving for V_2 results in:

$$V_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{V}{R_1} + I$$

$$V_2 = \left(\frac{R_1 R_2}{R_1 + R_2} \right) \left(\frac{V}{R_1} + I \right)$$

Knowing the value of V_2 , the current I_1 can be obtained as:

$$I_1 = \frac{V - V_2}{R_1} = \frac{V}{R_1} - \frac{V_2}{R_1}$$

Replacing from V_2 results in:

$$I_1 = \frac{V}{R_1} - \frac{1}{R_1} \left(\frac{R_1 R_2}{R_1 + R_2} \right) \left(\frac{V}{R_1} + I \right)$$

$$I_1 = \frac{1}{R_1} V - \left(\frac{R_2}{R_1 + R_2} \right) I$$

Dependent Sources

The value of dependent sources may change based on their type of dependency to other parameters of the circuit. These dependencies may be to voltage of nodes or current of branches in the circuit. Therefore, a dependent voltage source may be controlled by voltage of a node or current of a branch. A dependent current source may be controlled by voltage a node or current of a branch (Fig. 3.42).

The fact that the source output voltage or current is dependent on circuit parameters makes no difference in their operation such as KVL and KCL, as they are still voltage or current sources, but their values may depend on other circuit values. A dependent voltage source still keeps the voltage of the terminal nodes constant to the value dictated by the control parameter, and a dependent current source still keeps the current of the branch constant by changing the voltage.

Note 3.32 The only consideration is to correctly account for the dependency of the voltage or current generated by the circuit parameters.

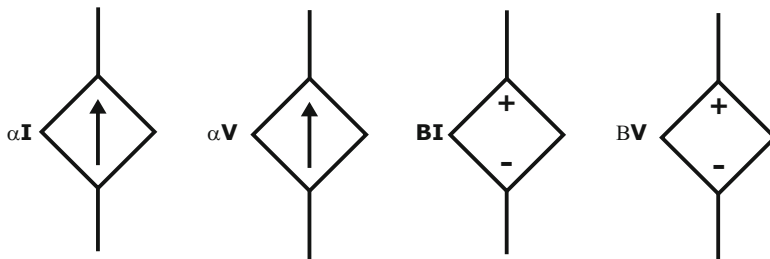
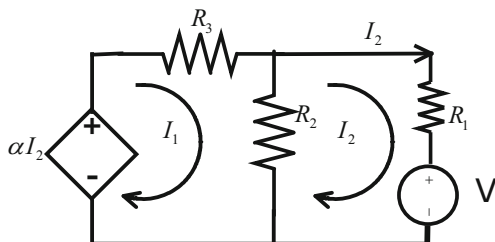


Fig. 3.42 Dependent current and voltage sources. The dependency of the voltage or current generated from each source can be on a current or a voltage measured at any part of the circuit

Fig. 3.43 Circuit of Example 3.26



Note 3.33 Changes in the circuit or equivalent analysis of elements should be such that the dependency of voltage or current is not eliminated from the simplified circuit, i.e., keeping track of the circuit parameters.

Note 3.34 A dependent voltage source or a dependent current source cannot be turned off unless the control parameter forces the source to be off.

Example 3.26 In circuit of Fig. 3.43, find I_1 .

Solution The circuit has a current controlled voltage source. The control current is the current through resistor R_1 . In the circuit simplification, this current, I_2 , must be tracked to be able to calculate the value of the voltage source. Therefore, although R_2 and R_3 are in parallel, it is not recommended to simplify these two elements.

The circuit has two loops I and II. KVL must be written for each of these loops (since there is no current source connected to any nodes around the loops).

KVL ①.

$$-\alpha I_2 + V_{R_3} + V_{R_2} = 0$$

Note 3.35 The value of dependent voltage source becomes negative because following the suggested direction of current in loop ①, the current enters the negative terminal of the voltage source.

Voltage drop across R_3 , $V_{R_3} = I_1 R_3$, and across R_2 , $V_{R_2} = (I_1 - I_2) R_2$. The currents I_1 and I_2 are in opposite direction over the resistor R_2 , and following the KVL in loop ①, the positive voltage drop suggests that the value of current is $I_1 - I_2$.

Therefore, loop ① KVL results in:

$$-\alpha I_2 + I_1 R_3 + (I_1 - I_2) R_2 = 0 \quad (3.2)$$

KVL ②.

$$V_{R_1} + V + V_{R_2} = 0$$

$V_{R_1} = I_2 R_1$, and $V_{R_2} = (I_2 - I_1) R_2$. In loop ②, the KVL is following the current I_2 and hence the positive voltage drop across the same resistor suggests $I_2 - I_1$.

Therefore, loop ② KVL results in:

$$I_2 R_1 + (I_2 - I_1) R_2 = -V \quad (3.3)$$

Simplifying (3.2) and (3.3) and solving for I_1 and I_2 result in:

$$\begin{cases} -\alpha I_2 + I_1 R_3 + (I_1 - I_2) R_2 = 0 \\ I_2 R_1 + (I_2 - I_1) R_2 = -V \end{cases}$$

Solving for I_1 and I_2 results in:

$$\begin{cases} (R_3 + R_2) I_1 - (\alpha + R_2) I_2 = 0 \\ -R_2 I_1 + (R_1 + R_2) I_2 = -V \end{cases}$$

Solving for I_1 from the first equation, $I_1 = \frac{(\alpha + R_2)}{(R_3 + R_2)} I_2$, and replacing in the second equation yield:

$$I_2 = \frac{V}{R_1 + R_2 - R_2 \frac{(\alpha + R_2)}{(R_3 + R_2)}}$$

α is a control parameter that is set to obtain specific characteristics from the circuit. In fact, α is mostly determined by transistors used in the circuit. MOSFETs and BJTs have different dependencies and generate dependable sources.

Example 3.27 Find voltage V_T and current I_1 in the circuit of Fig. 3.44.

Circuit Simplification Techniques

Voltage Division

Consider a circuit with a series of resistors connected to a voltage source. The voltage drop across each resistor is proportional to its resistance. Since the current passing through all elements is similar, the total current can be obtained as:

$$i = \frac{v}{\sum R}.$$

Therefore, the voltage drop of a resistor R_k is $v_k = \frac{R_k}{\sum R} v$ (Fig. 3.45).

Example 3.28 Find the voltage drop across all elements of the circuit shown in Fig. 3.46.

Solution The circuit has two sets of resistors in series. A $10\ \Omega$ and $2\parallel 8 = 1.6\ \Omega$ connected in series to a $100\ \text{V}$ voltage source. The voltage across 10 and $1.6\ \Omega$ resistors is:

$$V_{10\ \Omega} = \frac{10}{10 + 1.6} 100 = 86.21\ \text{V}$$

Fig. 3.44 Circuit of Example 3.27

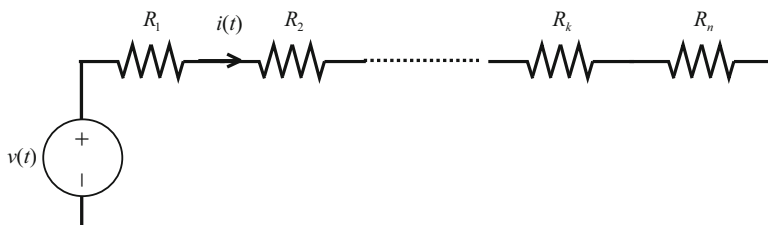
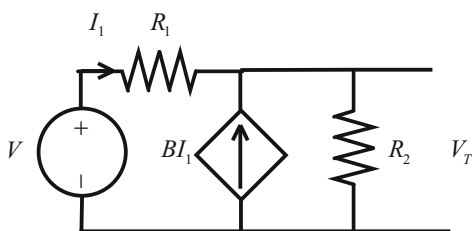


Fig. 3.45 Series connection of resistors and voltage division

Fig. 3.46 Circuit of Example 3.27

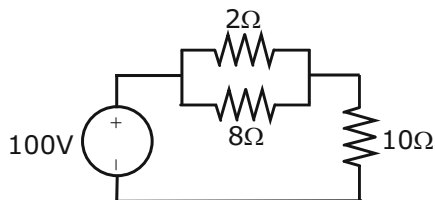


Fig. 3.47 Parallel connection of resistors and current division

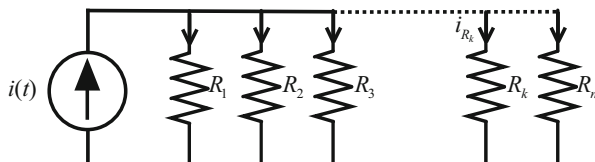
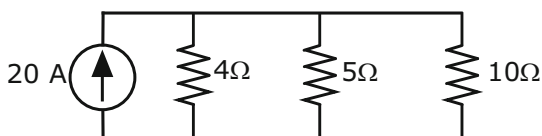


Fig. 3.48 Circuit of Example 3.29



$$V_{1.6\Omega} = \frac{1.6}{1.6 + 10} 100 = 13.79 \text{ V}$$

Current Division

The current shared among several elements in parallel depends on the conductance of each parallel branch (Fig. 3.47).

Consider a circuit with n resistors in parallel fed from a current source $i(t)$. The current drawn from the source is the summation of all branch currents, such that:

$$i = \sum_{m=1}^n i_{R_m}$$

$$i = \frac{v}{R_1} + \frac{v}{R_2} + \dots + \frac{v}{R_n} = v \left(\frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n} \right)$$

The voltage across the circuit is obtained from:

$$v = \frac{i}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n} \right)}$$

The current passing through any resistor is $i_{R_k} = \frac{v}{R_k} = \frac{\frac{1}{R_k}}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n} \right)} i$.

Example 3.29 Find the current of each branch in the circuit of Fig. 3.48.

Solution

The current of $4\ \Omega$ is $I_{R_4\ \Omega} = \frac{\frac{1}{4}}{(\frac{1}{4} + \frac{1}{5} + \frac{1}{10})} 20 = 9.1\ \text{A}$.

The current of $5\ \Omega$ is $I_{R_5\ \Omega} = \frac{\frac{1}{5}}{(\frac{1}{4} + \frac{1}{5} + \frac{1}{10})} 20 = 7.27\ \text{A}$.

The current of $10\ \Omega$ is $I_{R_{10\ \Omega}} = \frac{\frac{1}{10}}{(\frac{1}{4} + \frac{1}{5} + \frac{1}{10})} 20 = 3.63\ \text{A}$.

Source Conversion

A voltage source in series to a resistor can be converted to a current source and the same resistance in parallel. The size of the current source is obtained by the value of voltage source divided by the resistance (Fig. 3.49).

Note 3.36 The source conversion and Thevenin to Norton equivalent are also true for dependent sources.

Example 3.30 Convert the Thevenin to Norton and vice versa in circuits of Figs. 3.50, 3.51, and 3.52.

Thevenin Equivalent Circuit

Any circuit from a desired set of terminals can be modeled as a voltage source and a series resistance. This Thevenin equivalent circuit has two components V_{th} (an independent voltage source with fixed value) and R_{th} (or an impedance in case of RLC circuit) as shown in Fig. 3.53.

Thevenin equivalent circuits are defined for all circuits including resistive and circuits with inductors and capacitors and dependent sources.

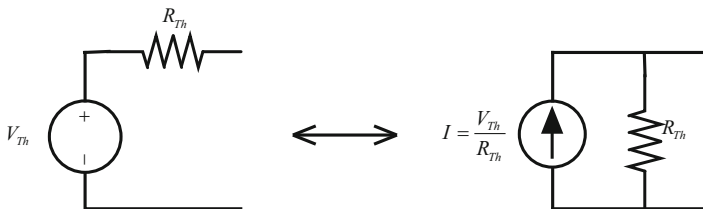


Fig. 3.49 Thevenin and Norton equivalent conversions

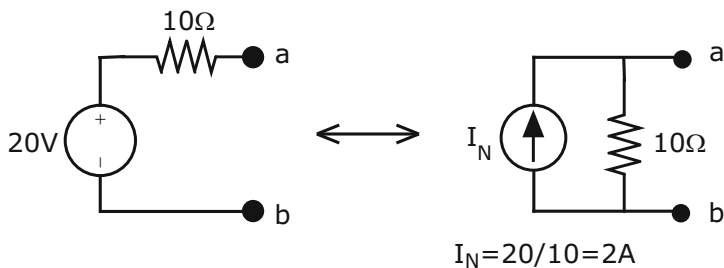


Fig. 3.50 Circuit of Example 3.30

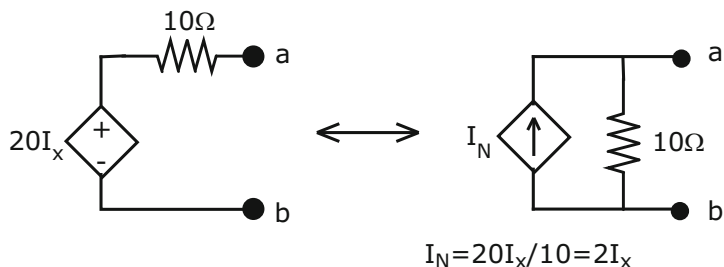


Fig. 3.51 Another circuit of Example 3.30

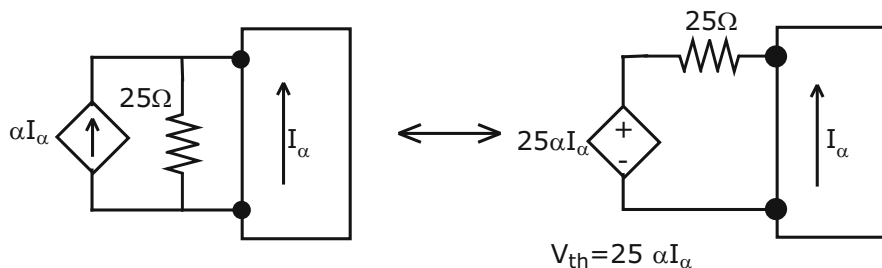


Fig. 3.52 Circuit of Example 3.30. I_α is the current of an element in the circuit. The amount of current generated by the current source depends on I_α

Note 3.37 To obtain the V_{th} across terminal ports a and b , the circuit must be disconnected from the load at these terminals. The voltage difference built at the terminals is measured as V_{th} .

Note 3.38 To obtain the R_{th} at terminal ports a and b , only independent sources must be turned off. This means a zero voltage source (short circuit) and a zero current source (open circuit).

Fig. 3.53 Thevenin equivalent from points a and b

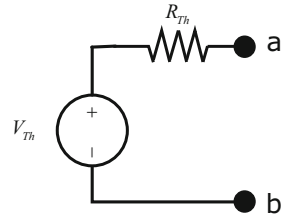


Fig. 3.54 Figure Thevenin of Example 3.31

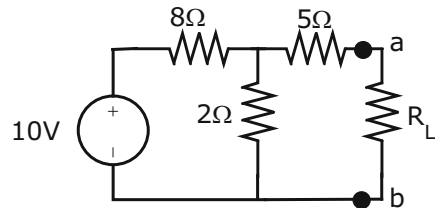
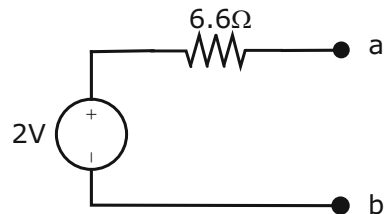


Fig. 3.55 Thevenin equivalent of circuit shown in Fig. 3.54



Example 3.31 Find the Thevenin equivalent of the circuit in Fig. 3.54.

Solution Finding V_{th} requires that R_L be disconnected from the circuit. If so, the 5Ω resistor branch is disconnected and does not pass any current. Therefore, the voltage drop across this resistor becomes zero and the terminal voltage becomes equivalent of the voltage drop across the 2Ω resistor.

The KVL for loop suggests that, $-10 + 8I_1 + 2I_1 = 0$. Therefore, $I_1 = \frac{10}{10} = 1$ A. The voltage of 2Ω resistor becomes $V_{th} = V_{2\Omega} = 2 \times 1 = 2$ V.

Finding R_{th} needs a 0V voltage source (short circuit) and zero A current source (open source).

Removing the voltage source leaves the 8Ω and 2Ω in parallel and the result in series with 5Ω . $R_{th} = (8\parallel 2) + 5 = \frac{8 \times 2}{8 + 2} + 5 = 6.6\Omega$.

The equivalent Thevenin circuit is shown as follows (Fig. 3.55):

Example 3.32 Find the Thevenin equivalent of the circuit shown in Fig. 3.56.

Solution Since the load must be disconnected from the terminals, there is no current passing the $5\text{ k}\Omega$ resistor. Hence, the voltage drop across this element is zero. Therefore, $V_{th} = V_x = V_1$ which is the voltage of node 1. Writing a KCL for node 1 determines the voltage.

Fig. 3.56 Equivalent circuit of Example 3.32

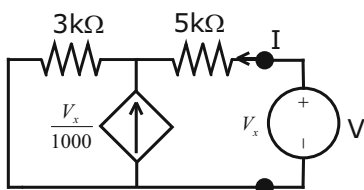
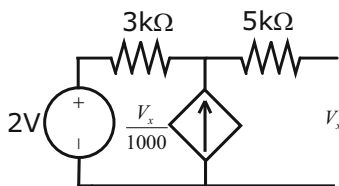


Fig. 3.57 All independent sources have been zeroed out, and since there is a dependent source remaining in the circuit, an external source with voltage V needs to be connected to terminal points to excite the circuit for impedance measurement

KCL ①. Passive element $1\text{ k}\Omega$ resistor drains the current out of the node, and the dependent current source forces the current into the node (direction of the current sources). Therefore,

$$\frac{V_1 - 2}{3000} - \frac{V_x}{1000} = 0$$

Considering $V_x = V_1$ results in:

$$\frac{V_x - 2}{3000} - \frac{V_x}{1000} = 0$$

$$\frac{-2V_x}{3000} - \frac{2}{1000} = 0$$

$$\frac{-2V_x}{3000} = \frac{2}{1000}$$

$$V_x = V_{th} = -3\text{ V}.$$

To obtain R_{th} , the independent voltage source must be turned off, and the dependent current source must remain in the circuit. The circuit is shown in Fig. 3.57.

As the circuit shows, the resistance measured at terminals depends on the value of the dependent current source. To excite this dependent source, an external voltage source V is connected to the terminals. The current drawn from the source I is measured, and then the resistance shown at the terminal can be obtained from $R_{th} = \frac{V}{I}$.

$$\text{KCL } \textcircled{1} . \frac{V_1}{3000} - \frac{V_x}{1000} + \frac{V_1 - V_x}{5000} = 0$$

From the circuit $V_x = V$, replacing in KCL results in $\frac{V_1}{3000} - \frac{V}{1000} + \frac{V_1 - V}{5000} = 0$.

$$V_1 \left(\frac{1}{3000} + \frac{1}{5000} \right) = V \left(\frac{1}{1000} + \frac{1}{5000} \right)$$

$$V_1 \left(\frac{1}{3000} + \frac{1}{5000} \right) = V \left(\frac{1}{1000} + \frac{1}{5000} \right)$$

$$V_1 \left(\frac{8}{15,000} \right) = V \left(\frac{6}{5000} \right), \text{ or } V_1 = \frac{9}{4} V$$

$$I = \frac{V - V_1}{5000} = \frac{V - \frac{9}{4}V}{5000} = \frac{-5}{4} V \frac{1}{5000}$$

$$V = -4000I \rightarrow R_{th} = -4000\Omega.$$

Example 3.33 Find the Thevenin of the circuit shown in Fig. 3.58.

To obtain the Thevenin impedance, an external source must be used. Connecting voltage source V at the terminals creates a two-loop system.

$$\text{KVL } \textcircled{1} . -7i + 3i + 5(i_1 - i) = 0$$

$$\text{KVL } \textcircled{2} . V + 5(i - i_1) = 0$$

From the first KVL, $i_1 = \frac{12}{8}i$. Replacing in the second equation, $V = \frac{20}{8}i$.

Therefore, the Thevenin equivalent circuit is a simple resistor with resistance of $\frac{20}{8} \Omega$.

Norton Equivalent Circuit

Any circuit from a set of desired terminal can be represented by a current source in parallel to a resistor. The current source shows the short circuit current that might have passed the terminals if it was shorted, hence called the short circuit current

Fig. 3.58 Circuit of Example 3.33

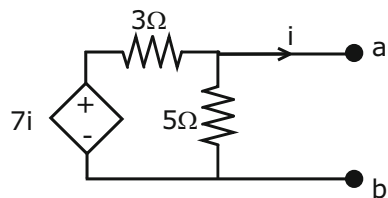


Fig. 3.59 Norton equivalent

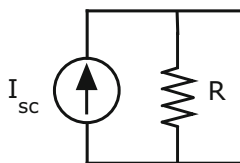
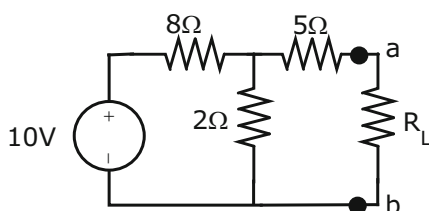


Fig. 3.60 Circuit of Example 3.34



source, and the parallel resistance shows the equivalent resistance when all the independent sources are turned off (Fig. 3.59).

Note 3.39 In Norton equivalent circuit, the load resistance across the terminals must become a short circuit. The current passing this short circuit is the equivalent of the Norton current source.

Note 3.40 The Norton resistance is obtained similar to the Thevenin resistance.

Example 3.34 Find the Norton equivalent of the following circuit (Fig. 3.60).

Solution A short circuit of the load leaves the R_L out of the circuit. Hence, the I_{sc} has similar value as of the current passing through the 5Ω resistor.

Total current of the circuit I can be calculated by $I = \frac{10}{R_{eq}}$. The equivalent resistance is parallel of $5\parallel 2$ in series with the 8Ω . $R_{eq} = 8 + (5\parallel 2) = 9.428\Omega$ which makes the total current $I = 1.06\text{ A}$.

The current of 5Ω resistor is obtained from the current division as:

$$I_{sc} = I_{5\Omega} = \frac{2}{2+5} 1.06 = 0.303\text{ A}.$$

Finding R_{th} needs a zero V voltage source (short circuit) and zero A current source (open source). The circuit of Fig. 3.61 is obtained. Removing the voltage source leaves the 8 and 2Ω in parallel and the result in series with 5Ω . $R_{th} = (8\parallel 2) + 5 = \frac{8 \times 2}{8+2} + 5 = 6.6\Omega$.

Norton and Thevenin Equivalent

Thevenin and Norton circuits can be converted to each other as follows (Fig. 3.62).

Fig. 3.61 Norton equivalent of the circuit
Fig. 3.60

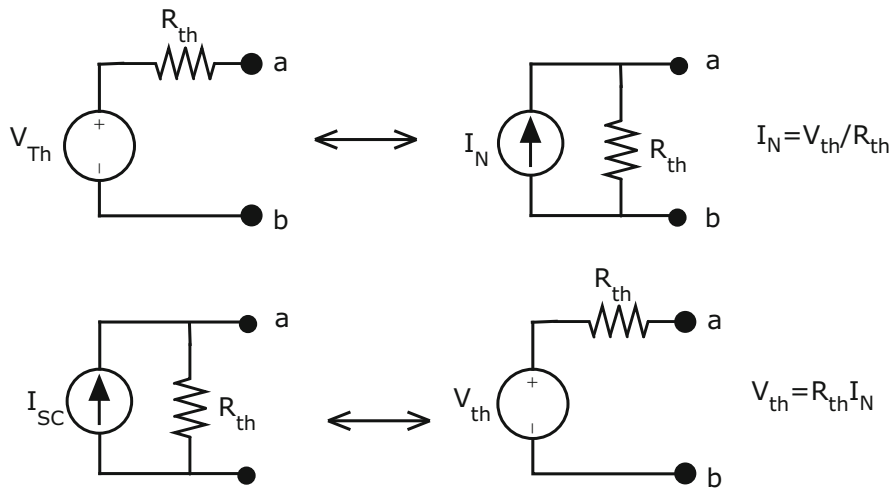
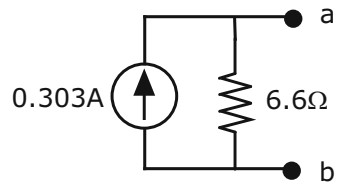


Fig. 3.62 Thevenin and Norton transformation

Power Calculations

Consumption of Power

Consider a resistor R as part of a circuit. DC current I passing through this resistor will generate a DC voltage drop V across the resistor. In this condition, the resistor starts to consume power and dissipate heat. The amount of power loss is directly proportional to the amount of voltage and current and is obtained by:

$$P_{\text{Loss}} = VI$$

The dissipated power is measured in watts. Considering the Ohm's law in resistors, $V = RI$, the power dissipation can be obtained as:

$$P_{\text{Loss}} = VI = (RI)I = RI^2 = \frac{V^2}{R}$$

The resistor is a passive element, which means the amount of energy stored in it over a cycle is zero. Resistors cannot store electric energy. However, they can be utilized to store thermal energy in a special type of ceramics.

Note 3.41 Power consumption has a positive sign, e.g., a +100 W load consumes power equivalent to 100 W.

Generation of Power

A DC power source feeds the circuit with a DC voltage and DC current. The product of total current drawn from the source by the voltage of the source determines the amount of power that the source has generated and fed to the circuit. Since the current is outgoing of the terminals of the source, the power is generated and is considered a negative value.

Note 3.42 Power generation has a negative sign, e.g., a −100 W sources generates and feeds the circuit by 100 W.

Example 3.35 Consider a 100 W power load connected to a power source. The load consumes +100 W, and the source generates −100 W. If the voltage is 20 V, the load has 5 A current entering to the terminal, and the source has 5 A current exiting the terminal.

Example 3.36 A battery unit is utilized to drive an electric vehicle, shown in Fig. 3.63. The battery unit can both generate and absorb power. The battery is discharged to propel the vehicle forward on an uphill road. It is charged through regenerative braking when the vehicle's energy is harvested to be stored on a downhill road.

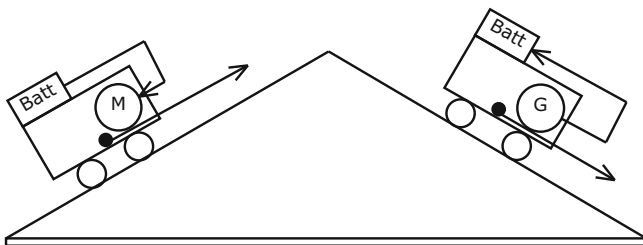


Fig. 3.63 An electric vehicle discharges the battery in the uphill and charges it in the downhill road

Fig. 3.64 A Thevenin equivalent circuit delivers power to a resistive load. The power delivery is maximum when the Thevenin resistance and the load resistance are equal

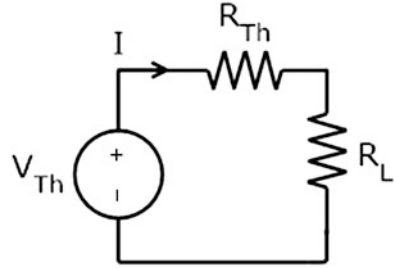
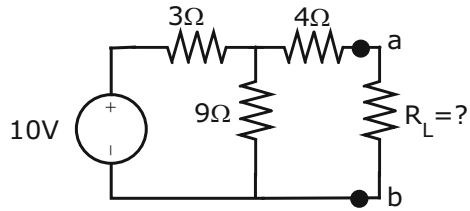


Fig. 3.65 Circuit of Example 3.37



Maximum Power Transfer to Load in Pure Resistive Circuits

Consider a Thevenin equivalent of a resistive circuit connected to a load resistance as shown in Fig. 3.64. A voltage source and a Thevenin resistance in series forces the current I through the circuit.

The load current is obtained from $I_L = \frac{V_{th}}{R_{th} + R_L}$. Therefore, the power delivery to the load is:

$$P_L = R_L I_L^2 = R_L \left(\frac{V_{th}}{R_{th} + R_L} \right)^2$$

To maximize the power delivery to the load, $\frac{dP_L}{dR_L} = 0$.

$$\frac{dP_L}{dR_L} = \frac{V_{th}^2 (R_{th} + R_L)^2 - 2(R_{th} + R_L)R_L V_{th}^2}{(R_{th} + R_L)^4} = \frac{V_{th}^2 (R_{th} + R_L) - 2R_L V_{th}^2}{(R_{th} + R_L)^3}$$

$$\frac{dP_L}{dR_L} = \frac{V_{th}^2 R_{th} - V_{th}^2 R_L}{(R_{th} + R_L)^3} = 0.$$

Hence, the condition to transfer maximum power from the source to the load is:

$$R_{th} = R_L.$$

Example 3.37 Find the load resistance that can absorb maximum power from the circuit of Fig. 3.65.

Solution Thevenin equivalent of the circuit with respect to load terminals can be found by disconnecting the load and turning the independent sources off. This results in:

$$R_L = R_{th} = 4 + (3 \parallel 9) = \frac{25}{4} \Omega$$

Problems

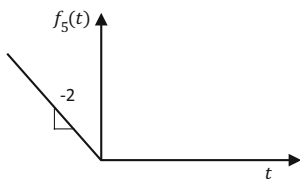
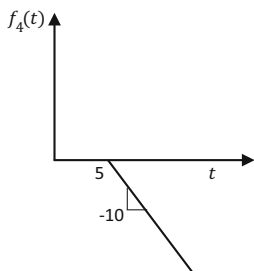
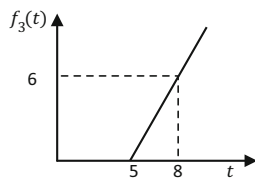
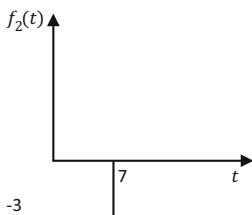
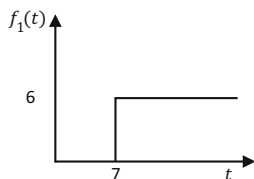
3.1. Sketch the following functions:

- (a) $f(t) = 10u(t)$
- (b) $f(t) = 10u(t + 7)$
- (c) $f(t) = 10u(t - 7)$
- (d) $f(t) = 10u(3t - 15)$
- (e) $f(t) = -10u(t)$
- (f) $f(t) = 10u(-t)$

3.2. Sketch the following functions:

- (a) $f(t) = 2tu(t)$
- (b) $f(t) = 2(t - 3)u(t - 3)$
- (c) $f(t) = 2(t - 3)u(t - 7)$
- (d) $f(t) = -2tu(t)$
- (e) $f(t) = 2tu(-t)$

3.3. Find the mathematical expression of the following signals:



3.4. Sketch the following functions:

(a) $f(t) = u(t - 1) - u(t - 2)$

(b) $f(t) = u(t - 1) + u(t - 2) - u(t - 3)$

(c) $f(t) = 3u(t - 1) + 2u(t - 2) - 5u(t - 4)$

(d) $f(t) = tu(t) - (t - 2)u(t - 2)$

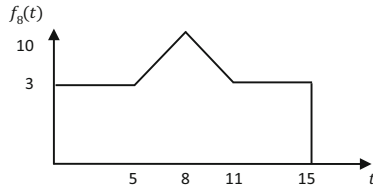
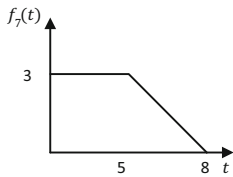
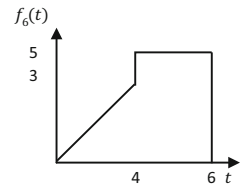
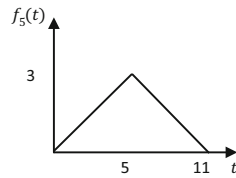
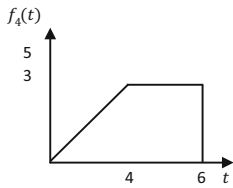
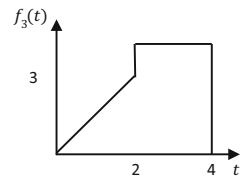
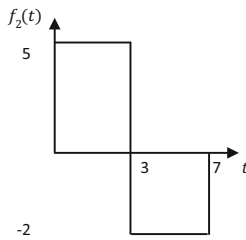
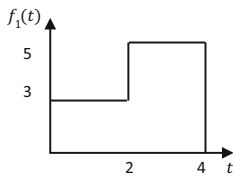
(e) $f(t) = tu(t) - (t - 3)u(t - 3) - (t - 4)u(t - 4) + (t - 5)u(t - 5)$

(f) $f(t) = 3(t - 2)u(t - 2) - 6(t - 4)u(t - 4) + 3(t - 6)u(t - 6)$

(g) $f(t) = 3tu(t - 5)$

(h) $f(t) = 5(t - 1)u(t - 3) + 3(t - 2)u(t - 4)$

3.5. Find mathematical expression of the following signals:



3.6. Sketch the following functions and determine the amplitude and their damping factor.

- (a) $f(t) = 3e^{-t}u(t)$
- (b) $f(t) = 2e^{+5t}u(t)$
- (c) $f(t) = 2e^{-3(t-1)}u(t-1)$
- (d) $f(t) = e^{-t}u(t) - e^{-(t-5)}u(t-5)$
- (e) $f(t) = 3e^{-(t-2)}u(t-1)$

3.7. In the following signals, determine the amplitude, phase shift, and frequency in (rad/s) and in Hz.

- (a) $f(t) = A \sin(\omega t + \phi)$
- (b) $f(t) = 200 \sin(377t + 10)$
- (c) $f(t) = 110\sqrt{2} \sin(120\pi t + 60)$
- (d) $f(t) = \sin(t)$
- (e) $f(t) = 20 \sin(100t + \frac{\pi}{6})$

3.8. Do the following math operations.

- (a) $\int u(t)dt$
- (b) $\int 10u(t-1)dt$
- (c) $\int 10u(t) - 10u(t-2)dt$
- (d) $\int tu(t)dt$
- (e) $\int 3(t-1)u(t-1)dt$
- (f) $\int u(-t)dt$
- (g) $\int -tu(-t)dt$
- (h) $\int 10 \sin(377t + 20)dt$
- (i) $\int 110\sqrt{2} \cos(100\pi t + \frac{\pi}{6})dt$
- (j) $\int e^{-5t}dt$
- (k) $\int \frac{1}{5}e^{-3t} + \frac{3}{4}e^{2t}dt$
- (l) $\int \delta(t)dt$
- (m) $\int (t^2+5) \delta(t)dt$
- (n) $\int (t^2+5) \delta(t-2)dt$

3.9. Do the following math operations:

- (a) $\frac{d}{dt}(u(t))$
- (b) $\frac{d}{dt}(u(t-1))$
- (c) $\frac{d}{dt}(2u(t) + 3u(t-1))$
- (d) $\frac{d}{dt}(3(t-1)u(t-1) + 5(t-3)u(t-3))$
- (e) $\frac{d}{dt}(e^{-3t}u(t))$
- (f) $\frac{d}{dt}(e^{-3t}u(t-2))$
- (g) $\frac{d}{dt}(e^{-5t} + e^{5t}u(t))$

(h) $\frac{d}{dt}(100 \sin 10\pi t)$

(i) $\frac{d}{dt}(200 \cos 1000t + 60)$

3.10. Do the following math operations:

(a) $2 \sin(10t + 60) + 3 \cos(10t + 30)$

(b) $\sin(100t + 10) - \sin(100t - 80)$

(c) $\cos(20\pi t) + \cos(20\pi t + 10) - \sin(20\pi t + 30)$

3.11. Find the amplitude, damping factor, frequency of oscillation, and initial phase in the following signals:

(a) $f(t) = 10 + \sin t$

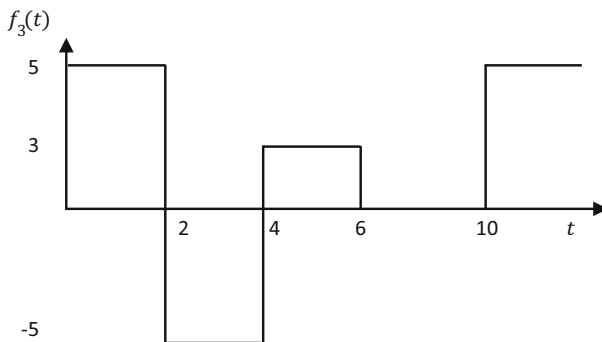
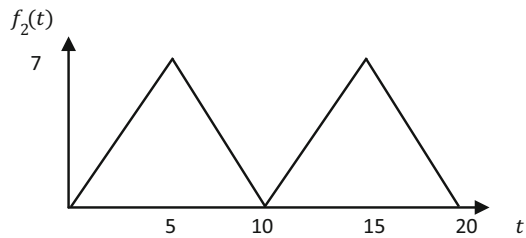
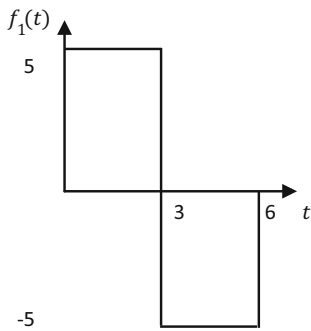
(b) $f(t) = 5 + 100\sqrt{2} \sin 377t$

(c) $f(t) = \sin^2 5t$

(d) $f(t) = \sin^2 5t - \cos^2 5t$

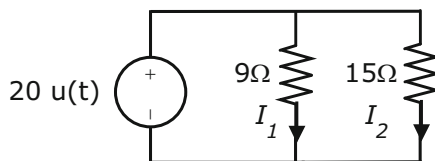
(e) $f(t) = 1 + \sin^2 10t$

3.12. Find the average and rms of the following waveforms:

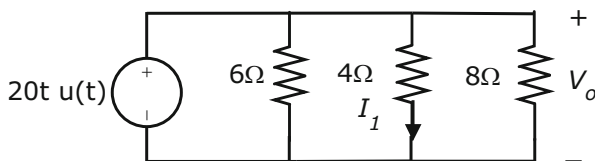


- (a) $f_4(t) = 110 \sin 377t$
 (b) $f_5(t) = 10 + \sin t$
 (c) $f_6(t) = 5 + 100\sqrt{2} \sin 377t$
 (d) $f_7(t) = \sin^2 5t$
 (e) $f_8(t) = \sin^2 5t - \cos^2 5t$
 (f) $f_9(t) = 1 + \sin^2 10t$

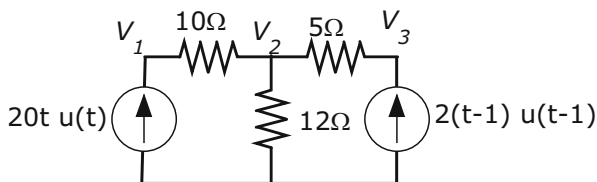
3.13. Find I_1, I_2 .



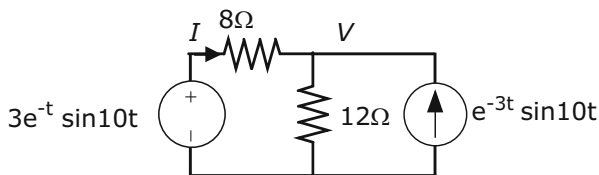
3.14. Find I_1, V_0 .



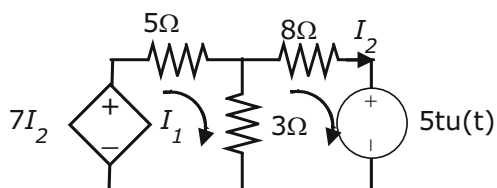
3.15. Find V_1, V_2, V_3 .



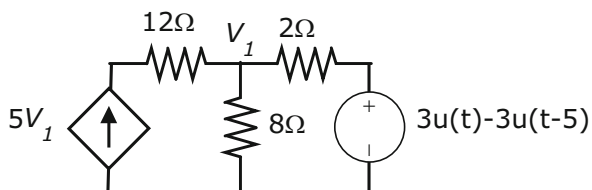
3.16. Find V_1, I .



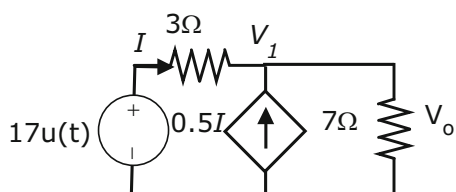
3.17. Find I_1, I_2 .



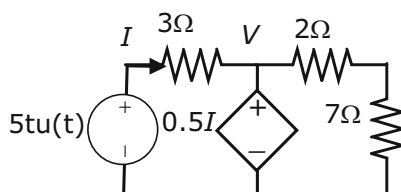
3.18. Find V_1 .



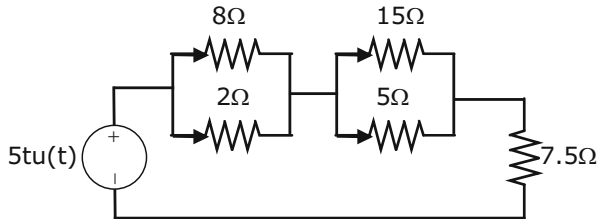
3.19. Find I, V_o .



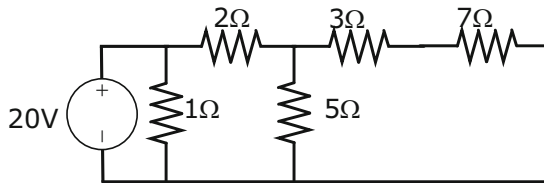
3.20. Find I, V .



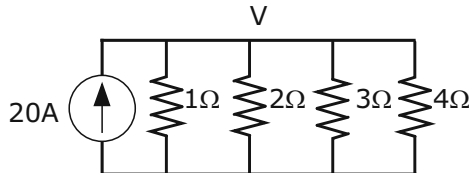
- 3.21. Find the voltage drop and current of all resistors. Find the power taken from the source and the power dissipated in each resistor.



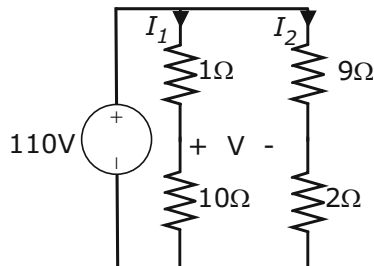
- 3.22. Find the voltage drop and current of all resistors. Find the power taken from the source and the power dissipated in each resistor.



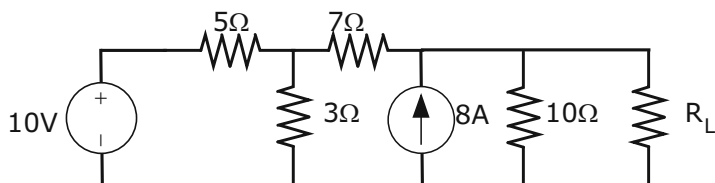
- 3.23. Find the voltage drop and current of all resistors. Find the power taken from the source and the power dissipated in each resistor.



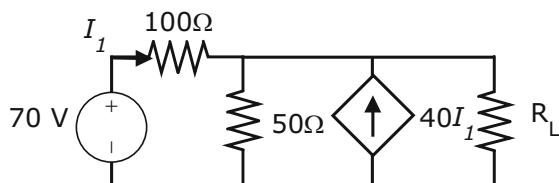
- 3.24. Find the voltage drop and current of all resistors. Find the power taken from the source and the power dissipated in each resistor.



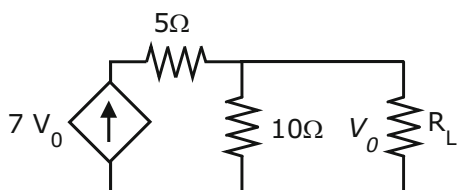
- 3.25. Find Thevenin and Norton equivalent of the following circuit across the load terminals.



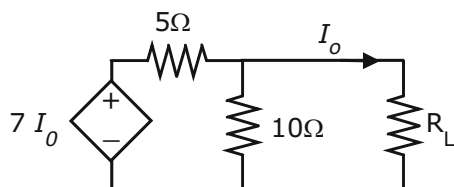
- 3.26. Find Thevenin and Norton equivalent of the following circuit across the load terminals.



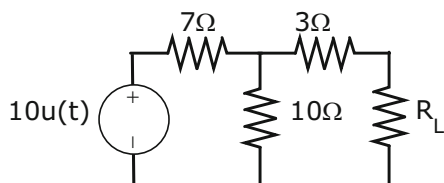
- 3.27. Find Thevenin and Norton equivalent of the following circuit across the load terminals.



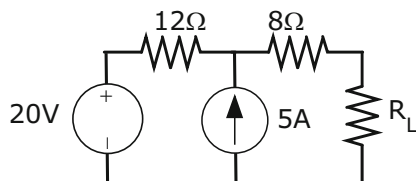
- 3.28. Find Thevenin and Norton equivalent of the following circuit across the load terminals.



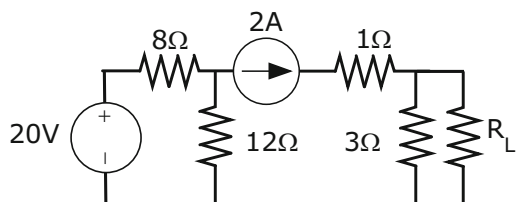
- 3.29. Find the load impedance at which the maximum power is transferred from the source to the load.



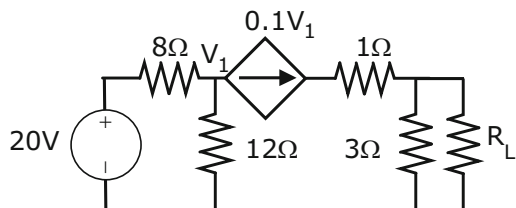
- 3.30. Find the load impedance at which the maximum power is transferred from the source to the load.



- 3.31. Find the load impedance at which the maximum power is transferred from the source to the load.



- 3.32. Find the load impedance at which the maximum power is transferred from the source to the load.



Chapter 4

Circuit Response Analysis



Introduction

The flow of current in the circuit branches and drop of voltage across circuit elements depend on their behavior and their ability to store energy. For instance, the voltage drop across a resistor is in phase with its current passing through. But that is not the same in a capacitor or an inductor. This makes the circuit KVL and KCL equations integrodifferential equations. The order of these equations depends on the number of energy-storing elements. In this chapter, the circuit elements are introduced and their equations are discussed. The order of a circuit is discussed, and responses of first- and second-order circuits to their initial condition and to external sources are analyzed.

Resistors

Consider a resistor shown in Fig. 4.1 that has current $i(t)$ passing through which results in a voltage drop $v(t)$. The relation of time-varying voltage and time-varying current to the resistance of the resistor follows Ohm's law as follows:

$$v(t) = Ri(t)$$

Ohm's law of a resistor indicates that the voltage drop across the resistor linearly depends on the current passing through. This also demonstrates that the voltage and current waveforms across a resistor are inphase. Figure 4.2 shows the voltage and current of a resistor. As the figure shows, the zero crossing of the two signals (voltage and current) is the same, and the peaks occur at the same time. The current is a scaled waveform of the voltage.



Fig. 4.1 Schematic of a resistor. The voltage drop $v(t)$ dependency across a resistor R when passing current $i(t)$

Fig. 4.2 Voltage and current of a resistor are inphase. They may have different values (peaks)

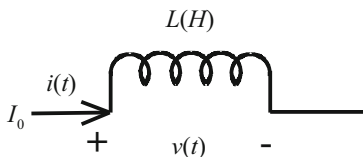
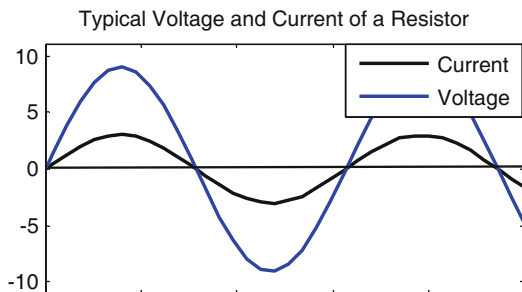


Fig. 4.3 Circuit schematic of a charged inductor with initial current I_0 . The voltage and current of the inductor are related through a differential equation as: $v(t) = L \frac{di(t)}{dt}$

Inductors

Consider an inductor with inductance of $L(H)$ Henrys. The inductor also has an initial current of I_0 as shown in Fig. 4.3.

The relation of the voltage drop and current of the inductor is obtained by:

$$v(t) = L \frac{di(t)}{dt}$$

where $i(t)$ is the current passing through the inductor and $v(t)$ is the voltage drop across the inductor. As the equations demonstrate, in sinusoidal waveforms, the voltage and current are 90° out of phase where the current waveform lags the voltage waveform. Figure 4.4 shows the voltage and current of an inductor.

Considering an initial current of I_0 passing through the inductor, the current can be expressed as:

$$i(t) = I_0 u(t)$$

Replacing this current in the inductor's equation results in the initial voltage of:

Fig. 4.4 The voltage and current of an inductor are 90° out of phase. The current lags the voltage. It is recommended to take voltage as a reference and then measure the phase shift of the current

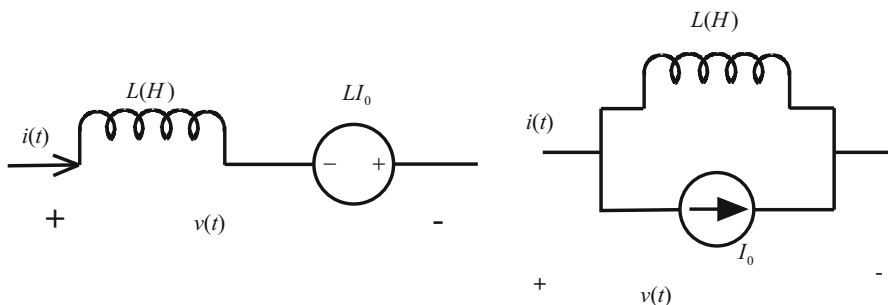
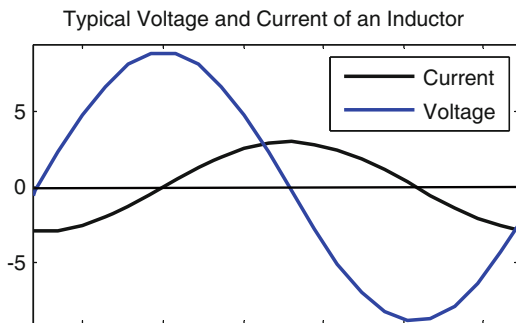
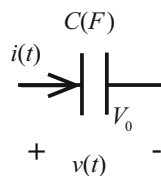


Fig. 4.5 A charged inductor with initial current I_0 . Initial condition can be modeled as a voltage source in series to the inductor. The value of the voltage source is $LI_0\delta(t)$, and the polarity of the voltage source is selected such that the current out of this source follows the same direction as the initial current. The model of series inductor and the voltage source is best for KVL analysis. Model of a charged inductor can also be shown as a current source in parallel to the inductor. This model is best for KCL analysis

Fig. 4.6 Circuit schematic of a charged capacitor. The voltage and current are related through a differential equation as $i(t) = C \frac{dv(t)}{dt}$

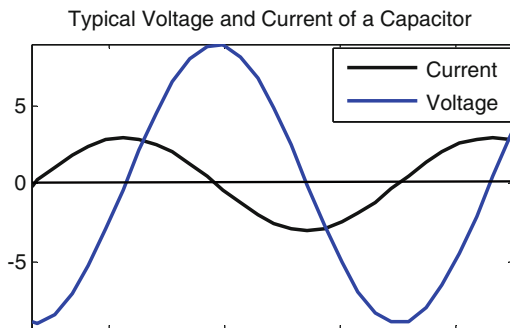


$$V_0 = L \frac{d(I_0 u(t))}{dt} = LI_0 \delta(t)$$

The model of a charged inductor can be presented as an inductor without charge in parallel or series to a source that represents the initial condition (Fig. 4.5). The initial condition can also be modeled as a current source with the value of I_0 which is connected in parallel to the inductor. Figure 4.6 shows the inductor without charge and a current source to represent the initial charge.

The inductor is an energy-storing element. The net energy stored in an ideal inductor when a sinusoidal voltage is applied is zero. The inductor is fully charged in

Fig. 4.7 The voltage and current of a capacitor 90° out of phase. The current leads the voltage. It is recommended to take voltage as a reference and then measure the phase shift of the current



a positive cycle and is fully discharged in a negative cycle. The inductor is therefore called a passive element.

The amount of stored energy (in Joules) in an inductor is proportional to its inductance $L(H)$ and the square of current passing through I^2 as follows:

$$W = \frac{1}{2}LI^2 \text{ (J)}$$

Capacitors

Consider a capacitor with capacitance C (F) Farads that is initially charged at voltage V_0 as shown in Fig. 4.6. The current $i(t)$ generates voltage $v(t)$ which are related as:

$$i(t) = C \frac{dv(t)}{dt}$$

This shows that the current and voltage across a capacitor when sinusoidal signals are applied are 90° out of phase, with the current waveform leading the voltage waveform. The phase shift is shown in Fig. 4.7.

The initial charge of the capacitor can be modeled as a separate source in series or in parallel to the no-charged capacitor. Considering a series voltage source, the model of the charged capacitor is shown in Fig. 4.8. Considering the voltage and current relation of a capacitor and its initial charge as:

$$v(t) = V_0u(t)$$

The initial current of the capacitor becomes:

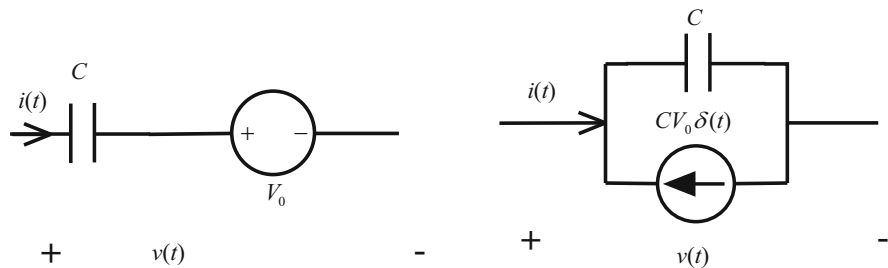


Fig. 4.8 Model of a charged capacitor when the initial charge of the capacitor is shown as a series-connected voltage source. This model is best for KVL analysis. The initial charge can also be demonstrated as a current source with amplitude $CV_0\delta(t)$. The direction of current is reversed as when the capacitor starts to discharge; it sends the current out of its positive terminal. This model is suitable for KCL analysis

$$I_0 = C \frac{d(V_0 u(t))}{dt} = CV_0 \delta(t)$$

An ideal capacitor receives charge in a positive cycle and is fully charged. In the negative cycle, it is fully discharged and charged with opposite polarity. Therefore, in a period, the net stored charge in this capacitor is zero. For this reason, a capacitor is a passive element.

The amount of stored energy (in Joules) in a capacitor is proportional to its capacitance C (F), and the square of applied voltage V^2 as follows:

$$W = \frac{1}{2} CV^2 \text{ (J)}$$

Order of a Circuit

Recalling from circuit element definitions, it was determined that resistors just dissipate energy as heat, but inductors and capacitors store energy. The net energy storage of these elements over one cycle was zero. This makes these elements to be categorized as passive elements.

The relation of the voltage and current of energy-storing elements is expressed by differential and integral equations. Therefore, each energy-storing element has the potential of increasing the order of a differential equation written for a circuit. These differential and integral equations are obtained through mesh and node analysis, i.e., KVL and KCL.

To determine the order of a given circuit:

- In the first step to determine the order of a circuit, all possible simplifications of capacitors and inductors must be considered. It means that the equivalent of series

Fig. 4.9 The circuit has one energy-storing element. Therefore, it is a first-order circuit. This means that the differential equations that show the voltage or current relations is a first order equation

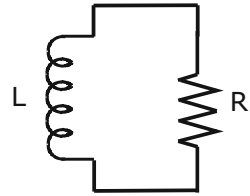


Fig. 4.10 The circuit is simplified already and has two energy-storing elements. Therefore, it is a second-order circuit. This means that the differential equations that shows the voltage or the current of the circuit is a second order equation

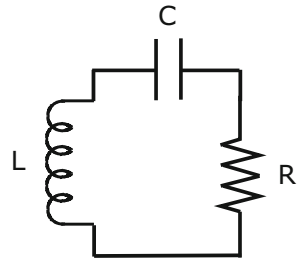
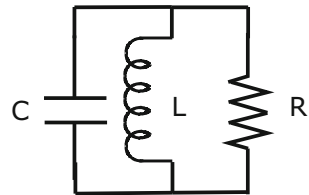


Fig. 4.11 The circuit is simplified already and has two energy-storing elements. Therefore, it is a second-order circuit



connection of capacitors/inductors or parallel connection of capacitors/inductors must be obtained. At this stage the highest possible order of a circuit is the count number of equivalent energy-storing elements.

- If a loop of the simplified circuit is formed by bunch of capacitors and cannot be simplified further, for each of these capacitive loops the order is reduced by one.
- If a node of the circuit is only connected to bunch of inductors and cannot be simplified further, for each of these inductive nodes the order is reduced by one.

This can be expressed as:

Order of a circuit = number of energy-storing elements – number of capacitive loops – number of inductive nodes.

Example 4.1 Find the order of the following circuits (Figs. 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, 4.15, and 4.16).

First-Order Circuits

A first-order circuit contains an equivalent of one energy-storing element either as a capacitor or an inductor. These circuits are analyzed in two conditions:

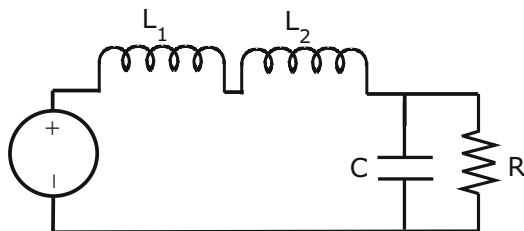


Fig. 4.12 Two series inductors L_1 and L_2 can be reduced to one equivalent inductor. Then, the number of energy-storing elements becomes two. One is the equivalent inductor and the other one is the capacitor. There is no capacitive loop or inductive node

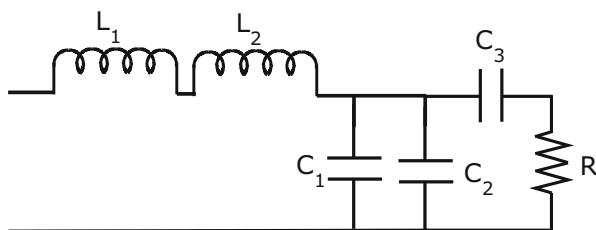


Fig. 4.13 Two inductors L_1 and L_2 in series are equivalent to one inductor. Two capacitors C_1 , C_2 in parallel are equivalent to one capacitor. Then, the number of energy-storing elements becomes the equivalent inductor, equivalent capacitor, and C_3 , a total of three. No inductive node and no capacitive loop exist. Therefore, the order is 3

Fig. 4.14 The inductors form an inductive node (the node that only inductors are connected to). The order of the circuit is 3 energy-storing elements – 1 inductive node – 0 capacitive loop = 2. The order is 2

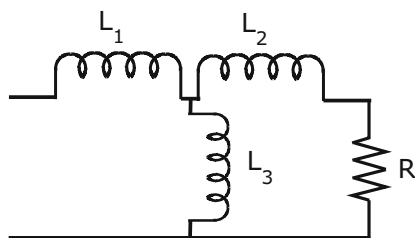
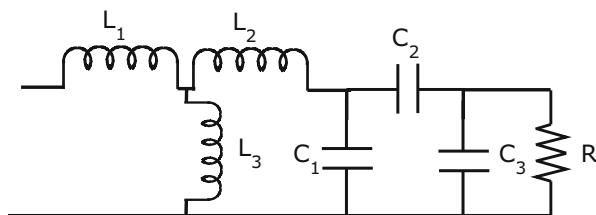


Fig. 4.15 Inductors form an inductive node; capacitors form a capacitive loop. The order of the circuit is 6 energy storing elements – 1 inductive node – 1 capacitive loop = 4



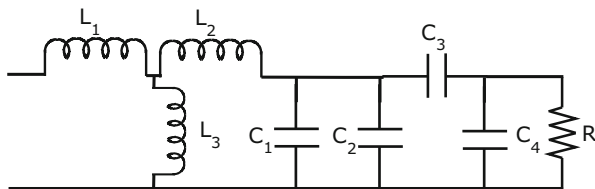


Fig. 4.16 Two capacitors C_1, C_2 in parallel are equivalent to one capacitor. The inductors form an inductive node; the equivalent capacitor and the other two form a capacitive loop. Therefore, the order of the circuit is 6 energy-storing elements (C_1, C_2 are equivalent to one capacitor) – 1 inductive node – 1 capacitive loop = 4

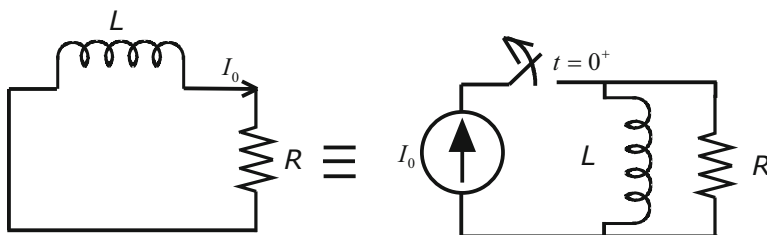


Fig. 4.17 RL is a first-order circuit. The initial condition is obtained by charging the inductor with an external source and then disconnecting the source at a desired time

1. First is when the circuit is driven by its initial charge of energy-storing elements. The circuit response is identified as the voltage and/or current profiles. The response that is generated as a result of initial charges is called *natural response*.
2. The second is when the response is generated as a result of an external source. The response is called a *forced response*, and depending on the type of the source applied, the name is adapted too, e.g., a step function source generates the step response, an impulse source generates the impulse response, and a sinusoidal source generates the steady-state sinusoidal response (when the transients are damped).

Natural Response: RL Circuits

Consider a series connection of a charged inductor and a resistor as shown in Fig. 4.17. The initial charge of the inductor I_0 is the current of the inductor at the moment of switching. To create the initial conditions, the inductor can be connected to a current source to establish the current I_0 . The objective is to find the inductor current before and after the switching.

The inductor intends to keep the current constant even after the switching. Therefore, the current direction is the same both before and after the switching. However, the inductor must change its polarity to feed the current the same direction when being discharged. Therefore, the polarity of inductor before and after the

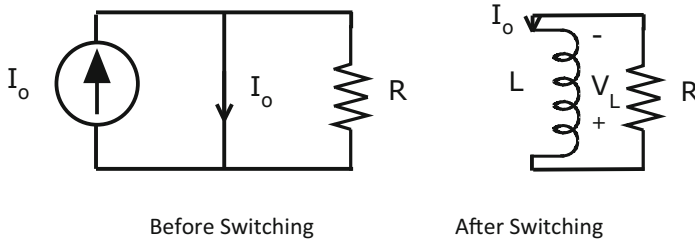


Fig. 4.18 The circuit schematics before switching and after switching are shown. Before switching the inductor is fully charged. It becomes a short circuit. Once the source is disconnected, the inductor is discharged, and the current drops through the resistor. From the before switching circuit Topology, the initial conditions are obtained. From the after switching circuit topology, the time constant and the final value of the circuit parameters are obtained

switching event changes. Knowing this effect, the circuit must be analyzed in two events, before switching and after switching.

Before switching, the switch has been closed for long time, and this provides enough time for the energy-storing elements to be fully charged. The inductor when charged with DC current becomes a short circuit. The short circuit across the resistor takes all the sourced current (in this example). Therefore, the initial current is obtained.

After switching, the charged inductor is connected to the circuit, and with the absence of the source, it is discharged to the resistor (Fig. 4.18).

The KVL (considering the polarity change in inductor) can be written as:

$$-V_L - V_R = 0$$

Ohm's law indicates $V_L = L \frac{di(t)}{dt}$ and $V_R = Ri(t)$. Therefore:

$$-L \frac{di(t)}{dt} - Ri(t) = 0$$

For simplicity, the time dependency of the current is removed from the equations.

$$-L \frac{di}{dt} - Ri = 0$$

Solving for i results in:

$$\begin{aligned} L \frac{di}{dt} &= -Ri \\ L \frac{di}{i} &= -Rdt \rightarrow \frac{di}{i} = -\frac{R}{L} dt \end{aligned}$$

Taking integral of both sides $\int_{t_0}^t \frac{di}{i} = \int_{t_0}^t -\frac{R}{L} dt$ results in:

$$\ln i(t) \Big|_{t_0}^t = -\frac{R}{L} t \Big|_{t_0}^t$$

$$\ln i(t) - \ln i(t_0) = -\frac{R}{L} (t - t_0)$$

Considering $t_0 = 0$ and the initial condition $i(t_0 = 0) = I_0$.

$$\ln i(t) - \ln I_0 = -\frac{R}{L} t = -\frac{t}{L/R} = -\frac{t}{\tau}.$$

$$\ln a - \ln b = \ln \frac{a}{b}$$

Therefore,

$$\ln \frac{i(t)}{I_0} = -\frac{t}{\tau}.$$

$\ln a = x \leftrightarrow a = e^x$. Therefore:

$$\frac{i(t)}{I_0} = e^{-\frac{t}{\tau}}$$

$$i(t) = I_0 e^{-\frac{t}{\tau}}, t \geq 0.$$

This equation requires the initial condition I_0 which is obtained from the before switching circuit analysis and the time constant τ which is obtained from the circuit of after the switching.

Example 4.2 Consider an RL first-order circuit shown in Fig. 4.19. The switch has been closed for long time. The switch opens at $t = 0^+$ second. Find the current of the inductor $i(t)$.

Solution There is need for the initial conditions and time constant.

Initial condition is obtained from the circuit before switching shown in Fig. 4.20. The inductor, connected to a 20 A dc source, is charged and becomes a short circuit. Therefore, it bypasses the 5Ω resistor out of the circuit, letting all the source current pass through the inductor. Therefore, the current of inductor before switching reaches 20 A. Since the direction of current $i(t)$ and the initial current I_0 match, the initial current is a positive number.

After switching, the circuit becomes a discharging RL circuit. The time constant from the ports of the inductor is measured to be $\tau = \frac{L}{R} = \frac{100\text{m}}{5} = 0.02 \text{ s}$.

Fig. 4.19 Circuit of the Example 4.2

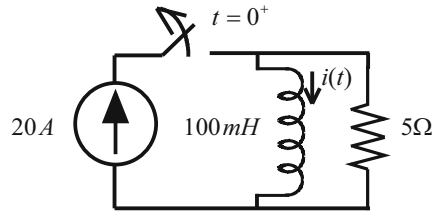


Fig. 4.20 The circuit schematic before the switching event. A fully-charged inductor becomes a short circuit

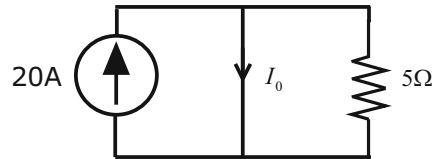
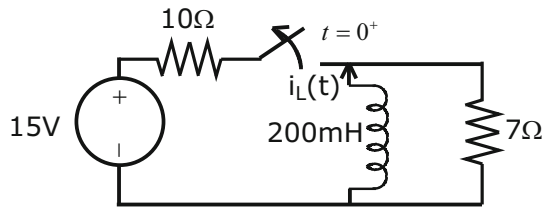


Fig. 4.21 Circuit schematic of Example 4.3



Inserting the initial condition and the time constant into the inductor current template $i(t) = I_0 e^{-t/\tau}$ results in:

$$i(t) = 20e^{-\frac{t}{0.02}} = 20e^{-50t}, t \geq 0 \text{ (A)}$$

The voltage across the resistor is obtained as:

$$V_R = -5i(t) = -100e^{-50t} \text{ (V)}$$

As it can be seen from the voltage, a sudden discharge of inductors generates high voltages. Of course, this example analyzed one time charge-discharge of the inductor. This process in real-world applications might be repeated periodically, hence, generating a train of high voltages.

Example 4.3 Consider a RL circuit shown in Fig. 4.21. The switch has been in closed position for long time. It opens at time $t = 0^+$. Find the inductor current $i(t)$ for positive time.

Solution Inductor current before switching reaches a steady current as shown in Fig. 4.22. The charged inductor becomes a short circuit bypassing the $7\ \Omega$ resistor. The initial current forced by the voltage source is measured against the desired direction of $i(t)$. Hence, it measures a negative value.

Fig. 4.22 Circuit before the switching event. A fully-charged inductor becomes a short circuit

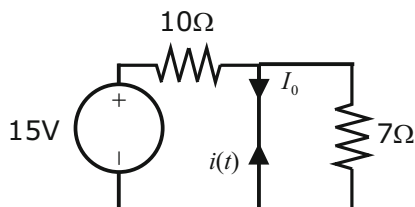
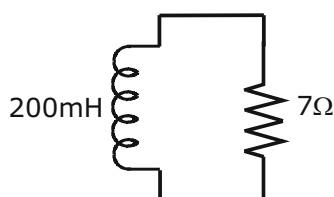


Fig. 4.23 Circuit after the switching event. The source is disconnected, and the inductor current is being discharged through the resistor



$$I_0 = -\frac{15}{10} = -1.5 \text{ A}$$

Figure 4.23 shows the circuit after the switching event. The switch opens and disconnects the source from the RL component. The circuit's time constant becomes $\tau = \frac{L}{R} = \frac{200}{7}$ ms.

$$\text{The current is } i(t) = -1.5e^{-\frac{t}{\frac{200e-3}{7}}} = -1.5e^{-35t}, t \geq 0.$$

The voltage across the 7Ω resistor becomes:

$$V_R = 7i(t) = 10.5e^{-35t} \text{ (V)}$$

Natural Response: RC First-Order Circuit

Consider a circuit with a capacitor as an energy-storing element. A first-order RC circuit is shown in Fig. 4.24 in which when the switch is closed forms a node involving a voltage source, a capacitor, and a resistor. The switch isolates the source from the rest of the circuit. It also allows for charging of the capacitor.

Consider that the switch is closed for a long period of time to fully charge the capacitor. In reality, a time more than 4 times of the time constant guarantees almost a full charge. The switch is open at time $t = 0^+$ disconnecting the source from the circuit. The capacitor is now discharged through the resistor. The objective is to obtain the voltage profile across the capacitor after the switching event. The circuit is analyzed both in before switching and after switching topologies.

Fig. 4.24 A first-order RC circuit with a switching event

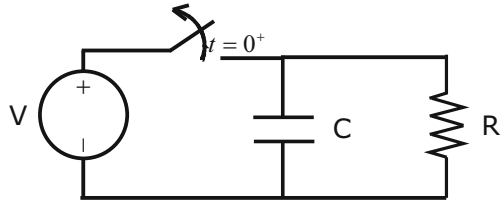


Fig. 4.25 Before the switching event the capacitor is fully charged and becomes an open circuit

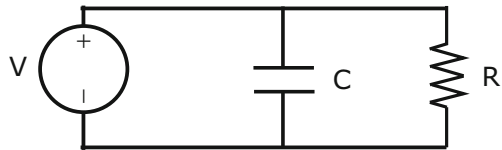
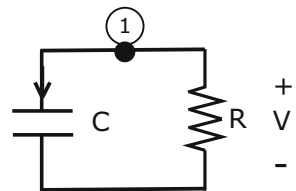


Fig. 4.26 After the switching, the source is disconnected, and the initial voltage stored in the capacitor is discharged through the resistor



Before switching, the circuit is shown in Fig. 4.25. A fully charged capacitor becomes an open circuit forcing the current to fully pass through the resistor (in this circuit). The voltage across the capacitor in this case is the voltage of the resistor (because they are connected in parallel). The initial voltage is, therefore, the source voltage V_0 .

The charged capacitor tends to keep the voltage at the terminals constant before and after the switching event. The capacitor accomplishes this task by changing the current direction. Therefore, the charging current to the terminals of the capacitor can suddenly change direction to exit the terminal, hence, keeping the voltage constant.

After switching, the source is disconnected, and the resistor drains the capacitor and converts the stored energy to heat. A KCL at node ① of circuit shown in Fig. 4.26 results in:

$$-i_C(t) - i_R(t) = 0$$

From Ohms' law, $i_C(t) = C \frac{dv(t)}{dt}$ and $I_R = \frac{v(t)}{R}$. For the simplicity, the time dependency is omitted.

$$C \frac{dv}{dt} = -\frac{v}{R}$$

$$\frac{dv}{v} = -\frac{dt}{RC}$$

Taking integral of the equation results in:

$$\int_{t_0}^t \frac{dv}{v} = \int_{t_0}^t -\frac{dt}{RC}$$

Therefore:

$$\begin{aligned} \ln v(t) \Big|_{t_0}^t &= -\frac{1}{RC} t \Big|_{t_0}^t \\ \ln v(t) - \ln v(t_0) &= -\frac{1}{RC} (t - t_0) \end{aligned}$$

Considering the initial time as zero $t_0 = 0$, $v(t_0 = 0) = V_0$, and the time constant, $\tau = RC$,

$$\begin{aligned} \ln v(t) - \ln V_0 &= -\frac{1}{RC} t = -\frac{t}{\tau} \\ \ln a - \ln b &= \ln \frac{a}{b} \end{aligned}$$

Therefore,

$$\begin{aligned} \ln \frac{V(t)}{V_0} &= -\frac{t}{\tau} \\ \ln a = x &\leftrightarrow a = e^x \\ \frac{v(t)}{V_0} &= e^{-\frac{t}{\tau}} \\ v(t) &= V_0 e^{-\frac{t}{\tau}}, t \geq 0. \end{aligned}$$

Comparing the results obtained from RL and RC first-order circuits, it can be concluded that the natural response $x(t)$ of the first-order circuits is obtained by:

$$x(t) = X_0 e^{-\frac{t}{\tau}}, t \geq 0.$$

where X_0 is the initial condition of the parameter $x(t)$ obtained from the before switching circuit and τ is the time constant which is obtained from the after switching circuit.

Example 4.4 In the circuit shown in Fig. 4.27, the switch has been in closed position for a long period of time. At time $t=0^+$ the switch is opened. Find the voltage $v(t)$ across the capacitor for $t \geq 0$.

Solution Since the circuit has been connected to the voltage source for a long period of time, the capacitor is charged through the 50 k Ω resistor. However, as the capacitor is fully charged, it becomes an open circuit letting only the 200 k Ω resistor

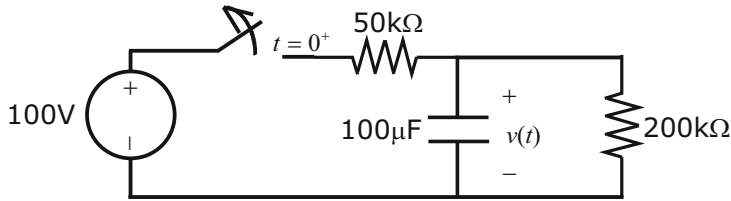
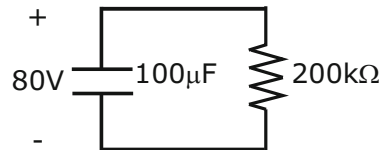


Fig. 4.27 Circuit of Example 4.4

Fig. 4.28 After the switching event, a charged capacitor is discharged through the resistor



in the circuit. Since the capacitor and 200 kΩ resistor are in parallel, they share the same voltage. Therefore, the initial voltage of the capacitor will be the same the voltage drop across this resistor. Since these resistors are connected in series, a voltage divider between the 50 kΩ and 200 kΩ shows the desired voltage as:

$$V_0 = V_{200\text{ k}\Omega} = \frac{200\text{ k}}{50\text{ k} + 200\text{ k}} 100 = 80\text{ V}$$

After the switching event, the voltage source and the 50 kΩ resistor are disconnected from the circuit. The circuit topology is shown in Fig. 4.28.

The time constant is obtained as:

$$\tau = RC = 200\text{ k} \times 100\text{ }\mu = 200e3 \times 100e-6 = 20000e-3 = 20\text{ s}.$$

The voltage response across the capacitor is obtained by:

$$v(t) = V_0 e^{-\frac{t}{\tau}}, t \geq 0.$$

$$v(t) = 80e^{-\frac{t}{20}} = 80e^{-0.05t}, t \geq 0.\text{V}$$

Forced Response of First-Order Circuits

Forced response of circuits is obtained by utilizing external sources which may or may not include initial charges. The forced response due to a step function is known as the step response, and a forced response due to an impulse is known as the impulse response. The circuit may switch between two charged elements, and this may

introduce a step in the circuit. However, when an input is applied in the circuit the primary objective is always to find the voltage of capacitors or current of inductors. From these values, voltages, currents, power, and energy of other elements may be calculated.

Step Response of RL Circuit

Consider a first-order RL circuit connected to a voltage source through a switch. Initial condition of the inductor is known as I_0 which is obtained from the circuit topology before the switching event. The switch changes its position such that the RL circuit experiences a new driving force other than the initial charge of the inductor itself. Figure 4.29 shows a switch which is closed at $t = 0^+$.

A KVL in the loop can be written as follows:

$$-V_S + v_L(t) + v_R(t) = 0$$

The DC source forces a current through the circuit, and the charging inductor ultimately becomes a short circuit. However, the purpose of this study is to determine the inductor current profile variation that starts from initial I_0 and reaches final value $I_f = \frac{V_S}{R}$.

KVL is rewritten as follows:

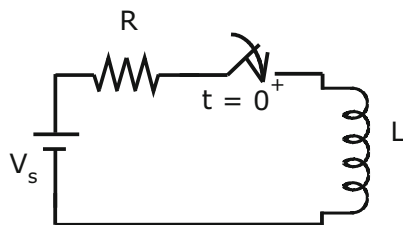
$$-V_S + L \frac{di(t)}{dt} + Ri(t) = 0$$

Then:

$$\frac{di(t)}{dt} = \frac{1}{L} (V_S - Ri(t))$$

$$\frac{di(t)}{dt} = \frac{-R}{L} \left(i(t) - \frac{V_S}{R} \right)$$

Fig. 4.29 An RL circuit is connected to the source at time $t = 0^+$. The source and the switch resemble a step function applied to the circuit. The response is the current of the inductor



$$\frac{di(t)}{\left(i(t) - \frac{V_S}{R}\right)} = \frac{-R}{L} dt$$

Integrating both sides from t_0 to t yields:

$$\begin{aligned} \int_{t_0}^t \frac{di(t)}{\left(i(t) - \frac{V_S}{R}\right)} &= \int_{t_0}^t \frac{-R}{L} dt \\ \ln \left(i(t) - \frac{V_S}{R} \right) \Big|_{t_0}^t &= \frac{-R}{L} t \Big|_{t_0}^t \\ \ln \left(i(t) - \frac{V_S}{R} \right) - \ln \left(I_0 - \frac{V_S}{R} \right) &= \frac{-R}{L} (t - t_0) \\ \ln a - \ln b &= \ln \frac{a}{b} \end{aligned}$$

Considering $t_0 = 0$, and the $i(t = 0) = I_0$ results in:

$$\begin{aligned} \ln \frac{\left(i(t) - \frac{V_S}{R}\right)}{\left(I_0 - \frac{V_S}{R}\right)} &= -\frac{t}{\tau} \\ \ln a = x &\leftrightarrow a = e^x \\ \frac{\left(i(t) - \frac{V_S}{R}\right)}{\left(I_0 - \frac{V_S}{R}\right)} &= e^{-\frac{t}{\tau}} \end{aligned}$$

Solving for $i(t)$:

$$i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R} \right) e^{-\frac{t}{\tau}}, t \geq 0.$$

Recalling from circuit values, $I_f = \frac{V_S}{R}$ and the initial condition, the step response can be obtained from:

$$i(t) = I_f + (I_0 - I_f) e^{-\frac{t}{\tau}}, t \geq 0.$$

Example 4.5 The switch in the circuit shown in Fig. 4.30 has been in position a for a long time. At time $t = 0^+$, it changes to position b . Find the current of the inductor $i(t)$.

Solution The circuit in position a is connected to a -16 V voltage source. The terminals of the source force the current through inductor in opposite direction of the desired $i(t)$. The initial current is obtained as:

Fig. 4.30 Circuit of Example 4.5

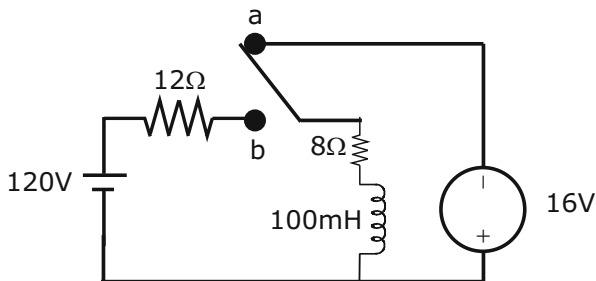
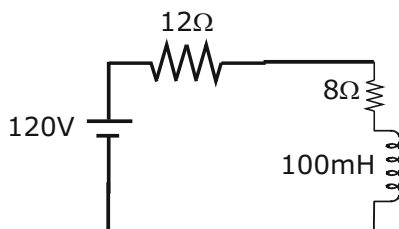


Fig. 4.31 The circuit schematic after the switching event



$$I_0 = -\frac{16}{8} = -2 \text{ A.}$$

The reason for this calculation is that the inductor is charged and becomes a short circuit. Therefore, in position *a* the circuit has 8Ω resistor and a -16 V source.

In position *b*, the circuit is connected to a voltage source of 120V with internal resistance of 12Ω . The circuit topology in switch position *b* is shown in Fig. 4.31. The final/ultimate current of the circuit is obtained when the inductor is fully charged i.e. short circuit again.

The final value of the current can be obtained as follows:

$$I_f = \frac{120}{12 + 8} = 6\text{A}$$

The inductor in position *b* is also a short circuit. The only time that the inductor is not short circuit is the time that the current transient starts from -2 A to reach 6 A .

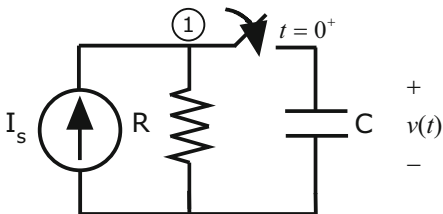
In position *b* the circuit time constant is calculated to be $\tau = \frac{L}{R} = 5 \text{ ms}$.

The transient inductor current is therefore obtained as:

$$i(t) = I_f + (I_0 - I_f)e^{-\frac{t}{\tau}}, t \geq 0.$$

$$i(t) = 6 + (-2 - 6)e^{-\frac{t}{5 \times 10^{-3}}} = 6 - 8e^{-200t}, t \geq 0.$$

Fig. 4.32 Forced response of a RC circuit. The current source is connected to the RC circuit through the switch and forms a step in current



Forced Response of First-Order RC Circuit

Consider an RC circuit shown in Fig. 4.32 with a switch to provide an option of connecting to a source. A sudden switching event introduces a step function into the system. The response is seen as a current that changes the voltage across the capacitor. Initial charge of the capacitor V_0 experiences a transient and shifts to a final value of V_f .

Before the switching event, the capacitor has initial voltage. After the switching event, the circuit topology configures a node that has voltage $v(t)$. A KCL analysis at this node ① results in the voltage variation as follows:

$$\text{KCL } \textcircled{1}. -I_s + i_R(t) + i_C(t) = 0.$$

Considering Ohm's law results in:

$$\begin{aligned} -I_s + \frac{v(t)}{R} + C \frac{dv(t)}{dt} &= 0 \\ \frac{-1}{RC} (v(t) - RI_s) &= \frac{dv(t)}{dt} \\ \frac{dv(t)}{(v(t) - RI_s)} &= \frac{-1}{RC} dt \end{aligned}$$

Integrating both sides from t_0 to t yields:

$$\begin{aligned} \int_{t_0}^t \frac{dv(t)}{(v(t) - RI_s)} &= \int_{t_0}^t \frac{-1}{RC} dt \\ \ln(v(t) - RI_s) \Big|_{t_0}^t &= \frac{-1}{RC} t \Big|_{t_0}^t \\ \ln(v(t) - RI_s) - \ln(V_0 - RI_s) &= \frac{-1}{RC} (t - t_0). \\ \ln a - \ln b &= \ln \frac{a}{b} \end{aligned}$$

Considering $t_0 = 0$, and the initial voltage $v(t = 0) = V_0$ results in:

$$\ln \frac{(v(t) - RI_s)}{(V_0 - RI_s)} = -\frac{t}{\tau}.$$

$$\ln a = x \leftrightarrow a = e^x$$

$$\frac{(v(t) - RI_s)}{(V_0 - RI_s)} = e^{-\frac{t}{\tau}}$$

Solving for $v(t)$:

$$v(t) = RI_s + (V_0 - RI_s)e^{-\frac{t}{\tau}}, t \geq 0.$$

Recalling from circuit values, $V_f = RI_s$ and the initial condition, the step response can be obtained from:

$$v(t) = V_f + (V_0 - V_f)e^{-\frac{t}{\tau}}, t \geq 0.$$

Example 4.6 Consider a first-order RC circuit as shown in Fig. 4.33. The switch has been in position ① for a long time, and at time $t = 0^+$, it changes to position ②. Find the voltage $v(t)$ across the capacitor for all time $t \geq 0$.

Solution When the switch is in position ①, the capacitor is connected to a voltage source and receives a steady-state voltage equal to the value set by the voltage divider of $10 \text{ k}\Omega$ and $90 \text{ k}\Omega$. When the capacitor is charged, it becomes an open circuit, and since it is connected in parallel to the $10 \text{ k}\Omega$ resistor, they share the same voltage.

Therefore, the initial voltage is:

$$V_0 = \frac{10 \text{ k}}{10 \text{ k} + 90 \text{ k}} 200 = 20 \text{ V}$$

When the switch is in position ② for a long time, the capacitor voltage will reach another ultimate value set by the circuit connected to the 150 V source. A voltage divider in this circuit results in the final voltage V_f . As the source polarity is opposite

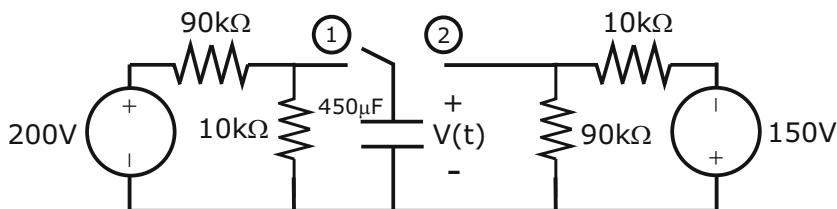


Fig. 4.33 Circuit of Example 4.6

of the measured voltage across the capacitor, the final voltage becomes a negative value.

$$V_f = \frac{90 \text{ k}}{90 \text{ k} + 10 \text{ k}}(-150) = -135 \text{ V}$$

The time constant of the circuit is obtained from the circuit topology of after switching. To obtain the time constant across the capacitor, equivalent resistance across the resistor can be used. From the circuit, two resistors of $10 \text{ k}\Omega$ and $90 \text{ k}\Omega$ are connected in parallel when the -150 V source is removed. Therefore, the time constant is:

$$\tau = R_{\text{eq}}C = (10 \text{ k} \parallel 90 \text{ k})450 \mu = 4.05 \text{ ms}$$

The circuit response if therefore obtained as follows:

$$v(t) = V_f + (V_0 - V_f)e^{-\frac{t}{\tau}}, t \geq 0.$$

Replacing the calculated values, results in:

$$v(t) = -135 + (20 - (-135))e^{-\frac{t}{4.05 \times 10^{-3}}}, t \geq 0.$$

$$v(t) = -135 + 155e^{-246.9t}, t \geq 0.$$

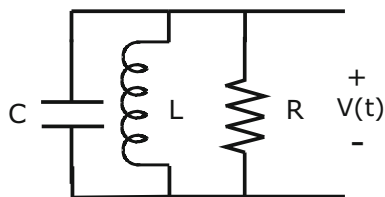
The circuit time constant is always obtained from the equivalent resistance across the terminals of the capacitor or the terminals of inductor.

Second-Order Circuits

Second-order circuits have equivalent of two energy-storing elements. As introduced earlier, the circuit elements have the relations of their voltage and current defined as either a linear function $v = Ri$ in resistors or differential or integral equations as $v = L \frac{di}{dt}$ and $i = \frac{1}{L} \int v dt$ in inductors and $i = C \frac{dv}{dt}$ and $v = \frac{1}{C} \int i dt$ in capacitors. One objective was to analyze circuits and find the voltage of capacitors and current of inductors. From this analysis, other parameters can be identified. To analyze higher-order circuits, KVL for each loop and/or KCL for each node may be written to form differential equations based on the desired parameter $v(t)$ or $i(t)$.

Accordingly, circuits can be analyzed and parameters can be identified. However, two typical *RLC* circuits where all elements are connected in series or parallel are of more interest because of the properties they show. Another analysis can be obtained by considering the initial charges of the elements or by analyzing the effect of an external source.

Fig. 4.34 Parallel connection of *RLC* elements. The initial conditions induce a voltage across the elements. The response of interest is the $v(t)$



Natural Response of RLC Parallel Circuits

Consider a circuit consisting of a resistor, an equivalent inductor, and an equivalent capacitor connected in parallel as shown in Fig. 4.34.

The inductor has an initial current I_0 and the capacitor has an initial voltage V_0 . There is an interest in finding the voltage across the terminals, $v(t)$. Since there is no external source and the voltage response is generated purely from the effect of initial charges, the voltage response is called natural response.

Parallel connection of these elements forms a node that has voltage $v(t)$. Therefore, at this node, the balance of currents must hold. KCL results in:

$$i_R(t) + i_L(t) + i_C(t) = 0$$

These currents leave the node because they are through passive elements and should naturally drain the current out of the node. To find the voltage at the terminals, each of these currents must be written in terms of $v(t)$. Therefore, replacing the equations, term by term, results in:

$$\frac{v(t)}{R} + \frac{1}{L} \int v(t)dt + C \frac{dv(t)}{dt} = 0$$

An integrodifferential equation has been obtained based on variable $v(t)$. To solve this equation for $v(t)$, one time differential must be taken from the equation to convert the entire equation to a differential equation. This is one time differential because there exists one time integral in the equation. Taking differential from the equation results in a second order differential equations as follows:

$$\begin{aligned} \frac{d}{dt} \left(\frac{v(t)}{R} + \frac{1}{L} \int v(t)dt + C \frac{dv(t)}{dt} \right) &= 0 \\ \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) + C \frac{d^2 v(t)}{dt^2} &= 0 \end{aligned}$$

Sorting the equation based on the order of derivative results in:

$$C \frac{d^2 v(t)}{dt^2} + \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) = 0$$

Dividing by the coefficient of the highest-order differential term $C \frac{d^2 v(t)}{dt^2}$ (in this Eq. C) results in a *monic* polynomial. Making the polynomial monic results in:

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = 0$$

The circuit has resulted in a second-order differential equation, which was expected, because the circuit was a second-order circuit. A general solution of this linear second-order differential equation contains terms $v(t) = e^{\lambda t}$ with two values for the λ . (Likewise, in a third-order circuit, it is expected to have three values for λ). Replacing this general solution in the equation helps finding the two values for λ as follows:

$$\frac{d^2 (e^{\lambda t})}{dt^2} + \frac{1}{RC} \frac{d(e^{\lambda t})}{dt} + \frac{1}{LC} e^{\lambda t} = 0$$

Replacing the derivatives $\frac{d(e^{\lambda t})}{dt} = \lambda e^{\lambda t}$ and $\frac{d^2 (e^{\lambda t})}{dt^2} = \lambda^2 e^{\lambda t}$ in the equation results in:

$$\lambda^2 e^{\lambda t} + \frac{1}{RC} \lambda e^{\lambda t} + \frac{1}{LC} e^{\lambda t} = 0$$

Factoring the exponential term $e^{\lambda t}$ out results in:

$$e^{\lambda t} \left(\lambda^2 + \frac{1}{RC} \lambda + \frac{1}{LC} \right) = 0$$

Since λ has physical limitations and cannot reach $-\infty$, then $e^{\lambda t} \neq 0$. Therefore, in a parallel *RLC* circuit:

$$\lambda^2 + \frac{1}{RC} \lambda + \frac{1}{LC} = 0$$

This is also called the *characteristics equation*.

The roots of the characteristics equation λ_1 and λ_2 determine the $v(t)$ response. Consider:

$$\alpha = \frac{1}{2RC}$$

and

$$\omega_0^2 = \frac{1}{LC}$$

where α is the damping factor and ω_0 ($\frac{\text{rad}}{\text{s}}$) is the resonant frequency. Therefore, the characteristics equation can be written as:

$$\lambda^2 + 2\alpha\lambda + \omega_0^2 = 0.$$

This quadratic equation has two roots as λ_1 and λ_2 . These roots are obtained as follows:

$$\lambda_{1,2} = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The value of $(\alpha^2 - \omega_0^2)$ might be positive, zero, or negative. In each case, the value of the roots changes which ultimately changes the $v(t)$ response.

(a) If $\alpha^2 - \omega_0^2 > 0$, there are two distinct real roots as $\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$. The response is *overdamped* and becomes:

$$v(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Initial conditions must be used to find A_1 and A_2 . Since two parameters are required to be determined, two equations must be formed.

Considering the initial voltage of the capacitor, one of the equations can be found as follows:

$$v(t=0) = V_0$$

$$v(t=0) = A_1 + A_2 = V_0.$$

Also evaluating the KCL $i_R(t) + i_L(t) + i_C(t) = 0$ at time $t = 0$ results in:

$$i_R(t=0) + i_L(t=0) + i_C(t=0) = 0$$

Since the initial voltage of the capacitor and initial current of the inductor are known, this results in:

$$\frac{V_0}{R} + I_0 + C \frac{dv(0)}{dt} = 0$$

This equation can be used to determine the second condition as:

$$\frac{dv(0)}{dt} = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$$

Therefore, the derivative of the $v(t)$ at time $t = 0$ must hold. Therefore:

$$\begin{aligned} \frac{dv(t=0)}{dt} &= \frac{d(A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t})}{dt} \bigg|_{t=0} = (A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t}) \bigg|_{t=0} \\ &= A_1 \lambda_1 + A_2 \lambda_2 \end{aligned}$$

Hence, the second equation is

$$A_1 \lambda_1 + A_2 \lambda_2 = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$$

A_1 and A_2 can be found.

- (b) If $\alpha^2 - \omega_0^2 = 0$, there are two equal real roots $\lambda_1 = \lambda_2 = -\alpha$. The response is *critically damped* and becomes:

$$v(t) = B_1 t e^{-\alpha t} + B_2 e^{-\alpha t}$$

Initial conditions must be used to find B_1 and B_2 . Since two parameters are required to be determined, two equations must be formed as well.

First equation is formed from the initial voltage of the capacitor as follows:

$$v(t=0) = V_0$$

$$v(t=0) = B_1 = V_0.$$

Writing a KCL at time $t = 0$ results in:

$$i_R(t=0) + i_L(t=0) + i_C(t=0) = 0$$

Since the initial voltage of the capacitor and initial current of the inductor are known, this results in:

$$\frac{V_0}{R} + I_0 + C \frac{dv(0)}{dt} = 0$$

This equation provides the second condition as:

$$\frac{dv(0)}{dt} = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$$

Derivative of the $v(t)$ at time $t = 0$ must hold. Therefore:

$$\begin{aligned}\frac{dv(t=0)}{dt} &= \left. \frac{d(B_1te^{-\alpha t} + B_2e^{-\alpha t})}{dt} \right|_{t=0} \\ &= (-\alpha B_1te^{-\alpha t} + B_1e^{-\alpha t} - \alpha B_2e^{-\alpha t}) \Big|_{t=0} = B_1 - \alpha B_2\end{aligned}$$

Math Reminder: $\frac{d(f_1(t)f_2(t))}{dt} = \frac{df_1(t)}{dt}f_2(t) + f_1(t)\frac{df_2(t)}{dt}$.

Therefore:

$$\begin{aligned}B_1 - \alpha B_2 &= V_0 - \alpha B_2 = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right) \\ B_2 &= \frac{1}{\alpha} \left(V_0 \left(1 + \frac{1}{RC} \right) + \frac{I_0}{C} \right)\end{aligned}$$

(c) If $\alpha^2 - \omega_0^2 < 0$, there are two complex conjugate roots. Considering the damping frequency ω_d as $\omega_d^2 = \omega_0^2 - \alpha^2$, then $-(\omega_0^2 - \alpha^2) = -\omega_d^2 > 0$. Therefore, $\lambda_1 = -\alpha + j\omega_d$, $\lambda_2 = -\alpha - j\omega_d$. The response is *underdamped* and becomes:

$$v(t) = C_1e^{(-\alpha+j\omega_d)t} + C_2e^{(-\alpha-j\omega_d)t}$$

Expanding the exponential functions, the response becomes:

$$v(t) = e^{-\alpha t}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

Initial conditions must be used to find C_1 and C_2 . Since two parameters are required to be determined, two equations must be formed as well.

First equation is formed from the initial voltage of the capacitor as follows:

$$\begin{aligned}v(t=0) &= V_0 \\ v(t=0) &= C_1 = V_0.\end{aligned}$$

Writing a KCL at time $t = 0$ results in:

$$i_R(t=0) + i_L(t=0) + i_C(t=0) = 0$$

Since the initial voltage of the capacitor and current of the inductor are known, this results in:

$$\frac{V_0}{R} + I_0 + C \frac{dv(0)}{dt} = 0$$

This equation provides the second condition as:

$$\frac{dv(0)}{dt} = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$$

Derivative of the $v(t)$ at time $t = 0$ must hold. Therefore:

$$\left. \frac{dv(t=0)}{dt} = \frac{de^{-\alpha t}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))}{dt} \right|_{t=0}$$

$$\begin{aligned} \left. \frac{dv(t)}{dt} = \left(-\alpha e^{-\alpha t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)) + e^{-\alpha t} (-C_1 \omega_d \sin(\omega_d t) + C_2 \omega_d \cos(\omega_d t)) \right) \right|_{t=0} \\ = -C_1 \alpha + C_2 \omega_d \end{aligned}$$

Therefore:

$$-C_1 \alpha + C_2 \omega_d = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$$

C_1 and C_2 can be found.

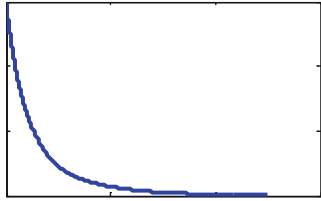
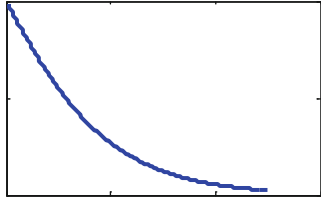
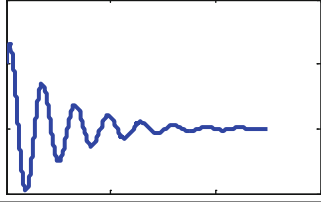
Summary of RLC Parallel Circuit

Note 4.1 In summary, the values of R , L , and C will determine α and ω_0 . The sign of $\alpha^2 - \omega_0^2$ determines the type of response.

Note 4.2 The response in Overdamped circuits indicates that the raise of output voltage has a slow growth and has been damped so much that the oscillations are eliminated.

Note 4.3 The response in Critically damped circuits is the limit of damping at which the system response starts to oscillate by epsilon decrement of damping factor with respect to the resonant frequency i.e. first signs of oscillations are about to start.

Note 4.4 The response in Underdamped circuits starts to show the damped oscillations. The sign of oscillations in an underdamped circuit is the existence of the first peak.

$\alpha^2 - \omega_0^2 > 0$	Overdamped	$v(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ $A_1 + A_2 = V_0$ $A_1 \lambda_1 + A_2 \lambda_2 = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$	
$\alpha^2 - \omega_0^2 = 0$	Critically damped	$v(t) = B_1 t e^{-\alpha t} + B_2 e^{-\alpha t}$ $B_1 = V_0$ $B_2 = \frac{1}{\alpha} \left(V_0 \left(1 + \frac{1}{RC} \right) + \frac{I_0}{C} \right)$	
$\alpha^2 - \omega_0^2 < 0$	Underdamped	$v(t) = e^{-\alpha t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$ $C_1 = V_0$ $-C_1 \alpha + C_2 \omega_d = \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right)$	

Example 4.7 Consider an RLC circuit in parallel as shown in Fig. 4.34. At $R = 25 \, \Omega$, $L = 10 \, \text{mH}$, and $C = 1 \, \mu\text{F}$, find the voltage response $v(t)$ if the initial charge of capacitor is $V_0 = 150 \, \text{V}$ and the initial current of the inductor $I_0 = 4 \, \text{A}$.

Solution In a parallel RLC circuit, the voltage response can be obtained using the characteristics equation $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$. Considering the circuit element values, the characteristics equation becomes:

$$\lambda^2 + \frac{1}{25 \times 1e-6} \lambda + \frac{1}{10m \times 1e-6} = 0$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 25 \times 1e-6} = 20,000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10e-3 \times 1e-6}} = 10,000 \, \text{rad/s}$$

These values indicate that $\alpha^2 - \omega_0^2 > 0$; therefore the system is *overdamped*. The response according to the table becomes:

$$v(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

where $\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -20000 \pm \sqrt{20000^2 - 10000^2} = -20000 \pm 17320.5 = -2679.5 \text{ rad/s} \text{ \& } -37320.5 \text{ rad/s}$.

To find the constants A_1 and A_2 , the following equations can be used:

$$\begin{aligned} A_1 + A_2 &= V_0, \rightarrow A_1 + A_2 = 150 \\ A_1\lambda_1 + A_2\lambda_2 &= \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right) \rightarrow -2679.5A_1 - 37320.5A_2 = \frac{-1}{1e-6} \left(\frac{150}{25} + 4 \right) \\ &= -10e6 \end{aligned}$$

This results in $A_1 = -127.07 \text{ V}$ and $A_2 = 277.072 \text{ V}$. Hence,

$$v(t) = -127.07e^{-2679.5t} + 277.072e^{-37320.5t} \text{ (V)}$$

Example 4.8 Consider the previous example when the resistor is adjusted to $R = 100 \Omega$. Find the damping coefficient and resonant frequency and the voltage response $v(t)$.

