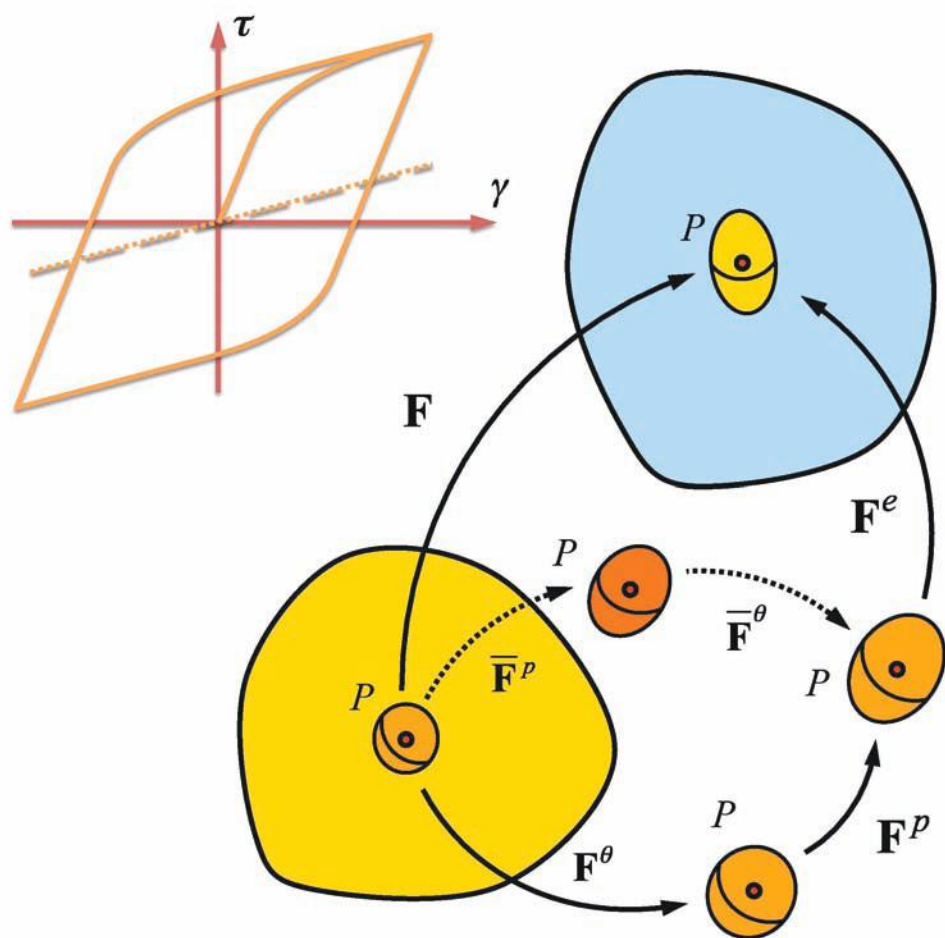


The Mechanical and Thermodynamical Theory of Plasticity



Mehrdad Negahban



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CRC Press

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Boca Raton London New York

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CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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Version Date: 20120419

International Standard Book Number-13: 978-1-4665-6321-6 (eBook - PDF)

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With Love

To my parents

Meriam and Ezat

for letting me become what I wanted to be and for helping me in this process

To my wife

Setareh

for reminding me that some will never understand

To my children

Arman and Shahdi

for the strength you give me

and

with the hope that
you each find what you want to become
you reach to achieve it
and
you enjoy the process

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Acknowledgments

The author would like to acknowledge the help of many people who provided insight, sometimes unknowingly, and people who helped in the preparation of this book. These include colleagues and students. Many of the students taking my classes pointed out corrections and provided suggestions, which was very much appreciated. Particular thanks go to Ashwani Kumar Goel, Lili Zhang, Kyle Strabala, Benjamin Polly, and Saeed Eghtedar Doust, who each read part or all of the text and provided corrections and comments, and Yen-an Wang for doing the steel and aluminum tests. I owe a particular debt of gratitude to **Ashwani Kumar Goel**, who did most of the simulations, and **Lili Zhang**, who carefully read and checked all the equations and representations; they were true partners in this endeavor.

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Preface

The text

The text of this book came out of a number of courses and lectures presented over the past fifteen years on continuum mechanics, nonlinear mechanics, continuum thermodynamics, viscoelasticity, plasticity, crystal plasticity, and thermodynamic plasticity given to a varied audience, including students from Material Science, Engineering Mechanics, Mechanical Engineering, and Civil Engineering. As a result, the book covers a large variety of subjects and includes presentations at many levels. The following is a short description of the chapters:

Chapter 1, Plasticity in the 1-D bar: An A to Z description of continuum mechanics, plasticity, thermoplasticity, rate-dependent plasticity, balance laws, jump conditions, and problem solving all presented in the context of a one-dimensional bar. The chapter is organized to parallel the multidimensional theory and can be used both for a short or undergraduate course on the subject and for a reading review or introduction for students wanting to study the multidimensional theory. The presentation includes an overview of equilibrium problems, dynamic loading problems, and wave propagation and impact problems.

Chapter 2, Vectors and tensors: An overview of linear algebra, vectors, tensors, and vector calculus. The chapter presents the notation and commonly used results and tools, including how to calculate derivatives of functions of specified vector and tensor variables. For most readers who have had continuum mechanics, this chapter is for familiarization with the notation.

Chapter 3, Kinematics of deformation and description of temperature: An extensive description of the deformation and temperature fields, their gradients, and associated rates. The chapter also provides a description of basic deformations. For readers who have had continuum mechanics, this chapter should be a review and a source of calculated relations between the different variables.

Chapter 4, Elastic, plastic, and thermal deformation: A description of the decomposition of the deformation gradient into elastic, plastic, and thermal parts and the description of their associated measures of strain and strain rate. The chapter also considers the idea of infinitesimal deformations. This is the starting point for multidimensional plasticity, particularly for students starting with a background in continuum mechanics.

Chapter 5, Traction, stress, and heat flux: An extensive description of traction, stress, heat flux, and heat flux vector. The chapter covers the associated Cauchy's theorems and introduces the concepts of nominal stress and heat flux vector and related quantities such as the stress invariants. The part of the chapter on traction and stress should be a review for readers familiar with nonlinear continuum mechanics.

Chapter 6, Balance laws and jump conditions: An overview of the balance laws of continuum thermodynamics and the associated transport theorems and jump conditions. The chapter both develops the finite deformation and infinitesimal versions of

the laws, and presents the concept of heat flow. This chapter should be a review for a reader familiar with nonlinear continuum mechanics.

Chapter 7, Infinitesimal plasticity: A description of the infinitesimal theories of plasticity, thermoplasticity, and rate-dependent plasticity with a presentation of thermodynamically consistent models. The chapter uses mechanical analogs to develop the basic ideas and representation theory to arrive at general models for anisotropic materials. Special attention is given to the classical yield functions and flow rules. An example of a quadratic free-energy function with both elastic and back-stress components is presented to show how to construct a model with both kinematic and isotropic hardening. This chapter could be a core chapter for a first course on thermodynamically based plasticity.

Chapter 8, Solutions for infinitesimal plasticity: An overview of problem solutions for infinitesimal plasticity, including homogeneous deformations, inhomogeneous deformations, nonuniform loading problems, axially or spherically symmetric problems, and associated heat generations. The problems include many of the classical solutions and some new ones.

Chapter 9, First gradient thermomechanical materials: A description of first gradient material models and associated restriction with an introduction to state variables and their role in such a structure. This is a description of the constitutive theory that all the large deformation models are based on. This should be a review for readers familiar with constitutive theory and may be skipped or assigned as reading for classes on finite deformation theory.

Chapter 10, Elastic and thermoelastic solids: A general development of the nonlinear theory of thermoelasticity, including discussions on material symmetry, internal constraints, and superposition of deformations and wave propagation. The chapter is constructed such that it can be an initial part of a nonlinear thermoelasticity and thermoplasticity class.

Chapter 11, Finite deformation mechanical theory of plasticity: A development of the finite deformation mechanical theory of plasticity. The chapter focuses on the constraints imposed by rigid body motions and by material symmetry and looks at representation theory and its role in constructing general mechanical models for plasticity.

Chapter 12, Thermoplastic solids: A development of the temperature-dependent and thermodynamically based theories of finite deformation plasticity. This chapter provides general models and examples of models, particularly some that mimic infinitesimal models. This is a core chapter for any course on thermoplasticity at finite deformations.

Chapter 13, Viscoelastic solids: A development of linear and nonlinear viscoelasticity based on a similar construction as used in plasticity. This chapter provides a view of viscoelasticity that builds on the same structure as plasticity. The chapter is a preamble to the development of rate-dependent plasticity and shows how the two structures are similar. The chapter also provides the thermodynamical constraints and related results that can be used for replacing the elastic range with a viscoelastic range.

Chapter 14, Rate-dependent plasticity: A development of finite deformation rate-dependent plasticity and its generalization to multiple relaxation processes. The chapter on viscoelasticity and this chapter construct the core for a course on rate effects in thermodynamically based models of this type.

Chapter 15, Crystal plasticity: A description of crystals, crystal classes, and finite deformation crystal plasticity. The chapter is a focused look at the mechanics and thermodynamics of crystal plasticity, including constructing rate-dependent and rate-independent models. The chapter can be used for a short special topics course on this subject or as part of a finite deformation plasticity class.

Appendix A, Representation of functions: A summary of representation theory for functions of scalars, vectors, and tensors.

Appendix B, Representation of fourth-order tensor constants: A summary of representations for constant fourth-order tensors having different symmetries.

Appendix C, Basic equations in curvilinear systems: A summary of the basic kinematic variables and balance laws in a general curvilinear system and their special representations for rectangular, cylindrical, and spherical systems.

The book can be used for many different courses and comes with problems in most chapters. Many of the chapters, particularly the later ones, are written to be used independently from one another. As a result, the reader may note some duplication that is included intentionally to allow this type of use. The following are some possible organizations of the chapters for courses:

Continuum Mechanics:

1. **Chapter 1:** Selected section on bar deformation, strain, transport theorems, balance laws, equilibrium, steady-state, dynamic problems, and jump conditions.
2. **Chapter 2:** Matrices, vectors, tensors, notation, algebra, and calculus.
3. **Chapter 15:** Example of using the dual bases in crystal classes.
4. **Chapter 3:** Kinematics of motion and deformation.
5. **Chapter 5:** Traction, stress, heat flow, and heat flux vector. The Cauchy theorem and the true and nominal stresses and heat fluxes.
6. **Chapter 15:** Example of resolved shear stress in crystals.
7. **Chapter 6:** Transport theorems, balance laws, and jump conditions.
8. **Chapter 9:** Selected sections on first gradient materials (might skip and only do elasticity).
9. **Chapter 10:** Selected sections on linear and nonlinear elasticity.

Theory of plasticity:

1. **Chapter 1:** Deformations in a one-dimensional bar, plasticity model, yield function, flow rule, consistency condition, thermodynamic plasticity, rate-dependent plasticity, balance laws and jump conditions, drawing, and wave propagation in an elastic-plastic bar.
2. **Chapter 3:** Review of kinematics and a description of deformation gradient and superposition of deformations.
3. **Chapter 4:** Decomposition of the deformation into elastic, plastic, and thermal parts. The relation between the finite measures of strain and their rates.
4. **Chapter 5:** Stress invariants and heat flow.
5. **Chapter 6:** Review of balance laws and the thermodynamic constraints for infinitesimal deformations and heat flow.

6. **Chapter 7:** Infinitesimal plasticity, mechanical analog and example models, yield functions, flow rule, consistency condition, thermodynamic models, and anisotropic materials.
7. **Chapter 8:** Selected example problems of homogeneous deformation (extension, simple shear, biaxial extension, etc.), inhomogeneous deformations (bending, torsion, etc.), inhomogeneous loading problems (deflection of a bar), axially symmetric and spherically symmetric problems, etc.

Nonlinear thermoelastic and thermoplastic materials:

1. **Chapter 3:** Review of kinematics and a description of deformation gradient and superposition of deformations.
2. **Chapter 4:** Decomposition of the deformation into elastic, plastic, and thermal parts. The relation between the finite measures of strain and their rates.
3. **Chapter 5:** Stress invariants, heat flow, nominal and true stresses, and heat flux vector.
4. **Chapter 6:** Review of balance laws and the thermodynamic constraints for finite deformations and heat flow.
5. **Chapter 10:** Thermoelasticity, rigid body motions, material symmetry, constraint conditions, superimposed deformations, and wave propagation.
6. **Chapter 12:** Thermoplasticity, mechanical analog, example models, thermodynamically consistent models, materials with quadratic free energy in elastic and plastic strain measures, isotropic and anisotropic materials.

Nonlinear plastic and thermoplastic materials:

1. **Chapter 3:** Review of kinematics and a description of deformation gradient and superposition of deformations.
2. **Chapter 4:** Decomposition of the deformation into elastic, plastic, and thermal parts. The relation between the finite measures of strain and their rates.
3. **Chapter 5:** Stress invariants, heat flow, nominal and true stresses, and heat flux vector.
4. **Chapter 6:** Review of balance laws and the thermodynamic constraints for finite deformations and heat flow.
5. **Chapter 12:** Thermoplasticity, mechanical analog, example models, thermodynamically consistent models, materials with quadratic free energy in elastic and plastic strain measures, and isotropic and anisotropic materials.
6. **Chapter 15:** Rate independent crystal plasticity.

Nonlinear plastic and viscoplastic materials:

1. **Chapter 3:** Review of kinematics and a description of deformation gradient and superposition of deformations.
2. **Chapter 4:** Decomposition of the deformation into elastic, plastic, and thermal parts. The relation between the finite measures of strain and its rate.
3. **Chapter 5:** Stress invariants, heat flow, nominal and true stresses, and heat flux vectors.
4. **Chapter 6:** Review of balance laws and the thermodynamic constraints for finite deformations and heat flow.

5. **Chapter 12:** Thermoplasticity, mechanical analog, example models, thermodynamically consistent models, materials with quadratic free energy in elastic and plastic strain measures, and isotropic and anisotropic materials.
6. **Chapter 14:** Rate-dependent plasticity and thermoplasticity.
7. **Chapter 15:** Rate-independent and rate-dependent crystal plasticity.

Viscoelastic and viscoplastic materials:

1. **Chapter 3:** Review of kinematics and a description of deformation gradient and superposition of deformations.
2. **Chapter 4:** Decomposition of the deformation into elastic, plastic, and thermal parts. The relation between the finite measures of strain and their rates.
3. **Chapter 5:** Stress invariants, heat flow, nominal and true stresses, and heat flux vectors.
4. **Chapter 6:** Review of balance laws and the thermodynamic constraints for finite deformations and heat flow.
5. **Chapter 12:** Thermoplasticity, mechanical analog, example models, thermodynamically consistent models, materials with quadratic free energy in elastic and plastic strain measures, and isotropic and anisotropic materials.
6. **Chapter 13:** Viscoelasticity, mechanical analogs, one-dimensional response, thermodynamic models, isotropy, and anisotropy.
7. **Chapter 14:** Rate-dependent plasticity and thermoplasticity.
8. **Chapter 15:** Rate-dependent crystal plasticity.

The subject of crystals and crystal classes has been used by the author to motivate using general descriptions of vectors and tensors. The same was done to motivate mathematics by using examples from one-dimensional viscoelasticity. This particularly works well for students of Materials Science.

References

There is a vast variety of books on the subject of plasticity. The following is a partial list of books the author recommends be used as references when studying plasticity.

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- Elasticity and Plasticity of Large Deformations, An Introduction, by Albrecht Bertram, 2005, Springer.
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Chapter 1

Plasticity in the 1-D bar

As a first step toward understanding plasticity, we will study plasticity in the context of the one-dimensional theory of straight bars. This approach will allow us to concentrate on the ideas central to continuum mechanics and plasticity without needing to deal with the many complications that appear when dealing with multidimensional theories. With this in mind, the one-dimensional theory presented in this chapter is developed on the same theoretical foundations that the multidimensional theory is based on. As such, we will avoid introducing simplifications or developments that do not easily generalize to multiple dimensions. Once we have become familiar with the basic concepts central to the development, then in the following chapters, we will study the multidimensional theory. It is my hope that this method of presentation will aid the reader to better understand plasticity. By following the presentation in this chapter, the reader also should develop a strong understanding of how plasticity fits into continuum mechanics and continuum thermodynamics, which then should help the reader in understanding the structure of the developments in the later chapters.

Even though plasticity normally refers to “rate-independent and temperature-independent plasticity,” we will not limit ourselves to this. After we introduce the basic concepts for a bar, we will look at the mechanical theory of rate-independent plasticity, followed by temperature-dependent plasticity, and then rate-dependent plasticity. To complete the presentation, we will look at the balance laws and jump conditions and consider specific problems that use these laws in their solution process.

1.1 Introduction to plastic response

Before we proceed to look at the theory of bars, let us take a look at plasticity. Plasticity refers to a specific type of material response normally characterized, for example, by the cyclic loading and unloading response of the aluminum shown in Figure 1.1. The schematic loading diagram for elastic plastic response is shown in Figure 1.2. The figure shows the load-extension plot for a uniform bar undergoing a homogeneous deformation. Plasticity normally is characterized by an initial elastic response which is reversible and continues until the initial yield load (B). When loading beyond the initial yield load, the elastic response is followed by an elastic-plastic response (BC) that includes further elastic deformation accompanied by plastic flow. Plastic flow is not recoverable so that when unloading (CD), after elastic-plastic deformation (BC), the elastic part of the deformation is recovered, but the plastic part (AD) remains even after full removal of the load. On compression, the local elastic response will continue to the yield point (E), after which combined elastic deformation and plastic flow in compression will start (EF). If the load is reversed again, a new elastic region (FGH) will be encountered, inside which the local response is fully reversible. For this elastic range, once we reach either yield point (tension or compression), plastic flow again is initiated until the loading is reversed. Once we enter any of the elastic regions, the response is locally reversible

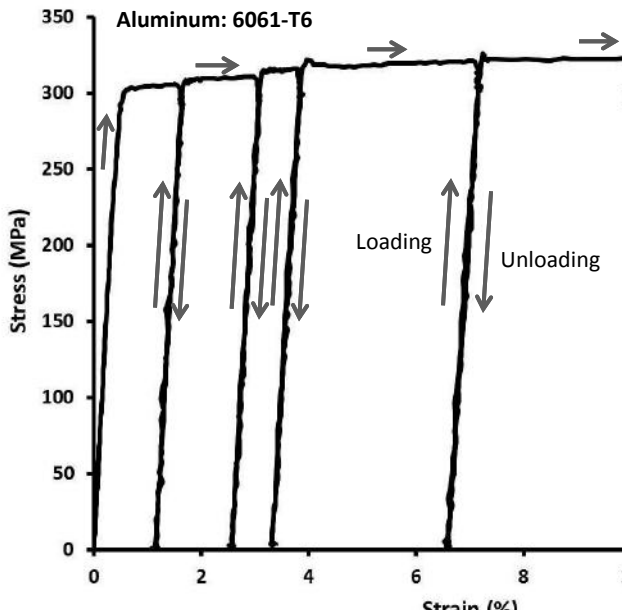


Figure 1.1: The cyclic loading and unloading of 6061-T6 aluminum in tension.

so that the loading and unloading graphs will fall on the same line. At each loading point (I), there is a current elastic response range (IJK), which is defined by the current yield points (I and K). This elastic range is assumed to always exist at each loading point, even if the current elastic range is not entered.

1.2 The bar and the continuum assumption

This entire chapter will be devoted to the study of bars under uniaxial deformation. Figure 1.3 shows the straight bar we will be considering. For this bar, it will be assumed that points on a given cross section of the bar move together and, therefore, all remain on the same cross section. Any lateral motions of the particles will be ignored. As a result, the motion of all points on a given cross section can be described by the history of the axial displacement of the cross section. In this way, for all practical purposes, the bar is replaced by a line. The resulting theory is self-consistent, and not a specialization of a multidimensional theory. At the conclusion of our study, we will have arrived at a model for the response of this bar, consisting of a system of equations and initial and boundary conditions that can be used to study how this bar responds to mechanical and thermal loads. At this point, all that we consider is of a general nature and applies to all material response types and is not restricted to plasticity. As a constitutive assumption, the assumption of plastic response only affects certain parts of the development, which should become clear to the reader as we proceed through the chapter.

In continuum theories, it is assumed that at each point in the body there is matter, so that the matter that creates the body is continuously connected. This assumption allows one to set up a mathematical structure that covers the body by a continuous coordinate system. As a result, one deals with functions of continuous variables that can be integrated

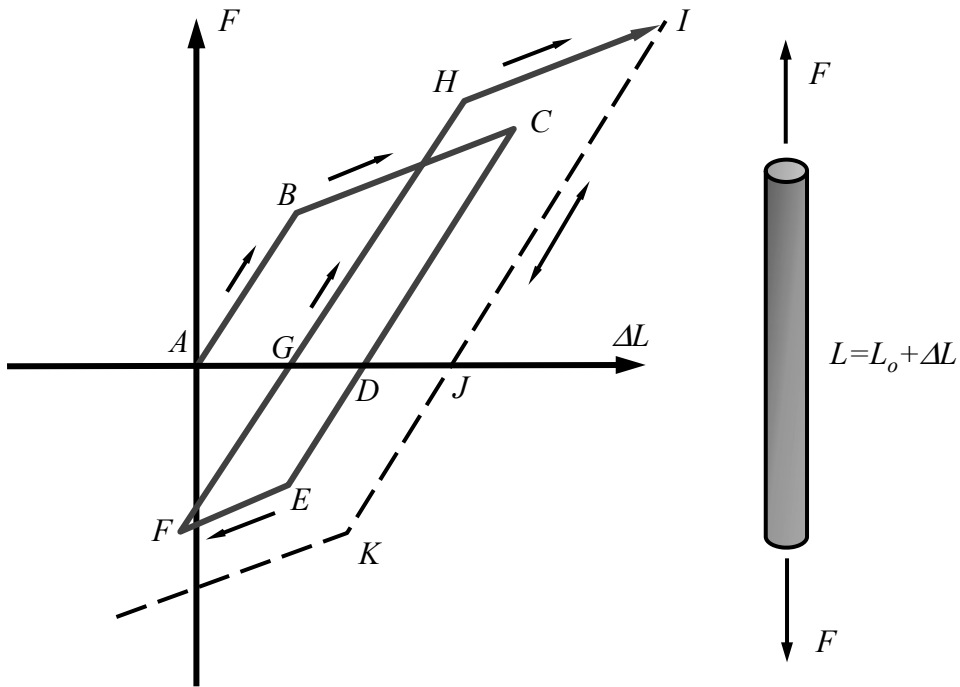


Figure 1.2: Schematic of the elastic-plastic response of a uniform bar. In the schematic, F is the axial load, L is the current length, L_o is the initial length, and $\Delta L = L - L_o$ is the extension of the bar from its initial length.

and differentiated. The continuum assumption does not create substantial limitations as long as we stay away from studying objects that have dimensions that are of the order of the discontinuities in the structure. It turns out that one can consider a body as a continuum at many different scales. For example, a metal part normally can be considered a continuum when the part is large compared to the grain size. Yet, we can study each grain as a continuum as long as we are looking at events occurring on a scale much larger than the lattice dimensions. The basic point to remember is that we should make sure that the continuum assumption is valid at the scale at which we are studying the material response.

Most properties of a continuum are assumed to be distributed throughout the body. For example, let us look at mass density. To define mass density at a point in the bar, we consider a sequence of segments of the bar, each smaller than the last, and all containing the point under consideration, as shown in Figure 1.4. For each segment, we define an average density equal to the mass of the segment divided by its length. We then assume that we can set up a limit process that sequentially takes smaller and smaller segments, each still containing the point under consideration. At the limit, we arrive at a quantity that defines the density of the bar at the desired point. As will be shown later in the chapter, quantities such as velocity, acceleration, stretch, linear momentum, internal energy, and kinetic energy also will be considered as distributed throughout the bar. The assumption that the bar is a continuum accommodates such distributions, which later will be useful in obtaining differential equations for the response of the bar.

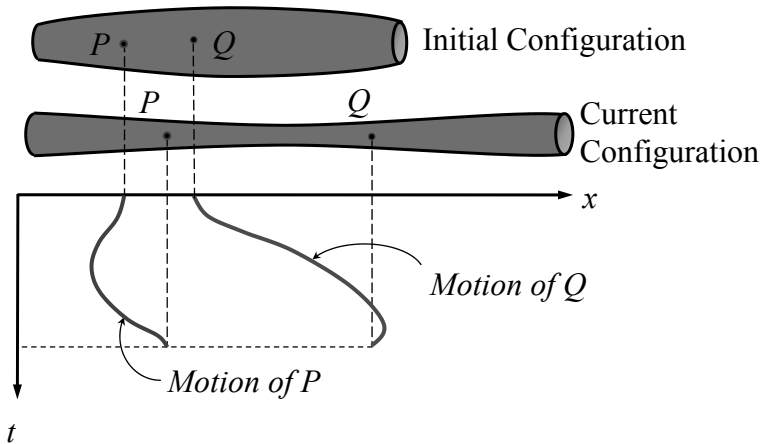


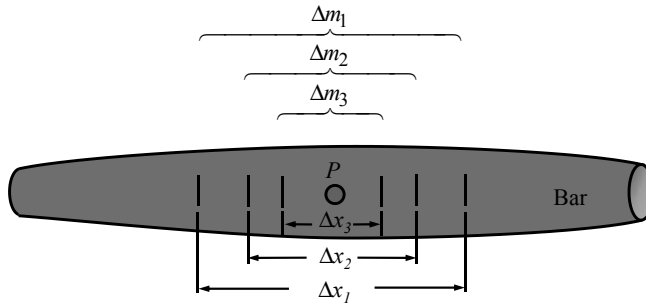
Figure 1.3: A straight bar undergoing a uniaxial deformation.

1.3 Motion and temperature of points on a bar

The next thing to do is to describe the primitive quantities of mechanical and thermal importance. Each point on our straight bar will be allowed to move along the axis of the bar and to change its temperature. These two quantities, position and temperature, will be the primitive quantities associated with each particle in our bar.

Let us first start by describing motion. Figure 1.5 shows the bar in several of its configurations. In our discussion, a *particle* will refer to a specific piece of matter that moves and occupies different *points* in space. Each configuration represents the relative location of particles in the bar. As such, each *configuration* is an ordered set of distinct points, each point representing a distinct particle in the bar. For simplicity, the reader can assume that each configuration represents the placement of particles in a shape actually taken by the bar as it is being subjected to some external loading. In general, configurations need not be related to actual shapes the body takes. Two special configurations we frequently use are the *reference configuration*, which is the configuration used to compare all other configurations against, and the *current configuration*, which is the configuration of the body at the current time. The *initial configuration* of the bar frequently is used as the reference configuration, even though this is not required.

In Figure 1.5, two particles P and Q on the bar are singled out for examination. The reader may imagine ink marks put on the bar to identify each point. The letters P and Q are the names we have selected for these particles. These names uniquely identify these two particles. We will first look at the motion of these specific particles and then come up with a different way of naming particles on the bar so that we easily can study how any of the infinite particles of the bar move. In the figure, below the configurations of the bar, a diagram of the position of each particle as a function of time is given. Here we let t denote time, let $x_P(t)$ denote the position of P as a function of time, and let $x_Q(t)$ denote the position of Q as a function of time. As in particle mechanics, the time derivative of the



$$\rho = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x}$$

Figure 1.4: The limit process used to calculate density. The limit process consists of a sequence of smaller and smaller segments containing the point under consideration.

position of each particle is its *velocity*, and the second time derivative of the position of each particle is its *acceleration*. If $v_P(t)$ and $a_P(t)$ denote, respectively, the velocity and acceleration of P , then we have

$$v_P(t) \equiv \frac{dx_P}{dt}(t),$$

$$a_P(t) \equiv \frac{dv_P}{dt}(t).$$

In a similar manner, one can define the velocity and acceleration of Q as

$$v_Q(t) \equiv \frac{dx_Q}{dt}(t),$$

$$a_Q(t) \equiv \frac{dv_Q}{dt}(t).$$

We need to define the position, velocity, and acceleration of each of the infinite particles in the body. To do this, we need to give a unique name to each particle in the body. Since there are an infinite number of particles (as we have assumed the bar to be a continuum), we have to come up with an infinite number of names, each distinct from the others. On the surface, this is an impossible task, but we can do it by using the real line. The real line has the nice property of being continuous, with each point on it having a unique number associated with it. We will use the position of each point on the real line in the reference configuration as the name of the associated particle. For example, in place of using names like P and Q to distinguish particles on our bar, we now are going to select a configuration of the bar, lay a real line on the bar in this configuration, and use the number associated with the coordinate of each particle as the name for the particle. This is shown in Figure 1.5. The special configuration that is used to give names to all the particles of the bar is called the *reference configuration* and need not be a real configuration of the bar. We will denote the reference configuration by κ_0 . Let the locations of P and Q in the reference configuration be X_P and X_Q , respectively. Now, in place of P and Q , we will call these same particles X_P and X_Q . As a result of this naming system, we now have names for all the particles in the bar. Next, we need to describe the position of each particle as a function of time.

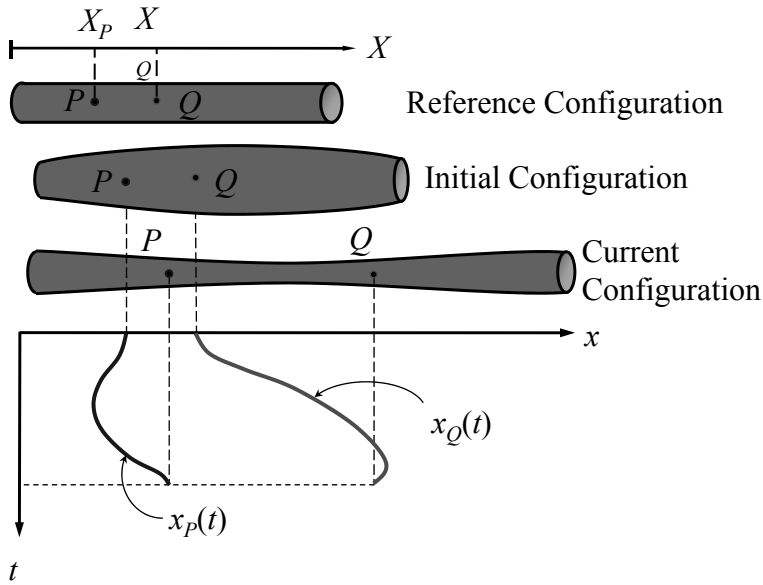


Figure 1.5: Reference, initial, and current configurations of our bar. Motion of points P and Q on the bar.

We used $x_P(t)$ to denote the position of P as a function of time. Now let us introduce a function $x(X, t)$, which gives the position at time t of the particle with name X . This function is called the *motion* of the bar since it tells us where every particle on the bar is as one moves forward in time. So our point P at time t is at the location

$$x_P(t) = x(X_P, t),$$

and our point Q is at location

$$x_Q(t) = x(X_Q, t).$$

Note that uppercase “ X ” is being used for position in the reference configuration, and lower case “ x ” is being used for position at the current time t . Since the reference configuration need not be a real configuration taken by the bar, there might not be a time at which the configuration of the body overlays the reference configuration. The velocity of any particle X is the rate of change of its position. If $v(X, t)$ denotes the velocity at time t of particle X , then

$$\boxed{v(X, t) \equiv \frac{\partial x(X, t)}{\partial t}}. \quad (1.1)$$

Note that the position of particle X is given by the function $x(X, t)$, where X is held fixed. This explains why we use the partial derivative to get the velocity. For example, the velocity of particle P will be given by

$$v_P(t) = v(X_P, t) = \left. \frac{\partial x(X, t)}{\partial t} \right|_{X=X_P},$$

where “ $|_{X=X_P}$ ” refers to the fact that we first take the partial derivative and then set X equal to X_P . Acceleration is also a quantity defined only for specific particles. Focusing

on the velocity of a single particle, the time derivative of the velocity is the acceleration. Therefore, if $a(X, t)$ denotes the acceleration of particle X at time t , then

$$a(X, t) \equiv \frac{\partial v(X, t)}{\partial t} = \frac{\partial^2 x(X, t)}{\partial t^2}. \quad (1.2)$$

As above, the acceleration of specific particle P will be given by

$$a_P(t) = a(X_P, t) = \left. \frac{\partial v(X, t)}{\partial t} \right|_{X=X_P} = \left. \frac{\partial^2 x(X, t)}{\partial t^2} \right|_{X=X_P}.$$

The type of time derivative employed above in calculating the velocity and acceleration is called a *material time derivative*. The special feature associated with this derivative is that we keep the particle under consideration fixed as we change time. To distinguish this derivative from others, we will employ a special notation “ D/Dt ” for material time derivative. We now can write the velocity and acceleration, respectively, as

$$v(t) \equiv \frac{Dx}{Dt}(t), \quad (1.3)$$

and

$$a(t) \equiv \frac{Dv}{Dt}(t) = \frac{D^2x}{Dt^2}(t). \quad (1.4)$$

In this notation, we assume that lowercase “ x ” is the position of particles, and that the time derivatives are evaluated as one follows the motion of a single particle.

The description of a function in terms of the variable set (X, t) is called a *material description*, or a *Lagrangian description*. As opposed to the material, or Lagrangian, description, the *spatial*, or *Eulerian*, description of a function is in terms of the variable set (x, t) , where x is the current position. For example, if the velocity is given in terms of a function $v(x, t)$, it would be a spatial or Eulerian description of the velocity. To obtain the acceleration from the spatial description of velocity, we need to find a way of taking the derivative of this function holding X fixed (i.e., holding the particle fixed). This can be done by using $x = x(X, t)$, which yields the current location of particle X . Using the chain rule, one can therefore write

$$a(x, t) = \frac{Dv(x, t)}{Dt} = \frac{\partial v(x, t)}{\partial x} \frac{Dx}{Dt} + \frac{\partial v(x, t)}{\partial t}.$$

The acceleration can now be calculated after some reorganization and using (1.3) to obtain

$$a(x, t) = \frac{\partial v(x, t)}{\partial t} + v(x, t) \frac{\partial v(x, t)}{\partial x}, \quad (1.5)$$

or in shorthand notation as

$$a = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}.$$

In this expression for calculating acceleration, the first term on the right-hand side is the derivative of velocity holding the current location fixed (looking at the velocity of different particles as they pass through a point). The second term is known as the convected term. In the second term on the right-hand side, v gives the speed at which particles are moving through the point under consideration, and the spatial derivative of velocity $\frac{\partial v}{\partial x}$ describes how velocity is changing from point-to-point in space (at a fixed time). The combination appearing in the second term on the right-hand side gives the correction needed to account for the fact that the first term is not a material time derivative.

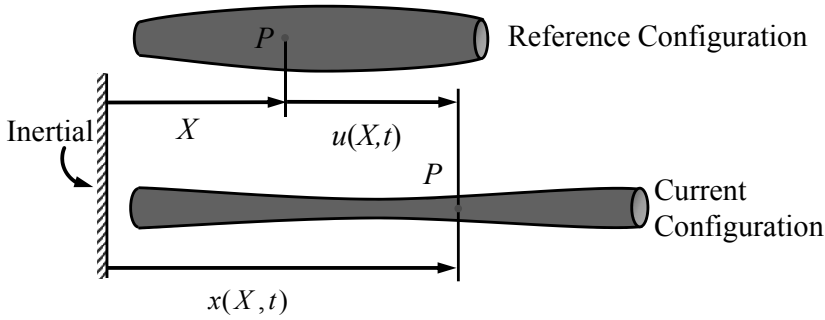


Figure 1.6: The displacement function $u(X, t)$ provides the amount each particle moves forward from its position in the reference configuration.

There is nothing special about the spatial description (other than the convenience of using it in some problems, for example, in fluid mechanics, where it is easier to describe velocity as a function of position in space as opposed to a particle). Other descriptions also can be devised. As long as one can find a way of taking the material time derivative, appropriate expressions for velocity and acceleration can be calculated for these descriptions.

An alternate way of describing motion is to use a displacement function. The *displacement function* is denoted by $u(X, t)$ and provides the amount of movement of particle X by time t . Figure 1.6 shows the displacement function. The relation between the motion and the displacement function is given by

$$\boxed{x(X, t) \equiv X + u(X, t).} \quad (1.6)$$

Since X is held constant during a material time derivative, the expressions for velocity and acceleration in terms of the displacement function are

$$\boxed{v(X, t) = \frac{\partial u(X, t)}{\partial t},} \quad (1.7)$$

$$\boxed{a(X, t) = \frac{\partial^2 u(X, t)}{\partial t^2}.} \quad (1.8)$$

Similar to position, each particle in the bar has a hotness described by a variable we call temperature. Temperature will be denoted by the symbol θ . We use exactly the same method to describe temperature in our bar as we did above to describe position. Therefore, we have the choice of a material or spatial description, among many others. In a material description, we give temperature as a function of position in the reference configuration, X , and time t (i.e., $\theta(X, t)$). In a spatial description of temperature, we provide temperature as a function of current position, x , and time t (i.e., $\theta(x, t)$).

Position, velocity, and acceleration describe important kinematic characteristics of a body at each point in time. If one knows how points of the body are moving, it is possible to figure out how the body is being distorted. The next section will focus on this and on how temperature is changing from point to point in the body. This will be done by describing stretch ratio, strain, velocity gradient, and temperature gradient.

Section 1.3-Problems:

- 1.3-1:** You are given the motion of points on a bar as $x(X, t) = X + \frac{t^3}{3}X$. Calculate the velocity and acceleration of each point on the bar. If the bar is described by $0 \leq X \leq L$, evaluate the motion of both ends of the bar. Also, evaluate the velocity and acceleration of these two end points.
- 1.3-2:** The velocity of points on a bar is given by $v(X, t) = 3t^2X$. You know that the bar in the reference configuration is described by $0 \leq X \leq L$, and that at time $t = 0$ you have $x(X, 0) = 2X$. Calculate the position and acceleration of points on the bar, and as special cases obtain the motion of its two end points. Does this body ever take its reference configuration?
- 1.3-3:** The acceleration of a body is given by $a(X, t) = 2tX^2 + 3t^2X$. You know that the bar in its reference configuration is described by $0 \leq X \leq L$, and that it starts from rest in the reference configuration. Calculate the position and velocity of each point of the bar and describe the motion of its end points and its initial midpoint. Does the midpoint stay in the middle of the bar?
- 1.3-4:** The spatial description of the velocity of a bar is given by $v(x, t) = 2tx$. Calculate the acceleration of points on the bar. Also, evaluate the motion of the bar knowing that the bar is in its reference configuration at the initial time $t = 0$. Verify that this motion gives the velocity you started with and the acceleration you calculated.
- 1.3-5:** Assume that velocity is described in terms of the variables ξ and t , where ξ is a function of x and t . Follow similar steps as described in evaluating (1.5) to derive the expression for evaluating acceleration using the (ξ, t) description. As an example, consider $v = (1 - t)\xi$ and $\xi = (1 + t)x$. For this example, first calculate acceleration using the equation you derived, and then recalculate acceleration by using $v = (1 - t^2)x$ in equation (1.5) and show that you obtain the same result.

1.4 Stretch ratio, strain, velocity gradient, temperature gradient

In continuum mechanics the models used to describe the forces developed in a body are related to how particles of the body move relative to each other. Stretch ratio, strain, and velocity gradient are parameters that measure this relative motion. Models developed to describe flow of heat in a body are based on the relative temperature of material points. Temperature gradient is the parameter that measures the relative temperature of points. In this section, we look at how these parameters are defined.

Stretch ratio is the ratio of the current separation of two close particles to their separation in the reference configuration. Figure 1.7 shows the length of the segment of bar between particles P and Q . These segments are shown both in the reference configuration and in the current configuration. These lengths are denoted by ΔX and Δx , respectively. If P and Q are not close, then the ratio of Δx to ΔX is an average measure of how much particles around P and Q have been separated. To make this a more accurate measure of the stretch,

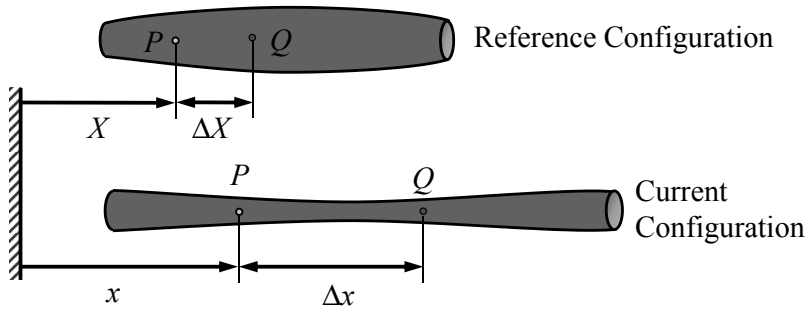


Figure 1.7: Stretching of the line element between P and Q as the body deforms.

we let Q approach P and arrive at the following expression for the stretch ratio λ given by

$$\lambda = \lim_{\Delta X \rightarrow 0} \frac{\Delta x}{\Delta X}.$$

The stretch ratio λ is in actuality a function of position and time since each segment of our bar may be undergoing a different extension process. The location of P in the reference configuration is given by X and the location of Q is given by $X + \Delta X$. The location of P in the current configuration can be obtained from the motion function as being equal to $x(X, t)$, and the location of Q is given as $x(X + \Delta X, t)$. This gives Δx as being equal to $x(X + \Delta X, t) - x(X, t)$. Our notion of the stretch ratio can now be written in terms of the motion function x as

$$\lambda(X, t) = \lim_{\Delta X \rightarrow 0} \frac{x(X + \Delta X, t) - x(X, t)}{\Delta X}.$$

Recognizing the partial derivative, one arrives at an expression for the stretch ratio of particles in the close vicinity of particle X given by

$$\lambda(X, t) \equiv \frac{\partial x(X, t)}{\partial X}. \quad (1.9)$$

One also can use the displacement function $u(X, t)$ to calculate the stretch ratio. Recalling that $x(X, t) = X + u(X, t)$, it is easy to see that

$$\lambda(X, t) = 1 + \frac{\partial u(X, t)}{\partial X}. \quad (1.10)$$

The *strain* is the increase in the separation of two close particles (from their separation in the reference configuration) divided by the separation of the two particles in the reference configuration. We will denote strain by ϵ . In terms of the line segments shown in Figure 1.7, the strain of the segment between points P and Q is given by

$$\epsilon \approx \frac{\Delta x - \Delta X}{\Delta X} = \frac{\Delta x}{\Delta X} - 1.$$

Since we want the two points to be close, we take the limit as ΔX goes to zero and arrive at the expression for strain given as

$$\epsilon(X, t) = \frac{\partial x(X, t)}{\partial X} - 1 = \lambda(X, t) - 1. \quad (1.11)$$

As in the case of the stretch ratio, the strain is a function of both time and space, as can be seen in (1.11). If one uses the displacement function in place of the motion function, it easily can be shown that

$$\epsilon(X, t) \equiv \frac{\partial u(X, t)}{\partial X}. \quad (1.12)$$

Another measure of strain is the *logarithmic strain*, which is the natural logarithm of stretch ratio. We will denote the logarithmic strain by ϵ_l and define it as

$$\epsilon_l \equiv \ln(\lambda) = \ln(1 + \epsilon). \quad (1.13)$$

The logarithmic strain is also known as the *true strain*. The reader is cautioned that the term “true” when used here is slightly misleading and should not be taken as an adjective to distinguish the logarithmic strain as the “true” measure of strain and the other measures of strain as false. Since the strain ϵ , the stretch ratio λ , and the logarithmic strain ϵ_l are all related by simple invertible algebraic relations, these three measures of deformation are fully equivalent and any expression written in terms of one can be easily cast in terms of any of the others. A characteristic of the logarithmic strain is that its increment is the change in length for the current increment per unit current length. That is,

$$\Delta \epsilon_l \approx \frac{\Delta x(t + \Delta t) - \Delta x(t)}{\Delta x(t)},$$

where $\Delta x(t) = x(X + \Delta X, t) - x(X, t)$ is the length at time t of the segment that was between X and $X + \Delta X$ in the reference configuration.

A bar is said to be undergoing a *homogeneous deformation* if the stretch (or strain) is the same for all points in the bar.

Another important measure of how the bar is deforming is given by the rate at which points are moving away from one another. One measure of the rate at which particles are moving away from one another is obtained by taking the derivative of the stretch or strain with respect to time. For given particles P and Q , the stretch rate is the rate at which these particles are separating divided by the separation of these particles in the reference configuration. Since the particles are held fixed while taking the derivative with respect to time, one arrives at an expression for the rate of stretching $\dot{\lambda}$ given as

$$\dot{\lambda} = \frac{\partial \lambda(X, t)}{\partial t} = \frac{\partial^2 x(X, t)}{\partial t \partial X}. \quad (1.14)$$

It is easy to see that the *strain rate* $\dot{\epsilon}$ is given by

$$\dot{\epsilon} = \frac{\partial \epsilon(X, t)}{\partial t} = \frac{\partial^2 u(X, t)}{\partial t \partial X}. \quad (1.15)$$

Another commonly used measure of the rate at which particles are separating is the velocity gradient. The *velocity gradient* is the rate of increase of the separation at the current time of two close particles divided by the current separation of the particles. One can show that the velocity gradient L is given by

$$L \equiv \frac{\partial v(x, t)}{\partial x} = \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\epsilon}}{1 + \epsilon}. \quad (1.16)$$

A comparison of the definitions of the velocity gradient and the logarithmic strain shows that velocity gradient is the rate of change of the logarithmic strain and one can, therefore, write

$$L = \dot{\epsilon}_l. \quad (1.17)$$

The spatial variation of temperature normally controls how heat flows in a body. At a local scale, the variation of temperature is given by the *temperature gradient* g , which is the variation of temperature in the bar per unit current length of the bar. This can be written by

$$g \equiv \frac{\partial \theta(x, t)}{\partial x}. \quad (1.18)$$

The material variation of temperature can be measured by calculating G , which is the variation of temperature per unit length of the bar in the reference configuration. This can be evaluated from the material description of temperature by the relation

$$G \equiv \frac{\partial \theta(X, t)}{\partial X}. \quad (1.19)$$

It can be shown that the relation between these two temperature gradients is given by

$$G = \lambda g. \quad (1.20)$$

Section 1.4-Problems:

- 1.4-1:** You are given the displacement of a bar to be a pure function of time (i.e., $u(X, t) = f(t)$). Calculate the stretch, strain, and velocity gradient. What kind of motion is the bar undergoing?
- 1.4-2:** The motion of a bar is given by $x(X, t) = X + t^3 X^2$. Calculate the stretch ratio, strain, and velocity gradient. Compare the stretch at each point to the average stretch of the entire bar evaluated by the ratio of the current length of the bar to the initial length of the bar at time $t = 0$. Assume the bar is described by $0 \leq X \leq L$ in the reference configuration.
- 1.4-3:** The stretch ratio of a bar is described by $\lambda(X, t) = 2 + 2tX$. Calculate the motion of the bar, assuming the bar is described by $0 \leq X \leq L$ in the reference configuration and that at all times $x(0, t) = b$. Calculate the length of the bar as a function of time and evaluate it at time $t = 0$.
- 1.4-4:** Starting from the definition of the velocity gradient in words, derive the two expressions

$$L = \frac{\dot{\epsilon}}{1 + \epsilon} = \frac{\dot{\lambda}}{\lambda},$$

and then show that

$$L = \dot{\epsilon}_l.$$

- 1.4-5:** The spatial description of the velocity of a bar is given by $v(x, t) = 2tx$. Calculate the velocity gradient and strain rate.
- 1.4-6:** Calculate the relation between g and G .
- 1.4-7:** A bar has a temperature variation described by $\theta(x, t) = \theta_o + 2x^2 t$. Calculate the temperature gradient g for this bar. If the motion of the bar is given by $x(X, t) = X + Xt$, calculate G .

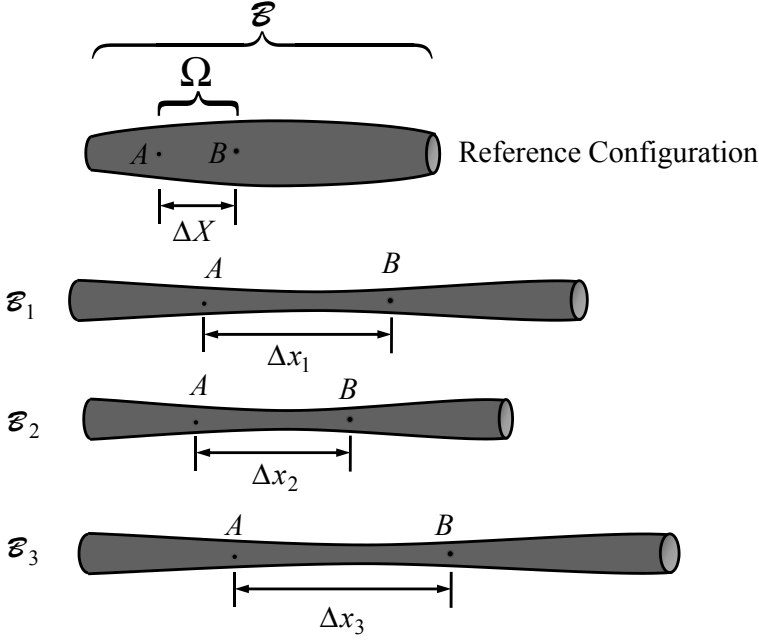


Figure 1.8: Several configurations of the material bar.

1.5 Superposition of deformations

We have not yet clearly defined the elastic and plastic strains. Before we do so, let us look at superposition of deformations, and later we will use the results we obtain to motivate the definitions of elastic and plastic strains. Consider a body that deforms from configuration $\mathcal{B}_1$ to $\mathcal{B}_2$ and then from $\mathcal{B}_2$ to $\mathcal{B}_3$ as shown in Figure 1.8. Let λ^{1-2} denote the stretch ratio comparing $\mathcal{B}_2$ to $\mathcal{B}_1$, let λ^{2-3} denote the stretch ratio comparing $\mathcal{B}_3$ to $\mathcal{B}_2$, and let λ^{1-3} denote the stretch ratio comparing $\mathcal{B}_3$ to $\mathcal{B}_1$ so that we have

$$\lambda^{1-2} = \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta x_2}{\Delta x_1}, \quad \lambda^{2-3} = \lim_{\Delta x_2 \rightarrow 0} \frac{\Delta x_3}{\Delta x_2}, \quad \lambda^{1-3} = \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta x_3}{\Delta x_1}, \quad (1.21)$$

where Δx_i is the length in configuration $\mathcal{B}_i$ of the segment between X and $X + \Delta X$ in the reference configuration, as shown in Figure 1.8. As can be seen, the following relation exists between these stretch ratios

$$\boxed{\lambda^{1-3} \equiv \lambda^{1-2} \lambda^{2-3}}. \quad (1.22)$$

Therefore, the relative stretches multiply to give the total stretch. The strains associated with each of these stretch ratios are

$$\epsilon^{1-2} = \lambda^{1-2} - 1, \quad \epsilon^{2-3} = \lambda^{2-3} - 1, \quad \epsilon^{1-3} = \lambda^{1-3} - 1. \quad (1.23)$$

Introducing this into the relation between the stretch ratios (1.22), one will get

$$1 + \epsilon^{1-3} = (1 + \epsilon^{1-2})(1 + \epsilon^{2-3}) = 1 + \epsilon^{1-2} + \epsilon^{2-3} + \epsilon^{1-2}\epsilon^{2-3}.$$

Therefore, we have the relation between the strains as

$$\boxed{\epsilon^{1-3} = \epsilon^{1-2} + \epsilon^{2-3} + \epsilon^{1-2}\epsilon^{2-3}}. \quad (1.24)$$

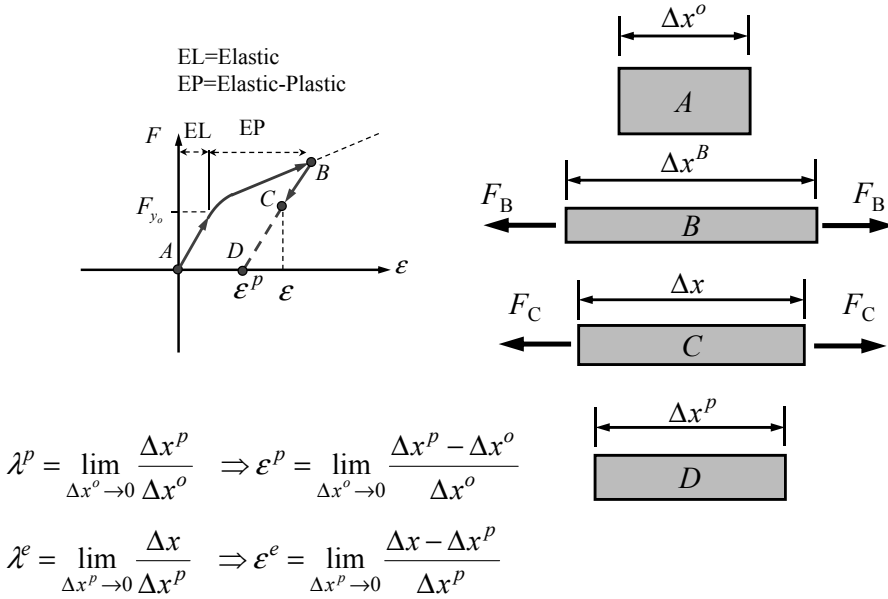


Figure 1.9: Separation of strain into elastic and plastic parts.

Obviously, if $|\epsilon^{1-2}| \ll 1$ and $|\epsilon^{2-3}| \ll 1$, then the term $\epsilon^{1-2}\epsilon^{2-3}$ can be neglected and one can write

$$\epsilon^{1-3} \approx \epsilon^{1-2} + \epsilon^{2-3}.$$

One also can define relative logarithmic strains as follows:

$$\epsilon_l^{1-2} = \ln(\lambda^{1-2}), \quad \epsilon_l^{2-3} = \ln(\lambda^{2-3}), \quad \epsilon_l^{1-3} = \ln(\lambda^{1-3}). \quad (1.25)$$

It follows from (1.25) that there is the *exact relation*

$$\boxed{\epsilon_l^{1-3} = \epsilon_l^{1-2} + \epsilon_l^{2-3}}. \quad (1.26)$$

It, therefore, follows that logarithmic strains simply add when deformations are superimposed.¹ This characteristic is independent of the size of the deformations. The reader will note that the difference between the normal (nominal) strain and the logarithmic strain vanishes when the nominal strain is small compared to unity, and, therefore, one can interchange the two strains under these conditions.

1.6 Elastic, plastic, and thermal strains

Figure 1.9 shows the schematic load-strain response of a material point on a bar, and the configurations of a small segment around this point at three locations on this diagram (i.e., points A, B, and C). Point A is the initial unloaded configuration, point C is the

¹It is prudent to mention at this point that $\ln([A][B]) \neq \ln([A]) + \ln([B])$ for matrices $[A]$ and $[B]$. Therefore, even though one may define a logarithmic strain in three dimensions, it will not be endowed with the same convenient characteristic of transforming multiplications into additions. As a result, the use of logarithmic strains in three-dimensional analysis may not provide benefits as substantial as their use provides in one-dimensional analysis.

current configuration after loading beyond the initial yield to configuration B and then partial unloading to C , and point D is the configuration if full unloading is allowed to occur, assuming that C to D follows without plastic flow. As is shown in the figure, one can define a total stretch, a plastic stretch, and an elastic stretch as follows:

$$\lambda = \lim_{\Delta x^o \rightarrow 0} \frac{\Delta x}{\Delta x^o}, \quad \lambda^p = \lim_{\Delta x^o \rightarrow 0} \frac{\Delta x^p}{\Delta x^o}, \quad \lambda^e = \lim_{\Delta x^p \rightarrow 0} \frac{\Delta x}{\Delta x^p}, \quad (1.27)$$

with the relationship

$$\boxed{\lambda = \lambda^e \lambda^p}. \quad (1.28)$$

The associated strains are defined as

$$\epsilon = \lambda - 1, \quad \epsilon^p = \lambda^p - 1, \quad \epsilon^e = \lambda^e - 1. \quad (1.29)$$

As shown above for superposition of deformations, the relation between the strains is

$$\boxed{\epsilon = \epsilon^e + \epsilon^p + \epsilon^e \epsilon^p}, \quad (1.30)$$

which can be approximated for small elastic and plastic strains as

$$\epsilon \approx \epsilon^e + \epsilon^p.$$

This approximation is good when both the elastic and plastic strains are less than 0.01 (1% strain) since the nonlinear term $\epsilon^e \epsilon^p$ will be at least two orders of magnitude smaller than ϵ , ϵ^p , and ϵ^e . It is normal that the elastic strain be less than 0.01, but the plastic strain can increase to much larger values, and under these conditions evaluation of the elastic strain from this relation can result in substantial error.

The associated logarithmic strains are defined as

$$\epsilon_l = \ln(\lambda), \quad \epsilon_l^p = \ln(\lambda^p), \quad \epsilon_l^e = \ln(\lambda^e), \quad (1.31)$$

which have the exact relation

$$\boxed{\epsilon_l = \epsilon_l^p + \epsilon_l^e}. \quad (1.32)$$

Unfortunately, this additive decomposition of the logarithmic elastic and plastic strains does not extend to three-dimensional strains. It can be shown that for small elastic and plastic strains there is no distinction between the normal definition of the elastic and plastic strains and their logarithmic counterparts. This is particularly true if both the elastic and plastic strains are below 0.01 (1% strain).

An alternate definition for the elastic and plastic strains can be given as

$$\bar{\epsilon}^p = \lim_{\Delta x^o \rightarrow 0} \frac{\Delta x^p - \Delta x^o}{\Delta x^o} = \epsilon^p, \quad \bar{\epsilon}^e = \lim_{\Delta x^o \rightarrow 0} \frac{\Delta x - \Delta x^p}{\Delta x^o} = \epsilon - \epsilon^p. \quad (1.33)$$

Obviously, for this definition the sum of the elastic and plastic strains gives the total strain so that

$$\epsilon = \bar{\epsilon}^p + \bar{\epsilon}^e \quad (1.34)$$

is an exact relation. The only thing to note for such a definition is that both the elastic and plastic strains are measured per unit length in the initial unloaded configuration.

For the case of small strains, as described above, all definitions of elastic and plastic strains are equivalent.

Thermal strains can be included using the same idea of superposition. If λ^θ denotes the thermal stretch ratio, then the nominal thermal strain and logarithmic thermal strain are given by

$$\epsilon^\theta = \lambda^\theta - 1, \quad \epsilon_l^\theta = \ln(\lambda^\theta). \quad (1.35)$$

We can think of the total stretch as being a result of the superposition of plastic stretch, elastic stretch, and thermal stretch, resulting in the relation

$$\lambda = \lambda^e \lambda^p \lambda^\theta. \quad (1.36)$$

This results in the relation between the strains given by

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta + \epsilon^e \epsilon^p + \epsilon^e \epsilon^\theta + \epsilon^p \epsilon^\theta + \epsilon^e \epsilon^p \epsilon^\theta. \quad (1.37)$$

For small elastic, plastic, and thermal strains, this can be approximated as

$$\epsilon \approx \epsilon^e + \epsilon^p + \epsilon^\theta.$$

One still obtains the exact relation between the logarithmic strains given by

$$\epsilon_l = \epsilon_l^e + \epsilon_l^p + \epsilon_l^\theta. \quad (1.38)$$

Again, the reader is cautioned that this additive character of the logarithmic strains does not carry over to the multidimensional case.

1.7 Examples of constitutive models

Before we study how to model different characteristics associated with plasticity, we will look at several examples of models used to describe material response. Such models are known as constitutive equations. Constitutive refers to the part that is related to a specific material. Constitutive assumptions are those assumptions in the theory that deal with the development of models to describe the behavior of specific materials. Unlike the balance laws, which are the same for all materials, constitutive models are different from material to material. Typically, we need to make constitutive models for how the material carries load, transmits heat, and changes entropy and free energy. That is, we need to provide models for axial load F , specific entropy η , heat flux q , and specific free energy ψ . We will proceed to look at several examples of constitutive equations.

1.7.1 Elastic

An *elastic* material is one for which the axial load is given in terms of the current value of stretch. This can be written as

$$F = F^\dagger(X, \lambda), \quad (1.39)$$

where “ $\dagger$ ” is used to designate the model we are using to characterize the axial load. The dependence on position X in this model is included to allow variation of material and geometric properties along the bar. This variable can be eliminated if the bar is made of a homogeneous material. Figure 1.10a shows the response of a typical elastic material with constant cross-sectional area. In this case, the bar is assumed to be unstretched in its reference configuration. Note that, therefore, a stretch of unity, which is equivalent to a strain of zero, represents the undeformed material at zero load. Since a stretch of zero represents the compression of the material to zero, there are no realistic stretches less than zero (i.e., $\epsilon > -1$). To avoid the possibility of negative stretches, the model shown has the load going to negative infinity as one approaches a stretch of zero. Typical constitutive models for nonlinear elastic response in axial loading would be

$$F = A_o E_o \ln(\lambda), \quad \text{or} \quad F = A_o G_o \left(\lambda - \frac{1}{\lambda^2} \right),$$

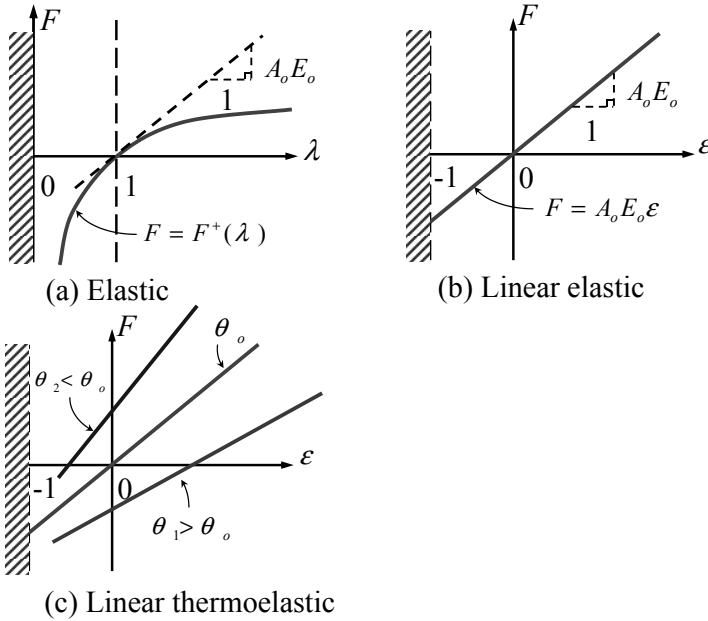


Figure 1.10: The schematic response diagrams for nonlinear elastic, linear elastic, and linear thermoelastic materials.

where A_o is the unloaded cross-sectional area, E_o is the elastic modulus at zero strain, and G_o is the shear modulus at zero strain. As would be expected, the values of A_o , E_o , and G_o may depend on location (i.e., may depend on X). Examples of such variations would be seen when dealing with a tapered bar, or when dealing with a bar with varying material properties. If the model for axial load does not depend on location, then the bar is considered homogeneous (with respect to the model for F). For small strains, one can expand the constitutive equation into a Taylor series around the stretch of unity to get

$$F = F^\dagger(1) + \frac{\partial F^\dagger}{\partial \lambda} \Big|_{\lambda=1} (\lambda - 1) + \dots$$

Recognizing that the load is zero at unit stretch, at small strains one can approximate any nonlinear model by truncating the Taylor series to get

$$F = \frac{\partial F^\dagger}{\partial \lambda} \Big|_{\lambda=1} \epsilon = A_o E_o \epsilon.$$

A material that follows this model is called a *linear elastic* material. As shown in Figure 1.10b, a *linear elastic material* is characterized by a straight line. Therefore, the load at zero stretch is $-A_o E_o$, which is finite. This allows the material to be compressed to zero volume with finite load, which is not physically possible. Yet, in spite of this shortcoming, the linear elastic material model has the distinct advantage of yielding linear systems of equations for small strains, which substantially simplifies the solution process.

1.7.2 Thermoelastic

A *thermoelastic* material is one for which the axial load depends on stretch and temperature, in addition to position. This can be written as

$$F = F^\dagger(X, \lambda, \theta). \quad (1.40)$$

The simplest model for a thermoelastic material is the linear form shown in Figure 1.10c, and is given by

$$F = A_o E_o [\epsilon - \alpha(\theta - \theta_o)],$$

where $A_o E_o(X)$ is defined as above, $\alpha(X)$ is the linear coefficient of thermal expansion, $\theta(X, t)$ is current temperature, and $\theta_o(X)$ is temperature in the reference configuration. In this form of the model, it is assumed that the bar is unloaded in the reference configuration. The reference configuration is assumed to be at the indicated reference temperature. This form of the model also can be obtained by a double Taylor series expansion around the stretch in the reference configuration $\lambda_o(X) = 1$ and the reference temperature $\theta_o(X)$.

1.7.3 Viscous fluid

A bar behaving like a *viscous fluid* will have an axial force given in terms of velocity gradient and temperature so that

$$F = F^\dagger(L, \theta) = F^\dagger\left(\frac{\dot{\lambda}}{\lambda}, \theta\right). \quad (1.41)$$

The simplest model of this kind is the linear form

$$F = A\mu L,$$

where A is the current cross-sectional area and μ is the current viscosity, both possibly functions of the current temperature. Viscous effects also can be introduced in other ways.

1.7.4 Elastic-plastic

An *elastic-plastic material* is one that behaves elastically in a given range of strains and undergoes “permanent” (i.e., plastic) deformations when one moves outside the elastic range. Typical plots of the response of an elastic-plastic material are given in Figure 1.11. Plasticity is considered an internal variable model, since additional variables can be introduced with their own evolution equations to capture the observed response. One of the simplest models for plasticity is given by

$$F = A_o E_o (\epsilon - \epsilon^p), \quad (1.42)$$

where the plastic strain ϵ^p is the internal variable, which is given by its own evolution equation called the *flow rule*. The evolution equation typically takes the form

$$\dot{\epsilon}^p = \begin{cases} \beta \dot{\epsilon} & \text{if } f(\epsilon, \epsilon^p, \xi) = 0 \text{ and } \frac{\partial f}{\partial \epsilon} \dot{\epsilon} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.43)$$

where f is the *yield function* ($f \leq 0$), defining the bounds of the elastic response, and ξ is a *hardening parameter*, defining the width of the elastic range. When $f = 0$, the material

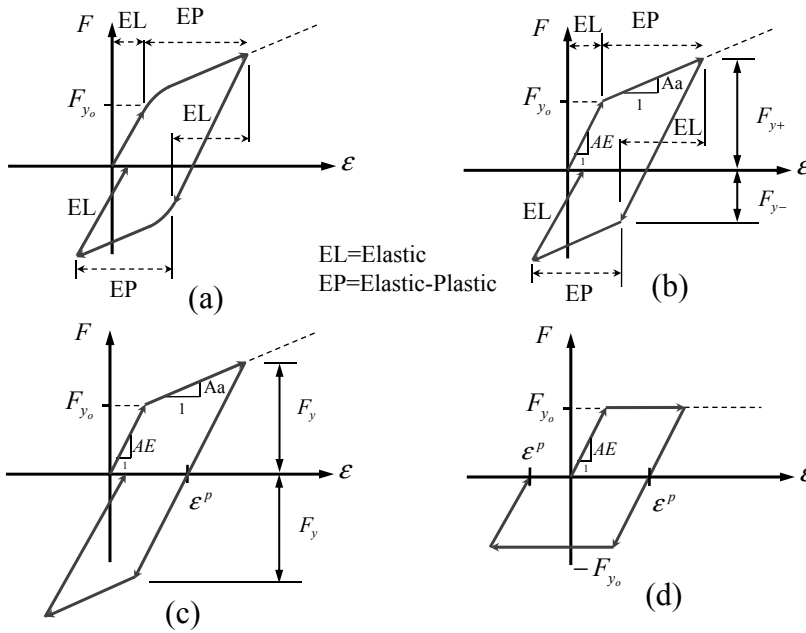


Figure 1.11: Schematics of different models for the elastic-plastic response of materials: (a) a typical response, (b) approximation of the response with linear elastic and linear elastic-plastic segments, (c) elastic-plastic response with isotropic hardening showing equal yield loads in tension and compression, and (d) elastic-perfectly plastic response.

response is said to be on the yield surface, and when $f < 0$ the material response is said to be within the yield surface. When on the yield surface ($f = 0$) and when $\frac{\partial f}{\partial \epsilon} \dot{\epsilon} < 0$, the material is said to be unloading. When on the yield surface ($f = 0$) and $\frac{\partial f}{\partial \epsilon} \dot{\epsilon} = 0$, the material is said to be undergoing neutral loading. When on the yield surface ($f = 0$) and $\frac{\partial f}{\partial \epsilon} \dot{\epsilon} > 0$, the material is said to be undergoing elastic-plastic loading. During plastic flow of the material, the model must be forced to stay on the yield surface ($f = 0$), requiring the additional condition $\dot{f} = 0$ that results in the relation

$$\frac{\partial f}{\partial \epsilon} \dot{\epsilon} + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial f}{\partial \xi} \dot{\xi} = 0. \quad (1.44)$$

This is known as the consistency condition. The hardening parameter ξ itself is an internal variable and must be given by an evolution law. The response drawn in Figure 1.11c can be obtained by using a yield function of the form

$$f = |F| - F_y,$$

with a *flow rule* given by

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E} \dot{\epsilon} & \text{if } f = 0 \text{ and } \frac{\partial f}{\partial \epsilon} \dot{\epsilon} > 0, \\ 0 & \text{for all other cases,} \end{cases}$$

where

$$F_y = F_{y_0} + A \frac{aE}{E-a} \xi,$$

Section 1.7-Problems:

- 1.7-1:** Derive the form of the constitutive equation for a linear thermoelastic material given by

$$F = A_o E_o [\epsilon - \alpha(\theta - \theta_o)].$$

Start by expanding in a Taylor series a general nonlinear model of the form $F = F^\dagger(X, \lambda, \theta)$ around the reference stretch λ_o and temperature θ_o . Explain how one can extract the appropriate material parameters $A_o E_o$ and α from experimental data.

- 1.7-2:** By eliminating time in favor of strain history, show how time dependence can be eliminated in the proposed model for plasticity.

- 1.7-3:** The temperature field in a bar is given as $\theta(X, t) = \theta_o + X^2 t$. Calculate the heat flux in the bar in terms of spatial variables if you know the motion of the bar is $x(X, t) = X + X t$. Assume heat flow to follow Fourier's law for heat conduction.

1.8 Mechanical theory of rate-independent plasticity

In this section, we will look at the general mechanical theory of rate-independent plasticity. In the next section, we will look at several specific examples to better understand how the elements of this theory are related. After this, we will look at how to include temperature and rate effects.

The general theory of plasticity is constructed from several elements. These elements include a yield function, a flow rule, a hardening model, and an equation to calculate load. The basic idea is that one can calculate the axial load from the knowledge of elastic strain, plastic strain, and one or more hardening parameters. Let us write this as

$$F = F^\dagger(X, \epsilon^e, \epsilon^p, \xi_i), \quad (1.45)$$

where F is the axial load, the superscript “†” is to designate the mathematical model for F , and its arguments indicate that our model depends on the position in the reference configuration, elastic strain, plastic strain, and a number of hardening parameters denoted by ξ_i . Position in the reference configuration is included in the model to capture any variation in material properties as one moves along the bar. As shown above, a typical example of such a model is

$$F = A E \epsilon^e, \quad (1.46)$$

where A is the *cross-sectional area in the reference configuration*, E is the elastic modulus, and both may depend on X and possibly on the hardening parameters.

The next element of our model is a yield function. The yield function, also a function of the same arguments as load, is used to evaluate the current range of the elastic response. In isothermal plasticity, it is assumed that at each loading stage there is a range of deformations that can occur with only changes to the elastic strain so that in this “current elastic range” there is no change to the plastic strain or hardening parameters. The bounds of the elastic range are established by the selection of an appropriate yield function $f(X, \epsilon^e, \epsilon^p, \xi_i)$ and the condition

$$f(X, \epsilon^e, \epsilon^p, \xi_i) \leq 0, \quad (1.47)$$

which must hold at all times. The yield surface, the bounds on the current elastic range, is given by setting the yield function equal to zero, $f = 0$. We say we are on the yield surface when $f = 0$, and within the yield surface when $f < 0$. When we are within the yield surface (i.e., $f < 0$), all deformations occur without changes in the plastic strain and hardening parameters. When we are on the yield surface (i.e., $f = 0$), depending on the nature of the next increment of loading, there might be plastic flow, neutral loading, or unloading into the elastic range. Plastic flow is when there has to be a change in the plastic strain or else there will be a violation of the condition $f \leq 0$. Neutral loading on the yield surface refers to increments of loading that move along the yield surface (i.e., f remains equal to zero during the increment) without needing any change to the plastic strain and hardening parameters. Neutral loading will not occur in one-dimensional mechanical loading, but it is included here to introduce the possibility of such loadings occurring in more complicated models. Unloading from the yield surface refers to increments of load that start from a loading point on the yield surface ($f = 0$) and with the next increment end up at a loading point within the yield surface ($f < 0$). Unloading occurs without any change to the plastic strain and hardening parameters. When on the yield surface, the nature of the next increment of loading is determined by evaluating the effect of the next increment on the value of the yield function while assuming no change in the plastic strain and hardening parameters. That is, if the increment can be accommodated by a fully elastic response without violating the condition $f \leq 0$, then there is no plastic flow and both the plastic strain and hardening parameters remain unchanged. As we saw in our previous examples, one model for the yield function can be written as

$$f = |F| - F_y, \quad (1.48)$$

where F_y is the yield load, which may depend on plastic strain and the hardening parameters. This type of yield function, which has the same magnitudes for the yield loads in tension and compression, normally is referred to as an *isotropic hardening* model. Since during neutral loading and during plastic flow the loading remains on the yield surface (i.e., $f = 0$), the loading process should occur in a way that requires $\dot{f} = 0$. This is automatically satisfied during neutral loading, but becomes a restriction during plastic flow. Given the form of the yield function, during plastic flow we can write $\dot{f} = 0$ as

$$\frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}^e + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \dot{\xi}_i = 0. \quad (1.49)$$

This constraint is known as the *consistency condition*.

In plasticity, the determination of whether there will be plastic flow in the next increment of loading is critical to establishing the flow rule. To lay out a general theoretical structure that will not need modification with the introduction of new elements, such as temperature, let us propose the following rule for determining the conditions under which plastic flow occurs:

If a sufficiently small loading increment can be accommodated without plastic flow (i.e., $f \leq 0$ after the loading increment), then there is no need to consider plastic flow. On the other hand, if ignoring plastic flow results in $f > 0$, then plastic flow occurs in such a way that at the end of the increment of loading we will have $f \leq 0$. In this latter case, if the increment of loading is taken small enough, at the end of the increment, we will have $f = 0$.

This rule is restricted to small increments of loading, where loading refers to a change in strain.

Let “ $\diamond$ ” on a character denote a time derivative, holding the plastic strain and the hardening parameters constant. For example, the rate of change of the yield function, in the

absence of plastic flow, as a result of the application of a strain rate $\dot{\epsilon}$ and temperature rate $\dot{\theta}$, is given by

$$\dot{f} = \frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}^e, \quad (1.50)$$

where

$$\dot{\epsilon}^e = \frac{\dot{\epsilon}}{1 + \epsilon^p} \quad (1.51)$$

if we take $\dot{\epsilon}^p = 0$. This latter equation can be shown by looking at

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^e \epsilon^p,$$

to get

$$\epsilon^e = \frac{\epsilon - \epsilon^p}{1 + \epsilon^p},$$

which results in

$$\dot{\epsilon}^e = \frac{\dot{\epsilon} - \dot{\epsilon}^p}{1 + \epsilon^p} - \frac{\epsilon - \epsilon^p}{(1 + \epsilon^p)^2} \dot{\epsilon}^p,$$

which simplifies to the above expression when $\dot{\epsilon}^p = 0$. For infinitesimal strains, we can use the approximate relation

$$\dot{\epsilon}^e \approx \dot{\epsilon}.$$

One can think of $\dot{f}$ as a *trial value* for $\dot{f}$ assuming a purely elastic response. Obviously, if $f = 0$ and $\dot{f} > 0$, the assumption of a purely elastic loading step would result in $f > 0$ after the completion of even an infinitesimally small load step, a violation of the constraint $f \leq 0$. Therefore, one should assume plastic flow when $f = 0$ and $\dot{f} > 0$. For all other loadings, the constraint $f \leq 0$ can be maintained with a pure elastic response. For example, if $f = 0$ and $\dot{f} \leq 0$, then a pure elastic loading increment would result in $f \leq 0$ ($f = 0$ if $\dot{f} = 0$, and $f < 0$ if $\dot{f} < 0$) for an infinitesimally small loading step. Or, for example, if $f < 0$ and $\dot{f} > 0$, then a small enough pure elastic loading increment would keep $f \leq 0$. Using this idea one can write the plastic flow rule as

$$\dot{\epsilon}^p = \begin{cases} \beta \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.52)$$

where β is a function of $(X, \epsilon^e, \epsilon^p, \xi_i)$. The term $\beta \dot{\epsilon}$ represents the part of the strain rate that ends up as plastic flow. Similar expressions normally are selected for each of the hardening parameters, normally connecting the hardening parameter to $\dot{\epsilon}^p$ as opposed to $\dot{\epsilon}$. A typical form, but not the only form, for the hardening rule could be written as

$$\dot{\xi}_i = \omega_i \dot{\epsilon}^p, \quad (1.53)$$

where again the coefficients ω_i are functions of $(X, \epsilon^e, \epsilon^p, \xi_i)$.

For infinitesimal strains, the consistency condition during plastic flow (i.e., $\dot{f} = 0$) can now be written as

$$\frac{\partial f}{\partial \epsilon^e} (\dot{\epsilon} - \dot{\epsilon}^p) + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i \dot{\epsilon}^p = 0.$$

Reorganization of this results in

$$\left(\frac{\partial f}{\partial \epsilon^e} - \frac{\partial f}{\partial \epsilon^p} - \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i \right) \dot{\epsilon}^p = \frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}.$$

Comparing this to the flow rule gives

$$\beta = \frac{\frac{\partial f}{\partial \epsilon^e}}{\frac{\partial f}{\partial \epsilon^e} - \frac{\partial f}{\partial \epsilon^p} - \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i}. \quad (1.54)$$

Obviously, the result only holds when the denominator is not equal to zero.

The reader will note that these last results, even though very general, are not the most general form. Particularly, we can get different results for the flow rule if we change the form of the hardening parameter.

Section 1.8-Problems:

- 1.8-1:** Calculate β for large elastic and plastic strains by using the consistency condition with the general relation

$$\dot{\epsilon}^e = \frac{\dot{\epsilon} - \dot{\epsilon}^p}{1 + \epsilon^p} - \frac{\epsilon - \epsilon^p}{(1 + \epsilon^p)^2} \dot{\epsilon}^p.$$

1.9 Mechanical models for plasticity

Let us now concentrate on specific models. This group of models essentially defines what might be considered classical plasticity. We start from the simplest model and build up to more complicated models. All the models considered fall into the small strain theory so that we have the relation

$$\epsilon = \epsilon^e + \epsilon^p.$$

These models can be extended for use under large deformations by replacing the nominal strains with logarithmic strains. This will not be an option when we look at multi-dimensional models.

1.9.1 Elastic perfectly plastic

Let us first examine a response that is termed an *elastic perfectly plastic* response. Figure 1.13 shows the characteristics of this response type. Essentially, for this response type, the yield load is constant and is the same in tension and compression so that we can write the yield function as

$$f = |F| - F_{yo} = \begin{cases} F - F_{yo} & \text{if } F \geq 0, \\ -F - F_{yo} & \text{if } F < 0, \end{cases} \quad (1.55)$$

where F_{yo} is the initial yield load in tension. The load in the elastic range is a linear function of the elastic strain and is given by

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.56)$$

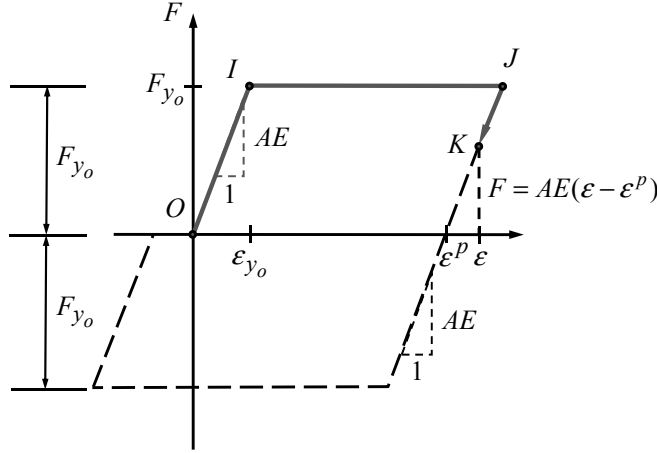


Figure 1.13: Elastic perfectly plastic response.

where E is the elastic modulus and A is the initial cross-sectional area. For this response type, the two material parameters F_{y_o} and E fully describe the material. The rate of change of the yield function is given by

$$\dot{f} = \begin{cases} \dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p) & \text{if } F \geq 0, \\ -\dot{F} = -AE(\dot{\epsilon} - \dot{\epsilon}^p) & \text{if } F < 0. \end{cases} \quad (1.57)$$

Holding plastic flow equal to zero one gets the value of the trial derivative $\overset{\diamond}{f}$ of the yield function to be given as

$$\overset{\diamond}{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0. \end{cases} \quad (1.58)$$

The consistency condition $\dot{f} = 0$ during plastic flow requires that

$$\dot{f} = \begin{cases} \dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p) = 0 & \text{if } F \geq 0, \\ -\dot{F} = -AE(\dot{\epsilon} - \dot{\epsilon}^p) = 0 & \text{if } F < 0, \end{cases} \quad (1.59)$$

which results in the relation $\dot{\epsilon}^p = \dot{\epsilon}$ during plastic flow. Therefore, the model for an elastic perfectly plastic response can be summarized as

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.60)$$

$$f = |F| - F_{y_o}, \quad (1.61)$$

$$\overset{\diamond}{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases} \quad (1.62)$$

$$\dot{\epsilon}^p = \begin{cases} \dot{\epsilon} & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (1.63)$$

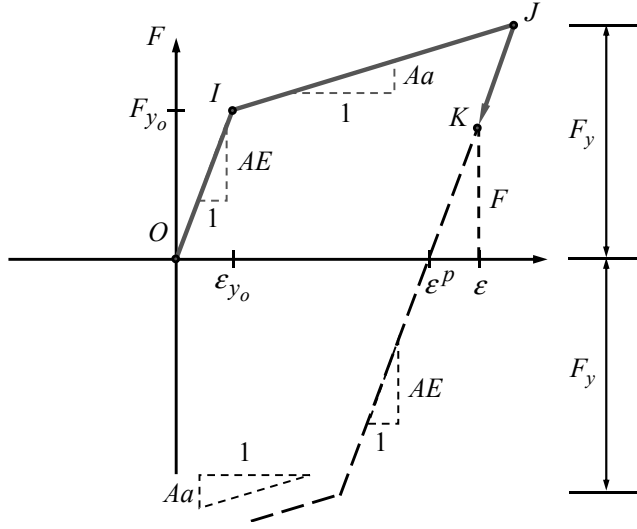


Figure 1.14: Elastic-plastic response with isotropic hardening.

Obviously, since AE is positive, one can determine the sign of $\dot{f}$ from $\dot{\epsilon}$ and the sign of F without the need to multiply $\dot{\epsilon}$ by AE . Yet, for the consistency of the presentation AE is included, even if in practice it is not used.

1.9.2 Elastic-plastic with isotropic hardening: strain hardening

The next model that we will look at is called *elastic-plastic with isotropic hardening*. Figure 1.14 shows a schematic of this type of response. As can be seen, the yield load gradually increases with plastic flow, indicated by the slope Aa on the graph. The term *isotropic hardening* refers to the fact that the yield load in tension is the same as that in compression, indicating that yielding in either tension or compression results in an increase in both limits of the current elastic response range. The yield function in this case is shown in Figure 1.15 and can be written as

$$f = |F| - F_y(\xi) = \begin{cases} F - F_y(\xi) & \text{if } F \geq 0, \\ -F - F_y(\xi) & \text{if } F < 0, \end{cases} \quad (1.64)$$

where, as indicated, F_y is a function of one hardening parameter ξ . As in the case of elastic-perfectly plastic response, the load is given by the equation

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p). \quad (1.65)$$

The rate of change of the yield function is given by

$$\dot{f} = \begin{cases} \dot{F} - \frac{dF_y}{d\xi} \dot{\xi} = AE(\dot{\epsilon} - \dot{\epsilon}^p) - \frac{dF_y}{d\xi} \dot{\xi} & \text{if } F \geq 0, \\ -\dot{F} - \frac{dF_y}{d\xi} \dot{\xi} = -AE(\dot{\epsilon} - \dot{\epsilon}^p) - \frac{dF_y}{d\xi} \dot{\xi} & \text{if } F < 0. \end{cases} \quad (1.66)$$

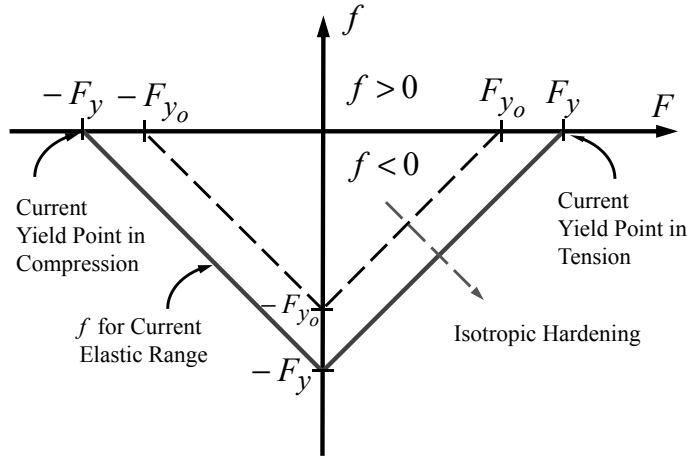


Figure 1.15: The yield function $f = |F| - F_y$ in load space for isotropic hardening.

Assuming no hardening in the absence of plastic flow ($\dot{\xi} = 0$), one can calculate $\dot{f}$ as

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0. \end{cases} \quad (1.67)$$

For the hardening rule, let us select the relation

$$\dot{\xi} = |\dot{\epsilon}^p|, \quad (1.68)$$

which is known as *strain hardening*. For now, we focus our attention on monotonic loading that increases with time. As the loading increases beyond the initial yield, we have $\dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p) = AE(\dot{\epsilon} - \beta\dot{\epsilon})$. Since the slope of the stress-strain curve is given by $\dot{F}/\dot{\epsilon}$, beyond the initial yield we should have $\dot{F}/\dot{\epsilon} = Aa$, which results in the relation

$$\frac{AE(\dot{\epsilon} - \beta\dot{\epsilon})}{\dot{\epsilon}} = Aa.$$

This results in

$$\beta = \frac{E - a}{E}. \quad (1.69)$$

Since the consistency condition should hold during plastic flow, and since during this monotonic loading we will have $\dot{\xi} = \dot{\epsilon}^p = \beta\dot{\epsilon}$, it follows

$$EA(\dot{\epsilon} - \beta\dot{\epsilon}) - \frac{dF_y}{d\xi}\beta\dot{\epsilon} = 0,$$

which, since $\dot{\epsilon} \neq 0$, yields the relation

$$EA(1 - \beta) - \frac{dF_y}{d\xi}\beta = 0.$$

Substituting for β and solving for $\frac{dF_y}{d\xi}$ results in

$$\frac{dF_y}{d\xi} = \frac{AaE}{E - a}.$$

Since we now know that $\frac{dF_y}{d\xi}$ is constant, the model for F_y must be a linear function of ξ , which passes through the initial yield point $F_y = F_{yo}$ at $\xi = 0$. Therefore, we have

$$F_y = F_{yo} + \frac{AaE}{E-a}\xi. \quad (1.70)$$

We now can summarize the elastic-plastic model constructed by using isotropic hardening with a strain hardening parameter as follows:

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.71)$$

$$f = |F| - F_y, \quad (1.72)$$

$$F_y = F_{yo} + \frac{AaE}{E-a}\xi, \quad (1.73)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases} \quad (1.74)$$

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E}\dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.75)$$

$$\dot{\xi} = |\dot{\epsilon}^p|. \quad (1.76)$$

Even though we did not fit the model to the form of the yield surface in compression, the current model will exhibit all the characteristics shown in Figure 1.11c. As a word of caution, this is not generally true, and one must check the response to nonmonotonic loading to make sure the model captures all the aspects of the actual observed response.

1.9.3 Elastic-plastic with isotropic hardening: work hardening

As another example, let us take an elastic-plastic model with isotropic hardening as discussed in the previous example, but replace the strain hardening assumption with a *work hardening* model based on the work done by the plastic strain. That is, we will replace the assumption $\dot{\xi} = |\dot{\epsilon}^p|$ with the model

$$\dot{\xi} = F\dot{\epsilon}^p. \quad (1.77)$$

Let us also assume a linear hardening relation for the yield load of the form

$$F_y = F_{yo} + \gamma\xi, \quad (1.78)$$

where γ is a material constant to be determined. Let us use the same isotropic hardening model $f = |F| - F_y$ given above and the same linear elastic model $F = AE\epsilon^e$. As before, the rate of change of the yield function is given by

$$\dot{f} = \begin{cases} \dot{F} - \frac{dF_y}{d\xi}\dot{\xi} = AE(\dot{\epsilon} - \dot{\epsilon}^p) - \gamma\dot{\xi} & \text{if } F \geq 0, \\ -\dot{F} - \frac{dF_y}{d\xi}\dot{\xi} = -AE(\dot{\epsilon} - \dot{\epsilon}^p) - \gamma\dot{\xi} & \text{if } F < 0. \end{cases} \quad (1.79)$$

As indicated, assuming no hardening in the absence of plastic flow, one can calculate $\dot{f}$ as

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0. \end{cases} \quad (1.80)$$

Therefore, for monotonically increasing loading, once the initial yield load is passed, the consistency condition requires that

$$AE(\dot{\epsilon} - \beta\dot{\epsilon}) - \gamma F\beta\dot{\epsilon} = 0.$$

Since $\dot{\epsilon} > 0$, after reorganization, one can show that

$$\beta = \frac{AE}{AE + \gamma F}. \quad (1.81)$$

Since F is not constant, one must conclude that in this case β is not constant. For this monotonic loading, beyond the initial yield, $F = F_y = F_{yo} + \gamma\xi$, so the hardening rule can be written as

$$\dot{\xi} = (F_{yo} + \gamma\xi)\dot{\epsilon}^p.$$

Separation of variables results in

$$\frac{\dot{\xi}}{F_{yo} + \gamma\xi} = \dot{\epsilon}^p,$$

which easily can be integrated to yield

$$F_y = F_{yo}e^{\gamma\epsilon^p},$$

after the substitution of the initial condition $F_y = F_{yo}$ at $\epsilon^p = 0$. Using the flow rule $\dot{\epsilon}^p = \beta\dot{\epsilon}$, substituting for β and F , separation of variables and integration results in

$$\epsilon = \epsilon^p + \epsilon_{yo}e^{\gamma\epsilon^p},$$

where $\epsilon_{yo} = F_{yo}/AE$ is the strain at the initial yield. As can be seen, the assumption of a linear relation between the yield load and the hardening parameter, in the case of work hardening, results in an exponential relation between the yield load and plastic strain. We now can summarize the model for elastic-plastic response with isotropic hardening using the work hardening parameter as follows:

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.82)$$

$$f = |F| - F_y, \quad (1.83)$$

$$F_y = F_{yo} + \gamma\xi, \quad (1.84)$$

$$\overset{\diamond}{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases} \quad (1.85)$$

$$\dot{\epsilon}^p = \begin{cases} \frac{AE}{AE + \gamma F}\dot{\epsilon} & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.86)$$

$$\dot{\xi} = F\dot{\epsilon}^p. \quad (1.87)$$

As will be shown in the section problems, the work hardening model can be used to generate a plasticity model that is exactly equivalent to the above model based on strain hardening with the response shown in Figure 1.14. For this case, to satisfy the linearity of the elastic-plastic response, the dependence of the yield load on the hardening parameter ξ becomes nonlinear.

1.9.4 Elastic-plastic with kinematic hardening

Up to this point, the models that we have looked at have had identical yield loads in tension and compression (i.e., they had isotropic hardening). Many materials do not

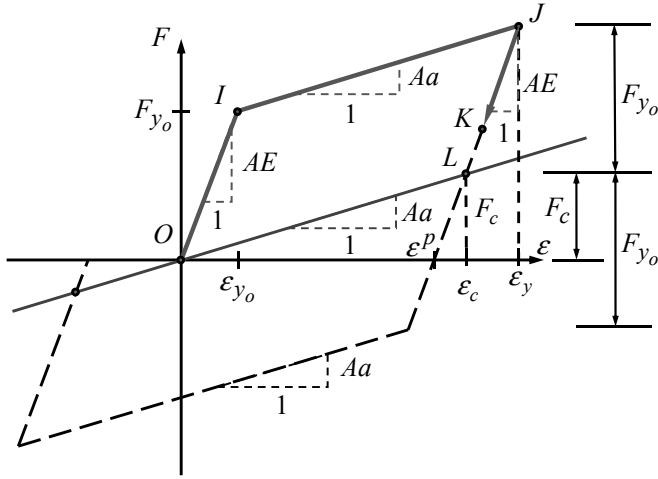


Figure 1.16: Elastic-plastic response with kinematic hardening.

show this symmetry in the yield load, particularly after plastic flow. The Bauschinger effect describes one such case where the yield load in compression gets reduced as a result of yielding in tension. A response that has this characteristic is shown in Figure 1.16. This type of response is known as *kinematic hardening*. In this case, the yield function takes the form shown in Figure 1.17 and can be written as

$$f = |F - F_c| - F_{yo} = \begin{cases} F - F_c - F_{yo} & \text{if } F - F_c \geq 0, \\ -F + F_c - F_{yo} & \text{if } F - F_c < 0, \end{cases}$$

where F_c is a function of ϵ^p . During the plastic flow from point I to J , the load is given by $F = F_{yo} + Aa(\epsilon - \epsilon_{yo})$, and also given by $F = AE(\epsilon - \epsilon^p)$. Equating $\dot{F}$ for these two equations yields

$$Aa\dot{\epsilon} = AE(\dot{\epsilon} - \dot{\epsilon}^p),$$

which gives

$$\dot{\epsilon}^p = \frac{E - a}{E} \dot{\epsilon}.$$

Therefore,

$$\beta = \frac{E - a}{E}.$$

The center of the current elastic range is given by point L . The load F_c at L can be calculated using $F_c = Aa\epsilon_c$ or using $F_c = AE(\epsilon_c - \epsilon^p)$. Equating these two equations and solving for ϵ_c gives

$$\epsilon_c = \frac{E}{E - a} \epsilon^p.$$

Substitution back into $F_c = Aa\epsilon_c$ provides the expression for F_c as

$$F_c = \frac{AaE}{E - a} \epsilon^p.$$

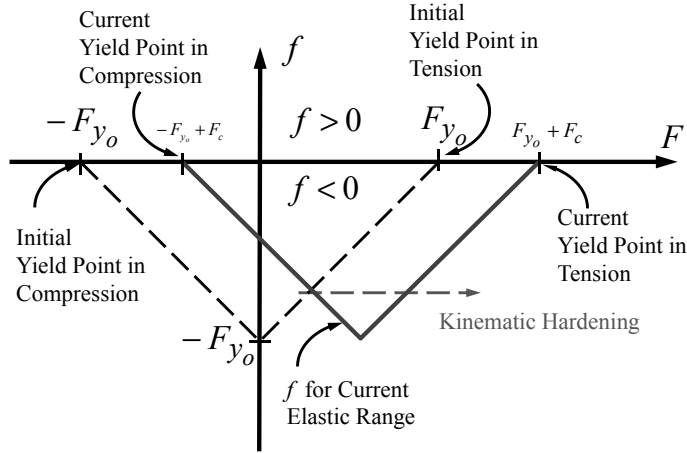


Figure 1.17: The yield function $f = |F - F_c| - F_{y_o}$ in load space for kinematic hardening.

We now can summarize the elastic-plastic model with pure kinematic hardening as

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.88)$$

$$f = |F - F_c| - F_{y_o}, \quad (1.89)$$

$$F_c = \frac{AaE}{E - a}\epsilon^p, \quad (1.90)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F - F_c \geq 0, \\ -AE\dot{\epsilon} & \text{if } F - F_c < 0, \end{cases} \quad (1.91)$$

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E}\dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (1.92)$$

1.9.5 Elastic-plastic with arbitrary hardening

We could proceed to develop more and more complicated models, each time slightly altering the hardening characteristics, but it is as easy to look at arbitrary hardening, as shown in Figure 1.18. The characteristics of this response are (a) a linear elastic response described by $F = AE(\epsilon - \epsilon^p)$, (b) an elastic-plastic response in monotonic loading from I to J shown in the figure that is represented by a function $F = F_1(\epsilon)$, and (c) a function $F_c = F_2(\epsilon)$ in monotonic loading, describing the curve OL that gives the center of the elastic range. F_1 and F_2 are in terms of ϵ only because the experimental data are plotted in terms of a load-strain plot, which would make it convenient to evaluate these two functions in terms of ϵ . In reality, since the center of the elastic range does not change with elastic loading, the location of the center should depend on the plastic strain, so that we can write $F_c(\epsilon^p)$. Also, the size of the elastic zone should depend on the hardening parameter and, therefore, $F_{iso}(\xi)$. We soon will see how $F_c(\epsilon^p)$ and $F_{iso}(\xi)$ can be obtained from $F_1(\epsilon)$ and $F_2(\epsilon)$. Since the center of the current elastic range is at a load F_c , the yield function can be written as

$$f = |F - F_c| - F_{iso} = \begin{cases} F - F_c - F_{iso} & \text{if } F - F_c \geq 0, \\ -F + F_c - F_{iso} & \text{if } F - F_c < 0. \end{cases} \quad (1.93)$$

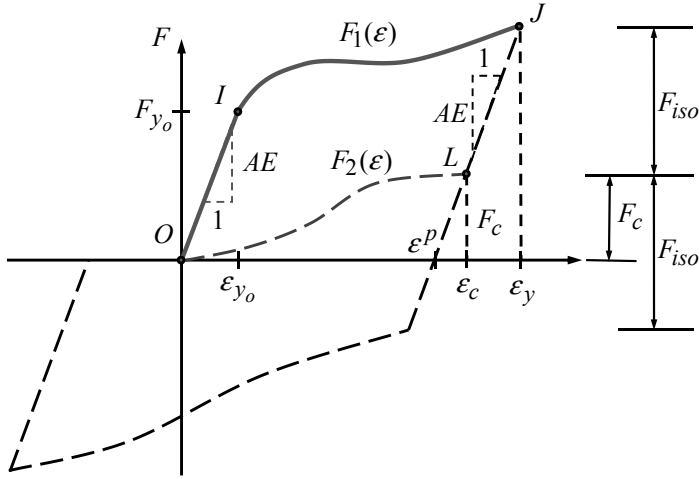


Figure 1.18: Elastic-plastic response with arbitrary hardening.

As a result, $\dot{f}$ and $\dot{f}$ can be written as

$$\dot{f} = \begin{cases} AE(\dot{\epsilon} - \dot{\epsilon}^p) - \frac{dF_c}{d\epsilon^p} \dot{\epsilon}^p - \frac{dF_{iso}}{d\xi} \dot{\xi} & \text{if } F - F_c \geq 0, \\ -AE(\dot{\epsilon} - \dot{\epsilon}^p) + \frac{dF_c}{d\epsilon^p} \dot{\epsilon}^p - \frac{dF_{iso}}{d\xi} \dot{\xi} & \text{if } F - F_c < 0, \end{cases} \quad (1.94)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F - F_c \geq 0, \\ -AE\dot{\epsilon} & \text{if } F - F_c < 0. \end{cases} \quad (1.95)$$

Since $\dot{\epsilon}^p = \beta\dot{\epsilon}$ during plastic flow from point I to J in the figure, we have $\dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p) = AE(1 - \beta)\dot{\epsilon}$. The slope of the section from I to J is given by $\dot{F}/\dot{\epsilon}$, which also should be equal to $dF_1(\epsilon)/d\epsilon$. Therefore,

$$\frac{dF_1}{d\epsilon} = AE(1 - \beta).$$

Solving for the current value of β yields

$$\beta = 1 - \frac{1}{AE} \frac{dF_1(\epsilon_y)}{d\epsilon_y}, \quad (1.96)$$

where ϵ_y is the total strain at the current yield point J . Using the relation $\dot{\epsilon}^p = \beta\dot{\epsilon}$ and integrating from I to J yields

$$\epsilon^p = \epsilon_y - \frac{1}{AE} F_1(\epsilon_y),$$

where use has been made of the initial condition at I given as $\epsilon^p = 0$ at $\epsilon_y = \epsilon_{yo}$. This relation easily can be obtained directly from taking the current yield strain ϵ_y and subtracting from it the elastic strain due to the linear elastic unloading from $F_1(\epsilon_y)$ to zero. Note that point J can be taken as any point on the loading path after the initial yield so that one can consider this equation a general relation giving the plastic strain in terms of the current yield strain. Now let us move our attention to the curve from O to L , which is the loci of the centers of the current elastic ranges. Obviously, during the loading process, we are on the yield surface, so to get the load, we need to calculate $F_1(\epsilon)$ at the current yield strain $\epsilon = \epsilon_y$. On the

other hand, to get the load at the center of the current elastic range, we need to calculate $F_2(\epsilon)$ at the strain $\epsilon = \epsilon_c$. Therefore, we need to find a way of calculating ϵ_c . This easily can be done by recognizing the relation $F_2(\epsilon_c) = AE(\epsilon_c - \epsilon^p)$. Therefore, during the described plastic flow, we have

$$\dot{F}_c = \frac{dF_2(\epsilon_c)}{d\epsilon_c} \dot{\epsilon}_c = AE(\dot{\epsilon}_c - \dot{\epsilon}^p),$$

which gives

$$\dot{\epsilon}_c = \frac{AE}{AE - \frac{dF_2(\epsilon_c)}{d\epsilon_c}} \dot{\epsilon}^p. \quad (1.97)$$

Using the initial condition $\epsilon_c = 0$ at $\epsilon^p = 0$, this equation can be integrated numerically to get the relation between ϵ_c and ϵ^p for a given function $F_2(\epsilon)$. Obviously, an alternate method is to directly solve the equation $F_2(\epsilon_c) = AE(\epsilon_c - \epsilon^p)$ for the given function $F_2(\epsilon)$. Once the relation between ϵ_c and ϵ^p is established, one can use the relation

$$\dot{F}_c = \frac{dF_c(\epsilon^p)}{d\epsilon^p} \dot{\epsilon}^p = \frac{dF_2(\epsilon_c)}{d\epsilon_c} \dot{\epsilon}_c$$

to find

$$\frac{dF_c(\epsilon^p)}{d\epsilon^p} = \frac{AE \frac{dF_2(\epsilon_c)}{d\epsilon_c}}{AE - \frac{dF_2(\epsilon_c)}{d\epsilon_c}}. \quad (1.98)$$

This can be used now to find the function $F_c(\epsilon^p)$. The only remaining function needed is $F_{iso}(\xi)$. Let us develop this model based on strain hardening so that

$$\dot{\xi} = |\dot{\epsilon}^p|. \quad (1.99)$$

The consistency condition $\dot{f} = 0$ must hold during plastic flow from I to J and $\dot{\xi} = \dot{\epsilon}^p$ so that we must have

$$AE(1 - \beta)\dot{\epsilon} - \frac{dF_c(\epsilon^p)}{d\epsilon^p} \beta \dot{\epsilon} - \frac{dF_{iso}(\xi)}{d\xi} \beta \dot{\epsilon} = 0.$$

This can be written as

$$\frac{dF_1(\epsilon_y)}{d\epsilon_y} - \frac{dF_c(\epsilon^p)}{d\epsilon^p} \beta - \frac{dF_{iso}(\xi)}{d\xi} \beta = 0,$$

which can be reorganized as

$$\frac{dF_{iso}(\xi)}{d\xi} = \frac{1}{\beta} \frac{dF_1(\epsilon_y)}{d\epsilon_y} - \frac{dF_c(\epsilon^p)}{d\epsilon^p}. \quad (1.100)$$

This can be integrated using the initial conditions $F_{iso} = F_{yo}$ at $\epsilon_y = \epsilon_{yo}$ and $\epsilon^p = \xi = 0$. Therefore, given any material response characterized by EA , $F_1(\epsilon)$ and $F_2(\epsilon)$, we have established a method to calculate the function $F_c(\epsilon^p)$ characterizing the kinematic hardening, the function $F_{iso}(\xi)$ characterizing the isotropic hardening, and the coefficient β in the flow rule.

Section 1.9-Problems:

- 1.9-1:** One can make power-law models of plasticity based on assuming a yield function of the form

$$f = |F| - F_{yo}(1 + a\xi)^n. \quad (1.101)$$

Combining this yield function with the work hardening assumption $\dot{\xi} = F\dot{\epsilon}^p$, show that you will get

$$\beta = \frac{1}{1 + an\epsilon_{yo}(1 + a\xi)^{n-1}|F|}, \quad (1.102)$$

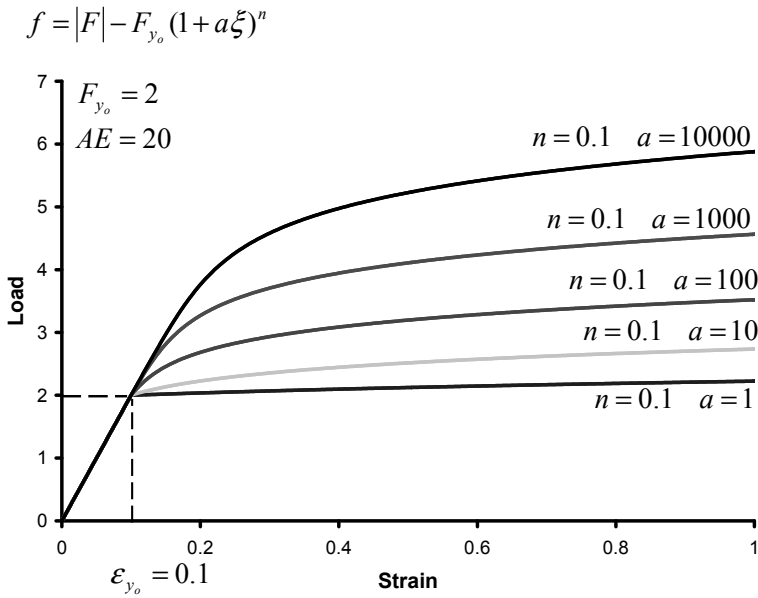


Figure 1.19: Monotonic elastic-plastic response for power-law isotropic hardening.

and verify that you obtain the response shown in Figure 1.19 for monotonic loading and in Figure 1.20 for cyclic loading. Examine what happens if you reset $\xi = 0$ whenever $|F| < \frac{1}{2}F_{y_0}$ and show that this results in the response given in Figure 1.21.

1.9.6 Elastic-plastic with combined isotropic and kinematic hardening

Let us now use the method developed for arbitrary hardening to make a model for the combined isotropic and kinematic hardening case shown in Figure 1.22. As can be seen in the figure,

$$F_1(\epsilon) = F_{y_0} + Aa(\epsilon - \epsilon_{y_0}), \quad F_2(\epsilon) = Ab\epsilon,$$

which results in

$$\frac{dF_1(\epsilon)}{d\epsilon} = Aa, \quad \frac{dF_2(\epsilon)}{d\epsilon} = Ab.$$

As a result of the relations obtained in the last section, we have

$$\begin{aligned} \beta &= 1 - \frac{1}{AE} \frac{dF_1(\epsilon_y)}{d\epsilon_y} = 1 - \frac{Aa}{AE} = \frac{E - a}{E}, \\ \frac{dF_c(\epsilon^p)}{d\epsilon^p} &= \frac{AE \frac{dF_2(\epsilon_c)}{d\epsilon_c}}{AE - \frac{dF_2(\epsilon_c)}{d\epsilon_c}} = \frac{(AE)(Ab)}{AE - Ab} = \frac{AEb}{E - b}, \\ \frac{dF_{iso}(\xi)}{d\xi} &= \frac{1}{\beta} \frac{dF_1(\epsilon_y)}{d\epsilon_y} - \frac{dF_c(\epsilon^p)}{d\epsilon^p} = \frac{E}{E - a} Aa - \frac{AEb}{E - b} = AE \left(\frac{a}{E - a} - \frac{b}{E - b} \right). \end{aligned}$$

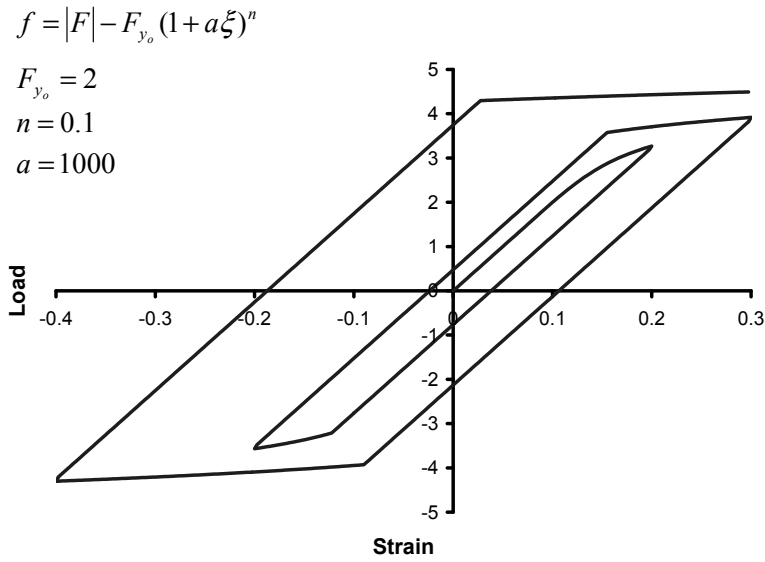


Figure 1.20: Cyclic elastic-plastic response for power-law isotropic hardening.

Integration of the last two terms yields

$$F_c(\epsilon^p) = \frac{AEb}{E-b} \epsilon^p,$$

$$F_{iso}(\xi) = AE \left(\frac{a}{E-a} - \frac{b}{E-b} \right) \xi + F_{y_o}.$$

We now can summarize the model for elastic-plastic response having combined isotropic and kinematic hardening, using a strain hardening parameter, as follows:

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p), \quad (1.103)$$

$$f = |F - F_c| - F_{iso}, \quad (1.104)$$

$$F_c(\epsilon^p) = \frac{AEb}{E-b} \epsilon^p, \quad (1.105)$$

$$F_{iso}(\xi) = AE \left(\frac{a}{E-a} - \frac{b}{E-b} \right) \xi + F_{y_o}, \quad (1.106)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F - F_c \geq 0, \\ -AE\dot{\epsilon} & \text{if } F - F_c < 0, \end{cases} \quad (1.107)$$

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E} \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.108)$$

$$\dot{\xi} = |\dot{\epsilon}^p|. \quad (1.109)$$

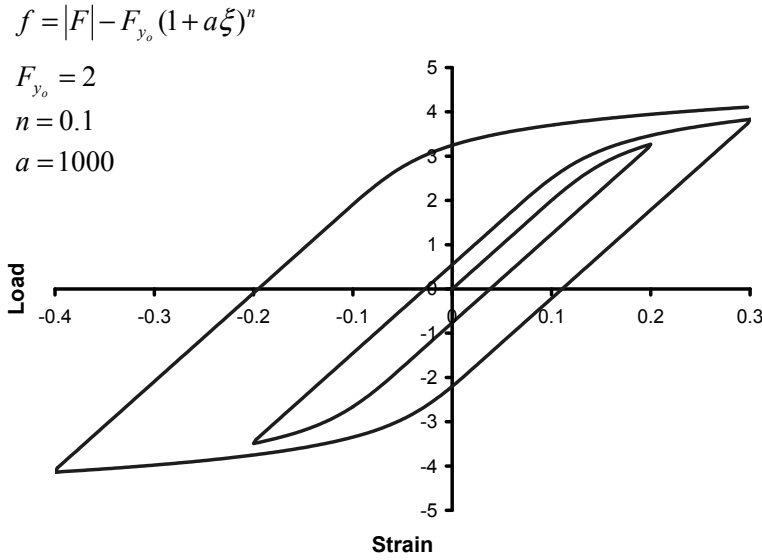


Figure 1.21: Cyclic elastic-plastic response for power-law isotropic hardening augmented by resetting the hardening parameter to zero every time $|F| < \frac{1}{2}F_{y_o}$.

1.9.7 Isotropic hardening with changing elastic modulus

Up to now, the models we have looked at have had constant elastic modulus. Though a convenient assumption, most materials show a change in their elastic modulus with plastic flow, in particular a reduction in the modulus is common. For example, Figure 1.23 shows the measured change of elastic modulus with plastic flow for aluminum and mild steel. To see how such a dependence affects our modeling, we will look at an isotropic hardening model for which the elastic modulus changes with plastic deformation. Let us assume that the elastic modulus is given by a function of the hardening parameter ξ so that we have a function $E(\xi)$ describing how this modulus changes. Let the hardening parameter be selected such that $\xi = 0$ describes the material before plastic deformation so that $E(0)$ is the initial elastic modulus before plastic flow. As before, let us select a linear elastic model to describe the current elastic response. Let this be given by

$$F = AE(\epsilon - \epsilon^p),$$

so that the loading rate $\dot{F}$ is given by

$$\dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p) + A(\epsilon - \epsilon^p) \frac{dE}{d\xi} \dot{\xi} = AE(\dot{\epsilon} - \dot{\epsilon}^p) + \frac{F}{E} \frac{dE}{d\xi} \dot{\xi}.$$

For isotropic hardening, we can write the yield function as

$$f = |F| - F_y(\xi) = \begin{cases} F - F_y(\xi) & \text{if } F \geq 0, \\ -F - F_y(\xi) & \text{if } F < 0, \end{cases}$$

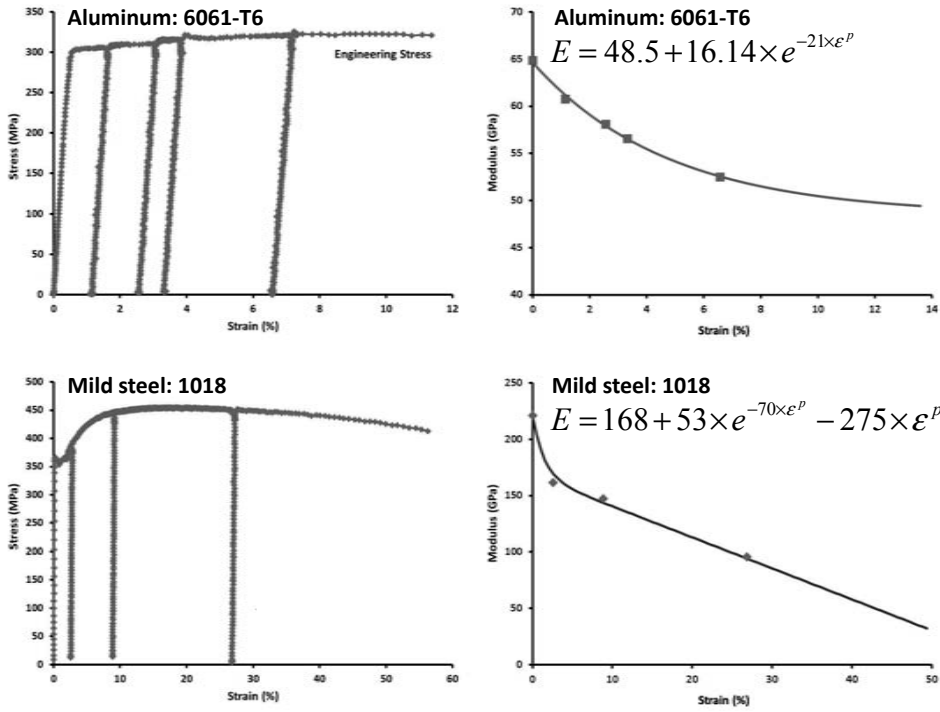


Figure 1.23: The changing of the elastic modulus as a function of plastic strain for 6061-T6 aluminum and 1018 mild steel measured in tension by periodically unloading the sample during the loading process.

During this same part of the loading, the consistency condition $\dot{f} = 0$ requires that

$$AE(1 - \beta) + \frac{F_y}{E} \frac{dE}{d\xi} \beta - \frac{dF_y}{d\xi} \beta = 0,$$

which, after substituting for β and solving for $\frac{dF_y}{d\xi}$, results in

$$\frac{dF_y}{d\xi} = \frac{Aa}{E - a} \left(E - \frac{F_y}{AE} \frac{dE}{d\xi} \right). \quad (1.112)$$

One can show that these relations, which are obtained for monotonic loading, hold also during arbitrary loading, the proof being left as a section problem. Given the form of $\frac{dE}{d\xi}$, this relation can be integrated, either analytically or numerically, to get F_y . The constant of integration can be evaluated from the condition $F_y = F_{y0}$ at $\xi = 0$.

We now can summarize the model for isotropic hardening with a changing elastic modulus

based on a strain hardening parameter as follows:

$$F = AE(\epsilon - \epsilon^p), \quad (1.113)$$

$$f = |F| - F_y, \quad (1.114)$$

$$\frac{dF_y}{d\xi} = \frac{Aa}{E-a} \left(E - \frac{F_y}{AE} \frac{dE}{d\xi} \right), \quad (1.115)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases} \quad (1.116)$$

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E - \frac{F_y}{AE} \frac{dE}{d\xi}} \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.117)$$

$$\dot{\xi} = |\dot{\epsilon}^p|. \quad (1.118)$$

Even though we did not fit the model to the form of the yield surface in compression, the current model will exhibit all of the characteristics shown in Figure 1.11c.

Section 1.9-Problems:

- 1.9-2:** Using the work hardening model $\dot{\xi} = F\dot{\epsilon}^p$ and assuming isotropic hardening, develop a plasticity model that exactly reproduces the response, shown in Figure 1.11c. In the process show that for this model

$$\beta = \frac{E-a}{E}, \quad F_y = \sqrt{\frac{2AaE}{E-a}} \xi + F_{yo}^2.$$

Can one conclude from this example that there is no difference between work hardening and strain hardening?

- 1.9-3:** Consider combined isotropic and kinematic hardening given by $F = EA(\epsilon - \epsilon^p)$, $F_1(\epsilon) = F_{yo} + Aa(\epsilon - \epsilon_{yo})$, and $F_2(\epsilon) = Ab\epsilon$. Develop the resulting model based on the work hardening assumption $\dot{\xi} = F\dot{\epsilon}^p$.
- 1.9-4:** Sketch the form of the current and initial yield functions in load space for combined isotropic and kinematic hardening.
- 1.9-5:** Write a program that calculates the load given any arbitrary strain history $\epsilon(t)$ for a general rate-independent and temperature-independent model of plasticity. Write the program such that the user can provide an equation for the load, the yield function, and the hardening rule.
- 1.9-6:** Use the program developed in the previous problem to evaluate the responses for the following models:
1. Isotropic hardening model with strain hardening,
 2. Kinematic hardening model with strain hardening,
 3. Combined isotropic and kinematic hardening with strain hardening.

For your simulations, use the demonstration parameters:

$$A = 1, \quad E = 20, \quad a = 2, \quad b = 1, \quad F_{yo} = 2.$$

Use the following strains as input:

1. $\epsilon = \epsilon_{max} \sin(2\pi t)$ for both $\epsilon_{max} = 0.1$ and $\epsilon_{max} = 0.5$,
2. $\epsilon = t \sin(2\pi t)$.

- 1.9-7:** Show that the model developed for isotropic hardening with a plastic flow-dependent elastic modulus exhibits the correct response characteristics in non-monotonic loading (i.e., in cyclic loading with yielding both in tension and compression).
- 1.9-8:** Develop a model for kinematic hardening, exhibiting a plastic flow-dependent elastic modulus using strain hardening.
- 1.9-9:** Develop an isotropic hardening model, exhibiting a plastic flow-dependent elastic modulus using work hardening.
- 1.9-10:** Develop a model for and simulate the response of the 6061-T6 aluminum shown in Figures 1.1 and 1.23 using a kinematic hardening model. Take the yield stress as 303 MPa, the elastic modulus in tension given as $E = 48.5 + 16.1e^{-21\epsilon^p}$ GPa, and the hardening slope as 303 MPa.

1.10 Temperature-dependent plasticity

The general theory of thermoplasticity is constructed from several elements. The first element is the separation of the total strain into elastic, plastic, and thermal parts. As described above, this results in the relation

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta + \epsilon^e \epsilon^p + \epsilon^e \epsilon^\theta + \epsilon^p \epsilon^\theta + \epsilon^e \epsilon^p \epsilon^\theta, \quad (1.119)$$

which for small elastic, plastic, and thermal strains can be approximated by

$$\epsilon \approx \epsilon^e + \epsilon^p + \epsilon^\theta. \quad (1.120)$$

An alternate separation is based on use of the logarithmic strain, which gives the exact relation

$$\epsilon_l = \epsilon_l^e + \epsilon_l^p + \epsilon_l^\theta. \quad (1.121)$$

In addition to the separation of total strain into parts, there are several other elements in the theory. These elements include a yield function, a flow rule, a hardening model, an equation to calculate load, and an equation to calculate thermal strain. The standard model used for thermal expansion is given as

$$\dot{\epsilon}^\theta = \alpha \lambda^\theta \dot{\theta}, \quad (1.122)$$

where α is the linear coefficient of thermal expansion. In solid mechanics, since the coefficient of thermal expansion and range of temperatures considered are normally both small, we use the approximate form of this model given by

$$\dot{\epsilon}^\theta \approx \alpha \dot{\theta}. \quad (1.123)$$

This approximation can be made exact if we take α as the product of the linear coefficient of thermal expansion with the thermal stretch. The thermal strain can be calculated by integrating the thermal strain rate through the expression

$$\epsilon^\theta = \int_{t_o}^t \dot{\epsilon}^\theta dt = \int_{\theta_o}^\theta \alpha \lambda^\theta d\theta, \quad (1.124)$$

where θ_o is the reference temperature at which the thermal strain is taken to be zero. Obviously, if we focus on small thermal strains (i.e., $\lambda^\theta \approx 1$) and take α to be constant, one can integrate this to get

$$\epsilon^\theta = \alpha(\theta - \theta_o). \quad (1.125)$$

The basic idea behind the calculation of axial load is that it depends on elastic strain, plastic strain, thermal strain, temperature, and one or more hardening parameters. Let us write this as

$$F = F^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi_i), \quad (1.126)$$

where F is the axial load. The superscript “ $\dagger$ ” is to designate the mathematical model for F and its arguments indicate that our model depends on the position in the reference configuration, elastic strain, plastic strain, thermal strain, temperature, and a number of hardening parameters denoted by ξ_i . A typical example of such a model is given by

$$F = AE\epsilon^e,$$

where A is the cross-sectional area in the reference configuration and E is the elastic modulus. The cross-sectional area may depend on X , and the modulus may depend on X , θ , ϵ^p , and any number of the hardening parameters.

The next element of our model is a yield function. The yield function, also a function of the same arguments as load, is used to evaluate the current range of the elastic response. In isothermal plasticity, it is assumed that at each loading stage there is a range of deformations that can occur with only changes to the elastic strain so that in this “current elastic range” there is no change to the plastic strain or hardening parameters. In nonisothermal plasticity, the same is assumed, but allow for both elastic and thermal strains in the current thermoelastic range, with no change in the plastic strain and hardening parameters during a pure thermoelastic response. At each temperature, bounds on the elastic range are established by the selection of an appropriate yield function $f(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi_i)$ and the constraint

$$f(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi_i) \leq 0, \quad (1.127)$$

which must hold at all times. The yield surface, the bounds on the current thermoelastic range, is given by setting the yield function equal to zero (i.e., $f = 0$ defines the current yield surface). We say we are on the yield surface when $f = 0$, and within the yield surface when $f < 0$. When within the yield surface (i.e., $f < 0$), all deformations occur without changes in the plastic strain and hardening parameters. When on the yield surface (i.e., $f = 0$), depending on the nature of the next increment of loading, there might be plastic flow, neutral loading, or unloading into the thermoelastic range. Plastic flow refers to when there has to be a change in the plastic strain or else there will be a violation of the constraint $f \leq 0$. Neutral loading on the yield surface refers to increments of loading that move along the yield surface (i.e., f remains equal to zero during the increment) without needing any change to the plastic strain and hardening parameters. Unloading from the yield surface refers to increments of load that start from a loading point on the yield surface ($f = 0$) and with the next increment end up at a loading point within the yield surface ($f < 0$). Unloading occurs without any changes to the plastic strain and hardening parameters. When

on the yield surface, the nature of the next loading increment is determined by evaluating the effect of the next increment on the value of the yield function, assuming no change in the plastic strain and hardening parameters. That is, if the increment can be accommodated by a fully thermoelastic response without violating the condition $f \leq 0$, then there should be no plastic flow, and both the plastic strain and hardening parameters should remain unchanged. As we saw in our previous examples, one model for the yield function can be written as

$$f = |F - F_c| - F_{iso},$$

where F_c is a result of kinematic hardening and F_{iso} reflects isotropic hardening. Both of these functions may depend on plastic strain, the hardening parameters, and temperature. The *consistency condition* during plastic flow requires that $\dot{f} = 0$, so that we must have

$$\frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}^e + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial f}{\partial \theta} \dot{\theta} + \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \dot{\xi}_i = 0. \quad (1.128)$$

In plasticity, the determination of whether there will be plastic flow in the next increment of loading is critical to establishing the flow rule. Many different situations may occur, particularly with the addition of temperature. An interesting example is when we are currently on the yield surface (i.e., $f = 0$) and the load is increasing so that in an isothermal process we would initiate plastic flow in the following increment of loading. Yet, if the yield surface is growing faster because of changing the temperature rather than because of loading, there will not only not be plastic flow in the following increment, but as a result of the temperature change we should move away from the yield surface, ending up inside the yield surface (i.e., $f < 0$). The reverse also can happen when we end up initiating plastic flow during mechanical unloading due to the contraction of the yield surface. Clearly, the addition of temperature has opened up a new set of options that need consideration. To avoid the need for separate consideration of all the special conditions possible, we follow the same single rule as for the mechanical theory:

If a sufficiently small loading increment can be accommodated without plastic flow (i.e., $f \leq 0$ after the loading increment), then there is no need to consider plastic flow in the increment. On the other hand, if ignoring plastic flow results in $f > 0$, then plastic flow occurs in such a way that at the end of the increment of loading $f \leq 0$. In this latter case, if the increment of loading is small enough, at the end of the increment $f = 0$.

The only difference between this and the mechanical theory is that a loading increment here refers to a change in strain and/or temperature. As before, this rule is restricted to small increments of loading.

As for the mechanical theory, we will again use “ $\diamond$ ” on a character to denote a time derivative holding the plastic strain and hardening parameters constant. For example, the rate of change of the yield function in the absence of plastic flow and hardening as a result of the application of a strain rate $\dot{\epsilon}$ and temperature rate $\dot{\theta}$ is given by

$$\diamond f = \frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}^e + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial f}{\partial \theta} \dot{\theta}, \quad (1.129)$$

where

$$\dot{\epsilon}^e = \frac{\dot{\epsilon} - (1 + \epsilon^e + \epsilon^p + \epsilon^e \epsilon^p) \dot{\epsilon}^\theta}{1 + \epsilon^p + \epsilon^\theta + \epsilon^e \epsilon^p}, \quad (1.130)$$

which for small elastic, plastic, and thermal strains can be approximated by

$$\overset{\circ}{\epsilon}^e \approx \dot{\epsilon} - \dot{\epsilon}^\theta.$$

One can think of $\overset{\circ}{f}$ as a *trial value* for $\dot{f}$ assuming a pure thermoelastic response. Obviously, if $f = 0$ and $\overset{\circ}{f} > 0$, the assumption of a pure thermoelastic loading step would result in $f > 0$, a violation of the constraint $f \leq 0$. Therefore, one should assume plastic flow when $f = 0$ and $\overset{\circ}{f} > 0$; otherwise, for all other loadings the condition $f \leq 0$ can be maintained with a pure thermoelastic response. For example, if $f = 0$ and $\overset{\circ}{f} \leq 0$, then a pure thermoelastic loading increment would result in $f \leq 0$ ($f = 0$ if $\overset{\circ}{f} = 0$, and $f < 0$ if $\overset{\circ}{f} < 0$). Or, for example, if $f < 0$ and $\overset{\circ}{f} > 0$, then a small enough pure thermoelastic loading increment would keep $f \leq 0$. Using this idea, one can write the flow rule for plastic strain as

$$\dot{\epsilon}^p = \begin{cases} \beta_{\dot{\theta}} \dot{\theta} + \beta_{\dot{\epsilon}} \dot{\epsilon} & \text{if } f = 0 \text{ and } \overset{\circ}{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.131)$$

where $\beta_{\dot{\theta}}$ and $\beta_{\dot{\epsilon}}$ are functions of $(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi_i)$. The term $\beta_{\dot{\theta}} \dot{\theta}$ represents the contribution of temperature change to plastic flow and the term $\beta_{\dot{\epsilon}} \dot{\epsilon}$ represents the contribution of changes in strain to plastic flow. Similar expressions normally are selected for each of the hardening parameters, normally connecting hardening to $\dot{\epsilon}^p$ as opposed to $\dot{\epsilon}$. A typical expression might be given as

$$\dot{\xi}_i = \omega_i \dot{\epsilon}^p, \quad (1.132)$$

where again each ω_i is a function of $X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi_i$. From this, we obviously conclude that

$$\overset{\circ}{\xi}_i = 0.$$

For the case of infinitesimal strains, the consistency condition during plastic flow (i.e., $\dot{f} = 0$) can now be written as

$$\frac{\partial f}{\partial \epsilon^e} (\dot{\epsilon} - \dot{\epsilon}^p - \dot{\epsilon}^\theta) + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial f}{\partial \theta} \dot{\theta} + \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \dot{\xi}_i = 0.$$

After substitution for $\dot{\epsilon}^\theta = \alpha \dot{\theta}$ and $\dot{\xi}_i = \omega_i \dot{\epsilon}^p$ and some reorganization, we arrive at

$$\left(\frac{\partial f}{\partial \epsilon^e} - \frac{\partial f}{\partial \epsilon^p} - \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i \right) \dot{\epsilon}^p = \left(\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \epsilon^\theta} \alpha - \frac{\partial f}{\partial \epsilon^e} \alpha \right) \dot{\theta} + \frac{\partial f}{\partial \epsilon^e} \dot{\epsilon}.$$

Comparison of this and the flow rule, therefore, requires that

$$\beta_{\dot{\theta}} = \frac{\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \epsilon^\theta} \alpha - \frac{\partial f}{\partial \epsilon^e} \alpha}{\frac{\partial f}{\partial \epsilon^e} - \frac{\partial f}{\partial \epsilon^p} - \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i}, \quad (1.133)$$

and

$$\beta_{\dot{\epsilon}} = \frac{\frac{\partial f}{\partial \epsilon^e}}{\frac{\partial f}{\partial \epsilon^e} - \frac{\partial f}{\partial \epsilon^p} - \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \omega_i}. \quad (1.134)$$

Obviously, these results only hold when the denominator is not equal to zero.

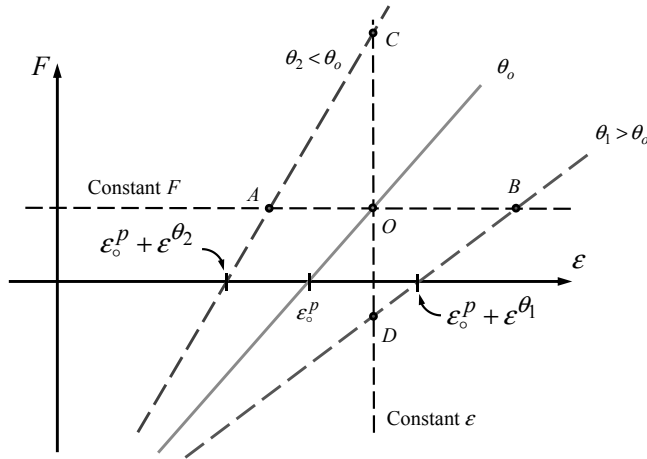


Figure 1.24: Thermoelastic response at constant strain and constant force.

The reader will note that these results are not unique since they depend on the form of the hardening law selected. In the next section, we will look at the infinitesimal theory of thermoplasticity.

Section 1.10-Problems:

- 1.10-1:** Write the consistency condition for the case of large strains and derive the expressions for $\beta_{\dot{\theta}}$ and $\beta_{\dot{\epsilon}}$.

1.11 An infinitesimal theory of thermoplasticity

A change in temperature can have several effects on the load response of a material. For example, in most materials there is a reduction of the elastic modulus and a reduction in the yield load with an increase of temperature. Another commonly known effect is the increase of strain due to thermal expansion. All these effects need to be considered simultaneously when a material is subjected to thermomechanical loading. As in the last section, we will separate the strain into an elastic part, a plastic part, and a thermal part. We will restrict our attention to infinitesimal strains so that we have

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta. \quad (1.135)$$

As shown in Figure 1.24, to describe the thermal strain we select a reference temperature θ_0 at which we take thermal strain to be zero. As we change temperature, thermal strain is then evaluated by the rule

$$\dot{\epsilon}^\theta = \alpha \dot{\theta}, \quad (1.136)$$

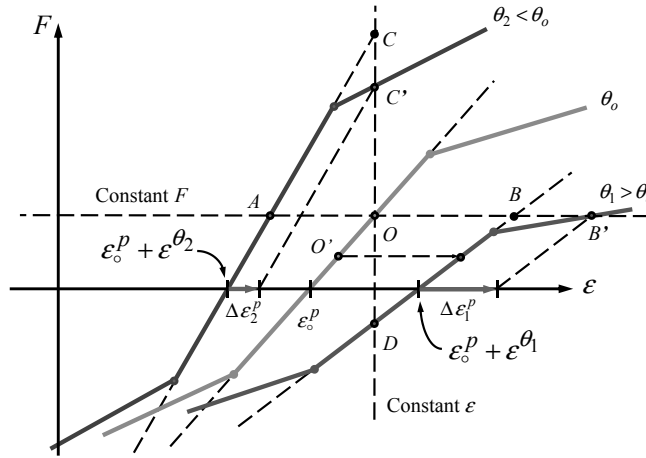


Figure 1.25: Temperature-dependent elastic-plastic response.

where α is the coefficient of thermal expansion. This can be integrated to obtain

$$\epsilon^\theta = \int_{t_o}^t \dot{\epsilon}^\theta dt = \int_{\theta_o}^\theta \alpha d\theta. \quad (1.137)$$

If the coefficient of thermal expansion is constant, one can integrate this to get

$$\epsilon^\theta = \alpha(\theta - \theta_o).$$

Before we proceed to study temperature-dependent plasticity, let us consider thermoelasticity. As our rule states, plastic flow only will be initiated when a pure thermoelastic response would not be possible without forcing the yield function to become positive. Therefore, we need to know what is meant by a pure thermoelastic response. Figure 1.24 shows the elastic response at three temperatures. First, at the reference temperature θ_o the thermal strain is taken to be zero so that the intercept of the loading curve and the strain axis indicates the current plastic strain (ϵ_o^p). Now, let us only assume thermoelastic response. If the temperature is increased to temperature θ_1 , there will be thermal expansion, forcing the intercept of the loading curve and the strain axis to move to the right ($\epsilon_o^p + \epsilon^{\theta_1}$). In addition, the modulus decreases, resulting in a lower slope for the loading curve. As shown in the figure, if we consider a constant strain condition starting from point O on the loading curve at temperature θ_o , the force will drop from point O at θ_o to point D at θ_1 . On the other hand, if the force is kept constant, then the strain will increase from point O to point B . In a similar way, if the temperature is reduced from θ_o to θ_2 , the intercept of the loading curve and the strain axis will move to the left ($\epsilon_o^p + \epsilon^{\theta_2}$) and the slope will increase. In such a case, constant strain would result in an increase in the force from O to C , and constant force would result in a reduction in strain from point O to A .

Let us now consider Figure 1.25, which shows the same conditions as in Figure 1.24, but for elastic-plastic response. If we again start from point O on the loading curve at temperature θ_o and then raise the temperature to θ_1 , as can be seen in the figure, for these conditions and constant strain, we end up again at point D , so that we only undergo thermoelastic unloading. On the other hand, under the indicated conditions and at constant force, we move from O to B' , as opposed to B . Obviously, if we had assumed pure thermoelastic

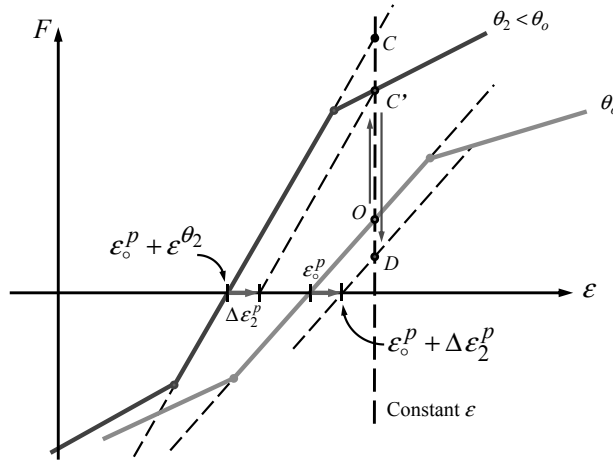


Figure 1.26: A thermal cycle at constant strain that goes from initial temperature θ_o to a lower temperature θ_2 and then back to θ_o . The process starts at point O at temperature θ_o , moves to point C' at temperature θ_2 , and then moves to point D when returned back to temperature θ_o .

loading, we would end up at point B , which is above the yield surface at temperature θ_1 , and would result in $f > 0$, clearly indicating that we need to introduce plastic flow. As indicated in the figure, a plastic strain increment of $\Delta \epsilon_1^p$ is needed to get us onto the current yield surface at the indicated force (point B'). A similar situation occurs when cooling from temperature θ_o to θ_2 . Under constant strain, pure thermoelastic response would require us to go to point C , but since this is above the yield surface, we end up at point C' . Again, an increment of plastic strain $\Delta \epsilon_2^p$ is needed to make this possible. In this case, constant force conditions can be accommodated by a pure thermoelastic response (from O to A). Obviously, the statements above are dependent on the starting point O since, for example, if we hold the sample at constant force, starting from O' and raise the temperature to θ_1 , the response could be accommodated by thermoelastic response.

To better understand how the material responds, consider the constant strain process shown in Figure 1.26. The process starts at point O at temperature θ_o . The temperature is then cycled to a lower temperature θ_2 and back to θ_o , holding the strain constant. As can be seen, when lowering the temperature to θ_2 a pure thermoelastic response would take us to point C , but because C is above the yield surface at θ_2 , we end up at C' with an induced plastic strain increment $\Delta \epsilon_2^p$. When raising the temperature back to θ_o , because of the induced plastic strain increment $\Delta \epsilon_2^p$, the current elastic response is shifted by $\Delta \epsilon_2^p$ to the right, resulting in a drop in force from C' to D on the current elastic response curve.

In most cases, the material is not restricted to processes that are constant strain or constant force. Therefore, the model we develop will not only need to be able to capture the examples considered above, but also will need to be of use under arbitrary loading histories. For the purposes of our discussion, a loading history will be described by how temperature and strain change over time. In this way, we are selecting temperature and strain as the independent variables and force as the dependent variable. With this in mind, the methodology we have constructed can be used directly to develop the following somewhat general model.

At each temperature, the elastic response will be selected as a linear relation

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p - \epsilon^\theta), \quad (1.138)$$

where the modulus E is assumed to depend on temperature. A general yield function of the form developed for arbitrary hardening will be selected so that

$$f = |F - F_c| - F_{iso}, \quad (1.139)$$

where F_c is taken to depend on the plastic strain and temperature and F_{iso} is taken to be a function of the hardening parameter and temperature. The rate of change of the yield function is, therefore, given by

$$\dot{f} = \begin{cases} \dot{F} - \dot{F}_c - \dot{F}_{iso} & \text{if } F - F_c \geq 0, \\ -\dot{F} + \dot{F}_c - \dot{F}_{iso} & \text{if } F - F_c < 0, \end{cases} \quad (1.140)$$

to yield

$$\dot{f} = \begin{cases} AE(\dot{\epsilon} - \dot{\epsilon}^p - \alpha\dot{\theta}) + A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta)\dot{\theta} - \frac{\partial F_c}{\partial \theta}\dot{\theta} - \frac{\partial F_c}{\partial \epsilon^p}\dot{\epsilon}^p - \frac{\partial F_{iso}}{\partial \theta}\dot{\theta} - \frac{\partial F_{iso}}{\partial \xi}\dot{\xi} & \text{if } F - F_c \geq 0, \\ -AE(\dot{\epsilon} - \dot{\epsilon}^p - \alpha\dot{\theta}) - A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta)\dot{\theta} + \frac{\partial F_c}{\partial \theta}\dot{\theta} + \frac{\partial F_c}{\partial \epsilon^p}\dot{\epsilon}^p - \frac{\partial F_{iso}}{\partial \theta}\dot{\theta} - \frac{\partial F_{iso}}{\partial \xi}\dot{\xi} & \text{if } F - F_c < 0. \end{cases}$$

Assuming there is no plastic flow or hardening during a thermoelastic response, one will get

$$\overset{\diamond}{f} = \begin{cases} AE(\dot{\epsilon} - \alpha\dot{\theta}) + A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta)\dot{\theta} - \frac{\partial F_c}{\partial \theta}\dot{\theta} - \frac{\partial F_{iso}}{\partial \theta}\dot{\theta} & \text{if } F - F_c \geq 0, \\ -AE(\dot{\epsilon} - \alpha\dot{\theta}) - A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta)\dot{\theta} + \frac{\partial F_c}{\partial \theta}\dot{\theta} - \frac{\partial F_{iso}}{\partial \theta}\dot{\theta} & \text{if } F - F_c < 0. \end{cases} \quad (1.141)$$

The plastic flow rule will be taken as

$$\epsilon^p = \begin{cases} \beta_\theta\dot{\theta} + \beta_\epsilon\dot{\epsilon} & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (1.142)$$

and the hardening rule will be taken to be plastic strain hardening given as

$$\dot{\xi} = |\dot{\epsilon}^p|. \quad (1.143)$$

We can write the rate of change of the yield function as

$$\dot{f} = \begin{cases} \overset{\diamond}{f} - AE\dot{\epsilon}^p - \frac{\partial F_c}{\partial \epsilon^p}\dot{\epsilon}^p - \frac{\partial F_{iso}}{\partial \xi}|\dot{\epsilon}^p| & \text{if } F - F_c \geq 0, \\ \overset{\diamond}{f} + AE\dot{\epsilon}^p + \frac{\partial F_c}{\partial \epsilon^p}\dot{\epsilon}^p - \frac{\partial F_{iso}}{\partial \xi}|\dot{\epsilon}^p| & \text{if } F - F_c < 0, \end{cases} \quad (1.144)$$

therefore eliminating the need to reevaluate all the terms used to calculate $\overset{\diamond}{f}$ when constructing $\dot{f}$. Assuming positive plastic flow for loads above F_c and negative plastic flow for

loads below F_c , one can conclude that the consistency condition $\dot{f} = 0$ during plastic flow ($f = 0$ and $\dot{f} > 0$) at loads above F_c requires that

$$AE(\dot{\epsilon} - \dot{\epsilon}^p - \alpha\dot{\theta}) + A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta)\dot{\theta} - \frac{\partial F_c}{\partial \theta}\dot{\theta} - \frac{\partial F_c}{\partial \epsilon^p}\dot{\epsilon}^p - \frac{\partial F_{iso}}{\partial \theta}\dot{\theta} - \frac{\partial F_{iso}}{\partial \xi}\dot{\epsilon}^p = 0.$$

Reorganization yields

$$\left(AE + \frac{\partial F_c}{\partial \epsilon^p} + \frac{\partial F_{iso}}{\partial \xi}\right)\dot{\epsilon}^p = AE\dot{\epsilon} + \left[-AE\alpha + A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta) - \frac{\partial F_c}{\partial \theta} - \frac{\partial F_{iso}}{\partial \theta}\right]\dot{\theta}.$$

Comparison to the flow rule yields the relations

$$\beta_{\dot{\theta}} = \frac{-AE\alpha + A\frac{dE}{d\theta}(\epsilon - \epsilon^p - \epsilon^\theta) - \frac{\partial F_c}{\partial \theta} - \text{sign}(F - F_c)\frac{\partial F_{iso}}{\partial \theta}}{AE + \frac{\partial F_c}{\partial \epsilon^p} + \frac{\partial F_{iso}}{\partial \xi}}, \quad (1.145)$$

and

$$\beta_{\dot{\epsilon}} = \frac{AE}{AE + \frac{\partial F_c}{\partial \epsilon^p} + \frac{\partial F_{iso}}{\partial \xi}}, \quad (1.146)$$

where $\text{sign}(F - F_c)$ has been added to the last term of the numerator of $\beta_{\dot{\theta}}$ so that it also will satisfy the condition when $F - F_c < 0$. The function sign will be defined as

$$\text{sign}(\phi) = \begin{cases} 1 & \text{if } \phi \geq 0, \\ -1 & \text{if } \phi < 0. \end{cases} \quad (1.147)$$

A similar development can be provided for plastic work hardening. This is left as a homework.

Section 1.11-Problems:

- 1.11-1:** Show that $\text{sign}(F - F_c)$ in the expression for $\beta_{\dot{\theta}}$ is needed for the result to work for plastic flow both when $F - F_c \geq 0$ and when $F - F_c < 0$.
- 1.11-2:** Develop a temperature-dependent model for the elastic-plastic response of a material using the plastic work hardening parameter in place of the strain hardening parameter.
- 1.11-3:** Write a program to integrate the equations of temperature-dependent elastic-plastic response given in this section. Test your program by looking at constant strain and constant load paths as described in Figures 1.25 and 1.26. Select $E = 10 - 0.01(\theta - 25)$, $\frac{\partial F_{iso}}{\partial \xi} = 0.1$, $\frac{\partial F_c}{\partial \epsilon^p} = 0.1$, $\frac{\partial F_{iso}}{\partial \theta} = -0.1$, $\frac{\partial F_c}{\partial \theta} = -0.1$, and $\alpha = 0.01$, where θ is given in degrees Celsius.
- 1.11-4:** Write a program to integrate the nonisothermal rate-independent plasticity model described in this section. Evaluate the response of the model for the loading:
1. $\epsilon = t$, $\theta = 0.1t$,
 2. $\epsilon = 0.1t$, $\theta = t$,
 3. $\epsilon = \sin(2\pi t)$, $\theta = 50 \sin(\frac{2\pi t}{10})$.

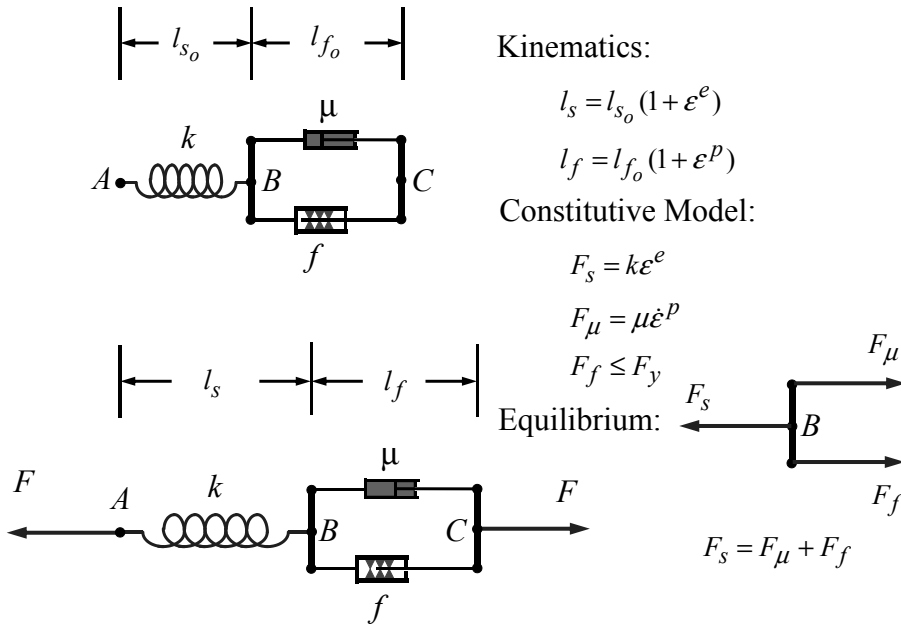


Figure 1.27: A mechanical analog constructed from a spring, viscous damper, and stick-slip friction element developed to describe the elastic-viscoplastic response.

For your simulations, use the parameters:

$$A = 1, \quad E_o = 20, \quad a_o = 2, \quad F_{y_o} = 2, \quad \alpha = 0.001,$$

$$\frac{dE}{d\theta} = -0.01E_o, \quad \frac{dF_{y_o}}{d\theta} = -0.01F_{y_o}, \quad \frac{da}{d\theta} = -0.01a_o,$$

where the given values for E_o , a_o , F_{y_o} are constants representing, respectively, the values of E , a , F_{y_o} at the reference temperature θ_o .

1.12 Rate-dependent models for plasticity

The models we have considered up to this point are rate independent in the sense that time appeared in the models in a homogeneous manner so that scaling time up or down would not affect the load-strain response of the material. For many materials, this is a reasonable assumption for slow to moderate loading rates and for temperatures much below the melting point. But, there are conditions for which speeding up the loading process or slowing it down does result in a substantially different responses. For these conditions, we need to develop a different type of model. Many different types of rate dependence can be considered. Covering all of these possibilities is out of the scope of our discussion. We will concentrate only on a set of models which are developed to describe elastic-viscoplastic

response. Elastic-viscoplastic response is characterized by an elastic range in which the response is purely elastic, outside which one observes a rate-dependent plastic flow.

Before we develop a general model for what we call the elastic-viscoplastic material, let us look at the mechanical analog described in Figure 1.27, which shows a mechanical system that exhibits many of the features that are characteristic of elastic-viscoplastic material response. This mechanical analog is constructed from a spring, a viscous damper, and a stick-slip friction element. The load F_s in the spring is modeled by the equation $F_s = k\epsilon^e$, where k represents the stiffness and ϵ^e represents the strain in the spring. The load F_μ in the viscous element is modeled by the equation $F_\mu = \mu\dot{\epsilon}^p$, where μ represents the viscosity and $\dot{\epsilon}^p$ represents the rate of straining of this element. The load F_f in the stick-slip friction element is always less than or equal to F_y , with the condition that the element is locked if $F_f < F_y$ and extends as required by the viscous damper when $F_f = F_y$. The total strain in the system is given by

$$\epsilon = \epsilon^e + \epsilon^p.$$

The applied load F on the system is given by

$$F = F_s = F_\mu + F_f.$$

As long as the applied load is less than F_y , the stick-slip friction element is locked and the viscous damper cannot extend. Therefore, the viscous damper will have a zero load in it. Under this condition, the spring and the friction element carry all of the load, and the total strain is only due to the spring. Once the applied load F reaches F_y , the friction element becomes free to move, but since the load in the viscous damper is zero it will not extend. This follows from the assumption that the extension of the stick-slip element is controlled by the viscous damper, which will only extend when its load is nonzero. Once the load F applied on the system becomes greater than F_y , equilibrium requires that $F_\mu = F - F_y$, initiating extension of the viscous damper and the stick-slip friction element through the relation

$$\dot{\epsilon}^p = \frac{1}{\mu}(F - F_y).$$

Therefore, the larger the applied load F , the more rapid the plastic flow. This is the source of the rate dependence in our mechanical analog. If at any stage the applied load F is reduced to a value below F_y , the stick-slip element locks, leaving the plastic strain constant. As the applied load is reduced to zero, the elastic strain goes to zero, leaving only the total strain equal to the plastic strain. Our model for elastic-viscoplastic response will be developed in a way to exhibit these same characteristics.

Now that we have seen a mechanical analog with the essential features that we want in our elastic-viscoplastic model, let us proceed to develop the actual model. First, we will assume the separation of the total strain to be given by

$$\epsilon = \epsilon^e + \epsilon^\theta + \epsilon^p, \quad (1.148)$$

where the three terms on the right-hand side have the same interpretations as before. We assume the applied load is given by a model of the form

$$F = F^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi). \quad (1.149)$$

The simplest example of this would be

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p - \epsilon^\theta).$$

To this we add a yield function of the form

$$f = f^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi). \quad (1.150)$$

An example of such a model would be the combined isotropic and kinematic hardening model given by

$$f = |F - F_c| - F_{iso}.$$

Unlike before, we allow f to take both negative and positive values. When $f < 0$, we consider the material to be in the current thermoelastic range and, therefore, assume no plastic flow. When $f > 0$, plastic flow is initiated at a rate that is in relation to the amount the load is greater than the load at $f = 0$. Simply, in this model, $f = 0$ will provide the load above which viscoplastic flow will occur. The flow rule will be given by

$$\dot{\epsilon}^p = \begin{cases} \dot{\epsilon}^{p\dagger}(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \theta, \xi) & \text{if } f > 0, \\ 0 & \text{if } f \leq 0. \end{cases} \quad (1.151)$$

A simple example for such a flow rule would be

$$\dot{\epsilon}^p = \begin{cases} \text{sign}(F - F_c) \frac{f}{\mu} & \text{if } f > 0, \\ 0 & \text{if } f \leq 0, \end{cases}$$

where μ is a constant describing the viscosity associated with the plastic flow, f is used as the parameter describing how much above the yield surface the current load is, and the sign of $F - F_c$ is included so that one obtains positive plastic flow at the upper yield point and negative plastic flow at the lower yield point. A hardening rule, such as plastic strain hardening or plastic work hardening, must be added to this.

As an example, let us consider a temperature-independent model that is characterized by the following equations

$$\begin{aligned} F &= AE(\epsilon - \epsilon^p), \\ f &= |F| - F_y, \\ F_y &= F_{yo} + \int_{t_o}^t \dot{F}_y dt, \\ \dot{F}_y &= \frac{AEa}{E-a} \dot{\xi}, \\ \dot{\epsilon}^p &= \begin{cases} \text{sign}(F) \frac{f}{\mu} & \text{if } f > 0, \\ 0 & \text{if } f \leq 0, \end{cases} \\ \dot{\xi} &= |\dot{\epsilon}^p|, \end{aligned} \quad (1.152)$$

where A , E , a , F_{yo} , and μ are constants. Let us now consider some of the characteristics of this simple model, which is based on isotropic hardening. Obviously, as long as the total strain ϵ increases faster than the plastic strain ϵ^p , the rate of load $\dot{F} = AE(\dot{\epsilon} - \dot{\epsilon}^p)$ will be positive, indicating continuously increasing load conditions. Therefore, it follows that if the total strain rate becomes equal to the plastic strain rate we will have constant load. This condition is given by

$$\dot{\epsilon} = \dot{\epsilon}^p = \text{sign}(F) \frac{|F| - F_y}{\mu} = \frac{F}{|F|} \frac{|F| - F_y}{\mu} = \frac{F}{\mu} \left(1 - \frac{F_y}{|F|} \right).$$

Let us now look at the case of monotonic loading at constant strain rate $\dot{\epsilon}_o$. Up to the initial yield load F_{yo} , there will be no plastic flow so that $\dot{F} = AE\dot{\epsilon}_o$. After passing the initial yield

load, the equation for plastic flow becomes

$$\dot{\epsilon}^p = \frac{F - F_y}{\mu}.$$

Differentiating with respect to time gives

$$\ddot{\epsilon}^p = \frac{1}{\mu}(\dot{F} - \dot{F}_y) = \frac{1}{\mu} \left[AE(\dot{\epsilon}_o - \dot{\epsilon}^p) - \frac{AEa}{E-a} \dot{\epsilon}^p \right].$$

Reorganizing this yields

$$\ddot{\epsilon}^p = \frac{AE^2}{\mu(E-a)} \left[\frac{E-a}{E} \dot{\epsilon}_o - \dot{\epsilon}^p \right].$$

This can be integrated using separation of variables to get

$$\ln \left(\frac{E-a}{E} \dot{\epsilon}_o - \dot{\epsilon}^p \right) = -\frac{AE^2}{\mu(E-a)} t + \ln(C).$$

Reorganization yields

$$\dot{\epsilon}^p = \frac{E-a}{E} \dot{\epsilon}_o - C e^{-\frac{AE^2}{\mu(E-a)} t}.$$

Integrating this once more yields

$$\epsilon^p = \frac{E-a}{E} \dot{\epsilon}_o t + \frac{C\mu(E-a)}{AE^2} e^{-\frac{AE^2}{\mu(E-a)} t} + C',$$

where C and C' are constants of integration, which must be fit to the initial conditions. Let us take $t = 0$ when $\epsilon = \epsilon_{yo}$, so that at $t = 0$, we have $F = F_{yo}$ and $f = 0$. Our initial conditions will therefore be

$$\epsilon^p = 0, \quad \dot{\epsilon}^p = 0$$

at $t = 0$. The initial condition on $\dot{\epsilon}^p$ directly follows from setting $f = 0$ into the flow rule. Introducing these initial conditions and solving for the integration constants results in

$$\epsilon^p = \frac{E-a}{E} \dot{\epsilon}_o \left[t - \frac{\mu(E-a)}{AE^2} \left(1 - e^{-\frac{AE^2}{\mu(E-a)} t} \right) \right].$$

If we introduce this into the expression for load, after some reorganization, we find that

$$F = F_{yo} + Aa\dot{\epsilon}_o t + \frac{\mu(E-a)^2}{E^2} \dot{\epsilon}_o \left(1 - e^{-\frac{AE^2}{\mu(E-a)} t} \right).$$

The rate of change of load with respect to strain ($\dot{F}/\dot{\epsilon}_o$) gives the slope in the load-strain diagram and is given by

$$\frac{\dot{F}}{\dot{\epsilon}_o} = AE \left[1 - \frac{E-a}{E} \left(1 - e^{-\frac{AE^2}{\mu(E-a)} t} \right) \right].$$

Examining this expression we see that the slope of the load-strain diagram is AE at the initial yield point and Aa as time goes to infinity, with a smooth exponential transition from one to the other. If one denotes the “equilibrium response” as the one described by $F = F_y$ (i.e., $f = 0$), an examination of the expression for load reveals that as time goes to infinity the response approaches a straight line parallel to the yield surface at a distance $\frac{\mu(E-a)^2}{E^2} \dot{\epsilon}_o$ above the equilibrium response. As can be seen, as the strain rate is reduced, the response approaches the equilibrium response indicated. As also can be seen, an increase

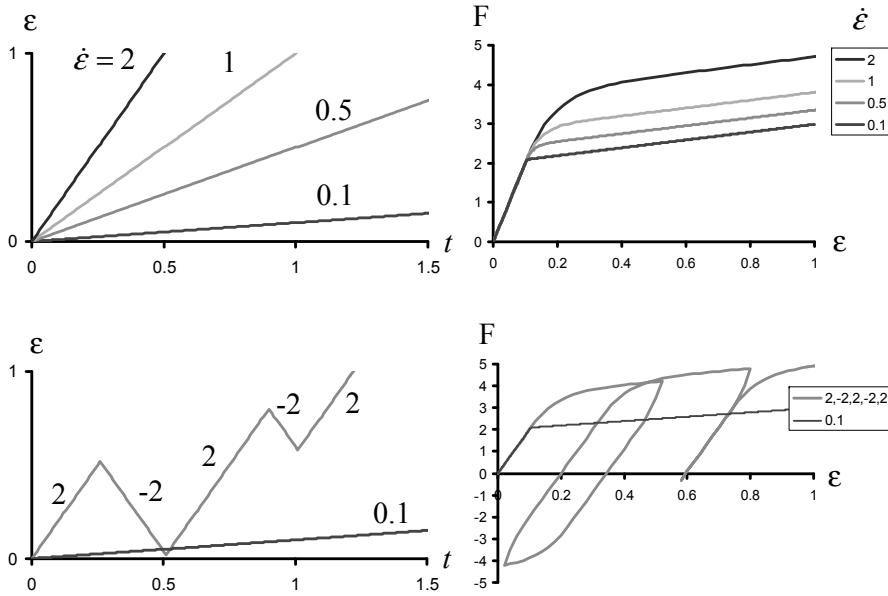


Figure 1.28: Elastic-viscoplastic response for two strain histories. Top: constant strain rate for several different strain rates as indicated on the strain time plot on the left. Bottom: loading/unloading cycle as indicated on the strain-time plot.

in the viscosity or strain rate increases this shift. Figure 1.28 shows the constant strain rate response just described, and a loading/unloading cycle as indicated for the parameters $A = 1$, $E = 20$, $a = 1$, $F_{yo} = 2$, and $\mu = 1$. Another characteristic of the response is seen when subjecting the material to a constant strain rate $\dot{\epsilon}_o$ from time 0 to t_1 and then changing the rate to $\dot{\epsilon}_1$. Figure 1.29 shows this response for several different values of $\dot{\epsilon}_1$.

Section 1.12-Problems:

- 1.12-1:** Develop a program to integrate the example model developed in this section for rate-dependent plasticity and use it to verify the response shown in Figures 1.28 and 1.29.
- 1.12-2:** Develop a rate-dependent plasticity model based on a combined isotropic and kinematic hardening model. Develop a program to evaluate the response of this model using the parameters $A = 1$, $E = 20$, $a = 2$, $b = 1$, $F_{yo} = 2$, and $\mu = 1$. Plot the response of the model for the loading histories indicated in Figures 1.28 and 1.29. Also, evaluate the response of this model to $\epsilon = t \sin(2\pi t)$.
- 1.12-3:** Construct a power-law viscoplasticity model based on taking the yield function for combined isotropic and kinematic hardening and using the flow rule

$$\dot{\epsilon}^p = \begin{cases} \text{sign}(F - F_c) \frac{F^n}{\mu} & \text{if } f > 0, \\ 0 & \text{if } f \leq 0, \end{cases} \quad (1.153)$$

where n is an exponent selected to characterize the nonlinear load dependence of the flow rule. Simulate the response of the material for monotonic loading.

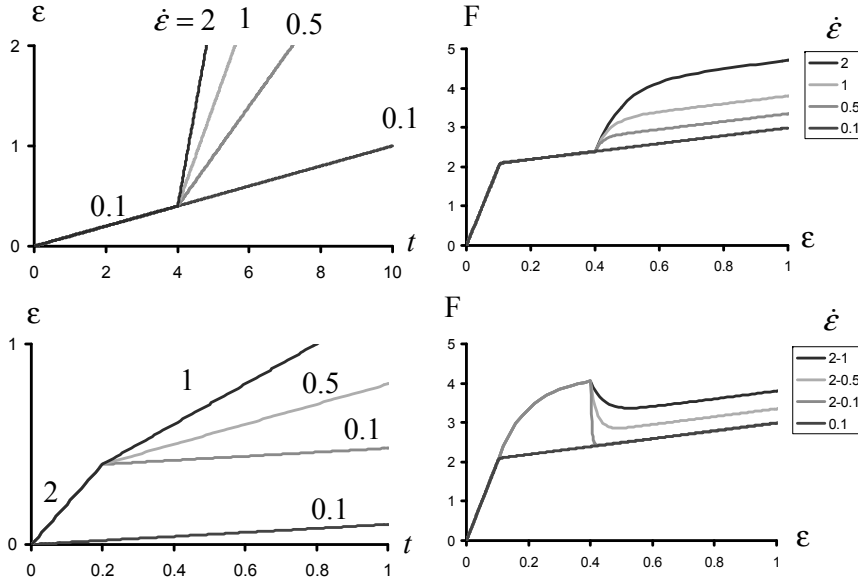


Figure 1.29: Effect of changing strain rate on elastic-viscoplastic response. Top: effect of increasing strain rate. Bottom: effect of reducing strain rate.

Study the effect of changing the power n on the observed response of the material for different monotonic strain rates.

1.13 Load control as opposed to strain control

Not all loading conditions are strain control, even though strain and temperature always can be used as the independent variables. In practice sometimes load is controlled, sometimes strain is controlled, and sometimes other parameters are controlled. In conditions where load is controlled, it is easier to look at the model from the viewpoint of load and temperature being the independent variables. The only problem that might arise in doing so occurs when strain must change under conditions of constant load, as would happen in perfect plasticity. This creates little problem if one shifts from load control to strain control when needed.

As before, we consider a theory described by an equation for load in terms of the position in the reference configuration, total strain, plastic strain, thermal strain, temperature, and the hardening parameter. This can be written as

$$F = F^\dagger(X, \epsilon, \epsilon^p, \epsilon^\theta, \theta, \xi). \quad (1.154)$$

For such a case, the rate of load is given by

$$\dot{F} = \frac{\partial F}{\partial \epsilon} \dot{\epsilon} + \frac{\partial F}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial F}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial F}{\partial \theta} \dot{\theta} + \frac{\partial F}{\partial \xi} \dot{\xi}. \quad (1.155)$$

Since load is now the independent variable, we solve for $\dot{\epsilon}$ to get

$$\dot{\epsilon} = \frac{1}{\frac{\partial F}{\partial \epsilon}} \left(\dot{F} - \frac{\partial F}{\partial \epsilon^p} \dot{\epsilon}^p - \frac{\partial F}{\partial \epsilon^\theta} \dot{\epsilon}^\theta - \frac{\partial F}{\partial \theta} \dot{\theta} - \frac{\partial F}{\partial \xi} \dot{\xi} \right). \quad (1.156)$$

This can be integrated to obtain the current value of ϵ , the only problem being when $\frac{\partial F}{\partial \epsilon} = 0$, which occurs when the strain can change without changing the load under the conditions of constant plastic strain, thermal strain, temperature, and hardening. This is highly unlikely since this can only happen when the isothermal elastic mechanical response has a zero modulus.

Next, let us look at a yield function that depends on the position in the reference configuration, load, temperature, plastic strain, thermal strain and a hardening parameter. We will write this as

$$f = f^\dagger(X, F, \theta, \epsilon^p, \epsilon^\theta, \xi). \quad (1.157)$$

The rate of change of the yield function can be written as

$$\dot{f} = \frac{\partial f}{\partial F} \dot{F} + \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial f}{\partial \xi} \dot{\xi}. \quad (1.158)$$

For given $\dot{F}$ and $\dot{\theta}$, thermoelastic response occurs with $\dot{\epsilon}^p = \dot{\xi} = 0$ and $\dot{\epsilon}^\theta = \alpha \dot{\theta}$, and yields

$$\dot{f} = \frac{\partial f}{\partial F} \dot{F} + \left(\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \epsilon^\theta} \alpha \right) \dot{\theta}. \quad (1.159)$$

The condition for plastic flow, as before, is given by $f = 0$ and $\dot{f} > 0$.

The plastic flow rule for load control can be written as

$$\dot{\epsilon}^p = \begin{cases} \beta_{\dot{F}} \dot{F} + \beta_{\dot{\theta}} \dot{\theta} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (1.160)$$

The two coefficients $\beta_{\dot{F}}$ and $\beta_{\dot{\theta}}$ need to be selected so as to satisfy the consistency condition $\dot{f} = 0$. Let us now look at evaluating the forms of these two coefficients. The consistency condition can be written as

$$\frac{\partial f}{\partial F} \dot{F} + \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial f}{\partial \xi} \frac{d\xi}{d\epsilon^p} \dot{\epsilon}^p = 0,$$

where we have taken $\dot{\xi} = \frac{d\xi}{d\epsilon^p} \dot{\epsilon}^p$ following the assumption that the hardening parameter can be directly evaluated from the rate of plastic strain. Solving for $\dot{\epsilon}^p$ yields

$$\dot{\epsilon}^p = - \frac{\frac{\partial f}{\partial F} \dot{F} + \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}^\theta}{\frac{\partial f}{\partial \epsilon^p} + \frac{\partial f}{\partial \xi} \frac{d\xi}{d\epsilon^p}}. \quad (1.161)$$

Comparison of this expression with that given for the flow rule above yields

$$\beta_{\dot{F}} = - \frac{\frac{\partial f}{\partial F}}{\frac{\partial f}{\partial \epsilon^p} + \frac{\partial f}{\partial \xi} \frac{d\xi}{d\epsilon^p}}, \quad \beta_{\dot{\theta}} = - \frac{\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \epsilon^\theta} \alpha}{\frac{\partial f}{\partial \epsilon^p} + \frac{\partial f}{\partial \xi} \frac{d\xi}{d\epsilon^p}}. \quad (1.162)$$

1.14 Numerical integration of constitutive equations

The complicated nature of the constitutive models for plasticity makes analytical evaluations of these models for most loading conditions cumbersome. Therefore, it is desirable to be able to set up algorithms for calculating these responses numerically. In this section, we will look at how load can be calculated numerically given the history of total strain and, when needed, the history of temperature. Therefore, we take the input to be the functions $\epsilon(t)$ and $\theta(t)$, and the output response will be the load $F(t)$. Three basic methods will be considered here. The first is a direct integration method using the given models. This is a direct implementation of the analytical methods. As will be seen, for this numerical method to provide consistently good results, very small time increments need to be taken. On the other hand, the second and third methods, based on minimizing the error, are capable of obtaining very accurate results, even with fairly large time steps.

In what follows, since both time and temperature might be considered, time is used as the common parameter between all variables. As is common in the description of numerical procedures, the subscript “ i ” will be used to indicate the current value of a variable, the subscript “ $i - 1$ ” will be used to denote the value of a variable in the previous step, and the subscript “ $i + 1$ ” will be used to describe the value of a variable in the next step. The index i will be incremented each time the algorithm is completed, and the process starts again from the beginning. For example, for the current time t_i and the current increment of time Δt_i , we have $t_{i+1} = t_i + \Delta t_i$, or for the previous step we would have had $t_i = t_{i-1} + \Delta t_{i-1}$. To evaluate the associated changes of strain and temperature in an increment of time, the following relations will be used:

$$\begin{aligned}\Delta \epsilon_i &= \epsilon_{i+1} - \epsilon_i = \epsilon(t_i + \Delta t_i) - \epsilon(t_i) \approx \dot{\epsilon}_i \Delta t_i, \\ \Delta \theta_i &= \theta_{i+1} - \theta_i = \theta(t_i + \Delta t_i) - \theta(t_i) \approx \dot{\theta}_i \Delta t_i.\end{aligned}$$

Obviously, the validity of the approximations is related to how small Δt_i is in relation to how close to linear are the functions $\epsilon(t)$ and $\theta(t)$. Sometimes Δt will be used in place of Δt_i , particularly when Δt_i is constant.

1.14.1 Rate-independent and temperature-independent plasticity

In temperature-independent and rate-independent plasticity, the input is the strain history given by a function $\epsilon(t)$, where t is a monotonic parameter such as time that is used to describe the sequence of events. For simplicity, we will consider t to be time, even though it may be any monotonic variable. As an example, let us examine linear isotropic hardening, using the strain hardening rule and a constant elastic modulus. As the reader recalls, the

model is given as

$$F = AE\epsilon^e = AE(\epsilon - \epsilon^p),$$

$$f = |F| - F_y,$$

$$F_y = F_{yo} + \frac{AaE}{E-a}\xi,$$

$$\overset{\diamond}{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases}$$

$$\dot{\epsilon}^p = \begin{cases} \frac{E-a}{E}\dot{\epsilon} & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ 0 & \text{for all other cases,} \end{cases}$$

$$\dot{\xi} = |\dot{\epsilon}^p|.$$

For this model, the input parameters are

$$A, \quad E, \quad F_{yo}, \quad a.$$

The initial conditions are

$$t = 0, \quad \epsilon^p = 0, \quad \xi = 0.$$

The algorithm to calculate the load as a function of time (or strain) for either fixed or variable Δt with initial conditions $t_1 = 0$, $\epsilon_1^p = 0$, and $\xi_1 = 0$ will be as follows:

1. Calculate the current strain $\epsilon_i = \epsilon(t_i)$.
2. Calculate the current load $F_i = AE(\epsilon_i - \epsilon_i^p)$.
3. Store t_i , ϵ_i , and F_i .
4. Calculate the yield load $F_y = F_{yo} + \frac{AaE}{E-a}\xi_i$.
5. Calculate the yield function $f = |F_i| - F_y$.
6. If $f \geq 0$, calculate $\overset{\diamond}{f} = \text{sign}(F_i)AE\dot{\epsilon}_i$.
7. Update the plastic strain and hardening parameter

- If $f \geq 0$ and $\overset{\diamond}{f} > 0$, calculate $\dot{\epsilon}_i^p = \frac{E-a}{E}\dot{\epsilon}_i$ and

$$\begin{aligned} \Delta\epsilon^p &= \dot{\epsilon}_i^p \Delta t, & \epsilon_{i+1}^p &= \epsilon_i^p + \Delta\epsilon^p, \\ \Delta\xi &= |\Delta\epsilon^p|, & \xi_{i+1} &= \xi_i + \Delta\xi. \end{aligned}$$

- Else

$$\begin{aligned} \epsilon_{i+1}^p &= \epsilon_i^p, \\ \xi_{i+1} &= \xi_i. \end{aligned}$$

8. Increment time

$$t_{i+1} = t_i + \Delta t.$$

9. Increment i and return to the first step.

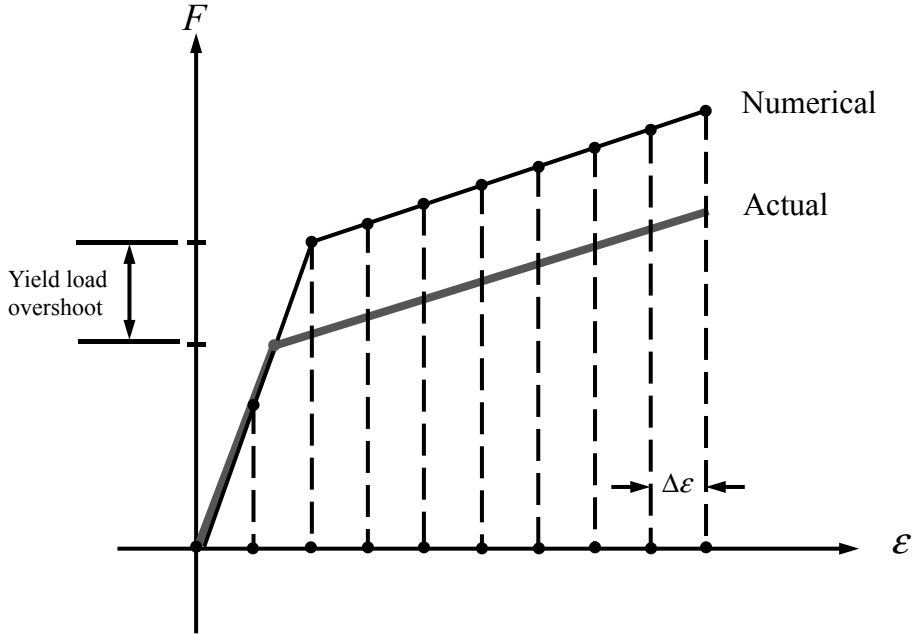


Figure 1.30: Overshoot of yield load due to large strain increments.

We have made at least one alteration to the analytical process that was described before. The reader will note that the condition of yield has changed from $f = 0$ and $\dot{f} > 0$ to $f \geq 0$ and $\dot{f} > 0$. This change is to avoid any confusion in the algorithm if the load overshoots the yield surface during a given increment. If the load does overshoot the yield surface, this algorithm will not return the load onto the yield surface, resulting in a propagating error in the form of a vertical shift of the plastic loading portion of the response, as shown in Figure 1.30. The amount of this error can be reduced simply by reducing the step size (i.e., reducing Δt). Therefore, this algorithm converges onto the analytical response function as the step size approaches zero. Unfortunately, the small increment size needed to avoid substantial overshooting makes this algorithm less than ideal for practical use. The following two methods provide solutions to this problem, each utilizing a slightly different idea.

One solution to the problem of the last algorithm is the use of adaptive step sizing. This can be achieved by many different methods. The method we will look at is based on returning the yield function to the yield surface ($f = 0$) through elastic unloading. We will call this the *elastic return* algorithm. For pure isotropic hardening the yield function is given as $f = |F| - F_y = |AE(\epsilon - \epsilon^p)| - F_y$. For elastic response, the plastic strain and hardening parameter are fixed so that a first-order Taylor series expansion would be given by

$$f(\epsilon + \Delta\epsilon) = f(\epsilon) + \frac{\partial f}{\partial \epsilon} \Delta\epsilon.$$

We now force $f(\epsilon + \Delta\epsilon) = 0$, setting up what is known as a Newton step, and solve for the increment of strain to get

$$\Delta\epsilon = -\frac{f(\epsilon)}{\frac{\partial f}{\partial \epsilon}} = -\text{sign}(F) \frac{f}{AE}.$$

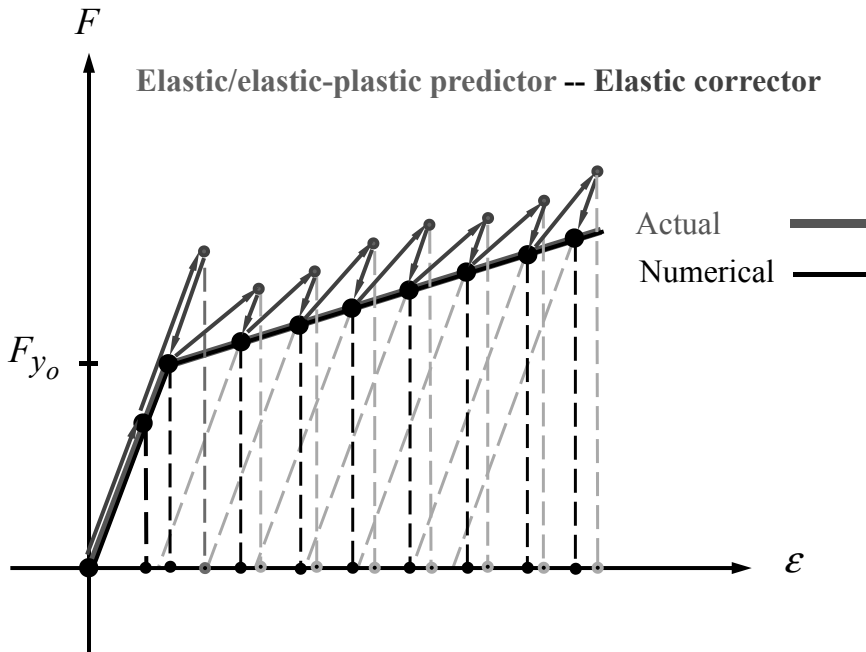


Figure 1.31: Elastic return algorithm based on an elastic/elastic-plastic predictor and an elastic corrector used to bring the response back onto the yield surface (only during plastic flow).

This one additional step, when needed, can be repeated until the yield function is as close to zero as desired. The time increment can be calculated for this step from the relation $\Delta\epsilon = \dot{\epsilon}\Delta t$. Let us require that at all times $f < m$ for m being a small positive number. As a practical point, numerically, we consider $f = 0$ to be satisfied when $-m < f < m$. The algorithm for this process can now be written as

1. Calculate the current strain $\epsilon_i = \epsilon(t_i)$.
2. Calculate the current strain rate $\dot{\epsilon}_i = \dot{\epsilon}(t_i)$.
3. Calculate the current load $F_i = AE(\epsilon_i - \epsilon_i^p)$.
4. Calculate the yield load $F_y = F_{y0} + \frac{AaE}{E-a}\xi_i$.
5. Calculate the yield function $f = |F_i| - F_y$.
6. If $f \geq m$,
 - Reduce the time increment by elastic unloading onto the yield surface using

$$\Delta t_c = -\text{sign}(F_i) \frac{f}{AE\dot{\epsilon}_{i-1}}.$$

- Update the time to reflect the smaller step

$$t_i = t_i + \Delta t_c.$$

- Calculate the current strain $\epsilon_i = \epsilon(t_i)$.
 - Calculate the current strain rate $\dot{\epsilon}_i = \dot{\epsilon}(t_i)$.
 - Calculate the current load $F_i = A([\epsilon_i - \epsilon_i^p])$.
 - Calculate the yield load $F_y = F_{yo} + \frac{AaE}{E-a}\xi_i$.
 - Calculate the yield function $f = |F_i| - F_y$.
 - If $|f| \geq m$, repeat this elastic correction again.
7. Store t_i , ϵ_i , and F_i .
8. If $f \geq -m$, calculate $\overset{\circ}{f} = \text{sign}(F_i)AE\dot{\epsilon}_i$.
9. Update the plastic strain and hardening parameter.
- If $f \geq -m$ and $\overset{\circ}{f} > 0$, calculate $\dot{\epsilon}_i^p = \frac{E-a}{E}\dot{\epsilon}_i$ and

$$\begin{aligned}\Delta\epsilon^p &= \dot{\epsilon}_i^p \Delta t, & \epsilon_{i+1}^p &= \epsilon_i^p + \Delta\epsilon^p, \\ \Delta\xi &= |\Delta\epsilon^p|, & \xi_{i+1} &= \xi_i + \Delta\xi.\end{aligned}$$

- Else

$$\begin{aligned}\epsilon_{i+1}^p &= \epsilon_i^p, \\ \xi_{i+1} &= \xi_i.\end{aligned}$$

10. Increment time

$$t_{i+1} = t_i + \Delta t.$$

11. Increment i and return to first step.

A description of this algorithm is shown in Figure 1.31.

A second solution to the problem faced in the first algorithm is to select the plastic strain increment in a way not to allow the yield function in plastic flow to move away from the yield surface. We will call this the *plastic return* algorithm. The idea is to take a fully elastic step and if the yield function becomes positive, to then change the plastic strain until the yield function returns to zero, as shown in Figure 1.32. Again, the idea of the Newton method is used. We expand the yield function in a Taylor series holding the total strain fixed to get

$$f(\epsilon^p + \Delta\epsilon^p) = f(\epsilon^p) + \frac{\partial f}{\partial \epsilon^p} \Delta\epsilon^p.$$

Setting the new value of the yield function equal to zero and solving for the increment of plastic strain we get

$$\Delta\epsilon^p = -\frac{f(\epsilon^p)}{\frac{\partial f}{\partial \epsilon^p}}.$$

In the spirit of the Newton method, we repeat this step until the yield function is sufficiently close to zero. We consider f to be equal to zero when $|f| < m$ for a given small positive number m . The algorithm for this method can be written as

1. Calculate the new strain $\epsilon_{i+1} = \epsilon(t_i + \Delta t)$.
2. Calculate the current strain rate $\dot{\epsilon}_i = \dot{\epsilon}(t_i)$.
3. Calculate the new load based on an elastic increment $F_{i+1} = AE(\epsilon_{i+1} - \epsilon_i^p)$.
4. Calculate the yield load $F_y = F_{yo} + \frac{AaE}{E-a}\xi_i$.

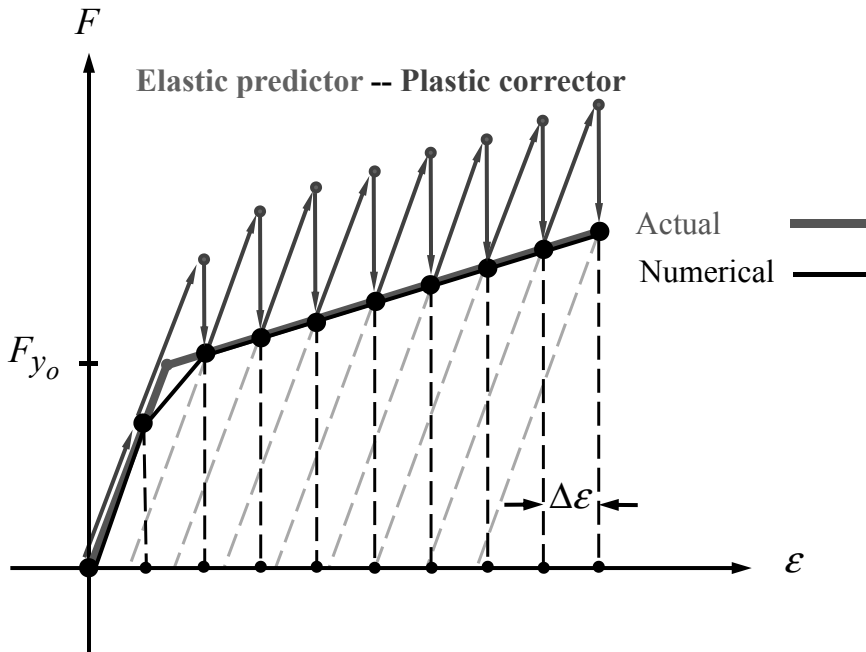


Figure 1.32: Plastic return algorithm based on an elastic predictor and plastic corrector used to keep the response on the yield surface.

5. Calculate the yield function $f = |F_{i+1}| - F_y$.

6. If $f \leq m$, the step is elastic. Set

$$\begin{aligned}\epsilon_{i+1}^p &= \epsilon_i^p, \\ \xi_{i+1} &= \xi_i.\end{aligned}$$

7. If $f > m$, initiate plastic flow:

- Calculate the plastic strain for the Newton step

$$\Delta\epsilon^p = \text{sign}(F_i) \frac{(E - a)f}{AE^2}.$$

- Update the plastic strain and hardening parameter

$$\begin{aligned}\epsilon_{i+1}^p &= \epsilon_i^p + \Delta\epsilon^p, \\ \xi_{i+1} &= \xi_i + |\Delta\epsilon^p|.\end{aligned}$$

- Calculate the load $F_{i+1} = AE(\epsilon_{i+1} - \epsilon_{i+1}^p)$.

- Calculate the yield load $F_y = F_{y0} + \frac{AaE}{E-a}\xi_{i+1}$.

- Calculate the yield function $f = |F_{i+1}| - F_y$.

- If $|f| < m$, continue to the next step; otherwise, repeat this Newton step, but first set

$$\begin{aligned}\epsilon_i^p &= \epsilon_{i+1}^p, \\ \xi_i^p &= \xi_{i+1}^p.\end{aligned}$$

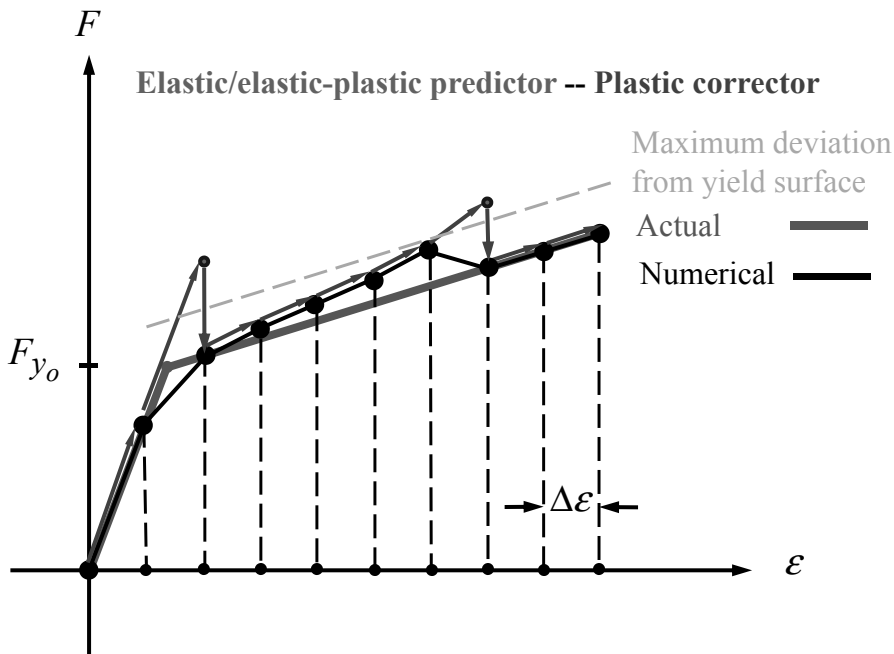


Figure 1.33: Plastic return algorithm based on an elastic/elastic-plastic predictor and plastic corrector used only when response moves outside a given range of the yield surface.

8. Increment time

$$t_{i+1} = t_i + \Delta t.$$

9. Store t_{i+1} , ϵ_{i+1} , and F_{i+1} .

10. Increment i and return to the first step.

Figure 1.33 shows a variation of the plastic return algorithm which uses an elastic or elastic-plastic step based on the condition $f \geq 0$ and $\dot{f} > 0$ for the elastic-plastic step. This part is identical with the first algorithm. The difference is that if the response moves beyond a given range of the yield surface during plastic flow, the plastic strain is modified based on a Newton method to bring the response back onto the yield surface.

The algorithms shown have all dealt with errors that can occur during plastic flow. In particular, we have presented methods to address the yield overshoot and the problem of drifting away from the yield surface during plastic flow. Another set of problems may arise when unloading from the yield surface. An example of such a problem is shown in Figure 1.34. As shown in the figure, a substantially larger load is predicted by the numerical approximation than actually would be seen. The problem is a result of missing the actual point of unloading M . One solution is to use extremely small increments. This problem is exaggerated when the slope of the elastic response is substantially larger than the slope during plastic flow. Since this is normally the case, a method needs to be introduced to catch the true initial unloading point M . This can be done by using any algorithm that will look for the initial point that $\dot{f}$ changes sign from positive to negative or becomes zero. The

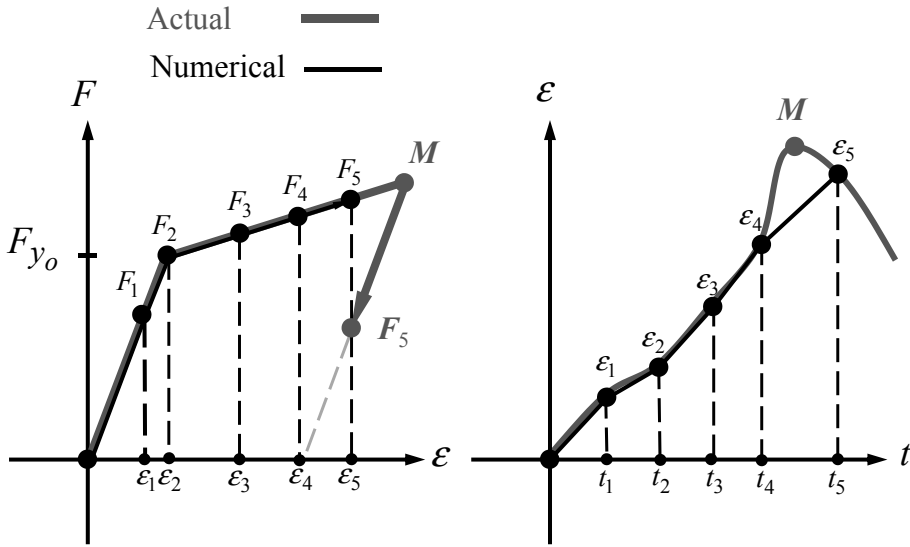


Figure 1.34: Error introduced when the true unloading point is not accurately identified.

algorithm would only be initiated when a step starts with $f = 0$ and $\dot{f} > 0$, but ends with $\dot{f} < 0$.

1.14.2 Rate-independent and temperature-dependent plasticity

Let us consider general thermoplasticity for the case that our inputs are $\epsilon(t)$ and $\theta(t)$ (functions for strain and temperature). From these functions, it is assumed that we can calculate the associated rates $\dot{\epsilon}(t)$ and $\dot{\theta}(t)$. The initial conditions are

$$t = 0, \quad \epsilon^p = 0, \quad \xi = 0, \quad F_{iso} = F_{yo}, \quad F_c = 0.$$

The algorithm used to integrate the response is given by:

1. Calculate the current strain and temperature $\epsilon_i = \epsilon(t_i)$ and $\theta_i = \theta(t_i)$.
2. Calculate the current values of the moduli

$$\begin{aligned} E &= E(\theta_i), & \frac{dE}{d\theta} &= \frac{dE}{d\theta}(\theta_i), \\ \frac{\partial F_{iso}}{\partial \theta} &= \frac{\partial F_{iso}}{\partial \theta}(\xi_i, \theta_i), & \frac{\partial F_{iso}}{\partial \epsilon^p} &= \frac{\partial F_{iso}}{\partial \epsilon^p}(\xi_i, \theta_i), \\ \frac{\partial F_c}{\partial \theta} &= \frac{\partial F_c}{\partial \theta}(\epsilon_i^p, \theta_i), & \frac{\partial F_c}{\partial \epsilon^p} &= \frac{\partial F_c}{\partial \epsilon^p}(\epsilon_i^p, \theta_i). \end{aligned}$$

3. Calculate the current load from

$$F_i = AE(\epsilon_i - \epsilon_i^p - \epsilon_i^\theta).$$

4. Store t_i , ϵ_i , θ_i , and F_i .

5. Calculate the yield function $f = |F_i - F_{c_i}| - F_{iso_i}$.
6. Calculate the current strain rate and temperature rate $\dot{\epsilon}_i = \dot{\epsilon}(t_i)$ and $\dot{\theta}_i = \dot{\theta}(t_i)$.
7. Calculate the rate of thermal expansion

$$\dot{\epsilon}_i^\theta = \alpha(\theta_i)\dot{\theta}_i.$$

8. If $f \geq 0$, calculate $\dot{f}$ from

$$\dot{f} = \text{sign}(F_i - F_{c_i}) \left[AE(\dot{\epsilon}_i - \alpha\dot{\theta}_i) + A \frac{dE}{d\theta} (\epsilon_i - \epsilon_i^p - \epsilon_i^\theta) \dot{\theta}_i - \frac{\partial F_c}{\partial \theta} \dot{\theta}_i \right] - \frac{\partial F_{iso}}{\partial \theta} \dot{\theta}_i.$$

9. Calculate the plastic strain rate and the rate of change of the hardening parameter

- If $f \geq 0$ and $\dot{f} > 0$, then

$$\dot{\epsilon}_i^p = \frac{AE\dot{\epsilon}_i + [-AE\alpha + A \frac{dE}{d\theta} (\epsilon_i - \epsilon_i^p - \epsilon_i^\theta) - \frac{\partial F_c}{\partial \theta} - \text{sign}(F_i - F_{c_i}) \frac{\partial F_{iso}}{\partial \theta}] \dot{\theta}_i}{(AE + \frac{\partial F_c}{\partial \epsilon^p} + \frac{\partial F_{iso}}{\partial \xi})},$$

$$\dot{\xi}_i = \frac{d\xi}{d\epsilon^p} \dot{\epsilon}_i^p.$$

- Else

$$\dot{\epsilon}_i^p = 0, \quad \dot{\xi}_i = 0.$$

10. Update the variables

$$\begin{aligned} \epsilon_{i+1}^p &= \epsilon_i^p + \dot{\epsilon}_i^p \Delta t, \quad \xi_{i+1} = \xi_i + \dot{\xi}_i \Delta t, \\ F_{c_{i+1}} &= F_{c_i} + \frac{\partial F_c}{\partial \epsilon^p} \dot{\epsilon}_i^p \Delta t + \frac{\partial F_c}{\partial \theta} \dot{\theta}_i \Delta t, \quad F_{iso_{i+1}} = F_{iso_i} + \frac{\partial F_{iso}}{\partial \xi} \dot{\xi}_i \Delta t + \frac{\partial F_{iso}}{\partial \theta} \dot{\theta}_i \Delta t, \\ t_{i+1} &= t_i + \Delta t. \end{aligned}$$

11. Increment i and return to first step.

As stated in the last subsection, the algorithm shown here will converge to the exact response once the step size is reduced to zero. Therefore, the algorithm is theoretically good, yet it suffers from the need to use very small steps, particularly when the slope of the elastic response is much larger than the slope of the plastic response. The reason for this is that the elastic response must control the step size so that the yield point is not substantially overshoot. As can be seen from Figure 1.30, if the step size is too large, the algorithm overshoots the yield load by a considerable amount and since the algorithm has no way of returning the load back onto the yield surface, the overshoot becomes a permanent error for the remainder of the loading process. Also, as shown in Figure 1.34, a large step size also might introduce large errors during unloading because of missing the actual point of unloading. As before, the proposed algorithm can be modified to eliminate these problems. The introduction of a thermoelastic/thermoelastic-plastic predictor and a thermoelastic corrector would create an algorithm consistent with that of the elastic return algorithm described above (in this case, thermoelastic return). The introduction of a thermoelastic predictor and a plastic corrector, when needed, would create an algorithm consistent with the plastic return algorithm. These two modifications would address the overshoot of the yield surface. To address the problems that can occur at unloading, as stated before, any program that can search to find the actual point of unloading would solve the problem. One idea would be to look at $\dot{f}$ and search for the point at which it changes from being strictly positive to being zero or negative. Obviously, this would be done only in a step that starts with $f \geq 0$ and $\dot{f} > 0$, and ends with $\dot{f} < 0$.

1.14.3 Load control as opposed to strain control

When load is the control parameter, one can directly develop the model based on load and temperature as the independent variables. We can put the steps to calculate the response into a numerical algorithm with the initial conditions

$$t = 0, \quad \epsilon = \epsilon_o, \quad \epsilon^p = 0, \quad \epsilon^\theta = 0, \quad \xi = 0.$$

It is assumed that the load and temperature histories are given by functions $F(t)$ and $\theta(t)$, from which we can extract $\dot{F}(t)$ and $\dot{\theta}(t)$ as needed, and in which $F(0)$ is consistent with the initial values indicated. The algorithm used to integrate the response is given by

1. Calculate the current load and temperature $F_i = F(t_i)$ and $\theta_i = \theta(t_i)$.
2. Calculate the current load rate and temperature rate $\dot{F}_i = \dot{F}(t_i)$ and $\dot{\theta}_i = \dot{\theta}(t_i)$.
3. Calculate the rate of thermal expansion

$$\dot{\epsilon}_i^\theta = \alpha \dot{\theta}_i.$$

4. Calculate the yield function f .
5. If $f \geq 0$, calculate $\overset{\diamond}{f}$ from

$$\overset{\diamond}{f} = \frac{\partial f}{\partial F} \dot{F}_i + \frac{\partial f}{\partial \theta} \dot{\theta}_i + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}_i^\theta.$$

6. Calculate the plastic strain rate and hardening rate:

- If $f \geq 0$ and $\overset{\diamond}{f} > 0$, then

$$\dot{\epsilon}_i^p = - \frac{\frac{\partial f}{\partial F} \dot{F}_i + \frac{\partial f}{\partial \theta} \dot{\theta}_i + \frac{\partial f}{\partial \epsilon^\theta} \dot{\epsilon}_i^\theta}{\frac{\partial f}{\partial \epsilon^p} + \frac{\partial f}{\partial \xi} \frac{d\xi}{d\epsilon^p}}, \quad \dot{\xi}_i = \frac{d\xi}{d\epsilon^p} \dot{\epsilon}_i^p$$

- Else

$$\dot{\epsilon}_i^p = 0, \quad \dot{\xi}_i = 0.$$

7. Calculate the strain rate

$$\dot{\epsilon}_i = \frac{1}{\frac{\partial F}{\partial \epsilon}} \left(\dot{F}_i - \frac{\partial F}{\partial \epsilon^p} \dot{\epsilon}_i^p - \frac{\partial F}{\partial \epsilon^\theta} \dot{\epsilon}_i^\theta - \frac{\partial F}{\partial \theta} \dot{\theta}_i - \frac{\partial F}{\partial \xi} \dot{\xi}_i \right).$$

8. Update the strain, plastic strain, hardening parameter, and time

$$\epsilon_{i+1} = \epsilon_i + \dot{\epsilon}_i \Delta t, \quad \epsilon_{i+1}^p = \epsilon_i^p + \dot{\epsilon}_i^p \Delta t, \quad \xi_{i+1} = \xi_i + \dot{\xi}_i \Delta t, \quad t_{i+1} = t_i + \Delta t.$$

9. Increment i and return to the first step.

This algorithm will converge to the exact response as the step size is reduced to zero, but one can modify it so that it will converge to the exact response even without small step sizes. As described above, the concepts behind the elastic return and plastic return algorithms can be used to eliminate overshooting of the yield load, and one can introduce an algorithm to find the point of load reversal to eliminate problems associated with inaccurate capturing of the unloading point. These modifications are left as homework.

1.14.4 Rate-dependent plasticity

The algorithm used for rate-dependent plasticity is similar to the first algorithm used for rate-independent plasticity with the removal of the need for calculation of $\dot{f}$. The initial conditions are

$$t = 0, \quad \epsilon^p = 0, \quad \xi = 0, \quad F_{iso} = F_{yo}, \quad F_c = 0.$$

The algorithm to integrate the response is given by:

1. Calculate the current strain $\epsilon_i = \epsilon(t_i)$.
2. Calculate the current load from

$$F_i = AE(\epsilon_i - \epsilon_i^p).$$

3. Store t_i , ϵ_i , and F_i .
4. Calculate the yield function $f = |F_i - F_{ci}| - F_{iso_i}$.
5. Calculate the current strain rate $\dot{\epsilon}_i = \dot{\epsilon}(t_i)$.
6. Update the plastic strain rate and hardening rate:

- If $f \geq 0$, then

$$\dot{\epsilon}_i^p = \text{sign}(F_i) \frac{f}{\mu}, \quad \dot{\xi}_i = \frac{d\xi}{d\epsilon^p} \dot{\epsilon}_i^p.$$

- Else

$$\dot{\epsilon}_i^p = 0, \quad \dot{\xi}_i = 0.$$

7. Update plastic strain, hardening parameter, etc.:

$$\begin{aligned} \epsilon_{i+1}^p &= \epsilon_i^p + \dot{\epsilon}_i^p \Delta t, & \xi_{i+1} &= \xi_i + \dot{\xi}_i \Delta t, \\ F_{c_{i+1}} &= F_{c_i} + \frac{dF_c}{d\epsilon^p} \dot{\epsilon}_i^p \Delta t, & F_{iso_{i+1}} &= F_{iso_i} + \frac{dF_{iso}}{d\xi} \dot{\xi}_i \Delta t, \\ t_{i+1} &= t_i + \Delta t. \end{aligned}$$

8. Increment i and return to the first step.

Section 1.14-Problems:

- 1.14-1:** Write out the algorithm for isotropic hardening using the plastic return method modified to capture the unloading point.
- 1.14-2:** Write out the algorithm for the kinematic hardening model, first without modifications, next with the elastic return method, and then with the plastic return method.
- 1.14-3:** Write out the algorithm for the combined isotropic and kinematic hardening model, first without modifications, next with the elastic return method, and then with the plastic return method.
- 1.14-4:** Write out the algorithm, including the thermoelastic return for the case of rate-independent thermoplasticity.
- 1.14-5:** Write out the algorithm, including the plastic return method for the case of rate-independent thermoplasticity.

- 1.14-6:** Write out the algorithm, including the plastic return method and the capturing of the unloading point for the case of rate-independent thermoplasticity.
- 1.14-7:** Write out the algorithm, including the elastic return method for the case of load control.
- 1.14-8:** Write out the algorithm, including the plastic return method for the case of load control.
- 1.14-9:** Write out the algorithm, including the plastic return method and the capturing of the unloading point for the case of load control.
- 1.14-10:** Set up a spreadsheet to implement each of the following algorithms:
1. The first algorithm developed for rate-independent and temperature-independent plasticity.
 2. The algorithm for temperature-dependent plasticity.
 3. The algorithm for load control.
 4. The algorithm for rate-dependent plasticity.
- 1.14-11:** Write a program to implement each of the following algorithms:
1. The first algorithm developed for rate-independent and temperature-independent plasticity.
 2. The algorithm for temperature-dependent plasticity.
 3. The algorithm for load control.
 4. The algorithm for rate-dependent plasticity.
- 1.14-12:** Write a program that implements the plastic return method for each of the following:
1. Rate-independent and temperature-independent plasticity.
 2. Temperature-dependent plasticity.
 3. The case of load control.
- 1.14-13:** Modify the programs developed in the previous problem to include the idea of finding the unloading point.
- 1.14-14:** Write a program based on the algorithm developed for rate-dependent plasticity and study the propagation of error by using this program with different step sizes and also with different loadings.

1.15 The balance laws

The conservation of mass, balance of linear momentum, balance of energy, and the entropy production inequality are the basic balance laws that we will consider. The balance laws are supposed to represent the collective experience of scientists and engineers over several centuries of studying how *all bodies* respond. There are aspects of the response of

bodies that are governed by what the body is made from and there are other aspects of the response of a body which are independent of what a body is made from. The balance laws fall into the latter category. They represent a set of laws (rules), which all material types follow.

The balance laws are most naturally written for fixed portions of matter. We already have developed all the necessary tools needed to keep track of a fixed set of particles on our bar.

The second thing to realize is that the laws of mechanics and thermodynamics must hold not only for the bar $\mathcal{B}$, but also for every open segment Ω of the bar. Since the bar itself can be considered its own largest segment, we will state the laws in terms of Ω .

The first law of continuum thermodynamics is the conservation of mass. The law of *conservation of mass* states that the mass of every segment $\Omega \subseteq \mathcal{B}$ remains constant during any process. That is, the total mass of the particles inside the segment of a bar described by Ω can never change. This is consistent with the Newtonian origins of our laws, which hold only for particles that have speeds far below that of light.

The second law of continuum thermodynamics is the *balance of linear momentum*. This law states that the total linear momentum of each segment of the bar changes at a rate equal to the total applied force on that same segment of the bar.

The third law of continuum thermodynamics for our bar is the balance of work and energy. The *balance of work and energy* states that the sum of the internal plus kinetic energy of each segment of the bar changes at a rate equal to the rate at which work is done on that same segment of the bar plus the rate at which heat is added to the same segment of the bar.

Finally, the fourth law of continuum thermodynamics for our bar is the *entropy production inequality*. This law states that the entropy of any segment of the bar increases at a rate equal to or faster than the rate at which entropy is being added to that same segment by heat.

Our next task will be to put these basic laws into a more mathematical form suitable for further analysis. We will do this by starting with the conservation of mass.

1.15.1 Calculating physical properties from their distributions

Mass, linear momentum, internal energy, kinetic energy, and entropy are physical properties of our bar. These properties are distributed throughout the particles of the bar. It is assumed that each of the distributed properties has a function ϕ that gives its value per unit mass of the bar, and that can be integrated over the bar to calculate its total value for the bar.

Given the distribution function per unit mass ϕ of any physical property, one can calculate how much of that property exists in any segment of the bar by integrating over the interval of that segment of the bar. Let us consider a segment of the bar contained between the points x_A and x_B in the current configuration (shown in Figure 1.35), and let Ω denote this segment of the bar. Any property Φ of the segment Ω can be calculated from its distribution function ϕ through the relation

$$\Phi = \int_{\Omega} \phi dm,$$

where dm is the differential element of mass. Since we are more familiar with integration over spatial variables, as opposed to mass, we can use mass density ρ , which is the mass per unit current length, to change the domain of integration to get

$$\Phi \equiv \int_{\Omega} \phi \rho dx = \int_{x_A}^{x_B} \phi \rho dx. \quad (1.163)$$

Let us first consider calculating the *mass* of a segment of our bar. It is obvious that the total mass M of Ω is given by

$$M \equiv \int_{\Omega} dm = \int_{x_A}^{x_B} \rho dx, \quad (1.164)$$

where in this case $\Phi = M$ and $\phi = 1$.

The *linear momentum* of each particle of the bar is given by its mass times its velocity. If the linear momentum of the segment Ω of the bar is denoted by P , then one can calculate it from

$$P \equiv \int_{\Omega} v dm = \int_{x_A}^{x_B} v \rho dx, \quad (1.165)$$

where in this case $\Phi = P$ and $\phi = v$.

Internal energy represents the amount of energy (excluding kinetic energy) existing in the material under consideration. The internal energy of any segment of the bar will be denoted by IE . The internal energy is calculated from the relation

$$IE \equiv \int_{\Omega} e dm = \int_{x_A}^{x_B} e \rho dx, \quad (1.166)$$

where in this case $\Phi = IE$, and $\phi = e$ is the specific internal energy (internal energy per unit mass).

The *kinetic energy* of each particle of the bar is given by one half the mass times its speed squared. If the kinetic energy of the segment Ω is denoted by KE , then it can be calculated from

$$KE \equiv \int_{\Omega} \frac{v^2}{2} dm = \int_{x_A}^{x_B} \frac{v^2}{2} \rho dx, \quad (1.167)$$

where in this case $\Phi = KE$, and $\phi = \frac{v^2}{2}$ is the specific kinetic energy (kinetic energy per unit mass).

The total *entropy* E of each segment Ω of the bar can be calculated from the relation

$$E \equiv \int_{\Omega} \eta dm = \int_{x_A}^{x_B} \eta \rho dx, \quad (1.168)$$

where in this case $\Phi = E$, and $\phi = \eta$ is the specific entropy (entropy per unit mass).

1.15.2 The material time derivative of physical properties

The balance laws are most naturally written for fixed portions of matter. Maybe an example from particle mechanics can better explain the idea. Newton's second law is a rule that holds for all particles (irrespective of what they are made from). It tells us how the forces acting on a particle result in the motion of that particle. This law only can be applied to one specific particle at a time.

Central to our discussion of the balance laws of thermodynamics is the ability to define the material time derivative of physical properties of the bar (or its segments). That is, the ability to follow changes that occur in a specified set of particles of the bar.

The first thing to do is to fix the matter under consideration. This is sometimes considered as establishing a *control mass*. The simplest way to do this is to identify a fixed segment of the bar in the reference configuration, and identify the particles in this segment as those we wish to write our laws for. Then, we can follow how the particles in this segment

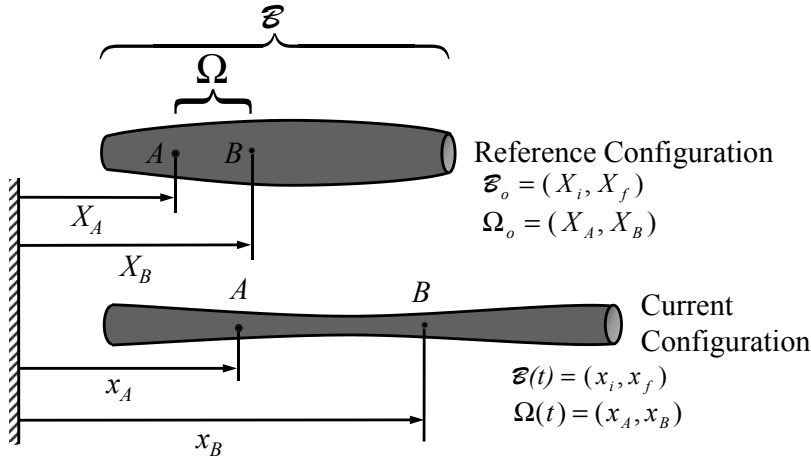


Figure 1.35: The bar $\mathcal{B}$ and a segment of the bar Ω . The line segments $\mathcal{B}_o$ and $\mathcal{B}(t)$ are occupied, respectively, by the bar in the reference configuration and in the current configuration. The line segments Ω_o and $\Omega(t)$ are occupied, respectively, by the bar segment Ω in the reference configuration and in the current configuration.

move and change temperature as the body is subjected to mechanical and thermal loads. The balance laws given in the next section will help us predict how these changes will occur. Figure 1.35 shows the reference configuration of our bar. The bar is contained in the interval between X_i and X_f . We can describe the bar by the interval it occupies $\mathcal{B}_o$, which is equal to the open interval (X_i, X_f) . Each particle of the bar is denoted by its location in the reference configuration, so that the particles of the bar are described by all $X \in \mathcal{B}_o = (X_i, X_f)$. A segment of the bar is an open interval Ω_o , which is contained in $\mathcal{B}_o$. If Ω_o represents the points between X_A and X_B , then $\Omega_o = (X_A, X_B)$.

In the current configuration the body is defined by the points between x_i and x_f . Let us represent the body in the current configuration by the interval $\mathcal{B}(t)$ of all points between x_i and x_f . The particles in $\mathcal{B}(t)$ are the same particles which were in $\mathcal{B}_o$, but their positions in space have changed. In a similar manner, the particles in the interval Ω_o are contained in the interval (x_A, x_B) in the current configuration. We will let $\Omega(t)$ denote the open interval (x_A, x_B) . The interval $\Omega(t)$ is the image of Ω_o , where the mapping from Ω_o onto $\Omega(t)$ is given by the function $x(X, t)$. Note that in the last section we used $\mathcal{B}$ and Ω to denote, respectively, the matter in the body and in a segment of the body. This should be contrasted with $\mathcal{B}_o$, $\mathcal{B}(t)$, Ω_o , and $\Omega(t)$, which are line segments.

As described in the last section, let us consider Φ which denotes a physical quantity (such as mass or momentum) of a given open segment of the body $\Omega(t)$, and let us denote by ϕ the distribution function of Φ per unit mass of the bar. The material time derivative of Φ is given by

$$\frac{D\Phi}{Dt} = \frac{D}{Dt} \left(\int_{\Omega(t)} \phi \rho dx \right),$$

where the reader recalls that D/Dt denotes the material time derivative. Using the mapping describing the motion, $x(t) = x(X, t)$, we can change the domain of integration to the

reference configuration and move D/Dt into the integral to get

$$\frac{D}{Dt} \left(\int_{\Omega(t)} \phi \rho dx \right) = \frac{D}{Dt} \left[\int_{\Omega_o} \phi \rho \frac{\partial x(X, t)}{\partial X} dX \right] = \int_{\Omega_o} \frac{D}{Dt} \left[\phi \rho \frac{\partial x(X, t)}{\partial X} \right] dX.$$

Changing the integral back to $\Omega(t)$ results in the relation

$$\boxed{\frac{D\Phi}{Dt} = \int_{\Omega(t)} \frac{D\phi\rho\lambda}{Dt} \lambda^{-1} dx,} \quad (1.169)$$

where the reader recalls that λ is the stretch ratio given by

$$\lambda(X, t) = \frac{\partial x(X, t)}{\partial X}.$$

1.15.3 Conservation of mass

Conservation of mass requires that the mass of every fixed amount of matter (control mass) remain constant over time. For any segment Ω of the bar $\mathcal{B}$, which currently occupies the line segment $\Omega(t)$, conservation of mass requires that

$$\boxed{\frac{DM}{Dt} = \frac{D}{Dt} \left(\int_{\Omega(t)} \rho dx \right) = 0.} \quad (1.170)$$

Setting $\Phi = M$ and $\phi = 1$, from (1.169) it follows that

$$\int_{\Omega(t)} \frac{D\rho\lambda}{Dt} \lambda^{-1} dx = 0, \quad \forall \Omega(t) \subseteq \mathcal{B}(t).$$

Since this integral must be zero over every possible segment of the bar, one can argue that if the argument of the integral is continuous, then it must be zero at each point of the bar. If this was not true, one could always devise a segment of the bar such that the argument of the integral is strictly positive, and the requirement that the integral be zero over every interval will be violated. Figure 1.36 shows the difference between a continuous and a discontinuous function and the idea behind the argument. The result of this argument is that we can write

$$\frac{D\rho\lambda}{Dt} = 0, \quad \forall X \in \mathcal{B}_o, \quad (1.171)$$

where λ^{-1} is removed from the argument since for physically realistic motions it is always positive. This is considered to be the “local” form of the law of conservation of mass. One can immediately integrate this to get

$$\rho\lambda = \text{constant in time}, \quad \forall X \in \mathcal{B}_o.$$

The constant of integration can be evaluated using the density of the bar in the reference configuration. If the density of the bar in the reference configuration is given by ρ_o , then the expression for the conservation of mass will be

$$\boxed{\rho\lambda = \rho_o, \quad \forall X \in \mathcal{B}_o,} \quad (1.172)$$

since the stretch ratio is unity for the reference configuration. Note that since ρ_o can be a function of location in the bar, X , and the stretch ratio λ can be both a function of position

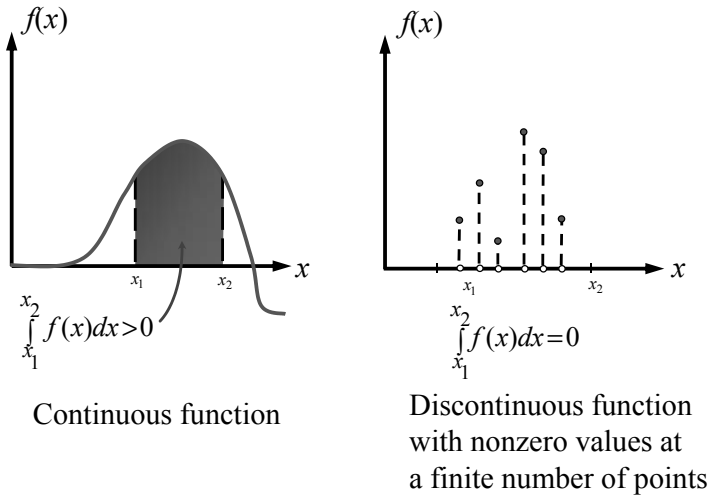


Figure 1.36: The difference between a continuous and discontinuous function. A discontinuous function can have nonzero values even though its integral is still zero. For a continuous function that has a nonzero value at one point, one always can find an interval over which the integral of the function is also nonzero.

X and time t , the current density ρ is a function of both position X and time t . Therefore, this law can be written in the explicit form

$$\rho(X, t)\lambda(X, t) = \rho_o(X), \quad \forall X \in \mathcal{B}_o.$$

A different approach for obtaining a relation from (1.171) uses the product rule to get

$$\frac{D\rho\lambda}{Dt} = \frac{D\rho}{Dt}\lambda + \rho\frac{D\lambda}{Dt}.$$

Substituting this into (1.171), reorganizing, and using a slight variation of the expression for velocity gradient derived above (i.e., $L = \frac{1}{\lambda}\frac{D\lambda}{Dt}$), we arrive at

$$\boxed{\frac{D\rho}{Dt} + \rho L = 0, \quad \forall x \in \mathcal{B}(t).} \quad (1.173)$$

This is the form of the law of conservation of mass, commonly used in fluid mechanics under the name “equation of continuity” (an odd name for conservation of mass!). If one were to assume that density and velocity are given in terms of spatial variables (x, t) , then one can use a procedure similar to that used to arrive at the expression for acceleration in (1.5) to get

$$\frac{D\rho}{Dt} + \rho L = \frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial x} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial x},$$

where we have used $L = \frac{\partial v}{\partial x}$. This yields another alternate form for the law of conservation of mass given as

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial x} = 0, \quad \forall x \in \mathcal{B}(t).} \quad (1.174)$$

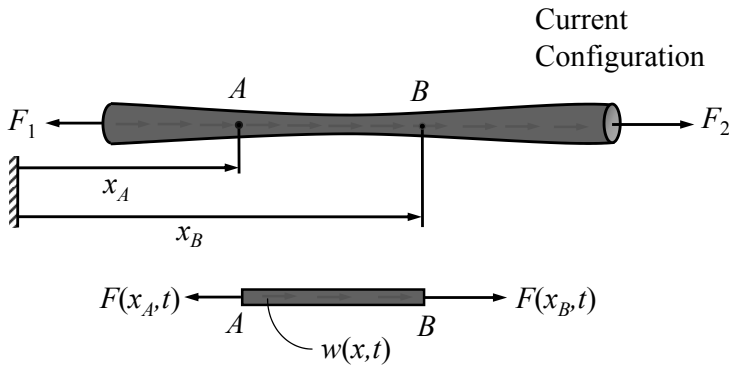


Figure 1.37: The loads applied on a straight bar undergoing a uniaxial deformation. Also shown are the loads applied on a segment of this bar.

An interesting consequence of the conservation of mass is that the product $\rho\lambda$ is not a function of time and, therefore, can be taken out of the material time derivative in (1.169) to get a new expression for this rule in the form

$$\boxed{\frac{D\Phi}{Dt} = \int_{\Omega(t)} \frac{D\phi}{Dt} \rho dx.} \quad (1.175)$$

1.15.4 Balance of linear momentum

The balance of linear momentum states that the sum of all forces applied on any segment of the bar is equal to the rate of change of linear momentum of that segment. If $\sum F$ denotes the sum of all forces applied on a given segment and if P denotes the segment's linear momentum, then this law states that

$$\boxed{\sum F = \frac{DP}{Dt}.} \quad (1.176)$$

Figure 1.37 shows the model we will use to characterize the loads applied to our bar or to a segment. The figure shows the bar in its current configuration. The force transmitted through the bar is denoted by $F(x, t)$ and has positive value as shown in the figure (i.e., tension). There is traction on the lateral surfaces of the bar. This traction is a distributed force parallel to the axis of the bar, and is denoted by w . This distributed force w is given in terms of the load per unit length of the bar in the current configuration. Therefore, the total axial load on a segment Ω of the bar at time t , taking forces directed to the right as positive, is given by

$$\boxed{\sum F \equiv F(x_B, t) - F(x_A, t) + \int_{\Omega(t)} w(x, t) dx.} \quad (1.177)$$

We now can write the law of balance of linear momentum for our bar as

$$F(x_B, t) - F(x_A, t) + \int_{\Omega(t)} w(x, t) dx = \frac{D}{Dt} \left[\int_{\Omega(t)} v(x, t) \rho(x, t) dx \right], \quad \forall \Omega(t) \subseteq \mathcal{B}(t). \quad (1.178)$$

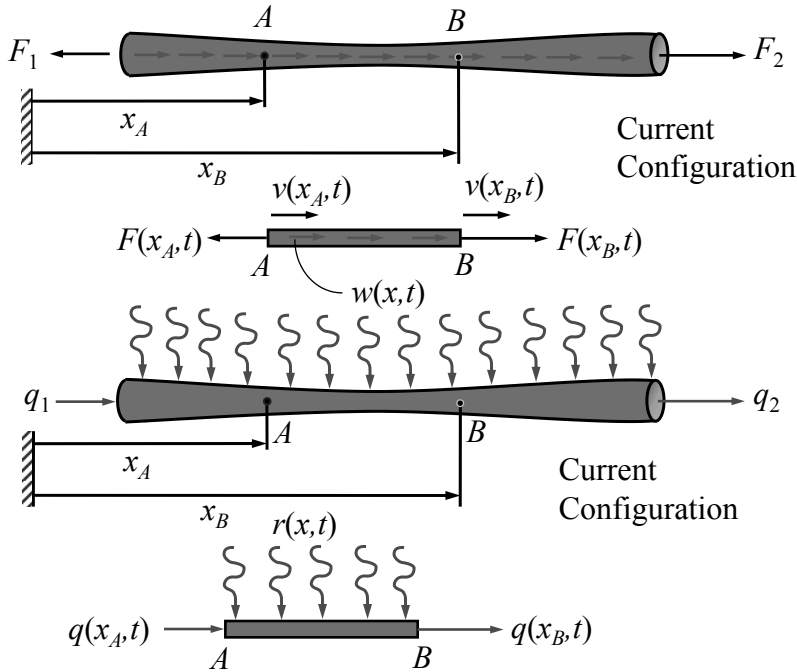


Figure 1.38: The work done on a segment of the bar and the heat flow into the same segment.

This equation is known as the integral form of the law of balance of linear momentum.

Assuming the axial force $F(x, t)$ is continuous, and that the arguments of the integrals are continuous, it can be shown that

$$\boxed{\frac{\partial F(x, t)}{\partial x} + w(x, t) = \rho(x, t)a(x, t), \quad \forall x \in \mathcal{B}(t).} \quad (1.179)$$

We will write this as

$$\frac{\partial F}{\partial x} + w = \rho a, \quad \forall x \in \mathcal{B}(t),$$

where the dependence on (x, t) is implied. An alternate form for the local form of the balance of linear momentum can be written as

$$\boxed{\frac{\partial F}{\partial X} + w_o = \rho_o a, \quad \forall X \in \mathcal{B}_o,} \quad (1.180)$$

where F , w_o , and a depend on (X, t) , F is the load in the current configuration written in terms of material variables, ρ_o is the density in the reference configuration and as such depends only on X , and $w_o = \lambda w$ is the distributed load in the current configuration evaluated per unit length in the reference configuration. The proof of this is left as a homework problem.

1.15.5 Balance of work and energy

The balance of energy states that the energy of any segment of the body is controlled by the work being done on that segment and by the heat flowing into it. If $\dot{W}$ denotes the

work being done on a segment, $\dot{Q}$ denotes the heat flowing into the segment, IE denotes the internal energy of the segment, and KE denotes the kinetic energy of the segment, then the law of balance of work and energy states that

$$\dot{W} + \dot{Q} = \frac{D}{Dt}(IE + KE). \quad (1.181)$$

Figure 1.38 shows the mechanical forces that can do work on the bar and also on the sources of heat into it. Work can be done by the axial loads and the traction applied on the lateral surfaces of the bar. The rate of doing work for each force is its power, which one can obtain by the dot product of the force into the velocity of the point on which it is applied. Therefore, the rate of doing work by the forces applied on the segment shown in the figure is

$$\dot{W} \equiv F(x_B, t)v(x_B, t) - F(x_A, t)v(x_A, t) + \int_{\Omega(t)} w(x, t)v(x, t)dx. \quad (1.182)$$

As shown in the figure, heat can flow along the axis of the bar or through its lateral surfaces. The rate of heat flow along the axis of the bar is called the heat flux and is denoted by q . The rate of heat flow into the bar through its lateral surfaces, evaluated per unit length of the bar in the current configuration, is denoted by r and is called the heat generation/radiation/convection, depending on its source. The total rate at which heat is entering the bar now can be calculated by adding all of the sources of heat into our segment to get

$$\dot{Q} \equiv q(x_A, t) - q(x_B, t) + \int_{\Omega(t)} r(x, t)dx. \quad (1.183)$$

The general equation of balance of work and energy for our bar now can be written as

$$\begin{aligned} & F(x_B, t)v(x_B, t) - F(x_A, t)v(x_A, t) + \int_{\Omega(t)} w(x, t)v(x, t)dx + q(x_A, t) - q(x_B, t) + \int_{\Omega(t)} r(x, t)dx \\ &= \frac{D}{Dt} \left[\int_{\Omega(t)} e(x, t)\rho(x, t)dx + \int_{\Omega(t)} \frac{v^2(x, t)}{2}\rho(x, t)dx \right], \quad \forall \Omega(t) \subseteq \mathcal{B}(t). \end{aligned} \quad (1.184)$$

If the product of axial force and velocity (Fv) and heat flux (q) are both continuous, and if assuming sufficient continuity in the arguments of the integrals, then what results is in the differential form of the law of balance of energy and energy given by

$$\rho(x, t)\frac{De}{Dt}(x, t) = F(x, t)L(x, t) - \frac{\partial q(x, t)}{\partial x} + r(x, t), \quad \forall x \in \mathcal{B}(t). \quad (1.185)$$

1.15.6 The entropy production inequality

The entropy production inequality is the second law of thermodynamics. This law states that the entropy in any segment of the bar increases at a rate equal to or greater than the rate at which entropy is being added to the bar by heat. If we let E denote the entropy of any segment of the bar and let $\dot{E}F$ denote the rate at which entropy flows into this segment by heat, then the entropy production inequality can be written as

$$\frac{DE}{Dt} \geq \dot{E}F. \quad (1.186)$$

Entropy is added to a body when heat is added to it. The amount of entropy added to a particle by heat is equal to the heat divided by the absolute temperature of the particle

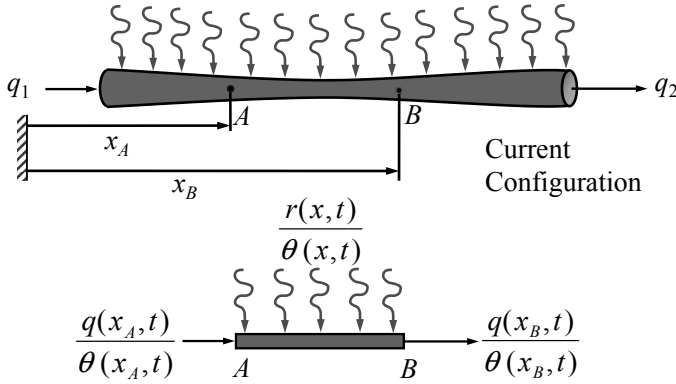


Figure 1.39: Entropy flow into a segment of the bar.

the heat is added to. Since heat is added to any segment of the bar by axial conduction and flow through the lateral surfaces, the expression for $\dot{E}F$ is

$$\dot{E}F \equiv \frac{q(x_A, t)}{\theta(x_A, t)} - \frac{q(x_B, t)}{\theta(x_B, t)} + \int_{\Omega(t)} \frac{r(x, t)}{\theta(x, t)} dx. \quad (1.187)$$

Figure 1.39 shows the entropy flow into a segment of the bar. Using the expression for entropy of a segment of a bar given in (1.168), one can write the integral form of the entropy production inequality as

$$\frac{D}{Dt} \left[\int_{\Omega(t)} \eta(x, t) \rho(x, t) dx \right] \geq \frac{q(x_A, t)}{\theta(x_A, t)} - \frac{q(x_B, t)}{\theta(x_B, t)} + \int_{\Omega(t)} \frac{r(x, t)}{\theta(x, t)} dx, \quad \forall \Omega(t) \subseteq \mathcal{B}(t). \quad (1.188)$$

If one assumes the variable $\frac{q(x, t)}{\theta(x, t)}$ is continuous, uses equation (1.175), and follows steps similar to those presented in the previous balance laws, one can obtain the following local (differential) form of the entropy production inequality given by

$$\rho(x, t) \frac{D\eta}{Dt}(x, t) \geq -\frac{\partial}{\partial x} \left[\frac{q(x, t)}{\theta(x, t)} \right] + \frac{r(x, t)}{\theta(x, t)}, \quad \forall x \in \mathcal{B}(t). \quad (1.189)$$

An alternate form of this law, known as the Clausius-Duhem inequality, can be obtained if one assumes strictly positive temperatures, an expression for internal energy can be given by $e = \psi + \eta\theta$, where ψ is free energy, and one introduces the law of balance of work and energy. This form is given as

$$\rho(x, t) \frac{D\psi}{Dt}(x, t) - F(x, t)L(x, t) + \rho(x, t)\eta(x, t) \frac{D\theta}{Dt}(x, t) + \frac{q(x, t)}{\theta(x, t)} g(x, t) \leq 0, \quad \forall x \in \mathcal{B}(t), \quad (1.190)$$

where the reader recalls that g denotes the temperature gradient.

Section 1.15-Problems:

- 1.15-1:** The velocity of particles of a bar of density $\rho = \rho^*(1 + tx)$, for constant ρ^* , are given as $v = 3x^2$. Calculate the current linear momentum and kinetic energy of the bar if it currently occupies the interval between $x = a$ and $x = b$.

- 1.15-2:** The velocity of particles on a bar is given by $v = X^2 t$. The density of the bar in the current configuration is given by $\rho_o/(Xt^2 + 1)$. The bar is in its reference configuration at time $t = 0$ and occupies the interval between $X = 0$ and $X = L$. Calculate the mass of the bar and its current linear momentum and kinetic energy.
- 1.15-3:** Given the motion of a bar as $x = X + Xt^2$, and its current density as $\rho = \rho_o/(1 + t^2)$, calculate the current value of the linear momentum of the bar segment initially between $X = X_A$ and $X = X_B$. Do this first by direct integration over the x and next by changing the variable of integration to X . Calculate the rate of change of linear momentum for this bar segment both by direct differentiation and by using equation (1.169).
- 1.15-4:** Given the motion of a bar as $x = X + X^2 t^2$, and its density in the reference configuration as $\rho_o = \rho^*(1 + X^2)$, calculate the current density of the bar.
- 1.15-5:** If the density of a bar is held uniform and constant, what type of velocity field is the bar undergoing?
- 1.15-6:** What kind of density field can be sustained given that the velocity gradient is constant? What will the density field look like if the density in the reference configuration is uniform and equal to ρ_o ?
- 1.15-7:** Derive the balance of linear momentum given in equation (1.180), starting from equation (1.179).
- 1.15-8:** A uniform bar of density ρ_o in the reference configuration is made of an elastic material for which the axial load is given by $F = A_o E \epsilon$, where $A_o E$ is constant. Assuming that $w_o = 0$, solve equation (1.180) by using separation of variables.
- 1.15-9:** Derive the entropy production inequality given in equation (1.189), starting from equation (1.188).
- 1.15-10:** Derive the entropy production inequality given in equation (1.190), starting from equation (1.189).

1.16 Thermodynamic restrictions on constitutive equations

In continuum thermodynamics, we also must make sure our constitutive assumptions are such that they obey the entropy production inequality. There are at least two ways of approaching this requirement. One would be to develop a model and test it to see that it never violates the entropy production inequality. The second way would be to make very general assumptions about the response functions and then see what the entropy production inequality requires in regards to admissible forms of the most general response functions. Let us start with the latter and develop the most general model for a thermoelastic material. Then we will follow this with a look at temperature-dependent elastic-plastic response, and then at rate-dependent plastic response.

1.16.1 Thermoelasticity using (ϵ, θ, g)

We shall define a thermoelastic material to be one in which the response functions depend on reference position, current strain, current temperature, and current temperature gradient. We have implicitly assumed that the response is indifferent to rigid body motions. These assumptions can be summarized as

$$\begin{aligned} \psi &= \psi^\dagger(X, \epsilon, \theta, g), \\ F &= F^\dagger(X, \epsilon, \theta, g), \\ \eta &= \eta^\dagger(X, \epsilon, \theta, g), \\ q &= q^\dagger(X, \epsilon, \theta, g), \end{aligned} \quad (1.191)$$

where the dependence of ψ , F , η , q , ϵ , θ , and g on (X, t) is omitted from the notation to simplify the presentation. As can be seen, we also have used the assumption of equipresence, since all functions depend on the same variables. The reader recalls that the entropy production inequality (1.190) requires that

$$\rho \frac{D\psi}{Dt} - FL + \rho\eta \frac{D\theta}{Dt} + \frac{q}{\theta}g \leq 0.$$

Using (1.191), we can take the material time derivative of free energy using the chain rule to get

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial\epsilon} \frac{D\epsilon}{Dt} + \frac{\partial\psi}{\partial\theta} \frac{D\theta}{Dt} + \frac{\partial\psi}{\partial g} \frac{Dg}{Dt}.$$

Introducing this into the entropy production inequality and reorganizing yields

$$\left[\rho \frac{\partial\psi}{\partial\epsilon} - F\lambda^{-1} \right] \frac{D\epsilon}{Dt} + \rho \left[\eta + \frac{\partial\psi}{\partial\theta} \right] \frac{D\theta}{Dt} + \rho \left[\frac{\partial\psi}{\partial g} \right] \frac{Dg}{Dt} + \left[\frac{q}{\theta}g \right] \leq 0,$$

which must hold for every admissible (X, ϵ, θ, g) , and for all possible processing conditions (i.e., every possible $(\frac{D\epsilon}{Dt}, \frac{D\theta}{Dt}, \frac{Dg}{Dt})$). The forms that we assumed in (1.191) for the response functionals make it obvious that the terms in the square brackets are not functions of the rates $(\frac{D\epsilon}{Dt}, \frac{D\theta}{Dt}, \frac{Dg}{Dt})$. If these rates can be selected arbitrarily, and if the heat flux is bounded at $g = 0$, then one can conclude that

$$\begin{aligned} F &= \rho_o \frac{\partial\psi}{\partial\epsilon}, \\ \eta &= -\frac{\partial\psi}{\partial\theta}, \\ \frac{\partial\psi}{\partial g} &= 0, \\ \frac{q}{\theta}g &\leq 0. \end{aligned} \quad (1.192)$$

The proof of this is left as a homework problem. The third relation states that the free energy is not a function of the temperature gradient. Combining this with the first and second relations, one must conclude that axial force and entropy cannot be functions of the temperature gradient. Therefore, for a thermoelastic material, heat flux is the only response function that depends on the temperature gradient. Note that the last relation requires that the heat flux always be zero or that it be an odd function of the temperature gradient with negative value for a positive temperature gradient. Therefore, the entropy production inequality requires that heat in a thermoelastic material must always flow from hot to cold, or not flow at all.

Let us now consider an example model of a thermoelastic material given by a specific free energy of the form

$$\psi = \frac{AE}{2\rho_o} [\epsilon - \alpha(\theta - \theta_o)]^2 + \psi^\theta,$$

where E and ψ^θ are functions of temperature, α is the coefficient of thermal expansion, and θ_o is the reference temperature. The load and entropy for this model will then be

$$\begin{aligned} F &= AE[\epsilon - \alpha(\theta - \theta_o)], \\ \eta &= -\frac{A}{2\rho_o} \frac{dE}{d\theta} [\epsilon - \alpha(\theta - \theta_o)]^2 + \frac{AE\alpha}{\rho_o} [\epsilon - \alpha(\theta - \theta_o)] - \frac{d\psi^\theta}{d\theta} \\ &= -\frac{F^2}{2\rho_o AE^2} \frac{dE}{d\theta} + \frac{F\alpha}{\rho_o} - \frac{d\psi^\theta}{d\theta}. \end{aligned}$$

Section 1.16-Problems:

1.16-1: Consider the inequality

$$\sum_{i=1}^n \mathcal{G}_i(\alpha_1, \dots, \alpha_m) \beta_i + \mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) \leq 0, \quad (1.193)$$

which must hold for all admissible parameters α_i and for all parameters β_i . In this equation each $\mathcal{G}_i$ is an arbitrary function of the variable set $(\alpha_1, \dots, \alpha_m)$. If it is determined that all α_i and β_i are independent, and that each β_i is independent of the other β_i 's, show that the inequality can only be satisfied if

$$\mathcal{G}_i(\alpha_1, \dots, \alpha_m) = 0, \quad \forall i = 1, \dots, n \quad (1.194)$$

and

$$\mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) \leq 0. \quad (1.195)$$

1.16-2: Apply the results of the previous problem to obtain the results given in (1.192).

1.16-3: A thermoelastic material is defined by the free-energy function of the form

$$\psi = \frac{E(\theta)A_o}{2\rho_o} [\epsilon - \alpha(\theta - \theta_o)]^2 + \psi^\theta(\theta),$$

where $E(\theta) = E_o - a\theta$ for a constant a , and $\psi^\theta(\theta)$ is a general function of temperature. Calculate the axial load and entropy for this material.

1.16.2 Thermoelasticity using $(\epsilon^e, \epsilon^\theta, \theta, g)$

The development presented in the last section is probably the most common presentation of thermoelasticity. An equivalent, but slightly different, representation of thermoelasticity can be obtained when starting from the constitutive assumptions

$$\begin{aligned} \psi &= \psi^\dagger(X, \epsilon^e, \epsilon^\theta, \theta, g), \\ F &= F^\dagger(X, \epsilon^e, \epsilon^\theta, \theta, g), \\ \eta &= \eta^\dagger(X, \epsilon^e, \epsilon^\theta, \theta, g), \\ q &= q^\dagger(X, \epsilon^e, \epsilon^\theta, \theta, g), \end{aligned} \quad (1.196)$$

where we have used the separation of strain into its thermal and mechanical parts. To keep the expressions fully nonlinear, the separation of strain is obtained through the relation $\lambda = \lambda^e \lambda^\theta$, which results in the expression

$$\epsilon = \epsilon^e + \epsilon^\theta + \epsilon^e \epsilon^\theta.$$

The key point in this alternate representation is that we have not only separated the strain into its mechanical and thermal parts, but by explicitly introducing ϵ^θ into the representations, we also have separated the contribution of temperature associated with the thermal strain from the other effects of temperature, such as hardening or softening of the material. For this set of variables, the material time derivative of the free energy is given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon^e} \dot{\epsilon}^e + \frac{\partial \psi}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \frac{\partial \psi}{\partial g} \dot{g}. \quad (1.197)$$

Unlike $\dot{\epsilon}$ and $\dot{\theta}$, which can be considered independent rates, the values $\dot{\epsilon}^e$ and $\dot{\epsilon}^\theta$ and $\dot{\theta}$ are not independent. To work with independent rates, we use the kinematic relation

$$\dot{\epsilon}^e = \frac{1}{\lambda^\theta} \dot{\epsilon} - \frac{\lambda^e}{\lambda^\theta} \dot{\epsilon}^\theta,$$

and use the model for thermal expansion given by

$$\dot{\epsilon}^\theta = \alpha \lambda^\theta \dot{\theta},$$

where α is the linear coefficient of thermal expansion, and may be a function of $(X, \epsilon^e, \epsilon^\theta, \theta)$. We substitute these relations into equation (1.197), and then introduce the material time derivative of the specific free energy into the Clausius-Duhem inequality. This results in the expression

$$\left[\rho \frac{\partial \psi}{\partial \epsilon^e} \frac{1}{\lambda^\theta} - \frac{F}{\lambda} \right] \dot{\epsilon} + \rho \left[\eta + \frac{\partial \psi}{\partial \theta} + \alpha \left(\lambda^\theta \frac{\partial \psi}{\partial \epsilon^\theta} - \lambda^e \frac{\partial \psi}{\partial \epsilon^e} \right) \right] \dot{\theta} + \rho \left[\frac{\partial \psi}{\partial g} \right] \dot{g} + \left[\frac{1}{\theta} qg \right] \leq 0. \quad (1.198)$$

Since $\dot{\epsilon}$, $\dot{\theta}$, and $\dot{g}$ are assumed to be independent, and since none of the terms in the square brackets depend on them, we can argue that if this inequality is to hold for all $(\dot{\epsilon}, \dot{\theta}, \dot{g})$, then we must have

$$\begin{aligned} F &= \rho \frac{\lambda}{\lambda^\theta} \frac{\partial \psi}{\partial \epsilon^e}, \\ \eta &= -\frac{\partial \psi}{\partial \theta} - \alpha \left(\lambda^\theta \frac{\partial \psi}{\partial \epsilon^\theta} - \lambda^e \frac{\partial \psi}{\partial \epsilon^e} \right), \\ \frac{\partial \psi}{\partial g} &= 0, \\ \frac{1}{\theta} qg &\leq 0. \end{aligned}$$

The third equation requires that the free energy be independent of the temperature gradient, and through the first two equations one will conclude that this is also true for the load and entropy, leaving only the heat flux to depend on the temperature gradient. The first two equations can be rewritten as

$$F = \rho \frac{\partial \psi}{\partial \epsilon^e} \lambda^e, \quad (1.199)$$

$$\eta = -\frac{\partial \psi}{\partial \theta} - \frac{\alpha}{\rho} (F^\theta - F), \quad (1.200)$$

where, by analogy to the mechanical load, we have defined a “thermodynamic load” coming through the derivative of the free energy with respect to the thermal strain and defined by the expression

$$F^\theta \equiv \rho \frac{\partial \psi}{\partial \epsilon^\theta} \lambda^\theta. \quad (1.201)$$

This load is commonly zero. This follows from the common assumption that free energy is a function of temperature, but not through thermal strain.

As an example, consider a material that has a quadratic dependence of free energy on the elastic strain given by

$$\psi = \frac{AE}{2\rho_o} \epsilon^e{}^2 + \psi^\theta,$$

where E and ψ^θ are functions of temperature. This example is similar, but not the same, as the example presented in the last section. The equations for load and entropy for this example model are given by

$$\begin{aligned} F &= \frac{AE}{\lambda^\theta} \epsilon^e, \\ \eta &= -\frac{A\lambda^{\theta 2}}{2\rho_o} \times \frac{F^2}{A^2 E^2} \times \frac{dE}{d\theta} - \frac{d\psi^\theta}{d\theta} + \frac{\alpha}{\rho} F, \end{aligned} \quad (1.202)$$

where we have used the relation $\rho\lambda = \rho_o$ obtained from the conservation of mass. Since thermal strains are normally small, as a first approximation the thermal stretch can be taken equal to one. This provides the approximate expression for load given by

$$F \approx AE\epsilon^e.$$

Noting that the stress (F/A) is normally several orders of magnitude smaller than the elastic modulus, for this model a reasonable approximation for the entropy might be given by

$$\eta \approx -\frac{d\psi^\theta}{d\theta} + \frac{\alpha}{\rho} F.$$

1.16.3 Thermoplasticity using $(\epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta, g)$

As we did in the development of thermoelasticity, here we first will define temperature-dependent plasticity in terms of the functional dependence of the constitutive models for specific free energy, specific entropy, load, and heat flux. It will be assumed that for temperature-dependent plasticity each of these constitutive functions may depend on the reference position, strain, plastic strain, thermal strain, hardening parameter, temperature, and temperature gradient. This will be written as

$$\begin{aligned} \psi &= \psi^\dagger(X, \epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ F &= F^\dagger(X, \epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ \eta &= \eta^\dagger(X, \epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ q &= q^\dagger(X, \epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta, g). \end{aligned} \quad (1.203)$$

This selection results in the material time derivative of the free energy to be given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} \dot{\epsilon} + \frac{\partial \psi}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial \psi}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial \psi}{\partial \xi} \dot{\xi} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \frac{\partial \psi}{\partial g} \dot{g}.$$

Substitution of this into the Clausius-Duhem inequality, after use of $L = \dot{\epsilon}/\lambda$, $\dot{\epsilon}^\theta = \alpha\lambda^\theta \dot{\theta}$, and some organization, becomes

$$\left[\rho \frac{\partial \psi}{\partial \epsilon} - \frac{F}{\lambda} \right] \dot{\epsilon} + \rho \left[\eta + \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \epsilon^\theta} \lambda^\theta \alpha \right] \dot{\theta} + \rho \frac{\partial \psi}{\partial g} \dot{g} + \rho \left[\frac{\partial \psi}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial \psi}{\partial \xi} \dot{\xi} \right] + \frac{1}{\theta} qg \leq 0.$$

First, let us consider response within the current thermoelastic range so that $\dot{\epsilon}^p = 0$ and $\dot{\xi} = 0$. As above, one can argue that the independence of the coefficients from the rates,

and the independence of the rates from each other require, if the law is to be true for all processes, that

$$\begin{aligned} F &= \rho_o \frac{\partial \psi}{\partial \epsilon}, \\ \eta &= -\frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \epsilon^\theta} \lambda^\theta \alpha, \\ \frac{1}{\theta} qg &\leq 0, \end{aligned} \quad (1.204)$$

and the condition that the free energy does not depend on the temperature gradient, which in view of the other relations indicates that load and entropy also do not depend on the temperature gradient. Since each point on the yield surface is close to an elastic point, one can argue that if there is sufficient continuity in the response functions, then the relations obtained for purely thermoelastic response also should extend to the region of plastic flow. One must conclude that in addition to these relations one has the condition

$$\rho \left[\frac{\partial \psi}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial \psi}{\partial \xi} \dot{\xi} \right] \leq 0. \quad (1.205)$$

Let us now consider an example for which the specific free energy is given by

$$\psi = \frac{AE}{2\rho_o} \epsilon^{e2} + \psi^\theta,$$

where E and ψ^θ are functions of temperature. Obviously, the expressions for load and entropy depend on the model used for evaluating the elastic strain. Let us first use the expression for small deformations given by

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta.$$

In this case, we have

$$\psi = \frac{AE}{2\rho_o} (\epsilon - \epsilon^p - \epsilon^\theta)^2 + \psi^\theta,$$

which results in

$$\begin{aligned} F &= AE(\epsilon - \epsilon^p - \epsilon^\theta) = AE\epsilon^e, \\ \eta &= -\frac{A}{2\rho_o} \frac{dE}{d\theta} \epsilon^{e2} + \frac{AE\alpha}{\rho_o} \epsilon^e - \frac{d\psi^\theta}{d\theta} \\ &= -\frac{F^2}{2\rho_o AE^2} \frac{dE}{d\theta} + \frac{F\alpha}{\rho_o} - \frac{d\psi^\theta}{d\theta}, \end{aligned}$$

where we have used the approximation $\lambda^\theta \approx 1$, consistent with the small strain assumption. The restriction (1.205) can be written as

$$-\rho \frac{AE}{\rho_o} \epsilon^e \dot{\epsilon}^p = -\frac{F}{\lambda} \dot{\epsilon}^p \leq 0.$$

This forces us to conclude that for this model the rate of plastic work ($F\dot{\epsilon}^p$) must always be zero or positive.