Solution In a parallel RLC circuit, the voltage response can be obtained using the characteristics equation $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$. Considering the circuit element values, the characteristics equation becomes:

$$\begin{aligned} \lambda^2 + \frac{1}{100 \times 1e-6}\lambda + \frac{1}{10m \times 1e-6} &= 0 \\ \alpha = \frac{1}{2RC} &= \frac{1}{2 \times 100 \times 1e-6} = 5,000 \\ \omega_0 = \frac{1}{\sqrt{LC}} &= \frac{1}{\sqrt{10e-3 \times 1e-6}} = 10,000 \text{ rad/s} \end{aligned}$$

These values indicate that $\alpha^2 - \omega_0^2 < 0$; therefore, the system is underdamped. This means the response would oscillate and the amplitudes of oscillations are damped.

The damping frequency $\omega_d^2 = \omega_0^2 - \alpha^2$ can be obtained as $\omega_d^2 = 10000^2 - 5000^2$ and $\omega_d = 8660.2 \text{ rad/s}$.

The response according to the table becomes:

$$\begin{aligned} v(t) &= e^{-\alpha t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)) \\ v(t) &= e^{-5000t} (C_1 \cos(8660.2t) + C_2 \sin(8660.2t)) \\ C_1 &= V_0 = 150 \text{ V} \end{aligned}$$

$$\begin{aligned}
 -C_1\alpha + C_2\omega_d &= \frac{-1}{C} \left(\frac{V_0}{R} + I_0 \right) \\
 -150 \times 5000 + C_2 8660.2 &= \frac{-1}{1e-6} \left(\frac{150}{100} + 4 \right) \\
 C_2 &= -721.69 \text{ V}
 \end{aligned}$$

Therefore the answer is:

$$v(t) = e^{-5000t} (150 \cos(8660.2t) - 721.69 \sin(8660.2t)) \text{ (V)}$$

Example 4.9 In the previous example, consider an unknown resistance value. Adjust the resistance to obtain a critically damped circuit, and using the initial conditions, find the system response.

Solution To have a critical damping $\alpha^2 - \omega_0^2 = 0$. Therefore, $\left(\frac{1}{2RC}\right)^2 = \left(\frac{1}{\sqrt{LC}}\right)^2$

$$\left(\frac{1}{2R \times 1e-6}\right)^2 = \left(\frac{1}{\sqrt{10e-3 \times 1e-6}}\right)^2$$

$$\frac{1}{2R \times 1e-6} = 10000 \rightarrow R = 50 \text{ } \Omega$$

In critical damping $\alpha = \omega_0 = 10000 \text{ rad/s}$

$$v(t) = B_1 t e^{-\alpha t} + B_2 e^{-\alpha t}$$

$$v(t) = B_1 t e^{-10000t} + B_2 e^{-10000t}$$

$$B_1 = V_0 = 150 \text{ V}$$

$$B_2 = \frac{1}{\alpha} \left(V_0 \left(1 + \frac{1}{RC} \right) + \frac{I_0}{C} \right) = \frac{1}{10000} \left(150 \left(1 + \frac{1}{50 \times 1e-6} \right) + \frac{4}{1e-6} \right) = 70.015 \text{ V}$$

$$v(t) = 150 t e^{-10000t} + 70.015 e^{-10000t} \text{ (V)}$$

Example 4.10 Consider the RLC parallel circuit in the past three examples. Knowing the voltage response $v(t)$ find the current of each element in case where $R = 25 \text{ } \Omega$, $R = 100 \text{ } \Omega$, $R = 500 \text{ } \Omega$.

Solution Ohm's law should be imposed.

The current of resistors are:

$$\text{When } R = 25 \text{ } \Omega, \text{ over-damp } i(t) = \frac{v(t)}{R} = \frac{-127.07e^{-2679.5t} + 277.072e^{-37320.5t}}{25} \text{ A}$$

$$\text{When } R = 100 \text{ } \Omega, \text{ under-damp } i(t) = \frac{v(t)}{R} = \frac{e^{-5000t} (150 \cos(8660.2t) - 721.69 \sin(8660.2t))}{100} \text{ A}$$

$$\text{When } R = 500 \text{ } \Omega, \text{ critically-damp } i(t) = \frac{v(t)}{R} = \frac{150 t e^{-10000t} + 430.015 e^{-10000t}}{500} \text{ A}$$

The current of inductor is:

$$i_L(t) = \frac{1}{L} \int v(t) dt$$

When $R = 25 \, \Omega$, over-damp:

$$\begin{aligned} i_L(t) &= \frac{1}{10e-3} \int (-127.07e^{-2679.5t} + 277.072e^{-37320.5t}) dt \\ &= 100 \left(\frac{-127.07}{-2679.5} e^{-2679.5t} + \frac{277.072}{-37320.5} e^{-37320.5t} \right) \\ i_L(t) &= (4.71e^{-2679.5t} - 0.74e^{-37320.5t}) \text{ A} \end{aligned}$$

When $R = 100 \, \Omega$, under-damp:

$$\begin{aligned} i_L(t) &= \frac{1}{10e-3} \int (e^{-5000t} (150 \cos(8660.2t) - 721.69 \sin(8660.2t))) dt \\ &= (e^{-5000t} (6.202 \cos(8660.2t) - 29.84 \sin(8660.2t))) \text{ A} \end{aligned}$$

When $R = 500 \, \Omega$, critically-damp:

$$\begin{aligned} i_L(t) &= \frac{1}{10e-3} \int (150te^{-10000t} + 430.015e^{-10000t}) dt \\ i_L(t) &= 6.202te^{-10000t} + 17.78e^{-10000t} \text{ A} \end{aligned}$$

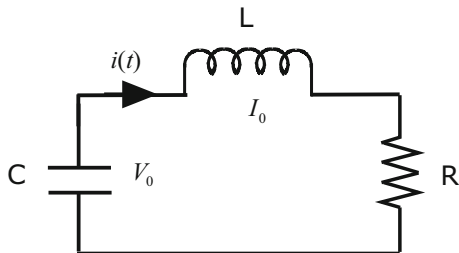
The current of the capacitor is:

$$i_C(t) = C \frac{dv(t)}{dt}$$

When $R = 25 \, \Omega$, over-damp:

$$\begin{aligned} i_C(t) &= (1e-6) \frac{d}{dt} (-127.07e^{-2679.5t} + 277.072e^{-37320.5t}) \\ i_C(t) &= (1e-6) (-127.07 \times -2679.5e^{-2679.5t} + 277.072 \times -37320.5e^{-37320.5t}) \\ i_C(t) &= (0.34e^{-2679.5t} - 10.34e^{-37320.5t}) \text{ A} \end{aligned}$$

When $R = 100 \, \Omega$, under-damp:

Fig. 4.35 *RLC series circuit*

$$i_C(t) = (1e - 6) \frac{d}{dt} (e^{-5000t} (150 \cos(8660.2t) - 721.69 \sin(8660.2t)))$$

$$i_C(t) = (1e - 6) (-5000e^{-5000t} (150 \cos(8660.2t) - 721.69 \sin(8660.2t)) + e^{-5000t} (150 \times -8660.2 \sin(8660.2t) - 721.69 \times 8660.2 \cos(8660.2t)))$$

$$i_C(t) = (e^{-5000t} (-7 \cos(8660.2t) + 2.318 \sin(8660.2t))) \text{A}$$

When $R = 500 \, \Omega$, critically-damp:

$$i_C(t) = (1e - 6) \frac{d}{dt} (150te^{-10000t} + 430.015e^{-10000t})$$

$$i_C(t) = (1e - 6) (150e^{-10000t} - 150 \times 10000te^{-10000t} + 430.015 \times -10000e^{-10000t})$$

$$i_C(t) = (1e - 6) (150e^{-10000t} - 150 \times 10000te^{-10000t} + 430.015 \times -10000e^{-10000t})$$

$$i_C(t) = (4.3e^{-10000t} - 1.5te^{-10000t}) \text{A}$$

Natural Response of RLC Series Circuits

Consider a circuit consisting of a loop of R , L , and C where the initial charge of capacitor is V_0 and the initial charge of inductor is I_0 . The circuit is shown in Fig. 4.35. The loop has current $i(t)$ that drops a voltage across each element.

KVL in this loop indicates:

$$v_R(t) + v_L(t) + v_C(t) = 0$$

Ohm's law indicates the voltage drop across each element as follows:

$$R i(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt = 0$$

The KVL equation results in an integrodifferential equation over $i(t)$. To solve for current, there is a need to take one time differential from the equation because there exists single integral in the equation.

Taking one time differential from the equation results in:

$$\frac{d}{dt} \left(R i(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt = 0 \right)$$

$$R \frac{di(t)}{dt} + L \frac{d^2 i(t)}{dt^2} + C i(t) = 0$$

Sorting the equation based on the order of derivative results in:

$$L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} + C i(t) = 0$$

Dividing by the coefficient of the highest-order differential term (in this eq. L) results in a *monic* polynomial. Dividing the characteristics equation by L results in:

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0$$

The circuit has resulted in a second-order differential equation which was expected because the circuit was second order (two energy-storing elements). A general solution of this linear second-order differential equation is $i(t) = e^{\lambda t}$ with two roots for the variable λ . To find the roots, the general solution needs to be replaced into the equation which results in:

$$\frac{d^2 (e^{\lambda t})}{dt^2} + \frac{R}{L} \frac{d(e^{\lambda t})}{dt} + \frac{1}{LC} e^{\lambda t} = 0$$

Considering the derivatives $\frac{d(e^{\lambda t})}{dt} = \lambda e^{\lambda t}$ and $\frac{d^2 (e^{\lambda t})}{dt^2} = \lambda^2 e^{\lambda t}$ and replacing these in the differential equation results in:

$$\lambda^2 e^{\lambda t} + \frac{R}{L} \lambda e^{\lambda t} + \frac{1}{LC} e^{\lambda t} = 0$$

Factoring the exponential term out results in:

$$e^{\lambda t} \left(\lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} \right) = 0$$

Since λ has physical limitations and cannot reach $-\infty$, then $e^{\lambda t} \neq 0$. Therefore, characteristics equation needs to be zero:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

The roots of this equation determine templates of $i(t)$ response. Considering:

$$\alpha = \frac{R}{2L}$$

and

$$\omega_0^2 = \frac{1}{LC}$$

where α is the damping factor and $\omega_0 \left(\frac{\text{rad}}{\text{s}} \right)$ is the resonant frequency. Therefore, the characteristics equation can be written as:

$$\lambda^2 + 2\alpha\lambda + \omega_0^2 = 0.$$

This quadratic equation has two roots as λ_1 and λ_2 . These roots are obtained as follows:

$$\lambda_{1,2} = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The value of $(\alpha^2 - \omega_0^2)$ might be positive, zero, or negative. In each case, the value of the roots changes which ultimately changes the $i(t)$ response.

- (a) If $\alpha^2 - \omega_0^2 > 0$, there are two distinct real roots for $\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$. The response is *overdamped* and becomes:

$$i(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Initial conditions must be used to find A_1 and A_2 . Since two parameters are required to be determined, two equations must be formed as well.

First equation is formed from the initial current of inductor as follows:

$$i(t = 0) = I_0$$

$$i(t = 0) = A_1 + A_2 = I_0.$$

Writing a KVL at time $t = 0$ results in:

$$v_R(t=0) + v_L(t=0) + v_C(t=0) = 0$$

Since the initial voltage and current are known, this results in:

$$RI_0 + L \frac{di(0)}{dt} + CV_0 = 0$$

This equation provides the second condition as:

$$\frac{di(0)}{dt} = \frac{-1}{L} (RI_0 + V_0)$$

Derivative of the $i(t)$ at time $t = 0$ must hold. Therefore:

$$\begin{aligned} \frac{di(t=0)}{dt} &= \left. \frac{d(A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t})}{dt} \right|_{t=0} = (A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t}) \Big|_{t=0} \\ &= A_1 \lambda_1 + A_2 \lambda_2 \end{aligned}$$

Therefore:

$$A_1 \lambda_1 + A_2 \lambda_2 = \frac{-1}{L} (RI_0 + V_0)$$

A_1 and A_2 can be found.

- (b) If $\alpha^2 - \omega_0^2 = 0$, there are two real repeated roots $\lambda_1 = \lambda_2 = -\alpha$. The response is *critically damped* and becomes:

$$i(t) = B_1 t e^{-\alpha t} + B_2 e^{-\alpha t}$$

Initial conditions must be used to find B_1 and B_2 . Since two parameters are required to be determined, two equations must be formed as well.

The first equation is formed from the initial current of inductor as follows:

$$i(t=0) = I_0$$

$$i(t=0) = B_1 = I_0.$$

Writing a KVL at time $t = 0$ results in:

$$v_R(t=0) + v_L(t=0) + v_C(t=0) = 0$$

Since the initial voltage of the capacitor and current of the inductor are known, this results in:

$$RI_0 + L \frac{di(0)}{dt} + CV_0 = 0$$

This equation provides the second condition as:

$$\frac{di(0)}{dt} = \frac{-1}{L} (RI_0 + V_0)$$

Derivative of $i(t)$ at time $t = 0$ must hold. Therefore:

$$\begin{aligned} \frac{di(t=0)}{dt} &= \left. \frac{d(B_1 t e^{-\alpha t} + B_2 e^{-\alpha t})}{dt} \right|_{t=0} \\ &= (-\alpha B_1 t e^{-\alpha t} + B_1 e^{-\alpha t} - \alpha B_2 e^{-\alpha t}) \Big|_{t=0} = B_1 - \alpha B_2. \\ \frac{d(f_1(t)f_2(t))}{dt} &= \frac{df_1(t)}{dt} f_2(t) + f_1(t) \frac{df_2(t)}{dt}. \end{aligned}$$

Therefore:

$$\begin{aligned} B_1 - \alpha B_2 &= I_0 - \alpha B_2 = \frac{-1}{L} (RI_0 + V_0) \\ B_2 &= \frac{1}{\alpha} \left(I_0 \left(1 + \frac{R}{L} \right) + \frac{V_0}{L} \right) \end{aligned}$$

- (c) If $\alpha^2 - \omega_0^2 < 0$, there are two complex conjugate roots. Consider $\omega_d^2 = \omega_0^2 - \alpha^2$, then $-(\omega_0^2 - \alpha^2) = -\omega_d^2 > 0$. Therefore, $\lambda_1 = -\alpha + j\omega_d$, $\lambda_2 = -\alpha - j\omega_d$. The response is *underdamped* and becomes:

$$i(t) = C_1 e^{(-\alpha + j\omega_d)t} + C_2 e^{(-\alpha - j\omega_d)t}$$

Expanding the exponential functions, the response becomes:

$$i(t) = e^{-\alpha t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

Initial conditions must be used to find C_1 and C_2 . Since two parameters are required to be determined, two equations must be formed as well.

The first equation is formed from the initial current of inductor as follows:

$$\begin{aligned} i(t=0) &= I_0 \\ i(t=0) &= C_1 = I_0. \end{aligned}$$

Writing a KVL at time $t = 0$ results in:

$$v_R(t = 0) + v_L(t = 0) + v_C(t = 0) = 0$$

Since the initial voltage of the capacitor and current of inductor are known, this results in:

$$RI_0 + L \frac{di(0)}{dt} + CV_0 = 0$$

This equation provides the second condition as:

$$\frac{di(0)}{dt} = \frac{-1}{L}(RI_0 + V_0)$$

Derivative of the $i(t)$ at time $t = 0$ must hold. Therefore:

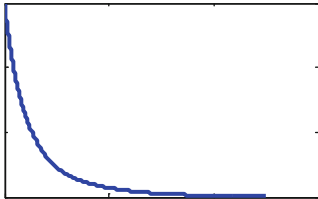
$$\begin{aligned} \frac{di(t = 0)}{dt} &= \left. \frac{de^{-\alpha t}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))}{dt} \right|_{t=0} \\ &= (-\alpha e^{-\alpha t}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)) \\ &\quad + e^{-\alpha t}(-C_1 \omega_d \sin(\omega_d t) + C_2 \omega_d \cos(\omega_d t))) \Big|_{t=0} \\ &= -C_1 \alpha + C_2 \omega_d \end{aligned}$$

Therefore:

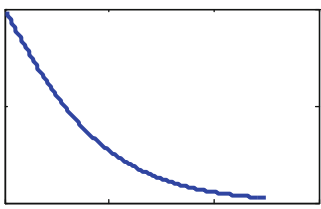
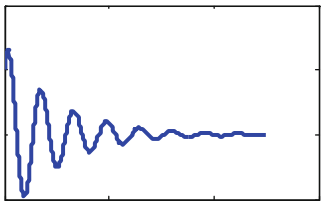
$$-C_1 \alpha + C_2 \omega_d = \frac{-1}{L}(RI_0 + V_0)$$

C_1 and C_2 can be found.

Summary of RLC Series Circuit

$\alpha^2 - \omega_0^2 > 0$	Overdamped	$i(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ $A_1 + A_2 = I_0$ $A_1 \lambda_1 + A_2 \lambda_2 = \frac{-1}{L}(RI_0 + V_0)$	

(continued)

$\alpha^2 - \omega_0^2 = 0$	Critically damped	$i(t) = B_1 t e^{-\alpha t} + B_2 e^{-\alpha t}$ $B_1 = I_0$ $B_2 = \frac{1}{\alpha} \left(I_0 \left(1 + \frac{R}{L} \right) + \frac{V_0}{L} \right)$	
$\alpha^2 - \omega_0^2 < 0$	Underdamped	$i(t) = e^{-\alpha t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$ $C_1 = I_0$ $-C_1 \alpha + C_2 \omega_d = \frac{-1}{L} (R I_0 + V_0)$	

Problems

4.1 Find the voltage across a $10 \, \Omega$ resistor if the current flowing through is:

- (a) $i(t) = 10u(t)$
- (b) $i(t) = 10tu(t)$
- (c) $i(t) = 200 \sin(60\pi t + 10)$
- (d) $i(t) = t^2 u(t)$
- (e) $i(t) = e^{-3t} \sin 10\pi t$

4.2 Find the current of a $100 \, \Omega$ resistor if the applied voltage is as follows:

- (a) $v(t) = 120\sqrt{2} \sin(100\pi t + 10)$
- (b) $v(t) = u(t) + u(t - 2) - 2u(t - 3)$
- (c) $v(t) = 10tu(t) - 20(t - 1)u(t - 1) + 10(t - 3)u(t - 3)$
- (d) $v(t) = e^{-3t} \sin 100\pi t$
- (e) $v(t) = 100te^{-10t}u(t)$

4.3 Find the voltage induced across a $L = 100 \, \text{mH}$ inductor when the current is:

- (a) $i(t) = 10u(t)$
- (b) $i(t) = 10tu(t)$
- (c) $i(t) = 200 \sin(60\pi t + 10)$
- (d) $i(t) = t^2 u(t)$
- (e) $i(t) = e^{-3t} \sin 10\pi t$

4.4 Find the current through a $L = 100 \, \text{mH}$ inductor if the voltage applied across it is:

- (a) $v(t) = 120\sqrt{2} \sin(100\pi t + 10)$
- (b) $v(t) = u(t) + u(t - 2) - 2u(t - 3)$
- (c) $v(t) = 10tu(t) - 20(t - 1)u(t - 1) + 10(t - 3)u(t - 3)$
- (d) $v(t) = e^{-3t} \sin 100\pi t$
- (e) $v(t) = 100te^{-10t}u(t)$

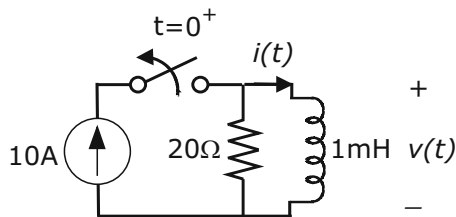
4.5 Find the voltage across a $C = 100 \mu\text{F}$ capacitor when the current is:

- (a) $i(t) = 10u(t)$
- (b) $i(t) = 10tu(t)$
- (c) $i(t) = 200 \sin(60\pi t + 10)$
- (d) $i(t) = t^2 u(t)$
- (e) $i(t) = e^{-3t} \sin 10\pi t$

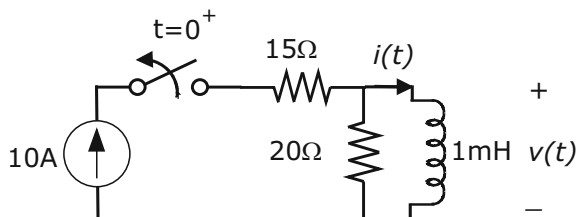
4.6 Find the current through a $C = 100 \mu\text{F}$ capacitor if the voltage applied across it is:

- (a) $v(t) = 120\sqrt{2} \sin(100\pi t + 10)$
- (b) $v(t) = u(t) + u(t - 2) - 2u(t - 3)$
- (c) $v(t) = 10tu(t) - 20(t - 1)u(t - 1) + 10(t - 3)u(t - 3)$
- (d) $v(t) = e^{-3t} \sin 100\pi t$
- (e) $v(t) = 100te^{-10t}u(t)$

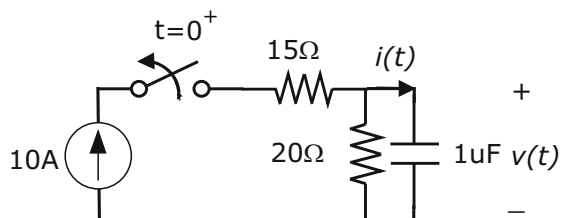
4.7 The switch has been closed for long time. At time $t = 0^+$, it is opened. Find $v(t)$, $i(t)$.



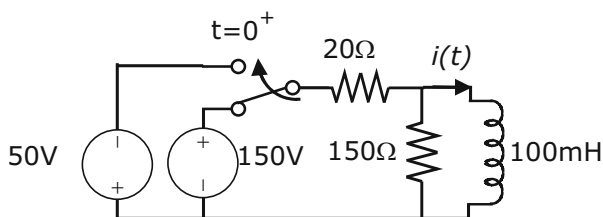
4.8 The switch has been closed for long time. At time $t = 0^+$, it is opened. Find $v(t)$, $i(t)$.



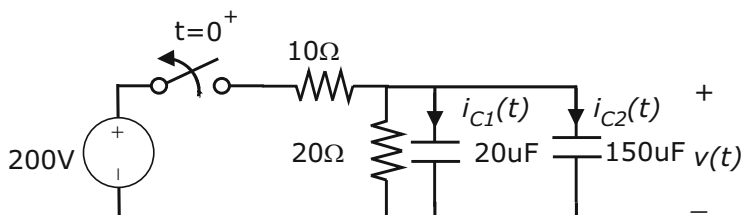
- 4.9 The switch has been closed for long time. At time $t = 0^+$, it is opened. Find $v(t)$, $i(t)$.



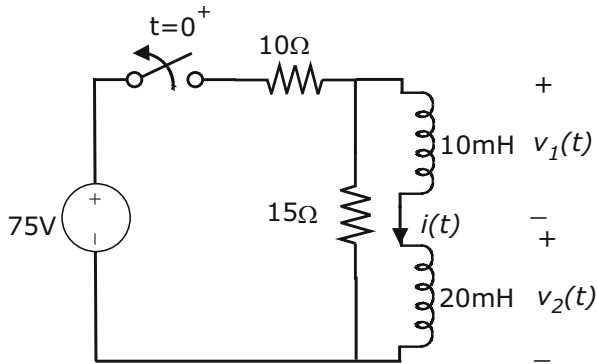
- 4.10 The switch has been closed for long time. At time $t = 0^+$ it is opened. Find $v(t)$, $i(t)$.



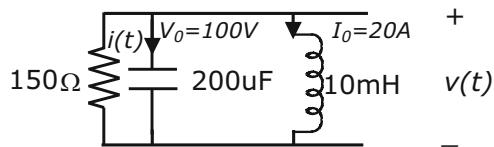
- 4.11 The switch has been closed for long time. At time $t = 0^+$, it is opened. Find $v(t)$, $i_{C1}(t)$, $i_{C2}(t)$.



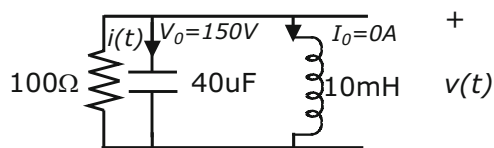
- 4.12 The switch has been closed for long time. At time $t = 0^+$ it is opened. Find $i(t)$, $v_1(t)$, $v_2(t)$.



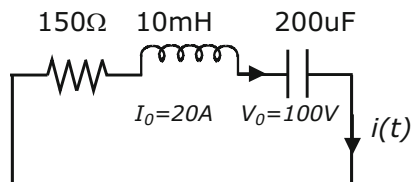
- 4.13 Find the characteristics equations, characteristics roots, damping conditions, and the response in the following circuit.



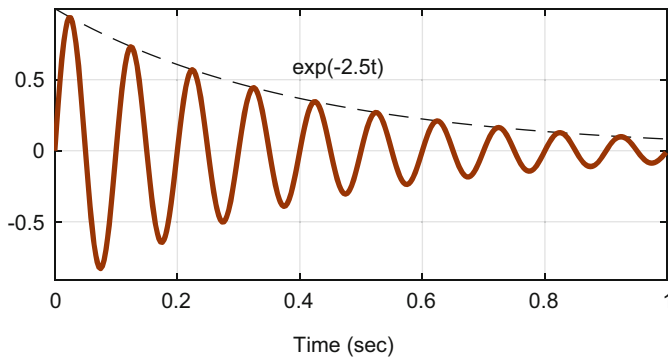
- 4.14 Find the characteristics equations, characteristics roots, damping conditions, and the response in the following circuit.



- 4.15 Find the characteristics equations, characteristics roots, damping conditions, and the response in the following circuit.



- 4.16 Characteristics roots of a parallel RLC circuit are $\lambda_{1,2} = -1000 \pm j5000 \frac{\text{rad}}{\text{s}}$. Find the system's natural voltage response, if the initial conditions are $I_0 = -25 \text{ A}$, $V_0 = 150 \text{ V}$.
- 4.17 Characteristics roots of a parallel RLC circuit are $\lambda_1 = -1000 \frac{\text{rad}}{\text{s}}$, $\lambda_2 = -5000 \frac{\text{rad}}{\text{s}}$. Find the system voltage response if the circuit has initial conditions as $V_0 = 100 \text{ V}$, $\frac{d}{dt}v(0^+) = -12000 \frac{\text{V}}{\text{s}}$.
- 4.18 Characteristics roots of a series RLC circuit are $\lambda_{1,2} = -1000 \pm j15000 \frac{\text{rad}}{\text{s}}$. Find the system current response, if the initial conditions are $I_0 = -250 \text{ A}$, $V_0 = -200 \text{ V}$.
- 4.19 The response of a parallel RLC circuit recorded from the oscilloscope is as follows. Find the characteristics roots and characteristics equations.



- 4.20 Natural responses of some RLC circuits are as follows. Find the characteristics roots, characteristics equations, and initial conditions of the circuit.
- $i(t) = 150e^{-200t} \cos 1000t$
 - $v(t) = 100e^{-2000t} \sin(30000t + 30)$
 - $i(t) = 200e^{-300t} + 150e^{-120t}$
 - $v(t) = 200te^{-2000t} + 20e^{-2000t}$
- 4.21 Design an RLC series circuit such that the natural current response becomes:

$$i(t) = 100e^{-100t} \cos 750t \text{ A}$$

Select the values of R , L , and C and the initial conditions that result in the desired response.

Note that here might be multiple solutions for this design. Therefore, select the range to be no smaller than mH, μF in the inductor and capacitor.

Chapter 5

Steady-State Sinusoidal Circuit Analysis



Introduction

Sinusoidal waveforms as explained in Chap. 3 have an amplitude r , a frequency ω , and a phase shift or phase angle ϕ and are expressed as:

$$f(t) = r \sin(\omega t + \theta)$$

The same function can be presented as a phasor. The phasor conveys important information regarding a signal, its *amplitude* and *phase angle*, considering that the frequency throughout the operation is fixed. The amplitude and phase information resemble polar coordinates. A polar coordinate can be transformed into rectangle coordinates as well. This reciprocal transform can be achieved as follows:

- $R \rightarrow P$. Consider a complex value $a+jb$ in rectangle coordinates of real and imaginary. This value in polar coordinate has an amplitude of $r = \sqrt{a^2 + b^2}$ and an angle of $\theta = \tan^{-1} \frac{b}{a}$.
- $P \rightarrow R$. Consider a number $r \angle \theta$ in polar coordinates. This number in rectangle coordinates has a real axis value of $a = r \cos \theta$ and an imaginary value of $b = r \sin \theta$.

$$r \angle \theta \leftrightarrow r \cos \theta + jr \sin \theta$$

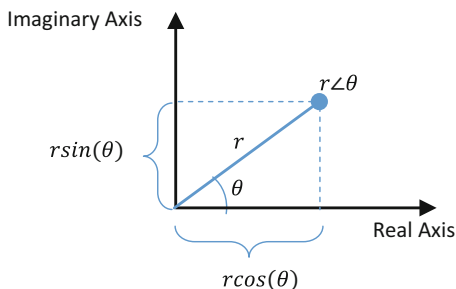
$$r \cos \theta + jr \sin \theta \leftrightarrow re^{j\theta}$$

$$r \angle \theta \leftrightarrow re^{j\theta}$$

The conversion of the rectangle and polar coordinates is shown in Fig. 5.1.

Example 5.1 Consider a sinusoidal voltage source $v(t) = 169.7 \sin(377t + 30^\circ)$ being connected to a circuit. This means that the source voltage alternates from

Fig. 5.1 The polar coordinate of $r \angle \theta$ can be projected on the real and imaginary axes to identify the real and imaginary values in a rectangular coordinate



$+169.7$ to -169.7 V each half cycle. The angular frequency 377 (rad/s) translates into $f(\text{Hz}) = \frac{\omega(\text{rad/s})}{2\pi} = \frac{377}{2\pi} = 60$ Hz. This means that the period of the waveform is $T = \frac{1}{f} = \frac{1}{60} = 16.6$ ms. Each half cycle is 8.3 ms. In a polar coordinate, the voltage will have an amplitude and a phase as $169.7 \angle 30^\circ$.

Example 5.2 Express the function $v(t) = 169.7 \sin(377t + 30^\circ)$ in phasor.

Solution The polar coordinate is $169.7 \angle 30^\circ$. Therefore, the phasor becomes $169.7e^{j30^\circ}$.

Example 5.3 Find the phasor expression of the following numbers:

- (a) $10 + j10$
- (b) $2 + j\sqrt{3}$

Solution

(a)

$$10 + j10 \leftrightarrow \sqrt{10^2 + 10^2} \angle \tan^{-1} \frac{10}{10} = \sqrt{200} \angle 45^\circ \leftrightarrow \sqrt{200} e^{j45^\circ}$$

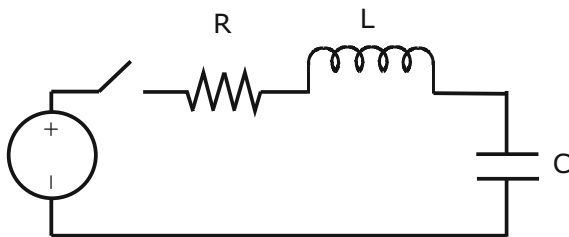
(b)

$$2 + j\sqrt{3} \leftrightarrow \sqrt{2^2 + \sqrt{3}^2} \angle \tan^{-1} \frac{\sqrt{3}}{2} = \sqrt{7} \angle 50.76^\circ \leftrightarrow \sqrt{7} e^{j50.76^\circ}$$

How to Use Phasor in Circuit Analysis

An electric circuit might be excited by a variety of waveforms. The representation of quantities in phasor is useful in that it preserves the frequency information, and it can be independent of the time and the trigonometric functions. The circuit quantities are either voltage, current, or impedance values that all can be expressed in phasor. Among all, impedance of components once excited by a sinusoidal waveform might

Fig. 5.2 Resistive-Inductive-Capacitive or RLC circuit with all components connected in series



be frequency dependent. Therefore, there are some steps to analyze the circuit response (i.e., the voltage drops and current flows) under sinusoidal excitations.

The first step is to determine the stage of the response. This means determining whether the response has reached a stable operation or its amplitude is changing due to the nature of the circuit. The response stages are discussed in details in the next section.

The second step is to identify the equivalent of circuit elements and their impedance when a sinusoidal source at the frequency $\omega = 2\pi f$ is utilized. This is explained in details in this chapter.

The third step is to use the circuit analysis laws such as KVL and KCL, etc. to calculate the circuit values. Several examples are provided in this chapter to show the circuit analysis under sinusoidal excitation.

Circuit Response Stages

Consider an RLC circuit consisting of a sinusoidal source and a switch to connect the circuit to the source at a desired time (Fig. 5.2).

The circuit is considered to have no initial condition, meaning that the capacitors and inductors are fully discharged. The switch is closed at $t = 0^+$ connecting the source to the circuit. The voltage source characteristics suggest that potentially an unlimited amount of current is available to fill the capacitors and inductors while maintaining the voltage. The current $i(t)$ is therefore influenced by the circuit topology and the switching angle that represents the voltage amplitude. In general and depending on the switching time, colliding with the instantaneous amplitude of the sinusoidal waveform, the current response can be divided into three periods:

- First 1–3 cycles show the sub-transient response.
- The next 10–15 cycles show the transient response.
- Once the amplitude settles to a fixed peak value, the steady-state response starts.

Please note that the frequency of the waveform is fixed during the sub-transient, transient, and steady-state responses. However, only their amplitude is different due to the nature of the circuit components. Figure 5.3 shows sub-transient, transient, and steady-state parts of the current response.

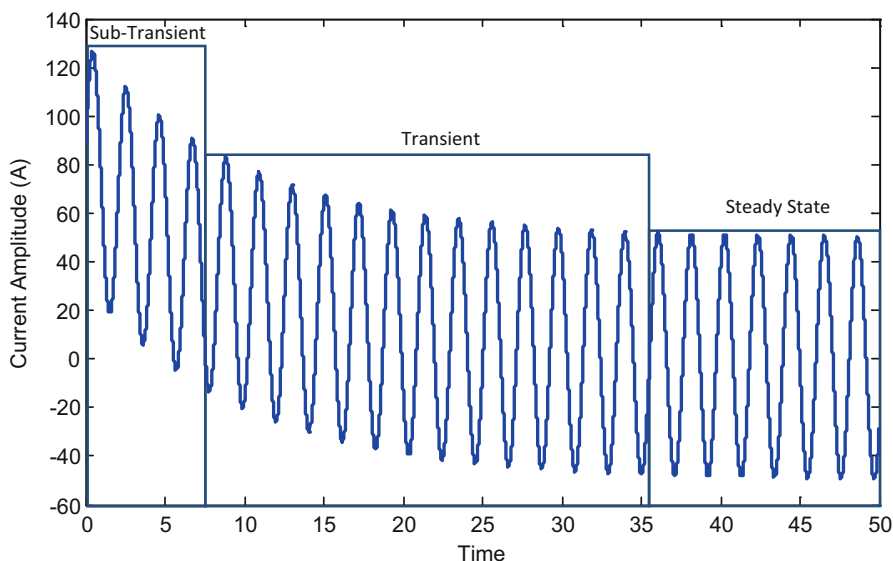


Fig. 5.3 A transient response to a sinusoidal excitation. Three main stages may exist as sub-transient, transient, and steady state

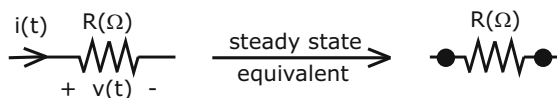


Fig. 5.4 Equivalent of a pure resistor when excited by a sinusoidal source at frequency ω remains a resistor with the same resistance value

This chapter analyzes the system response in steady state, where all the switching transient responses are already damped and a fixed amplitude of voltage and current is reached. Under this condition, the equivalents of resistors, inductors, and capacitors are calculated.

Resistors in Steady State

Consider a voltage source $v(t) = V_m \sin(\omega t + \theta)$ at peak value of V_m and angular frequency of ω (rad/s) is applied across a resistor. The current is obtained according to Ohm's law as follows (Fig. 5.4):

$$v(t) = Ri(t)$$

and

$$i(t) = \frac{v(t)}{R}$$

With the given sinusoidal voltage source, the current becomes:

$$i(t) = \frac{V_m \sin(\omega t + \theta)}{R} = \frac{V_m}{R} \sin(\omega t + \theta) = I_m \sin(\omega t + \theta).$$

where I_m is the current. The current in phasor form I can also be obtained as follows:

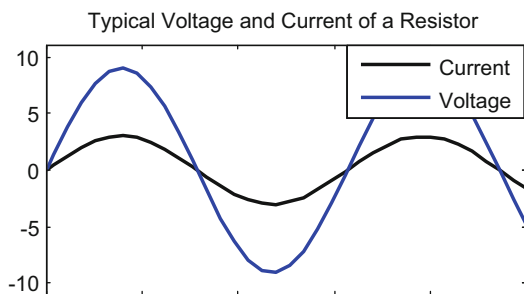
$$I = \frac{V}{R} = \frac{V_m \angle \theta}{R} = \frac{V_m}{R} \angle \theta = I_m \angle \theta.$$

As it can be seen from the voltage and current phasors, resistors cannot change the phase of the current. Both voltage and current had phase angle of θ but showed different amplitudes. A resistor scales the current according to the amount of its resistance. Therefore, a resistor under steady-state condition shows a pure resistive impedance with zero degrees phase shift from voltage to current.

Power Factor of Resistive Circuits

Power factor is defined as the $\cos(\widehat{v, i})$ cos of the phase shift between voltage and current passing through the circuit. Therefore, the power factor can be $0 \leq \text{PF} \leq 1$, where 1 shows a pure resistive circuit and 0 shows a pure inductive or capacitive circuit. Since the phase shift in current with respect to voltage (reference) is zero (Fig. 5.5), the power factor (PF) of a resistive circuit is 1. $\text{PF} = \cos(0) = 1$.

Fig. 5.5 The voltage and current waveforms in a pure resistive circuit. The zero crossings are the same for both the voltage and the current. This means, they are inphase. Only their amplitudes are scaled by the value of the resistance as $V = RI$



Inductors in Steady State

The relation of voltage and current in an inductor is determined by $v(t) = L \frac{di(t)}{dt}$. Considering a sinusoidal current through an inductor under steady-state condition as $i(t) = I_m \sin(\omega t + \theta)$, the voltage drop across the inductor is obtained as follows:

$$\begin{aligned} v(t) &= L \frac{di(t)}{dt} = L \frac{d(I_m \sin(\omega t + \theta))}{dt} = LI_m \omega \cos(\omega t + \theta) \\ &= L\omega I_m \sin(\omega t + \theta + 90^\circ) = L\omega I_m \angle(\theta + 90^\circ) \end{aligned}$$

This shows that the current is lagging the voltage by 90° . This phase shift is shown by a j factor in Fig. 5.6 and is presented in the following equalities.

$$V = LI_m \omega e^{j(\theta + 90^\circ)} = LI_m \omega e^{j\theta} e^{j90^\circ} = j\omega L(I_m e^{j\theta}) = j\omega L(I_m \angle \theta) = j\omega L(I_m \angle \theta)$$

According to these equations, inductors impede the flow of current in the presence of sinusoidal (time-varying) waveform excitations. The impedance of an inductor at inductance L (H) operating at angular frequency of ω (rad/s) is $j\omega L$ measured in (Ω). Phasor representation of Ohm's law for an inductor under steady-state sinusoidal condition is as follows:

$$V = j\omega LI = X_L I$$

It can be concluded that an inductor L (H) in steady-state sinusoidal shows a reactance of $X_L = j\omega L$ (Ω) (Fig. 5.7).

Fig. 5.6 In a pure inductive circuit, the voltage and current are 90° apart. This means that the peak of one occurs at the zero of the other. Since the current is lagging the voltage and considering voltage as a reference, the current occurs with a time delay equal to $t_{\text{delay}} = \frac{\pi}{2} T$ s

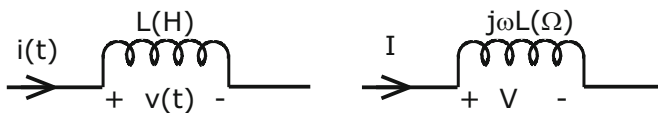
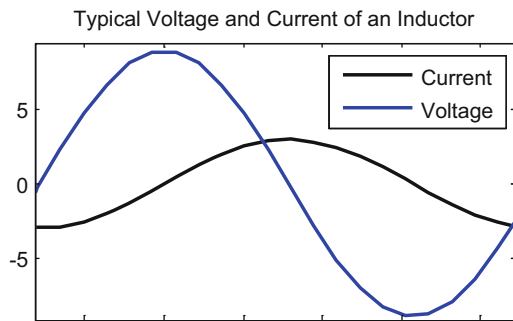


Fig. 5.7 Equivalent of an inductor with inductance L (H) when excited by a sinusoidal source at frequency ω becomes an inductor with impedance $j\omega L$ measured in (Ω)

Note 5.1 The inductance L is measured in Henrys (H), but $j\omega L$ is measured in Ohms (Ω).

Power Factor of Inductive Circuits

Since the phase shift of current with respect to voltage is 90° , the power factor becomes zero as $\text{PF} = \cos 90^\circ = 0$. It can be concluded that a pure inductive circuit has power factor equal to zero.

Capacitors in Steady-State Sinusoidal

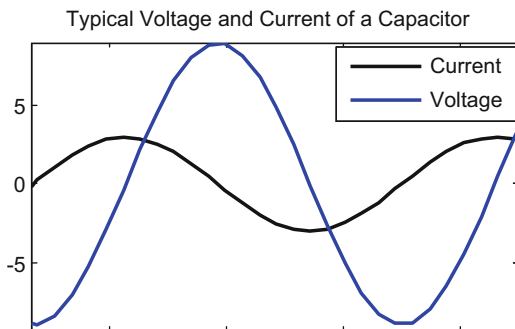
The relation of voltage and current in a capacitor is determined by $v(t) = \frac{1}{C} \int i(t) dt$. Considering a sinusoidal current through the capacitor under steady-state condition as $i(t) = I_m \sin(\omega t + \theta)$, the voltage drop across the capacitor is obtained as follows:

$$\begin{aligned} v(t) &= \frac{1}{C} \int i(t) dt = \frac{1}{C} \int (I_m \sin(\omega t + \theta)) dt = \frac{-I_m}{\omega C} \cos(\omega t + \theta) \\ &= \frac{I_m}{\omega C} \sin(\omega t + \theta - 90^\circ) = \frac{I_m}{\omega C} \angle(\theta - 90^\circ) \end{aligned}$$

This shows that the current is leading the voltage by 90° . This phase shift is shown by a $-j$ factor as shown in Fig. 5.8 and is presented in the following equalities.

$$V = \frac{I_m}{\omega C} e^{j(\theta + 90^\circ)} = \frac{I_m}{\omega C} e^{j\theta} e^{-j90^\circ} = -j \frac{1}{\omega C} (I_m e^{j\theta}) = \frac{1}{j\omega C} (I_m e^{j\theta}) = \frac{1}{j\omega C} (I_m \angle \theta)$$

Fig. 5.8 In a pure capacitive circuit, the voltage and current are 90° apart. This means that the peak of one waveform occurs at the zero of the other waveform. Since the current is leading the voltage and considering voltage as a reference, the current occurs with a time ahead of the voltage equal to $t_{\text{lead}} = \frac{\pi}{2} T$ s



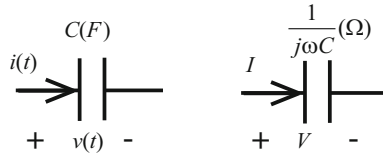


Fig. 5.9 Equivalent of a capacitor with capacitance C (F) when excited by a sinusoidal source at frequency ω becomes a capacitor with impedance $\frac{1}{j\omega C}$ measured in (Ω)

According to these equations, capacitor impedes the flow of current in the presence of sinusoidal (periodic) waveform excitations. The impedance of a capacitor at capacitance C (F) operating at angular frequency of ω (rad/s) is $\frac{1}{j\omega C}$ measured in (Ω) . Phasor representation of Ohm's law for a capacitor under steady-state sinusoidal condition is as follows:

$$V = \frac{1}{j\omega C} I$$

It can be concluded that a capacitor C (F) in steady-state sinusoidal has a reactance of $X_C = \frac{1}{j\omega C}$ (Ω) (Fig. 5.9).

Power Factor of Capacitive Circuits

Since the phase shift in current with respect to the voltage is -90° , the power factor $\text{PF} = \cos(-90^\circ) = 0$. It can be concluded that a pure capacitive circuit has $\text{PF} = 0$.

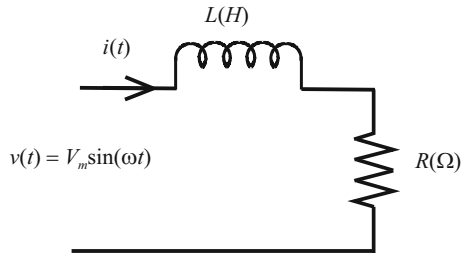
Resistive-Inductive Circuits

Consider a series connection of an inductor L (H) and a resistor $R(\Omega)$ to a voltage source $v(t) = V_m \sin(\omega t)$ Fig. 5.10. This circuit operating at frequency ω shows total impedance of $R(\Omega)$ in the resistor and $j\omega L$ (Ω) in the inductor. Total ohmic value of impedance $z(j\omega)$ in this series circuit is the summation of both elements (because of series connection) as follows:

$$z(j\omega) = R + j\omega L = R + jX_L(\Omega)$$

Impedance of this circuit has a real part known as *resistance* R and an imaginary part known as *reactance* X_L . In polar coordinates, the impedance of this circuit can be written as:

Fig. 5.10 A resistive-inductive RL circuit excited by external sinusoidal source $V(t)$



$$Z = \sqrt{R^2 + X_L^2} \angle \left(\tan^{-1} \frac{X_L}{R} \right) = z \angle \varphi$$

where φ is the phase angle of the impedance, which determines the amount of phase shift in current with respect to voltage. This phase is a positive value because the values of resistance and reactance observed from an inductor are positive. As the phase angle is positive for an RL circuit, the circuit generates a lag type of behavior. This means that the current lags the voltage, when the voltage is set as a reference. Ohm's law indicates that the voltage drop across the impedance $Z = z \angle \varphi$ when current $I = I_m \angle \theta$ is passing through the impedance can be calculated as $V = ZI$.

This can be expanded using the phasor as follows:

$$V = ZI = z \angle \varphi I_m \angle \theta = z I_m \angle (\varphi + \theta)$$

Therefore, the current knowing the voltage of the source can be obtained by dividing the voltage over the impedance as follows:

$$I = \frac{V}{Z}$$

Considering the the phasors values:

$$I = \frac{V_m \angle 0}{z \angle \varphi} = \frac{V_m}{z} \angle -\varphi$$

This current can also be obtained by utilizing the circuit element values, as follows:

$$I = \frac{V_m}{\sqrt{R^2 + X_L^2}} \angle - \left(\tan^{-1} \frac{X_L}{R} \right)$$

Power Factor of Resistive-Inductive Circuits

Consider the circuit of Fig. 5.10. The phase shift in current with respect to voltage is:

$$\varphi = -\left(\tan^{-1} \frac{X_L}{R}\right)$$

Therefore, the power factor of the RL circuit is:

$$\text{PF} = \cos\left(-\tan^{-1} \frac{X_L}{R}\right)$$

Since the circuit generates a lagging current, the power factor can be read as $\text{PF} = \cos\left(-\tan^{-1} \frac{X_L}{R}\right)$ lag.

Power factor can also be obtained from the ratio of resistance over the impedance amplitude. Therefore:

$$\text{PF} = \frac{R}{|Z|} = \frac{R}{\sqrt{R^2 + X_L^2}}$$

Power factor can also be obtained from the division of the voltage measured across the resistor over the entire circuit voltage drop as follows:

$$\text{PF} = \frac{V_R}{V_m}$$

The outcome of this discussion is that:

- The phase of current in an RL circuit is a negative value.
- The phase of impedance in an RL circuit is a positive value.
- An RL circuit shows a phase lag in current with respect to voltage.

Example 5.4 A resistive-inductive circuit is excited at the frequency of 60 Hz. The equivalent resistance is $R = 10 \, \Omega$ and the inductance of $L = 100 \, \text{mH}$. What is the power factor of the circuit?

Solution

$$\text{PF} = \frac{R}{|Z|} = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{10}{\sqrt{10^2 + (2\pi 60 100 \times 10^{-3})^2}} = \frac{10}{39} = 0.256$$

Vector Analysis of RL Circuits

The impedance $z(j\omega) = R + jX_L$ in rectangle form has a real part R and an imaginary part X_L . The factor j indicates a right-angle (90°) rotation from the real axis. These impedance values show amplitude of unit vectors on the real and the imaginary axis. In polar coordinates, this impedance shows an amplitude z and a phase ϕ . The impedance in both rectangle and polar coordinates is shown as follows (Fig. 5.10).

This analysis can be expanded to KVL in the loop as follows:

$$-V_s + V_R + V_L = 0$$

Note that these voltages are presented as vectors or phasors.

Consider the source voltage $V_m \angle 0$ as a reference. The current of an RL circuit has a time delay in reaching peak values respect to the voltage, by angle ϕ . Vectors of voltage and current are shown in Fig. 5.11.

The phase shift between two vectors generates shifted rectangle coordinates on the current vector. Therefore, to obtain the KVL in the loop, the voltage drop across the resistor generates an inphase value of $V_R = RI$ with the current and a vector with 90° rotation (perpendicular to the current vector) that demonstrates the voltage drop across the inductor as $V_L = j\omega LI$ (Fig. 5.12).

Example 5.5 The rms voltage drop measured across elements of a series RL circuit is $V_R = 25 \text{ V}$, $V_L = 10 \text{ V}$. Find the power factor of the circuit.

Fig. 5.11 Impedance of a series RL circuit is split into a real part resistance R and an imaginary part reactance of ωL vertical to the resistance on the imaginary axis. The factor j also confirms the angle of reactance respect to the resistance

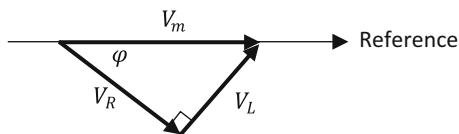
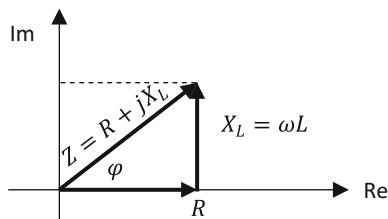


Fig. 5.12 Summation of voltage drops in the RL circuit. The voltage drop across the resistor has a phase shift respect to the reference because of the phase of the current that is passing through the resistor. The voltage drop across the inductor has 90° phase shift from the vector of the resistor voltage drop. The factor j in this voltage drop makes the phase shift

Solution The source voltage is the vector summation of the resistive and inductive voltages. Considering that the inductive voltage has a j factor, the source voltage is:

$$V_m = V_R + jV_L$$

$$V_m = 25 + j10$$

Therefore:

$$\text{PF} = \frac{V_R}{\sqrt{V_R^2 + V_L^2}} = \frac{25}{\sqrt{25^2 + 10^2}} = \frac{25}{26.92} = 0.928$$

Since the circuit has inductive effect in it, the power factor becomes $\text{PF} = 0.928$ lag.

Resistive-Capacitive Circuits

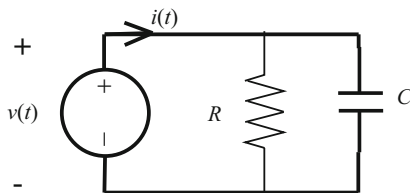
Consider a parallel connection of a capacitor C (F) at reactance of $X_C = \frac{1}{\omega C}$ (Ω) and a resistance R (Ω) to a voltage source $v(t) = V_m \sin(\omega t)$ as shown in Fig. 5.13. This circuit operating at frequency ω shows total admittance of $\frac{1}{R}$ (Ω^{-1}) from the resistor and $j\omega C$ (Ω^{-1}) in capacitor. Total ohmic value of admittance $Y(j\omega)$ in this parallel circuit is the summation of both elements as follows (Fig. 5.13):

$$Y(j\omega) = \frac{1}{R} + \frac{1}{jX_C} = G - jB \text{ } (\Omega^{-1})$$

Using Admittance

Admittance of this circuit has a real part known as *conductance* G and an imaginary part known as *susceptance* $B = \omega C$. A transformation to polar coordinates, the admittance of this circuit can be written as:

Fig. 5.13 A parallel resistive-capacitive circuit that is excited by a sinusoidal voltage source $v(t)$. The current $i(t)$ is flowing in the circuit



$$Y = \sqrt{G^2 + B^2} \angle \left(\tan^{-1} \frac{B}{G} \right) = y \angle \psi$$

where ψ is the phase angle of the circuit, which shows a shift in current with respect to voltage. This phase is a positive value because the value of susceptance observed from a capacitor is positive. As the phase angle is positive for an RC circuit, the circuit generates a lead type of behavior. This means that the current leads the voltage, when the voltage is set as a reference. Ohm's law indicates that the voltage drop across the admittance $Y = y \angle \psi$ when current $I = I_m \angle \theta$ is passing through the admittance can be calculated as $V = \frac{I}{Y}$. Therefore, the current knowing the voltage of the source can be obtained by the product of the voltage and the admittance.

$$I = YV$$

Note that these values are expressed in phasors.

$$I = y \angle \psi \quad V_m \angle 0 = y V_m \angle \psi$$

This current can also be obtained utilizing the rectangle coordinate values as follows:

$$I = \frac{V_m}{\sqrt{G^2 + B^2}} \angle \left(\tan^{-1} \frac{B}{G} \right)$$

Using Impedance

The current phasor using the impedance values can be obtained as follows:

$$\begin{aligned} Z(j\omega) &= R \parallel \frac{1}{jC\omega} = \frac{R \frac{1}{jC\omega}}{R + \frac{1}{jC\omega}} = \frac{R}{1 + j\omega RC} = \frac{R \angle 0}{\sqrt{1 + (\omega RC)^2} \angle \tan^{-1}(\omega RC)} \\ &= \frac{R}{\sqrt{1 + (\omega RC)^2}} \angle -\tan^{-1}(\omega RC) \\ Z(j\omega) &= z \angle -\varphi \end{aligned}$$

where φ is the phase angle of the circuit, which shows a shift in current with respect to voltage. The phase of a resistive-capacitive circuit is a negative value because the value of reactance observed from a capacitor is negative, resembling a lead circuit. Ohm's law indicates that the voltage drop across the impedance $Z = z \angle -\varphi$ when current $I = I_m \angle \theta$ is passing through the admittance can be calculated as $V = ZI$. Therefore, the current knowing the voltage of the source can be obtained by dividing the voltage by the impedance:

$$I = \frac{V}{Z}$$

Note that these values are expressed in phasors. Therefore,

$$I = \frac{V_m \angle 0}{z \angle -\varphi} = \frac{V_m}{z} \angle \varphi$$

Power Factor of Resistive-Capacitive Circuits

The phase shift in current with respect to voltage is $(-\tan^{-1}(\omega RC))$. Therefore, the power factor of an RC circuit is $\text{PF} = \cos(-\tan^{-1}(\omega RC))$. Since the circuit generates a leading current with respect to the voltage, the power factor can be read as $\text{PF} = \cos(-\tan^{-1}(\omega RC))$ lead.

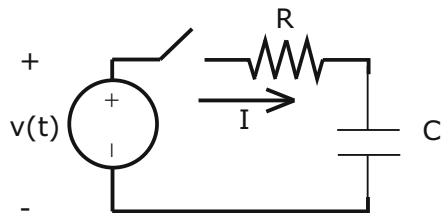
The outcome of this discussion is that:

- The phase angle of current in an RC circuit is positive.
- The phase angle of admittance in an RC circuit is positive.
- The phase angle of impedance in an RC circuit is negative.
- An RC circuit shows a phase lead in current with respect to a reference voltage.

Vector Analysis of RC Circuits

The impedance of a RC circuit shown in Fig. 5.14 from the terminals of source is $z(j\omega) = R - jX_C$ in rectangle form. This has a real part R and an imaginary part $-X_C$. The factor $-j$ indicates a right angle (-90°) rotation from the real axis. In polar coordinates, this impedance shows an amplitude z and a phase φ . The impedance in both rectangle and polar coordinates is shown as follows (Fig. 5.14).

Fig. 5.14 RC circuit excited by an external source. The current also has a sinusoidal form but at a different phase angle



This analysis can be expanded to KVL written as:

$$-V_s + V_R + V_C = 0$$

Considering the vectors of these voltages, their balance becomes:

$$V_s = \sqrt{V_R^2 + (-V_C)^2}$$

Or:

$$V_s = \sqrt{V_R^2 + V_C^2}$$

Considering the source voltage $V_m \angle 0$ as a reference, the current of a RC circuit has a time lead in reaching peak values respect to the voltage, by angle $-\varphi$. Vectors of voltage and current are shown in Fig. 5.15.

The phase shift between two vectors generates a shifted rectangle coordinates on the current vector. Therefore, to obtain the KVL in the loop, the voltage drop across the resistor generates an inphase value of $V_R = RI$ with the current and a vector with -90° rotation (perpendicular to the current vector) that demonstrates the voltage drop across the capacitor as $V_C = -j\frac{1}{\omega C}I$ (Fig. 5.16).

Fig. 5.15 The impedance of an RC circuit has two components of R on the reference axis and a capacitive component as $-\frac{1}{j\omega C}$, or $-jX_C$. The equivalent impedance is a vector summation of R and $-jX_C$ as $Z = R - jX_C$

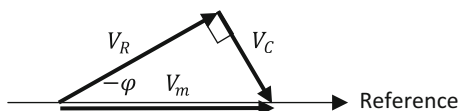
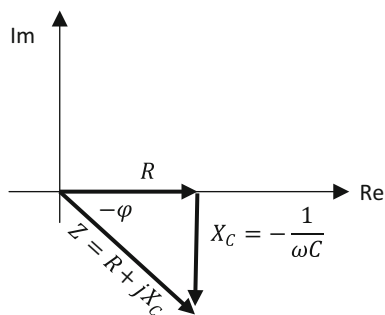


Fig. 5.16 The voltage drop across the resistor and the capacitor is -90° apart. Consider that the voltage of the resistor and the source is $-\varphi$ apart

Example 5.6 Find the impedance of the circuit shown in Fig. 5.17 at frequency ω .

Solution

$$\begin{aligned}
 z(j\omega) &= R + \left(\frac{1}{j\omega C_1} \parallel \left(j\omega L + \frac{1}{j\omega C_2} \right) \right) \\
 z(j\omega) &= R + \left(\frac{\frac{1}{j\omega C_1} \left(j\omega L + \frac{1}{j\omega C_2} \right)}{\frac{1}{j\omega C_1} + \left(j\omega L + \frac{1}{j\omega C_2} \right)} \right) \\
 z(j\omega) &= R + \left(\frac{-\frac{1}{C_1 C_2 \omega^2} (1 - LC_2 \omega^2)}{\frac{-1}{C_1 C_2 \omega^2} (jC_2 \omega - LC_1 C_2 \omega^2 + j\omega C_1)} \right) \\
 z(j\omega) &= \left(R - \frac{LC_1 \omega^2 (1 - LC_2 \omega^2)}{(-LC_1 C_2 \omega^2)^2 + (\omega(C_1 + C_2))^2} \right) \\
 &\quad - j \left(\frac{\omega(C_1 + C_2)(1 - LC_2 \omega^2)}{(LC_1 C_2 \omega^2)^2 + (\omega(C_1 + C_2))^2} \right)
 \end{aligned}$$

Example 5.7 Find impedance of the circuit shown in Fig. 5.18 at frequency ω .

Fig. 5.17 Circuit of Example 5.6

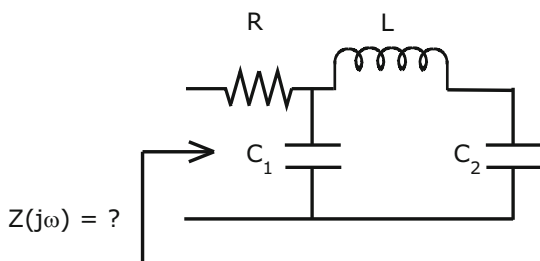
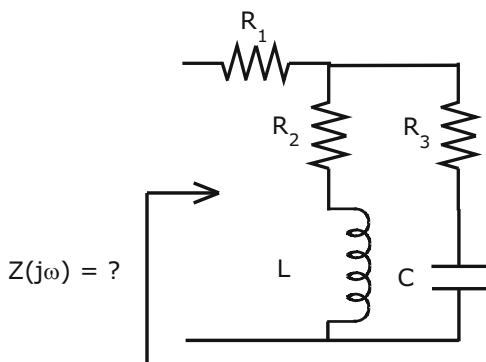


Fig. 5.18 Circuit of Example 5.7



$$Z(j\omega) = R_1 + \frac{1}{\frac{1}{R_2 + j\omega L} + \frac{1}{R_3 + \frac{1}{j\omega C}}}$$

$$Z(j\omega) = R_1 + \frac{1}{\frac{R_2 - j\omega L}{R_2^2 + \omega^2 L^2} + \frac{R_3 + \frac{j}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}}}$$

$$Z(j\omega) = R_1 + \frac{1}{\frac{R_2 - j\omega L}{R_2^2 + \omega^2 L^2} + \frac{R_3 + \frac{j}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}}}$$

Consider:

$$A = R_2^2 + \omega^2 L^2$$

$$B = R_3^2 + \frac{1}{\omega^2 C^2}$$

Therefore:

$$Z(j\omega) = R_1 + \frac{1}{\frac{R_2 - j\omega L}{A} + \frac{R_3 + \frac{j}{\omega C}}{B}}$$

$$Z(j\omega) = R_1 + \frac{1}{\frac{BR_2 - jB\omega L + AR_3 + j\frac{A}{\omega C}}{AB}}$$

$$Z(j\omega) = R_1 + \frac{AB}{BR_2 - jB\omega L + AR_3 + j\frac{A}{\omega C}}$$

$$Z(j\omega) = R_1 + \frac{AB}{(BR_2 + AR_3) + j\left(\frac{A}{\omega C} - B\omega L\right)}$$

Consider:

$$M = BR_2 + AR_3$$

$$N = \frac{A}{\omega C} - B\omega L$$

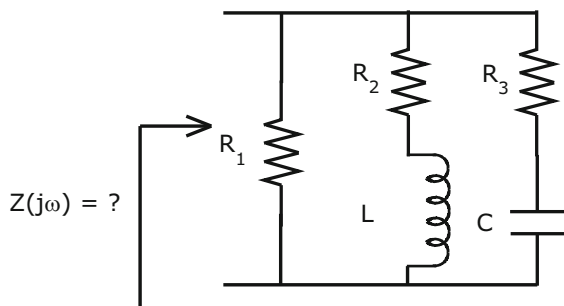
Therefore:

$$Z(j\omega) = R_1 + \frac{AB}{M + jN}$$

$$Z(j\omega) = R_1 + \frac{AB(M - jN)}{M^2 + N^2}$$

$$Z(j\omega) = \left(R_1 + \frac{ABM}{M^2 + N^2} \right) - j \frac{ABN}{M^2 + N^2}$$

Fig. 5.19 Circuit of Example 5.8



Example 5.8 Find impedance of the circuit shown in Fig. 5.19 at frequency ω .

Solution

$$Z(j\omega) = \frac{1}{Y(j\omega)}$$

$$Y(j\omega) = \frac{1}{R_1} + \frac{1}{R_2 + j\omega L} + \frac{1}{R_3 + \frac{1}{j\omega C}}$$

It is recommended that each part of $Y(j\omega)$ is simplified first and then the real and imaginary parts separated and determined.

Therefore:

$\frac{1}{R_1}$ is already simplified.

$\frac{1}{R_2 + j\omega L}$ becomes:

$$\begin{aligned} \frac{1}{R_2 + j\omega L} &= \frac{1}{R_2 + j\omega L} \frac{R_2 - j\omega L}{R_2 - j\omega L} \\ &= \frac{R_2 - j\omega L}{R_2^2 + \omega^2 L^2} \end{aligned}$$

$\frac{1}{R_3 + \frac{1}{j\omega C}}$ becomes:

$$\frac{1}{R_3 + \frac{1}{j\omega C}} = \frac{1}{R_3 - j\frac{1}{\omega C}} = \frac{1}{R_3 - j\frac{1}{\omega C}} \frac{R_3 + j\frac{1}{\omega C}}{R_3 + j\frac{1}{\omega C}} = \frac{R_3 + j\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}}$$

As a result, the admittance of the circuit becomes:

$$Y(j\omega) = \frac{1}{R_1} + \frac{R_2 - j\omega L}{R_2^2 + \omega^2 L^2} + \frac{R_3 + j\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}}$$

$$Y(j\omega) = \left(\frac{1}{R_1} + \frac{R_2}{R_2^2 + \omega^2 L^2} + \frac{R_3}{R_3^2 + \frac{1}{\omega^2 C^2}} \right) + j \left(\frac{-\omega L}{R_2^2 + \omega^2 L^2} + \frac{\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}} \right)$$

$$Z(j\omega) = \frac{1}{\left(\frac{1}{R_1} + \frac{R_2}{R_2^2 + \omega^2 L^2} + \frac{R_3}{R_3^2 + \frac{1}{\omega^2 C^2}} \right) + j \left(\frac{-\omega L}{R_2^2 + \omega^2 L^2} + \frac{\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}} \right)}$$

$$Z(j\omega) = \frac{\left(\frac{1}{R_1} + \frac{R_2}{R_2^2 + \omega^2 L^2} + \frac{R_3}{R_3^2 + \frac{1}{\omega^2 C^2}} \right) - j \left(\frac{-\omega L}{R_2^2 + \omega^2 L^2} + \frac{\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}} \right)}{\left(\frac{1}{R_1} + \frac{R_2}{R_2^2 + \omega^2 L^2} + \frac{R_3}{R_3^2 + \frac{1}{\omega^2 C^2}} \right)^2 + \left(\frac{-\omega L}{R_2^2 + \omega^2 L^2} + \frac{\frac{1}{\omega C}}{R_3^2 + \frac{1}{\omega^2 C^2}} \right)^2}$$

Steady-State Analysis of Circuits

Circuits operating in steady-state sinusoidal condition show specific impedance that determines the current of branches and voltage of nodes. To obtain these parameters, the circuit equivalent in a given angular frequency must be evaluated. Then KVL and KCL can be used to analyze the circuit. It is important to note that all voltages, currents, and impedances follow vector and phasor analysis such that pure ohmic values are inphase with the reference, and the inductive impedance is projected on positive imaginary, and the capacitive impedance is projected on the negative imaginary axis.

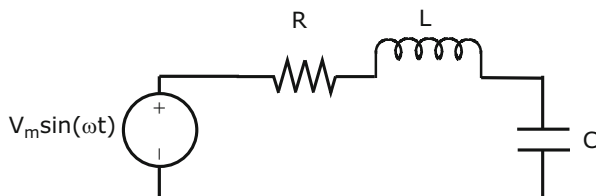
RLC Series

Consider a series connection of R , L , and C to a voltage source $v(t) = V_m \sin(\omega t)$ as shown in Fig. 5.20. The impedance of the circuit observed from the source terminals is measured to be (Fig. 5.20):

$$z(j\omega) = R + jX_L - jX_C = R + j\omega L + \frac{1}{j\omega C} = R + j\omega L - \frac{j}{\omega C} = R + j \left(\omega L - \frac{1}{\omega C} \right)$$

Vector representation of the impedance is obtained by having R on positive real axis, ωL on the positive imaginary axis because of $+j$, and $\frac{1}{\omega C}$ on the negative imaginary axis because of $-j$.

Fig. 5.20 An RLC series excited by a sinusoidal voltage source



The amplitude of impedance is obtained as $|z| = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$, and the phase of impedance is obtained as $\angle z = \varphi = \tan^{-1} \frac{X_L - X_C}{R}$.

The impedance of inductor and capacitor cancel each other up to some extent, depending on their values. If $|\omega L| > |\frac{1}{\omega C}|$, the circuit becomes more inductive, and if $|\omega L| < |\frac{1}{\omega C}|$, the circuit becomes more capacitive.

When the overall impedance becomes more inductive, the phase of impedance becomes a positive value, and when the overall impedance becomes more capacitive, the phase of impedance becomes negative.

Amplitude of impedance is also influenced by the inductor and capacitor reactance. The impedance value is minimum (pure resistive) when the impedance of inductor equals the impedance of capacitor or $X_L = X_C$.

$$|z|_{\min} \Big|_{X_L = X_C} = R$$

At this point ($X_L = X_C$), the impedance becomes purely resistive. As the impedance value decreases to its minimum value, the circuit current increases to its maximum value. This operating point at which the energy stored in capacitors and inductors of the circuit cancel each other is called **resonance**.

KVL indicates that the summation of voltage drops in a loop is zero. It should be noted that the voltages in steady-state analysis represent vectors either on the real axis or on the imaginary axis with positive and negative values.

For instance, the KVL in loop ① is written as follows:

$$\begin{aligned} -V_m \angle 0 + V_R + jV_L - jV_C &= 0 \\ \sqrt{V_R^2 + (V_L - V_C)^2} &= V_m \end{aligned}$$

Considering the loop current phasor I , the KVL can be written as:

$$RI + jX_L I - jX_C I = V_m \angle 0$$

The summation of amplitudes suggests that:

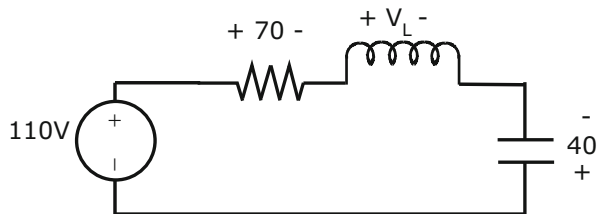
$$\sqrt{(RI)^2 + (X_L I - X_C I)^2} = |V_m|$$

Factoring I out results in:

$$|I| \sqrt{R^2 + (X_L - X_C)^2} = |V_m|$$

Therefore, the current amplitude of the loop is obtained from dividing the voltage by impedance amplitude as:

Fig. 5.21 Circuit of Example 5.9



$$|I| = \frac{|V_m|}{\sqrt{R^2 + (X_L - X_C)^2}}$$

The phase of current as explained earlier can be obtained from the circuit as:

$$\theta = -\tan^{-1} \frac{X_L - X_C}{R}$$

Example 5.9 Find the voltage of inductor and the power factor in the following circuit.

Solution As Fig. 5.21 shows, the voltage drop on each element is known except for the inductor. The balance of voltages can be written as:

$$\sqrt{V_R^2 + (V_L - V_C)^2} = V_m$$

$$\sqrt{70^2 + (V_L - 40)^2} = 110$$

$$(V_L - 40)^2 = 110^2 - 70^2 = 7200$$

$$V_L - 40 = 84.85$$

$$V_L = 124.85 \text{ V}$$

Power factor can be obtained from:

$$\text{PF} = \frac{V_R}{V} = \frac{70}{110} = 0.63 \text{ lag.}$$

The circuit is lag because the voltage drop across the inductor is larger than the voltage drop across the capacitor. This has resulted from larger inductive reactance with respect to the capacitive reactance for the same circuit current. Therefore, $X_L > X_C$. The voltage across the inductor can be higher than the amplitude of voltage source. This may occur as a result of energy exchange between the inductor and capacitor. The phase delay naturally occurred in capacitor and inductor causes the capacitor and inductor voltages reach peak value with a time

delay. Therefore, a simple summation of voltages without considering their phase angle is not an accurate KVL.

Example 5.10 Considering a series connection of RLC circuit to a voltage source of $v(t) = 167 \sin(377t+10)\text{V}$ draws a current as $i(t) = 7.07 \sin(377t+70)\text{A}$. Find the impedance, resistance, and reactance of the circuit. Is this circuit more capacitive or more inductive? Find the power factor of the circuit.

Solution The voltage and current phasors are obtained as $V = 167 \angle 10$ and $I = 7.07 \angle 70$. Knowing that the impedance is $Z = \frac{V}{I}$:

$$Z = \frac{167 \angle 10}{7.07 \angle 70} = 23.62 \angle -60 \Omega$$

Since the phase of impedance is negative, the circuit is more capacitive. Therefore, the impedance in rectangle coordinates shows the resistance and reactance as follows:

$$\begin{aligned} Z = \frac{167 \angle 10}{7.07 \angle 70} &= 23.62 \angle -60 = 23.62 \cos(-60) + j23.62 \sin(-60) \\ &= \underbrace{11.81}_R - \underbrace{j20.45}_{-jX_C} \Omega \end{aligned}$$

The real part of impedance shows the resistance, and the imaginary part shows the reactance. Since the reactance is negative, the circuit is more capacitive.

Power factor of the circuit is obtained from the ratio of the resistance 11.81Ω over the impedance amplitude 23.62Ω . $\text{PF} = \frac{11.81}{23.62} = 0.5$ lead. The circuit is lead because it shows more capacitive behavior.

Power factor can also be obtained from $\text{PF} = \cos(\widehat{V, I}) = \cos(-60) = 0.5$.

RLC Parallel

Consider a parallel connection of R, L, and C to a current source $i(t) = I_m \sin(\omega t)$. The circuit forms a node that is considered $v(t)$ volts (Fig. 5.22).

A KCL at node ① can be written as follows:

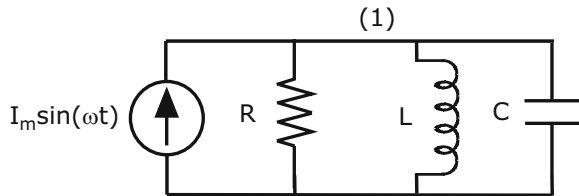
$$-i(t) + i_R + i_L + i_C = 0$$

Phasor current values can be replaced in the KCL. This yields:

$$-I_m \angle 0 + \frac{V}{R} + \frac{V}{j\omega L} + \frac{V}{1/j\omega C} = 0$$

Solving for V (phasor representation of voltage) can be found as:

Fig. 5.22 Parallel RLC circuit excited by a sinusoidal current source



$$V \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right) = I_m$$

$$V = \frac{I_m}{\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right)} = \frac{I_m}{\left(\frac{1}{R} + \frac{-j}{\omega L} + j\omega C \right)}$$

$$V = \frac{I_m}{\sqrt{\left(\frac{1}{R} \right)^2 + \left(\omega C - \frac{1}{\omega L} \right)^2}} \angle \left(-\tan^{-1} R \left(\omega C - \frac{1}{\omega L} \right) \right)$$

Power factor of this circuit is obtained by:

$$\text{PF} = \cos \left(\tan^{-1} R \left(\frac{1}{\omega L} - \omega C \right) \right)$$

The circuit has maximum voltage when the admittance reaches minimum value. At this point, the $\omega C - \frac{1}{\omega L} = 0$ results in $\omega^2 CL = 1$ or $\omega = \frac{1}{\sqrt{LC}}$. Power factor at resonance reaches unity, $\text{PF} = 1$. The maximum voltage can be obtained as $V = RI_m$.

Example 5.11 Find the current drawn from the source in the given circuit (Fig. 5.23).

Solution The source is operating at angular frequency of $\omega = 3000$ rad/s. Therefore, the ohmic value of inductor becomes $jX_L = j\omega L = j3000 \times \frac{1}{3} = j1000 \Omega$, and the ohmic value of capacitor impedance becomes $-jX_C = -j\frac{1}{\omega C} = -j\frac{1}{3000 \times \frac{1}{6e-6}} = -j2000 \Omega$.

The circuit is shown in Fig. 5.24.

The current of the source is the division of voltage over the impedance measured at the terminals of source. This is:

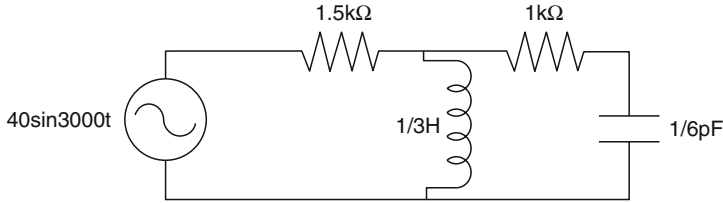


Fig. 5.23 Circuit of Example 5.11

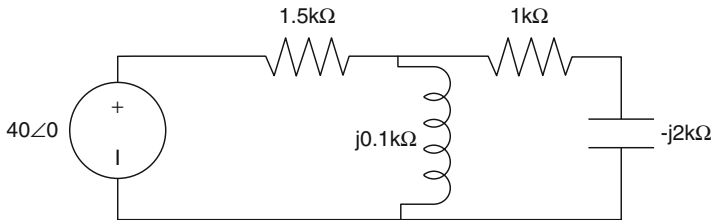


Fig. 5.24 Circuit of Example 5.23 when impedance values are calculated at the frequency of the source, 3000 rad/s

$$\begin{aligned}
 I &= \frac{40\angle 0}{1500 + j1000 \parallel (1000 - j2000)} = \frac{40\angle 0}{1500 + \frac{j1000 \times (1000 - j2000)}{j1000 + (1000 - j2000)}} \\
 &= \frac{40\angle 0}{1500 + \frac{j1000 \times (1000 - j2000)}{(1000 - j1000)}} = \frac{40\angle 0}{1500 + \frac{j1 \times (1000 - j2000)}{(1 - j1)}} \\
 &= \frac{40\angle 0}{1500 + \frac{(1 + j1)j1 \times (1000 - j2000)}{(1 + j1)(1 - j1)}} = \frac{40\angle 0}{2000 + j1500} = \frac{40\angle 0}{2500\angle 36.87} \\
 &= 16\angle(-36.87) \text{ mA}
 \end{aligned}$$

Example 5.12 Consider the circuit shown in Fig. 5.25. Using node KCL, find the voltage of each node.

Solution The circuit has two nodes ① and ②, at voltage of $v_1(t)$ and $v_2(t)$, respectively, in time domain and V_1 and V_2 as phasors. Node ① involves five elements including the 1A source which is forcing the current to the node and four passive elements that drain the current out of the node. Node ② has also five elements including a $0.5\angle -90^\circ$ which is leaving the node. Considering that all currents leaving the node through passive elements are positive and all currents entering the node are negative, KCLs for these nodes can be written as follows:

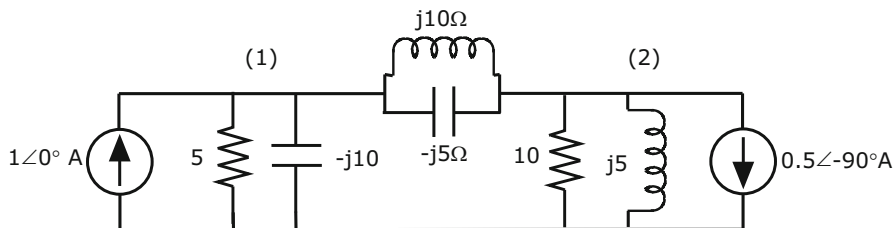


Fig. 5.25 Circuit of Example 5.12

$$\text{KCL} \textcircled{1}. -1\angle 0^\circ + \frac{V_1}{5} + \frac{V_1}{-j10} + \frac{V_1 - V_2}{-j5} + \frac{V_1 - V_2}{j10} = 0$$

$$\text{KCL} \textcircled{2}. 0.5\angle -90^\circ + \frac{V_2}{10} + \frac{V_2}{j5} + \frac{V_2 - V_1}{-j5} + \frac{V_2 - V_1}{j10} = 0$$

Simplifying these equations results in:

$$(0.2 + j0.2)V_1 - j0.1V_2 = 1\angle 0^\circ$$

$$-j0.1V_1 + (0.1 - j0.1)V_2 = -0.5\angle -90^\circ = -(-j)0.5 = j0.5$$

Solving for V_1 and V_2 results in:

$$V_1 = (1 - j2) = 2.24\angle -63.4^\circ \text{ V}$$

$$V_2 = (-2 + j4) = 4.47\angle 116.6^\circ \text{ V}$$

Considering angular frequency of $\omega = 377 \text{ rad/s}$, the time representation of these voltages becomes:

$$v_1(t) = 2.24 \sin(377t - 63.4^\circ) \text{ V}$$

$$v_2(t) = 4.47 \sin(377t + 116.6^\circ) \text{ V}$$

Example 5.13 Using nodal analysis, find $v_1(t)$ and $v_2(t)$, knowing the angular frequency $\omega = 1000 \text{ rad/s}$.

Solution The circuit has two nodes shown in Fig. 5.26 as ① and ②. Node ① involves two current sources of 20 and 50 mA, where one leaves the node ($+50\angle -90^\circ \text{ mA}$) and one enters the node hence (-20 mA). The (50 mA) source enters the node ② hence becomes a ($-50\angle -90^\circ \text{ mA}$). The current through each passive element follows Ohm's law as $I = YV$.

$$\text{KCL} \textcircled{1} : -20m + 50m\angle -90^\circ + V_1 j50m + (V_1 - V_2) \times -j25m = 0$$

Fig. 5.26 Circuit of Example 5.13

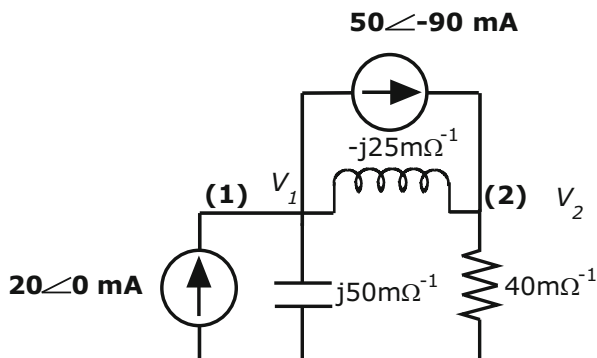
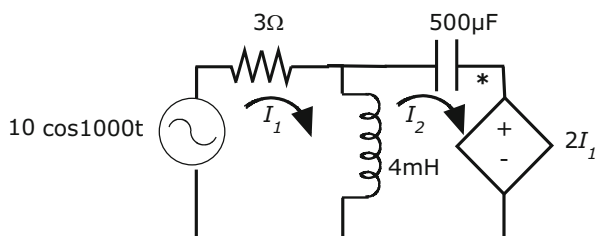


Fig. 5.27 Circuit of Example 5.14



$$\text{KCL} \textcircled{2} : -50m\angle -90 + (V_2 - V_1) \times -j25m + V_2 40m = 0$$

Solving for V_1 and V_2 results in:

$$V_1 = 1.062\angle 23.3^\circ \text{ V} \rightarrow v_1(t) = 1.062 \sin(1000t + 23.3^\circ) \text{ V}$$

$$V_2 = 1.593\angle -50.0^\circ \text{ V} \rightarrow v_2(t) = 1.593 \sin(1000t - 50^\circ) \text{ V}$$

Example 5.14 In the given circuit, find the current of each loop (Fig. 5.27).

Solution The voltage source feeds the circuit at a voltage of 10 V and angular frequency of 1000 rad/s. The 4 mH inductor shows impedance of $j4 \Omega$, ($j\omega L$), and the 500 μF capacitance shows $-j2 \Omega$, ($1/j\omega C$). The current-dependent voltage source has a dependency on the current in loop ①. Therefore, in writing KVL, it is treated like an independent voltage source with value as a function of I_1 as $2I_1$. The circuit has two loops with current phasors I_1 and I_2 circulating clockwise (the direction is optional). In each loop, the KVL suggests some voltage drops starting from the * sign, as follows:

$$\text{KVL} \textcircled{1} : 3I_1 + j4(I_1 - I_2) - 10\angle 0 = 0$$

$$\text{KVL} \textcircled{2} : j4(I_2 - I_1) - j2I_2 + 2I_1 = 0$$

Equations can be simplified into:

Fig. 5.28 Circuit of Example 5.15

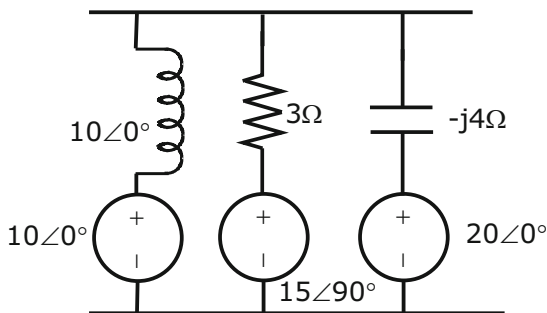
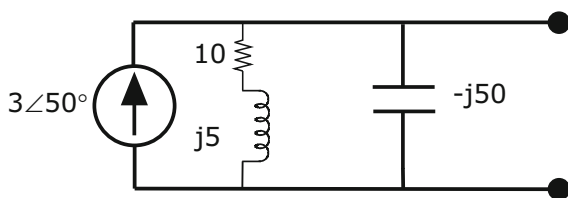


Fig. 5.29 Circuit of Example 5.16



$$(3 + j4)I_1 - j4I_2 = 10$$

$$(2 - j4)I_1 + j2I_2 = 0$$

Solving for I_1 and I_2 results in:

$$I_1 = 1.24 \angle 29.7^\circ \text{ A} \rightarrow i_1(t) = 1.24 \cos(3000t + 29.7^\circ) \text{ A}$$

$$I_2 = 2.77 \angle 56.3^\circ \text{ A} \rightarrow i_2(t) = 2.77 \cos(3000t + 56.3^\circ) \text{ A}$$

Example 5.15 Find I_1 and I_2 phasors (Fig. 5.28).

Solution $I_1 = 4.87 \angle -164.6^\circ \text{ A}$, and $I_2 = 7.17 \angle -144.9^\circ \text{ A}$.

Example 5.16 Find the voltage across the terminals of current source (Fig. 5.29).

Solution The current source is connected to two impedances of $-j50 \Omega$ and $10 + j5 \Omega$ in parallel. The voltage at the terminals is $V = ZI$. Therefore:

$$\begin{aligned} V &= 3 \angle 30^\circ (-j50 \parallel (10 + j5)) \\ V &= 3 \angle 30^\circ \left(\frac{-j5(10 + j5)}{-j5 + (10 + j5)} \right) = 3 \angle 30^\circ \left(\frac{-j(10 + j5)}{2} \right) \\ &= 3 \angle 30^\circ \left(\frac{1 \angle -90^\circ \times 11.183 \angle 26.56^\circ}{2} \right) \end{aligned}$$

$$V = 3 \times 11.183 \times \frac{1}{2} \angle (-30 - 90 + 26.56) = 16.773 \angle -33.4^\circ \text{V}$$

Resonance

Consider a circuit (RLC) that is connected to a variable/adjustable frequency source. In each half cycle, the capacitors and inductors are charged and in their opposite cycles will be discharged. There is a phase delay in this charge and discharge observed in the inductor and capacitor. As the frequency changes, the time delay in this charge and discharge process changes. This also demonstrates different amounts of impedance observed from inductors and from capacitors. At a specific frequency, all inductive effects cancel all capacitive effects, resulting in a pure resistive circuit.

At resonant, the amount of voltage or current depending on the circuit topology reach an extreme amount (either maximum or minimum). For instance, it was observed that the current in a series RLC circuit reached maximum at the resonance frequency of $\omega = \frac{1}{\sqrt{LC}}$. It was also observed that the voltage of a parallel RLC circuit reached its maximum value at the resonant frequency of $\omega = \frac{1}{\sqrt{LC}}$.

To obtain the resonant frequency, a frequency must be identified that makes imaginary part of the impedance or imaginary part of the reactance zero.

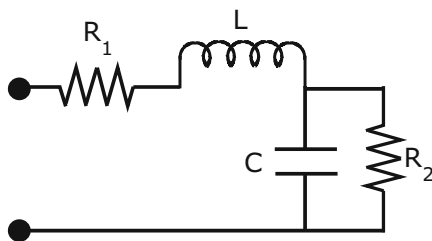
Example 5.17 Find the input impedance of the following circuit, and calculate the resonant frequency (Fig. 5.30).

Solution The circuit shows a parallel of C and R_2 in series with R_1 and L . At operating frequency of ω , the input impedance from the terminal of this circuit is measured as:

$$Z(j\omega) = R_1 + j\omega L + \left(\frac{1}{j\omega C} \parallel R_2 \right)$$

In this case, each term is simplified individually. A common denominator does not help, so it can be avoided. Hence:

Fig. 5.30 Circuit of Example 5.17



$$Z(j\omega) = R_1 + j\omega L + \left(\frac{\frac{1}{j\omega C} R_2}{\frac{1}{j\omega C} + R_2} \right)$$

$$Z(j\omega) = R_1 + j\omega L + \left(\frac{R_2}{1 + j\omega C R_2} \right)$$

$$Z(j\omega) = R_1 + j\omega L + \frac{1 - j\omega C R_2}{1 - j\omega C R_2} \left(\frac{R_2}{1 + j\omega C R_2} \right)$$

$$\begin{aligned} Z(j\omega) &= R_1 + j\omega L + \left(\frac{R_2(1 - j\omega C R_2)}{1 + (\omega C R_2)^2} \right) \\ &= R_1 + j\omega L + \left(\frac{R_2}{1 + (\omega C R_2)^2} - \frac{j\omega C R_2^2}{1 + (\omega C R_2)^2} \right) \end{aligned}$$

Now, the fractions can be separated into real and imaginary, and each part can be collected as follows:

$$Z(j\omega) = \left(R_1 + \frac{R_2}{1 + (\omega C R_2)^2} \right) + j\omega \left(L - \frac{C R_2^2}{1 + (\omega C R_2)^2} \right)$$

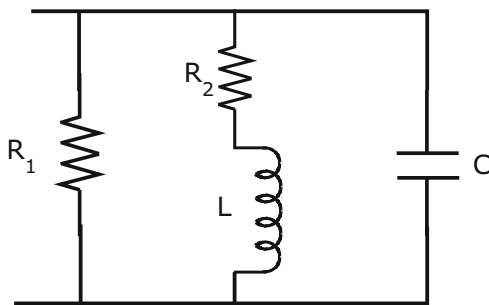
To operate at resonance, the imaginary part of impedance must be zero. Therefore:

$$\begin{aligned} \left. \text{Im}(Z(j\omega)) \right|_{\omega = \omega_0} &= 0 \\ j\omega_0 \left(L - \frac{C R_2^2}{1 + (\omega_0 C R_2)^2} \right) &= 0 \end{aligned}$$

Solving for ω_0 , and considering $\omega_0 \neq 0$, yields:

$$\begin{aligned} L - \frac{C R_2^2}{1 + (\omega_0 C R_2)^2} &= 0 \\ L(1 + (\omega_0 C R_2)^2) &= C R_2^2 \\ \omega_0^2 &= \frac{1}{(C R_2)^2} \left(\frac{C R_2^2}{L} - 1 \right) = \frac{1}{LC} - \frac{1}{C^2 R_2^2} \\ \omega_0 &= \sqrt{\frac{1}{LC} - \frac{1}{C^2 R_2^2}} \left(\frac{\text{rad}}{\text{s}} \right). \end{aligned}$$

Fig. 5.31 Circuit of Example 5.18



Example 5.18 Find the input admittance of circuit shown in Fig. 5.31, and find the resonant frequency.

Solution Admittance of circuit from terminal at the frequency ω has R_1 , $R_2 + j\omega L$, and $\frac{1}{j\omega C}$ in parallel.

$$Y(j\omega) = \frac{1}{R_1} + \frac{1}{R_2 + j\omega L} + \frac{1}{\frac{1}{j\omega C}}$$

Simplifying each term yields:

$$Y(j\omega) = \frac{1}{R_1} + \frac{R_2 - j\omega L}{R_2 - j\omega L R_2 + j\omega L} + j\omega C$$

$$Y(j\omega) = \frac{1}{R_1} + \left(\frac{R_2 - j\omega L}{R_2^2 + \omega^2 L^2} \right) + j\omega C$$

Splitting the fraction into real-imaginary sections yields:

$$Y(j\omega) = \frac{1}{R_1} + \left(\frac{R_2}{R_2^2 + \omega^2 L^2} - \frac{j\omega L}{R_2^2 + \omega^2 L^2} \right) + j\omega C$$

$$Y(j\omega) = \frac{1}{R_1} + \frac{R_2}{R_2^2 + \omega^2 L^2} + j\omega \left(C - \frac{L}{R_2^2 + \omega^2 L^2} \right)$$

At the resonant frequency of $\omega = \omega_0$:

$$I_m(Y(j\omega)) \Big|_{\omega = \omega_0} = 0$$

Considering $\omega_0 \neq 0$, the resonant frequency becomes:

$$C - \frac{L}{R_2^2 + \omega_0^2 L^2} = 0$$

Solving for ω_0 yields:

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R_2^2}{L^2} \left(\frac{\text{rad}}{\text{s}} \right)}.$$

Power in Sinusoidal Steady-State Operation

The term power is defined as the product of voltage and current. The amount of voltage and current can be instantaneous, RMS, or DC. Therefore, instantaneous power or apparent power and DC power can be measured as:

$$S = \text{instantaneous power} = \text{apparent power} = v(t) \cdot i(t)$$

Apparent Power

Consider an RLC circuit in which the current has a phase shift φ with respect to the voltage as shown in Fig. 5.32. The current can be projected to inphase and vertical components with respect to the voltage. The inphase component of current with respect to voltage has the value of $I \cos \varphi$, and the vertical components have the value of $I \sin \varphi$. Therefore, it can be represented as $(V, I \cos \varphi) = 0$ and $(V, I \sin \varphi) = \pm 90$. This angle is $+90$ for a lag system and is -90 for a lead system (Fig. 5.32).

$$S = v(t)i(t)$$

$$S = V_m \sin(\omega t) I_m \sin(\omega t - \varphi)$$

where in V_m and I_m are the peak values of voltage and current respectively.

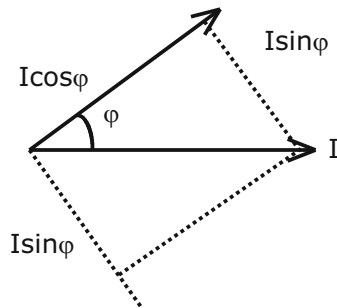
$$S = \frac{1}{2} V_m I_m (\cos \varphi - \cos(2\omega t - \varphi))$$

$$S = \frac{1}{2} V_m I_m \cos \varphi - \frac{1}{2} V_m I_m \cos(2\omega t - \varphi)$$

Utilizing rms values as $V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$ and $I_{\text{rms}} = \frac{I_m}{\sqrt{2}}$, the apparent power becomes:

$$S = V_{\text{rms}} I_{\text{rms}} \cos \varphi - V_{\text{rms}} I_{\text{rms}} \cos(2\omega t - \varphi)$$

Fig. 5.32 Projection of current with a phase shift on its reference



The apparent power has two components:

- $V_{\text{rms}} I_{\text{rms}} \cos \varphi$ is a constant value power and contributes to what is known as *active power*.
- $V_{\text{rms}} I_{\text{rms}} \sin \varphi$ is a pulsating power at the frequency twice as that of the voltage and current. This power has an average value of zero over one cycle. Therefore, it does not contribute to the actual work done by the circuit.

As a reminder, the voltage waveform is considered as reference, and the current might be leading, inphase, or lagging the voltage.

- **The phase of a lead current is positive.**
- **The phase of an inphase current is zero.**
- **The phase of a lag current is negative.**

In phasors' presentation, the apparent power is calculated by:

$$S = V_{\text{rms}} I_{\text{rms}}^* = \frac{1}{2} V_m I_m^* \text{ (VA)}$$

Considering real and imaginary parts of current, apparent power can be converted as follows:

$$\begin{aligned}
 S &= V_{\text{rms}} I_{\text{rms}} (\cos \varphi + j \sin \varphi) \\
 S &= \underbrace{V_{\text{rms}} I_{\text{rms}} \cos \varphi}_{P: \text{ Active}} + j \underbrace{V_{\text{rms}} I_{\text{rms}} \sin \varphi}_{Q: \text{ Reactive}} \\
 S &= P + jQ \\
 P &= V_{\text{rms}} I_{\text{rms}} \cos \varphi \\
 P &= \frac{1}{2} VI \cos \varphi
 \end{aligned}$$

where V, I are the peak values.

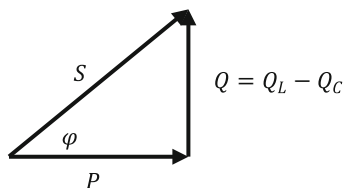


Fig. 5.33 Power triangle. The balance of the active, reactive, and apparent power forms a triangle which is also known as power triangle. Considering that Q_L is the consumption of reactive power in an inductor, Q_C is the generation of reactive power in a capacitor, and $Q = Q_L - Q_C$ as the balance of reactive power, then the Q value might be more consumed, i.e., $Q > 0$; more generation, i.e., $Q < 0$; or balanced, i.e., $Q = 0$

And:

$$Q = V_{\text{rms}} I_{\text{rms}} \sin \phi$$

$$Q = \frac{1}{2} VI \sin \phi$$

where V, I are the peak values.

Apparent power is a vector summation of active power and reactive power as shown in Fig. 5.33.

The amplitude of apparent power is $|S| = \sqrt{P^2 + Q^2}$, and the phase is obtained as:

$$\frac{Q}{P} = \frac{V_{\text{rms}} I_{\text{rms}} \sin \phi}{V_{\text{rms}} I_{\text{rms}} \cos \phi} = \tan \phi$$

$$\phi = \tan^{-1} \frac{Q}{P}$$

Active Power

The mean power consumed in a circuit is obtained by the product of voltage and the inphase component of the current. Measured in watts (W), the mean power is also called active power and is shown as:

$$P = V(I \cos \phi) = (VI) \cos \phi = S \cos \phi \text{ (W)}$$

Considering an impedance in resistive and reactive parts, the power that is consumed in the resistive part is active power. This power is converted into torque in electric machines or generates heat in space heaters. The active power does the work in electromechanical systems. For this reason, $\cos \phi$ is called the power factor meaning the portion of the apparent power that does the active work.

Consider an impedance of an RL circuit shown in Fig. 5.34.

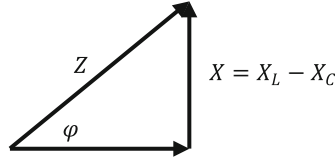


Fig. 5.34 The balance of capacitive and inductive reactance and the value of resistance determine the impedance. The balance of inductive and capacitive reactance is $X = X_L - X_C$

The angle between the impedance and the resistance determines the current phase shift as follows:

$$\cos \varphi = \frac{R}{Z}$$

Therefore, the active power can be calculated as:

$$P = VI \cos \varphi = VI \frac{R}{Z} = \frac{V}{Z} IR = IIR = RI^2 \text{ (W)}$$

Reactive Power

Perpendicular to the inphase component of the current, the power developed into the imaginary part of the impedance is known as reactive power. Measured in volt-ampere reactive (VAR), the value for this power can be obtained as:

$$Q = V(I \sin \varphi) = (VI) \sin \varphi = S \sin \varphi \text{ (VAR)}$$

Consider the impedance shown in Fig. 5.34, the value of $\sin \varphi$ can be obtained as follows:

$$\sin \varphi = \frac{X}{Z}$$

Therefore, the reactive power can be calculated as:

$$Q = VI \sin \varphi = VI \frac{X}{Z} = \frac{V}{Z} IX = IIX = XI^2 \text{ (VAR)}$$

As the equation illustrates, in an impedance consisting of resistance and reactance, only the reactance generates or consumes the reactive power.

Note 5.2 The power consumption has positive value, and the power generation has a negative value.

Note 5.3 Apparent power can also be obtained from the product of $Z(j\omega)$ and the current I as follows:

$$S = Z(j\omega)I^2 = (R + jX)I^2 = RI^2 + jXI^2 = P + jQ$$

Note 5.4 As the sign of each element RI^2 and XI^2 is positive, it means that the circuit has been a resistive-inductive circuit which consume active power (in resistor) and consume reactive power (in inductor).

Reactive power of an RC circuit. Consider an RC series circuit that has resistance R and reactance $-X$; therefore, the impedance is obtained as:

$$Z(j\omega) = R - jX$$

Apparent power of this circuit can be obtained as:

$$S = Z(j\omega)I^2 = (R - jX)I^2 = RI^2 - jXI^2 = P - jQ$$

It can be observed that the resistive-capacitive circuits consume active power $+RI^2$ and generate reactive power $-XI^2$ in the capacitor.

Note 5.5 Reactive power is generated in capacitors and is consumed in inductors.

Note 5.6 Ideal reactive elements L, C do not consume or generate active power.

Note 5.7 Resistors do not consume or generate reactive power.

Non-ideal Inductors

Ideal inductors have no internal resistance. Therefore, they only demonstrate a reactance. However, the resistance of wires used to make the inductors sometimes cannot be ignored. Therefore, the non-ideal inductors have impedance similar to an RL circuit. To demonstrate the internal resistance of a non-ideal inductor, a quality factor Q_f is defined (Fig. 5.35).

Fig. 5.35 A non-ideal inductor shown by its internal resistance and the inductance

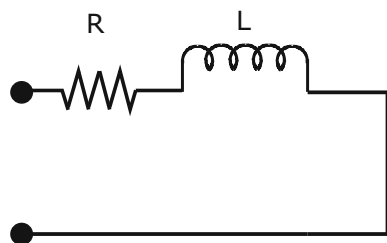
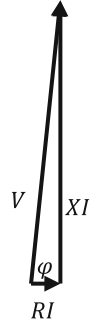


Fig. 5.36 The small amount of internal resistance R creates a phase angle φ . Voltage drops of RI and XI are perpendicular



Quality Factor (Q_f)

Reciprocal of power factor is known as quality factor $Q_f = \frac{1}{\text{PF}}$. This value shows the merit of a coil. From calculations, the power factor of an ideal inductor was calculated to be $\text{PF} = 0$, as the resistive part was zero. Therefore, the quality factor of an ideal inductor is $Q_f = \frac{1}{0} = \infty$. Actual (non-ideal) inductors have internal resistance that makes their power factors non-zero as $\text{PF} = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_L^2}}$ (Fig. 5.36).

Considering the circuit of Fig. 5.53, the quality factor is obtained as:

$$Q_f = \frac{1}{\text{PF}} = \frac{Z}{R} = \frac{\sqrt{R^2 + X_L^2}}{R}$$

For inductors that have large reactance, i.e. $L\omega \gg R$, the quality factor can be obtained as:

$$Q_f = \frac{\sqrt{R^2 + X_L^2}}{R} \cong \frac{\sqrt{X_L^2}}{R} = \frac{L\omega}{R}$$

Example 5.19 In an RL series circuit, the current and voltage are measured to be $i(t) = 7.5 \sin(377t + 120)$ A and $v(t) = 150 \sin(377t + 150)$ V. Find:

- The impedance of the circuit
- The value of the resistance
- The value of reactance and the inductance in (H)
- The amount of active power
- The amount of reactive power
- The power factor

Solution According to Ohm's law, the voltage and current phasor are related as follows:

$$V = ZI$$

$$Z = \frac{150 \angle 150}{7.5 \angle 120} = 20 \angle 30$$

As the impedance has a positive phase, the circuit is a resistive-inductive. The real part of the impedance shows the resistance, and the imaginary part of the impedance shows the reactance as follows:

$$Z = 20 \angle 30 = 20 \cos 30 + j20 \sin 30 = 17.32 + j10 \, \Omega$$

$$R = 17.32 \, \Omega$$

$$X_L = 10 \, \Omega$$

$$X_L = L\omega = 10$$

$$L = \frac{10}{377} = 26.5 \, \text{mH}$$

Apparent power utilizing the peak values of voltage and current can be calculated as:

$$S = \frac{1}{2} VI^* = \frac{1}{2} (150 \angle 150) (7.5 \angle 120)^* = \frac{1}{2} (150 \angle 150) 7.5 \angle -120 = 562.5 \angle 30 \, \text{VA}$$

Converting to rectangle coordinates yields:

$$S = 562.5 \cos 30 + j562.5 \sin 30 = 487.1 + j281.25$$

$$P = 487.1 \, \text{W}, Q = 281.25 \, \text{VAR}$$

The Power factor is:

$$\text{PF} = \cos 30 = 0.86 \, \text{lag.}$$

Quality factor of the inductor becomes:

$$Q_f = \frac{1}{\text{PF}} = \frac{1}{0.86} = 1.15$$

Example 5.20 The voltage drop of a coil when a DC of 9 A is passing through is measured to be 4.5 V. The same coil when an AC sinusoidal of 9 A at 25 Hz is passing through drops 24 V. Find the impedance, power, PF, and Q_f at a voltage of 150 V, 60 Hz.

Solution The DC and voltage drop result in resistance value as:

$$R = \frac{V}{I} = \frac{4.5}{9} = 0.5 \Omega$$

At 25 Hz, the impedance value is obtained as:

$$Z = \frac{V}{I} = \frac{24}{9} = 2.66 \Omega$$

The reactance value can be calculated as:

$$\begin{aligned} Z &= \sqrt{R^2 + X_L^2} \\ X_L &= \sqrt{Z^2 - R^2} = \sqrt{2.66^2 - 0.5^2} = 2.61 \Omega \\ L &= \frac{X_L}{\omega} = \frac{2.61}{2\pi 25} = 17.4 \text{ mH} \end{aligned}$$

At 60 Hz frequency:

$$\begin{aligned} X_L &= 17.4 \text{ m} \times 2\pi \times 60 = 6.56 \Omega \\ Z &= \sqrt{R^2 + X_L^2} = \sqrt{0.5^2 + 6.56^2} = 6.57 \Omega \\ I &= \frac{V}{Z} = \frac{150}{6.57} = 22.8 \text{ A} \end{aligned}$$

The impedance phase is $\varphi = \tan^{-1} \frac{6.56}{0.5} = 85.64^\circ$. Therefore, the current phase angle is -85.64° .

Apparent power utilizing rms parameters becomes:

$$\begin{aligned} S &= VI^* = (150 \angle 0) (22.8 \angle -85.64)^* = (150 \angle 0) 22.8 \angle +85.64 \\ &= 3420 \angle 85.64 \text{ VA} \end{aligned}$$

$$S = 3420 \cos 85.64 + j3420 \sin 85.64 = 256 + j3410$$

$$P = 256 \text{ W}, Q = 3410 \text{ VAR}$$

$$\text{PF} = \cos 85.64 = 0.076$$

$$Q_f = \frac{1}{0.076} = 13.15$$

Example 5.21 A black box series circuit has the following voltage and current measurements, $v(t) = 200\sqrt{2} \sin(377t + 10)^\circ \text{ V}$ and $i(t) = 10\sqrt{2} \cos(377t - 35)^\circ \text{ A}$. Find the circuit elements and their values.

Solution The circuit current is given in cos form which needs to be converted to a sin function as follows:

$$\begin{aligned} i(t) &= 10\sqrt{2}\cos(377t - 35) = 10\sqrt{2}\sin(377t - 35 + 90) \\ &= 10\sqrt{2}\sin(377t + 55) \end{aligned}$$

Impedance of the circuit is obtained as follows:

$$Z = \frac{V}{I} = \frac{200\sqrt{2}\angle 10}{10\sqrt{2}\angle 55} = 20\angle -45\ \Omega$$

As the angle of impedance is a negative value, the circuit is a RC circuit. The circuit is more capacitive than inductive. The resistance becomes $R = 20 \cos(-45) = 14.14\ \Omega$. The amount of reactance is $-X_C = 20 \sin(-45) = -14.14\ \Omega$.

The operating frequency $\omega = 377$ rad/s results in:

$$X_C = \frac{1}{C\omega} \rightarrow C = \frac{1}{\omega X_C} = \frac{1}{377 \times 14.14} = 1.87e-4\ F = 187\ \mu F.$$

Example 5.22 A circuit when current of $4 - j5$ A is passing through drops a voltage of $180 + j90$ V. Find the impedance of the circuit, power consumption, or generation of each element, and determine whether the system is lead or lag and the power factor.

Solution The circuit impedance can be obtained by:

$$Z = \frac{V}{I} = \frac{180 + j90}{4 - j5} = \frac{201.24\angle 26.56}{6.40\angle -51.34} = 31.444\angle 77.9\ \Omega$$

Power is calculated by:

$$S = VI^* = 201.24\angle 26.56 \times 6.40\angle +51.34 = 1287.9\angle 77.9\ \text{VA}$$

$$S = 1287.9 \cos 77.9 + j1287.9 \sin 77.9\ \text{VA}$$

$$S = 268.08 + j1259.28\ \text{VA}$$

Non-ideal Capacitors

The dielectric material used in capacitors is ideally loss-free. It means that the charge applied to terminals of capacitors (i.e., plates) stays on the plates for infinite time. There is no internal current leak and internal discharge. However, that might not be true for existing dielectric materials. There is current, although minimal, passing through the material and hence discharge.

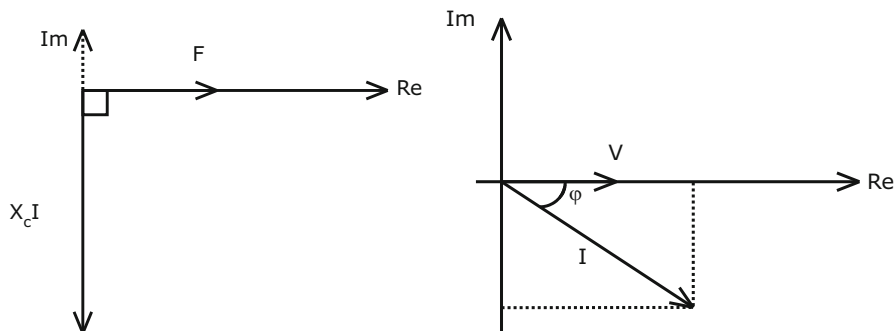


Fig. 5.37 Non-ideal capacitors. The internal resistance creates a phase angle φ between the voltage across the terminals of the capacitor and the current flow in the capacitor

The model for this internal discharge is a resistor that can be added either in series or in parallel to an ideal capacitor forming an RC circuit. The current in this circuit deviates from ideal 90° and hence generates an inphase and a perpendicular component (Fig. 5.37).

Model as RC Series

Consider the internal resistance R_{se} in series to an ideal capacitor. The angle of current with respect to the voltage as a reference is φ as opposed to 90° . Consider the angle deviation from 90° as $\beta = 90 - \varphi$.

$$\tan \beta = \frac{I R_{se}}{I X_C} = \frac{R_{se}}{X_C} = R_{se} C \omega$$

$$R_{se} = X_C \tan \beta$$

Power loss in this model can be calculated as:

$$P = R_{se} I^2 = X_C I^2 \tan \beta \text{ (W)}$$

Power factor can be obtained as follows:

$$\text{PF} = \sin \beta = \cos \varphi$$

Proof. From Fig. 5.38, it can be written as:

Fig. 5.38 Internal resistance of a non-ideal capacitor when presented in series

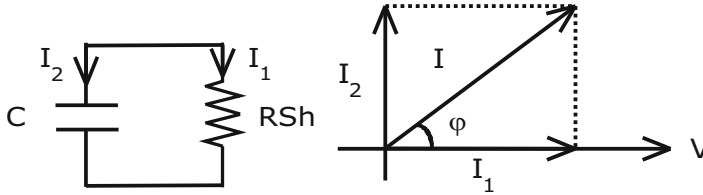
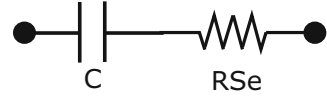


Fig. 5.39 Internal resistance of a non-ideal capacitor when presented in parallel

$$\begin{aligned}\cos \varphi &= \frac{R_{se}}{\sqrt{R_{se}^2 + X_C^2}} = \frac{X_C \tan \beta}{\sqrt{(X_C \tan \beta)^2 + X_C^2}} = \frac{\tan \beta}{\sqrt{\tan^2 \beta + 1}} = \frac{\frac{\sin \beta}{\cos \beta}}{\sqrt{\left(\frac{\sin \beta}{\cos \beta}\right)^2 + 1}} \\ &= \frac{\frac{\sin \beta}{\cos \beta}}{\sqrt{\frac{\sin^2 \beta + \cos^2 \beta}{\cos^2 \beta}}} = \sin \beta \\ \tan \beta &= \frac{\sin \beta}{\cos \beta} = \frac{\cos \varphi}{\sqrt{1 - \cos^2 \varphi}} = \frac{PF}{\sqrt{1 - PF^2}}\end{aligned}$$

Model as RC Parallel

Consider the internal resistance R_{sh} in parallel to an ideal capacitor. Figure 5.39 shows the balance of current in each branch and the real and imaginary values of current phasor.

$$\begin{aligned}\tan \beta &= \frac{I_1}{I_2} = \frac{V/R_{sh}}{V/X_C} = \frac{X_C}{R_{sh}} = \frac{1}{R_{sh} C \omega} \\ \tan \beta &= \frac{1}{R_{sh} C \omega} \\ R_{sh} &= \frac{1}{C \omega \tan \beta}\end{aligned}$$

Power loss in this model can be calculated as:

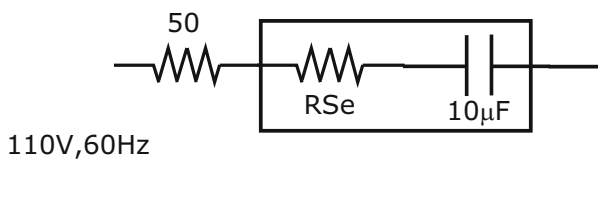


Fig. 5.40 Circuit of Example 5.23

$$P = \frac{V^2}{R_{sh}} = \frac{V^2}{X_C} \tan \beta \text{ (W)}$$

Example 5.23 A non-ideal capacitor with capacitance of $10 \mu\text{F}$ and phase deviation of $\beta = 10^\circ$ is connected in series with a 50Ω resistance shown in Fig. 5.40. The RC circuit is connected to a 120 V , 60 Hz source.

Find:

1. The increase in resistance due to the insertion of the capacitor
2. Power loss of the capacitor
3. Power factor PF

Solution The impedance of capacitor at 60 Hz is:

$$X_C = \frac{1}{C\omega} = \frac{1}{10e-6 \times 2\pi 60} = 265.25 \Omega$$

$$\varphi = 90 - \beta = 90 - 10 = 80^\circ$$

Considering a series RC circuit:

$$R_{se} = X_C \tan \beta = 265.25 \tan 10 = 46.77 \Omega$$

Therefore, the circuit has 50Ω in series with a 46.77Ω which is a total of 96.77Ω . Total impedance becomes:

$$Z(j\omega) = 96.77 - j265.26 \Omega$$

The impedance in polar form shows the amplitude (to calculate current) and the angle (to calculate the PF).

$$Z(j\omega) = 282.36 \angle -69.95^\circ \Omega$$

To find the current (needed for loss calculations):

$$I = \frac{V}{|Z|} = \frac{110}{282.36} = 0.389 \text{ A}$$

The power loss of the capacitor is obtained by:

$$P = R_{sc} I^2 = 46.77 (0.389)^2 = 7.098 \text{ (W)}$$

The power factor is obtained by:

$$\text{PF} = \cos \varphi = \cos 80 = 0.173 \text{ lead}$$

Dielectric Heating

Formation of non-ideal capacitors and loss in dielectric can be used in industry for heating. Voltage and frequency of operation can be adjusted to generate the amount of heat needed. The power loss, required voltage, and frequency using the parallel model are obtained as follows:

$$P = \frac{V^2}{R_{sh}}$$

$$V = \sqrt{PR_{sh}}$$

Considering $R_{sh} = \frac{1}{C\omega \tan \beta}$, the power loss is a function of V^2 and frequency ω .

$$P = V^2 C \omega \tan \beta$$

Adjusting the frequency and voltage from V_1, ω_1 to V_2, ω_2 for similar material to generate a desired amount of heat can be identified as follows:

$$P \propto V_1^2 \omega_1 = V_2^2 \omega_2$$

Example 5.24 Design a dielectric heating oven that can generate 200 W of power in a block material with $d = 1$ inch thickness, area of $A = 1$ square foot, with relative permittivity of $\epsilon = 5$, at PF = 0.05 and $f = 30$ MHz. If the voltage is limited to 200 V, find the required frequency to generate the same power.

Solution The block of material forms a capacitor with capacitance of $C = \epsilon_0 \epsilon \frac{A}{d}$:

$$d = 1 \text{ in} = 2.54 \text{ cm} = 2.54e-2 \text{ m}$$

$$A = 30.48e-2 \times 30.48e-2 \text{ m}^2$$

The capacitance of the system is $C = \epsilon_0 \epsilon_d^A = 5 \times 8.85e - 12 \times \frac{0.0929}{2.54e-2} = 161.84$ pF = $161.84e - 12$ F

$$\text{PF} = \cos \varphi = \sin \beta = 0.05$$

$$\tan \beta \cong 0.05$$

$$R_{\text{sh}} = \frac{1}{C\omega \tan \beta} = \frac{1}{161.84e - 12 \times 2\pi \times 30e6 \times 0.05} = 655.60 \Omega$$

$$V = \sqrt{PR_{\text{sh}}} = \sqrt{200 \times 655.60} = 362.1 \text{ V}$$

When the voltage is limited to 200 V, the frequency and voltage balance are as follows:

$$P \propto V_1^2 \omega_1 = V_2^2 \omega_2$$

$$362.1^2 \times 2\pi 30 \text{ MHz} = 200^2 \times 2\pi f_2$$

The adjusted frequency becomes:

$$f_2 = 98.337 \text{ MHz}$$

Thevenin Equivalent Circuits in Sinusoidal Steady State

Similar to Thevenin equivalent defined in RLC circuits, when the circuit operates at the steady-state sinusoidal, the impedance value of elements is considered. Therefore, the circuit can be presented as a voltage source in series to an impedance (Fig. 5.41).

The value of the voltage source and the impedance is obtained as follows. When the load is disconnected from the circuit the voltage measured at the terminals becomes the Thevenin voltage. To obtain the Thevenin impedance, the value of independent sources must become zero. It means an independent voltage source becomes a short circuit and an independent current source becomes an open circuit. Then, Thevenin impedance can be measured at the terminals.

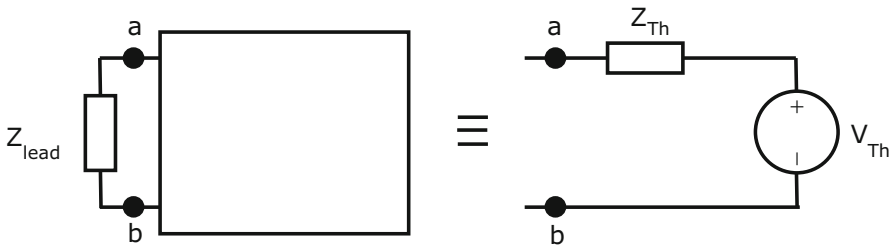


Fig. 5.41 Thevenin equivalent circuit from ports a and b across the rest of the circuit that is known as load

Fig. 5.42 Circuit of Example 5.25

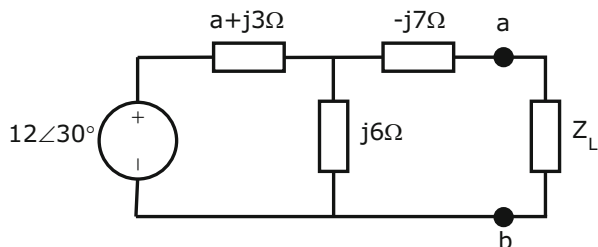
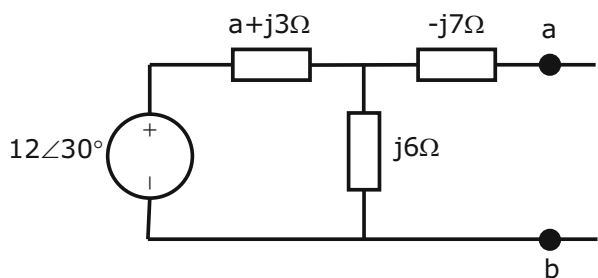


Fig. 5.43 Thevenin voltage calculations require that the load disconnected from the ports a and b



Example 5.25 Find the Thevenin equivalent of the following circuit measured at terminals a and b (Fig. 5.42).

Solution To obtain the Thevenin voltage, the Z_L must be disconnected from the circuit.

The voltage measured at the terminals a and b is the same as the voltage drop across the impedance $j6 \Omega$ because there is no current passing through the impedance $-j7 \Omega$; hence, it drops zero volts.

The voltage of $j6 \Omega$ is obtained through a voltage divider between $9 + j3 \Omega$ and $j6 \Omega$, as follows:

$$\begin{aligned} V_{th} &= \frac{j6}{j6 + (9 + j3)} 12 \angle 30^\circ = \frac{6 \angle 90^\circ 12 \angle 30^\circ}{12.72 \angle 45^\circ} = \frac{6 \times 12}{12.72} \angle (90 + 30 - 45) \\ &= 5.66 \angle 75^\circ \text{ V} \end{aligned}$$

To obtain the Thevenin impedance, the $12 \angle 30^\circ \text{ V}$ source must become zero (Fig. 5.43).

The impedance observed at the terminal is a parallel connection of $9 + j3 \Omega$ and $j6 \Omega$ in series to $-j7 \Omega$ as follows:

Fig. 5.44 Thevenin equivalent circuit from the ports a and b

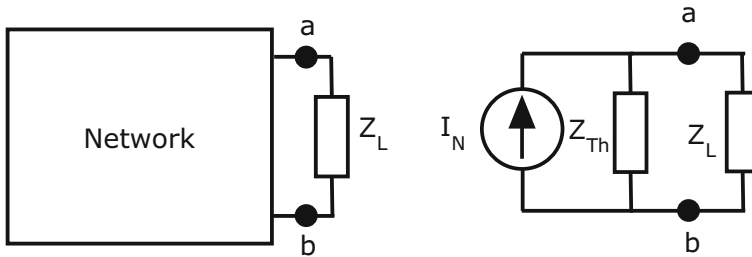
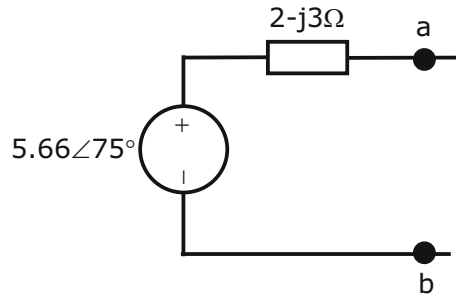


Fig. 5.45 Norton equivalent of a circuit from ports a and b

$$Z_{th} = ((9 + j3) \parallel j6) - j7 = \frac{(9 + j3)j6}{(9 + j3) + j6} - j7 = \frac{(3 + j1)j2}{1 + j1} - j7$$

$$Z_{th} = \frac{1 - j1}{1 - j1} \frac{j6 - 2}{1 + j1} - j7 = \frac{4 + j8}{2} - j7 = 2 - j3 \, \Omega$$

Therefore, the circuit from terminals has an equivalent as shown in Fig. 5.44.

Norton Equivalent and Source Conversion

Norton model presents equivalent of any circuit at a desired terminal through a current source in parallel to an impedance. The amount of current source (also known as short circuit current source) is obtained by measuring the current that passes through when shorting the circuit terminals. The impedance is obtained similar to Thevenin equivalent. Figure 5.45 shows a Norton equivalent of a network.

Example 5.26 Find the Norton equivalent of the following circuit (Fig. 5.46).

Solution Circuit load must be replaced with a short circuit at the terminals a and b . The short circuit current is actually the current through $-j7 \, \Omega$ impedance. The source current is calculated to be:

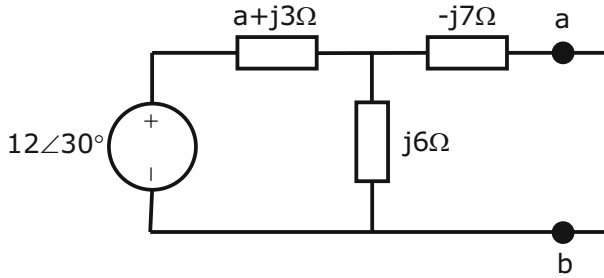
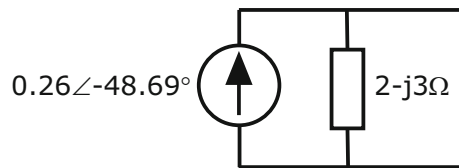


Fig. 5.46 Circuit of Example 5.26

Fig. 5.47 Norton equivalent circuit



$$I = \frac{12\angle 30}{(9 + j3) + (j6 \parallel -j7)} = \frac{12\angle 30}{(9 + j3) + \left(\frac{j6 \times -j7}{j6 - j7}\right)} = \frac{12\angle 30}{(9 + j3) + \left(\frac{j6 \times -j7}{j6 - j7}\right)}$$

$$= \frac{12\angle 30}{(9 + j3) + j42} = \frac{12\angle 30}{45.89\angle 78.69} = 0.26\angle -48.69 \text{ A}$$

The impedance observed at the terminal is a parallel connection of $9 + j3 \Omega$ and $j6 \Omega$ in series to $-j7 \Omega$ as follows:

$$Z_{th} = ((9 + j3) \parallel j6) - j7 = \frac{(9 + j3)j6}{(9 + j3) + j6} - j7 = \frac{(3 + j1)j2}{1 + j1} - j7$$

$$Z_{th} = \frac{1 - j1}{1 - j1} \frac{j6 - 2}{1 + j1} - j7 = \frac{4 + j8}{2} - j7 = 2 - j3 \Omega$$

Therefore, the circuit from terminals has an equivalent as shown in Fig. 5.47.

Thevenin to Norton Conversion. Equivalent circuits of Thevenin and Norton can be converted to each other. Following Fig. 5.48 shows the Thevenin to Norton conversion.

Following Fig. 5.49 shows the Norton to Thevenin conversion.

Example 5.27 Convert all Thevenin to Norton and all Norton to Thevenin (Fig. 5.50).

Solution The current can be calculated by $I_{sc} = \frac{120 - j150}{1 + j2} = -36 - j78 \text{ A}$ (Fig. 5.51)

Solution The voltage can be calculated by $V = (1 - j5)(70 + j15) = 145 - j335 \text{ V}$

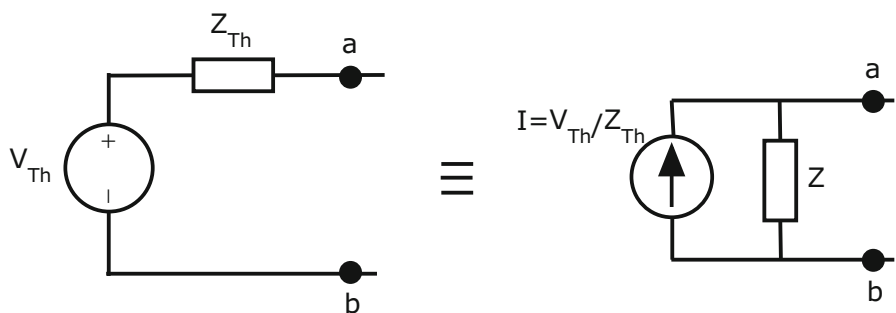


Fig. 5.48 Conversion of Thevenin to Norton equivalent circuit

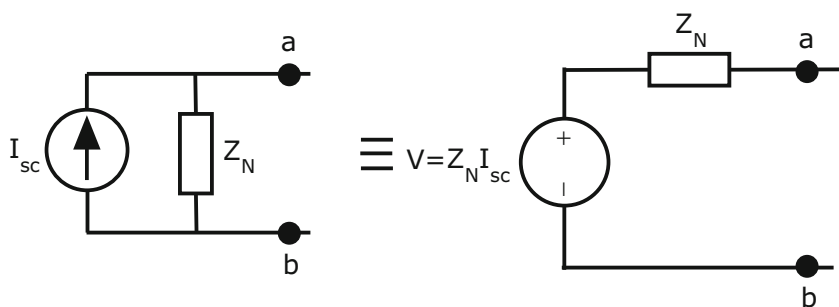


Fig. 5.49 Conversion of Norton to Thevenin equivalent circuit

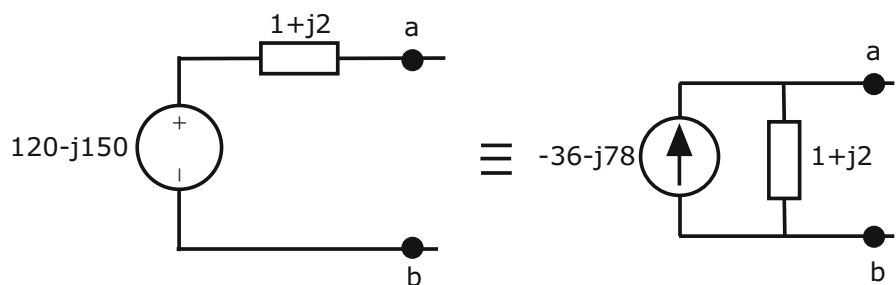


Fig. 5.50 Conversion of Thevenin to Norton

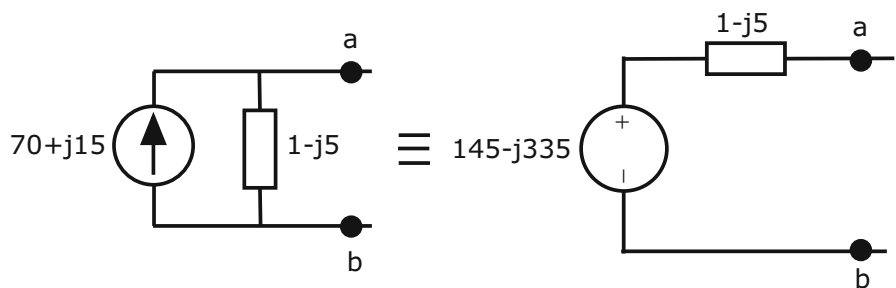


Fig. 5.51 Conversion of Norton to Thevenin

Maximum Power Transfer

Consider a circuit that is shown by its equivalent Thevenin model (V_{th} , $Z_{th}(j\omega)$) and is connected to a load impedance of $Z_L(j\omega)$. The power delivered to the load from the source can be calculated as follows:

Consider the real and imaginary part of each impedance as $Z_{th} = R_{th} + jX_{th}$ and $Z_L = R_L + jX_L$. The amount of power delivered to the load is calculated as:

$$P = R_L I^2 = R_L \left(\frac{|V|}{|Z_{th} + Z_L|} \right)^2 = R_L \left(\frac{|V|}{|R_{th} + R_L + j(X_{th} + X_L)|} \right)^2$$

$$= \frac{R_L |V|^2}{|(R_{th} + R_L)^2 + (X_{th} + X_L)^2|}$$

To maximize the power delivery to the load (Fig. 5.52),

$$\frac{dP}{dR_L} = 0 \quad \text{and} \quad \frac{dP}{dX_L} = 0$$

Imposing $\frac{dP}{dR_L} = 0$ yields:

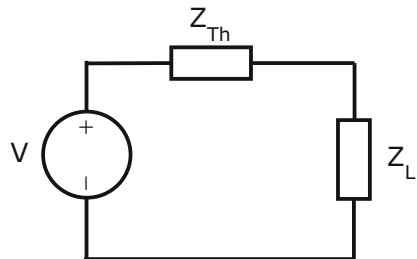
$$\frac{|V|^2 |(R_{th} + R_L)^2 + (X_{th} + X_L)^2| - R_L |V|^2 |2(R_{th} + R_L)|}{|(R_{th} + R_L)^2 + (X_{th} + X_L)^2|^2} = 0$$

$$\mathbf{R_L = R_{th}}$$

Imposing $\frac{dP}{dX_L} = 0$ yields:

$$\frac{-2R_L |V|^2 (X_{th} + X_L)}{|(R_{th} + R_L)^2 + (X_{th} + X_L)^2|^2} = 0$$

Fig. 5.52 Thevenin equivalent circuit is feeding a load. The maximum power is transferred from the source to the load when the Thevenin impedance and the load impedance become pairs of complex conjugate. This means $Z_L = Z_{th}^*$



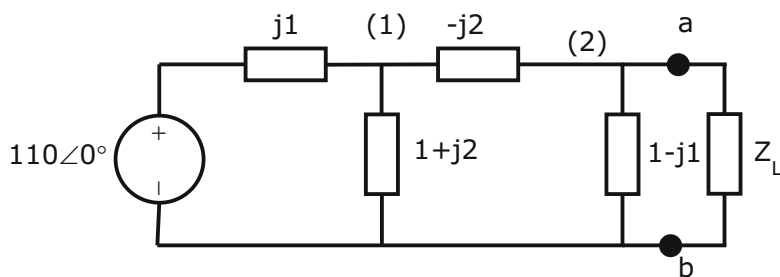


Fig. 5.53 Circuit of Example 5.28

$$X_L = -X_{th}$$

It can be concluded that to transfer maximum power in AC sinusoidal steady-state condition, the load impedance must be complex conjugate of the Thevenin impedance as:

$$@P_{\max}: Z_L = Z_{th}^*$$

Note 5.8 To obtain the load impedance that transfers maximum power from the source to the load through an existing circuit, the load impedance must be complex conjugate of the Thevenin impedance.

Example 5.28 Considering the following circuit, find the load impedance to cause maximum power transfer from the source to the load (Fig. 5.53).

Solution To transfer maximum power to the load, $Z_L = Z_{th}^*$ must hold. Therefore, it is required to find the Thevenin equivalent of the circuit from the load terminals.

The Thevenin voltage can be obtained from the voltage across the $1 - j \Omega$ impedance at node ② (when the load is disconnected). $V_2 = V_{th} = V_1 - j \Omega$.

KCL ①. All elements connected to node (1) are passive, hence draining the current out of the node.

$$\frac{V_1 - 110\angle 0}{j} + \frac{V_1}{1 + j2} + \frac{V_1 - V_{th}}{j2} = 0$$

KCL ②. Since the load is disconnected (to get the Thevenin impedance), there are two elements in current balance as follows:

$$\frac{V_{th} - V_1}{j2} + \frac{V_{th}}{1 - j} = 0$$

From ①:

$$V_1 \left(\frac{1}{j} + \frac{1}{1+j2} + \frac{1}{j2} \right) - \frac{V_{th}}{j2} = \frac{110 \angle 0}{j} \rightarrow V_1 \left(-j + \frac{1+j2}{5} + \frac{-j}{2} \right) + j \frac{V_{th}}{2} = -j110$$

$$V_1(0.2 - j1.1) + j0.5V_{th} = -j110$$

From ②:

$$V_{th} \left(\frac{1}{j2} + \frac{1}{1-j} \right) - \frac{V_1}{j2} = 0 \rightarrow V_{th} \left(\frac{-j}{2} + \frac{1+j}{2} \right) + j \frac{V_1}{2} = 0$$

$$0.5V_{th} + j0.5V_1 = 0 \rightarrow V_{th} = -jV_1$$

Replacing in ①:

$$V_1(0.2 - j1.1) + j0.5(-jV_1) = -j110$$

$$V_1(0.7 - j1.1) = -j110 \rightarrow V_1 = \frac{-j110}{0.7 - j1.1} = 71.17 - j45.29$$

$$V_1 = 84.36 \angle -32.47^\circ$$

$$V_{th} = 84.36 \angle -122.47^\circ$$

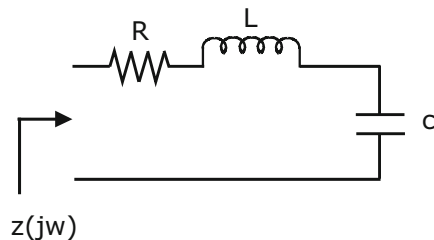
The Norton impedance can be obtained from the circuit when the independent sources are turned off. From the terminals, the circuit shows:

$$Z_{th} = (j \parallel (1 + j2) - j2) \parallel (1 - j) = 0.29 - j0.66 \, \Omega$$

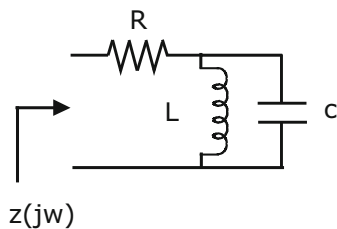
Therefore, the load impedance should be $Z_L = Z_{th}^* = 0.29 + j0.66 \, \Omega$.

Problems

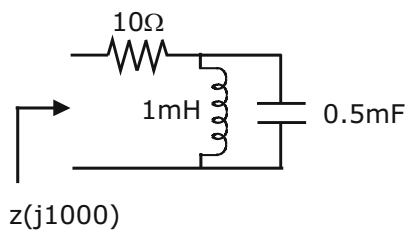
5.1. Find the impedance and admittance of the circuit.



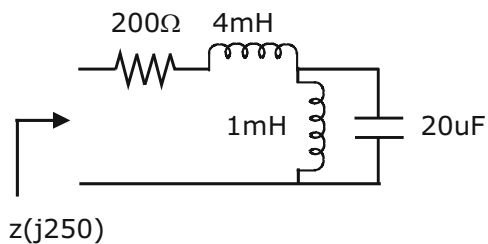
5.2. Find the impedance and admittance of the circuit.



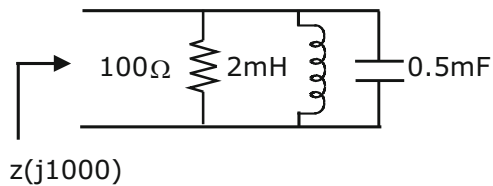
5.3. Find the impedance and admittance of the circuit.



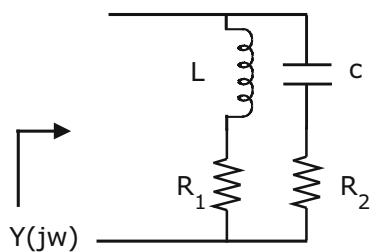
5.4. Find the impedance and admittance of the circuit.



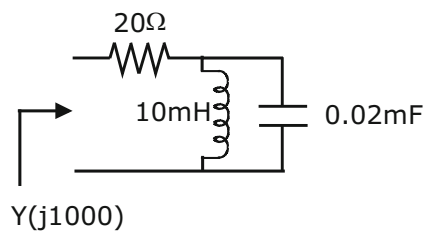
5.5. Find the impedance and admittance of the circuit.



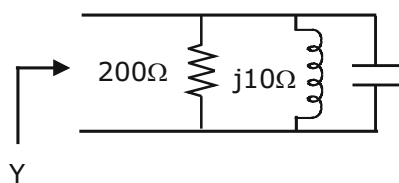
5.6. Find the admittance of the circuit.



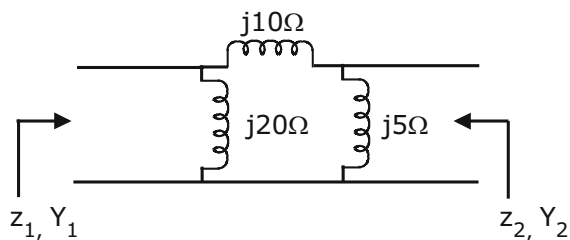
5.7. Find the admittance of the circuit.



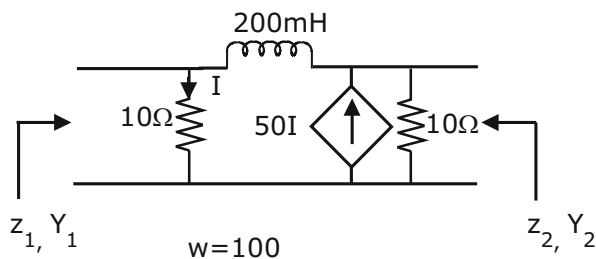
5.8. Find the admittance of the circuit.



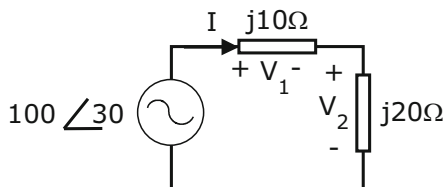
5.9. Find the admittance of the circuit.



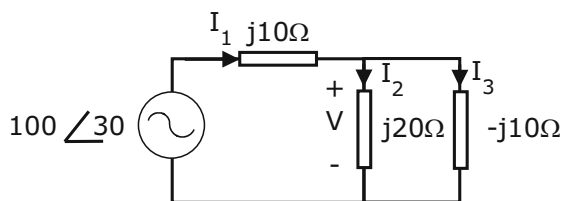
5.10. Find the admittance of the circuit.



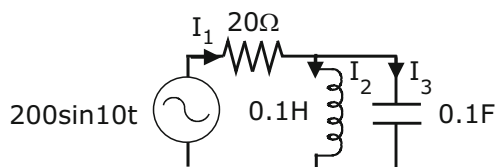
5.11. Find I , V_1 , and V_2 .



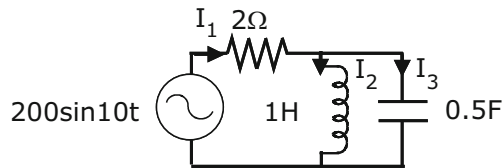
5.12. Find I_1 , I_2 , I_3 , and V .



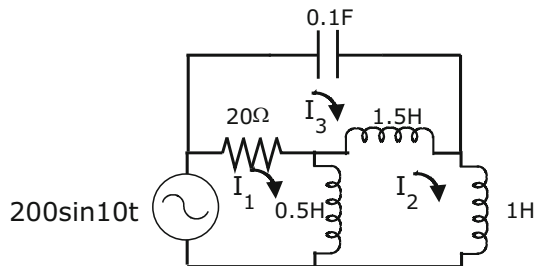
5.13. Find I_1 , I_2 , and I_3 .



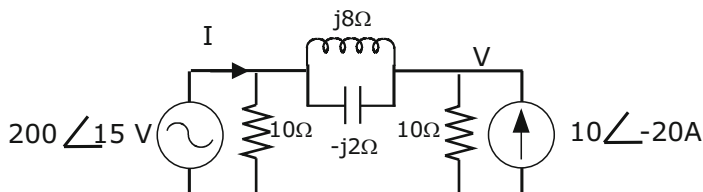
5.14. Find I_1 , I_2 , and I_3 .



5.15. Find I_1 , I_2 , and I_3 .



5.16. Find I and V .



5.17. The current $i(t) = 300 \sin(377t + 50^\circ)$ when passing through an impedance shows a voltage drop of $v(t) = 480 \sin(377t + 10^\circ)$.

- Find the impedance of the element.
- Find the resistance and reactance of the element.
- Determine whether the impedance is more capacitive or more inductive.

5.18. Find the current through a $Z = 100 \angle 30^\circ \Omega$ impedance when it is connected to a voltage of $V = 120\sqrt{2} \angle 10^\circ \text{ V}$.

5.19. A circuit creates 60° phase shift in the current with respect to the voltage. What is the power factor?

5.20. The voltage and current of an element are recorded as:

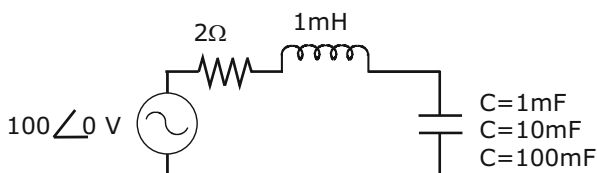
$$v(t) = 110\sqrt{2} \sin(100\pi t + 30^\circ) \text{ V}$$

$$i(t) = 20 \sin(100\pi t - 30^\circ) \text{ A}$$

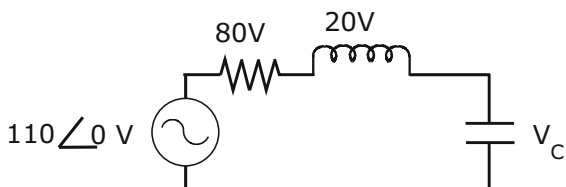
- (a) Find the impedance
 (b) Find the resistance
 (c) Find the reactance
 (d) Is the circuit more inductive or more capacitive?
- 5.21. The power factor of a circuit is $\text{PF} = 0.6$ lag. If the phase of the voltage is $+25^\circ$, what is the phase of its current?
- 5.22. The power factor of a circuit is $\text{PF} = 0.6$ lead. If the phase of the voltage is $+25^\circ$, what is the phase of its current?
- 5.23. In a series RLC circuit, the voltage drop across each element is measured as:

$$V_R = 20 \text{ V}, V_L = 75 \text{ V}, V_C = 50 \text{ V}.$$

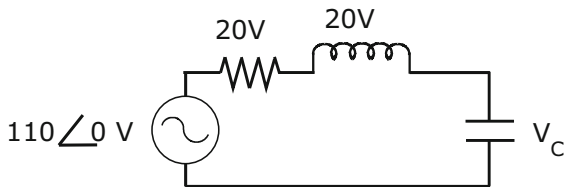
- (a) How much is the source voltage?
 (b) What is the circuit's power factor?
- 5.24. In the following circuit, find the voltage of the inductor for different values of the capacitor $C = 1 \text{ mF}$, $C = 10 \text{ mF}$, $C = 100 \text{ mF}$.



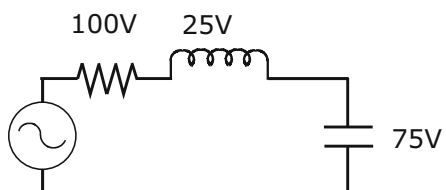
5.25. Find V_C .



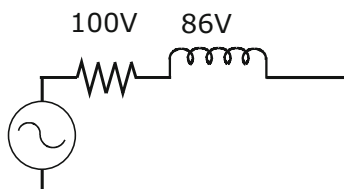
5.26. Find V_C .



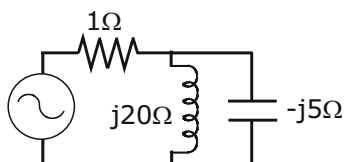
5.27. Find PF.



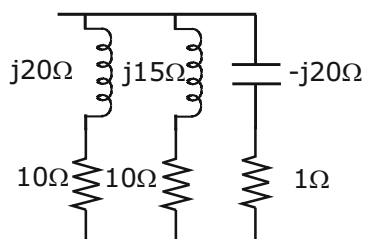
5.28. Find PF.



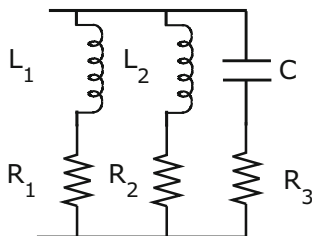
5.29. Find PF.



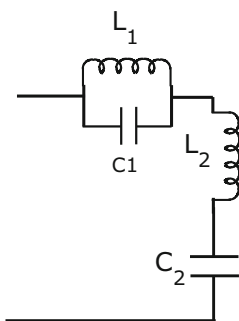
5.30. Find PF.



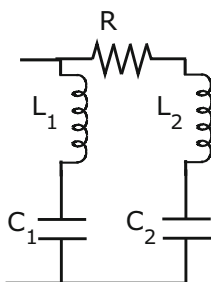
5.31. Find PF.



5.32. Find PF.



5.33. Find PF.



5.34. Find the apparent power, active power, reactive power, and power factor in the following cases. Determine the generation or consumption of the power.

- (a) $Z = 1 + j5 \, \Omega$, $I = 15 \angle -30^\circ \text{ A}$
- (b) $Z = 1 + j5 \, \Omega$, $I = 15 \angle +30^\circ \text{ A}$
- (c) $V_{\text{rms}} = 100 \angle 30^\circ \text{ V}$, $I_{\text{rms}} = 20 \angle -40^\circ \text{ A}$
- (d) $V_{\text{rms}} = 100 \angle 30^\circ \text{ V}$, $I_{\text{rms}} = 20 \angle +40^\circ \text{ A}$
- (e) $Z = 75 \angle 20^\circ \, \Omega$, $V_{\text{rms}} = 220 \angle +10^\circ \text{ V}$

- (f) $Z = 50 \angle -20^\circ \Omega$, $V_{\text{rms}} = 110 \angle +35^\circ \text{ V}$
- (g) $V = 100 \angle 30^\circ \text{ V}$, $I = 20 \angle -40^\circ \text{ A}$
- (h) $V = 100 \angle 30^\circ \text{ V}$, $I = 20 \angle +40^\circ \text{ A}$
- (i) $Z = 75 \angle 20^\circ \Omega$, $V = 220 \angle +10^\circ \text{ V}$
- (j) $Z = 50 \angle -20^\circ \Omega$, $V = 110 \angle +35^\circ \text{ V}$

5.35. Find the requested variable in each case.

- (a) $P = 1000 \text{ W}$, $Q = 1500 \text{ VAR}$, $S = ?$, $\text{PF} = ?$, Lead or Lag?
- (b) $P = 1000 \text{ W}$, $Q = -1500 \text{ VAR}$, $S = ?$, $\text{PF} = ?$, Lead or Lag?
- (c) $S = 100 \text{ kVA}$, $P = 80 \text{ kW}$, $Q = ?$, $\text{PF} = ?$, Lead or Lag?
- (d) $S = 200 \text{ kVA}$, $Q = 50 \text{ kVA}$, $Q = ?$, $\text{PF} = ?$, Lead or Lag?
- (e) $S = 500 \text{ kVA}$, $\text{PF} = 0.6$ Lead, $Q = ?$, $P = ?$
- (f) $S = 500 \text{ kVA}$, $\text{PF} = 0.6$ Lag, $Q = ?$, $P = ?$
- (g) $V_p = 200 \angle 30^\circ \text{ V}$, $I_p = 10 \angle -40^\circ$, $S = ?$, $P = ?$, $Q = ?$, $\text{PF} = ?$, Lead or Lag?
- (h) $V_p = 200 \angle 30^\circ \text{ V}$, $I_p = 10 \angle 40^\circ$, $S = ?$, $P = ?$, $Q = ?$, $\text{PF} = ?$, Lead or Lag?

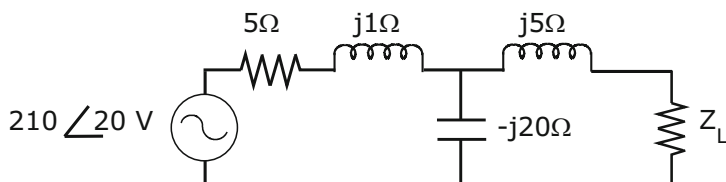
5.36. In a household operating at the $110 \text{ V}_{\text{rms}}$, 60 Hz , there are multiple loads taking current from the grid. The current taken by these loads is:

$$I_1 = 50 \angle 0^\circ \text{ A}, I_2 = 20 \angle -45^\circ \text{ A}, I_3 = 12 \angle -30^\circ \text{ A}, I_4 = 35 \angle 0^\circ \text{ A}$$

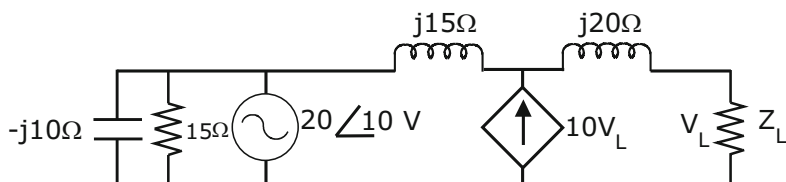
- (a) Find the apparent, active, and reactive power of each load.
 - (b) Find the total active power and total reactive power of the loads.
 - (c) Find the total apparent power the household.
 - (d) Find the power factor of the house.
 - (e) Is this load lead or lag? Why?
- 5.37. One phase of an industrial load operating at $220 \text{ V}_{\text{rms}}$ has a load of $S = 2000 + j15,000 \text{ VA}$.
- (a) Find the power factor of this load. Lead or lag? Why?
 - (b) In parallel to this load, a capacitor is used to compensate for the power factor. Find the capacitor power required to change the power factor to 0.8 and to 0.97.
- 5.38. An induction motor shows an inductive load of $S = 300 \text{ kVA}$ at power factor $\text{PF} = 0.67$.
- (a) Find the right amount of capacitor to be connected in parallel at the terminal of this machine to bring the power factor to $\text{PF} = 0.97$.
 - (b) Find the active power and reactive power taken from the grid before and after the power factor correction.
- 5.39. A mix of wood chips and glue is pressed between two parallel plates at voltage of 600 V . The structure forms of a plywood structure at a capacity of $1 \mu\text{F}$.

Find the required frequency at which a power loss of 500 W is applied to the glue and woodchips to melt the glue. Consider the phase shift in the current with respect to the voltage of the capacitor to be $\phi = 10^\circ$. Find the frequency at which the power loss is 1200 W. At 1200 W, find the equivalent voltage if the frequency is fixed at 1 kHz.

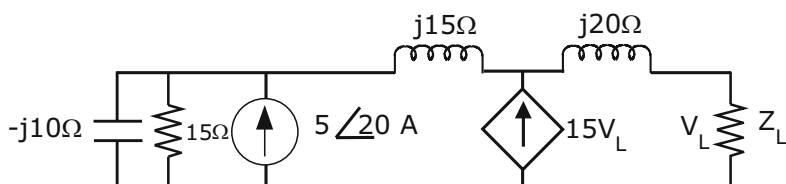
5.40. Find Thevenin and Norton.



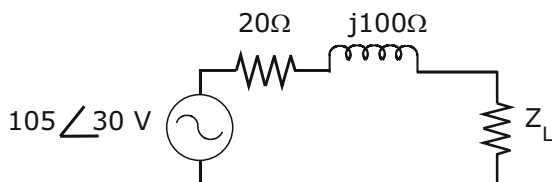
5.41. Find Thevenin and Norton.



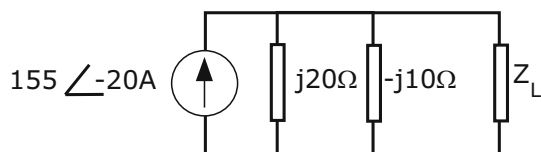
5.42. Find Thevenin and Norton.



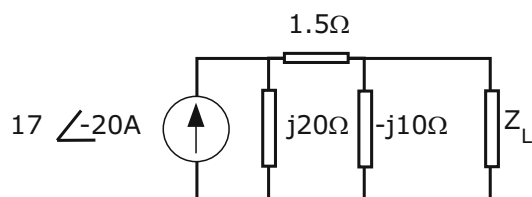
5.43. Find Thevenin and Norton.



5.44. Find Thevenin and Norton.



5.45. Find Thevenin and Norton.



5.46. Find the load impedance at which the circuits of previous problems (40–45) transfer maximum power from the source to the load.

Chapter 6

Mutual Inductance



Introduction

Electric circuits specifically when they are excited by AC sources can transfer energy either by direct electric connection or through magnetic coupling. Consider an inductor with N turns of winding. The current i passing through this inductor generates a magnetic flux ϕ around the windings. This flux creates a magnetic field that starts from the North Pole and ends at the South Pole. When the direction of current changes, the location of north and south poles changes which causes a change in direction of flux but still from the North Pole to the South Pole (Fig. 6.1). The time variations in the flux generates a voltage in the coil that is measured by:

$$v = N \frac{d\phi}{dt}$$

As explained earlier, the flux is a current dependent variable. Therefore:

$$v = N \frac{d\phi}{di} \frac{di}{dt}$$

The total amount of flux variation accounted for number of turns in a coil determines its inductance. This means that higher number of turns or higher area to generate more flux increases the inductance value. This can be calculated by:

$$L = N \frac{d\phi}{di}$$

Replacing the inductance in the voltage equation reveals the OHM's law that was discussed earlier. Hence:

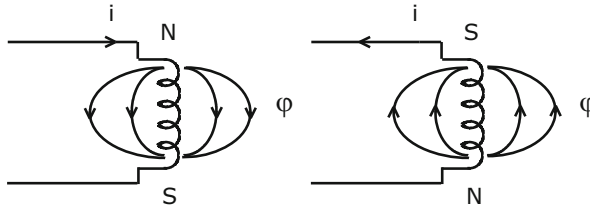
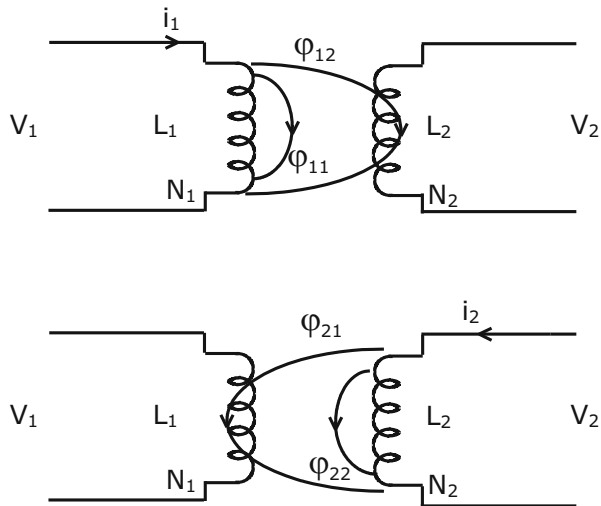


Fig. 6.1 Magnetic field generated because of the current passing through a single inductor. The direction of the field is always from the North Pole to the South Pole. When the direction of current changes, the direction of the magnetic field also changes.

Fig. 6.2 When the magnetic fields generated by one coil is cut by the windings of another adjacent coil, there is voltage induced in the second coil. The direction of the current entering the first coil and the direction of windings in the second coil determine the polarity of the induced voltage. The split of magnetic flux that is passing through air and circulating the main coil generates the self-inductance, and the part that is passing the adjacent coil builds the mutual inductance



$$v = L \frac{di}{dt}$$

Self-Inductance and Mutual Inductance

The value of inductance that is utilized in this equation is known as **self-inductance**. This means that the voltage induced in a coil is solely generated by the time varying current of the same coil. However, if the flux generated by another coil passes through the windings of the inductor, a voltage is induced in the coil due to a **mutual inductance** which may exist between coils.

Consider two adjacent coils (Fig. 6.2) with self-inductances of L_1 and L_2 Henrys, and N_1 and N_2 turns. Coil 1 is connected to v_1 source that establishes current i_1 .

The flux ϕ_1 generated by coil 1 has two parts. Part 1 links the coil 1, ϕ_{11} , due to self-inductance and part 2 that links coil 2, ϕ_{12} , due to mutual inductance. Hence:

$$\phi_1 = \phi_{11} + \phi_{12}$$

Induced Voltage

The induced voltages at the terminal of each coil depends on total amount of flux linking the coil and the number of turns the coil has as follows:

Total flux $\phi_1 = \phi_{11} + \phi_{12}$ links coil 1 and is generated by current i_1 :

$$v_1 = N_1 \frac{d\phi_1}{dt} \rightarrow v_1 = N_1 \left(\frac{d\phi_1}{di_1} \frac{di_1}{dt} \right) = \left(N_1 \frac{d\phi_1}{di_1} \right) \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

The voltage in the second coil is generated by the current in the first coil through the linking flux ϕ_{12} . Considering M_{21} as the mutual inductance on coil 2 influenced by coil 1:

$$v_2 = N_2 \frac{d\phi_{12}}{dt} \rightarrow v_2 = N_2 \left(\frac{d\phi_{12}}{di_1} \frac{di_1}{dt} \right) = \left(N_2 \frac{d\phi_{12}}{di_1} \right) \frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

Hence $v_2 = M_{21} \frac{di_1}{dt}$ determines that the induced voltage in coil 2 is influenced by the current variation in coil 1 through mutual inductance, M_{21} . It also indicates that the voltage is induced in coil 2 as a result of current passing coil 1.

Now consider that the current is passing coil 2 and coil 1 voltage is measured (Fig. 6.2). The current i_2 generates a total flux of ϕ_2 which has two parts of ϕ_{22} that links the coil 2 and ϕ_{21} that links the coil 1. Total flux is:

$$\phi_2 = \phi_{22} + \phi_{21}$$

The induced voltages at the terminal of each coil depend on total amount of flux linking the coil and the number of turns the coil has as follows:

Total flux $\phi_2 = \phi_{22} + \phi_{21}$ links coil 2 and is generated by current i_2 :

$$v_2 = N_2 \frac{d\phi_2}{dt} \rightarrow v_2 = N_2 \left(\frac{d\phi_2}{di_2} \frac{di_2}{dt} \right) = \left(N_2 \frac{d\phi_2}{di_2} \right) \frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

The voltage induced in first coil is generated by the current in the second coil through the linking flux ϕ_{21} . Therefore:

Fig. 6.3 Direction of both currents entering the dot • induces the polarity that supports the direction of current i_2

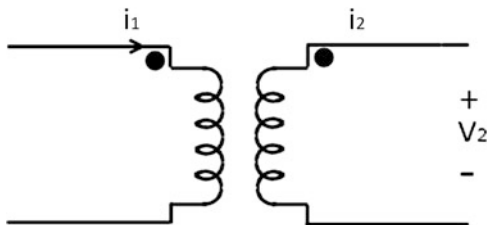
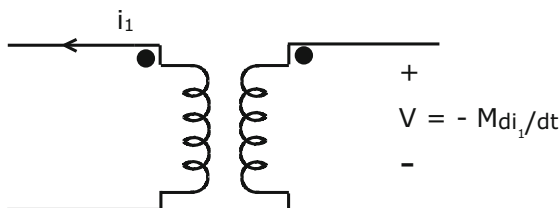


Fig. 6.4 Reverse direction of the current i_1 induces the reverse polarity of voltage v_2



$$v_1 = N_1 \frac{d\phi_{21}}{dt} \rightarrow v_1 = N_1 \left(\frac{d\phi_{21}}{di_2} \frac{di_2}{dt} \right) = \left(N_1 \frac{d\phi_{21}}{di_2} \right) \frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

$$v_1 = M_{12} \frac{di_2}{dt}$$

Mutual inductance is a reciprocal quantity, meaning that the same voltage will be induced in coil 1 if the current is passed through coil 2. This means $M_{21} = M_{12} = M$, all measured in Henys (H).

Mutual inductance exists when two or more coils are physically located such that the flux generated by one coil finds appropriate path to link the adjacent coils. If this path does not exist or the flux is not time varying, the mutual inductance disappears.

Mutual inductance is often shown by • signs located on one of the terminals at each port. The location of dot and the direction of current in and out of the dot determine the polarity of the induced voltage due to mutual inductance.

- The current entering the dotted terminal of a coil induces positive voltage at the dotted terminal of the second coil (Fig. 6.3).
- The current leaving the dotted terminal of a coil induces negative voltage in the dotted terminal of the second coil (Figs. 6.4, 6.5, and 6.6).

Therefore, to determine the polarity of induced voltages in each coil, the location of dotted terminals and the direction of current in the other coil must be considered.

Example 6.1 Determine the polarity of the induced voltage in the circuit of Fig. 6.6.

Solution Induced voltage due to mutual inductance in the first loop can be modeled as a voltage source that has positive polarity in the loop circulating clockwise (Fig. 6.7).

Fig. 6.5 The amount of voltage induced is directly proportional to the value of the mutual inductance m and the rate of the change of the voltage inducing current, i_1 , or $v = -M \frac{di_1}{dt}$

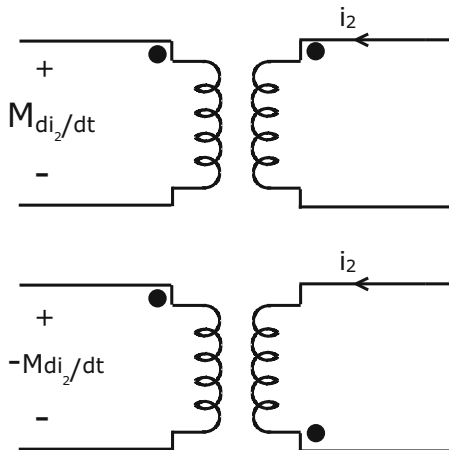


Fig. 6.6 Direction of currents is in the dotted mutual inductance. The voltages they induce must support the direction of these currents

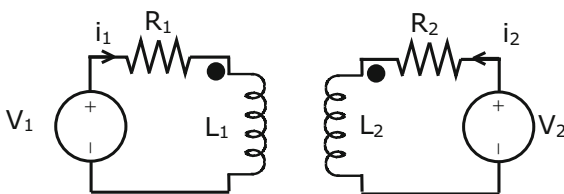
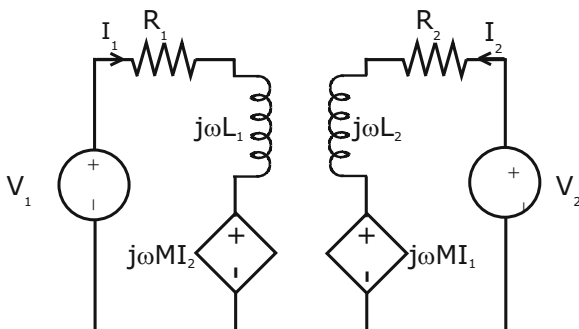


Fig. 6.7 Figure of circuit in Example 6.1



The amount of induced voltage source in the first loop is $+j\omega MI_2$ because the current is entering the dotted terminal in the second loop. The second loop has an induced voltage of $+j\omega MI_1$ with positive polarity because the current I_1 enters the dotted terminal of the first loop.

Mutual inductance generates a current controlled voltage source. In two loops, when the dotted terminals are removed, the circuit can be shown as Fig. 6.7 and calculated as follows:

KVL ①:

$$-V_1 + R_1 I_1 + j\omega L_1 I_1 + j\omega M I_2 = 0$$

KVL ②:

$$-V_2 + R_2 I_2 + j\omega L_2 I_2 + j\omega M I_1 = 0$$

Solving for I_1 and I_2 using Cramer's method is as follows:

First, the equations have to be simplified as follows:

$$\begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$\begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix}$ is the matrix of coefficients (lets consider it as A),

$$A = \begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix}$$

$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ is the matrix of unknown variables (let consider it as X),

$$X = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

and $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is the matrix of known inputs (let consider it as B).

$$B = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Therefore, the equation reads as:

$$AX = B$$

Solving for X results in:

$$X = A^{-1}B$$

Matrix analysis of circuits will be covered in Chap. 12. Unknown variables are obtained as:

Fig. 6.8 Circuit of Example 6.2. The same current passes through the series inductors, and the current enters each of these dotted nodes

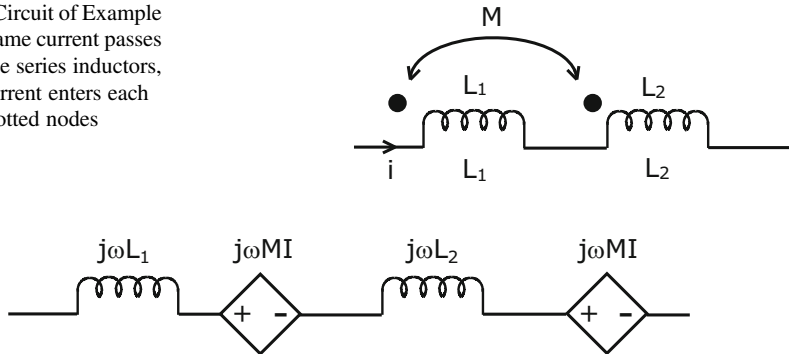


Fig. 6.9 Equivalent circuit of Example 6.2. Induced voltages are shown as voltage sources. One should pay attention to the direction of the current and the polarity of the dependent voltage sources (showing the induced voltage)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{(R_1 + j\omega L_1)(R_2 + j\omega L_2) - (j\omega M)(j\omega M)} \begin{bmatrix} R_2 + j\omega L_2 & -j\omega M \\ -j\omega M & R_1 + j\omega L_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I_1 = \frac{R_2 V_1 + j\omega(L_2 V_1 - M V_2)}{M^2 \omega^2 + R_1 R_2 - L_1 L_2 \omega^2 + j\omega(L_1 R_2 + L_2 R_1)}$$

$$I_2 = \frac{R_1 V_2 + j\omega(L_1 V_2 - M V_1)}{M^2 \omega^2 + R_1 R_2 - L_1 L_2 \omega^2 + j\omega(L_1 R_2 + L_2 R_1)}$$

Example 6.2 Consider two mutually coupled inductors connected in series as shown in Fig. 6.8. Find the equivalent inductance of the circuit.

Solution As shown the current enters the dotted terminal of inductor 1; therefore the voltage induced in inductor 2 shows a positive voltage with respect to the dotted terminal of second inductor. As the current of first inductor enters the dotted terminal, the inductor shows a positive polarity voltage (Fig. 6.9).

A KVL of the circuit shows:

$$V = j\omega L_1 I + j\omega M I + j\omega L_2 I + j\omega M I$$

$$V = j\omega(L_1 + M + L_2 + M)I = j\omega L_{eq} I$$

$$L_{eq} = L_1 + L_2 + 2M$$

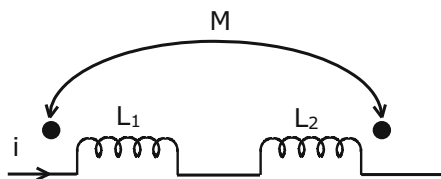


Fig. 6.10 Circuit of Example 6.3. The same current passes through the series inductors, but the current has opposite direction to each of the dotted nodes, enters one and exits the other

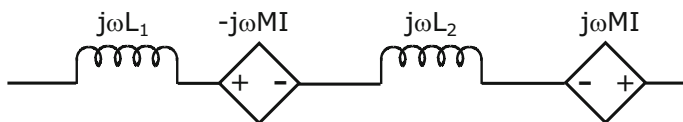


Fig. 6.11 Equivalent circuit of Example 6.2. Induced voltages are shown as voltage sources. One should pay attention to the direction of the current and the polarity of the dependent voltage sources (showing the induced voltage)

Example 6.3 Consider two mutually coupled inductors connected in series as shown in Fig. 6.3. Find the equivalent inductance of the circuit (Fig. 6.10).

Solution As shown the current enters the dotted terminal of inductor 1; therefore the voltage induced in inductor 2 shows a positive voltage with respect to the dotted terminal of second inductor. The source positive terminal is connected to the dotted terminal. The same principle applies to the first inductor as the current leaves the dotted terminal of second inductor it induces negative voltage in the first inductor (Fig. 6.11).

A KVL of the circuit shows:

$$V = j\omega L_1 I + (-j\omega MI) + j\omega L_2 I - j\omega MI$$

$$V = j\omega(L_1 - M + L_2 - M)I = j\omega L_{eq} I$$

$$L_{eq} = L_1 + L_2 - 2M$$

Example 6.4 Consider the circuit show in Fig. 6.12 with mutual inductance between the inductors. Find the current in each circuit.

Solution Considering the mutual inductance and the dotted terminals and the direction of currents, the voltage induced in the first circuit shows a negative voltage because the current i_2 leaves the dotted terminal of the second inductor.

Fig. 6.12 Circuit of Example 6.4

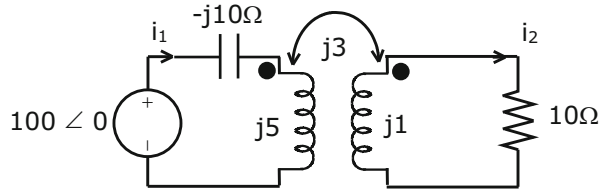
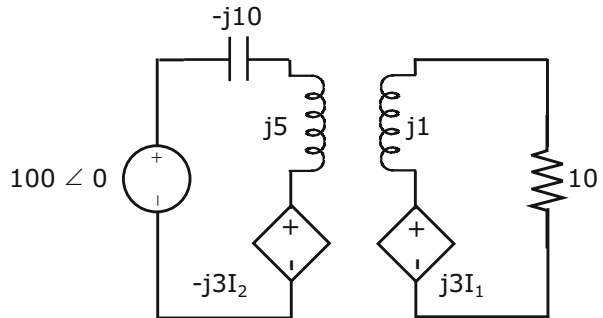


Fig. 6.13 The induced voltage of circuit in Fig. 6.12 is expanded to the dependent voltage sources. The mutual inductance previously shown as dots is now presented at the circuit level by voltage sources



In the second loop, the induced voltage is a positive polarity with respect to the dotted line because the current in the first loop enters the dotted terminal of the first inductor. The equivalent circuit is shown as follows (Fig. 6.13).

KVL ①.

$$-100 - j10I_1 + j5I_1 - j3I_2 = 0$$

KVL ②.

$$10I_2 - j3I_1 + jI_2 = 0$$

Using matrix approach:

$$\begin{bmatrix} -j5 & -j3 \\ -j3 & 10 + j \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -j5 & -j3 \\ -j3 & 10 + j \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$I_1 = 3.33 + j19.06 \text{ A} = 19.34 \angle 80^\circ \text{ A}$$

$$I_2 = -5.56 + j1.55 \text{ A} = 5.77 \angle 164.4^\circ \text{ A}$$

Fig. 6.14 Circuit of example 6.5

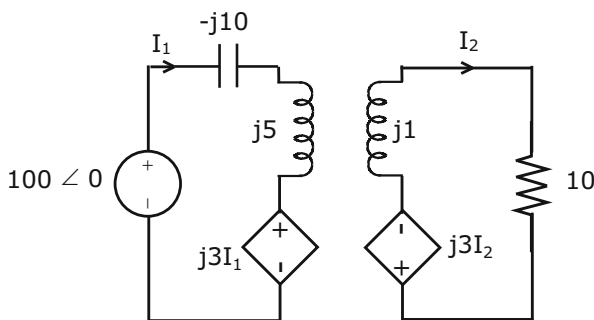
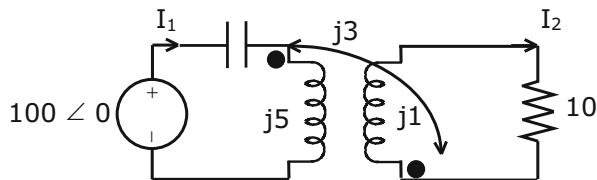


Fig. 6.15 The induced voltage of circuit in Fig. 6.14 is expanded to the dependent voltage sources. The mutual inductance previously shown as dots is now presented at the circuit level by voltage sources. Pay close attention to the direction of the currents entering or leaving the mutual inductance dots and the polarity of the voltage sources

Example 6.5 Consider the circuit of previous example with a reverse connection of mutual inductance (dotted terminals are connected in reverse). Find the current of each circuit (Fig. 6.14).

Solution Since the current enters the dotted terminal in both inductors, they induce positive voltage to the mutually coupled inductors. This polarity is measured positive with respect to the dotted terminal. The equivalent circuit is shown in Fig. 6.15.

Therefore, the KVLs can be written as follows:

KVL ①.

$$-100 - j10I_1 + j5I_1 + j3I_2 = 0$$

KVL ②.

$$10I_2 + j3I_1 + jI_2 = 0$$

Using matrix approach:

$$\begin{bmatrix} -j5 & j3 \\ j3 & 10+j \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -j5 & j3 \\ j3 & 10+j \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$I_1 = 3.33 + j19.06 \text{ A} = 19.34 \angle 80^\circ \text{ A}$$

$$I_2 = 5.56 - j1.55 \text{ A} = 5.77 \angle -15.57^\circ \text{ A}$$

Energy Stored in Coupled Circuits

Inductors store energy depending on the current amplitude and their self-inductance calculated by $W = \frac{1}{2}LI^2$. Considering the power of a coil as a product of voltage and current, the energy is obtained as follows:

$$P(t) = v(t)i(t) = L \frac{di(t)}{dt} i(t)$$

$$W = \int P(t)dt = \int L \frac{di(t)}{dt} i(t)dt = L \int i(t)di(t) = \frac{1}{2}LI^2$$

Consider a mutually coupled coil as shown in Fig. 6.16. Mutual inductance M induces additional voltage, which changes the power equation as follows:

In loop ①, the voltage is measured as:

$$v_1 = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

The power measured from loop ① is obtained as:

Fig. 6.16 Mutual inductance M between two coils at inductance L_1 and L_2

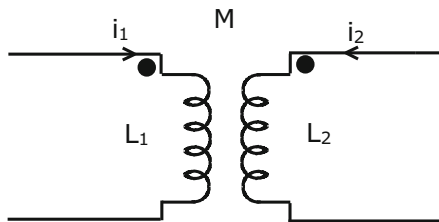
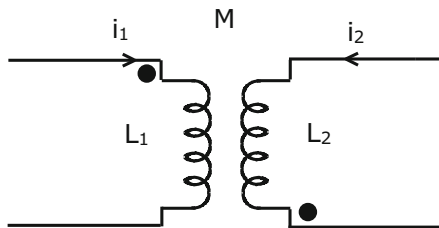


Fig. 6.17 Mutual inductance M with opposite dotted terminals between two coils of L_1 and L_2



$$P_1(t) = v_1(t)i_1(t) = \left(L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \right) i_1(t)$$

$$W_1 = \int \left(L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \right) i_1(t) dt = \frac{1}{2} L_1 I_1^2 + M I_1 I_2$$

Considering the overall circuit, there are three energy storing elements as self-inductance L_1 , self-inductance L_2 , and the mutual inductance M .

W_1 considers the energy stored in self-inductance L_1 and the mutual inductance M . The energy stored in the self-inductance L_2 is measured as:

$$W_2 = \int L_2 \frac{di_2(t)}{dt} i_2(t) dt = \frac{1}{2} L_2 I_2^2$$

The energy stored in entire circuit is obtained as:

$$W = W_1 + W_2$$

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

Example 6.6 Find the stored energy in the following circuit (Fig. 6.17).

Solution In loop ①, the voltage is measured as:

$$v_1 = L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt}$$

The power measured from loop ① is obtained as:

$$P_1(t) = v_1(t)i_1(t) = \left(L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt} \right) i_1(t)$$

$$W_1 = \int \left(L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt} \right) i_1(t) dt = \frac{1}{2} L_1 I_1^2 - M I_1 I_2$$

Considering the overall circuit, there are three energy storing elements as self-inductance L_1 , self-inductance L_2 , and the mutual inductance M .

W_1 considers the energy stored in self-inductance L_1 and the mutual inductance M . The energy stored in the self-inductance L_2 is measured as:

$$W_2 = \int L_2 \frac{di_2(t)}{dt} i_2(t) dt = \frac{1}{2} L_2 I_2^2$$

The energy stored in entire circuit is obtained as:

$$W = W_1 + W_2$$

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2$$

Limit of Mutual Inductance

Considering that the energy stored in passive inductors cannot be negative, $\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2 \geq 0$. As a complete square expression, the maximum limit of the mutual inductance is shown as:

$$M \leq \sqrt{L_1 L_2}$$

As mentioned earlier, the mutual inductance depends on the geometry of the coils with respect to each other, magnetic core, and their orientation. A maximum mutual inductance is reached with the linking flux of one coil entirely pass through the second coil. This may occur when two coils are concentric, e.g., one coil is wrapped around the other. As the geometries move away from each other, the linking flux between the coils is reduced. The ratio of

$$k = \frac{\phi_{12}}{\phi_{11} + \phi_{12}} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}}$$




From Coil 1
From Coil 2

is a factor $0 \leq k \leq 1$. The mutual inductance is obtained as:

$$M = k \sqrt{L_1 L_2}$$

Considering the minimum and maximum values of k , the mutual inductance can reach:

$$0 \leq M \leq \sqrt{L_1 L_2}$$

Turn Ratio

Consider two coils wrapped around a core. The ratio of their inductance is

$$\frac{L_2}{L_1} \propto \frac{N_2^2}{N_1^2} = n^2$$

where $N_{1,2}$ is the number of turns a coil has and n is the turn ratio. Therefore, the ratio of inductance is proportional to the square of turn ratio.

Equivalent Circuit of Mutual Inductance

Consider the following circuit in which two inductors L_1 and L_2 have mutual inductance of M . This mutual inductance might be positive or negative value depending on the dotted terminals and the direction of current, which also determines the polarity of the induced voltage. Therefore, in general, the mutual inductance can be either a positive or negative number. This mutual inductance can be shown as an equivalent T or Π inductive circuit.

T Equivalent Circuit

Considering the circuit shown in Fig. 6.18, KVLs in loops ① and ② show:

$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$V_2 = j\omega M I_1 + j\omega L_2 I_2$$

In matrix form, these equations can be written as:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Fig. 6.18 Mutual inductance for the purpose of T, Π equivalent circuit

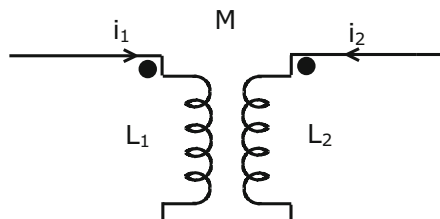
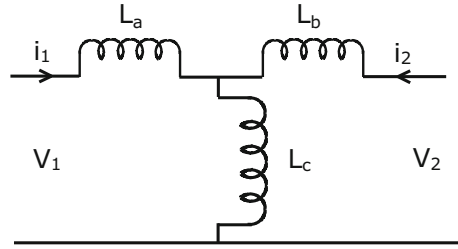


Fig. 6.19 T equivalent circuit of mutual inductance



Considering the same current direction and terminal voltages, a T equivalent circuit is shown in Fig. 6.19. KVLs in loop ① and ② can be written as:

$$V_1 = j\omega(L_a + L_b)I_1 + j\omega L_c I_2$$

$$V_2 = j\omega L_c I_1 + j\omega(L_b + L_c)I_2$$

In matrix form the equivalent circuit becomes:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_b) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Comparing the matrices, the equivalent L_a , L_b , and L_c can be obtained as follows:

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \equiv \begin{bmatrix} j\omega(L_a + L_b) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix}$$

Therefore:

$$L_a = L_1 - M$$

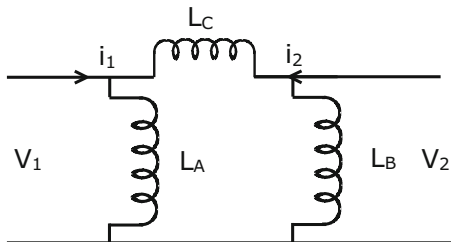
$$L_b = L_2 - M$$

$$L_c = M$$

II Equivalent Circuit

The admittance matrix can be obtained from the matrix form KVL written or the mutual inductance circuit Fig. 6.18, as follows (solving for current in the KVL):

Fig. 6.20 Π equivalent circuit of mutual inductance



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

This yields:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{L_2}{j\omega(L_1 L_2 - M^2)} & \frac{-M}{j\omega(L_1 L_2 - M^2)} \\ \frac{-M}{j\omega(L_1 L_2 - M^2)} & \frac{L_1}{j\omega(L_1 L_2 - M^2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Consider a Π equivalent circuit as shown in Fig. 6.20.

KCL in nodes ① and ② shows:

$$I_1 = \frac{V_1}{j\omega L_A} + \frac{V_1 - V_2}{j\omega L_C}$$

$$I_2 = \frac{V_2}{j\omega L_B} + \frac{V_2 - V_1}{j\omega L_C}$$

In matrix form, these equations can be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega L_A} + \frac{1}{j\omega L_C} & \frac{-1}{j\omega L_C} \\ \frac{-1}{j\omega L_C} & \frac{1}{j\omega L_B} + \frac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The equivalent admittance matrix and the original circuit admittance must be equal. This yields:

$$\begin{bmatrix} \frac{L_2}{j\omega(L_1 L_2 - M^2)} & \frac{-M}{j\omega(L_1 L_2 - M^2)} \\ \frac{-M}{j\omega(L_1 L_2 - M^2)} & \frac{L_1}{j\omega(L_1 L_2 - M^2)} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{j\omega L_A} + \frac{1}{j\omega L_C} & \frac{-1}{j\omega L_C} \\ \frac{-1}{j\omega L_C} & \frac{1}{j\omega L_B} + \frac{1}{j\omega L_C} \end{bmatrix}$$

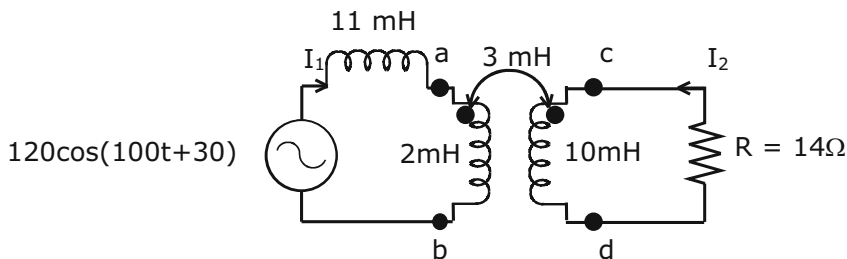


Fig. 6.21 Circuit of Example 6.7. The mutual inductance can be replaced by its T equivalent

Therefore,

$$L_A = \frac{L_1 L_2 - M^2}{L_2 - M}$$

$$L_B = \frac{L_1 L_2 - M^2}{L_1 - M}$$

$$L_C = \frac{L_1 L_2 - M^2}{M}.$$

Note 6.1 The loop current directions are preserved in the equivalent circuit.

Note 6.2 The terminal voltage polarities are preserved in the equivalent circuit.

Note 6.3 Positive or negative values of $\pm M$ must be considered depending on the original circuit.

Note 6.4 Either T or Π circuits can be utilized, depending on the original circuit.

Example 6.7 Consider the circuit shown in Fig. 6.21. Using the T equivalent circuit of the mutual inductance, find the current of each circuit (Fig. 6.21).

Solution The mutual inductance part of the circuit shown with terminals a, b, c, d can be replaced with its T equivalent. Arrangement of dotted terminals and the direction of currents result in positive mutual inductance value. The circuit and current directions are shown on the Fig. 6.22.

The equivalent inductance observed at the terminals a, b becomes $(2 \text{ mH} - 3 \text{ mH}) = -1 \text{ mH}$. The impedance at $\omega = 1000 \text{ rad/s}$ shows equivalent impedance of a capacitor as $-j1 \Omega$.

KVL in loop ①:

$$-120 \angle 30 + j10I_1 + j3(I_1 + I_2) = 0$$

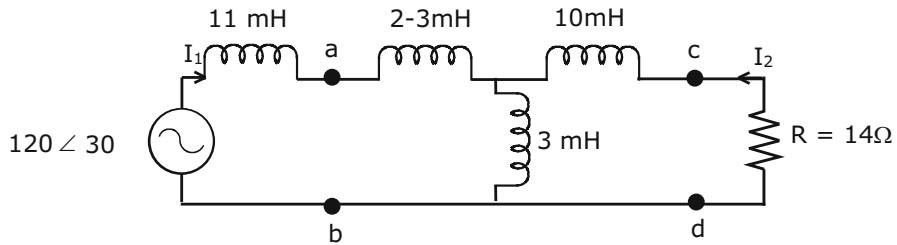


Fig. 6.22 Mutual inductance is replaced by the T equivalent. The values of the inductance are shown

KVL in loop ②:

$$(14 + j7)I_2 + j3(I_1 + I_2) = 0$$

Using the matrix approach, the currents are obtained as:

$$\begin{bmatrix} j13 & j3 \\ j3 & 14 + j10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 120\angle 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 103.92 + j60 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} j13 & j3 \\ j3 & 14 + j10 \end{bmatrix}^{-1} \begin{bmatrix} 103.92 + j60 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 4.99 - j8.01 \\ -1.64 + j0.103 \end{bmatrix} = \begin{bmatrix} 9.43\angle -58.07 \\ 1.64\angle -266.4 \end{bmatrix} \quad A$$

Ideal Mutual Inductance

Consider the circuit shown in Fig. 6.23, in which, as mentioned earlier, the mutual inductance is influenced by the geometry of the windings and the core material. Coil ① (primary) has N_1 turns and coil ② (secondary) has N_2 turns. If the magnetic material used in the circuit provides an ideal path such that the flux of one coil fully passes the second coil, the induced voltages can be written as (Figs. 6.23 and 6.24):

$$v_1 = N_1 \frac{d\phi}{dt}$$

and

$$v_2 = N_2 \frac{d\phi}{dt}$$

Fig. 6.23 A mutual inductance with primary and secondary and a coupling through magnetic material

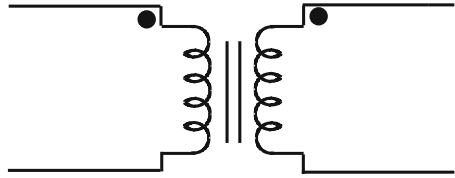


Fig. 6.24 The turn ratio is $N_1 : N_2$ or normalized as $1 : n$. Accordingly, the applied voltage ratios are proportional to the turn ratio, and the current flowing is reverse proportional to the turn ratio

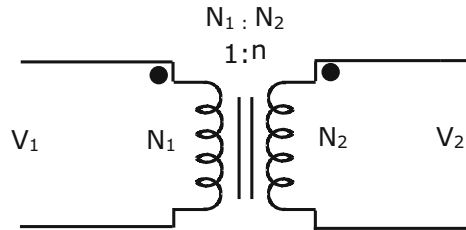
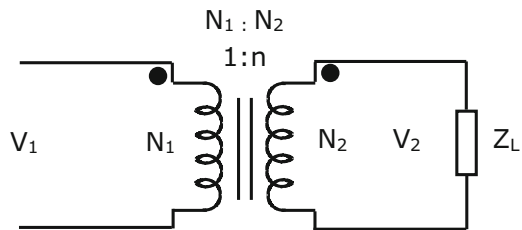


Fig. 6.25 Ideal transformer with load



Ideal Transformer

As the value of self-inductance in both coils and the value of mutual inductance reach very large numbers, the core has to provide path for the flux to link both coils equally. The loading on secondary increases the current at the secondary circuit. This generates flux and in the core which in turn increases the current taken from the primary (Fig. 6.25).

Considering the induced voltage in both primary and secondary, the ratios become:

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n$$

In ideal transformer, the loss in each coil is negligible. The power input at primary $P_{\text{in}} = v_1 i_1$ is therefore delivered at the secondary $P_{\text{out}} = v_2 i_2$. This yields:

$$v_1 i_1 = v_2 i_2$$

Therefore, in phasor form, the ratio of voltages becomes:

$$\frac{V_2}{V_1} = \frac{I_1}{I_2} = n$$

The impedance Z_L at the secondary can be calculated as:

$$V_2 = Z_L I_2$$

Replacing from the primary voltage and current ratios, it results in:

$$nV_1 = Z_L \frac{1}{n} I_1$$

This impedance observed (transferred) to primary is measured as:

$$V_1 = \frac{Z_L}{n^2} I_1 \Rightarrow Z_1 = \frac{Z_L}{n^2}$$

Note 6.5 Turn ratio $n = 1$ shows an isolation transformer. The voltage at the primary and secondary is the same. The currents of primary and secondary are also similar. The impedance is observed similar on both sides.

Note 6.6 Turn ratio $n > 1$ shows a step-up transformer. In this case, the secondary voltage is increased by n times, and, at constant power, the current is scaled down by $\frac{1}{n}$ times. The impedance transferred to secondary is n^2 times larger.

Note 6.7 Turn ratio $n < 1$ shows a step-down transformer. In this case, the secondary voltage is dropped by n times, and the current is increased at secondary by $\frac{1}{n}$ times. The impedance at the secondary is still n^2 times that of the primary.

Note 6.8 The impedance value is higher at the high voltage side.

Note 6.9 Negative mutual inductance is observed as a 180° out of phase signal.

Equivalent circuit of ideal transformer. Consider an ideal transformer with impedance of primary and secondary windings. The equivalent circuit when the transformer is seen from the primary side meaning that all voltages and impedances are transferred to the primary is shown in Fig. 6.26. The impedance of the secondary Z_2 when transferred to the primary becomes $\frac{Z_2}{n^2}$, and the voltage of secondary V_2 when transferred to the primary becomes $\frac{V_2}{n}$.

The transformer equivalent circuit can also be seen from the secondary side. The values of the primary must be transferred to the secondary. Therefore, the impedance of the primary Z_1 when transferred to the secondary becomes $n^2 Z_1$, and the voltage V_1 becomes nV_1 . The equivalent circuit seen from the secondary side is shown in Fig. 6.27.

Fig. 6.26 Equivalent circuit of an ideal transformer seen from the primary side

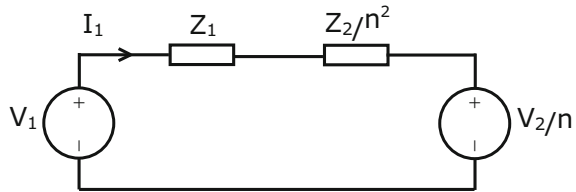


Fig. 6.27 Equivalent circuit of a transformer seen from the secondary side

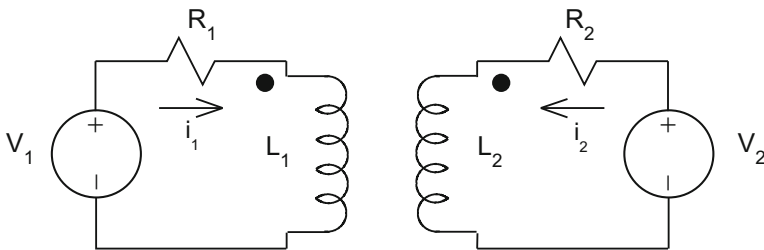
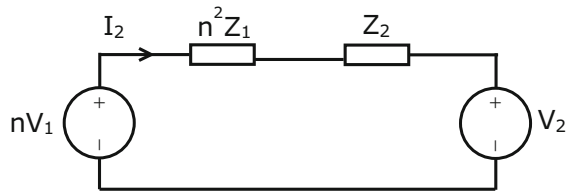


Fig. 6.28 Circuit of Example 6.8

Example 6.8 In the mutual inductance circuit shown in Fig. 6.28, find the currents in each loop. Find i_1 and i_2 .

$$\text{KVL : } -V_1 + R_1 I_1 + j\omega L_1 I_1 + j\omega M I_2 = 0$$

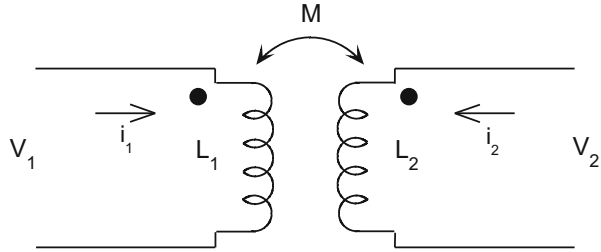
$$\text{KVL : } j\omega L_2 I_2 + j\omega M I_1 - V_2 + R_2 I_2 = 0$$

$$\begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 + R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I_1 = \frac{\begin{vmatrix} V_1 & j\omega M \\ V_2 & j\omega L_2 + R_2 \end{vmatrix}}{\begin{vmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 + R_2 \end{vmatrix}}$$

$$I_2 = \frac{\begin{vmatrix} R_1 + j\omega L_1 & V_1 \\ j\omega M & V_2 \end{vmatrix}}{\begin{vmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 + R_2 \end{vmatrix}}$$

Fig. 6.29 Mutual inductance circuit of Example 6.9



Example 6.9 Find the total energy stored in the mutual inductance circuit shown in Fig. 6.29.

$$V_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$P_1(t) = V_1(t) \cdot i_1(t) = \left(L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right) \cdot i_1$$

$$W_1(t) = (L_1, M) = \int \left(L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right) i_1 dt$$

$$\text{Let :} \quad i_1(t) = I_1, \quad i_2(t) = I_2$$

$$W_1(t) = \int L_1 i_1 di + M i_1 di_2$$

$$= \frac{1}{2} I_1^2 (\text{energy in } L_1, \text{ self inductance}) + M I_1 I_2 (\text{energy in mutual inductance})$$

$$W_2(\text{energy in } L_2 \text{ only}) = \frac{1}{2} L_2 I_2^2$$

$$\begin{aligned} W_{\text{Total}} &= W_1 + W_2 \\ &= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \end{aligned}$$

Example 6.10 In the circuit of Fig. 6.30, find the equivalent inductance from ports A and B.

Solution Using the T equivalent circuit, and considering the connection of two mutual inductances, the equivalent inductance becomes:

$$\begin{aligned} L_{\text{eq}} &= (L_1 - M) \parallel (L_2 - M) + M \\ &= \frac{(L_1 - M)(L_2 - M)}{(L_1 - M) + (L_2 - M)} + M \end{aligned}$$

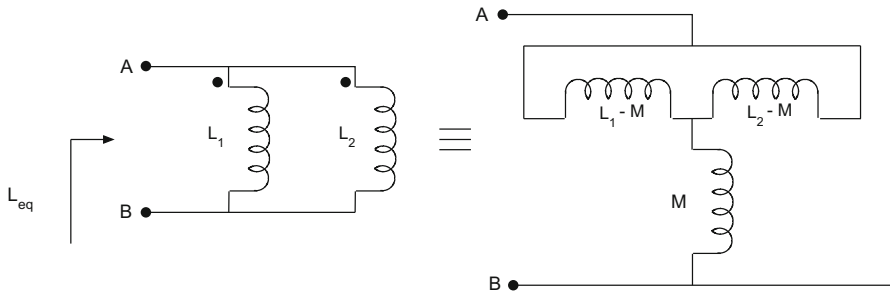


Fig. 6.30 Circuit of Example 6.10

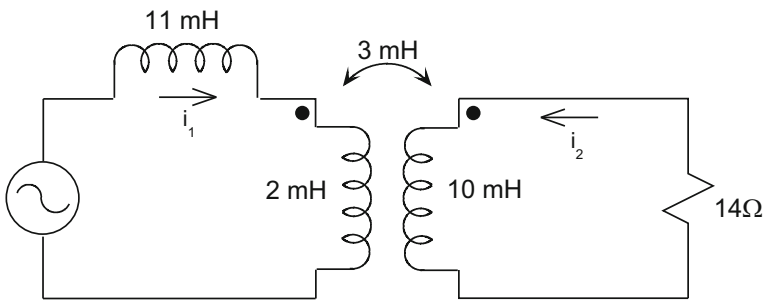


Fig. 6.31 Circuit of Example 6.11

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

Example 6.11 In the circuit shown in Fig. 6.31, find I_1 and I_2 using equivalent circuits.

Solution The mutual inductance can be replaced by its T equivalent circuit. The circuit with replaced T model becomes (Fig. 6.32):

KVL in loop ①:

$$-120 \angle 30^\circ + j10I_1 + j3(I_1 + I_2) = 0$$

KVL in loop ②:

$$j7I_2 + j3(I_1 + I_2) + 14I_2 = 0$$

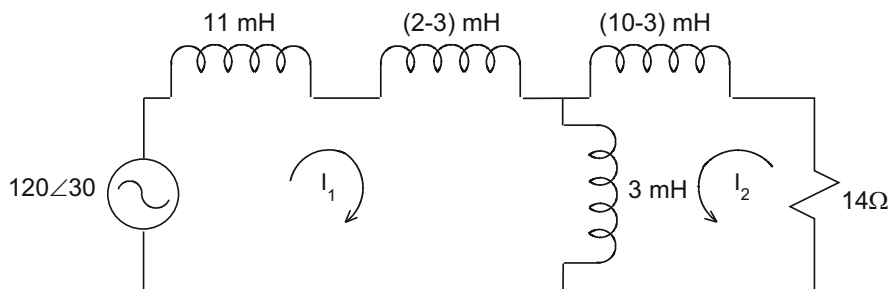
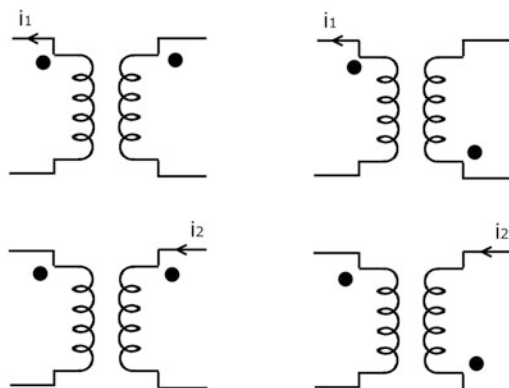


Fig. 6.32 Circuit 6.31 when the mutual inductance is replaced by the T equivalent

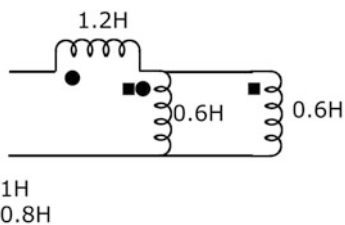
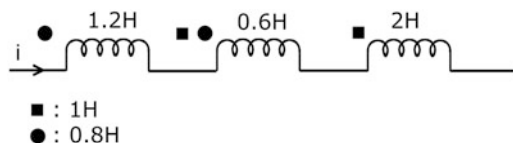
$$\begin{aligned}
 \begin{bmatrix} j13 & j3 \\ j3 & j10 + 14 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} 120\angle 30^\circ \\ 0 \end{bmatrix} \\
 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} j13 & j3 \\ j3 & j10 + 14 \end{bmatrix}^{-1} \begin{bmatrix} 120\angle 30^\circ \\ 0 \end{bmatrix} \\
 I_1 &= \frac{\begin{vmatrix} 120\angle 30^\circ & j3 \\ 0 & j10 + 14 \end{vmatrix}}{\begin{vmatrix} j13 & j3 \\ j3 & j10 + 14 \end{vmatrix}} \\
 &= \frac{(60\sqrt{3} + j60)(14 + j10)}{j13(14 + j10) - (j3)^2} \\
 &= \frac{840\sqrt{3} + j600\sqrt{3} + j840 - 600}{j182 - 130 + 9} \\
 &= \frac{854.92 + 1879.23j}{j182 - 121} = \frac{2064.56\angle 65.54^\circ}{218.55\angle 123.62^\circ} \\
 &= 9.45\angle -58.07^\circ \text{ A} \\
 I_2 &= \frac{\begin{vmatrix} j13 & 60\sqrt{3} + j60 \\ j3 & 0 \end{vmatrix}}{\begin{vmatrix} j13 & j3 \\ j3 & j10 + 14 \end{vmatrix}} \\
 &= \frac{-j3(60\sqrt{3} + j60)}{218.55\angle 123.62^\circ} = \frac{-j180\sqrt{3} + 180}{218.55\angle 123.62^\circ} \\
 &= \frac{360\angle -120^\circ}{218.55\angle 123.62^\circ} = 1.54\angle -243.62^\circ \text{ A}
 \end{aligned}$$

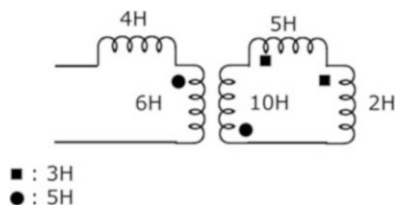
Problems

- 6.1. A $2mH$ inductor has $N = 100$ turns. If the coil current is $i(t) = 3 \sin 120\pi t$, find the flux variations in the core and the induced voltage at the terminals.
- 6.2. Two inductors are located such that maximum of flux linkage may occur. If the inductance of the coils is $L_1 = 25H$, $L_2 = 4H$, what is the maximum possible mutual inductance between the coils?
- 6.3. Determine the polarity of the induced voltage at the terminals of any of the following mutual inductance circuits.

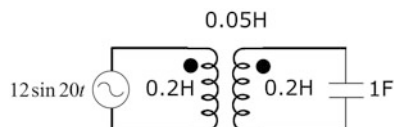


- 6.4. Calculate total inductance of the following circuits.

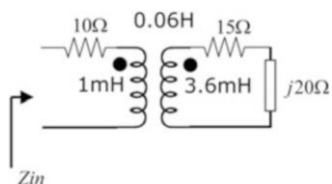




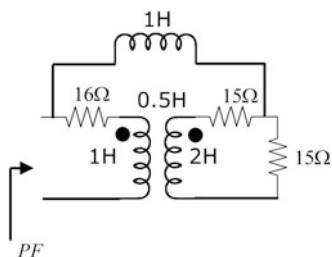
6.5. Find V_o across the capacitor in the following circuit.



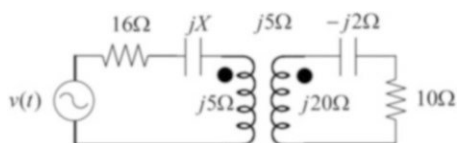
6.6. Find the input impedance in the following circuit.



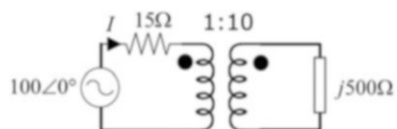
6.7. Find the PF of the circuit.