Next, let us look at the nonlinear relation for strain. That is, let us consider the case when $\lambda = \lambda^e \lambda^p \lambda^\theta$ so that

$$\epsilon^e = \frac{1 + \epsilon}{(1 + \epsilon^p)(1 + \epsilon^\theta)} - 1.$$

For this case, one can see that

$$\frac{\partial \epsilon^e}{\partial \epsilon} = \frac{1}{(1 + \epsilon^p)(1 + \epsilon^\theta)}, \quad \frac{\partial \epsilon^e}{\partial \epsilon^p} = -\frac{1 + \epsilon}{(1 + \epsilon^p)^2(1 + \epsilon^\theta)}, \quad \frac{\partial \epsilon^e}{\partial \epsilon^\theta} = -\frac{1 + \epsilon}{(1 + \epsilon^p)(1 + \epsilon^\theta)^2}.$$

Therefore, the resulting expression for load is

$$F = AE\epsilon^e \frac{\partial \epsilon^e}{\partial \epsilon} = \frac{AE}{(1 + \epsilon^p)(1 + \epsilon^\theta)} \epsilon^e = \frac{AE}{\lambda^p \lambda^\theta} \epsilon^e = \frac{AE\lambda^e}{\lambda} \epsilon^e.$$

The expression for entropy is given by

$$\eta = -\frac{A}{2\rho_o} \frac{dE}{d\theta} \epsilon^{e2} + \frac{AE\alpha}{\rho_o} \frac{(1 + \epsilon)}{(1 + \epsilon^p)(1 + \epsilon^\theta)} \epsilon^e - \frac{d\psi^\theta}{d\theta} = -\frac{(\lambda^p \lambda^\theta F)^2}{2AE^2 \rho_o} \frac{dE}{d\theta} + \frac{F\lambda\alpha}{\rho_o} - \frac{d\psi^\theta}{d\theta}.$$

The restriction on the plastic strain rate becomes

$$-\rho \frac{AE}{\rho_o} \frac{(1 + \epsilon)}{(1 + \epsilon^p)^2(1 + \epsilon^\theta)} \epsilon^e \dot{\epsilon}^p = -F \frac{\dot{\epsilon}^p}{\lambda^p} = -FL^p \leq 0,$$

which again requires that $F\dot{\epsilon}^p$ be zero or positive.

Section 1.16-Problems:

- 1.16-4:** Consider a temperature-dependent elastic-plastic model given by the free-energy function of the form

$$\psi = \frac{AE}{2\rho_o} (\epsilon - \epsilon^p - \epsilon^\theta)^2 + \psi^\theta(\theta),$$

where AE is constant and ψ^θ is a function of temperature only. Calculate the expressions for load and entropy and show that

$$F\dot{\epsilon}^p \geq 0.$$

Argue that this requires that the yield in compression to occur at negative loads (i.e., there cannot be a strong enough Bauschinger effect that would force the yield at the lower end of the elastic range to occur above the strain axis).

- 1.16-5:** Show that for all general models of the form $\psi^\dagger(\epsilon^e, \theta)$ where the free energy only depends on the elastic strain we cannot have negative plastic flow at positive loads. That is, these models exclude the possibility of a strong Bauschinger effect in which we see reverse yielding upon unloading happening at loads larger than zero.

1.16.4 Thermoplasticity using $(\epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, g)$

One can construct a theory of thermoplasticity which directly uses the elastic strain in place of the total strain in the arguments of the constitutive functions. Even though this does not change the nature of the modeling, *this totally changes the meaning of the partial derivatives of the free energy*. The constitutive assumptions in this case can be written as

$$\begin{aligned} \psi &= \psi^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ F &= F^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ \eta &= \eta^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, g), \\ q &= q^\dagger(X, \epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, g). \end{aligned} \tag{1.206}$$

This results in the material time derivative of the specific free energy given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon^e} \dot{\epsilon}^e + \frac{\partial \psi}{\partial \epsilon^p} \dot{\epsilon}^p + \frac{\partial \psi}{\partial \epsilon^\theta} \dot{\epsilon}^\theta + \frac{\partial \psi}{\partial \xi} \dot{\xi} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \frac{\partial \psi}{\partial g} \dot{g}.$$

The externally controllable rates are $(\dot{\epsilon}, \dot{\theta}, \dot{g})$, so using the relation $\lambda = \lambda^\theta \lambda^e \lambda^p$ we replace the $\dot{\epsilon}^e$ through the relation

$$\dot{\epsilon} = \dot{\epsilon}^\theta \lambda^e \lambda^p + \lambda^\theta \dot{\epsilon}^e \lambda^p + \lambda^\theta \lambda^e \dot{\epsilon}^p,$$

which can be written as

$$\frac{\dot{\epsilon}}{\lambda} = \frac{\dot{\epsilon}^\theta}{\lambda^\theta} + \frac{\dot{\epsilon}^e}{\lambda^e} + \frac{\dot{\epsilon}^p}{\lambda^p}, \quad (1.207)$$

and we replace the thermal strain rate with the model

$$\dot{\epsilon}^\theta = \alpha \lambda^\theta \dot{\theta}.$$

Introduction of this into the Clausius-Duhem inequality and some reorganization yield

$$\begin{aligned} & \left[\rho \frac{\partial \psi}{\partial \epsilon^e} \lambda^e - F \right] \frac{\dot{\epsilon}}{\lambda} + \rho \left[\eta + \frac{\partial \psi}{\partial \theta} + \frac{\alpha}{\rho} \left(\rho \frac{\partial \psi}{\partial \epsilon^\theta} \lambda^\theta - \rho \frac{\partial \psi}{\partial \epsilon^e} \lambda^e \right) \right] \dot{\theta} \\ & + \left[\rho \frac{\partial \psi}{\partial \epsilon^p} \lambda^p - \rho \frac{\partial \psi}{\partial \epsilon^e} \lambda^e \right] \frac{\dot{\epsilon}^p}{\lambda^p} + \rho \frac{\partial \psi}{\partial \xi} \dot{\xi} + \rho \frac{\partial \psi}{\partial g} \dot{g} + \frac{1}{\theta} qg \leq 0. \end{aligned} \quad (1.208)$$

Following standard arguments for the thermoelastic range we obtain

$$F = \rho \frac{\partial \psi}{\partial \epsilon^e} \lambda^e, \quad (1.209)$$

$$\eta = -\frac{\partial \psi}{\partial \theta} + \frac{\alpha}{\rho} (F - F^\theta), \quad (1.210)$$

$$\rho \frac{\partial \psi}{\partial g} = 0, \quad (1.211)$$

$$\frac{1}{\theta} qg \leq 0, \quad (1.212)$$

where

$$F^\theta \equiv \rho \frac{\partial \psi}{\partial \epsilon^\theta} \lambda^\theta. \quad (1.213)$$

Assuming continuity in these functions when extended to the current yield surface, we obtain the relation

$$-(F - F^b) \frac{\dot{\epsilon}^p}{\lambda^p} + \rho \frac{\partial \psi}{\partial \xi} \dot{\xi} \leq 0, \quad (1.214)$$

where F^b is a “load” associated with the derivative of the free energy with respect to plastic strain, which we define similar to the load by the relation

$$F^b \equiv \rho \frac{\partial \psi}{\partial \epsilon^p} \lambda^p. \quad (1.215)$$

This thermodynamic load sometimes is associated with the back-stress in the material, which is the thermodynamic force driving kinematic hardening in a material.

One can construct a fairly general kinematic hardening model based on this structure using models for free energy that do not depend on the hardening parameter. For this class of models, the constraint of the Clausius-Duhem inequality on the rate of plastic flow can be written as

$$\boxed{-\Delta F \dot{\epsilon}^p \leq 0}, \quad (1.216)$$

where $\boxed{\Delta F \equiv F - F^b}$ is the amount of load above the load associated with kinematic hardening. It easily can be seen that the sign of $\dot{\epsilon}^p$ and ΔF must be the same, so that at loads above F^b the plastic flow is positive and at loads below F^b the plastic flow is negative. A direct consequence is that we may have negative plastic flow at positive loads as long as they are below F^b . As a special example of this type of a model, we will consider a model for free energy that is quadratic in the elastic and plastic strains and is given by the expression

$$\psi = \frac{AE}{2\rho_o} \epsilon^{e2} + \frac{AE^b}{2\rho_o} \epsilon^{p2} + \psi^\theta, \quad (1.217)$$

where E , E^b , and ψ^θ depend on temperature. For this model, we will have

$$F = \frac{AE\lambda^e}{\lambda} \epsilon^e, \quad (1.218)$$

$$F^b = \frac{AE^b\lambda^p}{\lambda} \epsilon^p, \quad (1.219)$$

$$\eta = -\frac{\lambda^{\theta2}}{2\rho_o A} \left[\lambda^{p2} \left(\frac{F}{E} \right)^2 \frac{dE}{d\theta} + \lambda^{e2} \left(\frac{F^b}{E^b} \right)^2 \frac{dE^b}{d\theta} \right] - \frac{d\psi^\theta}{d\theta} + \frac{\alpha}{\rho} F. \quad (1.220)$$

One may select a yield function of the form

$$f = |F - F^b| - F_{iso}, \quad (1.221)$$

where F_{iso} represents the isotropic hardening effect and will be assumed to be a function of temperature and the hardening parameter ξ . As can be seen, F^b in the thermodynamic model has a similar role as F_c in the mechanical model as it represents the location of the center of the elastic range. An appropriate work hardening rule for this parameter would be

$$\dot{\xi} = \Delta F \dot{\epsilon}^p, \quad (1.222)$$

which, as a result of the restriction imposed by the Clausius-Duhem inequality, always will provide a positive $\dot{\xi}$.

Section 1.16-Problems:

- 1.16-6:** Construct the flow rule for thermoplasticity based on the quadratic free energy given in equation (1.217). Use the yield function $f = |F - F^b| - F_{iso}$ with

$$F_{iso} = F_{yo} + a_\xi(\xi - \xi_o) + a_\theta(\theta - \theta_o), \quad (1.223)$$

where F_{yo} , a_ξ and a_θ are material constants evaluated at the reference temperature θ_o and initial value of the hardening parameter ξ_o . For your model use the hardening rule $\dot{\xi} = \Delta F \dot{\epsilon}^p$.

1.16.5 Rate-dependent thermoplasticity

The Clausius-Duhem inequality for rate-dependent plasticity is identical to temperature-dependent plasticity with the exception that the plastic strain rate does not depend on any of the rates. This removes the need to look separately at the thermoelastic response and then appeal to the continuity of the functions to extend the thermoelastic results to include the plastic flow portion. The results are identical to those obtained for temperature-dependent plasticity and will not be repeated here.

Section 1.16-Problems:

- 1.16-7:** Show that the results obtained for rate-dependent plasticity are identical to those obtained for temperature-dependent plasticity.

1.17 Heat generation and flow

Plastic flow generates a substantial amount of heat which either flows out from the point or increases the temperature at the point. We will use the balance of work and energy to study the effect of plastic flow on the heating or on the heat flow at the point. The balance of energy is given by the equation

$$\rho \dot{e} = FL + r - \frac{\partial q}{\partial x}, \quad (1.224)$$

where the last two terms on the right-hand side account for the rate of heat flowing into the point due to radiation and heat flux. We will define

$$\rho \dot{h} \equiv r - \frac{\partial q}{\partial x}, \quad (1.225)$$

so that $\dot{h}$ represents the specific rate of heat entering the material point. Using the balance of energy, we can see that

$$\dot{h} = \dot{e} - \frac{1}{\rho} FL. \quad (1.226)$$

Integrating this over the time interval from t_o to t yields

$$\Delta h = e - e_o - \int_{t_o}^t \frac{1}{\rho} FL ds, \quad (1.227)$$

where Δh is the net amount of heat added to the point in the given time interval, and e_o is the specific internal energy at time t_o . Introducing the relation $e = \psi + \eta\theta$ yields

$$\Delta h = \psi - \psi_o + \eta\theta - \eta_o\theta_o - \int_{t_o}^t \frac{1}{\rho} FL ds, \quad (1.228)$$

where ψ_o , η_o , and θ_o are, respectively, the specific free energy, specific entropy, and temperature at the initial time t_o . In rate form, this equation can be written as

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta} - \frac{1}{\rho} FL. \quad (1.229)$$

Assuming the specific free energy is given by a function

$$\psi(X, t) = \psi^\dagger[X, \epsilon, \epsilon^p, \epsilon^\theta, \xi, \theta], \quad (1.230)$$

we can write

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - FL = \left(\rho\frac{\partial\psi}{\partial\epsilon} - \frac{F}{\lambda}\right)\dot{\epsilon} + \rho\left(\eta + \frac{\partial\psi}{\partial\theta} + \frac{\partial\psi}{\partial\epsilon^\theta}\lambda^\theta\alpha\right)\dot{\theta} + \rho\frac{\partial\psi}{\partial\epsilon^p}\dot{\epsilon}^p + \rho\frac{\partial\psi}{\partial\xi}\dot{\xi}, \quad (1.231)$$

where we are using $\dot{\epsilon}^\theta = \alpha\lambda^\theta\dot{\theta}$. From the Clausius-Duhem inequality, we know that

$$\boxed{F = \rho_o \frac{\partial\psi}{\partial\epsilon}}, \quad \boxed{\eta = -\frac{\partial\psi}{\partial\theta} - \frac{\partial\psi}{\partial\epsilon^\theta}\lambda^\theta\alpha}. \quad (1.232)$$

Therefore, the terms in the parentheses in equation (1.231) add to zero and we get

$$\dot{\psi} + \eta\dot{\theta} - \frac{1}{\rho}FL = \frac{\partial\psi}{\partial\epsilon^p}\dot{\epsilon}^p + \frac{\partial\psi}{\partial\xi}\dot{\xi}. \quad (1.233)$$

It follows that

$$\boxed{\dot{h} = \dot{\eta}\theta + \frac{\partial\psi}{\partial\epsilon^p}\dot{\epsilon}^p + \frac{\partial\psi}{\partial\xi}\dot{\xi}}. \quad (1.234)$$

This equation can be used to study the relation between heat flow and temperature change. For example, to study adiabatic conditions, we can set $\dot{h} = 0$ and calculate $\dot{\theta}$, or to study the heat flow needed to sustain isothermal conditions, we can set $\dot{\theta} = 0$ and solve for $\dot{h}$. In the case of an *isothermal process*, we can integrate this expression to get the amount of heat added to the point given as

$$\boxed{\Delta h = \theta(\eta - \eta_o) + \int_{t_o}^t \left(\frac{\partial\psi}{\partial\epsilon^p}\dot{\epsilon}^p + \frac{\partial\psi}{\partial\xi}\dot{\xi} \right) dt}, \quad (1.235)$$

where η_o is the entropy at time t_o . For an *isentropic process* (one for which entropy is constant), the amount of heat added to the point is given by

$$\boxed{\Delta h = \int_{t_o}^t \left(\frac{\partial\psi}{\partial\epsilon^p}\dot{\epsilon}^p + \frac{\partial\psi}{\partial\xi}\dot{\xi} \right) dt}. \quad (1.236)$$

We will study this further in the context of a special example.

Consider a specific example using $\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta$, the approximation $\dot{\epsilon}^\theta = \alpha\dot{\theta}$, and the specific free energy given by

$$\psi = \frac{1}{2\rho_o}AE\epsilon^{e2} + \psi^\theta, \quad (1.237)$$

where A is the initial cross-sectional area, E is the elastic modulus and ψ^θ is the specific free energy at zero elastic strain. In this expression, A and ρ_o are constants, and E and ψ^θ are assumed to depend on temperature. For this model,

$$\begin{aligned} \frac{\partial\psi}{\partial\epsilon} &= \frac{1}{\rho_o}AE\epsilon^e, & \frac{\partial\psi}{\partial\epsilon^p} &= -\frac{1}{\rho_o}AE\epsilon^e, & \frac{\partial\psi}{\partial\epsilon^\theta} &= -\frac{1}{\rho_o}AE\epsilon^e, \\ \frac{\partial\psi}{\partial\xi} &= 0, & \frac{\partial\psi}{\partial\theta} &= \frac{1}{2\rho_o}A\frac{dE}{d\theta}\epsilon^{e2} + \frac{d\psi^\theta}{d\theta}. \end{aligned} \quad (1.238)$$

From this we get

$$F = AE\epsilon^e, \quad (1.239)$$

$$\eta = -\frac{1}{2\rho_o} A \frac{dE}{d\theta} \epsilon^{e2} - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho_o} AE\epsilon^e \alpha, \quad (1.240)$$

$$\frac{\partial \psi}{\partial \epsilon^p} = -\frac{F}{\rho_o}. \quad (1.241)$$

Assuming the coefficient of thermal expansion α is given as a function of temperature, the rate of change of specific entropy is given by

$$\begin{aligned} \dot{\eta} &= -\frac{A}{\rho_o} \frac{dE}{d\theta} \epsilon^e \dot{\epsilon}^e - \frac{A}{2\rho_o} \frac{d^2 E}{d\theta^2} \epsilon^{e2} \dot{\theta} - \frac{d^2 \psi^\theta}{d\theta^2} \dot{\theta} + \frac{A}{\rho_o} \frac{dE}{d\theta} \epsilon^e \alpha \dot{\theta} + \frac{A}{\rho_o} E \alpha \dot{\epsilon}^e + \frac{A}{\rho_o} E \epsilon^e \frac{d\alpha}{d\theta} \dot{\theta} \\ &= -\frac{F}{\rho_o E} \frac{dE}{d\theta} \dot{\epsilon}^e - \frac{F^2}{2\rho_o A E^2} \frac{d^2 E}{d\theta^2} \dot{\theta} - \frac{d^2 \psi^\theta}{d\theta^2} \dot{\theta} + \frac{F}{\rho_o E} \frac{dE}{d\theta} \alpha \dot{\theta} + \frac{A}{\rho_o} E \alpha \dot{\epsilon}^e + \frac{F}{\rho_o} \frac{d\alpha}{d\theta} \dot{\theta} \\ &= -\left(\frac{F}{\rho_o E} \frac{dE}{d\theta} - \frac{AE\alpha}{\rho_o} \right) \dot{\epsilon} - \left(\frac{F^2}{2\rho_o A E^2} \frac{d^2 E}{d\theta^2} + \frac{d^2 \psi^\theta}{d\theta^2} - \frac{F}{\rho_o} \frac{d\alpha}{d\theta} + \frac{AE\alpha^2}{\rho_o} - \frac{2F\alpha}{\rho_o E} \frac{dE}{d\theta} \right) \dot{\theta} \\ &\quad + \left(\frac{F}{\rho_o E} \frac{dE}{d\theta} - \frac{AE\alpha}{\rho_o} \right) \dot{\epsilon}^p. \end{aligned} \quad (1.242)$$

It follows that $\dot{h}$ is given by

$$\begin{aligned} \dot{h} &= -\frac{\theta}{\rho_o} \left(\frac{F}{E} \frac{dE}{d\theta} - AE\alpha \right) \dot{\epsilon} - \frac{\theta}{\rho_o} \left(\frac{F^2}{2AE^2} \frac{d^2 E}{d\theta^2} + \rho_o \frac{d^2 \psi^\theta}{d\theta^2} - F \frac{d\alpha}{d\theta} + AE\alpha^2 - \frac{2F\alpha}{E} \frac{dE}{d\theta} \right) \dot{\theta} \\ &\quad + \frac{1}{\rho_o} \left(\frac{\theta F}{E} \frac{dE}{d\theta} - \theta AE\alpha - F \right) \dot{\epsilon}^p. \end{aligned} \quad (1.243)$$

In the elastic range, we set $\dot{\epsilon}^p = 0$. For example, the thermoelastic temperature change in an adiabatic process will be given by setting $\dot{h} = 0$ to get

$$\dot{\theta} = -\frac{\frac{F}{E} \frac{dE}{d\theta} - AE\alpha}{\frac{F^2}{2AE^2} \frac{d^2 E}{d\theta^2} + \rho_o \frac{d^2 \psi^\theta}{d\theta^2} - F \frac{d\alpha}{d\theta} + AE\alpha^2 - \frac{2F\alpha}{E} \frac{dE}{d\theta}} \dot{\epsilon}. \quad (1.244)$$

On the other hand, the heat flow to keep a thermoelastic system under isothermal conditions can be obtained by setting $\dot{\theta} = 0$ to get

$$\dot{h} = -\frac{\theta}{\rho_o} \left(\frac{F}{E} \frac{dE}{d\theta} - AE\alpha \right) \dot{\epsilon}. \quad (1.245)$$

During plastic flow, we can use $\dot{\epsilon}^p = \beta_\epsilon \dot{\epsilon} + \beta_\theta \dot{\theta}$ to get

$$\begin{aligned} \dot{h} &= -\frac{1}{\rho_o} \left[\theta \left(\frac{F}{E} \frac{dE}{d\theta} - AE\alpha \right) (1 - \beta_\epsilon) + \beta_\epsilon F \right] \dot{\epsilon} \\ &\quad - \frac{1}{\rho_o} \left[\theta \left(\frac{F^2}{2AE^2} \frac{d^2 E}{d\theta^2} + \rho_o \frac{d^2 \psi^\theta}{d\theta^2} - F \frac{d\alpha}{d\theta} + AE\alpha^2 - \frac{2F\alpha}{E} \frac{dE}{d\theta} \right) \right. \\ &\quad \left. - \beta_\theta \left(\frac{\theta F}{E} \frac{dE}{d\theta} - \theta AE\alpha - F \right) \right] \dot{\theta}. \end{aligned} \quad (1.246)$$

Again, the temperature rise due to adiabatic conditions will be given by

$$\dot{\theta} = -\frac{\theta \left(\frac{F}{E} \frac{dE}{d\theta} - AE\alpha \right) (1 - \beta_\epsilon) + \beta_\epsilon F}{\theta \left(\frac{F^2}{2AE^2} \frac{d^2 E}{d\theta^2} + \rho_o \frac{d^2 \psi^\theta}{d\theta^2} - F \frac{d\alpha}{d\theta} + AE\alpha^2 - \frac{2F\alpha}{E} \frac{dE}{d\theta} \right) - \beta_\theta \left(\frac{\theta F}{E} \frac{dE}{d\theta} - \theta AE\alpha - F \right)} \dot{\epsilon}. \quad (1.247)$$

On the other hand, the heat flow needed to keep the system under isothermal conditions is given by

$$\dot{h} = -\frac{1}{\rho_o} \left[\theta \left(\frac{F}{E} \frac{dE}{d\theta} - AE\alpha \right) (1 - \beta_\epsilon) + \beta_\epsilon F \right] \dot{\epsilon}. \quad (1.248)$$

An interesting point to note is that for the specific free energy given in equation (1.237), both the free energy and entropy are only a function of elastic strain and temperature. Since the load is given in terms of the elastic strain, any process that starts and ends at the same load and temperature will have both the same initial and final free energy and the same initial and final entropy. As a result of this, equation (1.228) can be written as

$$\Delta h = - \int_{t_o}^{t_f} \frac{1}{\rho} F L ds, \quad (1.249)$$

where Δh is the net heat flow added to the point as a result of going through this special process that has the same initial and final load and temperature. Since $L = \dot{\epsilon}/\lambda$ and, as a result of conservation of mass, we know that $\rho\lambda = \rho_o$, this also can be written for this special process as

$$\Delta h = -\frac{1}{\rho_o} \int_{t_o}^t F \dot{\epsilon} ds. \quad (1.250)$$

The integral represents the working of the load. This equation states, for our special model that has a specific free energy that is only a function of elastic strain and temperature, that any process that starts and ends at the same values of load and temperature generates a net amount of heat per unit length, flowing out of the point, and equal to the working of the force over the process. Figure 1.40 shows that the expression for the working is equivalent to the area under the load-strain plot. Also in this figure we have shown several processes that start and end at the same load, showing the working for each.

Section 1.17-Problems:

- 1.17-1:** Start from the kinematic assumption $\lambda = \lambda^\theta \lambda^e \lambda^p$, where $\dot{\epsilon}^\theta = \alpha \lambda^\theta \dot{\theta}$ for a coefficient of thermal expansion $\alpha = \alpha^\dagger(\theta)$ given as a function of temperature. Take the quadratic free energy given by

$$\psi = \frac{1}{2\rho_o} AE \epsilon^{e2} + \psi^\theta,$$

where E and ψ^θ are functions of temperature only, and ρ_o and A are constants.. Find the expression for the rate of heat flow under isothermal conditions and the rate of temperature rise during adiabatic conditions.

- 1.17-2:** Start from the kinematic assumption $\lambda = \lambda^\theta \lambda^e \lambda^p$, where $\dot{\epsilon}^\theta = \alpha \lambda^\theta \dot{\theta}$ for a coefficient of thermal expansion $\alpha = \alpha^\dagger(\theta)$ given as a function of temperature. Take the quadratic free energy given by

$$\psi = \frac{1}{2\rho_o} AE \epsilon^{e2} + \frac{1}{2\rho_o} AE^b \epsilon^{p2} + \psi^\theta,$$

where E , E^b , and ψ^θ are functions of temperature only, and ρ_o and A are constants. Find the expression for the rate of heat flow under isothermal conditions and the rate of temperature rise during adiabatic conditions.

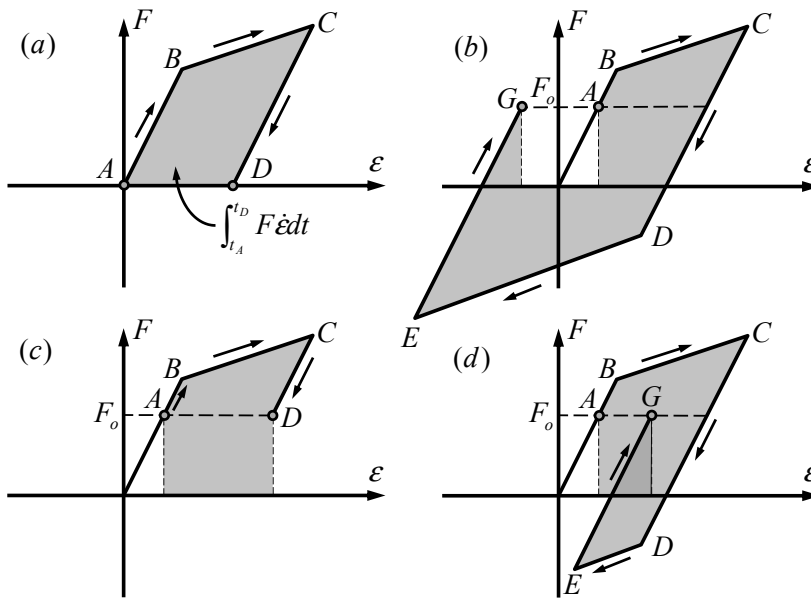


Figure 1.40: Working of the force for the special case that free energy is only a function of elastic strain and temperature. The working is calculated over cycles that start and end at the same load and temperature.

1.18 Equilibrium and quasi-equilibrium problems

In this section, we will look at several problems to illustrate how to use the balance laws in the solution of boundary value problems. We will concentrate on equilibrium and quasi-equilibrium problems in which acceleration can be set equal to zero. As a result, the governing mechanical balance laws for our bar can be given as

$$\boxed{\rho\lambda = \rho_o}, \quad \boxed{\frac{\partial F}{\partial X} + w_o = 0.} \quad (1.251)$$

We now will proceed to look first at the problem of a tapered elastic bar to show how these equations are used in the simpler setting of elasticity. We will follow this by looking at an elastic-plastic bar showing isotropic hardening, followed by the study of the elastic-viscoplastic case.

For each example in this section, we will be considering a bar that is fixed at the left end and is being loaded by the application of a force history or displacement history at the right end, and with a distributed load on the lateral surface. Figure 1.41 shows this bar, which can be of variable cross-sectional area and material properties. The balance of linear momentum is written as

$$\frac{\partial F(X, t)}{\partial X} = -w_o(X, t), \quad (1.252)$$

where the arguments of each function are explicitly written to eliminate any confusion that

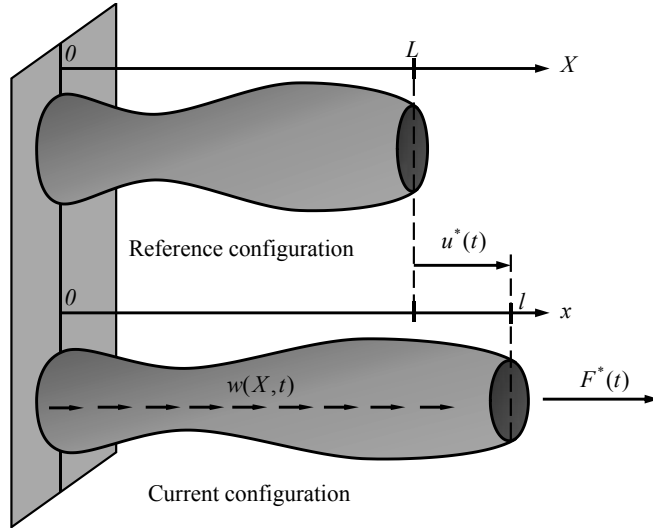


Figure 1.41: A nonuniform bar fixed at the left end and under either displacement control or load control at the right end, and with a distributed load on the lateral surface.

might exist in the following steps. Integrating this relation once gives

$$F(X, t) = - \int_0^X w_o(Y, t) dY + C(t),$$

where $C(t)$ is the “constant” of integration that must be fit to the boundary conditions. We have the boundary condition on the right end of the bar ($X = L$) given by $F(L, t) = F^*(t)$. This gives the value of $C(t)$ as

$$C(t) = F^*(t) + \int_0^L w_o(Y, t) dY.$$

Substitution back into the expression for $F(X, t)$ results in

$$F(X, t) = F^*(t) + \int_X^L w_o(Y, t) dY. \quad (1.253)$$

The material time derivative of this expression is given by

$$\dot{F}(X, t) = \dot{F}^*(t) + \int_X^L \dot{w}_o(Y, t) dY. \quad (1.254)$$

For all the problems considered, we will assume infinitesimal strains. The kinematical separation of strain will, therefore, be given by

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta. \quad (1.255)$$

The relation between strain and displacement is given by

$$\epsilon(X, t) = \frac{\partial u(X, t)}{\partial X}.$$

Integration of this expression results in the relation

$$u(X, t) = \int_0^X \epsilon(Y, t) dY + u(0, t).$$

The material time derivative of this expression yields

$$v(X, t) = \int_0^X \dot{\epsilon}(Y, t) dY + v(0, t).$$

Using the relationship between strain, plastic strain, elastic strain, and thermal strain, we get

$$v(X, t) = \int_0^X \dot{\epsilon}^e(Y, t) dY + \int_0^X \dot{\epsilon}^p(Y, t) dY + \int_0^X \dot{\epsilon}^\theta(Y, t) dY + v(0, t). \quad (1.256)$$

For our particular problem, the left end is fixed so that $v(0, t) = 0$. Also, the motion of the right end is given by $v(L, t) = \dot{u}^*(t)$ so that we have

$$\dot{u}^*(t) = \int_0^L \dot{\epsilon}^e(Y, t) dY + \int_0^L \dot{\epsilon}^p(Y, t) dY + \int_0^L \dot{\epsilon}^\theta(Y, t) dY. \quad (1.257)$$

Now that we have established the general solution to this problem for quasi-static loading, let us look at several specific problems.

1.18.1 Elastic bar

Let us first look at a fully elastic bar with varying cross-sectional area and elastic modulus. The expression for a linear elastic bar is given by

$$F = AE\epsilon.$$

This can be written as

$$F(X, t) = A(X)E(X)\epsilon(X, t). \quad (1.258)$$

Therefore, the material time derivative of this expression yields the relation

$$\dot{\epsilon}(X, t) = \frac{\dot{F}(X, t)}{A(X)E(X)}.$$

Substitution for $\dot{F}$ from (1.254) results in the expression

$$\dot{\epsilon}(X, t) = \frac{1}{A(X)E(X)} [\dot{F}^*(t) + \int_X^L \dot{w}_o(Y, t) dY].$$

Substitution of this into the expression for velocity given in (1.256) results in

$$v(X, t) = \int_0^X \frac{1}{A(Y)E(Y)} dY \dot{F}^*(t) + \int_0^X \frac{1}{A(Y)E(Y)} \int_Y^L \dot{w}_o(Z, t) dZ dY. \quad (1.259)$$

This can be integrated with respect to time to get the displacement given by

$$u(X, t) = \int_0^X \frac{1}{A(Y)E(Y)} dY F^*(t) + \int_0^X \frac{1}{A(Y)E(Y)} \int_Y^L w_o(Z, t) dZ dY, \quad (1.260)$$

where we already have introduced the boundary condition $u(0, t) = 0$. The expression for the right end displacement is given by

$$u^*(t) = \int_0^L \frac{1}{A(Y)E(Y)} dY F^*(t) + \int_0^L \frac{1}{A(Y)E(Y)} \int_Y^L w_o(Z, t) dZ dY. \quad (1.261)$$

If the load on the right end is specified, we can directly use the applied load F^* and the distributed load w_o to calculate the displacement of each point using (1.260). If in place of the load, the displacement u^* of the right end and the distributed load w_o are given, we can first use (1.261) to get the load F^* and then calculate the displacement of each point from (1.260). Obviously, for both cases the internal load at each point can be calculated using (1.253).

1.18.2 Elastic-plastic bar with isotropic hardening

We will now consider an elastic-plastic bar which exhibits isotropic hardening. The model including the strain hardening parameter written for the case of load control is given by

$$\begin{aligned} F(X, t) &= A(X)E(X)\epsilon^e(X, t), \\ f(X, t) &= |F(X, t)| - F_y(X, t), \\ F_y(X, t) &= A(X) \left[\sigma_{yo}(X) + \frac{a(X)E(X)}{E(X)-a(X)} \xi(X, t) \right], \\ \dot{f}(X, t) &= \begin{cases} \dot{F}(X, t) & \text{if } F(X, t) \geq 0, \\ -\dot{F}(X, t) & \text{if } F(X, t) < 0, \end{cases} \\ \dot{\epsilon}^p(X, t) &= \begin{cases} \frac{E(X)-a(X)}{A(X)a(X)E(X)} \dot{F}(X, t) & \text{if } f(X, t) = 0 \text{ and } \dot{f}(X, t) > 0, \\ 0 & \text{for all other cases,} \end{cases} \\ \dot{\xi}(X, t) &= |\dot{\epsilon}^p(X, t)|. \end{aligned}$$

Using the expression given in (1.254) for $\dot{F}$, we can write the elastic and plastic strain rates as

$$\begin{aligned} \dot{\epsilon}^e(X, t) &= \frac{1}{A(X)E(X)} \left[\dot{F}^*(t) + \int_X^L \dot{w}_o(Y, t) dY \right], \\ \dot{\epsilon}^p(X, t) &= \begin{cases} \frac{E(X)-a(X)}{A(X)a(X)E(X)} \left[\dot{F}^*(t) + \int_X^L \dot{w}_o(Y, t) dY \right] & \text{if } f(X, t) = 0 \text{ and } \dot{f}(X, t) > 0, \\ 0 & \text{for all other cases.} \end{cases} \end{aligned}$$

Substitution of these expressions into (1.256) results in

$$\begin{aligned} v(X, t) &= \int_0^X \frac{1}{A(Y)E(Y)} dY \dot{F}^*(t) + \int_0^X \frac{1}{A(Y)E(Y)} \int_Y^L \dot{w}_o(Z, t) dZ dY \\ &\quad + \int_0^X \frac{E(Y)-a(Y)}{A(Y)a(Y)E(Y)} \left[\dot{F}^*(t) + \int_Y^L \dot{w}_o(Z, t) dZ \right] \phi[f(Y, t), \dot{f}(Y, t)] dY, \end{aligned} \quad (1.262)$$

where the function ϕ is defined as

$$\phi(a, b) = \begin{cases} 1 & \text{if } a = 0 \text{ and } b > 0, \\ 0 & \text{for all other cases.} \end{cases}$$

The displacement rate of the right end of the bar can be written as

$$\begin{aligned} \dot{u}^*(t) = & \int_0^L \left\{ \frac{1}{A(Y)E(Y)} + \frac{E(Y) - a(Y)}{A(Y)a(Y)E(Y)} \phi[f(Y, t), \dot{f}(Y, t)] \right\} dY \dot{F}^*(t) \\ & + \int_0^L \left\{ \frac{1}{A(Y)E(Y)} + \frac{E(Y) - a(Y)}{A(Y)a(Y)E(Y)} \phi[f(Y, t), \dot{f}(Y, t)] \right\} \int_Y^L \dot{w}_o(Z, t) dZ dY. \end{aligned} \quad (1.263)$$

If the applied load F^* and surface traction w_o are given, then equation (1.262) can be used to evaluate the velocity at each point. Integration of this velocity over time gives the displacement at each point. If the end displacement u^* and surface traction w_o are given, then (1.263) can be used to calculate $\dot{F}^*$ first and then using this in (1.262) the velocity of each point can be evaluated. The reader is reminded that as a step in the solution process one must calculate the value of the hardening parameter at each point in the bar and for each time step. This requires that we calculate

$$\xi(X, t) = \int_0^t |\dot{\epsilon}^p(X, s)| ds, \quad (1.264)$$

using the expression for the plastic strain rate given above.

1.18.3 Elastic-viscoplastic bar

The same problem can be solved for an elastic-viscoplastic bar. Let the response of the bar be given by

$$\begin{aligned} F(X, t) &= A(X)E(X)\epsilon^e(X, t), \\ f(X, t) &= |F(X, t)| - F_y(X, t), \\ F_y(X, t) &= A(X) \left[\sigma_{yo}(X) + \frac{a(X)E(X)}{E(X) - a(X)} \xi(X, t) \right], \\ \dot{\epsilon}^p(X, t) &= \begin{cases} \text{sign}[F(X, t)] \frac{f(X, t)}{\mu(X)} & \text{if } f(X, t) \geq 0, \\ 0 & \text{if } f(X, t) < 0, \end{cases} \\ \dot{\xi}(X, t) &= |\dot{\epsilon}^p(X, t)|. \end{aligned}$$

The solution process is identical to the previous example, resulting in the expression

$$\begin{aligned} v(X, t) = & \int_0^X \frac{1}{A(Y)E(Y)} dY \dot{F}^*(t) + \int_0^X \frac{1}{A(Y)E(Y)} \int_Y^L \dot{w}_o(Z, t) dZ dY \\ & + \int_0^X \text{sign}[F(Y, t)] \frac{f(Y, t)}{\mu(Y)} H[f(Y, t)] dY, \end{aligned} \quad (1.265)$$

where the function H is defined as

$$H(a) = \begin{cases} 1 & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

The displacement rate of the right end of the bar can be written as

$$\dot{u}^*(t) = \int_0^L \frac{1}{A(Y)E(Y)} dY \dot{F}^*(t) + \int_0^L \frac{1}{A(Y)E(Y)} \int_Y^L \dot{w}_o(Z, t) dZ dY$$

$$+ \int_0^L \text{sign}[F(Y, t)] \frac{f(Y, t)}{\mu(Y)} H[f(Y, t)] dY. \quad (1.266)$$

If the applied load F^* and surface traction w_o are given, then equation (1.265) can be used to evaluate the velocity at each point. Integration of this velocity over time gives the displacement at each point. If the end displacement u^* and surface traction w_o are given, then (1.266) can be used to calculate $\dot{F}^*$ first and then using this in (1.265) the velocity of each point can be evaluated. As before, in the solution process the hardening parameter needs to be evaluated using (1.264).

Section 1.18-Problems:

- 1.18-1:** Write a program to evaluate the load and displacement in an elastic bar with variable cross-sectional area and material properties. Consider both types of boundary conditions discussed.
- 1.18-2:** Write a program to evaluate the load and displacement in an elastic-plastic bar with variable cross-sectional area and material properties. Use the model given in this section.
- 1.18-3:** Write the equations to be used in the solution of an elastic-plastic bar for combined isotropic and kinematic hardening. Assume variable cross-sectional area and material properties.
- 1.18-4:** Write a program to evaluate the load and displacement in an elastic-viscoplastic bar with variable cross-sectional area and material properties. Use the model given in this section.

1.19 Dynamic loading problems: Numerical solution

Problems dealing with time-dependent loading and including inertia effects are normally sufficiently complex that only a few analytical solutions can be developed easily. Two such solutions will be presented in the sections following the discussion of jump conditions. One deals with the steady-state drawing of bars, and the other deals with plastic shock waves traveling through a bar. For now, let us look at developing a numerical method for the solution of a general dynamics problem.

The general problem we will consider is similar to that shown in Figure 1.41, with the exception that we also will consider inertia effects. Figure 1.42 shows the bar in its reference and current configurations. The governing equations for our problem are given by

$$\boxed{\rho\lambda = \rho_o}, \quad \boxed{\frac{\partial F}{\partial X} + w_o = \rho_o a.}$$

We will seek to find the displacement $u(X, t)$, the velocity $v(X, t)$, the acceleration $a(X, t)$, the load $F(X, t)$, and the density $\rho(X, t)$. To expose the variables, let us write these balance laws as

$$\rho(X, t)\lambda(X, t) = \rho_o(X), \quad \frac{\partial F(X, t)}{\partial X} + w_o(X, t) = \rho_o(X)a(X, t), \quad (1.267)$$

which must hold for each point X in the reference configuration, $X \in (0, L)$, and subject to the initial conditions

$$u(X, 0) = u_o(X), \quad v(X, 0) = v_o(X), \quad (1.268)$$

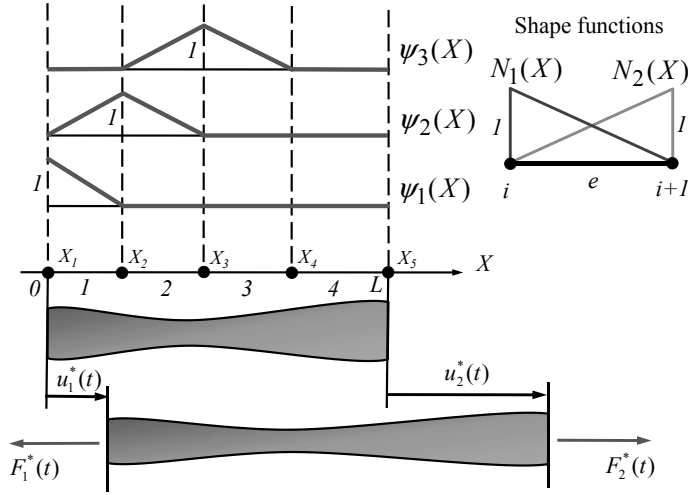


Figure 1.42: Dynamic loading boundary conditions, finite element discretization, linear base functions $\psi_i(X)$, and linear shape functions $N_i(X)$.

and the displacement boundary conditions

$$u(0, t) = u_1^*(t), \quad u(L, t) = u_2^*(t), \quad (1.269)$$

or a mixed displacement and load boundary condition such as

$$u(0, t) = u_1^*(t), \quad F(L, t) = F_2^*(t). \quad (1.270)$$

For this problem, we will use a finite element formulation in space and a finite difference formulation in time. We will use a set of trial functions $\phi(X)$ that are zero at each point in the bar where there is a displacement boundary condition. We will construct the variational formulation by multiplying the balance of linear momentum by $\phi(X)$ and integrating it over the bar to get

$$\int_0^L \left[\frac{\partial F(X, t)}{\partial X} + w_o(X, t) \right] \phi(X) dX = \int_0^L \rho_o(X) a(X, t) \phi(X) dX.$$

We next use integration by parts to get

$$\begin{aligned} F(L, t) \phi(L) - F(0, t) \phi(0) - \int_0^L F(X, t) \frac{d\phi(X)}{dX} dX + \int_0^L w_o(X, t) \phi(X) dX \\ = \int_0^L \rho_o(X) a(X, t) \phi(X) dX, \end{aligned} \quad (1.271)$$

where for the boundary conditions selected $\phi(0) = 0$. $\phi(L)$ is also zero if a displacement boundary condition is imposed at $X = L$. For now, we will not introduce these values into the equations. Next, as shown in Figure 1.42, we separate the reference configuration of the bar into ne elements and n nodes, where the position of node i in the reference configuration is denoted by X_i . We will introduce a set of n base functions $\psi_i(X)$ constructed such that at each point in the bar we have

$$\sum_{i=1}^n \psi_i(X) = 1, \quad (1.272)$$

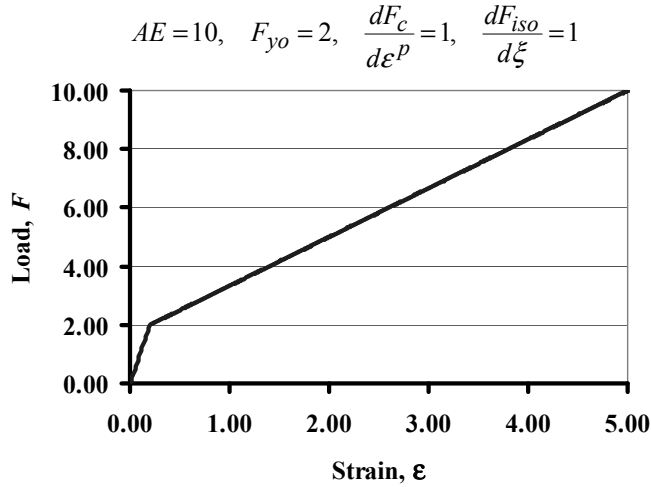


Figure 1.43: Load-strain response used in the numerical example.

and

$$\psi_i(X_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1.273)$$

Each trial function will be constructed from the base functions by an expression given

$$\phi(X) = \sum_{i=1}^n \phi_i \psi_i(X), \quad (1.274)$$

where ϕ_i are constant coefficients which later will be selected appropriately to construct a system of equations. In this same way, we will construct the finite element approximation for the displacement, velocity, and acceleration functions as

$$u(X, t) = \sum_{i=1}^n u_i(t) \psi_i(X), \quad v(X, t) = \sum_{i=1}^n v_i(t) \psi_i(X), \quad a(X, t) = \sum_{i=1}^n a_i(t) \psi_i(X), \quad (1.275)$$

where $u_i(t)$, $v_i(t)$, and $a_i(t)$ are the values of the displacement, velocity, and acceleration, respectively, at node i and at time t . These are related by the relation

$$a_i(t) = \frac{dv_i(t)}{dt} = \frac{d^2 u_i(t)}{dt^2}. \quad (1.276)$$

Substitution of these finite element approximations of the functions into the variational formulation (also called the weak form) of the balance of linear momentum yields

$$\begin{aligned} & F(L, t) \phi_n \psi_n(L) - F(0, t) \phi_1 \psi_1(0) - \sum_{i=1}^n \phi_i \int_0^L F(X, t) \frac{d\psi_i(X)}{dX} dX \\ & + \sum_{i=1}^n \phi_i \int_0^L w_o(X, t) \psi_i(X) dX = \sum_{i=1}^n \phi_i \int_0^L \rho_o(X) \psi_i(X) \sum_{j=1}^n a_j(t) \psi_j(X) dX. \end{aligned}$$

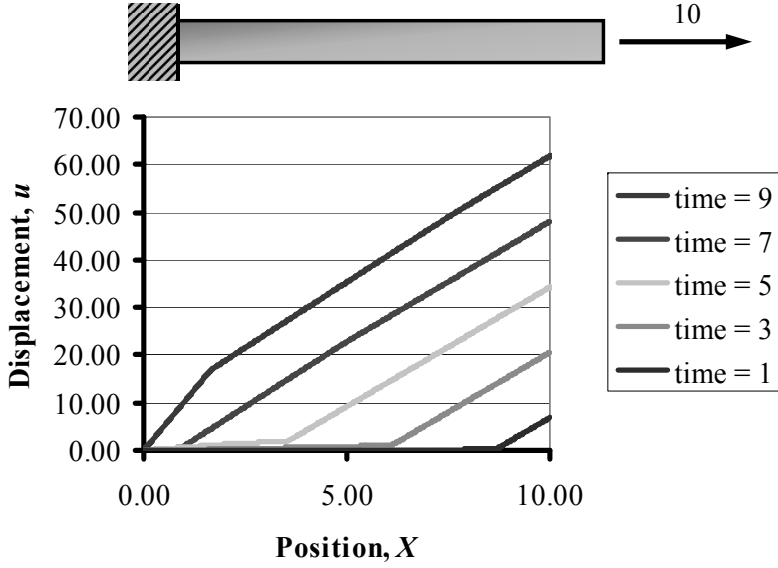


Figure 1.44: Displacement along the bar at different times for a bar loaded by a sudden constant load (calculated with 1000 two-node linear line elements with $\Delta t = 0.0001$).

Next, we systematically select a set of trial functions, each equal to one of the base functions. The result will be a set of n equations of the form

$$\begin{aligned}
 & -F(0, t) - \int_0^L F(X, t) \frac{d\psi_1(X)}{dX} dX + \int_0^L w_o(X, t) \psi_1(X) dX \\
 & = \sum_{j=1}^n \int_0^L \rho_o(X) \psi_1(X) \psi_j(X) dX a_j(t), \\
 & - \int_0^L F(X, t) \frac{d\psi_2(X)}{dX} dX + \int_0^L w_o(X, t) \psi_2(X) dX = \sum_{j=1}^n \int_0^L \rho_o(X) \psi_2(X) \psi_j(X) dX a_j(t), \\
 & - \int_0^L F(X, t) \frac{d\psi_3(X)}{dX} dX + \int_0^L w_o(X, t) \psi_3(X) dX = \sum_{j=1}^n \int_0^L \rho_o(X) \psi_3(X) \psi_j(X) dX a_j(t), \\
 & \dots \\
 & \dots \\
 & F(L, t) - \int_0^L F(X, t) \frac{d\psi_n(X)}{dX} dX + \int_0^L w_o(X, t) \psi_n(X) dX \\
 & = \sum_{j=1}^n \int_0^L \rho_o(X) \psi_n(X) \psi_j(X) dX a_j(t),
 \end{aligned}$$

where $F(0, t)$ and $F(L, t)$ are only included when displacement boundary conditions are not present. This system of equations can be written as

$$[M][a(t)] = [c(t)] + [F^*(t)], \quad (1.277)$$

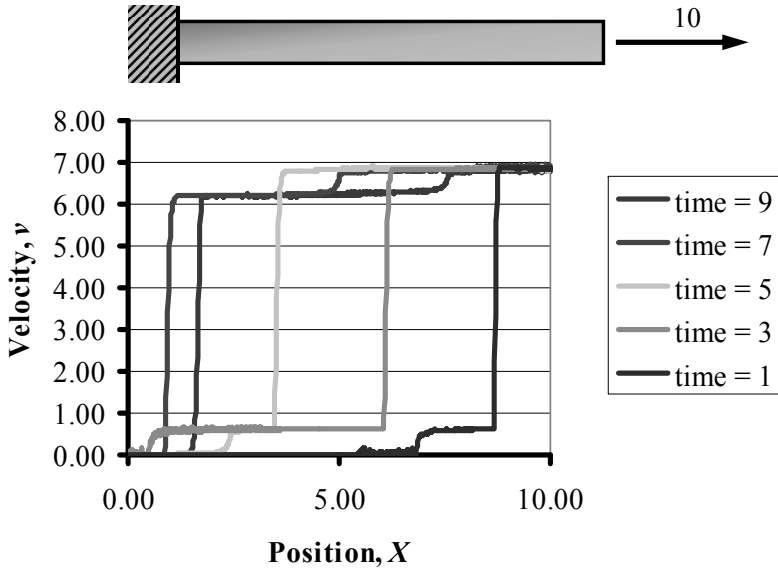


Figure 1.45: Velocity along the bar at different times for a bar loaded by a sudden constant load (calculated with 1000 two-node linear line elements with $\Delta t = 0.0001$).

where $[M]$ is a constant mass matrix given by the components

$$M_{ij} = \int_0^L \rho_o(X) \psi_i(X) \psi_j(X) dX, \quad (1.278)$$

$[a(t)]$ is the column vector of nodal values of the current acceleration $a_i(t)$, $[c(t)]$ is the column vector given by the components

$$c_i = - \int_0^L F(X, t) \frac{d\psi_i(X)}{dX} dX + \int_0^L w_o(X, t) \psi_i(X) dX, \quad (1.279)$$

and $[F^*(t)]$ is the column vector containing all zeros, other than at the row associated with the load boundary condition. In these rows, the load is inserted consistent with equation (1.278).

This system of equations can be solved by many different integration methods. The goal is to calculate the acceleration from this equation and then integrate it to find the velocity and displacement. The simplest method to formulate this problem is based on the Euler method that uses the approximations

$$a_i(t) = \frac{v_i(t + \Delta t) - v_i(t)}{\Delta t}, \quad v_i(t) = \frac{u_i(t + \Delta t) - u_i(t)}{\Delta t}. \quad (1.280)$$

The reader is cautioned that the Euler method is not the preferred method for numerical integration since it suffers from stability problems that require very small time increments. We will discuss this later. The basic idea behind the integration process is as follows. First, using the current values of the variables we solve $[M][a(t)] = [c(t)] + [F^*(t)]$ for the current acceleration $[a(t)]$. Next, we update both the displacement and velocity by using the expressions

$$[u(t + \Delta t)] = [u(t)] + [v(t)]\Delta t, \quad [v(t + \Delta t)] = [v(t)] + [a(t)]\Delta t.$$

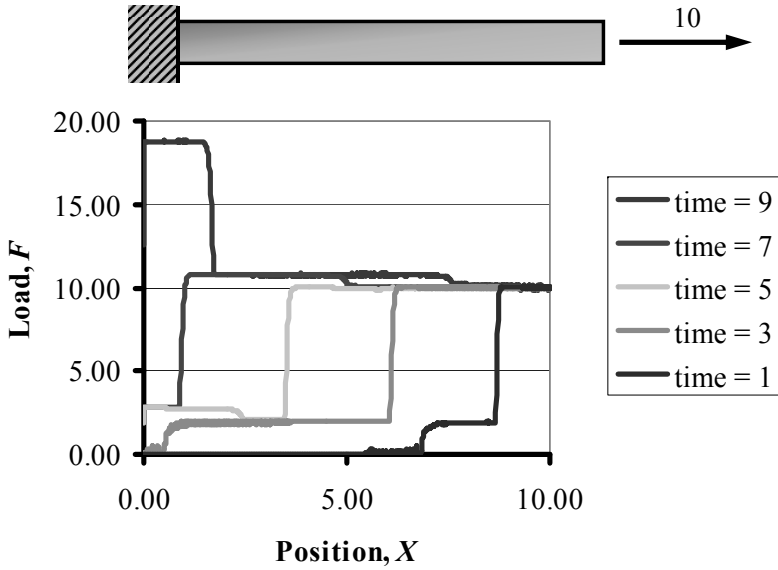


Figure 1.46: Load along the bar at different times for a bar loaded by a sudden constant load (calculated using 1000 two-node linear line elements with $\Delta t = 0.0001$).

This process is repeated as one marches through time.

A slight modification to the above method can be introduced that substantially will improve the numerical solution. First, since the mass matrix $[M]$ does not depend on time, we can calculate and invert it before starting the time integration. Therefore, the equations will be written as

$$[a(t)] = [M]^{-1}\{[c(t)] + [F^*(t)]\}.$$

By doing this, we replace the need to solve a set of simultaneous equations at each step of the time integration, replacing it by a simple matrix multiplication. We still will need to calculate $[c(t)]$ and $[F^*(t)]$ at each time, and then multiply $[M]^{-1}$ into it to get the acceleration. One additional change that can be introduced without much computational cost, and which substantially reduces the error in the time integration, is to replace the Euler integration method by a midpoint method, which is a second-order method (also known as second order Runge-Kutta method). This can be done by calculating the acceleration at the midpoint of the time step. This is particularly easy to implement since the right-hand side does not depend on the velocity, even though it does depend on the displacement. To implement the process, we first calculate the displacement at the midpoint of the time interval based on the known velocity at the beginning of the time interval. That is,

$$\left[u\left(t + \frac{1}{2}\Delta t\right)\right] = [u(t)] + \frac{1}{2}[v(t)]\Delta t.$$

Using this displacement and the time $t + \frac{1}{2}\Delta t$, we calculate $[c(t + \frac{1}{2}\Delta t)]$ and $[F^*(t + \frac{1}{2}\Delta t)]$, and then evaluate the acceleration at the midpoint by

$$\left[a\left(t + \frac{1}{2}\Delta t\right)\right] = [M]^{-1}\left\{\left[c\left(t + \frac{1}{2}\Delta t\right)\right] + \left[F^*\left(t + \frac{1}{2}\Delta t\right)\right]\right\}.$$

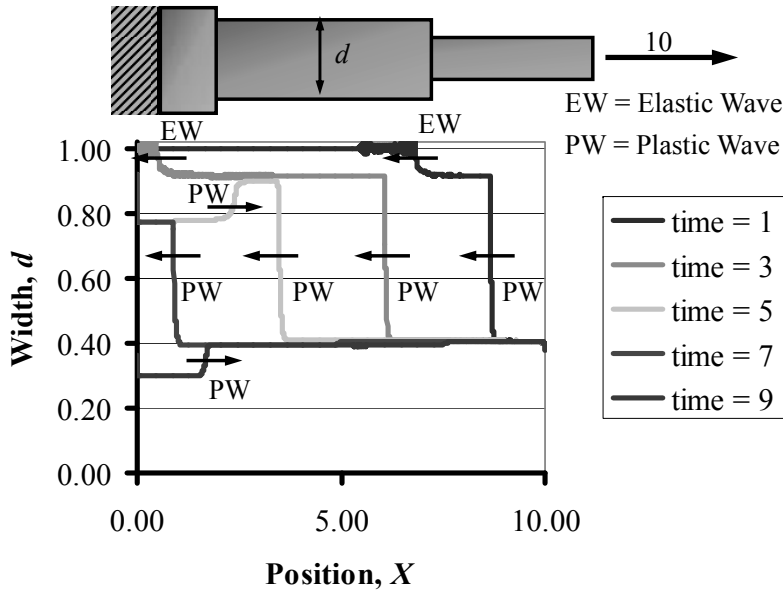


Figure 1.47: The width of the bar as a function of the position in reference configuration and time with the assumption of constant volume (calculated using 1000 two-node linear line elements with $\Delta t = 0.0001$).

Using this acceleration, we advance the velocity using the midpoint relation

$$[v(t + \Delta t)] = [v(t)] + \left[a \left(t + \frac{1}{2} \Delta t \right) \right] \Delta t.$$

Finally, we advance the displacement using the average of the velocity at time t and $t + \Delta t$ using the relation

$$[u(t + \Delta t)] = [u(t)] + \frac{1}{2} \{ [v(t)] + [v(t + \Delta t)] \} \Delta t.$$

These two modifications substantially reduce the computational time and increase the accuracy of the results. Alternate methods of time integration, such as the Stoermer's rule, can be used to accomplish the time integration more accurately, but pursuing these methods is outside the scope of this book.

Since the finite element method is used to resolve the spatial dependence of the variables, one has the choice of selecting higher-order elements to improve the spatial resolution of the variables, or to use more elements. In the example that follows, we have chosen to use two-node linear line elements, accomplishing the accuracy needed by increasing the number of elements used. This is not necessarily the most efficient method for solving the problem, but it satisfies our current objectives without introducing additional complexity. Those familiar with the finite element method should find it easy to implement higher-order elements.

As an example, here we look at the results of the implementation of this numerical algorithm for a bar of length $L = 10$, constant cross-sectional area $A = 1$ and density $\rho_o = 1$. It will be assumed that the bar behaves locally as an elastic-plastic material with

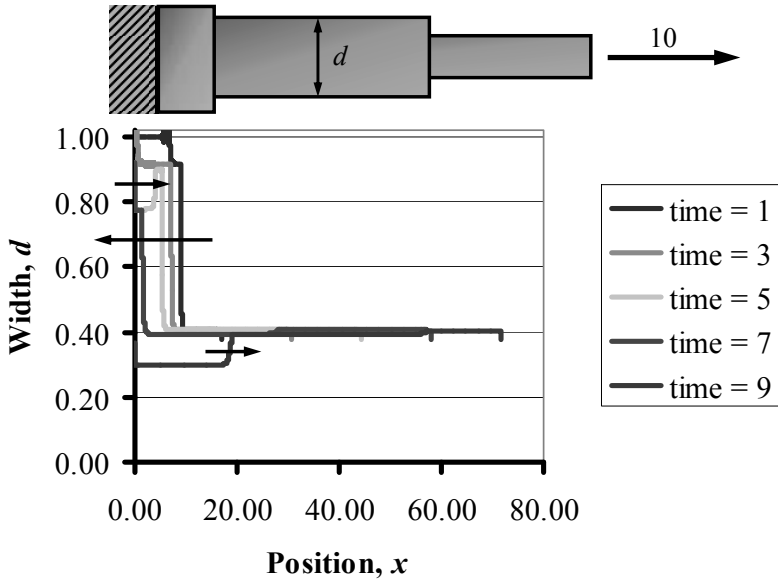


Figure 1.48: The width of the bar as a function of the position in the current configuration and time with the assumption of constant volume (calculated using 1000 two-node linear line elements with $\Delta t = 0.0001$).

combined isotropic and kinematic hardening described by the material constants

$$E = 10, \quad F_{yo} = 2.0, \quad \frac{dF_c}{d\epsilon^p} = 1.0, \quad \frac{dF_{iso}}{d\xi} = 1.0,$$

and using the strain hardening parameter $\dot{\xi} = |\dot{\epsilon}^p|$. Figure 1.43 shows the load-strain plot for these particular demonstration parameters. The numerical results are shown for the bar separated into 100 equal line elements, using the penalty method to impose a zero displacement boundary condition at the left end of the bar, with a load of 10 applied to the right-hand end of the bar, and using a zero distributed load. Figure 1.44 shows the displacement for nine different times. As can be seen, as time progresses a larger and larger portion of the bar is deformed. Figure 1.45 shows the velocity of points in the bar for the same nine times. As can be seen in this figure, there are three distinct regions at each time, each characterized by a constant velocity field. Closest to the applied load, there is a region of high velocity, next is a region of much lower velocity, and finally there is a region which has not yet been affected by the load and therefore is stationary. As a segment with constant velocity indicates rigid body motion of that segment, we can see in the figure that deformations are concentrated in very small regions where the transition from one velocity to another occur. This also is seen in Figure 1.46 for the load, which shows similar characteristics to the velocity. As will be shown in the following sections, the elastic wave speed v_e^* and the plastic wave speed v_p^* can be calculated for this problem from the equations

$$v_e^* = \sqrt{\frac{AE}{\rho_o}} = \sqrt{10} = 3.16, \quad v_p^* = \sqrt{\frac{F_-}{\rho_o \epsilon_-}} = \sqrt{\frac{10}{5}} = 1.41,$$

where $F_- = 10$ is the force behind the plastic wave, and $\epsilon_- = 5$ is the strain associated

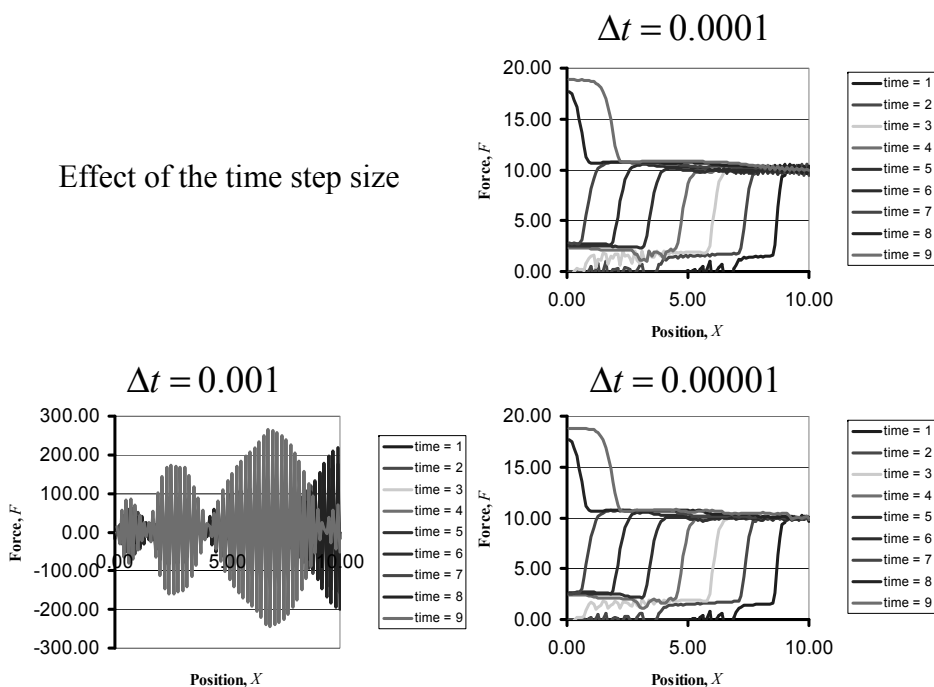


Figure 1.49: Effect of the time step size on the simulated load along the bar (calculated using 100 two-node linear line elements).

with this load as given in Figure 1.43. If we look at the response, in view of these two wave speeds, we see that the region at the right end is behind the plastic wave that moves to the left at a speed of 1.41. Ahead of the plastic wave is an elastic wave that defines the left end of the central region and which is moving to the left at a speed of 3.16. For sufficiently large times both waves reflect off the left end and start their motion to the right. This is shown most clearly in Figure 1.46, where the reflection of the elastic wave is a plastic wave with much lower speed and the reflection of the initial plastic wave increases the load from about 10 to 18. Figures 1.47 and 1.48 show the width of the bar assuming no volumetric changes from an initial cross-section of width 1 and depth 1. Figure 1.47 shows the width in terms of position in the reference configuration, and Figure 1.48 shows the width in terms of the position in the current configuration. As can be seen, for the parameters and loading selected, the bar starts at a length of 10 and in nine time units ends up at a length of 72. When the load is applied, the elastic wave, traveling ahead of the plastic wave, reduces the width from 1 to 0.92, and then the plastic wave further reduces it to 0.4. Once the elastic wave reflects from the wall, it becomes a plastic wave and reduces the width from 0.92 to 0.78. After the plastic wave reflects from the fixed end, as a result of the jump in the load, it reduces the width from 0.4 to 0.3.

The Euler method is very sensitive to the time step size, as can be seen in Figure 1.49 showing how the time step size of 0.001 results in oscillations, while step sizes of 0.0001 and smaller show smaller oscillations in the load. The same is true in the velocity and the displacement. Figure 1.50 shows that the displacement has fewer oscillations, yet both the

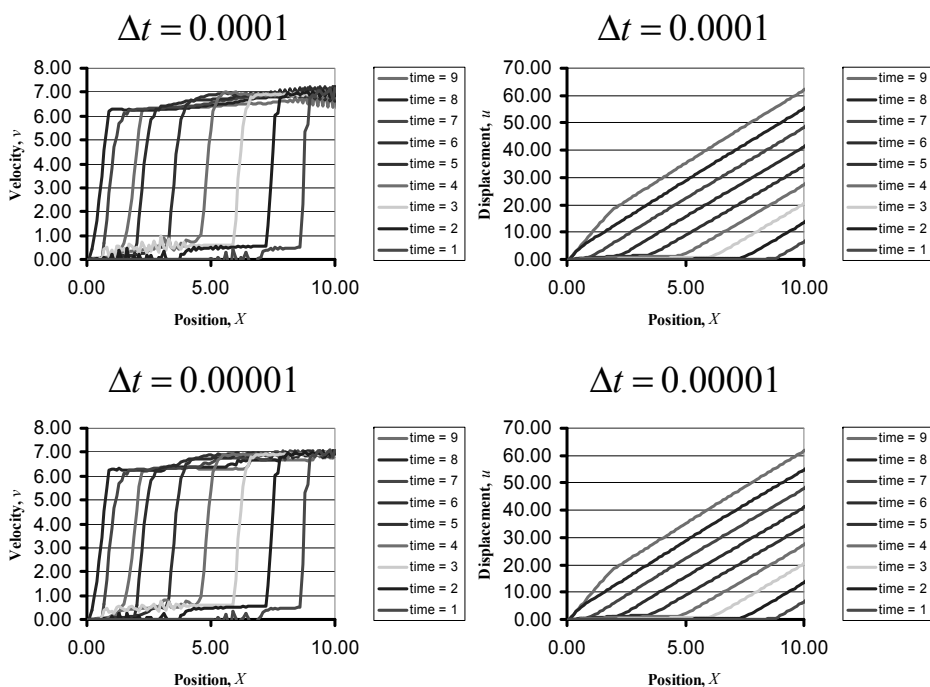


Figure 1.50: Effect of the time step size on reducing oscillations in the simulation of displacement and velocity (calculated using 100 two-node linear line elements).

velocity and displacement show fewer oscillations as the time step becomes smaller.

1.20 Dealing with discontinuities: Jump conditions

We need to be aware of discontinuities and how they affect our analysis of the response of a bar. For example, all the differential forms of the balance laws presented above depend on continuity in some of the field variables. In some cases, it is simpler to model a variable which changes over a short interval of space as a discontinuity rather than fully analyze the shape of the change. In this section, we will look at how to derive special jump conditions across a discontinuity in order to deal with a moving discontinuity in the bar.

Figure 1.51 shows a typical setting where there is a discontinuity moving with a speed of v^* in the bar. The differential balance equations can be applied separately to the right and left sides of the discontinuity, where the fields are continuous, but they cannot be integrated through the discontinuity. The jump conditions that we will develop can be used to relate the values of the field variable on the two sides of the discontinuity, allowing the solutions obtained on one side of the discontinuity to be related to those obtained for the other side.

Before we proceed to obtain the jump conditions, let us generalize and summarize the

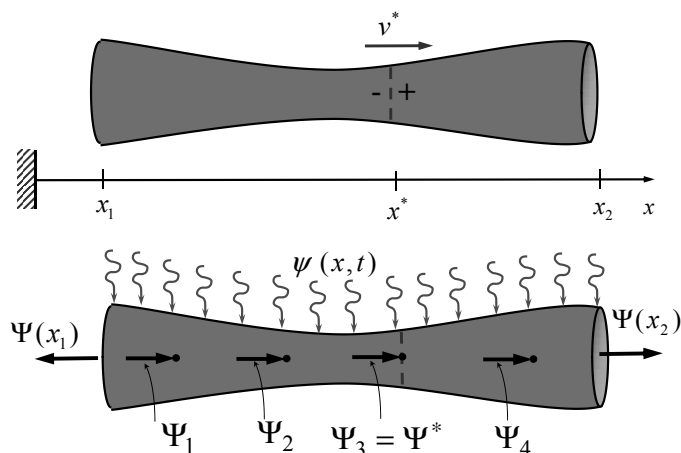


Figure 1.51: A moving discontinuity in a bar.

balance laws as follows. All of the balance laws can be written as

$$\boxed{\frac{D}{Dt} \left(\int_{x_1(t)}^{x_2(t)} \phi \rho dx \right) \begin{matrix} \geq \\ = \end{matrix} \int_{x_1(t)}^{x_2(t)} \psi dx + \Psi(x_2) - \Psi(x_1) + \sum \Psi_i,} \quad (1.281)$$

which must hold for every material bar segment $(x_1, x_2) \subseteq \mathcal{B}(t)$, where $\mathcal{B}(t)$ is the current configuration of the bar. The specific balance laws can be obtained from this generic form by the following substitution:

Generic	ϕ	ψ	Ψ
Conservation of mass	1	0	0
Balance of linear momentum	v	w	F
Balance of energy	$e + \frac{v^2}{2}$	$r + wv$	$Fv - q$
Entropy production inequality	η	$\frac{r}{\theta}$	$-\frac{q}{\theta}$

Figure 1.51 also shows a material segment and the loading on it. Equation (1.281) is more general than the balance equations previously presented in that the last term $\sum \Psi_i$ has been added to allow point sources at any point inside the bar. For example, a point load now may be added on the bar at any point along it. This point load must be added to the sum of forces in the balance of linear momentum, and must be accounted for in calculating the rate of work done on the bar in the balance of energy.

We have assumed that at point x^* we have a discontinuity that is moving at a speed v^* , as shown in Figure 1.51. The integral in the left-hand side of our generic equation of balance can be separated into two integrals as follows:

$$\int_{x_1(t)}^{x_2(t)} \phi \rho dx = \int_{x_1(t)}^{x^*(t)} \phi \rho dx + \int_{x^*(t)}^{x_2(t)} \phi \rho dx.$$

The reader will recall from calculus that the derivative of an integral with variable limits can be calculated from the relation

$$\frac{d}{dt} \left[\int_{a(t)}^{b(t)} f(x, t) dx \right] = f(b, t) \frac{db(t)}{dt} - f(a, t) \frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx.$$

Applying this to calculate the time derivative in our generic balance equation yields

$$\frac{D}{Dt} \left(\int_{x_1(t)}^{x_2(t)} \phi \rho dx \right) = \phi_- \rho_- v^* - \phi_1 \rho_1 v_1 + \int_{x_1}^{x^*} \frac{\partial \phi \rho}{\partial t} dx + \phi_2 \rho_2 v_2 - \phi_+ \rho_+ v^* + \int_{x^*}^{x_2} \frac{\partial \phi \rho}{\partial t} dx,$$

where a subscript “-” indicates the value of the quantity for a particle just to the left of the discontinuity (just to the left of x^*), and a “+” indicates the value of the quantity for a particle just to the right of the discontinuity. We should note that $x_1(t)$ and $x_2(t)$ are the positions of specific particles of the bar, and, therefore, when we take their time derivatives, we get the velocities of these particles, which we denote by v_1 and v_2 , respectively, but $x^*(t)$ is the position of the discontinuity and not any specific particle so that its time derivative is the speed at which the discontinuity moves (i.e., v^*). Substituting this relation into our generic balance law results in

$$\phi_- \rho_- v^* - \phi_1 \rho_1 v_1 + \phi_2 \rho_2 v_2 - \phi_+ \rho_+ v^* + \int_{x_1}^{x_2} \frac{\partial \phi \rho}{\partial t} dx \geq \int_{x_1}^{x_2} \psi dx + \Psi(x_2) - \Psi(x_1) + \sum \Psi_i,$$

$$=$$

which must hold for any segment (x_1, x_2) of the bar that contains the discontinuity. To arrive at the jump conditions, we let x_1 and x_2 approach x^* . The values of the two integrals go to zero, $\phi_1 \rho_1 \rightarrow \phi_- \rho_-$, $\phi_2 \rho_2 \rightarrow \phi_+ \rho_+$, and the only sources remaining in the interval will be the ones at x^* . This process results in our jump conditions

$$\phi_+ \rho_+ (v_+ - v^*) - \phi_- \rho_- (v_- - v^*) \geq \Psi(x_+^*) - \Psi(x_-^*) + \Psi^*,$$

$$=$$

where v_- and v_+ are, respectively, the velocities of the particles to the left and right of the discontinuity. Using the notation

$$|[\Upsilon]| \equiv \Upsilon_+ - \Upsilon_-, \quad (1.282)$$

one can write the generic jump condition as

$$|[\phi \rho (v - v^*)]| \geq |[\Psi]| + \Psi^*.$$

$$=$$

$$(1.283)$$

Section 1.20-Problems:

1.20-1: Write the jump conditions for each of the balance laws and show that you obtain

$$|[\rho(v - v^*)]| = 0, \quad (1.284)$$

$$|[\rho v(v - v^*)]| = |[F]| + F^*, \quad (1.285)$$

$$\left| \left[\rho \left(e + \frac{v^2}{2} \right) (v - v^*) \right] \right| = |[Fv - q]| + (Fv - q)^*, \quad (1.286)$$

$$|[\rho \eta(v - v^*)]| \geq - \left| \left[\frac{q}{\theta} \right] \right| - \frac{q^*}{\theta^*}. \quad (1.287)$$

Expand the equations for the conservation of mass and balance of linear momentum.

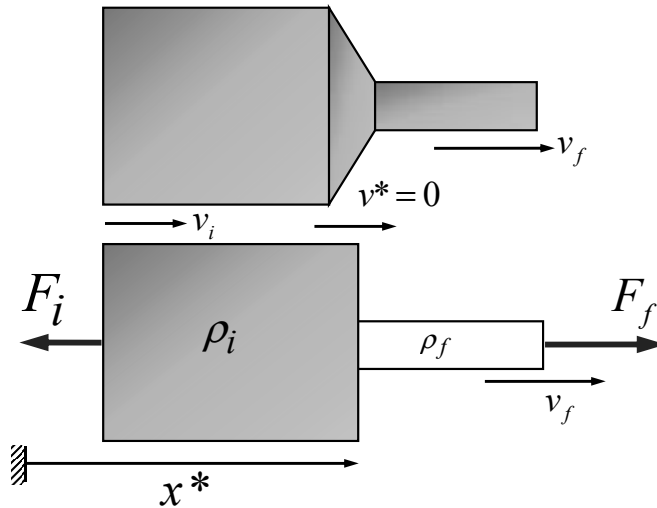


Figure 1.52: Steady-state plastic drawing of a bar modeled by a stationary jump.

- 1.20-2:** There is a material discontinuity at x^* such that the material properties abruptly change. Write out the jump conditions across this discontinuity.
- 1.20-3:** In the steady-state drawing process, shown in Figure 1.52, material enters the drawing zone at a speed v_i and is drawn down by stretching it so that the material leaves at a speed $v_f = nv_i$. Since the drawing zone is small, approximate it by a discontinuity as shown in the figure. If the material behaves in a way such that $F = AE \ln(\lambda)$ and the reference density of the material is given by ρ_o , calculate the initial and final densities, the initial and final stretches, and the initial and final load.
- 1.20-4:** A large mass B is impacted against end A of a bar, as shown in Figure 1.53. It is assumed that the mass generates a sufficient impulse to give the particle at A a velocity v_o equal to that of B . As a result of the impact, a wave propagates from A into the otherwise stationary linear elastic bar. Calculate the initial velocity v^* of the wave front just after impact. Can you say anything about how the wave front velocity will change as it moves forward? Is there an impact velocity v_o above which your analysis is not valid?
- 1.20-5:** Write the jump conditions when a bar is in static equilibrium.

1.21 Plastic drawing of bars

Let us consider the drawing of a bar. As is shown in Figure 1.52, material comes in at a velocity of v_i , is drawn down over a small interval, and is removed with a velocity of $v_f = nv_i$. As a first approximation we can consider the drawing down to occur abruptly,

indicating a jump. We will consider the steady-state problem in which the spatial location of the drawdown does not change so that the velocity of the discontinuity is $v^* = 0$. Also, it will be assumed that outside the drawing zone (i.e., the jump), the deformation is homogeneous with a constant load F_i and density ρ_i before the drawing zone and a constant load F_f and density ρ_f after the drawing zone. It will be assumed that the load in each segment of the bar is given by a constitutive equation of the form

$$F = AE\epsilon_l^e = AE(\epsilon_l - \epsilon_l^p), \quad (1.288)$$

to yield

$$F = AE[\ln(\lambda) - \ln(\lambda^p)] = AE \ln \left(\frac{\lambda}{\lambda^p} \right), \quad (1.289)$$

where A is the initial cross-sectional area (before loading) and E is the elastic modulus at zero load.

The conservation of mass on the two sides of the jump require that

$$\boxed{\rho_i \lambda_i = \rho_o}, \quad \boxed{\rho_f \lambda_f = \rho_o}. \quad (1.290)$$

Balance of linear momentum in each of these regions is satisfied since the acceleration is zero (constant velocity) and the loads are constant.

The jump condition due to the conservation of mass is

$$\rho_+(v_+ - v^*) - \rho_-(v_- - v^*) = 0.$$

Introducing the above assumptions into this equation yields

$$\boxed{\rho_f v_f = \rho_i v_i}. \quad (1.291)$$

The jump condition for the balance of linear momentum is

$$\rho_+ v_+ (v_+ - v^*) - \rho_- v_- (v_- - v^*) = F_+ - F_-.$$

Substitution of the above assumptions into this equation gives

$$\rho_f v_f^2 - \rho_i v_i^2 = AE \ln \left(\frac{\lambda_f}{\lambda_f^p} \right) - AE \ln \left(\frac{\lambda_i}{\lambda_i^p} \right). \quad (1.292)$$

Substitution for the density from (1.290) into (1.291) yields the relation

$$\boxed{\frac{\lambda_f}{\lambda_i} = n}. \quad (1.293)$$

Substitution of the expression for density from (1.290) into the jump condition for the balance of linear momentum given in (1.292) yields

$$\frac{\rho_o}{\lambda_f} v_f^2 - \frac{\rho_o}{\lambda_i} v_i^2 = AE \ln \left(\frac{\lambda_f \lambda_i^p}{\lambda_f^p \lambda_i} \right).$$

Introduction of (1.293) and reorganization yields

$$\boxed{\frac{\rho_o v_i^2 (n - 1)}{\lambda_i} = AE \ln \left(n \frac{\lambda_i^p}{\lambda_f^p} \right)}. \quad (1.294)$$

If one assumes both the initial and final loads to be in the elastic range (i.e., $\lambda_i^p = \lambda_f^p = 1$), then one can solve (1.294) directly for the initial stretch and in view of (1.293) one gets

$$\lambda_i = \frac{\rho_o v_i^2 (n-1)}{AE \ln(n)}, \quad \lambda_f = \frac{\rho_o v_i^2 n (n-1)}{AE \ln(n)}. \quad (1.295)$$

Using these one can find the initial and final loads as $F_i = AE \ln(\lambda_i)$ and $F_f = AE \ln(\lambda_f)$. The initial and final densities can be obtained as $\rho_i = \rho_o / \lambda_i$ and $\rho_f = \rho_o / \lambda_f$.

Now let us assume the material comes into the drawing process under elastic conditions and is only subjected to plastic flow during the drawing so that $\lambda_i^p = 1$. In this case, using relation (1.293) for λ_i , one can rewrite (1.294) as

$$\lambda_f \ln \left(\frac{n}{\lambda_f^p} \right) = \frac{\rho_o v_i^2 n (n-1)}{AE}. \quad (1.296)$$

This represents one relation for the two unknowns, λ_f and λ_f^p , in terms of the process parameters v_i and $v_f = nv_i$, and the other parameters represented by ρ/AE . The other relation needed is obtained from the load-strain plot. The material enters the drawing zone as an elastically loaded one, it is monotonically loaded to beyond its yield point, and, therefore, comes out of the drawing zone at the current yield load. Since we have load given by (1.289), we can write

$$\frac{F}{AE} = \ln(\lambda) - \ln(\lambda^p).$$

Substitution into (1.296), and considering that the load is the current yield load, results in

$$\lambda_f [\ln(n) + \frac{F_y}{AE} - \ln(\lambda_f)] = \frac{\rho_o v_i^2 n (n-1)}{AE}. \quad (1.297)$$

To obtain λ_f , we need to search on the monotonic loading plot to find the point at which the current stretch and yield load satisfy (1.297) for the given parameters. Obviously, some process parameters might be such that satisfaction of this relation is not possible by the loading characteristics of the material's load-strain diagram. Once we find a point (λ_f, F_y) on the monotonic stretch-load plot that satisfies (1.297), we can get the stretch from the relation $\lambda_i = \lambda_f / n$, and then we can get the initial load from evaluating the load associated with λ_i on the monotonic stretch-load plot. One can estimate the reduction in the cross-sectional area through the assumption of negligible volume changes in the plastic flow, a common assumption in metal plasticity. This requires that $\lambda_f^p A_{fo} = A$, where A_{fo} is the final cross-sectional area after unloading, and, as stated above, A is the initial cross-sectional area before loading. The area A_{fo} accounts for the elastic spring-back after unloading.

1.22 Elastic and plastic (shock) waves in a bar

A direct application of the jump conditions is seen in wave propagation. Here we want to look at the movement of a disturbance in the form of a discontinuity into a stationary bar. This type of wave is referred to as a shock wave. Our example is characterized by the impact problem where we hit a massive rigid object to one end of an unloaded stationary bar and investigate the propagation of the disturbance through the bar. Figure 1.53 shows the problem under consideration. Let us only consider the mechanical problem. The jump conditions of concern at the leading edge of the disturbance are those resulting from the conservation of mass and balance of linear momentum. These are

$$\begin{aligned} \rho_+(v_+ - v^*) - \rho_-(v_- - v^*) &= 0, \\ \rho_+ v_+ (v_+ - v^*) - \rho_- v_- (v_- - v^*) &= F_+ - F_-. \end{aligned} \quad (1.298)$$

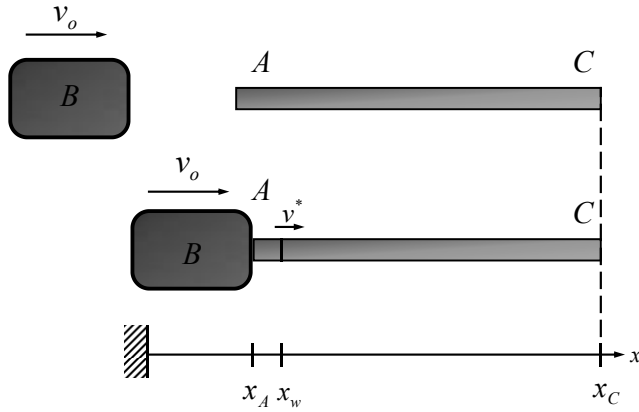


Figure 1.53: Impact of a rigid mass against one end of a stationary bar and the propagation of the disturbance through the bar.

Note that, as shown in the figure, we are assuming no point loads at the discontinuity. In front of the disturbance, the bar is unloaded and stationary so that $F_+ = 0$ and $v_+ = 0$. The density is the same as that of the reference configuration ρ_o if the initial unloaded configuration is taken as the reference. Substitution of these values into the two jump conditions yields

$$\begin{aligned} -\rho_o v^* &= \rho_-(v_- - v^*), \\ \rho_-(v_- - v^*) &= F_-. \end{aligned}$$

Solving the first equation for ρ_- , substituting it into the second equation, and solving for F_- gives

$$\begin{aligned} \rho_- &= -\frac{\rho_o v^*}{v_- - v^*}, \\ F_- &= -\rho_o v^* v_-. \end{aligned}$$

In addition to these relations, we know from conservation of mass that behind the disturbance $\rho_- \lambda_- = \rho_o$ so that

$$\lambda_- = \frac{\rho_o}{\rho_-} = 1 - \frac{v_-}{v^*}. \quad (1.299)$$

Therefore, one has the strain ϵ_- given by

$$\epsilon_- = -\frac{v_-}{v^*}. \quad (1.300)$$

Evaluation of v_- from the expression for F_- , substitution into the expression for ϵ_- and reorganization gives

$$v^* = \sqrt{\frac{F_-}{\rho_o \epsilon_-}}. \quad (1.301)$$

Back substitution to get v_- yields

$$v_- = \sqrt{\frac{F_- \epsilon_-}{\rho_o}}. \quad (1.302)$$

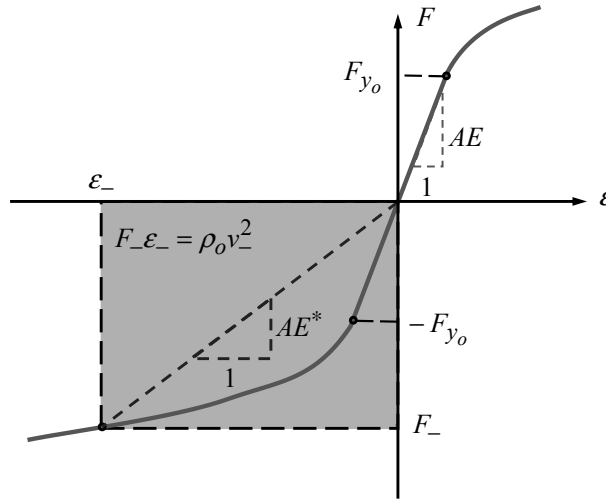


Figure 1.54: Relation between impact velocity and the product $F_- \epsilon_-$, and the interpretation of the effective slope AE^* .

There will be an elastic wave propagating through the bar if the load $F_- < F_{yo}$. In this case, $F_- = AE \epsilon_-$, and, as a result, the speed of the disturbance traveling into the bar is given by

$$v^* = \sqrt{\frac{AE \epsilon_-}{\rho_o \epsilon_-}} = \sqrt{\frac{AE}{\rho_o}},$$

which is a constant. Therefore, the speed of a linear elastic disturbance moving through a bar is constant, irrespective of the speed of the impacting mass. It is worth mentioning that since the density for a bar is given in terms of mass per unit length of the bar, the value of ρ_o/A will give the average density per unit volume of the bar at each cross-section. This should remove any concern the reader might have about the difference between this expression and those found in other texts since the density in terms of mass per unit volume is used in most cases other than for the study of bars, and as a result the elastic wave speed is expressed in the form $\sqrt{E/\rho}$. The maximum speed that one can impact the bar and still get an elastic wave is determined by the load reaching the initial yield load in compression. Letting $F_- = -F_{yo}$ in the expression for F_- , and substituting the elastic wave speed for v^* , we get

$$-F_{yo} = -\rho_o v_- \sqrt{\frac{AE}{\rho_o}},$$

which yields the maximum impact velocity v_o for a linear elastic wave as

$$(v_o)_{max-elastic} = \sqrt{\frac{F_{yo}^2}{\rho_o AE}}.$$

If the maximum impact velocity for the elastic response is exceeded, then the impact load will result in plastic flow in the bar, and we will get both elastic and plastic waves propagating through the bar. To calculate the initial speed of the plastic wave, we use the expression for v_- with the impact velocity v_o to get the value of the product $F_- \epsilon_-$ as

$$\boxed{F_- \epsilon_- = \rho_o v_o^2.} \quad (1.303)$$

The product $F_- \epsilon_-$ is depicted in Figure 1.54 on the load-strain plot as the area of the colored rectangular region. From the monotonic load-strain plot in compression, we evaluate the compressive load F_- and compressive strain ϵ_- that will yield the product $F_- \epsilon_- = \rho_o v_o^2$. Next, we evaluate $AE^* = F_- / \epsilon_-$ to get the wave speed

$$v^* = \sqrt{\frac{AE^*}{\rho_o}}.$$

As can be seen in Figure 1.54, for a typical elastic-plastic response, AE^* is smaller than AE , which results in the plastic wave having a slower speed than the elastic wave. Since plastic flow is dissipative, the load driving the wave drops as the wave propagates into the bar, resulting in a gradual increase of the ratio $AE^* = F_- / \epsilon_-$ toward the value AE of the elastic response. Therefore, the plastic wave increases its speed as it propagates into the bar until it reaches that of the elastic wave and then continues to move into the bar with the constant speed of the elastic wave. The driving force of such a wave would be the initial yield load of the material. It should be noted that this yield load would reflect the true transition from elastic to elastic-plastic response and, therefore, would be the limit of elastic response. As such, this load might be past the linear elastic limit since it also would include any nonlinear elastic portion of the response. To follow the transition of the plastic wave to an elastic wave, it is necessary to solve the balance of linear momentum for the growing region behind the wave, coupled with the equations of the rigid impacting mass, and subject both to the jump conditions at the wave front and interaction conditions between the rigid mass and the bar.

Section 1.22-Problems:

- 1.22-1:** Set up a numerical procedure to solve the impact problem discussed for plastic impact. Write a program to implement this procedure and use it to follow the motion of a plastic wave.

1.23 General comment on selection of moduli

Throughout this chapter, we have looked at models that idealize material response. Most materials don't behave like the models presented, but under special conditions their response can be approximated by such models. A point that cannot be overemphasized is that the models and their associated material parameters must always be selected with the application of the model in mind. For example, when studying impact on an elastic bar, we should select E to be the axial wave speed modulus, while in the quasi-static compression of the same bar we would select E to be the Young's modulus. For the same material, the axial wave speed modulus might be double the Young's modulus so the resulting error from taking the wrong modulus might be substantial. The axial wave modulus is based on assuming zero strains in the lateral direction and can be measured in a confined compression test, while the Young's modulus assumes zero stress in the lateral direction and normally is measured in a tension test.

Another common assumption is to ignore rate effects. In such cases, we should take care to select the material parameters associated with the rate of loading in the application. For example, ultrasonic moduli for the same parameter normally are higher than their quasi-static counterparts. In glassy polymers, this difference can be as much as 20%.

1.24 Notation and summary

The following table shows the symbols used in the text and their definitions.

$\dot{}$		material time derivative
$\diamond$		time derivative for constant plastic strain and hardening parameter
a		acceleration or the tangent modulus during plastic flow
A		cross-sectional area (normally same as A_o)
A_o		cross-sectional area in reference configuration
$\mathcal{B}$		bar
$\mathcal{B}_o$		reference configuration of bar
$\mathcal{B}(t)$		current configuration of bar
$\frac{D}{Dt}$		material time derivative
e		specific internal energy
E		entropy or elastic modulus (normally same as E_o)
E_o		elastic modulus
EF		entropy flow due to heat into a segment Ω of the bar
f		yield function or generic continuous function
$\diamond f$		$\dot{f}$ calculated assuming constant plastic strain and hardening
F		axial load
F^b		back load (thermodynamic load associated with ϵ^p , similar to F_c)
F_c		center of elastic range (controlled by kinematic hardening)
F_{iso}		radius of elastic range (controlled by isotropic hardening)
F_y		yield load
F^θ		thermodynamic load associated with ϵ^θ
g		temperature gradient ($d\theta = gdx$; $g = \partial\theta(x, t)/\partial x$)
G		temperature gradient ($d\theta = GdX$; $G = \partial\theta(X, t)/\partial X$)
		shear modulus
G_o		shear modulus
IE		internal energy
k		coefficient of thermal conductivity
KE		kinetic energy
L		velocity gradient tensor [$dv = Ldx$, $L = \partial v(x, t)/\partial x = \dot{\lambda}/\lambda$]
m		mass
M		mass
n		hardening exponent
P		linear momentum
q		heat flux
$\dot{Q}$		heat flow into a segment Ω of the bar (heat added per unit time)
r		radiation or heat generation density (per unit current length)
t		time
u		displacement
v		velocity
V		volume
w		distributed load (load per unit current length of bar)
W		work done on Ω
w_o		distributed load (per unit length in reference configuration of bar)
x		position in current configuration κ
$x(X, t)$		motion function
X		position in reference configuration κ_o

α		coefficient of thermal expansion
β		scaling factor for plastic flow
η		specific entropy (entropy per unit mass)
θ		temperature
κ		current configuration or bulk modulus
κ_o		reference configuration
λ		stretch ratio
λ^e		elastic stretch ratio
λ^p		plastic stretch ratio
λ^θ		thermal stretch ratio
μ		viscosity
ϵ		strain
ϵ^e		elastic strain
ϵ_l		logarithmic strain
ϵ_l^e		logarithmic elastic strain
ϵ^p		plastic strain
ϵ_l^p		logarithmic plastic strain
ϵ^θ		thermal strain
ϵ_l^θ		logarithmic thermal strain
ξ		hardening parameter
ρ		density in terms of mass per unit current length ($\rho\lambda = \rho_o$)
ρ_o		density in reference configuration
ψ		specific free energy (free energy per unit mass)
Φ		generic symbol for a property of Ω
Ω		segment of the bar
Ω_o		Ω in reference configuration
$\Omega(t)$		Ω in current configuration

Models for plasticity:

1. Kinematic separation of strain:

- Stretch: $\lambda = \lambda^e \lambda^p \lambda^\theta$
- Strain: $\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta + \epsilon^e \epsilon^p + \epsilon^e \epsilon^\theta + \epsilon^p \epsilon^\theta + \epsilon^e \epsilon^p \epsilon^\theta$
- Infinitesimal strain: $\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta$
- Logarithmic strain: $\epsilon_l = \epsilon_l^e + \epsilon_l^p + \epsilon_l^\theta$

2. Axial load:

- Linear elastic: $F = AE\epsilon^e$
- Nonlinear elastic: $F = AE\epsilon_l^e$ or $F = AG(\lambda^e - \frac{1}{\lambda^{e2}})$

3. Yield function:

- No hardening: $f = |F| - F_{yo}$
- Isotropic hardening: $f = |F| - F_y$
- Kinematic hardening: $f = |F - F_c| - F_{yo}$
- Combined isotropic and kinematic hardening: $f = |F - F_c| - F_{iso}$

4. Hardening parameters:

- Plastic strain hardening: $\dot{\xi} = |\dot{\epsilon}^p|$ with initial condition $\xi_o = 0$

- Plastic work hardening: $\dot{\xi} = F\dot{\epsilon}^p$ with initial condition $\xi_o = 0$
5. Plastic flow condition: $f = 0$ and $\dot{f} > 0$
- Yield surface: $f = 0$
 - $\dot{f}$: $\dot{f}$ calculated based on assuming zero plastic strain rate and zero hardening rate
6. Flow rules:
- Temperature- and rate-independent plasticity:

$$\dot{\epsilon}^p = \begin{cases} \beta \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0 \\ 0 & \text{for all other cases} \end{cases}$$

- Temperature-dependent and rate-independent plasticity:

$$\dot{\epsilon}^p = \begin{cases} \beta_{\dot{\theta}} \dot{\theta} + \beta_{\epsilon} \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0 \\ 0 & \text{for all other cases} \end{cases}$$

- Rate-dependent plasticity:

$$\dot{\epsilon}^p = \begin{cases} \dot{\epsilon}^{p\dagger}(X, \epsilon^e, \epsilon^p, \epsilon^{\theta}, \theta, \xi) & \text{if } f > 0 \\ 0 & \text{if } f \leq 0 \end{cases}$$

for example:

$$\dot{\epsilon}^p = \begin{cases} \text{sign}(F - F_c) \frac{f}{\mu} & \text{if } f > 0 \\ 0 & \text{If } f \leq 0 \end{cases}$$

- Load control:

$$\dot{\epsilon}^p = \begin{cases} \beta_{\dot{F}} \dot{F} + \beta_{\dot{\theta}} \dot{\theta} & \text{if } f = 0 \text{ and } \dot{f} > 0 \\ 0 & \text{for all other cases} \end{cases}$$

7. Consistency condition:

- Temperature- and rate-independent plasticity: $\dot{f} = 0$ during plastic flow
- Temperature-dependent and rate-independent plasticity: $\dot{f} = 0$ during plastic flow
- Rate-dependent plasticity: No consistency condition
- Load control: $\dot{f} = 0$ during plastic flow

Different forms of the balance laws:

1. Conservation of mass:

- $\frac{D}{Dt}(\int_{\Omega(t)} \rho dx) = 0$
- $\rho \lambda = \rho_o$

- $\frac{D\rho}{Dt} + \rho L = 0$
- $\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$

2. Balance of linear momentum:

- $F(x_B, t) - F(x_A, t) + \int_{\Omega(t)} w(x, t) dx = \frac{D}{Dt} [\int_{\Omega(t)} v(x, t) \rho(x, t) dx]$
- $\frac{\partial F(x, t)}{\partial x} + w(x, t) = \rho(x, t) a(x, t)$
- $\frac{\partial F}{\partial X} + w_o = \rho_o a$

3. Balance of work and energy:

- $F(x_B, t) v(x_B, t) - F(x_A, t) v(x_A, t) + \int_{\Omega(t)} w(x, t) v(x, t) dx + q(x_A, t) - q(x_B, t) + \int_{\Omega(t)} r(x, t) dx = \frac{D}{Dt} [\int_{\Omega(t)} e(x, t) \rho(x, t) dx + \int_{\Omega(t)} \frac{v^2(x, t)}{2} \rho(x, t) dx]$
- $\rho(x, t) \frac{De}{Dt}(x, t) = F(x, t) L(x, t) - \frac{\partial q(x, t)}{\partial x} + r(x, t)$

4. The entropy production inequality:

- $\frac{D}{Dt} [\int_{\Omega(t)} \eta(x, t) \rho(x, t) dx] \geq \frac{q(x_A, t)}{\theta(x_A, t)} - \frac{q(x_B, t)}{\theta(x_B, t)} + \int_{\Omega(t)} \frac{r(x, t)}{\theta(x, t)} dx$
- $\rho(x, t) \frac{D\eta}{Dt}(x, t) \geq -\frac{\partial}{\partial x} [\frac{q(x, t)}{\theta(x, t)}] + \frac{r(x, t)}{\theta(x, t)}$
- $\rho(x, t) \frac{D\psi}{Dt}(x, t) - F(x, t) L(x, t) + \rho(x, t) \eta(x, t) \frac{D\theta}{Dt}(x, t) + \frac{q(x, t)}{\theta(x, t)} g(x, t) \leq 0$

Generalized balance laws and jump conditions:

1. Generalized balance laws:

$$\frac{D}{Dt} \left(\int_{x_1(t)}^{x_2(t)} \phi \rho dx \right) \overset{\geq}{=} \int_{x_1}^{x_2} \psi dx + \Psi(x_2) - \Psi(x_1) + \sum \Psi(x_i)$$

Generic	ϕ	ψ	Ψ
Conservation of mass	1	0	0
Balance of linear momentum	v	w	F
Balance of energy	$e + \frac{v^2}{2}$	$r + wv$	$Fv - q$
Entropy production inequality	η	$\frac{r}{\theta}$	$-\frac{q}{\theta}$

2. Jump conditions:

$$|[\phi \rho(v - v^*)]| \overset{\geq}{=} |[\Psi]| + \sum \Psi(x^*)$$

Chapter 2

Vectors and tensors

In this chapter, we look primarily at background material in mathematics and, in the process, introduce the associated methods and notation. We will cover matrix algebra, vectors and coordinate systems, tensors, and the associated algebra and calculus.

2.1 Matrix algebra

In this section, we look at matrix algebra as it relates to continuum mechanics and thermodynamics. As such, our review will be limited and will focus on those aspects needed for understanding the later chapters.

The section starts with a section on index or indicial notation, which is a simple method of removing summation symbols from the notation. Although simple, this notation brings substantial ease to writing many complex expressions. The second part of the section looks directly at matrix operations, their algebra, notation, definitions, and a variety of different results. Included in this review is the description of such operations as trace and determinant, definitions of similar matrices and isotropic invariants, the eigenvalue problem, matrix polynomials, the Cayley-Hamilton theorem, and polar decomposition.

2.1.1 Indicial notation

Indicial notation is used to simplify the writing of complex expressions that have one or more summations over indexed variables. As such, indicial notation not only helps simplify notation in continuum mechanics, but also in many other areas that frequently deal with summations, such as in the finite element method. The convention is very simple, *an index that is repeated once represents a summation*. For example, in the following expression the left-hand side is the index or indicial notation of the summation in the middle, and the right-hand side is the fully expanded expression.

$$a_i b_i = \sum_{i=1}^3 a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (2.1)$$

In this book, the summation will always be from 1 to 3, if not otherwise noted. The index “ i ” in this expression is called a *dummy index* since $a_i b_i = a_j b_j = a_k b_k$ all represent the same summation described by the right-hand side of the equation above. Therefore, the dummy index can be changed to a new dummy index without changing the expression. It will be necessary to *avoid repeating an index more than once* (i.e., it is not acceptable to write $a_i b_i c_i$). The most obvious application of such a system of notation can be seen when

writing the system of equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= y_1, \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= y_2, \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= y_3. \end{aligned} \quad (2.2)$$

This system can be written as

$$\sum_{j=1}^3 A_{ij}x_j = y_i, \quad (2.3)$$

or in indicial notation it is simply written as

$$A_{ij}x_j = y_i. \quad (2.4)$$

In the above expression, “ i ” is called a *free index*. By selecting $i = 1$, we get the first equation in the system, by selecting $i = 2$, we get the second equation in the system, and so forth. On the other hand, “ j ” in this expression is called a dummy index since it simply represents a summation. Now consider a second system of equations

$$\begin{aligned} B_{11}y_1 + B_{12}y_2 + B_{13}y_3 &= z_1, \\ B_{21}y_1 + B_{22}y_2 + B_{23}y_3 &= z_2, \\ B_{31}y_1 + B_{32}y_2 + B_{33}y_3 &= z_3. \end{aligned} \quad (2.5)$$

This can be written as $B_{ki}y_i = z_k$ in indicial notation. We can combine the two systems to get

$$B_{ki}A_{ij}x_j = z_k. \quad (2.6)$$

Note that this last equation includes two summations since it contains two repeated indices (i and j).

The Kronecker delta and the permutation symbol frequently are used to write expressions when using index notation. The *Kronecker delta* is denoted by δ_{ij} and is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (2.7)$$

The Kronecker delta frequently is used to replace one index with another. This results from the fact that

$$a_i\delta_{ij} = a_j. \quad (2.8)$$

This property can be shown by direct expansion of the expression.

The permutation symbol primarily is used for operations such as the cross product or when evaluating determinants. The *alternator or permutation symbol* is denoted by ϵ_{ijk} and is defined as follows:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation (i.e., } ijk = 123, 231, 312), \\ -1 & \text{if } ijk \text{ is an odd permutation (i.e., } ijk = 321, 213, 132), \\ 0 & \text{in all other cases (i.e., } ijk = 113, 232, 332, \text{etc.).} \end{cases} \quad (2.9)$$

Section 2.1-Problems:

- 2.1-1:** For $(a_1, a_2, a_3) = (1, 2, 1)$, $(b_1, b_2, b_3) = (2, 1, 2)$, and $(c_1, c_2, c_3) = (1, 2, 2)$, calculate the following expressions: (a) a_ib_i , (b) a_ic_i , (c) b_ic_i , (d) $a_ib_ic_j$, (e) $a_ib_jc_j$, (f) $a_ib_jc_k\epsilon_{ijk}$, (g) $a_ib_j\delta_{ij}$, (h) $a_i\delta_{jk}\epsilon_{ijk}$, and (i) $\epsilon_{ijk}a_ib_j$.

2.1-2: Simplify the following expressions: (a) $\delta_{ij}\epsilon_{ijk}$, (b) $A_{ij}(\delta_{ik} + \delta_{jk})$, (c) $\epsilon_{ijk}\epsilon_{ijm}\delta_{km}$, (d) $A_{mj}a_j + A_{mk}a_k$, (e) $B_{mj}\delta_{jm}$, (f) $A_{mn}\delta_{mi}\delta_{nj} + A_{kl}\delta_{kj}\delta_{li}$, (g) $A_{mn}\delta_{mn}$, and (h) $a_i(\delta_{ij}b_j + \delta_{ji}b_j)$.

2.1-3: Prove the following identities: (a) $\delta_{ii} = 3$, (b) $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, (c) $\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$, and (d) $\epsilon_{ijk}\epsilon_{ijk} = 6$.

2.1.2 Basic matrix operations

The following is a short review of matrix notation and matrix operations as needed for the understanding of vectors and tensors. In most cases the presentation is restricted to 3 by 3 matrices. The presentation assumes that the reader already has a basic understanding of matrix algebra.

A matrix is a two-dimensional array of elements. The location of each element is identified by a row number and a column number. A matrix will be identified by square brackets (e.g., $[A]$ or $[B]$). The elements of matrix $[A]$ will be denoted by A_{ij} where i denotes the row number and j denotes the column number. For example, one can write

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \quad (2.10)$$

A *real* matrix is one that has only real numbers (as opposed to complex numbers) for its members. Matrix multiplication has the following properties:

$$([A][B])[C] = [A]([B][C]), \quad (2.11)$$

$$[A]([B] + [C]) = [A][B] + [A][C], \quad (2.12)$$

$$([A] + [B])[C] = [A][C] + [B][C], \quad (2.13)$$

$$\alpha([A][B]) = (\alpha[A])[B] = [A](\alpha[B]), \quad (2.14)$$

for any scalar α .

The *transpose* of a matrix is an operation which switches the rows and columns of a matrix and is denoted by adding a superscript “T” to the matrix (e.g., $[A]^T$). The elements of the transpose of a matrix will be denoted by A_{ij}^T and are related to the elements of $[A]$ by the relation $A_{ij}^T = A_{ji}$. It can be shown by inspection that

$$([A]^T)^T = [A], \quad (2.15)$$

$$(\alpha[A])^T = \alpha[A]^T, \quad (2.16)$$

$$([A] + [B])^T = [A]^T + [B]^T, \quad (2.17)$$

$$([A][B])^T = [B]^T[A]^T, \quad (2.18)$$

where α is any scalar.

A matrix containing only one column is known as a *column matrix* and normally will be denoted by a lowercase character (e.g., $[x]$). The transpose of the column matrix is a row matrix (e.g., $[x]^T$).

The linear system of equations

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = c_1, \quad (2.19)$$

$$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = c_2, \quad (2.20)$$

$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = c_3, \quad (2.21)$$

can be represented by the matrix operation

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad (2.22)$$

or symbolically can be written as $[A][x] = [c]$. In this context, $[A]$ is known as the coefficient matrix and $[c]$ is known as the right-hand side.

A *square matrix* is a matrix that has the same number of rows and columns. A square matrix $[A]$ is *symmetric* if $[A] = [A]^T$. A square matrix $[A]$ is *skew-symmetric* if $[A] = -[A]^T$. It is easy to see that any square matrix $[A]$ can be separated into a symmetric part

$$[A]_{sym} = \frac{1}{2}([A] + [A]^T), \quad (2.23)$$

and a skew-symmetric part

$$[A]_{skew} = \frac{1}{2}([A] - [A]^T), \quad (2.24)$$

where

$$[A] = [A]_{sym} + [A]_{skew}. \quad (2.25)$$

A square matrix is *upper triangular* if it has nonzero elements only on and above the diagonal, and it is *lower triangular* if it has nonzero elements only on and below the diagonal. A *diagonal matrix* is one that only has non-zero elements on the diagonal. The *identity matrix* $[I]$ is a square diagonal matrix where all the diagonal elements are unity. That is, for 3 by 3 matrices we will have

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.26)$$

The identity matrix transforms any column matrix into itself. Therefore, for every column matrix $[x]$ one gets

$$[I][x] = [x]. \quad (2.27)$$

The same is true for a square matrix. That is, for square matrix $[A]$, one can show that

$$[I][A] = [A][I] = [A]. \quad (2.28)$$

For a square matrix $[A]$, the *trace* of $[A]$ is the sum of the diagonal terms and is written as $tr([A])$ so that

$$tr([A]) = A_{ii}. \quad (2.29)$$

One can show that $tr([A]) = tr([A]^T)$ and that

$$tr([A][B]) = tr([B][A]), \quad (2.30)$$

since $tr([A][B]) = A_{ij}B_{ji} = B_{ji}A_{ij} = B_{ij}A_{ji} = tr([B][A])$.

The *determinant* of a 3 by 3 square matrix $[A]$ is given by

$$det([A]) = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}A_{il}A_{jm}A_{kn}. \quad (2.31)$$

The determinant also is written as

$$\begin{aligned} \det([A]) &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\ &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ &\quad - A_{31}A_{22}A_{13} - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12}. \end{aligned} \quad (2.32)$$

In general, for an n by n matrix, the determinant is given by the sum of the product of the elements of any row or column by their cofactors. That is, if the determinant is calculated using row i , then

$$\det([A]) = A_{ij}M_{ij} \quad (\text{no sum on } i), \quad (2.33)$$

where M_{ij} is the cofactor of element A_{ij} . The *cofactor* of element A_{ij} is calculated using the $(n-1)$ by $(n-1)$ matrix $[B]$, which is obtained from $[A]$ by eliminating the i th row and j th column associated with element A_{ij} and using the equation

$$M_{ij} = (-1)^{i+j} \det([B]). \quad (2.34)$$

If one is calculating the determinant based on using column j , then

$$\det([A]) = A_{ij}M_{ij} \quad (\text{no sum on } j). \quad (2.35)$$

The following are some properties of determinants:

- A matrix that has a row or column that consists of only zero elements has a determinant equal to zero.
- If a matrix has two identical rows or two identical columns, then its determinant is zero.
- If a matrix is upper or lower triangular, then its determinant is the product of the diagonal terms.
- If matrix $[B]$ is constructed from matrix $[A]$ by multiplying a row or a column by the scalar α , then $\det([B]) = \alpha \det([A])$.
- If matrix $[B]$ is constructed from $[A]$ by the interchange of either two rows or two columns, then $\det([B]) = -\det([A])$.
- If matrix $[B]$ is constructed from $[A]$ by the addition of a multiple of one row to another row or the addition of a multiple of one column to another column, then $\det([B]) = \det([A])$.
- The determinant of the product of two matrices $[A]$ and $[B]$ is the product of the determinant of $[A]$ and the determinant of $[B]$ so that

$$\boxed{\det([A][B]) = \det([A])\det([B])}. \quad (2.36)$$

- The determinant of $[A][B]$ is the same as the determinant of $[B][A]$.
- The determinant of a matrix is equal to the determinant of its transpose so that

$$\boxed{\det([A]) = \det([A]^T)}. \quad (2.37)$$

- For an n by n matrix $[A]$ and scalar α

$$\boxed{\det(\alpha[A]) = \alpha^n \det([A])}. \quad (2.38)$$

For every linear transformation $[A][x] = [y]$ that transforms $[x]$ into $[y]$, there may be an inverse transformation given by $[A]^{-1}[y] = [x]$ that transforms $[y]$ into $[x]$. In this notation, the matrix $[A]^{-1}$ denotes the inverse of $[A]$. The condition $\det([A]) \neq 0$ is a necessary and sufficient condition for the square matrix $[A]$ to have an inverse. A square matrix $[A]$ and its inverse have the property

$$\boxed{[A]^{-1}[A] = [A][A]^{-1} = [I]}. \quad (2.39)$$

This follows from the fact that $[x] = [A]^{-1}[y] = [A]^{-1}[A][x]$, which must hold for all $[x]$. The inverse of a square matrix is unique if it exists. This follows from the fact that if $[X]$ and $[Y]$ are both the inverse of $[A]$, then they satisfy the relations $[A][X] = [X][A] = [I]$ and $[A][Y] = [Y][A] = [I]$. Using these relations we can write $[X] = [X]([A][Y]) = ([X][A])[Y] = [Y]$, which shows the uniqueness of the inverse. The determinant of $[A]^{-1}$ is given by

$$\boxed{\det([A]^{-1}) = \frac{1}{\det([A])}}. \quad (2.40)$$

Two matrices $[A]$ and $[B]$ are *similar* if there exists an invertible matrix $[P]$ such that

$$\boxed{[B] = [P]^{-1}[A][P]}. \quad (2.41)$$

The *scalar invariants* of a matrix are scalar functions of a matrix which have the same value for a matrix and for all the matrices that are similar to it under a designated group of transformations $[P]$. One calls such scalar functions “form invariant” under the designated group of transformations. For example, the determinant function of a matrix is form invariant under the group of all invertible matrices. This can be shown by

$$\det([B]) = \det([P]^{-1}[A][P]) = \det([P]^{-1})\det([A])\det([P]) = \det([A]). \quad (2.42)$$

As another example consider the trace function. One can show that the trace of a matrix is form invariant under the group of all invertible matrices. This can be shown by

$$\text{tr}([B]) = \text{tr}([P]^{-1}[A][P]) = \text{tr}([P][P]^{-1}[A]) = \text{tr}([A]). \quad (2.43)$$

The three *isotropic invariants* of a matrix $[A]$ are given by

$$\boxed{I_1 = \text{tr}([A])}, \quad (2.44)$$

$$\boxed{I_2 = \text{tr}([A]^2)}, \quad (2.45)$$

$$\boxed{I_3 = \text{tr}([A]^3)}. \quad (2.46)$$

It is easy to show that all three isotropic invariants are scalar invariants of the matrix $[A]$ under the full group of invertible transformations.

A linear system of equations given by $[A][x] = [0]$ is known as a *homogeneous system of equations*. A homogeneous system of equations obviously has the *trivial solution* $[x] = [0]$. A homogeneous system has *nontrivial solutions* if and only if $\det([A]) = 0$. Given a set of solutions $[x_1], [x_2], \dots, [x_n]$ of the system of equations $[A][x] = [0]$, it can be shown by direct

substitution that $\alpha_1[x_1] + \alpha_2[x_2] + \dots + \alpha_n[x_n]$ is also a solution to $[A][x] = [0]$, where each α_i is a scalar multiplier.

The linear system of equations $[A][x] = [c]$ has zero, one, or an infinite number of solutions. It can be shown that if this system has more than one solution it has infinite solutions. If one lets $[x_1]$ and $[x_2]$ be two solutions of this system, then by direct substitution one can show that $[x] = [x_1] + \alpha([x_1] - [x_2])$ is also a solution for any scalar α . A linear system of equations $[A][x] = [c]$ has a unique solution if and only if $\det([A]) \neq 0$.

A square matrix $[R]$ is said to be *orthogonal* if $[R]^{-1} = [R]^T$, such that one has

$$[R]^T[R] = [R][R]^T = [I]. \quad (2.47)$$

The determinant of an orthogonal matrix can be evaluated by taking the determinant of $[R][R]^T$. That is, $\det([R][R]^T) = \det([R])\det([R]^T) = \det^2([R])$, which should be equal to the determinant of the identity matrix, which is unity. As a result, for an orthogonal matrix $[R]$ one obtains

$$\det([R]) = \pm 1. \quad (2.48)$$

The group of matrices called *proper orthogonal* contains all the orthogonal matrices that have a positive determinant (i.e., $\det([R]) = 1$). The product of two orthogonal matrices is orthogonal. This can be shown by direct evaluation. Let $[R_1]$ and $[R_2]$ be two orthogonal matrices, then the matrix $[Q] = [R_1][R_2]$ will be orthogonal since $[Q][Q]^T = ([R_1][R_2])([R_1][R_2])^T = [R_1][R_2][R_2]^T[R_1]^T = [R_1][R_1]^T = [I]$.

For a square matrix $[A]$, one can form powers of $[A]$ defined as

$$[A]^0 = [I], \quad [A]^1 = [A], \quad [A]^2 = [A][A], \quad \dots \quad (2.49)$$

A polynomial function in $[A]$ is denoted as $f([A])$, where for a given polynomial function $f(x) = \alpha_o + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n$ we define $f([A])$ as

$$f([A]) = \alpha_o[I] + \alpha_1[A] + \alpha_2[A]^2 + \dots + \alpha_n[A]^n. \quad (2.50)$$

A *root of this polynomial* is any matrix $[A]$ that satisfies the equation $f([A]) = [0]$.

Section 2.1-Problems:

2.1-4: Consider the following matrices

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad [B] = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad [C] = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 4 & 2 \\ 4 & 2 & 4 \end{bmatrix}.$$

Calculate the following:

1. $[A]^T$, $[A]_{sym}$, $[A]_{skew}$, $tr([A])$.
2. $[B]^T$, $[B]_{sym}$, $[B]_{skew}$, $tr([B])$.
3. $[C]^T$, $[C]_{sym}$, $[C]_{skew}$, $tr([C])$.
4. $tr([A][B])$, $tr([A][B][C])$.
5. $\det([A])$, $\det([B])$, $\det([A][B])$, $\det([C])$, $\det([A][B][C])$, $\det([A]^{-1})$.
6. The isotropic invariants of $[A]$.
7. The isotropic invariants of $[B]$.
8. The isotropic invariants of $[C]$.

2.1-5: Consider the following matrices

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate the following:

1. $[A]^T$, $[A]_{sym}$, $[A]_{skew}$, $tr([A])$.
2. $[B]^T$, $[B]_{sym}$, $[B]_{skew}$, $tr([B])$.
3. $[C]^T$, $[C]_{sym}$, $[C]_{skew}$, $tr([C])$.
4. $tr([A][B])$, $tr([A][B][C])$.
5. $det([A])$, $det([B])$, $det([A][B])$, $det([C])$, $det([A][B][C])$, $det([A]^{-1})$.
6. The isotropic invariants of $[A]$.
7. The isotropic invariants of $[B]$.
8. The isotropic invariants of $[C]$.
9. The similarity transformation of $[A]$ by $[B]$.
10. Is $[A]$, $[B]$, or $[C]$ orthogonal?
11. Is $[A]$, $[B]$, or $[C]$ proper orthogonal?

2.1-6: Show that the trace of a skew-symmetric matrix $[A]$ is equal to zero.

2.1-7: Show that $tr([A][B]) = 0$ if $[A]$ is symmetric and $[B]$ is skew-symmetric.

2.1.3 The eigenvalue problem

Given a matrix $[A]$, in the eigenvalue problem, we seek to find all column matrices $[x]$ and associated scaling factors λ for which

$$[A][x] = \lambda[x]. \quad (2.51)$$

Such column matrices $[x]$ represent directions, which are left unchanged by the matrix $[A]$ in the multiplication $[A][x]$, and as such represent a special characteristic of the matrix $[A]$. A scalar λ for which the equation $[A][x] = \lambda[x]$ holds is known as an *eigenvalue* of $[A]$, and the associated column matrix $[x]$ is known as an *eigenvector* of $[A]$. Other names for eigenvalues are *principal value* and *characteristic value*. The eigenvalue problem plays a central role in many areas of mechanics and its importance will become obvious later when it is used to study the properties of matrices and tensors. One can reorganize this equation so as to get

$$([A] - \lambda[I])[x] = [0], \quad (2.52)$$

which is a homogeneous system of equations. A homogeneous system has a nontrivial solution (a solution other than $[x] = [0]$) only if the determinant of the coefficient matrix is zero. Therefore, the eigenvalue problem has a nontrivial solution only if

$$det([A] - \lambda[I]) = 0. \quad (2.53)$$

This last equation is known as the *characteristic equation*. For an n by n matrix $[A]$ the characteristic equation is a polynomial equation in λ of order n . Therefore, *there are at*

most n distinct eigenvalues. For each distinct eigenvalue λ , one can solve the equation $[A][x] = \lambda[x]$ to find its associated eigenvector $[x]$, up to a scalar multiplier. Since the eigenvalues are roots of the *characteristic polynomial* $\det([A] - \lambda[I])$, one can always write

$$\det([A] - \lambda[I]) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda), \quad (2.54)$$

where each λ_i is an eigenvalue of $[A]$. By setting $\lambda = 0$, one obtains

$$\det([A]) = \lambda_1 \lambda_2 \dots \lambda_n, \quad (2.55)$$

which states that the determinant of an n by n matrix is the product of its n eigenvalues. For a 3 by 3 matrix the characteristic equation can be written as

$$-\lambda^3 + I \lambda^2 - II \lambda + III = 0, \quad (2.56)$$

where I , II , and III are *invariants* of $[A]$, known as the *isotropic invariants*, and are defined as

$$I = \text{tr}([A]), \quad (2.57)$$

$$II = \frac{1}{2} \{ \text{tr}^2([A]) - \text{tr}([A]^2) \}, \quad (2.58)$$

$$III = \det([A]). \quad (2.59)$$

The following are some characteristics associated with eigenvalues:

- The characteristic equations of a matrix and its transpose are the same (i.e., $\det([A] - \lambda[I]) = \det([A]^T - \lambda[I])$), and, therefore, a matrix and its transpose have the same eigenvalues.
- A zero eigenvalue is only associated with a singular matrix (i.e., $\det([A]) = 0$).
- Given an invertible matrix $[A]$ with eigenvalue λ , the inverse $[A]^{-1}$ will have an associated eigenvalue λ^{-1} .
- If λ is an eigenvalue of the matrix $[A]$, then λ^n will be an eigenvalue of the matrix $[A]^n$.
- Given square matrices $[A]$ and $[B]$, the eigenvalues of $[A][B]$ and $[B][A]$ are the same.

For eigenvalue λ of matrix $[A]$ and its associated eigenvector $[x]$, it is simple to show by direct substitution that $\alpha[x]$ is also an eigenvector of $[A]$ associated with the same eigenvalue λ , where α is any nonzero scalar multiplier. Therefore, each eigenvalue of a matrix is associated with an infinite number of eigenvectors, each differing from the others by a scalar multiplier.

The eigenvalues of a real symmetric matrix $[A]$ are real. Before showing this let us review several points regarding complex numbers. A *complex number* a is given by $a = a_1 + ia_2$, where $i^2 = -1$. The *complex conjugate* of a is $\bar{a} = a_1 - ia_2$. The product $a\bar{a} = a_1^2 + a_2^2$ is strictly real and is the square of the magnitude of a . The addition $a + \bar{a} = 2a_1$ is strictly real and the subtraction $a - \bar{a} = 2ia_2$ is only imaginary. Finally, we note that $\overline{ab} = \bar{a}\bar{b}$. Now let us proceed to show that the eigenvalues of a real symmetric matrix are real. An eigenvalue of $[A]$ satisfies the equation $[A][x] = \lambda[x]$ and therefore

$$[\bar{x}]^T [A] [x] = \lambda [\bar{x}]^T [x]. \quad (2.60)$$

Also, by taking the transpose of $[A][x] = \lambda[x]$ one obtains $[x]^T[A] = \lambda[x]^T$. Evaluating the complex conjugate of this expression gives $[\bar{x}]^T[A] = \bar{\lambda}[\bar{x}]^T$, which results in

$$[\bar{x}]^T[A][x] = \bar{\lambda}[\bar{x}]^T[x]. \quad (2.61)$$

Subtraction of (2.61) from (2.60) yields

$$(\lambda - \bar{\lambda})[\bar{x}]^T[x] = 0. \quad (2.62)$$

Since $[\bar{x}]^T[x]$ is purely real and strictly positive, the only way for this equation to be satisfied is when $\lambda - \bar{\lambda} = 0$, which states that the imaginary part of λ is zero. Therefore, the eigenvalues of a real symmetric matrix are real.

Eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal. If λ_1 and λ_2 are distinct eigenvalues of $[A]$ and $[x^{(1)}]$ and $[x^{(2)}]$ are their respective eigenvectors, then we have $[A][x^{(1)}] = \lambda_1[x^{(1)}]$ and $[A][x^{(2)}] = \lambda_2[x^{(2)}]$. It follows that

$$[x^{(2)}]^T[A][x^{(1)}] = \lambda_1[x^{(2)}]^T[x^{(1)}], \quad (2.63)$$

$$[x^{(1)}]^T[A][x^{(2)}] = \lambda_2[x^{(1)}]^T[x^{(2)}]. \quad (2.64)$$

Transposing the second and subtraction from the first yield

$$(\lambda_1 - \lambda_2)[x^{(2)}]^T[x^{(1)}] = 0. \quad (2.65)$$

Since the eigenvalues are distinct, one is forced to conclude that $[x^{(2)}]^T[x^{(1)}] = 0$, which for nonzero eigenvectors leads one to the conclusion that $[x^{(1)}]$ and $[x^{(2)}]$ are perpendicular.

One always can select a set of orthogonal eigenvectors for a real symmetric matrix $[A]$. If all the eigenvalues are distinct, from our previous discussion, it follows that all the eigenvectors will be orthogonal. Let us focus on 3 by 3 matrices. If $\lambda_1 = \lambda_2 \neq \lambda_3$, then $[x^{(3)}]$ is orthogonal to $[x^{(1)}]$ and $[x^{(2)}]$. One can select $[x^{(1)}]$ and $[x^{(2)}]$ as any two orthogonal column matrices that are in the plane orthogonal to $[x^{(3)}]$. If $\lambda_1 = \lambda_2 = \lambda_3$, then $[A]$ is diagonal with identical elements equal to the eigenvalue on the diagonal. In this case, any three orthogonal vectors can be selected as the eigenvectors.

A real nonsymmetric matrix $[A]$ may have real eigenvalues and pairs of complex conjugate eigenvalues.

Eigenvalues of similar matrices are the same. This can be shown by substitution of $[B] = [P]^{-1}[A][P]$ into the characteristic equation of $[B]$ to get $\det([P]^{-1}[A][P] - \lambda[I]) = \det\{[P]^{-1}([A] - \lambda[I])[P]\} = \det([P]^{-1})\det([A] - \lambda[I])\det([P]) = \det([A] - \lambda[I])$. The eigenvector $[x^*]$ of $[B]$ is related to the eigenvector $[x]$ of $[A]$ by the relation $[x^*] = [P]^{-1}[x]$. This can be shown by substitution for $[B]$ into the relation $[B][x^*] = \lambda[x^*]$ to obtain $[P]^{-1}[A][P][x^*] = \lambda[x^*]$, which when reorganized gives the above relation.

A real symmetric matrix $[A]$ is similar to a matrix which is diagonal, and which has the eigenvalues of $[A]$ as the elements on the diagonal. To show this let us consider a 3 by 3 matrix and select the eigenvectors such that we have three mutually orthogonal eigenvectors $[x^{(1)}]$, $[x^{(2)}]$, and $[x^{(3)}]$, each normalized to have a magnitude of unity. Now set

$$[P]^{-1} = [[x^{(1)}][x^{(2)}][x^{(3)}]]. \quad (2.66)$$

That is, construct $[P]^{-1}$ from the three eigenvectors, each defining one column of $[P]^{-1}$. First, one can show by direct multiplication of $[P]^{-1}$ and $[P]^{-T}$ that $[P]^{-1}$ is orthogonal and therefore

$$[P]^{-1} = [P]^T. \quad (2.67)$$

Next, it can be shown by direct multiplication that

$$[P][A][P]^T = \begin{bmatrix} \lambda_1 [x^{(1)}]^T [x^{(1)}] & \lambda_2 [x^{(1)}]^T [x^{(2)}] & \lambda_3 [x^{(1)}]^T [x^{(3)}] \\ \lambda_1 [x^{(2)}]^T [x^{(1)}] & \lambda_2 [x^{(2)}]^T [x^{(2)}] & \lambda_3 [x^{(2)}]^T [x^{(3)}] \\ \lambda_1 [x^{(3)}]^T [x^{(1)}] & \lambda_2 [x^{(3)}]^T [x^{(2)}] & \lambda_3 [x^{(3)}]^T [x^{(3)}] \end{bmatrix}. \quad (2.68)$$

It directly follows from the selection of normalized orthogonal eigenvectors that

$$[P][A][P]^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \quad (2.69)$$

Therefore, $[A]$ is similar to a diagonal matrix with the eigenvalues of $[A]$ as its diagonal elements. It is easy to see that the three isotropic invariants can be written in terms of the eigenvalues by the following expressions:

$$I = \text{tr}([A]) = \lambda_1 + \lambda_2 + \lambda_3, \quad (2.70)$$

$$II = \frac{1}{2} \{ \text{tr}^2([A]) - \text{tr}([A]^2) \} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \quad (2.71)$$

$$III = \det([A]) = \lambda_1 \lambda_2 \lambda_3. \quad (2.72)$$

The ability to set a real symmetric matrix similar to a diagonal matrix can be used to define the *square root* of a real symmetric matrix as

$$\sqrt{[A]} = [P]^T \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} [P]. \quad (2.73)$$

One also can define the *exponent* of a matrix by the relation

$$e^{[A]} = [P]^T \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix} [P]. \quad (2.74)$$

This definition of the exponent easily can be shown to give

$$e^{[A]} = [I] + [A] + \frac{1}{2!}[A]^2 + \frac{1}{3!}[A]^3 + \dots, \quad (2.75)$$

consistent with the scalar expansion $e^x = 1 + x + \frac{1}{2!}x^2 + \dots$

The generalization of most of the above to n by n matrices is straightforward.

A *positive definite* matrix is a matrix for which $[x]^T [A] [x]$ is positive for all real nontrivial $[x]$. A necessary and sufficient condition for a matrix to be positive definite is that all of its eigenvalues be positive. It can be shown that any matrix $[C] = [F]^T [F]$ obtained from a real matrix $[F]$ is positive definite. This easily can be shown by using $[F]$ to define the mapping $[y] = [F][x]$, and noting that $[y]^T [y] = ([F][x])^T [F][x] = [x]^T ([F]^T [F])[x] = [x]^T [C][x]$ is the magnitude of $[y]$ squared, and therefore positive.

Section 2.1-Problems:

- 2.1-8:** Write the characteristic equation and calculate the eigenvalues and eigenvectors for each of the following matrices:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

- 2.1-9:** For the following matrix, first calculate the exact value of $e^{[A]}$ and then calculate it using the three-term polynomial approximation given by $e^{[A]} \approx [I] + [A] + \frac{1}{2!}[A]^2$.

$$[A] = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

- 2.1-10:** Check to see if any of the following matrices are positive definite.

$$\begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 4 \\ 5 & 6 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

- 2.1-11:** Calculate $\sqrt{[A]}$ for matrix $[A]$ given by

$$[A] = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

- 2.1-12:** Calculate the square root and the exponent of matrix $[A]$ given by

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.1.4 Cayley-Hamilton theorem

The Cayley-Hamilton theorem says that every square matrix satisfies its own characteristic equation. In other words, every matrix is a root of its characteristic polynomial $f(\lambda) = \det([A] - \lambda[I])$. This can be written for a 3 by 3 matrix $[A]$ as

$$-[A]^3 + I [A]^2 - II [A] + III [I] = [0]. \quad (2.76)$$

The Cayley-Hamilton theorem can be used to evaluate the inverse of a 3 by 3 matrix by the relation

$$[A]^{-1} = \frac{1}{III}([A]^2 - I [A] + II [I]), \quad (2.77)$$

which is obtained from (2.76) by multiplication of $[A]^{-1}$ and reorganization. One can use the Cayley-Hamilton theorem to reduce the power of matrix expressions in a polynomial by using the relation

$$[A]^n = I [A]^{n-1} - II [A]^{n-2} + III [A]^{n-3}, \quad (2.78)$$

which can be obtained from (2.76) by multiplication by $[A]^{n-3}$ and reorganization. As a result of this relation we always can rewrite a polynomial function

$$f([A]) = \alpha_o[I] + \alpha_1[A] + \dots + \alpha_n[A]^n \quad (2.79)$$

with constant coefficients α_i in the form

$$f([A]) = g_o(I, II, III)[I] + g_1(I, II, III)[A] + g_2(I, II, III)[A]^2, \quad (2.80)$$

where each g_i is a scalar valued function of the three invariants I , II , and III . An alternate form would be

$$f([A]) = q_o(I, II, III)[I] + q_1(I, II, III)[A] + q_2(I, II, III)[A]^{-1}, \quad (2.81)$$

where each q_i is a scalar valued function of the three isotropic invariants.

Section 2.1-Problems:

2.1-13: Use the Cayley-Hamilton theorem to calculate the inverse of the matrix

$$[A] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

2.1-14: Reduce the order of the following polynomial by one.

$$f([A]) = [I] + [A] + [A]^2 + [A]^3.$$

2.1.5 Polar decomposition theorem

The *polar decomposition theorem* states that every invertible square matrix $[F]$ can be decomposed uniquely into an orthogonal matrix $[R]$ and either of two symmetric positive definite matrices $[U]$ and $[V]$ in the form

$$[F] = [R][U] = [V][R]. \quad (2.82)$$

In the polar decomposition, $[U]$ is known as the right symmetric factor, $[V]$ is known as the left symmetric factor, and $[R]$ is known as the orthogonal factor. One can prove polar decomposition by direct construction of the matrices. First, let us construct $[U]$. If the decomposition can be done, it follows directly that $[C] = [F]^T[F] = ([R][U])^T[R][U] = [U]^2$. The matrix $[C]$ is symmetric since $([F]^T[F])^T = [F]^T[F]$, and can be used to define $[U] = \sqrt{[C]}$. Next, let us define $[R] = [F][U]^{-1}$. It is easy to check and see if $[R]$ is orthogonal by $[R]^T[R] = ([F][U]^{-1})^T[F][U]^{-1} = [U]^{-1}[C][U]^{-1} = [I]$. In a similar way, one can construct $[V] = \sqrt{[B]}$, where $[B] = [F][F]^T$.

Section 2.1-Problems:

2.1-15: Calculate the polar decomposition of the following matrices:

$$\begin{aligned}
 1. \quad & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
 2. \quad & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\
 3. \quad & \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
 4. \quad & \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
 \end{aligned}$$

2.2 Vectors

In this section, we introduce the general concept of a vector and its representation. We start with the description of frames and coordinates. This is followed by a description of vectors and basic vector operations. We then look at base vectors and vector components, and relate components for different sets of base vectors. This is followed by a description of curvilinear coordinates, which is a system for developing a consistent set of base vectors for a given continuous coordinate system.

2.2.1 Frames and coordinates

The reference frame contains the concept of what is meant by stationary and what is meant by not deforming (rigid). As the reader truly appreciates, the concepts of stationary and rigid are both closely tied to the point of view of the observer and the event being observed. For example, in most experiments the background of the event is taken as the reference frame. In many ways, a reference frame is like a rigid body on which we frequently inscribe an imaginary coordinate system to easily locate points in space. A point to remember is that frames are not real objects, and as such should not influence the event being observed. We will write frames with script characters such as $\mathcal{A}$ or $\mathcal{F}$, as shown in Figure 2.1.

An *inertial frame* is central to classical mechanics. The reader may recall that Newtonian mechanics is based on the concept of the existence of an inertial frame, a frame that has no accelerating points. In such a frame, the second law of Newton for a particle takes the very simple form of $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the resultant force applied on the particle, m is the mass of the particle, and $\mathbf{a}$ is the acceleration of the particle measured in this frame. We do not always use inertial frames. The reader may recall that in dynamics we frequently work with rotating and translating frames. To do so, we need to take into account the relation between the motion of the frame that is being used and an inertial frame.

A *coordinate system* is a way of uniquely defining every point on a frame. Coordinate systems normally consist of an imaginary grid put on the frame that allows the unique identification of each point on the frame by a n-tuple of scalars. In three-dimensional space, each point is given by three scalar values. The grid lines of the coordinate system are called coordinate lines.

A *Cartesian coordinate* system is one for which all coordinate lines are mutually orthogonal at each point of the grid. Figure 2.2 shows three of the most common Cartesian coor-

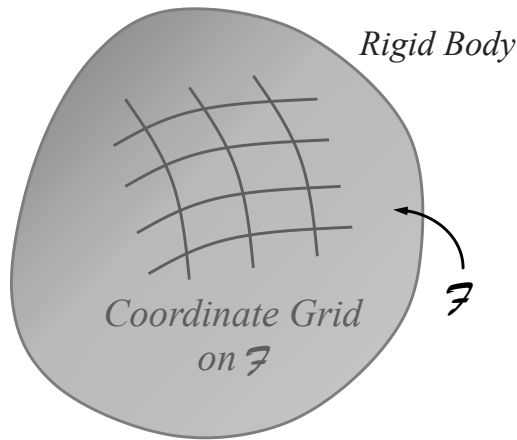


Figure 2.1: The frame is an imaginary rigid body used to locate points in space.

dinate systems. These are the rectangular, cylindrical, and spherical systems [i.e., (x, y, z) , (r, θ, z) , and (ρ, θ, ϕ) , respectively].

2.2.2 Vectors and basic vector operations

A *vector* is a mathematical object with both a magnitude and a direction, and which obeys the parallelogram law of addition shown in Figure 2.3. Vectors normally are denoted by boldfaced (mostly lowercase) characters such as $\mathbf{a}$ or $\mathbf{v}$, as opposed to scalars that are denoted by plane characters. In handwritten documents, one normally would use a tilde under a character to denote a boldfaced character. This also is done to denote a vector.

The *magnitude* of a vector is its length and is denoted by vertical lines on the two sides of the vector. For example, the magnitude of vector $\mathbf{u}$ is written as $|\mathbf{u}|$. A vector of unit length is called a *unit vector*. Unit vectors normally are denoted by putting a hat on a boldfaced character such as $\hat{\mathbf{e}}$.

The *dot product* (scalar product) of two vectors is a scalar. As shown in Figure 2.3, the dot product of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \circ \mathbf{v}$ and is given by

$$\mathbf{u} \circ \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\phi), \quad (2.83)$$

where ϕ is the angle between $\mathbf{u}$ and $\mathbf{v}$. From this definition, it is obvious that the elements of the dot product commute since $\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$. The *cross product* of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector. The cross product is denoted by $\mathbf{u} \times \mathbf{v}$, and its magnitude is defined by

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin(\phi). \quad (2.84)$$

The direction of $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and given by the right-hand rule. The magnitude of the cross product is equal to the area in the parallelogram constructed by using the two vectors as the sides of the parallelogram. It follows directly from the definition of the cross product that this product does not commute since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

The *triple scalar product* between the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ and defined by

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \circ \mathbf{c}. \quad (2.85)$$

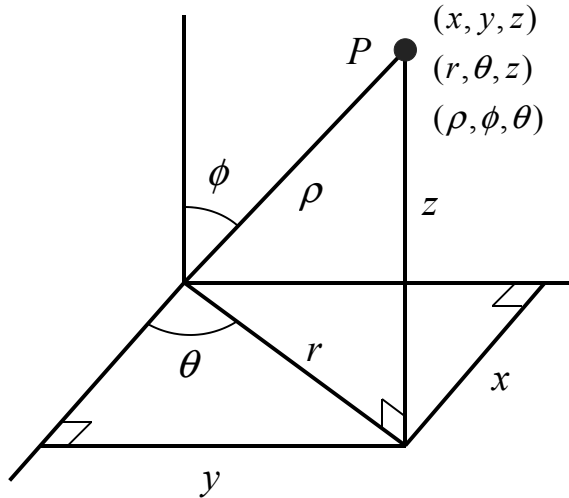


Figure 2.2: The rectangular, cylindrical and spherical coordinate systems.

This product has the properties:

- $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}]$,
- $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}]$,
- $[\alpha \mathbf{a}, \mathbf{b}, \mathbf{c}] = \alpha [\mathbf{a}, \mathbf{b}, \mathbf{c}]$,
- $[\mathbf{a} + \mathbf{d}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{d}, \mathbf{b}, \mathbf{c}]$.

The *triple vector product* between vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}). \quad (2.86)$$

This product has the property

$$\boxed{\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{a} \circ \mathbf{b})\mathbf{c}.} \quad (2.87)$$

Section 2.2-Problems:

2.2-1: Prove the following identities:

1. $\mathbf{a} \circ (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \circ (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \circ (\mathbf{c} \times \mathbf{a})$,
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{a} \circ \mathbf{b})\mathbf{c}$,
3. $(\mathbf{a} \times \mathbf{b}) \circ (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \circ \mathbf{c})(\mathbf{b} \circ \mathbf{d}) - (\mathbf{a} \circ \mathbf{d})(\mathbf{b} \circ \mathbf{c})$, (Lagrange's identity),
4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$, (Jacobi's identity),
5. $\boxed{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \circ (\mathbf{c} \times \mathbf{d})\mathbf{b} - \mathbf{b} \circ (\mathbf{c} \times \mathbf{d})\mathbf{a}.}$

2.2-2: Prove the following identities:

1. $[\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}, \mathbf{e} \times \mathbf{f}] = [\mathbf{b}, \mathbf{e}, \mathbf{f}][\mathbf{a}, \mathbf{c}, \mathbf{d}] - [\mathbf{a}, \mathbf{e}, \mathbf{f}][\mathbf{b}, \mathbf{c}, \mathbf{d}]$,

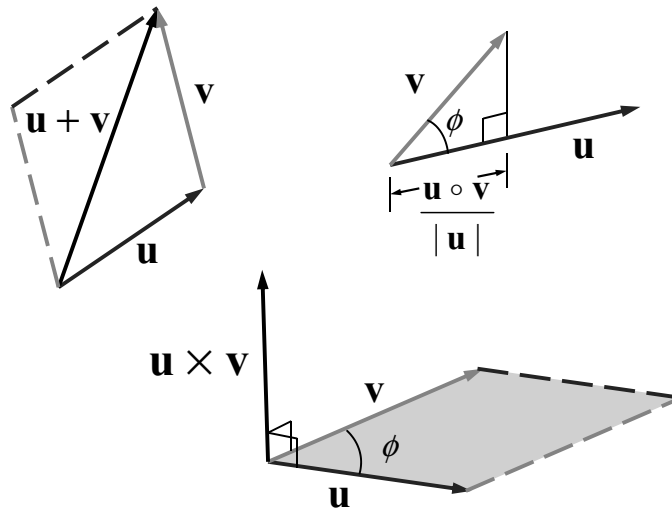


Figure 2.3: Vector addition, the dot product, and the cross product.

$$2. \quad \mathbf{a}[\mathbf{b}, \mathbf{c}, \mathbf{d}] - \mathbf{b}[\mathbf{c}, \mathbf{d}, \mathbf{a}] + \mathbf{c}[\mathbf{d}, \mathbf{a}, \mathbf{b}] - \mathbf{d}[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{0},$$

$$3. \quad [\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = |\mathbf{a}, \mathbf{b}, \mathbf{c}|^2,$$

$$4. \quad [\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} \mathbf{a} \circ \mathbf{x} & \mathbf{a} \circ \mathbf{y} & \mathbf{a} \circ \mathbf{z} \\ \mathbf{b} \circ \mathbf{x} & \mathbf{b} \circ \mathbf{y} & \mathbf{b} \circ \mathbf{z} \\ \mathbf{c} \circ \mathbf{x} & \mathbf{c} \circ \mathbf{y} & \mathbf{c} \circ \mathbf{z} \end{vmatrix}.$$

2.2.3 Base vectors

Base vectors are a set of preselected vectors used to describe or compare other vectors. In an n -dimensional space, one can always find a set of n linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n$ such that every vector $\mathbf{u}$ can be written in the form

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 + \dots + u_n \mathbf{e}_n = u_i \mathbf{e}_i \quad (\text{no sum on } n), \quad (2.88)$$

where each u_i in this expression is a component of $\mathbf{u}$. Figure 2.4 shows a two-dimensional flat space on which $\mathbf{e}_1$ and $\mathbf{e}_2$ are selected for the base. Also, in this figure is the method of constructing the arbitrary vector $\mathbf{u}$ from vectors parallel to this base using scalar multiplication and vector addition. Note that base vectors need neither be orthogonal nor of unit length, even though many of the coordinate systems we use have a system of mutually orthogonal unit vectors for their base. A base constructed of mutually orthogonal vectors is called an *orthogonal* or a *Cartesian* base. If an orthogonal base is constructed only of unit vectors, the base is known as an *orthonormal* base.

The operations of scalar multiplication, vector addition, the dot product, and the cross product easily can be done using the base vectors and components as is described in the following:

- Scalar multiplication: $\alpha \mathbf{u} = \alpha(u_i \mathbf{e}_i) = (\alpha u_i) \mathbf{e}_i$.
- Vector addition: $\mathbf{u} + \mathbf{v} = u_i \mathbf{e}_i + v_i \mathbf{e}_i = (u_i + v_i) \mathbf{e}_i$.

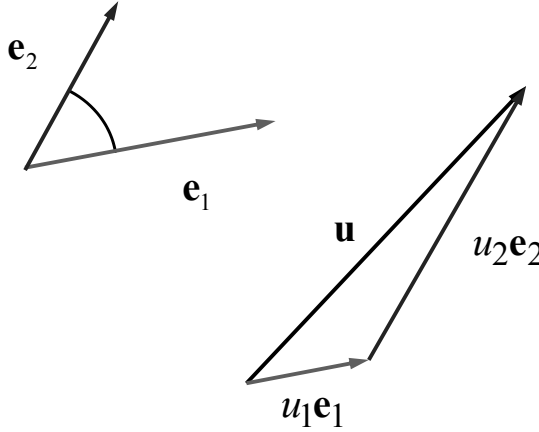


Figure 2.4: Base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can be used to describe any vector $\mathbf{u}$ using the relation $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$.

- Dot product: $\mathbf{u} \circ \mathbf{v} = (u_i \mathbf{e}_i) \circ (v_j \mathbf{e}_j) = u_i v_j (\mathbf{e}_i \circ \mathbf{e}_j) = u_i v_j g_{ij}$, where

$$g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j \quad (2.89)$$

is known as the *metric tensor*.

- Cross product: $\mathbf{u} \times \mathbf{v} = (u_i \mathbf{e}_i) \times (v_j \mathbf{e}_j) = u_i v_j (\mathbf{e}_i \times \mathbf{e}_j) = u_i v_j e_{ijk} \mathbf{e}_k$, where e_{ijk} is defined through the relation

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k. \quad (2.90)$$

Note that $\mathbf{e}_i \times \mathbf{e}_j$ define components of a skew-symmetric matrix since $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_j \times \mathbf{e}_i$.

Unlike what its name seems to imply, the metric tensor $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ is not a real tensor, but it plays a very central role in tensor operations, as will be shown later. One thing to note about the metric tensor is that it is symmetric since the elements in the dot product commute (i.e., $\mathbf{e}_i \circ \mathbf{e}_j = \mathbf{e}_j \circ \mathbf{e}_i$). Also, if the base is constructed of mutually orthogonal vectors the metric tensor is diagonal, and if the base is orthonormal, then the metric tensor becomes equal to the Kronecker delta (i.e., $g_{ij} = \delta_{ij}$). Also, for an orthonormal base e_{ijk} becomes the permutation symbol (i.e., $e_{ijk} = \epsilon_{ijk}$).

Section 2.2-Problems:

- 2.2-3:** Show that for an orthonormal base $g_{ij} = \delta_{ij}$ and $e_{ijk} = \epsilon_{ijk}$.

2.2.4 Change of base

We do not always work in a single base. For example, sometimes we may be provided with the components of a vector in one base but need to know its components in another base. In

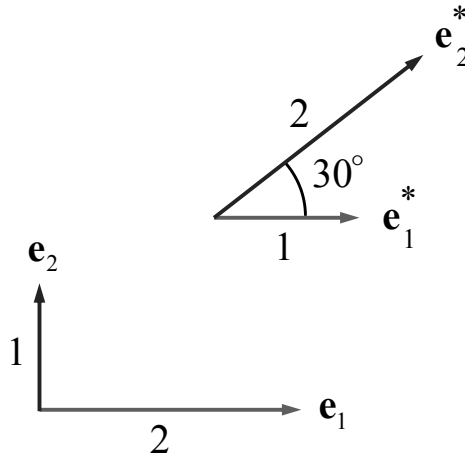


Figure 2.5: An example for describing the transformation matrix between two bases.

such cases we need to know how to change the base, and what happens to the components of the vector with this change. Two bases $\mathbf{e}_i$ and $\mathbf{e}_i^*$ can be related by a transformation rule

$$\mathbf{e}_i^* = l_{ij} \mathbf{e}_j, \quad (2.91)$$

where l_{ij} is known as the transformation matrix. For example, consider the two bases given in Figure 2.5. Assume that the third base vector in each case is the same for both bases and pointing out of the page. For these bases, we have the relations

$$\begin{aligned} \mathbf{e}_1^* &= 0.5\mathbf{e}_1, \\ \mathbf{e}_2^* &= \cos(30^\circ)\mathbf{e}_1 + 2\sin(30^\circ)\mathbf{e}_2, \\ \mathbf{e}_3^* &= \mathbf{e}_3. \end{aligned}$$

This results in the components of the transformation matrix given as

$$[l] = \begin{bmatrix} 0.5 & 0 & 0 \\ \cos(30^\circ) & 2\sin(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The relation between the components of a vector in two different bases can be obtained from the knowledge of the transformation matrix between these bases. Consider vector $\mathbf{u}$, which is given in the base $\mathbf{e}_i$ as $\mathbf{u} = u_i \mathbf{e}_i$ and is given in the base $\mathbf{e}_i^*$ as $\mathbf{u} = u_i^* \mathbf{e}_i^*$. One can use the transformation matrix between the two bases to write $\mathbf{u} = u_i \mathbf{e}_i = u_j^* \mathbf{e}_j^* = u_j^* l_{ji} \mathbf{e}_i$. Therefore,

$$u_i = l_{ji} u_j^*. \quad (2.92)$$

Letting l_{ij}^{-1} denote the components of the inverse matrix of l_{ij} , one also can write $l_{ip}^{-1} u_i = l_{ip}^{-1} l_{ji} u_j^* = \delta_{pj} u_j^* = u_p^*$. Therefore,

$$u_i^* = l_{ji}^{-1} u_j. \quad (2.93)$$

Let us again consider the two bases given in Figure 2.5 and take the vector described by $\mathbf{u} = \mathbf{e}_1^* + 2\mathbf{e}_2^*$. It follows that the components of $\mathbf{u}$ in the $\mathbf{e}_i$ base are given by

$$\begin{aligned} u_1 &= l_{11}u_1^* + l_{21}u_2^* + l_{31}u_3^* \\ &= (0.5)(1) + [\cos(30^\circ)](2) + (0)(0) = 0.5 + 2\cos(30^\circ), \\ u_2 &= l_{12}u_1^* + l_{22}u_2^* + l_{32}u_3^* \\ &= (0)(1) + [2\sin(30^\circ)](2) + (0)(0) = 4\sin(30^\circ), \\ u_3 &= l_{13}u_1^* + l_{23}u_2^* + l_{33}u_3^* \\ &= (0)(1) + (0)(2) + (1)(0) = 0. \end{aligned}$$

These transformations also can be written in matrix form as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \\ u_3^* \end{bmatrix}, \quad (2.94)$$

or in symbolic matrix form as $[u] = [l]^T[u^*]$, where $[u]$ and $[u^*]$ are assumed to be column matrices. Following this same procedure, one can write the inverse transformation as $[u^*] = [l]^{-T}[u]$.

The relation between the metric tensor of two bases can be obtained by direct evaluation of the metric tensor through

$$\boxed{g_{ij}^* = \mathbf{e}_i^* \circ \mathbf{e}_j^* = (l_{ik}\mathbf{e}_k) \circ (l_{jl}\mathbf{e}_l) = l_{ik}l_{jl}g_{kl}.} \quad (2.95)$$

The matrix representation of this will be $[g^*] = [l][g][l]^T$.

Normally one selects a set of mutually orthogonal unit vectors as the base vectors. Such a set would be called an orthonormal base. As shown in Figure 2.6, the bases for the three most commonly used orthonormal systems are

- Rectangular coordinates: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.
- Cylindrical coordinates: $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$.
- Spherical coordinates: $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta$.

Unlike the rectangular system, the base vectors for the cylindrical and spherical coordinate systems change with location on the frame. Yet, the base vectors for cylindrical and spherical coordinate systems always remain an orthonormal set at each point on the frame.

For a right-handed Cartesian base of unit vectors, the metric tensor is simply given by $g_{ij} = \hat{\mathbf{e}}_i \circ \hat{\mathbf{e}}_j = \delta_{ij}$, and the dot and cross products are given by

- $\mathbf{u} \circ \mathbf{v} = (u_i\hat{\mathbf{e}}_i) \circ (v_j\hat{\mathbf{e}}_j) = u_iv_j(\hat{\mathbf{e}}_i \circ \hat{\mathbf{e}}_j) = u_iv_j\delta_{ij} = u_iv_i$.
- $\mathbf{u} \times \mathbf{v} = (u_i\hat{\mathbf{e}}_i) \times (v_j\hat{\mathbf{e}}_j) = u_iv_j(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) = u_iv_j\epsilon_{ijk}\hat{\mathbf{e}}_k$.

In this expression ϵ_{ijk} is the permutation symbol previously defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for } ijk = 123, 231, 312, \\ -1 & \text{for } ijk = 321, 213, 132, \\ 0 & \text{for all other cases.} \end{cases}$$

As an example, consider the product $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \epsilon_{12k}\hat{\mathbf{e}}_k = \epsilon_{121}\hat{\mathbf{e}}_1 + \epsilon_{122}\hat{\mathbf{e}}_2 + \epsilon_{123}\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$.

One always can find the metric tensor of a base if one knows the transformation matrix relating an orthonormal base to the base under consideration. Consider two bases: one a Cartesian base $\hat{\mathbf{e}}_i$ of unit vectors and the other an arbitrary base $\mathbf{e}_i^*$ which is related to the first by

$$\mathbf{e}_i^* = l_{ij}\hat{\mathbf{e}}_j. \quad (2.96)$$

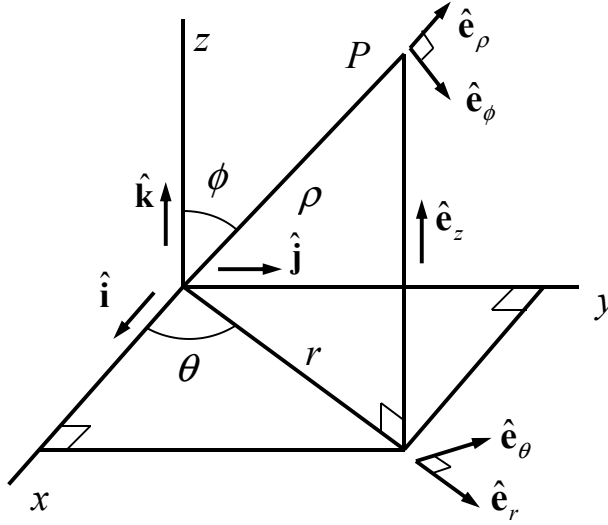


Figure 2.6: The bases for the rectangular, cylindrical and spherical coordinate systems.

The metric tensor of the second base will be given by

$$g_{ij}^* = \mathbf{e}_i^* \circ \mathbf{e}_j^* = l_{ip} \hat{\mathbf{e}}_p \circ l_{jq} \hat{\mathbf{e}}_q = l_{ip} l_{jq}. \quad (2.97)$$

Therefore, one can find the metric tensor of a non-orthogonal base from the transformation matrix relating it to an orthogonal base of unit vectors. In matrix form this can be written as $[g^*] = [l][l]^T$.

The components of the transformation matrix between an orthonormal base $\hat{\mathbf{e}}_i$ and an arbitrary base $\mathbf{e}_i^*$ can be found using the relation

$$\mathbf{e}_i^* \circ \hat{\mathbf{e}}_j = (l_{im} \hat{\mathbf{e}}_m) \circ \hat{\mathbf{e}}_j = l_{im} \delta_{mj} = l_{ij}. \quad (2.98)$$

In matrix form this can be written as

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1^* \circ \hat{\mathbf{e}}_1 & \mathbf{e}_1^* \circ \hat{\mathbf{e}}_2 & \mathbf{e}_1^* \circ \hat{\mathbf{e}}_3 \\ \mathbf{e}_2^* \circ \hat{\mathbf{e}}_1 & \mathbf{e}_2^* \circ \hat{\mathbf{e}}_2 & \mathbf{e}_2^* \circ \hat{\mathbf{e}}_3 \\ \mathbf{e}_3^* \circ \hat{\mathbf{e}}_1 & \mathbf{e}_3^* \circ \hat{\mathbf{e}}_2 & \mathbf{e}_3^* \circ \hat{\mathbf{e}}_3 \end{bmatrix}. \quad (2.99)$$

The reader will recall that the volume of the parallelepiped constructed by any three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given as $|(\mathbf{a} \times \mathbf{b}) \circ \mathbf{c}|$. One can calculate the volume of the parallelepiped constructed from the base vectors $\mathbf{e}_i^*$ by recalling the identity

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} \mathbf{a} \circ \mathbf{x} & \mathbf{a} \circ \mathbf{y} & \mathbf{a} \circ \mathbf{z} \\ \mathbf{b} \circ \mathbf{x} & \mathbf{b} \circ \mathbf{y} & \mathbf{b} \circ \mathbf{z} \\ \mathbf{c} \circ \mathbf{x} & \mathbf{c} \circ \mathbf{y} & \mathbf{c} \circ \mathbf{z} \end{vmatrix}. \quad (2.100)$$

Therefore, if $\mathbf{e}_i^*$ is a right-handed system, one can write

$$(\mathbf{e}_1^* \times \mathbf{e}_2^*) \circ \mathbf{e}_3^* = \det([l]). \quad (2.101)$$

Recalling that $[g^*] = [l][l]^T$, one can show that for any arbitrary base $\mathbf{e}_i^*$ the volume of the parallelepiped constructed by the base vectors is

$$|(\mathbf{e}_1^* \times \mathbf{e}_2^*) \circ \mathbf{e}_3^*| = \sqrt{\det([g^*])}. \quad (2.102)$$

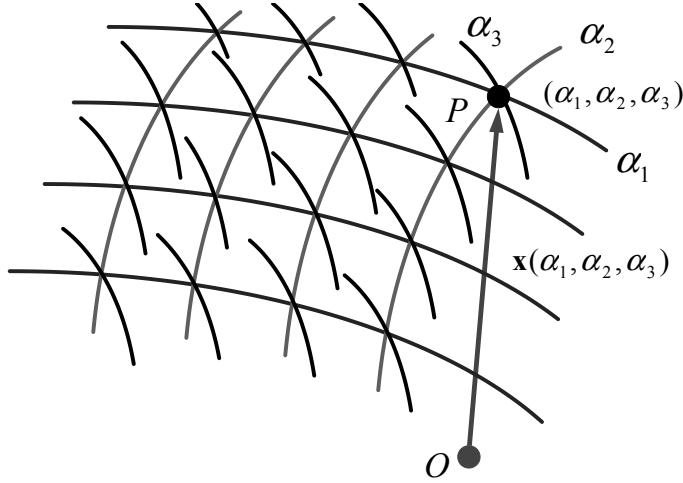


Figure 2.7: A curvilinear coordinate system is a grid that describes each point on the frame by a unique triplet of numbers that change in a continuous manner from point to point on the frame.

This shows that the volume of the parallelepiped constructed by any base is given by the metric tensor of that base.

2.2.5 Curvilinear coordinates

Coordinate systems like cylindrical and spherical coordinate systems are considered curvilinear coordinate systems. A *curvilinear coordinate system* is a general grid in space, as shown in Figure 2.7, where each point is uniquely defined by a triplet $(\alpha_1, \alpha_2, \alpha_3)$ that varies continuously as one moves from point to point on the frame. Selecting a unique reference point O on the frame as the origin, the position of each point in space can be defined by a vector going from the origin to the point under consideration. This vector is called the position vector. One can define a vector valued function $\mathbf{x}(\alpha_1, \alpha_2, \alpha_3)$ that gives the position of each point on the frame in terms of its coordinates $(\alpha_1, \alpha_2, \alpha_3)$.

The base vectors of a curvilinear system are selected so that the components of $d\mathbf{x}$ in this base are $d\alpha_i$. That is, in curvilinear coordinates we always have

$$d\mathbf{x} = d\alpha_i \mathbf{e}_i. \quad (2.103)$$

This is possible by selecting the base vectors as

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \alpha_i}, \quad (2.104)$$

for $i = 1, 2, 3$ since we know from the chain rule that one can always write

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \alpha_i} d\alpha_i = d\alpha_i \frac{\partial \mathbf{x}}{\partial \alpha_i} = d\alpha_i \mathbf{e}_i. \quad (2.105)$$

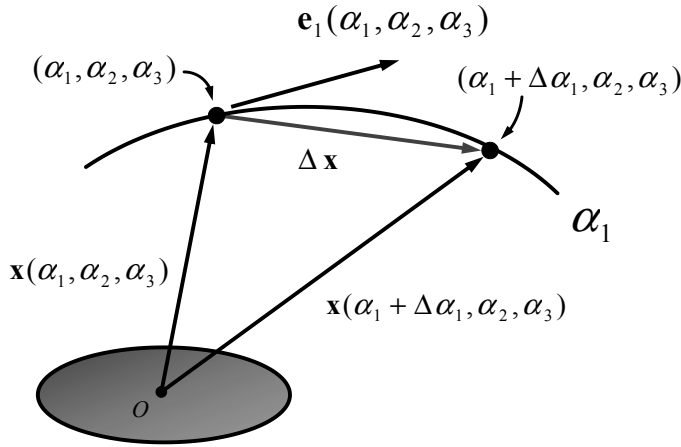


Figure 2.8: Change in position as one moves from point to point along a given coordinate line.

As can be seen from the above, the definition of a curvilinear system is tied very closely to the existence of the partial derivatives of the position vector with respect to the coordinates. It turns out that the base vectors of a curvilinear system are tangent to the coordinate lines at each point on the frame. For example, the direction of $\mathbf{e}_1$ can be shown to be tangent to the coordinate line α_1 by considering Figure 2.8 and the definition of $\mathbf{e}_1$ given by

$$\mathbf{e}_1 = \frac{\partial \mathbf{x}}{\partial \alpha_1} = \lim_{\Delta \alpha_1 \rightarrow 0} \frac{\Delta \mathbf{x}}{\Delta \alpha_1}.$$

From the figure, the reader will recognize that $\Delta \mathbf{x}$ becomes tangent to the coordinate line α_1 as one lets $\Delta \alpha_1$ go to zero. Since this partial derivative obtains its direction from the direction of $\Delta \mathbf{x}$, one must conclude that $\mathbf{e}_1$ is tangent to the coordinate line associated with constant α_2 and α_3 . The base vectors can change with location in space and, therefore, may be functions of the coordinates $(\alpha_1, \alpha_2, \alpha_3)$.

Let us now consider the cylindrical coordinate system as an example. The cylindrical coordinate system is defined by the coordinates (r, θ, z) . The position vector in this coordinate system is given by

$$\mathbf{x} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z. \quad (2.106)$$

Therefore, the curvilinear base vectors of this coordinate system are given by

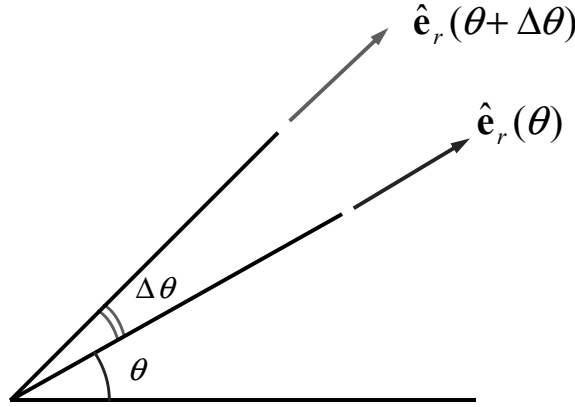
$$\mathbf{e}_1 = \frac{\partial \mathbf{x}}{\partial r} = \hat{\mathbf{e}}_r, \quad (2.107)$$

$$\mathbf{e}_2 = \frac{\partial \mathbf{x}}{\partial \theta} = r \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = r\hat{\mathbf{e}}_\theta, \quad (2.108)$$

$$\mathbf{e}_3 = \frac{\partial \mathbf{x}}{\partial z} = \hat{\mathbf{e}}_z, \quad (2.109)$$

so that the metric tensor is given by

$$[g] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.110)$$

Figure 2.9: The dependence of $\hat{\mathbf{e}}_r$ on θ .

The reader first should note that, as shown in Figure 2.9, the vector $\hat{\mathbf{e}}_r$ is a function of the coordinate θ . Next, the reader should note that the curvilinear base $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\hat{\mathbf{e}}_r, r\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ is not the same as the commonly used base for the cylindrical coordinate system that is $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$. This difference results from the fact that

$$d\mathbf{x} = dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta + dz\hat{\mathbf{e}}_z. \quad (2.111)$$

Let us now consider relating two curvilinear coordinate systems. Let one coordinate system be described by the coordinates $(\alpha_1, \alpha_2, \alpha_3)$ and the other by the coordinates $(\beta_1, \beta_2, \beta_3)$. The same position vector $\mathbf{x}$ can be given in terms of either coordinate system, and each coordinate can be considered a function of the other. Therefore, if $\mathbf{e}_i$ is the curvilinear base for the α_i coordinate system, and if $\mathbf{e}_i^*$ is the curvilinear base for the β_i coordinate system, then the two bases will be related through the chain rule given by

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \alpha_i} = \frac{\partial \mathbf{x}}{\partial \beta_j} \frac{\partial \beta_j}{\partial \alpha_i} = \mathbf{e}_j^* \frac{\partial \beta_j}{\partial \alpha_i}, \quad (2.112)$$

and

$$\mathbf{e}_i^* = \frac{\partial \mathbf{x}}{\partial \beta_i} = \frac{\partial \mathbf{x}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \beta_i} = \mathbf{e}_j \frac{\partial \alpha_j}{\partial \beta_i}. \quad (2.113)$$

Using $\mathbf{e}_i^* = l_{ij}\mathbf{e}_j$ to define the transformation matrix l_{ij} , one must conclude that for curvilinear systems one has

$$l_{ij} = \frac{\partial \alpha_j}{\partial \beta_i}, \quad l_{ij}^{-1} = \frac{\partial \beta_j}{\partial \alpha_i}. \quad (2.114)$$

2.3 Tensors

We have spent the majority of this chapter reviewing vectors and discussing their representation and algebra. This was done in part to better understand vectors, and in part as a springboard for introducing tensors. It turns out that tensors are constructed from vectors, and, as such, their mathematical characteristics are inherited directly from the mathematical characteristics of vectors. As a result, the reader frequently can expand his or her understanding of tensors simply by better understanding vectors, vector operations, and vector calculus.

Let us see what we mean by our statement that tensors are constructed from vectors. It turns out that a scalar is considered to be a tensor of order zero, and a vector is considered a tensor of order one. The reader may think of a scalar as a vector to the power zero and a vector as a vector to the power one. Now we need to examine what we mean by a tensor of order two and higher. To put two tensors of order one (vectors) together to get a tensor of order two we need to define the tensor product. The tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$ will be written as $\mathbf{u} \otimes \mathbf{v}$. Unlike the dot product that takes two vectors and returns a scalar, and unlike the cross product that takes two vectors and returns a new vector, the tensor product takes two vectors and returns a mathematical object that retains the characteristics of both vectors. In a way, the tensor product $\mathbf{u} \otimes \mathbf{v}$ is an ordered pairing of the two vectors $\mathbf{u}$ and $\mathbf{v}$ that retains all the characteristics of both $\mathbf{u}$ and $\mathbf{v}$.

Let us now examine this new mathematical object $\mathbf{u} \otimes \mathbf{v}$. This mathematical object is a second-order tensor, and later will be used as the base to construct all second-order tensors. First, let us provide the basic algebra that goes along with the tensor product. This algebra is described through the rules:

1. $\alpha(\mathbf{u} \otimes \mathbf{v}) = (\alpha\mathbf{u}) \otimes \mathbf{v} = \mathbf{u} \otimes (\alpha\mathbf{v}) = (\mathbf{u} \otimes \mathbf{v})\alpha$,
2. $\mathbf{u} \otimes \mathbf{v} + \mathbf{w} \otimes \mathbf{x} = \mathbf{w} \otimes \mathbf{x} + \mathbf{u} \otimes \mathbf{v}$,
3. $(\mathbf{u} + \mathbf{v}) \otimes \mathbf{w} = \mathbf{u} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}$,
4. $\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$.

The reader will note that in all of the operations introduced above the order is preserved by the tensor product. The vectors in the tensor product $\mathbf{u} \otimes \mathbf{v}$ do not commute, which results in $\mathbf{u} \otimes \mathbf{v}$ not being equal to $\mathbf{v} \otimes \mathbf{u}$. Let us next examine how the tensor product interacts with the dot and cross products. The rules for calculating these products are

1. $(\mathbf{u} \otimes \mathbf{v}) \circ \mathbf{w} = \mathbf{u}(\mathbf{v} \circ \mathbf{w})$,
2. $\mathbf{u} \circ (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \circ \mathbf{v})\mathbf{w}$,
3. $(\mathbf{u} \otimes \mathbf{v}) \times \mathbf{w} = \mathbf{u} \otimes (\mathbf{v} \times \mathbf{w})$,
4. $\mathbf{u} \times (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$.

As can be seen, the dot and cross products are operations between the closest vectors to the product. In the case of a dot product, the vectors closest to the dot product are dotted to give a scalar and the resulting object is a vector. In the case of the cross product, the vectors closest to the product are crossed into each other to obtain a new vector and the resulting object is a new tensor of order two. Other products between vectors also can be applied between a vector and the second-order tensor $\mathbf{u} \otimes \mathbf{v}$ in the same way as described above. Before we proceed to give the general description of a second-order tensor, let us introduce the following two properties, which frequently are used to define the tensor product. These are

$$(\mathbf{u} \otimes \mathbf{v}) \circ \mathbf{w} = \mathbf{u}(\mathbf{v} \circ \mathbf{w}) = \mathbf{u}(\mathbf{w} \circ \mathbf{v}) = (\mathbf{u} \otimes \mathbf{w}) \circ \mathbf{v}, \quad (2.115)$$

and

$$\mathbf{w} \circ (\mathbf{u} \otimes \mathbf{v}) = (\mathbf{w} \circ \mathbf{u})\mathbf{v}. \quad (2.116)$$

Also, let us mention that the tensor product also is known as the *dyadic product*.

Second- and higher-order tensors normally are denoted by upper case bold-faced characters. A general second-order tensor always can be written as the sum of the tensor products of pairs of vectors. That is, any second-order tensor $\mathbf{T}$ can be written as

$$\mathbf{T} = \sum_{p=1}^n \mathbf{u}_p \otimes \mathbf{v}_p, \quad (2.117)$$

where $\mathbf{u}_p$ and $\mathbf{v}_p$ are sets of vectors. In general, all vectors in this description can be written in a single base. To obtain such a representation, let us write each vector in terms of components in a given base so that $\mathbf{u}_p = u_{pi}\mathbf{e}_i$ and $\mathbf{v}_p = v_{pj}\mathbf{e}_j$. Substitution into the above equation for $\mathbf{T}$ yields

$$\mathbf{T} = \sum_{p=1}^n u_{pi}v_{pj}\mathbf{e}_i \otimes \mathbf{e}_j. \quad (2.118)$$

Letting $T_{ij} = \sum_{p=1}^n u_{pi}v_{pj}$, one can write any second-order tensor as

$$\boxed{\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j}, \quad (2.119)$$

where $\mathbf{e}_i$ are base vectors, and T_{ij} are the components of $\mathbf{T}$ in this base. Note that there are nine components T_{ij} , each the scalar coefficient of one of the nine bases $\mathbf{e}_i \otimes \mathbf{e}_j$. In a similar way, if all of the vectors are described in the base $\mathbf{e}_i^*$, the tensor $\mathbf{T}$ can be written as

$$\mathbf{T} = T_{ij}^*\mathbf{e}_i^* \otimes \mathbf{e}_j^*, \quad (2.120)$$

where T_{ij}^* are the components of the same tensor $\mathbf{T}$, but for the new base $\mathbf{e}_i^*$. There is nothing that requires us to represent the vectors on the left and right sides of the tensor product using the same base. For example, if we use $\mathbf{e}_i$ as the base for the vectors on the left side of the tensor product, and we use $\mathbf{e}_i^*$ as the base for the vectors on the right side of the tensor product, then we get a mixed representation of $\mathbf{T}$ in the form

$$\mathbf{T} = \bar{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j^*, \quad (2.121)$$

where $\bar{T}_{ij}$ are the components of $\mathbf{T}$ in this mixed base. In any case, all representations should be of the same tensor so that we must require that

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T_{ij}^*\mathbf{e}_i^* \otimes \mathbf{e}_j^* = \bar{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j^*.$$

This fact is the foundation on which we will build the rules for transformation of bases.

Transformation of bases can be done exactly as is done for vectors. Consider the transformation rule from $\mathbf{e}_i$ to $\mathbf{e}_i^*$ and its inverse, respectively, given as

$$\mathbf{e}_i^* = l_{ij}\mathbf{e}_j, \text{ and } \mathbf{e}_i = l_{ij}^{-1}\mathbf{e}_j^*. \quad (2.122)$$

One can change the first, the second, or both bases of the second-order tensor by the operations

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T_{ij}l_{ip}^{-1}\mathbf{e}_p^* \otimes \mathbf{e}_j = T_{ij}l_{jq}^{-1}\mathbf{e}_i \otimes \mathbf{e}_q^* = T_{ij}l_{ip}^{-1}l_{jq}^{-1}\mathbf{e}_p^* \otimes \mathbf{e}_q^*. \quad (2.123)$$

Using this, one can construct relations between the components of $\mathbf{T}$ in its different representations. These relations are, for example, given by

$$T_{pq}^* = T_{ij}l_{ip}^{-1}l_{jq}^{-1}, \quad (2.124)$$

and

$$\bar{T}_{iq} = T_{ij}l_{jq}^{-1}. \quad (2.125)$$

In a similar way one can show that

$$T_{pq} = l_{ip}l_{jq}T_{ij}^*. \quad (2.126)$$

Tensors of order three and higher can be constructed by a similar approach as above, but using more vectors. For example, a third-order tensor $\mathbf{A}$ always can be written as

$$\mathbf{A} = \sum_{i=1}^n \mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i \quad (2.127)$$

in terms of the sum of the tensor products of triplets of vectors $\mathbf{u}_i$, $\mathbf{v}_i$, and $\mathbf{w}_i$. As a result, a third -order tensor can be represented in the base $\mathbf{e}_i$ as

$$\boxed{\mathbf{A} = A_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k,} \quad (2.128)$$

where A_{ijk} are the 27 components of $\mathbf{A}$ in this base. As in the case for second-order tensors, one can mix different bases to get representations of the tensor in a mixed base. These representations follow the same development as for second-order tensors. It is simple to see that an n th order tensor $\mathbf{B}$ will have a representation

$$\boxed{\mathbf{B} = B_{i_1 i_2 \dots i_n} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \dots \otimes \mathbf{e}_{i_n}} \quad (2.129)$$

in the base given by $\mathbf{e}_i$. This same tensor can be represented in the base $\mathbf{e}_i^*$ as

$$\mathbf{B} = B_{i_1 i_2 \dots i_n}^* \mathbf{e}_{i_1}^* \otimes \mathbf{e}_{i_2}^* \otimes \dots \otimes \mathbf{e}_{i_n}^*. \quad (2.130)$$

Using the transformation rule $\mathbf{e}_i^* = l_{ij}\mathbf{e}_j$, one can obtain the relation between the components of $\mathbf{B}$ in the two bases as

$$\boxed{B_{i_1 i_2 \dots i_n} = l_{j_1 i_1} l_{j_2 i_2} \dots l_{j_n i_n} B_{j_1 j_2 \dots j_n}^*} \quad (2.131)$$

and

$$\boxed{B_{i_1 i_2 \dots i_n}^* = l_{j_1 i_1}^{-1} l_{j_2 i_2}^{-1} \dots l_{j_n i_n}^{-1} B_{j_1 j_2 \dots j_n}.} \quad (2.132)$$

The tensor product of two tensors $\mathbf{A}$ and $\mathbf{B}$ follows a similar rule as vectors. That is, if $\mathbf{A}$ is a tensor of order m and $\mathbf{B}$ is a tensor of order n , then $\mathbf{A} \otimes \mathbf{B}$ is a tensor of order $m + n$. Describing these tensors in the base $\mathbf{e}_i$, the product takes the form

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= (A_{i_1 \dots i_m} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_m}) \otimes (B_{j_1 \dots j_n} \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_n}) \\ &= A_{i_1 \dots i_m} B_{j_1 \dots j_n} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_m} \otimes \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_n}. \end{aligned} \quad (2.133)$$

We now need to introduce the “normal” product of two tensors. We denote the “normal” product of two tensors $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{AB}$. This product is different from the tensor product of $\mathbf{A}$ and $\mathbf{B}$ that was denoted by $\mathbf{A} \otimes \mathbf{B}$. The product $\mathbf{AB}$ simply takes the vectors closest to each other in the two tensors $\mathbf{A}$ and $\mathbf{B}$ and sets a dot product between them to contract them into scalars, resulting in the order of the product being two less than the sum of the order of the two original tensors. For example, if $\mathbf{A}$ is a third-order tensor and $\mathbf{B}$ is a fourth-order tensor, then the product $\mathbf{AB}$ is of order five (i.e., $3+4-2=5$). As a specific example, consider $\mathbf{A} = \mathbf{a} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{d}$ and $\mathbf{B} = \mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g}$, where the product of $\mathbf{A}$ and $\mathbf{B}$ is

$$\begin{aligned} \mathbf{AB} &= (\mathbf{a} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{d})(\mathbf{e} \otimes \mathbf{f} \otimes \mathbf{g}) \\ &= (\mathbf{b} \circ \mathbf{e})\mathbf{a} \otimes \mathbf{f} \otimes \mathbf{g} + (\mathbf{d} \circ \mathbf{e})\mathbf{c} \otimes \mathbf{f} \otimes \mathbf{g}. \end{aligned} \quad (2.134)$$

The reader will note that even though the dot product “ $\circ$ ” is implied between the closest vectors, it is not explicitly written between the two tensors. The reason for this is that in tensor algebra the operator “ $\circ$ ” normally is reserved for an operation which produces a scalar result. In most cases, we deal with tensors written in terms of components and base vectors. In such cases, the product of two tensors simply results in a dot product between the closest base vectors. For example, in the case of the two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, we will have

$$\boxed{\mathbf{AB} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(B_{kl}\mathbf{e}_k \otimes \mathbf{e}_l) = A_{ij}B_{kl}(\mathbf{e}_j \circ \mathbf{e}_k)\mathbf{e}_i \otimes \mathbf{e}_l = A_{ij}B_{kl}g_{jk}\mathbf{e}_i \otimes \mathbf{e}_l.} \quad (2.135)$$

One also can mix bases without needing to do anything special. For example, if $\mathbf{A}$ is represented in base $\mathbf{e}_i$ and $\mathbf{B}$ is represented in base $\mathbf{e}_i^*$, then the product of these two tensors can be calculated as

$$\mathbf{AB} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(B_{kl}^*\mathbf{e}_k^* \otimes \mathbf{e}_l^*) = A_{ij}B_{kl}^*(\mathbf{e}_j \circ \mathbf{e}_k^*)\mathbf{e}_i \otimes \mathbf{e}_l^*. \quad (2.136)$$

Other mixes of the bases result in the same type of operations. Namely, take the dot product of the closest base vectors and leave the remaining base vectors in order of appearance with tensor products between them. The reader will note that this product does not commute and, therefore, $\mathbf{AB}$ is different from $\mathbf{BA}$.

The *identity* for second-order tensors will be denoted by $\mathbf{I}$ and has the property

$$\boxed{\mathbf{I}\mathbf{u} = \mathbf{u}\mathbf{I} = \mathbf{u}.} \quad (2.137)$$

Let us consider evaluating the components of the second-order identity tensor in a base given by the vectors $\mathbf{e}_i$. It can be seen that the components of $\mathbf{I} = I_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, in any base $\mathbf{e}_i$ can be found from the property of the identity through the operation

$$(I_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)u_p\mathbf{e}_p = I_{ij}g_{jp}u_p\mathbf{e}_i = u_i\mathbf{e}_i,$$

which must hold for every $\mathbf{u}$. Obviously, this requires that $I_{ij}g_{jp} = \delta_{ip}$, resulting in

$$\boxed{\mathbf{I} = g_{ij}^{-1}\mathbf{e}_i \otimes \mathbf{e}_j.} \quad (2.138)$$

The reader will note that the representation of the identity in another base can be found by using the transformation matrix $[l]$ between the bases. In the case of a Cartesian base of unit vectors, one will have

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j. \quad (2.139)$$

The product of the identity and any second-order tensor $\mathbf{A}$ will leave $\mathbf{A}$ unchanged, resulting in the relation

$$\boxed{\mathbf{AI} = \mathbf{IA} = \mathbf{A}.} \quad (2.140)$$

The same is true for the product of the second-order identity tensor and a tensor of any order.

One also can define the *transpose* of a tensor. Let a superscript “ T ” denote the transpose of a tensor. The calculation of the transpose is done through the three rules:

- $\boxed{\alpha^T = \alpha}$ for any scalar α ,
- $\boxed{\mathbf{u}^T = \mathbf{u}}$ for any vector $\mathbf{u}$,
- $\boxed{(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{B}^T \otimes \mathbf{A}^T}$ for any order tensors $\mathbf{A}$ and $\mathbf{B}$.

Therefore, for example, the transpose of the third-order tensor $\mathbf{T} = T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ is given by

$$\begin{aligned} \mathbf{T}^T &= (T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)^T = T_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)^T = T_{ijk}\mathbf{e}_k^T \otimes (\mathbf{e}_i \otimes \mathbf{e}_j)^T \\ &= T_{ijk}\mathbf{e}_k \otimes \mathbf{e}_j^T \otimes \mathbf{e}_i^T = T_{ijk}\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i = T_{kji}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \end{aligned} \quad (2.141)$$

The last step in this operation is simply a renaming of the dummy indices. A tensor $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$, and is skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.

The *trace* of a second-order tensor $\mathbf{A}$ will be denoted by $tr(\mathbf{A})$ and can be obtained by contracting the base vectors with a dot product. Therefore, for a tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ one will have

$$tr(\mathbf{A}) = A_{ij}\mathbf{e}_i \circ \mathbf{e}_j = A_{ij}g_{ij}. \quad (2.142)$$

If one lets $[A]$ denote the matrix of components of $\mathbf{A}$ in the base $\mathbf{e}_i$ and $[g]$ is the matrix of the metric tensor for this same base, then the trace of the tensor in matrix notation can be written as

$$tr(\mathbf{A}) = tr([A][g]^T) = tr([A][g]). \quad (2.143)$$

It is easy to show by direct evaluation that the trace of a second-order tensor and the trace of its transpose are the same. That is,

$$tr(\mathbf{A}) = tr(\mathbf{A}^T). \quad (2.144)$$

It also can be shown that for two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ one has the relation

$$tr(\mathbf{AB}) = tr(\mathbf{BA}). \quad (2.145)$$

The *determinant* of a second-order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ will be denoted by $det(\mathbf{A})$ and is given by

$$det(\mathbf{A}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}A_{im}g_{mp}A_{jn}g_{nq}A_{ko}g_{or}. \quad (2.146)$$

Using matrix notation, one can write this as

$$det(\mathbf{A}) = det([A][g]). \quad (2.147)$$

Since we know that the determinant of the product of two matrices is the product of the determinant of each, and that the determinant of the inverse of a matrix is the reciprocal of its determinant, one also can write the determinant of the tensor $\mathbf{A}$ as

$$det(\mathbf{A}) = det([g])det([A]). \quad (2.148)$$

A nonsingular second-order tensor $\mathbf{A}$ has an *inverse* that is denoted by $\mathbf{A}^{-1}$ with the property

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \quad (2.149)$$

A necessary and sufficient condition for a second-order tensor to have an inverse is that its determinant be nonzero. It can be shown that the components of the inverse of the second-order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ can be calculated in the same base from the equation

$$[A^{-1}] = [g]^{-1}[A]^{-1}[g]^{-1}. \quad (2.150)$$

The reader will note that we have defined tensor counterparts for many of the operations introduced originally for matrices. These operations are indistinguishable between matrices and tensors when working in a single orthonormal base with tensors of up to order two. The similarity of the operations stops when using mixed bases, nonorthonormal bases, or tensors of higher order. A tensor is orthogonal if its inverse and transpose are the same.

The “ $\circ$ ” product normally is reserved to denote an operation that results in a scalar. In this book, we will use this symbol between two tensors $\mathbf{A}$ and $\mathbf{B}$ of second order to denote the trace of the product of $\mathbf{A}^T$ and $\mathbf{B}$. That is,

$$\mathbf{A} \circ \mathbf{B} = tr(\mathbf{A}^T\mathbf{B}). \quad (2.151)$$

There are actually at least two ways to make a scalar from two second-order tensors. Some books distinguish between these two ways by denoting one by “:” and the other by “..” as defined by

$$\boxed{\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B}), \text{ and } \mathbf{A} .. \mathbf{B} = \text{tr}(\mathbf{A} \mathbf{B}).} \quad (2.152)$$

We will normally explicitly use “:” in place of using “o” to avoid confusion. These operators also are used when contracting two pairs of bases in higher-order tensor operations. For example, for the two third-order tensors $\mathbf{A}$ and $\mathbf{B}$ one can use these operators to describe the following operations

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= (A_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) : (B_{mno} \mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o) = A_{ijk} B_{mno} (\mathbf{e}_j \circ \mathbf{e}_m) (\mathbf{e}_k \circ \mathbf{e}_n) \mathbf{e}_i \otimes \mathbf{e}_o \\ &= A_{ijk} B_{mno} g_{jm} g_{kn} \mathbf{e}_i \otimes \mathbf{e}_o, \end{aligned}$$

and

$$\begin{aligned} \mathbf{A} .. \mathbf{B} &= (A_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) .. (B_{mno} \mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_o) = A_{ijk} B_{mno} (\mathbf{e}_k \circ \mathbf{e}_m) (\mathbf{e}_j \circ \mathbf{e}_n) \mathbf{e}_i \otimes \mathbf{e}_o \\ &= A_{ijk} B_{mno} g_{km} g_{jn} \mathbf{e}_i \otimes \mathbf{e}_o. \end{aligned}$$

One can define “:” and “...” in a similar way to act between the last three bases of the first tensor and the first three of the second tensor.

Several useful fourth-order tensors are $\mathbf{I}$, $\mathbf{T}$, $\mathbf{II}$, and $\mathbf{I} \otimes \mathbf{I}$, which are defined by the following properties in relation to any second-order tensor $\mathbf{A}$:

$$\boxed{\mathbf{I} : \mathbf{A} = \mathbf{A} : \mathbf{I} = \mathbf{A},} \quad (2.153)$$

$$\boxed{\mathbf{T} : \mathbf{A} = \mathbf{A} : \mathbf{T} = \mathbf{A}^T,} \quad (2.154)$$

$$\boxed{\mathbf{II} : \mathbf{A} = \mathbf{A} : \mathbf{II} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T),} \quad (2.155)$$

$$\boxed{(\mathbf{I} \otimes \mathbf{I}) : \mathbf{A} = \mathbf{A} : (\mathbf{I} \otimes \mathbf{I}) = \text{tr}(\mathbf{A}) \mathbf{I}.} \quad (2.156)$$

In component form in any base $\mathbf{e}_i$, these tensors can be written as

$$\mathbf{I} = g_{ik}^{-1} g_{jl}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad (2.157)$$

$$\mathbf{T} = g_{il}^{-1} g_{jk}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad (2.158)$$

$$\mathbf{II} = \frac{1}{2}(\mathbf{I} + \mathbf{T}) = \frac{1}{2}(g_{ik}^{-1} g_{jl}^{-1} + g_{il}^{-1} g_{jk}^{-1}) \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad (2.159)$$

$$\mathbf{I} \otimes \mathbf{I} = g_{ij}^{-1} g_{kl}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \quad (2.160)$$

Section 2.3-Problems:

- 2.3-1:** Show that for any second-order tensor $\mathbf{A}$ and any set of base vectors $\mathbf{e}_i$ with metric tensor $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$, we have the relations

$$\boxed{\text{tr}(\mathbf{A}) = \mathbf{e}_i \circ (\mathbf{A} \mathbf{e}_j) g_{ij}^{-1} = \mathbf{I} : \mathbf{A}.} \quad (2.161)$$

- 2.3-2:** Show that the components in the base $\mathbf{e}_i \otimes \mathbf{e}_j$ of the nonsingular second-order tensor $\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ can be calculated from the matrix equation

$$[A^{-1}] = [g]^{-1} [A]^{-1} [g]^{-1}, \quad (2.162)$$

where $[A^{-1}]$ is the components in matrix form of $\mathbf{A}^{-1}$ in the base $\mathbf{e}_i \otimes \mathbf{e}_j$.

- 2.3-3:** Show that the component forms of $\mathbf{I}$, $\mathbf{T}$, $\mathbf{II}$, and $\mathbf{I} \otimes \mathbf{I}$ given in (2.157) to (2.160) result in the properties given in (2.153) to (2.156).
- 2.3-4:** Prove the following identities in which $\mathbf{a} = a_i \mathbf{e}_i$, $\mathbf{b} = b_i \mathbf{e}_i$, $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_i \mathbf{e}_i$ are vectors, $\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, $\mathbf{B} = B_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, $\mathbf{C} = C_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, and $\mathbf{D} = D_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ are second-order tensors, and $\mathbf{E} = E_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ is a fourth-order tensor.

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{u} \otimes \mathbf{v}) = (\mathbf{a} \circ \mathbf{u})(\mathbf{b} \circ \mathbf{v}) = (\mathbf{a} \otimes \mathbf{v}) : (\mathbf{u} \otimes \mathbf{b}), \quad (2.163)$$

$$(\mathbf{a} \otimes \mathbf{b}) \times \mathbf{u} = \mathbf{a} \otimes (\mathbf{b} \times \mathbf{u}), \quad (2.164)$$

$$(\mathbf{a} \otimes \mathbf{b}) \times (\mathbf{u} \otimes \mathbf{v}) = -(\mathbf{a} \otimes \mathbf{u}) \times (\mathbf{b} \otimes \mathbf{v}), \quad (2.165)$$

$$\mathbf{a} \times (\mathbf{u} \otimes \mathbf{v}) = -[(\mathbf{v} \otimes \mathbf{u}) \times \mathbf{a}]^T, \quad (2.166)$$

$$\boxed{\mathbf{a} \times \mathbf{B} = -(\mathbf{B}^T \times \mathbf{a})^T}, \quad (2.167)$$

$$\boxed{\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A}}, \quad (2.168)$$

$$\boxed{\mathbf{A} : \mathbf{B} = \mathbf{A} \cdot \mathbf{B}^T}, \quad (2.169)$$

$$\boxed{tr(\mathbf{AB}) = \mathbf{A}^T : \mathbf{B} = \mathbf{A} : \mathbf{B}^T}, \quad (2.170)$$

$$\boxed{(\mathbf{a} \otimes \mathbf{b}) : \mathbf{C} = \mathbf{a} \circ (\mathbf{Cb}) = \mathbf{b} \circ (\mathbf{C}^T \mathbf{a})}, \quad (2.171)$$

$$\boxed{\mathbf{u} \circ (\mathbf{Av}) = (\mathbf{A}^T \mathbf{u}) \circ \mathbf{v}}, \quad (2.172)$$

$$\boxed{(\mathbf{Au}) \circ \mathbf{v} = \mathbf{u} \circ (\mathbf{A}^T \mathbf{v})}, \quad (2.173)$$

$$\boxed{(\mathbf{Au}) \circ (\mathbf{Bv}) = \mathbf{u} \circ (\mathbf{A}^T \mathbf{Bv}) = (\mathbf{B}^T \mathbf{Au}) \circ \mathbf{v}}, \quad (2.174)$$

$$\boxed{(\mathbf{AB}) : \mathbf{C} = \mathbf{A} : (\mathbf{CB}^T) = \mathbf{B} : (\mathbf{A}^T \mathbf{C})}, \quad (2.175)$$

$$\boxed{\mathbf{A} : (\mathbf{BCD}) = (\mathbf{B}^T \mathbf{AD}^T) : \mathbf{C}}, \quad (2.176)$$

$$\boxed{\mathbf{E} : (\mathbf{AB}) = (\mathbf{EB}^T) : \mathbf{A}}, \quad (2.177)$$

2.3.1 Extracting components of a tensor

From time to time, it is necessary to compute the components of a tensor for a given base when the tensor is described in another base. Let us say we are given the vector $\mathbf{a}$ and we would like to evaluate its components in the base $\mathbf{e}_i$ so that we can write this vector as $\mathbf{a} = a_i \mathbf{e}_i$. This easily can be done if we note that

$$(\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{a} = a_1 (\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{e}_1 + a_2 (\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{e}_2 + a_3 (\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{e}_3.$$

Since $(\mathbf{e}_2 \times \mathbf{e}_3)$ is orthogonal to both $\mathbf{e}_2$ and $\mathbf{e}_3$, the second and third terms in this expansion become zero and we can conclude that

$$a_1 = \frac{(\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{a}}{(\mathbf{e}_2 \times \mathbf{e}_3) \circ \mathbf{e}_1}.$$

The second and third components of vector $\mathbf{a}$ also can be evaluated by a similar process.

Let us formalize this process by defining a new base $\bar{\mathbf{e}}_i$ called dual or reciprocal base and defined as¹

$$\begin{aligned} \bar{\mathbf{e}}_1 &= \frac{\mathbf{e}_2 \times \mathbf{e}_3}{V}, \\ \bar{\mathbf{e}}_2 &= \frac{\mathbf{e}_3 \times \mathbf{e}_1}{V}, \\ \bar{\mathbf{e}}_3 &= \frac{\mathbf{e}_1 \times \mathbf{e}_2}{V}, \end{aligned} \quad (2.178)$$

where

$$V = (\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3. \quad (2.179)$$

The reader will note that the magnitude of the triple scalar product $|(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3|$ of the base vectors is the volume of the parallelepiped constructed from the base vectors. Therefore, V is either equal to or negative to the volume of this parallelepiped, depending on if the the base vectors create a right-handed or left-handed system. From the above development, one must conclude that the components of $\mathbf{a}$ in the base $\mathbf{e}_i$ are given by the relations

$$a_i = \mathbf{a} \circ \bar{\mathbf{e}}_i. \quad (2.180)$$

This same method can be used to calculate the components of a second-order tensor. For example, let $\mathbf{A}$ be a second-order tensor with components in the base $\mathbf{e}_i$ given by A_{ij} so that $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. Using this technique, these components can be evaluated by using the equation

$$A_{ij} = \bar{\mathbf{e}}_i \circ (\mathbf{A}\bar{\mathbf{e}}_j). \quad (2.181)$$

This idea can be expanded to evaluate the components of third- and higher-order tensors. For example, let $\mathbf{B} = B_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ be a third-order tensor. The components of $\mathbf{B}$ are easily evaluated through the operation

$$B_{ijk} = [(\mathbf{B}\bar{\mathbf{e}}_k)\bar{\mathbf{e}}_j] \circ \bar{\mathbf{e}}_i. \quad (2.182)$$

As can be seen, the process is to contract one base at a time. The extension to higher-order tensors is similar.

We can calculate the dual base through knowing the inverse of the metric tensor. If we calculate the dot product of the original base and its dual, we get the relation

$$\mathbf{e}_i \circ \bar{\mathbf{e}}_j = \delta_{ij}. \quad (2.183)$$

If $\bar{\mathbf{e}}_i = l_{ij}\mathbf{e}_j$ gives the transformation from the original base to its dual, then we can write $\mathbf{e}_i \circ \bar{\mathbf{e}}_j = \mathbf{e}_i \circ l_{jk}\mathbf{e}_k = g_{ik}l_{jk}$. We therefore have the relation

$$g_{ik}l_{jk} = \delta_{ij}, \quad (2.184)$$

which gives $l_{ij} = g_{ji}^{-1} = g_{ij}^{-1}$. We therefore can calculate the dual base from the relation

$$\bar{\mathbf{e}}_i = g_{ij}^{-1}\mathbf{e}_j. \quad (2.185)$$

As a point to note, the volume of the parallelepiped formed by the dual base $\bar{\mathbf{e}}_i$ is the inverse of the volume resulting from the original base. This follows from the relation

$$(\bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2) \circ \bar{\mathbf{e}}_3 = \frac{1}{V^3}[\mathbf{e}_2 \times \mathbf{e}_3, \mathbf{e}_3 \times \mathbf{e}_1, \mathbf{e}_1 \times \mathbf{e}_2] = \frac{1}{V^3}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]^2 = \frac{1}{V}, \quad (2.186)$$

¹The base $\mathbf{e}_i$ also is known as the covariant base, and the dual base $\bar{\mathbf{e}}_i$ is known as the contravariant base. For a development on tensors using covariant and contravariant components, the reader is directed to an article by Budiansky on tensors in the *Handbook of Applied Mathematics, Selected Results and Methods*, edited by Carl Pearson, Van Nostrand Reinhold Company, Inc., 1983. Even though unnecessary, the use of covariant and contravariant components can at times make the notation simpler, unfortunately sometimes at the expense of hiding the role of the base vectors.

a result of one of the identities in Section 2.

Section 2.3-Problems:

2.3-5: Show the identity

$$(\bar{\mathbf{e}}_i \times \bar{\mathbf{e}}_j) \circ \bar{\mathbf{e}}_k = \frac{1}{V} \epsilon_{ijk}, \quad (2.187)$$

where ϵ_{ijk} is the permutation symbol.

2.3-6: If $[A]$ denotes the matrix of components of $\mathbf{A}$ in the base $\mathbf{e}_i$, show that

$$\det([A]) = \frac{1}{V} [(\mathbf{A}\bar{\mathbf{e}}_1) \times (\mathbf{A}\bar{\mathbf{e}}_2)] \circ (\mathbf{A}\bar{\mathbf{e}}_3) \quad (2.188)$$

using the identity

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} \mathbf{a} \circ \mathbf{x} & \mathbf{a} \circ \mathbf{y} & \mathbf{a} \circ \mathbf{z} \\ \mathbf{b} \circ \mathbf{x} & \mathbf{b} \circ \mathbf{y} & \mathbf{b} \circ \mathbf{z} \\ \mathbf{c} \circ \mathbf{x} & \mathbf{c} \circ \mathbf{y} & \mathbf{c} \circ \mathbf{z} \end{vmatrix}.$$

2.3-7: Show the identity

$$[\mathbf{A}\mathbf{a}, \mathbf{A}\mathbf{b}, \mathbf{A}\mathbf{c}] = \det(\mathbf{A})[\mathbf{a}, \mathbf{b}, \mathbf{c}], \quad (2.189)$$

which must hold for any second-order tensor $\mathbf{A}$ and any three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Note that $\det(\mathbf{A}) = \det([g])\det([A])$ where $[A]$ is the matrix of components of $\mathbf{A}$ and $[g]$ is the metric tensor for the associated base.

2.3-8: Prove the identity

$$(\mathbf{A}\mathbf{a}) \times (\mathbf{A}\mathbf{b}) = \det(\mathbf{A})(\mathbf{a} \times \mathbf{b})\mathbf{A}^{-1}. \quad (2.190)$$

that must hold for every invertible tensor $\mathbf{A}$ and any two vectors $\mathbf{a}$ and $\mathbf{b}$.

2.3-9: Prove that a base and its reciprocal are related by the relation

$$\mathbf{e}_i \circ \bar{\mathbf{e}}_j = \delta_{ij}. \quad (2.191)$$

2.3.2 The eigenvalue problem for tensors

Similar to the eigenvalue problem defined for matrices, an eigenvalue problem can be defined for tensors. For a second-order tensor $\mathbf{A}$ one can search for all scalars λ and their associated first-order tensors $\mathbf{x}$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \quad (2.192)$$

As in the case of matrices, we call λ the eigenvalue and $\mathbf{x}$ its associated eigenvector. For manipulation purposes, one always can select an orthonormal base $\mathbf{e}_i$ to describe the tensors and in such a base one will get

$$(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(x_k\mathbf{e}_k) = \lambda x_i\mathbf{e}_i,$$

which reduces to

$$A_{ij}x_j\mathbf{e}_i = \lambda x_i\mathbf{e}_i.$$

Equating components on both sides results in

$$A_{ij}x_j = \lambda x_i.$$

This last equation is identical to the eigenvalue problem for matrices. Therefore, when we use an orthonormal base the eigenvalue problem for second-order tensors takes exactly the same form as that for matrices. If the base is not orthonormal, then the eigenvalue problem can be written as

$$A_{ij}g_{jk}x_k\mathbf{e}_i = \lambda x_i\mathbf{e}_i, \quad (2.193)$$

which is equivalent to solving the eigenvalue problem for the matrix $[B]$ with components $B_{ik} = A_{ij}g_{jk}$. Therefore, in all cases, one can reduce the eigenvalue problem for the tensor to an eigenvalue problem for a matrix.

Of particular interest for our development are the eigenvalues for a real symmetric tensor. This tensor will have a real and symmetric component matrix if one represents it in an orthonormal base. As such, from our study of matrices we know that this tensor will have real eigenvalues and that one always can select an orthonormal set of eigenvectors. Let λ_i be the three eigenvalues of the real symmetric tensor $\mathbf{A}$ and let $\hat{\mathbf{n}}_i$ be their associated eigenvectors such that

$$\begin{aligned} \mathbf{A}\hat{\mathbf{n}}_1 &= \lambda_1\hat{\mathbf{n}}_1, \\ \mathbf{A}\hat{\mathbf{n}}_2 &= \lambda_2\hat{\mathbf{n}}_2, \\ \mathbf{A}\hat{\mathbf{n}}_3 &= \lambda_3\hat{\mathbf{n}}_3. \end{aligned}$$

If one represents $\mathbf{A}$ in the base $\hat{\mathbf{n}}_i$ of its orthonormal eigenvectors, then one gets

$$(A_{ij}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j)\hat{\mathbf{n}}_k = \lambda_k\hat{\mathbf{n}}_k \quad (\text{no sum on } k),$$

or

$$A_{ik}\hat{\mathbf{n}}_i = \lambda_k\hat{\mathbf{n}}_k \quad (\text{no sum on } k).$$

This can be written as

$$\begin{aligned} A_{11}\hat{\mathbf{n}}_1 + A_{21}\hat{\mathbf{n}}_2 + A_{31}\hat{\mathbf{n}}_3 &= \lambda_1\hat{\mathbf{n}}_1, \\ A_{12}\hat{\mathbf{n}}_1 + A_{22}\hat{\mathbf{n}}_2 + A_{32}\hat{\mathbf{n}}_3 &= \lambda_2\hat{\mathbf{n}}_2, \\ A_{13}\hat{\mathbf{n}}_1 + A_{23}\hat{\mathbf{n}}_2 + A_{33}\hat{\mathbf{n}}_3 &= \lambda_3\hat{\mathbf{n}}_3. \end{aligned}$$

That is, $A_{11} = \lambda_1$, $A_{22} = \lambda_2$, $A_{33} = \lambda_3$ and all other components are zero. Therefore, a real symmetric tensor $\mathbf{A}$ has a diagonal matrix of components when represented in an orthonormal base of its eigenvectors. The tensor representation of $\mathbf{A}$ for such a base can be written as

$$\boxed{\mathbf{A} = \lambda_1\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \lambda_2\hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2 + \lambda_3\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3.} \quad (2.194)$$

This representation is known as the *spectral representation*.

For a symmetric tensor (matrix), the eigenvalue problem is equivalent to a stationary value problem in which one is looking for the direction $\hat{\mathbf{n}}$ for which $\mathbf{y} = \mathbf{A}\hat{\mathbf{n}}$ has a maximum or minimum magnitude. As a result, the eigenvalues represent the limits of how much the tensor $\mathbf{A}$ can stretch a vector (relative to its initial length). The eigenvalue problem indicates that these limit stretches occur along the eigenvectors, and that the tensor $\mathbf{A}$ leaves the directions of these vectors unchanged (i.e., $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$).

2.3.3 Length of tensors

From time to time, it becomes necessary to evaluate the length of a tensor. We frequently use the “ $|\cdot|$ ” to denote the length or magnitude. For a scalar, the measure of length is its magnitude. For a vector, we can define the length of the vector through a dot product. For

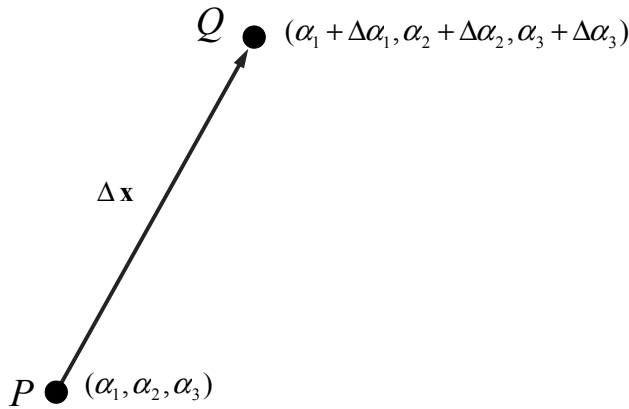


Figure 2.10: Two close points in space.

example, the length of vector $\mathbf{a}$ would be $|\mathbf{a}| = \sqrt{\mathbf{a} \circ \mathbf{a}}$. For a second-order tensor, we can define the length of the tensor $\mathbf{A}$ as

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}} = \sqrt{\text{tr}(\mathbf{A}\mathbf{A}^T)}. \quad (2.195)$$

We can generalize this to higher-order tensors through the selection of an appropriate scalar product so that the length of tensor $\mathbf{A}$ is given by $|\mathbf{A}| = \sqrt{\mathbf{A} \circ \mathbf{A}}$.

2.4 Tensor calculus

We now proceed to look at tensor calculus, including operators such as the gradient, the divergence, the Laplacian, and the curl and theorems such as the divergence and the Stokes' theorems. We will also show the difference between the del and gradient operators, develop differential identities, and define partial derivatives with respect to tensors.

2.4.1 Gradient

Let us start with the description of the gradient of a scalar field. Consider a function ϕ of the coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Let $\Delta\mathbf{x}$ denote the vector going from point P to a point Q that is close to P , as shown in Figure 2.10. The change in the value of ϕ from P to Q will be denoted by $\Delta\phi$. The *gradient* of ϕ will be denoted as $\mathbf{grad}(\phi)$ and will be defined as the unique vector that gives the differential change in ϕ , as Q approaches P , through the relation

$$d\phi = \mathbf{grad}(\phi) \circ d\mathbf{x}. \quad (2.196)$$

The gradient of a scalar field is a vector and can be written as

$$\mathbf{grad}(\phi) = \text{grad}(\phi)_i \mathbf{e}_i, \quad (2.197)$$

where $\text{grad}(\phi)_i$ are the components of $\mathbf{grad}(\phi)$ in the given base. Therefore, the gradient operator provides a means of calculating how the function changes as one moves by $d\mathbf{x}$.

The curvilinear components of the gradient can be calculated using the chain rule, $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$, and $g_{ij}^{-1} g_{jk} = \delta_{ik}$. This is done through the following steps

$$\begin{aligned} d\phi &= \frac{\partial\phi(\alpha_1, \alpha_2, \alpha_3)}{\partial\alpha_i} d\alpha_i = \frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1} g_{pq} d\alpha_q = \frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1} (\mathbf{e}_p \circ \mathbf{e}_q) d\alpha_q \\ &= \left(\frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1} \mathbf{e}_p \right) \circ (d\alpha_q \mathbf{e}_q). \end{aligned} \quad (2.198)$$

In curvilinear coordinates $d\mathbf{x} = d\alpha_q \mathbf{e}_q$, so one can write

$$d\phi = \mathbf{grad}(\phi) \circ d\mathbf{x} = \left(\frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1} \mathbf{e}_p \right) \circ d\mathbf{x}. \quad (2.199)$$

Since this must be true irrespective of the selection of $d\mathbf{x}$, we can conclude that $\mathbf{grad}(\phi)_p = \frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1}$, and, therefore,

$$\mathbf{grad}(\phi) = \frac{\partial\phi}{\partial\alpha_i} g_{ip}^{-1} \mathbf{e}_p = \frac{\partial\phi}{\partial\alpha_i} \bar{\mathbf{e}}_i, \quad (2.200)$$

where $\bar{\mathbf{e}}_i$ is the dual base to $\mathbf{e}_i$.

The definition for the gradient of a higher-order tensor $\mathbf{A}$ is given through the relation

$$d\mathbf{A} = \mathbf{grad}(\mathbf{A}) d\mathbf{x}. \quad (2.201)$$

That is, the change in the tensor $\mathbf{A}$ for an infinitesimal change $d\mathbf{x}$ in the position is equal to the product of the gradient of $\mathbf{A}$ times the change in the position. The result of this provides the definition of the gradient as

$$\mathbf{grad}(\mathbf{A}) = \frac{\partial\mathbf{A}}{\partial\alpha_i} g_{ip}^{-1} \otimes \mathbf{e}_p = \frac{\partial\mathbf{A}}{\partial\alpha_i} \otimes \bar{\mathbf{e}}_i. \quad (2.202)$$

The “del” operator, also known as the nabla operator, is denoted by “ ∇ ” and is closely related to the gradient. The ∇ operator is defined

$$\nabla = \mathbf{e}_p g_{pi}^{-1} \frac{\partial}{\partial\alpha_i} = \bar{\mathbf{e}}_i \frac{\partial}{\partial\alpha_i}. \quad (2.203)$$

For a scalar field ϕ , we get

$$\nabla(\phi) = \bar{\mathbf{e}}_i \frac{\partial\phi}{\partial\alpha_i} = \mathbf{grad}(\phi). \quad (2.204)$$

In the case of a vector field $\mathbf{u}$, we find

$$\nabla(\mathbf{u}) = \bar{\mathbf{e}}_i \otimes \frac{\partial\mathbf{u}}{\partial\alpha_i} = [\mathbf{grad}(\mathbf{u})]^T. \quad (2.205)$$

For a second- or higher-order tensor field $\mathbf{A}$ we will have

$$\nabla(\mathbf{A}) = \bar{\mathbf{e}}_i \otimes \frac{\partial\mathbf{A}}{\partial\alpha_i} = [\mathbf{grad}(\mathbf{A}^T)]^T. \quad (2.206)$$

Using the del operator, we can find the change of the field through the equations

$$d\phi = d\mathbf{x} \circ \nabla(\phi) = \nabla(\phi) \circ d\mathbf{x}, \quad (2.207)$$

$$d\mathbf{u} = d\mathbf{x} \nabla(\mathbf{u}) = [\nabla(\mathbf{u})]^T d\mathbf{x}, \quad (2.208)$$

$$d\mathbf{A} = d\mathbf{x} \nabla(\mathbf{A}) = [\nabla(\mathbf{A}^T)]^T d\mathbf{x}. \quad (2.209)$$

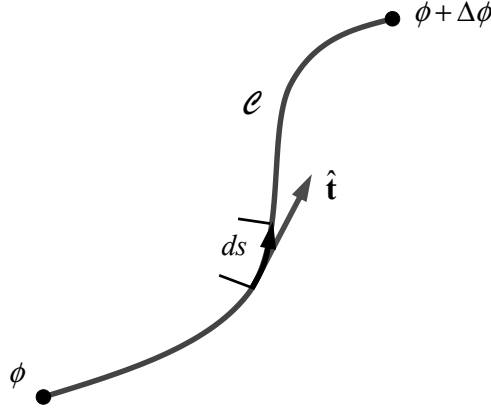


Figure 2.11: Calculation of the change in ϕ from point to point using the directional derivative and integrating over a curve $\mathcal{C}$.

We will denote the components of $\nabla(\phi)$ as $\nabla(\phi)_i$.

As an example, consider the cylindrical coordinate system given by

$$\alpha_1 = r, \quad \alpha_2 = \theta, \quad \alpha_3 = z, \quad (2.210)$$

$$\mathbf{e}_1 = \hat{\mathbf{e}}_r, \quad \mathbf{e}_2 = r\hat{\mathbf{e}}_\theta, \quad \mathbf{e}_3 = \hat{\mathbf{e}}_z, \quad (2.211)$$

$$[g] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.212)$$

For this coordinate system and a scalar field ϕ , one gets

$$\mathbf{grad}(\phi) = \nabla(\phi) = \frac{\partial \phi}{\partial \alpha_i} g_{ip}^{-1} \mathbf{e}_p = \begin{bmatrix} \frac{\partial \phi}{\partial r} & \frac{\partial \phi}{\partial \theta} & \frac{\partial \phi}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_r \\ r\hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z \end{bmatrix}, \quad (2.213)$$

which gives

$$\mathbf{grad}(\phi) = \nabla(\phi) = \frac{\partial \phi}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{\partial \phi}{\partial z} \hat{\mathbf{e}}_z. \quad (2.214)$$

The gradient also can be used to define the directional derivative of a scalar field. If the magnitude of $d\mathbf{x}$ is ds , then the change in ϕ per unit length along $d\mathbf{x}$ can be written as

$$\boxed{\frac{d\phi}{ds} = \mathbf{grad}(\phi) \circ \hat{\mathbf{t}} = \nabla(\phi) \circ \hat{\mathbf{t}}}, \quad (2.215)$$

where $\hat{\mathbf{t}}$ is a unit vector along $d\mathbf{x}$. From this expression, one must conclude that the change in ϕ per unit length is largest when $\hat{\mathbf{t}}$ is pointing along the direction of the gradient. The reader recalls from calculus that the gradient points in the direction of the largest variation of the function. One can use the directional derivative to calculate the change in ϕ between two points by integrating along a curve connecting the points. This is called a line integral.

As shown in Figure 2.11, the change in ϕ can be calculated by integrating along $\mathcal{C}$ using the relation

$$\Delta\phi = \int_{\mathcal{C}} \mathbf{grad}(\phi) \circ \hat{\mathbf{t}} ds = \int_{\mathcal{C}} \nabla(\phi) \circ \hat{\mathbf{t}} ds, \quad (2.216)$$

where $\hat{\mathbf{t}}$ is a unit vector that is tangent to the curve $\mathcal{C}$ at each point. The directional derivative of a vector field $\mathbf{u}$ and a higher order tensor field $\mathbf{A}$ can be calculated from

$$\frac{d\mathbf{u}}{ds} = \mathbf{grad}(\mathbf{u})\hat{\mathbf{t}} = \hat{\mathbf{t}}\nabla(\mathbf{u}), \quad (2.217)$$

$$\frac{d\mathbf{A}}{ds} = \mathbf{grad}(\mathbf{A})\hat{\mathbf{t}} = \hat{\mathbf{t}}\nabla(\mathbf{A}). \quad (2.218)$$

These provide the means of calculating the change in the field variable for points finitely apart by using

$$\Delta\mathbf{u} = \int_{\mathcal{C}} \mathbf{grad}(\mathbf{u})\hat{\mathbf{t}} ds = \int_{\mathcal{C}} \hat{\mathbf{t}}\nabla(\mathbf{u}) ds, \quad (2.219)$$

$$\Delta\mathbf{A} = \int_{\mathcal{C}} \mathbf{grad}(\mathbf{A})\hat{\mathbf{t}} ds = \int_{\mathcal{C}} \hat{\mathbf{t}}\nabla(\mathbf{A}) ds. \quad (2.220)$$

Section 2.4-Problems:

2.4-1: Prove the following relations for the gradient in terms of the del operator given by

$$\mathbf{grad}(\phi) = \nabla(\phi), \quad (2.221)$$

$$\mathbf{grad}(\mathbf{u}) = [\nabla(\mathbf{u})]^T, \quad (2.222)$$

$$\mathbf{grad}(\mathbf{A}) = [\nabla(\mathbf{A}^T)]^T, \quad (2.223)$$

$$(2.224)$$

where ϕ is a scalar field, $\mathbf{u}$ is a vector field, and $\mathbf{A}$ is a higher-order tensor field.

2.4.2 Divergence and Laplacian

The *divergence* of a vector field $\mathbf{u}$ is denoted by $div(\mathbf{u})$ and is defined as

$$div(\mathbf{u}) = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \mathbf{u} \circ \hat{\mathbf{n}} dS, \quad (2.225)$$

where V is a volume containing the point under consideration, S is the surface of V , and $\hat{\mathbf{n}}$ is the unit outward normal to S . It can be seen that $div(\mathbf{u})$ represents the net outward projection of $\mathbf{u}$ at the point under consideration. For example, if $\mathbf{u}$ was the velocity of a fluid, then $div(\mathbf{u})$ would represent the net outflux of the fluid from the point.

One can directly calculate the expression for the divergence in a given curvilinear coordinate system by applying its definition to a volume element created by using vectors along the three coordinate directions, as shown in Figure 2.12. The volume of this element is given by $V = \beta_1\beta_2\beta_3(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3$. To the first approximation, the change in the vector field $\mathbf{u}$ can

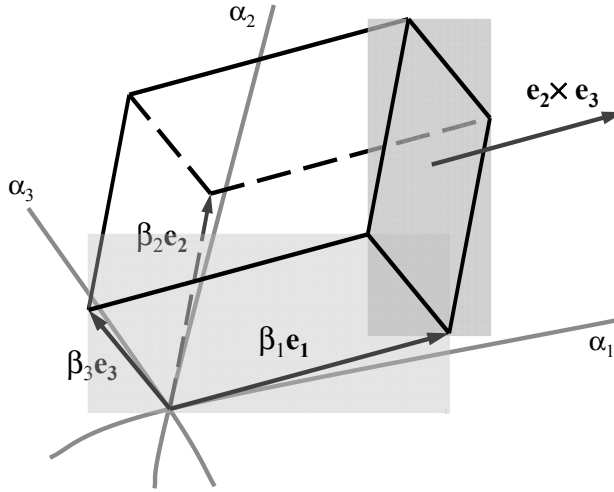


Figure 2.12: A volume element with sides along the coordinate directions.

be obtained using the gradient through the relation $\Delta \mathbf{u} \approx \mathbf{grad}(\mathbf{u})\Delta \mathbf{x} = \Delta \mathbf{x} \nabla(\mathbf{u})$. Also, to the first approximation, the value of $\hat{\mathbf{n}} dS$ for each side is given by the cross product of the vectors constructing the sides. Substituting this into the definition for the divergence and taking the limit as $\beta_1 \beta_2 \beta_3$ goes to zero, one can show that

$$\text{div}(\mathbf{u}) = \frac{1}{(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3} \{ [\mathbf{grad}(\mathbf{u})\mathbf{e}_1] \circ (\mathbf{e}_2 \times \mathbf{e}_3) + [\mathbf{grad}(\mathbf{u})\mathbf{e}_2] \circ (\mathbf{e}_3 \times \mathbf{e}_1) + [\mathbf{grad}(\mathbf{u})\mathbf{e}_3] \circ (\mathbf{e}_1 \times \mathbf{e}_2) \}.$$

We now can use the relation

$$\mathbf{grad}(\mathbf{u})\mathbf{e}_i = \left(\frac{\partial \mathbf{u}}{\partial \alpha_j} g_{jk}^{-1} \otimes \mathbf{e}_k \right) \mathbf{e}_i = \frac{\partial \mathbf{u}}{\partial \alpha_j} g_{jk}^{-1} g_{ki} = \frac{\partial \mathbf{u}}{\partial \alpha_j} \delta_{ji} = \frac{\partial \mathbf{u}}{\partial \alpha_i},$$

and the definition for the dual base to write

$$\text{div}(\mathbf{u}) = \left\{ \left[\frac{\partial \mathbf{u}}{\partial \alpha_1} \right] \circ \bar{\mathbf{e}}_1 + \left[\frac{\partial \mathbf{u}}{\partial \alpha_2} \right] \circ \bar{\mathbf{e}}_2 + \left[\frac{\partial \mathbf{u}}{\partial \alpha_3} \right] \circ \bar{\mathbf{e}}_3 \right\}.$$

It therefore follows that

$$\boxed{\text{div}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial \alpha_i} \circ \bar{\mathbf{e}}_i.} \quad (2.226)$$

It follows that

$$\boxed{\text{div}(\mathbf{u}) = \text{tr}[\mathbf{grad}(\mathbf{u})] = \text{tr}[\nabla(\mathbf{u})].} \quad (2.227)$$

This last result allows one to directly write the divergence in terms of the del operator with the notation $\text{div}(\mathbf{u}) = \nabla \circ \mathbf{u}$ having the meaning

$$\text{div}(\mathbf{u}) = \nabla \circ \mathbf{u} = \mathbf{e}_p g_{pi}^{-1} \circ \frac{\partial \mathbf{u}}{\partial \alpha_i} = \bar{\mathbf{e}}_i \circ \frac{\partial \mathbf{u}}{\partial \alpha_i}. \quad (2.228)$$

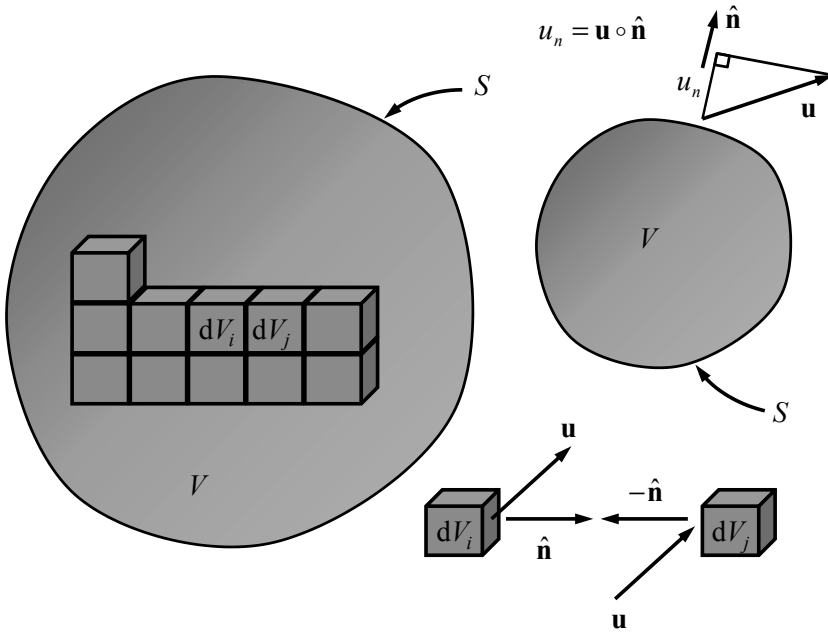


Figure 2.13: The drawing shows internal volumes and internal surfaces that are put together to construct the proof of the divergence theorem.

Let us now consider a vector field $\mathbf{u}$ given in a curvilinear coordinate system by $\mathbf{u} = u_i \mathbf{e}_i$. It follows that the divergence of $\mathbf{u}$ is given in component form as

$$\begin{aligned} \operatorname{div}(\mathbf{u}) &= \frac{\partial u_j \mathbf{e}_j}{\partial \alpha_i} g_{ip}^{-1} \circ \mathbf{e}_p = \left(\frac{\partial u_j}{\partial \alpha_i} \mathbf{e}_j + u_j \frac{\partial \mathbf{e}_j}{\partial \alpha_i} \right) g_{ip}^{-1} \circ \mathbf{e}_p \\ &= \frac{\partial u_j}{\partial \alpha_i} g_{jp} g_{ip}^{-1} + u_j \frac{\partial \mathbf{e}_j}{\partial \alpha_i} g_{ip}^{-1} \circ \mathbf{e}_p = \frac{\partial u_i}{\partial \alpha_i} + u_j \frac{\partial \mathbf{e}_j}{\partial \alpha_i} g_{ip}^{-1} \circ \mathbf{e}_p, \end{aligned}$$

which results in the relation

$$\operatorname{div}(\mathbf{u}) = \frac{\partial u_i}{\partial \alpha_i} + u_j \frac{\partial \mathbf{e}_j}{\partial \alpha_i} \circ \bar{\mathbf{e}}_i. \quad (2.229)$$

As an example, let us again consider the cylindrical coordinate system. In cylindrical coordinates $(\alpha_1, \alpha_2, \alpha_3) = (r, \theta, z)$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\hat{\mathbf{e}}_r, r\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$, and $\mathbf{u} = u_i \mathbf{e}_i = u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z$, to give

$$\operatorname{div}(\mathbf{u}) = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r}. \quad (2.230)$$

In a base where the base vectors are constant, as is the case for a rectangular system, the expression of the divergence simplifies to

$$\operatorname{div}(\mathbf{u}) = \frac{\partial u_i}{\partial \alpha_i}. \quad (2.231)$$

The divergence for higher-order tensors can be defined in a similar manner so that

$$\operatorname{div}(\mathbf{A}) = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \mathbf{A} \hat{\mathbf{n}} dS. \quad (2.232)$$

The proof of the divergence for higher-order tensors is similar to that for a vector. The result is

$$\text{div}(\mathbf{A}) = \frac{\partial \mathbf{A}}{\partial \alpha_i} \bar{\mathbf{e}}_i = [\nabla \circ (\mathbf{A}^T)]^T, \quad (2.233)$$

where the definition of $\nabla \circ (\mathbf{A})$ is

$$\nabla \circ (\mathbf{A}) = \bar{\mathbf{e}}_i \frac{\partial \mathbf{A}}{\partial \alpha_i}. \quad (2.234)$$

The *divergence theorem* relates the following surface and volume integrals

$$\int_S \mathbf{u} \circ \hat{\mathbf{n}} dS = \int_V \text{div}(\mathbf{u}) dV, \quad (2.235)$$

where, as shown in Figure 2.13, S is the surface surrounding the volume V , $\hat{\mathbf{n}}$ is the unit outward normal to S , and $\mathbf{u}$ is any continuous vector function of the coordinates. This figure also shows that the divergence theorem directly follows from the definition of the divergence since the contribution of all internal surfaces that are shared between differential elements are canceled, leaving only the contribution of the external surface S . The same arguments can be used to show that *any order tensor* function $\mathbf{A}$ of position $\mathbf{x}$ provides a similar relation given by

$$\int_S \mathbf{A} \hat{\mathbf{n}} dS = \int_V \text{div}(\mathbf{A}) dV. \quad (2.236)$$

The *Laplacian* is the divergence of the gradient of a scalar field. Letting $\nabla^2(\phi)$ denote the Laplacian, we have the relation

$$\nabla^2(\phi) = \text{div}[\mathbf{grad}(\phi)] = \text{div}[\nabla(\phi)]. \quad (2.237)$$

Section 2.4-Problems:

- 2.4-2:** Complete the intermediate steps required to obtain the expression for the divergence in a curvilinear system. Start from the integral definition.
- 2.4-3:** Derive the expression for the divergence operator in cylindrical coordinates.

2.4.3 Curl and circulation

The *curl* of a vector field $\mathbf{u}$ is denoted by $\text{curl}(\mathbf{u})$. The curl at any point is defined by its projection along any given unit vector $\hat{\mathbf{n}}$ through the definition

$$\text{curl}(\mathbf{u}) \circ \hat{\mathbf{n}} = \lim_{\Delta S \rightarrow 0} \frac{1}{\Delta S} \int_{\mathcal{C}} \mathbf{u} \circ \hat{\mathbf{t}} dl, \quad (2.238)$$

where, as shown in Figure 2.14, ΔS is the area of a surface containing the point under consideration, $\hat{\mathbf{n}}$ is the vector normal to the surface at this point, $\mathcal{C}$ is the curve defining the boundary of ΔS , $\hat{\mathbf{t}}$ is a unit tangent vector to the curve at each point, and dl is a differential element of this curve such that $\hat{\mathbf{t}} dl = d\mathbf{l}$ is the vector having a magnitude of dl and a direction tangent to the curve $\mathcal{C}$ at each point. The direction of $\hat{\mathbf{t}}$ is given by the right-hand rule in relation to $\hat{\mathbf{n}}$. As in the case of the divergence, let us obtain the expression for the curl in

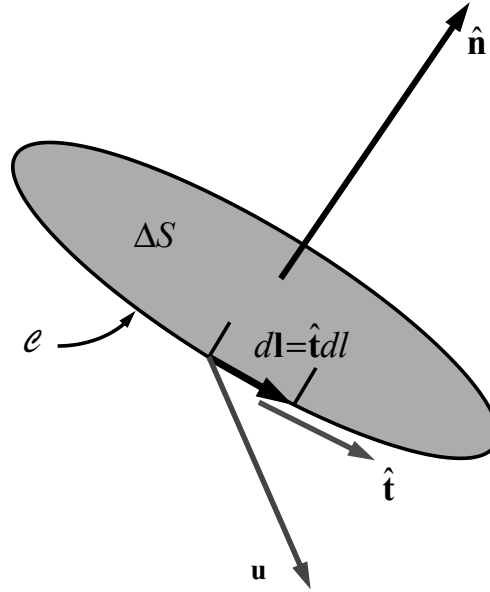


Figure 2.14: The drawing shows the surface element used in calculating the curl of a vector field $\mathbf{u}$.

a given curvilinear system by calculating projection of the curl along the normal to each plane side of a parallelepiped constructed by the vectors along the base directions. Again, we use the gradient to calculate the variation in $\mathbf{u}$ as we integrate around each curve. After integration and taking the limit we obtain

$$\begin{aligned} \text{curl}(\mathbf{u}) \circ (\mathbf{e}_2 \times \mathbf{e}_3) &= [\text{grad}(\mathbf{u})\mathbf{e}_2] \circ \mathbf{e}_3 - [\text{grad}(\mathbf{u})\mathbf{e}_3] \circ \mathbf{e}_2, \\ \text{curl}(\mathbf{u}) \circ (\mathbf{e}_3 \times \mathbf{e}_1) &= [\text{grad}(\mathbf{u})\mathbf{e}_3] \circ \mathbf{e}_1 - [\text{grad}(\mathbf{u})\mathbf{e}_1] \circ \mathbf{e}_3, \\ \text{curl}(\mathbf{u}) \circ (\mathbf{e}_1 \times \mathbf{e}_2) &= [\text{grad}(\mathbf{u})\mathbf{e}_1] \circ \mathbf{e}_2 - [\text{grad}(\mathbf{u})\mathbf{e}_2] \circ \mathbf{e}_1. \end{aligned}$$

Each one of these expressions can be related to a component of the curl. One then obtains a general expression for the curl given by

$$\begin{aligned} \text{curl}(\mathbf{u}) &= \frac{1}{(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3} \{ [([\text{grad}(\mathbf{u}) - \text{grad}(\mathbf{u})^T]\mathbf{e}_2) \circ \mathbf{e}_3] \mathbf{e}_1 \\ &\quad + [([\text{grad}(\mathbf{u}) - \text{grad}(\mathbf{u})^T]\mathbf{e}_3) \circ \mathbf{e}_1] \mathbf{e}_2 + [([\text{grad}(\mathbf{u}) - \text{grad}(\mathbf{u})^T]\mathbf{e}_1) \circ \mathbf{e}_2] \mathbf{e}_3 \}. \end{aligned} \quad (2.239)$$

Also, noting that $\text{grad}(\mathbf{u})\mathbf{e}_i = \frac{\partial \mathbf{u}}{\partial \alpha_i}$, one can rewrite these expressions as

$$\begin{aligned} \text{curl}(\mathbf{u}) &= \frac{1}{(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3} \left[\left(\frac{\partial \mathbf{u}}{\partial \alpha_2} \circ \mathbf{e}_3 - \frac{\partial \mathbf{u}}{\partial \alpha_3} \circ \mathbf{e}_2 \right) \mathbf{e}_1 + \left(\frac{\partial \mathbf{u}}{\partial \alpha_3} \circ \mathbf{e}_1 - \frac{\partial \mathbf{u}}{\partial \alpha_1} \circ \mathbf{e}_3 \right) \mathbf{e}_2 \right. \\ &\quad \left. + \left(\frac{\partial \mathbf{u}}{\partial \alpha_1} \circ \mathbf{e}_2 - \frac{\partial \mathbf{u}}{\partial \alpha_2} \circ \mathbf{e}_1 \right) \mathbf{e}_3 \right]. \end{aligned} \quad (2.240)$$

One can use the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{a} \circ \mathbf{b})\mathbf{c}$ to rewrite this expression

as

$$\begin{aligned} \text{curl}(\mathbf{u}) = & \frac{1}{(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3} \{ [\mathbf{grad}(\mathbf{u})\mathbf{e}_2] \times (\mathbf{e}_1 \times \mathbf{e}_3) + [\mathbf{grad}(\mathbf{u})\mathbf{e}_3] \times (\mathbf{e}_2 \times \mathbf{e}_1) \\ & + [\mathbf{grad}(\mathbf{u})\mathbf{e}_1] \times (\mathbf{e}_3 \times \mathbf{e}_2) \}, \end{aligned} \quad (2.241)$$

that also can be written as

$$\text{curl}(\mathbf{u}) = \frac{1}{(\mathbf{e}_1 \times \mathbf{e}_2) \circ \mathbf{e}_3} \left[\frac{\partial \mathbf{u}}{\partial \alpha_2} \times (\mathbf{e}_1 \times \mathbf{e}_3) + \frac{\partial \mathbf{u}}{\partial \alpha_3} \times (\mathbf{e}_2 \times \mathbf{e}_1) + \frac{\partial \mathbf{u}}{\partial \alpha_1} \times (\mathbf{e}_3 \times \mathbf{e}_2) \right]. \quad (2.242)$$

If $\bar{\mathbf{e}}_i$ is the dual or reciprocal base to $\mathbf{e}_i$, then we can write the curl as

$$\boxed{\text{curl}(\mathbf{u}) = -[\mathbf{grad}(\mathbf{u})\mathbf{e}_i] \times \bar{\mathbf{e}}_i = -\frac{\partial \mathbf{u}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i.} \quad (2.243)$$

We can use the operator $\nabla \times (\mathbf{u})$ defined as

$$\boxed{\nabla \times (\mathbf{u}) = \bar{\mathbf{e}}_i \times \frac{\partial \mathbf{u}}{\partial \alpha_i}.} \quad (2.244)$$

to write the curl as

$$\boxed{\text{curl}(\mathbf{u}) = \nabla \times (\mathbf{u}).} \quad (2.245)$$

In a right-handed orthonormal base $\hat{\mathbf{e}}_i$, the base and the dual are the same so that the expression of the curl simplifies to

$$\text{curl}(\mathbf{u}) = -\frac{\partial \mathbf{u}}{\partial \alpha_1} \times \hat{\mathbf{e}}_1 - \frac{\partial \mathbf{u}}{\partial \alpha_2} \times \hat{\mathbf{e}}_2 - \frac{\partial \mathbf{u}}{\partial \alpha_3} \times \hat{\mathbf{e}}_3 = -\frac{\partial \mathbf{u}}{\partial \alpha_i} \times \hat{\mathbf{e}}_i. \quad (2.246)$$

For a higher-order tensor, we can define the curl in a similar manner so that

$$\boxed{\text{curl}(\mathbf{A}) \circ \hat{\mathbf{n}} = \lim_{\Delta S \rightarrow 0} \frac{1}{\Delta S} \int_{\mathcal{C}} \mathbf{A} \hat{\mathbf{t}} dl.} \quad (2.247)$$

The steps of the proof are the same as for a vector and result in the definition for the curl as

$$\boxed{\text{curl}(\mathbf{A}) = -[\mathbf{grad}(\mathbf{A})\mathbf{e}_i] \times \bar{\mathbf{e}}_i = -\frac{\partial \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i.} \quad (2.248)$$

It can be shown that this results in

$$\boxed{\text{curl}(\mathbf{A}) = [\nabla \times (\mathbf{A}^T)]^T,} \quad (2.249)$$

where

$$\nabla \times (\mathbf{A}) = \bar{\mathbf{e}}_i \times \frac{\partial \mathbf{A}}{\partial \alpha_i}. \quad (2.250)$$

The *circulation* of a vector field around a closed curve $\mathcal{C}$ is evaluated by

$$\boxed{\Gamma = \int_{\mathcal{C}} \mathbf{u} \circ \hat{\mathbf{t}} dl,} \quad (2.251)$$

where the direction around the curve must be defined. Using the definition of the curl, it can be shown that the circulation can be evaluated from the integration of the normal

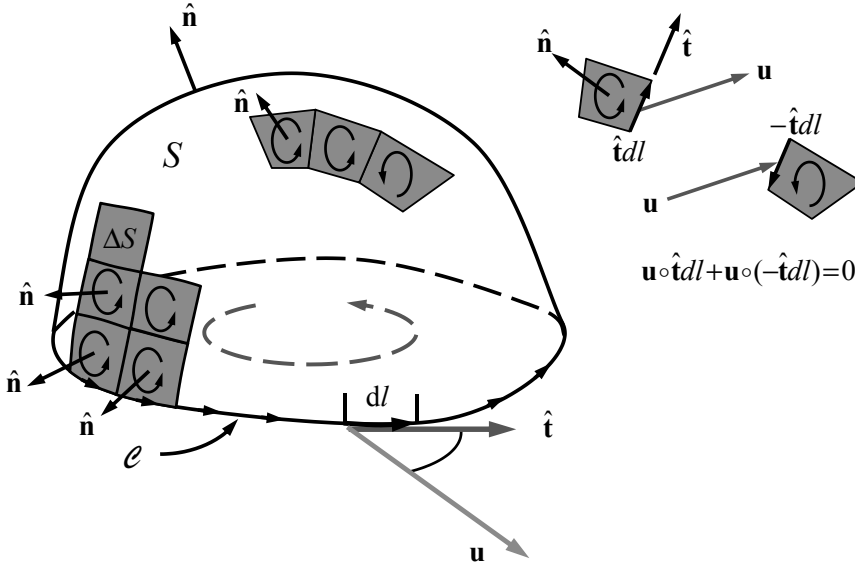


Figure 2.15: Stokes' theorem is constructed by looking at differential area elements and their interaction with neighboring elements.

component of the curl over any surface S with boundary $\mathcal{C}$, such as is shown in Figure 2.15. This relation is known as *Stokes' theorem*, and is given as

$$\int_{\mathcal{C}} \mathbf{u} \circ \hat{\mathbf{t}} dl = \int_S \text{curl}(\mathbf{u}) \circ \hat{\mathbf{n}} dS, \quad (2.252)$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface at each point (using the right-hand rule to obtain its direction from the direction of integration around the curve). The proof of Stokes' theorem follows directly from the definition of the curl and, as shown in the figure, the observation that the contribution from lines between adjacent differential surfaces elements inside S cancel, leaving only a contribution from the line integral over the curve $\mathcal{C}$. For a general tensor function $\mathbf{A}$ of position $\mathbf{x}$, we can show a similar relation exist. This is given as

$$\int_{\mathcal{C}} \mathbf{A} \hat{\mathbf{t}} dl = \int_S \text{curl}(\mathbf{A}) \hat{\mathbf{n}} dS. \quad (2.253)$$

Section 2.4-Problems:

2.4-4: If the curl of a second-order tensor $\mathbf{A}$ is defined as

$$\text{curl}(\mathbf{A}) = -[\nabla(\mathbf{A})\mathbf{e}_i] \times \bar{\mathbf{e}}_i = -\frac{\partial \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i, \quad (2.254)$$

show that

$$\text{curl}(\mathbf{A}) \circ \hat{\mathbf{n}} = \lim_{\Delta S \rightarrow 0} \frac{1}{\Delta S} \int_{\mathcal{C}} \mathbf{A} \hat{\mathbf{t}} dl. \quad (2.255)$$

From this conclude that

$$\int_C \mathbf{A} \hat{\mathbf{t}} dl = \int_S \text{curl}(\mathbf{A}) \circ \hat{\mathbf{n}} dS. \quad (2.256)$$

2.4-5: Derive the expression for the curl in a general curvilinear coordinate system from that written in the rectangular coordinate system (x_1, x_2, x_3) with base vectors $\hat{\mathbf{i}}_k$. As a result, argue that the value of the curl of a vector does not depend on the coordinate system it is evaluated in. To do this, start with the expression for the curl of the vector field $\mathbf{u}$ given in the rectangular coordinate system by

$$\text{curl}(\mathbf{u}) = \hat{\mathbf{i}}_k \times \frac{\partial \mathbf{u}}{\partial x_k}. \quad (2.257)$$

Define a curvilinear coordinate system $(\alpha_1, \alpha_2, \alpha_3)$ and assume you know the relation between this coordinate system and the rectangular system so that $\alpha_i(x_1, x_2, x_3)$. Assume the position vector $\mathbf{x}$ is a function of α_i and by using the chain rule show that

$$\hat{\mathbf{i}}_k = \mathbf{e}_m \frac{\partial \alpha_m}{\partial x_k}, \quad \mathbf{e}_m = \hat{\mathbf{i}}_k \frac{\partial x_k}{\partial \alpha_m}. \quad (2.258)$$

Calculate $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ and show that

$$g_{ij} = \frac{\partial x_k}{\partial \alpha_i} \frac{\partial x_k}{\partial \alpha_j}, \quad g_{ij}^{-1} = \frac{\partial \alpha_i}{\partial x_k} \frac{\partial \alpha_j}{\partial x_k}. \quad (2.259)$$

Use the chain rule to calculate $\frac{\partial \mathbf{u}}{\partial x_k}$ in terms of the curvilinear coordinate system and then replace for $\hat{\mathbf{i}}_k$ and $\frac{\partial \mathbf{u}}{\partial x_k}$ in the expression for the curl in rectangular coordinate systems to get

$$\text{curl}(\mathbf{u}) = \hat{\mathbf{i}}_k \times \frac{\partial \mathbf{u}}{\partial x_k} = \bar{\mathbf{e}}_k \times \frac{\partial \mathbf{u}}{\partial \alpha_k} = - \frac{\partial \mathbf{u}}{\partial \alpha_k} \times \bar{\mathbf{e}}_k, \quad (2.260)$$

where $\bar{\mathbf{e}}_k$ is the reciprocal or dual base to $\mathbf{e}_k$ and given by $\bar{\mathbf{e}}_k = g_{kl}^{-1} \bar{\mathbf{e}}_l$.

2.4-6: Considering the expressions for gradient, divergence and curl in the rectangular coordinate system (x_1, x_2, x_3) with base vectors $\hat{\mathbf{i}}_k$, prove the identities

$$\text{div}[\text{curl}(\mathbf{u})] = 0, \quad (2.261)$$

$$\text{curl}[\mathbf{grad}(\phi)] = \mathbf{0}, \quad (2.262)$$

$$\text{curl}[\mathbf{grad}(\mathbf{u})] = \mathbf{0}, \quad (2.263)$$

and since the values of gradient, divergence, and curl are independent of the coordinate in which they are calculated, conclude that these identities hold in general. Three facts that might be useful in your proof are (a) the symbol for a dummy index can be changed (i.e., $A_{ij}B_{ji} = A_{ji}B_{ij}$), (b) the order of partial derivatives can be changed (i.e., $\frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j} = \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_i}$), and (c) if a quantity is equal to its negative, then it must be equal to zero. For a scalar field ϕ , a vector field $\mathbf{u}$ and a general tensor field $\mathbf{A}$, take the expression of gradient, divergence, and

curl in the rectangular system to be given by

$$\mathbf{grad}(\phi) = \frac{\partial \phi}{\partial x_k} \hat{\mathbf{i}}_k, \quad (2.264)$$

$$\mathbf{grad}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial x_k} \otimes \hat{\mathbf{i}}_k, \quad (2.265)$$

$$\mathit{div}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial x_k} \circ \hat{\mathbf{i}}_k, \quad (2.266)$$

$$\mathit{curl}(\mathbf{u}) = - \frac{\partial \mathbf{u}}{\partial x_k} \times \hat{\mathbf{i}}_k, \quad (2.267)$$

$$\mathit{curl}(\mathbf{A}) = - \frac{\partial \mathbf{A}}{\partial x_k} \times \hat{\mathbf{i}}_k. \quad (2.268)$$

From this conclude that

$$\mathit{curl}[\nabla(\phi)] = \mathbf{0}, \quad (2.269)$$

$$\mathit{curl}[\nabla(\mathbf{u})] = \mathbf{0}. \quad (2.270)$$

2.4.4 Identities

The following is a set of identities that may be useful. In these expressions, ϕ is a scalar field, $\mathbf{u}$ and $\mathbf{v}$ are vector fields, and $\mathbf{A}$ and $\mathbf{B}$ are second-order tensor fields. For the product of a scalar and a vector or tensor field, we have

$$\mathit{div}(\phi \mathbf{u}) = \mathbf{u} \circ \mathbf{grad}(\phi) + \phi \mathit{div}(\mathbf{u}), \quad (2.271)$$

$$\mathit{div}(\phi \mathbf{A}) = \mathbf{A} \mathbf{grad}(\phi) + \phi \mathit{div}(\mathbf{A}), \quad (2.272)$$

$$\mathbf{grad}(\phi \mathbf{u}) = \mathbf{u} \otimes \mathbf{grad}(\phi) + \phi \mathbf{grad}(\mathbf{u}), \quad (2.273)$$

$$\mathbf{grad}(\phi \mathbf{A}) = \mathbf{A} \otimes \mathbf{grad}(\phi) + \phi \mathbf{grad}(\mathbf{A}), \quad (2.274)$$

$$\mathit{curl}(\phi \mathbf{u}) = \mathbf{grad}(\phi) \times \mathbf{u} + \phi \mathit{curl}(\mathbf{u}), \quad (2.275)$$

$$\mathit{curl}(\phi \mathbf{A}) = - \mathbf{A} \times \mathbf{grad}(\phi) + \phi \mathit{curl}(\mathbf{A}). \quad (2.276)$$

For expressions with two vectors, we have the identities

$$\mathit{div}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{grad}(\mathbf{u})\mathbf{v} + \mathbf{u} \mathit{div}(\mathbf{v}), \quad (2.277)$$

$$\mathit{div}(\mathbf{u} \times \mathbf{v}) = \mathit{curl}(\mathbf{u}) \circ \mathbf{v} - \mathbf{u} \circ \mathit{curl}(\mathbf{v}), \quad (2.278)$$

$$\mathbf{grad}(\mathbf{u} \circ \mathbf{v}) = \mathbf{v} \mathbf{grad}(\mathbf{u}) + \mathbf{u} \mathbf{grad}(\mathbf{v}), \quad (2.279)$$

$$\mathit{curl}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{grad}(\mathbf{u}) \times \mathbf{v} + \mathbf{u} \otimes \mathit{curl}(\mathbf{v}), \quad (2.280)$$

$$\mathit{curl}(\mathbf{u} \times \mathbf{v}) = \mathbf{grad}(\mathbf{u})\mathbf{v} - \mathbf{grad}(\mathbf{v})\mathbf{u} - \mathit{div}(\mathbf{u})\mathbf{v} + \mathit{div}(\mathbf{v})\mathbf{u}, \quad (2.281)$$

$$= \mathit{div}(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}). \quad (2.282)$$

For the product of a vector and a second-order tensor, we have the identities

$$\mathit{div}(\mathbf{A}\mathbf{u}) = \mathit{div}(\mathbf{A}^T) \circ \mathbf{u} + \mathbf{grad}(\mathbf{u}) : \mathbf{A}^T, \quad (2.283)$$

$$\mathit{div}(\mathbf{u}\mathbf{A}) = \mathit{div}(\mathbf{A}) \circ \mathbf{u} + \mathbf{grad}(\mathbf{u}) : \mathbf{A}, \quad (2.284)$$

$$\mathit{div}(\mathbf{A} \otimes \mathbf{u}) = \mathbf{grad}(\mathbf{A})\mathbf{u} + \mathbf{A} \mathit{div}(\mathbf{u}), \quad (2.285)$$

$$\mathit{div}(\mathbf{u} \otimes \mathbf{A}) = \mathbf{grad}(\mathbf{u})\mathbf{A}^T + \mathbf{u} \otimes \mathit{div}(\mathbf{A}). \quad (2.286)$$

For the product of two second-order tensors, we have the identities²

$$\text{div}(\mathbf{AB}) = \mathbf{grad}(\mathbf{A}) : \mathbf{B} + \mathbf{A} \text{div}(\mathbf{B}), \quad (2.287)$$

$$\mathbf{grad}(\mathbf{AB}) = \frac{\partial \mathbf{A}}{\partial \alpha_i} \mathbf{B} \otimes \bar{\mathbf{e}}_i + \mathbf{A} \mathbf{grad}(\mathbf{B}), \quad (2.288)$$

$$\text{curl}(\mathbf{AB}) = - \frac{\partial \mathbf{A}}{\partial \alpha_i} \mathbf{B} \times \bar{\mathbf{e}}_i + \mathbf{A} \text{curl}(\mathbf{B}). \quad (2.289)$$

For relations, including the fourth-order tensor $\mathbf{E}$, we have

$$\text{div}(\phi \mathbf{E}) = \mathbf{E} \mathbf{grad}(\phi) + \phi \text{div}(\mathbf{E}), \quad (2.290)$$

$$\text{div}(\mathbf{E} : \mathbf{A}) = \frac{\partial \mathbf{E}}{\partial \alpha_i} : (\bar{\mathbf{e}}_i \otimes \mathbf{A}) + \mathbf{E} : [\mathbf{grad}(\mathbf{A}^T)]^T, \quad (2.291)$$

where “:” is similar to “.” but between the last three bases of the first tensor and the first three bases of the second tensor. For the product of the operators, we have

$$\text{curl}[\text{curl}(\mathbf{u})] = \text{div}([\mathbf{grad}(\mathbf{u})]^T) - \nabla^2(\mathbf{u}), \quad (2.292)$$

$$= \text{div}([\mathbf{grad}(\mathbf{u})]^T - \mathbf{grad}(\mathbf{u})), \quad (2.293)$$

$$\text{div}[\text{curl}(\mathbf{u})] = 0, \quad (2.294)$$

$$\text{curl}[\mathbf{grad}(\phi)] = \mathbf{0}, \quad (2.295)$$

$$\text{curl}[\mathbf{grad}(\mathbf{u})] = \mathbf{0}. \quad (2.296)$$

The proof of these identities are fairly direct and follow with similar steps to those shown below for two examples cases. In each case, one starts from the general definitions for the gradient, divergence, and curl written as

$$\mathbf{grad}(\mathbf{A}) = \frac{\partial \mathbf{A}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i, \quad (2.297)$$

$$\text{div}(\mathbf{A}) = \frac{\partial \mathbf{A}}{\partial \alpha_i} \bar{\mathbf{e}}_i, \quad (2.298)$$

$$\text{curl}(\mathbf{A}) = - \frac{\partial \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i. \quad (2.299)$$

Then one calculates the identity by substituting the appropriate expression. For example, we can construct the proof

$$\begin{aligned} \text{curl}(\phi \mathbf{A}) &= - \frac{\partial \phi \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i \\ &= - \left[\frac{\partial \phi}{\partial \alpha_i} \mathbf{A} + \phi \frac{\partial \mathbf{A}}{\partial \alpha_i} \right] \times \bar{\mathbf{e}}_i \\ &= - \mathbf{A} \times \frac{\partial \phi}{\partial \alpha_i} \bar{\mathbf{e}}_i - \phi \frac{\partial \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i \\ &= - \mathbf{A} \times \mathbf{grad}(\phi) + \phi \text{curl}(\mathbf{A}). \end{aligned} \quad (2.300)$$

²One can define special operators for the terms expressed here in terms of derivatives in the curvilinear coordinates α_i , base $\mathbf{e}_i$ and reciprocal base $\bar{\mathbf{e}}_i$.