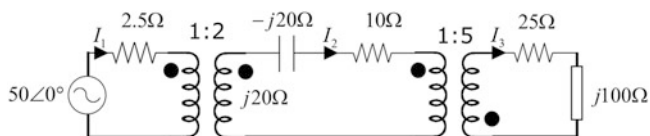
6.8. Find the reactance X such that the maximum power is transferred to the 10Ω load.



6.9. Find the current in the circuit.



6.10. Find the currents I_1 , I_2 , I_3 .



Chapter 7

Laplace Transform and Its Application in Circuits



Introduction

Most of the circuits introduced so far have been analyzed in time domain. This means that the input to the circuit, the circuit variables, and the responses have been presented as a function of time. All the input functions such as unit step, ramp, impulse, exponential, sinusoidal, etc. have been introduced as a time-dependent variable, and their effects on circuits have been identified directly as a function of time. This required utilization of differential equations and solutions in time domain. However, high-order circuits result in high-order differential equations, which, considering the initial conditions, sometimes are hard to solve. In addition, for circuits which are exposed to a spectrum of frequencies such as filters, the time domain analysis is a limiting factor.

To simplify the analysis of high-order circuits and incorporate the variable frequency nature of some circuits in effect, time domain analysis can be transformed into a frequency domain analysis. One of the transformations that can take the circuits from time domain to the frequency domain is the Laplace transform. In the frequency domain, the input to the circuit, the circuit itself, and the results are obtained using algebraic equations. The system response can be transformed back into the time domain through the inverse of Laplace transform.

The process of using the Laplace transform to analyze circuits is:

1. Find the Laplace of the input functions.
2. Represent the circuit in frequency domain.
3. Then find the desired response in the frequency domain.
4. Transform back to time domain.

Figure 7.1 demonstrates a set of differential equations that are represented in time domain. These equations are transformed to frequency domain using Laplace transform and are presented by algebraic equations. Laplace inverse takes the circuit back to the time domain.

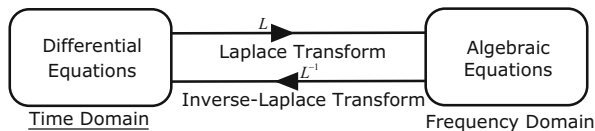


Fig. 7.1 Laplace transform from time to frequency domain and Laplace inverse transform from frequency domain to time domain

Mathematical Background

Laplace transform is defined for a given function $f(t)$, to be represented in frequency domain using the Laplace operator s . It reads, Laplace of $f(t)$ is $F(s)$ and is written as:

$$L\{f(t)\} = F(s)$$

Laplace transform is a two-sided transform which means it can be defined as time from $-\infty$ to $+\infty$. The Laplace transform is defined as:

$$L\{f(t)\} \triangleq \int_{-\infty}^{+\infty} f(t)e^{-st} dt$$

In this transform, the time domain representation of the function is transformed to a complex frequency domain $s = \sigma + j\omega$. However, this frequency remains a variable in all presentations of functions.

Because most of electric circuits are defined and operated in positive time, the Laplace transform in this book is defined as one-sided, e.g., in positive time. Therefore, a one-sided transform is defined as follows:

$$L\{f(t)\} = F(s) \triangleq \int_0^{+\infty} f(t)e^{-st} dt.$$

The Laplace transform exists for a given function $f(t)$ if and only if the integral $\int_0^{+\infty} f(t)e^{-st} dt \neq \infty$ exists.

Inverse of Laplace transform takes the functions from the frequency domain to the time domain. It is defined as:

$$f(t) = L^{-1}\{F(s)\}$$

Note 7.1 The functions in time domain are presented in lowercase alphabets such as f , and when the frequency domain is presented, capital letters are used such as F . This is helpful when the time variable t and frequency operator s are not shown in the transformations.

Laplace of Unit Step Function

$$f(t) = u(t)$$

According to the definition:

$$U(s) = L\{u(t)\} \triangleq \int_0^{\infty} u(t)e^{-st} dt$$

However, the value of $u(t)$ for time-positive $t \geq 0$ is 1. Therefore:

$$U(s) = \int_0^{\infty} 1 \cdot e^{-st} dt = \frac{1}{-s} e^{-st} \Big|_0^{\infty} = \frac{1}{-s} (e^{-\infty} - e^0) = \frac{1}{s}$$

Therefore:

$$L\{u(t)\} = \frac{1}{s}$$

Example 7.1 Find the Laplace transform of $10u(t)$.

Solution

$$U(s) = \int_0^{\infty} 10 \cdot e^{-st} dt = \frac{10}{-s} e^{-st} \Big|_0^{\infty} = -\frac{10}{s} (e^{-\infty} - e^0) = \frac{10}{s}$$

Laplace of Impulse Function

$$f(t) = \delta(t)$$

According to the definition:

$$\Delta(s) = L\{\delta(t)\} \triangleq \int_0^{\infty} \delta(t)e^{-st} dt$$

Impulse function has value at the time that makes the argument zero. Therefore, this function has value at $t = 0$. The integral becomes:

$$\Delta(s) = \int_0^{\infty} \delta(t)e^{-st} dt \Big|_{t=0} = 1$$

Therefore:

$$L\{\delta(t)\}=1$$

Laplace of Ramp Function

$$f(t) = r(t) = ktu(t)$$

According to the definition:

$$R(s) = L\{r(t)\} \triangleq \int_0^{\infty} r(t)e^{-st} dt$$

The value of function is $r(t) = ktu(t)$. Therefore:

$$R(s) = \int_0^{\infty} ktu(t)e^{-st} dt = k \left(t \frac{1}{-s} e^{-st} - \frac{1}{s^2} e^{-st} \right) \Big|_0^{\infty} = k \frac{1}{s^2}$$

Laplace of Exponential Function

$$f(t) = e^{at} u(t)$$

The $u(t)$ part of the function represents that the function is defined in positive time, i.e., for $t \geq 0$.

According to the definition:

$$F(s) = L\{f(t)\} \triangleq \int_0^{\infty} f(t)e^{-st} dt$$

Therefore:

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{-t(s-a)} dt = \frac{1}{-(s-a)} e^{-(s-a)t} \Big|_0^{\infty} \\ &= \frac{1}{-(s-a)} (e^{-\infty} - e^0) = \frac{1}{(s-a)} \end{aligned}$$

Hence:

$$L\{e^{at} u(t)\} = \frac{1}{(s-a)}$$

Find the Laplace and Laplace inverse of the following functions:

(a) *Step Function*

Example 7.2 *Laplace.* $L\{10u(t)\}$.

Solution The Laplace of unit step function at amplitude 10 is $F(s) = \frac{10}{s}$.

Example 7.3 *Laplace Inverse.* Find the original time domain function $f(t)$ if the Laplace transform is $F(s) = \frac{-3}{s}$.

Solution The form of Laplace function $\frac{1}{s}$ matches with that of the unit step, and -3 is just an amplitude/coefficient. Therefore, the original time domain function has been $f(t) = -3u(t)$.

Since the time domain function is obtained from the frequency domain function, it can be written as:

$$L^{-1}\left\{\frac{-3}{s}\right\} = -3u(t).$$

(b) *Ramp Function*

Example 7.4 *Laplace.* $L\{5tu(t)\}$.

Solution The Laplace of a ramp with slope 5 is $F(s) = \frac{5}{s^2}$.

Example 7.5 *Laplace Inverse.* Find the original time domain function $f(t)$ if the Laplace transform is $F(s) = \frac{10}{s^2}$.

Solution The form of Laplace $\frac{1}{s^2}$ matches with that of the ramp function $tu(t)$, and 10 is just an amplitude/coefficient/slope. Therefore, the original time domain function has been $f(t) = 10tu(t)$.

Therefore:

$$L^{-1}\left\{\frac{10}{s^2}\right\} = 10tu(t).$$

(c) *Exponential Function*

Example 7.6 *Laplace.* $L\{e^{-t}u(t)\}$.

Solution This is an exponential with damping factor of -1 . The Laplace transform is $F(s) = \frac{1}{s+1}$.

Note 7.2 Note that if the exponential function has a positive damping, the Laplace function will have a negative shift in frequency and vice versa.

Example 7.7 *Laplace Inverse.* Find the original time domain function $f(t)$ if the Laplace transform is $F(s) = \frac{3}{s+1}$.

Solution The form of Laplace $\frac{1}{s+1}$ matches with that of the exponential function $e^{\alpha t} u(t)$ where $\alpha = -1$ and 3 is just a coefficient. Therefore, the original time domain function has been $f(t) = 3e^{-t}u(t)$.

Therefore:

$$L^{-1}\left\{\frac{3}{s+1}\right\} = 3e^{-t}u(t).$$

(d) *Exponential Function*

Example 7.8 *Laplace.* $L\{10e^{20t}u(t)\}$.

Solution This is an exponential with damping of $+20$ and amplitude of 10. Therefore, $F(s) = \frac{10}{s-20}$.

Example 7.9 *Laplace Inverse.* Find the original time domain function $f(t)$ if the Laplace transform is $F(s) = \frac{5}{s+20}$.

Solution The form of Laplace $\frac{1}{s+20}$ matches with that of the exponential function $e^{\alpha t} u(t)$ where $\alpha = -20$ and 5 is just a coefficient. Therefore, the original time domain function has been $f(t) = 5e^{-20t}u(t)$.

Therefore:

$$L^{-1}\left\{\frac{5}{s+20}\right\} = 5e^{-20t}u(t).$$

Laplace of Sinusoidal Function

$$f(t) = \sin(\omega t)$$

According to the definition:

$$F(s) = L\{f(t)\} \triangleq \int_0^{\infty} f(t)e^{-st} dt$$

Therefore:

$$F(s) = \int_0^{\infty} \sin(\omega t) e^{-st} dt$$

However, $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$. Therefore:

$$\begin{aligned} \int_0^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{j2} e^{-st} dt &= \int_0^{\infty} \frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{j2} dt \\ &= \frac{1}{j2} \left(\frac{1}{-(s-j\omega)} e^{-(s-j\omega)t} - \frac{1}{-(s+j\omega)} e^{-(s+j\omega)t} \right) \Bigg|_0^{\infty} \\ &= \frac{-1}{j2} \left(\frac{1}{(s-j\omega)} - \frac{1}{(s+j\omega)} \right) = \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

Therefore:

$$L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

Laplace of Co-sinusoidal Function

$$f(t) = \cos(\omega t)$$

According to the definition:

$$F(s) = L\{f(t)\} \triangleq \int_0^{\infty} f(t) e^{-st} dt$$

Therefore:

$$F(s) = \int_0^{\infty} \cos(\omega t) e^{-st} dt$$

However, $\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$. Therefore:

$$\begin{aligned} \int_0^{\infty} \frac{e^{j\omega t} + e^{-j\omega t}}{2} e^{-st} dt &= \int_0^{\infty} \frac{e^{-(s-j\omega)t} + e^{-(s+j\omega)t}}{2} dt \\ &= \frac{1}{2} \left(\frac{1}{-(s-j\omega)} e^{-(s-j\omega)t} - \frac{1}{(s+j\omega)} e^{-(s+j\omega)t} \right) \Bigg|_0^{\infty} \\ &= \frac{1}{2} \left(\frac{1}{(s-j\omega)} + \frac{1}{(s+j\omega)} \right) = \frac{s}{s^2 + \omega^2} \end{aligned}$$

Therefore:

$$L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

Example 7.10 Find $L\{\sin 10t u(t)\} = \frac{10}{s^2 + 10^2} = \frac{10}{s^2 + 100}$.

Example 7.11 Find the original time domain function that resulted in $F(s) = \frac{50}{s^2 + 9}$.

Solution Looking at the format of the Laplace transform $\frac{50}{s^2 + 3^2}$ indicates a sinusoidal function at a frequency of $\omega = 3$ and an amplitude of $\frac{50}{3}$. Therefore, $f(t) = \frac{50}{3} \sin 3t u(t)$. Here, the $u(t)$ function indicates that the function $f(t)$ is defined at time positive.

Example 7.12 Find $L\{\cos 5t u(t)\}$.

Solution

$$L\{\cos 5t u(t)\} = \frac{s}{s^2 + 5^2} = \frac{s}{s^2 + 25}$$

Example 7.13 Find the original time domain function that resulted in $F(s) = \frac{10s}{s^2 + 16}$.

Solution The format of the Laplace transform $10 \frac{s}{s^2 + 4^2}$ indicates a co-sinusoidal function at a frequency of $\omega = 4$ and amplitude of 10. Therefore, $f(t) = 10 \cos 4t u(t)$. The part of $u(t)$ indicates that the function $f(t)$ is defined at time positive.

Example 7.14 Find the original time domain function that resulted in $F(s) = \frac{10s+3}{s^2 + 16}$.

Solution Looking at the numerator of the Laplace transform, if it is split into $10 \frac{s}{s^2 + 4^2} + \frac{3}{4} \frac{4}{s^2 + 4^2}$, it shows a summation of two parts: (1) cos function at a frequency of $\omega = 4$ and an amplitude of 10 and (2) sin function at a frequency of $\omega = 4$ and an amplitude of $\frac{3}{4}$. Therefore, $f(t) = (10 \cos 4t + \frac{3}{4} \sin 4t) u(t)$.

Example 7.15 Find $L\{10 \sin(5t+30)\}$.

Solution

$$\begin{aligned} L\{10 \sin(5t + 30)\} &= L\{10(\sin 5t \cos 30 + \cos 5t \sin 30)\} \\ &= 10 \frac{5}{s^2 + 25} 0.866 + 10 \frac{s}{s^2 + 25} 0.5 \\ &= \frac{43.3}{s^2 + 25} + \frac{5s}{s^2 + 25} = \frac{5s + 43.3}{s^2 + 25} \end{aligned}$$

Laplace of Hyperbolic Sinusoidal Function

$$f(t) = \sinh(\omega t)$$

According to the definition:

$$F(s) = L\{f(t)\} \triangleq \int_0^{\infty} f(t)e^{-st} dt$$

Therefore:

$$F(s) = \int_0^{\infty} \sinh(\omega t)e^{-st} dt$$

However, $\sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}$. Therefore:

$$\begin{aligned} \int_0^{\infty} \frac{e^{\omega t} - e^{-\omega t}}{2} e^{-st} dt &= \int_0^{\infty} \frac{e^{-(s-\omega)t} - e^{-(s+\omega)t}}{2} dt \\ &= \frac{1}{2} \left(\frac{1}{-(s-\omega)} e^{-(s-\omega)t} - \frac{1}{-(s+\omega)} e^{-(s+\omega)t} \right) \Bigg|_0^{\infty} \\ &= \frac{-1}{2} \left(\frac{1}{(s-\omega)} - \frac{1}{(s+\omega)} \right) = \frac{\omega}{s^2 - \omega^2} \end{aligned}$$

Therefore:

$$L\{\sinh(\omega t)\} = \frac{\omega}{s^2 - \omega^2}$$

Laplace of Hyperbolic Co-sinusoidal Function

$$f(t) = \cosh(\omega t)$$

According to the definition:

$$F(s) = L\{f(t)\} \triangleq \int_0^{\infty} f(t)e^{-st} dt$$

Therefore:

$$F(s) = \int_0^{\infty} \cosh(\omega t) e^{-st} dt$$

However, $\cosh(\omega t) = \frac{e^{\omega t} + e^{-\omega t}}{2}$. Therefore:

$$\begin{aligned} \int_0^{\infty} \frac{e^{\omega t} + e^{-\omega t}}{2} e^{-st} dt &= \int_0^{\infty} \frac{e^{-(s-\omega)t} + e^{-(s+\omega)t}}{2} dt \\ &= \frac{1}{2} \left(\frac{1}{-(s-\omega)} e^{-(s-\omega)t} - \frac{1}{(s+\omega)} e^{-(s+\omega)t} \right) \Bigg|_0^{\infty} \\ &= \frac{1}{2} \left(\frac{1}{(s-\omega)} + \frac{1}{(s+\omega)} \right) = \frac{s}{s^2 - \omega^2} \end{aligned}$$

Therefore:

$$L\{\cosh(\omega t)\} = \frac{s}{s^2 - \omega^2}$$

Example 7.16 Find $L\{\sinh 10t u(t)\} = \frac{10}{s^2 - 10^2} = \frac{10}{s^2 - 100}$.

Example 7.17 Find the original time domain function that resulted in $F(s) = \frac{50}{s^2 - 9}$.

Solution Looking at the format of the Laplace transform $\frac{50}{s^2 - 3^2}$ indicates a hyperbolic sinusoidal function at a frequency of $\omega = 3$ and an amplitude of $\frac{50}{3}$. Therefore, $f(t) = \frac{50}{3} \sinh 3t u(t)$.

Example 7.18 Find $L\{\cosh 5t u(t)\} = \frac{s}{s^2 - 5^2} = \frac{s}{s^2 - 25}$.

Example 7.19 Find the original time domain function that resulted in $F(s) = \frac{10s}{s^2 - 16}$.

Solution Looking at the format of the Laplace transform $10 \frac{s}{s^2 - 4^2}$ indicates a hyperbolic sinusoidal function at a frequency of $\omega = 4$ and an amplitude of 10. Therefore, $f(t) = 10 \cosh 4t u(t)$.

Example 7.20 Find the original time domain function that resulted in $F(s) = \frac{10s+3}{s^2 - 16}$.

Solution Looking at the format of the Laplace transform, if the function is split into $10 \frac{s}{s^2 - 4^2} + \frac{3}{4} \frac{4}{s^2 - 4^2}$, indicates the summation of two parts: (1) a cosh function at a frequency of $\omega = 4$ and an amplitude of 10 and (2) a sinh function at a frequency of $\omega = 4$ and an amplitude of $\frac{3}{4}$. Therefore, $f(t) = (10 \cosh 4t + \frac{3}{4} \sinh 4t) u(t)$.

Example 7.21 Find $L\{10 \sinh(5t+3)\}$.

Solution

$$\begin{aligned} L\{10\sinh(5t+3)\} &= L\{10(\sinh 5t \cosh 3 + \cosh 5t \sinh 3)\} \\ &= 10 \frac{5}{s^2 + 25} 10.06 + 10 \frac{s}{s^2 + 25} 10.01 \\ &= \frac{503}{s^2 + 25} + \frac{101.1s}{s^2 + 25} = \frac{101.1s + 503}{s^2 + 25} \end{aligned}$$

Laplace of Derivatives of Impulse

$$L\{\dot{\delta}(t)\}$$

Each derivative of impulse function generates a s factor. Therefore:

$$L\{\dot{\delta}(t)\} = s$$

$$L\{\ddot{\delta}(t)\} = s^2$$

In general:

$$L\{\delta^{(n)}(t)\} = s^n$$

Laplace of Differential Functions

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^+)$$

where $F(s)$ is the Laplace of function $f(t)$ in frequency domain and $f(0^+)$ is the time domain initial condition at $t = 0^+$.

Note 7.3

$$\frac{df(t)}{dt} \equiv \dot{f}(t)$$

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^+) - \dot{f}(0^+)$$

where $\dot{f}(0^+)$ is the differential of initial condition in time domain at $t = 0^+$.

Note 7.4

$$\frac{d^2 f(t)}{dt^2} \equiv \ddot{f}(t)$$

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^+) - s^{n-2}\dot{f}(0^+) - \dots - f^{(n-1)}(0^+)$$

Solve the following differential equations using Laplace transform.

Example 7.22 $\dot{v} + 3v = \delta(t)$, $v(0^+) = 1$.

Solution Taking Laplace of both sides results in:

$$\begin{aligned} L\{\dot{v} + 3v\} &= L\{\delta(t)\} \\ L\{\dot{v}\} + 3L\{v\} &= L\{\delta(t)\} \\ sV(s) - v(0^+) + 3V(s) &= 1 \end{aligned}$$

Replacing from initial condition:

$$sV(s) - 1 + 3V(s) = 1$$

Solving for $V(s)$:

$$V(s) = \frac{2}{s+3}$$

Original time domain function $v(t)$ has the format of exponential at damping factor -3 and coefficient 2 as follows:

$$v(t) = 2e^{-3t}u(t)$$

$u(t)$ determines the function which is defined for positive time.

Example 7.23 $\ddot{v} + 4\dot{v} + 3v = 0$, $v(0^+) = -1$, $\dot{v}(0^+) = 2$.

Solution Taking Laplace from both sides of the equation results in:

$$\begin{aligned} L\{\ddot{v} + 4\dot{v} + 3v\} &= 0 \\ (s^2V(s) - s v(0^+) - \dot{v}(0^+)) + 4(sV(s) - v(0^+)) + 3V(s) &= 0 \\ (s^2V(s) - s(-1) - (2)) + 4(sV(s) - (-1)) + 3V(s) &= 0 \\ V(s)(s^2 + 4s + 3) + s - 2 + 4 &= 0 \\ V(s) &= \frac{-s+2}{s^2+3s+4} = \frac{-s-2}{(s+1)(s+3)} \end{aligned}$$

Splitting the fraction into two and assuming coefficients A and B result in:

$$V(s) = \frac{A}{s+1} + \frac{B}{s+3} \equiv \frac{-s-2}{(s+1)(s+3)}$$

Common denominator and simplification yields:

$$V(s) = \frac{(A+B)s + 3A+B}{(s+1)(s+3)} \equiv \frac{-s-2}{(s+1)(s+3)}$$

This results in $A+B = -1$ and $3A+B = -2$.

This gives $A = -\frac{1}{2}$, $B = -\frac{1}{2}$. Therefore:

$$V(s) = \frac{-\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}}{s+3}$$

The original time domain function $v(t)$ has two exponential structures at damping factor -1 and -3 with coefficients 1 and -2 , respectively. Therefore:

$$v(t) = \left(-\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \right) u(t)$$

Laplace Operations

There are several operations that can be utilized to simplify the Laplace transformation from frequency to time and vice versa. These operations have indicators that are discussed in this section. Note that these indicators trigger specific operations which need to be carefully considered. Otherwise, the transformation may become wrong.

Linear Combination of Functions

$$L\{\alpha f(t) \pm \beta g(t)\}$$

Laplace of linear combination of several functions is the summation of Laplace of those functions.

Consider $L\{f(t)\} = F(s)$, and $L\{g(t)\} = G(s)$, then:

$$L\{\alpha f(t) \pm \beta g(t)\} = \alpha L\{f(t)\} \pm \beta L\{g(t)\} = \alpha F(s) \pm \beta G(s)$$

Note 7.5 This is true only for summation of functions and *not* in the product. Therefore:

$$L\{f(t)g(t)\} \neq L\{f(t)\}L\{g(t)\}$$

Example 7.24 Find $L\{e^{-5t} + \sin 3t\}$.

Solution

$$L\{e^{-5t} + \sin 3t\} = L\{e^{-5t}\} + L\{\sin 3t\} = \frac{1}{s+5} + \frac{3}{s^2+9}$$

Example 7.25 Find $L\{\cos 5t + 2 \sin 3t\}$.

Solution

$$\begin{aligned} L\{\cos 5t + 2 \sin 3t\} &= L\{\cos 5t\} + 2L\{\sin 3t\} \\ &= \frac{s}{s^2+25} + 2 \frac{3}{s^2+9} = \frac{7s^2+159}{(s^2+25)(s^2+9)} \end{aligned}$$

Example 7.26 Find $L\{-\cos 5t + 2 \sin 5t\}$.

Solution

$$\begin{aligned} L\{-\cos 5t + 2 \sin 5t\} &= -L\{\cos 5t\} + 2L\{\sin 5t\} \\ &= \frac{-s}{s^2+25} + 2 \frac{5}{s^2+25} = \frac{-s+10}{(s^2+25)} \end{aligned}$$

Example 7.27 Find the original time domain (Laplace inverse) of $L^{-1}\left\{\frac{-s+10}{(s^2+25)}\right\}$.

Solution Splitting the function into $\frac{-s}{(s^2+25)} + \frac{10}{(s^2+25)}$ reveals two known structures for $\cos$ as $\frac{s}{s^2+\omega^2}$ and for $\sin$ as $\frac{\omega}{s^2+\omega^2}$. Therefore:

$$L^{-1}\left\{\frac{-s}{(s^2+25)} + \frac{10}{(s^2+25)}\right\} = -\cos 5t + \frac{10}{5} \sin 5t, t \geq 0.$$

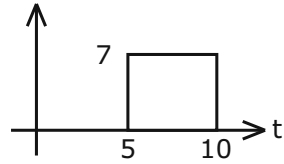
Shift in Time

$$L\{f(t+a)\}$$

A shift in time domain becomes an exponential in frequency domain.

$$L\{f(t+a)\} = e^{as}F(s)$$

Fig. 7.2 Pulse function of
Example 7.28



Example 7.28 Find the Laplace of the function shown in Fig. 7.2.

Solution The function is a summation of two shifted unit steps by 5 and 10 s, respectively, at an amplitude of 7 and -7 . Therefore, $f(t) = 7u(t - 5) - 7u(t - 10)$. The Laplace becomes:

$$\begin{aligned} L\{7u(t - 5) - 7u(t - 10)\} &= 7L\{u(t - 5)\} - 7L\{u(t - 10)\} \\ &= 7e^{-5s}\frac{1}{s} - 7e^{-10s}\frac{1}{s} = \frac{7}{s}(e^{-5s} - e^{-10s}) \end{aligned}$$

Example 7.29 Find the Laplace of the function $f(t) = 2tu(t - 3)$.

Solution As the function shows, only the step function is shifted by 3 s. To use the time shift operations, all time functions must have been shifted by the same 3 s. To obtain such function, the part that is not shifted will be shifted manually as follows:

$$\begin{aligned} f(t) &= 2(t - 3 + 3)u(t - 3) \\ f(t) &= 2(t - 3)u(t - 3) + 2(3)u(t - 3) \end{aligned}$$

Now the function is shifted by the same amount of 3 s, and the Laplace operation can be used as follows:

$$L\{2(t - 3)u(t - 3) + 2(3)u(t - 3)\} = \frac{2}{s^2}e^{-3s} + \frac{6}{s}e^{-3s}$$

Product by an Exponential

$$L\{e^{-at}f(t)\}$$

Appearance of an exponential and function $f(t)$ triggers a shift in frequency as follows:

$$L\{e^{-at}f(t)\} = F(s + a)$$

Note 7.6 This means that after taking the Laplace of $f(t)$ as $F(s)$, convert all $s \rightarrow (s + a)$.

Example 7.30 Find $L\{e^{-5t}u(t)\} = ?$

Solution The exponential e^{-5t} triggers a shift in frequency over the $L\{u(t)\} = \frac{1}{s}$ by $s \rightarrow s+5$. This results in:

$$L\{e^{-5t}u(t)\} = \frac{1}{s+5}$$

Example 7.31 Find $L\{e^{-5t} \sin 3t\} = ?$

Solution The exponential e^{-5t} triggers a shift in frequency over the $L\{\sin 3t\} = \frac{3}{s^2+9}$ by $s \rightarrow s+5$. This results in:

$$L\{e^{-5t} \sin 3t\} = \frac{3}{(s+5)^2 + 9}$$

Example 7.32 Find $L\{e^{-2t} \cosh t u(t-5)\} = ?$

Solution Only one part of the function is shifted by 5 s. The rest should be manually shifted as follows:

$$\begin{aligned} f(t) &= e^{-2(t-5+5)} \cosh(t-5+5) u(t-5) \\ f(t) &= e^{-2(t-5)} e^{-2(5)} [\cosh(t-5) \cosh(5) - \sinh(t-5) \sinh(5)] u(t-5) \\ f(t) &= [e^{-2(5)} \cosh(5) e^{-2(t-5)} \cosh(t-5) - e^{-2(5)} \sinh(5) e^{-2(t-5)} \sinh(t-5)] u(t-5) \end{aligned}$$

Laplace transform of $f(t)$ becomes:

$$F(s) = e^{-10} \cosh(5) \frac{s+2}{(s+2)^2 + 1} e^{-5s} + e^{-10} \sinh(5) \frac{1}{(s+2)^2 + 1} e^{-5s}$$

Product by Time Factors

$$L\{tf(t)\}$$

Factors of t trigger a differential in frequency domain as follows:

$$L\{tf(t)\} = -\frac{d}{ds} F(s)$$

In general, when multiplied by t^n :

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example 7.33 Find $L\{tu(t)\}$.

Solution t indicates a derivative operation $-\frac{d}{ds}$ of the Laplace of $u(t)$. Considering that $\{u(t)\} = \frac{1}{s}$, it can be concluded that $L\{tu(t)\} = -\frac{d}{ds} \left(\frac{1}{s}\right) = \frac{1}{s^2}$.

Example 7.34 Find $L\{t^2u(t)\}$.

Solution t^2 indicates a second derivative of the Laplace of the function $u(t)$. Therefore:

$$L\{t^2u(t)\} = -\frac{d}{ds} \left(-\frac{d}{ds} \left(\frac{1}{s} \right) \right) = -\frac{d}{ds} \left(\frac{1}{s^2} \right) = \frac{2}{s^3}$$

Example 7.35 Find $L\{t^n u(t)\}$.

Solution Accordingly:

$$L\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

Example 7.36 Find $L\{t^3 e^{-5t} u(t)\}$.

Solution The exponential e^{-5t} triggers a shift in frequency over the Laplace of the function t^3 . Therefore, $L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{3!}{s^4}$ shifted by $s \rightarrow s+5$ results in:

$$L\{t^3 e^{-5t}\} = \frac{3!}{(s+5)^4}$$

Divide by Time Factors

$$L\left\{\frac{1}{t}f(t)\right\}$$

The factor $\frac{1}{t}$ indicates an integral of the Laplace transform of the function $f(t)$ as follows:

$$L\left\{\frac{1}{t}f(t)\right\} = \int_0^s F(x)dx$$

Complementary Laplace Inverse Techniques

To obtain the time domain transform of the Laplace functions, it is best to convert the function to individual functions or templates with known Laplace inverse transforms. Some of the techniques that can help in identifying known templates from given frequency domain functions are introduced in this section.

Long Division

In fractions that have a polynomial in the numerator with higher order than the polynomial in the denominator, a long division results in functions that are simpler to break into known templates.

Example 7.37 Find the Laplace inverse of the function $L^{-1}\left\{\frac{s^5+2s^4+s+1}{s^2+2}\right\}$.

Solution Long division results in:

$$\frac{s^5 + 2s^4 + s + 1}{s^2 + 2} = s^3 + 2s^2 - 2s - 4 + \frac{5s + 9}{s^2 + 2}$$

Split the fraction into two terms:

$$\begin{aligned} \frac{s^5 + 2s^4 + s + 1}{s^2 + 2} &= s^3 + 2s^2 - 2s - 4 + \left(\frac{5s + 9}{s^2 + 2}\right) \\ &= s^3 + 2s^2 - 2s - 4 + \left(5\frac{s}{s^2 + (\sqrt{2})^2} + \frac{9}{\sqrt{2}}\frac{\sqrt{2}}{s^2 + (\sqrt{2})^2}\right) \end{aligned}$$

Now, all terms are presented in known templates:

$$\begin{aligned} L^{-1}\left\{s^3 + 2s^2 - 2s - 4 + 5\frac{s}{s^2 + (\sqrt{2})^2} + \frac{9}{\sqrt{2}}\frac{\sqrt{2}}{s^2 + (\sqrt{2})^2}\right\} \\ = \ddot{\delta} + 2\ddot{\delta} - 2\dot{\delta} - 4\delta + 5\cos\sqrt{2}t + \frac{9}{\sqrt{2}}\sin\sqrt{2}t, t \geq 0. \end{aligned}$$

Partial Fraction Expansion

Using this technique, fractions with high-order polynomials can be broken into simpler functions and known templates to find the Laplace inverse transforms. Consider a fraction with numerator and denominator polynomials where the order of numerator is less than the order of denominator:

$$F(s) = \frac{\text{Num}(s)}{\text{Den}(s)}$$

Depending on the form and order of the denominator, several cases may exist.

Case 1 Simple distinct roots. In this case, the denominator has distinct roots. The numerator coefficients can be calculated by:

$$F(s) = \frac{\text{Num}(s)}{\prod_{i=1}^n (s + p_i)}$$

$$F(s) = \sum_{i=1}^n \frac{A_i}{(s + p_i)}$$

Each A_i coefficient is obtained by a product of the original function by the denominator of the fraction $\frac{A_i}{(s+p_i)}$ or $(s + p_i)$ and evaluating the entire product at the root of the denominator as follows:

$$A_i = ((s + p_i) \cdot F(s)) \Big|_{s = -p_i}$$

Example 7.38 Do partial fraction expansion of function $F(s) = \frac{s-1}{s^2+5s+6}$.

Solution The denominator s^2+5s+6 has two roots of $s = -2$ and $s = -3$. Therefore:

$$F(s) = \frac{s-1}{s^2+5s+6} = \frac{A_1}{s+2} + \frac{A_2}{s+3}$$

To find each of the coefficients A_1 and A_2 , the procedure is as follows:

$$A_1 = (s+2) \frac{s-1}{s^2+5s+6} \Big|_{s=-2} = \frac{-2-1}{-2+3} = -3$$

$$A_2 = (s+3) \frac{s-1}{s^2+5s+6} \Big|_{s=-3} = \frac{-3-1}{-3+2} = 4$$

$$F(s) = \frac{s-1}{s^2+5s+6} = \frac{-3}{s+2} + \frac{4}{s+3}$$

Case 2 Denominator has repeated roots. Root p_j is repeated m times, and there are some other roots.

$$F(s) = \frac{\text{Num}(s)}{(s + p_j)^m \prod_{i=1}^{n-m} (s + p_i)}$$

The partial fraction expansion must include all power of $s + p_j$ from 1 to m and all distinct roots p_i as follows:

$$F(s) = \frac{B_1}{(s + p_j)^m} + \frac{B_2}{(s + p_j)^{m-1}} + \dots + \frac{B_m}{s + p_j} + \sum_{i=1}^{n-m} \frac{A_i}{(s + p_i)}$$

To find B coefficients:

$$B_1 = (s + p_j)^m F(s) \Big|_{s = -p_j}$$

$$B_2 = \frac{d(s + p_j)^m F(s)}{ds} \Big|_{s = -p_j}$$

In general form:

$$B_m = \frac{1}{(m-1)!} \frac{d^{m-1}(s + p_j)^m F(s)}{ds^{m-1}} \Big|_{s = -p_j}$$

Example 7.39 Do partial fraction expansion of function $F(s) = \frac{s-1}{(s+1)^3(s^2+5s+6)}$.

Solution Denominator has two distinct roots of $s = -2$ and $s = -3$ and three repeated roots of $s = -1$.

Therefore:

$$F(s) = \frac{s-1}{(s+1)^3(s^2+5s+6)} = \frac{B_1}{(s+1)^3} + \frac{B_2}{(s+1)^2} + \frac{B_3}{s+1} + \frac{A_1}{s+2} + \frac{A_2}{s+3}$$

$$B_1 = (s+1)^3 \frac{s-1}{(s+1)^3(s^2+5s+6)} \Big|_{s=-1} = \frac{-1-1}{(-1^2-5+6)} = \frac{-2}{2}$$

$$B_1 = -1$$

$$\begin{aligned} B_2 &= \frac{d}{ds} \left((s+1)^3 \frac{s-1}{(s+1)^3(s^2+5s+6)} \right) \Big|_{s=-1} = \frac{d}{ds} \left(\frac{s-1}{(s^2+5s+6)} \right) \Big|_{s=-1} \\ &= \frac{(s^2+5s+6) - (s-1)(2s+5)}{(s^2+5s+6)^2} \Big|_{s=-1} \end{aligned}$$

$$= \frac{(-1^2 - 5 + 6) - (-1 - 1)(-2 + 5)}{(-1^2 - 5 + 6)^2} = \frac{2 + 6}{4}$$

$$\mathbf{B}_2 = 2$$

$$\begin{aligned} \mathbf{B}_3 &= \frac{1}{2!} \frac{d^2}{ds^2} \left((s+1)^3 \frac{s-1}{(s+1)^3 (s^2 + 5s + 6)} \right) \Big|_{s=-1} \\ &= \frac{d}{ds} \left(\frac{(s^2 + 5s + 6) - (s-1)(2s+5)}{(s^2 + 5s + 6)^2} \right) \Big|_{s=-1} \\ &= \frac{d}{ds} \left(\frac{(-s^2 + 2s + 11)}{(s^2 + 5s + 6)^2} \right) \Big|_{s=-1} \\ &= \left(\frac{(-2s+2)(s^2 + 5s + 6)^2 - 2(2s+5)(s^2 + 5s + 6)(-s^2 + 2s + 11)}{(s^2 + 5s + 6)^4} \right) \Big|_{s=-1} \\ &= \left(\frac{(-2s+2)(s^2 + 5s + 6) - 2(2s+5)(-s^2 + 2s + 11)}{(s^2 + 5s + 6)^3} \right) \Big|_{s=-1} \\ &= \frac{4 \times 2 - 2 \times 3 \times 10}{2 \times 8} = \frac{-52}{2 \times 8} \end{aligned}$$

$$\mathbf{B}_3 = -3.75$$

$$\mathbf{A}_1 = (s+2) \frac{s-1}{(s+1)^3 (s+2)(s+3)} \Big|_{s=-2} = \frac{-2-1}{-1 \times +1} = 3$$

$$\mathbf{A}_2 = (s+3) \frac{s-1}{(s+1)^3 (s+2)(s+3)} \Big|_{s=-3} = \frac{-3-1}{-8 \times -1} = -0.5$$

$$F(s) = \frac{-1}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{-3.75}{s+1} + \frac{3}{s+2} + \frac{-0.5}{s+3}$$

Case 3. Complex Conjugate Roots In case the denominator has a set of repeated complex conjugate roots as $s = -\alpha \pm j\beta$ repeated m times as follows:

$$\begin{aligned} F(s) &= \frac{\text{Num}(s)}{(s + \alpha - j\beta)^m (s + \alpha + j\beta)^m} \\ F(s) &= \frac{k_1}{(s + \alpha - j\beta)^m} + \frac{k_1^*}{(s + \alpha + j\beta)^m} + \frac{k_2}{(s + \alpha - j\beta)^{m-1}} + \frac{k_2^*}{(s + \alpha + j\beta)^{m-1}} \\ &\quad + \cdots + \frac{k_m}{(s + \alpha - j\beta)} + \frac{k_m^*}{(s + \alpha + j\beta)} \end{aligned}$$

The process is similar to the one in real repeated roots. However, the coefficients $k_1 = |k_1| \angle \theta_1, \dots, k_m = |k_m| \angle \theta_m$ are complex conjugate values of $k_1^* = |k_1| \angle -\theta_1, \dots, k_m^* = |k_m| \angle -\theta_m$.

Application of Laplace in Electric Circuits

As illustrated earlier, Laplace is utilized to express and solve for circuit parameters in frequency domain. Laplace transforms of all circuit components including the input sources, the elements, and the responses are required to be completely expressed in frequency domain. Transformation of sources in Laplace domain directly applies the methods introduced in the beginning of this chapter. In this section, Laplace transform of the circuit elements is introduced.

Resistors in Frequency Domain

Consider a resistor shown in Fig. 7.3 with current $i(t)$ passing through and a voltage drop $v(t)$ across the element. Ohm's law indicates that:

$$v(t) = Ri(t)$$

The Laplace transform of this equation considering no initial condition (because resistors do not store energy) can be obtained as follows:

$$\begin{aligned} L\{v(t) = Ri(t)\} \\ V(s) = RI(s) \end{aligned}$$

Therefore, the resistance value in frequency domain remains similar to its time domain value and both of which are measured in Ohms Ω .

Inductors in Frequency Domain

Consider an inductor with inductance of L Henrys. Current $i(t)$ drops voltage $v(t)$ which is related as follows:

$$v(t) = L \frac{di(t)}{dt}$$

Considering an initial current of I_o through the inductor, the Laplace transform of the equation is obtained as follows:



Fig. 7.3 A resistor represented in frequency domain

$$L \left\{ v(t) = L \frac{di(t)}{dt} \right\}$$

$$V(s) = L(sI(s) - I_o)$$

$$V(s) = sLI(s) - LI_o$$

Interpreting this equation in a mesh and through KVL, the voltage drop $V(s)$ equals the voltage drop across an impedance sL times the current $I(s)$ in series with a voltage source that is generated by the existence of initial current LI_o . The inductor tends to keep the current constant by changing the voltage polarity. For this reason, the source indicating the initial current demonstrates a negative polarity at time $t = 0^+$. Figure 7.4 shows the frequency domain equivalent of an inductor charged with initial current I_o .

Note 7.7 The unit of inductance $L(H)$ in time domain is Henry, but the its Laplace transform in frequency domain is measured in ohms $sL(\Omega)$.

Solving for $I(s)$ results in admittance equivalent of an inductor in frequency domain:

$$V(s) = sLI(s) - LI_o$$

$$I(s) = \frac{1}{sL} V(s) + \frac{1}{s} I_o$$

This equation demonstrates that the current passing a charged inductor is equivalent to a parallel of inductor with admittance $\frac{1}{sL}$ and a current source which is generated because of the initial current as $\frac{1}{s}I_o$. Figure 7.5 shows the parallel equivalent of a charged inductor in frequency domain.

Capacitors in Frequency Domain

Consider a capacitor at capacitance C that is initially charged at voltage V_0 as shown in Fig. 7.6. The current $i(t)$ generates voltage $v(t)$ which is related as:

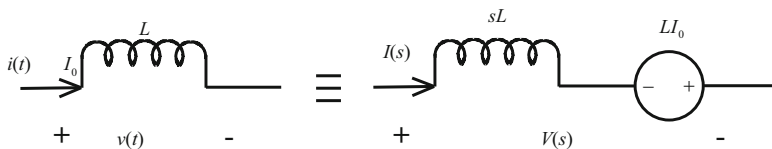


Fig. 7.4 Laplace transformation of a charged inductor from time domain to frequency domain. The initial charge of the inductor is shown as a voltage source in series connection to the inductor. This model is best for KVL analysis. Note that the polarity of the voltage source is reversed

Fig. 7.5 Laplace transformation of a charged inductor when the initial charge of the inductor is shown as a current source in parallel to the inductor. This model is best for KCL analysis

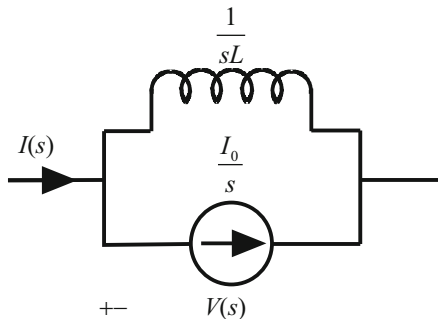
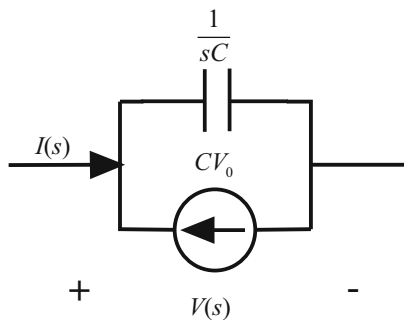


Fig. 7.6 Laplace transformation of a charged capacitor when the initial charge of the capacitor is shown as a current source in parallel to the capacitor. This model is best for KCL analysis. Note that the direction of the current source is reversed



$$i(t) = C \frac{dv(t)}{dt}$$

Taking Laplace of this equation considering the initial charge at time $v(t = 0^+) = V_0$ results in:

$$L\left\{i(t) = C \frac{dv(t)}{dt}\right\}$$

$$I(s) = C(sV(s) - v(0^+)) = CsV(s) - CV_0$$

Considering a parallel equivalent circuit, the charged capacitor in frequency domain is presented as:

$$I(s) = CsV(s) - CV_0$$

The equivalent circuit is shown in Fig. 7.6.

The voltage $V(s)$ can be obtained to show a charged capacitor in series form. The voltage is:

$$V(s) = \frac{1}{Cs}I(s) + \frac{1}{s}CV_0$$

The circuit is obtained by writing the KVL across the uncharged capacitor and the source representing the initial charge as a voltage source. Once this source is presented as a current source, a parallel circuit represents the initial charge. Considering a KVL as shown in the above equation, the equivalent circuit can be obtained as shown in Fig. 7.7.

Circuit Analysis Using Laplace Transform

Laplace can be used in circuit analysis in two ways, (1) solving differential equations that are obtained from circuits and (2) writing circuit KVL and KCL equations directly in Laplace. This section studies techniques to write circuit equations directly in Laplace.

To write KVL and KCL equations in Laplace domain, the following steps are recommended, although some of these steps can be skipped depending on the circuit.

Step 1: The circuit elements must be converted to their Laplace equivalent. Table 7.1 lists the equivalent of circuit elements in frequency domain.

Step 2: Convert current and voltage sources from time domain to Laplace domain. These sources preserve the type of the source, but their functions or values are transformed into frequency domain. For instance, a current source of $i(t) = 10u(t)$ remains a current source with the same direction, but its value becomes $I(s) = 10\frac{1}{s}$.

Step 3: Dependent sources remain a dependent source, and their functions are converted to Laplace domain. Their dependent parameter from time domain is converted to the Laplace of the same parameter.

Step 4: The voltage drop across a Laplace represented impedance is $V(s) = Z(s)I(s)$ and can be used in KVL.

Step 5: The current of a branch written in KCL is obtained from $I(s) = \frac{V(s)}{Z(s)}$.

Step 6: Writing KVL and KCL in Laplace domain should result in a set of algebraic equations. Solve these equations for the desired parameters.

Step 7: Take Laplace inverse of the circuit responses and find the time domain functions and values.

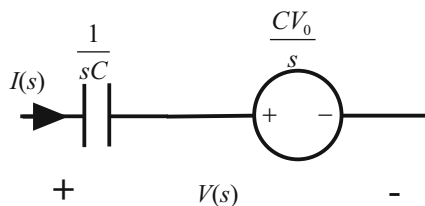


Fig. 7.7 Laplace transformation of a charged capacitor from time domain to frequency domain. The initial charge of the capacitor is shown as a voltage source in series connection to the inductor. This model is best for KVL analysis

Table 7.1 Table of circuit element conversion

Circuit element	Time domain (unit)	Laplace domain impedance (Ω)	Laplace domain admittance (Ω^{-1})
Resistor	$R(\Omega)$	R	$\frac{1}{R}$
Inductor	$L(H)$	sL	$\frac{1}{sL}$
Capacitor	$C(F)$	$\frac{1}{sC}$	sC

Example 7.40 Find the current flowing through the circuit of Fig. 7.8 in Laplace domain.

Solution Steps 1 and 5. Converting the circuit elements and the source function in Laplace domain requires the inductor 10 mH to be presented as $10e - 3s$ (Ω) and the resistor to remain as a 20Ω resistor. The source of $10u(t)$ becomes $\frac{10}{s}$. The current flowing through the circuit is $I(s)$. Since all impedances in the frequency domain are presented in Ω s, the voltage drop across each element is the product of the current $I(s)$ by the impedance. For instance, the voltage drop across the inductor is $10e - 3sI(s)$, and the voltage drop across the resistor is $20I(s)$ (Fig. 7.9).

Writing a KVL starting from the negative terminal of the source and circling clockwise results in:

$$-\frac{10}{s} + 10e - 3sI(s) + 20I(s) = 0$$

Solving for $I(s)$ results in:

$$I(s) = \frac{\frac{10}{s}}{10e - 3s + 20}$$

Simplifying the equation by factoring the $10e - 3$ out to make the denominator $\times$ ba monic polynomial results in:

$$I(s) = \frac{10}{0.01s(s + \frac{20}{0.01})} = \frac{1000}{s(s + 2000)}$$

Taking Laplace inverse by partial fraction results in:

$$I(s) = \frac{A}{s} + \frac{B}{(s + 2000)}$$

$$A = s \frac{1000}{s(s + 2000)} \Big|_{s=0} = \frac{1}{2}$$

$$B = (s + 2000) \frac{1000}{s(s + 2000)} \Big|_{s=-2000} = -\frac{1}{2}$$

Fig. 7.8 Circuit of Example 7.40 in time domain

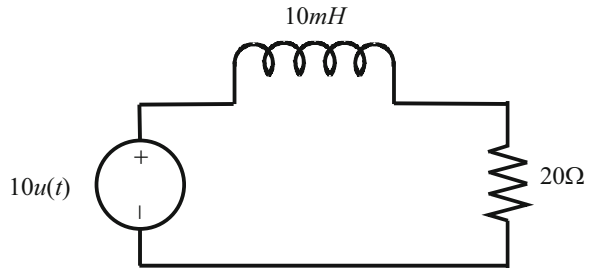
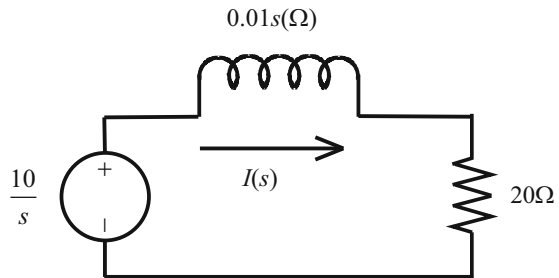


Fig. 7.9 Laplace transform of the circuit (Example 7.40)



Therefore, the function can be written as:

$$I(s) = \frac{0.5}{s} - \frac{0.5}{(s + 2000)}$$

The time domain, by taking Laplace inverse and using the templates introduced earlier, can be obtained as:

$$i(t) = 0.5u(t) - 0.5e^{-2000t}u(t) \text{ A}$$

Example 7.41 Considering the circuit of the previous example (7.41), find the voltage across the 20Ω resistor and the voltage across the 10 mH inductor both in frequency and time domains.

Solution The voltage of 20Ω in frequency domain can be obtained as:

$$V_{20 \Omega} = 20I(s) = 20\left(\frac{0.5}{s} - \frac{0.5}{(s + 2000)}\right) = \frac{10}{s} - \frac{10}{(s + 2000)}$$

And the voltage in time domain is:

$$v_{20 \Omega}(t) = 10u(t) - 10e^{-2000t}u(t) \text{ V}$$

The voltage across 10 mH inductor is obtained in Laplace domain as (remember the ohmic value of this inductor in Laplace domain is $10e - 3 s = 0.01s (\Omega)$):

$$V_{10 \text{ mH}}(s) = 0.01sI(s) = 0.01s \frac{1000}{s(s + 2000)} = \frac{10}{(s + 2000)}$$

In time domain, this current is:

$$v_{10\text{mH}}(t) = 10e^{-2000t}u(t) \text{ V}$$

Example 7.42 Considering the circuit in Fig. 7.10, find the current of each branch I_1 and I_2 in Laplace and in time domain.

Solution Circuit element and source equivalent in Laplace result in the following circuit (Fig. 7.11):

The current $I(s)$ can be obtained by dividing the voltage by the total impedance observed at the source terminals. Therefore:

$$\begin{aligned} I(s) &= \frac{20}{s^2 + 100} = \frac{20}{s^2 + 100} = \frac{20}{s^2 + 100} = \frac{20}{s^2 + 100} \\ &= \frac{20}{2s + \left(1 \parallel \frac{4}{s}\right)} = \frac{20}{2s + \frac{4}{1 + \frac{s}{4}}} = \frac{20}{2s + \frac{4}{1 + \frac{s}{4}}} = \frac{20}{2s + \frac{4}{1 + \frac{s}{4}}} \\ &= \frac{20(s + 4)}{(s^2 + 100)(2s^2 + 8s + 4)} \\ I(s) &= \frac{10(s + 4)}{(s^2 + 100)(s^2 + 4s + 2)} \end{aligned}$$

This fraction can be split into two partial fractions as follows:

$$I(s) = \frac{10(s + 4)}{(s^2 + 100)(s^2 + 4s + 2)} = \frac{As + B}{s^2 + 100} + \frac{Cs + D}{s^2 + 4s + 2}$$

where $A = -0.1017$, $B = 0.0071$, $C = 0.1017$, $D = 0.399$.

Therefore:

$$I(s) = \frac{-0.1017s + 0.0071}{s^2 + 100} + \frac{0.1017s + 0.399}{(s^2 + 4s + 2) = (s + 2)^2 - 2}$$

Splitting the fractions results in the known templates for $\sin$, $\cos$, $\sinh$, and $\cosh$ as follows:

Fig. 7.10 Circuit of Example 7.42 in time domain

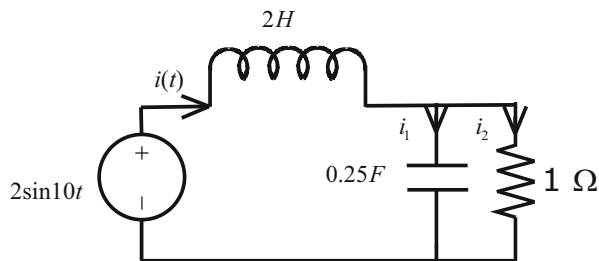
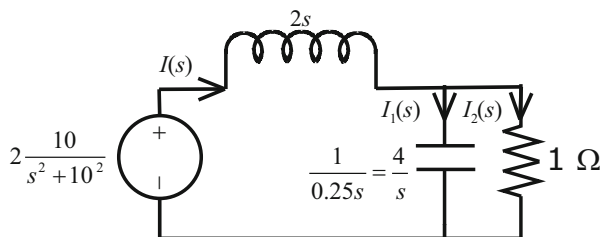


Fig. 7.11 Laplace transform of the circuit (Example 7.42)



$$I(s) = \frac{-0.1017s}{s^2 + 100} + \frac{0.0071}{s^2 + 100} + 0.1017 \frac{s + 3.932}{(s + 2)^2 - 2}$$

The coefficient of the last fraction 3.932 can be written as $2 + 1.932$ to show the same shift of frequency by 2 rad/s as the term $(s + 2)^2$ demonstrates.

$$I(s) = \frac{-0.1017s}{s^2 + 100} + \frac{0.0071}{s^2 + 100} + 0.1017 \frac{(s + 2) + 1.932}{(s + 2)^2 - 2}$$

$$I(s) = \frac{-0.1017s}{s^2 + 100} + \frac{0.0071}{s^2 + 100} + 0.1017 \left(\frac{(s + 2)}{(s + 2)^2 - 2} + \frac{1.932}{(s + 2)^2 - 2} \right)$$

Therefore, the time domain expression of the current can be obtained as follows (note that the shift in frequency becomes an exponential in time domain):

$$i(t) = \left(-0.1017 \cos 10t + \frac{0.0071}{10} \sin 10t + 0.1017 \left(e^{-2t} \cosh \sqrt{2}t - \frac{1.932}{\sqrt{2}} e^{-2t} \sinh \sqrt{2}t \right) \right) u(t) A$$

Current division between the branches of $\frac{4}{s} \Omega$ capacitor and the 1Ω resistor is as follows:

$$\begin{aligned}
 I_1(s) &= \frac{1}{\frac{4}{s} + 1} I(s) = \frac{s}{s+4} I(s) = \frac{s}{s+4} \frac{10(s+4)}{(s^2+100)(s^2+4s+2)} \\
 &= \frac{10s}{(s^2+100)(s^2+4s+2)} = \frac{As+B}{s^2+100} + \frac{Cs+D}{s^2+4s+2} \\
 A &= -0.0875, B = 0.3570, C = 0.0875, D = -0.0071.
 \end{aligned}$$

Therefore:

$$\begin{aligned}
 I_1(s) &= \frac{-0.875s}{s^2+100} + \frac{0.3570}{s^2+100} + 0.0875 \left(\frac{s+0.0811}{(s+2)^2-2} \right) \\
 I_1(s) &= \frac{-0.875s}{s^2+100} + \frac{0.3570}{s^2+100} + 0.0875 \left(\frac{s+2-1.9189}{(s+2)^2-2} \right) \\
 I_1(s) &= \frac{-0.875s}{s^2+100} + \frac{0.3570}{s^2+100} + 0.0875 \left(\frac{s+2}{(s+2)^2-2} - \frac{1.9189}{(s+2)^2-2} \right) \\
 i_1(t) &= \left(-0.875 \cos 10t + \frac{0.3570}{10} \sin 10t \right. \\
 &\quad \left. + 0.0875 \left(e^{-2t} \cosh \sqrt{2}t - \frac{1.9189}{\sqrt{2}} e^{-2t} \sinh \sqrt{2}t \right) \right) u(t) \text{ A}
 \end{aligned}$$

Current passing 1Ω resistor is obtained by current division as follows:

$$\begin{aligned}
 I_2(s) &= \frac{\frac{4}{s}}{\frac{4}{s} + 1} I(s) = \frac{4}{s+4} I(s) = \frac{4}{s+4} \frac{10(s+4)}{(s^2+100)(s^2+4s+2)} \\
 &= \frac{40}{(s^2+100)(s^2+4s+2)} = \frac{As+B}{s^2+100} + \frac{Cs+D}{s^2+4s+2} \\
 A &= -0.0143, B = -0.3499, C = 0.0143, D = 0.4070.
 \end{aligned}$$

Therefore:

$$\begin{aligned}
 I_2(s) &= \frac{-0.0143s}{s^2+100} + \frac{-0.3499}{s^2+100} + 0.0143 \left(\frac{s+28.4615}{(s+2)^2-2} \right) \\
 I_2(s) &= \frac{-0.0143s}{s^2+100} + \frac{-0.3499}{s^2+100} + 0.0143 \left(\frac{s+2}{(s+2)^2-2} + \frac{26.4615}{(s+2)^2-2} \right)
 \end{aligned}$$

$$i_2(t) = \left(-0.0143 \cos 10t - \frac{0.3499}{10} \sin 10t + 0.0143 \left(e^{-2t} \cosh \sqrt{2}t - \frac{26.4615}{\sqrt{2}} e^{-2t} \sinh \sqrt{2}t \right) \right) u(t) \text{ A}$$

Example 7.43 Considering the circuit in the previous example (7.42), shown as follows, find the voltage across the 1Ω resistor (Fig. 7.12).

Solution The current $I(s)$ passes through the parallel of 1Ω and $\frac{4}{s}$ impedances. Therefore, the voltage becomes:

$$V_o(s) = I(s) \left(1 \parallel \frac{4}{s} \right)$$

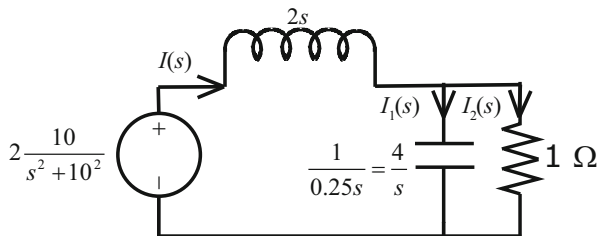
Replacing the values results in:

$$\begin{aligned} V_o(s) &= \frac{10(s+4)}{(s^2+100)(s^2+4s+2)} \left(1 \parallel \frac{4}{s} \right) \\ V_o(s) &= \frac{10(s+4)}{(s^2+100)(s^2+4s+2)} \frac{4}{(s+4)} = \frac{40}{(s^2+100)(s^2+4s+2)} \\ &= \frac{As+B}{s^2+100} + \frac{Cs+D}{s^2+4s+2} \\ A &= -0.0143, B = -0.3499, C = 0.0143, D = 0.4070. \end{aligned}$$

Therefore:

$$\begin{aligned} V_o(s) &= \frac{-0.0143s}{s^2+100} + \frac{-0.3499}{s^2+100} + 0.0143 \left(\frac{s+28.4615}{(s+2)^2-2} \right) \\ V_o(s) &= \frac{-0.0143s}{s^2+100} + \frac{-0.3499}{s^2+100} + 0.0143 \left(\frac{s+2}{(s+2)^2-2} + \frac{26.4615}{(s+2)^2-2} \right) \end{aligned}$$

Fig. 7.12 Laplace transform of the circuit (Example 7.43)



$$v_0(t) = \left(-0.0143 \cos 10t - \frac{0.3499}{10} \sin 10t + 0.0143 \left(e^{-2t} \cosh \sqrt{2}t - \frac{26.4615}{\sqrt{2}} e^{-2t} \sinh \sqrt{2}t \right) \right) u(t) \text{ V}$$

Example 7.44 Considering the circuit shown in Fig. 7.13, find the current in each loop, $I_1(s), I_2(s)$ as a function of input voltage $V_{in}(s)$.

Solution The circuit in Laplace form has two loops as follows:

KVL ①. $-V_{in}(s) + RI_1(s) + sL_1(I_1(s) - I_2(s)) = 0$

KVL ②. $\frac{1}{sC}I_2(s) + sL_2I_2(s) + sL_1(I_2(s) - I_1(s)) = 0$

Solving for $I_1(s)$, $I_2(s)$ results in:

$$(R + sL_1)I_1(s) - sL_1I_2(s) = V_{in}(s)$$

$$\left(\frac{1}{sC} + sL_1 + sL_2 \right) I_2(s) - sL_1I_1(s) = 0$$

Finding $I_1(s)$ from the second equation results in $I_1(s) = \frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) I_2(s)$, and replacing it in the first equation yields:

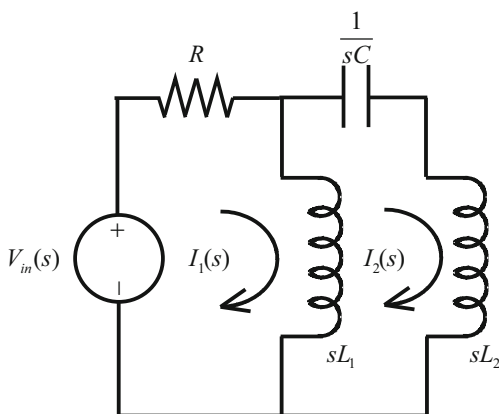
$$(R + sL_1) \left(\frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) I_2(s) \right) - sL_1I_2(s) = V_{in}(s)$$

Solving for $I_2(s)$ results in:

$$I_2(s) = \frac{1}{\left((R + sL_1) \left(\frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) \right) - sL_1 \right)} V_{in}(s)$$

Simplifying:

Fig. 7.13 Circuit of Example 7.44



$$I_2(s) = \frac{CL_1s^2}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R} V_{in}(s)$$

This function represents the current in the second loop as a function of input voltage source. Considering $I_1(s) = \frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) I_2(s)$, and replacing this from the second loop current, $I_1(s)$ becomes:

$$I_1(s) = \frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) \frac{1}{\left((R + sL_1) \left(\frac{1}{sL_1} \left(\frac{1}{sC} + sL_1 + sL_2 \right) \right) - sL_1 \right)} V_{in}(s)$$

Simplifying current $I_1(s)$ in terms of voltage $V_{in}(s)$ yields:

$$I_1(s) = \frac{C(L_1 + L_2)s^2 + 1}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R} V_{in}(s)$$

Example 7.45 Consider the circuit shown in Fig. 7.14. Find the voltage of nodes in terms of the input currents using the Laplace transform.

Solution Taking Laplace transform of the circuit, the circuit shown in Fig. 7.15 is obtained.

For simplicity in writing KCL equations (in this example), the Laplace operator s is omitted from the functions.

$$\text{KCL ①. } -I_1 + \frac{V_1}{R} + sC(V_1 - V_2) = 0$$

$$\text{KCL ②. } I_2 + \frac{V_2}{sL} + sC(V_2 - V_1) = 0$$

Simplifying results in:

$$\left(\frac{1}{R} + sC \right) V_1 - sCV_2 = I_1$$

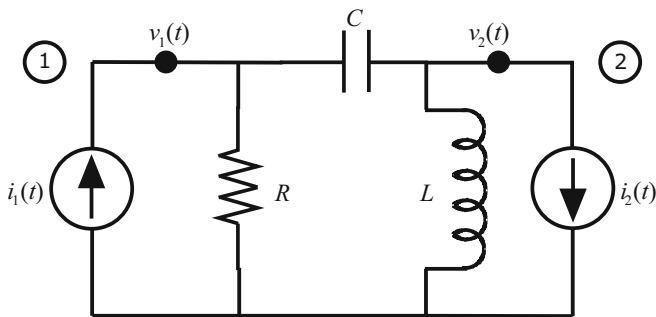


Fig. 7.14 Circuit of Example 7.45 in time domain

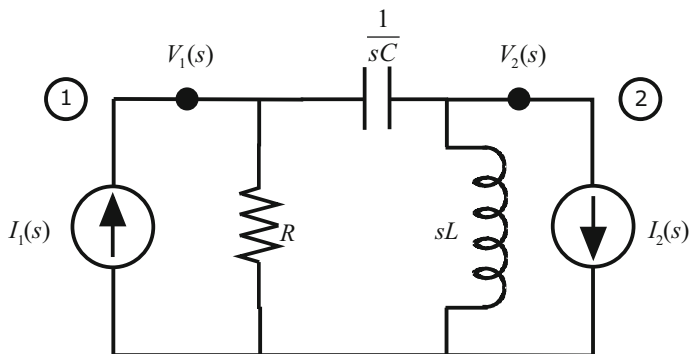


Fig. 7.15 Laplace transform of the circuit (Example 7.45)

$$-sCV_1 + \left(\frac{1}{sL} + sC\right)V_2 = -I_2$$

$$V_1 = \frac{R(CL(I_1 - I_2) + I_1)}{CLs^2 + RCs + 1} = \frac{R\left(1 + \frac{1}{LC}\right)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_1 - \frac{R}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_2$$

$$V_2 = -\frac{Ls(I_2 - CRsI_1 + CRsI_2)}{CLs^2 + RCs + 1} = \frac{Rs^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_1 - \frac{RLs\left(\frac{1}{RC} + s\right)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_2$$

As these equations show, the voltages of V_1 and V_2 are functions of both input currents I_1 and I_2 .

Laplace Transform

7.1 Find the Laplace transform of the following functions:

- (a) $f(t) = 10u(t)$
- (b) $f(t) = -20\delta(t)$
- (c) $f(t) = -5tu(t)$
- (d) $f(t) = 10e^{-30t}u(t)$
- (e) $f(t) = 3te^{-5t}u(t)$
- (f) $f(t) = 20 \sin 75tu(t)$
- (g) $f(t) = \sin^2 \omega tu(t)$
- (h) $f(t) = \cos^2 \omega tu(t)$
- (i) $f(t) = 10 \sin (5t+30)u(t)$
- (j) $f(t) = 110\sqrt{2} \cos 377t u(t)$
- (k) $f(t) = 12 \sinh 20t u(t)$
- (l) $f(t) = 10 \cosh 2\pi t u(t)$

7.2 Find the Laplace of the following function operations:

- (a) $f(t) = 2u(t) + 5tu(t)$
- (b) $f(t) = 10 \sin 20t + 5 \cos 20t$
- (c) $f(t) = 5 \sin 3t - \cos 15t$
- (d) $f(t) = 3e^{-10t} \sin 15t$
- (e) $f(t) = 75e^{-100t} \cos 314t$
- (f) $f(t) = 3e^{-10t} \sinh 15t$
- (g) $f(t) = 75e^{-100t} \cosh 314t$
- (h) $f(t) = 13tu(t - 5)$
- (i) $f(t) = 10te^{-3t}u(t - 4)$
- (j) $f(t) = 5t^3u(t)$

7.3 Find the Laplace inverse of the following functions:

- (a) $F(s) = 1 + s$
- (b) $F(s) = \frac{1}{s+5}$
- (c) $F(s) = \frac{s}{s+5}$
- (d) $F(s) = \frac{s-1}{s+5}$
- (e) $F(s) = \frac{1}{s^2+25}$
- (f) $F(s) = \frac{s+1}{s^2+25}$
- (g) $F(s) = \frac{s^2-1}{s^2+25}$
- (h) $F(s) = \frac{1}{s^2-25}$
- (i) $F(s) = \frac{s+1}{s^2-25}$
- (j) $F(s) = \frac{1}{s^2+7s+12}$
- (k) $F(s) = \frac{s+1}{s^2+7s+12}$
- (l) $F(s) = \frac{s^2+5s+6}{s^2+7s+12}$
- (m) $F(s) = \frac{1}{s^2+6s+100}$
- (n) $F(s) = \frac{s+1}{s^2+6s+100}$
- (o) $F(s) = \frac{s^2+1}{s^2+6s+100}$
- (p) $F(s) = \frac{s^2+s+1}{s^2+6s+100}$
- (q) $F(s) = \frac{1}{s(s^2+100)}$
- (r) $F(s) = \frac{1}{s(s+3)(s+7)}$
- (s) $F(s) = \frac{(s+1)(s+2)}{s(s+3)(s+7)}$
- (t) $F(s) = \frac{1}{s^3(s+3)^2(s+7)}$
- (u) $F(s) = \frac{1}{s} e^{-3s}$
- (v) $F(s) = \frac{1}{s^2} (e^{-3s} + e^{-5s})$
- (w) $F(s) = e^{-10s} \frac{s}{s+5}$
- (x) $F(s) = \frac{e^{-s}}{s^2+25}$
- (y) $F(s) = \frac{(s+1)e^{2s}}{s^2+25}$
- (z) $F(s) = \frac{20}{(s+3\pi)^3}$

7.4 Prove that:

$$(a) \mathcal{L}\{\cosh at \cos at\} = \frac{s^3}{s^4 + 4a^4}$$

$$(b) \mathcal{L}\{t^2 \cos \omega t\} = \frac{2s(s^2 - 3\omega^2)}{(s^2 + \omega^2)^3}$$

7.5 Find the convolution of $h(t) = f(t) * g(t)$, wherein $f(t)$ & $g(t)$ are:

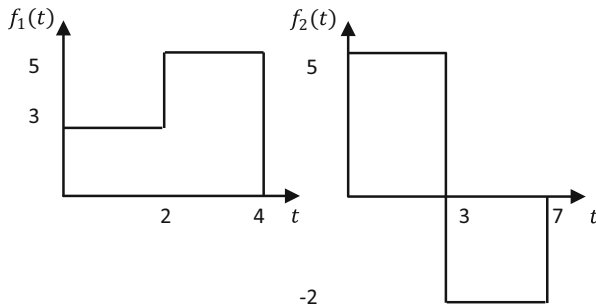
$$(a) f(t) = u(t) - u(t - 5), \quad g(t) = 3u(t - 4) - 3u(t - 5).$$

$$(b) f(t) = u(t) - u(t - 5), \quad g(t) = 3tu(t) - 3(t - 5)u(t - 5) - 3u(t - 5).$$

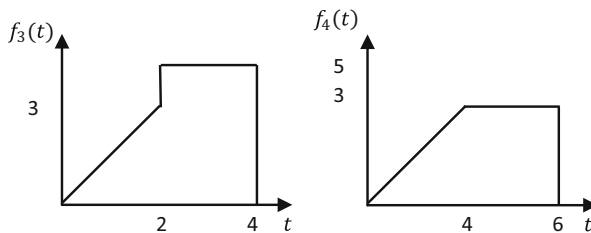
$$(c) f(t) = 5u(t) - u(t - 1) - u(t - 2) - 3u(t - 3), \quad g(t) = 3tu(t) - 6(t - 1)u(t - 1) + 3(t - 2)u(t - 2).$$

7.6 Do the following convolution integrals using Laplace and direct method:

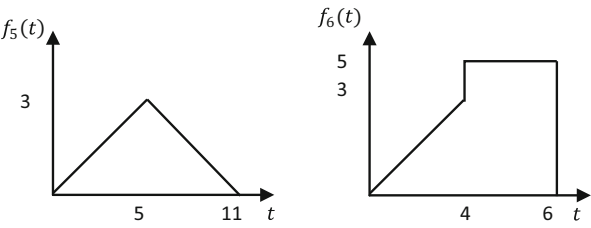
$$(a) f_1(t) * f_2(t)$$



$$(b) f_3(t) * f_4(t)$$



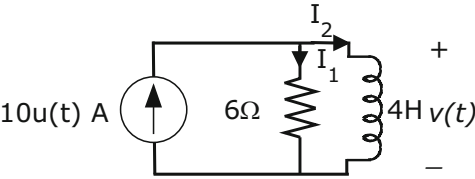
(c) $f_5(t) * f_6(t)$



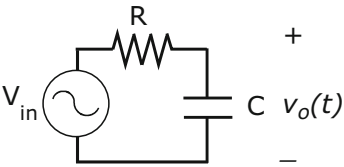
7.7 Solve the following differential equations using Laplace transform:

(a)	$\dot{y} + 9y = 0$	$y(0) = 5$
(b)	$\ddot{y} + 9y = 0$	$y(0) = 0, y'(0) = 2$
(c)	$\ddot{y} + 2\dot{y} = 10u(t)$	$y(0) = 1, y'(0) = -1$
(d)	$\ddot{y} + 2\dot{y} = 10tu(t)$	$y(0) = 1, y'(0) = -1$
(e)	$\ddot{y} - 2\dot{y} - 3y = \dot{\delta}$	$y(0) = 1, y'(0) = 7$
(f)	$4\ddot{y} + y = 0$	$y(0) = 1, y'(0) = -2$
(g)	$4\ddot{y} + y = \sin(t)$	$y(0) = 0, y'(0) = -1$
(h)	$\ddot{y} + 2\dot{y} + 5y = 0$	$y(0) = 2, y'(0) = -4$
(i)	$\ddot{y} + 2y = u(t) - u(t - 1)$	$y(0) = 0, y'(0) = 0$
(j)	$\ddot{y} + 3\dot{y} + 2y = \delta(t - a)$	$y(0) = 0, y'(0) = 0$

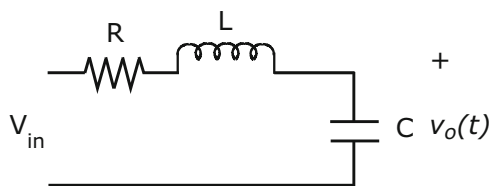
7.8 Using Laplace transform find $i_1(t)$, $i_2(t)$, $v(t)$.



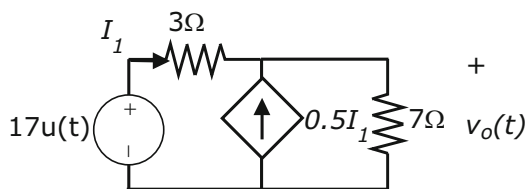
7.9 Using Laplace transform find $v_o(t)$.



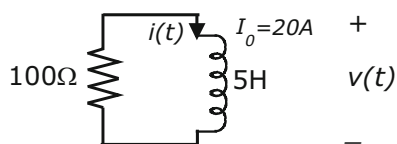
7.10 Using Laplace transform find $v_o(t)$.



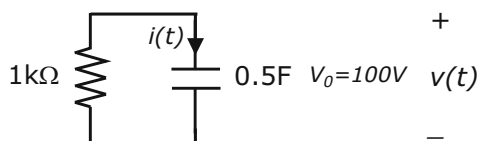
7.11 Using Laplace transform find I_1 , $V_o(t)$.



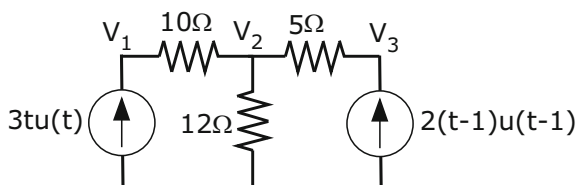
7.12 Using Laplace transform find $i(t)$, $v(t)$.



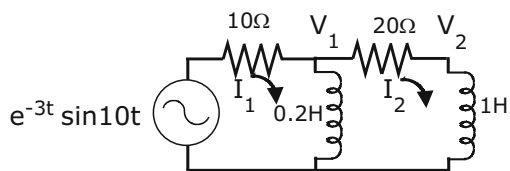
7.13 Using Laplace transform find $i(t)$, $v(t)$.



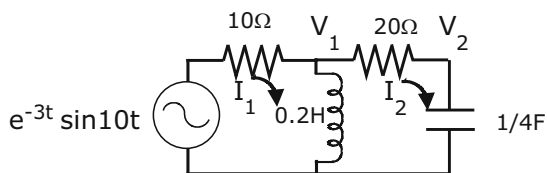
7.14 Using Laplace transform find $v_1(t)$, $v_2(t)$, $v_3(t)$.



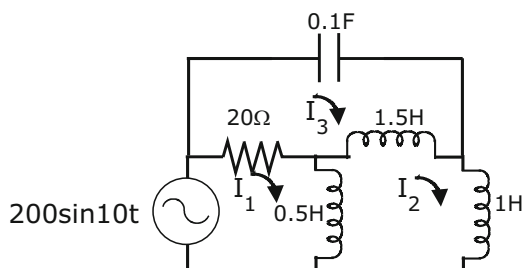
7.15 Using Laplace transform find $v_1(t)$, $v_2(t)$, $i_1(t)$, $i_2(t)$.



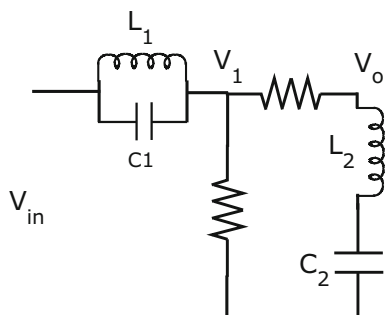
7.16 Using Laplace transform find $v_1(t)$, $v_2(t)$, $i_1(t)$, $i_2(t)$.



7.17 Using Laplace transform find $i_1(t)$, $i_2(t)$, $i_3(t)$.



7.18 Using Laplace transform find $v_1(t)$, $v_o(t)$.



Chapter 8

Transfer Functions



Definition of Transfer Function

Linear physical system with one or multiple set of input and output can be represented by mathematical functions that relate any of the outputs to any of the inputs. These functions are unique and are defined based on the systems governing equations. The transfer function of a system is defined as the Laplace transform of the output response over the Laplace transform of the input excitation. Transfer functions are defined for any desired set of input and output functions that may relate the input and output together. Considering the Laplace transform of the input function as $X(s)$ and the output as $Y(s)$, the transfer function $H(s)$ can be defined as:

$$H(s) \triangleq \frac{Y(s)}{X(s)}$$

In time domain, the transfer function $h(t)$ is defined through the convolution product (*) as follows:

$$y(t) = x(t) * h(t)$$

The Laplace transform of the convolution product is obtained as:

$$L\{x(t) * h(t)\} = X(s)H(s)$$

This chapter identifies transfer function of linear circuits in both the frequency and time domains. Figure 8.1 demonstrates the system response when a desired input is applied to the system's transfer function.

Example 8.1 A circuit has a response function of $y(t) = 2e^{-t} \sin(10t)$ when the input function of $x(t) = 3u(t)$ is applied. Find the transfer function in frequency domain.

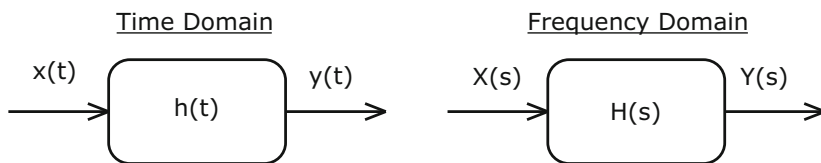


Fig. 8.1 Relation of the input, output, and the transfer function in both time domain and frequency domain. In time domain, convolution integral obtains the output for a given input signal $x(t)$ by $y(t) = x(t) * h(t)$. In frequency domain, a product of the system transfer function $H(s)$ by the given input signal $X(s)$ determines the output by $Y(s) = X(s)H(s)$

Solution According to the definition, transfer function is a division of Laplace transforms of the output response over the Laplace transform of the input signal. Hence, the transfer function is obtained as:

$$H(s) = \frac{L\{y(t)\}}{L\{x(t)\}} = \frac{L\{2e^{-t}\sin(10t)\}}{L\{3u(t)\}} = \frac{\frac{20}{(s+1)^2 + 100}}{\frac{3}{s}} = \frac{\frac{20}{3}s}{(s+1)^2 + 100}$$

Example 8.2 Transfer function of a system is given as $H(s) = \frac{1}{s+5}$. Find the output if the system is excited by input $x(t) = 5u(t)$.

Solution In frequency domain, the transfer function definition finds the output in Laplace domain to be

$$H(s) \triangleq \frac{Y(s)}{X(s)} \Rightarrow Y(s) = X(s)H(s)$$

The Laplace of input is obtained by:

$$X(s) = L\{5u(t)\}$$

Hence, the output becomes:

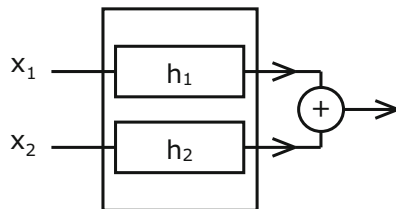
$$Y(s) = \frac{5}{s} \frac{1}{s+5} = \frac{5}{s(s+5)}$$

In time domain, the output can be obtained by:

$$y(t) = 5u(t) * e^{-5t}u(t).$$

In this chapter, a technique to obtain this convolution product is introduced.

Fig. 8.2 Two systems in parallel that receive two separate inputs. The overall system output is an algebraic summation of the separately excited system outputs



Multi-input-Multi-output Systems

In physical systems, an output may be influenced by several inputs. The transfer function of this system is the linear summation of all transfer functions excited by various inputs that contribute to a desired output. For instance, if inputs $x_1(t)$ and $x_2(t)$ directly influence the output $y(t)$, respectively, through transfer functions $h_1(t)$ and $h_2(t)$, the output is therefore obtained as:

$$y(t) = x_1(t) * h_1(t) + x_2(t) * h_2(t)$$

In Laplace domain, the output is obtained as:

$$Y(s) = X_1(s)H_1(s) + X_2(s)H_2(s)$$

Figure 8.2 demonstrates this system's configuration.

Note 8.1 In MIMO systems, individual transfer functions are obtained by measuring the output signal while applying an input signal to one input port and zeroing out the rest of inputs.

Note 8.2 In the MIMO system with two inputs, transfer function $H_1(s)$ is obtained when a signal is applied to $X_1(s)$ and $X_2(s)$ is kept zero.

$$H_1(s) = \left. \frac{Y(s)}{X_1(s)} \right|_{X_2(s) = 0}$$

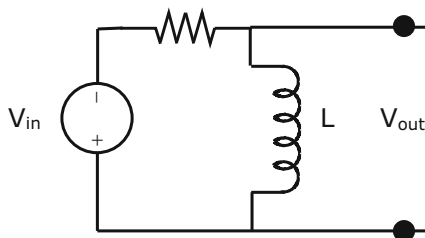
Similarly,

$$H_2(s) = \left. \frac{Y(s)}{X_2(s)} \right|_{X_1(s) = 0}$$

Obtaining Transfer Function of Electric Circuits

Transfer functions of electric circuits demonstrate a mathematical representation of the circuit behavior. A transfer function defined for an input and output set is unique and depends only on the circuit characteristics and not the input or output signals.

Fig. 8.3 RL circuit. The input function is forced by the voltage source, and the output is either the measured voltage across the inductor or the current in the circuit



However, circuit parameters such as voltage and current can be used as input and output signals to obtain the transfer functions. For instance, in the following circuit, V_{in} can be considered as an input signal, and I and V_{out} are considered as output signals. It is a totally desired signal consideration and has to match the reality of the circuit operation. In this circuit, a source is the actual driver of the current and the voltage measured across the inductor. Therefore, it makes sense to consider it as an input. The results are the current flowing in the circuit (Fig. 8.3) and the voltage appeared at the inductor.

Based on this logical approach, the following transfer functions can be defined:

$$H_1 \triangleq \frac{I}{V_{in}} \quad \text{or} \quad H_2 \triangleq \frac{V_{out}}{V_{in}}$$

Now the question is how to find these transfer functions. One of the best approaches is to solve the circuit in Laplace domain for the desired output defined in the transfer function. For instance, to find the transfer function H_1 , the circuit can be solved to obtain I , and to obtain the transfer function H_2 , the circuit can be solved for V_{out} .

When the desired output parameter is identified, it is required to do the changes and use replacements to express the desired output in terms of the circuit input signal. Then the ratio of the output over input can be easily obtained.

Example 8.3 In the given circuit of Fig. 8.3 find the transfer functions defined as $H_1 \triangleq \frac{I}{V_{in}}$ or $H_2 \triangleq \frac{V_{out}}{V_{in}}$.

Solution To find H_1 , as explained, the circuit is solved for I .

The circuit current is obtained using a KVL as follows:

$$-V_{in} + RI + sLI = 0$$

The current becomes:

$$I = \frac{V_{in}}{R + sL}$$

The output I is in terms of the input voltage V_{in} . Therefore, no further replacement is needed. A ratio of the output I over the input V_{in} can be obtained as:

$$H_1(s) = \frac{I}{V_{in}} = \frac{1}{R + sL} = \frac{\frac{1}{L}}{s + \frac{R}{L}}$$

To find $H_2(s)$, the circuit is solved for the desired output V_{out} as follows:

$$V_{out} = I \times sL$$

Now the current is not the type of input the transfer function was asked for. Rather, the input has to be the input voltage. Therefore, a replacement of I with its equivalent $\frac{V_{in}}{R+sL}$ seems necessary. This yields:

$$V_{out} = sL \frac{V_{in}}{R + sL}$$

The ratio of the output over input is obtained as:

$$H_2(s) = \frac{V_{out}}{V_{in}} = \frac{sL}{R + sL} = \frac{s}{s + \frac{R}{L}}$$

Example 8.4 Show the equivalent of the circuit a transfer functions obtained in example 8.3.

Solution See Fig. 8.4.

Example 8.5 Find the transfer function of the circuit shown in Fig. 8.5.

The circuit has two loops with current flowing I_1 and I_2 and the source of V_{in} . The transfer functions can be defined as

$H_1(s) = \frac{I_1(s)}{V_{in}(s)}$ and $H_2(s) = \frac{I_2(s)}{V_{in}(s)}$. This means that the transfer functions require that the circuit be solved for I_1 and I_2 . The results are shown as follows:

Fig. 8.4 Equivalent system transfer functions to obtain the current or the voltage when the input is the voltage

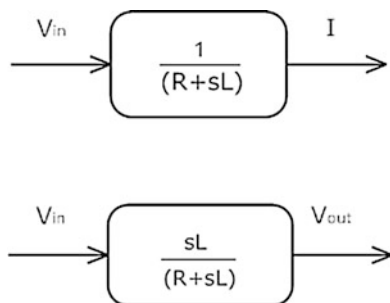


Fig. 8.5 Circuit of Example 8.5

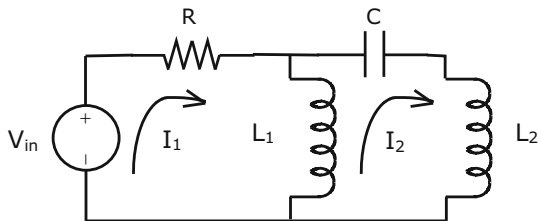
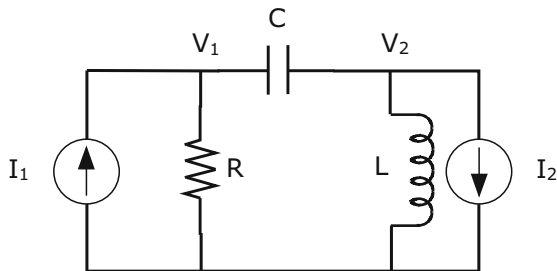


Fig. 8.6 Circuit of Example 8.6



$$I_1(s) = \frac{C(L_1 + L_2)s^2 + 1}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R} V_{in}(s)$$

$$I_2(s) = \frac{CL_1s^2}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R} V_{in}(s)$$

The only step remaining is to obtain the ratio of the output over input signals as defined in transfer functions. Therefore,

$$H_1(s) = \frac{I_1(s)}{V_{in}(s)} = \frac{C(L_1 + L_2)s^2 + 1}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R}$$

$$H_2(s) = \frac{I_2(s)}{V_{in}(s)} = \frac{CL_1s^2}{CL_1L_2s^3 + RC(L_1 + L_2)s^2 + L_1s + R}$$

Example 8.6 Considering the circuit shown in Fig. 8.6. Find the transfer functions defined as $H_1(s) \triangleq \frac{V_1(s)}{I_1(s)}$, $H_2(s) \triangleq \frac{V_1(s)}{I_2(s)}$, $H_3(s) \triangleq \frac{V_2(s)}{I_1(s)}$, and $H_4(s) \triangleq \frac{V_2(s)}{I_2(s)}$. The circuit has two inputs I_1 and I_2 and two outputs V_1 and V_2 were defined. Therefore, there can be four transfer functions defined (Fig. 8.6).

Solution Considering the transfer functions defined, one can see that the problem is asking for voltages V_1 and V_2 .

The solutions are as follows:

$$V_1 = \frac{R(CL(I_1 - I_2) + I_1)}{CLs^2 + RCs + 1} = \frac{R(1 + \frac{1}{LC})}{s^2 + \frac{R}{L}s + \frac{1}{LC}} I_1 - \frac{R}{s^2 + \frac{R}{L}s + \frac{1}{LC}} I_2$$

$$V_2 = -\frac{Ls(I_2 - CRsI_1 + CRsI_2)}{CLs^2 + RCs + 1} = \frac{Rs^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_1 - \frac{RLs(\frac{1}{RC} + s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}I_2$$

As the solutions demonstrate, each of the voltages is dependent to each of the input current sources. For instance, V_1 is a function of both I_1 and I_2 . To obtain transfer function $H_1(s) \triangleq \frac{V_1(s)}{I_1(s)}$, the second current source value should be zero, $I_2 = 0$, or:

$$H_1(s) \triangleq \frac{V_1(s)}{I_1(s)} \Big|_{I_2(s)=0}, H_2(s) \triangleq \frac{V_1(s)}{I_2(s)} \Big|_{I_1(s)=0}. \text{ Hence,}$$

$$H_1(s) = \frac{R(1 + \frac{1}{LC})}{s^2 + \frac{R}{L}s + \frac{1}{LC}}, H_2(s) = -\frac{R}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H_3(s) \triangleq \frac{V_2(s)}{I_1(s)} \Big|_{I_2(s)=0}, H_4(s) \triangleq \frac{V_2(s)}{I_2(s)} \Big|_{I_1(s)=0}. \text{ Hence,}$$

$$H_3(s) = \frac{Rs^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}, H_4(s) = -\frac{RLs(\frac{1}{RC} + s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Transfer Function Operations

Consider a large circuit with several branches, nodes, and sources which generates several responses. The circuit can be broken into manageable sections to find the transfer function. Each section's transfer function can be obtained. The overall transfer function is obtained through some operations defined for parallel, series, feedback, and feedforward connection of transfer functions. In this section, these operations are introduced and analyzed.

Series connection. Consider two or more transfer functions connected in series. The output of one transfer function is fed to the input of the other and so forth, as shown in the Fig. 8.7.

The overall transfer function is the product of all transfer functions as:

$$TF(s) = \prod_{i=1}^n TF_i(s)$$

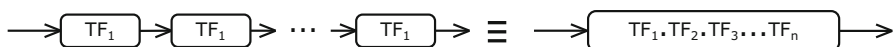


Fig. 8.7 Tandem connection or series connection of transfer functions

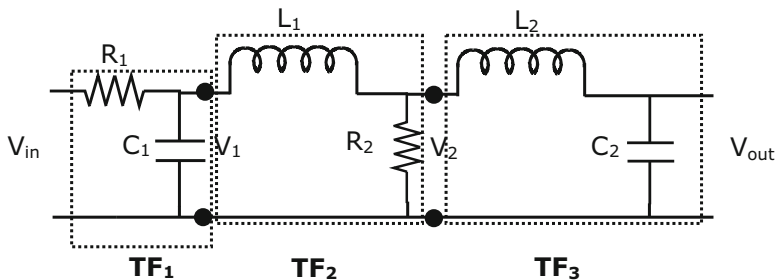


Fig. 8.8 A circuit is broken into three tandem-connected subsystems. The transfer function of overall system is the product of the individual transfer functions of subsystems

Example 8.7 Consider a multi-loop circuit as given in Fig. 8.8. Find the transfer function of the circuit defined as $H(s) = \frac{V_o}{V_{in}}$.

Solution The circuit can be split into three sections which are connected in series. Transfer function of each section is obtained to find the overall transfer function. It can be observed that:

$$\frac{V_o}{V_{in}} = \frac{V_o}{V_2} \times \frac{V_2}{V_1} \times \frac{V_1}{V_{in}}$$

Each of these transfer functions can be obtained as follows:

$$\begin{aligned} \frac{V_1}{V_{in}} &= \frac{\frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{1}{1 + R_1 C_1 s} = \frac{\frac{1}{R_1 C_1}}{s + \frac{1}{R_1 C_1}} \\ \frac{V_2}{V_1} &= \frac{R_2}{R_2 + sL_1} = \frac{\frac{R_2}{L_1}}{s + \frac{R_2}{L_1}} \\ \frac{V_o}{V_2} &= \frac{\frac{1}{sC_2}}{sL_2 + \frac{1}{sC_2}} = \frac{1}{1 + L_2 C_2 s^2} = \frac{\frac{1}{L_2 C_2}}{s^2 + \frac{1}{L_2 C_2}} \end{aligned}$$

Replacing into the overall transfer equation,

$$\frac{V_o}{V_{in}} = \frac{\frac{1}{R_1 C_1}}{s + \frac{1}{R_1 C_1}} \times \frac{\frac{R_2}{L_1}}{s + \frac{R_2}{L_1}} \times \frac{\frac{1}{L_2 C_2}}{s^2 + \frac{1}{L_2 C_2}} = \frac{\frac{R_2}{R_1 C_1 C_2 L_1 L_2}}{\left(s + \frac{1}{R_1 C_1}\right) \left(s + \frac{R_2}{L_1}\right) \left(s^2 + \frac{1}{L_2 C_2}\right)}$$

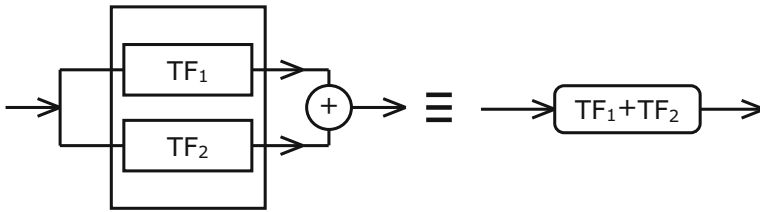


Fig. 8.9 Parallel connection of two systems receiving the same input signal

Parallel Connection

If two or more transfer functions are connected in parallel, their operation will depend on the final operation designed by the circuit. For instance, consider the following transfer functions (Fig. 8.9).

The equivalent transfer function becomes the summation of two systems as:

$$TF = TF_1 + TF_2$$

Feedback Connection

Connection of transfer functions as feedback suggests a measurement of the output compared with the desired reference to generate an error signal. This signal is used to correct the behavior of the system. In feedback systems, the output signals may be added or subtracted from the reference signal which generates positive or negative feedback, respectively.

Consider the transfer function of the plant G and the feedback signal transducer F as shown in Fig. 8.10.

Signal E is obtained as $E = R - CF$. Looking at the plant transfer function, G , the output signal $C = EG$. Replacing error into this equation yields:

$$C = (R - CF)G$$

The transfer function is defined as the ratio of output over input transfer functions $\frac{C}{R}$ obtained as follows:

$$C(1 + FG) = RG$$

$$\frac{C}{R} = \frac{G}{1 + FG}$$

The transfer function of a negative feedback in a single loop system is obtained by the ratio of feedforward gain G over 1 plus the loop gain FG . The feedforward gain is

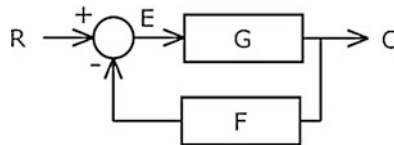


Fig. 8.10 Closed-loop connection of system G through system of F . Since the feedback signal is subtracted from the reference R , the system is called “a negative feedback closed-loop system”

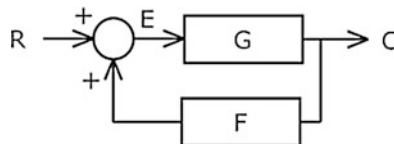
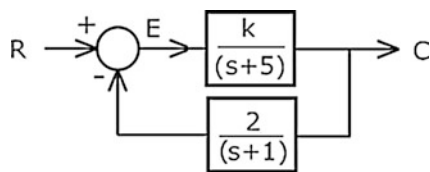


Fig. 8.11 Closed-loop connection of system G through system of F . Since the feedback signal is added to the reference R , the system is called “a positive feedback closed-loop system”

Fig. 8.12 Closed-loop system of Example 8.8



the gain observed in a forward path from the input to the output, and the loop gain is the product of all transfer functions existing in the loop circulating once.

In a positive feedback system as shown in Fig. 8.11.

The error signal becomes $E = R + CF$. Considering the output signal $C = EG$, and the replacement from the error yields:

$$C = (R + CF)G$$

The transfer function of a positive feedback system becomes:

$$\frac{C}{R} = \frac{G}{1 - FG}$$

Example 8.8 System diagram of a circuit is shown in Fig. 8.12. Find the transfer function of the closed loop system.

Solution The transfer function of the closed-loop system is obtained as $\frac{C}{R}$ by identifying the feedforward path and the loop gain.

Feedforward path is the direct connection from input R to output C identified as $\frac{K}{s+5}$, and the loop gain is identified as the product of system and the feedback as $\frac{K}{s+5} \cdot \frac{2}{s+1}$. Therefore, the closed-loop transfer function becomes:

$$\frac{C}{R} = \frac{\frac{K}{s+5}}{1 + \frac{K}{s+5} \frac{2}{s+1}} = \frac{K(s+1)}{s^2 + 6s + 5 + 2K}$$

Feedback and Change of Order of Circuit

The order of a circuit is determined as the higher power of s in the denominator of its transfer function. The feedback system may change the order of a circuit too. As explained in previous example, the plant $\frac{K}{s+5}$ is a first-order circuit. However, the closed-loop system $\frac{C}{R} = \frac{K(s+1)}{s^2+6s+5+2K}$ is a second-order system.

Example 8.9 Consider a circuit with input $x(t)$ and output $y(t)$, presented in its differential equation as follows:

$$\ddot{y} + 2\dot{y} + y = \ddot{x} + 6\dot{x} + x$$

Find the transfer function of the system $G(s) = \frac{Y(s)}{X(s)}$.

Solution The transfer function is the ratio of the Laplace of output over the Laplace of input signals. In this example, the Laplace transform of the differential equation is identified as:

$$(s^2 + 2s + 1)Y(s) = (s^2 + 6s + 1)X(s)$$

The transfer function becomes:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 6s + 1}{s^2 + 2s + 1}$$

Poles and Zeros

Consider a transfer function defined as the ratio of two polynomials, namely, the numerator and the denominator. These polynomials depending on their order might have several roots. Any root of the numerator polynomial makes the transfer function zero. Hence, the roots of the numerator polynomial are defined as zeros of the transfer function. Any root of the denominator polynomial makes the transfer function ∞ . The roots of denominator polynomial are called poles of the transfer functions. Poles and zeros are measured in frequency domain and have the unit of

rad/s These frequencies may become real numbers or complex conjugate numbers which indicate complex conjugate frequencies. The real and imaginary parts indicate various parts of a response as explained in Chap. 4, response of second-order systems.

Phase Plane

The type and location of poles and zeros determine many characteristics of the system including the system response to a desired input, and its stability among others. A complex conjugate plane that indicates the location of all poles with a cross $\times$ and the location of all zeros with a circle $\circ$ is called phase plane. As the system gains or parameters change, the location of poles and zeros changes and shows a trajectory that indicates system characteristics. This topic will be discussed in details in the rest of this chapter. The phase plane has one real and one imaginary axis that show the real and imaginary part of the poles and zeros. This distinction is important as it indicates the limits of stability and the effect of controller gains.

The real value axis can be divided into positive (right hand) and negative (left hand) side. This divides the phase plane into Right Half Plane (RHP) and Left Half Plane (LHP) regions (Fig. 8.13).

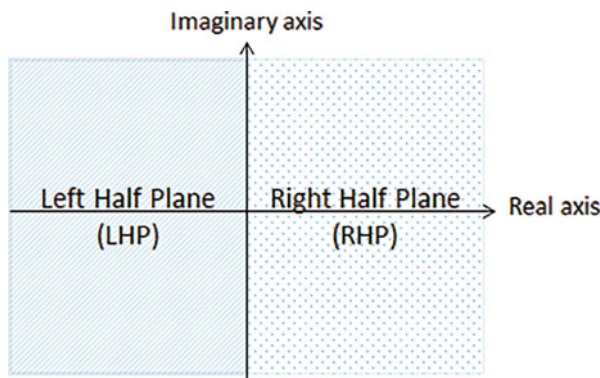


Fig. 8.13 Figure of a phase plane. The real and imaginary axes are shown. The plane is divided into left and right half planes that show entirely different system characteristics. This plane shows the location of poles and zeros and their trajectories should the system parameters change (due to controller effects or the system changes). Poles and zeros in RHP possess a positive real part and those in LHP possess a negative real part

Limit of Stability

The phase plane is one indicator that the system is stable, and if a change occurs in the system whether it remains stable. This indicator depends on the time domain response of a system.

For instance, consider a system response as $y(t) = 2e^{+3t}u(t)$. This exponential value increases as the time increases. This does not reach a steady value, and if this is the energy of a system, it shows a growing level of energy in the system and can virtually reach an infinite value which is out of control. This system is unstable because its internal energy does not remain bounded. It remains stable if the internal energy decreases or remains constant. For instance, consider the same system with a negative exponential argument as $y(t) = 2e^{-3t}u(t)$. In this system, as the time increases, the system loses energy and reaches a stable operation, the origin.

Translating these systems into frequency domain through Laplace transform reveals that the unstable system $Y(s) = \frac{2}{s-3}$ had a pole in the RHP (positive real part pole) and the stable system $Y(s) = \frac{2}{s+3}$ had a pole in LHP (negative real part pole).

It can be concluded that stable systems have no poles in RHP. They also should not have repeated poles at the origin.

To understand this analogy, consider a system as $G(s) = \frac{1}{s^2}$ which has two poles at the origin. The Laplace inverse of this system is $g(t) = tu(t)$. As the time increases, the value of $g(t)$ increases and reaches infinite. Therefore, since there is no limit for this output (or energy) increment, the system is considered unstable.

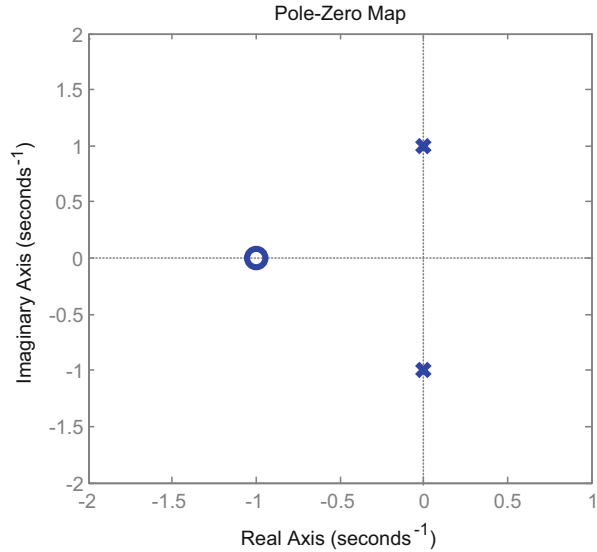
Phase of the system. The system depending on the location of poles and zeros may have a phase. The phase of a transfer function is identified as the phase of numerator minus the phase of denominator. For a stable system response as it is explained in details, the phase of the system should be less than -180° . Considering that this phase is a subtract of two numbers, the phase of numerator helps decrease the phase hence making it more stable and show better response. Zeros in RHP show a positive phase as their phase is calculated by $\tan^{-1} \frac{Im}{Re}$. This takes the system move away from its best phase angle and makes it closer to the limit -180° . Therefore, the systems with zeros in RHP are called non-minimum phase systems. Their response to a control action in time domain shows an initial decrease and moving away from the reference before moving towards the reference. An initial dip causes a lot of trouble for the control system. Examples of these systems can be found in power electronics boost converters or the water level controls in drum of power plant boilers.

Example 8.10 Find the poles and zeros of the following transfer functions.

1.

$$G(s) = \frac{s+1}{s^2+1}.$$

Fig. 8.14 Figure of poles and zeros of $G(s) = \frac{s+1}{s^2+1}$



This system has a first-order numerator which results in one zero as the root of $s+1 = 0$. Therefore, the zero of the transfer function is $s = -1$ rad/s. The denominator is a second-order system which results in two poles as the root of $s^2+1 = 0$. This yields two complex conjugate roots of $s = \pm j1$ rad/s.

The phase plane of these roots can be shown as Fig. 8.14

The system is stable as there is no poles in RHP.

2.

$$G(s) = \frac{s^2 + 4}{s^2 + 2s + 5}.$$

The system has two zeros and two poles. The zeros are obtained from the roots of $s^2+4 = 0$ as $s = \pm j2$ rad/s. The poles are obtained by the roots of $s^2 - 2s+5 = 0$ as $s = +1 \pm j2$ rad/s. As the poles have all positive real parts, the system is unstable (Fig. 8.15).

3.

$$G(s) = \frac{s^2 - 4}{s^2 + 2s + 5}$$

The system has two zeros at $s^2 - 4 = 0$ as $s = \pm 2$ rad/s. One of the zeros is located in RHP which makes the system non-minimum phase. The poles are obtained by the roots of $s^2+2s+5 = 0$ as $s = -1 \pm j2$ rad/s. As the poles have all negative real part, the system is stable. Therefore, the system is stable, but non-minimum phase (Fig. 8.16).

Fig. 8.15 Figure of poles and zeros of $G(s) = \frac{s^2+4}{s^2+2s+5}$

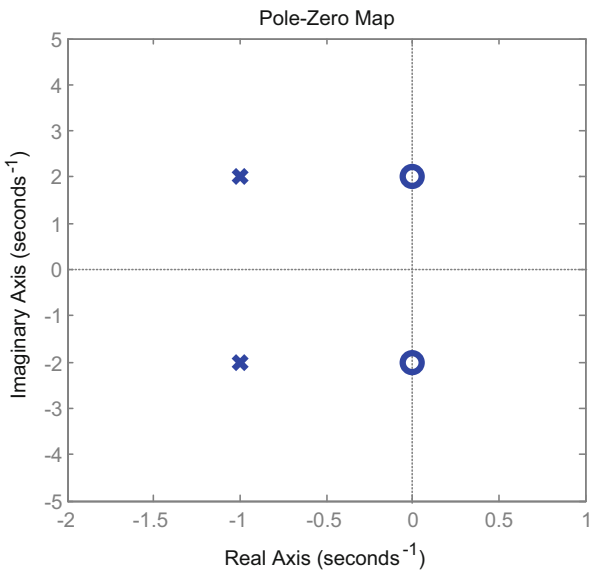
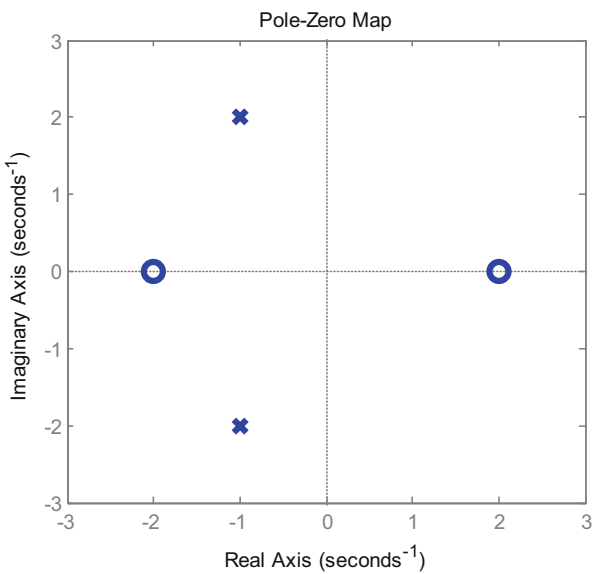


Fig. 8.16 Figure of poles and zeros of $G(s) = \frac{s^2-4}{s^2+2s+5}$



Initial Value and Final Value Theorems

A partial time domain response of a circuit can be approximated by its initial and final values. To obtain these values, the system response limit is obtained as the time reaches zero or infinite. Consider $y(t)$ as the system response. Therefore:

Initial value becomes $y(0) = \lim_{t \rightarrow 0} y(t)$.

Final value becomes $y(\infty) = \lim_{t \rightarrow \infty} y(t)$.

If all values are given in frequency domain, one way to obtain the initial and final values will be through the Laplace inverse transforms of the response and then evaluate the system initial and final values in time domain.

An alternative approach will be the use of initial and final value theorems, which directly utilizes the frequency domain system responses. According to the initial value theorem, the initial amount of a function is obtained by:

$$y(0) = \lim_{t \rightarrow 0} y(t) \equiv \lim_{s \rightarrow \infty} sY(s)$$

Note 8.3 Left-hand side of this theorem is in time domain, and the right-hand side is in frequency domain. Once the time reaches zero, the frequency must reach infinite.

Note 8.4 There is a factor s imposed in the frequency domain.

The final value, **which is also called the steady-state response**, is accordingly defined as:

$$y(\infty) = y_{ss} = \lim_{t \rightarrow \infty} y(t) \equiv \lim_{s \rightarrow 0} sY(s)$$

Note 8.5 Left-hand side of this theorem is in time domain, and the right-hand side is in frequency domain. Once the time reaches infinite, the frequency must reach zero.

Note 8.6 There is a factor s imposed in the frequency domain.

Example 8.11 Find the initial and final value of the following signals:

$$1. Y(s) = \frac{s^2 + 3s + 0.5}{s(s+1)^2}.$$

Solution Do not forget that the limit parameter values are switched from time and frequency and that the initial value is found as $y(t \rightarrow 0) = \lim_{s \rightarrow \infty} s \frac{s^2 + 3s + 0.5}{s(s+1)^2} = 1$.

Hint. When a polynomial of order n is reaching infinite, the polynomial is equivalent to its highest-order component, e.g., when $s \rightarrow \infty$, $s^2 + 3s + 1 \equiv s^2$.

Final value is found as: $y(t \rightarrow \infty) = \lim_{s \rightarrow 0} s \frac{s^2 + 3s + 0.5}{s(s+1)^2} = \frac{1}{2}$.

This means that the signal $y(t)$ was started at value 1 and reached to value 1/2 as shown in Fig. 8.17.

Fig. 8.17 Time representation of $y(t)$ also shows $y(0) = 1$ and $y(\infty) = 0.5$

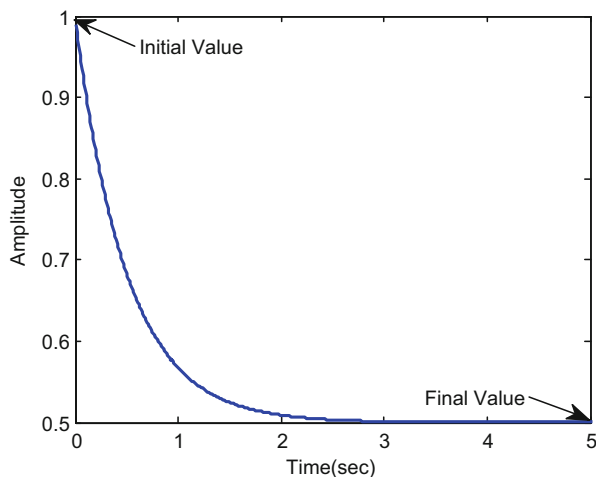
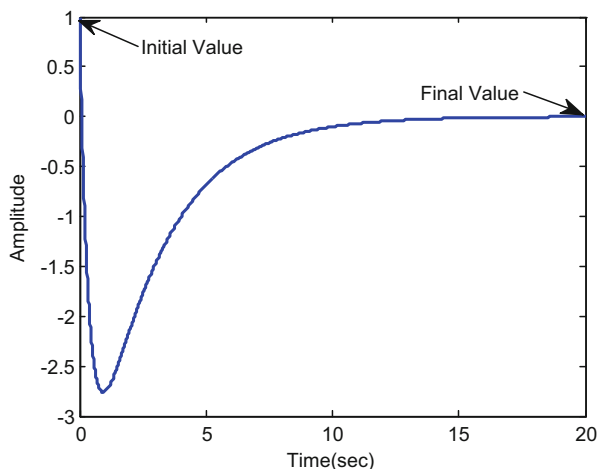


Fig. 8.18 Time representation of $y(t)$ also shows $y(0) = 1$ and $y(\infty) = 0$



$$Y(s) = \frac{s - 10}{s^2 + 3s + 1}.$$

Solution Initial value is obtained by: $y(t \rightarrow 0) = \lim_{s \rightarrow \infty} s \frac{s-10}{s^2+3s+1} = \lim_{s \rightarrow \infty} \frac{s^2-10s}{s^2+3s+1} = \lim_{s \rightarrow \infty} \frac{s^2}{s^2} = 1$

The final value is obtained by $y(t \rightarrow \infty) = \lim_{s \rightarrow 0} s \frac{s-10}{s^2+3s+1} = \lim_{s \rightarrow 0} \frac{-10}{1} = 0$.

The signal $y(t)$ starts from 1 and reaches 0 at very large time. Figure 8.18 shows the sketch of the response.

Order and Type of a System

A system may be presented by its governing differential equation. A standard form of differential equation has two polynomials that represent the input and output dynamics. The order of the system is the highest degree of differential in the output equation. Consider the following differential equation where y is the output and u is the input:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = b_0 u^{(m)}(t) + b_1 u^{(m-1)}(t) + \dots + b_m u(t)$$

The polynomial that is a function of y : output, is the output equation and the polynomial that involves u : input is the input equation.

Output equation: $y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t)$

Input equation: $b_0 u^{(m)}(t) + b_1 u^{(m-1)}(t) + \dots + b_m u(t)$

Order of a System: The order is defined as the highest order of the differential in the output equation or n . This system will have n poles.

The system transfer function can be obtained as the ratio of the Laplace of the output over the Laplace of the input as follows:

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

Type of a System: Type of the system $H(s)$ is the number of its poles at the origin.

First-Order Systems

A first-order system has one pole. In a standard form, it can be expressed as:

$$H(s) = \frac{k}{s + k}$$

The location of the pole is at the root of the denominator or,

$$\begin{aligned} s + k &= 0 \\ s &= -k \left(\frac{\text{rad}}{\text{s}} \right) \end{aligned}$$

The DC gain of this system is 1.

A first-order system cannot generate oscillations in the response because its pole can only be a real number. Hence, resulting a time response as an exponential function.

For instance:

$$H(s) = \frac{3}{s + 5}$$

This system has a pole at

$$s + 5 = 0$$

$$s = -5$$

It is a first-order system. Since this system has no poles at the origin, the system is type 0.

Another example (nonstandard 1st-order system):

Consider the system

$$H(s) = \frac{s + 0.5}{s - 4}$$

The pole of the system is located at:

$$s - 4 = 0$$

$$s = +4$$

The system has a zero at:

$$s + 0.5 = 0$$

$$s = -0.5$$

It is a first-order system. Since the system has no poles at the origin, the system is type 0.

$$H(s) = \frac{10}{s}$$

This system has a pole at:

$$s = 0$$

It is a first-order system. Since the pole is also located at the origin, the system is type 1.

Second-Order Systems

The highest-order differentials of the output equation of a second-order system is 2. Therefore, the denominator of second-order systems is a quadratic equation resulting two poles.

Standard Form Standard form of a 2nd-order system contains several parameters as follows:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The factor ζ is the damping factor of the system, and ω_n is the natural frequency or resonant frequency of the system.

A second-order system can be obtained by its damping factor and its natural frequency. For instance, the transfer function of a system with damping of 0.4 and the natural frequency of 400 rad/s is:

$$H(s) = \frac{400^2}{s^2 + 2 \times 0.4 \times 400 s + 400^2}$$

The DC gain of a standard form 2nd-order system is 1.

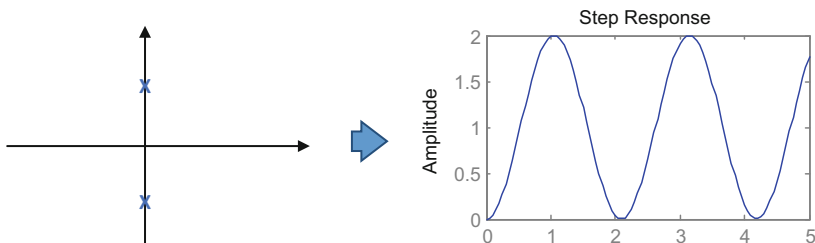
The location of poles in a 2nd-order system depends on the amount of damping. The step response according to the location of poles and the damping conditions are discussed as follows:

In general, the location of poles in a second-order system is:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case 1 When the damping is zero ($\zeta = 0$), the system is **oscillatory**:

$$\text{if } \zeta = 0, \text{ then } s_{1,2} = \pm j\omega_n$$



Case 2 When the damping is $0 < \zeta < 1$, the system is **underdamped**:

$$\text{if } 0 < \zeta < 1, \text{ then } s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

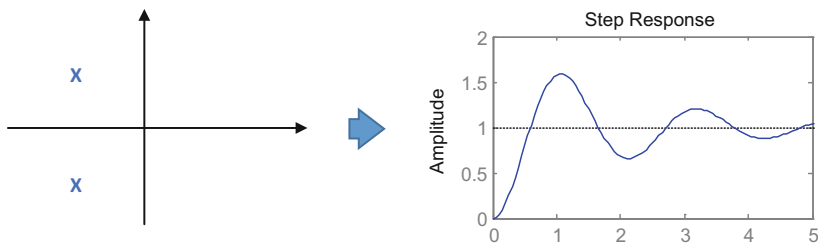
Considering:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Where ω_d is the damping frequency.

Therefore:

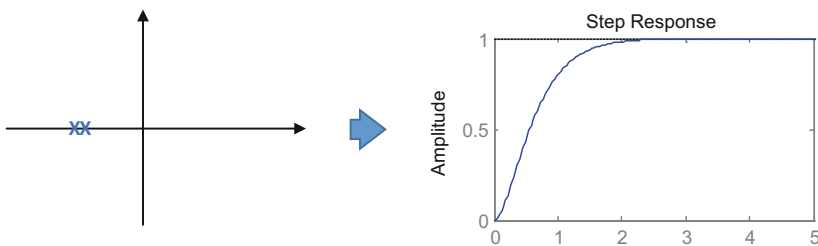
$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$



Case 3 When the damping is one $\zeta = 1$, the system is **critically damped**.

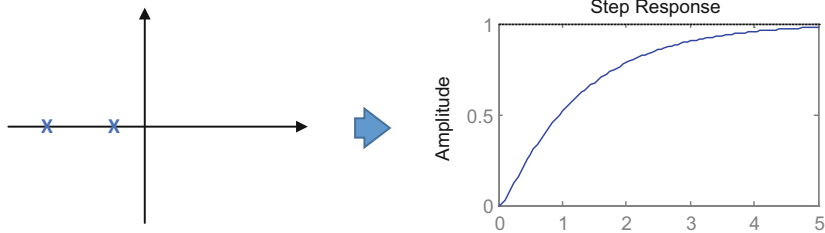
$$\text{if : } \zeta = 1, \text{ then : } s_{1,2} = -\omega_n$$

In this case, the system response has no oscillation.



Case 4 When the damping is larger than 1, $\zeta > 1$, the system is **overdamped**.

$$s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$



Nonstandard Form: A nonstandard 2nd-order system still has two poles but may have zeros. A transfer function of second-order system with a zero may exist as follows:

$$H(s) = \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(s) = \frac{s^2 + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step Response of Second-Order System

In a second-order underdamped system, the response generates an overshoot and settles over time to reach the reference. The response analysis involves several time and amplitude that are discussed in this section.

$$\underbrace{\int \int \int \dots \int}_{n \text{ Integrals}} (y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t)) = \underbrace{\int \int \int \dots \int}_{n \text{ Integrals}} (b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \dots + b_0 u(t))$$

$$y + a_1 \int y + a_2 \int \int y + \dots + a_n \underbrace{\int \int \int \dots \int}_{n \text{ Integrals}} y = b_m \underbrace{\int \int \int \dots \int}_{n-m \text{ Integrals}} u + \dots + b_0 \underbrace{\int \int \int \dots \int}_{n \text{ Integrals}} u$$

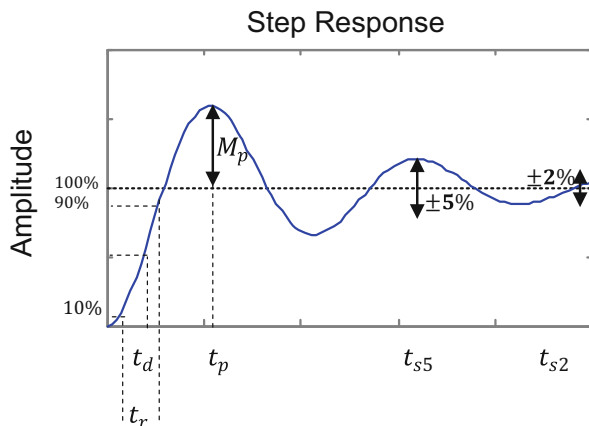
The step response of an underdamped second-order system shown in Fig. 8.19 can be predicted by knowing the damping factor ζ and the natural frequency ω_n . The measures that can result in these values are shown on the figure. They can be defined as follows:

Rise time (t_r): It is the time it takes for the response to reach from 10% to 90% of the reference.

Delay time (t_d): It is the time it takes for the response to reach 50% of the reference.

Peak time (t_p): It is the time of the first peak.

Fig. 8.19 Step response of an under damped system



Maximum overshoot (M_p): It is the maximum amount of response that exceeds the reference.

Settling time % (t_{s5}): It is the time that the response reaches within $\pm 5\%$ variation from the reference.

Settling time % (t_{s2}): It is the time that the response reaches within $\pm 2\%$ variation from the reference.

The values of these parameters can be obtained as follows:

$$t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

$$t_d = \frac{1 - 0.7\zeta}{\omega_d}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_{s5} = \frac{3}{\zeta\omega_n}$$

$$t_{s2} = \frac{4}{\zeta\omega_n}$$

Example 8.12 A second-order system has damping factor $\zeta = 0.6$, and the natural frequency of $\omega_n = 100$ rad/s. Find the transfer function of the system. Determine the step response overshoot and peak time.

Solution The standard second-order system is expressed as follows:

$$H(s) = \frac{100^2}{s^2 + 2 \times 0.6 \times 100 s + 100^2}$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{\frac{-\pi 0.6}{\sqrt{1-0.6^2}}} = 0.094$$

This means there is exists a 9.4% overshoot. The peak value becomes $1+0.094 = 1.094$:

$$t_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 100 \sqrt{1 - 0.6^2} = 80 \frac{\text{rad}}{\text{s}}$$

$$t_p = \frac{\pi}{80} = 0.0393 \text{ s}$$

Example 8.13 Step response of a system is shown as follows. Find the transfer function of the system (Fig. 8.20).

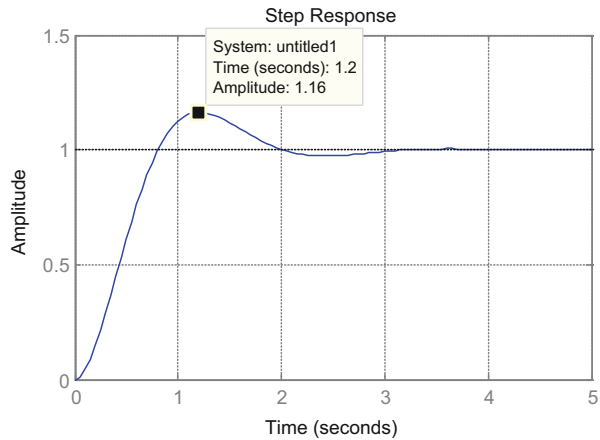
Solution The system step response reaches maximum value of 1.16. This means a 0.16 Maximum Peak or

$$M_p = 0.16$$

This results in:

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.16$$

Fig. 8.20 Step response of Example 8.13



Taking natural log of both sides results in:

$$\text{Ln} \left(e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \right) = \text{Ln } 0.16$$

$$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = -1.83$$

$$\frac{\pi^2\zeta^2}{1-\zeta^2} = 1.83^2$$

$$1 - \zeta^2 = \frac{\pi^2}{1.83^2} \zeta^2$$

$$1 = (2.944 + 1)\zeta^2$$

$$\zeta = 0.5$$

From the response, the time of peak is recorded as 1.2 s. Therefore,

$$t_p = \frac{\pi}{\omega_d} = 1.2$$

$$\frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 1.2$$

Replacing the damping value, found earlier, results in:

$$\frac{\pi}{\omega_n \sqrt{1-0.5^2}} = 1.2$$

$$\omega_n = \frac{\pi}{1.2 \times \sqrt{1-0.5^2}} = 3 \frac{\text{rad}}{\text{s}}$$

Knowing the damping $\zeta = 0.5$ and the natural frequency as $\omega_n = 3$, a standard transfer function becomes:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(s) = \frac{3^2}{s^2 + 2 \times 0.5 \times 3 s + 3^2}$$

$$H(s) = \frac{9}{s^2 + 6 s + 9}$$

Fig. 8.21 Figure of closed-loop controller



The Effect of Controller on Type-Zero Systems

Consider a plant (system) with one non-zero pole and either no or one zero. The type of a system refers to the number of poles located at the origin. The system can be expressed as $G(s) = \frac{k}{s+a}$, where k is the gain and a is the pole, decay factor, or the inverse of time constant. This system can be tuned and influenced by another block which is named a controller. The controller in a closed-loop system is shown in Fig. 8.21.

The closed-loop system has the following transfer function: $\frac{C}{R} = \frac{FF}{1+LG} = \frac{G_c \frac{k}{s+a}}{1+G_c \frac{k}{s+a}} = \frac{kG_c}{s+(a+kG_c)}$. The poles of the transfer function can be placed at any desired locations on the phase plane to obtain the desired system responses. The gain k not only tunes the system gain, but also it influences the location of poles.

The controller can be considered as follows:

- (a) **Simple gain controller**, proportional, $G_c = k_p$. The closed-loop system becomes:

$$\frac{C}{R} = \frac{kk_p}{s + (a + kk_p)}$$

This results in a straight forward pole placement and controller through both gains of k and k_c .

The system error shown on Fig. 8.21 can be calculated as $E = R - C$.

A division by R shows the error signal as:

$$\frac{E}{R} = 1 - \frac{C}{R}$$

Replacing from the transfer function results in:

$$\frac{E}{R} = 1 - \frac{kk_p}{s + (a + kk_p)} = \frac{s + a}{s + (a + kk_p)}$$

The error depends on the input R to the system.

1. Consider a unit step $r = u(t)$ or $R = \frac{1}{s}$

The error becomes:

$$E = R \frac{s+a}{s+(a+kk_p)} = \frac{1}{s} \frac{s+a}{s+(a+kk_p)}$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s} \frac{s+a}{s+(a+kk_p)} = \frac{a}{(a+kk_p)}$$

It is a limited error and reaches zero only if a very high gain k_p is applied. High-gain controllers are particularly not desirable as they tend to amplify noise too and expensive to implement in hardware.

2. Consider a ramp input as $r = tu(t)$ or $R = \frac{1}{s^2}$

The steady-state error becomes:

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s^2} \frac{s+a}{s+(a+kk_p)} = \frac{a}{(a+kk_p)} = \infty$$

This means that the proportional gain controller in a type zero system cannot limit the system error under ramp excitation.

- (b) **Effect of a proportional-integral (PI) controller on type-zero systems.** The control system is now considered to be $G_c = k_p + \frac{k_i}{s}$. The transfer function of the closed system considering this controller becomes:

$$\frac{C}{R} = \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))}$$

The error is always:

$$E = R \left(1 - \frac{C}{R} \right)$$

1. Consider a unit step $r = u(t)$ or $R = \frac{1}{s}$

The error becomes:

$$E = \frac{1}{s} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right)$$

The steady-state error is calculated as:

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right) = 1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} = 0$$

Note 8.7 It is observed that the integral part of the controller can force the system reach a zero error condition. Therefore, it can be concluded that the PI controller has zero control error in type-zero systems with unit step input.

2. Consider a ramp input as $r = tu(t)$ or $R = \frac{1}{s^2}$

The steady-state error becomes:

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s^2} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right) = \lim_{s \rightarrow 0} \frac{1}{s} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right) = -\frac{2k_p}{k_i}$$

Note 8.8 The system tracks the ramp input with a non-zero steady-state error.

3. Considering a hyperbolic input as $r = t^n u(t)$, $n \geq 2$, or $R = \frac{n!}{s^{n+1}}$

The steady-state error becomes:

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s^{n+1}} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right) = \lim_{s \rightarrow 0} \frac{1}{s^n} \left(1 - \frac{k(k_p + \frac{k_i}{s})}{s + (a + k(k_p + \frac{k_i}{s}))} \right) = \infty$$

Note 8.9 This means that the system cannot track the inputs with higher order than $tu(t)$.

Tracking Error Considering the Type and the Input as Reference Waveform

A summary of what explained in the steady-state error calculations while the type of the combined controller and the plant are considered against the input reference is summarized in the following table:

As a hint: The Laplace of input reference determines the type of the input.

- If the type of the controller and the system combined matches the type of the input, the error is limited.
- If the type of the controller and the system combined is less than the type of the input, the tracking is impossible.
- If the type of the controller and the system combined is higher than the type of the input, the error is zero.

	Tracking error		
System input	System type 0	System type 1	System type 2
Unit step	Limited error	Zero error	Zero error
Ramp	Cannot track. Error = ∞	Limited error	Zero error
Parabolic	Cannot track. Error = ∞	Cannot track. Error = ∞	Limited error

Convolution Integral

Consider a system presented by a function in time as $h(t)$. Once an input signal $x(t)$ is applied to this system function, an output signal $y(t)$ will be generated. Almost all systems operate based on this principle, and there is a need to identify the output signal. The use of this analysis is to determine the stability of system, the type, and the shape of the output. Most often the input signal is identified as a series of recorded data points which might make the analysis difficult. One technique to obtain the output is convolution integral (Fig. 8.22).

In time domain, the output is obtained as follows:

$$y(t) = h(t) * x(t) = x(t) * h(t)$$

Where in the $*$ represents the convolution.

Example 8.14 The system $h(t) = 2tu(t)$ receives an input signal of $x(t) = 3u(t)$ and explains how the input and output are related.

Solution The output is obtained through convolution integral as follows:

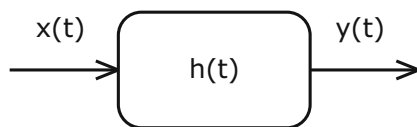
$$y(t) = x(t) * h(t)$$

$$y(t) = 2tu(t) * 3u(t)$$

Properties of convolution integral. Consider three signals $x(t)$, $h(t)$ and $y(t)$:

1. $x(t) * h(t) = h(t) * x(t)$
2. $x(t) * (h(t) * z(t)) = (x(t) * h(t)) * z(t)$
3. $x(t) * (h(t) + z(t)) = (x(t) * h(t)) + (x(t) * z(t))$

Fig. 8.22 Time domain representation of a system $h(t)$, input $x(t)$, and the output $y(t)$



It is of utmost importance to describe the convolution integral operation. This operation can be defined as an integral of product of the two given signals in a partial time steps as follows:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t - \lambda)d\lambda = \int_{-\infty}^{+\infty} x(t - \lambda)h(\lambda)d\lambda$$

Let's consider that the signals are defined for positive time, i.e., $t > 0$, and that the systems are “causal,” i.e., the more number of poles than zeros, then the integral can be modified to:

$$y(t) = x(t) * h(t) = \int_0^t x(\lambda)h(t - \lambda)d\lambda = \int_0^t x(t - \lambda)h(\lambda)d\lambda$$

To evaluate the integral, the functions $x(\lambda)$ and $h(t - \lambda)$ are rolled over each other, and their functions will be evaluated in a specific time shift. Following steps are needed:

1. Mirror one of the functions either $x(t)$ **or** $h(t)$ with respect to the vertical axis. This results in $x(-\lambda)$ **or** $h(-\lambda)$.
2. Shift the mirrored signal by t seconds to obtain $x(t - \lambda)$ **or** $h(t - \lambda)$. This time shift allows the mirrored signal to roll over the other signal.
3. Specific time partitions are obtained by observing the changes in product of the two signals $x(t - \lambda)h(\lambda)$ **or** $x(\lambda)h(t - \lambda)$, resulted by the time shift.
4. Evaluate the integral in the specified times until two signals have no overlap or the results remains similar (no new condition is generated).

Example 8.15 A system $h(t)$ is a pulse of amplitude 2 stretched from time 1 to 3 s. Find the output response due to an input signal $x(t) = u(t)$.

Solution It is possible to mirror either one of the signals x or h , in this solution. Let's mirror $x(t)$ with respect to the vertical axis. As the horizontal axis changes from $t \rightarrow \lambda$, the function $x(-\lambda) = u(-\lambda)$.

A shift by t seconds results in $x(t - \lambda) = u(t - \lambda)$.

Figure 8.23 shows the position of two signals when the time shift does not reach the $h(\lambda)$. Hence, for any time shift less than $t < 1$, the two signals generate 0 as product (Fig. 8.24).

$$t < 1, \int_0^t x(t - \lambda)h(\lambda)d\lambda = 0$$

Fig. 8.25 shows the position of two signals when time shift generates a collision.

$$1 < t < 3,$$

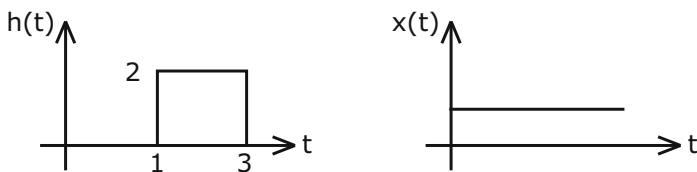


Fig. 8.23 Functions of $h(t)$ and $x(t)$ defined in Example 8.15

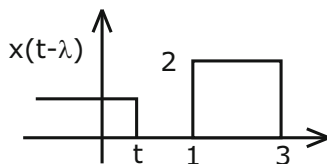


Fig. 8.24 The position of two functions when their product is zero. This means two functions do not have any common area. This occurs for all the time when $t < 1$

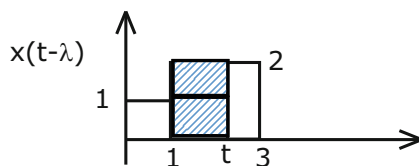


Fig. 8.25 Collision of two functions when their product is still a function of t . This variable can vary from $1 < t < 3$

The two signals have output from $1 < \lambda < t$. Therefore, the $\int_0^t x(t-\lambda)h(\lambda)d\lambda$ becomes

$$\int_1^t x(t-\lambda)h(\lambda)d\lambda = \int_1^t 1 \times 2 d\lambda = 2\lambda \Big|_1^t = 2(t-1)$$

$t > 3$, the product of two signals will change to a fixed value (0) (Fig. 8.26)

$$\int_3^t x(t-\lambda)h(\lambda)d\lambda = \int_3^t 0d\lambda = 0$$

Example 8.16 Find the convolution of the Fig. 8.26 two signals as shown in Fig. 8.27.

Solution Obtaining $x(-\lambda)$ and shifting by t seconds results in $x(t-\lambda)$. Rolling this signal over $h(t)$ in segments results in:

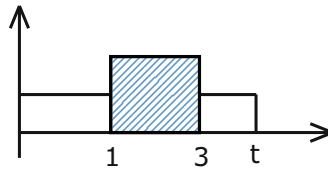


Fig. 8.26 Collision of two functions when their product is no longer a function of time. Note that for all times $t > 3$, the common area covered by both signals is fixed

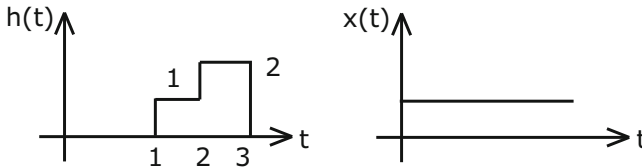


Fig. 8.27 Signals used in the Example 8.16

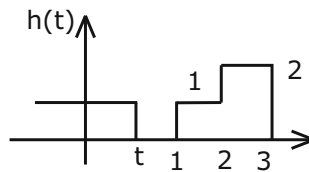


Fig. 8.28 The position of two functions when their product is zero. This means two functions do not have any common area. This occurs for all the time when $t < 1$

If $t < 1$, the two signals generate zero product as shown in Fig. 8.27. Therefore,

$$\int_0^t x(t - \lambda)h(\lambda)d\lambda = 0$$

If $1 < t < 2$, product of two signals generates a value as shown in Fig. 8.28. The integral is evaluated in the range of $1 < \lambda < t$ as follows:

$$\int_1^t x(t - \lambda)h(\lambda)d\lambda = \int_1^t 1 \times 1d\lambda = t - 1$$

As the time shift continues over 2 s, the value of $h(\lambda)$ changes. Therefore, a new limit of integral is needed.

If $2 < t < 3$, results in $1 < \lambda < 2$ & $2 < \lambda < t$. Figure 8.29 shows this functions. The convolution integral becomes:

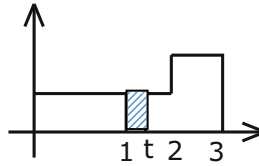


Fig. 8.29 As the function $h(t)$ has two steps, when the function $x(t)$ moves forward, the area it covers by colliding the $h(t)$ changes depending on the position of t . If $1 < t < 2$, then the area covered is as shown, and it depends on variable t

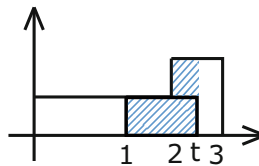
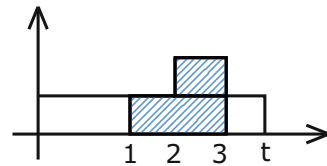


Fig. 8.30 When t passes point 2 but still below ($2 < t < 3$), part of the collision area becomes fixed and not a function of t , and part of it will still depend on t . For this reason, the area between 1 and 3 should split into two sections

Fig. 8.31 When $t > 3$, the collision of two functions is no longer a function of variable t



$$\int_1^2 x(t-\lambda)h(\lambda)d\lambda + \int_2^t x(t-\lambda)h(\lambda)d\lambda =$$

$$\int_1^2 1 \times 1d\lambda + \int_2^t 1 \times 2d\lambda = (2-1) + 2(t-2) = 2t-3$$

If $t > 3$, the value of the functions will not change, but the value of the integral will not depend on t .

Figure 8.30 shows the signals.

The convolution integral becomes (Fig. 8.31):

$$\int_1^2 x(t-\lambda)h(\lambda)d\lambda + \int_2^3 x(t-\lambda)h(\lambda)d\lambda =$$

$$\int_1^2 1 \times 1d\lambda + \int_2^3 1 \times 2d\lambda = (2-1) + 2(3-2) = 3$$

Therefore:

$$x(t) * h(t) = \begin{cases} 0 & t < 1 \\ 2t - 2 & 1 < t < 2 \\ 2t - 3 & 2 < t < 3 \\ 3 & t > 3 \end{cases}$$

Laplace of convolution integral. Laplace of convolution of two signals in time domain is equivalent to the product of their Laplace transforms in frequency domain, as follows:

$$L\{x(t) * h(t)\} = X(s)H(s)$$

Example 8.17 Redo Example 8.15 using Laplace transform.

Considering the waveforms of $h(t)$ and $x(t)$, their Laplace transforms becomes:

$$h(t) = 2u(t - 1) - 2u(t - 3), \rightarrow H(s) = \frac{2}{s}e^{-s} - \frac{2}{s}e^{-3s}$$

$$x(t) = u(t), \rightarrow H(s) = \frac{1}{s}$$

Therefore, the convolution integral becomes:

$$\begin{aligned} x(t) * h(t) &= L^{-1} \left\{ \left(\frac{2}{s}e^{-s} - \frac{2}{s}e^{-3s} \right) \left(\frac{1}{s} \right) \right\} = L^{-1} \left\{ \left(\frac{2}{s^2}e^{-s} - \frac{2}{s^2}e^{-3s} \right) \right\} \\ &= 2(t - 1)u(t - 1) - 2(t - 3)u(t - 3) \end{aligned}$$

Example 8.18 Redo Example 8.16 using Laplace transform.

In this example, the signals are $x(t) = u(t)$ and $h(t) = u(t - 1) + u(t - 2) - 2u(t - 3)$. Therefore, their Laplace transform becomes:

$$\begin{aligned} X(s) &= \frac{1}{s}, H(s) = \frac{1}{s}(e^{-s} + e^{-2s} - 2e^{-3s}) \\ x(t) * h(t) &= L^{-1} \left\{ \left(\frac{1}{s} \right) \left(\frac{1}{s}(e^{-s} + e^{-2s} - 2e^{-3s}) \right) \right\} \\ &= L^{-1} \left\{ \left(\frac{1}{s^2}(e^{-s} + e^{-2s} - 2e^{-3s}) \right) \right\} \\ &= (t - 1)u(t - 1) + (t - 2)u(t - 2) - 2(t - 3)u(t - 3). \end{aligned}$$

Example 8.19 Find the convolution of the functions shown in Fig. 8.32.

Functions $x(t)$ and $h(t)$ are given. It is simpler to mirror and shift $h(t)$ respect to the vertical axis. Then the function $h(t - \lambda)$ is rolled over the function $x(\lambda)$.

Fig. 8.32 Functions of Example 8.19

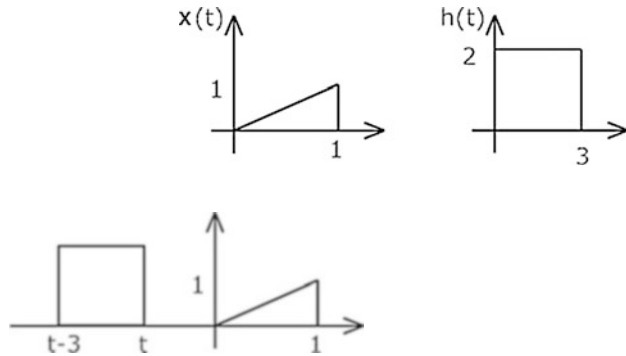


Fig. 8.33 The function $h(t)$ has been mirrored and shifted. The product of two functions is zero before they collide for the time $t < 0$

Fig. 8.34 Product of two signals is a function of t for the time $0 < t < 1$

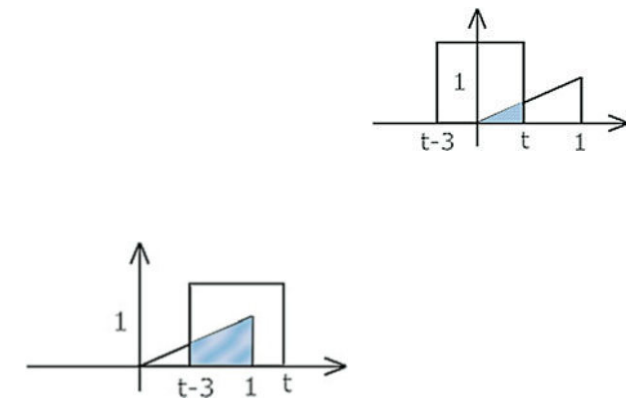


Fig. 8.35 Product of two signals is no longer a function of t when the function $h(t)$ has moved such that $1 < t < 3$

For $t < 0$, the two functions generate no product. Therefore, (Fig. 8.33)

$$y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = 0$$

For $0 < t < 1$, the two functions generate (Fig. 8.34).

$$y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = \int_0^t \lambda \times 2d\lambda = t^2$$

For $1 < t < 3$, the two functions generate a new product as (Fig. 8.35):

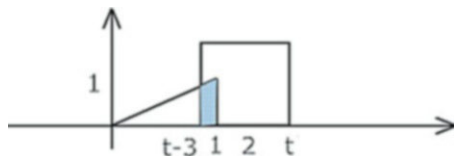


Fig. 8.36 The overlap part of function $h(t)$ still makes the product of two functions dependent on time t when $3 < t < 4$. Since the length of function $h(t)$ is 3, for the start point t , the end point becomes $(t - 3)$. Therefore, the product of these functions becomes non-zero starting from the point $(t - 3)$ to the point 1 at which the function $x(t)$ ends

$$y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = \int_0^1 \lambda \times 2d\lambda = 1$$

For $3 < t < 4$, the two signals have a new product as (Fig. 8.36):

$$y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = \int_{t-3}^1 \lambda \times 2d\lambda = 1 - (t-3)^2$$

For $t > 4$, the two signals have no product, and the output becomes zero.

$$y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = 0$$

$$\text{The function becomes: } f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 < t < 1 \\ 1 & 1 < t < 3 \\ 1 - (t-3)^2 & 3 < t < 4 \\ 0 & t > 4 \end{cases}$$

Example 8.20 Redo the previous example using Laplace transform.

Solution In this example mathematical representation of the signals become $x(t) = tu(t) - (t-1)u(t-1) - u(t-1)$, and $h(t) = 2u(t) - 2u(t-3)$. Therefore, their Laplace transforms becomes:

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}, H(s) = \frac{2}{s} - \frac{2}{s}e^{-3s}$$

The convolution becomes:

$$\begin{aligned}
 y(t) &= x(t) * h(t) = L^{-1} \left\{ \left(\frac{1}{s^2} - \frac{1}{s^2} e^{-s} - \frac{1}{s} e^{-s} \right) \left(\frac{2}{s} - \frac{2}{s} e^{-3s} \right) \right\} \\
 &= L^{-1} \left\{ \frac{2}{s^2} \left(\frac{1}{s} - \frac{1}{s} e^{-s} - e^{-s} \right) (1 - e^{-3s}) \right\} \\
 &= L^{-1} \left\{ 2 \left(\frac{1}{s^3} - \frac{e^{-3s}}{s^3} - \frac{1}{s^3} e^{-s} - \frac{1}{s^3} e^{-4s} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2} e^{-4s} \right) \right\} \\
 &= 2 \left(\frac{1}{2} t^2 u(t) - \frac{1}{2} (t-3)^2 u(t-3) - \frac{1}{2} (t-1)^2 u(t-1) \right. \\
 &\quad \left. - \frac{1}{2} (t-4)^2 u(t-4) - (t-1) u(t-1) + (t-4) u(t-4) \right)
 \end{aligned}$$

State Space Analysis

Analysis of electric circuits leads to differential equations that represent the voltage and current variations. The order of these differential equations depends on the order of the circuit. Considering an n^{th} order system, the differential equation results in one variable either the current or the voltage of components. To obtain the rest of variables, extra mathematical operations may be needed. However, the voltages of capacitors and current of inductors can be obtained individually. The variables that show the energy content of the system (voltage of capacitors and current of inductors) are called state variables. State space equations are another form of system dynamics representation, which also shows the dynamics of individual state variables. Therefore, the variation of all system dynamics can be observed all at the same time.

The general form of state space representation is:

$$\begin{aligned}
 \dot{x}(t) &= Ax(t) + Bu(t) \\
 y(t) &= Cx(t) + Du(t)
 \end{aligned}$$

Where $x(t)$ is the vector of state variables at the same size as the order of the system $n \times 1$, $u(t)$ is the vector of input variables of size $m \times 1$, $y(t)$ is the vector of output variables of size $p \times 1$, $A_{n \times n}$ is the system matrix, $B_{n \times m}$ is the input matrix, $C_{p \times n}$ is the output matrix, and $D_{p \times m}$ is the feedforward matrix. The system input, state, and output signals are as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}.$$

A variable is a state variable if with given initial value and with the dynamics, the future values of the variable can be predicted.

A state variable shows the internal energy variation in the system. For instance, the energy stored in a capacitor is shown in its voltage and the energy stored in an inductor is shown on its current. Therefore, studying the state variables provides an insight over the internal energy stored in the circuit.

According to passivity notion in stability analysis of system, a stable system dissipates more energy than it receives. Therefore, the net energy variation in a stable system is either zero or negative.

$$\text{If : } \Delta E_{\text{internal}} \leq 0, \text{ Then : Stable}$$

Obtaining State Space Equations from Differential Equations

State space representation of a system is not unique. This means that a system can be presented by various forms of state space equations, i.e., matrices A , B , C , and D may have various forms. Consider an n th order differential equation as follows:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = bu(t)$$

To obtain canonical form state space representation of this system, the following steps are needed:

Step 1: Consider

$$y(t) = x_1(t)$$

This yields:

$$x_1^{(n)}(t) + a_1 x_1^{(n-1)}(t) + \dots + a_n x_1(t) = bu(t)$$

Step 2: Consider the rest of the state variables except the last one as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \end{aligned}$$

Replacing these assumptions into the differential equation obtains $\dot{x}_n(t)$ as follows:

$$\begin{aligned}\dot{x}_n(t) &= -a_1x_{n-1}(t) - \dots - a_nx_1(t) + bu(t) \\ \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ -a_1 & \dots & -a_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ b \end{bmatrix} u(t) \\ y(t) &= [1 \cdot \dots \cdot 0] \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}\end{aligned}$$

Example 8.21 Find the state space representation of the following differential equation.

$$\ddot{y} + 5\ddot{y} + 10\dot{y} + 7y = u$$

Solution Consider:

$$y = x_1$$

This yields:

$$\ddot{x}_1 + 5\ddot{x}_1 + 10\dot{x}_1 + 7x_1 = u$$

This is a third-order differential equation, which means there is dynamics for three state variables, namely, x_1, x_2 , and x_3 . Therefore, there is a need to assume two of these dynamics as:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

Replacing into differential equation results in:

$$\dot{x}_3 + 5x_3 + 10x_2 + 7x_1 = u$$

Solving for $\dot{x}_3$ yields:

$$\dot{x}_3 = -7x_1 - 10x_2 - 5x_3 + u$$

In matrix form, the system is represented as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Obtaining Block Diagram of a State Space Equation

Block diagram representation of a state space equation is possible with the notion that each set of equation is a first-order differential equation. Therefore, dynamics of each state variable can be obtained by an integrator $\int$ or in Laplace domain as $\frac{1}{s}$ as shown in Fig. 8.37.

Total number of integrators must equal the number of state variables. For each state variable dynamic, an integrator demonstrates has the $\dot{x}$ in the input and the x in the output. A given $\dot{x}$ dynamic must be configured using the state variables, inputs, and the required gains available from the integrators. This is explained in an example as follows.

Example 8.22 Draw a block diagram representation of the following state space model.

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 - 5x_3 - 2u \\ \dot{x}_2 = -5x_1 - x_2 - x_3 + u \\ \dot{x}_3 = x_1 + 9x_2 - x_3 \end{cases}$$

The system has three state variables that suggest existence of three integrators. Figure 8.38 shows these integrators.

Fig. 8.37 The two functions have no non-zero product when $t > 4$

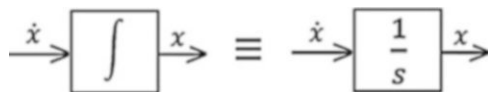
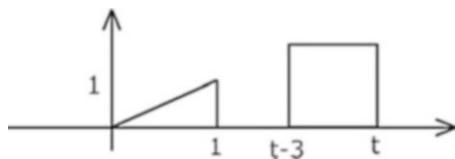


Fig. 8.38 Integrators in time domain and frequency domain. Considering the output of integrators as state variables, the input to these units expresses the formation of state space equations



Fig. 8.39 The system with three state variables has to have three integrators, each of which generate one of the state space equations

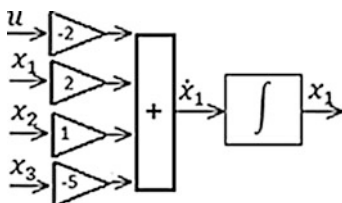


Fig. 8.40 Formation of the first state space equation is shown. The input to the equations is state variables or the inputs with proper gains

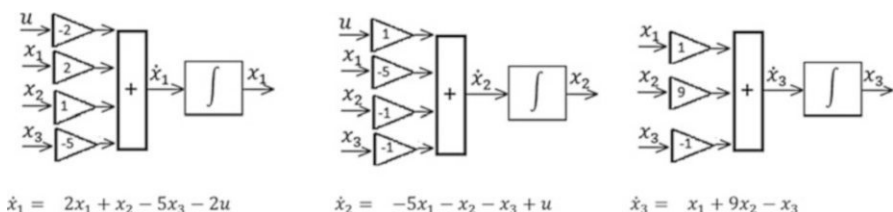


Fig. 8.41 Block diagram of three state space equations is formed according to the equations

As the Fig. 8.39 shows, input to the integrators is the derivative of the state variable. Therefore, these dynamics can be set up. Figure 8.40 shows the setup for the first dynamic $\dot{x}_1 = 2x_1 + x_2 - 5x_3 - 2u$.

Now, the signals x_1, x_2, x_3, u can be connected to the output of the right integrators. Same procedure is utilized to build the other state variable dynamics. Figure 8.41 shows the block diagram of the entire system.

$$\dot{x}_1 = 2x_1 + x_2 - 5x_3 - 2u; \quad \dot{x}_2 = -5x_1 - x_2 - x_3 + u \quad \dot{x}_3 = x_1 + 9x_2 - x_3$$

Example 8.23 Draw a block diagram representation of the following state space model.

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 2x_1 - x_2 + 5u \end{cases}$$

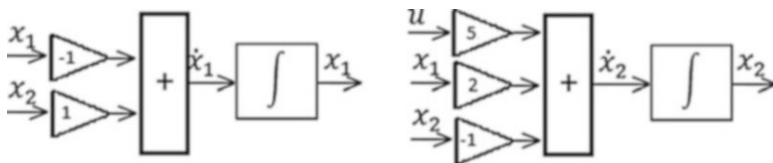


Fig. 8.42 Block diagram of the differential equation.

Solution Since there are two state variables, the system dynamics has two integrators that result in x_1 and x_2 . The input to these integrators is $\dot{x}_1$ and $\dot{x}_2$. Therefore, the diagrams can be implemented as shown in Fig. 8.42.

Obtaining State Space of Differential Equations that Involve Differential of the Input Signals

Consider a causal differential equation that involves n th order differential of the output and m th order differential of the output signal ($n < m$) as follows:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \dots + b_0 u(t)$$

There are several techniques to obtain the state space equation of this system. The technique introduced in this chapter is based on the block diagrams. To do that, this equation is solved for y by taking n integrals as follows:

Solving for y , the block diagram is formed as follows:

$$y = -a_1 \int y - a_2 \iint y - \dots - a_n \iiint \dots \int y + b_m \iiint \dots \int u + \dots + b_0 \iiint \dots \int u$$

To obtain y , the differential equation suggests taking n integrals. The output of the n th integrator is y .

Example 8.24 Find the state space representation of the following differential equation.

$$\ddot{y} + 5\ddot{y} + 6\dot{y} + 11y = 2\ddot{u} + \dot{u} + u$$

Solution This is a third-order differential equation, and since the second-order input equation $2\ddot{u} + \dot{u} + u$ has less order than the third-order output differential equation, the system is also causal. There is a need to take three times integral from both sides of the equation to obtain y . Solving for y yields:

$$y = \underbrace{-5 \int y}_{\textcircled{1}} - \underbrace{6 \iint y}_{\textcircled{2}} - \underbrace{11 \iiint y}_{\textcircled{3}} + \underbrace{2 \int u}_{\textcircled{4}} + \underbrace{\iint u}_{\textcircled{5}} + \underbrace{\iiint u}_{\textcircled{6}}$$

To build y through integrals of y and u , there is a need to feedback y considering the gains. Three integrators exist in this differential equation. In the next step, consider three integrator blocks with an adder in the input of each block, and the output of the far right integrator block is y .

Now, consider the differential equation for each of the right-hand side components of the equation (*) is built. To build $\textcircled{1}$, the feedback is taken from y , and considering gain -5 , it is added to the summation of the right integral block. When the system runs numerically, this loop forms the $-5 \int y$ component. The input signal $\textcircled{4}$ is added to the far right integrator by gain 2 to be integrated once by reaching signal y . Segment $\textcircled{5}$ is being added to the second integrator, and segment $\textcircled{6}$ is added to the left integrator. Figure 8.43 shows blockdiagram implementation.

Obtaining Transfer Function from State Space Representation

Finding the state space representation of a system might have several approaches. The representations these techniques achieve might look different, but they all show the same dynamics. The state space representations are not unique. However, the transfer function of a system is unique. There is a relation between the state space and the transfer function of a system expressed as follows:

Consider a state space system as:

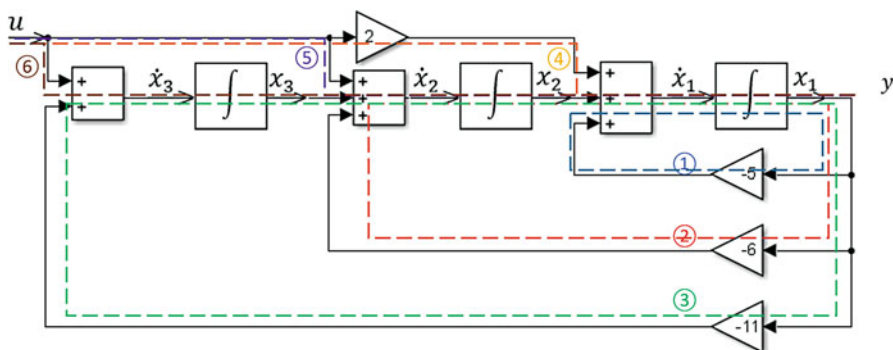


Fig. 8.43 Block diagram implementation of the system in example 8.24

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Taking the Laplace of this system results in:

$$\begin{aligned}sX(s) - X(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

The transfer function is obtained at the zero initial conditions. Therefore: $X(0) = 0$.

This results in:

$$sX(s) - AX(s) = BU(s)$$

Solving for $X(s)$ yields:

$$(sI - A)X(s) = BU(s)$$

And

$$X(s) = (sI - A)^{-1}BU(s)$$

Replacing this into the output equation results in:

$$\begin{aligned}Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ Y(s) &= \left(C(sI - A)^{-1}B + D \right) U(s)\end{aligned}$$

The system transfer function which is unique over various combinations of matrices A, B, C , and D can be obtained and presented as:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example 8.25 A state space representation of a dynamical system is given as:

$$\dot{X} = \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

Find the transfer function of the system.

Solution The transfer function is defined as:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

From the system, the following matrices can be obtained:

$$A = \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = 0 \text{ and } I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the transfer function is:

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

Simplifying, results in:

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ 3 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{\det \begin{bmatrix} s-1 & 1 \\ 3 & s+5 \end{bmatrix}} \begin{bmatrix} s+5 & -1 \\ -3 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & -1 \\ -3 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(s-1)(s+5) - 1 \times 3}$$

$$\frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 1 \times (s+5) + 0 \times -3 & 1 \times -1 + 0 \times (s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 4s - 8}$$

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{\begin{bmatrix} s+5 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 4s - 8} \\ \frac{Y(s)}{U(s)} &= \frac{\begin{bmatrix} (s+5) \times 0 & -1 \times 1 \end{bmatrix}}{s^2 + 4s - 8} \\ \frac{Y(s)}{U(s)} &= \frac{-1}{s^2 + 4s - 8}\end{aligned}$$

Example 8.26 A state space representation of a dynamical system is given as:

$$\begin{aligned}\dot{X} &= \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} X\end{aligned}$$

Find the transfer function of the system, and poles and zeros of the system.

Solution The transfer function is defined as:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

From the system, the following matrices can be obtained:

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D &= 0 \text{ and } I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

Therefore, the transfer function is:

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

Simplifying, results in:

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 3 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}\frac{Y(s)}{U(s)} &= [1 \quad 0] \frac{1}{\det \begin{bmatrix} s & -1 \\ 3 & s+5 \end{bmatrix}} \begin{bmatrix} s+5 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{Y(s)}{U(s)} &= \frac{[1 \quad 0] \begin{bmatrix} s+5 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s(s+5) - 1 \times -3} \\ \frac{Y(s)}{U(s)} &= \frac{[1 \times (s+5) + 0 \times -3 \quad 1 \times 1 + 0 \times s] \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 5s + 3} \\ \frac{Y(s)}{U(s)} &= \frac{\begin{bmatrix} s+5 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 5s + 3} \\ \frac{Y(s)}{U(s)} &= \frac{[(s+5) \times 0 \quad -1 \times 1]}{s^2 + 5s + 3} \\ \frac{Y(s)}{U(s)} &= \frac{-1}{s^2 + 5s + 3}\end{aligned}$$

There is no zeros in the transfer function.

There is two poles at $s^2+5s+3 = 0$.

The poles are $s = -2.5 \pm \frac{\sqrt{13}}{2}$.

Bode Diagram

Transfer functions are defined in Laplace domain using operation s . As the Laplace operator is a function frequency, the transfer function is influenced by change of operating frequencies. As all complex functions, the transfer function shows an amplitude and a phase respected to any operating frequency. The profile of amplitude and phase change with respect to the operating frequency is also called “Bode” plot or Bode diagram.

The simplest way of obtaining the Bode diagram is to evaluate the amplitude and phase at each frequency for a given range. The range of frequency starts from DC ($s = j0$) and is extended as long as there is variation in the phase and amplitude. However, this is computationally expensive. There are indicators in the transfer function that determine the amplitude and phase diagrams. This section is to identify these indicators and determine the Bode diagram accordingly.

Transfer Function Amplitude and Phase

As stated earlier, the effective range of frequencies needed to be checked for amplitude and phase might extend from zero to extremely high frequencies. To obtain a readable chart, the axis of frequencies is shown as logarithmic values.

The amplitude of transfer function might also be a very large number. To avoid operation with large numbers, the unit of amplitude is presented in decibel.

Decibel: Consider the transfer functions that determine the ration of power in the output (P_o) over the power in the input (P_i) as $H(s) = \frac{P_o}{P_i}$ or the ratio of voltage $H(s) = \frac{V_o}{V_i}$ or current $H(s) = \frac{I_o}{I_i}$. This is shown in Fig. 8.44.

Transfer function gain in decibel is defined as:

$$|H|\text{dB} = 10\log\frac{P_o}{P_i}$$

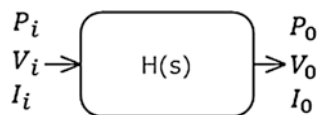
Considering that the power is proportional to the square of voltage or current as $P = RI^2 = \frac{V^2}{R}$, the transfer functions amplitude becomes:

$$\begin{aligned} |H|\text{dB} &= 10\log\frac{P_o}{P_i} = 10\log\frac{\frac{V_o^2}{R}}{\frac{V_i^2}{R}} \\ &= 10\log\frac{V_o^2}{V_i^2} \\ &= 20\log\frac{V_o}{V_i} \end{aligned}$$

Similarly, the current gain of an amplifier in dB is obtained as follows:

$$\begin{aligned} |H|\text{dB} &= 10\log\frac{P_o}{P_i} = 10\log\frac{RI_o^2}{RI_i^2} \\ &= 10\log\frac{I_o^2}{I_i^2} \\ &= 20\log\frac{I_o}{I_i} \end{aligned}$$

Fig. 8.44 Input – output signal types to a system $H(s)$



Example 8.27 The output power delivery of an amplifier is 10 W. Find the amplifier gain in dB if the input signal was 0.1 W.

Solution The ratio of $\frac{P_0}{P_i}$ is $\frac{10}{0.1}$. Therefore, the amplifier gain in dB is:

$$|H|_{\text{dB}} = 10 \log \frac{P_0}{P_i} = 10 \log \frac{10}{0.1} = 20(\text{dB})$$

Example 8.28 Calculate the gain of a circuit when the output voltage drops by 70.7% at a certain frequency.

Solution Since voltage gain is used, the gain in dB is obtained by:

$$|H|_{\text{dB}} = 20 \log \frac{V_0}{V_i} = 20 \log 0.707 = -3 \text{ dB}$$

Example 8.29 Find the power gain reduction of a circuit at the frequency in which the voltage gain is -3dB .

Solution When the voltage gain is -3dB , the voltage gain $A_V = \frac{V_0}{V_i}$ becomes:

$$\begin{aligned} -3 &= 20 \log \frac{V_0}{V_i} \\ \frac{V_0}{V_i} &= 10^{\frac{-3}{20}} = 0.707 \\ \frac{P_0}{P_i} &= \left(\frac{V_0}{V_i} \right)^2 = 0.5 \end{aligned}$$

It can be concluded that, at a certain frequency, when the voltage gain drops to 70.7 %, the power gain drops to 50 %. This frequency is also called half-power point frequency.

Bode Plot of A Transfer Function

A transfer function is a combination of several components. There is a template for each element of a transfer function. Therefore, in first step the transfer function must be broken into these elements and their frequency responses be combined.

Element 1 Gain or DC gain. $H(s) = K$. To obtain the gain of a transfer function, the poles and zeros must be written as:

$$H(s) = K \frac{\prod_{i=1}^m \left(1 + \frac{s}{z_m}\right)}{\prod_{j=1}^n \left(1 + \frac{s}{p_j}\right)}$$

The amplitude of this transfer function at a certain frequency $s = j\omega$ is obtained as:

$$|H(j\omega)| = K \frac{\prod_{i=1}^m \sqrt{1 + \left(\frac{\omega}{z_m}\right)^2}}{\prod_{j=1}^n \sqrt{1 + \left(\frac{\omega}{p_j}\right)^2}}$$

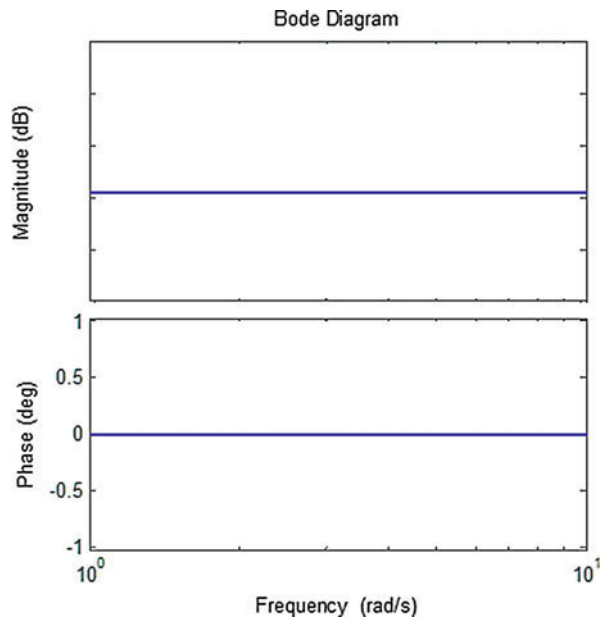
The DC gain in dB is:

$$20\log K$$

The phase for this element is zero.

Therefore, for a simple gain element, Bode plot is obtained as shown in Fig. 8.45.

Fig. 8.45 Amplitude and phase diagram of $H(s) = K$



Example 8.30 Find the Bode plot of the following function.

$$H(s) = 30$$

Solution The transfer function using $s = j\omega$ results in:

$$H(j\omega) = 30$$

The amplitude of this function is:

$$|H| = 20\log 30 = 29.54 \text{ dB}$$

The phase of this function is constant 0° . Figure 8.46 shows the Bode diagram.

Element 2 Zero at the origin. $H(s) = s$.

This element has the transfer function as $H(j\omega) = j\omega$, with the amplitude of $|H| = 20 \log \omega$ (dB) and the phase of $\angle H = 90^\circ$. Bode plot of this element is shown in Fig. 8.47.

Note 8.10 As the graph shows, the amplitude is increased 20dB in each decade of the frequency. It means when the frequency is increased tenfold, the gain is changed 20 dB.

Fig. 8.46 Amplitude and phase diagram of $H(s) = 30$

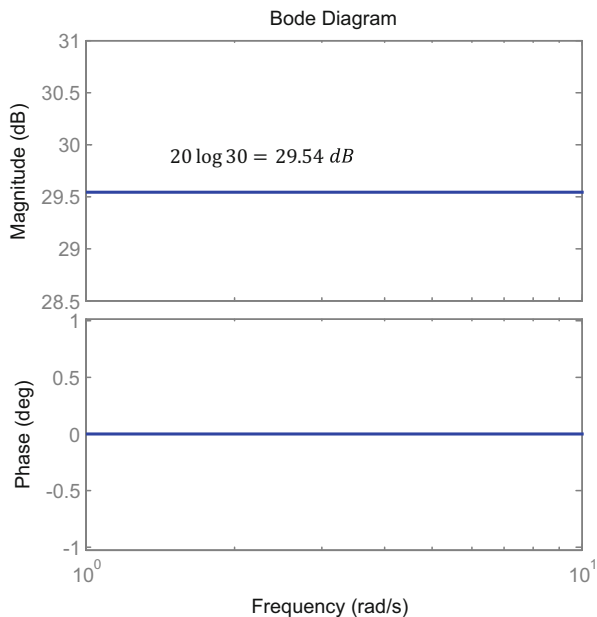
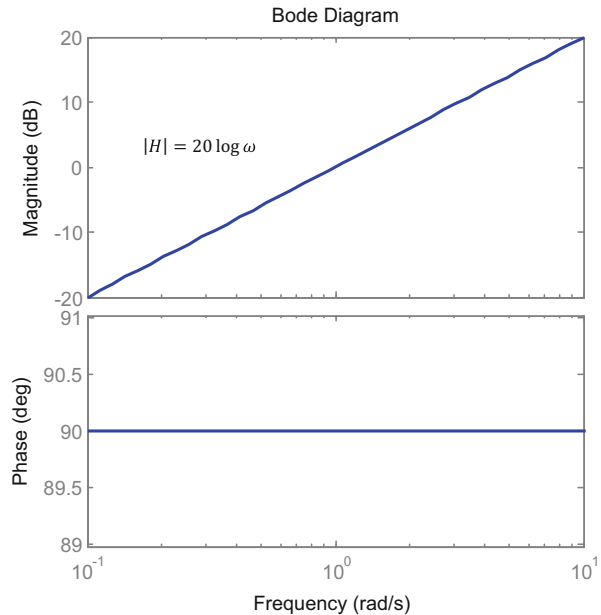


Fig. 8.47 Amplitude and phase diagram of $H(s) = s$



Note 8.11 The amplitude is a straight line with positive slope of 20 dB/dec passing the frequency of 1 rad/s or 10^0 in Fig. 8.47. The amplitude is negative for frequencies less than 1 rad/s.

Example 8.31 Find Bode plot of the following transfer function:

$$H(s) = 2s$$

Solution This function has two elements, one is the gain of 2 ①, and one is a zero at the origin, s^2 . This suggests two plots for the amplitude $|H|$ in Fig. 8.48:

Element 3 A repeated zero at the origin. $H(s) = s^N$, N times

The repeat of zero N times at the origin increases the slope by the same factor $20N$ dB/dec, and the phase is shifted to $N(90)$.

Element 4 A pole at the origin. $H(s) = \frac{1}{s}$. In this case the amplitude is inversely proportional to the increase of the frequency as $|H| = 20 \log \frac{1}{\omega}$. The phase is independent of frequency at -90° . The Bode plot is shown in Fig. 8.49. Amplitude and phase if $H(s) = \frac{1}{s}$.

Note 8.12 the slope of this amplitude is -20 dB per decade. It means the amplitude drops 20 dB when the frequency increases from 1 to 10 rad/s.

Fig. 8.48 Amplitude and phase diagram of $H(s)=2$ ①, and $H(s)=s$ ②, both seprataely and combined

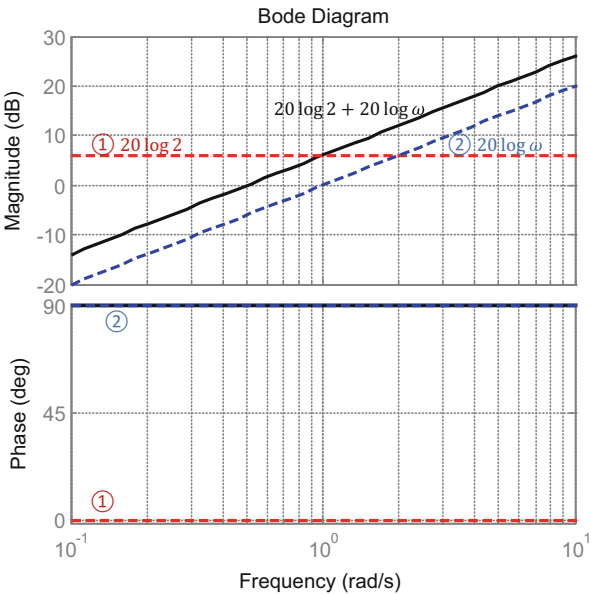


Fig. 8.49 Amplitude and phase diagram of $H(s) = \frac{1}{s}$

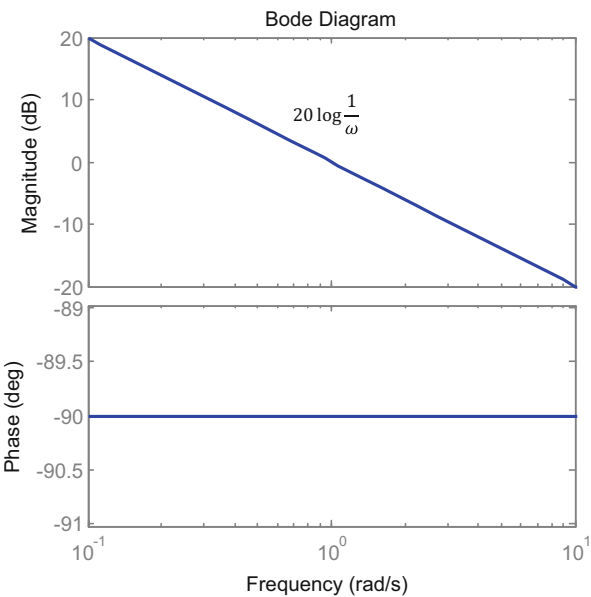
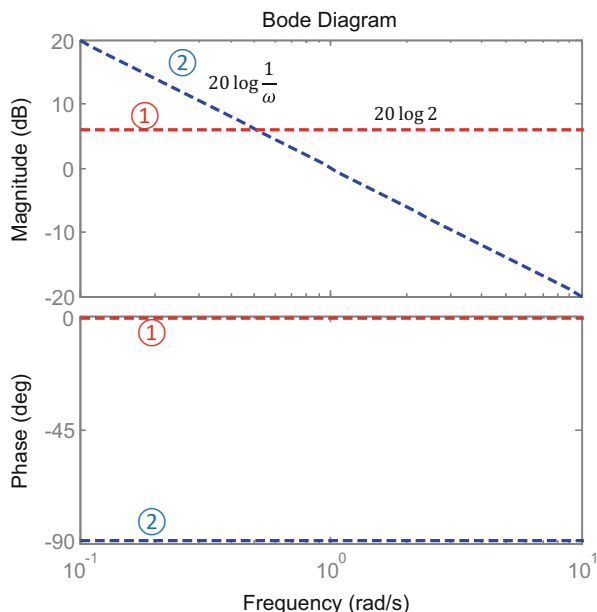


Fig. 8.50 Amplitude and phase diagram of $H(s) = \frac{1}{s}$ and $H(s) = 2$, separately



Note 8.13 The amplitude is still decreasing from near zero to 1 rad/s onward.

Note 8.14 The phase is constant and not changing with frequency.

Example 8.32 Find Bode plot of the function: $H(s) = \frac{2}{s}$.

Solution The function has two elements, 2 ① and $\frac{1}{s}$ ②. This results in two amplitudes as $20 \log 2$ and $20 \log \frac{1}{\omega}$. A separate plot of these two elements is shown in Fig. 8.50 and Fig. 8.51.

The combined elements have the following frequency response (Fig. 8.50):

Element 5 A repeated pole at the origin N times. $H(s) = \frac{1}{s^N}$.

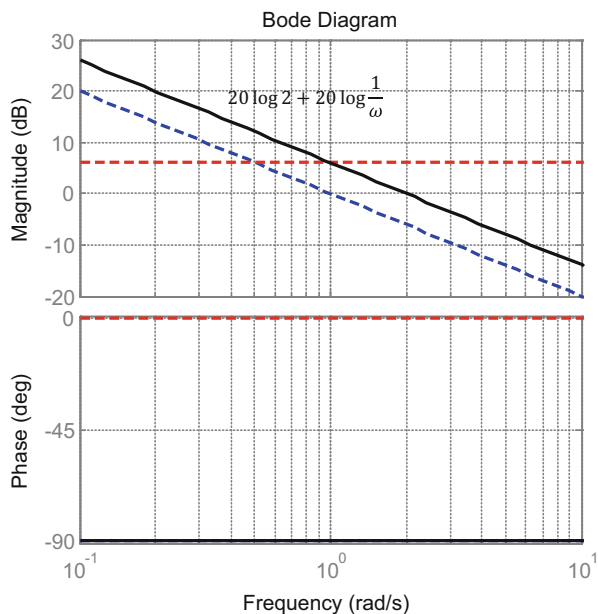
The Bode plot of this function has slope $-20N$ dB/dec, and the phase of this system is constant $-N(90^\circ)$.

Element 6 A unity gain transfer function with a zero at z . $H(s) = \left(1 + \frac{s}{z}\right)$.

The amplitude of this transfer function is evaluated for the frequencies higher than z rad/s. The phase of this transfer function needs to be evaluated in a decade below and a decade above the z rad/s.

Considering $s = j\omega$, the transfer function becomes:

Fig. 8.51 Amplitude and phase diagram of $H(s) = \frac{1}{s}$ and $H(s) = 2$, combined



$$H(j\omega) = \left(1 + \frac{j\omega}{z}\right)$$

The amplitude of transfer function in dB becomes:

$$|H| = 20 \log \sqrt{1 + \left(\frac{\omega}{z}\right)^2}$$

This function shows ≈ 0 dB gain for all frequencies from $\omega = \frac{1}{10}z$ up to the location of zero $\omega = z$ rad/s :

$$\left|H\left(\omega = \frac{1}{10}z\right)\right| = 20 \log \sqrt{1 + \left(\frac{\frac{1}{10}z}{z}\right)^2} = 0 \text{ dB}$$

At the location of zero, the gain becomes:

$$|H(\omega = z)| = 20 \log \sqrt{1 + \left(\frac{z}{z}\right)^2} = 3 \text{ dB}$$

The gain at ten times of the zero $\omega = 10z$ becomes:

$$|H(\omega = 10z)| = 20 \log \sqrt{1 + \left(\frac{10z}{z}\right)^2} = 20 \text{ dB}$$

It can be concluded that the gain is increased from the location of zero by slope of 20 dB/dec.

To analyze the phase of this function, consider $\omega = \frac{1}{10}z$, $\omega = z$ and $\omega = \frac{1}{10}z$. At each of these values, the phase becomes:

$$\begin{aligned} \angle H &= \tan^{-1} \frac{\frac{\omega}{z}}{1} \\ \angle H\left(\omega = \frac{1}{10}z\right) &= \tan^{-1} \frac{\frac{1}{10}}{1} = 5.7^\circ \end{aligned}$$

An asymptote of the phase variation for smaller frequencies can be considered as approximately zero. At the location of zero the phase reaches:

$$\angle H(\omega = z) = \tan^{-1} \frac{1}{1} = 45^\circ$$

In a decade higher, the phase of transfer function becomes:

$$\angle H(\omega = 10z) = \tan^{-1} \frac{10}{1} = 85^\circ$$

As an asymptote, the phase saturates to 90° when the frequency increases very high.

Therefore, the Bode plot of the function is shown in Fig. 8.52.

Example 8.33 Find the Bode diagram of the following function

$$H(s) = (s + 7)$$

Solution This function has a zero at $s = -7$, or $z = 7$. To convert this function to the form of $\left(1 + \frac{s}{z}\right)$, 7 is factored out as follows:

$$H(s) = 7 \left(1 + \frac{s}{7}\right)$$

Therefore, this function has an amplitude of 20 dB. Bode plot of each element is $(20 \log 7)$ ① & $20 \log \sqrt{1 + \left(\frac{\omega}{7}\right)^2}$ ② is obtained as shown in Fig. 8.53.

Fig. 8.52 Amplitude and phase diagram of $H(s) = \left(1 + \frac{s}{z}\right)$

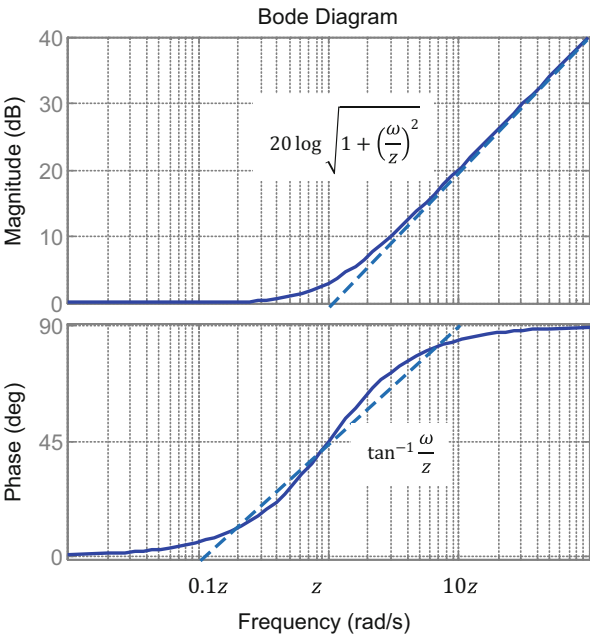


Fig. 8.53 Amplitude and phase diagram of $H(s) = 7$ and $H(s) = \left(1 + \frac{s}{z}\right)$, separately

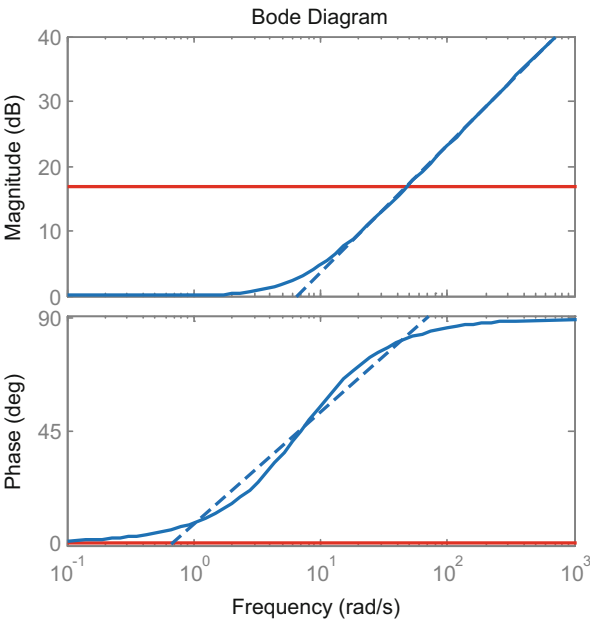
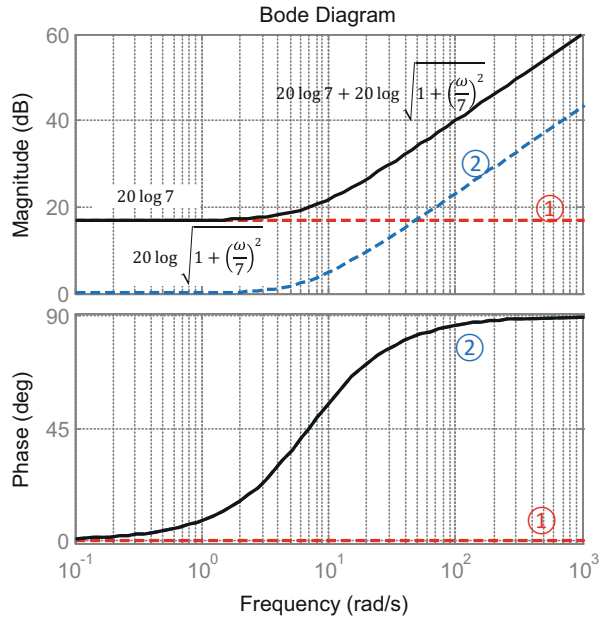


Fig. 8.54 Amplitude and phase diagram of $H(s) = 7$ and $H(s) = (1 + \frac{s}{7})$, combined



As it is observed, the amplitude of transfer function at zero $s = -7$ rad/s starts an increment of 20 dB/dec. The effective range of phase starts a decade below at $\frac{1}{10} 7$ rad/s and ends a decade higher at 10×7 rad/s. Combining the bode plot of these two elements is shown in Figs. 8.53 and 8.54.

Element 7 A repeated zero at $s = z$. $H(s) = (1 + \frac{s}{z})^N$. The repeated zero at z increases the slope of amplitude by factor N as $20N$ dB/dec, and the phase reaches $N(90^\circ)$ which suggests an increased phase slope of $N(45^\circ)$ deg /dec.

Element 8 A unity gain transfer function with a pole at $s = p$. $H(s) = \frac{1}{(1 + \frac{s}{p})}$

Amplitude of this transfer function can be obtained as:

$$|H| = 20 \log \frac{1}{\sqrt{1 + (\frac{\omega}{p})^2}}$$

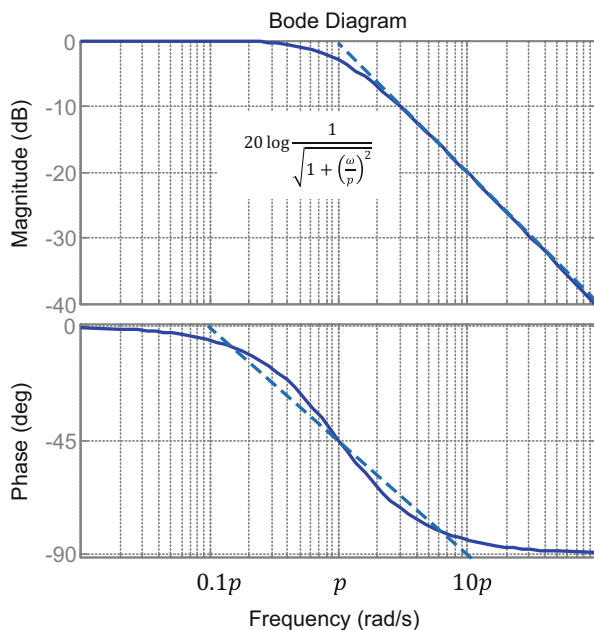
The phase of this transfer function can be obtained as:

$$\angle H = 0 - \tan^{-1} \frac{\omega}{p}$$

The evaluation points are a decade below and a decade above the location of pole. The amplitude is zero until the location of pole and starts to drop with slope

Fig. 8.55 Amplitude and phase diagram of

$$H(s) = \frac{1}{(1+\frac{s}{p})}$$



-20 dB/dec, and the phase starts to drop a decade below the pole location at $1/10 p$ and reaches -90° a decade above the pole at $10p$. At the pole location, the phase is -45° , which suggests a -45 deg /dec slope of the phase diagram. This can result in asymptotes and Bode diagram is shown in Fig. 8.55:

Example 8.34 Find the Bode diagram of the following function: $H(s) = \frac{1}{1+\frac{s}{7}}$.

Solution The transfer function has a pole at $\omega = 7$. Therefore, the phase is zero for frequencies below 7 rad/s and is decreased by a slope of -20 dB/dec. This means that the amplitude reaches -20 dB at frequency of $10 \times 7 = 70$ rad/s. The phase starts to decrease a decade below the pole location at $\frac{1}{10} 7 = 0.7$ rad/s and reaches -90° a decade higher at 70 rad/s. Figure 8.56 shows the Bode diagram asymptotes and its accurate graph. The phase reaches -45° at the location of the pole. Therefore, the phase drops by slope of -45 deg/dec and saturates at -90° .

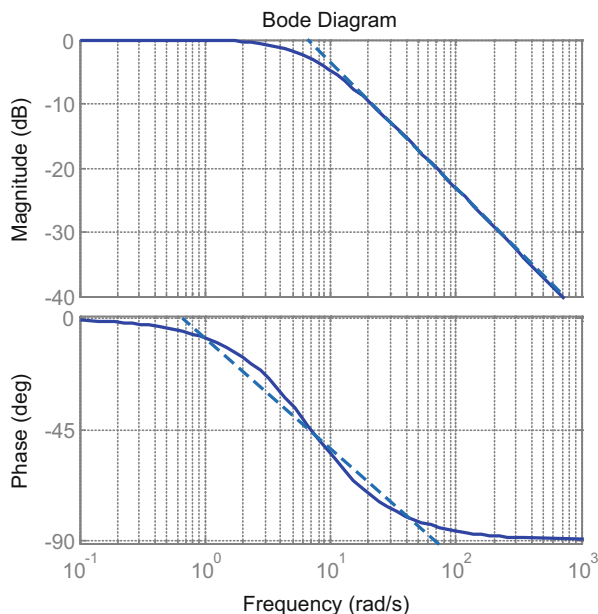
Element 9 Repeated pole at p . $H(s) = \frac{1}{(1+\frac{s}{p})^N}$. The amplitude of this function starts to drop at the location of pole p and drops with slope $-N(20)$ dB/dec. The phase starts to drop with slope $-N45$ deg /dec at a decade below the pole and saturates with $-N(90^\circ)$ at a decade above the pole.

Example 8.35 Find Bode plot of the following function. $H(s) = \frac{(s+50)}{(s+1)(s+10)}$.

Solution The first step in obtaining the Bode plot of a transfer function is to convert it to the form with gain factored out as follows:

Fig. 8.56 Amplitude and phase diagram of

$$H(s) = \frac{1}{1+\frac{s}{7}}$$



$$H(s) = \frac{(s+50)}{(s+1)(s+10)} = \frac{50(1+\frac{s}{50})}{10(s+1)(1+\frac{s}{10})} = \frac{5(1+\frac{s}{50})}{(s+1)(1+\frac{s}{10})}$$

There are four elements including gain 5 ①, zero at 50 of $(1+\frac{s}{50})$ ②, poles at 1 and 10 through $(s+1)$ ③ & $(1+\frac{s}{10})$ ④, respectively.

Therefore, four signals in amplitude and four signals in phase are obtained and then combined to get the transfer function response. The Bode plot elements and combined are shown in Fig. 8.57.

Example 8.36 Find Bode plot of $H(s) = \frac{4}{(s+5)^3}$.

Solution The transfer function has an amplitude of:

$$H(s) = \frac{4}{5^3(1+\frac{s}{5})^3} = \frac{4}{125} \frac{1}{(1+\frac{s}{5})^3}$$

Bode plot of each of these two elements $\frac{4}{125}$ ① and $\frac{1}{(1+\frac{s}{5})^3}$ ② is shown in Fig. 8.58.

As the figure shows, the triple poles at 5 rad/s shows a slope of -60 dB/dec, and the phase reaches $-3 \times 90^\circ = -270^\circ$ a decade above 5 or 50 rad/s. The slope of phase drop is $-45 \times 3 = -135$ deg/dec.

Fig. 8.57 Amplitude and phase diagram of $H(s) = 5$

①, $H(s) = (1 + \frac{s}{50})$ ②, $H(s) = \frac{1}{(s+1)}$ ③, $H(s) = \frac{1}{(1+\frac{s}{10})}$ ④, separately and combined

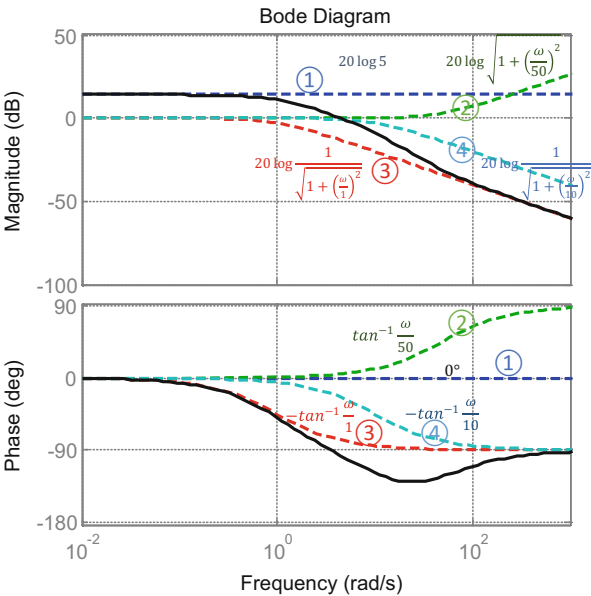


Fig. 8.58 Amplitude and phase diagram of $H(s) = \frac{4}{125}$ ①, $H(s) = \frac{1}{(1+\frac{s}{5})^3}$, separately and combined

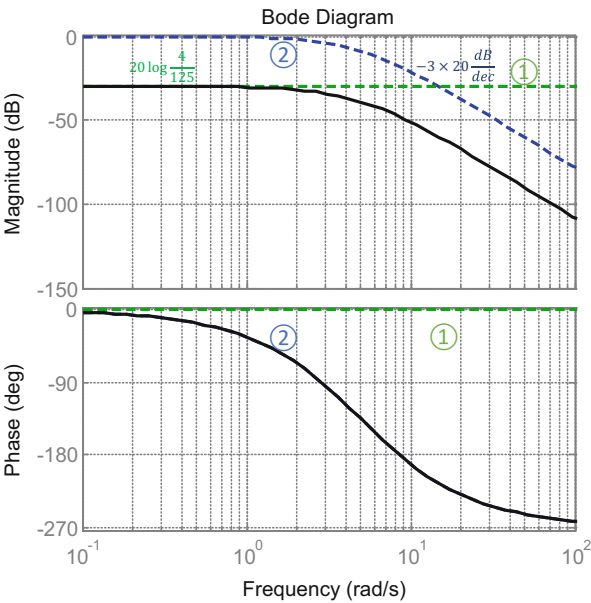
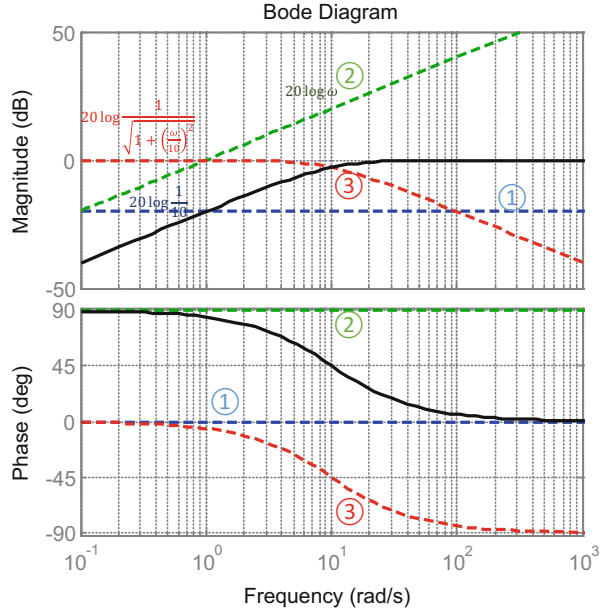


Fig. 8.59 Amplitude and phase diagram of $H(s) = \frac{1}{10}$ ①, $H(s) = 2$ ②, $H(s) = (1 + \frac{s}{10})$ ③ separately and combined



Example 8.37 Find Bode plot of $H(s) = \frac{s}{s+10}$.

Solution After converting the transfer function to the known template (factoring out the gain), it is known that there are three elements of gain $\frac{1}{10}$ ①, a zero at the origin s ②, and a pole at 10, $(1 + \frac{s}{10})$ ③.

Bode plot of these three elements is shown in Fig. 8.59.

Element 10 A unity-gain complex-conjugate zeros $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

$$H(s) = \left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)$$

In a complex conjugate roots condition, the damping factor $0 < \zeta < 1$. Therefore, there is an overshoot observed on the amplitude of the Bode diagram. The amplitude is 0 dB until the reach of resonant frequency ω_n . At this point the amplitude is increased by slope of 40 dB/dec.

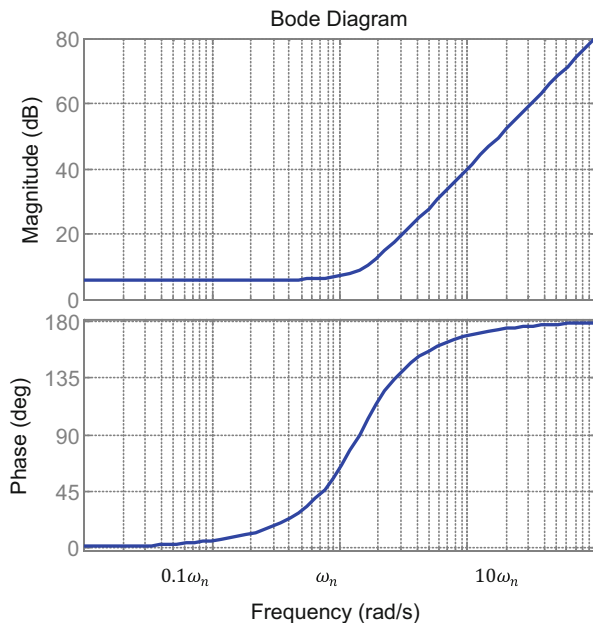
The phase starts to increase a decade below the resonant frequency ω_n with slope of 90 deg/dec. At the resonant frequency, the phase reaches 90°, and a decade above the resonant, it is saturated to 180°.

Figure 8.60 shows the actual and asymptotes of the Bode plot.

Effect of damping factor: As the damping factor becomes less than 1, there is overshoot observed in the time response of the system. This overshoot is increased in

Fig. 8.60 Amplitude and phase diagram of

$$H(s) = \left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)$$



damping factors near zero, when the system becomes oscillatory. This effect is shown in Fig. 8.61. As the damping factor drops the amplitude change at the resonant frequency becomes larger. The rate of phase change becomes faster.

Example 8.38 Find Bode plot of the following function: $H(s) = (1 + 0.02s + 0.01s^2)$.

Solution Compared to the template $H(s) = \left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)$, the parameters are found to be $\zeta = 0.1$ and $\omega_n = 10$. Therefore, sketch of the Bode diagram for the amplitude starts at 10 rad/s with an increase of slope +40 dB/dec.

The phase change starts from 0° at $\frac{1}{10} \times 10 = 1$ rad/s and ends to 180° at $10 \times 10 = 100$ rad/s. The slope of change is +90 deg/dec. Figure 8.62 shows the Bode diagram.

Element 11 Repeated component N times. $H(s) = \left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)^N$.

In this case, the resonant frequency and the damping factor do not change. However, the amplitude increases sharper with slope $N \times 40$ dB/dec, and the phase reaches final value of $N \times 180^\circ$ with slope of $N \times 90$ deg/dec.

Fig. 8.61 Amplitude and phase diagram of

$$H(s)=\left(1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^2}s^2\right)$$

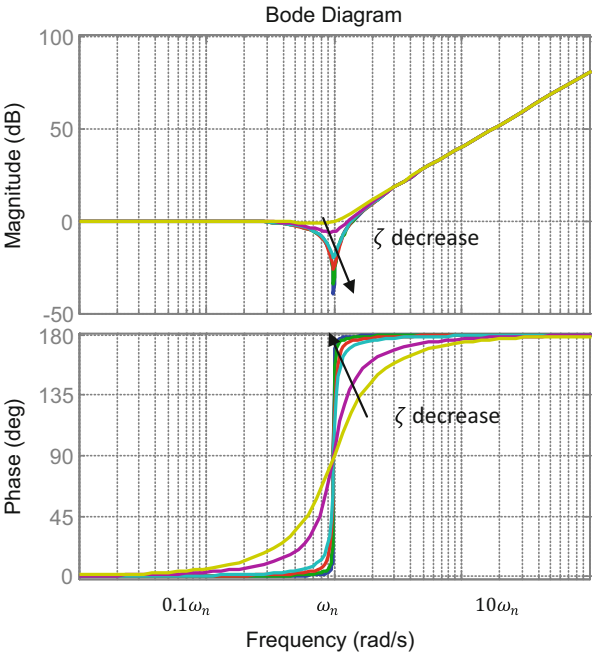
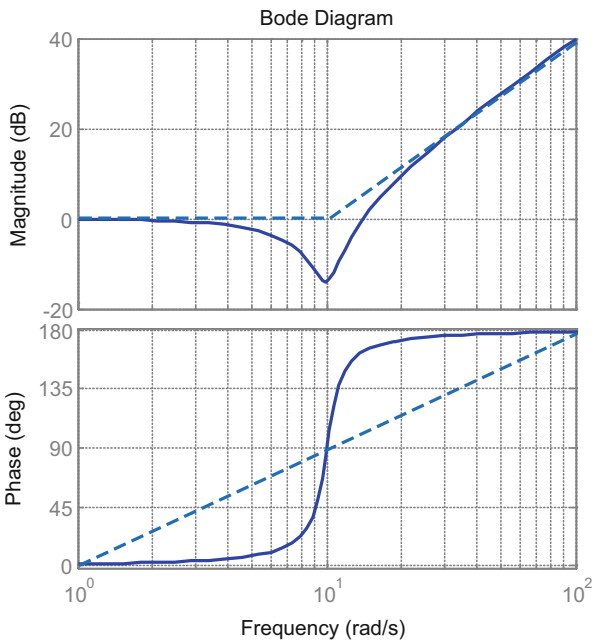


Fig. 8.62 Amplitude and phase diagram of $H(s) = (1 + 0.02s + 0.01s^2)$. This has a damping factor $\zeta = 0.1$ and resonant frequency of $\omega_n = 10$



Example 8.39 Find Bode plot of the following transfer function. $H(s) = (1 + 0.02s + 0.01s^2)^2$.

Solution Compared to the template $H(s) = \left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)$, the parameters are found to be $\zeta = 0.1$ and $\omega_n = 10$. Therefore, sketch of the Bode diagram for the amplitude starts at 10 rad/s with an increase of slope $+2 \times 40$ dB/dec.

The phase change starts from 0° at $\frac{1}{10} \times 10 = 1$ rad/s and ends to $2 \times 180^\circ$ at $10 \times 10 = 100$ rad/s. The slope of change is $+2 \times 90$ deg/dec. Figure 8.63 shows the Bode diagram.

Element 12

$$H(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2}.$$

This transfer function have two poles at $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. The amplitude is zero until the reach of resonant frequency ω_n , and then it starts to drop with the slope of -40 dB/dec. Phase also starts to drop from zero beginning at a decade below the resonant frequency, $\frac{1}{10}\omega_n$, continues to drop with slope -90 deg/dec and reaches a saturated value of -180° a decade above the resonant frequency $10\omega_n$. The amplitude and phase are shown in Fig. 8.64.

Fig. 8.63 Amplitude and phase diagram of $H(s) = (1 + 0.02s + 0.01s^2)^2$ and $H(s) = (1 + 0.02s + 0.01s^2)^2$.

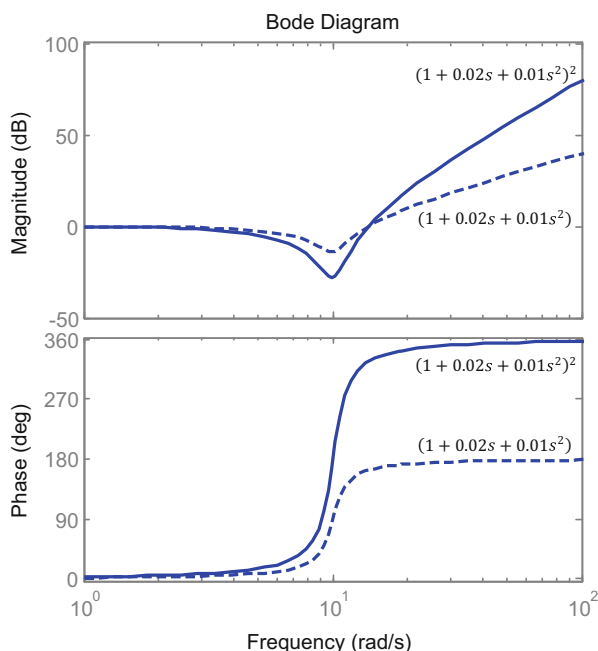
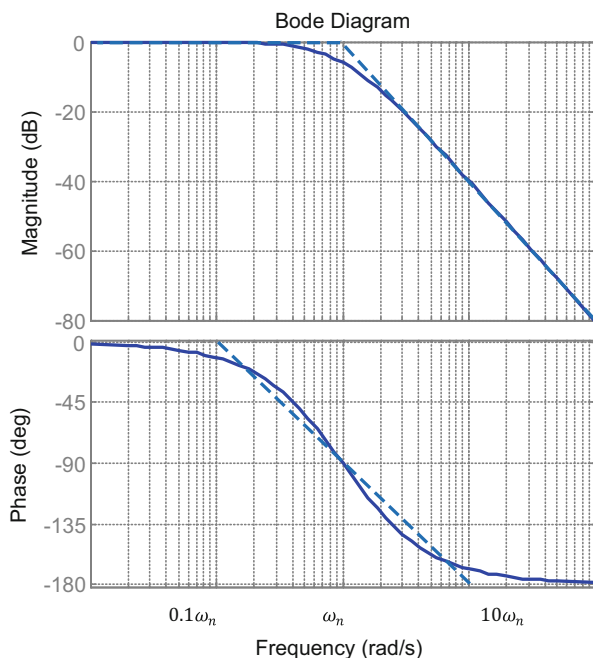


Fig. 8.64 Amplitude and phase diagram of

$$H(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$



Element 13 Repeated complex conjugate poles. $H(s) = \frac{1}{\left(1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2\right)^N}$.