Or, we can construct the proof

$$\begin{aligned}
 \mathbf{grad}(\mathbf{u} \circ \mathbf{v}) &= \frac{\partial \mathbf{u} \circ \mathbf{v}}{\partial \alpha_i} \bar{\mathbf{e}}_i \\
 &= \left[\frac{\partial \mathbf{u}}{\partial \alpha_i} \circ \mathbf{v} + \mathbf{u} \circ \frac{\partial \mathbf{v}}{\partial \alpha_i} \right] \bar{\mathbf{e}}_i \\
 &= (\mathbf{v} \circ \frac{\partial \mathbf{u}}{\partial \alpha_i}) \bar{\mathbf{e}}_i + (\mathbf{u} \circ \frac{\partial \mathbf{v}}{\partial \alpha_i}) \bar{\mathbf{e}}_i \\
 &= \mathbf{v} \left(\frac{\partial \mathbf{u}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i \right) + \mathbf{u} \left(\frac{\partial \mathbf{v}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i \right) \\
 &= \mathbf{v} \mathbf{grad}(\mathbf{u}) + \mathbf{u} \mathbf{grad}(\mathbf{v}).
 \end{aligned} \tag{2.301}$$

The proof of the reminder of the identities is assigned as problems.

Section 2.4-Problems:

- 2.4-7:** Prove the identities for a scalar field ϕ and a vector field $\mathbf{u}$.
- 2.4-8:** Prove the identities for a scalar field ϕ and a second-order tensor field $\mathbf{A}$.
- 2.4-9:** Prove the identities for a vector field $\mathbf{u}$ and a second-order tensor field $\mathbf{A}$.
- 2.4-10:** Prove the identities for $\mathbf{u} \times \mathbf{v}$.
- 2.4-11:** Prove the identities for $\mathbf{u} \otimes \mathbf{v}$.
- 2.4-12:** Prove the identities for two second-order tensor fields $\mathbf{A}$ and $\mathbf{B}$.
- 2.4-13:** Prove the identities

$$\nabla \circ (\phi \mathbf{u}) = \text{div}(\phi \mathbf{u}) = \mathbf{u} \circ \nabla(\phi) + \phi \nabla \circ (\mathbf{u}), \tag{2.302}$$

$$\nabla \circ (\phi \mathbf{A}) = \text{div}(\phi \mathbf{A}) = \nabla(\phi) \mathbf{A} + \phi \nabla \circ (\mathbf{A}), \tag{2.303}$$

$$\nabla(\phi \mathbf{u}) = \nabla(\phi) \otimes \mathbf{u} + \phi \nabla(\mathbf{u}), \tag{2.304}$$

$$\nabla(\phi \mathbf{A}) = \nabla(\phi) \otimes \mathbf{A} + \phi \nabla(\mathbf{A}), \tag{2.305}$$

$$\nabla \times (\phi \mathbf{u}) = \text{curl}(\phi \mathbf{u}) = \nabla(\phi) \times \mathbf{u} + \phi \nabla \times (\mathbf{u}), \tag{2.306}$$

$$\nabla \times (\phi \mathbf{A}) = [\text{curl}(\phi \mathbf{A}^T)]^T = \nabla(\phi) \times \mathbf{A} + \phi \nabla \times (\mathbf{A}). \tag{2.307}$$

- 2.4-14:** Prove the identities

$$\nabla \circ (\mathbf{u} \otimes \mathbf{v}) = \text{div}(\mathbf{v} \otimes \mathbf{u}) = \nabla \circ (\mathbf{u}) \mathbf{v} + \mathbf{u} \nabla(\mathbf{v}), \tag{2.308}$$

$$\nabla \circ (\mathbf{u} \times \mathbf{v}) = \text{div}(\mathbf{u} \times \mathbf{v}) = \nabla \times (\mathbf{u}) \circ \mathbf{v} - \mathbf{u} \circ \nabla \times (\mathbf{v}), \tag{2.309}$$

$$\nabla(\mathbf{u} \circ \mathbf{v}) = \mathbf{grad}(\mathbf{u} \circ \mathbf{v}) = \nabla(\mathbf{u}) \mathbf{v} + \nabla(\mathbf{v}) \mathbf{u}, \tag{2.310}$$

$$\nabla \times (\mathbf{u} \otimes \mathbf{v}) = [\text{curl}(\mathbf{v} \otimes \mathbf{u})]^T = \nabla \times (\mathbf{u}) \otimes \mathbf{v} - \mathbf{u} \otimes \nabla(\mathbf{v}), \tag{2.311}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \text{curl}(\mathbf{u} \times \mathbf{v}) = \nabla \circ (\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}), \tag{2.312}$$

$$= \mathbf{v} \nabla(\mathbf{u}) - \mathbf{u} \nabla(\mathbf{v}) - \nabla \circ (\mathbf{u}) \mathbf{v} + \nabla \circ (\mathbf{v}) \mathbf{u}. \tag{2.313}$$

- 2.4-15:** Prove the identities

$$\nabla \circ (\mathbf{A} \mathbf{u}) = \text{div}(\mathbf{A} \mathbf{u}) = \nabla \circ (\mathbf{A}) \circ \mathbf{u} + \nabla(\mathbf{u}) : \mathbf{A}, \tag{2.314}$$

$$\nabla \circ (\mathbf{u} \mathbf{A}) = \text{div}(\mathbf{u} \mathbf{A}) = \nabla \circ (\mathbf{A}^T) \circ \mathbf{u} + \nabla(\mathbf{u}) : \mathbf{A}^T, \tag{2.315}$$

$$\nabla \circ (\mathbf{A} \otimes \mathbf{u}) = \nabla \circ (\mathbf{A}) \otimes \mathbf{u} + \mathbf{A}^T \nabla(\mathbf{u}), \tag{2.316}$$

$$\nabla \circ (\mathbf{u} \otimes \mathbf{A}) = \nabla \circ (\mathbf{u}) \mathbf{A} + \mathbf{u} \nabla(\mathbf{A}). \tag{2.317}$$

2.4-16: Prove the identities

$$\nabla \circ (\mathbf{AB}) = \text{div}(\mathbf{B}^T \mathbf{A}^T) = \nabla \circ (\mathbf{A})\mathbf{B} + \mathbf{A} : \nabla(\mathbf{B}), \quad (2.318)$$

$$\nabla(\mathbf{AB}) = [\mathbf{grad}(\mathbf{B}^T \mathbf{A}^T)]^T = \nabla(\mathbf{A})\mathbf{B} + \bar{\mathbf{e}}_i \otimes \mathbf{A} \frac{\partial \mathbf{B}}{\partial \alpha_i}, \quad (2.319)$$

$$\nabla \times (\mathbf{AB}) = \nabla \times (\mathbf{A})\mathbf{B} + \bar{\mathbf{e}}_i \times \mathbf{A} \frac{\partial \mathbf{B}}{\partial \alpha_i}. \quad (2.320)$$

2.4-17: Prove the identities

$$\nabla \circ (\phi \mathbf{E}) = \nabla(\phi) \mathbf{E} + \phi \nabla \circ (\mathbf{E}), \quad (2.321)$$

$$\nabla \circ (\mathbf{E} : \mathbf{A}) = \nabla \circ (\mathbf{E}) : \mathbf{A} + \bar{\mathbf{e}}_i \circ \mathbf{E} : \frac{\partial \mathbf{A}}{\partial \alpha_i}, \quad (2.322)$$

$$\nabla \times [\nabla \times (\mathbf{u})] = \text{curl}[\text{curl}(\mathbf{u})] = \nabla \circ \{[\nabla(\mathbf{u})]^T - \nabla(\mathbf{u})\}. \quad (2.323)$$

2.4.5 Time derivatives and partial derivatives

Frequently, we need to calculate time derivatives of functions of tensor variables and would like to do this using the chain rule of differentiation. For example, consider the scalar function $f(\mathbf{A}, \mathbf{B})$ of second-order tensor arguments $\mathbf{A}$ and $\mathbf{B}$, which are each a function of time. We would like to write the time derivative of f as

$$\boxed{\frac{df(\mathbf{A}, \mathbf{B})}{dt} = \partial_{\mathbf{A}}(f) \circ \frac{d\mathbf{A}}{dt} + \partial_{\mathbf{B}}(f) \circ \frac{d\mathbf{B}}{dt}}, \quad (2.324)$$

where $\partial_{\mathbf{A}}(f)$ and $\partial_{\mathbf{B}}(f)$ are, respectively, the partial derivatives of f with respect to $\mathbf{A}$ and $\mathbf{B}$, and “ $\circ$ ” is the scalar product “:” for second-order tensors $\mathbf{A}$ and $\mathbf{B}$. As an example, consider the function $f = \mathbf{A} : \mathbf{B}^2$. We can calculate its time derivative as

$$\begin{aligned} \dot{f} &= \dot{\mathbf{A}} : \mathbf{B}^2 + \mathbf{A} : (\dot{\mathbf{B}}\mathbf{B} + \mathbf{B}\dot{\mathbf{B}}) = \mathbf{B}^2 : \dot{\mathbf{A}} + \text{tr}(\mathbf{A}^T \dot{\mathbf{B}}\mathbf{B}) + \text{tr}(\mathbf{A}^T \mathbf{B}\dot{\mathbf{B}}) \\ &= \mathbf{B}^2 : \dot{\mathbf{A}} + \text{tr}(\mathbf{B}\mathbf{A}^T \dot{\mathbf{B}}) + \text{tr}(\mathbf{A}^T \mathbf{B}\dot{\mathbf{B}}) = \mathbf{B}^2 : \dot{\mathbf{A}} + (\mathbf{AB}^T) : \dot{\mathbf{B}} + (\mathbf{B}^T \mathbf{A}) : \dot{\mathbf{B}} \\ &= \mathbf{B}^2 : \dot{\mathbf{A}} + (\mathbf{AB}^T + \mathbf{B}^T \mathbf{A}) : \dot{\mathbf{B}}. \end{aligned} \quad (2.325)$$

It follows that

$$\partial_{\mathbf{A}}(f) = \mathbf{B}^2, \quad (2.326)$$

$$\partial_{\mathbf{B}}(f) = \mathbf{AB}^T + \mathbf{B}^T \mathbf{A}. \quad (2.327)$$

As is the case for partial derivatives of scalar arguments, the meaning of the partial derivative is determined by the variables of the function. That is, for example, consider that we can write $f(x, y) = x^2 y$ as $g(x, y, z) = xyz$ for $z = x$. Even though $f(x, y) = g(x, y, z)$ for $z = x$, the partial derivatives are not similar since

$$\frac{\partial f(x, y)}{\partial x} = 2xy, \quad \frac{\partial f(x, y)}{\partial y} = x^2, \quad (2.328)$$

$$\frac{\partial g(x, y, z)}{\partial x} = yz = xy, \quad \frac{\partial g(x, y, z)}{\partial y} = xz = x^2, \quad \frac{\partial g(x, y, z)}{\partial z} = xy. \quad (2.329)$$

The reader will note that even though the partial derivatives are different for the two functions, the final time derivative calculated from either will result in the same value if we impose the condition $z = x$. That is,

$$\dot{f} = \frac{\partial f(x, y)}{\partial x} \dot{x} + \frac{\partial f(x, y)}{\partial y} \dot{y} = 2xy\dot{x} + x^2\dot{y}, \quad (2.330)$$

$$\begin{aligned} \dot{g} &= \frac{\partial g(x, y, z)}{\partial x} \dot{x} + \frac{\partial g(x, y, z)}{\partial y} \dot{y} + \frac{\partial g(x, y, z)}{\partial z} \dot{z} = yz\dot{x} + xz\dot{y} + xy\dot{z} \\ &= xy\dot{x} + x^2\dot{y} + xy\dot{x} = 2xy\dot{x} + x^2\dot{y}. \end{aligned} \quad (2.331)$$

As can be seen, partial derivatives are based on the argument list, not based on possible relations between the arguments. When relations between the arguments of a function exist, one can either use them to eliminate variables and define a new function of fewer variables, as would be the case when taking $g(x, y, z) = xyz$ for $z = x$ and rewriting it as $f(x, y) = x^2y$, or one may act as if the arguments are independent and calculate the partial derivatives and then impose the relation after calculating the derivatives. The same considerations are true when working with tensors. For example, the function $f(\mathbf{A}, \mathbf{B}) = \mathbf{A} : \mathbf{B}^2$ may be considered as the function $g(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{A} : (\mathbf{B}\mathbf{C})$ with the condition $\mathbf{C} = \mathbf{B}$. Using this relation, we can see that

$$\begin{aligned} \dot{g} &= \dot{\mathbf{A}} : (\mathbf{B}\mathbf{C}) + \mathbf{A} : (\dot{\mathbf{B}}\mathbf{C}) + \mathbf{A} : (\mathbf{B}\dot{\mathbf{C}}) \\ &= (\mathbf{B}\mathbf{C}) : \dot{\mathbf{A}} + (\mathbf{A}\mathbf{C}^T) : \dot{\mathbf{B}} + (\mathbf{B}^T \mathbf{A}) : \dot{\mathbf{C}} \\ &= \mathbf{B}^2 : \dot{\mathbf{A}} + (\mathbf{A}\mathbf{B}^T + \mathbf{B}^T \mathbf{A}) : \dot{\mathbf{B}}, \end{aligned} \quad (2.332)$$

which, as can be seen, yields the same final result after imposing the condition $\mathbf{C} = \mathbf{B}$, but provides the following partial derivatives

$$\partial_{\mathbf{A}}(g) = \mathbf{B}\mathbf{C}, \quad (2.333)$$

$$\partial_{\mathbf{B}}(g) = \mathbf{A}\mathbf{C}^T, \quad (2.334)$$

$$\partial_{\mathbf{C}}(g) = \mathbf{B}^T \mathbf{A}. \quad (2.335)$$

In the same way one may define partial derivatives for tensor valued function of tensor arguments. For example, consider the vector valued function $\mathbf{f}(\mathbf{A}, \mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a second-order tensor, and $\mathbf{x}$ is a vector. The time derivative of $\mathbf{f}$ is given by

$$\dot{\mathbf{f}} = \dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} = \mathbf{x}\dot{\mathbf{A}}^T + \mathbf{A}\dot{\mathbf{x}} = \mathbf{x}\mathbf{T} : \dot{\mathbf{A}} + \mathbf{A}\dot{\mathbf{x}}, \quad (2.336)$$

so that we have

$$\dot{\mathbf{f}} = \partial_{\mathbf{A}}(\mathbf{f}) : \dot{\mathbf{A}} + \partial_{\mathbf{x}}(\mathbf{f}) \cdot \dot{\mathbf{x}}, \quad (2.337)$$

for

$$\partial_{\mathbf{A}}(\mathbf{f}) = \mathbf{x}\mathbf{T}, \quad (2.338)$$

$$\partial_{\mathbf{x}}(\mathbf{f}) = \mathbf{A}, \quad (2.339)$$

where $\mathbf{T}$ is a fourth-order tensor with the property that $\mathbf{T} : \mathbf{B} = \mathbf{B}^T$ for any second-order tensor $\mathbf{B}$. The component form of $\mathbf{T}$ is given in any base $\mathbf{e}_i$ as $\mathbf{T} = g_{il}^{-1} g_{jk}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$.

Section 2.4-Problems:

2.4-18: Consider the scalar functions

$$I_1 = \text{tr}(\mathbf{A}), \quad I_2 = \text{tr}(\mathbf{A}^2), \quad I_3 = \text{tr}(\mathbf{A}^3). \quad (2.340)$$

Calculate the time derivative of each scalar functions with respect to time assuming $\mathbf{A}$ is a second-order tensor function of time and put each time derivative in the form:

$$\dot{I}_i = \partial_{\mathbf{A}}(I_i) : \dot{\mathbf{A}}. \quad (2.341)$$

2.4-19: Consider the scalar functions

$$J_1 = \text{tr}(\mathbf{A}), \quad J_2 = \frac{1}{2}[\text{tr}^2(\mathbf{A}) - \text{tr}(\mathbf{A}^2)], \quad J_3 = \det(\mathbf{A}). \quad (2.342)$$

Calculate the time derivative of each of the scalar functions with respect to time assuming $\mathbf{A}$ is a second-order tensor function of time, and put each time derivative in the form:

$$\dot{J}_i = \partial_{\mathbf{A}}(J_i) : \dot{\mathbf{A}}. \quad (2.343)$$

Hint: Use the Cayley-Hamilton theorem for J_3 .

2.4-20: Consider the scalar functions

$$f_1 = \text{tr}(\mathbf{A}\mathbf{B}), \quad f_2 = \text{tr}(\mathbf{A}^2\mathbf{B}), \quad f_3 = \text{tr}(\mathbf{A}\mathbf{B}^2), \quad f_4 = \text{tr}(\mathbf{A}^2\mathbf{B}^2). \quad (2.344)$$

Calculate the time derivative of each scalar functions with respect to time assuming $\mathbf{A}$ and $\mathbf{B}$ are second-order tensor functions of time. Put each time derivative in the form:

$$\dot{f}_i = \partial_{\mathbf{A}}(f_i) : \dot{\mathbf{A}} + \partial_{\mathbf{B}}(f_i) : \dot{\mathbf{B}}. \quad (2.345)$$

2.5 Notation

The following table shows the notation used in the text and their definitions.

$\circ$		dot product or scalar product
$\times$		cross product
$:$		parallel double dot product
$..$		sequential double dot product
$\otimes$		tensor product
$!$		factorial
$ \mathbf{A} $		magnitude or determinant of $\mathbf{A}$
$\mathbf{grad}$		gradient [$d\mathbf{A} = \mathbf{grad}(\mathbf{A})d\mathbf{x}$]
∇		nabla ($d\mathbf{A} = d\mathbf{x}\nabla(\mathbf{A})$; $\nabla(\mathbf{A}) = [\mathbf{grad}(\mathbf{A}^T)]^T$)
div		divergence
$\nabla \circ (\mathbf{A})$		$[div(\mathbf{A}^T)]^T$
∇^2		Laplacian
$curl$		curl
$\nabla \times (\mathbf{A})$		$curl(\mathbf{A}) = [\nabla \times (\mathbf{A}^T)]^T$
$[\mathbf{a}, \mathbf{b}, \mathbf{c}]$		triple scalar product $[(\mathbf{a} \times \mathbf{b}) \circ \mathbf{c}]$
$\partial_{\mathbf{A}}$		partial derivative with respect to $\mathbf{A}$
$[A]$		matrix A or matrix of components of tensor $\mathbf{A}$
$[A]^T$		transpose of $[A]$
$[A]^{-1}$		inverse of $[A]$

$[A]^n$		$[A]$ to the power n
$\sqrt{[A]}$		square root of $[A]$
$e^{[A]}$		exponent with matrix exponent $[A]$
$[A]_{sym}$		symmetric part of $[A]$
$[A]_{skew}$		skew part of $[A]$
$tr([A])$		trace of matrix $[A]$
$det([A])$		determinant of $[A]$
$\mathbf{A}$		tensor A
$\mathbf{A}^T$		transpose of tensor $\mathbf{A}$
$\mathbf{A}^{-1}$		inverse of tensor $\mathbf{A}$
$tr(\mathbf{A})$		trace of tensor $\mathbf{A}$ [$tr([A][g])$]
$det(\mathbf{A})$		determinant of $\mathbf{A}$ [$det([A][g])$]
$\mathbf{e}_i$		base vectors of α_i
$\bar{\mathbf{e}}_i$		reciprocal or dual base of $\mathbf{e}_i$
$\mathbf{e}_i^*$		base vectors of β_i
e_{ijk}		$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k$
g_{ij}		metric tensor ($\mathbf{e}_i \circ \mathbf{e}_j$)
$[g]$		matrix of metric tensor
g_{ij}^*		metric tensor ($\mathbf{e}_i^* \circ \mathbf{e}_j^*$)
$\hat{\mathbf{i}}_i$		rectangular base
I		first isotropic invariants ($tr([A])$)
$[I]$		identity matrix
$\mathbf{I}$		identity tensor
I_i		isotropic invariants
II		second isotropic invariants [$\frac{1}{2}\{tr^2([A]) - tr([A]^2)\}$]
III		third isotropic invariants [$det([A])$]
$\mathbf{I}$		$\mathbf{I} : \mathbf{A} = \mathbf{A} : \mathbf{I} = \mathbf{A}$
$\mathbf{II}$		$\mathbf{II} : \mathbf{A} = \mathbf{A} : \mathbf{II} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$
$\mathbf{I} \otimes \mathbf{I}$		$(\mathbf{I} \otimes \mathbf{I}) : \mathbf{A} = \mathbf{A} : (\mathbf{I} \otimes \mathbf{I}) = tr(\mathbf{A})\mathbf{I}$
l_{ij}		transformation from $\mathbf{e}_j$ to $\mathbf{e}_i^*$
$[x]$		eigenvector
$\hat{\mathbf{n}}$		unit normal
$\hat{\mathbf{t}}$		unit tangent
$\mathbf{T}$		$\mathbf{T} : \mathbf{A} = \mathbf{A} : \mathbf{T} = \mathbf{A}^T$
α_i		coordinates
β_i		coordinates
δ_{ij}		Kronecker delta
ϵ_{ijk}		permutation symbol
Γ		circulation
λ		eigenvalue

Chapter 3

Describing motion, deformation, and temperature

In this chapter, we will look at how to describe motion, deformation, and temperature at material points in a body. In the process, we define measures of strain for small and large deformations and variables such as the deformation, velocity, and temperature gradients.

To define material points of a body, we introduce the notion of a configuration, a mapping of each material particle into a location in space. The configuration provides a method of identifying the material points of a body and their relative placement and temperature. This information is used to evaluate the distortion of the body and the temperature distribution by evaluating associated gradient. The balance of the chapter then is devoted to the study of common tools used to describe distortion and temperature distribution, with examples provided for some of the common deformations.

3.1 Position, velocity, acceleration, and temperature

In the construction of the theory, we assume that particles (distinct segments of matter) can be marked and followed. In particular, we would like to follow the position and temperature of all the distinct particles of a body. To do this, as shown in Figure 3.1, we can mark points on the body and follow them through time. Let P be a specific particle in the body. The position and temperature of P with respect to time t are given, respectively, by functions $\mathbf{x}_P(t)$ and $\theta_P(t)$. The vector function $\mathbf{x}_P(t)$, known as the position vector of P , is a vector that goes from a designated point on a reference frame, in this case O , to the point under consideration, in this case P . In a similar manner, $\theta_P(t)$ is a scalar function that provides the temperature, with respect to some predefined scale, of particle P as a function of time.

The velocity is the rate of change of position with respect to time for a given particle. For particle P we denote its velocity by $\mathbf{v}_P(t)$ and define it as

$$\mathbf{v}_P(t) \equiv \frac{d\mathbf{x}_P(t)}{dt}. \quad (3.1)$$

The acceleration is the rate of change of the velocity of a particle with respect to time. For particle P we denote its acceleration by $\mathbf{a}_P(t)$ and define it as

$$\mathbf{a}_P(t) \equiv \frac{d\mathbf{v}_P(t)}{dt} = \frac{d^2\mathbf{x}_P(t)}{dt^2}. \quad (3.2)$$

3.2 Configurations of material bodies

The key problem in naming particles of a body is that there are an infinite number of

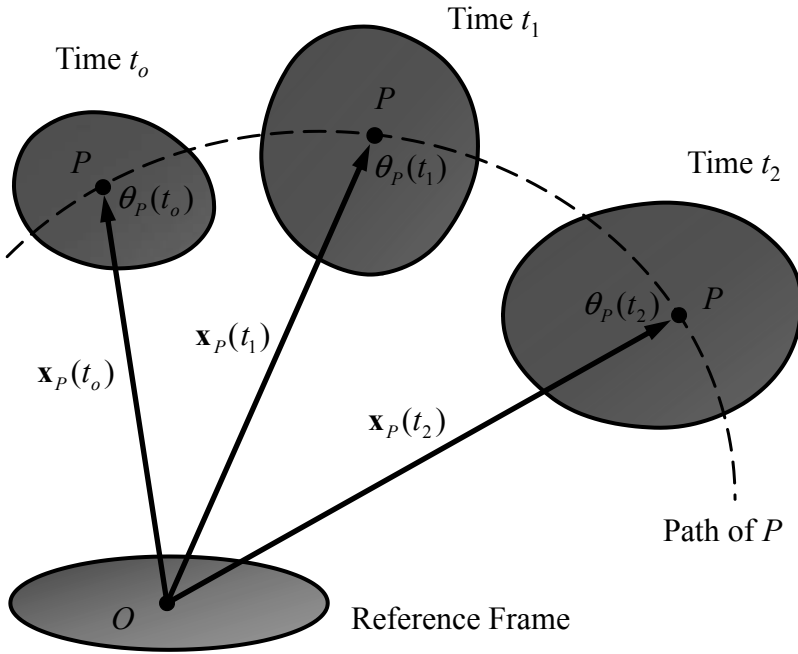


Figure 3.1: The motion of a point on a body.

them. Defining the points of a body by their locations in space at a particular time provides a method to solve this problem. The mapping that connects each particle in the body to its position in space is called the *configuration of the body at that time*. As the reader might imagine, the current configuration of the body is a mapping that connects each particle in the body to its current location. We will use κ to denote the current configuration.

It is common to take one of the configurations of the body as a *reference configuration*. This configuration serves as a means of distinguishing between material points in the body, and it also identifies the relative placement of them. In this way, the reference configuration represents the body and its points. We will use κ_o to denote the reference configuration. We will let $\mathbf{X}$ denote the position vector of points in the reference configuration. For the purpose of tracking particles, $\mathbf{X}$ replaces the name of the particle. The particle that occupies the location given by $\mathbf{X}$ in the reference configuration will be at location $\mathbf{x}(t)$ at time t . The body contains many particles and their motions can be given by a function $\mathbf{x}(\mathbf{X}, t)$ which gives the position at time t of the particle at location $\mathbf{X}$ in the reference configuration. Therefore, as shown in Figure 3.2,

$$\mathbf{x}_P(t) = \mathbf{x}(\mathbf{X}_P, t), \quad (3.3)$$

provides the current location of particle P at time t , where P occupies the position $\mathbf{X}_P$ in the reference configuration.

The function $\mathbf{x}(\mathbf{X}, t)$ is known as the motion function. As can be seen in Figure 3.2, one can calculate the current position $\mathbf{x}$ by adding the reference position $\mathbf{X}$ and the displacement $\mathbf{u}$. In this way, we can define the displacement function $\mathbf{u}(\mathbf{X}, t)$ that provides the displacement of all the points from their location in the reference configuration. The relation

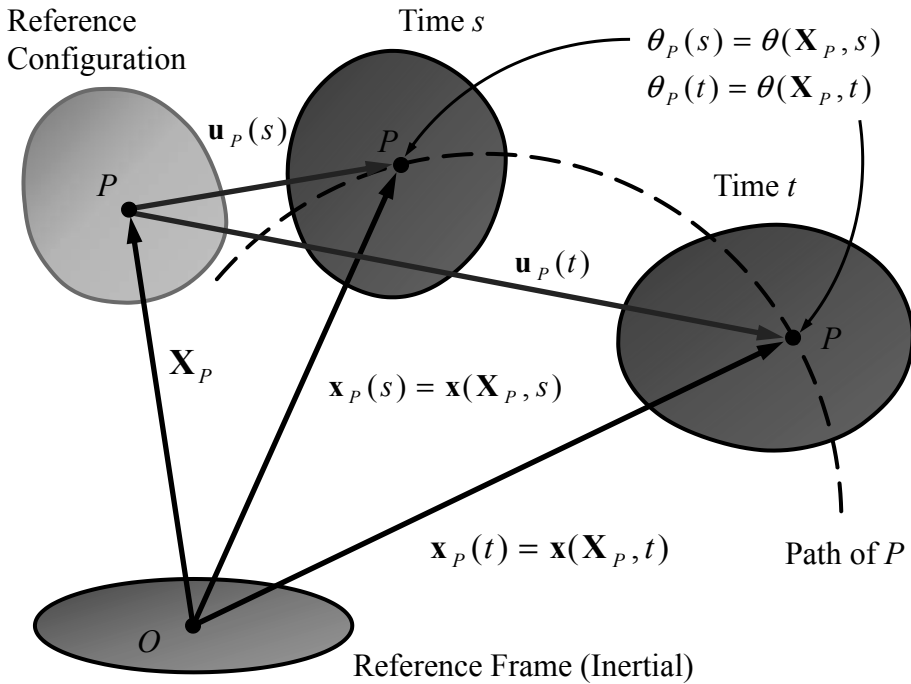


Figure 3.2: The motion function and the reference configuration.

between the motion function and the displacement function is

$$\mathbf{x}(\mathbf{X}, t) \equiv \mathbf{X} + \mathbf{u}(\mathbf{X}, t). \quad (3.4)$$

The velocity $\mathbf{v}$ of particle $\mathbf{X}$ can be calculated from the motion of the body by taking a time derivative holding $\mathbf{X}$ constant. Therefore,

$$\mathbf{v}(t) \equiv \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} = \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t}. \quad (3.5)$$

The acceleration $\mathbf{a}$ of particle $\mathbf{X}$ can be calculated from the motion of the body by relation

$$\mathbf{a}(t) \equiv \frac{\partial^2 \mathbf{x}(\mathbf{X}, t)}{\partial t^2} = \frac{\partial^2 \mathbf{u}(\mathbf{X}, t)}{\partial t^2}. \quad (3.6)$$

In calculating velocity and acceleration we keep $\mathbf{X}$ fixed and follow the particle over time. This type of derivative is known as a *material or Lagrangian time derivative*. If the velocity is written as $\mathbf{v}(\mathbf{X}, t)$, so that it is given in terms of a function of the reference position $\mathbf{X}$ and current time t , then we call this a *material or Lagrangian description* of the velocity. If we use this description, then the acceleration can be written as

$$\mathbf{a}(t) \equiv \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t}. \quad (3.7)$$

In a similar manner, we can define a temperature function $\theta(\mathbf{X}, t)$, which is a material description giving the current temperature of each point of the reference configuration. Using this function we can follow the temperature of each point in the body and define the temperature rate $\dot{\theta}$ as

$$\dot{\theta} \equiv \frac{\partial \theta(\mathbf{X}, t)}{\partial t}. \quad (3.8)$$

In most of the book we will use curvilinear coordinate systems to describe location in space. For each configuration, we can define a different coordinate system. We will let $(\alpha_1, \alpha_2, \alpha_3)$ be the curvilinear coordinate system used in the reference configuration and will let $(\beta_1, \beta_2, \beta_3)$ be the curvilinear coordinate system used for the current configuration. The current coordinates $(\beta_1, \beta_2, \beta_3)$ of a particle occupying the location $(\alpha_1, \alpha_2, \alpha_3)$ in the reference configuration can be described by a function $\beta_i^\dagger(\alpha_j, t)$. This relation is written as

$$\beta_i = \beta_i^\dagger(\alpha_j, t). \quad (3.9)$$

This mapping that follows the individual coordinates of specific particles is denoted as a material mapping. That is, it uses the material particles to connect the coordinates α_i to the coordinates β_i . As long as there is no tear or fold in the body, this mapping should be invertible. The reader might imagine that a tear will result in one location in the reference configuration (one particle) separating into two points on the two sides of the tear, making inversion impossible. In a similar manner, a fold will make two points into one. To simplify the notation, we will remove the “†” from the mapping and simply write it as $\beta_i(\alpha_j, t)$.

The current position of a particle can be obtained from its coordinates so that we can write

$$\mathbf{x}(t) = \mathbf{x}(\beta_i). \quad (3.10)$$

If we use the material mapping $\beta_i(\alpha_j, t)$, we can rewrite this as

$$\mathbf{x}(t) = \mathbf{x}[\beta_i(\alpha_j, t)]. \quad (3.11)$$

The velocity, which is a time derivative holding α_i fixed, therefore can be calculated as

$$\mathbf{v}(t) = \frac{\partial \mathbf{x}}{\partial \beta_i} \frac{\partial \beta_i}{\partial t} = v_i \mathbf{e}_i^*, \quad (3.12)$$

where

$$v_i = \frac{\partial \beta_i}{\partial t}, \quad \mathbf{e}_i^* = \frac{\partial \mathbf{x}}{\partial \beta_i}. \quad (3.13)$$

The base vectors $\mathbf{e}_i^*$ are the curvilinear base vectors for the β_i coordinate system.

The velocity can be written as a function of the current location and time by a function of the form

$$\mathbf{v}(t) = \mathbf{v}(\beta_i, t). \quad (3.14)$$

A description of this form, which gives a variable in terms of the current coordinates and time, is known as a *spatial or Eulerian description*. If we introduce the material mapping $\beta_i(\alpha_j, t)$ into this spatial description, we get the expression for velocity given by

$$\mathbf{v}(t) = \mathbf{v}[\beta_i(\alpha_j, t), t], \quad (3.15)$$

which provides us with the needed structure to take a material time derivative. The acceleration, which is the material time derivative of the velocity, therefore will be given by

$$\mathbf{a}(t) = \frac{\partial \mathbf{v}}{\partial \beta_i} \frac{\partial \beta_i}{\partial t} + \frac{\partial \mathbf{v}}{\partial t}. \quad (3.16)$$

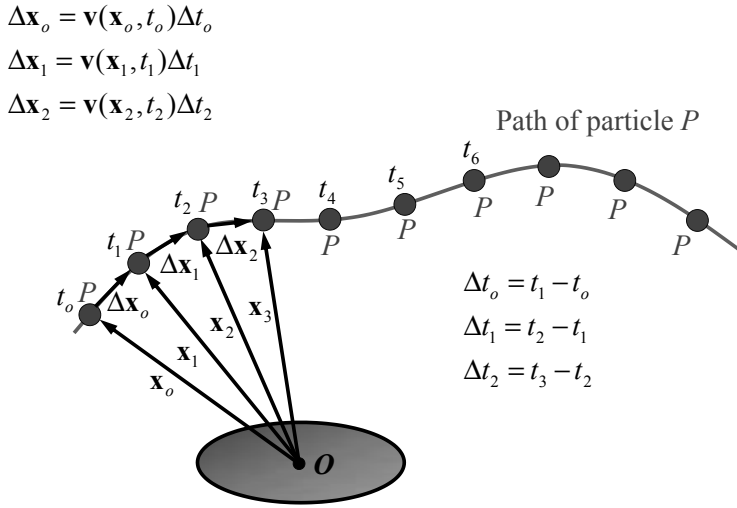


Figure 3.3: The pathline of particle P is constructed by connecting the different locations of P as one moves forward in time.

One can use the results from curvilinear coordinates to show that

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial \beta_i} \frac{\partial \beta_i}{\partial t} &= \frac{\partial \mathbf{v}}{\partial \beta_i} g_{ij}^{*-1} g_{jk}^* \frac{\partial \beta_k}{\partial t} \\ &= \frac{\partial \mathbf{v}}{\partial \beta_i} g_{ij}^{*-1} (\mathbf{e}_j^* \circ \mathbf{e}_k^*) \frac{\partial \beta_k}{\partial t} \\ &= \left(\frac{\partial \mathbf{v}}{\partial \beta_i} g_{ij}^{*-1} \otimes \mathbf{e}_j^* \right) \left(\frac{\partial \beta_k}{\partial t} \mathbf{e}_k^* \right) \\ &= [\nabla_{\mathbf{x}}(\mathbf{v})]^T \mathbf{v} = \mathbf{v} \nabla_{\mathbf{x}}(\mathbf{v}), \end{aligned}$$

where $g_{ij}^* = \mathbf{e}_i^* \circ \mathbf{e}_j^*$ is the metric tensor for the β_i coordinate system. Introduction of this into the above expression for acceleration gives

$$\boxed{\mathbf{a}(t) = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla_{\mathbf{x}}(\mathbf{v})}, \quad (3.17)$$

where the subscript $\mathbf{x}$ on the del operator has been added to distinguish between the two operators $d\mathbf{u} = d\mathbf{x} \circ \nabla_{\mathbf{x}}(\mathbf{u}) = d\mathbf{X} \circ \nabla_{\mathbf{X}}(\mathbf{u})$ associated with the two vector fields $\mathbf{x}$ and $\mathbf{X}$. We also can write this as

$$\boxed{\mathbf{a}(t) = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{grad}_{\mathbf{x}}(\mathbf{v}) \mathbf{v}}. \quad (3.18)$$

where $\mathbf{grad}_{\mathbf{x}}(*)$ is the gradient with respect to changes in the variable $\mathbf{x}$.

3.3 Streamlines and pathlines

We use streamlines and pathlines to visualize the flow of materials, particularly for fluids. A *pathline* is the line describing the path of a single particle over a given time interval. For

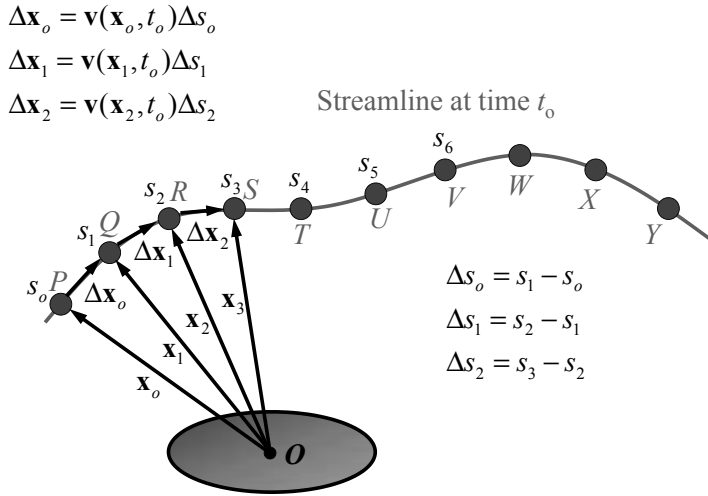


Figure 3.4: The streamline at time t_o is constructed by moving along the direction of the velocity from particle to particle.

example, we may introduce an ink-mark into a flow to see how it moves over time. Figure 3.3 shows how a pathline is constructed following a particle. If one is given a material description of the velocity, a *pathline* is obtained by integrating the equation

$$\boxed{d\mathbf{x} = \mathbf{v}(\mathbf{X}, t)dt}, \quad (3.19)$$

for a fixed value of $\mathbf{X}$ and subject to initial conditions for $\mathbf{x}$. Therefore, in drawing a pathline we are marching forward in time, following the locations of a single particle. A pathline also can be drawn given a spatial representation of the velocity. This can be done by starting from an initial location $\mathbf{x}_o$ at time t_o and integrating the relation

$$d\mathbf{x} = \mathbf{v}(\mathbf{x}, t)dt. \quad (3.20)$$

As can be seen, one needs to find a way of separating the variables before integration if an analytical method is to be used to integrate this system. In a numerical method, one can simply update $\mathbf{x}$ by adding $d\mathbf{x}$ to its previous value and evaluating the velocity $\mathbf{v}$ for this new value of $\mathbf{x}$ and for the new time, and then use the new $\mathbf{v}$ for the next integration step.

Streamlines, unlike pathlines, are calculated for a single fixed time. Streamlines follow the direction of the velocity at a single time, moving from particle to particle based on the particle velocity, as shown in Figure 3.4. As such, when calculating streamlines one takes time as constant. Let us say we intend to draw a streamline at time t_o . First, we select a point to start the streamline, say $\mathbf{x}_o$. Next, we move along the direction of the velocity of the particle at $\mathbf{x}_o$ to a close particle. Say we move the amount $\mathbf{v}(\mathbf{x}_o, t_o)ds$ to the new point $\mathbf{x}_1$. Then, from $\mathbf{x}_1$ we move along the direction of the velocity of the particle at location $\mathbf{x}_1$ to a close particle, say we move the amount $\mathbf{v}(\mathbf{x}_1, t_o)ds$ to $\mathbf{x}_2$, and so forth. The result of this process is a line called a streamline. If one has the spatial description of the velocity, one can formulate the evaluation of the *streamline* by integrating the equation

$$\boxed{d\mathbf{x} = \mathbf{v}(\mathbf{x}, t_o)ds}, \quad (3.21)$$

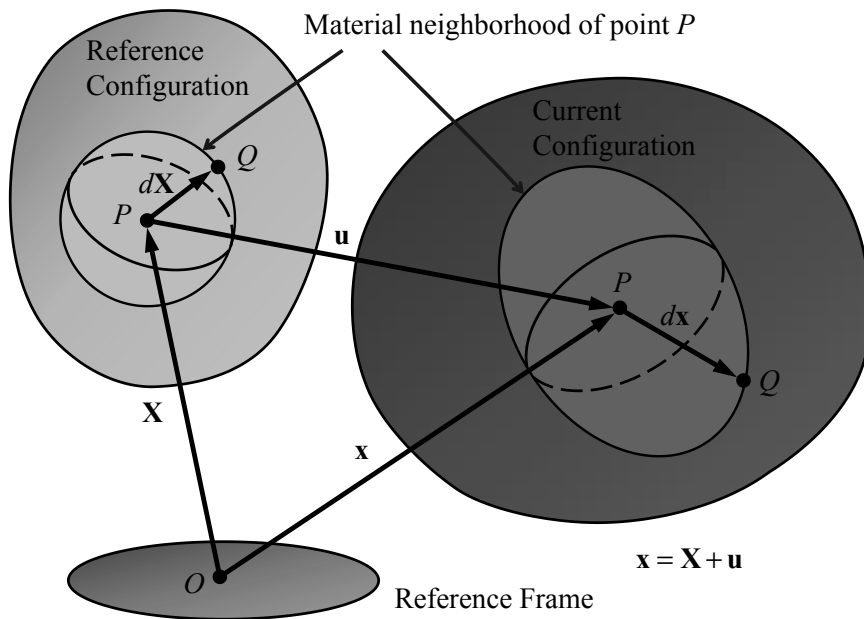


Figure 3.5: The distortion of the neighborhood of a point and the associated line elements $d\mathbf{X}$ and $d\mathbf{x}$.

subject to the initial condition $\mathbf{x} = \mathbf{x}_o$ at $s = s_o$. In this formulation, s is used to parameterize the streamline. As can be seen, s is not time.

One should note that if one has a steady-state motion then the streamlines and pathlines coincide. Since the streamlines are parameterized by s and the pathlines by time, one needs to establish the relation between s and t to show the equivalence of the two. This can be done through the relation

$$d\mathbf{x} = \mathbf{v}(\mathbf{x}, t)dt = \mathbf{v}(\mathbf{x}, t_o)ds.$$

3.4 Deformation gradient and temperature gradient

We now will look at describing the relative difference of position and temperature of points in the body. These will be described through gradients of position and temperature, providing a local measure of distortion and relative temperature distribution in a neighborhood of each point.

We will start by looking at the distortion of the neighborhood of a point as shown in Figure 3.5 for the given point P . To study distortion, we need to consider the relative placement of two points. For this purpose, as shown in the figure, we consider the two points P and Q that are very close together. The vector going from P to Q in the reference configuration will be denoted by $d\mathbf{X}$ and the vector going from P to Q in the current configuration will be denoted by $d\mathbf{x}$. The *deformation gradient* $\mathbf{F}$ is a second-order tensor which relates $d\mathbf{x}$ and $d\mathbf{X}$ through the relation

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}. \quad (3.22)$$

We also can define the displacement gradient $\mathbf{H}$ which gives the change in the displacement as we move slightly in the reference configuration, and is defined as

$$d\mathbf{u} = \mathbf{H}d\mathbf{X}. \quad (3.23)$$

Since we know that $\mathbf{x} = \mathbf{X} + \mathbf{u}$, it follows that $d\mathbf{x} = d\mathbf{X} + d\mathbf{u} = \mathbf{I}d\mathbf{X} + \mathbf{H}d\mathbf{X} = (\mathbf{I} + \mathbf{H})d\mathbf{X}$ so that we have the relation between the deformation gradient and the displacement gradient given by

$$\mathbf{F} = \mathbf{I} + \mathbf{H}. \quad (3.24)$$

We later will show that the deformation gradient contains all the information needed to calculate the strain of a small neighborhood around a point. In a similar manner, we can look at the change in temperature with respect to changes in position. Depending on which configuration is used, there are at least two commonly used temperature gradients. We will denote these by $\mathbf{G}$ and $\mathbf{g}$ and define them through the relations

$$d\theta = \mathbf{G}d\mathbf{X} \equiv \mathbf{g}d\mathbf{x}. \quad (3.25)$$

As can be seen, $\mathbf{G}$ is the gradient of temperature with respect to changes of position in the reference configuration and $\mathbf{g}$ is the gradient of temperature with respect to changes of position in the current configuration. We can use the relation $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ to substitute for $d\mathbf{x}$ in this equation. The result of this operation is a relation between the two temperature gradients given as

$$\mathbf{G} = \mathbf{g}\mathbf{F}. \quad (3.26)$$

We first will look at representing the deformation gradient in a general curvilinear coordinate system. Let $(\alpha_1, \alpha_2, \alpha_3)$ be the curvilinear coordinate system used to describe locations in the reference configuration, and let $(\beta_1, \beta_2, \beta_3)$ be the curvilinear coordinate system used to describe locations in the current configuration. Using the chain rule, one can write

$$d\mathbf{X} = \frac{\partial \mathbf{X}}{\partial \alpha_i} d\alpha_i, \quad (3.27)$$

and

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \beta_i} d\beta_i, \quad (3.28)$$

where $d\alpha_i$ are the components of $d\mathbf{X}$ in the curvilinear base given by

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{X}}{\partial \alpha_i}, \quad (3.29)$$

and $d\beta_i$ are the components of $d\mathbf{x}$ in the curvilinear base given by

$$\mathbf{e}_i^* \equiv \frac{\partial \mathbf{x}}{\partial \beta_i}. \quad (3.30)$$

That is,

$$d\mathbf{X} = d\alpha_i \mathbf{e}_i, \quad \text{and} \quad d\mathbf{x} = d\beta_i \mathbf{e}_i^*. \quad (3.31)$$

If we introduce the material mapping $\beta_i(\alpha_j, t)$ for a fixed time t into the expression for $d\mathbf{x}$, we will be able to use the chain rule to write

$$d\mathbf{x} = d\beta_i \mathbf{e}_i^* = \frac{\partial \beta_i}{\partial \alpha_j} d\alpha_j \mathbf{e}_i^*. \quad (3.32)$$

Using the metric tensor for the α_i coordinate system given by $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ allows us to reorganize this to get

$$\begin{aligned} d\mathbf{x} &= \frac{\partial \beta_i}{\partial \alpha_j} \mathbf{e}_i^* (g_{jk}^{-1} g_{kl}) d\alpha_l \\ &= \frac{\partial \beta_i}{\partial \alpha_j} \mathbf{e}_i^* g_{jk}^{-1} (\mathbf{e}_k \circ \mathbf{e}_l) d\alpha_l \\ &= \left(\frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k \right) (d\alpha_l \mathbf{e}_l). \end{aligned} \quad (3.33)$$

Noting that $d\mathbf{X} = d\alpha_i \mathbf{e}_i$, it is clear that $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ can be constructed by selecting

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k = \frac{\partial \beta_i}{\partial \alpha_j} \mathbf{e}_i^* \otimes \bar{\mathbf{e}}_j, \quad (3.34)$$

where $\bar{\mathbf{e}}_i$ is the dual base to $\mathbf{e}_i$. This is the equation for calculating the deformation gradient using general curvilinear coordinates α_i and β_i for, respectively, the reference and current configurations. In the special case where we have a Cartesian base $\hat{\mathbf{e}}_i$ of unit vectors, we will have $g_{ij} = \delta_{ij}$, and the expression for calculating the deformation gradient will simplify to

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} \hat{\mathbf{e}}_i^* \otimes \hat{\mathbf{e}}_j. \quad (3.35)$$

If we use the parallel rectangular coordinates X_i for the reference configuration and x_i for the current configuration, with the same orthonormal unit base vectors $\hat{\mathbf{i}}_k$ for both systems, we can write this as

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j. \quad (3.36)$$

For each of the two vector fields $\mathbf{X}$ and $\mathbf{x}$ we have a definition of a gradient and a corresponding nabla operator. These two gradients and nabla operators are defined for a vector field $\mathbf{u}$ by the expressions

$$d\mathbf{u} = \mathbf{grad}_{\mathbf{X}}(\mathbf{u}) d\mathbf{X} = d\mathbf{X} \nabla_{\mathbf{X}}(\mathbf{u}), \quad (3.37)$$

$$d\mathbf{u} = \mathbf{grad}_{\mathbf{x}}(\mathbf{u}) d\mathbf{x} = d\mathbf{x} \nabla_{\mathbf{x}}(\mathbf{u}), \quad (3.38)$$

where the subscripts $\mathbf{X}$ and $\mathbf{x}$ are used to clearly denote which is being used. We frequently use “**Grad**” to denote the gradient $\mathbf{grad}_{\mathbf{X}}$ with respect to changes in the reference position $\mathbf{X}$ and “**grad**” to denote the gradient $\mathbf{grad}_{\mathbf{x}}$ with respect to changes in the current position $\mathbf{x}$. In terms of the curvilinear coordinates α_i for describing $\mathbf{x}$ and β_i for describing $\mathbf{X}$, we can write these as

$$\mathbf{Grad}(\mathbf{u}) \equiv \mathbf{grad}_{\mathbf{X}}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial \alpha_i} g_{ip}^{-1} \otimes \mathbf{e}_p = \frac{\partial \mathbf{u}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i, \quad (3.39)$$

$$\mathbf{grad}(\mathbf{u}) \equiv \mathbf{grad}_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial \beta_i} g_{ip}^{*-1} \otimes \mathbf{e}_p = \frac{\partial \mathbf{u}}{\partial \beta_i} \otimes \bar{\mathbf{e}}_i^*. \quad (3.40)$$

We will use a similar notation to describe the divergence resulting from each gradient. We will write these using the notation “*Div*” and “*div*” to be defined as

$$\mathbf{Div}(\mathbf{u}) \equiv \mathbf{div}_{\mathbf{X}}(\mathbf{u}) = \text{tr}[\mathbf{Grad}(\mathbf{u})] = \frac{\partial \mathbf{u}}{\partial \alpha_i} g_{ip}^{-1} \circ \mathbf{e}_p = \frac{\partial \mathbf{u}}{\partial \alpha_i} \circ \bar{\mathbf{e}}_i, \quad (3.41)$$

$$\mathbf{div}(\mathbf{u}) \equiv \mathbf{div}_{\mathbf{x}}(\mathbf{u}) = \text{tr}[\mathbf{grad}(\mathbf{u})] = \frac{\partial \mathbf{u}}{\partial \beta_i} g_{ip}^{*-1} \circ \mathbf{e}_p = \frac{\partial \mathbf{u}}{\partial \beta_i} \circ \bar{\mathbf{e}}_i^*. \quad (3.42)$$

In a similar way, we have the expressions for the del operator given by

$$\nabla_{\mathbf{x}}(\mathbf{u}) = \mathbf{e}_i \otimes g_{ip}^{-1} \frac{\partial \mathbf{u}}{\partial \alpha_p} = \bar{\mathbf{e}}_i \otimes \frac{\partial \mathbf{u}}{\partial \alpha_i}, \quad (3.43)$$

$$\nabla_{\mathbf{x}}(\mathbf{u}) = \mathbf{e}_i^* \otimes g_{ip}^{*-1} \frac{\partial \mathbf{u}}{\partial \beta_p} = \bar{\mathbf{e}}_i^* \otimes \frac{\partial \mathbf{u}}{\partial \beta_i}. \quad (3.44)$$

Comparison of the definition for the gradient $\mathbf{Grad}(\mathbf{u})$ and $\nabla_{\mathbf{x}}(\phi)$ to the expression to calculate the deformation gradient $\mathbf{F}$ yields

$$\mathbf{F} \equiv \mathbf{Grad}(\mathbf{x}) = [\nabla_{\mathbf{x}}(\mathbf{x})]^T. \quad (3.45)$$

That is,

$$d\mathbf{x} = \mathbf{Grad}(\mathbf{x})d\mathbf{X} = d\mathbf{X}\nabla_{\mathbf{x}}(\mathbf{x}). \quad (3.46)$$

By a similar procedure one can show that

$$d\mathbf{X} = \mathbf{grad}(\mathbf{X})d\mathbf{x} = d\mathbf{x}\nabla_{\mathbf{x}}(\mathbf{X}). \quad (3.47)$$

The inverse of the deformation gradient follows from the recognition of

$$d\mathbf{x} = \mathbf{Grad}(\mathbf{x})d\mathbf{X} = \mathbf{Grad}(\mathbf{x})[\mathbf{grad}(\mathbf{X})d\mathbf{x}], \quad (3.48)$$

which must hold for all $d\mathbf{x}$, and which, therefore, states that

$$\mathbf{Grad}(\mathbf{x}) \mathbf{grad}(\mathbf{X}) = \mathbf{I}. \quad (3.49)$$

In a similar way one can show

$$\nabla_{\mathbf{x}}(\mathbf{x})\nabla_{\mathbf{x}}(\mathbf{X}) = \mathbf{I}. \quad (3.50)$$

Therefore, the inverse of the deformation gradient is given by

$$\mathbf{F}^{-1} = \mathbf{grad}(\mathbf{X}) = [\nabla_{\mathbf{x}}(\mathbf{X})]^T. \quad (3.51)$$

The volume ratio J is the ratio of current volume dV to volume in the initial configuration dV_o of a differential element of mass. It can be shown that

$$J(t) \equiv \frac{dV(t)}{dV_o} = \det[\mathbf{F}(t)]. \quad (3.52)$$

In a similar manner we can describe the temperature gradients in terms of curvilinear coordinate systems. The two temperature gradients are defined as

$$\mathbf{G} \equiv \mathbf{Grad}(\theta) = \nabla_{\mathbf{x}}(\theta) = \frac{\partial \theta}{\partial \alpha_i} g_{ip}^{-1} \mathbf{e}_p = \frac{\partial \theta}{\partial \alpha_i} \bar{\mathbf{e}}_i, \quad (3.53)$$

$$\mathbf{g} \equiv \mathbf{grad}(\theta) = \nabla_{\mathbf{x}}(\theta) = \frac{\partial \theta}{\partial \beta_i} g_{ip}^{*-1} \mathbf{e}_p^* = \frac{\partial \theta}{\partial \beta_i} \bar{\mathbf{e}}_i^*. \quad (3.54)$$

We now will look at an example of using curvilinear coordinate systems in calculating the deformation gradient. For the example we consider cylindrical coordinate systems for both the reference and current configurations. We will let (R, Θ, Z) denote the coordinates in the reference configuration, and will let (r, θ, z) denote the coordinates in the current

configuration. As already has been shown in Chapter 2, the base vectors for the curvilinear system (R, Θ, Z) used in the reference configuration will be given as

$$\mathbf{e}_1 = \hat{\mathbf{e}}_R, \quad \mathbf{e}_2 = R\hat{\mathbf{e}}_\Theta, \quad \mathbf{e}_3 = \hat{\mathbf{e}}_Z, \quad (3.55)$$

and for the coordinate system (r, θ, z) used in the current configuration will be given as

$$\mathbf{e}_1^* = \hat{\mathbf{e}}_r, \quad \mathbf{e}_2^* = r\hat{\mathbf{e}}_\theta, \quad \mathbf{e}_3^* = \hat{\mathbf{e}}_z. \quad (3.56)$$

In this case, the metric tensors will be given by

$$[g] = [\mathbf{e}_i \circ \mathbf{e}_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.57)$$

and

$$[g^*] = [\mathbf{e}_i^* \circ \mathbf{e}_j^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.58)$$

We also can show that

$$\left[\frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \right] = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R^2} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{1}{R^2} \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R^2} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix}. \quad (3.59)$$

Therefore, the deformation gradient is given by

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R^2} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{1}{R^2} \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R^2} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R^2} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{1}{R^2} \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R^2} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \hat{\mathbf{e}}_R \\ R\hat{\mathbf{e}}_\Theta \\ \hat{\mathbf{e}}_Z \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & r \frac{\partial \theta}{\partial \Theta} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\Theta \\ \hat{\mathbf{e}}_Z \end{bmatrix}. \end{aligned} \quad (3.60)$$

Therefore,

$$\begin{aligned} \mathbf{F} &= \frac{\partial r}{\partial R} \hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_R + \frac{1}{R} \frac{\partial r}{\partial \Theta} \hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_\Theta + \frac{\partial r}{\partial Z} \hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_Z + r \frac{\partial \theta}{\partial R} \hat{\mathbf{e}}_\theta \otimes \hat{\mathbf{e}}_R + \frac{r}{R} \frac{\partial \theta}{\partial \Theta} \hat{\mathbf{e}}_\theta \otimes \hat{\mathbf{e}}_\Theta \\ &\quad + r \frac{\partial \theta}{\partial Z} \hat{\mathbf{e}}_\theta \otimes \hat{\mathbf{e}}_Z + \frac{\partial z}{\partial R} \hat{\mathbf{e}}_z \otimes \hat{\mathbf{e}}_R + \frac{1}{R} \frac{\partial z}{\partial \Theta} \hat{\mathbf{e}}_z \otimes \hat{\mathbf{e}}_\Theta + \frac{\partial z}{\partial Z} \hat{\mathbf{e}}_z \otimes \hat{\mathbf{e}}_Z. \end{aligned} \quad (3.61)$$

As a special example consider a line element $d\mathbf{X}$ along the $\hat{\mathbf{e}}_R$ direction as is shown in Figure 3.6. This line element is given by $d\mathbf{X} = dR\hat{\mathbf{e}}_R$. It can be seen that this line element will change to

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \frac{\partial r}{\partial R} dR\hat{\mathbf{e}}_r + r \frac{\partial \theta}{\partial R} dR\hat{\mathbf{e}}_\theta + \frac{\partial z}{\partial R} dR\hat{\mathbf{e}}_z. \quad (3.62)$$

Section 3.4-Problems:

- 3.4-1:** Consider the problem of transformation of a flat surface by the deformation gradient. Given a normal $\hat{\mathbf{n}}_o$ and a distance d_o in the reference configuration,

For any vector $\mathbf{t}_o$ in the plane of dA_o and its corresponding vector $\mathbf{t} = \mathbf{F}\mathbf{t}_o$ in the plane of dA we know that $\mathbf{t}_o \circ \hat{\mathbf{n}}_o = 0$ and $\mathbf{t} \circ \hat{\mathbf{n}} = 0$. Show that this yields

$$\mathbf{t}_o \circ (\mathbf{F}^T \hat{\mathbf{n}}) = 0. \quad (3.65)$$

Argue that since this must hold for every $\mathbf{t}_o$ in dA_o , considering that $\mathbf{t}_o \circ \hat{\mathbf{n}}_o = 0$, we can conclude that $\hat{\mathbf{n}}_o$ and $\mathbf{F}^T \hat{\mathbf{n}}$ must be parallel and conclude that

$$\hat{\mathbf{n}}_o = \varpi \mathbf{F}^T \hat{\mathbf{n}} \quad (3.66)$$

for some scalar ϖ to be determined. Since $\hat{\mathbf{n}}_o$ and $\hat{\mathbf{n}}$ are unit vectors, show the relations

$$\hat{\mathbf{n}}_o = \frac{\mathbf{F}^T \hat{\mathbf{n}}}{\sqrt{\hat{\mathbf{n}} \circ [(\mathbf{F}\mathbf{F}^T) \hat{\mathbf{n}}]}}, \quad (3.67)$$

$$\hat{\mathbf{n}} = \frac{\mathbf{F}^{-T} \hat{\mathbf{n}}_o}{\sqrt{\hat{\mathbf{n}}_o \circ [(\mathbf{F}^T \mathbf{F})^{-1} \hat{\mathbf{n}}_o]}}. \quad (3.68)$$

3.4-4: Using the identity $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$, show that

$$\dot{\mathbf{F}^{-1}} = -\mathbf{F}^{-1} \dot{\mathbf{F}} \mathbf{F}^{-1}. \quad (3.69)$$

3.5 Stretch and strain tensors

As shown in Figure 3.7, the change in the length of a line element and the change in the angle between line elements are the two standard measures of strain at a material point, in this case point P . In general, it can be shown that a linear transformation, such as that described by $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, in three dimensions takes a sphere into an ellipsoidal shape. To achieve this change, radial lines of the sphere get extended/compressed and also can get rotated in space. In this section, we will study these changes and define the different commonly used tensorial measures of stretch and strain.

To study the extension of line elements, we will calculate the length of a line element. Let ds denote the length of the line element $d\mathbf{x}$. It follows from $ds^2 = d\mathbf{x} \circ d\mathbf{x}$ and $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ that

$$\begin{aligned} ds^2 &= (\mathbf{F}d\mathbf{X}) \circ (\mathbf{F}d\mathbf{X}) \\ &= (\mathbf{F}d\mathbf{X})^T \circ (\mathbf{F}d\mathbf{X}) \\ &= (d\mathbf{X}\mathbf{F}^T) \circ (\mathbf{F}d\mathbf{X}) \\ &= d\mathbf{X} \circ [(\mathbf{F}^T \mathbf{F}) d\mathbf{X}]. \end{aligned} \quad (3.70)$$

The product $\mathbf{F}^T \mathbf{F}$ is a measure of stretch at the material point and is called the right Cauchy stretch (or strain) tensor. We will use $\mathbf{C}$ to denote this product so that $\boxed{\mathbf{C} \equiv \mathbf{F}^T \mathbf{F}}$. The expression for ds^2 now can be written in terms of the right Cauchy stretch tensor $\mathbf{C}$ as

$$ds^2 = d\mathbf{X} \circ (\mathbf{C}d\mathbf{X}) = (\mathbf{C}d\mathbf{X}) \circ d\mathbf{X}. \quad (3.71)$$

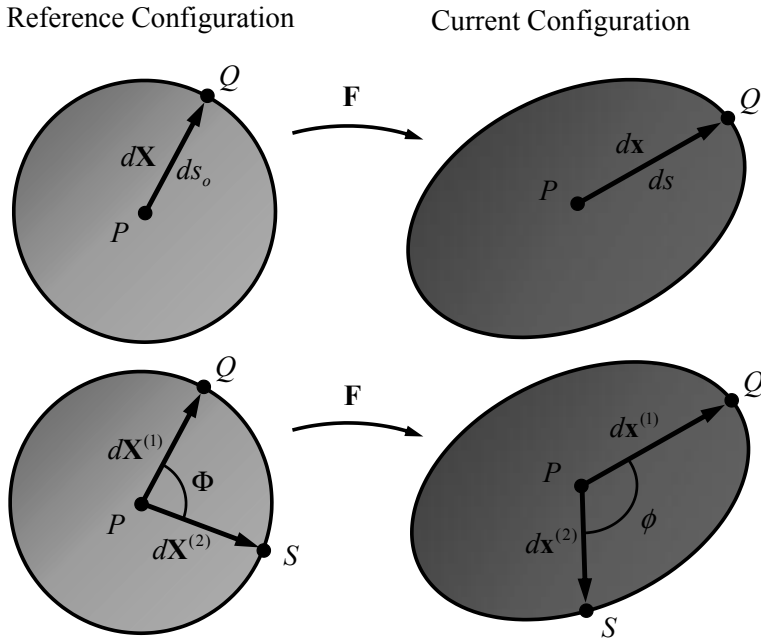


Figure 3.7: The change of length of a line element is one measure of strain at a point. The change in angle between two line elements is another measure of strain at the point.

Using this equation we can calculate the strain in the line element given by $d\mathbf{X}$ in the reference configuration. We will denote this strain as ϵ . It can be calculated from the equation

$$\epsilon \equiv \frac{ds - ds_o}{ds_o} = \frac{\sqrt{d\mathbf{X} \circ (\mathbf{C}d\mathbf{X})} - \sqrt{d\mathbf{X} \circ d\mathbf{X}}}{\sqrt{d\mathbf{X} \circ d\mathbf{X}}}. \quad (3.72)$$

As can be seen, the right Cauchy stretch tensor contains all the information needed to find how the length of any line element in the reference configuration changes when the material is deformed to the current configuration. The inverse of the right Cauchy stretch tensor $\mathbf{C}^{-1}$ is known as the *Finger strain tensor*.

We next will study changes of angles. Consider two line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ in the reference configuration as shown in Figure 3.7. Let Φ denote the angle between these line elements. Let $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ be, respectively, the line elements in the current configuration corresponding to $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ and let ϕ denote the angle between these line elements. The initial angle between these line elements can be found from

$$d\mathbf{X}^{(1)} \circ d\mathbf{X}^{(2)} = |d\mathbf{X}^{(1)}||d\mathbf{X}^{(2)}| \cos(\Phi). \quad (3.73)$$

The final angle between these line elements can be found from

$$d\mathbf{x}^{(1)} \circ d\mathbf{x}^{(2)} = |d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}| \cos(\phi) = (\mathbf{F}d\mathbf{X}^{(1)}) \circ (\mathbf{F}d\mathbf{X}^{(2)}) = d\mathbf{X}^{(1)} \circ (\mathbf{C}d\mathbf{X}^{(2)}). \quad (3.74)$$

It follows that the change in the angle between these line elements can be found from $\mathbf{C}$. Therefore, $\mathbf{C}$ contains all the information needed to calculate the change in length and the change in angle.

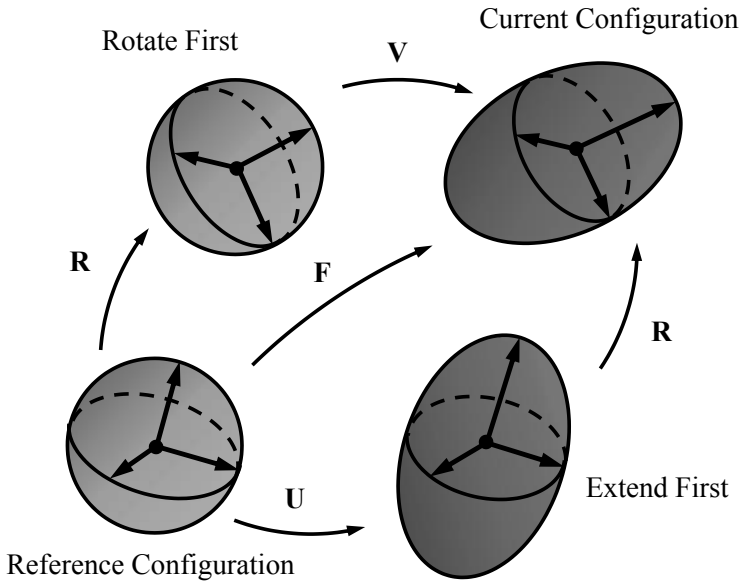


Figure 3.8: The polar decomposition decomposes the deformation gradient into an orthogonal factor $\mathbf{R}$ and symmetric factors $\mathbf{U}$ and $\mathbf{V}$. The two symmetric factors are identical in eigenvalues and have eigenvectors that are rotated by the value of the orthogonal factor.

One can formulate the inverse problem by first looking at line elements in the current configuration and finding how they have changed. Let ds_o denote the length of $d\mathbf{X}$. In this case,

$$\begin{aligned} ds_o^2 &= d\mathbf{X} \circ d\mathbf{X} = (\mathbf{F}^{-1} d\mathbf{x}) \circ (\mathbf{F}^{-1} d\mathbf{x}) = (\mathbf{F}^{-1} d\mathbf{x})^T \circ (\mathbf{F}^{-1} d\mathbf{x}) \\ &= (d\mathbf{x} \mathbf{F}^{-T}) \circ (\mathbf{F}^{-1} d\mathbf{x}) = d\mathbf{x} \circ [(\mathbf{F} \mathbf{F}^T)^{-1} d\mathbf{x}]. \end{aligned} \quad (3.75)$$

The tensor $\mathbf{B} \equiv \mathbf{F} \mathbf{F}^T$ is known as the *left Cauchy stretch (or strain) tensor*. Using this tensor, we can write the equation for ds_o as

$$ds_o^2 = d\mathbf{x} \circ (\mathbf{B}^{-1} d\mathbf{x}). \quad (3.76)$$

This shows that the left Cauchy strain tensor contains all the information needed to calculate ds_o from $d\mathbf{x}$. In a similar way, one can show that

$$d\mathbf{X}^{(1)} \circ d\mathbf{X}^{(2)} = d\mathbf{x}^{(1)} \circ (\mathbf{B}^{-1} d\mathbf{x}^{(2)}), \quad (3.77)$$

so that we can see that the reference angle between the line elements can be calculated from $\mathbf{B}^{-1}$.

It can be shown that orthogonal deformation gradients represent rigid body motions. For an orthogonal deformation gradient, the value of the right Cauchy strain tensor becomes the identity (i.e., $\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{I}$), which, in view of the above, results in no change in length of material line elements and no change of the angle between material line elements.

The polar decomposition theorem can be used to separate the deformation gradient into a pure rigid body rotation and a pure deformation as shown in Figure 3.8. The polar decomposition theorem in this case states that any deformation gradient $\mathbf{F}$ can be uniquely

decomposed into orthogonal and symmetric factors. This can be written as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad (3.78)$$

where $\mathbf{R}$ is orthogonal (i.e., $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$), and $\mathbf{U}$ and $\mathbf{V}$ are symmetric (i.e., $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{V} = \mathbf{V}^T$). The orthogonal factor $\mathbf{R}$ represents a rigid body rotation and the symmetric factors $\mathbf{U}$ and $\mathbf{V}$ represent pure triaxial extensions. As shown in the figure, this states that any deformation can be decomposed into a pure rigid body rotation and a pure triaxial extension along material lines representing the eigenvectors of the symmetric part. The difference between the two decompositions $\mathbf{F} = \mathbf{R}\mathbf{U}$ and $\mathbf{F} = \mathbf{V}\mathbf{R}$ is in the order of imposing the rigid body rotation and the pure triaxial extension. The decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ represents, as shown in the figure, first a pure triaxial extension along the principal directions of $\mathbf{U}$ followed by a rigid body rotation to bring the initial triaxial extension into alignment with the final current orientation of these axes. On the other hand, the decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$ represents first a rigid body rotation to bring the principal axis of extension into alignment with those of the current configuration followed by a pure triaxial extension. The relation between the principal axes of $\mathbf{U}$ and $\mathbf{V}$ is, therefore, given by the amount of the rigid body rotation. This can be seen from the fact that

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T. \quad (3.79)$$

Using polar decomposition we can express the right and left Cauchy strain tensors as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = (\mathbf{R}\mathbf{U})^T(\mathbf{R}\mathbf{U}) = \mathbf{U}\mathbf{U} = \mathbf{U}^2, \quad (3.80)$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{V}\mathbf{R})(\mathbf{V}\mathbf{R})^T = \mathbf{V}\mathbf{V} = \mathbf{V}^2, \quad (3.81)$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = (\mathbf{V}\mathbf{R})^T(\mathbf{V}\mathbf{R}) = \mathbf{R}^T\mathbf{B}\mathbf{R}, \quad (3.82)$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{R}\mathbf{U})(\mathbf{R}\mathbf{U})^T = \mathbf{R}\mathbf{C}\mathbf{R}^T. \quad (3.83)$$

The right Cauchy strain tensor has a value of identity for rigid body motions. It is sometimes convenient to use a strain measure that is zero for rigid body motions. One such strain tensor is the *Green strain* $\mathbf{E}$ that is defined as

$$\mathbf{E} \equiv \frac{1}{2}(\mathbf{C} - \mathbf{I}). \quad (3.84)$$

This strain also is known as the *Green-Lagrange strain*. On occasion it is useful to construct a strain similar to the Green strain tensor, but using $\mathbf{B}$ in place of $\mathbf{C}$. Only in certain places will this be needed so, to avoid confusion, in those places that a distinction is needed between the two, we will use $\mathbf{E}_R$ and $\mathbf{E}_L$ to denote the expressions

$$\mathbf{E}_R \equiv \frac{1}{2}(\mathbf{C} - \mathbf{I}), \quad (3.85)$$

$$\mathbf{E}_L \equiv \frac{1}{2}(\mathbf{B} - \mathbf{I}). \quad (3.86)$$

Another commonly used strain tensor is the *Almansi strain tensor* $\mathbf{A}$, which is defined as

$$\mathbf{A} \equiv \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}). \quad (3.87)$$

One may note that the Almansi strain and the Green strain are related through the relation $\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = \frac{1}{2}\mathbf{F}^{-T}(\mathbf{C} - \mathbf{I})\mathbf{F}^{-1}$ so that

$$\mathbf{A} = \mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1}. \quad (3.88)$$

Section 3.5-Problems:

- 3.5-1:** Show that a linear transformation takes a sphere into an ellipsoidal shape. In doing so, show that a sphere of radius R defined by the equation $\mathbf{x} \circ \mathbf{x} = R^2$ transforms by the linear transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$ into a general ellipsoidal shape given by the equation $\mathbf{y} \circ [(\frac{1}{b_1^2} \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \frac{1}{b_2^2} \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \frac{1}{b_3^2} \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)})\mathbf{y}] = 1$, where $\hat{\mathbf{n}}^{(i)}$ are unit vectors along the three perpendicular principal axes of the ellipsoidal shape and b_i are the corresponding radii.
- 3.5-2:** Show that a linear transformation takes a cube into a parallelepiped.
- 3.5-3:** Consider the polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$. Starting from the eigenvalue problem for second-order tensors, show that the principal axes of $\mathbf{U}$ and $\mathbf{V}$ are related by the orthogonal transformation $\mathbf{R}$.

3.6 Velocity gradient

We now will look at the velocity gradient and other measures of deformation rate. The *velocity gradient* is denoted by $\mathbf{L}$ and is defined as

$$\boxed{\mathbf{L} \equiv \mathbf{grad}(\mathbf{v}) = [\nabla_{\mathbf{x}}(\mathbf{v})]^T}, \quad (3.89)$$

where $\mathbf{v}$ is the velocity. To understand the meaning of $\mathbf{L}$ let us consider two close material points P and Q . We will let $d\mathbf{x}$ denote the vector going from P to Q in the current configuration, and we will let $d\mathbf{v}$ denote the change in the velocity when going from P to Q . From the definition of the gradient it is clear that the change in velocity $d\mathbf{v}$ can be calculated from

$$d\mathbf{v} = \mathbf{grad}(\mathbf{v})d\mathbf{x} = \mathbf{L}d\mathbf{x}, \quad (3.90)$$

or

$$d\mathbf{v} = \mathbf{grad}(\mathbf{v})\mathbf{Grad}(\mathbf{x})d\mathbf{X} = \mathbf{L}\mathbf{F}d\mathbf{X}, \quad (3.91)$$

where $d\mathbf{X}$ is the relative position of the two points in the reference configuration. A similar development shows that

$$d\mathbf{v} = \mathbf{Grad}(\mathbf{v})d\mathbf{X} = \mathbf{Grad}(\mathbf{v})\mathbf{grad}(\mathbf{X})d\mathbf{x} = \dot{\mathbf{F}}\mathbf{F}^{-1}d\mathbf{x}, \quad (3.92)$$

where

$$\boxed{\dot{\mathbf{F}} \equiv \frac{d\mathbf{F}}{dt}} \quad (3.93)$$

is the material time derivative of the deformation gradient. Comparison of these two expressions for $d\mathbf{v}$ results in the relation

$$\boxed{\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}}. \quad (3.94)$$

It can be shown that

$$\boxed{tr(\mathbf{L}) = \frac{\dot{J}}{J}}. \quad (3.95)$$

$\mathbf{L}$ commonly is separated into a symmetric tensor $\mathbf{D}$ and a skew-symmetric tensor $\mathbf{W}$ such that

$$\boxed{\mathbf{L} = \mathbf{D} + \mathbf{W}}, \quad (3.96)$$

where

$$\mathbf{D} \equiv \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad (3.97)$$

and

$$\mathbf{W} \equiv \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \quad (3.98)$$

$\mathbf{D}$ is known as the *rate of deformation or rate of strain tensor* and $\mathbf{W}$ is the *spin or vorticity tensor*.

We next will look at how line segments change their length with time. Let ds be the length of the vector $d\mathbf{x}$. We know that

$$ds^2 = d\mathbf{x} \circ d\mathbf{x}. \quad (3.99)$$

By taking a material time derivative of both sides of this equation one obtains

$$2 \frac{d(ds)}{dt} ds = 2 \frac{d(d\mathbf{x})}{dt} \circ d\mathbf{x} = 2 d\mathbf{v} \circ d\mathbf{x}. \quad (3.100)$$

Therefore, we can write

$$\frac{1}{ds} \frac{d(ds)}{dt} = \frac{d\mathbf{x}}{ds} \circ \left(\mathbf{L} \frac{d\mathbf{x}}{ds} \right) = \frac{d\mathbf{x}}{ds} \circ (\mathbf{L}^T \frac{d\mathbf{x}}{ds}). \quad (3.101)$$

This can be rewritten as

$$\frac{1}{ds} \frac{d(ds)}{dt} = \hat{\mathbf{n}} \circ \left[\frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \hat{\mathbf{n}} \right] = \hat{\mathbf{n}} \circ (\mathbf{D} \hat{\mathbf{n}}), \quad (3.102)$$

where $\hat{\mathbf{n}}$ is a unit vector along $d\mathbf{x}$. In a similar manner

$$\frac{d(d\mathbf{x}^{(1)} \circ d\mathbf{x}^{(2)})}{dt} = 2 d\mathbf{x}^{(1)} \circ (\mathbf{D} d\mathbf{x}^{(2)}), \quad (3.103)$$

where $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ are the vectors in the current configuration going from P to two material points Q and S that are close to P , as shown in Figure 3.7. As can be seen, the deformation rate $\mathbf{D}$ is closely related to the rate of change of length and of angles.

From the relation $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ we also can show that

$$\mathbf{D} = \frac{1}{2}\mathbf{F}^{-T}\dot{\mathbf{C}}\mathbf{F}^{-1}. \quad (3.104)$$

This follows from the relation

$$\dot{\mathbf{C}} = \dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}}. \quad (3.105)$$

3.7 Relative deformation

We now will look at relative deformation of a body as it deforms from one configuration to another. Consider a material undergoing a deformation. At each time s in the deformation let $(\beta_1(s), \beta_2(s), \beta_3(s))$ denote the curvilinear coordinates. The motion of a particle can be described by giving $\beta_i(\alpha_1, \alpha_2, \alpha_3, s)$, where α_i are the coordinates of the particle in the reference configuration. The deformation gradient $\mathbf{F}(s)$ can be defined as

$$\mathbf{F}(s) = \mathbf{Grad}[\mathbf{x}(s)] = \mathbf{grad}_{\mathbf{X}}[\mathbf{x}(s)], \quad (3.106)$$

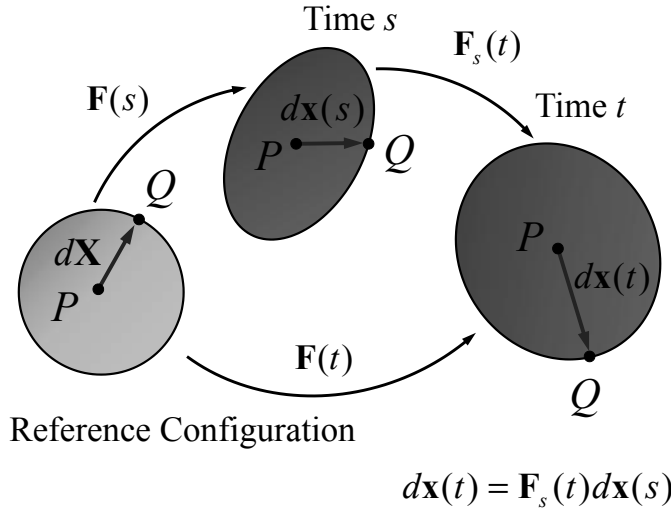


Figure 3.9: Consecutive deformation gradients superimpose to give the total deformation gradient.

where $\mathbf{x}(s)$ is the position of particles in the configuration at time s . This is defined in such a way that one will obtain

$$d\mathbf{x}(s) = \mathbf{F}(s)d\mathbf{X}. \quad (3.107)$$

The relative deformation of the body going from its configuration at time s to its configuration at time t can be measured using the deformation gradient $\mathbf{F}_s(t)$ shown in Figure 3.9, or its inverse $\mathbf{F}_t(s)$ which are defined as

$$\mathbf{F}_s(t) \equiv \text{grad}_{\mathbf{x}(s)}[\mathbf{x}(t)], \quad (3.108)$$

and

$$\mathbf{F}_t(s) \equiv \text{grad}_{\mathbf{x}(t)}[\mathbf{x}(s)]. \quad (3.109)$$

These are defined such that

$$d\mathbf{x}(t) = \mathbf{F}_s(t)d\mathbf{x}(s), \quad (3.110)$$

and

$$d\mathbf{x}(s) = \mathbf{F}_t(s)d\mathbf{x}(t). \quad (3.111)$$

It can be shown that these relative deformation gradients can be calculated from $\mathbf{F}(s)$ and $\mathbf{F}(t)$ by the relations

$$\mathbf{F}_s(t) = \mathbf{F}(t)\mathbf{F}^{-1}(s), \quad (3.112)$$

$$\mathbf{F}_t(s) = \mathbf{F}(s)\mathbf{F}^{-1}(t). \quad (3.113)$$

One can define the left and right Cauchy strain tensors derived from $\mathbf{F}_s(t)$ as

$$\mathbf{B}_s(t) = \mathbf{F}_s(t)\mathbf{F}_s^T(t), \quad (3.114)$$

and

$$\mathbf{C}_s(t) = \mathbf{F}_s^T(t)\mathbf{F}_s(t). \quad (3.115)$$

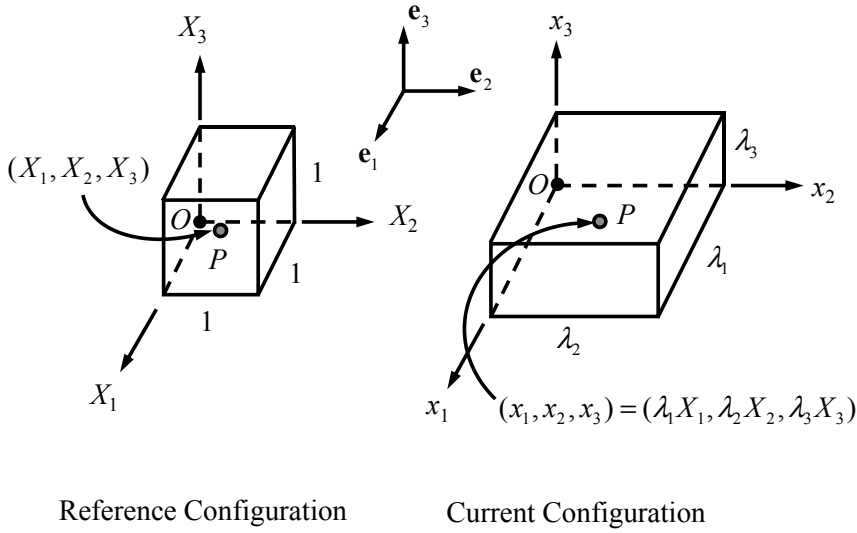


Figure 3.10: Triaxial extension along the coordinate directions.

3.8 Triaxial extension, simple shear, bending, and torsion

Let us now look at some examples of specific deformations. First, we will start with *triaxial extension*. This is a homogeneous deformation that takes a cube into a cuboid. Figure 3.10 shows the reference and current configurations for this deformation. As can be seen, rectangular coordinates X_i are used to define position in the reference configuration and rectangular coordinates x_i are used to describe position in the current configuration. The two coordinates are selected such that they are parallel. The same orthonormal curvilinear base vectors are used for both coordinates. In summary, we will be using

$$\begin{aligned}
 (\alpha_1, \alpha_2, \alpha_3) &= (X_1, X_2, X_3), \\
 (\beta_1, \beta_2, \beta_3) &= (x_1, x_2, x_3), \\
 \mathbf{e}_i &= \mathbf{e}_i^*, \\
 g_{ij} &= g_{ij}^* = \delta_{ij}.
 \end{aligned}$$

For this set of coordinate systems, the deformation can be written as

$$\begin{aligned}
 x_1 &= \lambda_1 X_1, \\
 x_2 &= \lambda_2 X_2, \\
 x_3 &= \lambda_3 X_3,
 \end{aligned} \tag{3.116}$$

where λ_i are constants and each one is a stretch ratio associated with extension along one of the coordinate directions. The deformation gradient can be calculated from the relation

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{3.117}$$

In matrix form, this can be written as

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

This results in

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

to yield

$$\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (3.118)$$

Since the deformation gradient is symmetric, the right and left Cauchy stretch tensors are identical and are given by

$$\begin{aligned} \mathbf{C} = \mathbf{B} = \mathbf{F}\mathbf{F}^T &= (\lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3)(\lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &= \lambda_1^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^2 \mathbf{e}_3 \otimes \mathbf{e}_3. \end{aligned} \quad (3.119)$$

The Green strain tensor is then given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\lambda_1^2 - 1)\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{2}(\lambda_2^2 - 1)\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{2}(\lambda_3^2 - 1)\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (3.120)$$

The Almansi strain tensor also can be calculated and is given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = \frac{1}{2}\left(1 - \frac{1}{\lambda_1^2}\right)\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{2}\left(1 - \frac{1}{\lambda_2^2}\right)\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{2}\left(1 - \frac{1}{\lambda_3^2}\right)\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (3.121)$$

The volume ratio can be calculated by the relation

$$J = \det(\mathbf{F}) = \lambda_1 \lambda_2 \lambda_3. \quad (3.122)$$

For example, if the volume is assumed to be constant, then $J = 1$ and one would have a relation between the stretch ratios of the form $\lambda_1 \lambda_2 \lambda_3 = 1$. If one assumes the motion at all times is described by triaxial extension along a fixed set of coordinate axes, then one can calculate the velocity gradient as

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \frac{\dot{\lambda}_1}{\lambda_1}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\dot{\lambda}_2}{\lambda_2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\dot{\lambda}_3}{\lambda_3}\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (3.123)$$

Since the velocity gradient is symmetric, it is equal to the deformation rate, and the spin tensor is zero.

As a second example, let us consider what is called *simple shear*. Figure 3.11 shows the in-plane deformation for simple shear. The out-of-plane deformation is assumed to be zero. Again, let us assume that we are using similar rectangular coordinates for both reference and current configurations, the same as we did for triaxial extension. The deformation associated with simple shear is written as

$$\begin{aligned} x_1 &= X_1 + \gamma X_2, \\ x_2 &= X_2, \\ x_3 &= X_3. \end{aligned} \quad (3.124)$$

As can be seen, particles at a given X_2 simply move to the right an amount γX_2 , just as would be the case when one shears a stack of cards by moving the top card relative to the bottom. As in the case of triaxial extension, the two coordinate systems are rectangular and

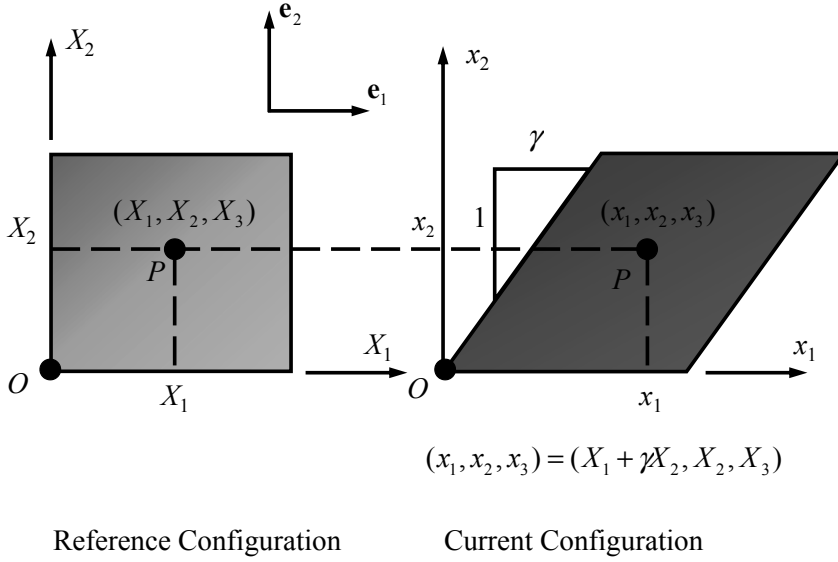


Figure 3.11: In-plane deformation for simple shear.

mutually parallel, so that the coordinates are identical to those described for rectangular systems and we, therefore, have

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{e}_j. \quad (3.125)$$

In matrix form, this can be written as

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

This results in

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

to yield deformation gradient

$$\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2. \quad (3.126)$$

The inverse of $\mathbf{F}$ is given by

$$\mathbf{F}^{-1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 - \gamma \mathbf{e}_1 \otimes \mathbf{e}_2. \quad (3.127)$$

The right Cauchy stretch tensor is therefore given by

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + (1 + \gamma^2) \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (3.128)$$

In matrix form, this can be written as

$$\mathbf{C} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

The left Cauchy stretch tensor is given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (1 + \gamma^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (3.129)$$

In matrix form, this can be written as

$$\mathbf{B} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

The Green strain tensor is given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{\gamma^2}{2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\gamma}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (3.130)$$

The Almansi strain tensor is given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = -\frac{\gamma^2}{2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\gamma}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (3.131)$$

Let us assume that for the time interval under consideration the flow is continuously described by simple shear. It is easy to see that the time rate of change of the deformation gradient is given by

$$\dot{\mathbf{F}} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2, \quad (3.132)$$

and the velocity gradient is given by

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2. \quad (3.133)$$

The deformation rate and spin tensors, respectively, are given by

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2}\dot{\gamma}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \frac{1}{2}\dot{\gamma}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (3.134)$$

The relative deformation gradient is given by

$$\mathbf{F}_s(t) = \mathbf{F}(t)\mathbf{F}^{-1}(s) = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + [\gamma(t) - \gamma(s)]\mathbf{e}_1 \otimes \mathbf{e}_2. \quad (3.135)$$

As a third example consider *pure bending*. Figure 3.12 shows the in-plane deformation associated with pure bending. As can be seen, the horizontal and vertical lines in the rectangular cross section are transformed, respectively, into circumferential and radial lines. This deformation can be described by

$$\begin{aligned} r &= r(X_2), \\ \theta &= \theta(X_1), \\ z &= z(X_3). \end{aligned} \quad (3.136)$$

To illustrate the importance of the selection of coordinates let us look at describing the current configuration first by using the rectangular coordinates (x_1, x_2, x_3) shown in the figure and next using the cylindrical coordinates (r, θ, z) . The current rectangular coordinates of a particle at location (X_1, X_2, X_3) can be described as

$$\begin{aligned} x_1 &= r(X_2) \cos[\theta(X_1)], \\ x_2 &= r(X_2) \sin[\theta(X_1)], \\ x_3 &= z(X_3). \end{aligned} \quad (3.137)$$

Since two mutually parallel rectangular coordinate systems are being used for the reference and current coordinate systems, the same form of the deformation gradient used for triaxial extension and simple shear can be used and as a result one obtains

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} -r \frac{d\theta}{dX_1} \sin \theta & \frac{dr}{dX_2} \cos \theta & 0 \\ r \frac{d\theta}{dX_1} \cos \theta & \frac{dr}{dX_2} \sin \theta & 0 \\ 0 & 0 & \frac{dz}{dX_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

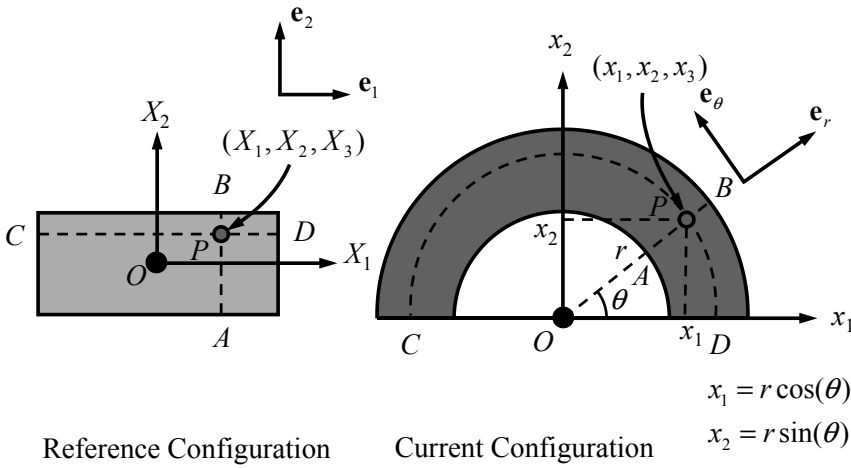


Figure 3.12: In-plane deformation for pure bending.

Now let us evaluate this same deformation gradient using the same rectangular system for the reference configuration and using the cylindrical system shown for the current configuration. As the metric tensor of the coordinate system selected for the reference configuration is equal to the Kronecker delta, one has

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k = \frac{\partial \beta_i}{\partial \alpha_j} \mathbf{e}_i^* \otimes \mathbf{e}_j. \quad (3.138)$$

Therefore,

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

which results in

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} 0 & \frac{dr}{dX_2} & 0 \\ r \frac{d\theta}{dX_1} & 0 & 0 \\ 0 & 0 & \frac{dz}{dX_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

From this expression one can see that the right Cauchy stretch tensor is given as

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \left(r \frac{d\theta}{dX_1} \right)^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \left(\frac{dr}{dX_2} \right)^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \left(\frac{dz}{dX_3} \right)^2 \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (3.139)$$

and the left Cauchy stretch tensor is given by

$$\mathbf{B} = \mathbf{F} \mathbf{F}^T = \left(\frac{dr}{dX_2} \right)^2 \mathbf{e}_r \otimes \mathbf{e}_r + \left(r \frac{d\theta}{dX_1} \right)^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \left(\frac{dz}{dX_3} \right)^2 \mathbf{e}_z \otimes \mathbf{e}_z. \quad (3.140)$$

Finally, let us look at *pure torsion*. As shown in Figure 3.13, cylindrical coordinates are selected for both the reference and current configurations. The deformation in pure torsion can be described as

$$\begin{aligned} r &= r(R), \\ \theta &= \Theta + \Delta\theta(Z), \\ z &= z(Z). \end{aligned} \quad (3.141)$$

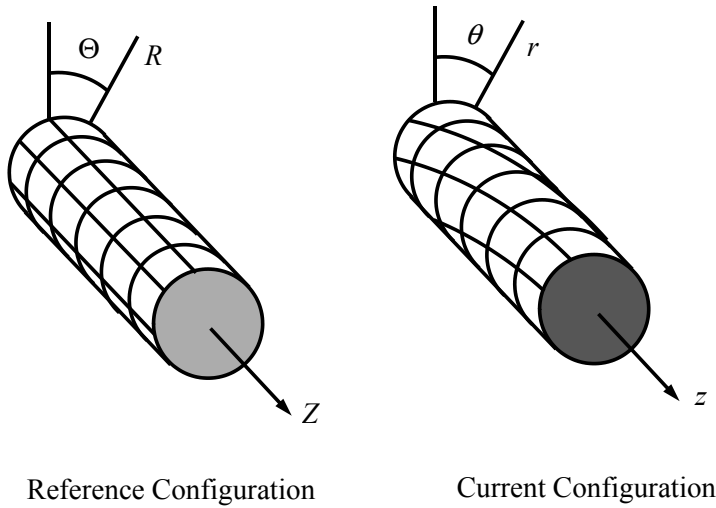


Figure 3.13: Deformation for pure torsion.

The deformation gradient is then given by

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & r \frac{\partial \theta}{\partial \Theta} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix},$$

which yields

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{dr}{dR} & 0 & 0 \\ 0 & \frac{r}{R} & r \frac{d\Delta\theta}{dZ} \\ 0 & 0 & \frac{dz}{dZ} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}.$$

Therefore, the right Cauchy stretch tensor is given by

$$\mathbf{C} = \left(\frac{dr}{dR} \right)^2 \mathbf{e}_R \otimes \mathbf{e}_R + \frac{r^2}{R^2} \mathbf{e}_\Theta \otimes \mathbf{e}_\Theta + \left[\left(\frac{dz}{dZ} \right)^2 + r^2 \left(\frac{d\Delta\theta}{dZ} \right)^2 \right] \mathbf{e}_Z \otimes \mathbf{e}_Z + \frac{r^2}{R} \frac{d\Delta\theta}{dZ} (\mathbf{e}_\Theta \otimes \mathbf{e}_Z + \mathbf{e}_Z \otimes \mathbf{e}_\Theta),$$

and the left Cauchy stretch tensor is given by

$$\mathbf{B} = \left(\frac{dr}{dR} \right)^2 \mathbf{e}_r \otimes \mathbf{e}_r + \left[\frac{r^2}{R^2} + r^2 \left(\frac{d\Delta\theta}{dZ} \right)^2 \right] \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \left(\frac{dz}{dZ} \right)^2 \mathbf{e}_z \otimes \mathbf{e}_z + r \frac{dz}{dZ} \frac{d\Delta\theta}{dZ} (\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta).$$

Section 3.8-Problems:

- 3.8-1:** The combination of triaxial extension and simple shear results in a deformation that can be described by

$$\begin{aligned} x_1 &= \lambda_1 X_1 + \gamma \lambda_2 X_2, \\ x_2 &= \lambda_2 X_2, \\ x_3 &= \lambda_3 X_3. \end{aligned}$$

For this deformation, evaluate the deformation gradient, the right and left Cauchy stretch tensors, the volume ratio, the Green strain tensor, and the Almansi strain tensor.

- 3.8-2:** Consider an isotropic material undergoing uniaxial extension along the $\hat{\mathbf{e}}_1$ direction described by the axial stretch ratio λ_a and having equal contraction in the $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ directions described by the transverse stretch ratio λ_t . For this material, describe the deformation and show that the deformation gradient is given by

$$\mathbf{F} = \lambda_a \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_t \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_t \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3, \quad (3.142)$$

and the velocity gradient

$$\mathbf{L} = \frac{\dot{\lambda}_a}{\lambda_a} \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \frac{\dot{\lambda}_t}{\lambda_t} \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \frac{\dot{\lambda}_t}{\lambda_t} \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3. \quad (3.143)$$

The Poisson's ratio is a parameter relating transverse to axial deformations. We denote the Poisson's ratio by " ν " and for finite uniaxial deformation define it as negative the transverse component of the velocity gradient divided by the axial component of the velocity gradient so that it is defined as

$$\nu \equiv - \frac{\lambda_a \dot{\lambda}_t}{\lambda_t \dot{\lambda}_a}. \quad (3.144)$$

An incompressible material is one for which the volume ratio remains constant. Calculate the volume ratio J for the deformation described and take its derivative to show

$$\dot{J} = \dot{\lambda}_a \lambda_t^2 + 2\lambda_a \lambda_t \dot{\lambda}_t. \quad (3.145)$$

Use this relation to show that for the uniaxial extension of an incompressible isotropic material we always get

$$\nu = \frac{1}{2}. \quad (3.146)$$

The Poisson's function is defined as negative the ratio of the transverse strain to the axial strain. Denoting the Poisson's function by " ν' ," we can write it for uniaxial extension as

$$\nu' \equiv - \frac{\epsilon_t}{\epsilon_a} = - \frac{\lambda_t - 1}{\lambda_a - 1}. \quad (3.147)$$

For an incompressible material that has a reference configuration equal to the initial configuration, the volume ratio is equal to unity. In this case, show that the Poisson's function for an incompressible isotropic material in uniaxial extension is given by

$$\nu' = \frac{1}{\sqrt{\lambda_a}(\sqrt{\lambda_a} + 1)}, \quad (3.148)$$

and plot this for axial strains from minus 50% to 100%. The incremental Poisson's ratio " ν'' " is defined as the negative of the ratio of the change in transverse strain to the change in axial strain so that it can be written as

$$\nu'' = - \frac{\dot{\lambda}_t}{\dot{\lambda}_a}. \quad (3.149)$$

Calculate this Poisson's ratio for an incompressible material to show that

$$\nu'' \equiv \frac{1}{2\lambda_a^{\frac{3}{2}}}, \quad (3.150)$$

and plot it for axial strains from minus 50% to 100%.

3.9 Small deformations

Let us now look at the expressions for the different strain tensors when the deformations are small. We will look at this for two cases. First, we will consider the case where the deformation gradient is only slightly different from the identity tensor $\mathbf{I}$. This represents the case for small stretch and small rigid body rotation. We will follow this by looking at small stretch, but with the possibility of large rigid body rotations.

We introduced the idea of small tensors without defining how to test a tensor for being small. The notion of small for a scalar is clear. For a vector $\mathbf{a}$, we will say it is small if for every unit vector $\hat{\mathbf{n}}$ we have $|\hat{\mathbf{n}} \circ \mathbf{a}| \ll 1$. Obviously, this requires that the projection of $\mathbf{a}$ along all directions be small, and, in particular, since this must be true for all unit vectors $\hat{\mathbf{n}}$, it must be true for a unit vector $\hat{\mathbf{n}}$ along $\mathbf{a}$, and therefore this definition for small requires that the magnitude of $\mathbf{a}$ be small. For a real second-order tensor $\mathbf{A}$, we will consider it to be small if for every two unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ we have $|(\mathbf{A}\hat{\mathbf{n}}_1) \circ \hat{\mathbf{n}}_2| \ll 1$. For a general real m th order tensor $\mathbf{E}$, we will consider it to be small if for all possible sets of unit vectors $\hat{\mathbf{n}}_1, \dots, \hat{\mathbf{n}}_m$ we have $|(\dots((\mathbf{E}\hat{\mathbf{n}}_1)\hat{\mathbf{n}}_2)\hat{\mathbf{n}}_3)\dots) \circ \hat{\mathbf{n}}_m| \ll 1$. For a general m th order tensor $\mathbf{E}$ let us define the projection norm $\|\mathbf{E}\|$ of an arbitrary order tensor as

$$\|\mathbf{E}\| \equiv \max_{\hat{\mathbf{n}}_1, \dots, \hat{\mathbf{n}}_m} |(\dots((\mathbf{E}\hat{\mathbf{n}}_1)\hat{\mathbf{n}}_2)\hat{\mathbf{n}}_3)\dots) \circ \hat{\mathbf{n}}_m|. \quad (3.151)$$

That is, $\|\mathbf{E}\|$ denotes the absolute maximum value that the projection can have.

Section 3.9-Problems:

3.9-1: Show that $\|\mathbf{E}\| = \|\mathbf{E}^T\|$.

3.9-2: Show that if for a real second-order tensor $\mathbf{E}$ we have $\|\mathbf{E}\| \ll 1$, then

$$\|\mathbf{E}^T\| \ll 1, \quad (3.152)$$

$$\|\mathbf{E}^2\| \ll \|\mathbf{E}\|, \quad (3.153)$$

$$\|\mathbf{E}\mathbf{E}^T\| \ll \|\mathbf{E}\|, \quad (3.154)$$

$$\|\mathbf{E}^T\mathbf{E}\| \ll \|\mathbf{E}\|. \quad (3.155)$$

3.9-3: Consider second-order tensors $\mathbf{A}$ and $\mathbf{B}$. Show that if $\|\mathbf{A}\| \ll 1$ and $\|\mathbf{B}\| \ll 1$, then $\|\alpha\mathbf{A} + \beta\mathbf{B}\| \ll 1$ for $|\alpha| \leq 1$ and $|\beta| \leq 1$.

3.9-4: Consider a real symmetric second-order tensor $\mathbf{E}$. Show that $\|\mathbf{E}\| \ll 1$ is equivalent to having all the magnitudes of the eigenvalues of $\mathbf{E}$ being much less than unity.

3.9-5: Consider two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$ for some orthogonal second-order tensor $\mathbf{Q}$. Show that

$$\|\mathbf{A}\| = \|\mathbf{B}\|. \quad (3.156)$$

3.9.1 Small stretch and small rigid body rotation

For the case of both small stretches and small rigid body rotations, the deformation gradient is only slightly different from the identity tensor $\mathbf{I}$. Let us write the deformation gradient as

$$\mathbf{F} = \mathbf{I} + \mathbf{H}, \quad (3.157)$$

where $\mathbf{H} = \mathbf{Grad}(\mathbf{u})$ is the displacement gradient relative to changes of position in the reference configuration so that $d\mathbf{u} = \mathbf{H}d\mathbf{X}$. For small stretches and small rigid body motions we will have $\mathbf{H}$ being small. That is, we require that $\|\mathbf{H}\| \ll 1$. In this case, the right Cauchy stretch tensor will be

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = (\mathbf{I} + \mathbf{H}^T)(\mathbf{I} + \mathbf{H}) = \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H} \approx \mathbf{I} + \mathbf{H} + \mathbf{H}^T = \mathbf{I} + 2\boldsymbol{\epsilon}, \quad (3.158)$$

where $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor defined by

$$\boldsymbol{\epsilon} \equiv \frac{1}{2}(\mathbf{H} + \mathbf{H}^T). \quad (3.159)$$

The Green strain tensor $\mathbf{E}$ is then given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}) \approx \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) = \boldsymbol{\epsilon}. \quad (3.160)$$

The left Cauchy stretch tensor $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{I} + \mathbf{H})(\mathbf{I} + \mathbf{H}^T) = \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T \approx \mathbf{I} + \mathbf{H} + \mathbf{H}^T = \mathbf{I} + 2\boldsymbol{\epsilon}. \quad (3.161)$$

As can be seen, to the same approximation, we will get $\mathbf{C} \approx \mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\epsilon}$.

To calculate the inverse of the deformation gradient $\mathbf{F}$, one may note that

$$(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}) = \mathbf{I} + \mathbf{H} - \mathbf{H} - \mathbf{H}^2 = \mathbf{I} - \mathbf{H}^2.$$

Therefore, in the case of small displacement gradient $\mathbf{H}$ we have $(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}) \approx \mathbf{I}$, which indicates that

$$\mathbf{F}^{-1} \approx \mathbf{I} - \mathbf{H}. \quad (3.162)$$

From this we can conclude that

$$\mathbf{C}^{-1} = (\mathbf{F}^T \mathbf{F})^{-1} = \mathbf{F}^{-1} \mathbf{F}^{-T} \approx (\mathbf{I} - \mathbf{H})(\mathbf{I} - \mathbf{H}^T) = \mathbf{I} - \mathbf{H} - \mathbf{H}^T + \mathbf{H}\mathbf{H}^T \approx \mathbf{I} - 2\boldsymbol{\epsilon}.$$

In a similar way one can show

$$\mathbf{B}^{-1} \approx \mathbf{C}^{-1} \approx \mathbf{I} - 2\boldsymbol{\epsilon}. \quad (3.163)$$

Therefore, the Almansi strain tensor is given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) \approx \boldsymbol{\epsilon}. \quad (3.164)$$

As can be seen, for small deformations we have

$$\mathbf{A} \approx \mathbf{E} \approx \boldsymbol{\epsilon}. \quad (3.165)$$

Section 3.9-Problems:

3.9-6: For infinitesimal deformations calculate the normal (axial) strain ϵ_n and the shear strains γ_1 and γ_2 on a surface with initial normal $\hat{\mathbf{n}}$ and tangents $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$. Start by assuming $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$, and $d\mathbf{X}^{(3)}$ are vectors in the reference configuration along the directions $\hat{\mathbf{n}}$, $\hat{\mathbf{t}}_1$, and $\hat{\mathbf{t}}_2$. Assume that dS_1 , dS_2 , and dS_3 are the lengths of $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$, and $d\mathbf{X}^{(3)}$, and that ds_1 , ds_2 , and ds_3 are the length of the corresponding vectors $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$, $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$, and $d\mathbf{x}^{(3)} = \mathbf{F}d\mathbf{X}^{(3)}$ in the current configuration. Using the relation

$$ds_1^2 = d\mathbf{X}^{(1)} \circ (\mathbf{C}d\mathbf{X}^{(1)}),$$

and $\mathbf{C} \approx \mathbf{I} + 2\boldsymbol{\epsilon}$, show that

$$\epsilon_n = \frac{ds_1 - dS_1}{dS_1} = \sqrt{1 + 2\hat{\mathbf{n}} \circ (\boldsymbol{\epsilon}\hat{\mathbf{n}})} - 1.$$

Knowing that for small x we know that

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x,$$

show that

$$\boxed{\epsilon_n \equiv \hat{\mathbf{n}} \circ (\boldsymbol{\epsilon}\hat{\mathbf{n}}).} \quad (3.166)$$

Assuming that γ_1 is the total angle change, in radians, between the vectors that were along $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, show that

$$\boxed{\gamma_1 \equiv 2 \hat{\mathbf{t}}_1 \circ (\boldsymbol{\epsilon}\hat{\mathbf{n}}),} \quad (3.167)$$

and if γ_2 is the total angle change between the vectors that were along $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(3)}$, show that

$$\boxed{\gamma_2 \equiv 2 \hat{\mathbf{t}}_2 \circ (\boldsymbol{\epsilon}\hat{\mathbf{n}}).} \quad (3.168)$$

Compare this to the components ϵ_{ij}^* of the infinitesimal strain in an orthonormal base $\hat{\mathbf{e}}_i^*$ given by

$$\boxed{\epsilon_{ij}^* = \hat{\mathbf{e}}_i^* \circ (\boldsymbol{\epsilon}\hat{\mathbf{e}}_j^*).} \quad (3.169)$$

3.9-7: Show that the Poisson's ratio, Poisson's function, and the incremental Poisson's ratio defined for an isotropic material under uniaxial deformation result in the same approximate value under infinitesimal deformations.

3.9.2 Small stretch and large rigid body rotation

In the last section, we considered absolutely small variations of the deformation gradient from the identity tensor $\mathbf{I}$. Let us now allow finite-sized rigid body motions, yet still limit the stretch to small quantities. From polar decomposition, we have $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite. To set up the process, let us define *symmetric* tensors $\mathbf{H}_U$ and $\mathbf{H}_V$ by

$$\boxed{\mathbf{U} = \mathbf{I} + \mathbf{H}_U,} \quad \boxed{\mathbf{V} = \mathbf{I} + \mathbf{H}_V.} \quad (3.170)$$

The assumption of small stretch will be interpreted as having $\|\mathbf{H}_U\| \ll 1$. It easily can be shown that if $\|\mathbf{H}_U\| \ll 1$, then one also will have $\|\mathbf{H}_V\| \ll 1$. Since we allow arbitrary rigid body motions, $\mathbf{R}$ may be any orthogonal tensor. In this case, the right Cauchy stretch tensor will be

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T \mathbf{F} = [\mathbf{R}(\mathbf{I} + \mathbf{H}_U)]^T \mathbf{R}(\mathbf{I} + \mathbf{H}_U) = (\mathbf{I} + \mathbf{H}_U) \mathbf{R}^T \mathbf{R} (\mathbf{I} + \mathbf{H}_U) \\ &= \mathbf{I} + 2\mathbf{H}_U + \mathbf{H}_U^2 \approx \mathbf{I} + 2\mathbf{H}_U,\end{aligned}$$

so that

$$\boxed{\mathbf{C} = \mathbf{I} + 2\boldsymbol{\epsilon}_U}, \quad (3.171)$$

where $\boldsymbol{\epsilon}_U$ is the infinitesimal strain tensor defined for symmetric tensor $\mathbf{U}$ by

$$\boxed{\boldsymbol{\epsilon}_U \equiv \frac{1}{2}(\mathbf{H}_U + \mathbf{H}_U^T) = \mathbf{H}_U}. \quad (3.172)$$

The Green strain tensor $\mathbf{E}$ is then given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(2\mathbf{H}_U + \mathbf{H}_U^2) \approx \mathbf{H}_U = \boldsymbol{\epsilon}_U, \quad (3.173)$$

so that

$$\boxed{\mathbf{E} = \boldsymbol{\epsilon}_U}. \quad (3.174)$$

The left Cauchy stretch tensor $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{I} + \mathbf{H}_V) \mathbf{R} \mathbf{R}^T (\mathbf{I} + \mathbf{H}_V)^T = \mathbf{I} + 2\mathbf{H}_V + \mathbf{H}_V^2 \approx \mathbf{I} + 2\mathbf{H}_V = \mathbf{I} + 2\boldsymbol{\epsilon}_V,$$

so that

$$\boxed{\mathbf{B} = \mathbf{I} + 2\boldsymbol{\epsilon}_V}, \quad (3.175)$$

where $\boldsymbol{\epsilon}_V$ is the infinitesimal strain tensor defined for the symmetric tensor $\mathbf{V}$ by

$$\boxed{\boldsymbol{\epsilon}_V \equiv \frac{1}{2}(\mathbf{H}_V + \mathbf{H}_V^T) = \mathbf{H}_V}. \quad (3.176)$$

Since $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$ it is easy to show that $\mathbf{H}_V = \mathbf{R}\mathbf{H}_U\mathbf{R}^T$ so that

$$\boxed{\boldsymbol{\epsilon}_V = \mathbf{R}\boldsymbol{\epsilon}_U\mathbf{R}^T}. \quad (3.177)$$

To calculate the inverse of the deformation gradient $\mathbf{F}$ one may note that $\mathbf{F}^{-1} = (\mathbf{R}\mathbf{U})^{-1} = \mathbf{U}^{-1}\mathbf{R}^T$. Since

$$(\mathbf{I} + \mathbf{H}_U)(\mathbf{I} - \mathbf{H}_U) = \mathbf{I} + \mathbf{H}_U - \mathbf{H}_U - \mathbf{H}_U^2 = \mathbf{I} - \mathbf{H}_U^2,$$

it follows that for small $\mathbf{H}_U$ we have $\mathbf{U}^{-1} \approx \mathbf{I} - \mathbf{H}_U$, and therefore

$$\boxed{\mathbf{F}^{-1} \approx (\mathbf{I} - \mathbf{H}_U) \mathbf{R}^T \approx \mathbf{R}^T (\mathbf{I} - \mathbf{H}_V)}. \quad (3.178)$$

The last relation on the right can be shown by a similar procedure. From this, we can conclude that

$$\mathbf{C}^{-1} = (\mathbf{F}^T \mathbf{F})^{-1} = \mathbf{F}^{-1} \mathbf{F}^{-T} \approx (\mathbf{I} - \mathbf{H}_U) \mathbf{R}^T \mathbf{R} (\mathbf{I} - \mathbf{H}_U) = \mathbf{I} - 2\mathbf{H}_U + \mathbf{H}_U^2 \approx \mathbf{I} - 2\boldsymbol{\epsilon}_U.$$

In a similar way, one can show

$$\boxed{\mathbf{B}^{-1} \approx \mathbf{I} - 2\boldsymbol{\epsilon}_V}. \quad (3.179)$$

Therefore, the Almansi strain tensor is given by

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) \approx \epsilon_V. \quad (3.180)$$

As can be seen, we have $\mathbf{A} \approx \mathbf{RER}^T$.

Section 3.9-Problems:

- 3.9-8:** Define $\mathbf{H} = \mathbf{Grad}(\mathbf{u})$ and $\bar{\mathbf{H}} = \mathbf{grad}(\mathbf{u})$, where $\mathbf{Grad}(\mathbf{u})$ is the displacement gradient with respect to changes of position in the reference configuration and $\mathbf{grad}(\mathbf{u})$ is the displacement gradient relative to changes of position in the current configuration so that $d\mathbf{u} = \mathbf{H}d\mathbf{X} = \bar{\mathbf{H}}d\mathbf{x}$. Show that $\bar{\mathbf{H}} = \mathbf{H}\mathbf{F}^{-1}$, and from this conclude that for small $\mathbf{H}$ we have $\bar{\mathbf{H}} \approx \mathbf{H}$.
- 3.9-9:** Given the polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ and defining $\mathbf{U} = \mathbf{I} + \mathbf{H}_U$ and $\mathbf{V} = \mathbf{I} + \mathbf{H}_V$, show that if $\|\mathbf{H}_U\| \ll 1$, then $\|\mathbf{H}_V\| \ll 1$.

3.10 Notation

The following table shows the symbols used in the text and their definitions.

“.”		material time derivative
$\nabla_{\mathbf{x}}(\theta)$		del (nabla) operator with respect to $\mathbf{x}$ [$d\theta = d\mathbf{x} \circ \nabla_{\mathbf{x}}(\theta)$] $\nabla_{\mathbf{x}}(\theta) = \mathbf{grad}(\theta)$
$\nabla_{\mathbf{X}}(\theta)$		del (nabla) operator with respect to $\mathbf{X}$ [$d\theta = d\mathbf{X} \circ \nabla_{\mathbf{X}}(\theta)$] $\nabla_{\mathbf{X}}(\theta) = \mathbf{Grad}(\theta)$
$\nabla_{\mathbf{x}}(\mathbf{u})$		del (nabla) operator with respect to $\mathbf{x}$ [$d\mathbf{u} = d\mathbf{x} \nabla_{\mathbf{x}}(\mathbf{u})$] $\nabla_{\mathbf{x}}(\mathbf{u}) = [\mathbf{grad}(\mathbf{u})]^T$
$\nabla_{\mathbf{X}}(\mathbf{u})$		del (nabla) operator with respect to $\mathbf{X}$ [$d\mathbf{u} = d\mathbf{X} \nabla_{\mathbf{X}}(\mathbf{u})$] $\nabla_{\mathbf{X}}(\mathbf{u}) = [\mathbf{Grad}(\mathbf{u})]^T$
$\nabla_{\mathbf{x}}(\mathbf{F})$		del (nabla) operator with respect to $\mathbf{x}$ [$d\mathbf{F} = d\mathbf{x} \nabla_{\mathbf{x}}(\mathbf{F})$] $\nabla_{\mathbf{x}}(\mathbf{F}) = [\mathbf{grad}(\mathbf{F}^T)]^T$
$\nabla_{\mathbf{X}}(\mathbf{F})$		del (nabla) operator with respect to $\mathbf{X}$ [$d\mathbf{F} = d\mathbf{X} \nabla_{\mathbf{X}}(\mathbf{F})$] $\nabla_{\mathbf{X}}(\mathbf{F}) = [\mathbf{Grad}(\mathbf{F}^T)]^T$
<i>div</i>		divergence with respect to $\mathbf{x}$
<i>Div</i>		divergence with respect to $\mathbf{X}$
$\mathbf{a}$		acceleration
$\mathbf{a}_P$		acceleration of point P
$\mathbf{A}$		Almansi or Eulerian strain $[\frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})]$
$\mathbf{B}$		left Cauchy stretch tensor $(\mathbf{F}\mathbf{F}^T)$
$\mathbf{C}$		right Cauchy stretch tensor $(\mathbf{F}^T\mathbf{F})$
$\mathbf{D}$		rate of deformation tensor $[\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)]$
$\mathbf{e}_i$		base vectors for curvilinear coordinate α_i
$\mathbf{e}_i^*$		base vectors for curvilinear coordinate β_i
$\mathbf{E}$		Green-Lagrange strain tensor $[\frac{1}{2}(\mathbf{C} - \mathbf{I})]$
$\mathbf{E}_L$		left Green-Lagrange strain tensor $[\frac{1}{2}(\mathbf{B} - \mathbf{I})]$
$\mathbf{E}_R$		right Green-Lagrange strain tensor $[\frac{1}{2}(\mathbf{C} - \mathbf{I})]$
$\mathbf{F}$		deformation gradient [$d\mathbf{x} = \mathbf{F}d\mathbf{X}$; $\mathbf{F} = \mathbf{Grad}(\mathbf{x})$]

$\mathbf{F}_s(t)$		deformation gradient of $\kappa(t)$ with respect to $\kappa(s)$ $d\mathbf{x}(t) = \mathbf{F}_s(t)d\mathbf{x}(s)$
$\mathbf{grad}(\theta)$		gradient with respect to $\mathbf{x}$ [$d\theta = \mathbf{grad}(\theta)d\mathbf{x}$; $\mathbf{grad}(\theta) = \nabla_{\mathbf{x}}(\theta)$]
$\mathbf{Grad}(\theta)$		gradient with respect to $\mathbf{X}$ [$d\theta = \mathbf{Grad}(\theta)d\mathbf{X}$; $\mathbf{Grad}(\theta) = \nabla_{\mathbf{X}}(\theta)$]
$\mathbf{grad}(\mathbf{u})$		gradient with respect to $\mathbf{x}$ [$d\mathbf{u} = \mathbf{grad}(\mathbf{u})d\mathbf{x}$] $\mathbf{grad}(\mathbf{u}) = [\nabla_{\mathbf{x}}(\mathbf{u})]^T$
$\mathbf{Grad}(\mathbf{u})$		gradient with respect to $\mathbf{X}$ [$d\mathbf{u} = \mathbf{Grad}(\mathbf{u})d\mathbf{X}$] $\mathbf{Grad}(\mathbf{u}) = [\nabla_{\mathbf{X}}(\mathbf{u})]^T$
$\mathbf{grad}(\mathbf{F})$		$\mathbf{grad}(\mathbf{F}) = [\nabla_{\mathbf{x}}(\mathbf{F}^T)]^T$
$\mathbf{Grad}(\mathbf{F})$		$\mathbf{Grad}(\mathbf{F}) = [\nabla_{\mathbf{X}}(\mathbf{F}^T)]^T$
g_{ij}		metric tensor associated with $\mathbf{e}_i$ ($\mathbf{e}_i \circ \mathbf{e}_j$)
g_{ij}^*		metric tensor associated with $\mathbf{e}_i^*$ ($\mathbf{e}_i^* \circ \mathbf{e}_j^*$)
$\mathbf{g}$		temperature gradient ($\mathbf{grad}(\theta)$)
$\mathbf{G}$		temperature gradient ($\mathbf{Grad}(\theta)$)
$\mathbf{H}$		displacement gradient ($\mathbf{Grad}(\mathbf{u})$)
$\mathbf{I}$		identity tensor
J		volume ratio [$det(\mathbf{F})$]
$\mathbf{L}$		velocity gradient tensor [$d\mathbf{v} = \mathbf{L}d\mathbf{x}$, $\mathbf{L} = \mathbf{grad}(\mathbf{v})$]
$\hat{\mathbf{n}}$		unit normal to surface
t		time
$\hat{\mathbf{t}}_i$		unit tangents to surface
$\mathbf{R}$		orthogonal factor in the polar decomposition of $\mathbf{F}$ $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, $\mathbf{U} = \mathbf{U}^T$, $\mathbf{V} = \mathbf{V}^T$
$\mathbf{u}$		displacement vector ($\mathbf{x} = \mathbf{X} + \mathbf{u}$)
$\mathbf{U}$		right symmetric factor in the polar decomposition of $\mathbf{F}$ (see $\mathbf{R}$)
$\mathbf{v}$		velocity
$\mathbf{v}^*$		velocity of discontinuity (jump surface) velocity of total motion in perturbation ($\mathbf{v}^* = \mathbf{v} + \delta\mathbf{v}$)
$\mathbf{v}(\mathbf{x}, t)$		velocity described in terms of Eulerian (spatial) variables
$\mathbf{v}(\mathbf{X}, t)$		velocity described in terms of Lagrangian (material) variables
$\mathbf{v}_P$		velocity of point P
$\mathbf{V}$		left symmetric factor in the polar decomposition of $\mathbf{F}$ (see $\mathbf{R}$)
$\mathbf{W}$		spin or vorticity tensor [$\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$]
$\mathbf{x}$		position in current configuration κ
$\mathbf{x}(\mathbf{X}, t)$		motion function
$\mathbf{X}$		position in reference configuration κ_o
α_i		coordinates used to describe points in κ_o
β_i		coordinates used to describe points in κ
$\beta_i(\alpha_j, t)$		current coordinate at time t of the particle at coordinate α_j in κ_o
ϵ		infinitesimal strain [$\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$]
ϵ_U		infinitesimal strain of $\mathbf{U}$ ($\mathbf{H}_U$)
ϵ_V		infinitesimal strain of $\mathbf{V}$ ($\mathbf{H}_V$)
κ		current configuration
κ_o		reference configuration
λ		stretch ratio
γ		shear strain
θ		temperature
$\theta(\mathbf{x}, t)$		Eulerian temperature function
$\theta(\mathbf{X}, t)$		Lagrangian temperature function

Chapter 4

Elastic, plastic, and thermal deformation

In this chapter, we look at the separation of the deformation into elastic, plastic, and thermal parts. In particular, we look at the deformation gradient and how it is multiplicatively decomposed into an elastic, a plastic, and a thermal deformation gradient. The physical meaning of the decomposition is different based on what range of loading and type of model is used. In a very loose way, one might associate with the elastic, thermal, and plastic parts, respectively, reversible mechanical, reversible thermal, and permanent deformations. These interpretations can change based on the model used and the conditions of the loading. In this chapter, we will focus only on the kinematics of such a decomposition. For the most part, we will not focus on assigning meaning to this decomposition.

We begin the chapter focusing on the pure mechanical case where the thermal deformation gradient is left out. From the deformation gradients, we construct strains and velocity gradients and then we relate these quantities. The idea of small deformations is considered next, showing that the infinitesimal counterparts of the elastic and plastic strains add to give the total strain. We also look at the case of small distortions but allowing large rigid body motions. Finally, we introduce the thermal deformation gradient to the decomposition.

4.1 Elastic and plastic deformation gradients

The separation of the deformation into elastic and plastic parts is shown in Figure 4.1. This separation is based on considering the deformation gradient $\mathbf{F}$ being composed of two parts

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \quad (4.1)$$

where $\mathbf{F}^e$ is the elastic deformation gradient and $\mathbf{F}^p$ is the plastic deformation gradient. The idea motivating this particular decomposition of the deformation gradient can be found in our description of the uniaxial response. At any stage of loading, it is assumed that the neighborhood of each material point can be unloaded (at least theoretically, if not always practically), leaving a stress-free configuration. The deformation gradient describing comparison of this intermediate and fictitious stress-free configuration to the reference configuration is taken as $\mathbf{F}^p$. As shown in Figure 4.1, the deformation gradient comparing the current configuration to the intermediate stress-free configuration is taken to be $\mathbf{F}^e$. Obviously, this definition has the characteristic of assigning information about the permanent deformation to $\mathbf{F}^p$, while the portion that can be recovered by elastic unloading is assigned to $\mathbf{F}^e$. This interpretation, even though convenient for understanding the initial motivation for introducing $\mathbf{F}^e$ and $\mathbf{F}^p$, does not in general hold.

There are several issues that need to be considered and clarified here. The intermediate configuration is not unique since any rigid body rotation of it will result in another configuration which is stress free, a potential alternate candidate for an intermediate configuration. This creates no practical difficulty if a proper method is prescribed to select only one of

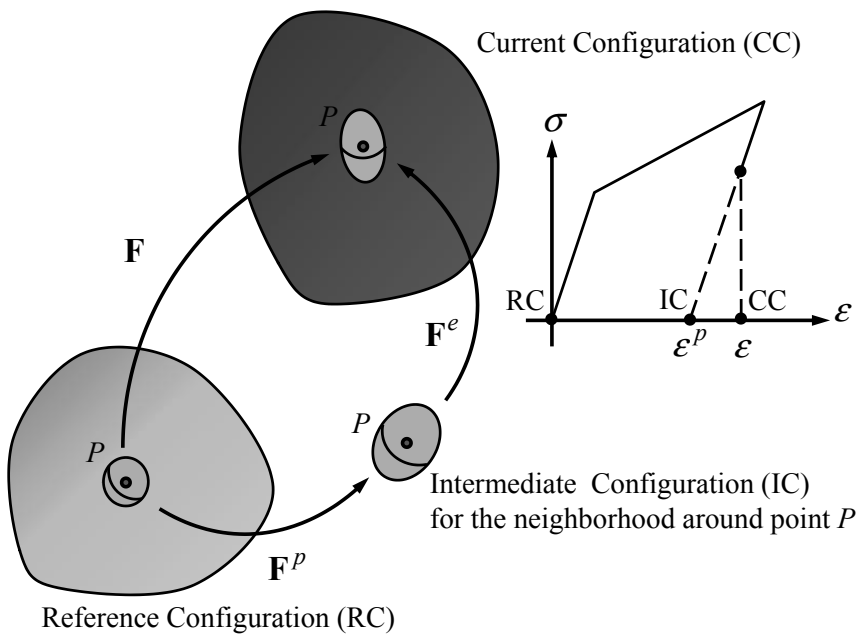


Figure 4.1: Decomposition of the deformation gradient $\mathbf{F}$ into a plastic deformation gradient $\mathbf{F}^p$ and an elastic deformation gradient $\mathbf{F}^e$.

these infinite choices, so as to have a uniquely defined $\mathbf{F}^p$. Also, it is important that this selection not conflict with any conditions that may be put on $\mathbf{F}^e$, since ultimately any deformation gradient $\mathbf{F}$ needs to be obtainable from the relation $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$. One method to solve this problem is simply not to impose any conditions on $\mathbf{F}^e$, defining it solely by the relation $\mathbf{F}^e = \mathbf{F} \mathbf{F}^{p-1}$. Even though this is a convenient method of defining $\mathbf{F}^e$, it is not universally accepted. An alternate idea that has been used substantially is to unload from the current configuration with a symmetric elastic deformation gradient $\mathbf{F}^e$. In such cases we take $\mathbf{F}^e = \mathbf{U}^e = \mathbf{V}^e$ and $\mathbf{R}^e = \mathbf{I}$, where $\mathbf{V}^e$ and $\mathbf{U}^e$ are symmetric positive-definite, as described by polar decomposition. This decomposition also has merit, as will be seen in what follows, yet in this case the plastic deformation gradient $\mathbf{F}^p$ must change with rigid body rotations to satisfy the relation $\mathbf{F} = \mathbf{V}^e \mathbf{F}^p$. This can be a disadvantage since we cannot assume plastic flow to be constant during rigid body motions, yet this feature can be accommodated. To avoid any unnecessary assumptions, *we will assume both the elastic and plastic deformation gradients may be nonsymmetric*, therefore having a portion that is attributed to rigid body rotation and a portion that is attributed to deformation. In general we can use polar decomposition to write

$$\boxed{\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e}, \quad \boxed{\mathbf{F}^p = \mathbf{R}^p \mathbf{U}^p = \mathbf{V}^p \mathbf{R}^p}. \quad (4.2)$$

Another issue to be considered here is the fact that one cannot necessarily put together the intermediate configurations for the neighborhood of each point in the body to get a continuous configuration of the entire body. The problem is that the intermediate configuration constructed for the neighborhood of one point might not fit properly into the intermediate configuration constructed for the neighborhood of an adjacent point. Figure 4.2 shows a

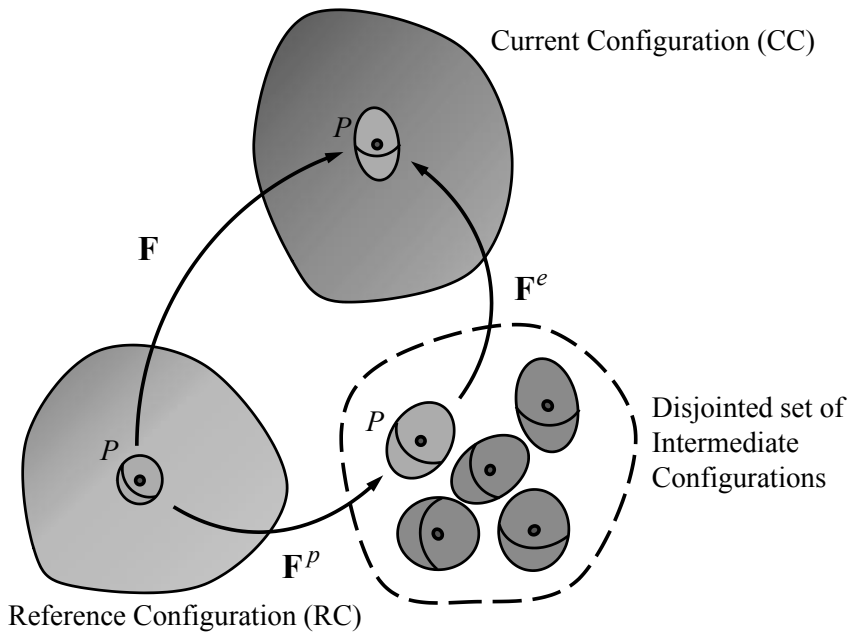


Figure 4.2: The intermediate configuration is not a continuous configuration.

schematic of this. Therefore, there might not be a single intermediate configuration for the entire body, which would mean that it *would not be meaningful* for $\mathbf{F}^p$ to be obtained from the gradient of a continuous displacement field.

4.2 Elastic and plastic strains

We normally calculate a strain measure from the deformation gradient. The left and right Cauchy stretch tensors and the Green and Almansi strains are the most commonly used of these measures of distortion. Let us proceed and calculate each of these strains. For each of the elastic and plastic deformation gradients we can define the following strains

$$\begin{array}{l} \mathbf{C}^e = \mathbf{F}^{eT} \mathbf{F}^e, \quad \mathbf{B}^e = \mathbf{F}^e \mathbf{F}^{eT}, \quad \mathbf{E}^e = \frac{1}{2}(\mathbf{C}^e - \mathbf{I}), \quad \mathbf{A}^e = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{e-1}), \\ \mathbf{C}^p = \mathbf{F}^{pT} \mathbf{F}^p, \quad \mathbf{B}^p = \mathbf{F}^p \mathbf{F}^{pT}, \quad \mathbf{E}^p = \frac{1}{2}(\mathbf{C}^p - \mathbf{I}), \quad \mathbf{A}^p = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{p-1}). \end{array} \quad (4.3)$$

To find the relation between these strains, let us first look at the right Cauchy stretch tensor. For the total deformation, we can see that we have the relation

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T \mathbf{F} = (\mathbf{F}^e \mathbf{F}^p)^T \mathbf{F}^e \mathbf{F}^p = \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{F}^e \mathbf{F}^p \\ &= \mathbf{F}^{pT} \mathbf{C}^e \mathbf{F}^p. \end{aligned} \quad (4.4)$$

It follows that

$$\mathbf{C}^e = \mathbf{F}^{p-T} \mathbf{C} \mathbf{F}^{p-1}. \quad (4.5)$$

For the plastic deformation gradient, we can show that

$$\begin{aligned}\mathbf{C}^p &= \mathbf{F}^{pT} \mathbf{F}^p = (\mathbf{F}^{e-1} \mathbf{F})^T \mathbf{F}^{e-1} \mathbf{F} = \mathbf{F}^T \mathbf{F}^{e-T} \mathbf{F}^{e-1} \mathbf{F} = \mathbf{F}^T (\mathbf{F}^e \mathbf{F}^{eT})^{-1} \mathbf{F} \\ &= \mathbf{F}^T \mathbf{B}^{e-1} \mathbf{F}.\end{aligned}\quad (4.6)$$

The similar three relations for the left Cauchy stretch tensors can be calculated as follows:

$$\begin{aligned}\mathbf{B} &= \mathbf{F} \mathbf{F}^T = \mathbf{F}^e \mathbf{F}^p (\mathbf{F}^e \mathbf{F}^p)^T = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^{pT} \mathbf{F}^{eT} \\ &= \mathbf{F}^e \mathbf{B}^p \mathbf{F}^{eT},\end{aligned}\quad (4.7)$$

$$\begin{aligned}\mathbf{B}^e &= \mathbf{F}^e \mathbf{F}^{eT} = \mathbf{F} \mathbf{F}^{p-1} (\mathbf{F} \mathbf{F}^{p-1})^T = \mathbf{F} \mathbf{F}^{p-1} \mathbf{F}^{p-T} \mathbf{F}^T \\ &= \mathbf{F} \mathbf{C}^{p-1} \mathbf{F}^T,\end{aligned}\quad (4.8)$$

$$\begin{aligned}\mathbf{B}^p &= \mathbf{F}^p \mathbf{F}^{pT} = \mathbf{F}^{e-1} \mathbf{F} (\mathbf{F}^{e-1} \mathbf{F})^T = \mathbf{F}^{e-1} \mathbf{F} \mathbf{F}^T \mathbf{F}^{e-T} \\ &= \mathbf{F}^{e-1} \mathbf{B} \mathbf{F}^{e-T}.\end{aligned}\quad (4.9)$$

For the Green strain, the three similar expressions are

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^{pT} \mathbf{C}^e \mathbf{F}^p - \mathbf{I}) = \frac{1}{2} \mathbf{F}^{pT} (\mathbf{C}^e - \mathbf{B}^{p-1}) \mathbf{F}^p \\ &= \mathbf{F}^{pT} (\mathbf{E}^e + \mathbf{A}^p) \mathbf{F}^p,\end{aligned}\quad (4.10)$$

$$\begin{aligned}\mathbf{E}^e &= \frac{1}{2}(\mathbf{C}^e - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^{p-T} \mathbf{C} \mathbf{F}^{p-1} - \mathbf{I}) = \frac{1}{2} \mathbf{F}^{p-T} (\mathbf{C} - \mathbf{C}^p) \mathbf{F}^{p-1} \\ &= \mathbf{F}^{p-T} (\mathbf{E} - \mathbf{E}^p) \mathbf{F}^{p-1},\end{aligned}\quad (4.11)$$

$$\begin{aligned}\mathbf{E}^p &= \frac{1}{2}(\mathbf{C}^p - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T \mathbf{B}^{e-1} \mathbf{F} - \mathbf{I}) = \frac{1}{2} \mathbf{F}^T (\mathbf{B}^{e-1} - \mathbf{B}^{-1}) \mathbf{F} \\ &= \mathbf{F}^T (\mathbf{A} - \mathbf{A}^e) \mathbf{F}.\end{aligned}\quad (4.12)$$

For the Almansi strain, these three expressions are

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{e-T} \mathbf{B}^{p-1} \mathbf{F}^{e-1}) = \frac{1}{2} \mathbf{F}^{e-T} (\mathbf{C}^e - \mathbf{B}^{p-1}) \mathbf{F}^{e-1} \\ &= \mathbf{F}^{e-T} (\mathbf{E}^e + \mathbf{A}^p) \mathbf{F}^{e-1},\end{aligned}\quad (4.13)$$

$$\begin{aligned}\mathbf{A}^e &= \frac{1}{2}(\mathbf{I} - \mathbf{B}^{e-1}) = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T} \mathbf{C}^p \mathbf{F}^{-1}) = \frac{1}{2} \mathbf{F}^{-T} (\mathbf{C} - \mathbf{C}^p) \mathbf{F}^{-1} \\ &= \mathbf{F}^{-T} (\mathbf{E} - \mathbf{E}^p) \mathbf{F}^{-1},\end{aligned}\quad (4.14)$$

$$\begin{aligned}\mathbf{A}^p &= \frac{1}{2}(\mathbf{I} - \mathbf{B}^{p-1}) = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{eT} \mathbf{B}^{-1} \mathbf{F}^e) = \frac{1}{2} \mathbf{F}^{eT} (\mathbf{B}^{e-1} - \mathbf{B}^{-1}) \mathbf{F}^e \\ &= \mathbf{F}^{eT} (\mathbf{A} - \mathbf{A}^e) \mathbf{F}^e.\end{aligned}\quad (4.15)$$

We also should recall that the following three standard relations between the Green and Almansi strains hold

$$\mathbf{A} = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1}, \quad \mathbf{A}^e = \mathbf{F}^{e-T} \mathbf{E}^e \mathbf{F}^{e-1}, \quad \mathbf{A}^p = \mathbf{F}^{p-T} \mathbf{E}^p \mathbf{F}^{p-1}.\quad (4.16)$$

4.3 Elastic and plastic velocity gradients

The next set of relations are between the velocity gradients. We will follow this by looking at the time rates of change of some of the strains considered above. First, let us define the

following expressions for the velocity gradient, deformation rate, and spin tensors for the elastic and plastic deformations as

$$\begin{aligned} \mathbf{L}^e &= \dot{\mathbf{F}}^e \mathbf{F}^{e-1}, & \mathbf{D}^e &= \frac{1}{2}(\mathbf{L}^e + \mathbf{L}^{eT}), & \mathbf{W}^e &= \frac{1}{2}(\mathbf{L}^e - \mathbf{L}^{eT}), \\ \mathbf{L}^p &= \dot{\mathbf{F}}^p \mathbf{F}^{p-1}, & \mathbf{D}^p &= \frac{1}{2}(\mathbf{L}^p + \mathbf{L}^{pT}), & \mathbf{W}^p &= \frac{1}{2}(\mathbf{L}^p - \mathbf{L}^{pT}). \end{aligned} \quad (4.17)$$

The material time derivatives of the three deformation gradients can be written as

$$\begin{aligned} \dot{\mathbf{F}} &= \overline{\dot{\mathbf{F}}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p, \\ \dot{\mathbf{F}}^e &= \overline{\dot{\mathbf{F}}^e} = \dot{\mathbf{F}}^e \mathbf{F}^{p-1} + \mathbf{F}^e \overline{\dot{\mathbf{F}}^{p-1}}, \\ \dot{\mathbf{F}}^p &= \overline{\dot{\mathbf{F}}^p} = \overline{\dot{\mathbf{F}}^{e-1}} \mathbf{F} + \mathbf{F}^{e-1} \dot{\mathbf{F}}, \end{aligned}$$

where “ $\overline{\cdot}$ ” indicates a material time derivative of all terms below the overline. From the relation $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$, one gets $\overline{\mathbf{F}\mathbf{F}^{-1}} = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\overline{\dot{\mathbf{F}}^{-1}} = \mathbf{0}$. This results in the three relations

$$\overline{\dot{\mathbf{F}}^{-1}} = -\mathbf{F}^{-1} \dot{\mathbf{F}} \mathbf{F}^{-1}, \quad (4.18)$$

$$\overline{\dot{\mathbf{F}}^{e-1}} = -\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^{e-1}, \quad (4.19)$$

$$\overline{\dot{\mathbf{F}}^{p-1}} = -\mathbf{F}^{p-1} \dot{\mathbf{F}}^p \mathbf{F}^{p-1}. \quad (4.20)$$

Substitution of these expressions into the above relations for the rates of the deformation gradients yields

$$\begin{aligned} \dot{\mathbf{F}}^e &= \overline{\dot{\mathbf{F}}^e} = \dot{\mathbf{F}}^e \mathbf{F}^{p-1} - \mathbf{F}^e \mathbf{F}^{p-1} \dot{\mathbf{F}}^p \mathbf{F}^{p-1} = \dot{\mathbf{F}}^e \mathbf{F}^{p-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1}, \\ \dot{\mathbf{F}}^p &= \overline{\dot{\mathbf{F}}^p} = -\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^{e-1} \mathbf{F} + \mathbf{F}^{e-1} \dot{\mathbf{F}} = -\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^{e-1} \dot{\mathbf{F}}. \end{aligned}$$

From these relations, it follows that

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}}\mathbf{F}^{-1} = (\dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p)(\mathbf{F}^e \mathbf{F}^p)^{-1} = (\dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p) \mathbf{F}^{p-1} \mathbf{F}^{e-1}, \\ \mathbf{L}^e &= \dot{\mathbf{F}}^e \mathbf{F}^{e-1} = (\dot{\mathbf{F}}^e \mathbf{F}^{p-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1})(\mathbf{F}^e \mathbf{F}^{p-1})^{-1} = (\dot{\mathbf{F}}^e \mathbf{F}^{p-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1}) \mathbf{F}^p \mathbf{F}^{-1}, \\ \mathbf{L}^p &= \dot{\mathbf{F}}^p \mathbf{F}^{p-1} = (-\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^{e-1} \dot{\mathbf{F}})(\mathbf{F}^{e-1} \mathbf{F})^{-1} = (-\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^{e-1} \dot{\mathbf{F}}) \mathbf{F}^{-1} \mathbf{F}^e. \end{aligned}$$

Reorganizing these terms and introducing the velocity gradients finally result in the relations

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}}^e \mathbf{F}^{e-1} + \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1} \mathbf{F}^{e-1} \\ &= \mathbf{L}^e + \mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1}, \end{aligned} \quad (4.21)$$

$$\begin{aligned} \mathbf{L}^e &= \dot{\mathbf{F}}^e \mathbf{F}^{-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1} \mathbf{F}^p \mathbf{F}^{-1} = \mathbf{L} - \mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} \\ &= \mathbf{L} - \mathbf{F} \mathbf{F}^{p-1} \mathbf{L}^p \mathbf{F}^p \mathbf{F}^{-1}, \end{aligned} \quad (4.22)$$

$$\begin{aligned} \mathbf{L}^p &= -\mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^{-1} \mathbf{F}^e + \mathbf{F}^{e-1} \dot{\mathbf{F}} \mathbf{F}^{-1} \mathbf{F}^e \\ &= \mathbf{F}^{e-1} (\mathbf{L} - \mathbf{L}^e) \mathbf{F}^e. \end{aligned} \quad (4.23)$$

Obviously, the second and third expressions easily follow from the first without independent calculation of these terms as we have done. The relation between the deformation rate and spin tensors directly follows. The expression for the deformation rate is

$$\begin{aligned} \mathbf{D} &= \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2}(\mathbf{L}^e + \mathbf{L}^{eT}) + \frac{1}{2}(\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} + \mathbf{F}^{e-T} \mathbf{L}^{pT} \mathbf{F}^{eT}) \\ &= \mathbf{D}^e + (\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{sym}, \end{aligned} \quad (4.24)$$

where the reader recalls that the subscript “*sym*” indicates that symmetric part of the tensor. Also, the reader should note that $(\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{sym} \neq \mathbf{F}^e \mathbf{D}^p \mathbf{F}^{e-1}$. In a similar manner, we conclude that

$$\begin{aligned}\mathbf{W} &= \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \frac{1}{2}(\mathbf{L}^e - \mathbf{L}^{eT}) + \frac{1}{2}(\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} - \mathbf{F}^{e-T} \mathbf{L}^{pT} \mathbf{F}^{eT}) \\ &= \mathbf{W}^e + (\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{skew},\end{aligned}\quad (4.25)$$

where the reader recalls that the subscript “*skew*” indicates that skew-symmetric part of the tensor. Also, the reader should note that $(\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{skew} \neq \mathbf{F}^e \mathbf{W}^p \mathbf{F}^{e-1}$. The following expressions are obtained directly from the above relations:

$$\mathbf{D}^e = \mathbf{D} - (\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{sym}, \quad (4.26)$$

$$\mathbf{W}^e = \mathbf{W} - (\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1})_{skew}, \quad (4.27)$$

$$\mathbf{D}^p = [\mathbf{F}^{e-1}(\mathbf{L} - \mathbf{L}^e)\mathbf{F}^e]_{sym}, \quad (4.28)$$

$$\mathbf{W}^p = [\mathbf{F}^{e-1}(\mathbf{L} - \mathbf{L}^e)\mathbf{F}^e]_{skew}. \quad (4.29)$$

The proof of the following relations are left for homework. For the rates of Green strain we have the relations

$$2\dot{\mathbf{E}} = \dot{\mathbf{C}} = \mathbf{F}^{pT}(\dot{\mathbf{C}}^e + \mathbf{C}^e \mathbf{L}^p + \mathbf{L}^{pT} \mathbf{C}^e) \mathbf{F}^p, \quad (4.30)$$

$$2\dot{\mathbf{E}}^e = \dot{\mathbf{C}}^e = \mathbf{F}^{p-T} \dot{\mathbf{C}} \mathbf{F}^{p-1} - (\mathbf{C}^e \mathbf{L}^p + \mathbf{L}^{pT} \mathbf{C}^e), \quad (4.31)$$

$$2\dot{\mathbf{E}}^p = \dot{\mathbf{C}}^p = \mathbf{F}^T [\mathbf{B}^{e-1}(\mathbf{L} - \mathbf{L}^e) + (\mathbf{L}^T - \mathbf{L}^{eT}) \mathbf{B}^{e-1}] \mathbf{F}. \quad (4.32)$$

Another set of relations can be developed for the rate of Almansi strain. These are

$$2\dot{\mathbf{A}} = -\dot{\overline{\mathbf{B}^{-1}}} = \mathbf{L}^T \mathbf{B}^{-1} + \mathbf{B}^{-1} \mathbf{L}, \quad (4.33)$$

$$2\dot{\mathbf{A}}^e = -\dot{\overline{\mathbf{B}^{e-1}}} = \mathbf{L}^{eT} \mathbf{B}^{e-1} + \mathbf{B}^{e-1} \mathbf{L}^e, \quad (4.34)$$

$$2\dot{\mathbf{A}}^p = -\dot{\overline{\mathbf{B}^{p-1}}} = \mathbf{L}^{pT} \mathbf{B}^{p-1} + \mathbf{B}^{p-1} \mathbf{L}^p, \quad (4.35)$$

$$\dot{\mathbf{A}} = \mathbf{F}^{-T} \dot{\mathbf{E}} \mathbf{F}^{-1} - (\mathbf{L}^T \mathbf{A} + \mathbf{A} \mathbf{L}), \quad (4.36)$$

$$\dot{\mathbf{A}}^e = \mathbf{F}^{e-T} \dot{\mathbf{E}}^e \mathbf{F}^{e-1} - (\mathbf{L}^{eT} \mathbf{A}^e + \mathbf{A}^e \mathbf{L}^e), \quad (4.37)$$

$$\dot{\mathbf{A}}^p = \mathbf{F}^{p-T} \dot{\mathbf{E}}^p \mathbf{F}^{p-1} - (\mathbf{L}^{pT} \mathbf{A}^p + \mathbf{A}^p \mathbf{L}^p), \quad (4.38)$$

$$\dot{\mathbf{A}} = \mathbf{F}^{e-T} (\dot{\mathbf{E}}^e + \dot{\mathbf{A}}^p) \mathbf{F}^{e-1} - (\mathbf{L}^{eT} \mathbf{A} + \mathbf{A} \mathbf{L}^e), \quad (4.39)$$

$$\dot{\mathbf{A}}^e = \mathbf{F}^{-T} (\dot{\mathbf{E}} - \dot{\mathbf{E}}^p) \mathbf{F}^{-1} - (\mathbf{L}^T \mathbf{A}^e + \mathbf{A}^e \mathbf{L}), \quad (4.40)$$

$$\dot{\mathbf{A}}^p = \mathbf{F}^{eT} (\dot{\mathbf{A}} - \dot{\mathbf{A}}^e) \mathbf{F}^e + \mathbf{F}^{eT} \mathbf{L}^{eT} \mathbf{F}^{e-T} \mathbf{A}^p + \mathbf{A}^p \mathbf{F}^{e-1} \mathbf{L}^e \mathbf{F}^e. \quad (4.41)$$

Section 4.3-Problems:

- 4.3-1:** Using the relations $\dot{\mathbf{C}} = 2\mathbf{F}^T \mathbf{D} \mathbf{F}$, $\dot{\mathbf{C}}^p = 2\mathbf{F}^{pT} \mathbf{D}^p \mathbf{F}^p$, and $\dot{\mathbf{C}}^e = 2\mathbf{F}^{eT} \mathbf{D}^e \mathbf{F}^e$, show the relations given in equations (4.30)–(4.32).
- 4.3-2:** Show the relations given in equations (4.33)–(4.35).
- 4.3-3:** Show the relations given in equations (4.36)–(4.38).
- 4.3-4:** Show the relations given in equations (4.39)–(4.41).

4.3-5: It seems reasonable to assume that for an isotropic material undergoing uniaxial extension the elastic and plastic deformation gradients are coaxial with the applied deformation. We assume that the uniaxial deformation is defined by

$$\mathbf{F} = \lambda_a \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_t \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_t \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3, \quad (4.42)$$

where λ_a is the axial stretch along the axis $\hat{\mathbf{e}}_1$, and λ_t is the transverse stretch along any line in the perpendicular plane $\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$. In a similar way, we define the elastic and plastic deformation gradients as

$$\mathbf{F}^e = \lambda_a^e \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_t^e \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_t^e \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3, \quad (4.43)$$

$$\mathbf{F}^p = \lambda_a^p \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \lambda_t^p \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \lambda_t^p \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3. \quad (4.44)$$

Show that the axial and transverse components of the velocity gradient in this case give

$$\frac{\dot{\lambda}_a}{\lambda_a} = \frac{\dot{\lambda}_a^e}{\lambda_a^e} + \frac{\dot{\lambda}_a^p}{\lambda_a^p}, \quad (4.45)$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\dot{\lambda}_t^e}{\lambda_t^e} + \frac{\dot{\lambda}_t^p}{\lambda_t^p}. \quad (4.46)$$

Define the associated Poisson's ratios as

$$\boxed{\nu = -\frac{\lambda_a \dot{\lambda}_t}{\lambda_t \dot{\lambda}_a}}, \quad \boxed{\nu^e = -\frac{\lambda_a^e \dot{\lambda}_t^e}{\lambda_t^e \dot{\lambda}_a^e}}, \quad \boxed{\nu^p = -\frac{\lambda_a^p \dot{\lambda}_t^p}{\lambda_t^p \dot{\lambda}_a^p}}. \quad (4.47)$$

Using these show that the axial and transverse components of the velocity gradient can be written as

$$\begin{bmatrix} 1 & 1 \\ \nu^e & \nu^p \end{bmatrix} \begin{Bmatrix} \frac{\dot{\lambda}_a^e}{\lambda_a^e} \\ \frac{\dot{\lambda}_a^p}{\lambda_a^p} \end{Bmatrix} = \frac{\dot{\lambda}_a}{\lambda_a} \begin{Bmatrix} 1 \\ \nu \end{Bmatrix}. \quad (4.48)$$

Use this to solve for the axial components of the elastic and plastic velocity gradients by inverting the coefficient matrix and multiplying it by both sides to get

$$\begin{Bmatrix} \frac{\dot{\lambda}_a^e}{\lambda_a^e} \\ \frac{\dot{\lambda}_a^p}{\lambda_a^p} \end{Bmatrix} = \frac{1}{\nu^e - \nu^p} \frac{\dot{\lambda}_a}{\lambda_a} \begin{bmatrix} -\nu^p & 1 \\ \nu^e & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ \nu \end{Bmatrix}, \quad (4.49)$$

and show that

$$\frac{\dot{\lambda}_a^e}{\lambda_a^e} = \frac{(\nu^p - \nu)}{(\nu^p - \nu^e)} \frac{\dot{\lambda}_a}{\lambda_a}, \quad (4.50)$$

$$\frac{\dot{\lambda}_a^p}{\lambda_a^p} = \frac{(\nu - \nu^e)}{(\nu^p - \nu^e)} \frac{\dot{\lambda}_a}{\lambda_a}. \quad (4.51)$$

For the incompressibility limit, show that $\nu \leq \frac{1}{2}$, $\nu^e \leq \frac{1}{2}$, and $\nu^p \leq \frac{1}{2}$. Study the expected response during extension for different values of the Poisson's ratios. In particular, consider the case when $\nu = \nu^p$, $\nu = \nu^e$, and the case that the plastic flow is incompressible so that $\nu^p = \frac{1}{2}$, and for this case discuss the condition when $\nu^e < \nu < \nu^p = \frac{1}{2}$ and $\nu < \nu^e < \nu^p = \frac{1}{2}$. As practical examples, consider the response of mild steel and of aluminum shown in Figure 4.3. The

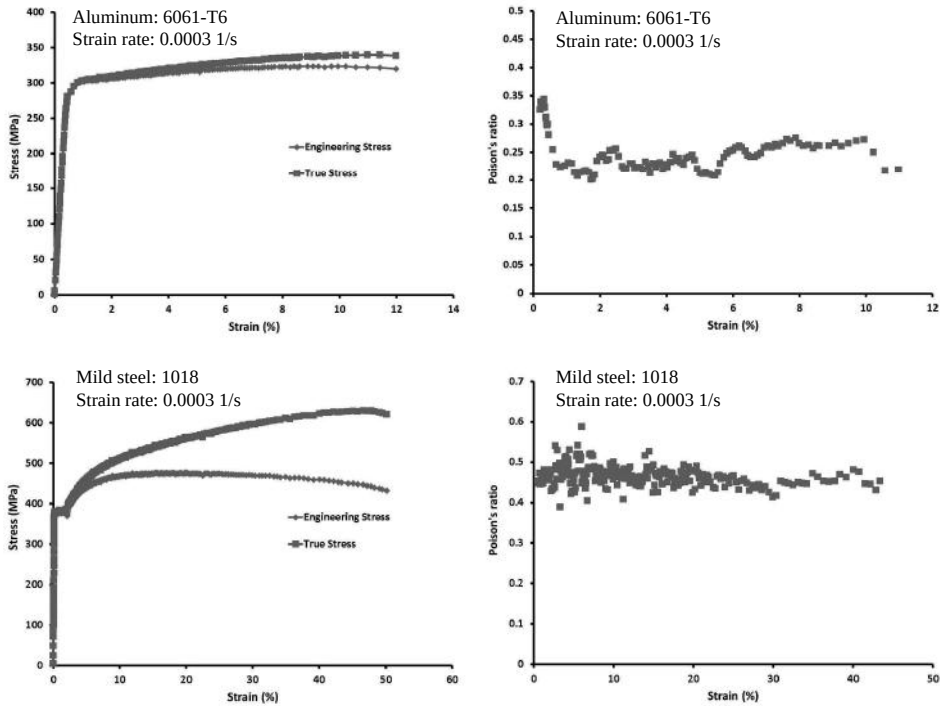


Figure 4.3: The uniaxial extension of 1018 mild steel and 6061-T6511 aluminum.

elastic Poisson's ratio of mild steel is approximately $\nu^e = 0.28$ and aluminum is approximately $\nu^e = 0.33$. Consider the measured Poisson's ratios shown in Figure 4.3 and argue that the incompressibility assumption for the plastic flow for aluminum would result in negative plastic flow as a result of extension, but might be appropriate for mild steel.

4.4 Infinitesimal elastic and plastic deformations

We now will consider the case of both small elastic deformation and small plastic deformation. For this case, let us take

$$\boxed{\mathbf{F}^e = \mathbf{I} + \mathbf{H}^e}, \quad \boxed{\mathbf{F}^p = \mathbf{I} + \mathbf{H}^p}, \quad (4.52)$$

where we shall take both $\mathbf{H}^e$ and $\mathbf{H}^p$ to be small such that

$$\|\mathbf{H}^e\| \ll 1, \text{ and } \|\mathbf{H}^p\| \ll 1. \quad (4.53)$$

That is, we assume the elastic and plastic displacement gradients are both small. The ex-

pression for the deformation gradient now can be approximated as follows:

$$\begin{aligned}\mathbf{F} &= \mathbf{F}^e \mathbf{F}^p = (\mathbf{I} + \mathbf{H}^e)(\mathbf{I} + \mathbf{H}^p) = \mathbf{I} + \mathbf{H}^e + \mathbf{H}^p + \mathbf{H}^e \mathbf{H}^p \\ &\approx \mathbf{I} + \mathbf{H}^e + \mathbf{H}^p.\end{aligned}\quad (4.54)$$

We conclude from $\mathbf{F} = \mathbf{I} + \mathbf{H}$ that the displacement gradient is given by

$$\boxed{\mathbf{H} \approx \mathbf{H}^e + \mathbf{H}^p}, \quad (4.55)$$

which yields the conclusion that $\mathbf{H}$ is also small. The right and left Cauchy stretch tensors are given as

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T \mathbf{F} = (\mathbf{I} + \mathbf{H}^{pT})(\mathbf{I} + \mathbf{H}^{eT})(\mathbf{I} + \mathbf{H}^e)(\mathbf{I} + \mathbf{H}^p) \\ &\approx \mathbf{I} + 2\boldsymbol{\epsilon}^e + 2\boldsymbol{\epsilon}^p,\end{aligned}\quad (4.56)$$

$$\begin{aligned}\mathbf{B} &= \mathbf{F} \mathbf{F}^T = (\mathbf{I} + \mathbf{H}^e)(\mathbf{I} + \mathbf{H}^p)(\mathbf{I} + \mathbf{H}^{pT})(\mathbf{I} + \mathbf{H}^{eT}) \\ &\approx \mathbf{I} + 2\boldsymbol{\epsilon}^e + 2\boldsymbol{\epsilon}^p,\end{aligned}\quad (4.57)$$

where we have used the definitions of infinitesimal elastic strain $\boldsymbol{\epsilon}^e$ and the infinitesimal plastic strain $\boldsymbol{\epsilon}^p$ given by the usual relations:

$$\boxed{\boldsymbol{\epsilon}^e = \frac{1}{2}(\mathbf{H}^e + \mathbf{H}^{eT})}, \quad \boxed{\boldsymbol{\epsilon}^p = \frac{1}{2}(\mathbf{H}^p + \mathbf{H}^{pT})}. \quad (4.58)$$

Comparing this to the relation obtained for small strains $\mathbf{C} \approx \mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\epsilon}$, we conclude that

$$\boxed{\boldsymbol{\epsilon} \approx \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p}. \quad (4.59)$$

It now is easy to show that

$$\boxed{\begin{aligned}\mathbf{E} &\approx \boldsymbol{\epsilon}, & \mathbf{E}^e &\approx \boldsymbol{\epsilon}^e, & \mathbf{E}^p &\approx \boldsymbol{\epsilon}^p, \\ \mathbf{A} &\approx \boldsymbol{\epsilon}, & \mathbf{A}^e &\approx \boldsymbol{\epsilon}^e, & \mathbf{A}^p &\approx \boldsymbol{\epsilon}^p,\end{aligned}}$$

which therefore result in $\mathbf{E} \approx \mathbf{A}$, $\mathbf{E} \approx \mathbf{E}^e + \mathbf{E}^p$ and $\mathbf{A} \approx \mathbf{A}^e + \mathbf{A}^p$.

The relations for the velocity gradients, deformation rates, and spin tensors are given by

$$\begin{aligned}\mathbf{L} &\approx \mathbf{L}^e + \mathbf{L}^p + (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e), \\ \mathbf{L}^e &\approx \mathbf{L} - \mathbf{L}^p - (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e), \\ \mathbf{L}^p &\approx \mathbf{L} - \mathbf{L}^e - [\mathbf{H}^e (\mathbf{L} - \mathbf{L}^e) - (\mathbf{L} - \mathbf{L}^e) \mathbf{H}^e], \\ \mathbf{D} &\approx \mathbf{D}^e + \mathbf{D}^p + (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e)_{sym}, \\ \mathbf{D}^e &\approx \mathbf{D} - \mathbf{D}^p - (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e)_{sym}, \\ \mathbf{D}^p &\approx \mathbf{D} - \mathbf{D}^e - [\mathbf{H}^e (\mathbf{L} - \mathbf{L}^e) - (\mathbf{L} - \mathbf{L}^e) \mathbf{H}^e]_{sym}, \\ \mathbf{W} &\approx \mathbf{W}^e + \mathbf{W}^p + (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e)_{skew}, \\ \mathbf{W}^e &\approx \mathbf{W} - \mathbf{W}^p - (\mathbf{H}^e \mathbf{L}^p - \mathbf{L}^p \mathbf{H}^e)_{skew}, \\ \mathbf{W}^p &\approx \mathbf{W} - \mathbf{W}^e - [\mathbf{H}^e (\mathbf{L} - \mathbf{L}^e) - (\mathbf{L} - \mathbf{L}^e) \mathbf{H}^e]_{skew}.\end{aligned}\quad (4.60)$$

In all the approximations in this section for the rate terms we have looked at the approximation to the order of $\mathbf{H}^e$. If $\mathbf{H}^e$ is very small, one also may ignore the first-order terms to get

$$\boxed{\mathbf{L} \approx \mathbf{L}^e + \mathbf{L}^p}, \quad \boxed{\mathbf{D} \approx \mathbf{D}^e + \mathbf{D}^p}, \quad \boxed{\mathbf{W} \approx \mathbf{W}^e + \mathbf{W}^p}. \quad (4.61)$$

It is easy to see that to this same approximation we have the relation

$$\boxed{\dot{\boldsymbol{\epsilon}} \approx \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p}. \quad (4.62)$$

Section 4.4-Problems:

- 4.4-1:** Prove the approximate relations for the velocity gradients, deformation rates, and spin tensors.

4.5 Large rigid body rotations

In this section, we first look at the general equations for the deformation gradient and the left and right Cauchy stretch tensors using the polar decomposition of the elastic and plastic deformation gradients. We follow this by looking at cases in which the elastic and plastic deformations are small, but include large rigid body motions. We also look at the case that the elastic deformation is small and the plastic deformation is moderate, but not large, with the possibility of large rigid body rotations.

In the case of large rigid body rotations, we will consider the polar decomposition of the associated deformation gradients as

$$\begin{aligned}
 \mathbf{F} &= \mathbf{R}\mathbf{U} = \mathbf{R}(\mathbf{I} + \mathbf{H}_U) \\
 &= \mathbf{V}\mathbf{R} = (\mathbf{I} + \mathbf{H}_V)\mathbf{R}, \\
 \mathbf{F}^e &= \mathbf{R}^e\mathbf{U}^e = \mathbf{R}^e(\mathbf{I} + \mathbf{H}_U^e) \\
 &= \mathbf{V}^e\mathbf{R}^e = (\mathbf{I} + \mathbf{H}_V^e)\mathbf{R}^e, \\
 \mathbf{F}^p &= \mathbf{R}^p\mathbf{U}^p = \mathbf{R}^p(\mathbf{I} + \mathbf{H}_U^p) \\
 &= \mathbf{V}^p\mathbf{R}^p = (\mathbf{I} + \mathbf{H}_V^p)\mathbf{R}^p,
 \end{aligned} \tag{4.63}$$

where $\mathbf{H}_U$, $\mathbf{H}_V = \mathbf{R}\mathbf{H}_U\mathbf{R}^T$, $\mathbf{H}_U^e$, $\mathbf{H}_V^e = \mathbf{R}^e\mathbf{H}_U^e\mathbf{R}^{eT}$, $\mathbf{H}_U^p$, and $\mathbf{H}_V^p = \mathbf{R}^p\mathbf{H}_U^p\mathbf{R}^{pT}$ are all symmetric, but not necessarily small. It follows directly that

$$\begin{aligned}
 \mathbf{C} &= \mathbf{I} + 2\mathbf{H}_U + \mathbf{H}_U^2 = \mathbf{R}^T(\mathbf{I} + 2\mathbf{H}_V + \mathbf{H}_V^2)\mathbf{R}, \\
 \mathbf{B} &= \mathbf{I} + 2\mathbf{H}_V + \mathbf{H}_V^2 = \mathbf{R}(\mathbf{I} + 2\mathbf{H}_U + \mathbf{H}_U^2)\mathbf{R}^T, \\
 \mathbf{C}^e &= \mathbf{I} + 2\mathbf{H}_U^e + \mathbf{H}_U^{e2} = \mathbf{R}^{eT}(\mathbf{I} + 2\mathbf{H}_V^e + \mathbf{H}_V^{e2})\mathbf{R}^e, \\
 \mathbf{B}^e &= \mathbf{I} + 2\mathbf{H}_V^e + \mathbf{H}_V^{e2} = \mathbf{R}^e(\mathbf{I} + 2\mathbf{H}_U^e + \mathbf{H}_U^{e2})\mathbf{R}^{eT}, \\
 \mathbf{C}^p &= \mathbf{I} + 2\mathbf{H}_U^p + \mathbf{H}_U^{p2} = \mathbf{R}^{pT}(\mathbf{I} + 2\mathbf{H}_V^p + \mathbf{H}_V^{p2})\mathbf{R}^p, \\
 \mathbf{B}^p &= \mathbf{I} + 2\mathbf{H}_V^p + \mathbf{H}_V^{p2} = \mathbf{R}^p(\mathbf{I} + 2\mathbf{H}_U^p + \mathbf{H}_U^{p2})\mathbf{R}^{pT}.
 \end{aligned} \tag{4.64}$$

These result in

$$\begin{aligned}
 \mathbf{E}_R &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \mathbf{H}_U + \frac{1}{2}\mathbf{H}_U^2 = \mathbf{R}^T(\mathbf{H}_V + \frac{1}{2}\mathbf{H}_V^2)\mathbf{R}, \\
 \mathbf{E}_L &= \frac{1}{2}(\mathbf{B} - \mathbf{I}) = \mathbf{H}_V + \frac{1}{2}\mathbf{H}_V^2 = \mathbf{R}(\mathbf{H}_U + \frac{1}{2}\mathbf{H}_U^2)\mathbf{R}^T, \\
 \mathbf{E}_R^e &= \frac{1}{2}(\mathbf{C}^e - \mathbf{I}) = \mathbf{H}_U^e + \frac{1}{2}\mathbf{H}_U^{e2} = \mathbf{R}^{eT}(\mathbf{H}_V^e + \frac{1}{2}\mathbf{H}_V^{e2})\mathbf{R}^e, \\
 \mathbf{E}_L^e &= \frac{1}{2}(\mathbf{B}^e - \mathbf{I}) = \mathbf{H}_V^e + \frac{1}{2}\mathbf{H}_V^{e2} = \mathbf{R}^e(\mathbf{H}_U^e + \frac{1}{2}\mathbf{H}_U^{e2})\mathbf{R}^{eT}, \\
 \mathbf{E}_R^p &= \frac{1}{2}(\mathbf{C}^p - \mathbf{I}) = \mathbf{H}_U^p + \frac{1}{2}\mathbf{H}_U^{p2} = \mathbf{R}^{pT}(\mathbf{H}_V^p + \frac{1}{2}\mathbf{H}_V^{p2})\mathbf{R}^p, \\
 \mathbf{E}_L^p &= \frac{1}{2}(\mathbf{B}^p - \mathbf{I}) = \mathbf{H}_V^p + \frac{1}{2}\mathbf{H}_V^{p2} = \mathbf{R}^p(\mathbf{H}_U^p + \frac{1}{2}\mathbf{H}_U^{p2})\mathbf{R}^{pT}.
 \end{aligned} \tag{4.65}$$

It easily can be shown that we get

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p = \mathbf{R}^e(\mathbf{I} + \mathbf{H}_U^e + \mathbf{H}_V^p + \mathbf{H}_U^e\mathbf{H}_V^p)\mathbf{R}^p. \tag{4.66}$$

It directly follows that the right Cauchy stretch tensor is given by

$$\begin{aligned}
 \mathbf{C} = \mathbf{F}^T\mathbf{F} &= \mathbf{R}^{pT}[\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p + 2(\mathbf{H}_U^e\mathbf{H}_V^p + \mathbf{H}_V^p\mathbf{H}_U^e) \\
 &\quad + \mathbf{H}_U^{e2} + \mathbf{H}_V^{p2} + \mathbf{H}_U^{e2}\mathbf{H}_V^p + \mathbf{H}_V^{p2}\mathbf{H}_U^e \\
 &\quad + 2\mathbf{H}_V^p\mathbf{H}_U^e\mathbf{H}_V^p + \mathbf{H}_V^p\mathbf{H}_U^{e2}\mathbf{H}_V^p]\mathbf{R}^p.
 \end{aligned} \tag{4.67}$$

The left Cauchy stretch tensor is given by

$$\begin{aligned}
 \mathbf{B} = \mathbf{F}\mathbf{F}^T &= \mathbf{R}^e[\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p + 2(\mathbf{H}_U^e\mathbf{H}_V^p + \mathbf{H}_V^p\mathbf{H}_U^e) \\
 &\quad + \mathbf{H}_U^{e2} + \mathbf{H}_V^{p2} + \mathbf{H}_U^e\mathbf{H}_V^{p2} + \mathbf{H}_V^p\mathbf{H}_U^{e2} \\
 &\quad + 2\mathbf{H}_U^e\mathbf{H}_V^p\mathbf{H}_U^e + \mathbf{H}_U^e\mathbf{H}_V^{p2}\mathbf{H}_U^e]\mathbf{R}^{eT}.
 \end{aligned} \tag{4.68}$$

In the following sections, we will look at the special cases where some of the stretches are small.

Section 4.5-Problems:

- 4.5-1:** Prove the expressions obtained for the right and left Cauchy stretch tensors given in (4.67) and (4.68).

4.5.1 Small elastic and plastic strain

There are cases in which both the elastic and plastic distortions are small, but which include large rigid body motions. Consider the case where both elastic and plastic stretches are only slightly different from unity so that $\mathbf{H}_U^e$ and $\mathbf{H}_V^p$ are small. That is,

$$\|\mathbf{H}_U^e\| \ll 1, \text{ and } \|\mathbf{H}_V^p\| \ll 1. \quad (4.69)$$

It is easy to show that these also require that $\|\mathbf{H}_V^e\| \ll 1$ and $\|\mathbf{H}_U^p\| \ll 1$.

Eliminating all second-order terms in $\mathbf{H}_U^e$ and $\mathbf{H}_V^p$ from the expressions given above results in the approximate relations:

$$\begin{aligned} \mathbf{E}_R^e &\approx \mathbf{H}_U^e = \mathbf{R}^{eT} \mathbf{H}_V^e \mathbf{R}^e, \\ \mathbf{E}_L^e &\approx \mathbf{H}_V^e = \mathbf{R}^e \mathbf{H}_U^e \mathbf{R}^{eT} = \mathbf{R}^e \mathbf{E}_R^e \mathbf{R}^{eT}, \\ \mathbf{E}_R^p &\approx \mathbf{H}_U^p = \mathbf{R}^{pT} \mathbf{H}_V^p \mathbf{R}^p, \\ \mathbf{E}_L^p &\approx \mathbf{H}_V^p = \mathbf{R}^p \mathbf{H}_U^p \mathbf{R}^{pT} = \mathbf{R}^p \mathbf{E}_R^p \mathbf{R}^{pT}, \\ \mathbf{C} &\approx \mathbf{R}^{pT} (\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p) \mathbf{R}^p = \mathbf{R}^{pT} (\mathbf{I} + 2\mathbf{E}_R^e + 2\mathbf{E}_L^p) \mathbf{R}^p, \\ \mathbf{B} &\approx \mathbf{R}^e (\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p) \mathbf{R}^{eT} = \mathbf{R}^e (\mathbf{I} + 2\mathbf{E}_R^e + 2\mathbf{E}_L^p) \mathbf{R}^{eT}, \\ \mathbf{E}_R &= \frac{1}{2} (\mathbf{C} - \mathbf{I}) \approx \mathbf{R}^{pT} (\mathbf{E}_R^e + \mathbf{E}_L^p) \mathbf{R}^p = \mathbf{R}^{pT} \mathbf{E}_R^e \mathbf{R}^p + \mathbf{E}_L^p, \\ \mathbf{E}_L &= \frac{1}{2} (\mathbf{B} - \mathbf{I}) \approx \mathbf{R}^e (\mathbf{E}_R^e + \mathbf{E}_L^p) \mathbf{R}^{eT} = \mathbf{E}_L^e + \mathbf{R}^e \mathbf{E}_L^p \mathbf{R}^{eT}, \end{aligned} \quad (4.70)$$

which hold for small $\mathbf{H}_U^e$ and $\mathbf{H}_V^p$, but include large $\mathbf{R}^e$ and $\mathbf{R}^p$.

Section 4.5-Problems:

- 4.5-2:** If $\|\mathbf{H}_U^e\| \ll 1$ and $\|\mathbf{H}_V^p\| \ll 1$, show that it follows that $\|\mathbf{H}_V^e\| \ll 1$ and $\|\mathbf{H}_U^p\| \ll 1$.

- 4.5-3:** Prove the approximations for the strain tensors given in equation (4.70).

4.5.2 Small elastic strain and moderate plastic strain

We now will consider the case where the plastic stretch is large, but still small enough that $\|\mathbf{H}_U^p\| < 1$. Yet, we still will assume that $\|\mathbf{H}_U^e\| \ll 1$. We will drop terms of second and higher order in $\mathbf{H}_U^e$, terms of order three and higher in $\mathbf{H}_U^p$, and product terms of order one or higher in $\mathbf{H}_U^e$ and order two or higher in $\mathbf{H}_U^p$. As a rule of thumb, the results should be good for $\|\mathbf{H}_U^p\|$ in the order of 0.1, and $\mathbf{H}_U^e$ in the order of 0.01. Under these assumptions, we will have the relations:

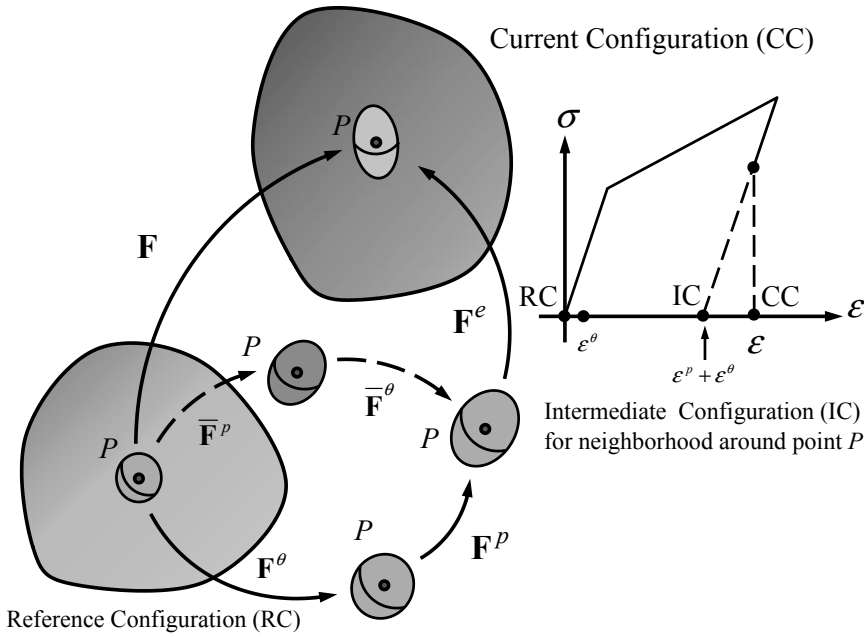


Figure 4.4: Decomposition of the deformation gradient $\mathbf{F}$ into a plastic deformation gradient $\mathbf{F}^p$, an elastic deformation gradient $\mathbf{F}^e$, and a thermal deformation gradient $\mathbf{F}^\theta$.

$$\begin{aligned}
 \mathbf{E}_R^e &\approx \mathbf{H}_U^e = \mathbf{R}^{eT} \mathbf{H}_V^e \mathbf{R}^e, \\
 \mathbf{E}_L^e &\approx \mathbf{H}_V^e = \mathbf{R}^e \mathbf{H}_U^e \mathbf{R}^{eT} = \mathbf{R}^e \mathbf{E}_R^e \mathbf{R}^{eT}, \\
 \mathbf{E}_R^p &= \mathbf{H}_U^p + \frac{1}{2} \mathbf{H}_U^{p2} = \mathbf{R}^{pT} (\mathbf{H}_V^p + \frac{1}{2} \mathbf{H}_V^{p2}) \mathbf{R}^p, \\
 \mathbf{E}_L^p &= \mathbf{H}_V^p + \frac{1}{2} \mathbf{H}_V^{p2} = \mathbf{R}^p (\mathbf{H}_U^p + \frac{1}{2} \mathbf{H}_U^{p2}) \mathbf{R}^{pT}, \\
 \mathbf{C} &\approx \mathbf{R}^{pT} [\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p + \mathbf{H}_V^{p2} + 2(\mathbf{H}_U^e \mathbf{H}_V^p + \mathbf{H}_V^p \mathbf{H}_U^e)] \mathbf{R}^p, \\
 \mathbf{B} &\approx \mathbf{R}^e [\mathbf{I} + 2\mathbf{H}_U^e + 2\mathbf{H}_V^p + \mathbf{H}_V^{p2} + 2(\mathbf{H}_U^e \mathbf{H}_V^p + \mathbf{H}_V^p \mathbf{H}_U^e)] \mathbf{R}^{eT}, \\
 \mathbf{E}_R &\approx \mathbf{R}^{pT} [\mathbf{E}_R^e + \mathbf{E}_L^p + (\mathbf{H}_U^e \mathbf{H}_V^p + \mathbf{H}_V^p \mathbf{H}_U^e)] \mathbf{R}^p, \\
 \mathbf{E}_L &\approx \mathbf{R}^e [\mathbf{E}_R^e + \mathbf{E}_L^p + (\mathbf{H}_U^e \mathbf{H}_V^p + \mathbf{H}_V^p \mathbf{H}_U^e)] \mathbf{R}^{eT}.
 \end{aligned} \tag{4.71}$$

Section 4.5-Problems:

4.5-4: Prove the above approximations given in equation (4.71).

4.6 Thermal deformation and thermal strain

There are several ways to introduce thermal effects kinematically. One way is to directly include the thermal effect into the plastic deformation so that the plastic deformation changes not only with yielding, but also with changes of temperature. If it is desirable to

make the thermal expansion explicit, a thermal deformation gradient can be introduced. Two ways of doing so are shown in Figure 4.4, in which either the thermal expansion is imposed first and then the plastic deformation gradient, or the plastic deformation is imposed first and then the thermal expansion. If $\mathbf{F}^\theta$ and $\overline{\mathbf{F}}^\theta$ represent the thermal deformation gradients for these two selections, we can write the total deformation gradient as

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta = \mathbf{F}^e \overline{\mathbf{F}}^\theta \overline{\mathbf{F}}^p. \quad (4.72)$$

This requires that we have the relation

$$\mathbf{F}^p \mathbf{F}^\theta = \overline{\mathbf{F}}^\theta \overline{\mathbf{F}}^p. \quad (4.73)$$

As will be shown in the chapters on thermoplasticity, viscoelasticity, and viscoplasticity, combining this requirement and the underlying assumptions of each theory provide a unique relation between the two decompositions. For the most part, in the remainder of this book, we will use the form $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$, except for in the chapter on crystal plasticity.

For infinitesimal deformations, we take $\|\mathbf{H}^e\| \ll 1$, $\|\mathbf{H}^p\| \ll 1$, and $\|\mathbf{H}^\theta\| \ll 1$, where $\mathbf{F}^e = \mathbf{I} + \mathbf{H}^e$, $\mathbf{F}^p = \mathbf{I} + \mathbf{H}^p$, $\mathbf{F}^\theta = \mathbf{I} + \mathbf{H}^\theta$. This results in

$$\mathbf{H} \approx \mathbf{H}^e + \mathbf{H}^p + \mathbf{H}^\theta, \quad (4.74)$$

which yields the conclusion that $\mathbf{H}$ is also small. The right and left Cauchy stretch tensors are given as

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T \mathbf{F} = (\mathbf{I} + \mathbf{H}^{\theta T})(\mathbf{I} + \mathbf{H}^{pT})(\mathbf{I} + \mathbf{H}^{eT})(\mathbf{I} + \mathbf{H}^e)(\mathbf{I} + \mathbf{H}^p)(\mathbf{I} + \mathbf{H}^\theta) \\ &\approx \mathbf{I} + 2\boldsymbol{\epsilon}^e + 2\boldsymbol{\epsilon}^p + 2\boldsymbol{\epsilon}^\theta, \end{aligned} \quad (4.75)$$

$$\begin{aligned} \mathbf{B} &= \mathbf{F} \mathbf{F}^T = (\mathbf{I} + \mathbf{H}^e)(\mathbf{I} + \mathbf{H}^p)(\mathbf{I} + \mathbf{H}^\theta)(\mathbf{I} + \mathbf{H}^{\theta T})(\mathbf{I} + \mathbf{H}^{pT})(\mathbf{I} + \mathbf{H}^{eT}) \\ &\approx \mathbf{I} + 2\boldsymbol{\epsilon}^e + 2\boldsymbol{\epsilon}^p + 2\boldsymbol{\epsilon}^\theta, \end{aligned} \quad (4.76)$$

where the infinitesimal thermal strain $\boldsymbol{\epsilon}^\theta$ is defined in the usual way by the relation

$$\boldsymbol{\epsilon}^\theta = \frac{1}{2}(\mathbf{H}^\theta + \mathbf{H}^{\theta T}). \quad (4.77)$$

It also follows that

$$\boldsymbol{\epsilon} \approx \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p + \boldsymbol{\epsilon}^\theta. \quad (4.78)$$

Section 4.6-Problems:

4.6-1: Prove the approximate relations for the right and left Cauchy stretch tensors given in equations (4.75) and (4.76).

4.6-2: Show that for $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$, the velocity gradient is given by

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} + \mathbf{F}^e \mathbf{F}^p \mathbf{L}^\theta \mathbf{F}^{p-1} \mathbf{F}^{e-1}, \quad (4.79)$$

and conclude that

$$tr(\mathbf{L}) = tr(\mathbf{L}^e) + tr(\mathbf{L}^p) + tr(\mathbf{L}^\theta). \quad (4.80)$$

4.6-3: Using the results of the previous problem show that

$$\frac{\dot{J}}{J} = \frac{\dot{J}^e}{J^e} + \frac{\dot{J}^p}{J^p} + \frac{\dot{J}^\theta}{J^\theta}. \quad (4.81)$$

4.6-4: Show that for $\mathbf{F} = \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$, the velocity gradient is given by

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e \bar{\mathbf{L}}^\theta \mathbf{F}^{e-1} + \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^p \bar{\mathbf{F}}^{\theta-1} \mathbf{F}^{e-1}, \quad (4.82)$$

and conclude that

$$tr(\mathbf{L}) = tr(\mathbf{L}^e) + tr(\bar{\mathbf{L}}^p) + tr(\bar{\mathbf{L}}^\theta). \quad (4.83)$$

4.7 Notation

The following table shows the symbols used in the text and their definitions.

$\mathbf{A}$		Almansi strain
$\mathbf{A}^e$		elastic Almansi strain
$\mathbf{A}^p$		plastic Almansi strain
$\mathbf{A}^\theta$		thermal Almansi strain
$\mathbf{B}$		left Cauchy stretch tensor
$\mathbf{B}^e$		elastic left Cauchy stretch tensor
$\mathbf{B}^p$		plastic left Cauchy stretch tensor
$\mathbf{B}^\theta$		thermal left Cauchy stretch tensor
$\mathbf{C}$		right Cauchy stretch tensor
$\mathbf{C}^e$		elastic right Cauchy stretch tensor
$\mathbf{C}^p$		plastic right Cauchy stretch tensor
$\mathbf{C}^\theta$		thermal right Cauchy stretch tensor
$\mathbf{D}$		deformation rate
$\mathbf{D}^e$		elastic deformation rate
$\mathbf{D}^p$		plastic deformation rate
$\mathbf{D}^\theta$		thermal deformation rate
$\mathbf{E}$		Green strain
$\mathbf{E}^e$		elastic Green strain
$\mathbf{E}^p$		plastic Green strain
$\mathbf{E}^\theta$		thermal Green strain
$\mathbf{F}$		deformation gradient
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}^p$		plastic deformation gradient
$\mathbf{F}^\theta$		thermal deformation gradient
$\mathbf{H}$		displacement gradient
$\mathbf{H}^e$		elastic displacement gradient
$\mathbf{H}^p$		plastic displacement gradient
$\mathbf{H}^\theta$		thermal displacement gradient
$\mathbf{H}_U^e$		displacement gradient of $\mathbf{U}^e$
$\mathbf{H}_V^e$		displacement gradient of $\mathbf{V}^e$
$\mathbf{H}_U^p$		displacement gradient of $\mathbf{U}^p$
$\mathbf{H}_V^p$		displacement gradient of $\mathbf{V}^p$
$\mathbf{I}$		identity
J		volume ratio
J^e		elastic volume ratio
J^p		plastic volume ratio

J^θ		thermal volume ratio
$\mathbf{L}$		velocity gradient
$\mathbf{L}^e$		elastic velocity gradient
$\mathbf{L}^p$		plastic velocity gradient
$\mathbf{L}^\theta$		thermal velocity gradient
$\mathbf{R}$		orthogonal factor in polar decomposition of $\mathbf{F}$
$\mathbf{R}^e$		orthogonal factor in polar decomposition of $\mathbf{F}^e$
$\mathbf{R}^p$		orthogonal factor in polar decomposition of $\mathbf{F}^p$
$\mathbf{R}^\theta$		orthogonal factor in polar decomposition of $\mathbf{F}^\theta$
$\mathbf{U}$		right symmetric factor in polar decomposition of $\mathbf{F}$
$\mathbf{U}^e$		right symmetric factor in polar decomposition of $\mathbf{F}^e$
$\mathbf{U}^p$		right symmetric factor in polar decomposition of $\mathbf{F}^p$
$\mathbf{U}^\theta$		right symmetric factor in polar decomposition of $\mathbf{F}^\theta$
$\mathbf{V}$		left symmetric factor in polar decomposition of $\mathbf{F}$
$\mathbf{V}^e$		left symmetric factor in polar decomposition of $\mathbf{F}^e$
$\mathbf{V}^p$		left symmetric factor in polar decomposition of $\mathbf{F}^p$
$\mathbf{V}^\theta$		left symmetric factor in polar decomposition of $\mathbf{F}^\theta$
$\mathbf{W}$		spin tensor
$\mathbf{W}^e$		elastic spin tensor
$\mathbf{W}^p$		plastic spin tensor
$\mathbf{W}^\theta$		thermal spin tensor
ϵ		infinitesimal strain
ϵ^e		elastic infinitesimal strain
ϵ^p		plastic infinitesimal strain
ϵ^θ		thermal infinitesimal strain

Chapter 5

Traction, stress, and heat flux

In this chapter, we look at traction and heat flow, and the associated variables of stress and heat flux. The concept of traction, which represents the distribution of load over an area, is developed first, followed by the introduction of Cauchy stress and Cauchy's theorem, which is one of the cornerstones of modern mechanics. This is followed by looking at special examples of common stress states, such as uniaxial stress, equal biaxial stress, and pure shear stress. The concepts of nominal and true stress are then developed. A similar development is then given for heat, normal heat flow, and heat flux.

5.1 The traction vector

The *traction vector* is a vector that gives the load per unit area applied on a designated surface. For example, in Figure 5.1 we show a body that is loaded and for which we would like to calculate the traction vector at point Q on surface A inside the body passing through point Q . There are two sides to the surface A and the traction vector is different on the two sides. For the purposes of the example, we will focus on the side that has normal $\hat{\mathbf{n}}$. We imagine removing the material on the side of A that $\hat{\mathbf{n}}$ points into, and replacing the effect of the removed material by load $\mathbf{P}$. In the figure, the load $\Delta\mathbf{P}$ is the part of the total load $\mathbf{P}$ that is applied through area ΔA . The traction vector at Q for the surface shown will be denoted by $\mathbf{t}^{(n)}$ and is the limit as the area goes to zero of the load divided by the area it is applied on. This is written as

$$\mathbf{t}^{(n)} \equiv \lim_{\Delta A \rightarrow 0} \frac{\Delta\mathbf{P}}{\Delta A}. \quad (5.1)$$

As can be seen, the traction gets its direction from the direction of the load and its magnitude from the limit value of the ratio of the magnitude of the load divided by the area over which it is applied. Also, the reader will note that for each surface passing through Q there could be a different traction vector. That is, for each normal $\hat{\mathbf{n}}$ we may have a different traction vector.

The relation between the traction on one side of a surface and on its opposite side can be found by constructing a volume element containing the surface as shown in Figure 5.2. Summing the forces applied on this volume and letting the sum equal the time rate of change of linear momentum will result in the relation

$$\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)} \quad (5.2)$$

if one lets first the thickness h go to zero, and then the area ΔA go to zero.

We can separate the traction into a part normal to the surface and one tangent to the surface. As shown in Figure 5.3, the *normal traction*, denoted by σ , is the portion of the

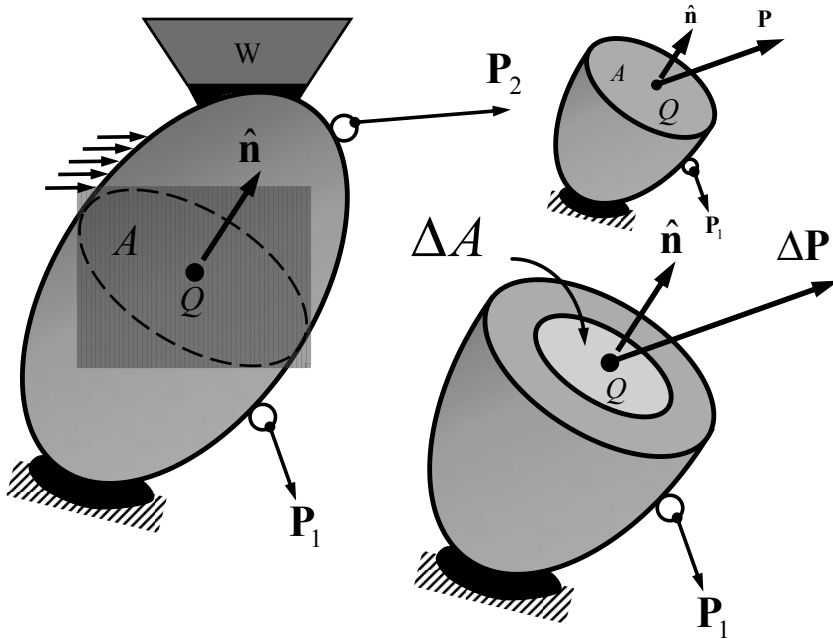


Figure 5.1: The figure shows a loaded body on the left. The traction vector is calculated on a surface of normal $\hat{\mathbf{n}}$ from the resultant load $\Delta \mathbf{P}$ applied on the area ΔA by taking the limit of the ratio of load to area as the area goes to zero.

traction that is normal to the surface and can be calculated by projecting the traction along the normal direction. Therefore,

$$\sigma = \mathbf{t}^{(n)} \circ \hat{\mathbf{n}}. \quad (5.3)$$

The *shear traction*, denoted by τ , is the portion of the traction vector that is tangent to the surface. This can be evaluated from either of the following two equations

$$\tau = |\mathbf{t}^{(n)} - (\mathbf{t}^{(n)} \circ \hat{\mathbf{n}})\hat{\mathbf{n}}|, \quad (5.4)$$

or

$$\tau = \sqrt{|\mathbf{t}^{(n)}|^2 - \sigma^2}. \quad (5.5)$$

The total resultant load on the surface of a body that only has distributed loads applied on it can be evaluated by integrating the traction vector over its surface area. Consider Figure 5.4 in which we have a body (or part of a body) with surface S , and let $\mathbf{P}$ denote the total load applied on this body by forces on its surface. This load can be evaluated from the integral

$$\mathbf{P} \equiv \int_S d\mathbf{P} = \int_S \mathbf{t}^{(n)} dA. \quad (5.6)$$

If one wants to calculate the resultant load for only a portion of the total surface area, one need only integrate the traction over the portion of the surface under consideration.

We have said that we may have a different traction for each surface we select through the point. Let us show this by a simple example. Figure 5.5 shows a bar under a uniform

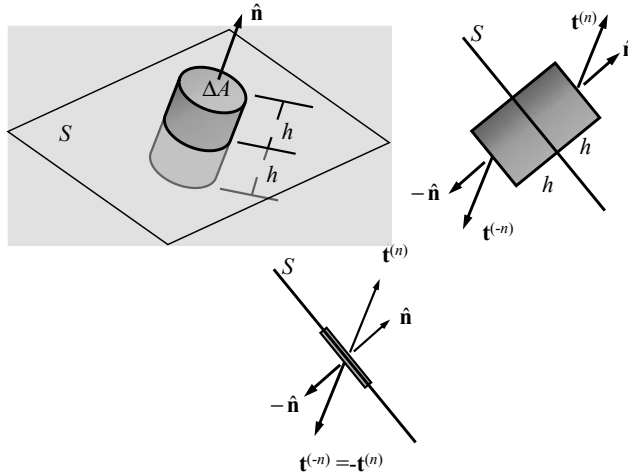


Figure 5.2: To evaluate the relation between the traction on two sides of a surface one can construct a volume element around the surface area ΔA and force the balance of linear momentum as first h goes to zero and then ΔA goes to zero.

uniaxial tensile load. As can be seen, depending on the normal $\hat{\mathbf{n}}$ the surface area that the load is applied on changes, yet for this example the load is always the same and along the bar. Let us assume that the load gets distributed evenly on the surface. If A denotes the area of the surface with normal $\hat{\mathbf{n}}$, and A_o is the cross-sectional area of the bar, then

$$\mathbf{t}^{(n)} = \frac{P}{A} \hat{\mathbf{e}}_1 = \frac{P}{A_o} \cos \theta \hat{\mathbf{e}}_1 = \cos \theta \mathbf{t}^{(e_1)},$$

where $\mathbf{t}^{(e_1)}$ is the traction on the end surface of the bar. As can be seen from this equation, the traction vector changes with the change of the orientation of the surface. Since in this case $\hat{\mathbf{n}} = \cos \theta \hat{\mathbf{e}}_1 + \sin \theta \hat{\mathbf{e}}_2$, we can find the normal traction as

σ

,

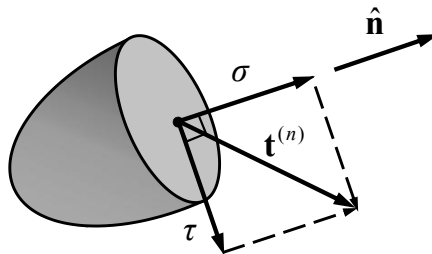


Figure 5.3: The normal traction σ and the shear traction τ are, respectively, the components of traction vector $\mathbf{t}^{(n)}$ along and perpendicular to the normal $\hat{\mathbf{n}}$.

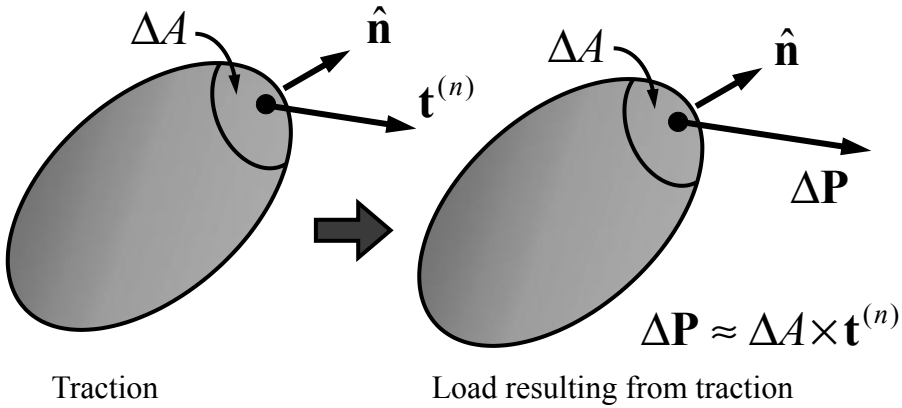


Figure 5.4: To calculate the resultant load applied on the surface of a body, one can integrate the traction vector over the surface area.

and the magnitude of the shear traction as

$$\tau = \sqrt{|\mathbf{t}^{(n)}|^2 - \sigma^2} = \sqrt{\frac{P^2}{A_o^2} \cos^2 \theta - \frac{P^2}{A_o^2} \cos^4 \theta} = \frac{P}{A_o} \cos \theta \sqrt{1 - \cos^2 \theta} = \frac{P}{A_o} \cos \theta \sin \theta.$$

5.2 The relation between tractions on different surfaces

Let us now look more closely at the relation between the tractions on different surfaces passing through the same point. One can establish this relation through the construction of a tetrahedron and relating the tractions on its different surfaces. For example, consider the tetrahedron shown in Figure 5.6. This tetrahedron is constructed from three vectors $h\xi_1\mathbf{e}_1$, $h\xi_2\mathbf{e}_2$, and $h\xi_3\mathbf{e}_3$ along the three coordinate directions. Three sides of the resulting tetrahedron are in planes of the coordinate lines, and one is skewed at an angle to these planes. Let the unit normal to the sides of the tetrahedron that are tangent to the coordinate planes be denoted by $\hat{\mathbf{n}}_i$ and the unit normal to the skewed surface be denoted as $\hat{\mathbf{n}}$. Also, let us let A_i denote the surface area of the surface with normal $\hat{\mathbf{n}}_i$ and let A denote the area of the surface with normal $\hat{\mathbf{n}}$.

Figure 5.6 shows, to the first approximation, the loads applied on the surfaces of the tetrahedron and due to the body forces. If the volume of the tetrahedron is denoted by V and its density by ρ , the balance of linear momentum (see Chapter 6 for further details) for this element states that

$$A\mathbf{t}^{(n)} + A_1\mathbf{t}^{(n_1)} + A_2\mathbf{t}^{(n_2)} + A_3\mathbf{t}^{(n_3)} + \rho V\mathbf{b} = \rho V\mathbf{a}, \quad (5.7)$$

where $\mathbf{b}$ is the body force and $\mathbf{a}$ is the acceleration. Let us divide this equation by the area A to obtain

$$\mathbf{t}^{(n)} + \frac{A_1}{A}\mathbf{t}^{(n_1)} + \frac{A_2}{A}\mathbf{t}^{(n_2)} + \frac{A_3}{A}\mathbf{t}^{(n_3)} + \rho\frac{V}{A}\mathbf{b} = \rho\frac{V}{A}\mathbf{a}. \quad (5.8)$$

Next, we will let h go to zero. We will show in the next paragraph that the ratios $\frac{A_i}{A}$ are

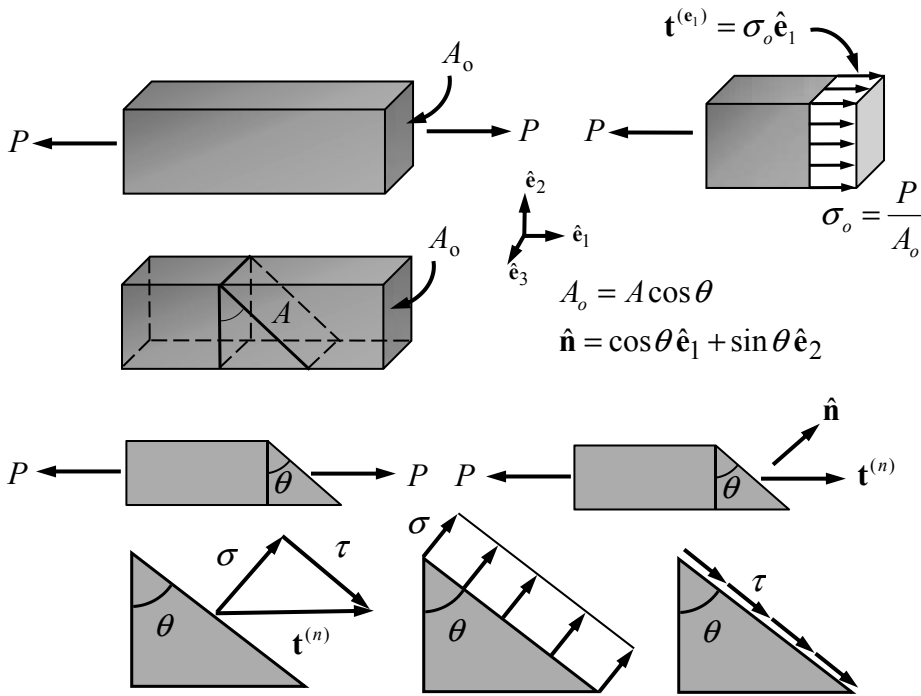


Figure 5.5: Calculation of traction in a bar under uniform uniaxial tensile load and under equilibrium.

independent of h and that $\frac{V}{A}$ goes to zero, resulting in the relation

$$\mathbf{t}^{(n)} = -\frac{A_1}{A} \mathbf{t}^{(n_1)} - \frac{A_2}{A} \mathbf{t}^{(n_2)} - \frac{A_3}{A} \mathbf{t}^{(n_3)} = -\frac{A_i}{A} \mathbf{t}^{(n_i)}. \quad (5.9)$$

This is the relation that must be satisfied by the tractions applied on the different surfaces passing through the same point. Even though we have looked at a tetrahedron constructed from three vectors that each is tangent to a coordinate direction, the result is not restricted to curvilinear coordinates and must hold for any tetrahedron. However, in this development it is assumed that the tetrahedron contains no point loads and that the variables are sufficiently continuous so that the approximations become more and more exact as h goes to zero. In the presence of point loads in the tetrahedron one needs to modify this relation. The reader is referred to the section on jump conditions for more information on dealing with such conditions.

Let us now look at the geometric relations that exist between the normals to the tetrahedron, the surface areas of its sides, and its volume. Let us assume the system to be a right-handed system. The following geometric relations exist for the areas and the normals:

$$\begin{aligned} \hat{\mathbf{n}}_1 &= -\frac{\mathbf{e}_2 \times \mathbf{e}_3}{|\mathbf{e}_2 \times \mathbf{e}_3|}, & A_1 &= \frac{h^2}{2} |\xi_2 \xi_3 \mathbf{e}_2 \times \mathbf{e}_3|, \\ \hat{\mathbf{n}}_2 &= -\frac{\mathbf{e}_3 \times \mathbf{e}_1}{|\mathbf{e}_3 \times \mathbf{e}_1|}, & A_2 &= \frac{h^2}{2} |\xi_3 \xi_1 \mathbf{e}_3 \times \mathbf{e}_1|, \\ \hat{\mathbf{n}}_3 &= -\frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}, & A_3 &= \frac{h^2}{2} |\xi_1 \xi_2 \mathbf{e}_1 \times \mathbf{e}_2|, \end{aligned} \quad (5.10)$$

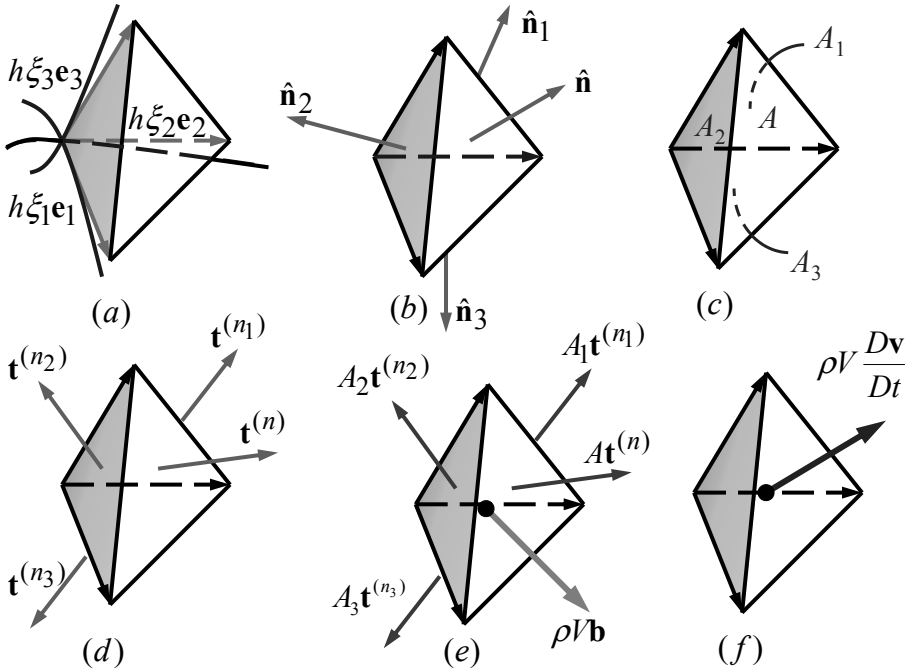


Figure 5.6: A diagram showing the tetrahedron used in proving Cauchy's theorem: (a) the tetrahedron constructed from the three vectors $h\xi_1\mathbf{e}_1$, $h\xi_2\mathbf{e}_2$, and $h\xi_3\mathbf{e}_3$, (b) the normals to the surfaces, (c) the areas of the surfaces, (d) the tractions on the sides, (e) the applied loads, and (f) the rate of change of linear momentum.

and

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{h^2}{2A}(\xi_2\mathbf{e}_2 - \xi_1\mathbf{e}_1) \times (\xi_3\mathbf{e}_3 - \xi_2\mathbf{e}_2) \\ &= \frac{h^2}{2A}(\xi_1\mathbf{e}_1 \times \xi_2\mathbf{e}_2 + \xi_2\mathbf{e}_2 \times \xi_3\mathbf{e}_3 + \xi_3\mathbf{e}_3 \times \xi_1\mathbf{e}_1) \\ &= -\frac{A_1}{A}\hat{\mathbf{n}}_1 - \frac{A_2}{A}\hat{\mathbf{n}}_2 - \frac{A_3}{A}\hat{\mathbf{n}}_3,\end{aligned}\quad (5.11)$$

$$A = \frac{h^2}{2}|\xi_1\mathbf{e}_1 \times \xi_2\mathbf{e}_2 + \xi_2\mathbf{e}_2 \times \xi_3\mathbf{e}_3 + \xi_3\mathbf{e}_3 \times \xi_1\mathbf{e}_1|. \quad (5.12)$$

This results in

$$\boxed{\hat{\mathbf{n}} = -\frac{A_i}{A}\hat{\mathbf{n}}_i.} \quad (5.13)$$

Upon examination, it becomes clear that the ratios $\frac{A_i}{A}$ do not depend on h . Let us proceed to evaluate the volume of the tetrahedron by the triple scalar product of the vectors creating its sides. That is,

$$V = \frac{h^3}{6}(\xi_1\mathbf{e}_1 \times \xi_2\mathbf{e}_2) \circ \xi_3\mathbf{e}_3. \quad (5.14)$$

Comparison of the expressions for V and A shows that the ratio $\frac{V}{A}$ is linearly dependent on h , so that it will go to zero as h tends to zero. Therefore, within the assumptions made, the relation derived in the last paragraph between the tractions on the sides of the tetrahedron must hold for both static and dynamic problems, with or without body forces. On the other hand, the relation does not hold when one has a point load in the tetrahedron. In such cases, the jump conditions derived in the next chapter must be used.

As a concluding remark, we note that if $\hat{\mathbf{n}} = -\hat{\mathbf{n}}_1$, so that the skewed surface becomes parallel to the surface with normal $\hat{\mathbf{n}}_1$, then we get

$$A\hat{\mathbf{n}} = -A\hat{\mathbf{n}}_1 = -A_1\hat{\mathbf{n}}_1 - A_2\hat{\mathbf{n}}_2 - A_3\hat{\mathbf{n}}_3,$$

which results in $A = A_1$ and $A_2 = A_3 = 0$. Substitution into the relation between the tractions results in

$$\mathbf{t}^{(-n_1)} = -\frac{A_1}{A}\mathbf{t}^{(n_1)} - \frac{A_2}{A}\mathbf{t}^{(n_2)} - \frac{A_3}{A}\mathbf{t}^{(n_3)} = -\mathbf{t}^{(n_1)}, \quad (5.15)$$

providing the same relation that was derived in the last section between the tractions on one side of a surface and those on the other side.

5.3 The stress tensor

As was shown in the last section, we have a different traction for each surface. The tetrahedron can be used to calculate the traction $\mathbf{t}^{(n)}$ on any arbitrary surface with normal $\hat{\mathbf{n}}$ from three tractions $\mathbf{t}^{(n_i)}$ on the three surfaces with normals $\hat{\mathbf{n}}_i$. Therefore, the traction on all surfaces passing through a point can be evaluated from the knowledge of tractions on three surfaces with non-coplanar normals. This fact allows us to construct a second-order tensor $\mathbf{T}$ called the *Cauchy stress tensor* that is used to calculate the traction on all surfaces. We define the Cauchy stress tensor through the relation

$$\boxed{\mathbf{T} \equiv g_{ij}^{-1}\hat{\mathbf{n}}_j \otimes \mathbf{t}^{(n_i)} = \bar{\mathbf{n}}_i \otimes \mathbf{t}^{(n_i)}}, \quad (5.16)$$

where $g_{ij} = \hat{\mathbf{n}}_i \circ \hat{\mathbf{n}}_j$ is the metric tensor created by the base constructed from the normals to the three sides of the tetrahedron on which the tractions $\mathbf{t}^{(n_i)}$ are applied, and $\bar{\mathbf{n}}_i = g_{ij}^{-1}\hat{\mathbf{n}}_j$ is its reciprocal or dual base. This tensor has the property that

$$\boxed{\mathbf{t}^{(n)} = \hat{\mathbf{n}}\mathbf{T} = \mathbf{T}^T\hat{\mathbf{n}}}. \quad (5.17)$$

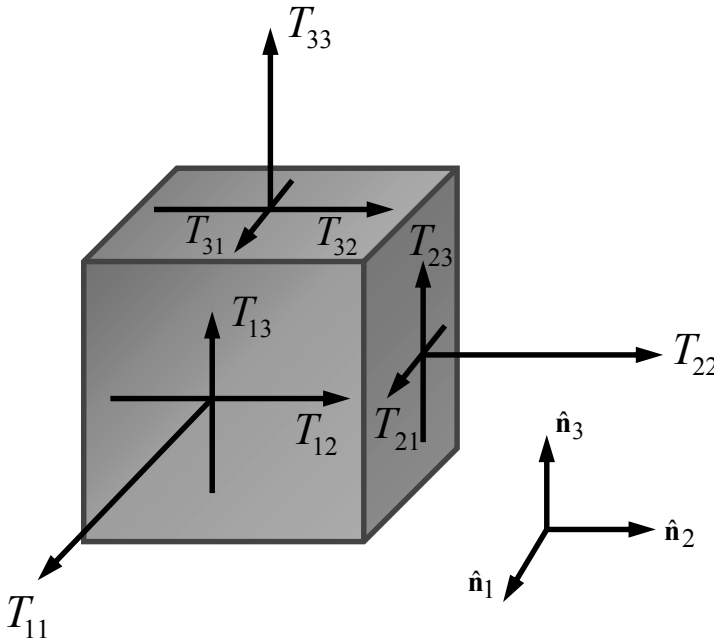
This last relation can be seen by direct substitution into the above equation to get

$$\begin{aligned} \hat{\mathbf{n}}\mathbf{T} &= \left(-\frac{A_k}{A}\hat{\mathbf{n}}_k\right)(g_{ij}^{-1}\hat{\mathbf{n}}_j \otimes \mathbf{t}^{(n_i)}) \\ &= -\frac{A_k}{A}g_{ij}^{-1}g_{kj}\mathbf{t}^{(n_i)} \\ &= -\frac{A_k}{A}\delta_{ik}\mathbf{t}^{(n_i)} \\ &= -\frac{A_i}{A}\mathbf{t}^{(n_i)} = \mathbf{t}^{(n)}. \end{aligned}$$

The combination of the fact that the traction on any surface can be calculated from the tractions on three (perpendicular) surfaces passing through a point, and the relation $\mathbf{t}^{(n)} = \hat{\mathbf{n}}\mathbf{T} = \mathbf{T}^T\hat{\mathbf{n}}$ is known as *Cauchy's fundamental theorem*. The reader will note that in the proof there is no need for the three planes to be perpendicular, as long as their three normals not be in a single plane.

Depending on the author, sometimes the transpose of $\mathbf{T}$ is defined as the Cauchy stress. If we denote the transpose of $\mathbf{T}$ by $\bar{\mathbf{T}}$, we can write

$$\boxed{\bar{\mathbf{T}} \equiv g_{ij}^{-1}\mathbf{t}^{(n_i)} \otimes \hat{\mathbf{n}}_j}, \quad (5.18)$$

Figure 5.7: Stress on a cube with normals $\hat{\mathbf{n}}_i$.

with the properties

$$\mathbf{t}^{(n)} = \bar{\mathbf{T}}\hat{\mathbf{n}} = \hat{\mathbf{n}}\bar{\mathbf{T}}^T. \quad (5.19)$$

In general, for first gradient materials the difference between $\mathbf{T}$ and $\bar{\mathbf{T}}$ disappears since we later will show that the Cauchy stress must be symmetric. For the purpose of consistency in presentation, in this book the term Cauchy stress will refer only to $\mathbf{T}$, as defined in equation (5.16), with the property described by Cauchy's theorem given by (5.17).

The Cauchy stress tensor gives the stress per unit current area of the surface. Therefore, the Cauchy stress is equivalent to the true stress, as opposed to the engineering stress, which provides load per unit original area.

If the tetrahedron is constructed such that the normals $\hat{\mathbf{n}}_i$ create an orthonormal set, then the components of the stress tensor become equal to the tractions on the associated three surfaces. As shown in Figure 5.7, if the stress tensor is represented as $\mathbf{T} = T_{ij}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j$, then the components T_{ij} are directly related to the traction on the surface of the cube shown and

$$\begin{aligned} \mathbf{t}^{(n_1)} &= \hat{\mathbf{n}}_1 \mathbf{T} = \hat{\mathbf{n}}_1 (T_{ij} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j) = T_{1j} \hat{\mathbf{n}}_j = T_{11} \hat{\mathbf{n}}_1 + T_{12} \hat{\mathbf{n}}_2 + T_{13} \hat{\mathbf{n}}_3, \\ \mathbf{t}^{(n_2)} &= \hat{\mathbf{n}}_2 \mathbf{T} = \hat{\mathbf{n}}_2 (T_{ij} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j) = T_{2j} \hat{\mathbf{n}}_j = T_{21} \hat{\mathbf{n}}_1 + T_{22} \hat{\mathbf{n}}_2 + T_{23} \hat{\mathbf{n}}_3, \\ \mathbf{t}^{(n_3)} &= \hat{\mathbf{n}}_3 \mathbf{T} = \hat{\mathbf{n}}_3 (T_{ij} \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j) = T_{3j} \hat{\mathbf{n}}_j = T_{31} \hat{\mathbf{n}}_1 + T_{32} \hat{\mathbf{n}}_2 + T_{33} \hat{\mathbf{n}}_3. \end{aligned} \quad (5.20)$$

5.4 Isotropic invariants and the deviatoric stress

As described in the section on tensors, the isotropic invariants are scalar invariants of a tensor. These invariants provide scalar measures of the tensor. In the case of the Cauchy

stress, the isotropic invariants are given by

$$\boxed{I_{1T} \equiv \text{tr}(\mathbf{T})}, \quad \boxed{I_{2T} \equiv \text{tr}(\mathbf{T}^2)}, \quad \boxed{I_{3T} \equiv \text{tr}(\mathbf{T}^3)}. \quad (5.21)$$

As discussed in the section on tensors, there are alternate forms of the isotropic invariants. The alternate set of isotropic invariants of the Cauchy stress are given by

$$\boxed{J_{1T} \equiv \text{tr}(\mathbf{T})}, \quad \boxed{J_{2T} \equiv \frac{1}{2}[\text{tr}^2(\mathbf{T}) - \text{tr}(\mathbf{T}^2)]}, \quad \boxed{J_{3T} \equiv \det(\mathbf{T})}. \quad (5.22)$$

These invariants, represented in terms of the components of stress T_{ij} in an orthonormal base, can be written as

$$\begin{aligned} I_{1T} &= T_{ii} = T_{11} + T_{22} + T_{33}, \\ I_{2T} &= T_{ij}T_{ji} = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2(T_{12}^2 + T_{23}^2 + T_{31}^2), \\ I_{3T} &= T_{ij}T_{jk}T_{ki} = T_{11}^3 + T_{22}^3 + T_{33}^3 \\ &\quad + 3(T_{11}T_{12}^2 + T_{22}T_{23}^2 + T_{33}T_{31}^2 + T_{22}T_{21}^2 + T_{33}T_{32}^2 + T_{11}T_{13}^2) + 6T_{12}T_{23}T_{31}, \end{aligned} \quad (5.23)$$

and

$$\begin{aligned} J_{1T} &= T_{ii} = T_{11} + T_{22} + T_{33}, \\ J_{2T} &= \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) = T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11} - (T_{12}^2 + T_{23}^2 + T_{31}^2), \\ J_{3T} &= \det(T_{ij}) = T_{11}T_{22}T_{33} + 2T_{12}T_{23}T_{31} - (T_{11}T_{23}^2 + T_{22}T_{31}^2 + T_{33}T_{12}^2). \end{aligned} \quad (5.24)$$

In terms of the principal stresses T_1 , T_2 , and T_3 , we can write these invariants as

$$\begin{aligned} I_{1T} &= T_1 + T_2 + T_3, \\ I_{2T} &= T_1^2 + T_2^2 + T_3^2, \\ I_{3T} &= T_1^3 + T_2^3 + T_3^3, \\ J_{1T} &= T_1 + T_2 + T_3, \\ J_{2T} &= T_1T_2 + T_2T_3 + T_3T_1, \\ J_{3T} &= T_1T_2T_3. \end{aligned} \quad (5.25)$$

The Cauchy stress can be separated into the average stress T_{ave} and the deviatoric stress $\mathbf{S}$ through the relation

$$\boxed{\mathbf{T} = T_{ave}\mathbf{I} + \mathbf{S}}. \quad (5.26)$$

The *average stress* T_{ave} is a scalar invariant of the stress tensor and is defined as

$$\boxed{T_{ave} \equiv \frac{1}{3}\text{tr}(\mathbf{T}) = \frac{1}{3}I_{1T}}, \quad (5.27)$$

which is related to the hydrostatic pressure p through the relation $\boxed{p \equiv -T_{ave}}$. The *spherical stress* is defined as $T_{ave}\mathbf{I}$, and the *deviatoric stress* $\mathbf{S}$ is defined as being the Cauchy stress minus the spherical stress so that

$$\boxed{\mathbf{S} \equiv \mathbf{T} - T_{ave}\mathbf{I}}. \quad (5.28)$$

The deviatoric stress has the characteristic that its trace is zero since $\text{tr}(\mathbf{S}) = \text{tr}(\mathbf{T} - T_{ave}\mathbf{I}) = \text{tr}(\mathbf{T}) - T_{ave}\text{tr}(\mathbf{I}) = 0$. The isotropic invariants of $\mathbf{S}$ are

$$\begin{aligned} I_{1S} &= \text{tr}(\mathbf{S}) = 0, \\ I_{2S} &= \text{tr}(\mathbf{S}^2) = \text{tr}(\mathbf{T}^2 - 2T_{ave}\mathbf{T} + T_{ave}^2\mathbf{I}) = I_{2T} - \frac{1}{3}I_{1T}^2, \\ I_{3S} &= \text{tr}(\mathbf{S}^3) = \text{tr}(\mathbf{T}^3 - 3T_{ave}\mathbf{T}^2 + 3T_{ave}^2\mathbf{T} - T_{ave}^3\mathbf{I}) = I_{3T} - I_{1T}I_{2T} + \frac{2}{9}I_{1T}^3. \end{aligned} \quad (5.29)$$

An alternate set of isotropic invariants is given as

$$\begin{aligned} J_{1S} &= \text{tr}(\mathbf{S}) = 0, \\ J_{2S} &= \frac{1}{2}[\text{tr}^2(\mathbf{S}) - \text{tr}(\mathbf{S}^2)] = -\frac{1}{2}\text{tr}(\mathbf{S}^2) = -\frac{1}{2}I_{2S}, \\ J_{3S} &= \det(\mathbf{S}). \end{aligned} \quad (5.30)$$

These invariants can be written in terms of the components of $\mathbf{S}$. For example, if $\mathbf{S}$ has components S_{ij} in an orthonormal base, the invariants can be written as

$$\begin{aligned} I_{1S} &= S_{ii} = S_{11} + S_{22} + S_{33} = 0, \\ I_{2S} &= S_{ij}S_{ji} = S_{11}^2 + S_{22}^2 + S_{33}^2 + 2(S_{12}^2 + S_{23}^2 + S_{31}^2), \\ I_{3S} &= S_{ij}S_{jk}S_{ki} = S_{11}^3 + S_{22}^3 + S_{33}^3 \\ &\quad + 3(S_{11}S_{12}^2 + S_{22}S_{23}^2 + S_{33}S_{31}^2 + S_{22}S_{21}^2 + S_{33}S_{32}^2 + S_{11}S_{13}^2) + 6S_{12}S_{23}S_{31}, \end{aligned} \quad (5.31)$$

and

$$\begin{aligned} J_{1S} &= S_{ii} = S_{11} + S_{22} + S_{33} = 0, \\ J_{2S} &= -\frac{1}{2}S_{ij}S_{ji} = -[\frac{1}{2}(S_{11}^2 + S_{22}^2 + S_{33}^2) + S_{12}^2 + S_{23}^2 + S_{31}^2], \\ J_{3S} &= \det(S_{ij}) = S_{11}S_{22}S_{33} + 2S_{12}S_{23}S_{31} - (S_{11}S_{23}^2 + S_{22}S_{31}^2 + S_{33}S_{12}^2). \end{aligned} \quad (5.32)$$

In terms of the principal values S_1 , S_2 , and S_3 of the deviatoric stresses, we can write these invariants as

$$\begin{aligned} I_{1S} &= S_1 + S_2 + S_3 = 0, \\ I_{2S} &= S_1^2 + S_2^2 + S_3^2, \\ I_{3S} &= S_1^3 + S_2^3 + S_3^3, \\ J_{1S} &= S_1 + S_2 + S_3 = 0, \\ J_{2S} &= S_1S_2 + S_2S_3 + S_3S_1 = -\frac{1}{2}(S_1^2 + S_2^2 + S_3^2), \\ J_{3S} &= S_1S_2S_3. \end{aligned} \quad (5.33)$$

The invariants of the deviatoric stress can be written in terms of the invariants of the Cauchy stress as

$$\begin{aligned} J_{2S} &= -\frac{1}{3}(J_{1T}^2 - 3J_{2T}), \\ J_{3S} &= \frac{1}{27}(2J_{1T}^3 - 9J_{1T}J_{2T} + 27J_{3T}). \end{aligned} \quad (5.34)$$

This yields

$$\begin{aligned} J_{2S} &= -\frac{1}{6}[(T_{11} - T_{22})^2 + (T_{22} - T_{33})^2 + (T_{33} - T_{11})^2 + 6(T_{12}^2 + T_{23}^2 + T_{31}^2)], \\ J_{3S} &= \frac{4}{9}T_{11}T_{22}T_{33} - \frac{1}{9}(T_{11}^2T_{22} + T_{11}^2T_{33} + T_{22}^2T_{11} + T_{22}^2T_{33} + T_{33}^2T_{11} + T_{33}^2T_{22}) \\ &\quad + \frac{2}{27}(T_{11}^3 + T_{22}^3 + T_{33}^3) - \frac{2}{3}(T_{11}T_{23}^2 + T_{22}T_{31}^2 + T_{33}T_{12}^2) \\ &\quad + \frac{1}{3}(T_{11}T_{12}^2 + T_{11}T_{13}^2 + T_{22}T_{21}^2 + T_{22}T_{23}^2 + T_{33}T_{31}^2 + T_{33}T_{32}^2) + 2T_{12}T_{23}T_{31}. \end{aligned} \quad (5.35)$$

5.5 Examples of elementary states of stress

In this section, we will study the properties of several elementary states of stress. These will include hydrostatic pressure, uniaxial tension, equal biaxial tension, plane stress, and pure shear. The goal is to become familiar with these stresses and with the general characteristics of simple stress states.

The first example we will consider is *hydrostatic pressure* as shown in Figure 5.8. The stress tensor for this state of stress is written as

$$\mathbf{T} = -p\mathbf{I} = -p(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3), \quad (5.36)$$

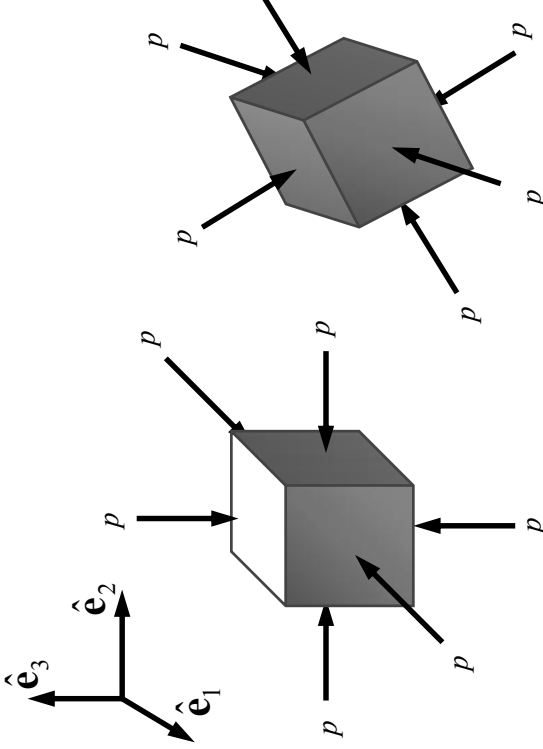


Figure 5.8: Stress state for hydrostatic pressure.

where p is the hydrostatic pressure and $\hat{\mathbf{e}}_i$ are a set of mutually orthogonal unit vectors. For this state of stress, the traction on any surface with normal $\hat{\mathbf{n}}$ can be calculated from

$$\begin{aligned}\mathbf{t}^{(n)} &= \hat{\mathbf{n}}\mathbf{T} = \hat{\mathbf{n}}[-p(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3)] \\ &= -p[(\hat{\mathbf{n}} \circ \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\hat{\mathbf{n}} \circ \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\hat{\mathbf{n}} \circ \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3] \\ &= -p\hat{\mathbf{n}}.\end{aligned}\quad (5.37)$$

Therefore, the traction on every surface is normal to the surface, pushing toward the surface with magnitude p . Since this stress state is real and symmetric with three identical eigenvalues $T_1 = T_2 = T_3 = -p$, it follows that all vectors are its eigenvectors.

For the second example, consider *uniaxial tension*. As shown in Figure 5.9, uniaxial tension refers to the condition in which there is only normal traction on a given surface and zero traction on all surfaces perpendicular to it. Let us assume there is only a normal traction σ on the surface with normal $\hat{\mathbf{h}}$. The stress tensor for this state of stress can be written as

$$\mathbf{T} = \sigma\hat{\mathbf{h}} \otimes \hat{\mathbf{h}}. \quad (5.38)$$

The traction $\mathbf{t}^{(n)}$ on any surface is given by $\mathbf{t}^{(n)} = \hat{\mathbf{n}}(\sigma\hat{\mathbf{h}} \otimes \hat{\mathbf{h}}) = \sigma(\hat{\mathbf{n}} \circ \hat{\mathbf{h}})\hat{\mathbf{h}} = \sigma\cos\phi\hat{\mathbf{h}}$. Therefore, for uniaxial tension, the traction on all surfaces will be along the direction $\hat{\mathbf{h}}$, irrespective of $\hat{\mathbf{n}}$. On any surface that has a normal perpendicular to $\hat{\mathbf{h}}$ the product $\hat{\mathbf{n}} \circ \hat{\mathbf{h}}$ equals zero and, therefore, the traction is zero. The stress tensor for uniaxial tension is symmetric and has eigenvalues $T_1 = \sigma$, $T_2 = T_3 = 0$, with $\hat{\mathbf{h}}$ as the eigenvector associated with T_1 and with the eigenvectors associated with T_2 and T_3 being any two vectors in the plane perpendicular to $\hat{\mathbf{h}}$. As the reader recalls, the eigenvectors associated with distinct eigenvalues for a real symmetric tensor are orthogonal.

For the third example, consider *equal biaxial tension* as shown in Figure 5.10. The stress tensor associated with this loading is

$$\mathbf{T} = \sigma(\hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_1 + \hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_2). \quad (5.39)$$

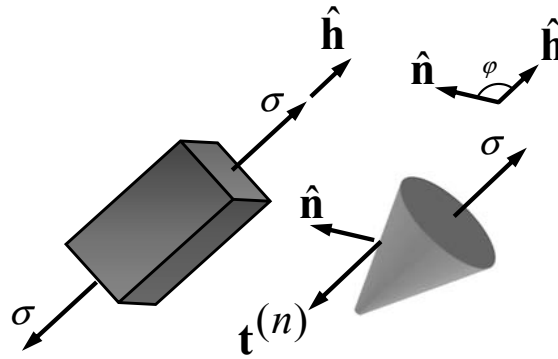


Figure 5.9: The stress state for uniaxial tension.

This stress state is one of a number of stress states that are known as plane-stress states. The characteristic of the plane-stress state is that one surface is traction free. For equal biaxial tension, the traction on any plane can be evaluated by

$$\mathbf{t}^{(n)} = \hat{\mathbf{n}}[\sigma(\hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_1 + \hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_2)] = \sigma(\hat{\mathbf{n}} \circ \hat{\mathbf{h}}_1)\hat{\mathbf{h}}_1 + \sigma(\hat{\mathbf{n}} \circ \hat{\mathbf{h}}_2)\hat{\mathbf{h}}_2. \quad (5.40)$$

As can be seen, the traction vector, irrespective of the orientation of $\hat{\mathbf{n}}$, is in the plane of $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$. Obviously, if $\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$, then the traction is zero. If $\hat{\mathbf{n}}$ is in the plane of $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$ and if $\hat{\mathbf{n}} \circ \hat{\mathbf{h}}_1 = \cos \phi$, then $\hat{\mathbf{n}} \circ \hat{\mathbf{h}}_2 = \sin \phi$. It follows that $\mathbf{t}^{(n)} = \sigma(\cos \phi \hat{\mathbf{h}}_1 + \sin \phi \hat{\mathbf{h}}_2) = \sigma \hat{\mathbf{n}}$, so that for all these surfaces the traction has only a normal component, and this component is equal to σ . If $\hat{\mathbf{n}}$ is skewed on an angle θ from the plane of $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$, then we can write $\hat{\mathbf{n}} = \cos \theta \hat{\mathbf{n}}_1 + \sin \theta \hat{\mathbf{n}}_2$, where $\hat{\mathbf{n}}_1 = \cos \phi \hat{\mathbf{h}}_1 + \sin \phi \hat{\mathbf{h}}_2$ is along the direction of the projection of $\hat{\mathbf{n}}$ onto the $\hat{\mathbf{h}}_1$ - $\hat{\mathbf{h}}_2$ plane, ϕ is the angle it makes with the $\hat{\mathbf{h}}_1$ direction, and $\hat{\mathbf{n}}_2$ is perpendicular to the $\hat{\mathbf{h}}_1$ - $\hat{\mathbf{h}}_2$ plane. Calculating the traction on these planes results in $\mathbf{t}^{(n)} = \cos \theta \sigma \hat{\mathbf{n}}_1$. The stress state for equal biaxial tension is symmetric and has eigenvalues $T_1 = T_2 = \sigma$ and $T_3 = 0$. Any two vectors in the plane of $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$ can be selected as the eigenvectors associated with T_1 and T_2 , and the eigenvector associated with T_3 is perpendicular to the plane of $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$.

Hydrostatic pressure, uniaxial tension, and equal biaxial tension are three special cases of *triaxial tension*. The stress tensor for triaxial tension can be written as

$$\mathbf{T} = T_1 \hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_1 + T_2 \hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_2 + T_3 \hat{\mathbf{h}}_3 \otimes \hat{\mathbf{h}}_3, \quad (5.41)$$

where $\hat{\mathbf{h}}_i$ represents a mutually orthogonal set of unit vectors as shown in Figure 5.11. This stress tensor is symmetric with eigenvalues T_i and associated eigenvectors $\hat{\mathbf{h}}_i$. As shown in the section on tensors, every real symmetric tensor has a spectral representation that has a form identical to that shown for triaxial tension. Therefore, every real symmetric stress tensor has a spectral representation that is identical to triaxial extension. As such, all real symmetric stress tensors are special cases of triaxial extension. As will be shown later, Cauchy stress is always symmetric, showing that all states of stress are special cases of triaxial extension.

Another symmetric stress state is *pure shear*. As shown in Figure 5.12, the state of pure shear can be written as

$$\mathbf{T} = \tau(\hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_2 + \hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_1), \quad (5.42)$$

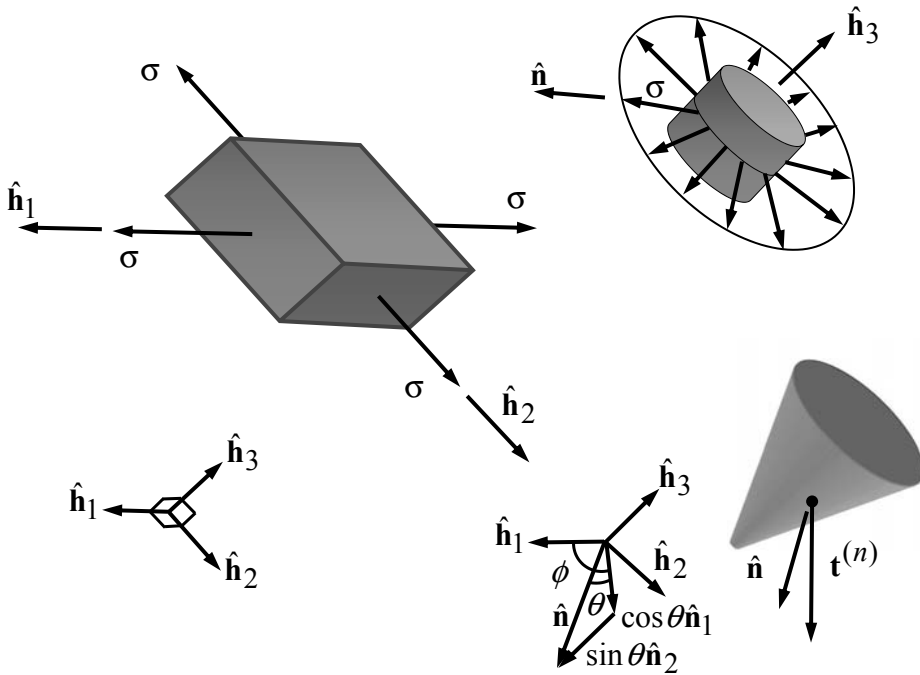


Figure 5.10: Stress state for equal biaxial tension.

where τ is the shear stress and $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$ are mutually orthogonal. Since pure shear is represented by a real symmetric tensor, it is a special case of triaxial tension. Pure shear has a spectral representation of the form

$$\mathbf{T} = \tau \hat{\mathbf{m}}_1 \otimes \hat{\mathbf{m}}_1 - \tau \hat{\mathbf{m}}_2 \otimes \hat{\mathbf{m}}_2, \quad (5.43)$$

where $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ are orthogonal and, as shown in Figure 5.12, $\hat{\mathbf{m}}_1$ is rotated counter-clockwise 45° from $\hat{\mathbf{h}}_1$ in the plane of shear. Therefore, the eigenvalues associated with pure shear are $T_1 = \tau$, $T_2 = -\tau$, and $T_3 = 0$, with their associated eigenvectors being $\hat{\mathbf{m}}_1$, $\hat{\mathbf{m}}_2$, and $\hat{\mathbf{h}}_3$, respectively. Pure shear is also a special case of plane-stress since there is no traction on the surface perpendicular to $\hat{\mathbf{h}}_3$.

5.6 True stress as opposed to engineering stress

In all our development up to now, we have implied that we have the body in the current configuration so that we calculate the traction by taking the load and dividing it by the area it is currently applied on. This is consistent with what is called “true stress.” Early in the development of mechanics it was noted that under most applications the body does not deform substantially and, therefore, there is little difference between the area in the current configuration and its image in the initial configuration (the configuration before the body was loaded). This idea resulted in a definition of stress that is based on load divided by area in the initial unloaded configuration. This is convenient since the unloaded shape

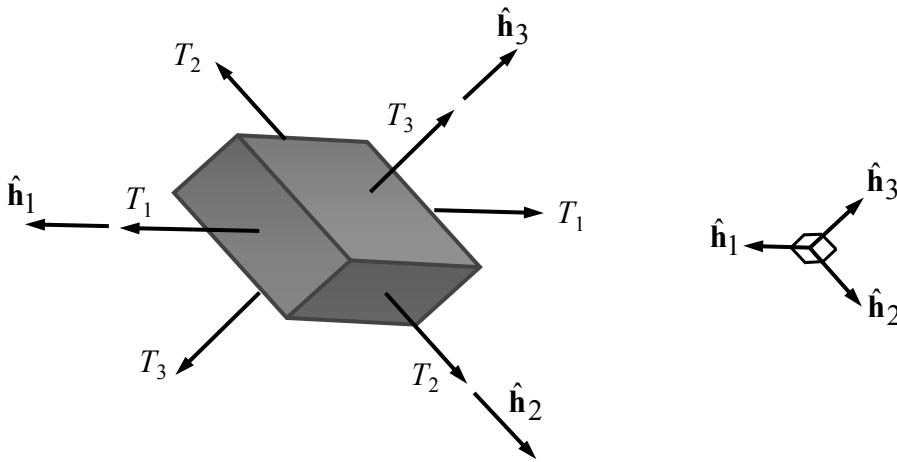


Figure 5.11: Stress state for triaxial tension.

of the body normally is known. The name for this stress is the “engineering stress” or the “nominal stress,” referring to the fact that if the deformations are small, and the rigid body motions negligible, then this stress is a very good approximation to the true stress. The reader should be reminded that not all loads result in small deformations of the body, so the engineering stress should only be used in place of the true stress when the approximation is justified. On the other hand, since there is a mapping between the different configurations of the body, one can find the exact relation between the engineering and true stresses, without a need to consider one as an approximation of the other.

The stress tensor that we have been considering, the one associated with the true stress, is known as the *Cauchy stress tensor*.

Let us develop another stress tensor based on the reference configuration. To do this, we need to define our objectives for such a stress tensor. Consider two configurations of a body as shown in Figure 5.13. In this case, one configuration is the current configuration and the other is the reference configuration. The material surface with area dA and normal $\hat{\mathbf{n}}$ in the current configuration has an area dA_o and normal $\hat{\mathbf{N}}$ in the reference configuration. The “true” traction $\mathbf{t}^{(n)}$ is defined by

$$\mathbf{t}^{(n)} = \frac{d\mathbf{P}}{dA}. \quad (5.44)$$

The “engineering” or “nominal” traction $\mathbf{t}_o^{(N)}$ will be defined as

$$\boxed{\mathbf{t}_o^{(N)} \equiv \frac{d\mathbf{P}}{dA_o}}, \quad (5.45)$$

where the reader will note that the same load is now divided by the associated area in the reference configuration. Associated with the true traction is the Cauchy stress $\mathbf{T}$ with the property

$$\mathbf{t}^{(n)} = \hat{\mathbf{n}}\mathbf{T}. \quad (5.46)$$

We will denote the *engineering stress or nominal stress* by $\mathbf{T}_o$ and define it in association

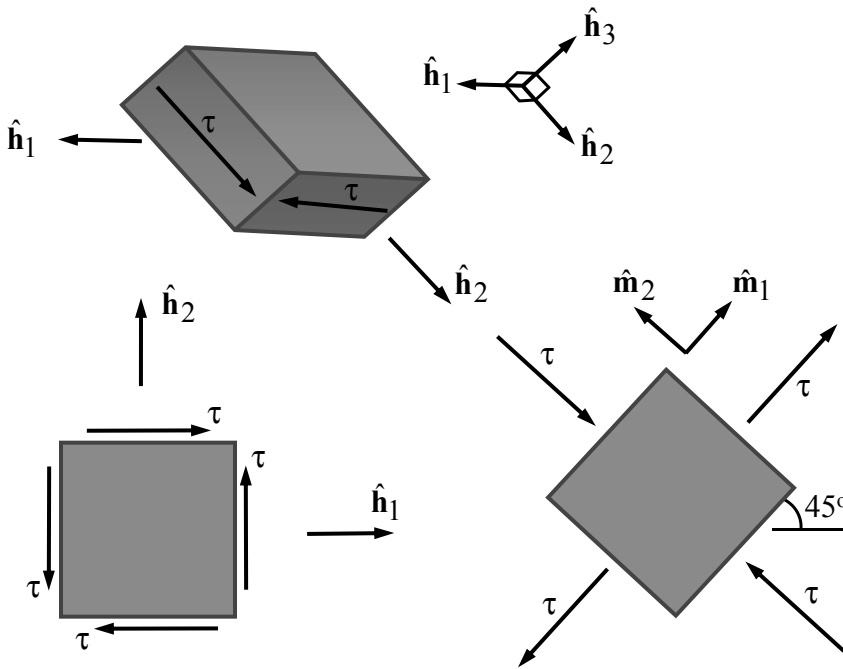


Figure 5.12: Stress state for pure shear.

to the engineering or nominal traction by the property

$$\mathbf{t}_o^{(N)} = \hat{\mathbf{N}} \mathbf{T}_o = \mathbf{T}_o^T \hat{\mathbf{N}}. \quad (5.47)$$

One can relate the two stress tensors $\mathbf{T}$ and $\mathbf{T}_o$ using the deformation gradient. The reader will note that the total load $\mathbf{P}$ applied on the body at the current time can be calculated from either of the two integrals

$$\mathbf{P} = \int_S \mathbf{t}^{(n)} dA = \int_{S_o} \mathbf{t}_o^{(N)} dA_o, \quad (5.48)$$

where S is the surface of the segment in the current configuration and S_o is its associated surface in the reference configuration. If the associated volumes are V and V_o , then the divergence theorem can be used to express this load in one of the two forms

$$\mathbf{P} = \int_V \text{div}(\mathbf{T}^T) dV = \int_{V_o} \text{Div}(\mathbf{T}_o^T) dV_o, \quad (5.49)$$

where $\text{div} = \text{div}_{\mathbf{x}}$ denotes the divergence based on position in the current configuration and $\text{Div} = \text{div}_{\mathbf{X}}$ denotes the divergence based on position in the reference configuration. Since the reference configuration is normally known and the current configuration unknown, it is desirable to use integration over the reference configuration to obtain the load from the stress, in this case using the stress $\mathbf{T}_o$. As will be shown in the next chapter on balance

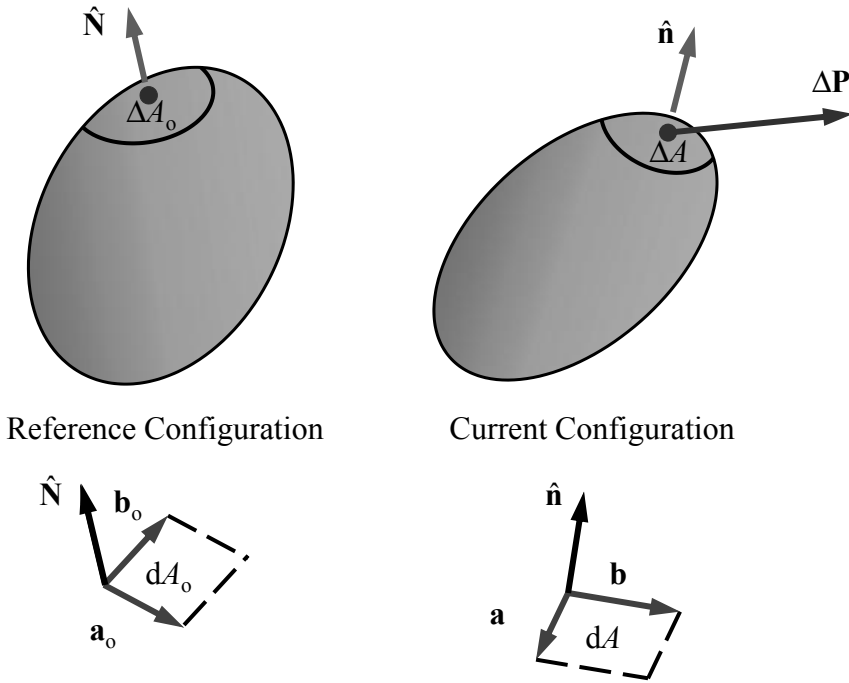


Figure 5.13: Areas in the reference and current configurations are, respectively, used in the calculation of the nominal and true traction.

laws, this stress will be used in the balance of linear momentum to obtain a special form of the law that is written for the body in the reference configuration.