The amplitude is zero until the reach of resonant frequency ω_n , and then it starts to drop with the slope of $-N \times 40$ dB/dec. Phase also starts to drop from zero beginning at a decade below the resonant frequency, $\frac{1}{10}\omega_n$, continues to drop with slope $-N \times 90$ deg/dec and reaches a saturated value of $-N \times 180^\circ$ a decade above the resonant frequency $10\omega_n$.

Example 8.40 Find Bode diagram of the following transfer function.

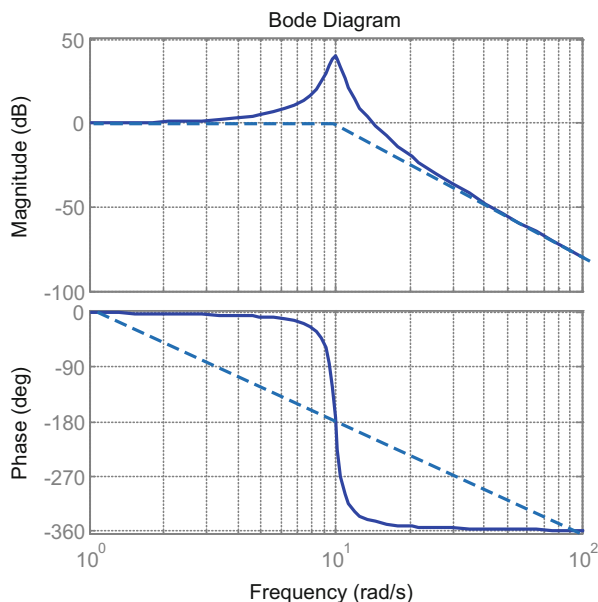
$$H(s) = \frac{1}{(1 + 0.02s + 0.01s^2)^2}$$

Solution Since this is a unity gain transfer function of a second-order unit, the entry gain (or the gain at low frequencies close to DC) is 0 dB. The amplitude starts to drop by slope of $-40 \times 2 = -80$ dB/dec at the resonant frequency of 10 rad/s. The phase diagram starts at a decade below the resonant 1 rad/s from 0° and reaches $-180 \times 2 = -360^\circ$ by the frequency a decade above the resonant, i.e., 100 rad/s. Figure 8.65 shows the amplitude and phase variations.

The transfer function shows a damping factor of $\zeta = 0.1$. Therefore, there is an overshoot observed in the time response. This overshoot is also shown in figure by

Fig. 8.65 Amplitude and phase diagram of

$$H(s) = \frac{1}{(1 + 0.02s + 0.01s^2)^2}$$



exceeding the approximate responses (dotted lines). The phase drop sharpness also depends on the damping, and as the damping factor is decreased, the slope of phase change and overshoot of the amplitude increase. This is shown in Figs. 8.65 and 8.66.

Example 8.41 Find Bode diagram $H(s) = \frac{(s+5)}{(s+7)(s+10)^2}$.

Solution First, the transfer function gain is obtained by factoring the gains out to form the element templates.

$$H(s) = \frac{(s+5)}{(s+7)(s+10)^2} = \frac{5}{7 \times 10^2} \frac{(1 + \frac{s}{5})}{(1 + \frac{s}{7})(1 + \frac{s}{10})^2}$$

Therefore, the transfer function has a gain of $\frac{5}{700}$ ① and three elements of $(1 + \frac{s}{5})$ ②, $(1 + \frac{s}{7})$ ③, and $(1 + \frac{s}{10})^2$ ④.

Individual Bode diagram of each element can be obtained as shown in Fig. 8.67:

Example 8.42 Find Bode diagram of $H(s) = \frac{100(s+10)}{(s+5)(s+20)}$.

Solution To obtain the DC gain, frequencies are factors out, and template is identified as:

$$H(s) = \frac{100(s+10)}{(s+5)(s+20)} = \frac{100 \times 10}{5 \times 20} \frac{(1 + \frac{s}{10})}{(1 + \frac{s}{5})(1 + \frac{s}{20})}$$

Fig. 8.66 Amplitude and phase diagram of

$$H(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

when damping decreases

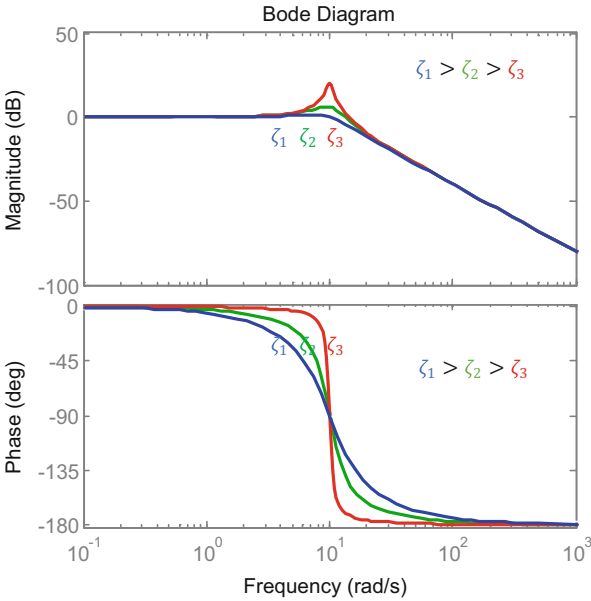


Fig. 8.67 Amplitude and phase diagram of $H(s) = \frac{5}{700}$ ①, $H(s) = (1 + \frac{s}{5})$ ②, $H(s) = (1 + \frac{s}{7})$ ③, and $H(s) = (1 + \frac{s}{10})^2$ ④ separately and combined

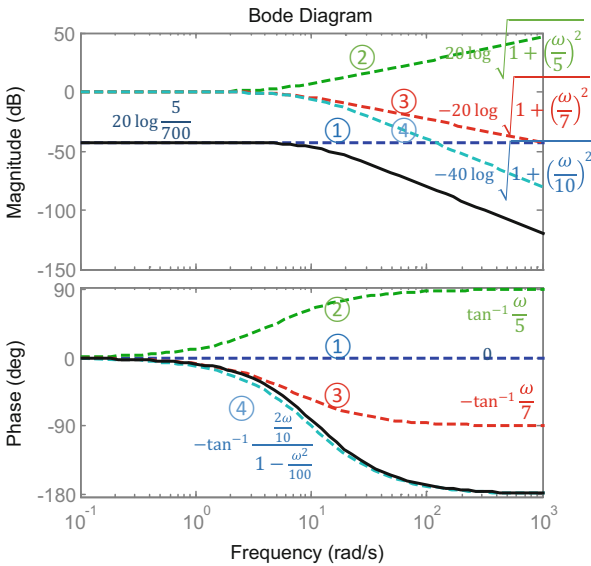
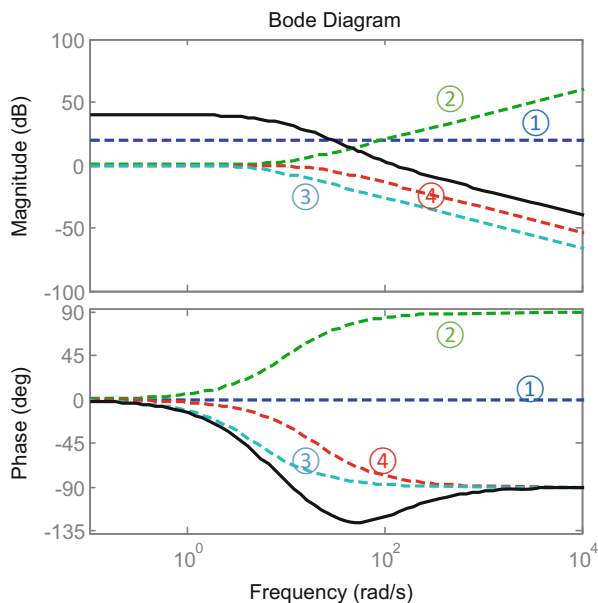


Fig. 8.68 Amplitude and phase diagram of $H(s) = \frac{100 \times 10}{5 \times 20}$ ①, $H(s) = (1 + \frac{s}{10})$ ②, $H(s) = (1 + \frac{s}{5})$ ③, and $H(s) = (1 + \frac{s}{20})$ ④, separately and combined



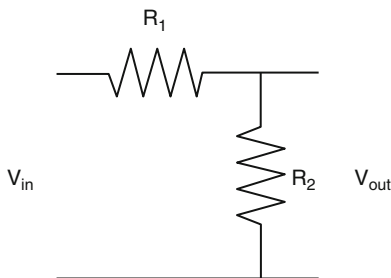
The function has four elements of $\frac{100 \times 10}{5 \times 20}$ as DC gain ①, zero of $(1 + \frac{s}{10})$ ②, two pole elements of $(1 + \frac{s}{5})$ ③, and $(1 + \frac{s}{20})$ ④.

A plot of these four elements and the overall summation of these elements is shown in 8.68.

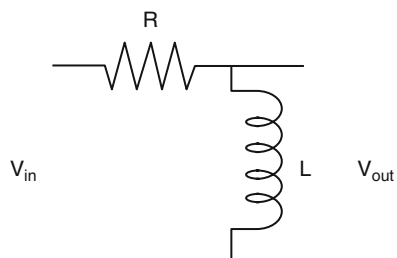
Problems

Find the transfer function of the following circuits as they are defined for each case:

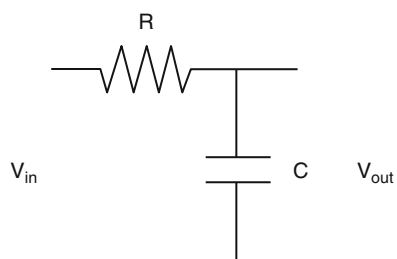
8.1. Find $H(s) = \frac{V_o}{V_{in}}$



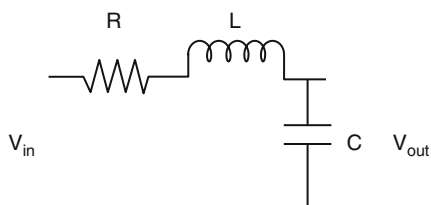
8.2. Find $H(s) = \frac{V_o}{V_{in}}$



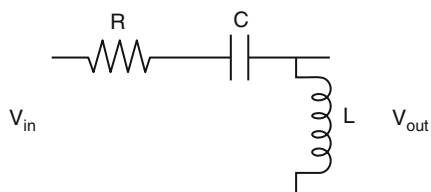
8.3. Find $H(s) = \frac{V_o}{V_{in}}$



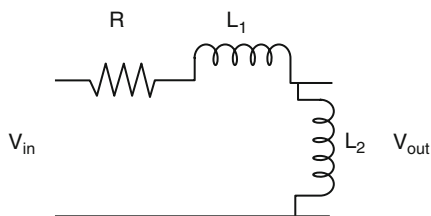
8.4. Find $H(s) = \frac{V_o}{V_{in}}$



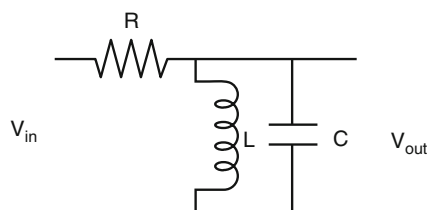
8.5. Find $H(s) = \frac{V_o}{V_{in}}$



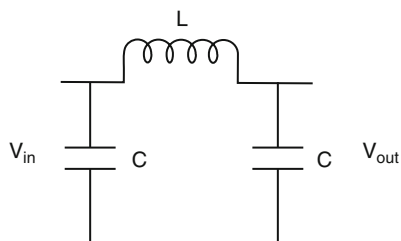
8.6. Find $H(s) = \frac{V_o}{V_{in}}$



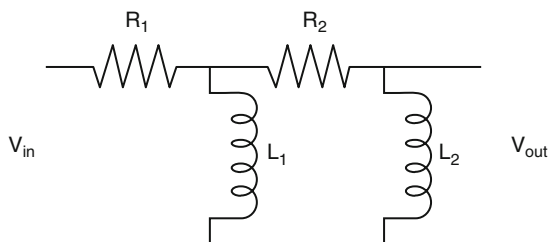
8.7. Find $H(s) = \frac{V_o}{V_{in}}$



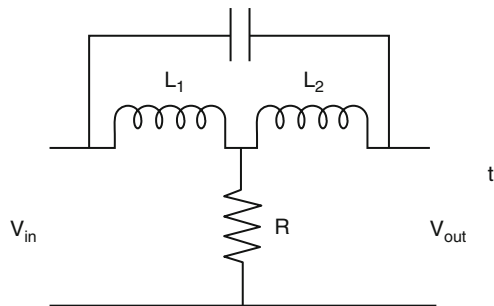
8.8. Find $H(s) = \frac{V_o}{V_{in}}$



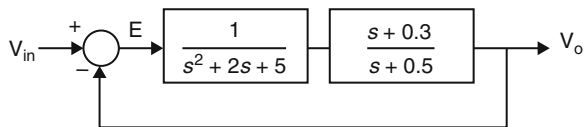
8.9. Find $H(s) = \frac{V_o}{V_{in}}$



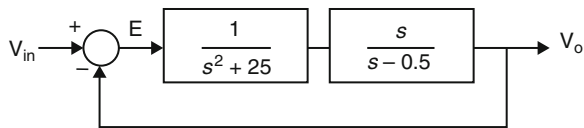
8.10. Find $H(s) = \frac{V_o}{V_{in}}$



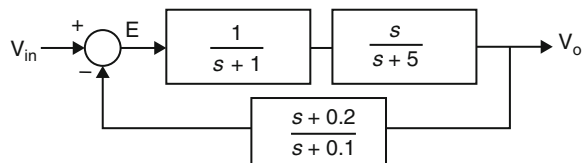
8.11. Find the transfer function $H(s) = \frac{V_o}{V_{in}}$.



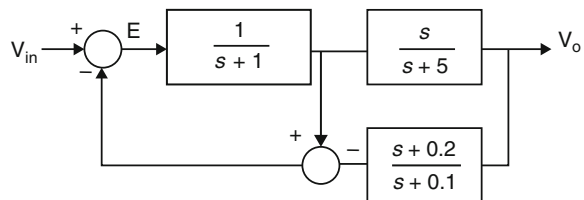
8.12. Find the transfer function $H(s) = \frac{V_o}{V_{in}}$.



8.13. Find the transfer function $H(s) = \frac{V_o}{V_{in}}$.



8.14. Find the transfer function $H(s) = \frac{V_o}{V_{in}}$.



- 8.15. Find poles and zeroes of the open loop and closed systems described in Problems 8.11–8.14.
- 8.16. Find the transfer function of the following systems, where in x is input and y is the output:
- (a) $\dot{y} + 5y = 5x$
 - (b) $\ddot{y} + 9y = \dot{x} + x$
 - (c) $\ddot{y} + 2\dot{y} = \ddot{x} + 2\dot{x} + 5x$
 - (d) $\ddot{y} + 7\ddot{y} + 2\dot{y} = \dot{x}$
 - (e) $\ddot{y} - 2\dot{y} - 3y = \ddot{x} + 2x$
- 8.17. Find poles and zeroes of the transfer functions described in Problem 8.16a–e.
- 8.18. A dynamic system with differential equation that relates the output y to the input x is:

$$\ddot{y} + 2\dot{y} = \ddot{x} + 2\dot{x} + 5x$$

Find the transfer function of a unity-negative feedback closed-loop system defined as $H(s) = \frac{Y(s)}{X(s)}$.

- 8.19. In the systems of problem 8.18:
- (a) Find the poles and zeros of the transfer functions.
 - (b) Determine whether the poles are in the right half plane or the left half plane.
 - (c) Determine the stability of the systems.
- 8.20. The systems of problem 8.16 are located in a unity-negative feedback closed-loop system.
- (a) Find the transfer function of the closed-loop system.
 - (b) Find the poles and zeros of the closed-loop system.
 - (c) Compare the poles obtained in problem 8.17 with the ones from the closed loop.
 - (d) Determine whether the systems have changed their stability.
- 8.21. Consider a gain K in the closed-loop negative feedback system (problem 8.16). Find the K or a limit for K such that the poles of the closed-loop systems obtained in problem 8.15 become stable.
- 8.22. Find the initial values and final values in the transfer functions of problem 8.15.
- 8.23. Find the initial values and final values in the transfer functions of problem 8.16.

8.24. Find the error value in the negative feedback closed-loop systems of problem 8.16 when:

- (a) A unit step function is applied in the input.
- (b) A ramp function is applied in the input.
- (c) $f(t) = t^2$ is applied in the input.

8.25. The systems $h(t)$ is defined in each case (below) and is excited by the given input $x(t)$. Find the output signals $y(t)$ using convolution integral and verify your results by its Laplace approach.

- (a) $h(t) = 2e^{-10t}$, $x(t) = u(t)$
- (b) $h(t) = 2u(t - 5) - 2u(t - 10)$, $x(t) = u(t)$
- (c) $h(t) = 2u(t - 5) - 2u(t - 10)$, $x(t) = tu(t) - 2(t - 2)u(t - 2) + (t - 3)u(t - 3)$
- (d) $h(t) = 2u(t - 5) - 2u(t - 10)$, $x(t) = u(t) + u(t - 1) - 2u(t - 2)$

8.26. Find the state space representation of the circuits shown in problems 8.1–8.10.

8.27. Find the state space representation of the following differential equations, where x is the input and y is the output. Put the models in matrix form $\dot{X} = AX + Bu$, $y = CX + Du$:

- (a) $\dot{y} + 5y = 5u$
- (b) $\ddot{y} - 2\dot{y} - 3y = 2u$
- (c) $\ddot{y} + 9y = \dot{u} + u$
- (d) $\ddot{y} + 2\dot{y} = \ddot{u} + 2\dot{u} + 5u$
- (e) $\ddot{y} + 7\dot{y} + 2y = \dot{u}$
- (f) $\ddot{y} - 2\dot{y} - 3y = \ddot{u} + 2u$

8.28. State space models found in Problem 8.27a–f are used in a unity feedback system. Find the state space model of the closed-loop system.

8.29. Show the state space representation of Problem 8.27 in block diagram format.

8.30. Determine the M_p , t_p , t_r , t_{s5} , t_{s2} , t_d , and sketch the step response of the following systems:

- (a) $H(s) = \frac{100}{s^2 + 10s + 100}$
- (b) $H(s) = \frac{1}{s^2 + 0.2s + 1}$
- (c) $H(s) = \frac{900}{s^2 + 12s + 900}$

8.31. Find transfer function of the following systems

$$(a) \begin{aligned} \dot{X} &= \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} X \end{aligned}$$

$$(b) \quad \dot{X} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]X$$

$$(c) \quad \dot{X} = \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]X$$

$$(d) \quad \dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]X$$

$$(e) \quad \dot{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -6 & -11 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad -1]X$$

8.32. Find Bode diagram of the following transfer functions:

$$(a) \quad H(s) = 2s$$

$$(b) \quad H(s) = \frac{2}{s}$$

$$(c) \quad H(s) = s+1$$

$$(d) \quad H(s) = \frac{1}{s+1}$$

$$(e) \quad H(s) = s+100$$

$$(f) \quad H(s) = \frac{1}{s+100}$$

$$(g) \quad H(s) = \frac{s}{s+100}$$

$$(h) \quad H(s) = \frac{s+3}{s+5}$$

$$(i) \quad H(s) = \frac{1}{s(s+20)}$$

$$(j) \quad H(s) = \frac{1}{s^2(s+20)}$$

$$(k) \quad H(s) = \frac{50}{s+0.1}$$

$$(l) \quad H(s) = \frac{100(s+40)}{(s+15)(s+25)}$$

$$(m) \quad H(s) = \frac{100(s+40)}{(s+15)(s+25)^2}$$

$$(n) \quad H(s) = \frac{(s+4)}{(s+5)^3(s+10)}$$

$$(o) \quad H(s) = \frac{100}{(s^2+15s+100)}$$

$$(p) \ H(s) = \frac{100(s+25)}{(s^2+15s+100)}$$

$$(q) \ H(s) = \frac{100}{(s^2+15s+100)(s+25)}$$

$$(r) \ H(s) = \frac{100}{(s^2+15s+100)^2(s+25)}$$

$$(s) \ H(s) = (s^2+10s+100)^2$$

Chapter 9

Passive Filters



Introduction

Frequency-selective circuits or “filters” are circuits that either pass or attenuate signals at a specific frequency or a range of desired frequencies. These filters can be implemented in hardware to be programmed in software that runs on a processor to accomplish similar tasks. For instance, consider old cassette players, or video players, in which several set of filters are required to make the quality of sound or video expected from high-quality sets, whereas consider a modern MP3 player, or a digital satellite receiver, in which the same or even better quality of signal processing is expected, but the sets have no actual hardware to accomplish the signal filtration. In this chapter, both hardware circuits and methods to implement the filters in software are discussed.

Passive and Active Filters

When a signal is passed through a filter, its amplitude may drop a small percentage because the circuit elements are not ideal and the signal may lose some power. Therefore, the filters are categorized into passive (with no amplification) or active (with amplification). This chapter discusses passive filters and Chap. 11 discusses active filters.

Category of Passive Filter Circuits

To determine the category of filters, the ratio of the output voltage to input voltage of a filter or its gain is analyzed over a range of frequency at the input. The gain variation respect to the frequency is called frequency response. Bode diagrams also

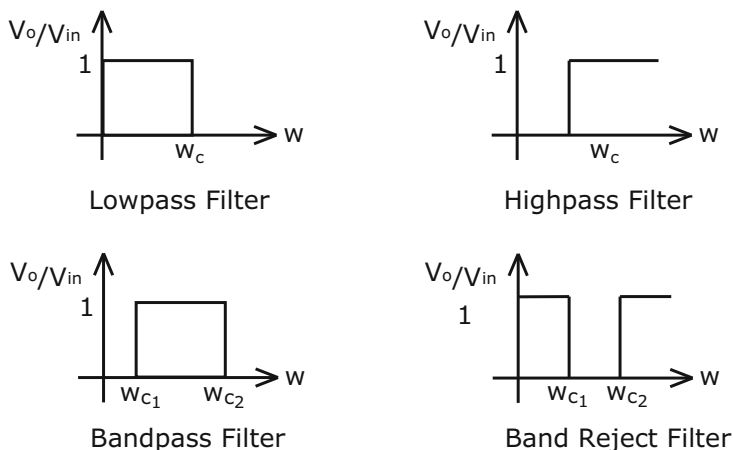


Fig. 9.1 Frequency response of ideal filters, their cutoff frequencies, passing band, and rejecting band. The filter passes a range of frequencies if at those frequencies the output receives majority of the input signal or ideally 100% of it ($\frac{V_o}{V_{in}} = 1$). The filter rejects a range of frequencies if at those frequencies the output is attenuated and the power of the signal is reduced significantly. Ideally, at the rejection band, the output is zero or ($\frac{V_o}{V_{in}} = 0$). Accordingly, the response of various filters can be recognized

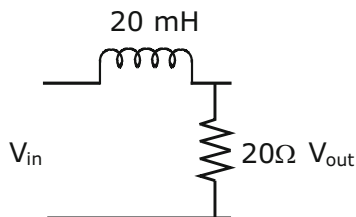
show the frequency response of a circuit. There are several frequencies that directly contribute to characteristics of a filter. Some of these frequencies are discussed here. All filters have a range of frequencies at which they fully pass, or they fully stop. The frequencies at which a filter switches from passing to no pass or vice versa, are called cutoff frequencies and are shown by ω_c . These frequencies are also called corner frequency. Based on the frequency response of a certain circuit, there are four major types of filters. First is the one to pass all frequencies below a certain frequency or low-pass filters, second is a filter to pass all signals above a certain frequency or high-pass filter, third is the filter to pass all signals in a desired band of frequency or band-pass filter, fourth is a filter to reject all signals in a band of frequency or band-reject filter. Figure 9.1 shows the ideal operation of these filters. It is desired that a filter drop the output voltage to zero at the no pass range and pass 100% of the signal in the pass range.

As it is shown, high-pass and low-pass filters have one cutoff frequency each. However, band-pass and band-reject filters have two cutoff frequencies. The band of frequencies determines the bandwidth of the filter. This is all the range of frequencies at which the filter passes the signals (signals at these frequencies pass through the filter). This bandwidth is obtained as the difference between two cutoff frequencies.

It is required to revisit the impedance of inductors and capacitors to better understand the operation of a filter.

Impedance of an inductor $L(H)$ respect to a signal at frequency ω is $Z = jX_L = j\omega L (\Omega)$. It shows that if the frequency of the signal is increased, the impedance of the inductor also increases. At low frequencies, the inductor shows a

Fig. 9.2 Circuit of Example 9.1



very low impedance and passes the low frequency signals and at high frequencies, the impedance of the inductor is increased to high values that can block the signals from passing. Of course, the use of term frequency is relative to the operation of the filter, and the cutoff frequency is determined based on the inductance and other circuit parameters.

Impedance of a capacitor C (F) is inversely proportional to the frequency a signal shown as $Z = -jX_C = \frac{-j}{C\omega}$. It means that a capacitor blocks low-frequency signals and passes high-frequency signals because at low frequencies the capacitor demonstrates a significant ohmic impedance that can block the signals and at high frequencies the ohmic impedance of the capacitor shows a short circuit that allows signals to pass.

Based on these facts, an inductor is shown with a low-frequency (LF) arrow on the top $\vec{L}^{LF}$, and a capacitor is shown with high-frequency (HF) arrow on the top $\vec{C}^{HF}$.

There are several techniques to determine the cutoff frequency of a filter. In this section, sinusoidal signal analysis is used, and in the next section, Laplace technique is used to analyze the filters.

Example 9.1 In an RL circuit, the input frequency is variable. Find the output voltage at the given frequencies when $V_{in} = 10$ at frequency $\omega = 1, 10, 100, 1000, 2000$, and 10000 ($\frac{\text{rad}}{\text{s}}$) (Fig. 9.2).

Solution The circuit shows a voltage divider between the 20Ω resistor and the $j\omega L$ inductor. The output voltage is:

$$V_o = \frac{20}{20 + j0.02\omega} V_{in}$$

The amplitude of the output voltage becomes:

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02\omega)^2}} 10$$

At $\omega = 1$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02)^2}} 10 \cong 10$$

At $\omega = 10$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02 \times 10)^2}} 10 \cong 10$$

At $\omega = 100$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02 \times 100)^2}} 10 = 9.95$$

At $\omega = 1000$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02 \times 1000)^2}} 10 = 7.07$$

At $\omega = 2000$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02 \times 2000)^2}} 10 = 4.47$$

At $\omega = 10000$

$$|V_o| = \frac{20}{\sqrt{20^2 + (0.02 \times 10000)^2}} 10 \cong 1$$

As it is observed, the output voltage decreases as the source frequency is increased. Up to $\omega = 1000$ rad/s, the output voltage is high and almost passes to the output without much change. At the frequency $\omega = 1000$ rad/s, the output becomes 70.7% of the input which is known as cutoff frequency (in actual filters. Ideally the signal would become zero). The output voltages above this frequency drop rapidly to reach very low values at high frequencies.

Filter Gains

At a certain frequency, filters can change the power of a signal with respect to the input power of the same signal. This gain can be positive, meaning the output power has increased, and can be negative, meaning that the output power has decreased. The gain of power G_P measured in decibel (dB) is as follows:

$$G_P = 10\log\frac{P_o}{P_{in}} \text{ (dB)}$$

where P_o is the output power and P_{in} is the input power. Considering the input voltage and output voltage signals at the same frequency, the power of these signals is proportional to the square of their voltages or:

$$P_o \propto V_o^2$$

$$P_{in} \propto V_{in}^2$$

Hence, the voltage gain of filter can be expressed as:

$$G_V = 10\log\frac{P_o}{P_{in}} = 10\log\frac{V_o^2}{V_{in}^2} = 10\log\left(\frac{V_o}{V_{in}}\right)^2 = 20\log\frac{V_o}{V_{in}} \text{ (dB)}$$

$$G_V = 20\log\frac{V_o}{V_{in}}$$

Measuring the gains based on the current measurement results in the following calculations:

$$P_o \propto I_o^2$$

$$P_{in} \propto I_{in}^2$$

Hence, the current gain of the filter can be expressed as:

$$G_I = 10\log\frac{P_{in}}{P_o} = 10\log\frac{I_o^2}{I_{in}^2} = 10\log\left(\frac{I_o}{I_{in}}\right)^2 = 20\log\frac{I_o}{I_{in}} \text{ (dB)}$$

$$G_I = 20\log\frac{I_o}{I_{in}}$$

Example 9.2 A 100 kHz signal at a voltage of 16 mV passes a filter and is measured at 8mV at the output. Find the gain of the filter.

Solution Gain in dB is obtained as follows:

$$G_V = 20\log\frac{V_o}{V_{in}}$$

$$G_V = 20\log\frac{8 \text{ m}}{16 \text{ m}} = -6 \text{ dB}$$

Example 9.3 Find the power gain of a 1dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = +1$$

$$\frac{P_o}{P_{in}} = 10^{\frac{1}{10}} = 1.26$$

It means that the output power has increased by 26%.

Example 9.4 Find the power gain of a + dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = +2$$

$$\frac{P_o}{P_{in}} = 10^{\frac{2}{10}} = 1.58$$

It means that the output power has increased by 58%.

Example 9.5 Find the power gain of a 3dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = +3$$

$$\frac{P_o}{P_{in}} = 10^{\frac{3}{10}} = 2$$

It means that the output power has increased by 100%.

Example 9.6 Find the power gain of a -1 dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = -1$$

$$\frac{P_o}{P_{in}} = 10^{\frac{-1}{10}} = 0.79$$

It means that the output power has dropped by 21%.

Example 9.7 Find the power gain of a -2dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = -2$$

$$\frac{P_o}{P_{in}} = 10^{\frac{-2}{10}} = 0.63$$

It means that the output power has dropped by 37%.

Example 9.8 Find the power gain of a -3 dB filter.

Solution

$$G_P = 10 \log \frac{P_o}{P_{in}} = -3$$

$$\frac{P_o}{P_{in}} = 10^{\frac{-3}{10}} = 0.5$$

It means that the output power has dropped by 50%.

Cutoff and Half-Power Point Frequencies

A summary of the Examples 9.1, 9.2, 9.3, 9.4, 9.5, 9.6, and 9.7 is given in the table as follows:

Power gain (dB)	Power ratio	% of output power change
+1	1.26	26% increase
+2	1.58	58% increase
+3	2	100% increase
-1	0.79	21% decrease
-2	0.63	37% decrease
-3	0.5	50% decrease

At the gain of -3 dB, the power of the signal is cut in half. Therefore, the frequency at which the power of a signal is dropped in half is called “the half-power point” frequency.

This is the frequency at which filters also show effective action of filtration. This frequency was also called the “cutoff” frequency.

The voltage and current gain at the cutoff frequency become:

$$-3 = 20 \log \frac{V_o}{V_{in}}$$

$$\frac{V_o}{V_{in}} = 10^{\frac{-3}{20}} = 0.707 \text{ or } \frac{1}{\sqrt{2}}$$

It means that if a signal is tuned at the cutoff frequency, its output voltage becomes 0.707 of the signal’s input voltage.

Low-Pass Filter

A LPF is designed to pass all frequencies below the cutoff frequency. It ideally has amplitude of 100% (of input appear in the output). However, in reality, the amplitude starts to drop before the cutoff frequency. In this case, the cutoff frequency is identified when the transfer function gain drops 3 dB or the output signal amplitude reaches 70.7% of the input signal.

Inductors and capacitors can be used to realize a low-pass filter. These circuit topologies may be different, but their operation characteristics are the same. They may be tuned to attenuate signals above a certain frequency or cutoff frequency.

There is an analysis procedure repeated for all types of filters, as follows:

- Find the transfer function of the circuit.
- Find the cutoff frequencies using the transfer function.
- Find the cutoff frequencies using the $\frac{1}{\sqrt{2}} H_{\max}$ rule.
- Find the bandwidth and resonant frequencies.
- Determine the frequency response in the form of Bode diagram (dB) and absolute value of amplitude.
- Double check the Bode diagram through evaluation of extreme frequency response $s \rightarrow 0$ & $s \rightarrow \infty$.

RL Low-Pass Filter

To pass low-frequency signals to the output of a filter, an inductor may be used to bridge the input to output ports. This circuit element parameter is tuned to obtain the desired cutoff frequency beyond the cutoff frequency, higher frequency signals generate high impedance and are automatically filtered out (Fig. 9.3).

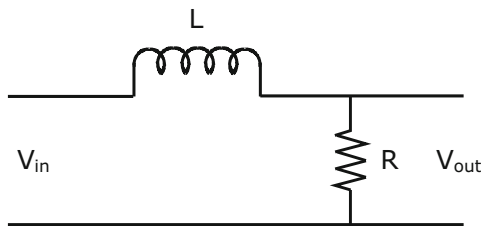


Fig. 9.3 An *RL* circuit to form a low-pass filter. Inductors pass low frequencies, and if connected between the source and the output, they provide a path to pass the low frequencies to the output, hence, forming a low-pass filter. A combination of resistance and inductance values determines the cutoff frequencies

Identifying the Cutoff Frequency from the Ratio of the Output over Input

At the cutoff frequency, the amplitude of the transfer function becomes 70.7% of the maximum or $\frac{1}{\sqrt{2}} H_{\max}$. $H(j\omega)$ is the transfer function of the filter or simply the ratio of the output voltage over the input voltage. Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output can reach 1. Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{V_o(j\omega)}{V_{in}(j\omega)} = \left| \frac{\frac{R}{L}}{j\omega_c + \frac{R}{L}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{\frac{R}{L}}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\frac{\left(\frac{R}{L}\right)^2}{\omega_c^2 + \left(\frac{R}{L}\right)^2} = \frac{1}{2}$$

$$\omega_c = \pm \frac{R}{L}$$

which positive frequency is acceptable. Therefore:

$$\omega_c = \frac{R}{L}$$

Note 9.1 The cutoff frequency of a first-order filter is the inverse of its time constant. In this RL circuit, the time constant is $\tau = \frac{L}{R}$, therefore, the cutoff frequency is $\omega_c = \frac{1}{\tau}$ or $\omega_c = \frac{1}{\frac{L}{R}} = \frac{R}{L}$.

Using Laplace Transform to Find the Cutoff Frequency

The output voltage is obtained by a voltage division between R in the output and L in the bridge to the output. The transfer function is as follows:

$$\frac{V_o}{V_{in}} = \frac{R}{R + sL} = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

This transfer function has no zeros and one pole at $s + \frac{R}{L} = 0$ or:

$$s = -\frac{R}{L}$$

The amplitude of Bode diagram drops at the location of pole by 3 dB. This qualifies the pole ($\frac{R}{L}$) to be the cutoff frequency. Therefore:

$$j\omega = -\frac{R}{L}$$

$$\omega_c = \frac{R}{L} \left(\frac{\text{rad}}{\text{s}} \right)$$

To obtain an approximate frequency response of the circuit, the amplitude can be checked in two extreme frequencies of $\omega = 0$ and $\omega = \infty$ as follows:

$$\left| \frac{V_o}{V_{in}}(j0) \right| = \lim_{s \rightarrow 0} \left| \frac{\frac{R}{L}}{s + \frac{R}{L}} \right| = 1$$

This means 100% of the input signal appears in the output.

$$\left| \frac{V_o}{V_{in}}(j\infty) \right| = \lim_{s \rightarrow \infty} \left| \frac{\frac{R}{L}}{s + \frac{R}{L}} \right| = 0$$

Bode plot of the transfer function $H(s) = \frac{\frac{R}{L}}{s + \frac{R}{L}}$ is obtained as Fig. 9.4. There is just one element of pole at $\frac{R}{L}$ with unity gain:

Utilizing the Bode diagram to identify the cutoff frequency requires identifying the frequency at which a -3 dB drop from the maximum amplitude occurs. Drawing a straight horizontal line of -3 dB cuts the amplitude and identifies the frequency as $\omega_c = \frac{R}{L}$.

The circuit phase changes from 0° to -90° , and at the location of pole (cutoff frequency), the phase reaches -45° .

Cutoff frequency can also be obtained from the ratio of the output over input when it reaches 70.7%.

Example 9.9 Design an RL LPF to cut frequencies higher than 1000 rad/s. Consider a 20 mH inductor (Fig. 9.6).

Solution An RL LPF circuit has the cutoff frequency of $\omega_c = \frac{R}{L}$ which is desired to be 1k rad/s.

Fig. 9.4 Frequency response of a non-ideal first order low-pass filter. For the frequencies above the cutoff frequency, the power of signals drops below an effective level. Hence, they are called rejected. Magnitude in this frequency response is shown in terms of decibel. At the cutoff frequency, a -3 dB (drop) is observed at the magnitude of the signals. Note that the phase shift reaches a -45° drop by the cutoff frequency and is saturated to -90° at frequencies a decade higher than ω_c (Fig. 9.5)

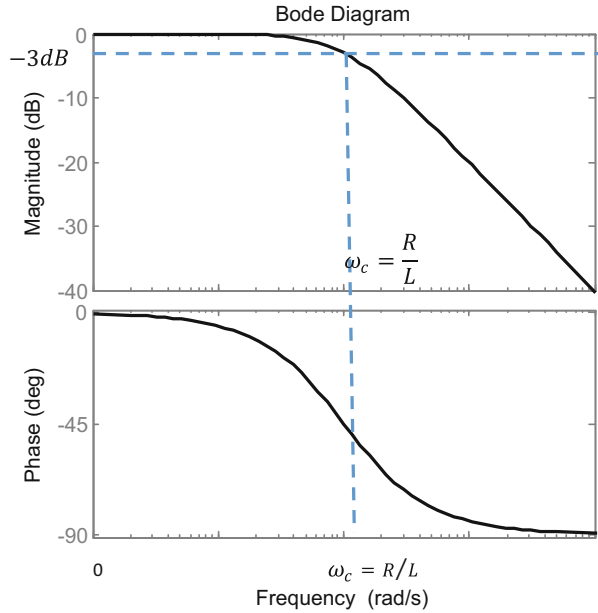


Fig. 9.5 The frequency response of a low-pass filter formed with an RL circuit. The magnitude of the signal is shown in absolute values. This shows the ratio of the voltages observed at the output. A 0.707 or 70.7% ratio determines the cutoff frequency. Note that the phase shift reaches a -45° drop by the cutoff frequency and is saturated to -90° at frequencies a decade higher than ω_c

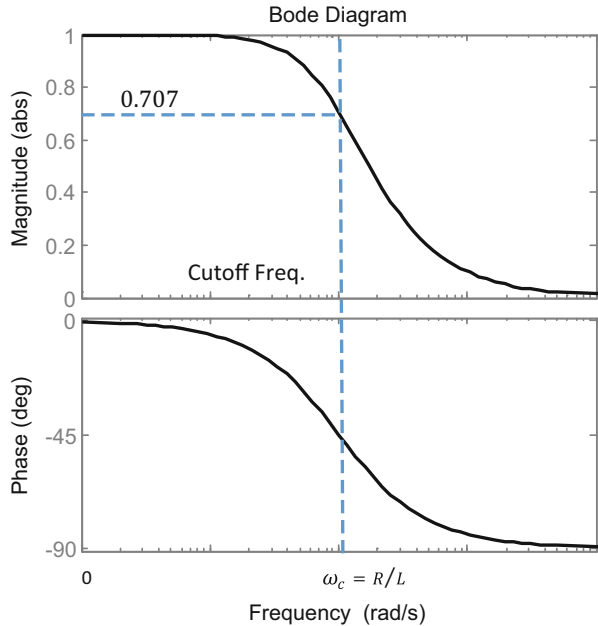
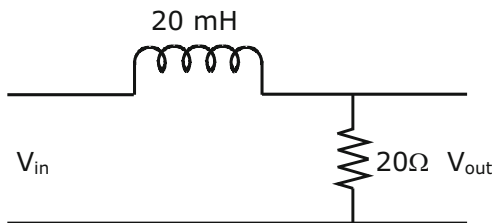


Fig. 9.6 Circuit of Example 9.1



$$\frac{R}{20e - 3} = 1000$$

This results in:

$$R = 20 \, \Omega$$

The transfer function of the circuit is:

$$H(s) = \frac{1000}{s + 1000}.$$

The frequency response of this transfer function is obtained by:

RC Low-Pass Filter

As explained earlier, capacitors conduct high frequencies. If a capacitor is connected such that the frequencies it is passing are connected to ground, then only low frequencies reach the output. Figure 9.7 shows an RC circuit that forms a low-pass filter.

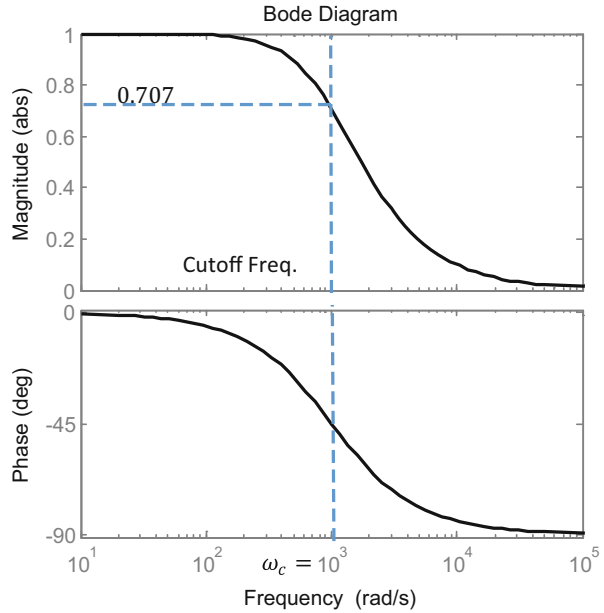
Identifying the Cutoff Frequency from the Ratio of the Output over Input

At the cutoff frequency, the amplitude of the transfer function becomes 70.7% of the maximum or $\frac{1}{\sqrt{2}} H_{\max}$. Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output can reach 1. Therefore:

Fig. 9.7 Frequency response of the circuit
Example 9.1



$$|H(j\omega)|_{\omega=\omega_c} = \frac{V_o(j\omega)}{V_{in}(j\omega)} = \left| \frac{\frac{1}{RC}}{j\omega_c + \frac{1}{RC}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{\frac{1}{RC}}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\frac{\left(\frac{1}{RC}\right)^2}{\omega_c^2 + \left(\frac{1}{RC}\right)^2} = \frac{1}{2}$$

$$\omega_c = \pm \frac{1}{RC}$$

which positive and negative frequencies show the same frequency amplitude.
Therefore:

$$\omega_c = \frac{1}{RC}$$

Note 9.2 The cutoff frequency of a first-order filter is the inverse of its time constant. In this RC circuit, the time constant is $\tau = RC$; therefore, the cutoff frequency is $\omega_c = \frac{1}{\tau}$ or $\omega_c = \frac{1}{RC}$.

Using Laplace Transform to Find the Cutoff Frequency

The transfer function is as follows:

$$\frac{V_o}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

This transfer function has no zeros and one pole at $s + \frac{1}{RC} = 0$ or:

$$s = -\frac{1}{RC}$$

The amplitude of Bode diagram drops at the location of pole by 3 dB. This qualifies the pole ($\frac{1}{RC}$) to be the cutoff frequency. Therefore:

$$j\omega = -\frac{1}{RC}$$

$$\omega_c = \frac{1}{RC} \left(\frac{\text{rad}}{\text{s}} \right)$$

To obtain an approximate frequency response of the circuit, the amplitude can be checked in two extreme frequencies of $\omega = 0$ and $\omega = \infty$ as follows:

$$\left| \frac{V_o}{V_{in}}(j0) \right| = \lim_{s \rightarrow 0} \left| \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right| = 1$$

This means 100% of the input signal appears in the output.

$$\left| \frac{V_o}{V_{in}}(j\infty) \right| = \lim_{s \rightarrow \infty} \left| \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right| = 0$$

Bode plot of the transfer function $H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$ is obtained as Fig. 9.8. There is just one element of pole at $\frac{1}{RC}$ with unity gain (Figs. 9.9 and 9.10):

Utilizing the Bode diagram to identify the cutoff frequency requires identifying the frequency at which a -3 dB drop from the maximum amplitude occurs. Drawing a straight line of -3 dB cuts the amplitude and identifies the frequency as $\omega = \frac{1}{RC}$.

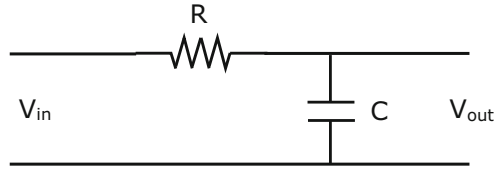
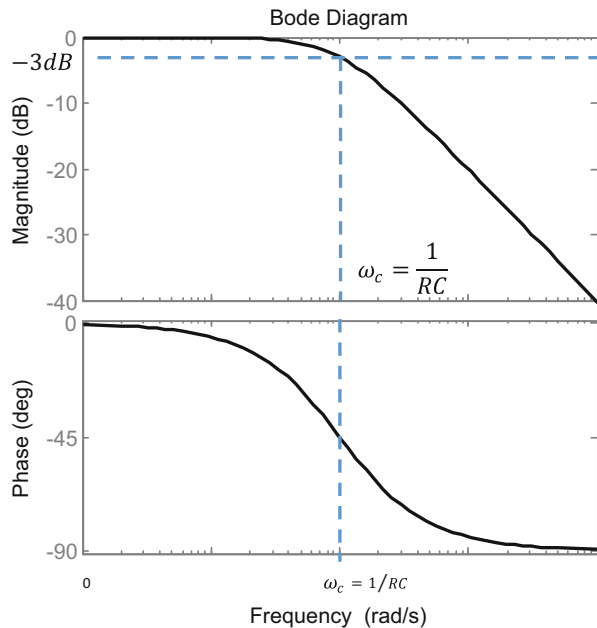


Fig. 9.8 An RC circuit to form a low-pass filter. Capacitors pass high frequencies. If they are connected to direct all high frequencies to the ground, the output equals a short circuit (or very low impedance) at high frequencies. Hence, the voltage measured at the output is nearly zero at high frequencies. A combination of the resistance and capacitance determines the cutoff frequency

Fig. 9.9 The frequency response of a low-pass filter either built by an RL or an RC circuit is similar. The parametric expression of the cutoff frequency might be different. Two different circuits might have identical frequency responses. Magnitude at this frequency response is shown in terms of decibel. At the cutoff frequency, a -3 dB (drop) is observed at the magnitude of the signals. Note that the phase shift reaches a -45° drop by the cutoff frequency and is saturated to -90° at frequencies a decade higher than ω_c



The circuit phase changes from 0° to -90° , and at the location of pole (cutoff frequency), the phase reaches -45° .

Cutoff frequency can also be obtained from the ratio of the output over input when it reaches 70.7%.

Example 9.10 Design an RC LPF to cut frequencies higher than 1000 rad/s . Consider a $20\text{ }\mu\text{F}$ capacitor (Fig. 9.11).

Solution An RC LPF circuit has the cutoff frequency of $\omega_c = \frac{1}{RC}$ which is desired to be $1k\text{ rad/s}$.

Fig. 9.10 The frequency response of a low-pass filter formed with an RC circuit. The magnitude of the signal is shown in absolute values. This shows the ratio of the voltages observed at the output. A 0.707 or 70.7% ratio determines the cutoff frequency. Note that the phase shift reaches a -45° drop by the cutoff frequency and is saturated to -90° at frequencies a decade higher than ω_c .

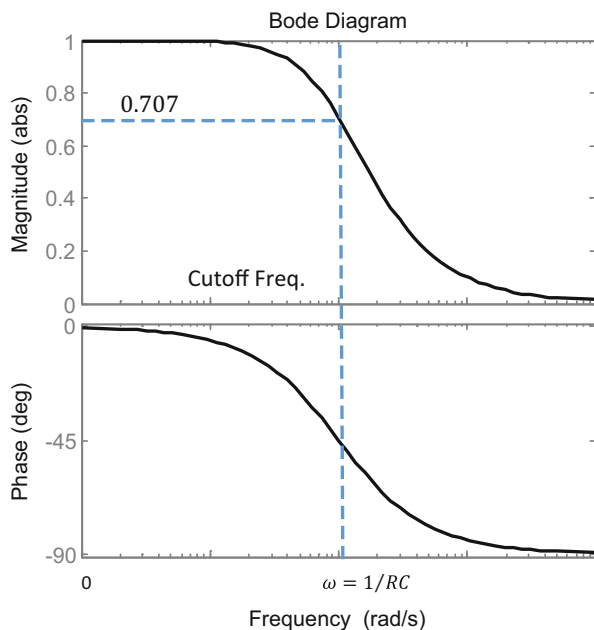
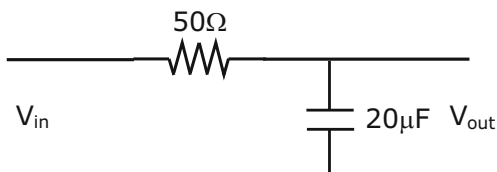


Fig. 9.11 Circuit for Example 9.2



$$\frac{1}{R20e-6} = 1000$$

This results in:

$$R = 50\Omega$$

The transfer function of the circuit is:

$$H(s) = \frac{1000}{s + 1000}.$$

The frequency response of this transfer function is obtained by:

Note 9.3 First-order low-pass filters with cutoff frequency of ω_c have a transfer function template of:

$$H(s) = \frac{\omega_c}{s + \omega_c}.$$

Example 9.11 Design a LPF to filter out frequencies higher than 200 Hz.

Solution The template for a LPF is:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

$$\omega_c = 2\pi f_c = 2\pi \times 200 = 1256.6 \frac{\text{rad}}{\text{s}}$$

Therefore, the filter becomes:

$$H(s) = \frac{1256.6}{s + 1256.6}$$

High-Pass Filter

A HPF is a circuit that eliminates frequencies below the cutoff frequency and allows to pass signals of higher frequency than the cutoff frequency. This characteristic can be obtained using an inductor- or a capacitor-based circuit.

RL HPF

A first-order *RL* HPF is shown in Fig. 9.12.

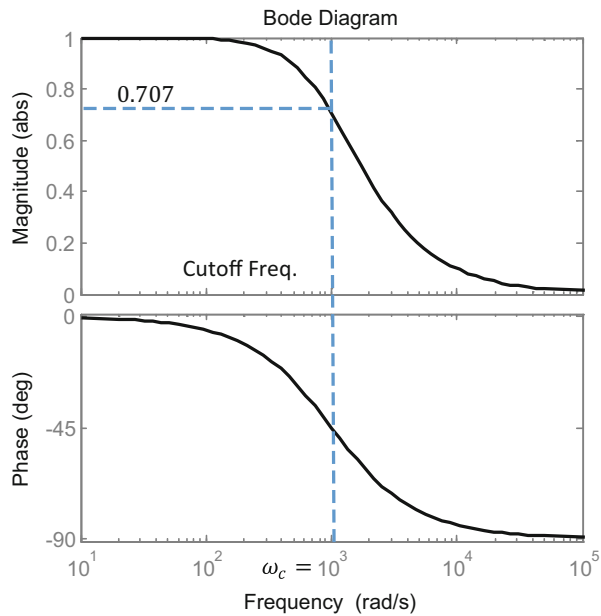
Identifying the Cutoff Frequency from the Ratio of the Output over Input

Obtaining cutoff frequency using the $\frac{1}{\sqrt{2}} H_{\max}$ rule:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output can reach 1. Therefore:

Fig. 9.12 Frequency response of the circuit for Example 9.2



$$|H(j\omega)|_{\omega=\omega_c} = \frac{V_o(j\omega)}{V_{in}(j\omega)} = \left| \frac{j\omega_c}{j\omega_c + \frac{R}{L}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{\omega_c}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\frac{\omega_c^2}{\omega_c^2 + \left(\frac{R}{L}\right)^2} = \frac{1}{2}$$

$$\omega_c = \pm \frac{R}{L}$$

which positive and negative frequencies show the same frequency amplitude. Therefore:

$$\omega_c = \frac{R}{L}$$

Using Laplace Transform to Find the Cutoff Frequency

The inductor passes low frequencies to the ground and allows a path for high-frequency signals to pass to the output. The circuit transfer function is obtained as:

$$H(s) = \frac{sL}{R + sL} = \frac{s}{s + \frac{R}{L}}$$

The circuit has one zero at the origin $s = 0$, and one pole of the circuit is $s + \frac{R}{L} = 0$ or $s = -\frac{R}{L}$. The cutoff frequency is therefore:

$$\omega_c = \frac{R}{L} \left(\frac{\text{rad}}{\text{s}} \right)$$

Replacing this frequency in the transfer function results in:

$$H(s) = \frac{s}{s + \omega_c}$$

Evaluating the transfer function on two extreme frequencies results in:

$$\left| \frac{V_o}{V_{in}}(j0) \right| = \lim_{s \rightarrow 0} \left| \frac{s}{s + \frac{R}{L}} \right| = 0$$

$$\left| \frac{V_o}{V_{in}}(j\infty) \right| = \lim_{s \rightarrow \infty} \left| \frac{s}{s + \frac{R}{L}} \right| = 1$$

This means that the circuit blocks all low frequencies and passes 100% of the high-frequency signals to the output.

Bode plot of the transfer function $H(s) = \frac{s}{s + \frac{R}{L}}$ is obtained as Fig. 9.13. There is just one element of pole at $\frac{R}{L}$ with unity gain (Fig. 9.14). Figure 9.15 shows the frequency response with normalized amplitude.

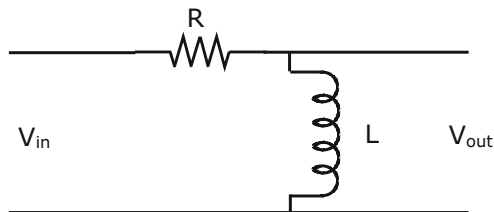


Fig. 9.13 An RL circuit to form a high-pass filter. Inductors pass low frequencies. If they are connected to direct all low frequencies to the ground, the output equals a short circuit (or very low impedance) at low frequencies. Hence, the voltage measured at the output is nearly zero at low frequencies. A combination of the resistance and inductance determines the cutoff frequency

Fig. 9.14 The frequency response of a high-pass filter by an RL circuit. Magnitude at this frequency response is shown in terms of decibel. At the cutoff frequency, a -3 dB (drop) is observed at the magnitude of the signals. Note that the phase shift starting at $+90^\circ$ reaches 45° by the cutoff frequency and is saturated to 0° at frequencies a decade higher than ω_c

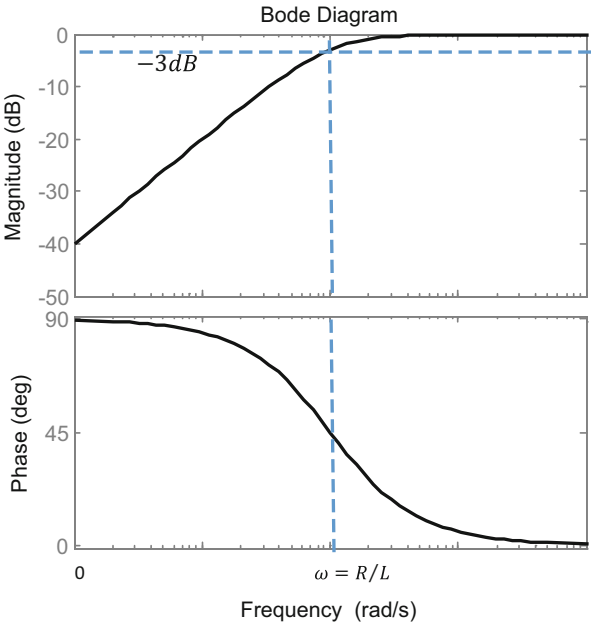
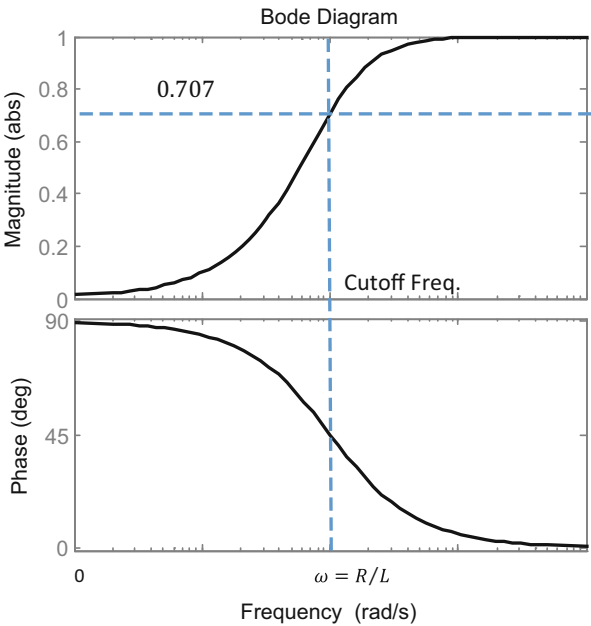


Fig. 9.15 The frequency response of a high-pass filter formed with an RL circuit. The magnitude of the signal is shown in absolute values. This shows the ratio of the voltages observed at the output. A 0.707 or 70.7% ratio determines the cutoff frequency. Note that the phase shift starting at $+90^\circ$ reaches 45° by the cutoff frequency and is saturated to 0° at frequencies a decade higher than ω_c



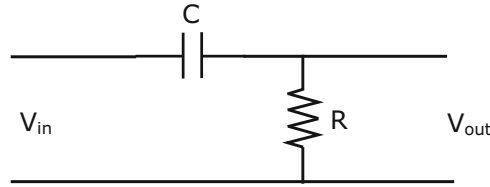


Fig. 9.16 An RC circuit to form a high-pass filter. Capacitors pass high frequencies. If they are connected directly between the source and the load, the output equals the source voltage at high frequencies. Hence, the voltage measured at the output is ideally the source voltage. A combination of the resistance and capacitance determines the cutoff frequency

RC High Pass Filter

Capacitors pass high-frequency signals and block low frequencies. Using a capacitor to bridge the input to the output of a resistive load provides path for high frequency signals to pass and low frequency signals to be rejected. Figure 9.16 shows an RC circuit that behaves as a high-pass filter.

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Obtaining cutoff frequency using the $\frac{1}{\sqrt{2}} H_{\max}$ rule:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output can reach 1. Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{V_o(j\omega)}{V_{in}(j\omega)} = \left| \frac{j\omega_c}{j\omega_c + \frac{1}{RC}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{\omega_c}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\frac{\omega_c^2}{\omega_c^2 + \left(\frac{1}{RC}\right)^2} = \frac{1}{2}$$

$$\omega_c = \pm \frac{1}{RC}$$

which positive frequency is acceptable. Therefore:

$$\omega_c = \frac{1}{RC}$$

Using Laplace Transform to Find the Cutoff Frequency

The circuit transfer function is obtained as:

$$H(s) = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{1 + RCs} = \frac{s}{s + \frac{1}{RC}}$$

The circuit has one zero at the origin $s = 0$, and one pole of the circuit is $s + \frac{1}{RC} = 0$, or $s = -\frac{1}{RC}$. The cutoff frequency is therefore:

$$\omega_c = \frac{1}{RC} \left(\frac{\text{rad}}{\text{s}} \right)$$

Replacing this frequency in the transfer function results in:

$$H(s) = \frac{s}{s + \omega_c}$$

Evaluating the transfer function on two extreme frequencies results in:

$$\begin{aligned} \left| \frac{V_o}{V_{in}}(j0) \right| &= \lim_{s \rightarrow 0} \left| \frac{s}{s + \frac{1}{RC}} \right| = 0 \\ \left| \frac{V_o}{V_{in}}(j\infty) \right| &= \lim_{s \rightarrow \infty} \left| \frac{s}{s + \frac{1}{RC}} \right| = 1 \end{aligned}$$

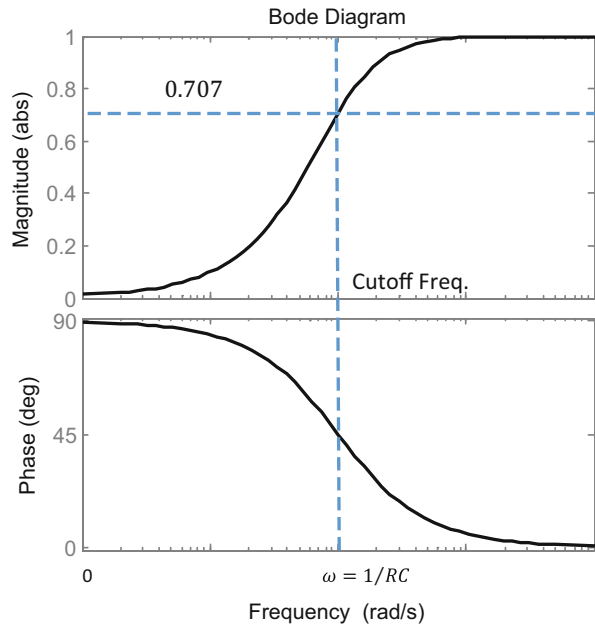
This means that the circuit blocks all low frequencies and passes 100% of the high-frequency signals to the output.

Bode plot of the transfer function $H(s) = \frac{s}{s + \frac{1}{RC}}$ is obtained as Fig. 9.17. There is just one element of pole at $\frac{1}{RC}$ with unity gain (Fig. 9.18).

Note 9.4 A first-order HPF with cutoff frequency of ω_c has a transfer function of:

$$H(s) = \frac{s}{s + \omega_c}.$$

Fig. 9.17 The frequency response of a high-pass filter by an RC circuit. Magnitude at this frequency response is shown in terms of absolute values. At the cutoff frequency, a -3 dB (drop) is observed at the magnitude of the signals. Note that the phase shift starting at $+90^\circ$ reaches 45° by the cutoff frequency and is saturated to 0° at frequencies a decade higher than ω_c



Example 9.12 Design a HPF to pass frequencies higher than 1500 Hz.

Solution Considering a first-order HPF template, the filter has a cutoff frequency of:

$$\omega_c = 2\pi f_c = 2\pi \times 1500 = 9429.7 \frac{\text{rad}}{\text{s}}$$

Therefore, the transfer function becomes:

$$H(s) = \frac{s}{s + 9429.7}$$

This circuit can be realized using RL or RC elements. Each of these frequencies results in a specific cutoff frequency expression as follows:

The cutoff frequency of an RL circuit is the inverse of time constant or $\omega_c = \frac{R}{L} = 9424.7$. Considering $L = 1\text{mH}$ results in $R = 9.424 \Omega$.

In an RC circuit, the cutoff frequency becomes $\omega_c = \frac{1}{RC} = 9424.7$. Considering $C = 1 \mu\text{F}$ results in $R = 106.10 \Omega$.

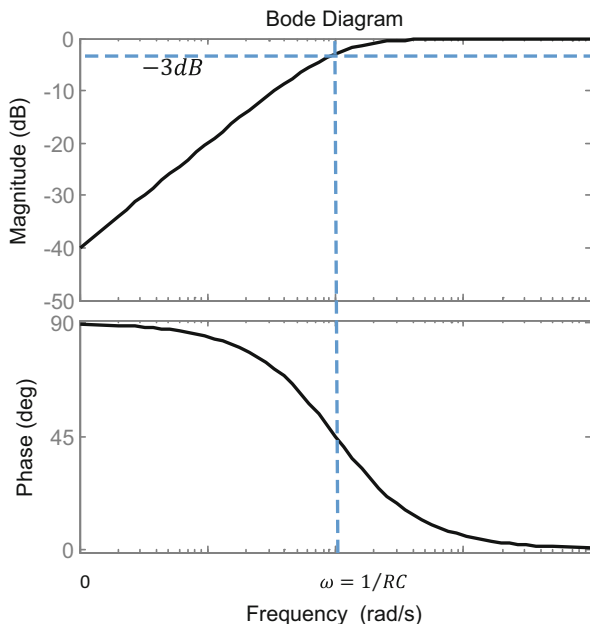


Fig. 9.18 The frequency response of a high-pass filter either built by an RL or an RC circuit is similar. The parametric expression of the cutoff frequency might be different. Two different circuits might have identical frequency responses. The frequency response of a high-pass filter by an RC circuit is shown. Magnitude at this frequency response is shown in terms of decibel. At the cutoff frequency, a -3 dB (drop) is observed at the magnitude of the signals. Note that the phase shift starting at $+90^\circ$ reaches 45° by the cutoff frequency and is saturated to 0° at frequencies a decade higher than ω_c .

Analysis of LC Circuits

Parallel LC Circuit

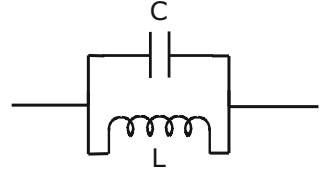
Consider a parallel connection of inductor L and capacitor C shown in Fig. 9.19.

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Impedance of this parallel unit is obtained as follows:

$$Z = \frac{1}{j\omega C} \parallel j\omega L = \frac{\frac{1}{j\omega C} j\omega L}{\frac{1}{j\omega C} + j\omega L} = \frac{\frac{1}{C} j\omega}{-\omega^2 + \frac{1}{LC}}$$

Fig. 9.19 A parallel LC circuit to be used as a building block forming other types of filters



This circuit has resonant frequency of $\frac{1}{\sqrt{LC}}$ at which it acts as an open circuit with impedance reaching infinite:

$$\lim_{\omega \rightarrow \frac{1}{\sqrt{LC}}} \left(\frac{1}{j\omega C} \parallel j\omega L \right) = \infty$$

This parallel unit reaches zero impedance at two extreme frequencies as follows:

$$\lim_{\omega \rightarrow 0} \left(\frac{1}{j\omega C} \parallel j\omega L \right) = 0 \quad \& \quad \lim_{\omega \rightarrow \infty} \left(\frac{1}{j\omega C} \parallel j\omega L \right) = 0$$

Parallel LC Circuit Using Laplace

Impedance of this parallel unit is obtained as follows:

$$Z = \frac{1}{sC} \parallel sL = \frac{\frac{1}{sC} sL}{\frac{1}{sC} + sL} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{LC}}$$

This circuit has resonant frequency of $\frac{1}{\sqrt{LC}}$ and a pole at the same frequency. Therefore, at the resonant frequency, this unit will act as an open circuit with impedance reaching infinite:

$$\lim_{s \rightarrow \frac{1}{\sqrt{LC}}} \left(\frac{1}{sC} \parallel sL \right) = \infty$$

This parallel unit reaches zero impedance at two extreme frequencies as follows:

$$\lim_{s \rightarrow 0} \left(\frac{1}{sC} \parallel sL \right) = 0 \quad \& \quad \lim_{s \rightarrow \infty} \left(\frac{1}{sC} \parallel sL \right) = 0$$

This unit can be utilized in circuits to bridge the input to output or connect across the output port. Either case, it should be used with this characteristics in mind that it becomes an open circuit at the resonant frequency.

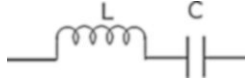


Fig. 9.20 A series LC circuit to be used as a building block forming other types of filters

Series LC Circuit

Consider a series connection of inductor L and capacitor C shown in Fig. 9.20.

Identifying the Cutoff Frequency from the Ratio of the Output Over Input

Impedance of this series unit is obtained as follows:

$$Z = \frac{1}{j\omega C} + j\omega L = \frac{1 - LC\omega^2}{j\omega C} = \frac{\frac{1}{LC} - \omega^2}{j\omega \frac{1}{L}}$$

This parallel unit reaches zero impedance at two extreme frequencies as follows:

$$\lim_{\omega \rightarrow 0} \left(\frac{1}{j\omega C} \parallel j\omega L \right) = \infty \quad \& \quad \lim_{\omega \rightarrow \infty} \left(\frac{1}{j\omega C} \parallel j\omega L \right) = \infty$$

This circuit has resonant frequency of $\frac{1}{\sqrt{LC}}$ at which it acts as a short circuit with impedance reaching zero:

$$\lim_{\omega \rightarrow \frac{1}{\sqrt{LC}}} \left(\frac{1}{j\omega C} + j\omega L \right) = 0$$

Series LC Circuit Using Laplace

Impedance of this series unit is obtained as follows:

$$Z = \frac{1}{sC} + sL = \frac{1 + LCs^2}{sC} = \frac{\frac{1}{LC} + s^2}{\frac{1}{L}s}$$

This parallel unit reaches zero impedance at two extreme frequencies as follows:

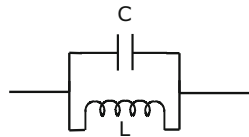
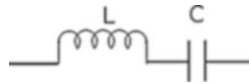
$$\lim_{s \rightarrow 0} \left(\frac{1}{sC} \parallel sL \right) = \infty \quad \& \quad \lim_{s \rightarrow \infty} \left(\frac{1}{sC} \parallel sL \right) = \infty$$

This circuit has resonant frequency of $\frac{1}{\sqrt{LC}}$ and a zero at the same frequency. Therefore, at the resonant frequency, this unit acts as a short circuit with impedance reaching zero.

$$\lim_{s \rightarrow \frac{1}{\sqrt{LC}}} \left(\frac{1}{sC} + sL \right) = 0$$

This unit can be utilized in circuits to bridge the input to output or connect across the output port. Either case, it should be used with this characteristics in mind that it becomes a short circuit at the resonant frequency.

Summary of LC Series and Parallel Circuits

Circuit	$\omega \rightarrow 0$	$\omega = \omega_0$	$\omega \rightarrow \infty$
	$Z = 0 \, \Omega$ Short	$Z = \infty \, \Omega$ Open	$Z = 0 \, \Omega$ Short
	$Z = \infty \, \Omega$ Open	$Z = 0 \, \Omega$ Short	$Z = \infty \, \Omega$ Open

Band-Pass Filters

A band-pass filter is designed to pass signals of frequencies in a specific range (bandwidth) and block the rest. A BPF reaches its highest amplitude at a frequency in the bandwidth BW known as resonant frequency ω_0 . There are two cutoff frequencies ω_{c1} , ω_{c2} to form the passband and determine the bandwidth as:

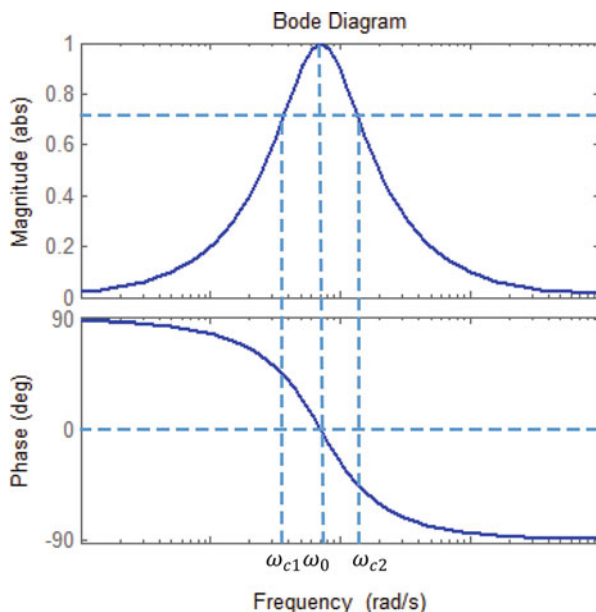
$$BW = \omega_{c2} - \omega_{c1}.$$

Therefore, a BPF has minimum a second-order circuit structure. Figure 9.21 shows the frequency response of a band-pass filter:

As an accurate measure, the resonant frequency ω_0 is a geometrical average of ω_{c1} , ω_{c2} as:

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$$

Fig. 9.21 A typical frequency response of a band-pass filter. The cutoff frequencies and the resonant frequency are shown. The frequency response reaches its maximum amplitude and phase 45° at the resonant frequency



An approximate considers the resonant an arithmetic average in the middle of bandwidth. Therefore, each of the cutoff frequencies is approximately located in $\frac{BW}{2}$ from the resonant ω_0 . Therefore:

$$\omega_{c1,2} = \omega_0 \mp \frac{BW}{2}$$

There are several circuits that behave similar to a band-pass filter. However, a second-order circuit with resonant frequency requires an inductor and a capacitor. Considering the parallel and series connection of these elements and their operation at the resonant frequency, two circuits are proposed.

Frequency response of BPF is shown in Fig. 9.22.

BPF Circuit 1: Using LC Series

Series connection of *LC* shows a short circuit at the resonant frequency. Given the required bandwidth, this filter can pass a desired range of frequencies. To obtain this characteristic, a series *LC* circuit is utilized to connect the input to the output, where the output is the voltage drop observed at a resistance in the terminals. Figure 9.23 shows the circuit.

Fig. 9.22 Magnitude and phase variation as a function of frequency and the damping factor ζ of a BPF. As the damping factor is increased, the bandwidth of the filter is increased, and rate of the phase shift transition at 45° point is decreased

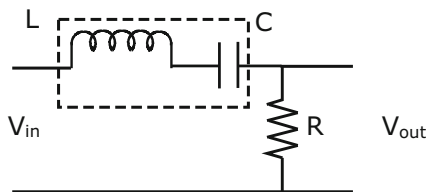
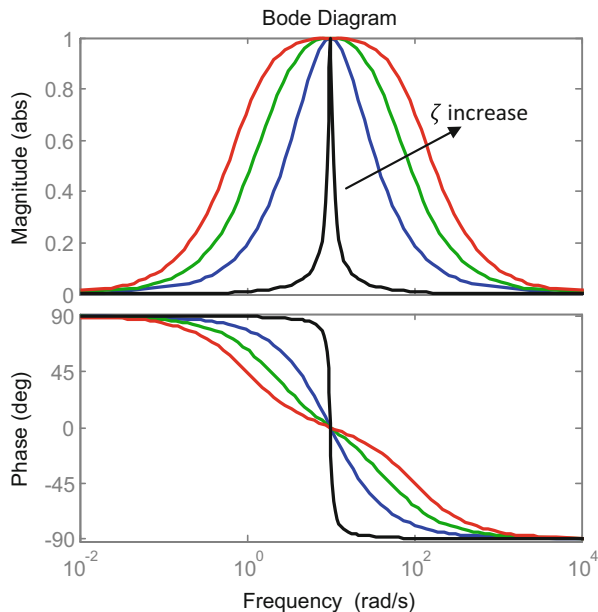


Fig. 9.23 Using the series LC block to build a BPF. The series LC circuit shows a short circuit at the resonant which results in passing input signals to the output. The impedance is increased as the input signal frequencies move away from resonant resulting in an attenuation in the output voltage. This results in a band-pass filter

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Cutoff frequencies can also be obtained utilizing the circuit parameters as follows:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output of the filter occurs at the resonant frequency ω_0 , obtained as follows:

$$H_{\max} = |H(j\omega)|_{\omega=\omega_0} = \left| \frac{R}{R + j\omega L + \frac{1}{j\omega C}} \right| = 1$$

Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \left| \frac{RC\omega}{\sqrt{(RC\omega)^2 + (LC\omega^2 - 1)^2}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{RC\omega_c}{\sqrt{(RC\omega_c)^2 + (LC\omega_c^2 - 1)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\omega_{c1,2} = \mp \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Replacing the BW and resonant frequency:

$$\omega_{c1,2} = \mp \frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_0^2}$$

Considering $Q = \frac{\omega_0}{BW}$:

$$\omega_{c1,2} = \omega_0 \left(\mp \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right)$$

Utilizing these cutoff frequencies, the bandwidth can be obtained as follows:

$$BW = \omega_{c2} - \omega_{c1} = \frac{R}{L}$$

$$Q = \frac{\omega_0}{BW} = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}} = \sqrt{\frac{L}{CR^2}}$$

Using Laplace Transform to Find the Cutoff Frequency

The circuit has a transfer function defined as:

$$\frac{V_o}{V_{in}} = \frac{R}{R + sL + \frac{1}{sC}} = \frac{RCs}{RCs + LCs^2 + 1} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

As the transfer function shows, there is one zero at the origin which results in no pass of low frequencies. There are two poles observed as ω_{c1} , ω_{c2} . At the location of these poles, the amplitude of transfer function drops by -3 dB from the peak $H_{\max}$ which is observed at the resonant ω_0 .

Compared to the denominator of a standard second-order system of $H(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$, it is obtained that:

$$\omega_0^2 = \frac{1}{LC}$$

$$2\zeta\omega_0 = \frac{R}{L} = \text{BW}$$

The quality of a filter depends on the shape and sharpness of the filtration. Known as Q , the quality factor is obtained as follows:

$$Q = \frac{\omega_0}{\text{BW}} = \frac{1}{2\zeta}$$

Therefore, the transfer function of a second-order BPF can be written as a template of:

$$H(s) = \frac{V_o}{V_{in}} = \frac{\text{BW}s}{s^2 + \text{BW}s + \omega_0^2}$$

The location of cutoff frequencies is obtained, as an approximate, as follows:

$$\omega_{c1,2} \approx \omega_0 \mp \frac{\text{BW}}{2}$$

In accurate calculations, the locations of poles are obtained as follows:

- If $0 < \zeta < 1$, the poles of the filter are located at $s_{1,2} = -\zeta\omega_0 \pm j\omega_d$. For very small damping factors, $\zeta \ll 1, \omega_d \approx \omega_0$. Therefore, poles and cutoff frequencies match $s_{1,2} \approx \omega_{c1,2}$. Note, ω_d is the damping frequency $\omega_d = \omega_0\sqrt{1 - \zeta^2}$.
- If $\zeta > 1$, the poles of the filter are located at $s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$, or $s_{1,2} = -\frac{\text{BW}}{2} \pm \omega_d$.

Example 9.13 Design a BPF with resonant frequency of 100 rad/s and bandwidth of 10 rad/s.

Solution $BW = 10$, & $\omega_0 = 100$, therefore:

$$H(s) = \frac{V_o}{V_{in}} = \frac{10s}{s^2 + 10s + 100^2} = \frac{10s}{s^2 + 10s + 10000}$$

Example 9.14 Damping of a BPF is designed to be $\zeta = 0.01$. What is the Q of the circuit?

Solution

$$Q = \frac{1}{2\zeta} = \frac{1}{2 \times 0.01} = 50$$

Example 9.15 Find the transfer function of a BPF with cutoff frequencies of $\omega_{c1} = 200$ rad/s, $\omega_{c2} = 1500$ rad/s.

Solution The bandwidth can be obtained as:

$$BW = \omega_{c2} - \omega_{c1} = 1500 - 200 = 1300 \frac{\text{rad}}{\text{s}}$$

Resonant frequency is obtained as:

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}} = \sqrt{200 \times 1500} = 547.7 \frac{\text{rad}}{\text{s}}$$

Knowing these two factors, the transfer function is obtained as:

$$H(s) = \frac{V_o}{V_{in}} = \frac{BWs}{s^2 + BWs + \omega_0^2} = \frac{1300s}{s^2 + 1300s + 547.7^2}$$

BPF Circuit 2: Using LC Parallel

Parallel connection of LC shows an open circuit at the resonant frequency. Given the required bandwidth, this filter can block a desired range of frequencies from being connected to ground. To obtain this characteristic, a parallel LC circuit is utilized to connect across the output. A resistor bridges the input to the output to prevent short circuit connection of signals at the resonant. Figure 9.24 shows the circuit.

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Cutoff frequencies can also be obtained utilizing the circuit parameters as follows:

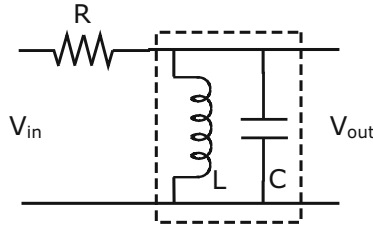


Fig. 9.24 Using a parallel LC block to build a BPF. The parallel LC unit exhibits an open circuit at the resonant frequency and a dropping impedance at other frequencies. Therefore, a band-pass behavior is obtained. The overall behavior is shown as a band-pass filter

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Maximum output of the filter occurs at the resonant frequency ω_0 , obtained as follows:

$$H_{\max} = |H(j\omega)|_{\omega=\omega_0} = \left| \frac{\frac{1}{j\omega C} \| j\omega L}{R + \frac{1}{j\omega C} \| j\omega L} \right| = 1$$

Therefore:

$$|H(j\omega)|_{\omega=\omega_c} = \left| \frac{\frac{j\omega}{-\omega^2 + \frac{1}{LC}}}{R + \frac{j\omega}{-\omega^2 + \frac{1}{LC}}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{\frac{\frac{1}{\omega_c}}{-\omega_c^2 + \frac{1}{LC}}}{\sqrt{R^2 + \left(\frac{\frac{1}{\omega_c}}{-\omega_c^2 + \frac{1}{LC}} \right)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\omega_{c1,2} = \mp \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC} \right)^2 + \frac{1}{LC}}$$

Replacing from the circuit values:

$$\omega_{c1,2} = \mp \frac{BW}{2} + \sqrt{\left(\frac{BW}{2} \right)^2 + \omega_0^2}$$

Considering $Q = \frac{\omega_0}{BW}$:

$$\omega_{c1,2} = \omega_0 \left(\mp \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right)$$

Utilizing these cutoff frequencies, the bandwidth can be obtained as follows:

$$\begin{aligned} \text{BW} &= \omega_{c2} - \omega_{c1} = \frac{1}{RC} \\ Q &= \frac{\omega_0}{\text{BW}} = \frac{\frac{1}{\sqrt{LC}}}{\frac{1}{RC}} = \sqrt{\frac{CR^2}{L}} \end{aligned}$$

Using Laplace Transform to Find the Cutoff Frequency

The circuit has a transfer function defined as:

$$\frac{V_o}{V_{in}} = \frac{sL \parallel \frac{1}{sC}}{R + (sL \parallel \frac{1}{sC})} = \frac{\frac{sL}{LCs^2+1}}{R + \frac{sL}{LCs^2+1}} = \frac{sL}{RLCs^2 + Ls + R} = \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

As the transfer function shows, there is one zero at the origin which results in no pass of low-frequency signals. There are two poles observed as ω_{c1} , ω_{c2} . At the location of these poles, the amplitude of transfer function drops by -3 dB from the peak $H_{\max}$ which is observed at the resonant ω_0 .

Compared to the denominator of a standard second-order system of $H(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$, it is obtained that:

$$\begin{aligned} \omega_0^2 &= \frac{1}{LC} \\ 2\zeta\omega_0 &= \frac{1}{RC} = \text{BW} \end{aligned}$$

The quality of a filter depends on the shape and sharpness of the filtration. Known as Q , the quality factor is obtained as follows:

$$Q = \frac{\omega_0}{\text{BW}} = \frac{1}{2\zeta}$$

Therefore, the transfer function of a second-order BPF can be written as a template of:

$$H(s) = \frac{V_o}{V_{in}} = \frac{\text{BW}s}{s^2 + \text{BW}s + \omega_0^2}$$

The location of cutoff frequencies is obtained, as an approximate, as follows:

$$\omega_{c1,2} \approx \omega_0 \mp \frac{\text{BW}}{2}$$

Cutoff frequencies can also be obtained utilizing the circuit parameters as follows:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

Repeating the same procedure as in *LC* series results in:

$$\omega_{c1,2} = \mp \frac{\text{BW}}{2} + \sqrt{\left(\frac{\text{BW}}{2}\right)^2 + \omega_0^2}$$

Replacing from the circuit values:

$$\omega_{c1,2} = \mp \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

Considering $Q = \frac{\omega_0}{\text{BW}}$:

$$\omega_{c1,2} = \omega_0 \left(\mp \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right)$$

Utilizing these cutoff frequencies, the bandwidth can be obtained as follows:

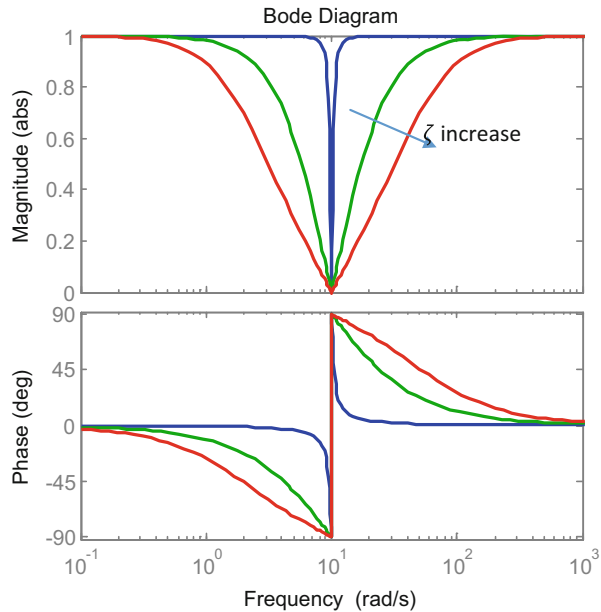
$$\text{BW} = \omega_{c2} - \omega_{c1} = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{\text{BW}} = \frac{\frac{1}{\sqrt{LC}}}{\frac{1}{RC}} = \sqrt{\frac{CR^2}{L}}$$

Band-Reject Filters

A band-reject filter is a circuit that passes signals outside a specific range and blocks the signals in the range, also known as bandwidths. Frequency response of a band-reject filter is shown in Fig. 9.25.

Fig. 9.25 Magnitude and phase variation as a function of frequency and the damping factor ζ of a BRF. As the damping factor is increased, the bandwidth of the filter is increased, and rate of the phase shift transition at 45° point is decreased



Considering the characteristics observed from the series and parallel LC circuits, a BRF behavior can be obtained from these circuits as follows:

BRF Circuit 1: LC Series

Consider the fact that a series LC circuit becomes short circuit at the resonant frequency, placing this circuit in parallel to the output results in a BRF. A resistor is utilized to bridge the input to the output and prevent a short circuit current at the resonant frequency signals. The filter is shown in Fig. 9.26.

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Cutoff frequencies can also be obtained utilizing the circuit parameters as follows:

$$|H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} H_{\max}$$

The ratio of the output voltage over the input voltage is obtained through a voltage divider as follows:

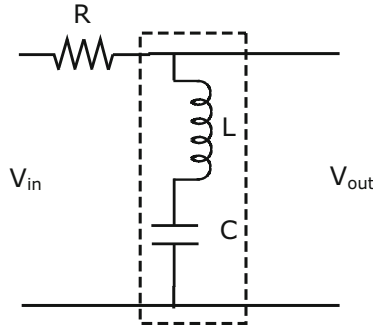


Fig. 9.26 LC series block is utilized to build a BRF. The series LC element exhibits a short circuit at the resonant frequency that imposes zero volts at the output. The impedance is increased once the input signal frequency moves away from the resonant frequency, hence, showing a band-reject response

$$|H(j\omega)| = \frac{V_o(j\omega)}{V_{in}(j\omega)} = \left| \frac{(j\omega L + \frac{1}{j\omega C})}{R + j\omega L + \frac{1}{j\omega C}} \right| = \left| \frac{1 - LC\omega^2}{1 - LC\omega^2 + j\omega RC} \right|$$

Maximum output of the filter occurs at DC and the resonant frequency, obtained as follows:

$$H_{\max} = |H(j\omega)|_{\omega=0} = \left| \frac{1 - LC\omega^2}{1 - LC\omega^2 + j\omega RC} \right| = 1$$

The cutoff frequencies are obtained as follows:

$$|H(j\omega)|_{\omega=\omega_c} = \left| \frac{1 - LC\omega_c^2}{1 - LC\omega_c^2 + j\omega_c RC} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{1 - LC\omega_c^2}{\sqrt{(1 - LC\omega_c^2)^2 + (\omega_c RC)^2}} = \frac{1}{\sqrt{2}}$$

Solving for ω_c results in:

$$\omega_{c1,2} = \mp \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Replacing from the circuit values, the cutoff frequencies are expressed as follows:

$$\omega_{c1,2} = \mp \frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_0^2}$$

Considering $Q = \frac{\omega_0}{\text{BW}}$, yields:

$$\omega_{c1,2} = \omega_0 \left(\mp \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right)$$

Utilizing these cutoff frequencies, the bandwidth can be obtained as follows:

$$\begin{aligned} \text{BW} &= \omega_{c2} - \omega_{c1} = \frac{R}{L} \\ Q &= \frac{\omega_0}{\text{BW}} = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}} = \sqrt{\frac{L}{CR^2}} \end{aligned}$$

Using Laplace Transform to Find the Cutoff Frequencies

The circuit has a transfer function defined as:

$$\frac{V_o}{V_{in}} = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{LCs^2 + 1}{RCs + LCs^2 + 1} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

As the transfer function shows, there are two zeros at $\frac{1}{\sqrt{LC}}$ which results in short circuit output of signals at the resonant frequency. There are two poles observed which relate to ω_{c1} , ω_{c2} . At the location of these cutoff frequencies, the amplitude of transfer function drops by -3 dB from the peak $H_{\max}$.

Compared to the denominator of a standard second-order system of $H(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$, it is obtained that:

$$\begin{aligned} \omega_0^2 &= \frac{1}{LC} \\ 2\zeta\omega_0 &= \frac{R}{L} = \text{BW} \end{aligned}$$

The quality of a filter depends on the shape and sharpness of the filtration. Known as Q , the quality factor is obtained as follows:

$$Q = \frac{\omega_0}{\text{BW}} = \frac{1}{2\zeta}$$

Therefore, the transfer function of a second-order BPF can be written as follows:

$$H(s) = \frac{V_o}{V_{in}} = \frac{s^2 + \omega_0^2}{s^2 + BWs + \omega_0^2}$$

The location of cutoff frequencies is obtained, as an approximate, as follows:

$$\omega_{c1,2} \approx \omega_0 \mp \frac{BW}{2}$$

In accurate calculations, the locations of poles are obtained as follows:

- **Underdamped: $0 < \zeta < 1$**

The poles of the filter are located at $s_{1,2} = -\zeta\omega_0 \pm j\omega_d$.

If $\zeta \ll 1$, then $\omega_d \approx \omega_0$. Poles and cutoff frequencies match $s_{1,2} \approx \omega_{c1,2}$.

- **ω_d is the damping frequency $\omega_d = \omega_0 \sqrt{1 - \zeta^2}$.**

- **Overdamped: $\zeta > 1$**

The poles of the filter are $s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$, or $s_{1,2} = -\frac{BW}{2} \pm \omega_d$.

Example 9.16 Design a BRF with resonant frequency of 100 rad/s and bandwidth of 10 rad/s.

Solution $BW = 10$, and $\omega_0 = 100$; therefore:

$$H(s) = \frac{V_o}{V_{in}} = \frac{s^2 + 100^2}{s^2 + 10s + 100^2} = \frac{s^2 + 100^2}{s^2 + 10s + 10000}$$

Example 9.17 Damping of a BRF is designed to be $\zeta = 0.2$. What is the Q of the circuit?

Solution

$$Q = \frac{1}{2\zeta} = \frac{1}{2 \times 0.2} = 2.5$$

Example 9.18 Find the transfer function of a BRF with cutoff frequencies of $\omega_{c1} = 200$ rad/s, $\omega_{c2} = 1500$ rad/s.

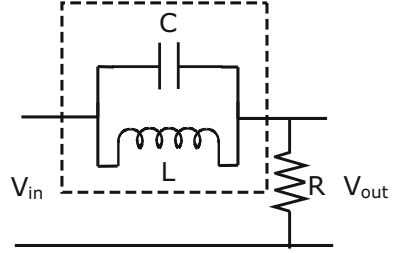
Solution The bandwidth can be obtained as:

$$BW = \omega_{c2} - \omega_{c1} = 1500 - 200 = 1300 \frac{\text{rad}}{\text{s}}$$

Resonant frequency is obtained as:

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}} = \sqrt{200 \times 1500} = 574.7 \frac{\text{rad}}{\text{s}}$$

Fig. 9.27 Using a parallel LC block to build a BRF. The parallel LC unit exhibits an open circuit at the resonant frequency and a dropping impedance at other frequencies. Therefore, a band-reject behavior is obtained



Knowing these two factors, the transfer function is obtained as:

$$H(s) = \frac{V_o}{V_{in}} = \frac{s^2 + \omega_0^2}{s^2 + BWs + \omega_0^2} = \frac{s^2 + 574.7^2}{s^2 + 1300s + 574.7^2}$$

BRF Circuit 2: Using LC Parallel

Parallel connection of LC demonstrates an open circuit at the resonant frequency. This filter blocks a desired range of frequencies in the bandwidth. It can also be used in the bridge connection from input to the output. Figure 9.27 shows the circuit.

Identifying the Cutoff Frequency from the Ratio of the Output over Input

Cutoff frequencies can also be obtained utilizing the circuit parameters.

The ratio of the output over input voltages is obtained through a voltage divider as follows:

$$|H(j\omega)| = \frac{V_o}{V_{in}} = \left| \frac{R}{R + \frac{1}{j\omega C} \parallel j\omega L} \right|$$

$$|H(j\omega)| = \left| \frac{R}{R + \frac{j\omega}{-\omega^2 + \frac{1}{LC}}} \right|$$

$$|H(j\omega)| = \left| \frac{1 - LC\omega^2}{1 - LC\omega^2 + j\omega \frac{1}{RC}} \right|$$

$$|H(j\omega)|_{\omega = \omega_c} = \frac{1}{\sqrt{2}} H_{max}$$

Repeating the same procedure as in LC parallel and considering the maximum amplitude, the cutoff frequencies can be obtained.

$$H_{\max} = |H(j\omega)|_{\omega=0} = \left| \frac{1 - LC\omega^2}{1 - LC\omega^2 + j\omega\frac{1}{RC}} \right| = 1$$

This results in:

$$\begin{aligned} |H(j\omega)|_{\omega=\omega_c} &= \left| \frac{1 - LC\omega_c^2}{1 - LC\omega_c^2 + j\omega_c\frac{1}{RC}} \right| = \frac{1}{\sqrt{2}} \\ \left| \frac{1 - LC\omega_c^2}{\sqrt{(1 - LC\omega_c^2)^2 + (\omega_c\frac{1}{RC})^2}} \right| &= \frac{1}{\sqrt{2}} \\ \omega_{c1,2} &= \mp \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \end{aligned}$$

Replacing the circuit equivalent values results in:

$$\omega_{c1,2} = \mp \frac{\text{BW}}{2} + \sqrt{\left(\frac{\text{BW}}{2}\right)^2 + \omega_0^2}$$

Considering $Q = \frac{\omega_0}{\text{BW}}$, yields:

$$\omega_{c1,2} = \omega_0 \left(\mp \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right)$$

The bandwidth can be obtained as follows:

$$\begin{aligned} \text{BW} &= \omega_{c2} - \omega_{c1} = \frac{1}{RC} \\ Q &= \frac{\omega_0}{\text{BW}} = \frac{\frac{1}{\sqrt{LC}}}{\frac{1}{RC}} = \sqrt{\frac{CR^2}{L}} \end{aligned}$$

Using Laplace Transform to Find the Cutoff Frequency

The circuit has a transfer function defined as:

$$\frac{V_o}{V_{in}} = \frac{R}{R + (sL \parallel \frac{1}{sC})} = \frac{R}{R + \frac{sL}{LCs^2 + 1}} = \frac{R(LCs^2 + 1)}{RLCs^2 + Ls + R} = \frac{RLCs^2 + R}{RLCs^2 + Ls + R}$$

$$\frac{V_o}{V_{in}} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

As the transfer function shows, there are two complex conjugate zeros at the resonant frequency. This guarantees that the output reaches zero at the resonant frequency.

There are two poles observed which are related to ω_{c1} , ω_{c2} . At the location of these cutoff frequencies, the amplitude of transfer function drops by -3 dB from the peak $H_{\max}$. Compared to the denominator of a standard second-order system of $H(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$, it is obtained that:

$$\omega_0^2 = \frac{1}{LC}$$

$$2\zeta\omega_0 = \frac{1}{RC} = \text{BW}$$

The quality of a filter depends on the shape and sharpness of the filtration. Known as Q , the quality factor is obtained as follows:

$$Q = \frac{\omega_0}{\text{BW}} = \frac{1}{2\zeta}$$

Therefore, the transfer function of a second-order BRF can be written as follows:

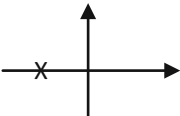
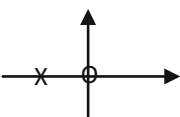
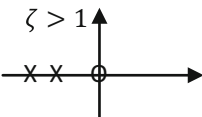
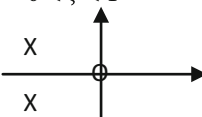
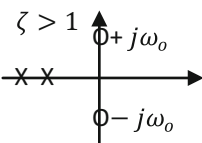
$$H(s) = \frac{V_o}{V_{in}} = \frac{s^2 + \omega_0^2}{s^2 + \text{BW}s + \omega_0^2}$$

The location of cutoff frequencies is obtained, as an approximate, as follows:

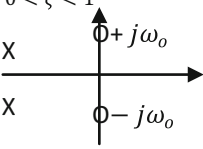
$$\omega_{c1,2} \approx \omega_0 \mp \frac{\text{BW}}{2}$$

Summary of Filters in Laplace

Transfer functions of filters discussed in this chapter are just examples of a larger body of circuits that can perform frequency selectivity and realize filters. These transfer functions and their characteristics are summarized in this section. The following table shows the filters and their location of poles and zeros.

Filter type	A transfer function	Circuit variations	Pole-zero map
Low-pass filter	$H(s) = \frac{\omega_c}{s + \omega_c}$	$RL: \omega_c = \frac{R}{L}$ $RC: \omega_c = \frac{1}{RC}$	 – A pole on the real axis
High-pass filter	$H(s) = \frac{s}{s + \omega_c}$	$RL: \omega_c = \frac{R}{L}$ $RC: \omega_c = \frac{1}{RC}$	 – A pole on the real axis. $s = -\omega_c$ – A zero at the origin. $s = 0$
Band-pass filter	$H(s) = \frac{BW s}{s^2 + BW s + \omega_0^2}$	LC: Series $BW = \frac{R}{L}$ $Q = \sqrt{\frac{L}{CR^2}}$ LC: Parallel $BW = \frac{1}{RC}$ $Q = \sqrt{\frac{CR^2}{L}}$	$\zeta > 1$  – Two poles on the real axis. $s_{1,2} = -\frac{BW}{2} \pm \omega_d$ – A zero at the origin. $s = 0$ $0 < \zeta < 1$  – Two comp. conj. poles. If $\zeta \ll 1 \rightarrow s_{1,2} \approx \omega_{c1,2} \pm j\omega_d$ – A zero at the origin
Band-reject filter	$H(s) = \frac{s^2 + \omega_0^2}{s^2 + BW s + \omega_0^2}$	LC: Series $BW = \frac{R}{L}$ $Q = \sqrt{\frac{L}{CR^2}}$ LC: Parallel $BW = \frac{1}{RC}$ $Q = \sqrt{\frac{CR^2}{L}}$	$\zeta > 1$  – Two poles on the real axis. $s_{1,2} = -\frac{BW}{2} \pm \omega_d$ – Two comp. conj. zeros on the imaginary axis. $s = \pm j\omega_0$

(continued)

Filter type	A transfer function	Circuit variations	Pole-zero map
			$0 < \zeta < 1$  – Two comp. conj. poles. If $\zeta \ll 1 \rightarrow s_{1,2} \approx \omega_{c1,2}$ – Two comp. conj. zeros on the imaginary axis. $s = \pm j\omega_0$

Higher-Order Filters

The frequency response of an ideal filter is different from what has been observed from an actual circuit. Inaccuracies are mostly introduced around the cutoff frequency as a simple structure first-order filter might not be able to reduce enough the power of a selected signal. This means the bandwidth of a second-order BPF or BRF might not be as accurate as it is desired. To solve this issue, there might be a need to repeat the same filter couple more times to obtain closer to ideal system response. However, this action changes the filter characteristics, shifts the cutoff frequencies and shrinks or expands the bandwidth.

Repeated LPF

Figure 9.28 shows a repeated low-pass filter, n times (Fig. 9.28).

Considering the transfer function of a low-pass filter as:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

The overall n th-order filter results in:

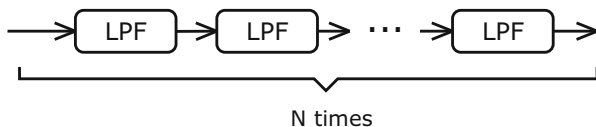


Fig. 9.28 Tandem connection of n identical low-pass filters. The cutoff frequency of the overall filter is shifted from the cutoff frequency of an individual filter

$$H(s) = \frac{\omega_c^n}{(s + \omega_c)^n}$$

The effect of this repeated filter is a shift in cutoff frequency, is obtained as follows:

At $\omega = \omega_{cn}$

$$\begin{aligned} -3 \text{ dB} &= 20 \log \left| \frac{\omega_c^n}{(s + \omega_c)^n} \right| \\ -3 \text{ dB} &= n 20 \log \left| \frac{\omega_c}{\sqrt{\omega_{cn}^2 + \omega_c^2}} \right| \\ -\frac{3}{n} \text{ dB} &= 20 \log \left| \frac{\omega_c}{\sqrt{\omega_{cn}^2 + \omega_c^2}} \right| \\ 10^{-\frac{3}{20n}} &= \frac{\omega_c}{\sqrt{\omega_{cn}^2 + \omega_c^2}} \end{aligned}$$

The cutoff frequency of n th-order LPF is obtained as follows:

$$\omega_{cn} = \omega_c \sqrt{10^{\left(\frac{3}{10n}\right)} - 1}$$

The cutoff frequency of a first-order filter if repeated n times is shifted by factor $\sqrt{10^{\left(\frac{3}{10n}\right)} - 1}$. The trade-off is higher-quality filter with sharper drop of power at frequencies outside of the desired range.