Let us now proceed to find the relation between $\mathbf{T}$ and $\mathbf{T}_o$. First, let us relate the normal vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{N}}$. As the reader will note, the material line along $\hat{\mathbf{N}}$ in the reference configuration is normal to the surface dA_o but is not normal to the surface dA in the current configuration. Therefore, one cannot use $\mathbf{F}\hat{\mathbf{N}}$ to obtain $\hat{\mathbf{n}}$. Let us say we have the two vectors $\mathbf{a}_o$ and $\mathbf{b}_o$ in the reference configuration and that the area of the parallelogram constructed from these vectors is given by dA_o so that $dA_o = |\mathbf{a}_o \times \mathbf{b}_o|$. The same material surface in the current configuration will be given by the two vectors $\mathbf{a} = \mathbf{F}\mathbf{a}_o$ and $\mathbf{b} = \mathbf{F}\mathbf{b}_o$, and its area is given by $dA = |\mathbf{a} \times \mathbf{b}|$. If $\hat{\mathbf{N}}$ is the unit normal to dA_o , and $\hat{\mathbf{n}}$ is the unit normal to the area dA , then

$$\hat{\mathbf{n}}dA = \mathbf{a} \times \mathbf{b} = (\mathbf{F}\mathbf{a}_o) \times (\mathbf{F}\mathbf{b}_o) = \det(\mathbf{F})\mathbf{F}^{-T}(\mathbf{a}_o \times \mathbf{b}_o) = \det(\mathbf{F})\mathbf{F}^{-T}(\hat{\mathbf{N}}dA_o), \quad (5.50)$$

where the next to last relation is an identity from Chapter 2. From this relation, one can conclude that

$$\hat{\mathbf{N}} = \frac{1}{\det(\mathbf{F})} \frac{dA}{dA_o} \hat{\mathbf{n}}\mathbf{F}, \quad (5.51)$$

$$\hat{\mathbf{n}} = \det(\mathbf{F}) \frac{dA_o}{dA} \mathbf{F}^{-T} \hat{\mathbf{N}}. \quad (5.52)$$

The ratio of the areas also can be related through this relation to obtain

$$\begin{aligned} \left(\frac{dA}{dA_o} \right)^2 &= \det(\mathbf{C}) \hat{\mathbf{N}} \circ (\mathbf{C}^{-1} \hat{\mathbf{N}}) \\ &= \frac{\det(\mathbf{B})}{\hat{\mathbf{n}} \circ (\mathbf{B} \hat{\mathbf{n}})}. \end{aligned} \quad (5.53)$$

Let us get back to our original goal of finding $\mathbf{T}_o$. To do this we note that

$$\boxed{\mathbf{t}^{(n)} = \frac{dA_o}{dA} \mathbf{t}_o^{(N)}}. \quad (5.54)$$

Introducing $\mathbf{t}^{(n)} = \hat{\mathbf{n}} \mathbf{T}$ and $\mathbf{t}_o^{(N)} = \hat{\mathbf{N}} \mathbf{T}_o$ into this expression results in

$$\hat{\mathbf{n}} \mathbf{T} = \frac{dA_o}{dA} \hat{\mathbf{N}} \mathbf{T}_o, \quad (5.55)$$

which yields

$$\det(\mathbf{F}) \frac{dA_o}{dA} (\mathbf{F}^{-T} \hat{\mathbf{N}}) \mathbf{T} = \frac{dA_o}{dA} \hat{\mathbf{N}} \mathbf{T}_o. \quad (5.56)$$

This simplifies to the expression

$$\det(\mathbf{F}) \hat{\mathbf{N}} (\mathbf{F}^{-1} \mathbf{T}) = \hat{\mathbf{N}} \mathbf{T}_o, \quad (5.57)$$

which must hold for every $\hat{\mathbf{N}}$. Therefore, one must conclude that

$$\boxed{\mathbf{T}_o = \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{T}}, \quad (5.58)$$

or

$$\boxed{\mathbf{T} = \frac{1}{\det(\mathbf{F})} \mathbf{F} \mathbf{T}_o}. \quad (5.59)$$

Depending on the author, either the the nominal (engineering) stress $\mathbf{T}_o$ or its transpose $\mathbf{T}_o^T$ is called the *first Piola-Kirchhoff stress tensor*. The more common definition for the first Piola-Kirchhoff stress tensor is to have it equal to the transpose of the nominal stress. As stated before, if the deformation and rigid body rotation are small so that the deformation gradient is close to identity, then the engineering stress is a close approximation to the Cauchy stress.

5.7 The Piola-Kirchhoff, rotated, and convected stresses

Several other stress tensors are used in mechanics. Of these are the rotated stress, the first Piola-Kirchhoff stress, the second Piola-Kirchhoff stress, and the convected stress. The *rotated stress* will be denoted by $\mathbf{T}^R$ and defined in terms of the Cauchy stress by the relation

$$\boxed{\mathbf{T}^R \equiv \mathbf{R}^T \mathbf{T} \mathbf{R}}, \quad (5.60)$$

where $\mathbf{R}$ is the orthogonal factor in the polar decomposition of $\mathbf{F} = \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{R}$. The rotated stress removes the rigid body rotation from the Cauchy stress and provides the traction (rotated appropriately) on the rotation-removed configuration through the relation

$$\mathbf{t}^{R(n_R)} = \hat{\mathbf{n}}^R \mathbf{T}^R = \mathbf{T}^{RT} \hat{\mathbf{n}}^R, \quad (5.61)$$

where $\hat{\mathbf{n}}^R = \mathbf{R}^T \hat{\mathbf{n}} = \hat{\mathbf{n}} \mathbf{R}$ is the normal in the intermediate rotation-removed configuration corresponding to the surface with normal $\hat{\mathbf{n}}$ in the current configuration. Here $\mathbf{t}^{R(n_R)} = \mathbf{R}^T \mathbf{t}^{(n)} = \mathbf{t}^{(n)} \mathbf{R}$. We also could call $\mathbf{T}^R$ the *rotation-removed stress* or the *unrotated stress*.

The *first Piola-Kirchhoff stress* $\bar{\mathbf{T}}_o$ is the transpose of the nominal stress so it is given by the equation

$$\bar{\mathbf{T}}_o \equiv \mathbf{T}_o^T = J \mathbf{T}^T \mathbf{F}^{-T} = J \bar{\mathbf{T}} \mathbf{F}^{-T}. \quad (5.62)$$

This results in the relation

$$\mathbf{t}_o^{(N)} = \bar{\mathbf{T}}_o \hat{\mathbf{N}} = \hat{\mathbf{N}} \bar{\mathbf{T}}_o^T. \quad (5.63)$$

The *second Piola-Kirchhoff stress* $\tilde{\mathbf{T}}$ is defined as

$$\tilde{\mathbf{T}} \equiv J \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-T} = \mathbf{T}_o \mathbf{F}^{-T}. \quad (5.64)$$

The *convected stress* is given as

$$\bar{\bar{\mathbf{T}}} \equiv \mathbf{F}^T \tilde{\mathbf{T}} \mathbf{F} = \mathbf{F}^T \mathbf{T} \mathbf{F}. \quad (5.65)$$

The *Kirchhoff stress* $\boldsymbol{\tau}$ is the product of the volume ratio and the Cauchy stress given by

$$\boldsymbol{\tau} \equiv J \mathbf{T}. \quad (5.66)$$

Section 5.7-Problems:

5.7-1: Show the identities

$$\mathbf{T} = \mathbf{R} \mathbf{T}^R \mathbf{R}^T, \quad (5.67)$$

$$\bar{\bar{\mathbf{T}}}^T = \frac{1}{J} \mathbf{C} \tilde{\mathbf{T}} \mathbf{C}. \quad (5.68)$$

5.8 Heat flux

We will introduce the concept of heat flux through a surface and then the heat flux vector. This will be done in a manner parallel to how at the start of this chapter the concept of traction vector was introduced and studied before we introduced the stress tensor.

As is schematically shown in Figure 5.14, let $\dot{Q}$ represent the amount of heat that is flowing through a surface A per unit time. The heat flux q_n through the surface with normal $\hat{\mathbf{n}}$ is defined as the limit of the ratio $\dot{Q}/A$ as A goes to zero around a point. That is, q_n is a scalar defined by

$$q_n \equiv \lim_{A \rightarrow 0} \frac{\dot{Q}}{A}. \quad (5.69)$$

The rate of heat flow $\dot{Q}$ out of a body with surface S can be calculated using q_n by integrating it over this surface. That is,

$$\dot{Q} \equiv \int_S q_n dA. \quad (5.70)$$

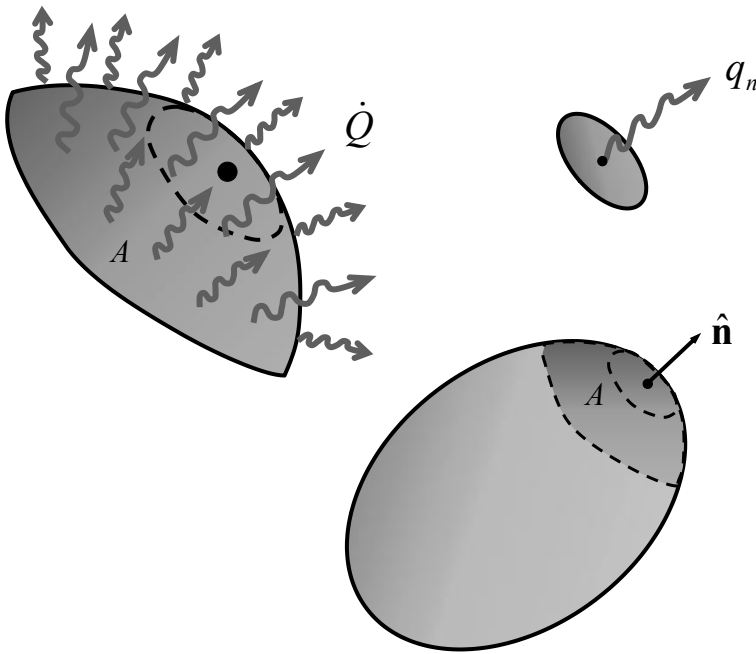


Figure 5.14: Schematic of heat flow $\dot{Q}$ through a surface A and of the heat flux q_n .

As was done for the traction, a tetrahedron can be constructed, similar to that given in Figure 5.6, and the heat flux through each of its three surfaces with normals $\hat{\mathbf{n}}_i$ can be denoted by q_{n_i} . Next, the balance of energy can be written for the tetrahedron. This law is described in more detail in Chapter 6. In general, this law states that the net heat flow into the point through conduction plus the net heat flow due to radiation plus the working of the stresses are equal to the rate of change of internal energy. This will be written for the tetrahedron as

$$-q_n A - q_{n_1} A_1 - q_{n_2} A_2 - q_{n_3} A_3 + tr(\mathbf{TL})V + \rho r V = \rho \dot{e} V, \quad (5.71)$$

where $tr(\mathbf{TL})$ is the working, r is the radiation or heat generation term, and e is the specific internal energy. As for the case of stress, this equation is first divided by A , and then we take the limit as h , the height of the tetrahedron, which goes to zero, noting that the ratios A_i/A are unaffected by the limit, but V/A goes to zero. The result of this process is

$$q_n = -q_{n_1} \frac{A_1}{A} - q_{n_2} \frac{A_2}{A} - q_{n_3} \frac{A_3}{A} = -q_{n_i} \frac{A_i}{A}. \quad (5.72)$$

Therefore, the heat flux through any surface with normal $\hat{\mathbf{n}}$ can be calculated from the heat flux through any other three surfaces at that same point, knowing the ratio of the respective areas of the three surfaces to the surface under consideration. Also, you will recall that the normal $\hat{\mathbf{n}}$ can be written as

$$\hat{\mathbf{n}} = -\frac{A_1}{A} \hat{\mathbf{n}}_1 - \frac{A_2}{A} \hat{\mathbf{n}}_2 - \frac{A_3}{A} \hat{\mathbf{n}}_3 = -\frac{A_i}{A} \hat{\mathbf{n}}_i, \quad (5.73)$$

so that the ratio of the surface areas is given by the components of $\hat{\mathbf{n}}$.

By examination of the results, it can be easily shown that a heat flux vector $\mathbf{q}$ at a point can be defined with the property that

$$q_n = \mathbf{q} \circ \hat{\mathbf{n}}. \quad (5.74)$$

Given the heat flux through three surfaces as described above and with normals to the surfaces given by $\hat{\mathbf{n}}_i$, the heat flux vector is defined by the relation

$$\mathbf{q} \equiv q_{n_i} g_{ij}^{-1} \hat{\mathbf{n}}_j = q_{n_i} \bar{\mathbf{n}}_i, \quad (5.75)$$

where $g_{ij} = \hat{\mathbf{n}}_i \circ \hat{\mathbf{n}}_j$ is the metric tensor for the base given by $\hat{\mathbf{n}}_i$, $\bar{\mathbf{n}}_i$ is the reciprocal of this base.

The primary property of the heat flux vector is that it allows us to calculate the heat flow through a surface with normal $\hat{\mathbf{n}}$ and differential surface area dA as

$$q_n dA = \mathbf{q} \circ \hat{\mathbf{n}} dA. \quad (5.76)$$

Therefore, the rate of heat flow $\dot{Q}$ out of surface S can be calculated by evaluating the integral

$$\dot{Q} = \int_S \mathbf{q} \circ \hat{\mathbf{n}} dA. \quad (5.77)$$

One can define an *engineering or nominal heat flux* $\mathbf{q}_o$ that has the property

$$q_n dA = \mathbf{q}_o \circ \hat{\mathbf{N}} dA_o, \quad (5.78)$$

where $\hat{\mathbf{N}}$ is the unit normal to the surface dA_o , where dA_o is the corresponding differential surface in the reference configuration for a differential surface dA in the current configuration. The same rate of heat flow $\dot{Q}$ now can be evaluated using this relation as

$$\dot{Q} = \int_{S_o} \mathbf{q}_o \circ \hat{\mathbf{N}} dA_o, \quad (5.79)$$

where S_o is the surface in the reference configuration corresponding to the surface S in the current configuration. Through procedures similar to that of calculating the relation between the Cauchy stress and the engineering stress, it can be shown that

$$\mathbf{q}_o = J \mathbf{F}^{-1} \mathbf{q}. \quad (5.80)$$

Since a vector is equal to its transpose, this also can be written as

$$\mathbf{q}_o = J \mathbf{q} \mathbf{F}^{-T}, \quad (5.81)$$

where $J = \det(\mathbf{F})$ is the volume ratio.

When the surface S covers the entire volume V , one can use the divergence theorem to show that the rate of heat flow out of the surface S surrounding the volume V in the current configuration is given by

$$\dot{Q} = \int_V \text{div}(\mathbf{q}) dA = \int_{V_o} \text{Div}(\mathbf{q}_o) dA_o, \quad (5.82)$$

where V_o is the corresponding volume in the reference configuration to the volume V in the current configuration. The surface surrounding V_o is S_o and contains the corresponding material particles that are currently on S .

Section 5.8-Problems:

- 5.8-1:** Show that the definition of the heat flux vector given in equation (5.75) results in the relation $q_n = \mathbf{q} \circ \hat{\mathbf{n}}$.
- 5.8-2:** Prove the relation given in equation (5.80) between the heat flux vector and the engineering heat flux vector.

5.9 Notation

The following table shows the symbols used in the text and their definitions.

$\mathbf{a}$		acceleration
A		area in current configuration
A_i		area or surface of tetrahedron perpendicular to unit normal $\hat{\mathbf{n}}_i$
A_o		area in reference configuration
$\mathbf{C}$		right Cauchy stretch tensor ($\mathbf{F}^T \mathbf{F}$)
$\mathbf{F}$		deformation gradient [$d\mathbf{x} = \mathbf{F}d\mathbf{X}$; $\mathbf{F} = \mathbf{Grad}(\mathbf{x})$]
g_{ij}		metric tensor of $\hat{\mathbf{n}}_i$
I_{iT}		isotropic invariants I_i of $\mathbf{T}$
$\mathbf{I}$		identity tensor
J		volume ratio [$\det(\mathbf{F})$]
J_{iT}		isotropic invariants J_i of $\mathbf{T}$
$\hat{\mathbf{n}}$		unit normal to the surface
$\hat{\mathbf{n}}_i$		unit normal to surface A_i
$\hat{\mathbf{N}}$		unit normal to surface in reference configuration
p		hydrostatic pressure ($-T_{ave}$)
$\mathbf{P}$		total load resulting from traction
q_n		heat flux through surface with normal $\hat{\mathbf{n}}$
$\mathbf{q}$		heat flux vector
$\mathbf{q}_o$		heat flux vector measured per unit area in reference configuration (nominal or engineering heat flux vector)
$\dot{Q}$		heat flow
$\mathbf{R}$		orthogonal factor in the polar decomposition of $\mathbf{F}$ ($\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, $\mathbf{U} = \mathbf{U}^T$, $\mathbf{V} = \mathbf{V}^T$)
$\mathbf{S}$		deviatoric stress
t		time
$\mathbf{t}^{(n)}$		traction vector on surface with unit normal $\hat{\mathbf{n}}$
$\mathbf{t}^{(n_i)}$		traction vector on surface of tetrahedron with unit normal $\hat{\mathbf{n}}_i$
$\mathbf{t}_o^{(N)}$		nominal traction on surface with normal $\mathbf{N}$
T_{ave}		average stress [$\frac{tr(\mathbf{T})}{3}$]
T_i		eigenvalues of $\mathbf{T}$
$\mathbf{T}$		Cauchy stress tensor
$\bar{\mathbf{T}}$		transpose of $\mathbf{T}$
$\mathbf{T}_o$		engineering (nominal) stress tensor

$\bar{\mathbf{T}}_o$		first Piola-Kirchhoff stress tensor ($\bar{\mathbf{T}}_o = \mathbf{T}_o^T$)
$\tilde{\mathbf{T}}$		second Piola-Kirchhoff stress tensor ($\tilde{\mathbf{T}} = \mathbf{T}_o \mathbf{F}^{-T}$)
$\bar{\mathbf{T}}$		convected stress tensor ($\bar{\mathbf{T}} = \mathbf{F}^T \mathbf{T}^T \mathbf{F} = \mathbf{F}^T \tilde{\mathbf{T}} \mathbf{F}$)
$\mathbf{T}^R$		rotated stress ($\mathbf{T}^R = \mathbf{R}^T \mathbf{T} \mathbf{R}$)
$\mathbf{U}$		right symmetric factor in the polar decomposition of $\mathbf{F}$ (see $\mathbf{R}$)
$\mathbf{V}$		left symmetric factor in the polar decomposition of $\mathbf{F}$ (see $\mathbf{R}$)
V		volume
ρ		density
σ		normal traction
τ		shear traction
$\boldsymbol{\tau}$		Kirchhoff stress ($\boldsymbol{\tau} = J \mathbf{T}$)

Chapter 6

Balance laws and jump conditions

The prediction of material response requires the combination of several elements. In general, these elements include mathematical models describing the material's response characteristics (constitutive equations), specific conditions describing the initial state of the matter (initial conditions), conditions describing how the specific body is being influenced by its surroundings (boundary conditions), and laws describing how to combine these elements (balance laws). The focus of this chapter will be on the balance laws. These laws have a special place in the theory of material response since they are the same for all materials, while other equations, such as the constitutive equations, change from material to material. This chapter also will look at the jump conditions, which are a specific form of the balance laws that hold across a discontinuity, such as a shock wave. The jump conditions relate variables on two sides of a discontinuity.

6.1 Introduction

The five laws that we will consider here, and which we collectively call the balance laws, include: the conservation of mass, the balance of linear momentum, the balance of angular momentum, the balance of work and energy, and the entropy production inequality. Each of the balance laws is a general statement on how all materials will respond over time, and can be used to calculate the specific response of a particular material body only when augmented by constitutive models for the specific material, and specific initial and boundary conditions describing the initial state of the material and the processing conditions.

The five balance laws can be described by the following statements. In the form presented, each balance law is a statement of how a material body made up of a fixed set of particles will behave. We will call such a material body that is made of a fixed set of particles a closed system or a control mass. Therefore, the primary focus of our presentation will be on a system in which particles are neither added nor subtracted from the system (i.e., the matter of the system remains the same). Alternate forms of the balance laws can be developed for open systems. These alternate forms are consistent with the forms presented for closed systems, and can be derived directly from the balance laws stated for a closed system.

1. **Conservation of mass:** The total mass of a material body remains constant if the particles in the body are kept fixed.

$$\frac{d}{dt}(\text{Mass of a body}) = 0.$$

2. **Balance of linear momentum:** The rate of change of linear momentum of a body at each instant is equal to the resultant force applied on that body.

$$\frac{d}{dt}(\text{Linear momentum}) = \sum \text{Forces applied on the body.}$$

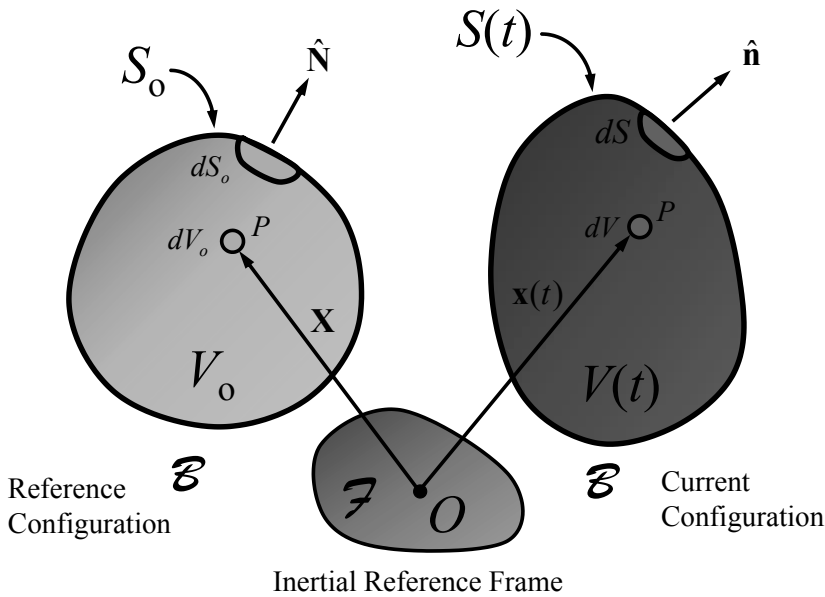


Figure 6.1: The material body in the reference configuration and the current configuration.

3. **Balance of angular momentum:** The rate of change of angular momentum about an inertial point O of a body at each instant is equal to the resultant moment applied on that body about the same point O .

$$\frac{d}{dt}(\text{Angular momentum about } O) = \sum \text{Moment applied on the body about } O.$$

4. **Balance of work and energy:** The rate of change of internal energy (IE) plus kinetic energy (KE) of a material body at each instant are equal to the rate at which heat flows into the body ($\dot{Q}$) plus the rate at which work is done on the body ($\dot{W}$).

$$\frac{d}{dt}(IE + KE) = \dot{Q} + \dot{W}.$$

5. **Entropy production inequality:** The rate of increase of entropy of a body at each instant is at least as large as the rate at which entropy is added to the body by heat flow into the body.

$$\frac{d}{dt}(\text{Entropy of the body}) \geq \text{Rate of entropy flow into the body by heat.}$$

In the following sections we will develop more specific forms of these balance laws using specific expressions for each of the terms. It turns out that each term in these balance laws is itself an integral over the body or the surface of the body. As shown in Figure 6.1, we will use $\mathcal{B}$ to denote the material body, $V(t)$ to denote the volume of space occupied by the material body at time t , and $S(t)$ to denote the surface area of $V(t)$. Also, V_o and S_o will denote

domain of integration changes with time. For example, consider the volume shown in Figure 6.2 that occupies the space V_o^* at time t_o and $V^*(t)$ at time t . Let $\Phi^*(t)$ denote the total amount of some quantity of interest inside $V^*(t)$, and let ϕ denote its density. Let us assume we can evaluate $\Phi^*(t)$ by using the integral of its density through the relation

$$\Phi^*(t) \equiv \int_{V^*(t)} \phi dV^*. \quad (6.1)$$

The rate of change of Φ^* is given by

$$\frac{d\Phi^*(t)}{dt} = \frac{d}{dt} \left(\int_{V^*(t)} \phi dV^* \right). \quad (6.2)$$

As can be seen, the rate of change of Φ^* depends both on how ϕ changes with time and also on how the volume $V^*(t)$ changes with time. For the purposes of developing the most general transport relations, we will let the volume $V^*(t)$ change in an arbitrary manner, only assuming that there is a mapping $\mathbf{u}^*(\mathbf{x}_o^*, t)$ that describes the displacement of each point $\mathbf{x}_o^*$ in V_o^* to a point $\mathbf{x}^*$ in $V^*(t)$. In particular, we assume that this displacement field maps the boundary of V_o^* onto the current boundary of $V^*(t)$. Since this displacement field is used only to describe $V^*(t)$ in terms of V_o^* , its form inside the volume is not particularly important as long as it is sufficiently smooth to allow the calculation of a Jacobian, needed later in the development, and that it map points on the boundary of V_o^* onto points on the boundary of $V^*(t)$.

The function ϕ can be written either in terms of $\mathbf{x}^*$ and t , or, using $\mathbf{u}^*(\mathbf{x}_o^*, t)$, it can be written in terms of $\mathbf{x}_o^*$ and t . Let these two functional forms of ϕ be described by the notation

$$\phi = \phi^\dagger(\mathbf{x}^*, t) = \phi^{\dagger\dagger}(\mathbf{x}_o^*, t).$$

Using the Jacobian $J^*(\mathbf{x}_o^*, t) = dV^*/dV_o^* = \det[\mathbf{grad}_{\mathbf{x}_o^*}(\mathbf{x}^*)] = \det[\nabla_{\mathbf{x}_o^*}(\mathbf{x}^*)]$ to change the domain of integration, it is easy to see that

$$\Phi^*(t) = \int_{V^*(t)} \phi^\dagger(\mathbf{x}^*, t) dV^* = \int_{V_o^*} \phi^{\dagger\dagger}(\mathbf{x}_o^*, t) J^*(\mathbf{x}_o^*, t) dV_o^*.$$

Since the initial volume V_o^* does not change with time, one can calculate the rate of change of Φ^* by using the expression

$$\frac{d\Phi^*(t)}{dt} = \frac{d}{dt} \left[\int_{V_o^*} \phi^{\dagger\dagger}(\mathbf{x}_o^*, t) J^*(\mathbf{x}_o^*, t) dV_o^* \right] = \int_{V_o^*} \frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t) J^*(\mathbf{x}_o^*, t)]}{\partial t} dV_o^*.$$

The reader will note that the total time derivative when moved into the argument becomes a partial time derivative that is taken while holding $\mathbf{x}_o^*$ constant. The first form of the transport theorem is complete once we reuse the Jacobian to change the domain of integration to $V^*(t)$ and obtain

$$\frac{d\Phi^*(t)}{dt} = \int_{V^*(t)} \frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t) J^*(\mathbf{x}_o^*, t)]}{\partial t} \frac{1}{J^*} dV^*. \quad (6.3)$$

The reader should note that even though we have used notation similar to what was used to describe motion and deformation, in this case *the mapping from V_o^* to $V^*(t)$ is totally arbitrary* and is not connected to any real material flow. Yet, if one selects the mapping to follow the motion of particles in a material body, then the volume $V^*(t)$ is the volume currently occupied by the particles that were initially in V_o^* and the partial derivative in the argument of the integral can be replaced by a material time derivative.

Next, let us take advantage of the mapping between V_o^* and $V^*(t)$ to obtain the relation

$$\begin{aligned} \frac{\partial \phi^{\dagger\dagger}(\mathbf{x}_o^*, t)}{\partial t} &= \frac{\partial \phi^\dagger[\mathbf{x}^*(\mathbf{x}_o^*, t), t]}{\partial t} \Big|_{\mathbf{x}_o^*} = \mathbf{grad}_{\mathbf{x}^*}[\phi^\dagger(\mathbf{x}^*, t)] \circ \frac{\partial \mathbf{x}^*(\mathbf{x}_o^*, t)}{\partial t} \Big|_{\mathbf{x}_o^*} + \frac{\partial \phi^\dagger(\mathbf{x}^*, t)}{\partial t} \Big|_{\mathbf{x}^*} \\ &\quad \frac{\partial \mathbf{x}^*(\mathbf{x}_o^*, t)}{\partial t} \Big|_{\mathbf{x}_o^*} \circ \nabla_{\mathbf{x}^*}[\phi^\dagger(\mathbf{x}^*, t)] + \frac{\partial \phi^\dagger(\mathbf{x}^*, t)}{\partial t} \Big|_{\mathbf{x}^*}, \end{aligned}$$

where we assume that ϕ may be any scalar, vector, or tensor field, and so the order of the gradient and nabla operators should be kept as shown. One can simply write this as

$$\begin{aligned} \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}_o^*} &= \mathbf{grad}_{\mathbf{x}^*}(\phi) \circ \frac{\partial \mathbf{x}^*}{\partial t} \Big|_{\mathbf{x}_o^*} + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*} \\ &= \frac{\partial \mathbf{x}^*}{\partial t} \Big|_{\mathbf{x}_o^*} \circ \nabla_{\mathbf{x}^*}(\phi) + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*}. \end{aligned}$$

Here $\frac{\partial \mathbf{x}^*}{\partial t} \Big|_{\mathbf{x}_o^*}$ is the current velocity of the point (not necessarily a material point) originally at location $\mathbf{x}_o^*$ as described by the motion $\mathbf{u}^*(\mathbf{x}_o^*, t)$. Denoting this velocity by $\mathbf{v}^*$ and substituting into the above expression yields

$$\begin{aligned} \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}_o^*} &= \mathbf{grad}_{\mathbf{x}^*}(\phi) \circ \mathbf{v}^* + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*} \\ &= \mathbf{v}^* \circ \nabla_{\mathbf{x}^*}(\phi) + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*}. \end{aligned} \quad (6.4)$$

If we replace $\mathbf{x}_o^*$ with the position in the reference configuration ($\mathbf{X}$), and replace $\mathbf{u}^*$ with the motion described by how the actual material points move ($\mathbf{u}$), then the left-hand side becomes the material time derivative (D/Dt) and the velocity $\mathbf{v}^*$ becomes the actual particle velocity ($\mathbf{v}$), giving the expression

$$\begin{aligned} \frac{D\phi}{Dt} &= \mathbf{grad}_{\mathbf{x}}(\phi) \circ \mathbf{v} + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}} \\ &= \mathbf{v} \circ \nabla_{\mathbf{x}}(\phi) + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}}, \end{aligned} \quad (6.5)$$

which is similar to the expression obtained for the spatial or Eulerian representation of acceleration. Equating $\mathbf{x}$ and $\mathbf{x}^*$, and then subtracting (6.5) from (6.4) yields

$$\begin{aligned} \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}_o^*} &= \frac{D\phi}{Dt} + \mathbf{grad}_{\mathbf{x}}(\phi) \circ (\mathbf{v}^* - \mathbf{v}) \\ &= \frac{D\phi}{Dt} + (\mathbf{v}^* - \mathbf{v}) \circ \nabla_{\mathbf{x}}(\phi), \end{aligned} \quad (6.6)$$

which gives the relation between the material time derivative and time derivatives taken when holding $\mathbf{x}_o^*$ constant.

Let us use the relation (6.4) to expand the argument of the integral in (6.3) to obtain

$$\begin{aligned} \frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t)J^*(\mathbf{x}_o^*, t)]}{\partial t} \frac{1}{J^*} &= \mathbf{grad}_{\mathbf{x}^*}(\phi) \circ \mathbf{v}^* + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*} + \frac{\phi}{J^*} \frac{\partial J^*}{\partial t} \Big|_{\mathbf{x}_o^*} \\ &= \mathbf{v}^* \circ \nabla_{\mathbf{x}^*}(\phi) + \frac{\partial \phi}{\partial t} \Big|_{\mathbf{x}^*} + \frac{\phi}{J^*} \frac{\partial J^*}{\partial t} \Big|_{\mathbf{x}_o^*}. \end{aligned}$$

Next, let us use the relation

$$\frac{\partial J^*}{\partial t} \Big|_{\mathbf{x}_o^*} = J^* \mathbf{F}^{*-T} : \dot{\mathbf{F}}^* = J^* \text{div}_{\mathbf{x}^*}(\mathbf{v}^*),$$

where $\mathbf{F}^* = \mathbf{grad}_{\mathbf{x}_o^*}(\mathbf{x}^*) = [\nabla_{\mathbf{x}_o^*}(\mathbf{x}^*)]^T$ is the deformation gradient associated with the mapping from V_o^* to $V^*(t)$. Substituting this relation yields

$$\begin{aligned} \frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t)J^*(\mathbf{x}_o^*, t)]}{\partial t} \frac{1}{J^*} &= \mathbf{grad}_{\mathbf{x}^*}(\phi) \circ \mathbf{v}^* + \frac{\partial\phi}{\partial t}|_{\mathbf{x}_o^*} + \phi \mathit{div}_{\mathbf{x}^*}(\mathbf{v}^*) \\ &= \mathbf{v}^* \circ \nabla_{\mathbf{x}^*}(\phi) + \frac{\partial\phi}{\partial t}|_{\mathbf{x}_o^*} + \phi \mathit{div}_{\mathbf{x}^*}(\mathbf{v}^*). \end{aligned}$$

We now will recall that for scalar α , vectors $\mathbf{a}$ and $\mathbf{v}$, and a second-order tensor $\mathbf{A}$ we have the identities

$$\begin{aligned} \mathit{div}(\alpha\mathbf{v}) &= \mathbf{grad}(\alpha) \circ \mathbf{v} + \alpha \mathit{div}(\mathbf{v}), \\ \mathit{div}(\mathbf{a} \otimes \mathbf{v}) &= \mathbf{grad}(\mathbf{a}) \mathbf{v} + \mathbf{a} \mathit{div}(\mathbf{v}), \\ \mathit{div}(\mathbf{A} \otimes \mathbf{v}) &= \mathbf{grad}(\mathbf{A})\mathbf{v} + \mathbf{A} \mathit{div}(\mathbf{v}). \end{aligned}$$

We use one of these identities to combine the first and last terms in our equations to form $\mathit{div}_{\mathbf{x}^*}(\phi\mathbf{v}^*)$ or $\mathit{div}_{\mathbf{x}^*}(\phi \otimes \mathbf{v}^*)$, depending on ϕ . This can be substituted into the integral to get

$$\frac{d\Phi^*(t)}{dt} = \int_{V^*(t)} \left[\frac{\partial\phi}{\partial t}|_{\mathbf{x}^*} + \mathit{div}_{\mathbf{x}^*}(\phi\mathbf{v}^*) \right] dV^*,$$

or

$$\frac{d\Phi^*(t)}{dt} = \int_{V^*(t)} \left[\frac{\partial\phi}{\partial t}|_{\mathbf{x}^*} + \mathit{div}_{\mathbf{x}^*}(\phi \otimes \mathbf{v}^*) \right] dV^*.$$

We will use the divergence theorem to replace the volume integral associated with the last term by a surface integral to get

$$\boxed{\frac{d\Phi^*(t)}{dt} = \int_{V^*(t)} \frac{\partial\phi}{\partial t}|_{\mathbf{x}^*} dV^* + \int_{S^*(t)} \phi\mathbf{v}^* \circ \hat{\mathbf{n}}^* dS^*,} \quad (6.7)$$

where $\hat{\mathbf{n}}^*$ is the unit outward normal to the surface $S^*(t)$. This is the second form of the transport theorem. The next form of the transport theorem is obtained by taking advantage of relation (6.6) between material and spatial time derivatives. If at the current time the material volume $V(t)$ and the arbitrarily moving volume $V^*(t)$ coincide, each material point can be associated to a point on $V^*(t)$. In this case, we can take $\mathbf{x} = \mathbf{x}^*$ for the current time and show that

$$\frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t)J^*(\mathbf{x}_o^*, t)]}{\partial t} \frac{1}{J^*} = \frac{D\phi}{Dt} + \phi \mathit{div}_{\mathbf{x}}(\mathbf{v}) + \mathit{div}_{\mathbf{x}}[\phi(\mathbf{v}^* - \mathbf{v})],$$

or

$$\frac{\partial[\phi^{\dagger\dagger}(\mathbf{x}_o^*, t)J^*(\mathbf{x}_o^*, t)]}{\partial t} \frac{1}{J^*} = \frac{D\phi}{Dt} + \phi \mathit{div}_{\mathbf{x}}(\mathbf{v}) + \mathit{div}_{\mathbf{x}}[\phi \otimes (\mathbf{v}^* - \mathbf{v})].$$

This results in the third form of the general transport theorem given by

$$\boxed{\frac{d\Phi^*(t)}{dt} = \int_{V(t)} \left[\frac{D\phi}{Dt} + \phi \mathit{div}_{\mathbf{x}}(\mathbf{v}) \right] dV + \int_{S(t)} \phi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS.} \quad (6.8)$$

As the development suggests, even though $d\Phi^*(t)/dt$ is the rate of change of Φ^* in $V^*(t)$, since $V(t)$ and $V^*(t)$ are the same at the current time, the integrals may be taken either over $V(t)$ or $V^*(t)$ and $S(t)$ or $S^*(t)$.

Let us now consider what the form of the transport relations will be if the arbitrary moving volume $V^*(t)$ is replaced by the material volume $V(t)$ that follows a fixed set of

particles. Let us use Φ to designate the integral of ϕ over the material volume $V(t)$. Since the boundary of $V(t)$ moves with the particles, the particle velocity and the velocity of the boundary are the same. That is, $\mathbf{v}^* = \mathbf{v}$ and the transport relations become

$$\boxed{\frac{D\Phi}{Dt} = \int_{V(t)} \frac{D(\phi J)}{Dt} \frac{1}{J} dV,} \quad (6.9)$$

$$\boxed{\frac{D\Phi}{Dt} = \int_{V(t)} \frac{\partial \phi}{\partial t} |_{\mathbf{x}} dV + \int_{S(t)} \phi \mathbf{v} \circ \hat{\mathbf{n}} dS,} \quad (6.10)$$

$$\boxed{\frac{D\Phi}{Dt} = \int_{V(t)} \left[\frac{D\phi}{Dt} + \phi \operatorname{div}_{\mathbf{x}}(\mathbf{v}) \right] dV.} \quad (6.11)$$

Also, for any arbitrarily moving volume $V^*(t)$ it can be shown that

$$\boxed{\frac{d\Phi^*(t)}{dt} = \frac{D\Phi(t)}{Dt} + \int_{S(t)} \phi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS,} \quad (6.12)$$

where as above $D\Phi(t)/Dt$ is the rate of change of Φ if we follow the integral of ϕ over a volume $V(t)$ that currently coincides with $V^*(t)$, but which changes in a way that follows the material particles in $V^*(t)$. The proof of this is left as a homework.

Section 6.2-Problems:

6.2-1: Show that

$$\frac{\partial J^*}{\partial t} |_{\mathbf{x}_0^*} = J^* \operatorname{div}_{\mathbf{x}^*}(\mathbf{v}^*).$$

6.2-2: Show that $d\Phi^*/dt$ for any arbitrarily moving volume $V^*(t)$ is given by the relation

$$\frac{d\Phi^*(t)}{dt} = \frac{D\Phi(t)}{Dt} + \int_{S(t)} \phi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS,$$

where, as above, $D\Phi(t)/Dt$ is the rate of change of Φ if we follow the integral of ϕ over a volume $V(t)$ that currently coincides with $V^*(t)$, but which changes in a way that follows the material particles in $V^*(t)$.

6.2-3: Show that

$$\boxed{\frac{D}{Dt} \int_{V(t)} dV = \int_{V(t)} \operatorname{tr}(\mathbf{D}) dV,}$$

where $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the deformation rate tensor.

6.3 Conservation of mass

Conservation of mass states that the mass in a body will not change if the particles in the body remain the same. This is mathematically written as

$$\boxed{\frac{d}{dt} \int_{\mathcal{B}} dm = 0,} \quad (6.13)$$

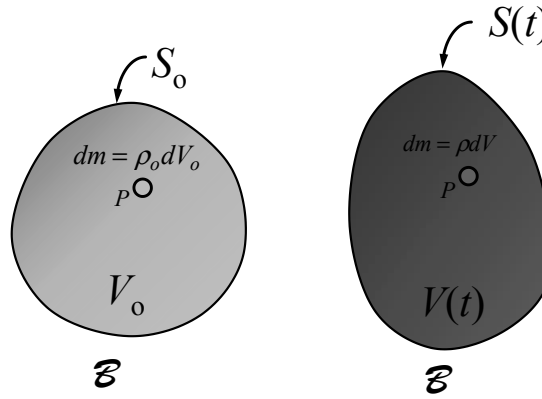


Figure 6.3: The relation between mass, volume, and density in the reference configuration (left) and the current configuration (right).

where dm is the differential element of mass. Using the relation between mass, density, and volume shown in Figure 6.3, this law can be rewritten as

$$\boxed{\frac{D}{Dt} \int_{V(t)} \rho dV = 0,} \quad (6.14)$$

where ρ is the current density, and $V(t)$ is the volume occupied by the body as it moves and deforms. This can be written as

$$\frac{D}{Dt} \int_{V_o} \rho J dV_o = 0, \quad (6.15)$$

since $J dV_o = dV$. As the domain of integration V_o does not change with time, one can move the time derivative into the integral to obtain

$$\int_{V_o} \frac{D(\rho J)}{Dt} dV_o = 0. \quad (6.16)$$

The volume V_o may be taken to be the volume of any part of the body and this relation should still hold. As was shown in Chapter 1 for a line integral, one can argue that if the argument of a volume integral is continuous and that the integral must be zero for all possible selections of its domain of integration, then the argument of the integral must be equal to zero at each and every point in space. Using this result one can conclude that if $D(\rho J)/Dt$ is continuous on the body, since this law must hold for every segment of the body, then $D(\rho J)/Dt$ must be equal to zero at all points in the body. This results in the differential form of the conservation of mass given as

$$\frac{D(\rho J)}{Dt} = 0, \quad (6.17)$$

which must hold at every point of the body. The same result can be derived using the transport theorem (6.9) to (6.14) followed by similar arguments as above. One can directly integrate (6.17) to obtain the relation

$$\boxed{\rho J = \rho_o,} \quad (6.18)$$

where ρ_o is the density in the reference configuration where we note that $J_o = 1$.

Application of the transport theorem (6.11) to equation (6.14) results in

$$\int_{V(t)} \left[\frac{D\rho}{Dt} + \rho \operatorname{div}(\mathbf{v}) \right] dV = 0,$$

where the reader recalls that div is the same as $\operatorname{div}_{\mathbf{x}}$ denoting the divergence with respect to movement in the current configuration. Again, this relation must hold for every volume $V(t)$ of the body so that if the argument of the integral is continuous over the body, this law requires that the argument be uniformly zero at all points in the body. Therefore, the differential form of the conservation of mass can be written as

$$\boxed{\frac{D\rho}{Dt} + \rho \operatorname{div}(\mathbf{v}) = 0}, \quad (6.19)$$

which must hold at every point in the body. Using the relation (6.5) to replace the material time derivative with the spatial representation results in

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0}. \quad (6.20)$$

A consequence of the conservation of mass is a new transport relation. Consider any quantity Ψ defined by

$$\Psi(t) = \int_B \psi dm = \int_{V(t)} \psi \rho dV, \quad (6.21)$$

where ψ is the distribution of Ψ per unit mass of the body. The material time derivative of Ψ can be calculated using (6.9) and is given by

$$\frac{D\Psi}{Dt} = \int_{V(t)} \frac{D(\psi \rho J)}{Dt} \frac{1}{J} dV.$$

As a consequence of conservation of mass, we know that the product $\rho J = \rho_o$ is a constant. Therefore, this results in a modified form of the transport theorem (6.9) given by

$$\boxed{\frac{D\Psi}{Dt} = \int_{V(t)} \frac{D\psi}{Dt} \rho dV}. \quad (6.22)$$

The reader will note that this transport relation is the result of combining the transport relation (6.9) with conservation of mass, and, as such, might not be as general as the original transport relation.

Section 6.3-Problems:

6.3-1: Show that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0.$$

6.3-2: Derive the following relation and interpret its meaning.

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV = - \int_{S(t)} \rho \mathbf{v} \circ \hat{\mathbf{n}} dS.$$

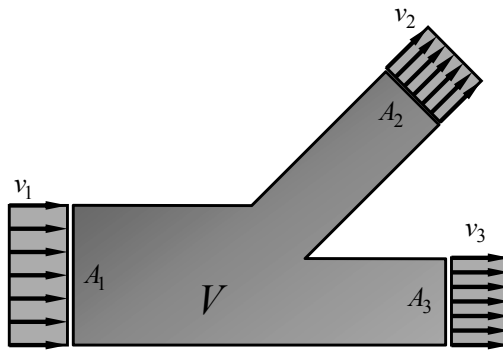


Figure 6.4: A section of a pipe.

- 6.3-3:** The section of the pipe shown has fluid flowing into it at a uniform speed of v_1 through the inlet with cross-sectional area A_1 , and fluid exiting it with uniform velocities v_2 and v_3 , respectively, through the outlets with cross-sectional areas A_2 and A_3 , as shown in Figure 6.4. If the section shown is stationary, and the density of the fluid is constant and equal to ρ , use the relation

$$\frac{d\Phi^*(t)}{dt} = \frac{D\Phi(t)}{Dt} + \int_{S(t)} \phi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS$$

applied to the conservation of mass to find the relation between the three velocities and the three cross-sectional areas.

- 6.3-4:** In a drawing process, the shape of the drawing zone remains the same even though it moves with velocity $\bar{v}$, as shown in Figure 6.5. To analyze this process, a volume is taken in the form of the drawing zone, as shown in the figure with dashed lines, that moves with the same speed as the drawing zone. Knowing that material is moving with a uniform velocity v_1 at the section with cross-sectional area A_1 , and is being drawn away with velocity v_2 at the section with cross-sectional area A_2 , find the relation between the three velocities and the cross-sectional areas using the conservation of mass and the relation given in the last problem, assuming density is constant for this material.

6.4 Balance of linear momentum

The law of balance of linear momentum states that the resultant of all applied forces on a material body is equal to the rate of change of linear momentum for that material body. As shown in Figure 6.6, one can apply both body forces and surface traction forces on a body. Body forces, such as the force due to gravity, are applied directly to each particle of the body, where as traction forces on the surface of the body are due to surface interaction

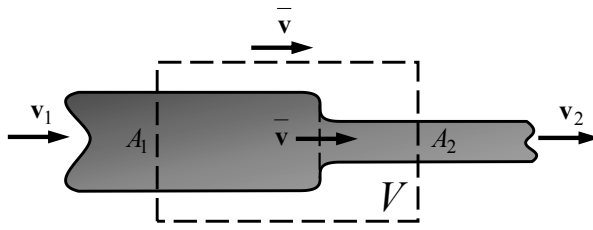


Figure 6.5: A self-similar drawing process in which the drawing zone is moving at a speed of $\bar{v}$.

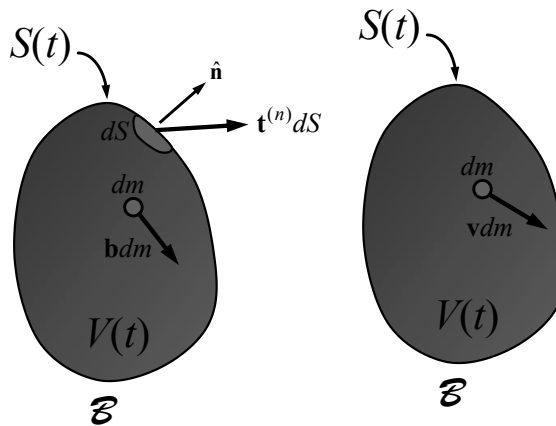


Figure 6.6: The left shows the surface force $\mathbf{t}^{(n)}dS$ that results from traction $\mathbf{t}^{(n)}$ applied on a surface area dS , and also shows the body force $\mathbf{b}dm$ that is applied on a particle of mass dm of the body. The right shows the linear momentum of each particle of mass dm and velocity $\mathbf{v}$ in the body.

between different bodies. The law of balance of linear momentum can be written as

$$\boxed{\int_{S(t)} \mathbf{t}^{(n)} dS + \int_B \mathbf{b} dm = \frac{d}{dt} \left(\int_B \mathbf{v} dm \right)}, \quad (6.23)$$

where the first integral represents the resultant force due to traction on the surface of the body, the second integral represents the resultant body force, and the third integral represents the linear momentum of the body. In this expression $\mathbf{b}$ is the body force per unit mass of the body (for example, $\mathbf{b}$ may be the acceleration of gravity $\mathbf{g}$). In the form written, this law only holds when considered from the point of view of an observer fixed to an *inertial frame of reference*. One easily can rewrite this, and all other balance laws, with respect to a noninertial observer. However, we would have to introduce terms resulting from the motion of the reference frame into the resulting expressions.

In the expression for the balance of linear momentum, the traction vector can be replaced by the Cauchy stress tensor through the relation $\mathbf{t}^{(n)} = \mathbf{T}^T \hat{\mathbf{n}}$, and the integration over mass

can be replaced by integration over volume through the relation $dm = \rho dV$. The result is the following expression for this balance law

$$\int_{S(t)} \mathbf{T}^T \hat{\mathbf{n}} dS + \int_{V(t)} \mathbf{b} \rho dV = \frac{D}{Dt} \int_{V(t)} \mathbf{v} \rho dV. \quad (6.24)$$

With the use of the special form of the transport theorem given in (6.22), the material time derivative can be moved into the argument of the integral to get

$$\boxed{\int_{S(t)} \mathbf{T}^T \hat{\mathbf{n}} dS + \int_{V(t)} \mathbf{b} \rho dV = \int_{V(t)} \frac{D\mathbf{v}}{Dt} \rho dV,} \quad (6.25)$$

where the material time derivative of the velocity can be replaced by the acceleration $\mathbf{a}$. Applying the divergence theorem we get

$$\int_{V(t)} \operatorname{div}(\mathbf{T}^T) dV + \int_{V(t)} \mathbf{b} \rho dV = \int_{V(t)} \mathbf{a} \rho dV, \quad (6.26)$$

where the reader recalls that div is the same as $\operatorname{div}_{\mathbf{x}}$ denoting the divergence with respect to movement in the current configuration. A differential form of this integral equation can be obtained from the realization that this equation also must hold for every subregion of $V(t)$. Therefore, if the argument of the integral resulting from the combination of the three integrals is continuous, then one must conclude that

$$\boxed{\operatorname{div}(\mathbf{T}^T) + \rho \mathbf{b} = \rho \mathbf{a}.} \quad (6.27)$$

This is the commonly used differential form of the law, but which can be written in the forms

$$\boxed{\operatorname{div}(\bar{\mathbf{T}}) + \rho \mathbf{b} = \rho \mathbf{a},} \quad (6.28)$$

$$\boxed{\nabla_{\mathbf{x}} \circ (\mathbf{T}) + \rho \mathbf{b} = \rho \mathbf{a},} \quad (6.29)$$

where we recall that $\bar{\mathbf{T}} = \mathbf{T}^T$ to give $\mathbf{t}^{(n)} = \bar{\mathbf{T}} \hat{\mathbf{n}} = \hat{\mathbf{n}} \mathbf{T}$, and $\nabla_{\mathbf{x}} \circ (\mathbf{T}) = [\operatorname{div}(\mathbf{T}^T)]^T = \operatorname{div}(\mathbf{T}^T)$.

In a *general curvilinear coordinate system*, we can write the differential form of this law as

$$\frac{\partial(T_{ij} \mathbf{e}_j^* \otimes \mathbf{e}_i^*)}{\partial \beta_k} \circ \bar{\mathbf{e}}_k^* + \rho b_j \mathbf{e}_j^* = \rho a_j \mathbf{e}_j^*, \quad (6.30)$$

where β_i are the coordinates used to describe the current configuration, $\mathbf{e}_i^*$ are its associated bases, $g_{kp}^* = \mathbf{e}_k^* \circ \mathbf{e}_p^*$ is the metric tensor associated with this base, and $\bar{\mathbf{e}}_i^* = g_{ij}^{*-1} \mathbf{e}_j^*$ is the associated dual base. For a *fixed base* $\mathbf{e}_i^*$, we can take the base vectors out of the derivative and the resulting form of this can be written as

$$\frac{\partial T_{ij}}{\partial \beta_i} \mathbf{e}_j^* + \rho b_j \mathbf{e}_j^* = \rho a_j \mathbf{e}_j^*, \quad (6.31)$$

which provides the component form of the equations in any fixed base (orthogonal or not) as

$$\frac{\partial T_{ij}}{\partial \beta_i} + \rho b_j = \rho a_j. \quad (6.32)$$

For a *Cartesian base*, $\mathbf{e}_i^* = \hat{\mathbf{e}}_i$, but varying in space, this can be written as

$$\frac{\partial(T_{ij} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i)}{\partial \beta_k} \circ \hat{\mathbf{e}}_k + \rho b_j \hat{\mathbf{e}}_j = \rho a_j \hat{\mathbf{e}}_j. \quad (6.33)$$

In addition, if we select a *rectangular coordinate system* $\beta_i = x_i$, that has fixed orthonormal base vectors $\mathbf{e}_i^* = \hat{\mathbf{i}}_i$, the base does not change from point-to-point so we get

$$\frac{\partial T_{ij}}{\partial x_i} + \rho b_j = \rho a_j. \quad (6.34)$$

The domain of integration in (6.26) will change with the deformation of the body. To avoid this change, it is sometimes desirable to work in the reference configuration. This can be done as follows. We can use relation (5.49) for the total load $\mathbf{P}$ applied on the surface of the body given as

$$\mathbf{P} = \int_V \text{div}(\mathbf{T}^T) dV = \int_{V_o} \text{Div}(\mathbf{T}_o^T) dV_o,$$

where $\mathbf{T}_o$ is the engineering stress (the transpose of which is normally termed the first Piola-Kirchhoff stress) given by relation (5.58) in terms of the Cauchy stress as

$$\mathbf{T}_o = \det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{T},$$

and Div is the same as $\text{div}_{\mathbf{X}}$ denoting the divergence with respect to motion in the reference configuration. Changing the last two volume integrals in (6.26) using the relation $dV = J dV_o$ and introducing conservation of mass ($\rho J = \rho_o$) yield the relation

$$\int_{V_o} \text{Div}(\mathbf{T}_o^T) dV_o + \int_{V_o} \mathbf{b} \rho_o dV_o = \int_{V_o} \mathbf{a} \rho_o dV_o. \quad (6.35)$$

This must hold not only for the entire body, but also for any segment of it. Therefore, if the functions in the arguments are continuous functions of position $\mathbf{X}$, then we obtain the alternate differential form of this balance law as

$$\boxed{\text{Div}(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o \mathbf{a}}, \quad (6.36)$$

which must hold for each point in the reference configuration. The index form of the equation is similar to that described for the current configuration and has the general form

$$\frac{(\partial T_{o-ij} \mathbf{e}_j \otimes \mathbf{e}_i)}{\partial \alpha_k} \circ \bar{\mathbf{e}}_k + \rho_o b_j \mathbf{e}_j = \rho_o a_j \mathbf{e}_j, \quad (6.37)$$

where α_i is the coordinate system used in the reference configuration, $\mathbf{e}_i$ and $\bar{\mathbf{e}}_i = g_{ij}^{-1} \mathbf{e}_j$ are, respectively, its associated base vectors and dual base, and g_{ij} is the associated metric tensor of $\mathbf{e}_i$. For a fixed rectangular system X_i , we can write the component equations as

$$\frac{\partial T_{o-ij}}{\partial X_i} + \rho_o b_j = \rho_o a_j. \quad (6.38)$$

It is sometimes desirable to work in a noninertial frame. To get the form of this balance law when using a noninertial frame $\mathcal{F}^*$, as opposed to the inertial frame $\mathcal{F}$ shown in Figure 6.7, we simply need to realize that the acceleration in the above equation can be replaced by

$$\boxed{\mathbf{a} = \mathbf{a}^* + 2 \boldsymbol{\omega} \times \mathbf{v}^* + \boldsymbol{\alpha} \times \mathbf{r}^* + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^*) + \ddot{\mathbf{R}}}, \quad (6.39)$$

where $\mathbf{a}$ is the acceleration of any particle with respect to the origin of the inertial frame $\mathcal{F}$; $\mathbf{r}^*$, $\mathbf{v}^*$, and $\mathbf{a}^*$ are, respectively, the position, velocity and acceleration of the same particle with respect to the origin of $\mathcal{F}^*$ as seen by an observer on $\mathcal{F}^*$; $\mathbf{R}$ is the position of the origin of $\mathcal{F}^*$ with respect to the origin of $\mathcal{F}$; $\ddot{\mathbf{R}}$ is the acceleration of the origin of $\mathcal{F}^*$ with respect to the origin of $\mathcal{F}$ as seen by an observer on $\mathcal{F}$; and $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are, respectively, the angular

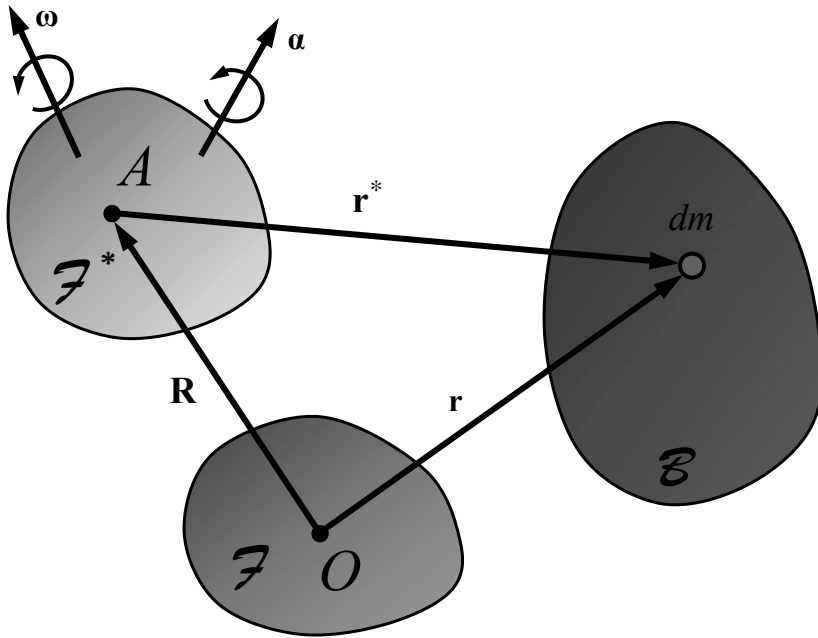


Figure 6.7: The relation between the expression of the balance of linear momentum for an inertial reference frame $\mathcal{F}$ and a noninertial reference frame $\mathcal{F}^*$ is obtained by relating the accelerations recorded by observers in the two frames.

velocity and acceleration of $\mathcal{F}^*$ with respect to $\mathcal{F}$. To obtain the appropriate equations one can simply substitute (6.39) for the acceleration in the balance law above. For example, the “generalized” form of the balance of linear momentum as given in (6.36) becomes

$$\boxed{\text{Div}(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o [\mathbf{a}^* + 2 \boldsymbol{\omega} \times \mathbf{v}^* + \boldsymbol{\alpha} \times \mathbf{r}^* + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^*) + \ddot{\mathbf{R}}]}. \quad (6.40)$$

Section 6.4-Problems:

6.4-1: An object of density ρ_s and volume V_s is sitting in a static fluid of density ρ_f as shown in Figure 6.8. Take the expression for the stress in a fluid to be given by $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$, where p is the pressure and μ is the viscosity. Given that the acceleration of gravity is g ,

1. Write the equation for the balance of linear momentum for this material under static conditions.
2. Show that one can solve this to get the stress expressed as

$$\mathbf{T} = -p_o \mathbf{I} + \rho_f g (\mathbf{x} \circ \hat{\mathbf{j}}) \mathbf{I}, \quad (6.41)$$

where p_o is the pressure on the surface of the fluid, $\mathbf{x}$ is the position vector measured from a point on the surface of the fluid, and $\hat{\mathbf{j}}$ is a unit vector in the vertical direction.

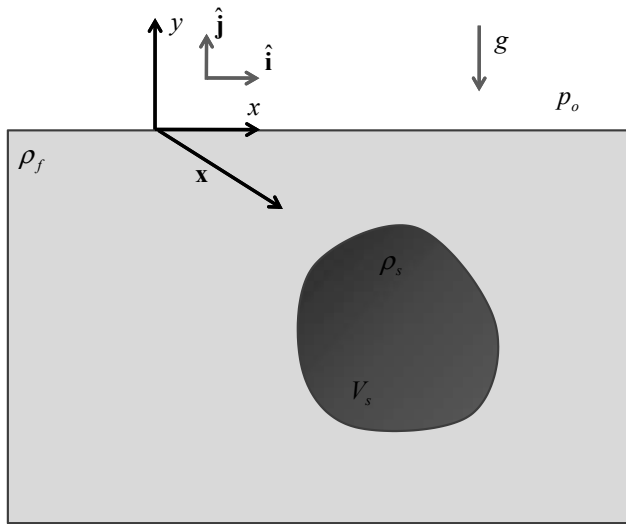


Figure 6.8: A solid object sitting in a static incompressible fluid.

3. Use this stress to calculate the load applied by the fluid on the solid using the aid of the divergence theorem.
4. From this buoyancy load, calculate the effective weight of the solid by taking its total weight and subtracting the buoyancy load.

6.4-2: The stress for an incompressible viscous fluid is given by $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$, where p is an indeterminate function and μ is the viscosity.

1. Show that the velocity $\mathbf{v} = \dot{\gamma}(\mathbf{x} \circ \hat{\mathbf{j}})\hat{\mathbf{i}}$ describes the a steady-state shearing process shown in Figure 6.9 and calculate the rate of deformation $\mathbf{D}$ for this motion.
2. Calculate the stress resulting from this deformation.
3. Assuming a constant body force $\mathbf{b} = -g\hat{\mathbf{j}}$, write the balance of linear momentum for this material and show that the steady-state shearing flow described in the figure satisfies this equation. Calculate p .

6.4-3: Consider an incompressible viscous fluid flowing between fixed parallel plates. Assume the stress is given by the equation $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$, where p is an indeterminate function and μ is the viscosity. Assume the two plates and coordinates are set up similar to those shown in Figure 6.9 and separated by a distance h . Study the steady-state flow of this fluid, assuming it is driven by constant body force $\mathbf{b} = b\hat{\mathbf{i}}$ created by an electric field pulling on the fluid along the direction of the plates, that the flow is described by a shearing motion given by $\mathbf{v} = v(y)\hat{\mathbf{i}}$, and that the fluid cannot slip on the surfaces of contact. Ignore the acceleration of gravity.

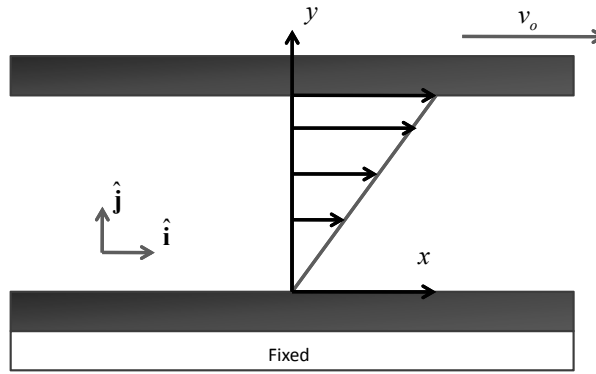


Figure 6.9: The simple shear flow of an incompressible fluid between two plates.

6.5 Balance of angular momentum

The law of balance of angular momentum states that the resultant moment applied on a body must equal the rate of change of angular momentum of that material body. This can be written as

$$\int_{S(t)} \mathbf{x} \times \mathbf{t}^{(n)} dS + \int_B \mathbf{x} \times \mathbf{b} dm = \frac{d}{dt} \left(\int_B \mathbf{x} \times \mathbf{v} dm \right), \quad (6.42)$$

where the first integral is the moment due to the traction on the surface of the body, the second integral is the moment due to body forces, and the third integral is the angular momentum of the body. In this expression, $\mathbf{x}$ represents the vector describing the position of the particle under consideration, either the position of the particle the load is applied on or the position of the particle for which angular momentum is to be calculated. This position must be measured relative to a point on an inertial reference frame.

Using the relation $\mathbf{t}^{(n)} = \mathbf{T}^T \hat{\mathbf{n}}$ and the divergence and transport theorems results in

$$\int_{V(t)} \text{div}(\mathbf{x} \times \mathbf{T}^T) dV + \int_{V(t)} \mathbf{x} \times \mathbf{b} \rho dV = \int_{V(t)} \frac{D(\mathbf{x} \times \mathbf{v})}{Dt} \rho dV.$$

Recognition of the relations $D\mathbf{x}/Dt = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ results in the expression

$$\int_{V(t)} \text{div}(\mathbf{x} \times \mathbf{T}^T) dV + \int_{V(t)} \mathbf{x} \times \mathbf{b} \rho dV = \int_{V(t)} \mathbf{x} \times \mathbf{a} \rho dV.$$

It can be shown that for any curvilinear coordinates β_i with bases $\mathbf{e}_i$ and metric $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ there is the relation

$$\text{div}(\mathbf{x} \times \mathbf{T}^T) = \mathbf{e}_i \times (\mathbf{T}^T \mathbf{e}_j) g_{ij}^{-1} + \mathbf{x} \times \text{div}(\mathbf{T}^T).$$

Using this relation and grouping terms together we get

$$\int_{V(t)} \mathbf{x} \times [\text{div}(\mathbf{T}^T) + \rho \mathbf{b} - \rho \mathbf{a}] dV + \int_{V(t)} \mathbf{e}_i \times (\mathbf{T}^T \mathbf{e}_j) g_{ij}^{-1} dV = \mathbf{0}.$$

The reader will note that the term in the square brackets in the argument of the first integral is zero as a result of the balance of linear momentum. Therefore, if the argument of the second integral is continuous, then one must conclude that the balance of angular momentum results in the requirement that

$$\mathbf{e}_i \times (\mathbf{T}^T \mathbf{e}_j) g_{ij}^{-1} = \mathbf{0}$$

at every point in the body. Expressing the Cauchy stress tensor $\mathbf{T}$ in component form $T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ and reorganization result in the relation

$$T_{ij} \mathbf{e}_i \times \mathbf{e}_j = \mathbf{0}.$$

Expanding this yields

$$T_{12} \mathbf{e}_1 \times \mathbf{e}_2 + T_{21} \mathbf{e}_2 \times \mathbf{e}_1 + T_{13} \mathbf{e}_1 \times \mathbf{e}_3 + T_{31} \mathbf{e}_3 \times \mathbf{e}_1 + T_{23} \mathbf{e}_2 \times \mathbf{e}_3 + T_{32} \mathbf{e}_3 \times \mathbf{e}_2 = \mathbf{0}.$$

This expression requires that $T_{ij} = T_{ji}$ so that the Cauchy stress $\mathbf{T}$ must be symmetric. That is,

$$\boxed{\mathbf{T} = \mathbf{T}^T}. \quad (6.43)$$

This reduces the number of independent components in the Cauchy stress tensor from nine to six. The result of this law is that

$$\boxed{\bar{\mathbf{T}} = \mathbf{T}}, \quad (6.44)$$

$$\boxed{\bar{\bar{\mathbf{T}}} = \mathbf{F}^T \bar{\mathbf{T}} \mathbf{F} = \mathbf{F}^T \mathbf{T} \mathbf{F}}, \quad (6.45)$$

$$\boxed{\tilde{\mathbf{T}} = J \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-T} = J \mathbf{F}^{-1} \bar{\mathbf{T}} \mathbf{F}^{-T}}. \quad (6.46)$$

Section 6.5-Problems:

6.5-1: Show that

$$\text{div}(\mathbf{x} \times \mathbf{T}^T) = \mathbf{e}_i \times (\mathbf{T}^T \mathbf{e}_j) g_{ij}^{-1} + \mathbf{x} \times \text{div}(\mathbf{T}^T).$$

6.5-2: Show that

$$\mathbf{e}_i \times (\mathbf{T}^T \mathbf{e}_j) g_{ij}^{-1} = \mathbf{0}$$

is equivalent to

$$T_{ij} \mathbf{e}_i \times \mathbf{e}_j = \mathbf{0}.$$

6.5-3: Show that the condition

$$T_{12} \mathbf{e}_1 \times \mathbf{e}_2 + T_{21} \mathbf{e}_2 \times \mathbf{e}_1 + T_{13} \mathbf{e}_1 \times \mathbf{e}_3 + T_{31} \mathbf{e}_3 \times \mathbf{e}_1 + T_{23} \mathbf{e}_2 \times \mathbf{e}_3 + T_{32} \mathbf{e}_3 \times \mathbf{e}_2 = \mathbf{0}$$

requires that $T_{ij} = T_{ji}$.

6.5-4: Show that the symmetry of Cauchy stress requires that the engineering stress $\mathbf{T}_o$ have the property

$$\boxed{\mathbf{F} \mathbf{T}_o = \mathbf{T}_o^T \mathbf{F}^T}, \quad (6.47)$$

$$\boxed{\tilde{\mathbf{T}} = \tilde{\mathbf{T}}^T}, \quad (6.48)$$

$$\boxed{\bar{\bar{\mathbf{T}}} = \bar{\bar{\mathbf{T}}}^T}. \quad (6.49)$$

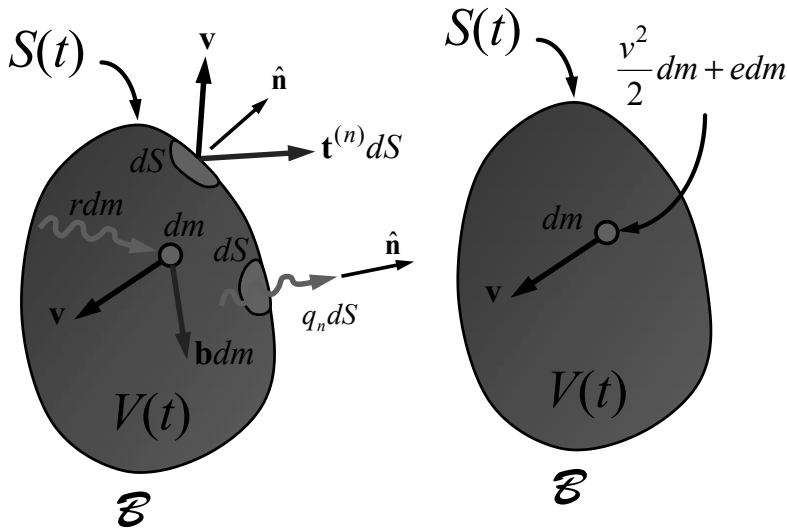


Figure 6.10: The left drawing shows the sources of heat flow into a body and ways one can do work on the body. The right drawing shows the kinetic and internal energy for each particle of the body.

6.6 Balance of work and energy

The law of balance of work and energy states that the rate at which heat flows into a body plus the rate at which work is being done on that body is equal to the rate at which the kinetic plus internal energy of the body changes. Figure 6.10 shows how heat may be added to a body through the surface or directly to each particle, and also shows that work can be done on a body by traction forces and by body forces. The total *kinetic energy* of a body is given by

$$KE \equiv \frac{1}{2} \int_{\mathcal{B}} \mathbf{v} \circ \mathbf{v} dm. \quad (6.50)$$

The total *internal energy* of a body is given by

$$IE \equiv \int_{\mathcal{B}} e dm, \quad (6.51)$$

where e is the internal energy per unit mass of the body known as the *specific internal energy*. The rate of doing work on the body $\dot{W}$ is due to the rate of doing work by the traction on the surface and by the body forces. Since the power of a force to do work is given by the dot product of the force and the velocity of the particle of the body on which the force is applied on, the rate of doing work on the body is given by

$$\dot{W} \equiv \int_{S(t)} \mathbf{t}^{(n)} \circ \mathbf{v} dS + \int_{\mathcal{B}} \mathbf{b} \circ \mathbf{v} dm. \quad (6.52)$$

The rate of heat flow *into* the body is given by

$$\dot{Q} \equiv - \int_{S(t)} \mathbf{q} \circ \hat{\mathbf{n}} dS + \int_{\mathcal{B}} r dm, \quad (6.53)$$

where $\mathbf{q}$ is the heat flux vector, $q_n = \mathbf{q} \circ \hat{\mathbf{n}}$ is the net heat flux out of the body, and r is the rate of heat addition per unit mass inside the body. The rate of heat addition r is due to heat flow to the body by sources other than the heat flux vector $\mathbf{q}$ and will include heat radiation and heat generation.

The law of balance of work and energy now can be written for a body as

$$\dot{Q} + \dot{W} = \frac{d}{dt}(KE + IE). \quad (6.54)$$

In view of the above expressions for the individual terms, this can be written as

$$- \int_{S(t)} \mathbf{q} \circ \hat{\mathbf{n}} dS + \int_{\mathcal{B}} r dm + \int_{S(t)} \mathbf{t}^{(n)} \circ \mathbf{v} dS + \int_{\mathcal{B}} \mathbf{b} \circ \mathbf{v} dm = \frac{d}{dt} \int_{\mathcal{B}} \left(\frac{1}{2} \mathbf{v} \circ \mathbf{v} + e \right) dm. \quad (6.55)$$

It follows, after the application of the divergence theorem, the transport theorem, the assumption of sufficient continuity in the variables and their derivatives, and standard arguments, that the differential form of the balance of work and energy can be written as

$$-div(\mathbf{q}) + \rho r + div(\mathbf{v} \mathbf{T}^T) + \rho \mathbf{b} \circ \mathbf{v} = \rho \frac{d}{dt} \left(\frac{1}{2} \mathbf{v} \circ \mathbf{v} + e \right).$$

Using the identity $div(\mathbf{v} \mathbf{T}^T) = tr(\mathbf{T} \mathbf{L}) + \mathbf{v} \circ div(\mathbf{T}^T)$, differentiation and reorganization yield

$$-div(\mathbf{q}) + \rho r + tr(\mathbf{T} \mathbf{L}) + \mathbf{v} \circ [div(\mathbf{T}^T) + \rho \mathbf{b} - \rho \mathbf{a}] = \rho \dot{e}.$$

As a result of the balance of linear momentum, the term in the square brackets is always equal to zero and one obtains the final form of the balance of energy as

$$-div(\mathbf{q}) + \rho r + tr(\mathbf{T} \mathbf{L}) = \rho \dot{e}. \quad (6.56)$$

Since we can show that $\nabla_{\mathbf{x}} \circ (\mathbf{q}) = [div(\mathbf{q}^T)]^T = div(\mathbf{q})$, this differential form of the balance of work and energy also can be written as

$$-\nabla_{\mathbf{x}} \circ (\mathbf{q}) + \rho r + tr(\mathbf{T} \mathbf{L}) = \rho \dot{e}. \quad (6.57)$$

The balance of energy can be written in terms of integrals over the shape of the body in the reference configuration and with the use of the engineering stress $\mathbf{T}_o$ and the “engineering or nominal heat flux” $\mathbf{q}_o$. Using the relation $dm = \rho_o dV_o$, the engineering traction vector $\mathbf{t}_o^{(N)}$ and the heat flux vector $\mathbf{q}_o$, the balance of work and energy (6.55) can be written as

$$- \int_{S_o} \mathbf{q}_o \circ \hat{\mathbf{N}} dS_o + \int_{V_o} r \rho_o dV_o + \int_{S_o} \mathbf{t}_o^{(N)} \circ \mathbf{v} dS_o + \int_{V_o} \mathbf{b} \circ \mathbf{v} \rho_o dV_o = \frac{d}{dt} \int_{V_o} \left(\frac{1}{2} \mathbf{v} \circ \mathbf{v} + e \right) \rho_o dV_o. \quad (6.58)$$

Introduction of $\mathbf{t}_o^{(N)} = \mathbf{T}_o^T \hat{\mathbf{N}}$, application of the divergence theorem, and standard assumptions of continuity of the arguments result in

$$-Div(\mathbf{q}_o) + \rho_o r + Div(\mathbf{v} \mathbf{T}_o^T) + \rho_o \mathbf{b} \circ \mathbf{v} = \rho_o \frac{d}{dt} \left(\frac{1}{2} \mathbf{v} \circ \mathbf{v} + e \right).$$

Using the relation $Div(\mathbf{v}\mathbf{T}_o^T) = tr(\mathbf{T}_o\dot{\mathbf{F}}) + \mathbf{v} \circ Div(\mathbf{T}_o^T)$, and taking advantage of the balance of linear momentum to eliminate terms result in the statement of the law of balance of work and energy as

$$-Div(\mathbf{q}_o) + \rho_o r + tr(\mathbf{T}_o\dot{\mathbf{F}}) = \rho_o \dot{e}, \quad (6.59)$$

which must hold for every point of the body in the reference configuration. This also can be written as

$$-\nabla_{\mathbf{x}} \circ (\mathbf{q}_o) + \rho_o r + tr(\mathbf{T}_o\dot{\mathbf{F}}) = \rho_o \dot{e}. \quad (6.60)$$

Section 6.6-Problems:

6.6-1: Derive the relations

$$div(\mathbf{v}\mathbf{T}^T) = tr(\mathbf{T}\mathbf{L}) + \mathbf{v} \circ div(\mathbf{T}^T),$$

and

$$Div(\mathbf{v}\mathbf{T}_o^T) = tr(\mathbf{T}_o\dot{\mathbf{F}}) + \mathbf{v} \circ Div(\mathbf{T}_o^T).$$

6.6-2: Fill in the steps to obtain (6.59) from (6.58).

6.6-3: Show that $tr(\mathbf{T}\mathbf{L}) = tr(\mathbf{T}\mathbf{D})$, where $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$ is the deformation rate tensor.

6.6-4: Show that

$$tr(\mathbf{T}_o\dot{\mathbf{F}}) = J tr(\mathbf{T}\mathbf{L}) = J tr(\mathbf{T}\mathbf{D}). \quad (6.61)$$

6.6-5: Show that

$$tr(\mathbf{T}_o\dot{\mathbf{F}}) = \frac{1}{2} tr(\tilde{\mathbf{T}}\dot{\mathbf{C}}), \quad (6.62)$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy stretch tensor and $\tilde{\mathbf{T}}$ is the *second Piola-Kirchhoff stress tensor* defined as

$$\tilde{\mathbf{T}} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} = \mathbf{T}_o\mathbf{F}^{-T}. \quad (6.63)$$

6.7 Entropy and the entropy production inequality

The entropy production inequality, also known as *the second law of thermodynamics*, states that the entropy in a material body of fixed mass increases at least as rapidly as entropy is added to the body through the addition of heat to the body, either by radiation directly into the body or by heat flow through the boundaries of the body. Figure 6.11 shows these sources of adding entropy to the body. The total *entropy* EN in a body is given by

$$EN \equiv \int_{\mathcal{B}} \eta dm, \quad (6.64)$$

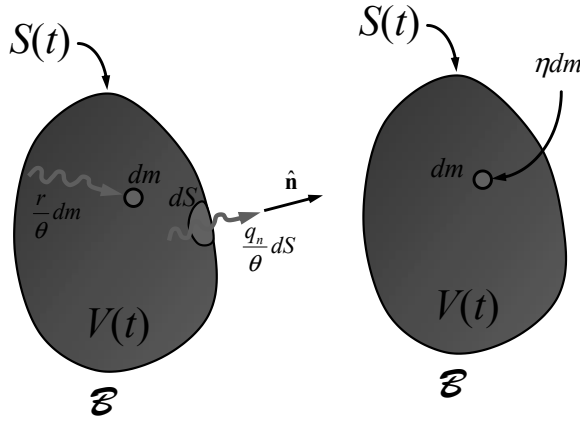


Figure 6.11: Entropy is added to a body through heat.

where η is the entropy per unit mass of the body, also known as the *specific entropy*. The rate at which entropy is added to the body by heat is given by

$$\int_{\mathcal{B}} \frac{r}{\theta} dm - \int_{S(t)} \frac{1}{\theta} \mathbf{q} \circ \hat{\mathbf{n}} dS, \quad (6.65)$$

where θ denotes the particle temperature of the point that the heat is added to the body, and is taken on a scale that is always positive.

The integral form of the entropy production inequality is given by

$$\int_{\mathcal{B}} \frac{r}{\theta} dm - \int_{S(t)} \frac{1}{\theta} \mathbf{q} \circ \hat{\mathbf{n}} dS \leq \frac{d}{dt} \int_{\mathcal{B}} \eta dm. \quad (6.66)$$

Using the divergence theorem, one can find the differential form of this law as

$$\rho \frac{r}{\theta} - \text{div}(\frac{1}{\theta} \mathbf{q}) \leq \rho \dot{\eta}. \quad (6.67)$$

This also can be written as

$$\rho \frac{r}{\theta} - \nabla_{\mathbf{x}} \circ (\frac{1}{\theta} \mathbf{q}) \leq \rho \dot{\eta}. \quad (6.68)$$

Free energy is defined by the relation $e = \psi + \eta\theta$, where ψ is free energy per unit mass, also known as the *specific free energy*. Free energy represents the portion of internal energy which is available for conversion into work or heat (not necessarily instantaneously). Through the expression relating internal energy, free energy, and entropy one can see that $\eta\theta$ represents the portion of the energy which is not accessible. One can introduce the relation $e = \psi + \eta\theta$ into the entropy production inequality after differentiation and substitution for $\dot{\eta}$. Assuming a strictly positive temperature scale and use of the balance of energy result in the inequality

$$\rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{grad}(\theta) \leq 0. \quad (6.69)$$

This inequality is also known as the *Clausius-Duhem inequality* and is the same as

$$\rho\dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q} \circ \nabla_{\mathbf{x}}(\theta) \leq 0. \quad (6.70)$$

As was the case for the previous laws, one can directly transfer the integrals over the current configuration to integrals over the reference configuration. This process gives the entropy production inequality (6.66) as

$$\int_{V_o} \frac{r}{\theta} \rho_o dV_o - \int_{S_o} \frac{1}{\theta} \mathbf{q}_o \circ \hat{\mathbf{N}} dS_o \leq \frac{d}{dt} \int_{V_o} \eta \rho_o dV_o. \quad (6.71)$$

Following similar steps as above, one arrives at the differential form of this law given by

$$\rho_o \frac{r}{\theta} - \text{Div}(\frac{1}{\theta} \mathbf{q}_o) \leq \rho_o \dot{\eta}, \quad (6.72)$$

which must hold at every point in the reference configuration. In a similar process to that described above, one can introduce the free energy and arrive at a statement of the Clausius-Duhem inequality to be applied on the reference configuration. This form of the Clausius-Duhem inequality is given by

$$\rho_o \dot{\psi} - \text{tr}(\mathbf{T}_o \dot{\mathbf{F}}) + \rho_o \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{Grad}(\theta) \leq 0. \quad (6.73)$$

Section 6.7-Problems:

- 6.7-1:** Fill in the steps to derive (6.67) from (6.66).
- 6.7-2:** Derive the Clausius-Duhem inequality (6.69) from (6.67). Explain why the temperature θ needs to be strictly positive to get this relation.
- 6.7-3:** Fill in the steps to derive (6.72) from (6.71).
- 6.7-4:** Fill in the steps to derive (6.73) from (6.72).

6.8 Heat flow and thermodynamic processes

In this section, we look at the interaction between mechanical work and heat generation. This is an important topic in many applications. For example, the heat generated by mechanical working of a material in many cases will result in a rise in temperature of the point if the heat is not removed from the point. There are two characteristic questions that one frequently sees in this regard. One question looks at how much heat needs to be removed to keep the point at constant temperature (isothermal process), and the other question looks at how much the temperature will change if we do not remove the heat (adiabatic process). In both cases, we are looking at the heat flowing to the point. Let us define $\dot{h}$ as the specific rate of heat flow into the point through heat conduction and through radiation. It can be shown that $\dot{h}$ is given by the relation

$$\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q}). \quad (6.74)$$

The balance of work and energy given in (6.56) can be written in terms of $\dot{h}$ as

$$\rho \dot{e} = \rho \dot{h} + \text{tr}(\mathbf{T}\mathbf{L}). \quad (6.75)$$

This can be reorganized to give

$$\dot{h} = \dot{e} - \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}). \quad (6.76)$$

If one integrates this over the time interval from t_o to t , then one gets

$$\Delta h = e - e_o - \int_{t_o}^t \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}) ds, \quad (6.77)$$

where Δh is the specific heat added over the time interval, and e_o is the specific internal energy at time t_o . In terms of the specific free energy and entropy, these equations can be written as

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta} - \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}), \quad (6.78)$$

$$\Delta h = \psi - \psi_o + \eta\theta - \eta_o\theta_o - \int_{t_o}^t \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}) ds, \quad (6.79)$$

where ψ_o , η_o , and θ_o are, respectively, the values of ψ , η , and θ at time t_o . As can be seen, any process that starts and ends at the same internal energy, or which starts and ends at the same free energy, entropy, and temperature, requires an amount of heat, per unit mass, flowing into the point given by

$$\Delta h = - \int_{t_o}^t \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}) ds. \quad (6.80)$$

An *adiabatic* process is thermally isolated so that $\dot{h}$ is zero. Therefore, an adiabatic process is characterized by the equation

$$\dot{e} = \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}), \quad (6.81)$$

or in terms of free energy by the equation

$$\dot{\psi} = -\dot{\eta}\theta - \eta\dot{\theta} + \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}). \quad (6.82)$$

An *isothermal* process happens at constant temperature so that $\dot{\theta}$ is zero. The equation describing the heat flow in an isothermal process is given as

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta - \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}). \quad (6.83)$$

A *rigid-body* process only undergoes rigid body motions. For such a process, the deformation gradient $\mathbf{F}$ is an orthogonal transformation. If we have $\mathbf{F} = \mathbf{Q}$ for an orthogonal transformation $\mathbf{Q}$ (i.e., $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$), then since $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ we can write

$$\mathbf{L} = \dot{\mathbf{Q}}\mathbf{Q}^T. \quad (6.84)$$

From the orthogonality of $\mathbf{Q}$, we can show that $\dot{\mathbf{Q}}\mathbf{Q}^T$ is skew symmetric, therefore making $\mathbf{L}$ skew symmetric. Since the Cauchy stress $\mathbf{T}$ is symmetric, and since we know the trace of a symmetric tensor multiplied by a skew symmetric tensor is zero, we can conclude that for a rigid-body process $tr(\mathbf{T}\mathbf{L}) = 0$. Therefore, the working of the stress is zero in a rigid body process and we can write the balance of work and energy as

$$\dot{h} = \dot{e}. \quad (6.85)$$

In terms of free energy, the equation governing rigid body processes can be written as

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}. \quad (6.86)$$

An *isentropic* process is one for which the entropy does not change. The balance of energy for an isentropic process can be written as

$$\dot{h} = \dot{\psi} + \eta\dot{\theta} - \frac{1}{\rho}tr(\mathbf{T}\mathbf{L}). \quad (6.87)$$

A process that is both adiabatic and isentropic will be such that

$$\dot{\psi} = \frac{1}{\rho}tr(\mathbf{T}\mathbf{L}) - \eta\dot{\theta}. \quad (6.88)$$

The *heat capacity* is the rate of heat flow divided by the rate of change of temperature. Denoting the heat capacity as c , we can write it as

$$c \equiv \frac{\dot{h}}{\dot{\theta}}. \quad (6.89)$$

There are many different definitions for heat capacity based on what the process conditions are. The two commonly used conditions are to hold the pressure or volume constant. The heat capacity at constant pressure and constant volume are, respectively, denoted by c_p and c_v . If the heat capacity is measured under a rigid body process we will denote it by c_{rb} and it is given by the relation

$$c_{rb} = \frac{\dot{e}}{\dot{\theta}} = \frac{\dot{\psi}}{\dot{\theta}} + \theta \frac{\dot{\eta}}{\dot{\theta}} + \eta. \quad (6.90)$$

The heat capacity at constant volume is a special case of heat capacity under rigid body motions. For processes occurring under hydrostatic pressure p the Cauchy stress is given by $\mathbf{T} = -p\mathbf{I}$ so that $tr(\mathbf{T}\mathbf{L}) = -p tr(\mathbf{L}) = -p\dot{J}/J$, where $J = det(\mathbf{F})$ is the volume ratio. Noting that by conservation of mass we have $\rho J = \rho_o$, we can write the rate of heat flow as

$$\dot{h} = \dot{e} + \frac{1}{\rho_o}p\dot{J}, \quad (6.91)$$

so that the heat capacity at constant pressure can be written as

$$c_p = \frac{\dot{e}}{\dot{\theta}} + \frac{p}{\rho_o} \frac{\dot{J}}{\dot{\theta}}. \quad (6.92)$$

Letting $\beta(p)$ denote the volumetric coefficient of thermal expansion at pressure p (i.e., $\dot{J} = \beta(p)J\dot{\theta}$), we can write this as

$$c_p = \frac{\dot{e}}{\dot{\theta}} + \frac{\beta(p)}{\rho_o}pJ = \frac{\dot{\psi}}{\dot{\theta}} + \theta \frac{\dot{\eta}}{\dot{\theta}} + \eta + \frac{\beta(p)}{\rho_o}pJ. \quad (6.93)$$

Section 6.8-Problems:**6.8-1:** Prove the relation

$$\boxed{tr(\mathbf{L}) = \frac{\dot{J}}{J}.} \quad (6.94)$$

6.8-2: Prove that if $\mathbf{Q}$ is an orthogonal second-order tensor, then $\dot{\mathbf{Q}}\mathbf{Q}^T$ is skew symmetric. Use this to prove that if $\mathbf{L} = \dot{\mathbf{Q}}\mathbf{Q}^T$, then $tr(\mathbf{TL}) = 0$.**6.9 Infinitesimal deformations**

For infinitesimal deformations, we assume the displacement gradient $\mathbf{H}$ to be small so that $\|\mathbf{H}\| \ll 1$. In such cases, we can assume the following kinematic approximation

$$\mathbf{E} \approx \boldsymbol{\epsilon}, \quad (6.95)$$

$$\mathbf{D} \approx \dot{\boldsymbol{\epsilon}}, \quad (6.96)$$

$$J \approx 1 + tr(\boldsymbol{\epsilon}), \quad (6.97)$$

$$J^{-1} \approx 1 - tr(\boldsymbol{\epsilon}), \quad (6.98)$$

$$\mathbf{F}^{-1} \approx \mathbf{I} - \mathbf{H}, \quad (6.99)$$

where

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T), \quad (6.100)$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\dot{\mathbf{H}} + \dot{\mathbf{H}}^T). \quad (6.101)$$

Using these approximations, the relation between the Cauchy stress $\mathbf{T}$ and the nominal stress $\mathbf{T}_o$ is given by

$$\begin{aligned} \mathbf{T} &= \frac{1}{J}\mathbf{FT}_o \approx [1 - tr(\boldsymbol{\epsilon})](\mathbf{I} + \mathbf{H})\mathbf{T}_o \approx \mathbf{T}_o - tr(\boldsymbol{\epsilon})\mathbf{T}_o + \mathbf{HT}_o \\ &\approx \mathbf{T}_o. \end{aligned} \quad (6.102)$$

In general, when working with infinitesimal deformations we will use $\boldsymbol{\sigma}$ to denote the nominal stress. As can be seen, the difference between the nominal and Cauchy stresses are negligible if $\|\mathbf{H}\|$ is in the order of 1% or is less than 1%. In the balance equations, we frequently have the term $tr(\mathbf{T}_o\dot{\mathbf{F}})$, which represents the working of the stress. Since the difference between the nominal stress and the Cauchy stress is negligible for infinitesimal deformations, and since the Cauchy stress is symmetric, we can assume that to the same approximation the nominal stress is symmetric. Noting that $\dot{\mathbf{H}} = \dot{\mathbf{F}}$, it can be shown that

$$\boxed{tr(\mathbf{T}_o\dot{\mathbf{F}}) \approx \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}.} \quad (6.103)$$

In general, we will use the balance laws written in the reference configuration when considering infinitesimal strains. We also will consider the reference configuration to be the

initial configuration. The conservation of mass, balance of linear momentum, balance of energy, and the Clausius-Duhem inequality for infinitesimal deformations can be, respectively, written as

$$\boxed{\rho[1 + tr(\epsilon)] = \rho_o,} \quad (6.104)$$

$$\boxed{Div(\sigma) + \rho_o \mathbf{b} = \rho_o \mathbf{a},} \quad (6.105)$$

$$\boxed{\rho_o \dot{e} = \rho_o r - Div(\mathbf{q}_o) + \sigma : \dot{\epsilon},} \quad (6.106)$$

$$\boxed{\rho_o \dot{\psi} - \sigma : \dot{\epsilon} + \rho_o \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0.} \quad (6.107)$$

The balance of energy, using $e = \psi + \eta\theta$, also can be written in terms of free energy and entropy as

$$\boxed{\rho_o \dot{\psi} + \rho_o \dot{\eta}\theta + \rho_o \eta \dot{\theta} = \rho_o r - Div(\mathbf{q}_o) + \sigma : \dot{\epsilon}.} \quad (6.108)$$

This also can be written as

$$\boxed{\rho_o \dot{\psi} + \rho_o \dot{\eta}\theta + \rho_o \eta \dot{\theta} = \rho_o \dot{h} + \sigma : \dot{\epsilon}.} \quad (6.109)$$

The integral form of this is given by

$$\boxed{\rho_o \Delta h = \rho_o(\psi - \psi_o) + \rho_o(\eta\theta - \eta_o\theta_o) - \int_{t_o}^t \sigma : \dot{\epsilon} dt,} \quad (6.110)$$

for ψ_o , η_o , and θ_o being the values, respectively, of ψ , η , and θ at time t_o .

Section 6.9-Problems:

- 6.9-1:** Show that $\mathbf{D} \approx \dot{\epsilon}$ for infinitesimal deformations.
- 6.9-2:** Show that $J^{-1} \approx 1 - tr(\epsilon)$ for infinitesimal deformations.
- 6.9-3:** Show that $tr(\mathbf{T}_o \dot{\mathbf{F}}) \approx \sigma : \dot{\epsilon}$ for infinitesimal deformations.
- 6.9-4:** Consider the problem of the uniform axial extension and pure torsion of a linear elastic circular bar by an axial force P and a torsional moment T as shown in Figure 6.12. Take the response of the bar to be given by the model for an isotropic linear elastic material written as

$$\sigma = \left(\kappa - \frac{2}{3} G \right) tr(\epsilon) \mathbf{I} + 2G\epsilon, \quad (6.111)$$

where κ is the bulk modulus and G is the shear modulus.

1. Calculate the deformation gradient, displacement gradient, and the infinitesimal strain ϵ .
2. Calculate the stress assuming the lateral surfaces are stress free.
3. Obtain the traction on the front surface.
4. Use the traction to calculate the axial load P and torsional moment T .
5. Show that this deformation satisfies the equilibrium equation when ignoring the body forces.

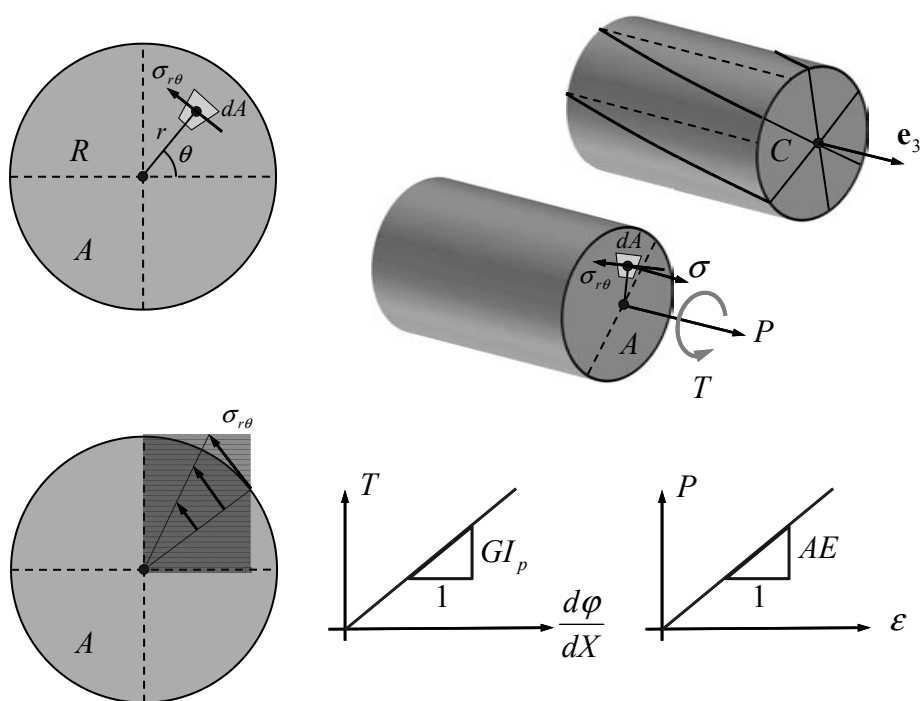


Figure 6.12: The uniform extension and torsion of an elastic body.

6.10 The generalized balance law

For the sake of economy in presentation and to include concentrated sources, we will write all of the balance laws in a single form. In the next section, we will derive all the jump conditions for this generic balance law. The jump conditions are the relations that connect the values of the parameters on the two sides of a surface on which the parameters are discontinuous.

All five balance laws can be written in the form

$$\boxed{\frac{D}{Dt} \left(\int_{V(t)} \phi \rho dV \right) \begin{matrix} \geq \\ \text{or} \\ = \end{matrix} \int_{V(t)} \psi \rho dV + \int_{S(t)} \xi dS + \sum \Upsilon,} \quad (6.112)$$

where $\xi = \bar{h} \circ \hat{\mathbf{n}}$. To get each balance law from this general form, we need to make one of the following substitutions:

Generic	ϕ	ψ	ξ	$\bar{h}$
Conservation of mass	1	0	0	0
Balance of linear momentum	$\mathbf{v}$	$\mathbf{b}$	$\mathbf{t}^{(n)}$	$\mathbf{T}^T$
Balance of angular momentum	$\mathbf{x} \times \mathbf{v}$	$\mathbf{x} \times \mathbf{b}$	$\mathbf{x} \times \mathbf{t}^{(n)}$	$\mathbf{x} \times \mathbf{T}^T$
Balance of work and energy	$e + \frac{1}{2} \mathbf{v} \circ \mathbf{v}$	$r + \mathbf{b} \circ \mathbf{v}$	$\mathbf{t}^{(n)} \circ \mathbf{v} - \mathbf{q} \circ \hat{\mathbf{n}}$	$\mathbf{v} \mathbf{T}^T - \mathbf{q}$
Entropy production inequality	η	$\frac{1}{\theta} r$	$-\frac{1}{\theta} \mathbf{q} \circ \hat{\mathbf{n}}$	$-\frac{1}{\theta} \mathbf{q}$

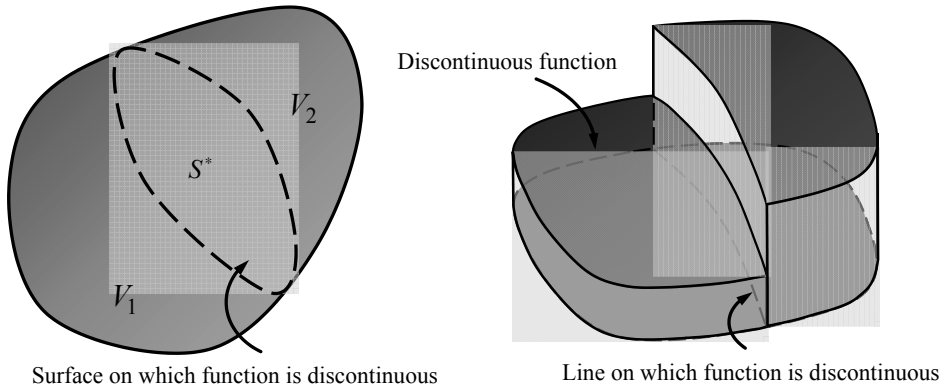


Figure 6.13: The left picture shows a body separated by a surface S^* on which the functions may be discontinuous. The right picture shows a function that is discontinuous over a line, but continuous over all other parts of the surface.

The term $\sum \Upsilon$ represents any concentrated sources present (as opposed to distributed parameters that are represented in the integrals). This term takes care of any concentrated sources such as applied concentrated forces, applied concentrated moments, etc. Application of the divergence theorem and the transport theorem results in the following form of the generalized balance law

$$\int_{V(t)} \frac{D\phi}{Dt} \rho dV \begin{matrix} \geq \\ \text{or} \\ = \end{matrix} \int_{V(t)} [\psi \rho + \text{div}(\vec{h})] dV + \sum \Upsilon. \quad (6.113)$$

In the absence of concentrated sources, assuming the arguments of the integrals are continuous, we obtain the generic differential form of the balance law to be given by

$$\boxed{\rho \frac{D\phi}{Dt} \begin{matrix} \geq \\ \text{or} \\ = \end{matrix} \rho \psi + \text{div}(\vec{h})}. \quad (6.114)$$

6.11 Jump conditions

Up to this point, we have assumed that all functions are continuous and sufficiently smooth over all the body so as to allow us to obtain the differential form of the balance laws from the integral forms. As the reader recalls, obtaining the differential forms required the use of the divergence theorem to change the surface integrals to volume integrals and the transport theorem to move the material time derivative from the outside to the argument of the integrals. Now we will look at the case when the functions are not sufficiently continuous to allow the use of the divergence theorem and transport theorem as derived above. The reader can imagine that in a volume any function can be discontinuous over an entire surface, say S^* as shown in Figure 6.13, yet continuous at all other points of the volume. Since the

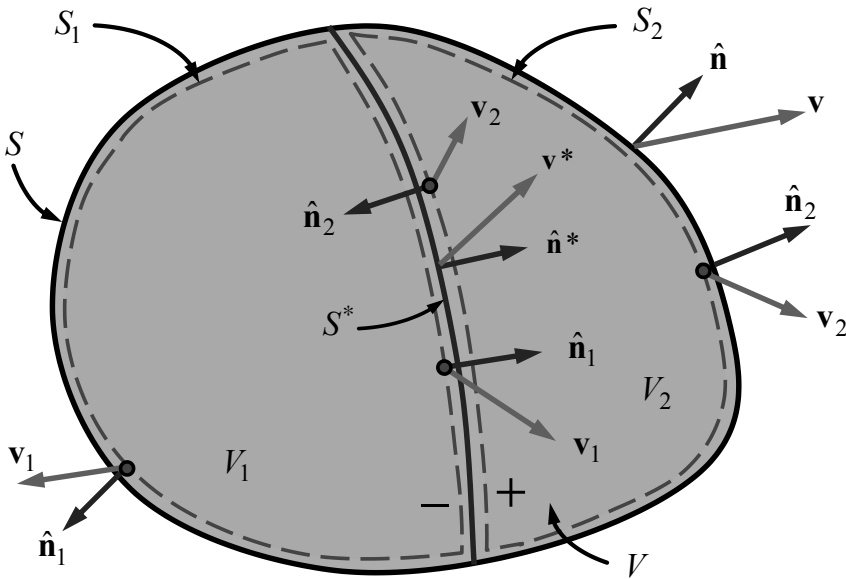


Figure 6.14: A body separated by a surface S^* on which the functions may be discontinuous.

drawing of a function on a volume is impossible, the idea is described in Figure 6.13 on the right where a function is shown that is discontinuous over a line, yet continuous over all other parts of the surface. The reader must use his/her imagination to extend this idea to a volume and a surface in this volume on which a function is discontinuous.

6.11.1 Divergence theorem

We first will look at the implications of a discontinuity in the functions on the divergence theorem. In particular, we consider the case where the body contains a surface S^* on which we allow the functions to be discontinuous. Figure 6.14 shows this surface separating a body into two segments with volumes V_1 and V_2 . Inside each segment it is assumed that the functions are sufficiently continuous for the divergence theorem to hold, confining the discontinuity only to the surface S^* .

As the reader recalls, for a continuous vector function $\mathbf{u}$ the divergence theorem states that

$$\int_S \mathbf{u} \circ \hat{\mathbf{n}} dS = \int_V \text{div}(\mathbf{u}) dV.$$

For the case where $\mathbf{u}$ is discontinuous on S^* , let $\mathbf{u} = \mathbf{u}_1$ in V_1 and $\mathbf{u} = \mathbf{u}_2$ in V_2 . Also, let $\hat{\mathbf{n}}_1$ denote the outward normal to V_1 , let $\hat{\mathbf{n}}_2$ denote the outward normal to V_2 , let $S_1 \cup S^*$ be the surface of V_1 , let $S_2 \cup S^*$ be the surface of V_2 , and let $S = S_1 \cup S_2$ be the surface of V . The divergence theorem holds for each segment of V on which $\mathbf{u}$ is continuous so that we have

$$\begin{aligned} \int_{S_1 \cup S^*} \mathbf{u}_1 \circ \hat{\mathbf{n}}_1 dS &= \int_{V_1} \text{div}(\mathbf{u}_1) dV, \\ \int_{S_2 \cup S^*} \mathbf{u}_2 \circ \hat{\mathbf{n}}_2 dS &= \int_{V_2} \text{div}(\mathbf{u}_2) dV. \end{aligned}$$

Adding these two relations gives

$$\int_S \mathbf{u} \circ \hat{\mathbf{n}} dS + \int_{S^*} (\mathbf{u}_1 - \mathbf{u}_2) \circ \hat{\mathbf{n}}^* dS = \int_V \text{div}(\mathbf{u}) dV, \quad (6.115)$$

where $\hat{\mathbf{n}}^*$ is the normal to S^* directed into V_2 . If we define $||[\mathbf{u}]|| \equiv \mathbf{u}_2 - \mathbf{u}_1$ to be the difference at each point on S^* between the values of $\mathbf{u}$ on its two sides, then this *generalized divergence theorem* can be written as

$$\boxed{\int_S \mathbf{u} \circ \hat{\mathbf{n}} dS = \int_V \text{div}(\mathbf{u}) dV + \int_{S^*} ||[\mathbf{u}]|| \circ \hat{\mathbf{n}}^* dS.} \quad (6.116)$$

One can use the same procedures to generalize Gauss' theorem also to get

$$\boxed{\int_S \mathbf{A} \hat{\mathbf{n}} dS = \int_V \text{div}(\mathbf{A}) dV + \int_{S^*} ||[\mathbf{A}]|| \hat{\mathbf{n}}^* dS,} \quad (6.117)$$

where $\mathbf{A}$ is any tensor field.

6.11.2 Transport theorem

Next let us look at Reynold's transport theorem. As the reader recalls, for any scalar function ϕ defined on the domain $V(t)$ with boundary $S(t)$, one can write (see Equation (6.12))

$$\frac{d\Phi^*(t)}{dt} = \frac{D\Phi(t)}{Dt} + \int_{S(t)} \phi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS,$$

where

$$\Phi^*(t) \equiv \int_{V^*(t)} \phi dV^*$$

for an arbitrarily moving volume $V^*(t)$, and

$$\Phi(t) \equiv \int_{V(t)} \phi dV$$

for a material volume $V(t)$ moving with the material body. Also, as a result of conservation of mass, we know that for any $\phi = \rho\psi$ we have (see Equation (6.22))

$$\frac{D\Phi(t)}{Dt} = \int_{V(t)} \frac{D\psi}{Dt} \rho dV.$$

Combining these relations we get

$$\frac{d\Phi^*(t)}{dt} = \int_{V(t)} \frac{D\psi}{Dt} \rho dV + \int_{S(t)} \rho\psi(\mathbf{v}^* - \mathbf{v}) \circ \hat{\mathbf{n}} dS. \quad (6.118)$$

This modified version of the transport theorem incorporates the result of imposing conservation of mass, and therefore is less general than the relations presented in Section 6.

Now consider a function Φ that can be calculated by the equation

$$\Phi = \int_{V_1} \psi_1 \rho_1 dV + \int_{V_2} \psi_2 \rho_2 dV,$$

where we will use Φ_1 to denote the first integral and Φ_2 to denote the second integral. As implied above, the functions ψ_1 and ρ_1 are continuous over V_1 and the functions ψ_2 and ρ_2 are continuous over V_2 . If we apply (6.118) to each of the two volumes V_1 and V_2 , we obtain

$$\frac{d\Phi_1^*(t)}{dt} = \int_{V_1(t)} \frac{D\psi_1}{Dt} \rho_1 dV + \int_{S^*(t)} \rho_1 \psi_1 (\mathbf{v}^* - \mathbf{v}_1) \circ \hat{\mathbf{n}}^* dS, \quad (6.119)$$

$$\frac{d\Phi_2^*(t)}{dt} = \int_{V_2(t)} \frac{D\psi_2}{Dt} \rho_2 dV + \int_{S^*(t)} \rho_2 \psi_2 (\mathbf{v}^* - \mathbf{v}_2) \circ (-\hat{\mathbf{n}}^*) dS, \quad (6.120)$$

where the reader notes that the surface integrals over S_1 and S_2 become zero since the surface velocity $\mathbf{v}^*$ and the particle velocity $\mathbf{v}$ are the same for these surfaces that are moving with the material points. Addition of the two equations gives the total material derivative of $\Phi = \Phi_1 + \Phi_2$ as

$$\boxed{\frac{D\Phi(t)}{Dt} = \int_{V(t)} \frac{D\psi}{Dt} \rho dV + \int_{S^*(t)} |[\rho\psi(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* dS,} \quad (6.121)$$

where $\rho = \rho_1$, $\mathbf{v} = \mathbf{v}_1$, and $\psi = \psi_1$ on V_1 , and $\rho = \rho_2$, $\mathbf{v} = \mathbf{v}_2$, and $\psi = \psi_2$ on V_2 . Since the volume of the surface S^* is zero, the contribution to the volume integral by including S^* is zero.

6.11.3 The jump conditions

Let us develop the form of the general balance law if there is discontinuity on the surface S^* in the functions $\rho\phi$ and $\xi = \hbar \circ \hat{\mathbf{n}}$. Application of the divergence theorem (6.116) and the transport theorem (6.121) to the generalized balance law (6.112) results in

$$\begin{aligned} \int_{V(t)} \frac{D\phi}{Dt} \rho dV + \int_{S^*(t)} |[\rho\phi(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* dS &\begin{matrix} \geq \\ or \\ = \end{matrix} \\ \int_{V(t)} \psi \rho dV + \int_{V(t)} \text{div}(\hbar) dV + \int_{S^*(t)} |[\hbar]| \circ \hat{\mathbf{n}}^* dS + \sum \Upsilon. &\quad (6.122) \end{aligned}$$

This form of the generalized balance law can be used to get the jump conditions through the process of collapsing the external surfaces onto S^* as shown in Figure 6.15. In doing this, we are eliminating all of the volume integrals, and at the limit all of the surface integrals will be over S^* . The generalized balance law becomes

$$\boxed{\int_{S^*(t)} |[\rho\phi(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* dS \begin{matrix} \geq \\ or \\ = \end{matrix} \int_{S^*(t)} |[\hbar]| \circ \hat{\mathbf{n}}^* dS + \sum \Upsilon^*,} \quad (6.123)$$

where $\sum \Upsilon^*$ represents all of the concentrated sources on S^* . The concentrated sources may be either point sources or distributed over S^* . Let us denote point sources on S^* by Υ_p^* and the intensity of distributed sources on S^* by v^* (per unit surface area of S^*). Therefore, the concentrated sources can be written as

$$\boxed{\sum \Upsilon^* = \sum \Upsilon_p^* + \int_{S^*(t)} v^* dS.}$$

If we consider distributed concentrated sources only, we can proceed to get point-wise jump conditions on the surface S^* . Since the balance law must hold for any segment of the

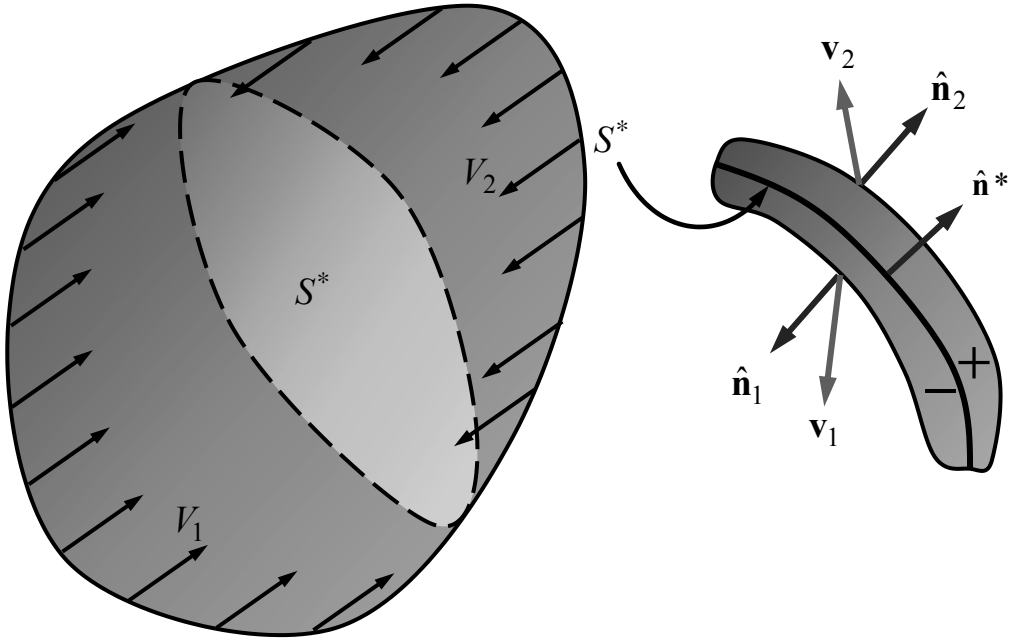


Figure 6.15: Collapsing the volume onto the surface of discontinuity (left). A surface cut showing the side view of the surface of discontinuity (right).

surface on which the functions are discontinuous, we get the jump conditions at each point on the surface S^* to be

$$|[\rho\phi(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* \begin{matrix} \geq \\ \text{or} \\ = \end{matrix} |[\mathbf{h}]| \circ \hat{\mathbf{n}}^* + v^*. \quad (6.124)$$

This is the most common form of the jump conditions, even though point sources are not included. If point sources exist, then the term $\sum \Upsilon_p^*$ needs to be taken into consideration and the integral form of the jump condition given above must be used.

We can now express the five jump conditions as

$$|[\rho(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* = 0, \quad (6.125)$$

$$|[\rho\mathbf{v} \otimes (\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* = |[\mathbf{T}^T]| \circ \hat{\mathbf{n}}^* + \mathbf{t}^{*(n^*)}, \quad (6.126)$$

$$|[\rho(\mathbf{x} \times \mathbf{v}) \otimes (\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* = |[\mathbf{x} \times \mathbf{T}^T]| \circ \hat{\mathbf{n}}^* + \mathbf{x} \times \mathbf{t}^{*(n^*)} + \frac{d\mathbf{c}^*}{dS}, \quad (6.127)$$

$$|[\rho(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* = |[\mathbf{v}\mathbf{T}^T - \mathbf{q}]| \circ \hat{\mathbf{n}}^* + \mathbf{t}^{*(n^*)} \circ \mathbf{v} + h^*, \quad (6.128)$$

$$|[\rho\eta(\mathbf{v} - \mathbf{v}^*)]| \circ \hat{\mathbf{n}}^* \geq -|[\frac{1}{\theta}\mathbf{q}]| \circ \hat{\mathbf{n}}^* + \frac{1}{\theta}h^*, \quad (6.129)$$

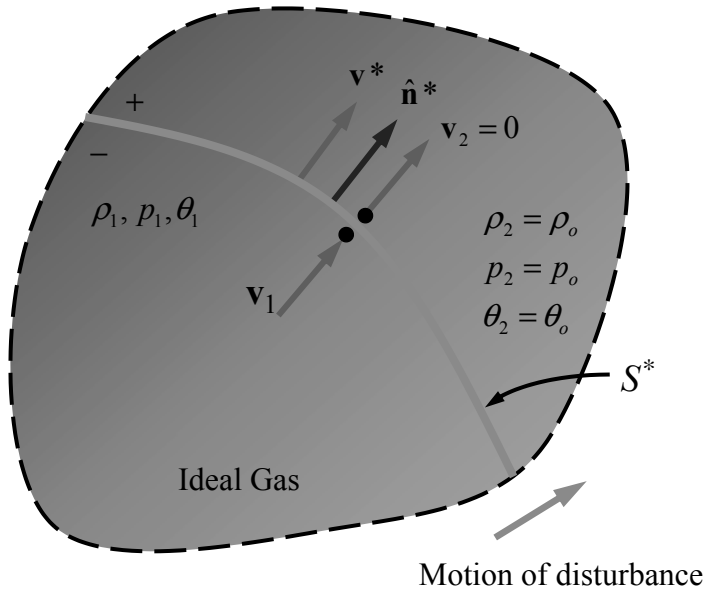


Figure 6.16: A disturbance moving in an ideal gas.

where $\mathbf{t}^{*(n^*)}$ is externally applied traction on S^* , $d\mathbf{c}^*/dS$ is externally applied point couples per unit area on S^* , h^* is externally generated heat flow per unit surface area on S^* , and h^*/θ represents any entropy added through h^* .

Section 6.11-Problems:

6.11-1: Consider the motion of a disturbance into a stationary ideal gas in equilibrium as shown in Figure 6.16. Go through the following steps to find the speed of propagation of the disturbance.

1. Show by using the ideal gas law and the law of conservation of mass that the constitutive equation for an ideal gas is given by

$$\mathbf{T} = -p\mathbf{I} = -\frac{\rho\theta}{\rho_o\theta_o}p_o\mathbf{I},$$

where p , ρ , and θ are, respectively, the current pressure, density, and temperature, and p_o , ρ_o , and θ_o are, respectively, these same quantities in the reference state.

2. Assuming all velocities are normal to the surface of propagation, show that jump conditions associated with the conservation of mass and balance of linear momentum, respectively, become

$$\begin{aligned} v^*(\rho_1 - \rho_o) &= \rho_1 v_1, \\ -\rho_1 v_1(v_1 - v^*) &= \frac{p_o}{\rho_o\theta_o}(\rho_1\theta_1 - \rho_o\theta_o), \end{aligned}$$

where v_1 , ρ_1 , and θ_1 are, respectively, the speed, density, and temperature of the gas behind the disturbance, v^* is the speed of propagation of the disturbance, and it is assumed that the gas ahead of the disturbance is at its reference state of pressure and temperature.

3. Combine the two jump conditions to get the velocity of the disturbance to be given by

$$v^* = \sqrt{\frac{p_o \rho_1 (\rho_o \theta_o - \rho_1 \theta_1)}{\rho_o^2 \theta_o (\rho_o - \rho_1)}}.$$

4. Show that the relation between density, pressure, and temperature is given by

$$\rho = \frac{\rho_o \theta_o}{p_o} \frac{p}{\theta},$$

and use this to show that

$$v^* = \sqrt{\frac{p_1 \theta_o (p_o - p_1)}{\rho_o (p_o \theta_1 - p_1 \theta_o)}}.$$

5. From this conclude that for isothermal conditions the disturbance moves at a speed

$$v^* = \sqrt{\frac{p_1}{\rho_o}}.$$

6. Show that the particle velocity behind the disturbance is given by

$$v_1 = \frac{(p_1 \theta_o - p_o \theta_1)}{p_1 \theta_o} v^*,$$

and conclude that for isothermal conditions the particle velocity v_1 goes to v^* as the pressure p_1 goes to infinity and that it goes to zero as the pressure p_1 goes to p_o . Discuss this result.

7. Consider an ideal gas confined inside a tube. Define its “elastic modulus” E as the change of axial stress in the gas per unit change in the axial strain in the gas. Show that for the isothermal propagation of a disturbance along the tube we have a wave speed

$$v^* = \sqrt{\frac{E}{\rho_o}},$$

identical to that of a solid elastic bar. Discuss the similarities and differences of elastic bar and our gas-filled tube.

- 6.11-2:** Consider the problem of the propagation of a shear wave in an elastic solid as shown in Figure 6.17. Assume the wave displaces the particles vertically and is moving into a stationary solid (at equilibrium) at a speed v^* . To the left of the surface of discontinuity assume the motion is given by $x_1 = X_1$, $x_2 = X_2 + u_2(X_1, t)$, and $x_3 = X_3$, where x_i is the current position of particles originally at position X_i , and u_2 is the vertical displacement of the particles. Study the propagation of this shear wave through the elastic solid of constitutive equation

$$\mathbf{T} = \kappa(J - 1)\mathbf{I} + G(\mathbf{B} - J^{2/3}\mathbf{I}),$$

where κ is the bulk modulus and G is the shear modulus. Follow the steps below to obtain the wave speed.

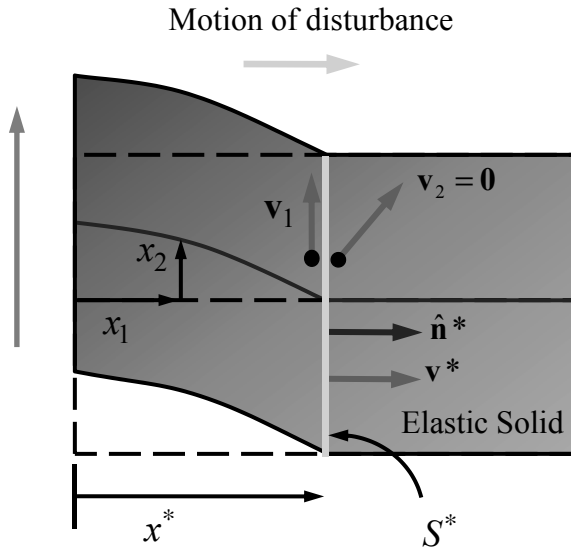


Figure 6.17: A disturbance in the form of a shear wave moving in an elastic solid.

1. Show that the left Cauchy stretch tensor for this deformation is given by $\mathbf{B} = \mathbf{I} + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \gamma^2 \mathbf{e}_2 \otimes \mathbf{e}_2$ where the shear strain γ is given by

$$\gamma = \frac{\partial u_2}{\partial X_1},$$

and show that this results in the Cauchy stress being given by

$$\mathbf{T} = G[\gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \gamma^2 \mathbf{e}_2 \otimes \mathbf{e}_2].$$

2. Show that the jump condition resulting from the conservation of mass yields

$$\rho_1 = \rho_2 = \rho_o.$$

3. Show that the jump condition resulting from the balance of linear momentum yields the relation

$$\rho_1 v^* v_1 = -G\gamma,$$

where v^* and v_1 are, respectively, the magnitudes of $\mathbf{v}^*$ and $\mathbf{v}_1$.

4. Constrain the motion so that to the left of the surface S^* the displacement is zero. Do this by requiring that $u_2(x^*, t) = 0$ at all times (the material does not tear). Use this condition to show that

$$\gamma v^* + v_1 = 0,$$

and then conclude that the wavefront speed v^* is constant and given by

$$v^* = \sqrt{\frac{G}{\rho_o}}.$$

5. Use this wave speed to show that the velocity of the particle to the left of the wavefront is given by

$$v_1 = -\frac{\tau}{\sqrt{\rho_o G}},$$

where $\tau = G\gamma$ is the driving shear stress.

6. Using the jump condition resulting from the conservation of mass, show that if the material abruptly changes, the disturbance cannot remain a simple shear deformation.
7. Discuss what happens when S^* reaches the free surface.

6.11-3: Write the jump conditions for a surface S^* that moves with the same velocity as the material particles. These are the conditions that must be satisfied when there is an abrupt change in material properties.