Example 9.19 A low-pass filter has cutoff frequency of 1000 rad/s. Find the cutoff frequency of this filter if it is repeated two and three times in cascade.

Solution For two times repeat of this filter, $n = 2$: (Fig. 9.29 shows the shift in frequencies)

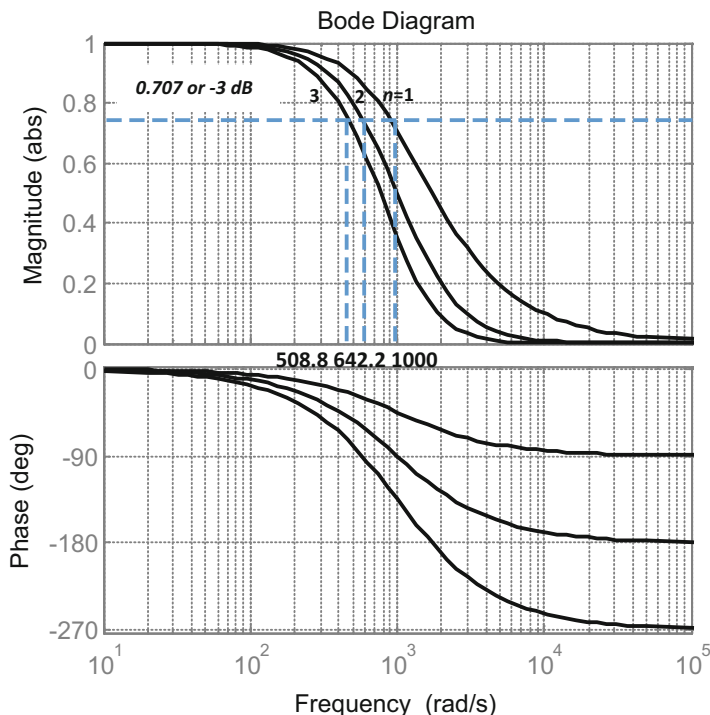


Fig. 9.29 Shift of the cutoff frequency in the case of two tandem connections and three tandem connections of low-pass filters. The cutoff frequency of one filter was 1000 rad/s, and using two of these filters in tandem reduces the cutoff frequency to 642.2 rad/s and three of the same filters in tandem reduces the cutoff frequency to 508.8 rad/s. The phase shifts also increase as the order of circuits in higher tandem filters increases. A single pole (in one filter) reaches -90° , two poles in a second-order filter drop the phase to -180° , and three poles in a third-order filter drop the phase to -270° .

$$\omega_{c2} = \omega_c \sqrt{10^{\left(\frac{3}{10n}\right)} - 1} = 1000 \sqrt{10^{\left(\frac{3}{10 \times 2}\right)} - 1} = 1000 \times 0.6422 = 642.2 \frac{\text{rad}}{\text{s}}$$

For $n = 3$,

$$\omega_{cn} = \omega_c \sqrt{10^{\left(\frac{3}{10n}\right)} - 1} = 1000 \sqrt{10^{\left(\frac{3}{10 \times 3}\right)} - 1} = 1000 \times 0.5088 = 508.8 \frac{\text{rad}}{\text{s}}$$

Example 9.20 It is desired to design a third-order low-pass filter that has the cutoff frequency of $\omega_c = 1000$ rad/s. Find an appropriate first-order LPF to be repeated three times.

Solution The cutoff frequency of the third order system is given and the frequency of a first order is needed. Hence,

$$\omega_{c3} = 1000.$$

$$n = 3.$$

Therefore:

$$1000 = \omega_c \sqrt{10^{\left(\frac{3}{10 \times 3}\right)} - 1}$$

$$\omega_c = \frac{1000}{0.508} = 1965.4 \frac{\text{rad}}{\text{s}}$$

Example 9.21 Find the cutoff frequency of the following filter.

$$H(s) = \frac{2000^4}{(s + 2000)^4}$$

Solution The transfer function shows a 4 times repeat of a low-pass filter with $\omega_c = 2000$. Therefore:

$$\omega_{c4} = 2000 \sqrt{10^{\left(\frac{3}{10 \times 4}\right)} - 1}$$

$$\omega_{c4} = 868.3 \frac{\text{rad}}{\text{s}}$$

Repeated HPF

Considering a cascade connection of HPFs as shown in Fig. 9.30, the transfer function of the overall system is obtained as follows (Fig. 9.30):

$$H(s) = \frac{s}{s + \omega_c}$$

Repeated transfer function is obtained as:

$$H(s) = \frac{s^n}{(s + \omega_c)^n}$$

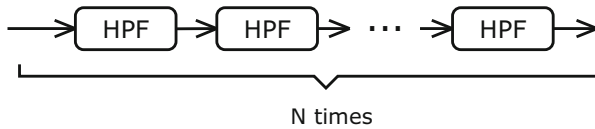


Fig. 9.30 Tandem connection of N identical high-pass filters. The cutoff frequency of the overall filter is shifted from the cutoff frequency of an individual filter

The cutoff frequency of the overall system can be obtained at the location of 3 dB drop gain as follows:

$$\begin{aligned}
 -3 \text{ dB} &= 20 \log \left| \frac{s^n}{(s + \omega_c)^n} \right| \\
 -3 \text{ dB} &= n 20 \log \left| \frac{\omega_{cn}}{\sqrt{\omega_{cn}^2 + \omega_c^2}} \right| \\
 -\frac{3}{n} \text{ dB} &= 20 \log \left| \frac{\omega_{cn}}{\sqrt{\omega_{cn}^2 + \omega_c^2}} \right| \\
 10^{-\frac{3}{20n}} &= \frac{\omega_{cn}}{\sqrt{\omega_{cn}^2 + \omega_c^2}}
 \end{aligned}$$

The cutoff frequency of n th-order HPF is obtained as follows:

$$\omega_{cn} = \frac{\omega_c}{\sqrt{10^{\left(\frac{3}{10n}\right)} - 1}}$$

Example 9.22 A high-pass filter has cutoff frequency of 1000 rad/s. Find the cutoff frequency of this filter if it is repeated two and three times in cascade (Fig. 9.31).

Solution For two times repeat of this filter, $n = 2$:

$$\omega_{c2} = \frac{\omega_c}{\sqrt{10^{\left(\frac{3}{10n}\right)} - 1}} = \frac{1000}{\sqrt{10^{\left(\frac{3}{10 \times 2}\right)} - 1}} = \frac{1000}{0.6422} = 1557.1 \frac{\text{rad}}{\text{s}}$$

For $n = 3$,

$$\omega_{cn} = \frac{\omega_c}{\sqrt{10^{\left(\frac{3}{10n}\right)} - 1}} = \frac{1000}{\sqrt{10^{\left(\frac{3}{10 \times 3}\right)} - 1}} = \frac{1000}{0.5088} = 1965.4 \frac{\text{rad}}{\text{s}}$$

Repeated BPF

If a band-pass filter is repeated n times, each of the cutoff frequencies is shifted to narrow the bandwidth of the filter. It results in a higher-quality filter. Low cutoff frequency is shifted as a high-pass filter, and high cutoff frequency is shifted as a low-pass filter (shown in Fig. 9.32).

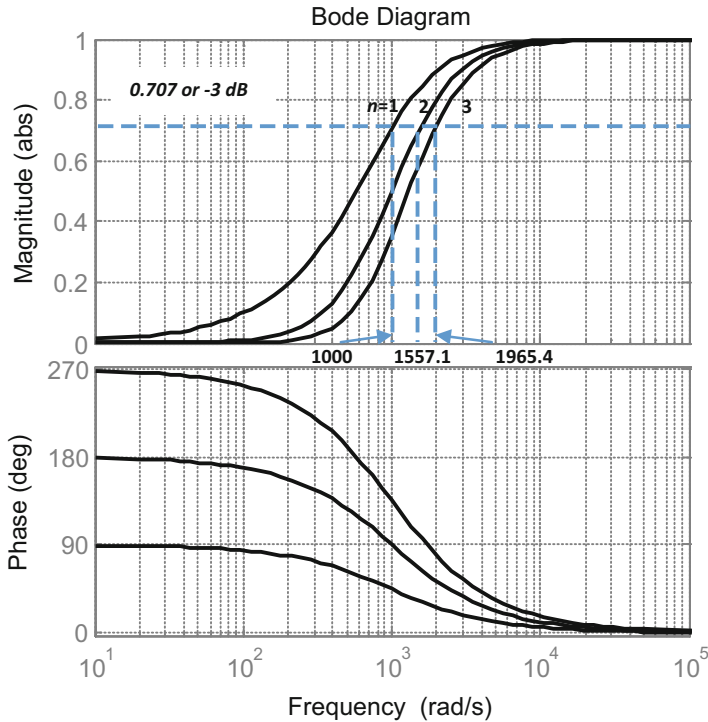


Fig. 9.31 Shift of the cutoff frequency in the case of two tandem connections and three tandem connections of high-pass filters. The cutoff frequency of one filter was 1000 rad/s, and using two of these filters in tandem increased the cutoff frequency to 1557.1 rad/s and three of the same filters in tandem increased the cutoff frequency to 1965.4 rad/s. The phase shifts also increase as the order of circuits in higher tandem filters increases. A single pole (in one filter) started from $+90^\circ$, two poles in a second-order filter started the phase from $+180^\circ$, and three poles in a third-order filter started the phase from $+270^\circ$.

Repeated BRF

If a band-reject filter is repeated n times, the cutoff frequency of high-pass filter part is shifted toward higher frequencies, and the cutoff frequency of low-pass filter part is shifted to lower frequencies. This results in wider bandwidth of the filter, i.e., a larger range of frequencies are blocked. Figure 9.33 shows the frequency response of a repeated band-reject filter (Fig. 9.33).

Fig. 9.32 The behavior of repeated BPF can be observed in both the low-pass and high-pass sides. As the cutoff frequency of low-pass filters is decreased and that of the high-pass filters were increased, it results in a narrower bandwidth of the BPF

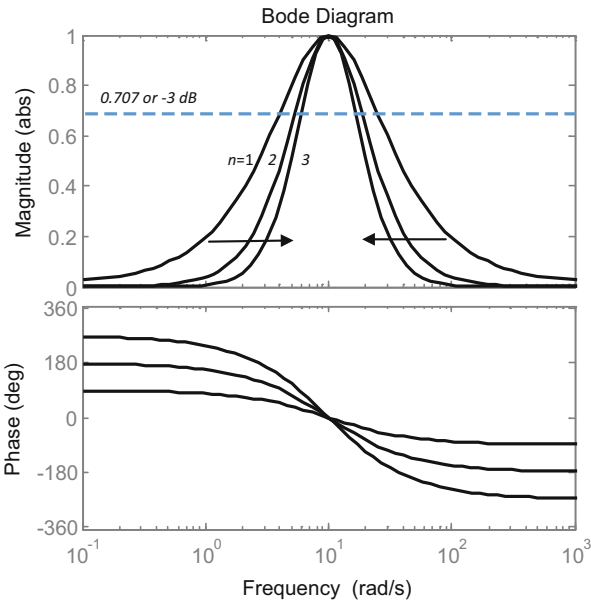
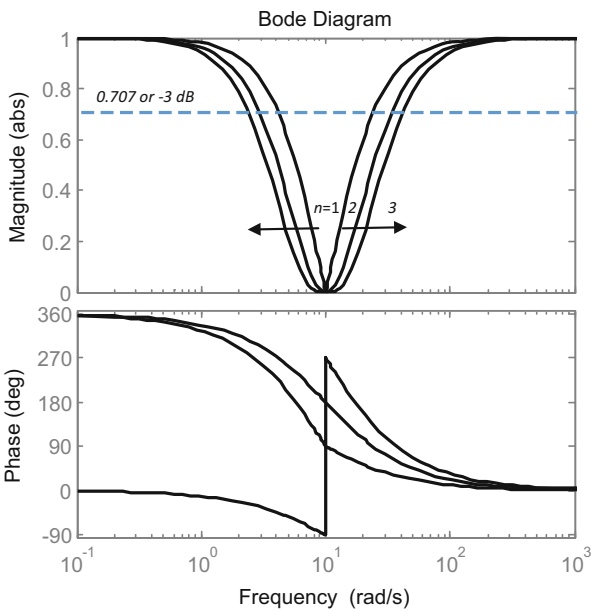


Fig. 9.33 The behavior of repeated BPF can be observed in both the low-pass and high-pass sides. As the cutoff frequency of low-pass filters is decreased and that of the high-pass filters were increased, it results in a wider bandwidth of the BRF



Problems

Low-Pass Filter

- 9.1. A low-pass filter has cutoff frequency of $f_c = 3000$ Hz.
- Design an RC circuit to realize this filter. Consider $C = 1 \mu\text{F}$. Draw the circuit and analyze its operation.
 - Design an RL circuit to realize this filter. Consider $L = 150$ mH. Draw the circuit and analyze its operation.
- 9.2. A low-pass filter has cutoff frequency of $f_c = 60$ Hz.
- Design an RC circuit to realize this filter. Consider $C = 10 \mu\text{F}$. Draw the circuit and analyze its operation.
 - Design an RL circuit to realize this filter. Consider $L = 10$ mH. Draw the circuit and analyze its operation.
- 9.3. In an RL LPF, the circuit parameters are $R = 2 \Omega$ and $L = 0.05$ mH. Find the cutoff frequency of the filter. Determine the output voltage amplitude and phase when a $V_{\text{in}} = 10$ V input is tuned at the following frequencies $\omega = 4000, 40,000, 400,000$ ($\frac{\text{rad}}{\text{s}}$).
- 9.4. In an RC LPF, the circuit parameters are $R = 2 \Omega$ and $C = 10 \mu\text{F}$. Find the cutoff frequency of the filter. Determine the output voltage amplitude and phase when a $V_{\text{in}} = 10$ V input is tuned at the cutoff frequency and a decade above and a decade below the cutoff.

Low-Pass Filter Using Laplace

- 9.5. Design a unity-gain LPF with cutoff frequency of $\omega_c = 15k$ rad/s. Find its transfer function.
- 9.6. Design a unity-gain LPF to cut the frequencies higher than the heartbeat of 1 Hz. The frequency of 1 Hz should fully pass to the output. Find the transfer function of the filter.
- 9.7. Determine the filter type of the following systems, and determine their cutoff frequency.
- $G(s) = \frac{200}{s+200}$
 - $G(s) = \frac{1}{1e^{-3s}+1}$
 - $G(s) = \frac{400}{2e^{-4s}+2}$
- 9.8. In Problem 9.7, find the poles and zeros of the transfer functions, and identify the correlation between the poles and the cutoff frequency and the filter types.
- 9.9. Determine and sketch the frequency response of the transfer functions in Problem 9.7.

High-Pass Filter

- 9.10. A high-pass filter has cutoff frequency of $f_c = 3000$ Hz.
- Design a RC circuit to realize this filter. Consider $C = 1 \mu\text{F}$. Draw the circuit and analyze its operation.
 - Design a RL circuit to realize this filter. Consider $L = 150$ mH. Draw the circuit and analyze its operation.
- 9.11. A high-pass filter has cutoff frequency of $f_c = 60$ Hz.
- Design a RC circuit to realize this filter. Consider $C = 10 \mu\text{F}$. Draw the circuit and analyze its operation.
 - Design a RL circuit to realize this filter. Consider $L = 10$ mH. Draw the circuit and analyze its operation.
- 9.12. In a RL HPF, the circuit parameters are $R = 2 \Omega$ and $L = 0.05$ mH. Find the cutoff frequency of the filter. Determine the output voltage amplitude and phase when a $V_{\text{in}} = 10$ V input is tuned at the following frequencies $\omega = 4000, 40,000, 400,000$ ($\frac{\text{rad}}{\text{s}}$).
- 9.13. In a RC HPF, the circuit parameters are $R = 2 \Omega$ and $C = 10 \mu\text{F}$. Find the cutoff frequency of the filter. Determine the output voltage amplitude and phase when a $V_{\text{in}} = 10$ V input is tuned at the cutoff frequency and a decade above and a decade below the cutoff.

High-Pass Filter Using Laplace

- 9.14. Design a unity gain HPF with cutoff frequency of $\omega_c = 15k$ rad/s. Find its transfer function.
- 9.15. Design a unity gain HPF to cut the frequencies lower than 60 Hz. The frequency of 60 Hz should fully pass to the output. Find the transfer function of the filter.
- 9.16. Determine the filter type of the following systems and determine their cutoff frequency.
- $G(s) = \frac{200s}{s+200}$
 - $G(s) = \frac{s}{1e^{-3s}+1}$
 - $G(s) = \frac{400s}{2e^{-4s}+2}$
- 9.17. In Problem 9.16, find the poles and zeros of the transfer functions and identify the correlation between the poles and the cutoff frequency and the filter types.
- 9.18. Determine and sketch the frequency response of the transfer functions in Problem 9.17.

Series and Parallel LC circuits

- 9.19. Tune a *LC* series circuit at the 60 Hz. Find a reasonable value (in existing range) for *L* and *C*. Analyze the operation of this circuit as a bridge between the input and output (of a filter). Analyze the operation of this circuit in parallel to the output (of a filter).
- 9.20. Tune a *LC* parallel circuit at the 60 Hz. Find a reasonable value (in existing range) for *L* and *C*. Analyze the operation of this circuit as a bridge between the input and output (of a filter). Analyze the operation of this circuit in parallel to the output (of a filter).

Band-Pass Filters

- 9.21. Design a BPF to show a bandwidth of 100 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.22. Design a BPF to show a bandwidth of 10 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.23. Design a BPF to show a bandwidth of 1 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.24. Design a BPF to show a bandwidth of 500 Hz and lower cutoff frequency of 10000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.25. Design a BPF to show a bandwidth of 500 Hz and resonant frequency of 40000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.26. Damping of a BPF is designed to be $\zeta = 0.5$. What is the *Q* of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.27. Damping of a BPF is designed to be $\zeta = 0.1$. What is the *Q* of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.28. Damping of a BPF is designed to be $\zeta = 0.05$. What is the *Q* of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.

Band-Pass Filters Using Laplace

- 9.29. Find transfer function of a BPF with bandwidth of 100 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.30. Find transfer function of a BPF with bandwidth of 10 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.31. Find transfer function of a BPF with bandwidth of 1 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.32. Find transfer function of a BPF with bandwidth of 0.5 Hz and cutoff frequency of 60 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.33. Determine the filter type, cutoff frequency, resonant frequency, quality factor, damping factor, and bandwidth of the following transfer functions:
- $G(s) = \frac{20s}{s^2 + 20s + 10000}$
 - $G(s) = \frac{200s}{s^2 + 200s + 9000}$
 - $G(s) = \frac{0.1s}{s^2 + 0.1s + (120\pi)^2}$
 - $G(s) = \frac{0.01s}{s^2 + 0.01s + (2\pi)^2}$
- 9.34. Realize the transfer functions of Problem 9.32 in *LC* series.
- 9.35. Realize the transfer functions of Problem 9.32 in *LC* parallel.
- 9.36. Damping of a BPF is designed to be $\zeta = 0.5$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.37. Damping of a BPF is designed to be $\zeta = 0.1$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.38. Damping of a BPF is designed to be $\zeta = 0.05$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.

- 9.39. Find the transfer function of a BPF with cutoff frequencies of $f_{c1} = 100$ Hz, $f_{c2} = 500$ Hz.
- 9.40. Find the transfer function of a BPF with cutoff frequencies of $\omega_{c1} = 58$ Hz, $f_{c2} = 60$ Hz.
- 9.41. Find the transfer function of a BPF with cutoff frequencies of $\omega_{c1} = 1000$ rad/s, $\omega_{c2} = 5000$ rad/s.

Band-Reject Filters

- 9.42. Design a BRF to show a bandwidth of 100 Hz and cutoff frequency of 400 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.43. Design a BRF to show a bandwidth of 10 Hz and cutoff frequency of 400 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.44. Design a BRF to show a bandwidth of 1 Hz and cutoff frequency of 400 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.45. Design a BRF to show a bandwidth of 0.5 Hz and cutoff frequency of 60 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components.
- 9.46. Damping of a BRF is designed to be $\zeta = 0.5$. What is the Q of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.47. Damping of a BRF is designed to be $\zeta = 0.1$. What is the Q of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.48. Damping of a BRF is designed to be $\zeta = 0.05$. What is the Q of the circuit? Design the filter if the resonant frequency is 100 Hz. Sketch the frequency response.

Band-Pass Filters Using Laplace

- 9.49. Find transfer function of a BRF with bandwidth of 100 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.

- 9.50. Find transfer function of a BRF with bandwidth of 10 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.51. Find transfer function of a BRF with bandwidth of 1 Hz and cutoff frequency of 5000 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.52. Find transfer function of a BRF with bandwidth of 0.5 Hz and cutoff frequency of 60 Hz. Find the transfer function in two cases of using a *LC* series and a *LC* parallel. Use existing range components. Determine the poles and zeros of the circuit and estimate the cutoff frequency from the values of the poles.
- 9.53. Determine the filter type, cutoff frequency, resonant frequency, quality factor, damping factor, and bandwidth of the following transfer functions:

$$(a) G(s) = \frac{s^2 + 10000}{s^2 + 20s + 10000}$$

$$(b) G(s) = \frac{s^2 + 9000}{s^2 + 200s + 9000}$$

$$(c) G(s) = \frac{s^2 + (120\pi)^2}{s^2 + 0.1s + (120\pi)^2}$$

$$(d) G(s) = \frac{s^2 + (2\pi)^2}{s^2 + 0.01s + (2\pi)^2}$$

- 9.54. Realize the transfer functions of Problem 9.52 in *LC* series.
- 9.55. Realize the transfer functions of Problem 9.52 in *LC* parallel.
- 9.56. Damping of a BRF is designed to be $\zeta = 0.5$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.57. Damping of a BRF is designed to be $\zeta = 0.1$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.58. Damping of a BRF is designed to be $\zeta = 0.05$. What is the Q of the circuit? Find the transfer function if the resonant frequency is 100 Hz. Sketch the frequency response.
- 9.59. Find the transfer function of a BRF with cutoff frequencies of $f_{c1} = 100$ Hz, $f_{c2} = 500$ Hz.
- 9.60. Find the transfer function of a BRF with cutoff frequencies of $\omega_{c1} = 58$ Hz, $f_{c2} = 60$ Hz.
- 9.61. Find the Transfer Function of a BRF with Cutoff Frequencies of $\omega_{c1} = 1000$ rad/s, $\omega_{c2} = 5000$ rad/s.

Overall Filtration Process

9.62. Determine whether each of the following input signals are passing or rejected in each of the filter listed.

(a) Input signals:

- (i) $V_1 = 5$
- (ii) $V_3 = 5 \sin 10t$
- (iii) $V_4 = 5 \sin 100t$
- (iv) $V_5 = 5 \sin 1000t$
- (v) $V_6 = 5 \sin 5000t$
- (vi) $V_7 = 5 \sin 10000t$

(b) Filters:

- (i) $H(s) = \frac{1000}{s+1000}$
- (ii) $G(s) = \frac{s}{s+1000}$
- (iii) $P(s) = \frac{4000s}{s^2+4000s+3000^2}$
- (iv) $Q(s) = \frac{s^2+3000^2}{s^2+2000s+3000^2}$

Higher-Order Filter

- 9.63. A LPF has cutoff frequency of $\omega_c = 100$ rad/s. The filter is repeated three times. Find the new cutoff frequency. Sketch an *RL* and an *RC* filter separately to realize the third-order filter. Sketch the frequency response of the first order and third order.
- 9.64. A third-order LPF has to have a cutoff frequency of $\omega_c = 100$ rad/s. Find the cutoff frequency of the first-order circuit. Sketch an *RL* and an *RC* filter separately to realize the third-order filter. Sketch the frequency response of the first order and third order.
- 9.65. A HPF has cutoff frequency of $\omega_c = 100$ rad/s. The filter is repeated 3 times. Find the new cutoff frequency. Sketch an *RL* and an *RC* filter separately to realize the third-order filter. Sketch the frequency response of the first order and third order.
- 9.66. A third-order HPF has to have a cutoff frequency of $\omega_c = 100$ rad/s. Find the cutoff frequency of the first-order circuit. Sketch an *RL* and an *RC* filter separately to realize the third-order filter. Sketch the frequency response of the first order and third order.

Higher-Order Filter Using Laplace

9.67. Determine the type, order, cutoff frequency, bandwidth, and quality factor of the following filters (whichever applies):

(a) $G(s) = \frac{10^3}{(s+10)^3}$

(b) $G(s) = \frac{s^4}{(s+1000)^4}$

(c) $G(s) = \frac{400s^2}{(s^2+20s+10000)^2}$

(d) $G(s) = \frac{(s^2+(120\pi)^2)^3}{(s^2+0.1s+(120\pi)^2)^3}$

Chapter 10

Operational Amplifiers



Operational Amplifiers

Years before microcontrollers and digital computers were introduced, control system operations, industrial computations, and even simulation of dynamical systems were made possible by use of analog computers. The heart of an industrial analog computer is a device called operational amplifier or “opamp.” These amplifiers consist many transistors to accomplish a theoretical infinite gain to either inputs.

In a simple form, an opamp is packaged in an eight-port device as shown in Fig. 10.1.

As the figure shows, the opamp has major ports as follows:

- Two input ports
 - Inverting
 - Noninverting
- One output port
- Positive power supply
- Negative power supply

The opamp is simply shown by these ports, and ideally power supplies are also eliminated from schematics. Figure 10.2 shows the ideal opamp.

Ideal Opamp

Ideally, opamps are considered devices that have:

1. Infinite gain. It means that the opamp has no limit in the amount of voltage it can generate in the output, and the supply voltages can be infinitely large.
2. Infinitely fast response. It means a large (close to infinite) slew rate.

Fig. 10.1 Pin layout of a typical 8-pin operational amplifier. This chip contains only one opamp. There are chips that contain two or more opamps

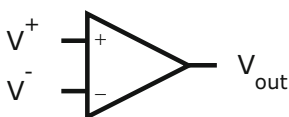
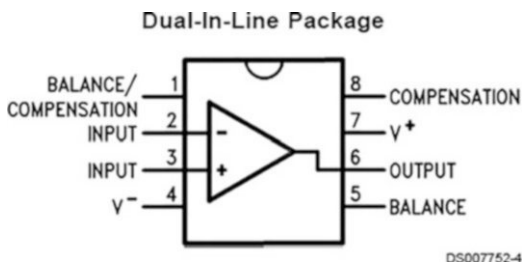


Fig. 10.2 Schematic of an ideal opamp without showing the power ports. In this type of application, it is considered that the opamp can take unlimited voltage levels and has no delay in response or what is known as slew rate

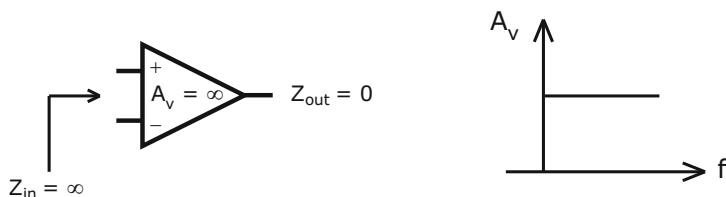
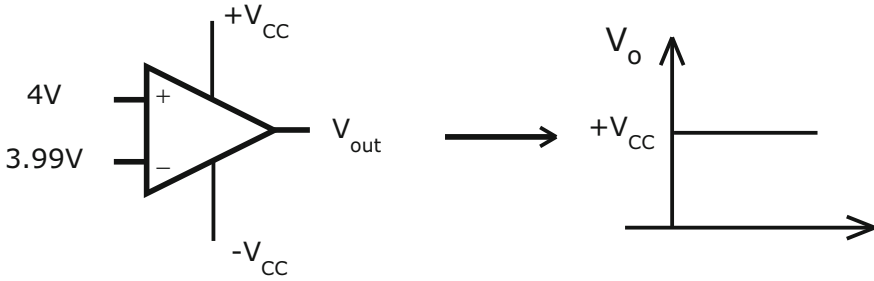


Fig. 10.3 Frequency response of an ideal opamp shows high gain in all across the frequency spectrum. However, the gain of an actual opamp drops as the operating frequency reaches physical limitations of the device. Hence, resulting a high gain over an unlimited range of frequencies

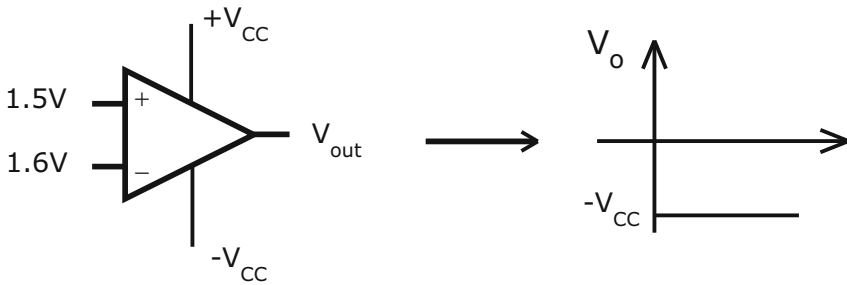
3. Infinite bandwidth. It means that the gain of opamp does not drop at infinitely large frequencies.
4. Infinite input impedance. It means that the input current to each input port (inverting and non-inverting) is zero. These opamps are voltage controlled devices.
5. Zero output impedance. It means that the output voltage is not dependent on the load impedance (Fig. 10.3).

Operation of an ideal opamp. Considering an infinitely fast response device with unlimited voltage levels, the voltage observed at the output port of an opamp becomes the value of its + supply if the + port voltage becomes slightly higher than the voltage applied at the - port.

For instance, $v^+ = 4 \text{ V}$ and $v^- = 3.99 \text{ V}$ result in $+V_{cc}$ in the output. $v^+ = 1.5 \text{ V}$, and $v^- = 1.6 \text{ V}$ results in $-V_{cc}$ in the output. As the opamp is ideal, the output voltage switches to $+V_{cc}$ or $-V_{cc}$ is infinitely fast (Fig. 10.4).



Note that v^+ is slightly larger than v^- .



Note that v^- is slightly larger than v^+ .

Fig. 10.4 Application as comparator. When inverting and noninverting pins have slightly different voltage, the output voltage switches to one of the supply voltages that is connected to the pin with higher voltage

Slew Rate

The speed of an opamp (actually measured in $\frac{V}{\mu s}$ or $\frac{V}{ms}$) to follow a reference signal with amplitude V at frequency f (Hz) is measured as slew rate, which can be found as:

$$\text{Slew Rate} = 2\pi fV \left(\frac{V}{s} \right)$$

$$\text{Slew Rate} = \frac{2\pi fV}{1e3} \left(\frac{V}{ms} \right)$$

$$\text{Slew Rate} = \frac{2\pi fV}{1e6} \left(\frac{V}{\mu s} \right)$$

This means that the variations in the reference signal can be amplified in the output of the opamp, this requires that the slew rate of the opamp be equal or higher

than $2\pi fV$. For instance, a 150 mV signal at frequency 10 kHz requires an opamp with minimum slew rate of

$$\text{Slew Rate} = \frac{(2\pi 10e3 \times 150e-3)}{1000}$$

$$\text{Slew Rate} = 0.942 \left(\frac{\text{V}}{\text{ms}} \right)$$

Opamp in Circuits

There are two rules that simplify the analysis of an opamp in a circuit.

Rule 1 There is a *virtual short circuit* between the two input ports v^+ and v^- . Figure 10.5 shows this simplification notation.

Rule 2 In spite of a short circuit between the ports of an opamp, there is no current flowing to either ports. Figure 10.6 shows this simplification notation.

Example 10.1 Find the output voltage of the following circuit (Fig. 10.7).

Fig. 10.5 The inverting and noninverting ports are virtually short circuit

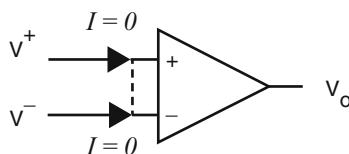
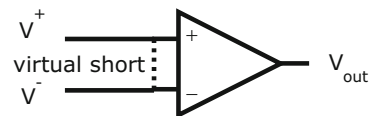
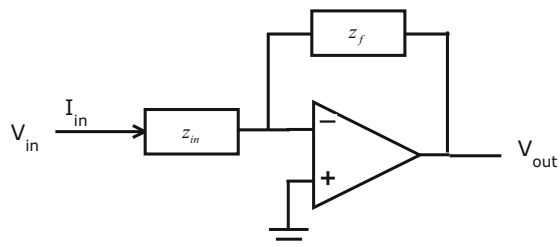


Fig. 10.6 The inverting and noninverting ports are virtually short circuit. However, the high input impedance of the ports makes the current entering each port equal zero

Fig. 10.7 Figure Example 10.1. Inverting amplifier



Solution Since the noninverting port is connected to ground it has zero potential or $v^+ = 0$. Applying rule 1, the voltage of the inverting port v^- due to the virtual short circuit between the inverting and noninverting ports takes up the voltage of the noninverting port. Therefore, virtually, $v^+ = v^- = 0$.

Imposing a zero volt potential at the inverting port, the current $I_{in} = \frac{V_{in}-0}{Z_{in}}$. Considering rule 2, there is no current entering the ports of opamp. Therefore, a KCL at the node ① indicates that the current must flow through the feedback impedance as follows:

$$I_{in} = I_f$$

$$I_f = \frac{0 - V_o}{Z_f}$$

Therefore,

$$I_{in} = I_f = \frac{V_{in} - 0}{Z_{in}} = \frac{0 - V_o}{Z_f}$$

The output voltage in terms of input voltage can be obtained as:

$$\frac{V_{in} - 0}{Z_{in}} = \frac{0 - V_o}{Z_f}$$

$$\frac{V_o}{V_{in}} = \frac{-Z_f}{Z_{in}}$$

This shows two important application of an opamp:

1. The input voltage can be amplified by the ratio of feedback impedance over the input impedance $\frac{Z_f}{Z_{in}}$ as long as it is not reached the supply voltage limits
2. The phase of the output voltage is 180° apart from the input voltage which is indicated by the $(-)$ sign. For this reason this circuit is also called inverting amplifier.

Example 10.2 Noninverting amplifier. Find the output voltage of the following circuit (Fig. 10.8).

Solution The noninverting port v^+ is connected to the input voltage V_{in} . Therefore, according to rule 1, the inverting port voltage is virtually $v^- = V_{in}$. This voltage forces current to ground through impedance Z_1 as

$$I_1 = \frac{V_{in}}{Z_1}$$

Fig. 10.8 Figure of circuit in Example 10.2, a noninverting amplifier

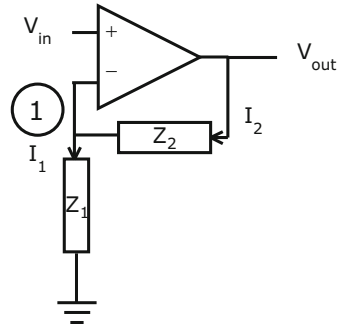
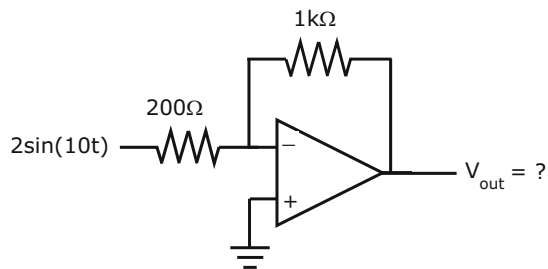


Fig. 10.9 Figure of circuit in Example 10.3, an inverting amplifier



According to rule 2, there is no current passing to the port. Therefore, a KCL at node ① is written as:

$$I_1 = I_2 = \frac{V_o - v^-}{Z_2} = \frac{V_o - V_{in}}{Z_2}$$

$$\frac{V_{in}}{Z_1} = \frac{V_o - V_{in}}{Z_2}$$

The ratio of output voltage over input voltage is obtained as:

$$\frac{V_o}{V_{in}} = 1 + \frac{Z_2}{Z_1}$$

As the equation shows, the output voltage depends on the ratio of feedback impedance over the ground impedance as $1 + \frac{Z_2}{Z_1}$, and it is inphase with the input voltage.

Example 10.3 Considering an ideal opamp and unlimited voltage supply, find the output voltage in the following circuit if the input is excited by a $V_{in} = 2 \sin(10 t)$ V (Fig. 10.9).

Solution The transfer function of an inverting amplifier is given by the ratio of the feedback over the input impedances. Therefore,

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} = -\frac{1000}{200} = -5$$

The output voltage becomes

$$V_o = -5V_{in} = -5 \times 2 \sin 10t = -10 \sin 10t \text{ V}$$

Example 10.4 Considering a supply voltage limit at $\pm V_{cc} = \pm 8 \text{ V}$. Find the output voltage from the circuit discussed in previous example.

Solution The output voltage of the circuit without any limit in supply voltage amplitude was expected to reach $\pm 10 \text{ V}$ peak. However, as the voltage is limited to $\pm 4 \text{ V}$, the peak voltage is clamped at 8 volts. The result is a sinusoidal clamped at 4 V peak.

$$V_o = \begin{cases} -4 & V_o < -4 \\ -10 \sin 10t & -4 < V_o < 4 \\ +4 & V_o > 4 \end{cases}$$

Figure 10.10 shows the output voltage of the amplifier. The figure shows that a large gain does not mean that the output voltage is amplified to a very large voltage.

Example 10.5 Consider the circuit shown in Fig. 10.11. Find the limit of input voltage amplitude before the output voltage is saturated (Fig. 10.11).

Solution The figure shows an inverting amplifier with the gain of

$$\frac{V_o}{V_{in}} = \frac{-Z_f}{Z_{in}} = -\frac{10 \text{ k}}{1 \text{ k}} = -10$$

Fig. 10.10 Figure of the clamped output in a saturated amplifier

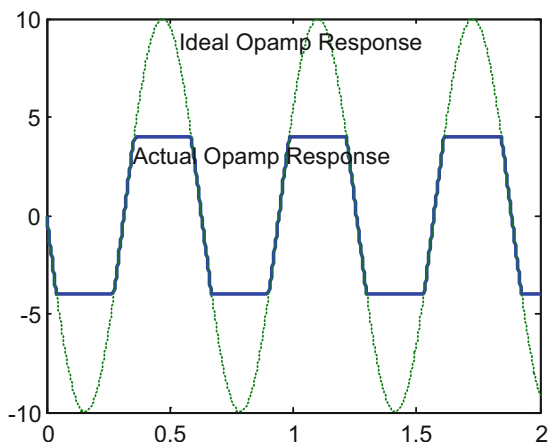


Fig. 10.11 Figure of circuit in Example 10.5

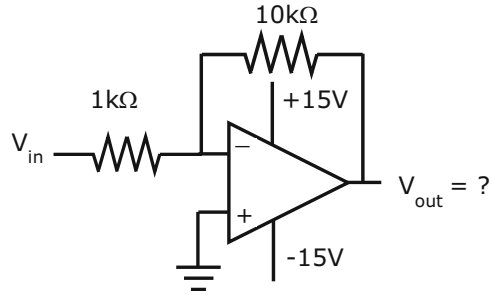
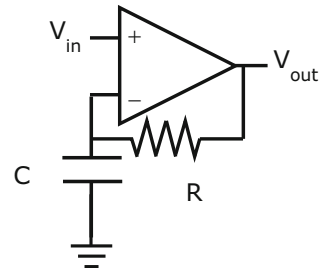


Fig. 10.12 Figure of circuit in Example 10.6



The input voltage is therefore amplified tenfold. The limit of power supply is shown as ± 15 V. This means any amplified voltage outside of this band is clamped. To reach output of 15 V with a gain of 10, the input voltage must not be increased above $V_{in} = \frac{15}{10} = 1.5$ V.

Example 10.6 Find the output voltage of the circuit shown in Fig. 10.12 when a unit step is applied in the input (Fig. 10.12).

Solution The figure shows a noninverting amplifier, in which the transfer function is obtained as:

$$\frac{V_o}{V_{in}} = 1 + \frac{Z_2}{Z_1}$$

From the circuit:

$$Z_2 = R$$

$$Z_1 = \frac{1}{sC}$$

Therefore,

$$\frac{V_o}{V_{in}} = 1 + RCs$$

Applying a unit step function

$$V_{\text{in}} = \frac{1}{s}$$

Results in:

$$V_o = \frac{1}{s}(1 + RCs)$$

Taking Laplace inverse obtains:

$$v_o(t) = u(t) + RC\delta(t)$$

Mathematical Operations

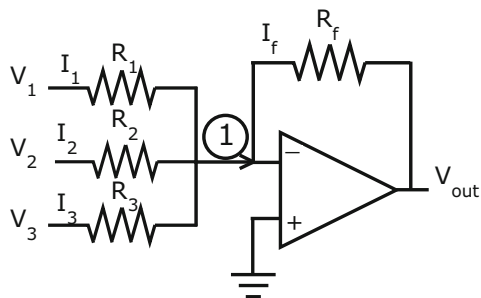
So far, opamps were used to scale a signal as an amplifier. Opamps can also be used to perform other mathematical operations such as addition, subtraction, differentiation, integration, unity follower, and comparator.

Adder

Operational amplifiers can add multiple signals with individually controlled gains. Consider an inverting amplifier with multiple input impedances sources by various voltages and connected to the inverting port. This circuit is shown in Fig. 10.13.

The voltage at node ① is virtually ground, because the noninverting port v^+ is connected to the ground. This causes currents to flow from the voltages to the node and from the node to the output. A summation of these currents flow to the feedback resistor as follows:

Fig. 10.13 Adder circuit with three inputs and inverting



$$I_1 = \frac{V_1 - 0}{R_1}$$

$$I_2 = \frac{V_2 - 0}{R_2}$$

$$I_3 = \frac{V_3 - 0}{R_3}$$

KCL at node ① shows:

$$I_1 + I_2 + I_3 = I_f$$

$$\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} = \frac{0 - V_o}{R_f}$$

Therefore,

$$V_o = -\frac{R_f}{R_1} V_1 - \frac{R_f}{R_2} V_2 - \frac{R_f}{R_3} V_3$$

If the input resistors are equal,

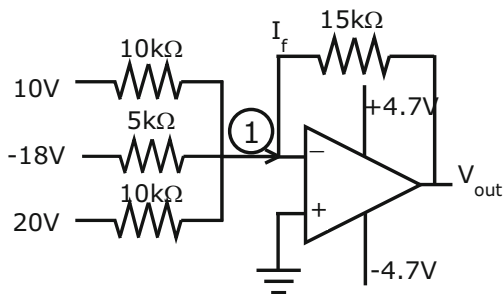
$$R_1 = R_2 = R_3 = R$$

Then,

$$V_o = -\frac{R_f}{R} (V_1 + V_2 + V_3)$$

Example 10.7 Find the output voltage of the circuit shown in Fig. 10.14 once without supply voltage limit and once considering the given supply voltage limits. Consider $V_1 = 10$ V, $V_2 = -18$ V, $V_3 = 20$ V.

Fig. 10.14 Circuit used in Example 10.7



Solution The output voltage is obtained as

$$V_o = -\frac{R_f}{R_1}V_1 - \frac{R_f}{R_2}V_2 - \frac{R_f}{R_3}V_3$$

$$V_o = -\frac{15\text{ k}}{10\text{ k}}10 - \frac{15\text{ k}}{5\text{ k}}(-18) - \frac{15\text{ k}}{10\text{ k}}20 = +9\text{ V}$$

When there is no saturation, the output voltage reaches +9 V. However, the supply voltage limit of 4.7 V limits the calculated output voltage of 9 V to 4.7 V. Therefore, the output does not exceed 4.7 V.

Example 10.8 In an adder circuit if all input resistors are equal $R_1 = R_2 = R_3 = R$ and the feedback resistor is $R_f = \frac{1}{3}R$, find the output voltage.

Solution The output voltage is calculated according to:

$$V_o = -\frac{R_f}{R_1}V_1 - \frac{R_f}{R_2}V_2 - \frac{R_f}{R_3}V_3$$

Replacing the values yields:

$$V_o = -\frac{1}{3}(V_1 + V_2 + V_3)$$

Example 10.9 Design a 4-bit digital to analog converter.

Solution A 4-bit binary number is known as $b_3b_2b_1b_0$. The number in base-10 equals

$$2^0b_0 + 2^1b_1 + 2^2b_2 + 2^3b_3$$

Considering the bits b_0 to b_3 as various inputs to the DAC, the gains need to be obtained for each of these inputs are $2^0, 2^1, 2^2, 2^3$, respectively. Therefore, the circuit is obtained as follows (Fig. 10.15):

Fig. 10.15 A 4-bit digital to analog converter

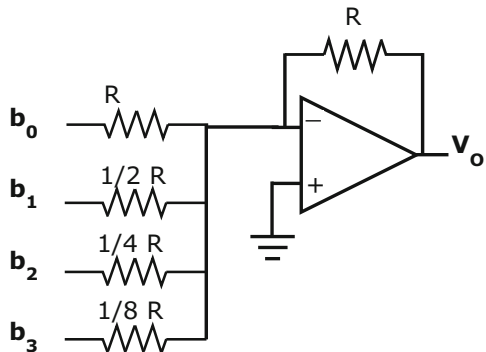
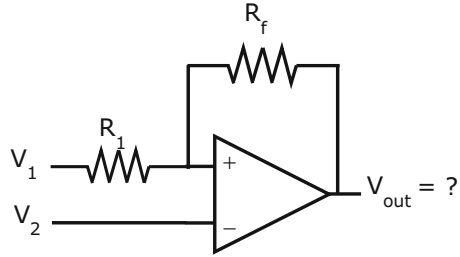


Fig. 10.16 Subtracting circuit



Subtraction

Consider the circuit shown in Fig. 10.16. Two input voltages are applied to the noninverting port and to a negative feedback amplifier through the inverting port. This results in the output to be a gained subtract of the two voltages.

Since there are two outputs, the superposition can be applied. For this purpose, let consider input voltages one by one while the other voltage is turned off, i.e., zero.

Effect of V_1 while $V_2 = 0$. The circuit becomes an inverting amplifier with the output to be:

$$\frac{V_{o1}}{V_1} = -\frac{R_f}{R_1}$$

Now, consider the effect of V_2 while $V_1 = 0$. The circuit becomes a noninverting amplifier with the gain calculated as:

$$\frac{V_{o2}}{V_2} = 1 + \frac{R_f}{R_1}$$

Therefore, the output voltage under the effect of two signals is the summation of

$$V_o = \frac{V_{o1}}{V_1} + \frac{V_{o2}}{V_2}$$

$$V_o = -\frac{R_f}{R_1} V_1 + \left(1 + \frac{R_f}{R_1}\right) V_2$$

If $\frac{R_f}{R_1} \gg 1$

$$V_o \approx -\frac{R_f}{R_1} V_1 + \left(\frac{R_f}{R_1}\right) V_2$$

Fig. 10.17 Circuit used in Example 10.10

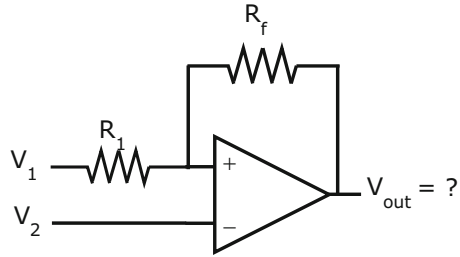
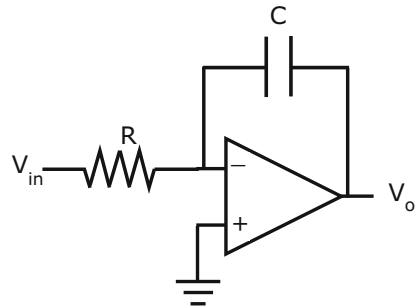


Fig. 10.18 An integrating circuit



$$V_o = \frac{R_f}{R_1}(V_2 - V_1)$$

Example 10.10 In a vehicle cruise control system, the measured signal must be subtracted from the reference signal. Using an opamp, design a circuit to accomplish this operation. (In this example, the speed is measured by an equivalent voltage.)

Solution There are two signals being subtracted. Figure 10.17 shows the circuit diagram.

Integrator

The inverting amplifier circuit was discussed earlier. The gain of the amplifier was obtained as the ratio of the feedback impedance over the input impedance. If the feedback impedance is replaced with an integrating element such as a capacitor with impedance $\frac{1}{sC}$, the resultant transfer function becomes an integrator (Fig. 10.18).

$$\frac{V_0}{V_{in}} = -\frac{1}{sR} = -\left(\frac{1}{RC}\right)\frac{1}{s}$$

$$v_0(t) = -\left(\frac{1}{RC}\right) \int v_{in}(t) dt$$

In this transfer function, $\frac{1}{RC}$ is the gain, and $\frac{1}{s}$ is the integrating agent. This transfer function takes the integral of the input signal and scales the integrated signal.

Example 10.11 Find the output voltage of the following circuit considering a sinusoidal input as $v_{in} = 0.1 \sin 377 t$ (Fig. 10.19).

$$v_0(t) = -\left(\frac{1}{RC}\right) \int v_{in}(t) dt$$

$$v_0 = \left(\frac{-1}{1e3 \times 0.5e-6}\right) \int 0.1 \sin 377 t dt$$

$$v_0 = -2000 \times 0.1 \int \sin 377 t dt$$

$$v_0 = \frac{200}{377} \cos 377 t \text{ V}$$

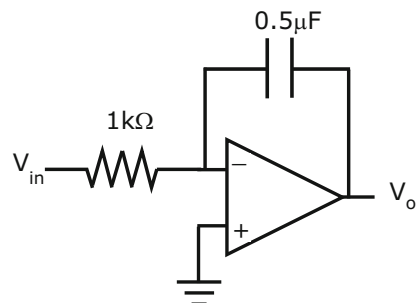
$$v_0 = 0.53 \cos 377 t \text{ V}$$

Example 10.12 Find the limit of gain in which the integrator circuit introduced in previous example remains unsaturated. The supply voltage amplitude is $\pm 4.7 \text{ V}$.

Solution The output voltage of the integrator when the sinusoidal voltage of frequency ω is applied (in time domain) is obtained as:

$$v_0 = -\left(\frac{1}{RC}\right) \int v \sin \omega t = \frac{v}{RC\omega} \cos \omega t$$

Fig. 10.19 Circuit used in Example 10.11



$$\frac{v}{RC\omega} = 4.7$$

Given $v = 0.1$ V, and $\omega = 377$ rad/s.

$$\begin{aligned}\frac{0.1}{RC \times 377} &= 4.7 \\ RC &= 56.43e-6\end{aligned}$$

Considering $C = 1$ μ F, results in $R = 56.43$ Ω .

Differentiator

Considering the gain of inverting amplifier, as the ratio of feedback to input impedance, and taking input impedance as $\frac{1}{sC}$ yield a Laplace operator in the numerator of the transfer function, making the system a differentiator. The differentiator circuit is shown in Fig. 10.20.

The transfer function is obtained as follows:

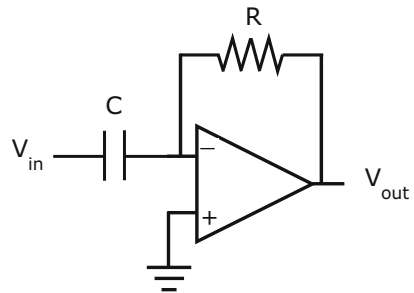
$$\frac{V_0}{V_{in}} = -\frac{R}{\frac{1}{sC}} = -RCs$$

Taking Laplace inverse of the function results in the output voltage as:

$$v_0(t) = -RC \frac{dv_{in}(t)}{dt}$$

Example 10.13 Considering $v_{in}(t) = 5e - 3 \sin 4000\pi t$, $R = 1$ M Ω , and $C = 1$ μ F, in a differentiator circuit, find the output voltage.

Fig. 10.20 A differentiator circuit



Solution

$$v_0(t) = -RC \frac{dv_{in}(t)}{dt}$$

$$v_0(t) = -1e6 \times 1e - 6 \frac{d5e - 3 \sin 4000\pi t}{dt}$$

$$v_0(t) = -20\pi \cos 4000\pi t \text{ V}$$

Comparator

This circuit compares two signals and sets the output as the supply voltage source associated to the highest of the two. For instance, the output is $+V_{cc}$ if the voltage applied to the noninverting port is higher than the voltage applied to the inverting port. The output becomes $-V_{cc}$ if the voltage applied to the inverting port is higher than the voltage applied to the noninverting port. This is shown as follows (Fig. 10.21):

If

$$v^+ > v^- \Rightarrow v_0 = +V_{cc}$$

$$v^+ < v^- \Rightarrow v_0 = -V_{cc}$$

This is due to the highest gain applied to the difference of

$$v^+ - v^-$$

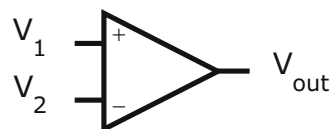
Infinite gain is saturated at the level of $\pm V_{cc}$ to result in the pulse of appropriate polarity. The speed of reaching the output to the highest level or “slew rate” is also very high that makes the comparator operate almost instantaneous.

Pulse Width Modulation (PWM)

Knowing the operation of comparator circuits, if a DC voltage is compared with a sawtooth signals, the result is a Pulse Width Modulation signal. The circuit is shown in Fig. 10.22.

As the figure shows, the sawtooth is compared with a DC level voltage. Since the DC voltage is connected to the noninverting port, when the DC voltage is higher than the value of sawtooth, the output voltage becomes the value of V_{cc} . As the sawtooth becomes larger than the DC, the output becomes zero. This results in a

Fig. 10.21 A comparator circuit



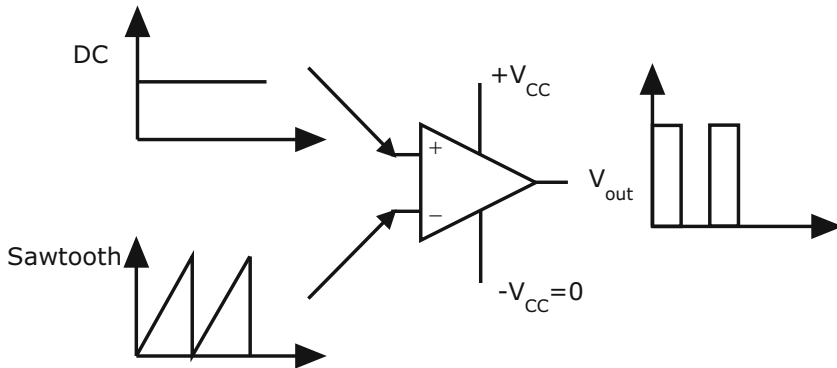


Fig. 10.22 Inputs of a comparator to generate controlled width pulses

pulse. To modulate the width of the pulse, the level of DC voltage can change. Increasing the level of DC extends the on time of the V_{cc} and therefore extends the width of the pulse. Lowering the value of DC also shortens the width of the pulse, making the train of pulses a controllable PWM.

Note It should be noted that the maximum value of DC voltage should not increase the amplitude of pulses, otherwise, the output voltage becomes a continuous step function. In this condition, no pulse is created.

Unit Follower

To isolate different stages of an amplifier, because of their power needs, buffer layers or isolation layers are utilized. Opamps can be used as a buffer stage to isolate the input layer from output, while the output follows the input signal at a higher power rating. Figure 10.23 shows a unit follower circuit.

In this circuit the noninverting input is virtually connected to the output layer while it is being isolated through the infinite input impedance of the opamp. The amount of power provided at the output depends on the power rating of the opamp.

In this circuit, the noninverting amplifier has a zero ohm feedback and an infinite ohm impedance to the ground. Therefore:

$$\frac{V_0}{V_{in}} = 1 + \frac{R_f}{R_1} = 1 + \frac{0}{\infty} = 1$$

$$V_0 = V_{in}$$

$$Z_{in} = \infty$$

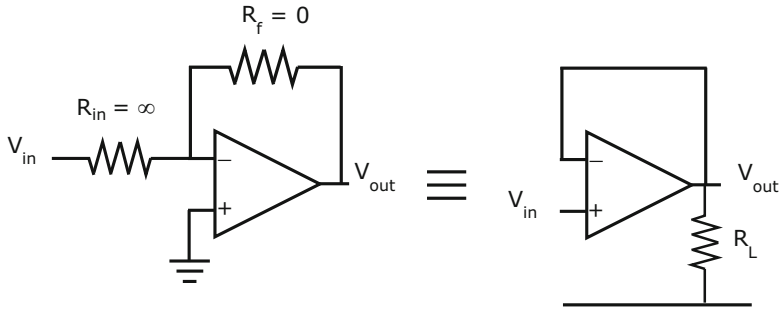


Fig. 10.23 A unit follower circuit

Function Builder

As it was explained, operational amplifiers were used to build functions in analog computers. What was explained earlier in this chapter was formation of simple algebraic equations, differentiators, and simple integrators. However, these functions can vary from simple algebraic operations to complicated differential equations.

Example 10.14 Using operational amplifiers, build a circuit to realize the following differential equation.

$$\dot{y} + 2y = \sin t$$

Solution The system requires an integrator and a summation. To convert the equation to an integral equation, there is a need to take integral of both sides as follows:

$$\begin{aligned} \int (\dot{y} + 2y) &= \int \sin t \\ y + 2 \int y &= \int \sin t \end{aligned}$$

Solving for y results in:

$$y = -2 \int y + \int \sin t$$

As one of the signals is negative and one is positive, there are multiple ways to implement the function. It is recommended that the signal $\sin t$ is passed through an inverting unity gain amplifier and then summed and integrated as shown in Fig. 10.24.

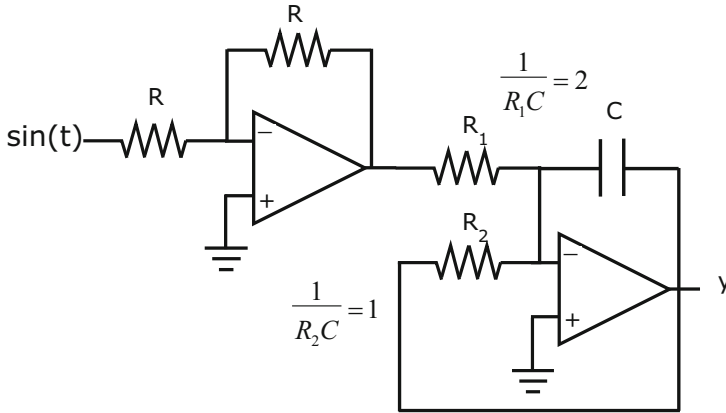


Fig. 10.24 Circuit realization for Example 10.14

The gains of the inverting integrator are, the coefficient for the $\int \sin t$ function as:

$$\frac{1}{R_2 C} = 1$$

And the gain for $2 \int y$ term is:

$$\frac{1}{R_1 C} = 2.$$

Example 10.15 Design a circuit that builds dynamics of the following system.

$$\ddot{y} + 2\dot{y} + y = u.$$

Solution This circuit requires two integrators to form the state space equations as follows:

$$y = x_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 + 2x_2 + x_1 = u$$

Which results in:

$$\dot{x}_2 = -x_1 - 2x_2 + u$$

Implementing these two equations through integrators is obtained as follows:

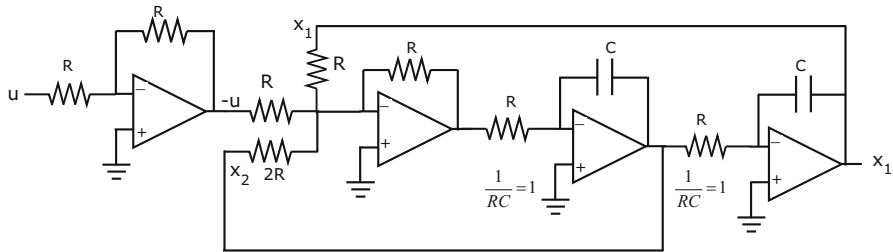


Fig. 10.25 Circuit realization of Example 10.15

$$\dot{x}_1 = x_2 \rightarrow x_1 = \int x_2$$

$$\dot{x}_2 = -x_1 - 2x_2 + u \rightarrow x_2 = \int -x_1 - 2x_2 + u$$

The implemented circuit is shown Fig. 10.25.

Negative Immittance Converter

The NIC has one input and one output port. The impedance inversion might occur at certain frequencies, but there is at least one frequency at which the impedance measured at the input port is negative of the impedance connected at the output port. The input and output ports are interchangeable, and the circuit is reciprocal. The NIC can be constructed from any two-port device with voltage gain higher than 2. Chapter 12 discusses 2-port networks.

Negative Impedance

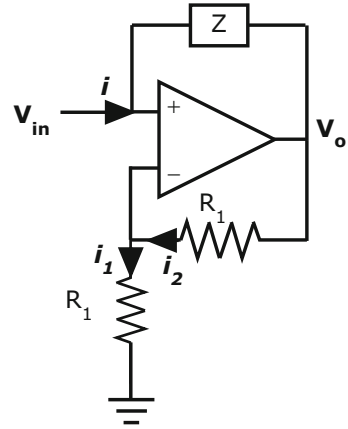
The circuit of Fig. 10.26 shows how an opamp can be utilized in impedance converting circuits.

The current i equals:

$$i = \frac{V_{in} - V_o}{Z}$$

The voltage of inverting port equals the V_{in} . Therefore, the current flows through resistors R_1 as i_1 and i_2 become:

Fig. 10.26 Opamp-based negative impedance converter



$$i_1 = \frac{V_{in}}{R_1}$$

$$i_2 = \frac{V_o - V_{in}}{R_1}$$

And

$$i_1 = i_2$$

Therefore,

$$V_o = 2V_{in}$$

Replacing this into the current equation results in:

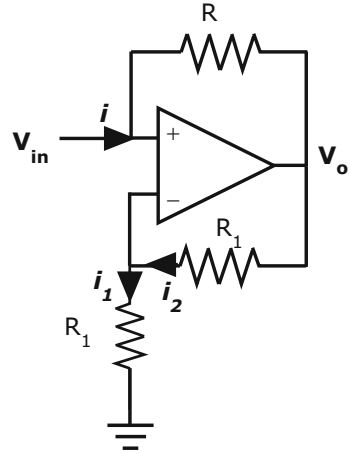
$$i = \frac{V_{in} - 2V_{in}}{Z}$$

$$\frac{V_{in}}{i} = -Z$$

The entire circuit shows the negative of the feedback impedance at the input. Therefore:

$$Z_{in} = -Z$$

Fig. 10.27 Opamp based negative resistance converter



Negative Resistance (Fig. 10.27)

Following the same approach as in the negative impedance converter, the input resistance becomes:

$$R_{in} = -R$$

Negative Capacitance (Fig. 10.28)

Following the same approach as in the negative impedance converter, the input impedance becomes:

$$Z_{in} = -Z = -\frac{1}{j\omega C}$$

$$Z_{in} = \frac{j}{\omega C}$$

Fig. 10.28 Negative capacitance converter

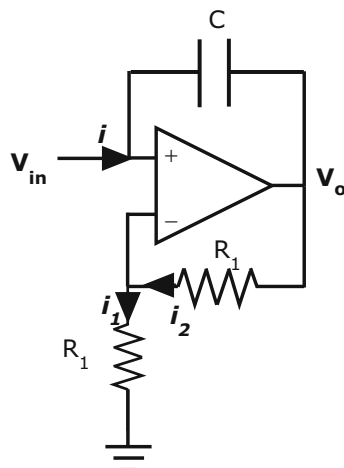
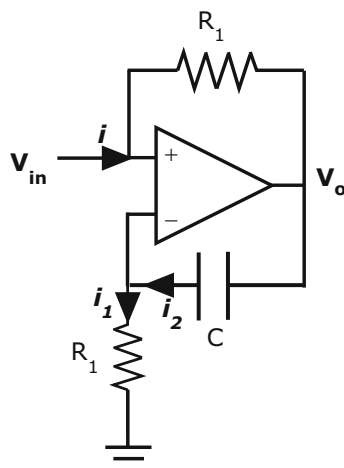


Fig. 10.29 Negative capacitance converter



Negative Inductance (Fig. 10.29)

The current i equals:

$$i = \frac{V_{in} - V_o}{R_1}$$

The voltage of inverting port equals the V_{in} . Therefore, the current flows through resistors R_1 as i_1 and i_2 becomes:

$$i_1 = \frac{V_{in}}{R_1}$$

$$i_2 = \frac{V_o - V_{in}}{\frac{1}{j\omega C}} = j\omega C (V_o - V_{in})$$

And

$$i_1 = i_2$$

Therefore,

$$V_o = V_{in} \left(1 + \frac{1}{j\omega R_1 C} \right)$$

Replacing this into the current equation results in:

$$i = \frac{V_{in} - V_{in} \left(1 + \frac{1}{j\omega R_1 C} \right)}{R_1}$$

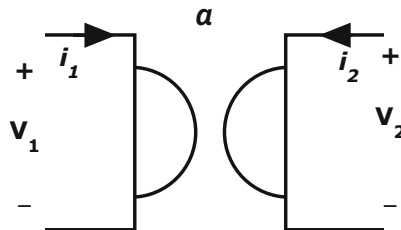
$$R_1 i = V_{in} \frac{-1}{j\omega R_1 C}$$

$$\frac{V_{in}}{i} = -j\omega R_1^2 C$$

$$Z_{in} = -j\omega R_1^2 C$$

Gyrator

A gyrator is an element with an input and an output port, known as a two-port device. There is specific relation between the input and output parameters.



The input current i_1 is a linear scale of the output voltage v_2 , and the output current i_2 is linearly proportional to the inverted input voltage v_1 . This means:

$$i_1 = \alpha v_2$$

$$i_2 = -\alpha v_1$$

The scaling factor α is also known as gyration conductance. The impedance conversion of an ideal gyrator is obtained as follows:

$$Z_1 = \frac{v_1}{i_1}$$

Replacing i_1 and v_1 values, results in:

$$Z_1 = \frac{v_1}{i_1} = \frac{-i_2}{\alpha v_2} = \frac{1}{\alpha^2} \frac{-i_2}{v_2}$$

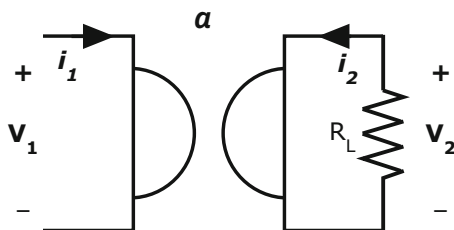
Considering the output admittance Y_2 and the direction of current that is out of the admittance, yields:

$$Y_2 = \frac{-i_2}{v_2}$$

The input impedance becomes:

$$Z_1 = \frac{1}{\alpha^2} Y_2$$

Example 10.16 Find the input impedance of a gyrator when a load resistance R_L is connected to the output port.



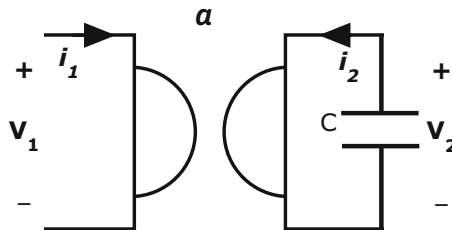
Solution The output admittance is:

$$Y_2 = \frac{1}{R_L}$$

Therefore:

$$Z_1 = \frac{1}{\alpha^2} \frac{1}{R_L}$$

Example 10.17 Find the input impedance of a gyrator when a capacitance C is connected to the output port.



Solution The admittance of the capacitor is

$$Y_2 = j\omega C$$

Replacing in the gyrator equation results in:

$$Z_1 = \frac{1}{\alpha^2} Y_2 = \frac{1}{\alpha^2} j\omega C = j\omega \frac{C}{\alpha^2}$$

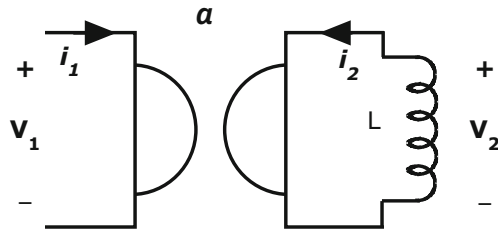
This imitates an inductor impedance of:

$$Z_1 = j\omega \frac{C}{\alpha^2} = j\omega L_{\text{eq}}$$

Therefore, at the input, the output capacitor imitates an inductor of value $\frac{C}{\alpha^2}$.

$$L_{\text{eq}} = \frac{C}{\alpha^2}$$

Example 10.18 Find the input impedance of a gyrator when an inductor L is connected to the output port.



Solution The output admittance of an inductor is:

$$Y_2 = \frac{1}{j\omega L}$$

Replacing in the gyrator equation results in:

$$Z_1 = \frac{1}{\alpha^2} Y_2 = \frac{1}{\alpha^2} \frac{1}{j\omega L} = \frac{1}{j\omega L \alpha^2}$$

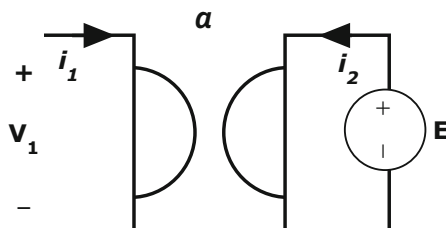
Therefore, at the input, the output inductor imitates a capacitor inductor of value $\alpha^2 L$.

$$Z_1 = \frac{1}{j\omega L \alpha^2} = \frac{1}{j\omega C_{\text{eq}}}$$

Therefore:

$$C_{\text{eq}} = \alpha^2 L$$

Example 10.19 Find the input equivalent of a gyrator when a voltage source E is connected to the output port.



Solution The input voltage and current relation is:

$$i_1 = \alpha v_2$$

$$i_2 = -\alpha v_1$$

When a voltage source is connected to the output, the voltage $v_2 = E$ regardless of the output current i_2 . Therefore,

$$i_1 = \alpha v_2 = \alpha E$$

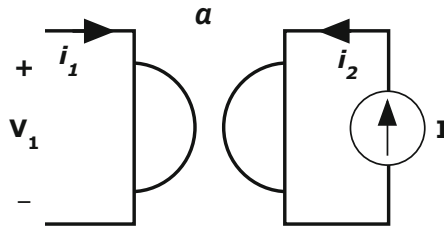
The input voltage is as follows:

$$v_1 = \frac{-1}{\alpha} i_2$$

Since the current i_2 of a voltage source E can be ideally any value, this equation imitates that the voltage in the input can take any value. However, the amount of current in the input is bounded by the value of the voltage source E . This fits into the operation of a current source in the input.

Therefore, the voltage source E is converted to a current source of value $I = \alpha E$.

Example 10.20 Find the input equivalent of a gyrator when a current source I is connected to the output port.



Solution The input voltage and current relation is:

$$i_1 = \alpha v_2$$

$$i_2 = -\alpha v_1$$

When a current source is connected to the output, the current $i_2 = I$ regardless of the output voltage v_2 . Therefore,

$$v_1 = \frac{-1}{\alpha} I$$

The output voltage across the current source can take any value. Therefore, the input current can take any value as follows:

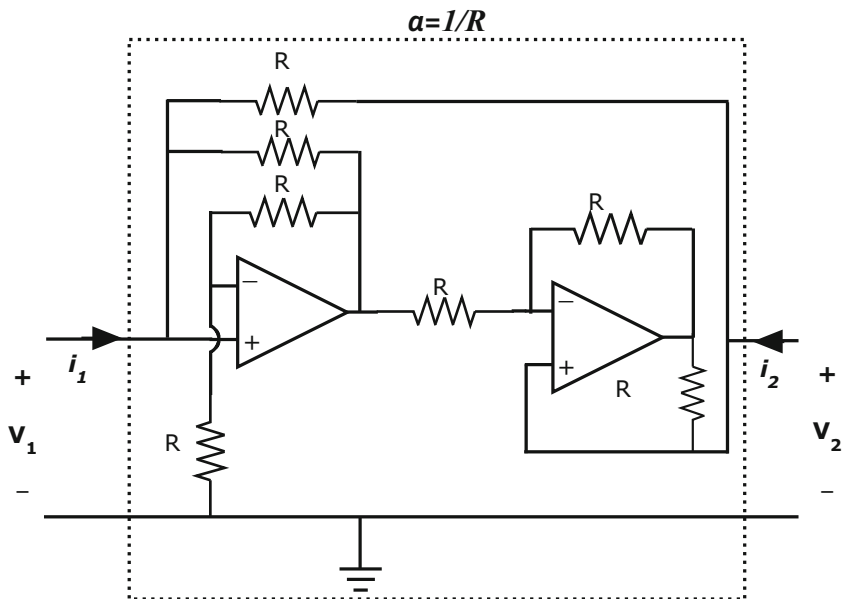
$$i_1 = \alpha v_2$$

Since the voltage v_2 of a current source I can be ideally any value, this equation imitates that the current in the input can take any value. However, the amount of voltage in the input is bounded by the value of the current source I . This fits into the operation of a voltage source in the input.

Therefore, the current source I is converted to a voltage source of value $V = \frac{1}{\alpha} I$.

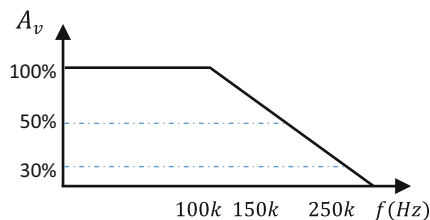
Realization of a Gyrator in Circuits

A gyrator can be realized using operational amplifiers. The circuit creates a voltage controlled current source (VCCS) when the input current is zero and the output current is controlled by the input voltage.



Problems

- 10.1. Design an inverting amplifier to obtain gain 100. What is the output voltage if the input is $v_{in}(t) = 0.2 \sin t$ V. Discuss the choices you made for the input and feedback impedance.
- 10.2. An inverting amplifier has the gain of 500. Calculate and sketch the output voltage when the input is 0.2 V at 1 kHz.
- 10.3. An inverting amplifier has the gain of 500 and the source voltage of ± 15 V. Calculate and sketch the output voltage when the input is 0.2 V at 1 kHz.
- 10.4. The frequency response of an inverting amplifier with gain 50 is shown. Find the output voltage at the given frequencies.



- (a) $v_{in} = 5$ at $f = 1$ kHz
 - (b) $v_{in} = 5$ at $f = 5$ kHz
 - (c) $v_{in} = 5$ at $f = 10$ kHz
 - (d) $v_{in} = 5$ at $f = 15$ kHz
 - (e) $v_{in} = 5$ at $f = 100$ kHz
- 10.5. The frequency of a 100 mV signal to be amplified is 40 kHz. Find the slew rate required from an operational amplifier.
 - 10.6. Slew rate of an operational amplifier is 2 V/ms. Is it suitable for the following cases:
 - (a) 10 V, 5 Hz
 - (b) 10 V, 10 Hz
 - (c) 10 V, 50 Hz
 - (d) 10 V, 500 Hz
 - 10.7. Find the slew rate required for each of the following signals.
 - (a) 1 V, 500 Hz
 - (b) 1 V, 1 kHz
 - (c) 1 V, 100 kHz
 - (d) 1 V, 10 MHz
 - (e) 5 V, 500 Hz
 - (f) 5 V, 10 kHz
 - (g) 5 V, 100 kHz
 - (h) 5 V, 10 MHz
 - (i) 15 V, 500 Hz
 - (j) 15 V, 1 kHz

(k) 15 V, 100 kHz

(l) 15 V, 10 MHz

10.8. Design a noninverting amplifier at gain of 51. What is the output voltage if the input is $v_{in}(t) = 0.2 \sin t$ V.

10.9. Design a noninverting amplifier at gain of 1001. What is the output voltage if the input is $v_{in}(t) = 0.2 \sin t$ V.

Add, Subtract

10.10. Design a circuit to perform the following operations, x_1 , x_2 , x_3 , inputs; y , output.

(a) $y = x_1 + 10x_2$

(b) $y = x_1 + 5x_2 - 6x_3$

(c) $y = 10(x_1 + x_2 - x_3)$

(d) $y = -10(x_1 + x_2 - x_3)$

Differentiators

10.11. Design a differentiator circuit using inductors to have gain of 30. What is the output of the amplifier if the input is $v_{in} = 0.5 \sin 500t$.

10.12. Design a differentiator circuit using a capacitor to have gain of 100. What is the output of the amplifier if the input is $v_{in} = 0.05 \sin 500t$.

10.13. Using opamps build the following signals, wherein x , x_1 , x_2 , input; y , output.

(a) $y = 20 \frac{dx}{dt}$

(b) $y = -0.1 \frac{dx_1}{dt} + \frac{dx_2}{dt}$

(c) $y = 150 \frac{d^2x}{dt^2}$

Integrators

10.14. Design an integrator circuit using inductors to have gain of 30. What is the output of the amplifier if the input is $v_{in} = 0.5 \sin 500t$.

10.15. Design an integrator circuit using a capacitor to have gain of 100. What is the output of the amplifier if the input is $v_{in} = 0.05 \sin 500t$.

10.16. Using opamps build the following signals, wherein x , x_1 , x_2 , input; y , output.

(a) $y = 12 \int x dt$

(b) $y = 50 \int x_1 dt + 0.1 \int x_2 dt$

(c) $y = 800 \int \int x dt$

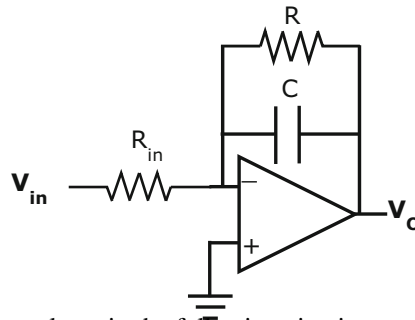
Build Analog Computers

10.17. Build an operational amplifier circuit to implement the following systems (x , input; y , output)

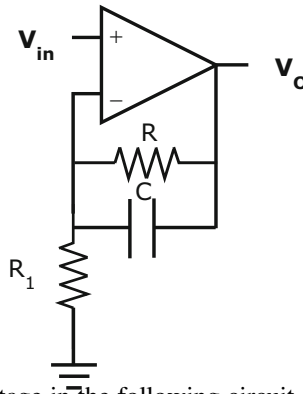
- (a) PID controller that takes error as $e = x - y$. A proportional, integral, and differential of the error is needed.
- (b) $\dot{y} + 3y = x$
- (c) $\ddot{y} + \dot{y} + 3y = x$
- (d) $\ddot{y} + 5\dot{y} + 6y = \int x dt$
- (e) $\ddot{y} + 11\dot{y} + 30y = \dot{x} + 5x$
- (f) $\ddot{y} + 11\dot{y} + 30y = \ddot{x} + \dot{x} + x$

10.18. Design an 8-bit digital to analog converter.

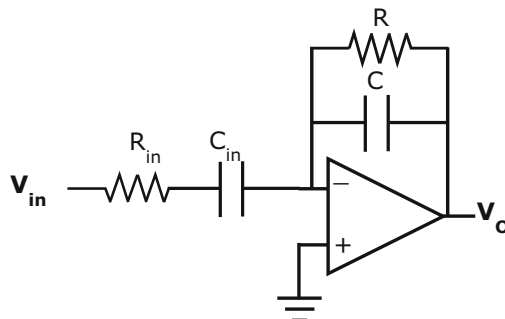
10.19. Find the output voltage in the following circuit.



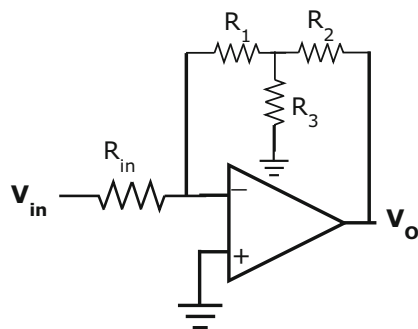
10.20. Find the output voltage in the following circuit.



10.21. Find the output voltage in the following circuit.



10.22. Find the output voltage in the following circuit.



Chapter 11

Active Filters



Introduction

Filters are frequency selective circuits. This means that they deliberately allow or block a range of frequencies while passing from input to output. Passive filters were circuits that accomplish this task by utilization of passive circuit elements such as R , L , and C . The maximum gain of the output signal is 100% of the input signal when operated at the resonant or at a frequency with orders of magnitude higher or lower than the cutoff frequency.

However, to amplify and filter the signals at the same time, an active element such as opamp can be utilized. These circuits are called active filters. The gain of these filters can be adjusted as it is explained in this chapter. There are several devices that can create active circuits: (1) a high gain operational amplifier (at least 60 dB), (2) a low gain voltage amplifier (20 dB or less), (3) a two-port device that can make the impedance connected on one terminal appear negative on the other terminal called negative immittance converter (NIC), and (4) a gyrator that converts capacitance to inductance and inductance to capacitance.

Active Low-Pass Filter

As it was explained earlier, the gain of an inverting amplifier utilizing opamp can be written as the negative ratio of the impedance at the feedback branch over the impedance at the input branch. Figure 11.1 is a reminder of how the operational amplifier gains when being used as an inverting amplifier can be obtained.

To design any type of filter, it should be noted that decreasing impedance at the feedback decreases the gain of the amplifier and decreasing the impedance at the input line increases the gain of the amplifier. The impedance on these branches can be adjusted by frequency which turns the circuit into an active filter.

Fig. 11.1 Inverting amplifier gain is directly proportional to the impedance at the feedback and inverse of the input impedance. The output voltage waveform is 180° out of phase from the input voltage

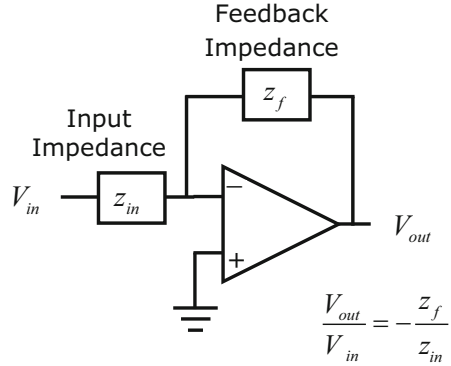
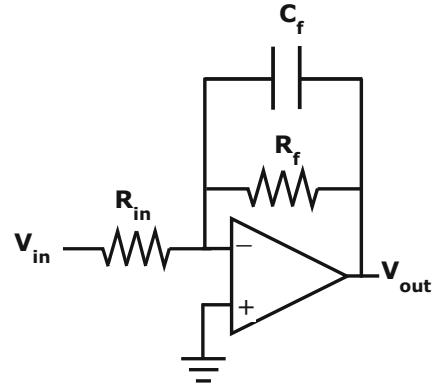


Fig. 11.2 Realizing an active LPF using a parallel RC circuit at the feedback. The capacitance in parallel to the resistor controls the impedance of the feedback at various frequencies. Hence, the gain at various frequencies can be controlled



Active Low-Pass Filters Using Feedback Impedance

To design a low-pass filter, if considered on the feedback line, the impedance must decrease as the frequency increases. In the meantime, it must be limited to a resistive circuit at DC to provide suitable amplification.

Consider a parallel RC circuit. As the frequency drops, the equivalent of an open loop capacitor and a parallel resistor is the resistance of the resistor. As the frequency increases, the capacitor becomes a short circuit and makes the equivalent circuit a short circuit. Once used on the feedback line, the change of impedance and its consequent amplifier gain change are aligned with what a LPF does. Figure 11.2 shows an active LPF circuit.

The transfer function of the filter is obtained by dividing the impedances of feedback over the input, as follows:

$$TF(s) = -\frac{R_f \parallel \frac{1}{sC_f}}{R_{in}}$$

$$TF(s) = -\frac{\frac{R_f \frac{1}{sC_f}}{R_f + \frac{1}{sC_f}}}{R_{in}} = -\frac{R_f}{R_{in}} \frac{1}{R_f C_f s + 1} = -\frac{R_f}{R_{in}} \frac{\frac{1}{R_f C_f}}{s + \frac{1}{R_f C_f}}$$

$$TF(s) = -K \frac{\omega_c}{s + \omega_c}$$

where $K = \frac{R_f}{R_{in}}$ is the gain of the filter and $\omega_c = \frac{1}{R_f C_f}$ is the cutoff frequency of the circuit.

Active Low-Pass Filters Using Input Impedance

The input impedance as stated before has a reverse effect on the gain of an inverted amplifier. To form a low-pass filter, the input impedance should be very low at low frequencies and should increase at higher frequencies. This means utilization of an inductor. To prevent high gain at low frequencies (inductor becomes a short circuit), a resistor needs to be connected in series with the inductor. Figure 11.3 shows the low-pass filter circuit realized with a RL series circuit in the input impedance.

The transfer function of the filter can be obtained as follows:

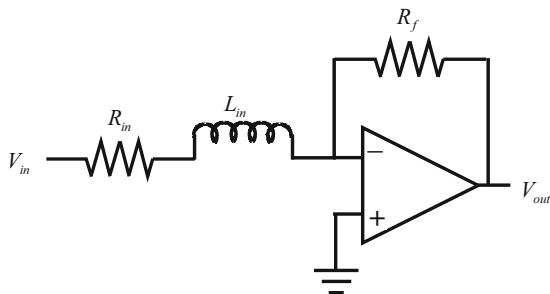
$$TF(s) = -\frac{R_f}{R_{in} + sL_{in}}$$

$$TF(s) = -\frac{R_f}{R_{in}} \frac{1}{1 + s\frac{L_{in}}{R_{in}}} = -\frac{R_f}{R_{in}} \frac{\frac{R_{in}}{L_{in}}}{s + \frac{R_{in}}{L_{in}}}$$

$$TF(s) = -K \frac{\omega_c}{s + \omega_c}$$

where $K = \frac{R_f}{R_{in}}$ is the gain of the filter and $\omega_c = \frac{R_{in}}{L_{in}}$ is the cutoff frequency of the circuit.

Fig. 11.3 Realizing an active LPF using a series RL circuit at the input



Active High-Pass Filters

In high-pass filters, the gain should increase as the frequency increases. That means if a feedback impedance is utilized, it should increase by frequency, and if an input impedance is utilized to realize the filter, the value of the input impedance should decrease as the frequency increases. At DC, the gain should be zero.

Active High-Pass Filters Using Feedback Impedance

Inductors' impedance increases by frequency. Therefore, a parallel connection of an RL circuit at the feedback performs as a high-pass filter. Figure 11.4 shows the high-pass filter realized by using a RL at the feedback. At low frequencies the inductor becomes a short and the gain of the amplifier is negligible. At high frequencies the inductor shows an extremely high impedance. The equivalent resistance in the feedback becomes the value of R_f which limits the gain of the amplifier. This type of frequency response resembles a high pass filter.

Transfer function of such filter can be obtained as follows:

$$TF(s) = -\frac{R_f \parallel sL_f}{R_{in}}$$

$$TF(s) = -\frac{\frac{R_f sL_f}{R_f + sL_f}}{R_{in}} = -\frac{R_f}{R_{in}} \frac{sL_f}{R_f + sL_f} = -\frac{R_f}{R_{in}} \frac{s}{s + \frac{R_f}{L_f}}$$

$$TF(s) = -K \frac{s}{s + \omega_c}$$

where $K = \frac{R_f}{R_{in}}$ is the gain of the filter and $\omega_c = \frac{R_f}{L_f}$ is the cutoff frequency of the circuit.

Fig. 11.4 Realizing an active HPF using a parallel RL circuit at the feedback

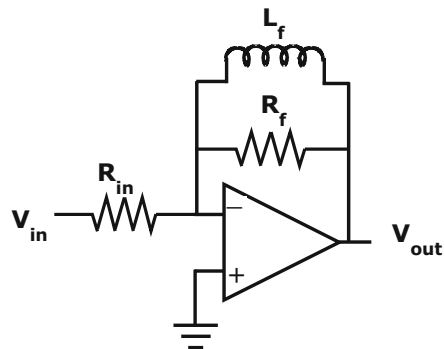
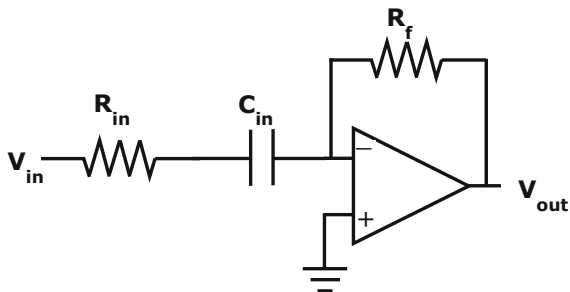


Fig. 11.5 Realizing an active HPF using a series RC circuit at the input



Active High-Pass Filters Using Input Impedance

To realize active high-pass filter by adjusting the input impedance of an inverting amplifier, a series RC circuit is utilized. The capacitor generates high impedance at DC, and this makes the gain of the amplifier become very small. As the frequency increases, the input impedance decrease and causes an increase in the gain. The filter circuit of an active high-pass filter is shown on Fig. 11.5.

The transfer function of the circuit can be obtained as follows:

$$TF(s) = -\frac{R_f}{R_{in} + \frac{1}{sC_{in}}}$$

$$TF(s) = -\frac{R_f}{R_{in} + \frac{1}{sC_{in}}} = -\frac{R_f}{R_{in}} \frac{s}{s + \frac{1}{R_{in}C_{in}}} = -K \frac{s}{s + \omega_c}$$

where $K = \frac{R_f}{R_{in}}$ is the gain of the filter and $\omega_c = \frac{1}{R_{in}C_{in}}$ is the cutoff frequency of the circuit.

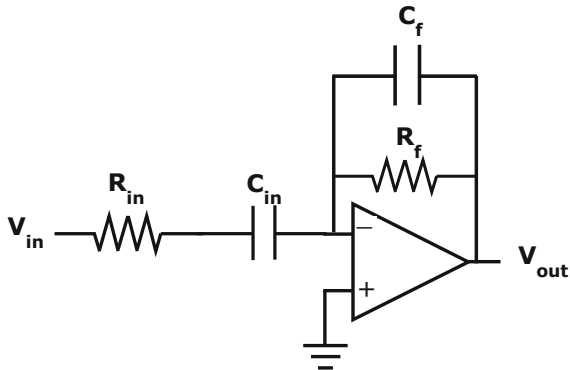
Active Band-Pass Filters

Band-pass filters can be made by either combining the low- and high-pass filters or by designing a new circuit approach. Figure 11.6 shows an active band-pass filter as a combination of both impedances on the feedback and in the input. Using RC circuits requires a parallel branch at the feedback and a series branch at the input. The gain is small at DC because of the high input impedance. The gain also drops at high frequencies because of the small impedance at the feedback.

The circuit transfer function is obtained as follows:

$$TF(s) = -\frac{R_f \parallel \frac{1}{sC_f}}{R_{in} + \frac{1}{sC_{in}}}$$

Fig. 11.6 Realizing an active BPF using a parallel RC at the feedback and a series RC circuit at input. This is a combination of LPF and HPF where the cutoff frequencies of the LPF and HPF are selected to form a BPF



$$\begin{aligned}
 TF(s) &= -\frac{\frac{R_f \frac{1}{sC_f}}{R_f + \frac{1}{sC_f}}}{\frac{R_{in}C_{in}s+1}{sC_{in}}} = -\frac{\frac{R_f \frac{1}{sC_f}}{\frac{R_f C_f s + 1}{sC_f}}}{\frac{R_{in}C_{in}s+1}{sC_{in}}} = -\frac{\frac{R_f \frac{1}{sC_f}}{\frac{R_f C_f s + 1}{sC_f}}}{\frac{R_{in}C_{in}s+1}{sC_{in}}} = -\frac{R_f}{\frac{R_{in}C_{in}s+1}{sC_{in}}} \\
 TF(s) &= -\frac{R_f C_{in}s}{(R_f C_f s + 1)(R_{in}C_{in}s + 1)} = -\frac{R_f C_{in}s}{R_f C_f R_{in}C_{in} \left(s + \frac{1}{R_f C_f}\right) \left(s + \frac{1}{R_{in}C_{in}}\right)} \\
 &= -\frac{R_f C_{in}s}{R_f C_f R_{in}C_{in} \left(s + \frac{1}{R_f C_f}\right) \left(s + \frac{1}{R_{in}C_{in}}\right)} \\
 &= -\frac{1}{C_f R_{in}} \frac{s}{\left(s^2 + \left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)s + \frac{1}{R_f C_f R_{in}C_{in}}\right)} \\
 &= -\frac{1}{C_f R_{in} \left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)} \frac{\left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)s}{\left(s^2 + \left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)s + \frac{1}{R_f C_f R_{in}C_{in}}\right)} \\
 &= -\frac{1}{\left(\frac{R_{in}}{R_f} + \frac{C_f}{C_{in}}\right)} \frac{\left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)s}{\left(s^2 + \left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)s + \frac{1}{R_f C_f R_{in}C_{in}}\right)} \\
 &= -k \frac{BWs}{(s^2 + BWs + \omega_0^2)}
 \end{aligned}$$

where $k = \frac{1}{\left(\frac{R_{in}}{R_f} + \frac{C_f}{C_{in}}\right)}$, bandwidth $BW = \left(\frac{1}{R_f C_f} + \frac{1}{R_{in}C_{in}}\right)$, and the natural frequency is

$$\omega_0^2 = \frac{1}{R_f C_f R_{in}C_{in}}.$$

Normally, to form a band-pass filter, the cutoff frequency of the high pass is lower than the cutoff frequency of the low pass. This creates a band of frequencies that pass

to the output. This circuit has no resonant frequency, and therefore the *unamplified maximum* value of the transfer function is less than 100%. Of course the operational amplifier can boost the maximum.

Active Band-Pass Filter Using a Combination of Low- and High-Pass Filters

A cascade combination of low-pass and high-pass filters can also perform as a band-pass filter as long as the cutoff frequencies of the low pass and high pass meet certain criterion. The filter is shown in Fig. 11.7.

The following procedures can be utilized to design and implement active band-pass filters (Fig. 11.8):

1. Design an active low-pass filter with unity gain and cutoff frequency of ω_{CLP} .
2. Design an active high-pass filter with unity gain and cutoff frequency of ω_{CHP} .

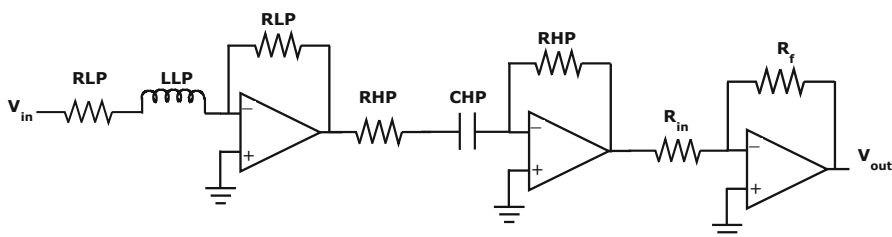
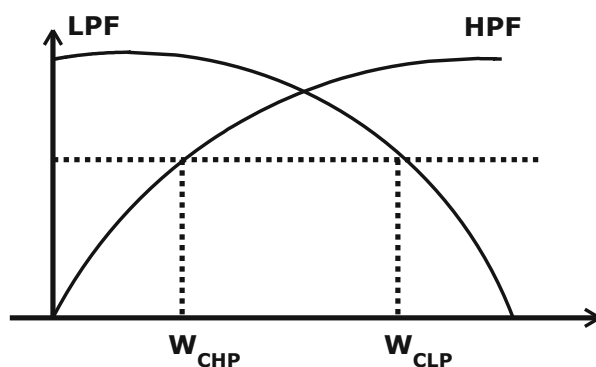


Fig. 11.7 Cascade connection of two active filters with LPF and HPF configurations and a final gain stage. Overall, the circuit operates as a BPF. The cutoff frequency of LPF is larger than the cutoff frequency of HPF

Fig. 11.8 In a BPF, the cutoff frequency of LPF can be considered larger than the cutoff frequency of HPF



To create a band-pass filter, the cutoff frequency of high-pass filter should be lower than the cutoff frequency of the low-pass filter, or:

$$\omega_{\text{CHP}} < \omega_{\text{CLP}}$$

3. The last stage is to apply gain to the filtered signal and amplify the signals in the desired band of frequencies.

Unity gain low-pass and high-pass filters are desired because they do not amplify the signal that is going to be filtered by the next stage. Once the signal is filtered, a suitable gain is applied to amplify it to a desired level.

Transfer Function of a Band-Pass Filter

The circuit transfer function is obtained by series connection of three blocks as shown in Fig. 11.9.

Transfer function of the overall system is obtained as follows:

$$TF(s) = -\frac{\frac{R_{\text{LP}}}{L_{\text{LP}}}}{s + \frac{R_{\text{LP}}}{L_{\text{LP}}}} \times -\frac{s}{s + \frac{1}{R_{\text{HP}}C_{\text{HP}}}} \times -\frac{R_f}{R_{\text{in}}}$$

$$TF(s) = -k \left(\frac{\omega_{\text{CLP}}}{s + \omega_{\text{CLP}}} \times \frac{s}{s + \omega_{\text{CHP}}} \right)$$

The gain of this filter is set by the last stage and equals $\frac{R_f}{R_{\text{in}}}$, and the cutoff frequencies ω_{CLP} and ω_{CHP} can be obtained from individual filters.

Active Band-Reject Filters

To obtain a band-reject filter, the combination of a low-pass and high-pass filters should change from series to parallel, and their summative effort should be amplified in the third stage. A band-reject filter circuit which is formed by adding the output of a low-pass filter to a high-pass filter is shown in Fig. 11.10.

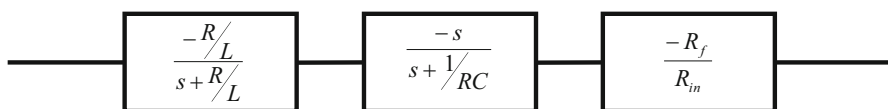


Fig. 11.9 Block diagram equivalent of the BPF which was a combination of transfer function of the LPF and HPF, and the gain stage

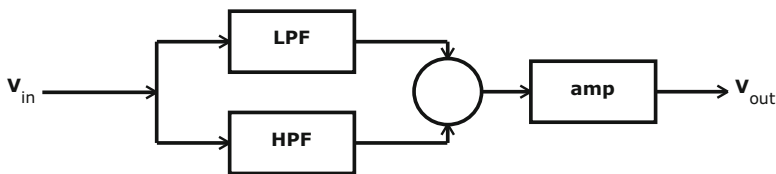
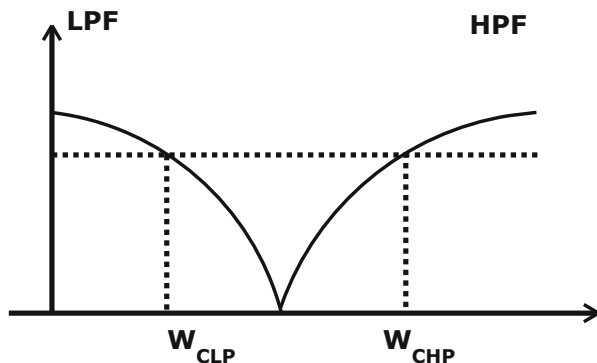


Fig. 11.10 Realizing an active BRF using a combination of LPF and HPF

Fig. 11.11 Realizing a BPF by a combination of LPF and HPF. The cutoff frequency of the LPF is lower than the cutoff frequency of the HPF. $\omega_{CHP} > \omega_{CLP}$



The following procedures can be utilized to design and implement active band-reject filters (Fig. 11.11):

1. Design an active low-pass filter with unity gain and cutoff frequency of ω_{CLP} .
2. Design an active high-pass filter with unity gain and cutoff frequency of ω_{CHP} .

To create a band-reject filter, the cutoff frequency of high-pass filter should be higher than the cutoff frequency of the low-pass filter, or:

$$\omega_{CHP} > \omega_{CLP}$$

3. The last stage is to apply gain to the filtered signal and amplify the signals in the desired band of frequencies.

The filter transfer function can be obtained as follows (Fig. 11.12):

$$TF(s) = - \left(\frac{\frac{1}{R_{LP}C_{LP}}}{s + \frac{1}{R_{LP}C_{LP}}} + \frac{s}{s + \frac{1}{R_{HP}C_{HP}}} \right) \times -\frac{R_f}{R_{in}}$$

$$TF(s) = k \left(\frac{\omega_{CLP}}{s + \omega_{CLP}} + \frac{s}{s + \omega_{CHP}} \right)$$

Fig. 11.12 Realization of a BRF by summation of a LPF and HPF

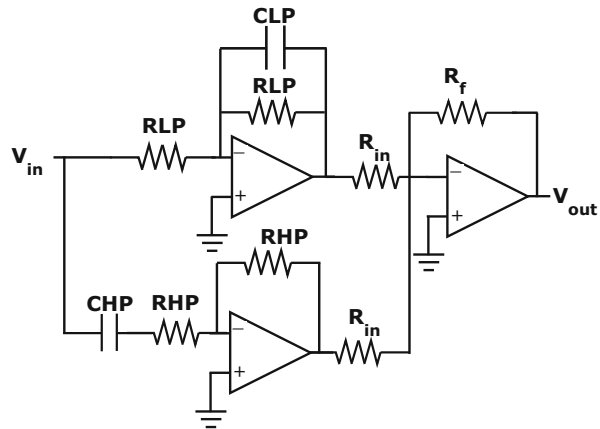
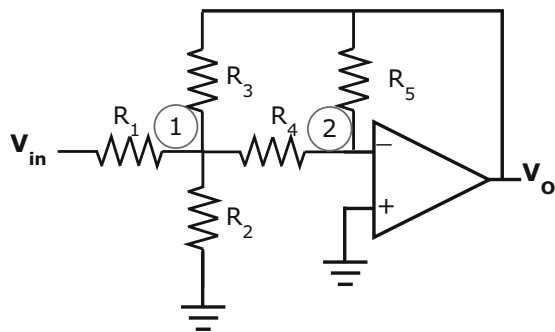


Fig. 11.13 Multiple feedback circuit



Multiple Feedback Opamp Circuits (MFB)

This circuit receives two sets of feedback from the output voltage. Figure 11.13 shows a MFB circuit in which the nodes 1 and 2 receive feedback through resistors R_3 and R_5 , respectively.

The circuit analysis shows that the voltage of node 2 is zero because it is virtually connected to the noninverting port of the opamp. Therefore, the current of I_{R_5} becomes:

$$I_{R_5} = \frac{0 - V_o}{R_5}$$

It is required to find the voltage of node 1. This results in the currents of resistors.

KCL in node 1 shows: (It is assumed that the input voltage V_{in} sources the current to the circuit through R_1 . Therefore:

$$-\frac{V_{in} - V_1}{R_1} + \frac{V_1 - V_o}{R_3} + \frac{V_1}{R_2} + \frac{V_1 - V_2}{R_4} = 0$$

$$V_2 = 0$$

$$V_1 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) = \frac{V_{in}}{R_1} + \frac{V_o}{R_3}$$

Since the input impedance of the Opamp is infinite:

$$I_{R_5} = I_{R_4}$$

$$-\frac{V_o}{R_5} = \frac{V_1}{R_4}$$

Or:

$$V_1 = -\frac{R_4}{R_5} V_o$$

Replacing in the KCL equation results in:

$$\begin{aligned} -\frac{R_4}{R_5} V_o \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) &= \frac{V_{in}}{R_1} + \frac{V_o}{R_3} \\ -\left(\frac{R_4}{R_5} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) + \frac{1}{R_3} \right) V_o &= \frac{V_{in}}{R_1} \\ V_o &= -\frac{V_{in}}{R_1 \left(\frac{R_4}{R_5} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) + \frac{1}{R_3} \right)} \end{aligned}$$

Considering:

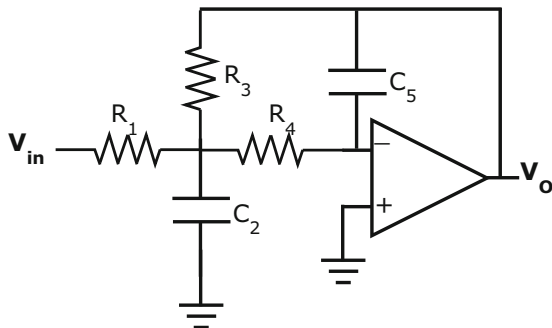
$$\begin{aligned} \frac{1}{R_1} &= G_1, \frac{1}{R_2} = G_2, \frac{1}{R_3} = G_3, \frac{1}{R_4} = G_4, \frac{1}{R_5} = G_5 \\ \frac{V_o}{V_{in}} &= \frac{-G_1 G_4}{G_5(G_1 + G_2 + G_3 + G_4) + G_3 G_4} \end{aligned}$$

Creating a Low-Pass Filter

The circuit can be converted to a low-pass filter with the following replacements:

- The conductance of G_2 is replaced by a capacitor of admittance $j\omega C_2$.
- The conductance of G_5 is replaced by a capacitor of admittance $j\omega C_5$.

Fig. 11.14 Low-pass filter using multiple feedback operational amplifier



Considering these replacements, the circuit of a LPF is shown in Fig. 11.14. The transfer function in steady state sinusoidal analysis becomes:

$$\frac{V_0}{V_{in}} = \frac{-G_1 G_4}{j\omega C_5 (G_1 + j\omega C_2 + G_3 + G_4) + G_3 G_4}$$

Simplification results in:

$$\frac{V_0}{V_{in}} = \frac{-G_1 G_4}{(G_3 G_4 - \omega^2 C_2 C_5) + j\omega C_5 (G_1 + G_3 + G_4)}$$

At low frequencies,

$$\omega \rightarrow 0, \quad \frac{V_0}{V_{in}} = \frac{-G_1 G_4}{(G_3 G_4 - 0^2 C_2 C_5) + j0 C_5 (G_1 + G_3 + G_4)} = -\frac{G_1}{G_3}$$

The DC gain of the amplifier is $\frac{G_1}{G_3}$.

At high frequencies

$$\omega \rightarrow \infty, \quad \frac{V_0}{V_{in}} = \frac{-G_1 G_4}{(G_3 G_4 - \infty^2 C_2 C_5) + j\infty C_5 (G_1 + G_3 + G_4)} = -\frac{G_1 G_4}{\infty} = 0$$

This shows the behavior of a LPF. Considering

$$C_2 = C_5 = C \text{ \& } G_1 = G_3 = G_4 = G$$

$$\omega_c = \frac{1}{RC}$$

The transfer function of the filter in Laplace domain can be expressed by considering $s = j\omega$ as follows:

$$\frac{V_0}{V_{in}} = \frac{-G_1 G_4}{s^2 C_2 C_5 + s C_5 (G_1 + G_3 + G_4) + G_3 G_4}$$

$$\frac{V_0}{V_{in}} = \frac{-\frac{G_1 G_4}{C_2 C_5}}{s^2 + s \frac{(G_1 + G_3 + G_4)}{C_2}} + \frac{G_3 G_4}{C_2 C_5}$$

Creating a High-Pass Filter

To build a HPF, the following elements need to be replaced.

- The conductance of G_1 is replaced by a capacitor of admittance $j\omega C_1$
- The conductance of G_3 is replaced by a capacitor of admittance $j\omega C_3$
- The conductance of G_4 is replaced by a capacitor of admittance $j\omega C_4$

The circuit configuration is shown in Fig. 11.15.

Considering the circuit element replacements, the ratio of voltages becomes

$$\frac{V_0}{V_{in}} = \frac{\omega^2 C_1 C_4}{G_5(j\omega C_1 + G_2 + j\omega C_3 + j\omega C_4) - \omega^2 C_3 C_4}$$

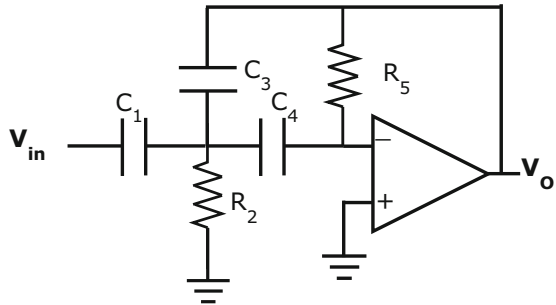
$$\frac{V_0}{V_{in}} = \frac{\omega^2 C_1 C_4}{(G_2 - \omega^2 C_3 C_4) + j\omega G_5(C_1 + C_3 + C_4)}$$

At low frequency,

$$\omega \rightarrow 0, \quad \frac{V_0}{V_{in}} = \frac{0^2 C_1 C_4}{(G_2 - 0^2 C_3 C_4) + j0 G_5(C_1 + C_3 + C_4)} = 0$$

At high frequency,

Fig. 11.15 High-pass filter using multiple feedback operational amplifier



$$\omega \rightarrow \infty, \frac{V_0}{V_{in}} = \frac{-\omega^2 C_1 C_4}{\sqrt{(G_2 - \omega^2 C_3 C_4)^2 + (\omega G_5 (C_1 + C_3 + C_4))^2}} = \frac{-\omega^2 C_1 C_4}{-\omega^2 C_3 C_4} = \frac{C_1}{C_3}$$

The zero DC gain, very low gains at low frequencies, and gain of $\frac{C_1}{C_3}$ at high frequency correspond with characteristics of a HPF. Considering:

$$C_1 = C_3 = C_4 = C \& G_2 = G_5 = G$$

The cutoff frequency is:

$$\omega_c = \frac{1}{RC}$$

The transfer function of the HPF is:

$$\begin{aligned} \frac{V_0}{V_{in}} &= \frac{-s^2 C_1 C_4}{s^2 C_3 C_4 + s G_5 (C_1 + C_3 + C_4) + G_2 G_5} \\ \frac{V_0}{V_{in}} &= \frac{-s^2 \frac{C_1 C_4}{C_3 C_4}}{s^2 + s \frac{G_5 (C_1 + C_3 + C_4)}{C_3 C_4} + \frac{G_2 G_5}{C_3 C_4}} \\ \frac{V_0}{V_{in}} &= \frac{-s^2 \frac{C_1}{C_3}}{s^2 + s \frac{G_5 (C_1 + C_3 + C_4)}{C_3 C_4} + \frac{G_2 G_5}{C_3 C_4}} \end{aligned}$$

Creating a Band-Pass Filter

The following replacements in the MFB circuit make it a BPF.

- The conductance of G_3 is replaced by a capacitor of admittance $j\omega C_3$.
- The conductance of G_4 is replaced by a capacitor of admittance $j\omega C_4$.

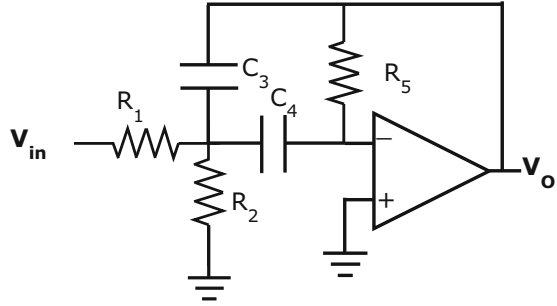
The circuit configuration is shown in Fig. 11.16.

The circuit transfer function becomes:

$$\begin{aligned} \frac{V_0}{V_{in}} &= \frac{-j\omega G_1 C_4}{G_5 (G_1 + G_2 + j\omega C_3 + j\omega C_4) - \omega^2 C_3 C_4} \\ \frac{V_0}{V_{in}} &= \frac{-j\omega G_1 C_4}{-\omega^2 C_3 C_4 + G_5 (G_1 + G_2) + j\omega G_5 (C_3 + C_4)} \end{aligned}$$

At low frequencies:

Fig. 11.16 Band-pass filter using multiple feedback operational amplifier



$$\omega \rightarrow 0, \quad \frac{V_0}{V_{in}} = \frac{-j\omega G_1 C_4}{-\omega^2 C_3 C_4 + G_5(G_1 + G_2) + j\omega G_5(C_3 + C_4)} = 0$$

At high frequencies:

$$\omega \rightarrow \infty, \quad \frac{V_0}{V_{in}} = \frac{-j\omega G_1 C_4}{-\omega^2 C_3 C_4 + G_5(G_1 + G_2) + j\omega G_5(C_3 + C_4)} = \frac{1}{\omega} = 0$$

Somewhere between these high and low frequencies, the amplitude increases and becomes non-zero. This resembles the behavior of a BPF.

The transfer function of the BPF becomes:

$$\begin{aligned} \frac{V_0}{V_{in}} &= \frac{-sG_1 C_4}{s^2 C_3 C_4 + sG_5(C_3 + C_4) + G_5(G_1 + G_2)} \\ \frac{V_0}{V_{in}} &= \frac{-s\frac{G_1}{C_3}}{s^2 + s\frac{G_5(C_3 + C_4)}{C_3 C_4} + \frac{G_5(G_1 + G_2)}{C_3 C_4}} \end{aligned}$$

The bandwidth of the filter is:

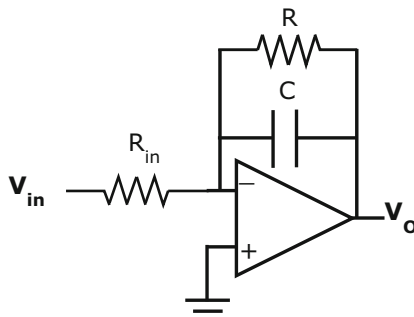
$$BW = \frac{G_5(C_3 + C_4)}{C_3 C_4}$$

The filter gain is:

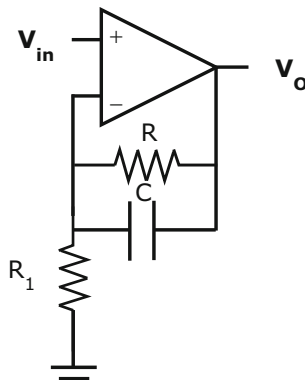
$$\text{Gain} = \frac{G_1 C_4}{G_5(C_3 + C_4)}$$

Problems

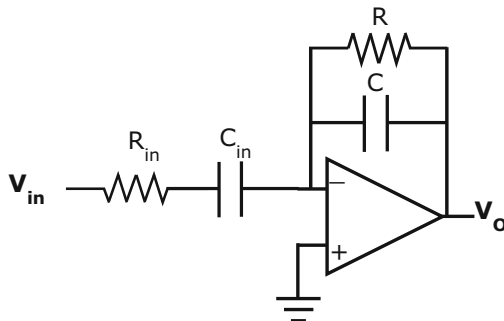
- 11.1. Design an inverting active low-pass filter to amplify the frequencies below 1kHz with gain of $K = 150$.
- 11.2. Design an inverting active high-pass filter to amplify the frequencies above 500Hz with gain of $K = 100$.
- 11.3. Design an active band-pass filter to amplify the frequencies between 500 and 1000Hz with gain of $K = 200$.
- 11.4. Design an active band-reject filter to eliminate the frequencies between $500 - 1000\text{Hz}$ with gain of $K = 200$.
- 11.5. Determine the type of filter in the following circuit.



- 11.6. Determine if the following circuit can be a filter?



- 11.7. Determine the type of filter in the following circuit.



- 11.8. Design a second-order active low-pass filter to amplify the frequencies below 1 kHz with gain of $K = 150$. Sketch the circuit and determine the component values.
- 11.9. Design a second-order active high-pass filter to amplify the frequencies above 500 Hz with gain of $K = 100$. Sketch the circuit and determine the component values.
- 11.10. Design a second-order active band-pass filter to amplify the frequencies between 500 and 1000 Hz with gain of $K = 200$. Sketch the circuit and determine the component values.
- 11.11. Design a second-order active band-reject filter to eliminate the frequencies between 500 and 1000 Hz with gain of $K = 200$. Sketch the circuit and determine the component values.

Chapter 12

Two-Port Networks



Introduction

A network is a combination of one or several electric circuits that together perform a specific action. The network might have several inputs that receive excitations and outputs to show the results. A port is a set of two terminals that allows for a source to connect and excite the network or connect a measurement device and record the response. For instance, a voltage source connected to a port of a network may cause currents to flow through the network and voltage drops to appear across the elements. To measure any of these currents or voltages, terminals may be extended to demonstrate some measurement locations, forming an output port. This forms a two-port network. Similarly, two-port networks can be easily expanded to multiple-port networks each showing a parameter in the circuit. Figure 12.1 shows a circuit and the process to consider it as a two-port network.

The benefit of showing a circuit as a two-port network is the capability of representing the network in mathematical forms. Similar to transfer functions, the mathematical terms are defined for a specified input port to a desired output port. The mathematical expressions present several characteristics such as impedance, admittance, transmission, or a combination of these terms in hybrid forms. Each of these characteristics is unique and is defined regardless of the type of the input and the type of the output. It means that only ports matter and the type of source or element connected to the port is irrelevant to the mathematical expression of the network. Once the network is presented in terms of impedance, admittance, or others, the network can be used in mathematical forms, and the response to several excitations can be obtained. Networks can also be connected in series or parallel and depending on their characteristics and their mathematical terms can be combined to represent a larger network. However, that network is also presented as a two-port network. Figure 12.2 shows a network matrix N and the set of input and output ports. In this network, the set of voltages at the input and output ports are V_1 , V_2 , and the set of currents at those ports are I_1 , I_2 . These voltages and currents might be AC or DC.

Fig. 12.1 A circuit showing an input terminal that connects to a source V_1 and an output port which is the measured voltage across the capacitor

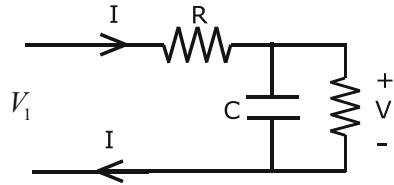
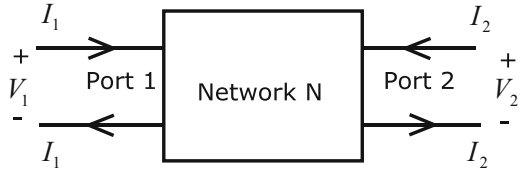


Fig. 12.2 Terminals of a two-port network system N



In a two-port network, the voltage and current in the input port will cause a voltage or current in the output port. This means that the presentations of the network have to fit in a 2×2 matrix. Depending on the grouping of the parameters, several presentations can be obtained.

For instance:

$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = N \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ where N shows the impedance of the network and hereafter is shown by Z :

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Z_{2 \times 2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = N \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, where N shows the admittance of the network and hereafter is shown by Y :

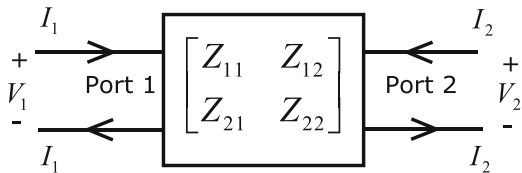
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = Y_{2 \times 2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = N \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$, where N shows a transmission matrix and hereafter is shown by T :

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = T_{2 \times 2} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

This chapter discusses these mathematical representations and the process to obtain them.

Fig. 12.3 System network is shown as impedance



Impedance Matrix of a Two-Port Network

In this representation of the network, the entire circuit is expressed by a $Z_{2 \times 2}$ matrix that defines transfer functions as the ratio of port voltages to the network's port currents. The elements of the matrix show how the voltage at the network ports varies by the current variations through the ports. It is considered that the direction of currents is entering the ports (Fig. 12.3).

The equations that represent an impedance network are as follows:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Z_{2 \times 2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{cases} V_1 = Z_{11}I_1 + Z_{12}I_2 \\ V_2 = Z_{21}I_1 + Z_{22}I_2 \end{cases}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Impedance matrix has four elements of Z_{11} , Z_{12} , Z_{21} , Z_{22} :

- Z_{11} shows how the voltage of port 1 is related to the current of port 1.
- Z_{12} shows how the voltage of port 1 is related to the current of port 2.
- Z_{21} shows how the voltage of port 2 is related to the current of port 1.
- Z_{22} shows how the voltage of port 2 is related to the current of port 2.

To measure each of the impedance matrix elements, only the voltage and current related to the element will have values, and the other parameters are considered zero. For instance:

$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0}$$

Note 12.1 This shows that to measure or calculate Z_{11} , only V_1 and I_1 are required while imposing $I_2 = 0$. This requires that the source at the second port be removed or disconnected to prevent current I_2 from flowing.

Similarly:

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0}$$

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0}$$

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0}.$$

Note. 12.2 In networks *without dependent sources*, the matrix elements Z_{12} and Z_{21} will become similar, i.e.:

$$Z_{12} = Z_{21}$$

These networks are called *reciprocal*.

Equivalent of an Impedance Network

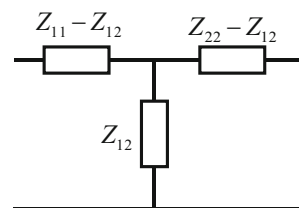
Most of the time, a two-port network can contain multiple loops and nodes. A matrix representation may also not be suitable when cascade networks exist or when part of the original network is missing. Therefore, the two-port network can be simplified to a *T*-equivalent network.

Reciprocal Networks

T Model

As mentioned earlier, an impedance network might be reciprocal if it does not contain dependent sources. In a reciprocal network, a two-loop network that shares a common element is presented. The element that is shared in both loops is either Z_{12} or Z_{21} , as $Z_{12} = Z_{21}$. Other elements in the first loop is $Z_{11} - Z_{12}$ and in the second loop $Z_{22} - Z_{12}$. This forms a *T* network as shown in Fig. 12.4.

Fig. 12.4 *T* equivalent for an impedance network



Nonreciprocal Networks

In a nonreciprocal network, there exists at least one dependent source that makes the Z_{12} and Z_{21} different. There are two approaches to model the impedance matrix by (1) having two separate loops or (2) having loops that share one element.

Separate Loop Model

In this model (shown in Fig. 12.5), the first loop current I_1 flows through Z_{11} . The loop contains a current-dependent voltage source with value of $Z_{12}I_2$. Therefore the KVL becomes:

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

In the second loop, the current I_2 flows through impedance Z_{22} . A current-dependent voltage source with value of $Z_{21}I_1$ is also added to form a KVL as:

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Element-Sharing Loops

When two loops share one element, the impedance matrix must be written such that a positive element exists in each of the four components, as described below.

Consider the original impedance matrix as follows:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

Consider a shared element Z_m exists in all four components of Z_{11} , Z_{12} , Z_{21} , Z_{22} . Therefore, the impedance matrix becomes:

$$Z = \begin{bmatrix} Z_1 + Z_m & Z_m + \alpha \\ Z_m + \beta & Z_2 + Z_m \end{bmatrix}$$

The equivalent element in loop 1 of the T model becomes impedance Z_1 and a current-dependent voltage source with value of αI_2 . The shared element between the loops becomes Z_m , and the second loop becomes the impedance Z_2 and a current-dependent voltage source with value of βI_1 . The equivalent circuit is shown in Fig. 12.6.

Fig. 12.5 Impedance network equivalent in separate loops. The dependent voltage sources show the dependency of the generated voltages

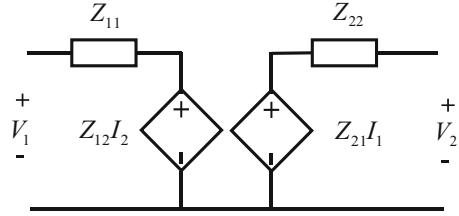
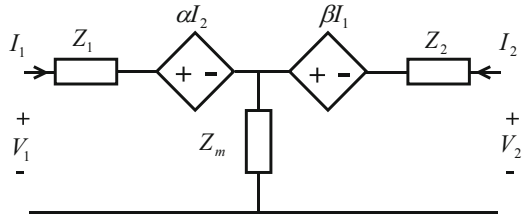


Fig. 12.6 Element-sharing impedance equivalent loops



Example 12.1 Find the impedance matrix of the circuit shown in Fig. 12.7.

Solution The two-port network shown in Fig. 12.1 forms a T network with a shared element. To obtain the impedance matrix, there are multiple methods. Some of them are explained in this example.

Method 1 Classical approach

In this method, the definition equations such as $Z_{11} = \frac{V_1}{I_1} \Big|_{I_2=0}$ are used. To obtain Z_{11} , port 2 is considered without a source; therefore, $I_2 = 0$. That makes the circuit as shown in Fig. 12.8.

Since $I_2 = 0$, a KVL in this loop results in:

$$V_1 = 10I_1 + 20I_1 = 30I_1$$

The ratio of $\frac{V_1}{I_1}$ is obtained as follows:

$$Z_{11} = \frac{V_1}{I_1} = 30$$

In the same circuit, the voltage generated at port 2 can be calculated as:

$$V_2 = 20I_1$$

The impedance element Z_{21} can be calculated as follows:

$$Z_{21} = \frac{V_2}{I_1} \Big|_{I_2=0} = \frac{20I_1}{I_1} = 20$$

Fig. 12.7 Circuit of Example 12.1

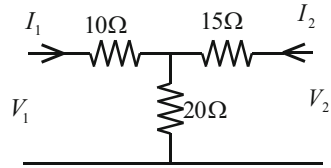


Fig. 12.8 Circuit when the output current is forced to zero

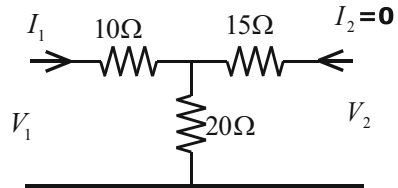
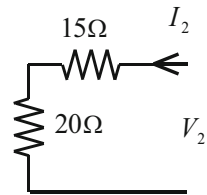


Fig. 12.9 Circuit when the input current is forced to zero



Now considering the current of port 1 as zero, $I_1 = 0$, the circuit is shown in Fig. 12.9.

In this case, Z_{22} and Z_{12} can be calculated as follows:

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0}$$

KVL in loop 2, when $I_1 = 0$, results in:

$$V_2 = 15I_2 + 20I_2 = 35I_2$$

The ratio results in:

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0} = \frac{35I_2}{I_2} = 35 \, \Omega$$

When $I_1 = 0$, the voltage in port 1 is calculated as:

$$V_1 = 20I_2$$

The element Z_{12} can be calculated as follows:

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0} = \frac{20I_2}{I_2} = 20 \Omega$$

Since there is no dependent source in the circuit, the impedance matrix is reciprocal, meaning that:

$$Z_{12} = Z_{21}$$

And by calculations, it was demonstrated that:

$$Z_{12} = Z_{21} = 20 \Omega.$$

The impedance matrix becomes:

$$Z = \begin{bmatrix} 30 & 20 \\ 20 & 35 \end{bmatrix}$$

Method 2 Using T matrix analysis

As Fig. 12.7 shows, a T network exists. Comparing side by side with the T network, Z_{11} can be obtained by adding all impedances existing in loop 1. It means:

In loop 1:

$$Z_{11} = 10 + 20 = 30\Omega$$

In loop 2:

$$Z_{22} = 15 + 20 = 35\Omega$$

The shared element between two loops is the $Z_{12} = Z_{21}$. Therefore:

$$Z_{12} = Z_{21} = 20\Omega$$

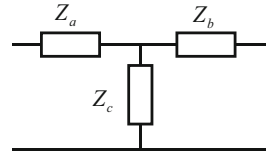
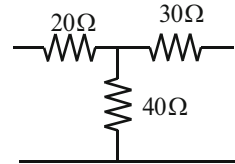
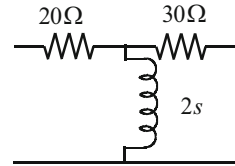
The impedance matrix becomes:

$$Z = \begin{bmatrix} 30 & 20 \\ 20 & 35 \end{bmatrix}$$

Alternative Approach in Impedance Matrix

Consider the circuit shown in Fig. 12.10.

The impedance matrix becomes:

Fig. 12.10 A T network

Fig. 12.11 Circuit of Example 12.2

Fig. 12.12 Circuit of Example 12.3


$$Z = \begin{bmatrix} Z_a + Z_c & Z_c \\ Z_c & Z_b + Z_c \end{bmatrix}$$

Example 12.2 Find the impedance matrix of the circuit shown in Fig. 12.11.

Solution

$$Z = \begin{bmatrix} 20 + 40 & 40 \\ 40 & 30 + 40 \end{bmatrix} = \begin{bmatrix} 60 & 40 \\ 40 & 70 \end{bmatrix}$$

Example 12.3 Find the impedance matrix of the circuit shown in Fig. 12.12.

Solution

$$Z = \begin{bmatrix} 20 + 2s & 2s \\ 2s & 30 + 2s \end{bmatrix}$$

Example 12.4 Find the impedance matrix of the circuit shown in Fig. 12.13.

Solution

$$Z = \begin{bmatrix} 16 + 12 & 12 \\ 12 & 0 + 12 \end{bmatrix} = \begin{bmatrix} 28 & 12 \\ 12 & 12 \end{bmatrix}$$

Fig. 12.13 Circuit of Example 12.4

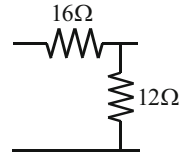


Fig. 12.14 Circuit of Example 12.5

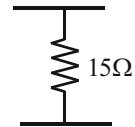
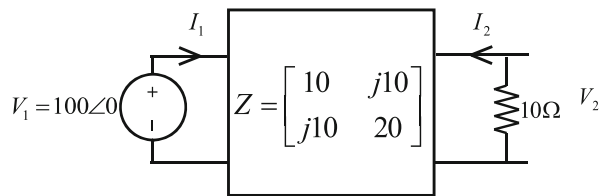


Fig. 12.15 Circuit of Example 12.6



Example 12.5 Find the impedance matrix of the circuit shown in Fig. 12.14.

Solution

$$Z = \begin{bmatrix} 0 + 15 & 15 \\ 15 & 0 + 15 \end{bmatrix} = \begin{bmatrix} 15 & 15 \\ 15 & 15 \end{bmatrix}$$

Example 12.6 The output of an impedance matrix (Fig. 12.13) feeds a 10 Ω resistor. If the input port voltage is $V_1 = 100 \angle 0$ V, find the currents I_1 and I_2 and the output voltage (Figs. 12.14 and 12.15).

Solution The impedance matrix shows the relations of the input and output port voltages and currents as:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10 & j10 \\ j15 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The output port voltage V_2 is a function of the load resistance and current as follows:

$$V_2 = -10I_2$$

Replacing $V_1 = 100 \angle 0$ and $V_2 = -10I_2$, the network equations become:

$$\begin{bmatrix} 100 \\ -10I_2 \end{bmatrix} = \begin{bmatrix} 10 & j10 \\ j15 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Bringing the $-10I_2$ back to the right side, the equations read as:

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & j10 \\ j15 & 20 + 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The currents can be found as:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 & j10 \\ j15 & 20 + 10 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{10 \times 30 - j15 \times j10} \begin{bmatrix} 30 & -j10 \\ -j15 & 10 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{450} \begin{bmatrix} 30 & -j10 \\ -j15 & 10 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3000}{450} \\ \frac{-j15 \times 100}{450} \end{bmatrix} = \begin{bmatrix} 6.67 \\ -j3.33 \end{bmatrix} A$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 6.67 \\ -j3.33 \end{bmatrix} A$$

Another approach to find currents is through Cramer's method as follows:

$$I_1 = \frac{\begin{vmatrix} 100 & j10 \\ 0 & 30 \end{vmatrix}}{\begin{vmatrix} 10 & j10 \\ j15 & 30 \end{vmatrix}} = \frac{3000}{450} = 6.67 A$$

$$I_2 = \frac{\begin{vmatrix} 10 & 100 \\ j15 & 0 \end{vmatrix}}{\begin{vmatrix} 10 & j10 \\ j15 & 30 \end{vmatrix}} = \frac{-j1500}{450} = -j3.33 A$$

Finding Impedance Matrix in Multi-loop Networks

In case the network contains any configuration except the T form, it may form more than two loops. However, there are still two ports that represent the entire network, i.e., the impedance matrix still has a 2×2 dimension.

Current and Voltage Considerations

- The voltage and current in the input port are V_1 and I_1 , with the current entering the port.
- The voltage and current in the output port are V_2 and I_2 , with the current entering the port.
- The direction of current in other loops is arbitrary.
- In case the impedance of input and output ports is not placed in the first and second equations, the rows and columns can be exchanged to shift the desired equations in their designated places. It is recommended to have the input equation in the first row and the output equation in the second row.

Finding the Matrix Dimension and Its Elements

- Consider a square matrix $Z_n \times n$ with its dimension matching the number of loops; i.e. for a three-loop system, $n = 3$, a $Z_3 \times 3$ matrix is obtained.
- The element Z_{ii} on the diagonal is obtained by the summation of all impedances in the loop i .
- The off-diagonal elements Z_{ij} are the shared elements between loop i and loop j .
 - If the current direction of these loops is similar through the shared element, a $+Z_{ij}$ is obtained.
 - If the current direction of these loops does not match through the shared element, a $-Z_{ij}$ is obtained.

General Form of KVL Equations

Following the abovementioned rules, a general impedance matrix equation is obtained with constant matrix elements being zero except for the V_1 and V_2 . For a network with n loops, the general form is:

$$\begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix}$$

A 2×2 matrix can be selected/partitioned from this general model that contains equations that have values other than zero. The system can be expressed as:

$$\begin{bmatrix} A_{2 \times 2} & M_{2 \times (n-2)} \\ N_{(n-2) \times 2} & D_{(n-2) \times (n-2)} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0_{(n-2) \times 1} \end{bmatrix}$$

Matrix Size Reduction

The size of split general equation can be reduced to the size of non-zero element constant matrix. In this case, V_1 and V_2 , i.e., the size is 2. The impedance matrix of a two-port representation of the multi-loop is as follows:

$$Z_{2 \times 2} = A_{2 \times 2} - M_{2 \times (n-2)} D^{-1}_{(n-2) \times (n-2)} N_{(n-2) \times 2}$$

Removing the matrix dimensions results in:

$$Z = A - MD^{-1}N$$

Example 12.7 Find the impedance matrix of the circuit shown in Fig. 12.16.

Solution There are multiple approaches in obtaining the impedance matrix of this network. In this example the impedance matrix is obtained by two methods. Later in this chapter, another method is introduced to simplify the solution.

Method 1 Using the definitions:

$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \Rightarrow V_1 = I_1(20 \parallel (5 + 15)) = 10I_1 \rightarrow Z_{11} = \frac{V_1}{I_1} = \frac{10I_1}{I_1} = 10$$

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} \Rightarrow V_2 = \frac{15}{5+15} V_1; (V_1 = 10I_1) \rightarrow Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} = \frac{\frac{15}{5+15} 10I_1}{I_1} = 7.5$$

$$\begin{aligned} Z_{22} &= \left. \frac{V_2}{I_2} \right|_{I_1=0} \Rightarrow V_2 = I_2(15 \parallel (20 + 5)) = 9.375I_2 \rightarrow Z_{22} \\ &= \frac{V_2}{I_2} = \frac{9.375I_2}{I_2} = 9.375 \end{aligned}$$

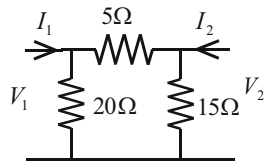
$$\begin{aligned} Z_{12} &= \left. \frac{V_1}{I_2} \right|_{I_1=0} \Rightarrow V_1 \\ &= \frac{20}{5+20} V_2; (V_2 = 9.375I_2) \rightarrow Z_{12} = \frac{V_1}{I_2} = \frac{\frac{20}{5+20}(9.375I_2)}{I_2} = 7.5 \end{aligned}$$

$$Z = \begin{bmatrix} 10 & 7.5 \\ 7.5 & 9.375 \end{bmatrix}$$

Method 2 Using a three-loop system and matrix reduction.

In circuit of Fig. 12.16, consider the current circulation in loop 1 clockwise, in loop 2 counterclockwise, and in loop 3 the same as loop 1, clockwise.

Fig. 12.16 Circuit of Example 12.7



Therefore, the elements of a general KVL by considering the assumed direction of currents are obtained as follows:

$$\begin{bmatrix} 20 & 0 & -20 \\ 0 & 15 & 15 \\ -20 & 15 & 5 + 20 + 15 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix}$$

Since the third row equation has a zero voltage source in the right-hand side of the equation, the third equation can be eliminated, and the remaining system becomes a 2×2 matrix, representing the impedance matrix of a two-port network.

The size reduction suggests the following formulation, wherein $n = 3$:

$$\begin{aligned} A_{2 \times 2} &= \begin{bmatrix} 20 & 0 \\ 0 & 15 \end{bmatrix} \\ M_{2 \times (n-2)} &= M_{2 \times 1} = \begin{bmatrix} -20 \\ 15 \end{bmatrix} \\ N_{(n-2) \times 2} &= N_{1 \times 2} = [-20 \quad 15] \\ D_{(n-2) \times (n-2)} &= D_{1 \times 1} = 40 \\ Z &= A - MD^{-1}N = \begin{bmatrix} 20 & 0 \\ 0 & 15 \end{bmatrix} - \begin{bmatrix} -20 \\ 15 \end{bmatrix} 40^{-1} [-20 \quad 15] \\ Z &= \begin{bmatrix} 10 & 7.5 \\ 7.5 & 9.375 \end{bmatrix} \end{aligned}$$

Example 12.8 Consider the circuit of Fig. 12.17, and find the impedance matrix.

Solution Considering the total impedance in loops 1 and 2, and the shared element, the impedance can be obtained as follows:

$$\text{Impedance of loop 1: } Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \Rightarrow Z_{11} = sL_1 + R$$

Impedance of the shared element considering that the current directions are similar:

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} \Rightarrow Z_{21} = R$$

$$\text{Impedance of loop 2: } Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0} \Rightarrow Z_{22} = sL_2 + R$$

Fig. 12.17 Circuit of Example 12.8

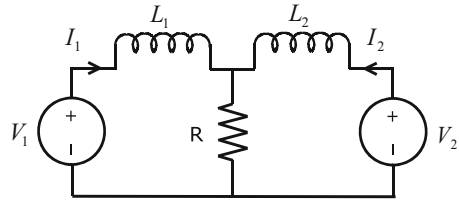
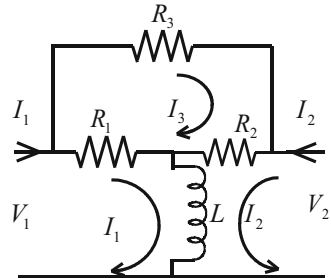


Fig. 12.18 Circuit of Example 12.9



Impedance of the shared element considering that the current directions are similar:

$$Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1=0} \Rightarrow Z_{12} = R$$

Therefore:

$$Z = \begin{bmatrix} 10 & 7.5 \\ 7.5 & 9.375 \end{bmatrix}$$

Example 12.9 In the circuit of Fig. 12.18, find the impedance matrix.

Solution Considering the direction of currents as determined on the figure (the direction of ports 1 and 2 is fixed, and the direction of current in loop 3 is arbitrary), the elements of a general KVL are obtained as follows:

$$\begin{bmatrix} R_1 + sL & sL & -R_1 \\ sL & R_2 + sL & R_2 \\ -R_1 & R_2 & R_1 + R_2 + R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix}$$

The elements of third row, shown in red color, are negative because the currents that pass the shared element between loops 1 and 3 are in opposite direction. Since the third row equation has a zero voltage source in the right-hand side of the equation, the third equation can be eliminated, and the remaining system becomes a 2×2 matrix, representing the impedance matrix of a two-port network.

The size reduction suggests the following formulation, wherein $n = 3$:

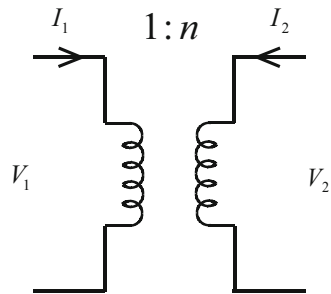
$$\begin{aligned}
 A_{2 \times 2} &= \begin{bmatrix} R_1 + sL & sL \\ sL & R_2 + sL \end{bmatrix} \\
 M_{2 \times (n-2)} &= M_{2 \times 1} = \begin{bmatrix} -R_1 \\ R_2 \end{bmatrix} \\
 N_{(n-2) \times 2} &= N_{1 \times 2} = \begin{bmatrix} -R_1 & R_2 \end{bmatrix} \\
 D_{(n-2) \times (n-2)} &= D_{1 \times 1} = R_1 + R_2 + R_3 \\
 Z &= A - MD^{-1}N = \begin{bmatrix} R_1 + sL & sL \\ sL & R_2 + sL \end{bmatrix} - \begin{bmatrix} -R_1 \\ R_2 \end{bmatrix} (R_1 + R_2 + R_3)^{-1} \begin{bmatrix} -R_1 & R_2 \end{bmatrix} \\
 Z &= \begin{bmatrix} R_1 + sL - \frac{R_1^2}{R_1 + R_2 + R_3} & sL + \frac{R_1 R_2}{R_1 + R_2 + R_3} \\ sL + \frac{R_1 R_2}{R_1 + R_2 + R_3} & R_2 + sL - \frac{R_2^2}{R_1 + R_2 + R_3} \end{bmatrix}
 \end{aligned}$$

Impedance Matrix Existence

The circuit and the two-port network must be such that the transfer functions of $Z_{11} = \frac{V_1}{I_1} \Big|_{I_2=0}$, $Z_{12} = \frac{V_1}{I_2} \Big|_{I_1=0}$, $Z_{21} = \frac{V_2}{I_1} \Big|_{I_2=0}$, and $Z_{22} = \frac{V_2}{I_2} \Big|_{I_1=0}$ exist. Transfer functions show the dynamics of a system that is defined to directly explain as the ratio of a desired output over a desired input parameter. In that sense, if the output is not influenced by a desired input, the transfer functions may not exist.

Example 12.10 For instance, consider an ideal transformer with transformation ratio n , shown in Fig. 12.19.

Fig. 12.19 Ideal transformer



Solution Considering the transformer as a two-port network, the output voltage V_2 is a function of its input voltage V_1 as follows:

$$V_2 = nV_1$$

The output current I_2 is also a function of its input current I_1 as:

$$I_2 = \frac{1}{n}I_1$$

As these equations show, the input voltage of an ideal transformer is not a function of the input current or the output current. Therefore, Z_{11} and Z_{12} do not exist. Similarly, the output voltage is not a function of the input and output currents. Therefore, Z_{21} and Z_{22} do not exist. A different form of transformation can be defined for a transformer, which is introduced later in this chapter.

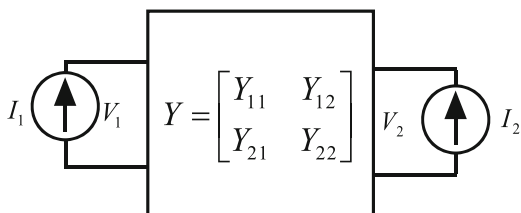
Admittance Matrix of a Two-Port Network

Admittance matrix shows the transfer functions existing in a two-port network defined as ratios of the currents over the voltages seen from each of the ports with respect to itself or the other ports. In this representation of the network, the entire circuit is expressed by a $Y_{2 \times 2}$ matrix that defines transfer functions as the ratio of port currents to the network's port voltages. The elements of the matrix show how the current at the network ports varies by the voltage variations through the ports. Current sources are utilized at the ports to control the amount of input currents, and the voltages are measured (Fig. 12.20).

The equations that represent an admittance network are as follows:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = Y_{2 \times 2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Fig. 12.20 Admittance matrix



$$\begin{cases} I_1 = Y_{11}V_1 + Y_{12}V_2 \\ I_2 = Y_{21}V_1 + Y_{22}V_2 \end{cases}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Admittance matrix has four elements of Y_{11} , Y_{12} , Y_{21} , Y_{22} :

- Y_{11} shows how the current of port 1 is related to the voltage of port 1.
- Y_{12} shows how the current of port 1 is related to the voltage of port 2.
- Y_{21} shows how the current of port 2 is related to the voltage of port 1.
- Y_{22} shows how the current of port 2 is related to the voltage of port 2.

To measure each of the admittance matrix elements, only the voltage and current related to the element should be considered, and the other parameters are considered zero. For instance:

$$Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0}$$

Note 12.3 This shows that to measure or calculate Y_{11} , only I_1 and V_1 are required while imposing $V_2 = 0$. This requires that the terminals of second port be short circuited.

Similarly,

$$Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0}$$

$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0}$$

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0}$$

Note 12.4 In networks that do not have *dependent sources*, the matrix elements Y_{12} and Y_{21} become similar, i.e.:

$$Y_{12} = Y_{21}$$

These networks are called reciprocal.

Equivalent of Admittance Network

Often, a two-port network can contain multiple loops and nodes. A matrix representation may also not be readily obtained when parallel networks exist or when part of the originally known network is missing. Therefore, the admittance matrix of a two-port network can be simplified to a Π -equivalent network.

Reciprocal Network

Π Model

As mentioned earlier, an admittance network might be reciprocal if it does not contain dependent sources. In a reciprocal network is a two-node network that shares a common element presented in Fig. 12.21. The shared element is either $-Y_{12}$ or $-Y_{21}$, as $Y_{12} = Y_{21}$. The not-shared element in the first node is $Y_{11} + Y_{12}$ and in the second node $Y_{22} + Y_{12}$. This forms a Π equivalent network.

Nonreciprocal Network

In a nonreciprocal network, there exists at least one dependent source that makes Y_{12} and Y_{21} different. The equivalent model presented in this chapter uses two voltage-dependent current sources at each node and shared element to model a nonreciprocal admittance matrix in a Π network.

Element-Sharing Nodes

The equivalent circuit of an element-sharing two-port network is shown in Fig. 12.22.

Consider the original admittance matrix as:

Fig. 12.21 Π equivalent of admittance network

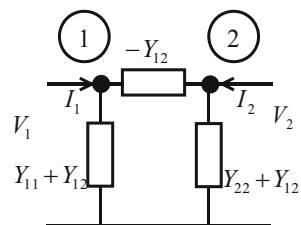


Fig. 12.22 An element-sharing node of an admittance network

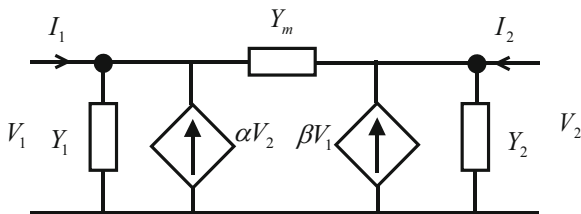
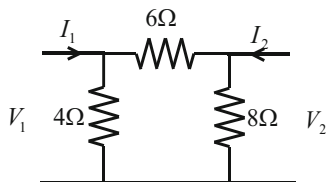


Fig. 12.23 Figure of circuit in Example 12.11



$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

Consider a shared element Y_m exists in all four components of Y_{11} , Y_{12} , Y_{21} , Y_{22} . Therefore, the admittance matrix can be written as:

$$Y = \begin{bmatrix} Y_1 + Y_m & -\alpha - Y_m \\ -\beta - Y_m & Y_2 + Y_m \end{bmatrix}$$

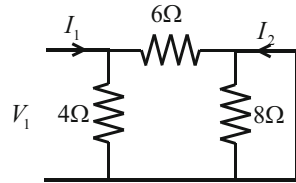
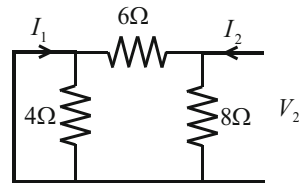
Parameters α and β show the existence of dependent sources in the Π -equivalent network and are used to model the nonreciprocal networks. The equivalent element in node 1 of the Π model becomes admittance Y_1 and a voltage-dependent current source with value of αV_2 . The shared element between the nodes becomes Y_m , and the second node has the admittance Y_2 and a voltage-dependent current source with value of βV_1 .

Example 12.11 In the circuit of Fig. 12.23, find the Y matrix.

Solution The two-port network shown in Fig. 12.23 forms a Π network with a shared element. To obtain the admittance matrix, there are multiple methods. Some of them are explained in this example.

Method 1 Classical approach

In this method, the definition equations such as $Y_{11} = \frac{I_1}{V_1} \Big|_{V_2=0}$ are used. To obtain Y_{11} , port 2 is considered short circuit; therefore, $V_2 = 0$ and the 8Ω resistor are shorted out. That makes the circuit as shown in Fig. 12.24.

Fig. 12.24 Short circuit**Fig. 12.25** When $V_1 = 0$ is imposed, a short circuit in node 1 is created

Since $V_2 = 0$, a KVL in node 1 results in:

$$V_1 = I_1(4 \parallel 6) = 2.4I_1$$

The ratio of $\frac{I_1}{V_1}$ is obtained as follows:

$$Y_{11} = \frac{I_1}{V_1} = \frac{I_1}{2.4I_1} = 0.417 \Omega^{-1}$$

In the same circuit, the current of I_2 , is in opposite direction to the current of 6Ω resistor. A current division finds the current as:

$$I_2 = -\frac{4}{4+6}I_1 = -0.4I_1$$

The admittance element Y_{21} can be calculated as follows:

$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0} = \frac{-0.4I_1}{V_1} = -0.4 \frac{I_1}{V_1} = -0.4 \times 0.417 = -0.166$$

Now considering the node voltage of port 1 as zero, $V_1 = 0$, the short circuit in the input node is shown in Fig. 12.25.

In this case, Y_{22} and Y_{12} can be calculated as follows:

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0}$$

KVL in loop 2, when $V_1 = 0$, results in shorting out the $4\ \Omega$ resistor. Therefore:

$$V_2 = I_2(6 \parallel 8) = 3.42I_2$$

The ratio $\frac{I_2}{V_2}$ is obtained as:

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0} = \frac{I_2}{3.42I_2} = 0.292$$

When $V_1 = 0$, the current in port 1 is similar to the opposite of the current that flows through the $6\ \Omega$ resistor, as follows:

$$I_1 = -\frac{8}{6+8}I_2 = -0.571I_2$$

The element Y_{12} can be calculated as follows:

$$Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0} = \frac{-0.571I_2}{V_2} = -0.571 \frac{I_2}{V_2} = -0.571 \times 0.292 = -0.166$$

Since there is no dependent source in the circuit, the impedance matrix is reciprocal, meaning that:

$$Y_{12} = Y_{21}$$

And by calculations, it was demonstrated that:

$$Y_{12} = Y_{21} = -0.166$$

The admittance matrix becomes:

$$Y = \begin{bmatrix} 0.417 & -0.166 \\ -0.166 & 0.292 \end{bmatrix}$$

Method 2 Using Π matrix analysis

As Fig. 12.21 shows, a Π network exists. Comparing side by side with the Π network explained in Fig. 12.19, the Y_{11} can be obtained by adding all admittances connected to node 1. It means:

In node 1:

$$Y_{11} = \frac{1}{4} + \frac{1}{6} = 0.416$$

In node 2:

$$Y_{22} = \frac{1}{6} + \frac{1}{8} = 0.292$$

The negative of shared admittance between two nodes is $Y_{12} = Y_{21}$. Therefore,

$$Y_{12} = Y_{21} = -\frac{1}{6} = -0.166$$

The admittance matrix becomes:

$$Y = \begin{bmatrix} 0.417 & -0.166 \\ -0.166 & 0.292 \end{bmatrix}$$

Alternative Approach in Admittance Matrix

Consider the circuit shown in Figs. 12.8 and 12.26.

The admittance matrix becomes:

$$Y = \begin{bmatrix} Y_a + Y_c & -Y_c \\ -Y_c & Y_b + Y_c \end{bmatrix}$$

Example 12.12 Find the admittance matrix of the circuit shown in Fig. 12.27.

Solution The element values in the Π network are provided in impedance values. Therefore, in their summation, their admittance should be considered as follows:

Fig. 12.26 A Π -equivalent circuit used in the admittance matrix

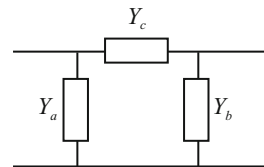


Fig. 12.27 Circuit of Example 12.12

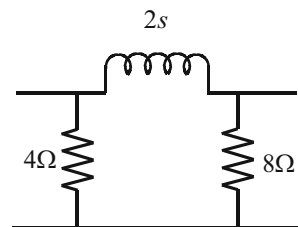


Fig. 12.28 Circuit of Example 12.13

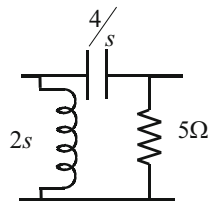
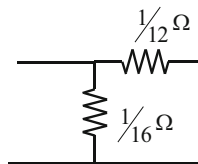


Fig. 12.29 Circuit of Example 12.14



$$Y = \begin{bmatrix} \frac{1}{4} + \frac{1}{2s} & -\frac{1}{2s} \\ -\frac{1}{2s} & \frac{1}{8} + \frac{1}{2s} \end{bmatrix}$$

Example 12.13 Find the admittance matrix of circuit shown in Fig. 12.28.

Solution

$$Y = \begin{bmatrix} \frac{1}{2s} + \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 5 + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2s} + \frac{s}{4} & -\frac{s}{4} \\ -\frac{s}{4} & 5 + \frac{s}{4} \end{bmatrix}$$

Example 12.14 Find the admittance matrix of the circuit shown in Fig. 12.29.

Solution

$$Y = \begin{bmatrix} 16 + 12 & -12 \\ -12 & 0 + 12 \end{bmatrix} = \begin{bmatrix} 28 & -12 \\ -12 & 12 \end{bmatrix}$$

Example 12.15 Find the admittance matrix of the circuit shown in Fig. 12.30.

Solution

$$Y = \begin{bmatrix} 0 + 15 & -15 \\ -15 & 0 + 15 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

Example 12.16 The output of an admittance matrix (Fig. 12.31) feeds a $0.1 \Omega^{-1}$ conductance. If the input port current source is $I_1 = 50 \angle 0^\circ \text{ A}$, find the currents V_1 and V_2 and the output current.

Fig. 12.30 Circuit of
Example 12.15

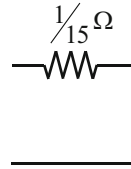
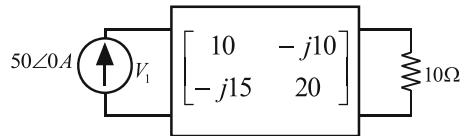


Fig. 12.31 Network of
Example 12.16



Solution The admittance matrix shows the relations of the input and output port voltage and currents as:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 & -j10 \\ -j15 & 20 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The output port voltage V_2 is a function of the load resistance and current as follows:

$$I_2 = -\frac{1}{0.1} V_2$$

Replacing $I_1 = 50 \angle 0$ and $I_2 = -10V_2$, the network equations become:

$$\begin{bmatrix} 50 \\ -10V_2 \end{bmatrix} = \begin{bmatrix} 10 & -j10 \\ -j15 & 20 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Bringing the $-\frac{1}{10} V_2$ back to the right side, the equations read as:

$$\begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & -j10 \\ -j15 & 20 + 10 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The currents can be found as:

$$\begin{aligned}
 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} 10 & -j10 \\ -j15 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 0 \end{bmatrix} \\
 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \frac{1}{10 \times 30 - j15 \times j10} \begin{bmatrix} 30 & +j10 \\ +j15 & 10 \end{bmatrix} \begin{bmatrix} 50 \\ 0 \end{bmatrix} \\
 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \frac{1}{450} \begin{bmatrix} 30 & +j10 \\ +j15 & 10 \end{bmatrix} \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1500}{450} \\ \frac{-j15 \times 50}{450} \end{bmatrix} = \begin{bmatrix} 3.33 \\ +j1.66 \end{bmatrix} A \\
 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} 3.33 \\ +j1.66 \end{bmatrix} A
 \end{aligned}$$

Another approach to find the currents is through Cramer's method.

$$\begin{aligned}
 I_1 &= \frac{\begin{vmatrix} \textcolor{red}{50} & -j10 \\ \textcolor{red}{0} & 30 \end{vmatrix}}{\begin{vmatrix} 10 & -j10 \\ -j15 & 30 \end{vmatrix}} = \frac{1500}{450} = 3.33 A \\
 I_2 &= \frac{\begin{vmatrix} 10 & \textcolor{red}{50} \\ -j15 & \textcolor{red}{0} \end{vmatrix}}{\begin{vmatrix} 10 & -j10 \\ -j15 & 30 \end{vmatrix}} = \frac{j750}{450} = +j1.66 A
 \end{aligned}$$

Finding Admittance Matrix in Multi-node Networks

In case the network contains any configuration other than standard Π form, it may form more than two nodes. However, there are still two ports that are utilized to represent the entire network, i.e., the admittance matrix still has a 2×2 dimension. A procedure to find the admittance matrix can be expressed as follows:

Current and Voltage Considerations

- The voltage and current in the input node are V_1 and I_1 , with the current entering the port.
- The voltage and current in the output node are V_2 and I_2 , with the current entering the port.
- The order of naming voltage of other nodes is arbitrary.

- In case the admittance of input and output nodes is not placed in the first and second equations, the rows and columns can be exchanged to shift the desired equations in their designated places. It is recommended to have the input node equation in the first row and the output node equation in the second row.

Finding the Matrix Dimension and Its Elements

- Consider a square matrix $Y_n \times n$ with its dimension matching the number of nodes, i.e., for a three-node system $n = 3$, a $Y_3 \times 3$ matrix is obtained.
- The element Y_{ii} on the diagonal is obtained by the summation of all admittances connected to node i .
- The off-diagonal elements Y_{ij} are the negative of shared elements between node i and node j .

General Form of KCL Equations

Following the abovementioned rules, a general admittance matrix equation is obtained with constant matrix elements being zero except for I_1 and I_2 . For a network with n nodes, the general form is:

$$\begin{bmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

A 2×2 matrix can be selected from this general model that contains equations that have values other than zero. The system can be expressed as:

$$\begin{bmatrix} A_{2 \times 2} & M_{2 \times (n-2)} \\ N_{(n-2) \times 2} & D_{(n-2) \times (n-2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0_{(n-2) \times 1} \end{bmatrix}$$

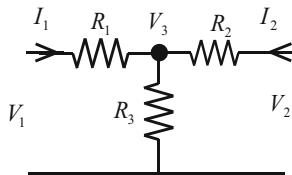
Matrix Size Reduction

The size of split general equation can be reduced to the size of non-zero element constant matrix. In this case, I_1 and I_2 , i.e., the size is 2.

The impedance matrix of a two-port representation of the multi-loop is as follows:

$$Y_{2 \times 2} = A_{2 \times 2} - M_{2 \times (n-2)} D^{-1}_{(n-2) \times (n-2)} N_{(n-2) \times 2}$$

Fig. 12.32 Circuit of Example 12.17



Removing the matrix dimensions, it results in:

$$Z = A - MD^{-1}N$$

Example 12.17 Find the admittance matrix of the circuit shown in Fig. 12.32.

Solution There are multiple approaches in obtaining the admittance matrix of this network. In this example, the admittance matrix is obtained by two methods. Later in this chapter, another method is introduced to simplify the solution.

Method 1 Using the definitions:

$$\begin{aligned}
 Y_{11} &= \left. \frac{I_1}{V_1} \right|_{V_2=0} \Rightarrow I_1 \\
 &= \frac{V_1}{R_1 + (R_2 \parallel R_3)} \rightarrow Y_{11} = \frac{I_1}{V_1} = \frac{1}{R_1 + (R_2 \parallel R_3)} = \frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \\
 Y_{21} &= \left. \frac{I_2}{V_1} \right|_{V_2=0} \Rightarrow I_2 = -\frac{R_3}{R_2 + R_3} I_1; \left(I_1 = \frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} V_1 \right) \rightarrow Y_{21} \\
 &= \frac{-\frac{R_3}{R_2 + R_3} \left(\frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} V_1 \right)}{V_1} \\
 &= -\frac{R_3}{R_2 + R_3} \left(\frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \right) \\
 Y_{21} &= -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \\
 Y_{22} &= \left. \frac{I_2}{V_2} \right|_{V_1=0} \Rightarrow I_2 = \frac{V_2}{R_2 + (R_1 \parallel R_3)} \rightarrow Y_{22} = \frac{I_2}{V_2} = \frac{R_1 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \\
 Y_{12} &= \left. \frac{I_1}{V_2} \right|_{V_1=0} \Rightarrow I_1 = -\frac{R_3}{R_1 + R_3} I_2; \left(I_2 = \frac{R_1 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} V_2 \right) \rightarrow Y_{12} = \frac{I_1}{V_2} \\
 &= -\frac{R_3}{R_1 + R_3} \left(\frac{R_1 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \right) = -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}
 \end{aligned}$$

$$Y = \begin{bmatrix} \frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} & -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \\ -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} & \frac{R_1 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \end{bmatrix}$$

Method 2 Using three-loop system and matrix reduction

In circuit of Fig. 12.32, consider the node voltages; the elements of a general KCL are obtained as follows:

$$\begin{bmatrix} \frac{1}{R_1} & 0 & -\frac{1}{R_1} \\ 0 & \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & -\frac{1}{R_2} & \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

Since the third row equation has a zero-value current source in the right-hand side, the third equation can be eliminated, and the remaining system becomes a 2×2 matrix, representing the admittance matrix of a two-port network.

The size reduction suggests the following formulation, wherein $n = 3$:

$$A_{2 \times 2} = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix}$$

$$M_{2 \times (n-2)} = M_{2 \times 1} = \begin{bmatrix} -\frac{1}{R_1} \\ -\frac{1}{R_2} \end{bmatrix}$$

$$N_{(n-2) \times 2} = N_{1 \times 2} = \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_2} \end{bmatrix}$$

$$D_{(n-2) \times (n-2)} = D_{1 \times 1} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$Y = A - MD^{-1}N = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix} - \begin{bmatrix} -\frac{1}{R_1} \\ -\frac{1}{R_2} \end{bmatrix} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1} \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_2} \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} & -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \\ -\frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} & \frac{R_1 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \end{bmatrix}$$

Admittance to Impedance Conversion

KVL and KCL equations can be converted to each other, if their systems are not singular. Accordingly, the admittance and impedance matrices can be converted to each other as follows:

$$\text{If } (\det(Y) \neq 0) \Rightarrow Z = Y^{-1}$$

and

$$\text{If } (\det(Z) \neq 0) \Rightarrow Y = Z^{-1}$$

Example 12.18 Consider a network with impedance matrix $Z = \begin{bmatrix} 10 & -j10 \\ -j15 & 20 \end{bmatrix}$.

Find the admittance matrix of the same network.

Solution

$$\begin{aligned} Y = Z^{-1} &= \begin{bmatrix} 10 & -j10 \\ -j15 & 20 \end{bmatrix}^{-1} \\ &= \frac{1}{10 \times 20 - (-j15) \times (-j10)} \begin{bmatrix} 20 & j10 \\ j15 & 10 \end{bmatrix} = \begin{bmatrix} 0.0574 & j0.0286 \\ j0.0429 & 0.0286 \end{bmatrix} \end{aligned}$$

Admittance Matrix Existence

The circuit and the two-port network must be such that the transfer functions of Y_{11} , Y_{12} , Y_{21} , and Y_{22} exist. Transfer functions show the dynamics of a system that is defined to directly explain the dependency of a desired output to a desired input. In that sense, if the output is not influenced by a desired input, the transfer functions may not exist.

Example 12.19 Consider a network with impedance matrix $Z = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$. Find the admittance matrix of the same network.

Solution $Y = Z^{-1} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}^{-1}$. However, $\det(Z) = 0$. Therefore, this circuit does not have admittance representation.

Example 12.20 Consider a network with admittance matrix $Y = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$. Find the impedance matrix of the same network.

Solution

$$Z = Y^{-1} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 0.102 & 0.0204 \\ 0.0204 & 0.2041 \end{bmatrix}$$

Example 12.21 Consider a network with impedance matrix $Z = \begin{bmatrix} 0.1 + j0.3 & -j0.2 \\ j0.4 & 0.5 + j0.1 \end{bmatrix}$. Find the admittance matrix of the same network.

Solution

$$Y = Z^{-1} = \begin{bmatrix} 0.1 + j0.3 & -j0.2 \\ j0.4 & 0.5 + j0.1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.47 - j2.94 & 1.09 - j0.411 \\ -2.19 + j0.82 & 1.43 - j1.16 \end{bmatrix}.$$

Note 12.5 A Π network shown in Fig. 12.33 has a Y matrix obtained as follows:

$$Y = \begin{bmatrix} \sum Y_{1j} & -Y_{12} \\ -Y_{21} & \sum Y_{2j} \end{bmatrix} = \begin{bmatrix} Y_a + Y_b & -Y_a \\ -Y_a & Y_a + Y_c \end{bmatrix}$$

Note 12.6 Adding an impedance element z (Ω) connected between the input port and the output port, as shown in Fig. 12.34, a new admittance matrix can be found as follows:

Fig. 12.33 A Π network
 Y_a, Y_b, Y_c

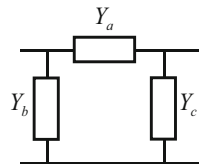


Fig. 12.34 An admittance network bypassed by an impedance Z

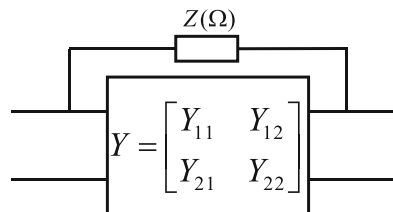
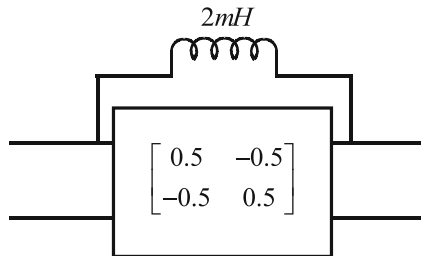


Fig. 12.35 Circuit of Example 12.22



The bypass impedance has an admittance matrix which is parallel to the given network admittance. Therefore, the new admittance matrix becomes:

$$Y_{\text{new}} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{11} \end{bmatrix} + \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} Y_{11} + \frac{1}{z} & Y_{12} - \frac{1}{z} \\ Y_{21} - \frac{1}{z} & Y_{11} + \frac{1}{z} \end{bmatrix}$$

Example 12.22 Admittance matrix of a two-port network is given as $Y = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$. The network is augmented by an inductor of $L = 2 \text{ mH}$ bypassing the input port to the output port, as shown in Fig. 12.35. Find the new admittance matrix.

Solution The new admittance matrix becomes:

$$\begin{aligned} Y_{\text{new}} &= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{11} \end{bmatrix} + \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} + \begin{bmatrix} \frac{1}{sL} & -\frac{1}{sL} \\ \frac{1}{sL} & \frac{1}{sL} \end{bmatrix} \\ &= \begin{bmatrix} 0.5 + \frac{500}{s} & -0.5 - \frac{500}{s} \\ -0.5 - \frac{500}{s} & 0.5 + \frac{500}{s} \end{bmatrix} \end{aligned}$$

Example 12.23 Impedance matrix of a two-port network is $Z = \begin{bmatrix} 1 & j0.2 \\ j0.5 & 1 + j0.5 \end{bmatrix}$. This network is augmented by a bypass resistor of $R = 2 \Omega$ from the input to the output, as shown in Fig. 12.34. Find the new impedance matrix (Fig. 12.36).

Solution Since the augmented element is a bypass from input to the output, there is a need to find the admittance matrix, as the added element can be easily integrated into the admittance matrix.

Fig. 12.36 Circuit of
Example 12.23

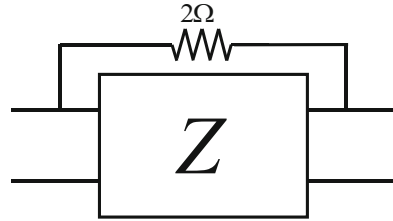
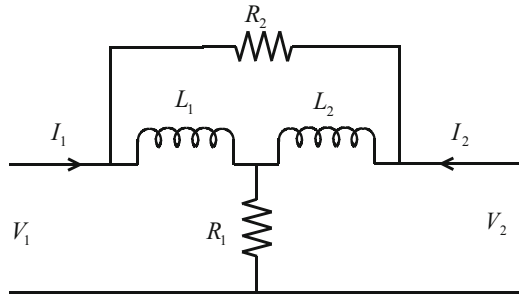


Fig. 12.37 Circuit of
Example 12.24



$$Y = Z^{-1} = \begin{bmatrix} 1 & j0.2 \\ j0.5 & 1 + j0.5 \end{bmatrix}^{-1} = \begin{bmatrix} 0.92 + j0.03 & -0.06 - j0.15 \\ -0.17 - j0.37 & 0.75 - j0.34 \end{bmatrix}$$

Considering the effect of added resistor in the Y matrix:

$$\begin{aligned} Y_{\text{new}} &= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{21} \end{bmatrix} + \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{bmatrix} \\ &= \begin{bmatrix} 0.92 + j0.03 & -0.06 - j0.15 \\ -0.17 - j0.37 & 0.75 - j0.34 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1.42 + j0.03 & -0.56 - j0.15 \\ -0.67 - j0.37 & 1.25 - j0.34 \end{bmatrix} \end{aligned}$$

To find the new:

$$Z_{\text{new}} = \begin{bmatrix} 1.42 + j0.03 & -0.06 - j0.15 \\ -0.17 - j0.37 & 1.25 - j0.34 \end{bmatrix}^{-1} = \begin{bmatrix} 0.68 - j0.001 & 0.01 + j0.08 \\ 0.03 + j0.21 & 0.71 + j0.21 \end{bmatrix}$$

Example 12.24 Find the Y for the circuit of Fig. 12.37.

Solution Consider the elements of L_1 , L_2 , R_1 forming a T connection. Therefore, the impedance matrix can be written as:

$$Z_T = \begin{bmatrix} sL_1 + R_1 & R_1 \\ R_1 & sL_2 + R_1 \end{bmatrix}$$

Now the element R_2 is considered as a bypass from the input to the output. Writing the Y matrix, this element can be integrated as follows:

$$Y = \frac{1}{s((L_1 + L_2)R_1 + L_1L_2s)} \begin{bmatrix} sL_2 + R_1 & -R_1 \\ -R_1 & sL_1 + R_1 \end{bmatrix}$$

Adding the bypass element, R_2 results in:

$$Y = \frac{1}{s((L_1 + L_2)R_1 + L_1L_2s)} \begin{bmatrix} sL_2 + R_1 & -R_1 \\ -R_1 & sL_1 + R_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{sL_2 + R_1}{s((L_1 + L_2)R_1 + L_1L_2s)} + \frac{1}{R_2} & \frac{-R_1}{s((L_1 + L_2)R_1 + L_1L_2s)} - \frac{1}{R_2} \\ \frac{-R_1}{s((L_1 + L_2)R_1 + L_1L_2s)} - \frac{1}{R_2} & \frac{sL_1 + R_1}{s((L_1 + L_2)R_1 + L_1L_2s)} + \frac{1}{R_2} \end{bmatrix}$$

Note 12.7 Adding a series element Z (Ω) to the input port of a two-port network adds the same impedance to the z_{11} of the impedance matrix as shown in Fig. 12.38.

$$Z_{\text{new}} = \begin{bmatrix} z_{11} + Z & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Note 12.8 Adding a series element Z (Ω) to the output port of a two-port network adds the same impedance to the z_{22} of the impedance matrix as shown in Fig. 12.39.

Fig. 12.38 An impedance Z added to the input of an impedance network

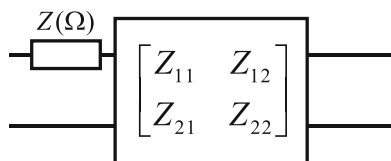


Fig. 12.39 An impedance Z added to the output of an impedance network

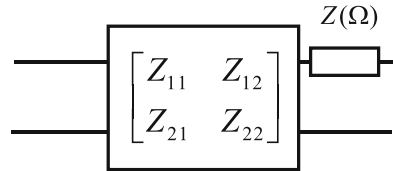


Fig. 12.40 Impedance Z is added in parallel to the admittance network

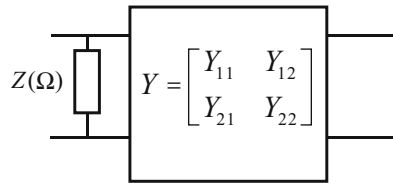
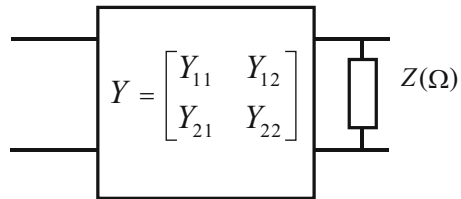


Fig. 12.41 Impedance Z is added in parallel to the output of an admittance network



$$Z_{\text{new}} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} + Z \end{bmatrix}$$

Note 12.9 Adding a parallel element $Z(\Omega)$ to the input port of a two-port network adds the same admittance $\frac{1}{Z}$ to the Y_{11} of the admittance matrix as shown in Fig. 12.40.

$$Y_{\text{new}} = \begin{bmatrix} Y_{11} + \frac{1}{Z} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

Note 12.10 Adding a parallel element $Z(\Omega)$ to the output port of a two-port network adds the same admittance $\frac{1}{Z}$ to the Y_{22} of the admittance matrix as shown in Fig. 12.41.

$$Y_{\text{new}} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} + \frac{1}{Z} \end{bmatrix}$$

Example 12.25 Admittance matrix of a circuit is given as $Y = \begin{bmatrix} 2s+4 & -1 \\ -1 & s+2 \end{bmatrix}$. Sketch the circuit.

Solution

- Admittance matrix can be presented as a Π network.

The elements of the network can be found as follows:

- Since the admittance matrix is reciprocal, the shared element is $Y_{12} = Y_{21} = 1 \Omega^{-1}$.
- Separating the shared element out, the rest of the system reads:

$$\begin{bmatrix} 2s+3 + \color{red}{1} & \color{red}{-1} \\ \color{red}{-1} & s+1 + \color{red}{1} \end{bmatrix}$$

- Form $2s+3$ in the input node to the ground.
 - Admittance $2s+3$ is a parallel of a 2 F capacitor and a $\frac{1}{3} \Omega$ resistor.
- Form $s+1$ in the output node to the ground.
 - Admittance $s+1$ is a parallel of a 1 F capacitor and a 1Ω resistor.

The circuit is shown in Fig. 12.42.

Example 12.26 Impedance of a two-port network is given as $Z = \begin{bmatrix} s+10 & 15 \\ 25 & \frac{1}{s} + 10 \end{bmatrix}$. Sketch the circuit.

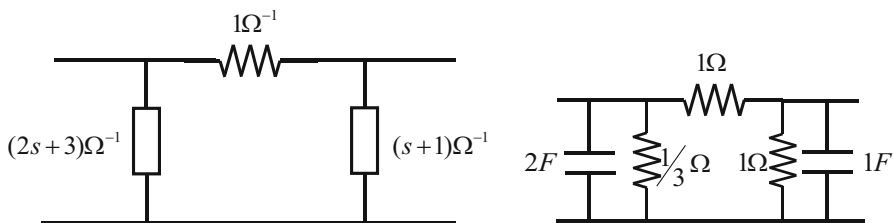
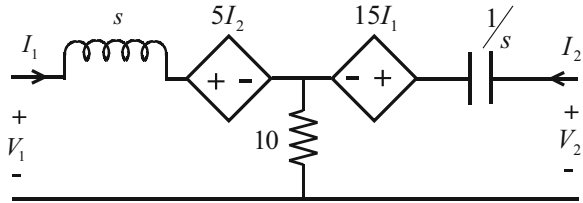


Fig. 12.42 The circuit found for the admittance matrix of Example 12.25

Fig. 12.43 The circuit found for the impedance matrix of Example 12.26



Solution

- Impedance matrix is better coordinated with a T network.

The elements on the T network can be found as follows:

- Since the network is not reciprocal, meaning $Z_{12} \neq Z_{21}$, then an arbitrary shared element needs to be found in all elements of the impedance matrix. One option for the shared element of the T network can be the $10\ \Omega$ resistor as follows:

$$Z = \begin{bmatrix} s + 10 & 5 + 10 \\ 15 + 10 & \frac{1}{s} + 10 \end{bmatrix}$$

- Besides the element s (taken from Z_{11} element), there is a dependent voltage source in loop 1, with value of $5I_2$ (taken from Z_{12} element).
- Besides the element $\frac{1}{s}$ (taken from Z_{22} element), there is a dependent voltage source in loop 2, with value of $15I_1$ (taken from Z_{21} element).

The circuit is shown in Fig. 12.43.

Example 12.27 Impedance of a two-port network is given as $Z = \begin{bmatrix} s + 1 & -2 \\ 3 & \frac{1}{s} + 1 \end{bmatrix}$.

Sketch the circuit.

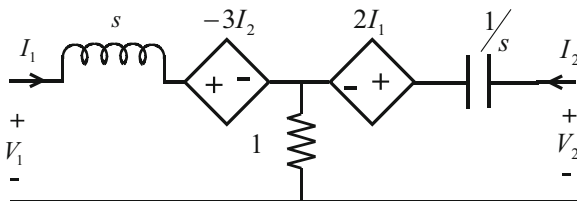
Solution

- Impedance matrix is better coordinated with a T network.

The elements on the T network can be found as follows:

- Since the network is not reciprocal, meaning $Z_{12} \neq Z_{21}$, then an arbitrary shared element needs to be found in all elements of the impedance matrix. One option for the shared element of the T network can be the $1\ \Omega$ resistor as follows:

Fig. 12.44 The circuit found for the impedance matrix of Example 12.27



$$Z = \begin{bmatrix} s + 1 & -3 + 1 \\ 2 + 1 & \frac{1}{s} + 1 \end{bmatrix}$$

- Besides the element s (taken from Z_{11} element), there is a dependent voltage source in loop 1, with value of $-3I_2$ (taken from Z_{12} element).
- Besides the element $\frac{1}{s}$ (taken from Z_{22} element), there is a dependent voltage source in loop 2, with value of $2I_1$ (taken from Z_{21} element).

The circuit is shown in Fig. 12.44.

Example 12.28 Impedance of a two-port network is given as

$$Z = \begin{bmatrix} s + 5 & \frac{1}{s} \\ 4 & s + \frac{1}{s} + 2 \end{bmatrix}. \text{ Sketch the circuit.}$$

Solution

- Impedance matrix is better coordinated with a T network.

The elements on the T network can be found as follows:

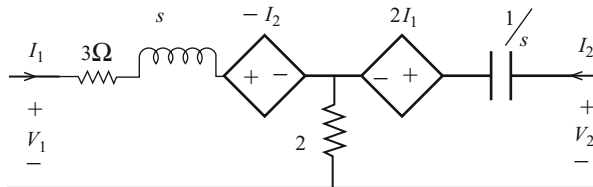
- Since the network is not reciprocal, meaning $Z_{12} \neq Z_{21}$, then an arbitrary shared element needs to be found in all elements of the impedance matrix. One option for the shared element of the T network can be the 2Ω resistor as follows:

$$Z = \begin{bmatrix} s + 3 + 2 & -1 + 2 \\ 2 + 2 & s + \frac{1}{s} + 2 \end{bmatrix}$$

- Besides the element $s + 3$ (taken from Z_{11} element), there is a dependent voltage source in loop 1, with value of $-I_2$ (taken from Z_{12} element).
- Besides the element $s + \frac{1}{s}$ (taken from Z_{22} element), there is a dependent voltage source in loop 2, with value of $2I_1$ (taken from Z_{21} element).

The circuit is shown in Fig. 12.45.

Fig. 12.45 The circuit found for the impedance matrix of Example 12.28



Nonreciprocal Admittance Matrix

Example 12.29 Admittance matrix of a two-port network is given as $Y = \begin{bmatrix} 10 & -15 \\ -2 & 7 \end{bmatrix}$. Sketch the circuit.

Solution Compare the admittance to what was introduced earlier as:

$$Y = \begin{bmatrix} Y_1 + Y_m & -\alpha - Y_m \\ -\beta - Y_m & Y_2 + Y_m \end{bmatrix}$$

- Since the elements of $Y_{12} \neq Y_{21}$, the system is nonreciprocal. The equivalent π circuit contains dependent sources.
- Consider a value for the shared element Y_m , for instance, $Y_m = +1$.
- The admittance matrix becomes:

$$Y = \begin{bmatrix} 9 + 1 & -14 - 1 \\ -1 - 1 & 6 + 1 \end{bmatrix}$$

- From element Y_{11} , there is a shared admittance $1 \Omega^{-1}$ connected between nodes 1 and 2, and there is $Y_1 = 9 \Omega^{-1}$ admittance from node 1 to common node.
- From element Y_{12} , there is the same shared element $1 \Omega^{-1}$ and the value of the voltage-controlled current source that feeds node 1 by value of $\alpha = +14V_2$.
- From element Y_{21} , there is the same shared element $1 \Omega^{-1}$ and the value of the voltage-controlled current source that feeds node 1 by value of $\beta = +1V_1$.
- From element Y_{22} , there is the shared admittance $1 \Omega^{-1}$ connected between nodes 2 and 1, and there is $Y_2 = 6 \Omega^{-1}$ admittance from node 2 to common node.

The circuit is shown in Fig. 12.46.

Example 12.30 Admittance matrix of a circuit is given as $Y = \begin{bmatrix} 2s + 4 & -2 \\ -3 & s + 2 \end{bmatrix}$. Sketch the circuit.

Fig. 12.46 A circuit found for the admittance matrix of Example 12.29

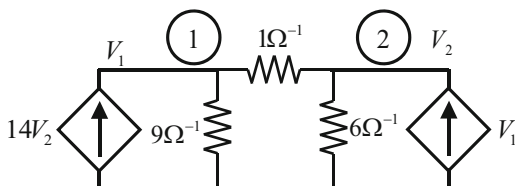
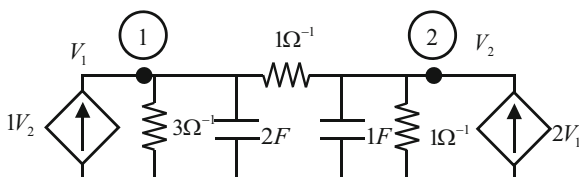


Fig. 12.47 A circuit found for the admittance matrix of Example 12.30



Solution

- Since the elements of $Y_{12} \neq Y_{21}$, the system is nonreciprocal. The equivalent π circuit contains dependent sources.
- Consider a value for the shared element Y_m , for instance, $Y_m = +1$.
- The admittance matrix becomes:

$$Y = \begin{bmatrix} 2s + 3 + 1 & -1 - 1 \\ -2 - 1 & s + 1 + 1 \end{bmatrix}$$

- From element Y_{11} , there is a shared admittance $1\Omega^{-1}$ connected between nodes 1 and 2, and there is $Y_1 = 2s + 3\Omega^{-1}$ admittance from node 1 to common node. This is a parallel of a $2F$ capacitor and conductance of $3\Omega^{-1}$.
- From element Y_{12} , there is the same shared element $1\Omega^{-1}$ and the value of the voltage-controlled current source that feeds node 1 by value of $\alpha = +1V_2$.
- From element Y_{21} , there is the same shared element $1\Omega^{-1}$ and the value of the voltage-controlled current source that feeds node 1 by value of $\beta = +2V_1$.
- From element Y_{22} , there is the shared admittance $1\Omega^{-1}$ connected between nodes 2 and 1, and there is $Y_2 = s + 1\Omega^{-1}$ admittance from node 2 to common node. That is a parallel of $1F$ capacitor and conductance of $1\Omega^{-1}$ (Fig. 12.47).

Example 12.31 Find the admittance matrix of the circuit shown in Fig. 12.48.

Solution The circuit clearly has a T network and a current source in parallel to the 4Ω resistor. A Norton to Thevenin conversion results in a circuit shown in Fig. 12.49.

Fig. 12.48 Figure of Example 12.31

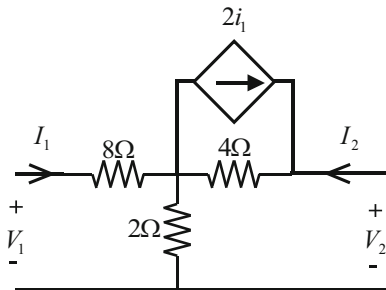
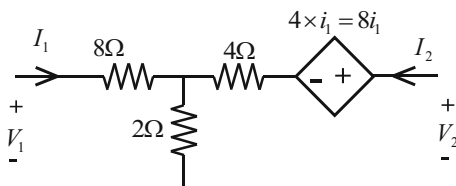


Fig. 12.49 Source conversion in circuit of Fig. 12.48



The circuit has a nonreciprocal Z matrix. The shared element is a 2Ω resistor, and in the first loop, there is 8Ω resistor and no current-dependent voltage source. Therefore, Z_{11} is $2+8$. However, since there is a 4Ω resistor in the second loop, Z_{22} becomes $2+4$. The value of $8i_1$ voltage source adds to the Z_{21} element, and it becomes $2+8$. Therefore, the impedance matrix is:

$$Z = \begin{bmatrix} 2+8 & 2 \\ 2+8 & 2+4 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 10 & 6 \end{bmatrix}$$

The admittance matrix becomes:

$$\begin{aligned} Y = Z^{-1} &= \begin{bmatrix} 10 & 2 \\ 10 & 6 \end{bmatrix}^{-1} \\ &= \frac{1}{10 \times 6 - 10 \times 2} \begin{bmatrix} 6 & -2 \\ -10 & 10 \end{bmatrix} = \begin{bmatrix} \frac{6}{40} & -\frac{2}{40} \\ -\frac{10}{40} & \frac{10}{40} \end{bmatrix} \\ Y &= \begin{bmatrix} 0.15 & -0.05 \\ -0.25 & 0.25 \end{bmatrix} \end{aligned}$$

Hybrid Parameters

In this system representation, a mix of impedance, voltage gain, current gain, and admittance is utilized. The system equations are as follows:

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Expansion of these equations demonstrates their units as follows:

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

As a result:

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0}$$

$$h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0}$$

$$h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0}$$

$$h_{22} = \left. \frac{I_2}{V_2} \right|_{I_1=0}$$

If the Z^{-1} and Y^{-1} matrices exist:

$$h_{11} = \frac{\det(Z)}{z_{22}}, h_{12} = \frac{z_{12}}{z_{22}}$$

$$h_{21} = -\frac{z_{21}}{z_{22}}, h_{22} = \frac{1}{z_{22}}$$

Inverse Hybrid Parameters

In this system representation, a mix of impedance, voltage gain, current gain, and admittance is utilized. The system equations are as follows:

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

Expansion of these equations demonstrates their units as follows:

$$I_1 = g_{11}V_1 + g_{12}I_2$$

$$V_2 = g_{21}V_1 + g_{22}I_2$$

As a result:

$$g_{11} = \left. \frac{I_1}{V_1} \right|_{I_2=0}$$

$$g_{12} = \left. \frac{I_1}{I_2} \right|_{V_1=0}$$

$$g_{21} = \left. \frac{V_2}{V_1} \right|_{I_2=0}$$

$$g_{22} = \left. \frac{V_2}{I_2} \right|_{V_1=0}$$

This shows that the:

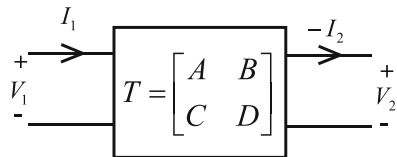
- Element g_{11} is an admittance equal to y_{11} .
- Element g_{12} is a current gain.
- Element g_{21} is a voltage gain.
- Element g_{22} is an impedance equal to z_{22} .

Transmission Matrix Parameters

Transmission matrix has some interesting characteristics in that large systems can be split into smaller sections. Once the transmission matrix of these sections is identified, they can be connected back together to define the entire system.

Transmission matrix T relates the voltage and current at the entry (or sending end) of a system to its parameters at the receiving end. Accordingly, the inverse of transmission matrix relates the parameters at the receiving end to the parameters at the sending end. Figure 12.50 shows the connection of the system and its transmission parameters.

Fig. 12.50 Transmission matrix



The equations for a system presented by its transmission matrix are as follows:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The expanded system is obtained as:

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

The components of the transition matrix can be found by either opening or shorting the receiving end and solving for the relations of the voltages and currents from the sending end to the receiving end.

$$A = \left. \frac{V_1}{V_2} \right|_{I_2=0}$$

$$B = - \left. \frac{V_1}{I_2} \right|_{V_2=0}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$D = - \left. \frac{I_1}{I_2} \right|_{V_2=0}$$

Note 12.11 The transmission matrix in a *reciprocal network* has a unique characteristic in that its determinant is always 1. This means:

$$AD - BC = 1$$

Presenting the Transmission Matrix Parameters in Terms of Impedance and Admittance Matrices

Parameters of the transmission matrix can be presented using the admittance and impedance matrices, provided that both Z and Y matrices exist. Otherwise, the transmission matrix cannot be defined and does not exist. To obtain the needed

impedance and admittance parameters, the ratios must be converted to the Z and/or Y parameters.

Note 12.12 In redefining the transmission matrix parameters in terms of impedance or admittance, the condition that the transmission matrix parameters are defined must be strictly enforced. For instance, A is defined where the current $I_2 = 0$. Therefore, its expansion can be found as:

$$A = \left. \frac{V_1}{V_2} \right|_{I_2=0} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$A = \frac{z_{11}}{z_{21}} \text{ or } A = -\frac{y_{22}}{y_{21}}$$

Accordingly:

$$B = -\left. \frac{V_1}{I_2} \right|_{V_2=0} = -\left. \frac{V_1}{I_1} \right|_{V_2=0} \left. \frac{I_1}{I_2} \right|_{V_2=0}$$

$$B = \frac{1}{y_{11}} \frac{z_{22}}{z_{21}}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$C = \frac{1}{z_{21}} \text{ or } C = \frac{1}{z_{11}} \frac{-y_{22}}{y_{21}}$$

$$D = -\left. \frac{I_1}{I_2} \right|_{V_2=0}$$

$$D = \frac{z_{22}}{z_{21}} \text{ or } D = \frac{y_{11}}{y_{21}}$$

Note 12.13 This approach is extremely useful when there are dependent sources in the circuit.

Example 12.32 Find the transmission matrix of the circuit shown in Fig. 12.51.

Solution The circuit forms a T matrix with a dependent source. The impedance matrix can be easily obtained as follows:

The T structure without the dependent source results in:

$$Z = \begin{bmatrix} 15 + 10 & 15 \\ 15 & 15 + 0 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 15 \end{bmatrix}$$

The effect of the dependent source is on the off-diagonal element in the second loop as follows:

Fig. 12.51 Circuit of Example 12.32

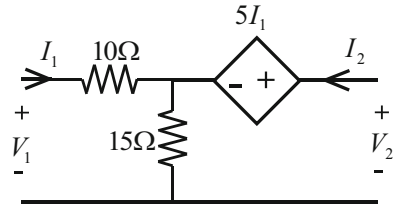
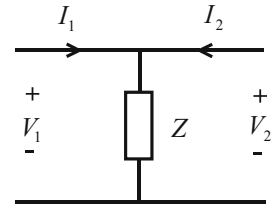


Fig. 12.52 Impedance in parallel



$$Z = \begin{bmatrix} 25 & 15 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 10 & 15 \end{bmatrix}$$

The admittance matrix is obtained by:

$$Y = Z^{-1}$$

$$Y = \begin{bmatrix} 0.0677 & -0.0677 \\ -0.0444 & 0.1111 \end{bmatrix}$$

The transmission matrix parameters become:

$$A = \frac{z_{11}}{z_{21}} = \frac{25}{15} = 1.667$$

$$B = \frac{1}{y_{11}} \frac{z_{22}}{z_{21}} = \frac{1}{0.0677} \frac{15}{10} = 22.156$$

$$C = \frac{1}{z_{21}} = \frac{1}{10} = 0.1$$

$$D = \frac{z_{22}}{z_{21}} = \frac{15}{10} = 1.5$$

Parallel Connection of an Element

Consider a circuit as shown in Fig. 12.52.

The transmission matrix of the network when the parallel element has impedance of z (Ω) can be obtained as follows:

Fig. 12.53 Circuit of Example 12.33

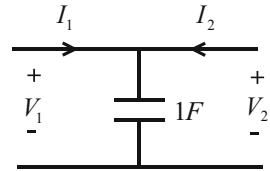
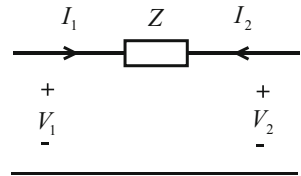


Fig. 12.54 Impedance in series



$$T = \begin{bmatrix} 1 & 0 \\ \frac{1}{z} & 1 \end{bmatrix}$$

Example 12.33 Find the transmission matrix of the circuit shown in Fig. 12.53.

Solution The capacitor C (F) has the impedance of $z = \frac{1}{sC}$ in Laplace. Therefore, the transmission impedance is:

$$T = \begin{bmatrix} 1 & 0 \\ \frac{1}{\frac{1}{sC}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix}$$

Series Connection of an Element

Consider a circuit as shown in Fig. 12.54.

The transmission matrix of the network when the series element has impedance of z (Ω) can be obtained as follows:

$$T = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

Example 12.34 Find the transmission matrix of the circuit in Fig. 12.55.

Solution The inductor L (H) has the impedance of $z = sL$ in Laplace. Therefore, the transmission impedance is:

$$T = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & sL \\ 0 & 1 \end{bmatrix}$$

Fig. 12.55 Circuit of Example 12.34

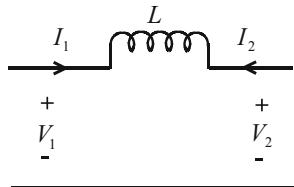


Fig. 12.56 Cascade connection of transmission matrices

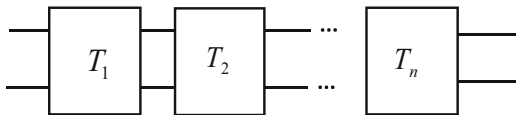
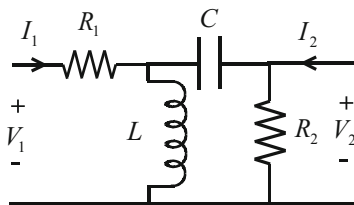


Fig. 12.57 Circuit of Example 12.35



Transmission Matrix of Cascade Systems

Consider cascade connection of several systems that have transmission matrices of $T_1, T_2, T_3, \dots, T_n$, as shown in Fig. 12.56.

The transmission matrix of the entire system is obtained as the product of the transmission of the subsystems in cascade connection. Therefore:

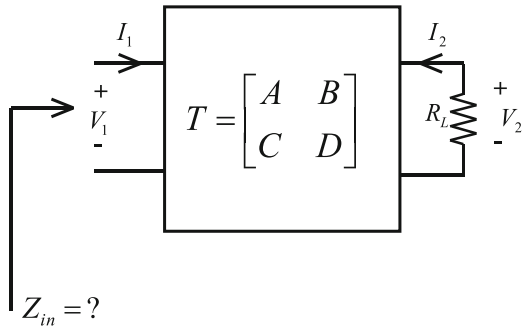
$$T = \prod_{i=1}^n T_i$$

Example 12.35 Find the transmission of the circuit shown in Fig. 12.57.

Solution The circuit can be broken into several parallel and series elements connected in cascade. In particular:

- R_1 is a series element with transmission matrix of $\begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix}$.
- L is a parallel element with transmission matrix of $\begin{bmatrix} 1 & 0 \\ \frac{1}{sL} & 1 \end{bmatrix}$.
- C is a series element with transmission matrix of $\begin{bmatrix} 1 & \frac{1}{sC} \\ 0 & 1 \end{bmatrix}$.
- R_2 is a parallel element with transmission matrix of $\begin{bmatrix} 1 & 0 \\ \frac{1}{R_2} & 1 \end{bmatrix}$.

Fig. 12.58 Circuit of
Example 12.36



Therefore, the transition matrix of the network is the product of the transmission matrices as follows:

$$T = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{sL} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{R_2} & 0 \\ 1 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \frac{(LC(R_1 + R_2))s^2 + (L + R_1R_2C)s + R_1}{R_2LCs^2} & \frac{R_1CLs^2 + Ls + R_1}{LCs^2} \\ \frac{R_1LC + Ls + R_1}{R_2LCs^2} & \frac{LCs^2 + 1}{LCs^2} \end{bmatrix}$$

Example 12.36 Find the input impedance of the circuit shown in Fig. 12.58.

Solution To find the input impedance of the circuit, the transmission matrix is used.

$$z_{in} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

Considering the relation of the voltage and current at the output or receiving end as:

$$V_2 = -R_L I_2$$

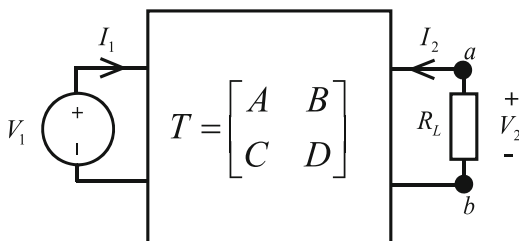
The replacement in the input impedance results in:

$$z_{in} = \frac{A(-R_L I_2) - BI_2}{C(-R_L I_2) - DI_2}$$

Simplifying the expression yields:

$$z_{in} = \frac{AR_L + B}{CR_L + D}$$

Fig. 12.59 A transmission matrix and its load connection



Finding Thevenin Equivalent Circuit from Transmission Matrix

Consider the circuit shown in Fig. 12.59.

With known input or sending end values of V_1 and I_1 , the circuit equations are obtained as:

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

To obtain the Thevenin voltage V_{th} , the load must be disconnected which imposes the current $I_2 = 0$. This results in:

$$V_1 = AV_2$$

Therefore:

$$V_{th} = V_2 = \frac{V_1}{A}$$

To obtain the Thevenin impedance, all independent sources in the circuit that includes the input voltage must be zero. Applying a voltage at the output indicates how much current is drawn. The ratio of the known applied voltage over the current determines the Thevenin impedance.

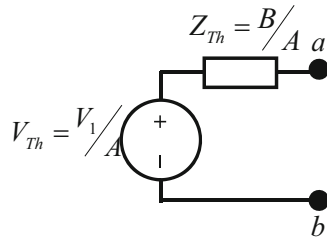
$$z_{th} = \left. \frac{(V_2 = 1)}{I_2} \right|_{V_1=0}$$

$$V_1 = AV_2 - BI_2 \rightarrow 0 = A - BI_2$$

$$z_{th} = \frac{1}{I_2} = \frac{B}{A}$$

Therefore, the Thevenin equivalent circuit is shown in Fig. 12.60.

Fig. 12.60 Thevenin equivalent of the transmission matrix (Fig. 12.60)



Example 12.37 A system has transmission matrix of $T = \begin{bmatrix} 2 & 1+j \\ 1 & 1+0.5j \end{bmatrix}$. Find the Thevenin equivalent when the input voltage is 50 V. At what load impedance the maximum power is transferred to the load.

Solution The Thevenin impedance and voltage are obtained as follows:

$$V_{th} = \frac{V_1}{A} = \frac{50}{2}$$

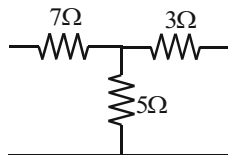
$$z_{th} = \frac{B}{A} = \frac{1+j}{2} \Omega$$

The load impedance must be the complex conjugate of the Thevenin impedance to transfer the maximum power. Therefore,

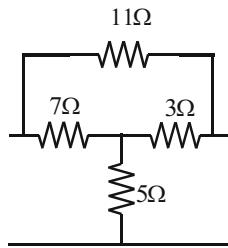
$$z_L = \frac{1-j}{2} \Omega$$

Problems

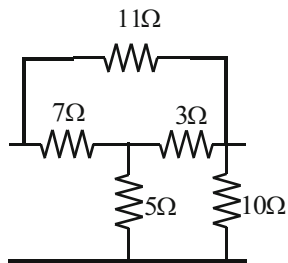
12.1. Find the impedance parameters of the following circuit.



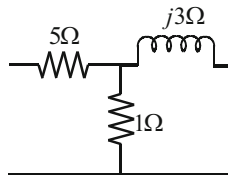
12.2. Find the impedance parameters of the following circuit.



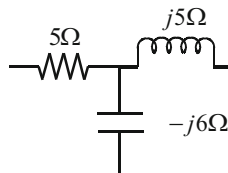
12.3. Find the impedance parameters of the following circuit.



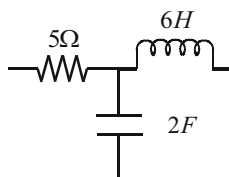
12.4. Find the impedance parameters of the following circuit.



12.5. Find the impedance parameters of the following circuit.



12.6. Find the impedance parameters of the following circuit.

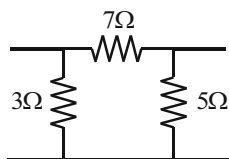


12.7. Find a two-port network that results in the following Z parameters.

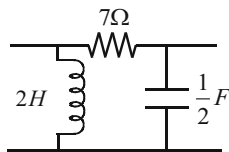
$$Z = \begin{bmatrix} \frac{3s+1}{s} & \frac{1}{s} \\ \frac{1}{s} & \frac{5s^2+1}{s} \end{bmatrix}$$

$$Z = \begin{bmatrix} 5s+4 & 4 \\ 4 & 3s+4 \end{bmatrix}$$

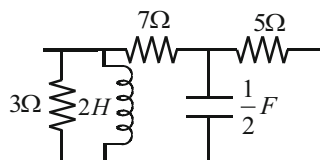
12.8. Find admittance parameters in the following circuits.



12.9. Find admittance parameters in the following circuits.



12.10. Find admittance parameters in the following circuits.

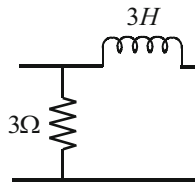


12.11. Find a two-port network that results in the following Y parameters.

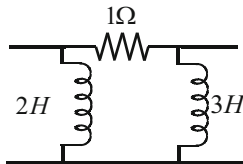
$$Y = \begin{bmatrix} \frac{2s+1}{s} & -2 \\ -2 & s+2 \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{s^2+1}{s} & -s \\ -s & \frac{s^2+1}{s} \end{bmatrix}$$

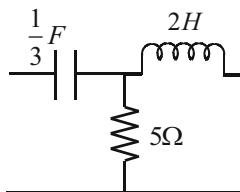
12.12. Find Y and Z .



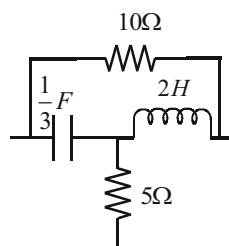
12.13. Find Y and Z .



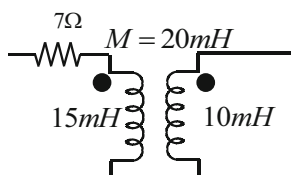
12.14. Find Y and Z .



12.15. Find Y and Z .



12.16. Find Y and Z .



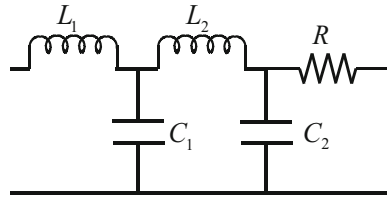
12.17. Find a two-port network that has the following impedance matrix.

$$Z = \begin{bmatrix} 2s + 1 & 5s + 4 \\ 5s + 1 & s + \frac{1}{s} \end{bmatrix}$$

12.18. Find a two-port network that has the following admittance matrix.

$$Y = \begin{bmatrix} \frac{s+1}{s} & s \\ -2 & 2s+7 \end{bmatrix}$$

12.19. Find transmission matrix of the following two-port network.



12.20. Find the impedance and admittance matrix of the circuit in previous problem when $L_1 = 1$ H, $L_2 = 3$ H, $C_1 = \frac{1}{2}$ F, $C_2 = \frac{1}{5}$ F, and $R = 5$ Ω .

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