6.12 Perturbing a motion

Sometimes it is useful to study the effect of a perturbation on a history that already satisfies the balance laws. For example, one might ask if the history is stable and perturbing it might answer this question, or one might ask how a wave might propagate in a deforming body. As such, the original history is known and assumed to satisfy the balance equations and we seek equations for the perturbation.

For our study, let us assume that the history of the known position and temperature are given by

$$\mathbf{x}(\mathbf{X}, t), \quad \theta(\mathbf{X}, t). \quad (6.130)$$

For this motion of the body, we have $\mathbf{b}$ and r and we get $\mathbf{v}$, $\mathbf{a}$, $\mathbf{F}$, $\mathbf{G}$, ψ , η , $\mathbf{T}$, and $\mathbf{q}$ at each point and time. We now construct an alternate history that is a perturbation of the original. We will denote it with a superscript “*” and write it as

$$\mathbf{x}^*(\mathbf{X}, t), \quad \theta^*(\mathbf{X}, t). \quad (6.131)$$

For this motion we have $\mathbf{b}^*$ and r^* and we then get $\mathbf{v}^*$, $\mathbf{a}^*$, $\mathbf{F}^*$, $\mathbf{G}^*$, ψ^* , η^* , $\mathbf{T}^*$, and $\mathbf{q}^*$ at each point and time. We will use “ δ ” to note the change, which we call the perturbation. As a result we have

$$\mathbf{x}^*(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) + \delta\mathbf{x}(\mathbf{X}, t), \quad \theta^*(\mathbf{X}, t) = \theta(\mathbf{X}, t) + \delta\theta(\mathbf{X}, t). \quad (6.132)$$

The same assumption will be made for the other variables so that we have $\mathbf{v}^* = \mathbf{v} + \delta\mathbf{v}$, $\mathbf{a}^* = \mathbf{a} + \delta\mathbf{a}$, $\mathbf{F}^* = \mathbf{F} + \delta\mathbf{F}$, $\mathbf{G}^* = \mathbf{G} + \delta\mathbf{G}$, $\psi^* = \psi + \delta\psi$, $\eta^* = \eta + \delta\eta$, $\mathbf{T}^* = \mathbf{T} + \delta\mathbf{T}$, and $\mathbf{q}^* = \mathbf{q} + \delta\mathbf{q}$.

Both motions are assumed to satisfy the balance laws. Here we look only at the differential form of these laws, and take the Cauchy stress to be symmetric so that we automatically satisfy the balance of angular momentum for both motions and assume the Clausius-Duhem inequality will be separately satisfied for specific models. Since the two motions are possibly distinct, it is easier to write the equations in the reference configuration, which is the same

for both. At each point $\mathbf{X}$ in the reference configuration and for all times t , we have the balance laws

$$\rho J = \rho_o, \quad (6.133)$$

$$\text{Div}(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o \mathbf{a}, \quad (6.134)$$

$$-\text{Div}(\mathbf{q}_o) + \rho_o r + \text{tr}(\mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \dot{e}, \quad (6.135)$$

for the initial motion and the balance laws

$$\rho^* J^* = \rho_o, \quad (6.136)$$

$$\text{Div}(\mathbf{T}_o^{*T}) + \rho_o \mathbf{b}^* = \rho_o \mathbf{a}^*, \quad (6.137)$$

$$-\text{Div}(\mathbf{q}_o^*) + \rho_o r^* + \text{tr}(\mathbf{T}_o^* \dot{\mathbf{F}}^*) = \rho_o \dot{e}^*, \quad (6.138)$$

for the second motion. We, respectively, subtract these equations to get

$$\rho^* J^* = \rho J, \quad (6.139)$$

$$\text{Div}(\mathbf{T}_o^{*T} - \mathbf{T}_o^T) + \rho_o \delta \mathbf{b} = \rho_o \delta \mathbf{a}, \quad (6.140)$$

$$-\text{Div}(\mathbf{q}_o^* - \mathbf{q}_o) + \rho_o \delta r + \text{tr}(\mathbf{T}_o^* \dot{\mathbf{F}}^* - \mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \delta \dot{e}. \quad (6.141)$$

We now use the relations for the nominal stress and heat flux given by

$$\mathbf{T}_o = J \mathbf{F}^{-1} \mathbf{T}, \quad \mathbf{q}_o = J \mathbf{F}^{-1} \mathbf{q}, \quad (6.142)$$

$$\mathbf{T}_o^* = J^* \mathbf{F}^{*-1} \mathbf{T}^*, \quad \mathbf{q}_o^* = J^* \mathbf{F}^{*-1} \mathbf{q}^*, \quad (6.143)$$

to get the balance laws as

$$\rho^* J^* = \rho J, \quad (6.144)$$

$$\text{Div}(J^* \mathbf{T}^{*T} \mathbf{F}^{*-T} - J \mathbf{T}^T \mathbf{F}^{-T}) + \rho_o \delta \mathbf{b} = \rho_o \delta \mathbf{a}, \quad (6.145)$$

$$-\text{Div}(J^* \mathbf{F}^{*-1} \mathbf{q}^* - J \mathbf{F}^{-1} \mathbf{q}) + \rho_o \delta r + \text{tr}(J^* \mathbf{F}^{*-1} \mathbf{T}^* \dot{\mathbf{F}}^* - J \mathbf{F}^{-1} \mathbf{T} \dot{\mathbf{F}}) = \rho_o \delta \dot{e}. \quad (6.146)$$

Let us now assume the perturbation is small. In particular, let us assume that $\mathbf{F}^* = \hat{\mathbf{F}} \mathbf{F}$ for $\hat{\mathbf{F}} = \mathbf{I} + \hat{\mathbf{H}}$ with the assumption that $\hat{\mathbf{H}}$ is small compared to the identity. We first note that $\hat{J} \approx 1 + \hat{\epsilon}_v$ so that

$$J^* = \hat{J} J \approx J + \hat{\epsilon}_v J. \quad (6.147)$$

We also note that $\hat{\mathbf{F}}^{-1} \approx \mathbf{I} - \hat{\mathbf{H}}$. We can, therefore, write nominal stress and heat flux vector as

$$\mathbf{T}_o^{*T} = J^* \mathbf{T}^{*T} \mathbf{F}^{*-T} = \hat{J} J \mathbf{T}^{*T} \hat{\mathbf{F}}^{-T} \mathbf{F}^{-T} \approx (1 + \hat{\epsilon}_v) J \mathbf{T}^{*T} (\mathbf{I} - \hat{\mathbf{H}}^T) \mathbf{F}^{-T}, \quad (6.148)$$

$$\mathbf{q}_o^* = J^* \mathbf{F}^{*-1} \mathbf{q}^* = \hat{J} J \mathbf{F}^{-1} \hat{\mathbf{F}}^{-1} \mathbf{q}^* \approx (1 + \hat{\epsilon}_v) J \mathbf{F}^{-1} (\mathbf{I} - \hat{\mathbf{H}}) \mathbf{q}^*. \quad (6.149)$$

Ignoring second- and higher-order terms, we get

$$\mathbf{T}_o^{*T} - \mathbf{T}_o^T \approx J \delta \mathbf{T}^T \mathbf{F}^{-T} + \hat{\epsilon}_v J \mathbf{T}^T \mathbf{F}^{-T} - J \mathbf{T}^T \hat{\mathbf{H}}^T \mathbf{F}^{-T}, \quad (6.150)$$

$$\mathbf{q}_o^* - \mathbf{q}_o \approx J \mathbf{F}^{-1} \delta \mathbf{q} + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{q} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{q}. \quad (6.151)$$

Since $\mathbf{F}^* = \hat{\mathbf{F}} \mathbf{F}$, we know that $\dot{\mathbf{F}}^* = \dot{\hat{\mathbf{F}}} \mathbf{F} + \hat{\mathbf{F}} \dot{\mathbf{F}} = \dot{\hat{\mathbf{H}}} \mathbf{F} + \hat{\mathbf{F}} \dot{\mathbf{F}} + \hat{\mathbf{H}} \dot{\mathbf{F}}$. From this we can calculate

$$\begin{aligned} J^* \mathbf{F}^{*-1} \mathbf{T}^* \dot{\mathbf{F}}^* &= \hat{J} J \mathbf{F}^{-1} \hat{\mathbf{F}}^{-1} \mathbf{T}^* (\dot{\hat{\mathbf{H}}} \mathbf{F} + \hat{\mathbf{F}} \dot{\mathbf{F}} + \hat{\mathbf{H}} \dot{\mathbf{F}}) \\ &\approx (1 + \hat{\epsilon}_v) J \mathbf{F}^{-1} (\mathbf{I} - \hat{\mathbf{H}}) \mathbf{T}^* (\dot{\hat{\mathbf{H}}} \mathbf{F} + \hat{\mathbf{F}} \dot{\mathbf{F}} + \hat{\mathbf{H}} \dot{\mathbf{F}}) \\ &\approx J \mathbf{F}^{-1} \mathbf{T}^* \dot{\mathbf{F}} + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{T}^* \dot{\mathbf{F}} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{T}^* \dot{\mathbf{F}} + J \mathbf{F}^{-1} \mathbf{T}^* \dot{\hat{\mathbf{H}}} \mathbf{F} + J \mathbf{F}^{-1} \mathbf{T}^* \hat{\mathbf{H}} \dot{\mathbf{F}} \\ &\quad + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{T}^* \dot{\hat{\mathbf{H}}} \mathbf{F} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{T}^* \dot{\hat{\mathbf{H}}} \mathbf{F}. \end{aligned} \quad (6.152)$$

As a result we have

$$\begin{aligned} \mathbf{T}_o^* \dot{\mathbf{F}}^* - \mathbf{T}_o \dot{\mathbf{F}} &\approx J \mathbf{F}^{-1} \delta \mathbf{T} \dot{\mathbf{F}} + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{T} \dot{\mathbf{F}} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{T} \dot{\mathbf{F}} + J \mathbf{F}^{-1} \mathbf{T} \hat{\mathbf{H}} \dot{\mathbf{F}} + J \mathbf{F}^{-1} \mathbf{T} \hat{\mathbf{H}} \dot{\mathbf{F}} \\ &+ J \mathbf{F}^{-1} \delta \mathbf{T} \hat{\mathbf{H}} \dot{\mathbf{F}} + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{T} \hat{\mathbf{H}} \dot{\mathbf{F}} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{T} \hat{\mathbf{H}} \dot{\mathbf{F}}, \end{aligned} \quad (6.153)$$

and

$$\begin{aligned} tr(\mathbf{T}_o^* \dot{\mathbf{F}}^* - \mathbf{T}_o \dot{\mathbf{F}}) &\approx J tr(\delta \mathbf{T} \mathbf{L}) + \hat{\epsilon}_v J tr(\mathbf{T} \mathbf{L}) - J tr(\hat{\mathbf{H}} \mathbf{T} \mathbf{L}) \\ &+ J tr(\mathbf{T} \hat{\mathbf{H}}) + J tr(\mathbf{T} \hat{\mathbf{H}} \mathbf{L}) \\ &+ J tr(\delta \mathbf{T} \hat{\mathbf{H}}) + \hat{\epsilon}_v J tr(\mathbf{T} \hat{\mathbf{H}}) - J tr(\hat{\mathbf{H}} \mathbf{T} \hat{\mathbf{H}}). \end{aligned} \quad (6.154)$$

For the internal energy $e^* = \psi^* + \eta^* \theta^*$ we have $\dot{e}^* = \dot{\psi}^* + \dot{\eta}^* \theta^* + \eta^* \dot{\theta}^*$ so that

$$\begin{aligned} \dot{e}^* &= \dot{\psi} + \delta \dot{\psi} + (\dot{\eta} + \delta \dot{\eta})(\theta + \delta \theta) + (\eta + \delta \eta)(\dot{\theta} + \delta \dot{\theta}) \\ &\approx \dot{\psi} + \delta \dot{\psi} + \dot{\eta} \theta + \eta \dot{\theta} + \theta \delta \dot{\eta} + \dot{\eta} \delta \theta + \eta \delta \dot{\theta} + \dot{\theta} \delta \eta + \delta \dot{\eta} \delta \theta + \delta \eta \delta \dot{\theta}. \end{aligned} \quad (6.155)$$

From this we get

$$\delta \dot{e} \approx \delta \dot{\psi} + \theta \delta \dot{\eta} + \dot{\eta} \delta \theta + \eta \delta \dot{\theta} + \dot{\theta} \delta \eta + \delta \dot{\eta} \delta \theta + \delta \eta \delta \dot{\theta}. \quad (6.156)$$

As a result, for the perturbation the conservation of mass becomes

$$\boxed{\rho^*(1 + \hat{\epsilon}_v) = \rho.} \quad (6.157)$$

The balance of linear momentum becomes

$$\boxed{Div[J \delta \mathbf{T}^T \mathbf{F}^{-T} + \hat{\epsilon}_v J \mathbf{T}^T \mathbf{F}^{-T} - J \mathbf{T}^T \hat{\mathbf{H}}^T \mathbf{F}^{-T}] + \rho_o \delta \mathbf{b} = \rho_o \delta \mathbf{a}.} \quad (6.158)$$

The balance of work and energy becomes

$$\begin{aligned} &-Div(J \mathbf{F}^{-1} \delta \mathbf{q} + \hat{\epsilon}_v J \mathbf{F}^{-1} \mathbf{q} - J \mathbf{F}^{-1} \hat{\mathbf{H}} \mathbf{q}) + J tr(\delta \mathbf{T} \mathbf{L}) + \hat{\epsilon}_v J tr(\mathbf{T} \mathbf{L}) - J tr(\hat{\mathbf{H}} \mathbf{T} \mathbf{L}) \\ &+ J tr(\mathbf{T} \hat{\mathbf{H}}) + J tr(\mathbf{T} \hat{\mathbf{H}} \mathbf{L}) + J tr(\delta \mathbf{T} \hat{\mathbf{H}}) + \hat{\epsilon}_v J tr(\mathbf{T} \hat{\mathbf{H}}) - J tr(\hat{\mathbf{H}} \mathbf{T} \hat{\mathbf{H}}) \\ &+ \rho_o \delta r = \rho_o (\delta \dot{\psi} + \theta \delta \dot{\eta} + \dot{\eta} \delta \theta + \eta \delta \dot{\theta} + \dot{\theta} \delta \eta + \delta \dot{\eta} \delta \theta + \delta \eta \delta \dot{\theta}). \end{aligned} \quad (6.159)$$

The balance laws can be expanded using the identities for divergence of products. Also, the laws can be written in the current configuration of the unperturbed motion. These are left for the homework.

Section 6.12-Problems:

6.12-1: Using the identities for the divergence of the products of scalars, vectors, and tensors, expand the balance laws derived for linear momentum and for work and energy.

6.12-2: Write the conservation of mass, balance of linear momentum, and balance of work and energy in the differential form for the intermediate configuration to get

$$\rho J = \rho_o, \quad (6.160)$$

$$div(\mathbf{T}^T) + \rho \mathbf{b} = \rho \mathbf{a}, \quad (6.161)$$

$$-div(\mathbf{q}) + \rho r + tr(\mathbf{T} \mathbf{L}) = \rho \dot{e}, \quad (6.162)$$

for the initial motion and

$$\rho^* \hat{J} = \rho, \quad (6.163)$$

$$\operatorname{div}(\hat{J} \mathbf{T}^{*T} \hat{\mathbf{F}}^{-T}) + \rho \mathbf{b}^* = \rho \mathbf{a}^*, \quad (6.164)$$

$$-\operatorname{div}(\hat{J} \hat{\mathbf{F}}^{-1} \mathbf{q}^*) + \rho r^* + \operatorname{tr}(\hat{J} \mathbf{T}^* \mathbf{L}^*) = \rho \dot{e}^*, \quad (6.165)$$

for the second flow, where $\hat{\mathbf{L}} = \dot{\hat{\mathbf{F}}} \hat{\mathbf{F}}^{-1}$ and $\operatorname{div}(\cdot) = \operatorname{div}_{\mathbf{x}}(\cdot)$ is the divergence with respect to the unperturbed configuration. Prove the relation $\mathbf{L}^* = \hat{\mathbf{L}} + \hat{\mathbf{F}} \mathbf{L} \hat{\mathbf{F}}^{-1}$, and show that

$$\rho^* \hat{J} = \rho, \quad (6.166)$$

$$\operatorname{div}(\hat{J} \mathbf{T}^{*T} \hat{\mathbf{F}}^{-T} - \mathbf{T}^T) + \rho \delta \mathbf{b} = \rho \delta \mathbf{a}, \quad (6.167)$$

$$-\operatorname{div}(\hat{J} \hat{\mathbf{F}}^{-1} \mathbf{q}^* - \mathbf{q}) + \rho \delta r + \operatorname{tr}(\hat{J} \mathbf{T}^* \hat{\mathbf{L}} + \hat{J} \mathbf{T}^* \hat{\mathbf{F}} \mathbf{L} \hat{\mathbf{F}}^{-1} - \mathbf{T} \mathbf{L}) = \rho \delta \dot{e}. \quad (6.168)$$

Assume that the perturbation is small so that $\hat{J} \approx 1 + \hat{\epsilon}_v$ and $\hat{\mathbf{F}}^{-1} \approx \mathbf{I} - \hat{\mathbf{H}}$ and show that for small perturbations we have the conservation of mass given by

$$\boxed{\delta \rho = -\hat{\epsilon}_v \rho}, \quad (6.169)$$

and the two other balance laws are given by

$$\operatorname{div}(\delta \mathbf{T}^T + \hat{\epsilon}_v \mathbf{T}^T - \mathbf{T}^T \hat{\mathbf{H}}^T) + \rho \delta \mathbf{b} = \rho \delta \mathbf{a}, \quad (6.170)$$

$$\begin{aligned} -\operatorname{div}(\delta \mathbf{q} + \hat{\epsilon}_v \mathbf{q} - \hat{\mathbf{H}} \mathbf{q}) + \rho \delta r + \operatorname{tr}[(\mathbf{T} + \delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T} - \hat{\mathbf{H}} \mathbf{T}) \dot{\hat{\mathbf{H}}}] \\ + \operatorname{tr}[(\mathbf{L} \mathbf{T} - \mathbf{T} \mathbf{L}) \hat{\mathbf{H}}] + \operatorname{tr}[(\delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T}) \mathbf{L}] = \rho \delta \dot{e}, \end{aligned} \quad (6.171)$$

where

$$\delta \dot{e} \approx \delta \dot{\psi} + \theta \delta \dot{\eta} + \dot{\eta} \delta \theta + \eta \delta \dot{\theta} + \dot{\theta} \delta \eta + \delta \dot{\eta} \delta \theta + \delta \eta \delta \dot{\theta}. \quad (6.172)$$

Use the identities

$$\operatorname{div}(\phi \mathbf{A}) = \mathbf{A} \operatorname{grad}(\phi) + \phi \operatorname{div}(\mathbf{A}), \quad (6.173)$$

$$\operatorname{div}(\mathbf{A} \mathbf{B}) = \operatorname{grad}(\mathbf{A}) : \mathbf{B} + \mathbf{A} \operatorname{div}(\mathbf{B}) \quad (6.174)$$

to show that the balance of linear momentum can be written as

$$\boxed{\operatorname{div}(\delta \mathbf{T}^T) + \mathbf{T}^T [\operatorname{grad}(\hat{\epsilon}_v) - \operatorname{div}(\hat{\mathbf{H}}^T)] + \hat{\epsilon}_v \operatorname{div}(\mathbf{T}^T) - \operatorname{grad}(\mathbf{T}^T) : \hat{\mathbf{H}}^T + \rho \delta \mathbf{b} = \rho \delta \mathbf{a}.} \quad (6.175)$$

Use the identities

$$\operatorname{div}(\phi \mathbf{u}) = \mathbf{u} \circ \operatorname{grad}(\phi) + \phi \operatorname{div}(\mathbf{u}), \quad (6.176)$$

$$\operatorname{div}(\mathbf{A} \mathbf{u}) = \operatorname{div}(\mathbf{A}^T) \circ \mathbf{u} + \operatorname{grad}(\mathbf{u}) : \mathbf{A}^T \quad (6.177)$$

to show that the balance of work and energy can be written as

$$\boxed{\begin{aligned} -\operatorname{div}(\delta \mathbf{q}) - \mathbf{q} \circ \operatorname{grad}(\hat{\epsilon}_v) - \hat{\epsilon}_v \operatorname{div}(\mathbf{q}) + \operatorname{div}(\hat{\mathbf{H}}^T) \circ \mathbf{q} + \operatorname{grad}(\mathbf{q}) : \hat{\mathbf{H}}^T \\ + \rho \delta r + \operatorname{tr}[(\mathbf{T} + \delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T} - \hat{\mathbf{H}} \mathbf{T}) \dot{\hat{\mathbf{H}}}] + \operatorname{tr}[(\mathbf{L} \mathbf{T} - \mathbf{T} \mathbf{L}) \hat{\mathbf{H}}] \\ + \operatorname{tr}[(\delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T}) \mathbf{L}] = \rho \delta \dot{e}. \end{aligned}} \quad (6.178)$$

Show that for homogeneous base histories we have the balance of linear momentum given by

$$\operatorname{div}(\delta \mathbf{T}^T) + \mathbf{T}^T[\operatorname{grad}(\hat{\epsilon}_v) - \operatorname{div}(\hat{\mathbf{H}}^T)] + \rho \delta \mathbf{b} = \rho \delta \mathbf{a}, \quad (6.179)$$

and the balance of work and energy given by

$$\begin{aligned} & -\operatorname{div}(\delta \mathbf{q}) - \mathbf{q} \circ \operatorname{grad}(\hat{\epsilon}_v) + \operatorname{div}(\hat{\mathbf{H}}^T) \circ \mathbf{q} + \rho \delta r \\ & + \operatorname{tr}[(\mathbf{T} + \delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T} - \hat{\mathbf{H}} \mathbf{T}) \hat{\mathbf{H}}] + \operatorname{tr}[(\mathbf{L} \mathbf{T} - \mathbf{T} \mathbf{L}) \hat{\mathbf{H}}] \\ & + \operatorname{tr}[(\delta \mathbf{T} + \hat{\epsilon}_v \mathbf{T}) \mathbf{L}] = \rho \delta \dot{e}. \end{aligned} \quad (6.180)$$

6.12-3: Consider the problem of a homogeneous preload on which we impose a steady attenuating wave. Use the results of the last section to obtain the equation of balance of linear momentum for the wave described by the perturbation given by

$$\delta \mathbf{u} = \mathbf{d} e^{-\alpha \mathbf{x} \circ \hat{\mathbf{n}}} \sin \left[\omega \left(t - \frac{\mathbf{x} \circ \hat{\mathbf{n}}}{c} \right) \right], \quad (6.181)$$

where c is the wave speed, ω is the circular frequency, $\hat{\mathbf{n}}$ is the direction of propagation of the wave, $\mathbf{d}$ is a constant vector representing the amplitude of oscillation on the reference surface defined as $\mathbf{x} \circ \hat{\mathbf{n}} = 0$, and α is the spatial attenuation coefficient. Do this by first showing that $\hat{\mathbf{H}} = \operatorname{grad}(\delta \mathbf{u})$ and that $\hat{\epsilon}_v = \operatorname{tr}(\hat{\mathbf{H}})$.

6.13 Initial and boundary conditions

For a given material body, the initial and boundary conditions define how the balance laws control the response of the body. Essentially, different problems are defined by the shape and material of the body, and the initial and boundary conditions.

To predict the response of a body, we need to have a good idea of the condition under which the body is at the first instance that we want to start the prediction process, the characteristic response of the body, and the influence of the surrounding on the body from the starting time onward. These are called, respectively, the initial conditions, the constitutive models, and the boundary conditions. Given these, and the balance laws, we are able to systematically build the solution (response) either analytically or numerically. The number of constitutive models is vast and varied and the remainder of the book is dedicated, to some extent, to developing different models, so we will leave their discussion to later.

The initial conditions are the sum of all the information needed to initiate the solution process from the initial time onward. Let us take the initial time as t_o . Typically, to start a solution process, we need to know the position, velocity, and temperature at each point of the body. This is the minimum information we need to predict the motion and temperature of the body for a first gradient theory. We can write this as

$$\mathbf{x}(\mathbf{X}, t_o) = \mathbf{x}_o(\mathbf{X}), \quad (6.182)$$

$$\mathbf{v}(\mathbf{X}, t_o) = \mathbf{v}_o(\mathbf{X}), \quad (6.183)$$

$$\theta(\mathbf{X}, t_o) = \theta_o(\mathbf{X}), \quad (6.184)$$

where $\mathbf{x}_o(\mathbf{X})$, $\mathbf{v}_o(\mathbf{X})$, and $\theta_o(\mathbf{X})$ are functions giving the initial values of, respectively, the position, velocity, and temperature for each point $\mathbf{X} \in V_o$. For the simpler theories, such

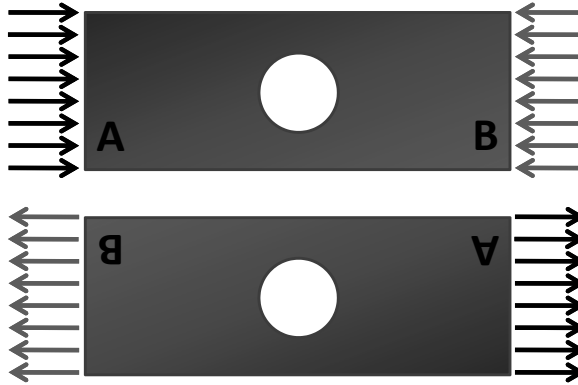


Figure 6.18: An example of a case when a large rotation totally changes the sense of the traction on a surface.

as for an elastic solid or viscous fluid, these provide all we need to know about the initial conditions of the body, but for more complex materials, we might need to know some or all of the history prior to the initial time for each point in the body. In essence, we need to be able to calculate the response from the constitutive equation and this might require us to know as much as all that has happened to each point of the body prior to the current time, which includes all times before t_o . We are fortunate, for example, that in the case of a thermoelastic material we can calculate the response from knowing the deformation gradient, temperature, and temperature gradient, which can be directly calculated at the initial time from $\mathbf{x}_o(\mathbf{X})$, $\mathbf{v}_o(\mathbf{X})$, and $\theta_o(\mathbf{X})$, and, as we proceed, the balance laws give us the new distributions of these variables that provide the needed information. Some material models require us to know the influence of the past through knowledge of internal parameters or state variables, such as the plastic strain and hardening parameter. For such models, these need to be added as initial conditions.

The boundary conditions provide the influence of the surrounding on the body from the initial time onward. For a first gradient material, this might be displacement, traction, temperature, and/or heat flux on the boundary. We could fully describe the displacement $\bar{\mathbf{u}}$ of points on the boundary S_{o1} by requiring that

$$\mathbf{u}(\mathbf{X}, t) = \bar{\mathbf{u}}(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o1}, \quad (6.185)$$

or we just could control the displacement along a direction $\hat{\mathbf{m}}$ on S_{o2} to be $\bar{u}$ by requiring that

$$\mathbf{u}(\mathbf{X}, t) \circ \hat{\mathbf{m}}(\mathbf{X}, t) = \bar{u}(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o2}. \quad (6.186)$$

In a similar manner, we might give conditions for the traction, temperature, and heat flux as

$$\mathbf{t}^{(n)}(\mathbf{X}, t) = \bar{\mathbf{t}}(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o3}, \quad (6.187)$$

$$\mathbf{t}^{(n)}(\mathbf{X}, t) \circ \hat{\mathbf{m}}(\mathbf{X}, t) = \bar{t}(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o4}, \quad (6.188)$$

$$\theta(\mathbf{X}, t) = \bar{\theta}(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o5}, \quad (6.189)$$

$$q_n(\mathbf{X}, t) = \bar{q}_n(\mathbf{X}, t), \text{ for } \mathbf{X} \in S_{o6}, \quad (6.190)$$

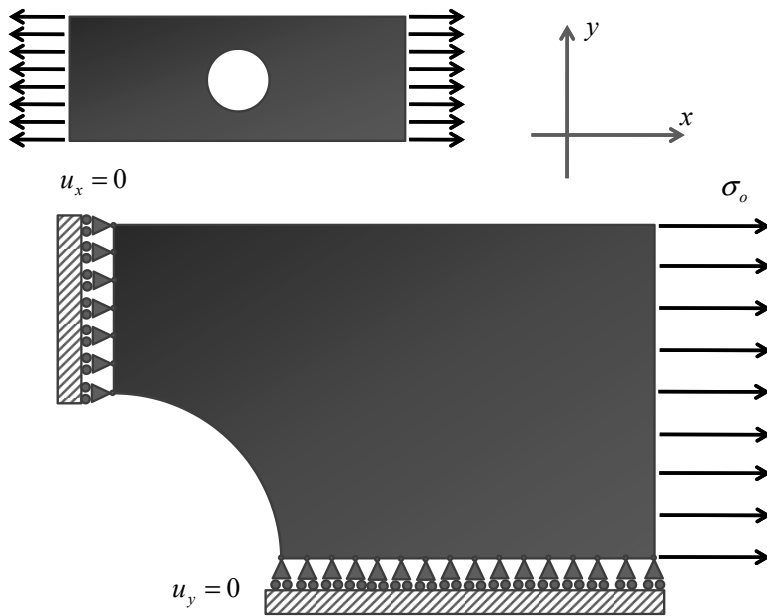


Figure 6.19: An example of modeling the extension of a plate using the symmetry of the problem to model only part of the plate and in the process introducing displacement boundary conditions that remove the possibility of rigid body motions.

where the S_{oi} are different parts of the initial surface S_o , and $\hat{\mathbf{m}}$ is a specified direction. Sometimes the conditions can be on the current surface of the body. We could sometimes know the total load on a part of the surface so that we might have a condition of the form

$$\int_{S_7} \mathbf{t}^{(n)} dS = \mathbf{P}_o \quad (6.191)$$

for the total applied load of $\mathbf{P}_o$ on the surface S_7 . In a similar manner, we might know the moment on a surface, the average displacement and rotation, or the total heat flux. The traction and heat flux boundary conditions frequently can be interactive, such as seen for the convection boundary condition where the heat flux depends on the surface temperature, or as seen for foundation support where the traction depends on the displacement of the surface. Some care needs to be taken when imposing boundary conditions since one can overspecify the influence of the boundary. For example, one can sometimes mix displacement and traction boundary conditions at a point as long as the conditions are not along the same direction. One should take special caution on imposing boundary conditions when there can be large deformations since the shape and place of the boundary can change substantially. For example, Figure 6.18 shows a plate before and after a 180° rotation. As can be seen, even though the direction of the traction on surface A is the same in both cases, in one case the problem represents the extension of the plate and in the other it represents the compression of the plate.

In statics there is the possibility of multiple solutions that satisfy the balance laws. To avoid this, particularly in numerical analysis, we need to make sure the body is secured and the temperature cannot float. This is achieved by making sure the body cannot undergo pure rigid body translations or rotations, and the temperature cannot be shifted. This normally

means that we need to put sufficient displacement and temperature boundary conditions to ensure the body cannot freely move, and the temperature cannot float. We need to make sure this is done in a way that does not alter the boundary conditions that characterize the problem. For example, in Figure 6.19, we see a plate being extended under the action of a uniform stress field. Even though the problem is technically well posed, if we are considering a static solution, we have translational undetermined degrees of freedom. To solve this, we can use the symmetry of the problem to reduce it to one quarter of the plate and introduce displacement boundary conditions that remove the possible translations.

Not all problems are forward predictions. Sometimes problems can be formulated where from the current response we try to predict the initial conditions, material properties, or the optimal shape. These problems, even under fairly normal conditions, might be ill posed. An example might be when multiple initial and boundary conditions can result in the same current state. Another is seen in the effort to characterize the material properties from how the material responds to a specific loading. In spite of the additional complexity associated with these problems, they can provide both interesting and valuable insight to the questions they are trying to answer.

In addition, sometimes the problem is formulated such that we are not seeking the solution to a specific problem, just the characteristics of the response of a body. An example of such a problem is seen when obtaining the normal modes of vibration of a linear elastic body.

Formulating problems is essentially an art that plays with the shape, the material, and the initial and boundary conditions to answer a question of interest. Some examples of this are given in Chapter 8 for elasticity and plasticity.

Section 6.13-Problems:

- 6.13-1:** Consider the problem shown in Figure 6.20, where the body of volume V is being deformed through the action of several components. Discuss the boundary conditions that you would use to model this problem. In doing so, assume that the surface S_1 is glued to a rigid block that is connected to two screw actuators that each can control the length of the actuator and how this length changes, the surface S_2 is rough and is acted on by a rigid block that is controlled through two hydraulic actuators that each can control the load in the actuator, the surface S_3 is glued to a rigid slider that sits on an incline and is connected by a spring to a fixed support, the surface S_4 is glued to a connection that has a weight W hanging from it, the surface S_5 is frictionless and is being pushed on by a pad that is loaded by the hydraulic actuator attached to the pivoting rigid arm, and the surface S_6 has a fluid bath of temperature θ^* touching it. As an example, you might model the influence of the weight to require that

$$\int_{S_4} \mathbf{t}^{(n)} dS = -W\hat{\mathbf{j}}.$$

6.14 Notation

The following table shows the symbols used in the text and their definitions.

$[[\mathbf{u}]]$		jump in a quantity ($\mathbf{u}_+ - \mathbf{u}_-$)
$\nabla_{\mathbf{x}}(\mathbf{A})$		del or nabla operator for $\mathbf{x}$ applied to $\mathbf{A}$ ($[\mathbf{grad}(\mathbf{A}^T)]^T$)

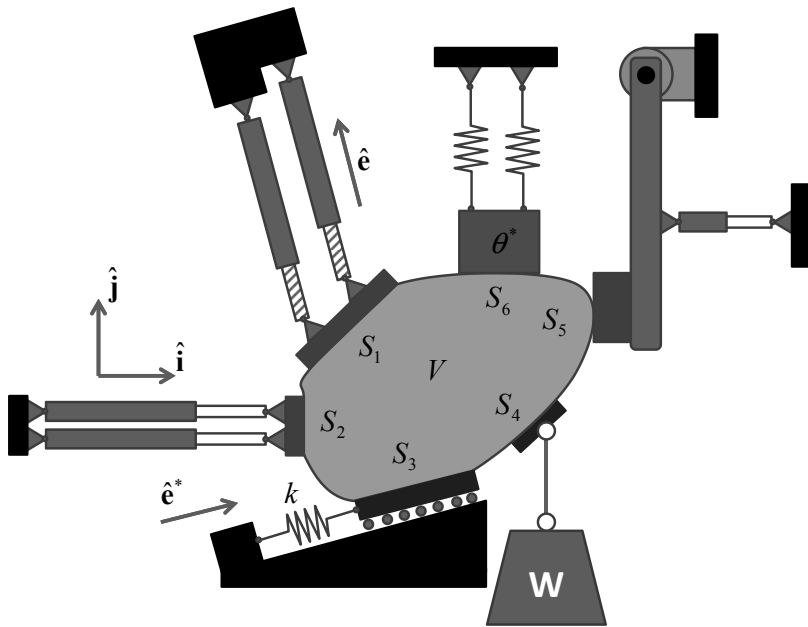


Figure 6.20: Example of a body being deformed under the action of several different loading systems.

$\nabla_{\mathbf{X}}(\mathbf{A})$		del or nabla operator for $\mathbf{X}$ applied to $\mathbf{A}$ ($[\mathbf{Grad}(\mathbf{A}^T)]^T$)
$\nabla_{\mathbf{x}} \circ (\mathbf{A})$		$[\text{div}(\mathbf{A}^T)]^T$
$\nabla_{\mathbf{X}} \circ (\mathbf{A})$		$[\text{Div}(\mathbf{A}^T)]^T$
$\mathbf{a}$		acceleration
$\mathbf{b}$		specific body force
$\mathcal{B}$		body
c		specific heat capacity
c_p		specific heat capacity at constant pressure
c_{rb}		specific heat capacity under rigid body processes
$\mathbf{C}$		right Cauchy stretch ($\mathbf{F}^T \mathbf{F}$)
div		divergence in current configuration
Div		divergence in reference configuration
$\frac{d}{dt}$		time derivative
$\frac{D}{Dt}$		material time derivative
$\mathbf{D}$		rate of deformation $[\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)]$
e		specific internal energy
$\mathbf{e}_i$		base vector in reference configuration
$\mathbf{e}_i^*$		base vector in current configuration
E		elastic modulus
EN		entropy of a body
$\mathbf{E}$		Green strain $[\frac{1}{2}(\mathbf{C} - \mathbf{I})]$
$\mathbf{F}$		deformation gradient ($d\mathbf{x} = \mathbf{F}d\mathbf{X}$)
$\hat{\mathbf{F}}$		$\mathbf{F}^* = \hat{\mathbf{F}}\mathbf{F} = \mathbf{F} + \delta\mathbf{F}$ in perturbation
$\mathcal{F}$		reference frame

g		acceleration of gravity
g_{ij}		metric tensor for base vectors in reference configuration
g_{ij}^*		metric tensor for base vectors in current configuration
grad		gradient with respect to $\mathbf{x}$
G		shear modulus
Grad		gradient with respect to $\mathbf{X}$
$\dot{h}$		specific heat flowing into a point [$r - \frac{1}{\rho} \text{div}(\mathbf{q})$]
$\mathbf{H}$		displacement gradient ($d\mathbf{u} = \mathbf{H}d\mathbf{X}$)
$\hat{\mathbf{H}}$		$\mathbf{F}^* = (\mathbf{I} + \hat{\mathbf{H}})\mathbf{F} = \mathbf{F} + \delta\mathbf{F}$ in perturbation
$\mathbf{I}$		identity
IE		internal energy of a body
J		volume ratio [$\det(\mathbf{F})$]
KE		kinetic energy of a body
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}}\mathbf{F}^{-1}$, $d\mathbf{v} = \mathbf{L}d\mathbf{x}$)
m		mass
$\hat{\mathbf{n}}$		unit normal vector to the surface in the current configuration
$\hat{\mathbf{N}}$		unit normal vector to the surface in the reference configuration
p		pressure
$\mathbf{P}$		force applied on a body
$q^{(\mathbf{n})}$		normal heat flux flowing out of surface with normal $\mathbf{n}$
$\mathbf{q}$		heat flux vector
$\mathbf{q}_o$		nominal heat flux vector ($J\mathbf{F}^{-1}\mathbf{q}$)
$\dot{Q}$		rate of heat flow into a body
$\mathbf{Q}$		orthogonal tensor representing rigid body rotation
r		specific radiation or heat generation
S_o		surface of body in reference configuration
$S(t)$		surface of body in current configuration
t		time
$\mathbf{t}^{(\mathbf{n})}$		traction vector on surface with normal $\mathbf{n}$
$\mathbf{t}_o^{(\mathbf{N})}$		engineering or nominal traction vector on surface with normal $\mathbf{N}$
$\mathbf{T}$		Cauchy stress
$\bar{\mathbf{T}}$		$\mathbf{T}^T$
$\tilde{\mathbf{T}}$		$\mathbf{F}^T \bar{\mathbf{T}} \mathbf{F}$
$\mathbf{T}_o$		nominal or engineering stress ($J\mathbf{F}^{-1}\mathbf{T}$)
$\tilde{\mathbf{T}}$		second Piola-Kirchhoff stress ($J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$)
$\mathbf{u}$		displacement from reference configuration ($\mathbf{x} = \mathbf{X} + \mathbf{u}$)
$\mathbf{v}$		velocity
V_o		volume of body in reference configuration
$V(t)$		volume of body in current configuration
$\dot{W}$		rate of doing work on a body
$\mathbf{x}$		current position
$\mathbf{X}$		reference position
α		angular acceleration
$\delta\mathbf{v}$		perturbation of $\mathbf{v}$ ($\mathbf{v}^* = \mathbf{v} + \delta\mathbf{v}$)
$\delta\mathbf{F}$		perturbation of $\mathbf{F}$ ($\mathbf{F}^* = \hat{\mathbf{F}}\mathbf{F} = \mathbf{F} + \delta\mathbf{F}$)
ϵ_v		infinitesimal volumetric strain [$\text{tr}(\epsilon)$]
$\hat{\epsilon}_v$		infinitesimal volumetric strain for $\hat{\mathbf{H}}$ [$\text{tr}(\hat{\epsilon})$]
ϵ		infinitesimal strain [$\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$]
$\hat{\epsilon}$		infinitesimal strain for $\hat{\mathbf{H}}$ [$\frac{1}{2}(\hat{\mathbf{H}} + \hat{\mathbf{H}}^T)$]
η		specific entropy

γ		shear strain
κ		bulk modulus
μ		viscosity or shear modulus
ω		angular velocity
Φ		total amount in volume V of a distributed quantity
Φ^*		total amount in volume V^* of a distributed quantity
ψ		specific free energy
τ		shear stress
ρ		density ($\rho J = \rho_o$)
ρ_o		reference density
σ		stress under infinitesimal deformations
θ		temperature

Chapter 7

Infinitesimal plasticity

This chapter will look at models for plasticity at small strains. As was noted in Chapter 4, at small strains the strain decomposes additively into elastic, plastic, and thermal strains. This provides substantial simplification in the resulting models and is why we first look at this special case before looking at the case for finite deformations. One might argue that the proper place for this chapter would be after the discussion of finite deformation theories, as a special limit case for small strains. This is true, and, as a result, we avoid extensive discussion of the details here, particularly in terms of material symmetry, since these are better addressed using the kinematics of finite deformation. These and other details are presented in more depth in the chapter on thermoplasticity.

Many materials, such as the aluminum shown in Figure 7.1, undergo small extensions before failure. In spite of the infinitesimal size of the extensions, as can be seen in the figure, these materials will undergo plastic flow under a substantial portion of their response. Yet, as also can be seen in the figure, during the entire loading the difference between the engineering and true stress is small and could be ignored. This chapter will focus on developing models for this type and range of response.

Infinitesimal plasticity historically was developed outside the thermodynamic framework so it contained arguments and restrictions that become superfluous when looking from a thermodynamic perspective. As such, these details are left out of the presentation.

The chapter begins with the examination of a mechanical analog for plasticity and the development of the elements of a simple model. This is followed by looking at the common assumptions made for these elements, which include the model for the elastic response, the yield function, the flow rule and the hardening parameter. This is then followed by the development of an example model from a specific selection from these choices, showing how one calculates the trial derivative of the yield function, and how one imposes the consistency condition to create a consistent flow rule.

Once we have looked at mechanical analogs and specific examples, we develop temperature dependent and thermodynamic models. For the thermodynamic model, we consider as an example materials with free-energy functions that are quadratic in the elastic and plastic strains.

For isotropic materials it is common to reduce the modeling of flow to models that are functions of scalar functions. In this context we consider scalar stress and hardening parameters. We also consider multielement mechanical analogs that are either in parallel or in series. The chapter is concluded by considering rate-dependent models and deformation plasticity.

7.1 A mechanical analog for plasticity

Mechanical analogs provide an ideal way of understanding the components of a model without directly dealing with the complexities that arise when looking at real response.

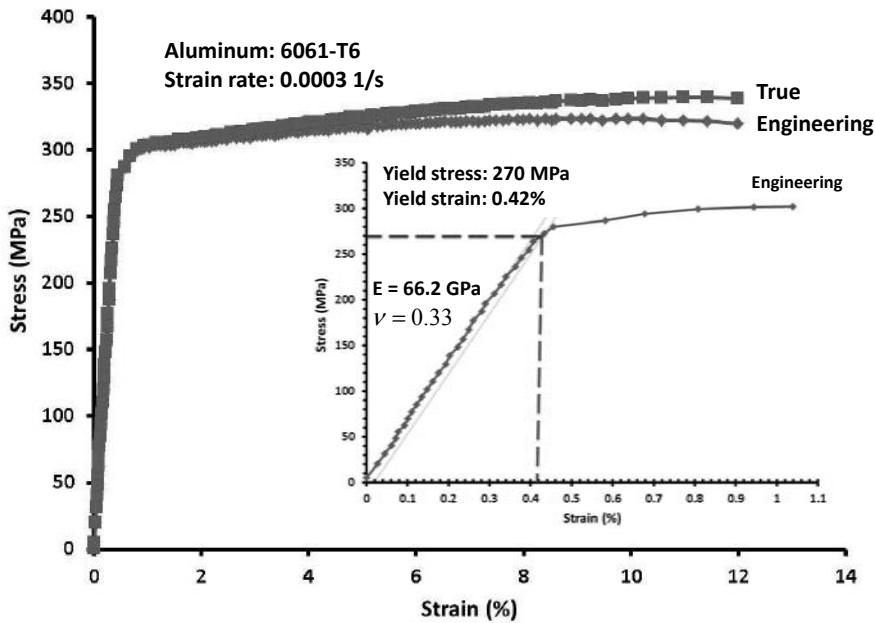


Figure 7.1: The monotonic extension of an aluminum.

In this section we will look at the mechanical analog shown in Figure 7.2. This is a typical mechanical analog for the elastic-plastic material response, being slightly more complex than the one shown in Figure 7.3 for elastic perfectly-plastic response.

The mechanical analog shown in Figure 7.2 is constructed from two springs and a stick-slip component. The characteristics of this system in monotonic loading are shown on the figure. As can be seen, with the increase of load there is an elastic response with slope k^e until reaching the yield load F_{yo} , after which yielding occurs, resulting in plastic flow and further extension, with the load increasing with a slope a . The mechanical analog is separated into two parts, an elastic element, consisting of a single spring with stiffness k^e , and a plastic element, consisting of a spring with stiffness k^b (known as the “back-stress spring”), and a stick-slip component that locks for loads less than F_{yo} . The stick-slip element may slip when the magnitude of the load in the stick-slip component becomes equal to F_{yo} , with slipping occurring in a manner that keeps the magnitude of the load in the stick-slip component equal to F_{yo} .

First, let us look at the kinematic assumptions. We will assume that the elastic and plastic elements initially have lengths L^e and L^p , respectively, such that the total initial length L is given by $L = L^e + L^p$. As we apply load on the system, we will assume that the elastic and plastic elements stretch by δ^e and δ^p , respectively, so that the total extension δ of the system is given by

$$\delta = \delta^e + \delta^p. \quad (7.1)$$

We will take the load in the spring of the elastic element as F^e , the load in the spring of the plastic element as F^b , and the load in the stick-slip component of the plastic element

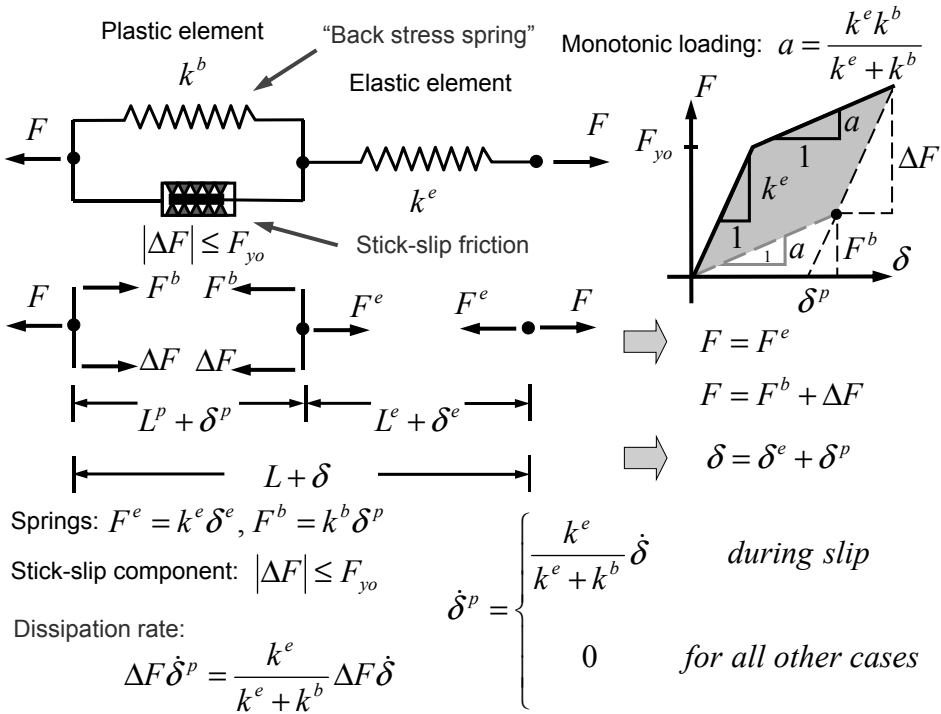


Figure 7.2: Mechanical analog for an elastic-plastic material with kinematic hardening.

as ΔF . Equilibrium at the junctions requires that

$$F = F^e, \quad (7.2)$$

$$F = F^b + \Delta F. \quad (7.3)$$

We further will assume that the two springs start from their unloaded states such that L^e and L^p represent the free lengths of the two springs. We shall take the springs to be linear such that the relation between the loads in the springs and their associated extensions is given by

$$F^e = k^e \delta^e, \quad (7.4)$$

$$F^b = k^b \delta^p. \quad (7.5)$$

The stick-slip component is assumed to be locked when its load ΔF is less than F_{yo} . When the load ΔF reaches the yield load F_{yo} , the stick-slip component may slip depending on whether or not the slip is needed to keep $|\Delta F| \leq F_{yo}$. The key to modeling plastic flow is in selecting the proper condition to initiate flow. Letting $f = |\Delta F| - F_{yo}$ be the yield function, it is clear that the first condition for yield is that $f = 0$. Yet, even though this is a necessary condition for flow, it is not sufficient. Let us examine this further. In general we can only have $f \leq 0$, so that the simultaneous conditions that result in $f = 0$ and $\dot{f} > 0$ are inadmissible. That is, the stick-slip component cannot support loads $|\Delta F| > F_{yo}$ so that conditions that might result in this should initiate slip to result in the extension/contraction of the back-stress spring to remove the excess load from the stick-slip component. Let us formalize the

conditions that result in slip of the stick-slip component as follows: *The stick-slip component is assumed to be locked until this assumption violates the condition $|\Delta F| \leq F_{yo}$.* In terms of the yield function, this states that there will be slip if and only if $f = 0$ and $\dot{f} > 0$ under the assumption that the stick-slip component is locked. The value of $\dot{f}$ under the assumption of a locked stick-slip component is termed the trial derivative of f and we denote it by $\overset{\diamond}{f}$. When the stick-slip component is locked, the value of $\dot{\delta}^p$ is zero. Therefore, we define $\dot{f}$ to be the value of $\dot{f}$ for $\dot{\delta}^p = 0$, under the current applied external conditions. We can write the yield function as

$$f = \begin{cases} \Delta F - F_{yo} = F - F^b - F_{yo} & \text{for } \Delta F \geq 0, \\ -\Delta F - F_{yo} = -F + F^b - F_{yo} & \text{for } \Delta F < 0. \end{cases} \quad (7.6)$$

Taking the derivative of this with respect to time results in

$$\dot{f} = \begin{cases} \dot{F} - \dot{F}^b & \text{for } \Delta F \geq 0, \\ -\dot{F} + \dot{F}^b & \text{for } \Delta F < 0. \end{cases} \quad (7.7)$$

If the stick-slip system is locked, the back-stress spring cannot extend or contract so that $\dot{F}^b = 0$. Therefore, we can write

$$\overset{\diamond}{f} = \begin{cases} \dot{F} & \text{for } \Delta F \geq 0, \\ -\dot{F} & \text{for } \Delta F < 0. \end{cases} \quad (7.8)$$

where $\dot{F}$ is the time derivative of the applied load given that $\dot{\delta}^p = 0$. If the applied load is being controlled, then $\overset{\diamond}{F} = \dot{F}$. On the other hand, if the displacement is being controlled, then, since $F = F^e = k^e \delta^e = k^e (\delta - \delta^p)$, we get $\overset{\diamond}{F} = k^e \dot{\delta}$. Once the condition of slip has been satisfied, the slip should commence in a way that keeps $|\Delta F| = F_{yo}$. Since this condition is given by $f = 0$, the slip should commence in a way that $\dot{f} = 0$. This is called the *consistency condition*. If we substitute the relations $F = F^e = k^e \delta^e = k^e (\delta - \delta^p)$ and $F^b = k^b \delta^p$ into the expression for $\dot{f}$, we arrive at the relation

$$\dot{f} = \begin{cases} k^e \dot{\delta} - (k^e + k^b) \dot{\delta}^p & \text{for } \Delta F \geq 0, \\ -k^e \dot{\delta} + (k^e + k^b) \dot{\delta}^p & \text{for } \Delta F < 0. \end{cases} \quad (7.9)$$

The two branches of the equation (for $\Delta F \geq 0$ and $\Delta F < 0$) give exactly the same result when setting $\dot{f} = 0$. By setting $\dot{f} = 0$ during the slip of the stick-slip component we get

$$\dot{\delta}^p = \frac{k^e}{k^e + k^b} \dot{\delta}. \quad (7.10)$$

We now can summarize the model for the change of δ^p as

$$\dot{\delta}^p = \begin{cases} \frac{k^e}{k^e + k^b} \dot{\delta} & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (7.11)$$

where $\overset{\diamond}{f}$ is the value of $\dot{f}$ evaluated under the condition $\dot{\delta}^p = 0$.

We now will proceed to study the response of the model. For monotonic tensile loading starting from zero strain and load in all components, as long as $|\Delta F| < F_{yo}$ we have $\dot{\delta}^p = 0$ and all the extension is accommodated by increasing δ^e . This yields

$$\dot{F} = k^e \dot{\delta}^e = k^e \dot{\delta}. \quad (7.12)$$

Once the two slip conditions are simultaneously satisfied, then we have the rate of loading given by

$$\dot{F} = k^e \dot{\delta} = k^e (\dot{\delta} - \dot{\delta}^p) = k^e \left(\dot{\delta} - \frac{k^e}{k^e + k^b} \dot{\delta} \right) = \frac{k^e k^b}{k^e + k^b} \dot{\delta}. \quad (7.13)$$

The slope in the load-extension plot is the ratio of $\dot{F}$ to $\dot{\delta}$, so that this slope during slip becomes

$$a = \frac{k^e k^b}{k^e + k^b}. \quad (7.14)$$

Let us denote by k^{equiv} the equivalent spring constant for two springs of spring constant k^e and k^b in series. It can be shown that k^{equiv} is

$$\frac{1}{k^{equiv}} = \frac{1}{k^e} + \frac{1}{k^b} = \frac{k^e + k^b}{k^e k^b}. \quad (7.15)$$

A comparison of this and the expression for the slope a shows that the response of the model during slip has an apparent stiffness of k^{equiv} , which is a stiffness equivalent to having the two springs in series.

Let us summarize the characteristics of the mechanical analog shown in Figure 7.2. When the magnitude of the overload ΔF is less than the yield (slip) load F_{yo} , the extension is fully accommodated by changing δ^e . Once the magnitude of the overload reaches F_{yo} , then, depending on the future direction of loading (extending), either the stick-slip component slips or remains locked. If it slips, then the slope of the load-extension plot is like that of a softer system constructed from putting the two springs in series.

The key concepts here are to understand the role of the elastic spring, the back-stress spring, and the overload. The elastic spring always is loaded by the full amount of the applied load and accommodates any needed extensions (contractions) when the stick-slip element is locked, but also extends (contracts) during the extension (contraction) of the stick-slip component, as needed to have a load equal to the applied load. The back-stress spring and the stick-slip element together carry the applied load. As the back-stress spring extends, the load in the back-stress spring changes, shifting the base from which we measure the overload ΔF (see the graph in Figure 7.2).

Finally, let us look at the dissipation in the system, which comes from slipping in the stick-slip component. The dissipation rate $\dot{H}$ in the stick-slip component is given by the force in this component times the rate of its extension. That is, it is given by $\dot{H} = \Delta F \dot{\delta}^p$. A reasonable expectation is that the dissipation always be positive or zero. Therefore, to have a physically meaningful model we should have

$$\dot{H} = \Delta F \dot{\delta}^p \geq 0. \quad (7.16)$$

In short, when ΔF is positive we should have an extension of the stick-slip system and when ΔF is negative we should have its contraction. The total dissipation in a process from time t_1 to time t_2 will be denoted by ΔH and is given by

$$\Delta H = \int_{t_1}^{t_2} \Delta F \dot{\delta}^p dt > 0. \quad (7.17)$$

For monotonic loading in tension, the slip occurs at constant $\Delta F = F_{yo}$ so that in such cases we have

$$\Delta H = F_{yo} \Delta \delta^p, \quad (7.18)$$

where $\Delta \delta^p = \delta_2^p - \delta_1^p$ is the extension of the stick-slip component from time t_1 to time t_2 . One can show that the dissipation in this case is equal to the colored area shown in the graph in Figure 7.2.

Section 7.1-Problems:

- 7.1-1:** Show that the colored area in the load-displacement curve in Figure 7.2 represents the dissipated portion of the work done on the mechanical analog.
- 7.1-2:** Prove that the load-displacement response from the mechanical analog shown in Figure 7.2 is not rate of loading dependent. That is, it does not depend on how fast we load/stretch the system.
- 7.1-3:** Use the model for the mechanical analog shown in Figure 7.2 to calculate the response of the material to loading followed by reverse loading until the plastic strain reaches zero. Draw a diagram of the response and calculate the dissipated portion of the work.
- 7.1-4:** Write a program to simulate the response of the mechanical analog shown in Figure 7.2 to a given extension history $\delta(t)$. Take for parameters the values $k^e = 10$, $k^b = 2$, and $F_{yo} = 10$. Test your program for an input $\delta(t) = A \sin(\omega t)$, for $A = 1$ and $\omega = 2\pi$, and for $A = 2$ and $\omega = 2\pi$. Have your program plot the heat dissipation and its rate as a function of time.
- 7.1-5:** Write a program to simulate the response of the mechanical analog shown in Figure 7.2 to a given load history $F(t)$. Take for parameters the values $k^e = 10$, $k^b = 2$, and $F_{yo} = 10$. Test your program for an input $F(t) = A \sin(\omega t)$, for $A = 10$ and $\omega = 2\pi$, and for $A = 20$ and $\omega = 2\pi$. Have your program plot the heat dissipation and its rate as a function of time.

7.2 Elastic perfectly plastic response

The elements of the theory of plasticity for multidimensional deformations are similar to the elements for the mechanical analog shown in the previous section, and also are similar to the models presented for the one-dimensional bar in Chapter 1. As an example, in this section we will consider the material response for an elastic perfectly plastic material characterized by the mechanical analog shown in Figure 7.3, which is slightly simpler than the mechanical analog shown in Figure 7.2. In the one-dimensional bar, we used this mechanical analog to motivate the construction of a model given as

$$\epsilon = \epsilon^e + \epsilon^p, \quad (7.19)$$

$$F = AE\epsilon^e, \quad (7.20)$$

$$f = |F| - F_{yo}, \quad (7.21)$$

$$\dot{f} = \begin{cases} AE\dot{\epsilon} & \text{if } F \geq 0, \\ -AE\dot{\epsilon} & \text{if } F < 0, \end{cases} \quad (7.22)$$

$$\dot{\epsilon}^p = \begin{cases} \dot{\epsilon} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (7.23)$$

chosen to use the classical J_2 model which uses the second invariant of the deviatoric stress expressed as

$$J_{2S} = -\frac{1}{2}tr(\mathbf{S}^2) = -\frac{1}{2}S_{ij}S_{ji} = -\frac{1}{2}[S_{11}^2 + S_{22}^2 + S_{33}^2 + 2(S_{12}^2 + S_{23}^2 + S_{31}^2)].$$

For uniaxial extension, $J_{2S} = -\sigma^2/3$. Therefore, at yield in uniaxial extension, we will have $J_{2S} = -\sigma_{yo}^2/3$. If we set the constant $J_{2S_{yo}}$ so that the yield surface fits yield in uniaxial extension, we will have the yield function given by

$$f = -\frac{\sigma_{yo}^2}{3} - J_{2S}.$$

Since there is only one constant in the yield function, establishing the constant fixes the yield under all conditions. For example, in pure shear with an applied shear stress τ , we get $J_{2S} = -\tau^2$ so that it follows that the relation between yield in tension and pure shear can be determined by setting $f = 0$ under pure shear to obtain

$$\tau_{yo} = \frac{\sigma_{yo}}{\sqrt{3}}.$$

After establishing the yield function, we need to determine the condition for plastic flow. As before, the condition of plastic flow is $f = 0$ and $\dot{f} > 0$. Before we calculate $\dot{f}$, let us calculate $\dot{f}$ since we need it later to impose the consistency condition. In terms of the nominal stress $\boldsymbol{\sigma}$, we can write

$$J_{2S} = -\frac{1}{3}(J_{1\sigma}^2 - 3J_{2\sigma}) = -\frac{1}{3}\left[\sigma_{ii}\sigma_{jj} - \frac{3}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})\right] = \frac{1}{6}\sigma_{ii}\sigma_{jj} - \frac{1}{2}\sigma_{ij}\sigma_{ji}.$$

The derivative of J_{2S} with respect to the σ_{kl} components of stress can be evaluated as

$$\frac{\partial J_{2S}}{\partial \sigma_{kl}} = \frac{1}{6}(\delta_{ik}\delta_{il}\sigma_{jj} + \sigma_{ii}\delta_{jk}\delta_{jl}) - \frac{1}{2}(\delta_{ik}\delta_{jl}\sigma_{ji} + \sigma_{ij}\delta_{jk}\delta_{il}).$$

This yields

$$\frac{\partial J_{2S}}{\partial \sigma_{kl}} = \frac{1}{3}\sigma_{ii}\delta_{kl} - \sigma_{kl} = -S_{kl}.$$

We now can use the chain rule of calculus to get the relation

$$\dot{f} = \frac{\partial f}{\partial \sigma_{kl}} \left(\frac{\partial \sigma_{kl}}{\partial \epsilon_{mn}} \dot{\epsilon}_{mn} + \frac{\partial \sigma_{kl}}{\partial \epsilon_{mn}^p} \dot{\epsilon}_{mn}^p \right),$$

and find that

$$\dot{f} = S_{kl}(\mathbf{E}_{klmn}\dot{\epsilon}_{mn} - \mathbf{E}_{klmn}\dot{\epsilon}_{mn}^p).$$

To get $\dot{f}$ we set $\dot{\epsilon}^p = 0$ to obtain

$$\dot{f} = S_{kl}\mathbf{E}_{klmn}\dot{\epsilon}_{mn}.$$

Once we have the condition for yield established, we can write the flow rule

$$\dot{\epsilon}^p = \begin{cases} \beta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases}$$

which states that plastic flow occurs along the direction of the deviatoric stress, with β being a scalar parameter to be calculated by using the consistency condition. Let us now look at the consistency condition $\dot{f} = 0$ during plastic flow. This requires that

$$\dot{f} = S_{kl}(\mathbf{E}_{klmn}\dot{\epsilon}_{mn} - \mathbf{E}_{klmn}\beta S_{mn}) = 0.$$

Solving for β , we obtain

$$\beta = \frac{S_{kl}\mathbf{E}_{klmn}\dot{\epsilon}_{mn}}{S_{kl}\mathbf{E}_{klmn}S_{mn}} = \frac{\mathbf{S} : \mathbf{E} : \dot{\epsilon}}{\mathbf{S} : \mathbf{E} : \mathbf{S}}.$$

The rate of dissipation d , measured in rate of heat generation per unit volume, for this system is given by $d = -\rho\dot{h} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p$ ($\dot{h}$ is heat flow into the material point, measured per unit mass). It can be shown that this results in the expression

$$d = -\rho\dot{h} = \beta \mathbf{S} : \mathbf{S} = \beta S_{ij}S_{ij} \quad (7.29)$$

during plastic flow. Since plastic flow is characterized by the conditions $f = 0$ and $\dot{f} > 0$, and since $f = -\frac{\sigma_{yo}^2}{3} - J_{2S} = 0$ and $J_{2S} = -\frac{1}{2}\mathbf{S} : \mathbf{S}$, we get $\mathbf{S} : \mathbf{S} = \frac{2}{3}\sigma_{yo}^2$ during plastic flow. Therefore, the dissipation rate, which is only nonzero during plastic flow, is given by

$$d = -\rho\dot{h} = \frac{2}{3}\beta\sigma_{yo}^2 = \frac{2}{3}\sigma_{yo}^2 \frac{\mathbf{S} : \mathbf{E} : \dot{\epsilon}}{\mathbf{S} : \mathbf{E} : \mathbf{S}} \geq 0. \quad (7.30)$$

We now have completed discussion of all the elements of our first three-dimensional plasticity model. As the reader may expect, there are different possible selections for each component of the model. Let us proceed by examining some of the common assumptions for the form of the elements discussed.

7.3 Common assumptions

In this section we will look at some of the commonly used models. We will do this separately for each of the elements described above.

7.3.1 Linear elastic response

The cornerstone of all infinitesimal theories of plasticity is the assumption of linear elastic response. In general, linear elastic response is described by the model

$$\boxed{\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon}^e}, \quad (7.31)$$

where $\mathbf{E}$, not to be confused with the Green strain, is the fourth-order tensor of elastic moduli. In a fixed orthonormal base, the components of $\mathbf{E}$ are E_{ijkl} , where each index can take any integer value between one and three. The 81 components of $\mathbf{E}$ are not all independent since they must satisfy several conditions. First, since nominal stress for infinitesimal deformations is symmetric, we must require that $\mathbf{E} : \boldsymbol{\epsilon}^e$ be symmetric. That is, $E_{ijkl}\epsilon_{kl}^e = E_{jikl}\epsilon_{kl}^e$ for all ϵ_{kl}^e . This can be interpreted as requiring the relation

$$\boxed{E_{ijkl} = E_{jikl}}. \quad (7.32)$$

Second, since strain is symmetric, $\sigma_{ij} = E_{ijkl}\epsilon_{kl}^e = E_{ijlk}\epsilon_{kl}^e = E_{ijlk}\epsilon_{kl}^e$. Therefore, one can write

$$\sigma_{ij} = \frac{1}{2}(E_{ijkl} + E_{ijlk})\epsilon_{kl}^e.$$

Even though it is not required that $\mathbf{E}_{ijkl}$ be symmetric in its two last indices, this equation states that only the symmetric part is needed for the calculation of stress. Obviously, it makes little sense to carry terms in $\mathbf{E}$ that do not contribute to the calculation of stress. For this reason we will require from the onset that

$$\boxed{\mathbf{E}_{ijkl} = \mathbf{E}_{ijlk}} \quad (7.33)$$

Further simplification can be made if it is assumed that stress is determined from a free energy. This will require that

$$\boxed{\mathbf{E}_{ijkl} = \mathbf{E}_{klij}} \quad (7.34)$$

Next, we will look at some special models for the elastic response. In particular, we will look at isotropic, transversely isotropic and orthotropic responses. The proofs of the relations will be left for later. A full description of representations for the crystal classes is given in the appendix.

Isotropic response

For isotropic response, samples taken along any direction should result in similar responses when subjected to similar loading. That is, there is no preferred direction in the material, which in turn states that there is no way of testing a sample to find out along which direction it was taken. For an isotropic material, the elastic modulus tensor is given by

$$\boxed{\mathbf{E}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}, \quad (7.35)$$

where λ and μ are the Lamé constants, not to be confused with stretch ratio and viscosity. The Lamé constants are given by

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad (7.36)$$

in terms of the elastic modulus E , shear modulus G , and Poisson's ratio ν . Substitution into the expression for stress yields the relation

$$\sigma_{ij} = \lambda \epsilon_{kk}^e \delta_{ij} + 2\mu \epsilon_{ij}^e, \quad (7.37)$$

which in tensor notation can be written as

$$\boxed{\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2\mu \boldsymbol{\epsilon}^e}. \quad (7.38)$$

Transversely isotropic

A transversely isotropic material has a preferred axis, with directions perpendicular to this axis being all similar. For a transversely isotropic material there are five independent components in $\mathbf{E}$. These are provided in the appendix for transverse isotropy along the direction $\mathbf{e}_3$, the third direction of the orthonormal base $\mathbf{e}_i$, and result in the expression for $\mathbf{E}$ given as

$$\begin{aligned} \mathbf{E} = & \mu_1 (\mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2 + \frac{1}{2} \mathbf{E}_6 \otimes \mathbf{E}_6) + \mu_2 \mathbf{E}_3 \otimes \mathbf{E}_3 \\ & + \mu_3 (\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1 - \frac{1}{2} \mathbf{E}_6 \otimes \mathbf{E}_6) + \mu_4 (\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2) \\ & + \mu_5 (\mathbf{E}_4 \otimes \mathbf{E}_4 + \mathbf{E}_5 \otimes \mathbf{E}_5), \end{aligned} \quad (7.39)$$

where μ_i are the material constants and $\mathbf{E}_i$ are the six base tensors for symmetric second-order tensors given in terms of an orthonormal base $\mathbf{e}_i$ as

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{e}_1 \otimes \mathbf{e}_1, & \mathbf{E}_2 &= \mathbf{e}_2 \otimes \mathbf{e}_2, & \mathbf{E}_3 &= \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{E}_4 &= \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2, & \mathbf{E}_5 &= \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, & \mathbf{E}_6 &= \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1. \end{aligned} \quad (7.40)$$

It is easy to see that

$$\begin{aligned} \mathbf{E}_1 : \boldsymbol{\epsilon}^e &= \epsilon_{11}^e, & \mathbf{E}_2 : \boldsymbol{\epsilon}^e &= \epsilon_{22}^e, & \mathbf{E}_3 : \boldsymbol{\epsilon}^e &= \epsilon_{33}^e, \\ \mathbf{E}_4 : \boldsymbol{\epsilon}^e &= \epsilon_{23}^e + \epsilon_{32}^e, & \mathbf{E}_5 : \boldsymbol{\epsilon}^e &= \epsilon_{13}^e + \epsilon_{31}^e, & \mathbf{E}_6 : \boldsymbol{\epsilon}^e &= \epsilon_{12}^e + \epsilon_{21}^e, \end{aligned} \quad (7.41)$$

yielding the expression for stress given as

$$\begin{aligned} \boldsymbol{\sigma} = & (\mu_1 \epsilon_{11}^e + \mu_3 \epsilon_{22}^e + \mu_4 \epsilon_{33}^e) \mathbf{E}_1 + (\mu_1 \epsilon_{22}^e + \mu_3 \epsilon_{11}^e + \mu_4 \epsilon_{33}^e) \mathbf{E}_2 \\ & + (\mu_2 \epsilon_{33}^e + \mu_4 \epsilon_{11}^e + \mu_4 \epsilon_{22}^e) \mathbf{E}_3 + \mu_5 (\epsilon_{23}^e + \epsilon_{32}^e) \mathbf{E}_4 + \mu_5 (\epsilon_{13}^e + \epsilon_{31}^e) \mathbf{E}_5 \\ & + \frac{1}{2} (\mu_1 - \mu_3) (\epsilon_{12}^e + \epsilon_{21}^e) \mathbf{E}_6. \end{aligned} \quad (7.42)$$

Orthotropic

An orthotropic material has three orthogonal preferred directions. For an orthotropic material, there are nine independent components in $\mathbf{E}$. We will take the preferred directions of orthotropy along the three coordinate directions $\mathbf{e}_i$, where $\mathbf{e}_i$ are base vectors for an orthonormal base. The representation of an orthotropic fourth-order tensors is provided under the name rhombic symmetry in the appendix and results in the expression for $\mathbf{E}$ given as

$$\begin{aligned} \mathbf{E} = & \mu_1 \mathbf{E}_1 \otimes \mathbf{E}_1 + \mu_2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \mu_3 \mathbf{E}_3 \otimes \mathbf{E}_3 + \mu_4 (\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1) \\ & + \mu_5 (\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1) + \mu_6 (\mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2) \\ & + \mu_7 \mathbf{E}_4 \otimes \mathbf{E}_4 + \mu_8 \mathbf{E}_5 \otimes \mathbf{E}_5 + \mu_9 \mathbf{E}_6 \otimes \mathbf{E}_6, \end{aligned} \quad (7.43)$$

where μ_i are the material constants and $\mathbf{E}_i$ are the six base tensors for second-order symmetric tensors given above. It can be shown that the expression for calculating stress for an orthotropic material is given by

$$\begin{aligned} \boldsymbol{\sigma} = & (\mu_1 \epsilon_{11}^e + \mu_4 \epsilon_{22}^e + \mu_5 \epsilon_{33}^e) \mathbf{E}_1 + (\mu_2 \epsilon_{22}^e + \mu_4 \epsilon_{11}^e + \mu_6 \epsilon_{33}^e) \mathbf{E}_2 \\ & + (\mu_3 \epsilon_{33}^e + \mu_5 \epsilon_{11}^e + \mu_6 \epsilon_{22}^e) \mathbf{E}_3 + \mu_7 (\epsilon_{23}^e + \epsilon_{32}^e) \mathbf{E}_4 + \mu_8 (\epsilon_{13}^e + \epsilon_{31}^e) \mathbf{E}_5 \\ & + \mu_9 (\epsilon_{12}^e + \epsilon_{21}^e) \mathbf{E}_6. \end{aligned} \quad (7.44)$$

Section 7.3-Problems:

- 7.3-1:** Using the expression for the elastic modulus for an isotropic elastic material show that the stress is expressed in the form

$$\sigma_{ij} = \lambda \epsilon_{kk}^e \delta_{ij} + 2\mu \epsilon_{ij}^e,$$

where λ and μ are known as the Lamé constants.

- 7.3-2:** Show that the Lamé constants given in the previous problem are related to the Young's modulus E , the shear modulus G , the bulk modulus κ , and Poisson's

ratio ν by the relations:

$$\lambda = \frac{G(2G - E)}{E - 3G} = \frac{3\kappa - 2G}{3} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = \frac{3\kappa(3\kappa - E)}{9\kappa - E} = \frac{3\kappa\nu}{1 + \nu}, \quad (7.45)$$

$$\mu = G = \frac{E}{2(1 + \nu)} = \frac{3E\kappa}{9\kappa - E} = \frac{3\kappa(1 - 2\nu)}{2(1 + \nu)}. \quad (7.46)$$

7.3-3: Derive the expression for stress in an orthotropic material given by equation (7.44) by using the fourth-order elastic modulus tensor for an orthotropic material given by equation (7.43).

7.3.2 Yield function

The yield function is a scalar function that defines the limit of elastic deformation. This function in a general form can be written as

$$f = f^\dagger(\mathbf{X}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}^p, \xi), \quad (7.47)$$

where ξ is a scalar hardening parameter. Considering that the stress can be determined from the elastic strain, a special, but equivalent, form of this function might be written as

$$f = f^\dagger(\mathbf{X}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \xi).$$

The most commonly used yield functions are written directly in terms of the stress. These are directly related to the Tresca and the von Mises yield criteria. The Tresca yield condition is a maximum shear stress criterion motivated by the notion that shear is the primary contributor to plastic flow. The Tresca yield function can be written as

$$f = |\tau_{max}| - \tau_{yo}, \quad (7.48)$$

where τ_{max} is the maximum shear stress and τ_{yo} is the initial yield stress in shear. Given the three unordered principal stresses σ_1 , σ_2 , and σ_3 , from the study of the Mohr circle we know that

$$|\tau_{max}| = \frac{1}{2} \max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|).$$

The von Mises yield function is based on J_{2S} and written as

$$f = \left| \sqrt{\frac{3}{2}} \mathbf{S} : \mathbf{S} \right| - \sigma_{yo}, \quad (7.49)$$

where σ_{yo} is the initial yield stress in tension. This can be written as $f = |\sqrt{-3J_{2S}}| - \sigma_{yo}$, showing the relation to J_{2S} . This is why the von Mises yield function is also known as the J_2 yield function and the associated plasticity theory is called the J_2 plasticity theory.

The Tresca and von Mises yield functions each define the yield surface ($f = 0$) through one material parameter. Therefore, for each we can only fit the yield under one condition. For example, if we fit the Tresca yield condition to pure shear, then it follows that the material will yield in tension or compression at a stress of $\sigma_{yo} = 2\tau_{yo}$. If we do the same

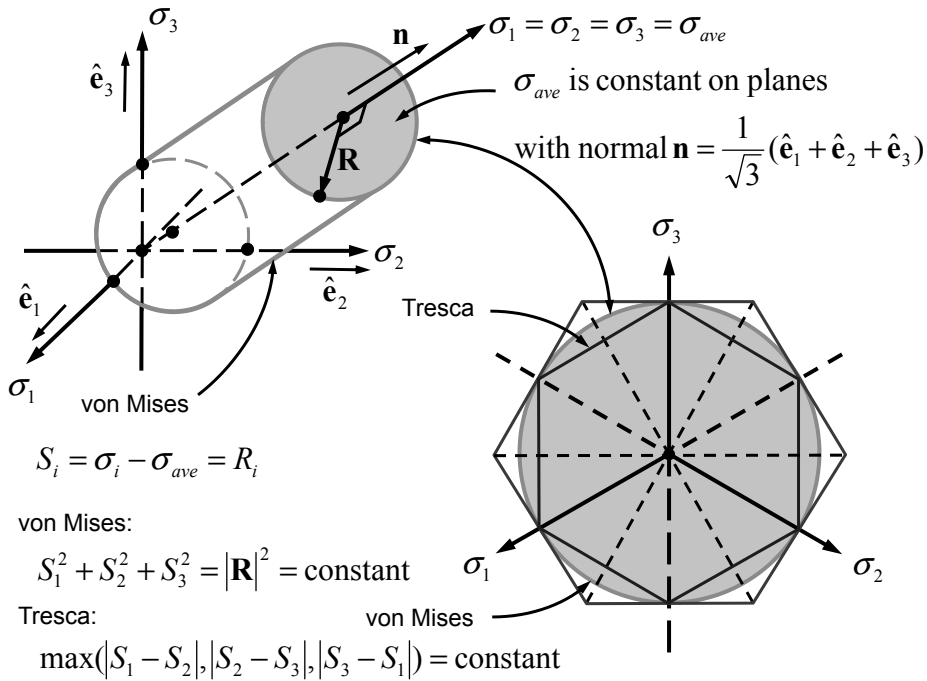


Figure 7.4: The von Mises yield condition is represented by a cylinder of circular cross-sectional area, which has its axis along the $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ direction in the principal stress space. The Tresca yield condition is also a cylinder along the direction of $\mathbf{n}$, but with a cross section of hexagonal shape.

for the von Mises yield function, that is fit it to yield in pure shear, then we find yield in uniaxial tension or compression occurs at $\sigma_{yo} = \sqrt{3}\tau_{yo}$.

The Tresca and von Mises yield conditions can be visualized in the space of principal stresses by infinite cylindrical surfaces, as shown in Figure 7.4. In the figure σ_i are the three principal stresses, and S_i are the associated principal deviatoric stresses. In this space, both yield conditions are represented by cylinders with axes along the line of hydrostatic stress $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{ave}$, which falls along the direction $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. The von Mises yield condition in this space has a circular cross section perpendicular to $\mathbf{n}$, and the Tresca yield condition has a hexagonal cross section. The principal values of the deviatoric stress are given in terms of σ_i as

$$S_i = \sigma_i - \sigma_{ave}. \quad (7.50)$$

Therefore, S_i can be interpreted as the components of a vector that connects a point on the hydrostatic axis $(\sigma_{ave}, \sigma_{ave}, \sigma_{ave})$ to a point $(\sigma_1, \sigma_2, \sigma_3)$ on the plane orthogonal to $\mathbf{n}$ and passing through $(\sigma_{ave}, \sigma_{ave}, \sigma_{ave})$. In the figure, this vector is denoted as $\mathbf{R}$. Therefore, as shown, lines of constant $S_1^2 + S_2^2 + S_3^2$ represent circles in planes perpendicular to $\mathbf{n}$.

Visualization of the Tresca and von Mises yield conditions are much simpler in plane stress. In Figure 7.5 we show the intersection of the cylinders representing the Tresca and von Mises yield conditions with the plane $\sigma_3 = 0$. As expected, since the cylinders are skew relative to the $\sigma_1 - \sigma_2$ plane, the intersection for the von Mises yield condition is no longer a

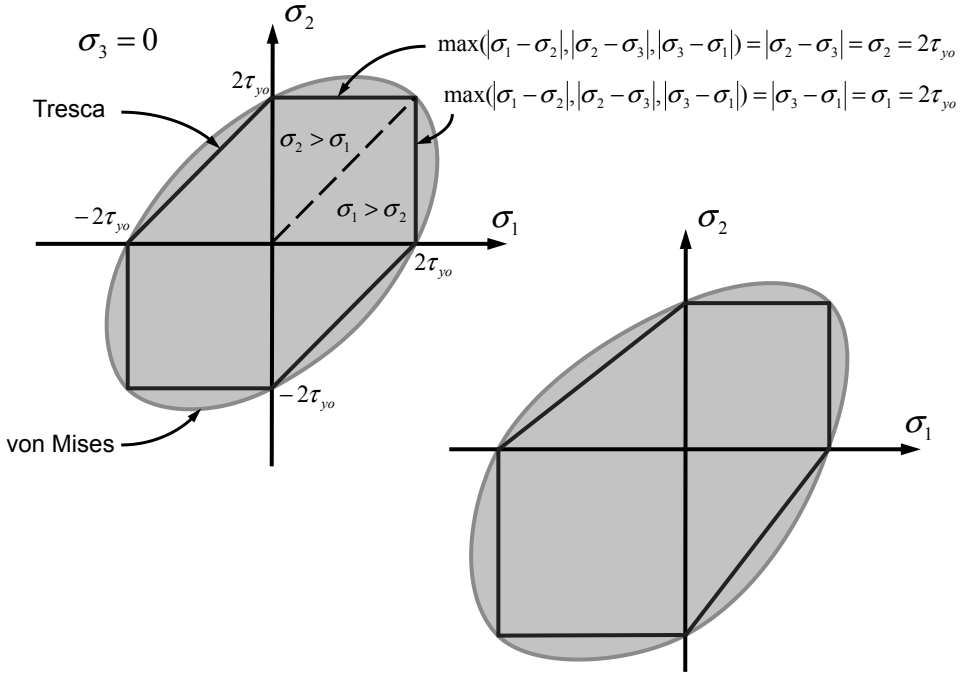


Figure 7.5: Under plane-stress conditions, the von Mises yield condition takes an elliptical shape, and the Tresca yield condition takes an extended hexagonal shape.

circle, but has taken an elliptical shape, and the intersection for the Tresca yield condition has taken an extended hexagonal shape. The reader will note that the two yield conditions are fit to yield under uniaxial extension, which explains the overlap of the two at points (σ_1, σ_2) equal to $(\sigma_{yo}, 0)$, $(0, \sigma_{yo})$, $(-\sigma_{yo}, 0)$, and $(0, -\sigma_{yo})$.

As presented, the two yield functions do not include any hardening. As such, they can only be used for characterizing elastic perfectly plastic response. The notions of kinematic and isotropic hardening easily can be added to these yield functions. One way to introduce kinematic hardening is to replace σ with the overstress $\Delta\sigma = \sigma - \sigma_c$, where σ_c is the stress associated with the center of the yield surface. For example, the von Mises yield function with kinematic hardening can be written as

$$f = \left| \sqrt{\frac{3}{2}} \Delta\mathbf{S} : \Delta\mathbf{S} \right| - \sigma_{yo}, \quad (7.51)$$

where $\Delta\mathbf{S}$ is the deviatoric part of $\Delta\sigma$ given by

$$\Delta\mathbf{S} = \Delta\sigma - \Delta\sigma_{ave}\mathbf{I} = \sigma - \sigma_c - \frac{1}{3}tr(\sigma - \sigma_c)\mathbf{I}.$$

We obviously now need to provide a method to calculate σ_c . One method would be to calculate it from the initial condition $\sigma_c = \mathbf{0}$ and a *Prager hardening rule* given by

$$\dot{\sigma}_c = c\dot{\epsilon}^p,$$

where c is a proportionality factor which should be fit to the response. If c is constant, we get the simple relation $\sigma_c = c\epsilon^p$. An alternate form is the *Ziegler hardening rule* given by

$$\dot{\sigma}_c = \dot{c}(\sigma - \sigma_c),$$

where $\dot{c}$ is a proportionality factor that needs to be prescribed. The similarity of σ_c and the back-stress σ^b should not be lost on the reader since the two are similar. The main difference is that σ^b assumes a “back-stress spring” so that it can be derived from a potential associated with the storage of energy in the back-stress element.

We can introduce isotropic hardening into the proposed models by replacing the initial yield stresses (either σ_{yo} or τ_{yo} , depending on the model) by a function of the hardening parameter. For example, the von Mises yield function with isotropic hardening can be written as

$$f = \left| \sqrt{\frac{3}{2}} \mathbf{S} : \mathbf{S} \right| - \sigma_{iso}, \quad (7.52)$$

where σ_{iso} is a scalar function of the hardening parameter ξ . For example, a simple model based on linear hardening can be written as

$$\sigma_{iso} = \sigma_{yo} + \frac{d\sigma_{iso}}{d\xi} \xi. \quad (7.53)$$

The effect of this model on the response will depend on the model used for the hardening parameter ξ . For example, as was shown in Chapter 1, the assumption of strain hardening and work hardening will result in different responses for the same yield function. See the following section on hardening parameters for examples for ξ .

Combined isotropic and kinematic hardening can be obtained through the application of both modifications to obtain

$$f = \left| \sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S} \right| - \sigma_{iso}. \quad (7.54)$$

There are many different equations that can be used as the yield function. In general, any scalar valued function of the variables indicated might be taken as the yield function. A further study of the form of the yield function will be done later when considering large deformations where material symmetry issues can be addressed. For now, we will focus on generalization of the yield condition using quadratic polynomials in stress. These yield conditions can be written as

$$\mathbf{A} : \sigma + \sigma : \mathbf{D} : \sigma = 1, \quad (7.55)$$

for a constant second-order tensor $\mathbf{A}$ and a constant fourth-order tensor $\mathbf{D}$ characterizing the shape of the yield surface. The symmetry conditions on $\mathbf{D}$ are similar to those on the fourth-order elastic modulus $\mathbf{E}$ so we can express it for an isotropic material as

$$\mathbf{D}_{ijkl} = \pi_1 \delta_{ij} \delta_{kl} + \pi_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (7.56)$$

where π_1 and π_2 are material constants. For this case, $\mathbf{A}$ is a constant isotropic second-order tensor, so it is given as $\mathbf{A} = \pi_o \mathbf{I}$. The resulting expression for the yield condition is

$$\pi_o tr(\sigma) + \pi_1 tr^2(\sigma) + 2\pi_2 tr(\sigma^2) = 1. \quad (7.57)$$

The associated yield function can be written as

$$f = \pi_o tr(\sigma) + \pi_1 tr^2(\sigma) + 2\pi_2 tr(\sigma^2) - 1. \quad (7.58)$$

As can be seen, the resulting expression is a function of the first and second invariants of the stress. This also can be rewritten as

$$f = \bar{\pi}_0 \sigma_{ave} + \bar{\pi}_1 \sigma_{ave}^2 + \bar{\pi}_2 \mathbf{S} : \mathbf{S} - 1, \quad (7.59)$$

showing the role of the average stress σ_{ave} and the deviatoric stress $\mathbf{S}$. The development of the forms associated with transverse isotropy and orthotropy are left for homework.

One can modify general quadratic polynomial yield conditions to accommodate isotropic and kinematic hardening using the methods described above.

Section 7.3-Problems:

- 7.3-4:** For the case of plane stress, calculate the Tresca yield surface and show that it is described by the lines given in Figure 7.5.
- 7.3-5:** For the case of plane stress, calculate the von Mises yield surface and show that it is described by the curve shown in Figure 7.5.
- 7.3-6:** The Drucker-Prager yield condition is sensitive to hydrostatic stress. It is given by the equation

$$\left| \sqrt{\frac{3}{2}} \mathbf{S} : \mathbf{S} \right| = \sigma_{yo} - a \sigma_{ave} \quad (7.60)$$

for $\sigma_{yo} - a \sigma_{ave} > 0$, where a is a positive material constant. Show that this yield condition defines a cone in the space of principal stresses, and that yielding becomes more difficult as the hydrostatic component of stress increases.

- 7.3-7:** Construct the general expression for a quadratic polynomial yield condition for transversely isotropic materials with axis of transverse isotropy along the third coordinate direction $\mathbf{e}_3$, for $\mathbf{e}_i$ being an orthonormal base. Note that the form of $\mathbf{D}$ is similar to that of $\mathbf{E}$, and that $\mathbf{A}$ for transverse isotropy (see Appendix A) is given by the form $\mathbf{A} = \pi_0 \mathbf{I} + \pi_1 \mathbf{e}_3 \otimes \mathbf{e}_3$.
- 7.3-8:** Construct the general expression for a quadratic polynomial yield condition for orthotropic materials with axis of orthotropy along the three coordinate directions $\mathbf{e}_i$, where $\mathbf{e}_i$ are base vectors for an orthonormal base. Note that the form of $\mathbf{D}$ is similar to that of $\mathbf{E}$, and that $\mathbf{A}$ for orthotropy (see Appendix A) is given by the form $\mathbf{A} = \pi_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \pi_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \pi_3 \mathbf{e}_3 \otimes \mathbf{e}_3$.

7.3.3 Hardening parameter

The hardening parameter is normally a monotonically increasing variable under all processing conditions and is used to characterize the extent of processing. The two most commonly used hardening parameters are plastic strain hardening and plastic work hardening. For both of these parameters, it is assumed that before deformation the hardening parameter ξ has the initial value of $\xi = 0$. For plastic strain hardening, the current value of the hardening parameter is evaluated from the integration of the equation

$$\dot{\xi} = |\dot{\epsilon}^p|, \quad (7.61)$$

where $\dot{\epsilon}^p$ is a scalar measure of the plastic strain rate normally defined as

$$\dot{\epsilon}^p = \sqrt{\dot{\epsilon}^p : \dot{\epsilon}^p}. \quad (7.62)$$

An alternate form for this is given based on ϵ^p that is a measure of the length of the plastic strain given by

$$\epsilon^p = \sqrt{\epsilon^p : \epsilon^p}, \quad (7.63)$$

for the norm $|\epsilon^p| = \sqrt{\epsilon^p : \epsilon^p}$. This definition results in the expression

$$\dot{\epsilon}^p = \frac{1}{2\sqrt{\epsilon^p : \epsilon^p}}(\dot{\epsilon}^p : \epsilon^p + \epsilon^p : \dot{\epsilon}^p) = \frac{1}{\sqrt{\epsilon^p : \epsilon^p}}\epsilon^p : \dot{\epsilon}^p = \frac{\epsilon^p}{|\epsilon^p|} : \dot{\epsilon}^p.$$

As can be seen, this expression at the initial plastic strain of $\epsilon^p = \mathbf{0}$ requires special care. One can avoid this totally by defining a plastic stretch measure based on the length of the right Cauchy stretch $\mathbf{C}^p$, which to the first approximation is given by $\mathbf{C}^p = \mathbf{I} + 2\epsilon^p$. If we take $\epsilon^p = \sqrt{\frac{1}{4}\mathbf{C}^p : \mathbf{C}^p} - \sqrt{\frac{3}{4}}$, then we get

$$\dot{\epsilon}^p = \frac{1}{2} \frac{\mathbf{C}^p : \dot{\mathbf{C}}^p}{\sqrt{\mathbf{C}^p : \mathbf{C}^p}} = \frac{\mathbf{C}^p : \dot{\epsilon}^p}{\sqrt{\mathbf{C}^p : \mathbf{C}^p}} = \frac{\mathbf{C}^p}{|\mathbf{C}^p|} : \dot{\epsilon}^p = \frac{\mathbf{I} + 2\epsilon^p}{|\mathbf{I} + 2\epsilon^p|} : \dot{\epsilon}^p. \quad (7.64)$$

All of these measures of strain rate result in approximately the same models at infinitesimal strains.

Plastic work hardening is another common measure used as a base for creating a hardening parameter. In this case, the value of the hardening parameter is calculated from the integration of the expression

$$\dot{\xi} = \sigma : \dot{\epsilon}^p. \quad (7.65)$$

An alternate work hardening parameter can be constructed by using the overstress $\Delta\sigma$. That is, using the work done by the overstress against the plastic flow. This hardening rule is written as

$$\dot{\xi} = \Delta\sigma : \dot{\epsilon}^p. \quad (7.66)$$

A word of caution regarding the work hardening parameter. It can be shown from thermodynamic considerations that $\xi = \sigma : \epsilon^p$ is appropriate when having pure isotropic hardening, but that one should use $\xi = \Delta\sigma : \epsilon^p$ when there is kinematic hardening. This is shown in the section on thermoplasticity.

7.3.4 Flow rule

The most common form of the flow rule is based on the assumption that plastic flow occurs along the direction of the deviatoric stress. This assumption is due to Reuss and written as

$$\dot{\epsilon}^p = \begin{cases} \beta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.67)$$

The value of β is selected to satisfy the consistency condition $\dot{f} = 0$ during plastic flow. Its particular form depends on the form of the yield function and the selected hardening parameter. A slight variation of this form, used when dealing with kinematic hardening (e.g., when there is a back-stress), is given by

$$\dot{\epsilon}^p = \begin{cases} \beta \Delta\mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.68)$$

There are other methods for developing the flow rule. One can select the direction of plastic flow along the gradient with respect to stress of a potential function. In such a case, a scalar function $g(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \xi)$ is selected as the potential function and the flow rule is written as

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} \beta \partial_{\boldsymbol{\sigma}}(g) & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.69)$$

If the function g is taken equal to the yield function f , this is called an *associated flow rule*, and if not, this is called a *nonassociated flow rule*.

A general flow rule that is linear in stress has the form

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} \mathbf{C} : \boldsymbol{\sigma} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (7.70)$$

where $\mathbf{C}$ is a fourth-order tensor. In general, for isotropic hardening $\mathbf{C}$ should be selected such that $\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p \geq 0$ for all admissible stresses. This is done to ensure that plastic flow is accompanied by dissipation of energy, so the condition will be termed the dissipation condition. For the proposed model, this constraint can be written as $\boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma} \geq 0$ for all admissible stresses. The symmetry conditions on $\mathbf{C}$ are similar to those on the elastic modulus tensor $\mathbf{E}$. Therefore, for an isotropic material, the components of $\mathbf{C}$ in an orthonormal base can be written as

$$C_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (7.71)$$

for material constant c_i . In this case, the flow rule can be written as

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} c_1 \text{tr}(\boldsymbol{\sigma}) \mathbf{I} + 2c_2 \boldsymbol{\sigma} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (7.72)$$

with the condition $\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p = c_1 \text{tr}^2(\boldsymbol{\sigma}) + 2c_2 \boldsymbol{\sigma} : \boldsymbol{\sigma} \geq 0$ for all admissible stresses during plastic flow. We will note that $c_1 \geq 0$ and $c_2 \geq 0$ is a sufficient condition to satisfy the inequality. This flow rule can be put in terms of the deviatoric stress as

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} (c_1 + \frac{2}{3}c_2) \text{tr}(\boldsymbol{\sigma}) \mathbf{I} + 2c_2 \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.73)$$

For this model, the trace of the plastic strain rate is $\text{tr}(\dot{\boldsymbol{\epsilon}}^p) = (3c_1 + 2c_2) \text{tr}(\boldsymbol{\sigma})$ yielding $\dot{\epsilon}_V^p = 3(3c_1 + 2c_2) \sigma_{ave}$. This will allow modeling volumetric plastic strains, as suggested in Figure 7.6 for mild steel where the volumetric strain continues to grow after yield even while the stress remains constant. Similar expressions can be calculated for anisotropic materials. When there is a back-stress, we can modify this general form to one that is linear in the overstress. This form of the flow rule is given by

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} \mathbf{C} : \Delta \boldsymbol{\sigma} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.74)$$

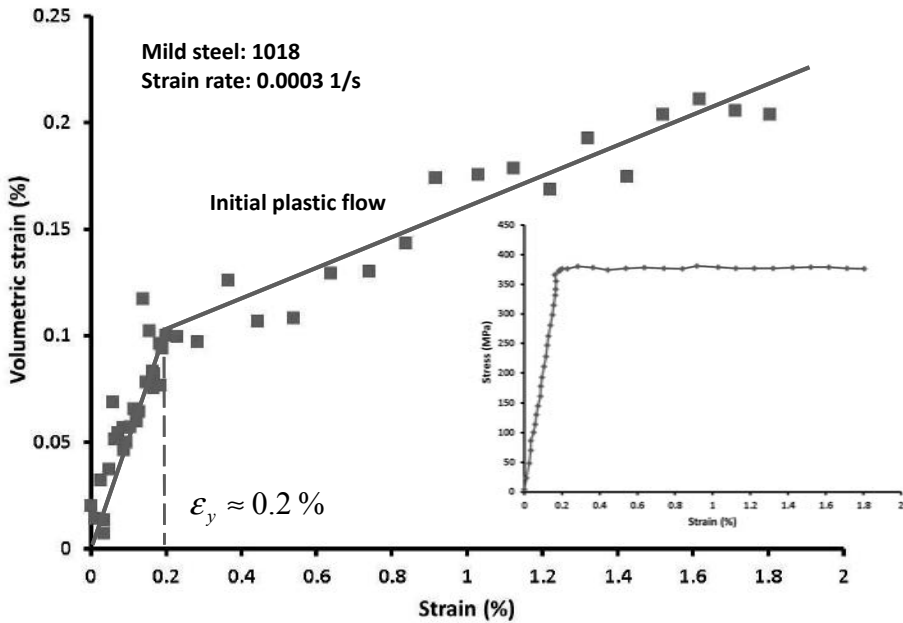


Figure 7.6: Volumetric strain during monotonic extension of a mild steel.

Section 7.3-Problems:

- 7.3-9:** Consider a transversely isotropic material. Construct the flow rule for such a material, assuming the flow rule is linear in stress. Write the dissipation condition for this flow rule.
- 7.3-10:** Consider an orthotropic material. Construct the flow rule for such a material assuming the flow rule is linear in stress. Write the dissipation condition for this flow rule.

7.3.5 Dissipation for a mechanical model

If we focus on mechanical models, it can be shown that in general the dissipation d in the material can be calculated from

$$d = -\rho \dot{h} = \Delta \sigma : \dot{\epsilon}^p, \quad (7.75)$$

where ρ is density, $\dot{h}$ is the specific heat flow into a point, $\Delta \sigma = \sigma - \sigma^b$ is the overstress, and $\dot{\epsilon}^p$ is the rate of plastic strain. In this expression, the dissipation d is the rate of heat dissipation per unit volume of the material. When there is no back-stress, the overstress $\Delta \sigma$ becomes the same as the applied stress σ .

7.4 Von Mises yield function with combined isotropic and kinematic hardening

As an example, let us consider a slightly more general, yet still relatively simple model constructed by taking linear elastic response, adding both isotropic and kinematic hardening to the von Mises yield function, and using the Prager kinematic hardening rule, linear isotropic hardening, the plastic work hardening parameter, and the modified Reuss assumption of plastic flow along the direction of the deviatoric stress. These assumptions can be summarized as

$$\begin{aligned}
 \boldsymbol{\sigma} &= \mathbf{E} : \boldsymbol{\epsilon}^e = \mathbf{E} : (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p), \\
 f &= |\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}| - \sigma_{iso}, \\
 \Delta \boldsymbol{\sigma} &= \boldsymbol{\sigma} - \boldsymbol{\sigma}_c, \\
 \Delta \mathbf{S} &= \Delta \boldsymbol{\sigma} - \Delta \sigma_{ave} \mathbf{I}, \\
 \boldsymbol{\sigma}_{c_o} &= \mathbf{0}, \quad \dot{\boldsymbol{\sigma}}_c = \mathbf{c} \dot{\boldsymbol{\epsilon}}^p, \\
 \sigma_{iso} &= \sigma_{y_o} + \frac{d\sigma_{iso}}{d\xi} \xi, \\
 \dot{\xi} &= \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p, \\
 \dot{\boldsymbol{\epsilon}}^p &= \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases}
 \end{aligned} \tag{7.76}$$

To complete the elements of this plasticity theory, we need to calculate $\dot{f}$ and β . Let us first calculate $\dot{f}$. From the expression for f , we have

$$\dot{f} = \frac{1}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \left(\frac{3}{2} \right) (\Delta \dot{\mathbf{S}} : \Delta \mathbf{S} + \Delta \mathbf{S} : \Delta \dot{\mathbf{S}}) - \frac{d\sigma_{iso}}{d\xi} \dot{\xi}.$$

This can be simplified to

$$\dot{f} = \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}} - \frac{d\sigma_{iso}}{d\xi} \dot{\xi}.$$

To calculate $\dot{f}$, we assume pure elastic response. For this, we take $\dot{\boldsymbol{\epsilon}}^p = \mathbf{0}$, $\dot{\xi} = 0$, and $\dot{\boldsymbol{\sigma}}_c = \mathbf{0}$ to obtain

$$\dot{f} = \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}}, \tag{7.77}$$

where $\Delta \dot{\mathbf{S}} = \dot{\mathbf{S}}$ is the value of $\dot{\mathbf{S}}$ assuming pure elastic response. Under stress control $\dot{\mathbf{S}} = \dot{\mathbf{S}}$ while under strain control $\dot{\mathbf{S}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}} - \frac{1}{3} \text{tr}(\mathbf{E} : \dot{\boldsymbol{\epsilon}}) \mathbf{I}$. The consistency condition during plastic flow ($f = 0$ and $\dot{f} > 0$) requires that $\dot{f} = 0$. This condition can be written as

$$\dot{f} = \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}} - \frac{d\sigma_{iso}}{d\xi} \dot{\xi} = 0.$$

We note that

$$\Delta \mathbf{S} : \Delta \dot{\mathbf{S}} = \Delta \mathbf{S} : (\Delta \dot{\boldsymbol{\sigma}} - \Delta \dot{\sigma}_{ave} \mathbf{I}) = \Delta \mathbf{S} : \Delta \dot{\boldsymbol{\sigma}} - \Delta \dot{\sigma}_{ave} \Delta \mathbf{S} : \mathbf{I} = \Delta \mathbf{S} : \Delta \dot{\boldsymbol{\sigma}}.$$

This also can be written as

$$\Delta \mathbf{S} : \Delta \dot{\mathbf{S}} = \Delta \mathbf{S} : (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_c) = \Delta \mathbf{S} : [\mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p) - c\dot{\boldsymbol{\epsilon}}^p].$$

Taking into consideration that during plastic flow $f = 0$ so that $|\sqrt{\frac{3}{2}\Delta \mathbf{S} : \Delta \mathbf{S}}| = \sigma_{iso}$, $\dot{\boldsymbol{\epsilon}}^p = \beta \Delta \mathbf{S}$, and $\dot{\xi} = \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p$, we obtain the consistency condition

$$\frac{3}{2} \frac{1}{\sigma_{iso}} \Delta \mathbf{S} : [\mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \beta \Delta \mathbf{S}) - c\beta \Delta \mathbf{S}] - \frac{d\sigma_{iso}}{d\xi} \Delta \boldsymbol{\sigma} : \beta \Delta \mathbf{S} = 0.$$

Solving for β results in

$$\beta = \frac{\Delta \mathbf{S} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{\Delta \mathbf{S} : \mathbf{E} : \Delta \mathbf{S} + c\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2\sigma_{iso}}{3} \frac{d\sigma_{iso}}{d\xi} \Delta \mathbf{S} : \Delta \mathbf{S}}. \quad (7.78)$$

In the *absence of kinematic hardening*, we obtain the special case of

$$\beta = \frac{\mathbf{S} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{\mathbf{S} : \mathbf{E} : \mathbf{S} + \frac{2\sigma_{iso}}{3} \frac{d\sigma_{iso}}{d\xi} \mathbf{S} : \mathbf{S}} \quad (7.79)$$

for pure isotropic hardening. In the *absence of isotropic hardening*, we obtain the special case

$$\beta = \frac{\Delta \mathbf{S} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{\Delta \mathbf{S} : \mathbf{E} : \Delta \mathbf{S} + c\Delta \mathbf{S} : \Delta \mathbf{S}} \quad (7.80)$$

for pure kinematic hardening.

7.4.1 Simple shear

To better understand the model, let us look at the response of this model to monotonically increasing simple shear, assuming isotropic elastic response. For simple shear, we have

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For isotropic elastic response, we have

$$\boldsymbol{\sigma} = \lambda[tr(\boldsymbol{\epsilon}) - tr(\boldsymbol{\epsilon}^p)]\mathbf{I} + 2\mu(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p).$$

Since for simple shear $tr(\boldsymbol{\epsilon}) = 0$ and from the flow rule we know $tr(\dot{\boldsymbol{\epsilon}}^p) = \beta tr(\Delta \mathbf{S}) = 0$ so that after integration we will have $tr(\boldsymbol{\epsilon}^p) = 0$, we can conclude that in simple shear we always have

$$\boldsymbol{\sigma} = 2\mu(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p).$$

Also, since $tr(\dot{\boldsymbol{\epsilon}}^p) = 0$, we can conclude that $tr(\dot{\boldsymbol{\sigma}}_c) = c tr(\dot{\boldsymbol{\epsilon}}^p) = 0$ so that we will have $\boldsymbol{\sigma}_c = \mathbf{S}_c$ and can write for simple shear

$$\Delta \mathbf{S} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_c.$$

From this we can express the flow rule as

$$\dot{\boldsymbol{\epsilon}}^p = \beta \Delta \mathbf{S} = \beta(\boldsymbol{\sigma} - \boldsymbol{\sigma}_c).$$

For the initial elastic response, we have $\epsilon^p = \mathbf{0}$ so that

$$[\sigma] = [\mathbf{S}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu\gamma & 0 \\ \mu\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\tau = \mu\gamma$. Since during the initial elastic response we have $\sigma_c = \mathbf{0}$, $\mathbf{S}_c = \mathbf{0}$, $\Delta\mathbf{S} = \mathbf{S}$, and $\sigma_{iso} = \sigma_{yo}$, we obtain

$$f = \sqrt{3}|\tau| - \sigma_{yo}.$$

Therefore, initial yield will be reached when $|\tau| = \sigma_{yo}/\sqrt{3}$.

We now will look at the zero components of ϵ^p and σ_c and argue that these two terms will have zero components similar to those of ϵ . Since $\dot{\epsilon}^p = \beta\Delta\mathbf{S}$, the form of $\Delta\mathbf{S} = \mathbf{S}$ at the initial yield excludes development of nonzero components of $\dot{\epsilon}^p$ corresponding to components of $\mathbf{S}$ that are zero. The same will, therefore, be true for $\dot{\sigma}_c$. Since components of ϵ^p and σ_c that have zero derivatives cannot change, the process that results at the initial yield excludes development of nonzero components of ϵ^p and σ_c where there are zeros in $\dot{\epsilon}^p$ and $\dot{\sigma}_c$. This in turn forces the zero terms of ϵ^p and σ_c to remain zero, ultimately showing that the plastic strain will be given by the form

$$[\epsilon^p] = \begin{bmatrix} 0 & \frac{\gamma^p}{2} & 0 \\ \frac{\gamma^p}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and σ_c will be given by the form

$$[\sigma_c] = \begin{bmatrix} 0 & \tau_c & 0 \\ \tau_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This results in

$$[\Delta\sigma] = [\Delta\mathbf{S}] = \begin{bmatrix} 0 & \tau - \tau_c & 0 \\ \tau - \tau_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We, therefore, have $\Delta\mathbf{S} : \Delta\mathbf{S} = 2(\tau - \tau_c)^2$, and the yield function becomes

$$f = |\sqrt{3(\tau - \tau_c)^2}| - \sigma_{iso},$$

resulting in the expression

$$\dot{f} = \frac{3}{|\sqrt{3(\tau - \tau_c)^2}|} (\tau - \tau_c)(\dot{\tau} - \dot{\tau}_c) - \dot{\sigma}_{iso}.$$

On the yield surface $|\sqrt{3(\tau - \tau_c)^2}| = \sigma_{iso}$. Substitution of this and the relations $\dot{\tau} - \dot{\tau}_c = \mu(\dot{\gamma} - \dot{\gamma}^p) - c\frac{\dot{\gamma}^p}{2} = \mu\dot{\gamma} - (2\mu + c)\frac{\dot{\gamma}^p}{2} = \mu\dot{\gamma} - \beta(2\mu + c)(\tau - \tau_c)$ and $\dot{\sigma}_{iso} = \frac{d\sigma_{iso}}{d\xi}\dot{\xi} = \frac{d\sigma_{iso}}{d\xi}\Delta\sigma$: $\dot{\epsilon}^p = 2\beta\frac{d\sigma_{iso}}{d\xi}(\tau - \tau_c)^2$ into the consistency condition $\dot{f} = 0$ yields

$$\beta = \frac{\sqrt{3}\mu\dot{\gamma}}{\sigma_{iso}[2\mu + c + \frac{2\sigma_{iso}}{3}\frac{d\sigma_{iso}}{d\xi}]}.$$

For pure kinematic hardening, without isotropic hardening, we obtain the simplified relation

$$\beta = \frac{\sqrt{3}\mu\dot{\gamma}}{\sigma_{yo}(2\mu + c)}.$$

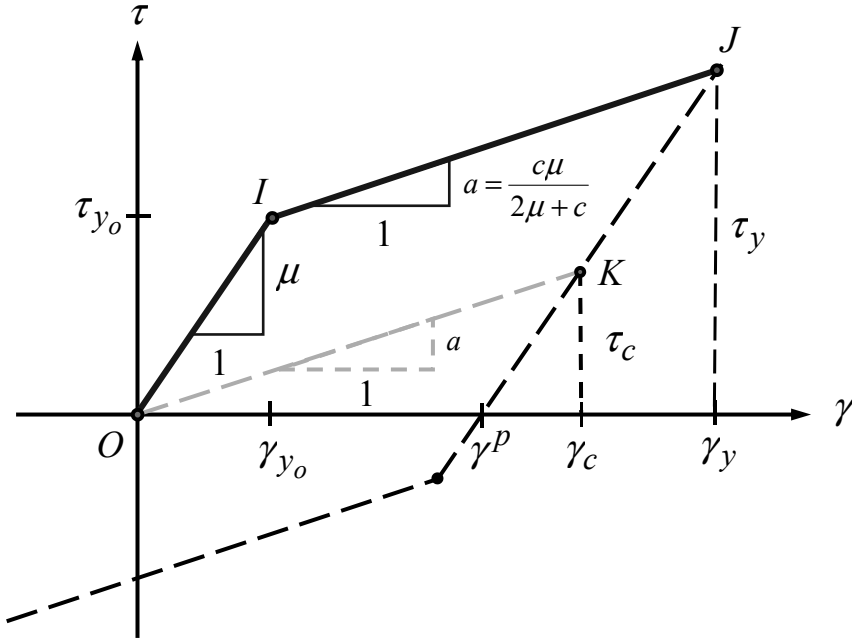


Figure 7.7: The response under simple shear of the case of pure kinematic hardening.

For this monotonic loading, using $\tau - \tau_c = \sigma_{y0}/\sqrt{3}$ during plastic flow, yields the relation for plastic flow as

$$\dot{\gamma}^p = 2\beta(\tau - \tau_c) = \frac{2\mu}{2\mu + c}\dot{\gamma},$$

which after integration gives the relation

$$\gamma^p = \frac{2\mu}{2\mu + c}(\gamma - \gamma_{y0}).$$

The expression for stress then can be determined as

$$\tau = \mu(\gamma - \gamma^p) = \frac{\mu}{2\mu + c}(c\gamma + 2\mu\gamma_{y0}).$$

The slope of the $\tau - \gamma$ diagram during monotonic loading after the initial yield is given, as shown in Figure 7.7, by $\frac{c\mu}{2\mu + c}$. The stress τ_c associated with the center for kinematic hardening is given by the relation

$$\dot{\tau}_c = c \frac{\dot{\gamma}^p}{2} = \frac{c\mu}{2\mu + c}\dot{\gamma},$$

which, as indicated in Figure 7.7, is parallel to the $\tau - \gamma$ loading segment.

For the case of pure isotropic hardening, we obtain the expression

$$\beta = \frac{\sqrt{3}\mu\dot{\gamma}}{\sigma_{iso}\left[2\mu + \frac{2\sigma_{iso}}{3}\frac{d\sigma_{iso}}{d\xi}\right]}.$$

The details of the response for this case are left as a homework problem.

Section 7.4-Problems:

7.4-1: For the case of simple shear and pure isotropic hardening show that

$$\dot{\gamma}^p = \frac{\mu \dot{\gamma}}{\mu + \frac{\tau}{\sqrt{3}} \frac{d\sigma_{iso}}{d\xi}}.$$

Substitute this into the expression for stress given by

$$\dot{\tau} = \mu(\dot{\gamma} - \dot{\gamma}^p),$$

and then separate variables and integrate to determine the relation between τ and γ . What relation would you use to evaluate the constant of integration?

7.4-2: In the model developed using the von Mises yield function with combined isotropic and kinematic hardening, add the assumption of isotropic elastic response and calculate the general form of β . Write a program to simulate the monotonic and cyclic loading under simple shear for this model.

7.4-3: In the model developed using the von Mises yield function with combined isotropic and kinematic hardening, replace the plastic work hardening relation by plastic strain hardening, and then calculate β for this model. Simplify this for the case of an isotropic elastic material. Study the response given by this model to both a monotonically increasing and cyclic simple shear.

7.5 Thermoplasticity

Thermal effects can be introduced into the infinitesimal theory through either modifications to the mechanical theory or through a complete redevelopment of the theory basing it on a thermodynamically consistent framework. We will focus first on a mechanical theory modified to make it temperature dependent and then continue on by developing thermodynamically based models.

The first thing to introduce into the development is the strain resulting from thermal expansion. We will use ϵ^θ to denote thermal strain and will assume that the total strain ϵ is given by the sum of the elastic, plastic, and thermal strains. Therefore, we assume that

$$\boxed{\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta.} \quad (7.81)$$

This we assume for both the modeling methods. Let us first look at modeling the thermal expansion, following this with a section on how we can modify the mechanical theory to include thermal effects, and then construct the thermodynamically based model.

7.5.1 Thermal expansion

The rate of change of thermal strain will be assumed to be proportional to the temperature rate and given by

$$\boxed{\dot{\epsilon}^\theta = \alpha \dot{\theta},} \quad (7.82)$$

where α is a second-order tensor representing the linear coefficient of thermal expansion. For an *isotropic material*, the linear coefficient of thermal expansion takes the form

$$\alpha = \alpha \mathbf{I}, \quad (7.83)$$

where α is a scalar function of temperature representing the linear coefficient of thermal expansion. For a *transversely isotropic material* with axis of transverse isotropy given along the unit vector $\hat{\mathbf{h}}$, the expression for α is given by

$$\alpha = \alpha_1 \mathbf{I} + \alpha_2 \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad (7.84)$$

where α_1 and α_2 are scalar functions of temperature. For an *orthotropic material* with directions of orthotropy given by the orthonormal base $\hat{\mathbf{h}}_i$, the thermal expansion is given by

$$\alpha = \alpha_1 \hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_1 + \alpha_2 \hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_2 + \alpha_3 \hat{\mathbf{h}}_3 \otimes \hat{\mathbf{h}}_3, \quad (7.85)$$

where α_i are scalar functions of temperature.

7.5.2 Temperature-dependent mechanical theory

One can introduce temperature dependence into the plasticity model by introducing the thermal strain into the kinematic relation for calculating the elastic strain and making the moduli temperature dependent. As we soon will see, the main difference with the pure mechanical model will be in the flow condition and flow rule. A typical model, including temperature dependence, can be written as

$$\begin{aligned} \epsilon &= \epsilon^e + \epsilon^p + \epsilon^\theta, \\ \sigma &= \mathbf{E} : \epsilon^e = \mathbf{E} : (\epsilon - \epsilon^p - \epsilon^\theta), \\ f &= |\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}| - \sigma_{iso}, \\ \Delta \sigma &= \sigma - \sigma^b, \\ \Delta \mathbf{S} &= \Delta \sigma - \Delta \sigma_{ave} \mathbf{I}, \\ \sigma^b &= \mathbf{E}^b : \epsilon^p, \\ \sigma_{iso} &= \sigma_{y_o} + \frac{d\sigma_{iso}}{d\xi} \xi, \\ \dot{\xi} &= \Delta \sigma : \dot{\epsilon}^p, \\ \dot{\epsilon}^\theta &= \alpha \dot{\theta}, \\ \dot{\epsilon}^p &= \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \end{aligned} \quad (7.86)$$

where the material coefficients $\mathbf{E}$, $\mathbf{E}^b$, σ_{iso} , σ_{y_o} , $\frac{d\sigma_{iso}}{d\xi}$, and α are all assumed to be functions of temperature, and β is selected so as to satisfy the consistency condition. For this example, we have selected a linear elastic response with a yield function for combined isotropic and kinematic hardening, with the kinematic hardening described by a back-stress σ^b , which is a linear function of the plastic strain, and an isotropic hardening rule, which is linear in the work hardening parameter.

To evaluate the form of β , we need to calculate the time derivative of the yield function. For the assumed form of the yield function, one can show that

$$\dot{f} = \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}} - \frac{d\sigma_{iso}}{d\xi} \dot{\xi} - \left[\frac{d\sigma_{y_o}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_{iso}}{d\xi} \right) \xi \right] \dot{\theta}, \quad (7.87)$$

which using the relation $\Delta \mathbf{S} : \dot{\Delta \mathbf{S}} = \Delta \mathbf{S} : \Delta \dot{\boldsymbol{\sigma}}$ can be written as

$$\begin{aligned} \dot{f} = & \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \left(\mathbf{E} : \dot{\boldsymbol{\epsilon}}^e + \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e \dot{\theta} - \mathbf{E}^b : \dot{\boldsymbol{\epsilon}}^p - \frac{d\mathbf{E}^b}{d\theta} : \boldsymbol{\epsilon}^p \dot{\theta} \right) \\ & - \frac{d\sigma_{iso}}{d\xi} \dot{\xi} - \left[\frac{d\sigma_{y_0}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_{iso}}{d\xi} \right) \xi \right] \dot{\theta}. \end{aligned} \quad (7.88)$$

This can be rewritten as

$$\dot{f} = \dot{f} - \left[\frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : (\mathbf{E} + \mathbf{E}^b) + \frac{d\sigma_{iso}}{d\xi} \Delta \boldsymbol{\sigma} \right] : \dot{\boldsymbol{\epsilon}}^p, \quad (7.89)$$

where

$$\dot{f} = \frac{3}{2} \frac{1}{|\sqrt{\frac{3}{2}} \Delta \mathbf{S} : \Delta \mathbf{S}|} \Delta \mathbf{S} : \left[\mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^\theta) + \left(\frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e - \frac{d\mathbf{E}^b}{d\theta} : \boldsymbol{\epsilon}^p \right) \dot{\theta} \right] - \left[\frac{d\sigma_{y_0}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_{iso}}{d\xi} \right) \xi \right] \dot{\theta}. \quad (7.90)$$

The consistency condition $\dot{f} = 0$ during plastic flow then will result in an expression for β given by

$$\beta = \frac{\sigma_{iso} \dot{f}}{\frac{3}{2} \Delta \mathbf{S} : (\mathbf{E} + \mathbf{E}^b) : \Delta \mathbf{S} + \sigma_{iso} \frac{d\sigma_{iso}}{d\xi} \Delta \mathbf{S} : \Delta \mathbf{S}}. \quad (7.91)$$

Knowing β completes the description of this example temperature-dependent model. Even though only an example, this model probably represents one of the most complex temperature-dependent mechanically based models. It includes both isotropic and kinematic hardening, with both the elastic and back-stress moduli depending on temperature, with the yield and isotropic hardening slope both depending on temperature, and is valid for isotropic and anisotropic material response obtained by selecting the forms of $\mathbf{E}$, $\mathbf{E}^b$, and $\boldsymbol{\alpha}$.

Section 7.5-Problems:

7.5-1: Derive the expression for β given by (7.91) filling the missing calculations starting from equation (7.87).

7.5-2: Derive the form of β given in (7.91) for a general quadratic yield function.

7.5.3 Thermodynamical model

The difference between a temperature-dependent model and a thermodynamical model is in the way we construct the model and the conditions we impose on it. For the thermodynamical model, we postulate a form for the free energy, entropy, stress, and heat flux, and then restrict the resulting model to always satisfy the entropy production inequality. The resulting models are similar, with the thermodynamical model providing expressions for entropy and stress in terms of derivatives of the free energy and, also, providing restrictions on the forms of the functions and connections between the mechanical and thermal parts of the response.

A fairly general model for free energy would include dependence on elastic, plastic, and thermal strains, hardening parameter, and also temperature and temperature gradient. That

is, $\psi(\epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta, \mathbf{G})$. We assume the same dependence for the stress, entropy, and heat flux vector. The Clausius-Duhem inequality provides a restriction on the response functions. This was given in equation (6.107) as

$$\rho_o \dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \rho_o \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0. \quad (7.92)$$

The selected form for the model of free energy yields a rate given by

$$\dot{\psi} = \partial_{\epsilon^e}(\psi) : \dot{\epsilon}^e + \partial_{\epsilon^p}(\psi) : \dot{\epsilon}^p + \partial_{\epsilon^\theta}(\psi) : \dot{\epsilon}^\theta + \partial_\xi(\psi) \dot{\xi} + \partial_\theta(\psi) \dot{\theta} + \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}. \quad (7.93)$$

Introduction of this into the Clausius-Duhem inequality, substituting for $\dot{\epsilon}^e$ by $\dot{\epsilon}^e = \dot{\boldsymbol{\epsilon}} - \dot{\epsilon}^p - \dot{\epsilon}^\theta$, substitution of $\dot{\epsilon}^\theta$ by $\dot{\epsilon}^\theta = \boldsymbol{\alpha} \dot{\theta}$, and reorganization yields

$$\begin{aligned} & [\rho_o \partial_{\epsilon^e}(\psi) - \boldsymbol{\sigma}] : \dot{\boldsymbol{\epsilon}} - \rho_o [\partial_{\epsilon^e}(\psi) - \partial_{\epsilon^p}(\psi)] : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi} \\ & + \rho_o \{ \eta + \partial_\theta(\psi) - [\partial_{\epsilon^e}(\psi) - \partial_{\epsilon^p}(\psi)] : \boldsymbol{\alpha} \} \dot{\theta} + \rho_o \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0. \end{aligned} \quad (7.94)$$

To expose the similarity between these results and other results in this book, we will introduce thermodynamic stresses $\boldsymbol{\sigma}^e$, $\boldsymbol{\sigma}^b$, and $\boldsymbol{\sigma}^\theta$ defined by

$$\boxed{\boldsymbol{\sigma}^e \equiv \frac{1}{2} \rho_o \{ \partial_{\epsilon^e}(\psi) + [\partial_{\epsilon^e}(\psi)]^T \}}, \quad (7.95)$$

$$\boxed{\boldsymbol{\sigma}^b \equiv \frac{1}{2} \rho_o \{ \partial_{\epsilon^p}(\psi) + [\partial_{\epsilon^p}(\psi)]^T \}}, \quad (7.96)$$

$$\boxed{\boldsymbol{\sigma}^\theta \equiv \frac{1}{2} \rho_o \{ \partial_{\epsilon^\theta}(\psi) + [\partial_{\epsilon^\theta}(\psi)]^T \}}. \quad (7.97)$$

Note that for a second-order symmetric tensor $\mathbf{A}$ we can show that $\mathbf{B} : \mathbf{A}$ can be written as

$$\mathbf{B} : \mathbf{A} = (\mathbf{B}_{sym} + \mathbf{B}_{skew}) : \mathbf{A} = \mathbf{B}_{sym} : \mathbf{A}. \quad (7.98)$$

As a result of this relation, we can rewrite the entropy production inequality as

$$\begin{aligned} & (\boldsymbol{\sigma}^e - \boldsymbol{\sigma}) : \dot{\boldsymbol{\epsilon}} - (\boldsymbol{\sigma}^e - \boldsymbol{\sigma}^b) : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi} + \rho_o \left\{ \eta + \partial_\theta(\psi) - \frac{1}{\rho_o} (\boldsymbol{\sigma}^e - \boldsymbol{\sigma}^\theta) : \boldsymbol{\alpha} \right\} \dot{\theta} \\ & + \rho_o \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0. \end{aligned} \quad (7.99)$$

In the elastic range, the plastic strain rate and hardening rate go to zero so the remaining equation becomes

$$\boxed{(\boldsymbol{\sigma}^e - \boldsymbol{\sigma}) : \dot{\boldsymbol{\epsilon}} + \rho_o \{ \eta + \partial_\theta(\psi) - \frac{1}{\rho_o} (\boldsymbol{\sigma}^e - \boldsymbol{\sigma}^\theta) : \boldsymbol{\alpha} \} \dot{\theta} + \rho_o \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0.} \quad (7.100)$$

The factors multiplied by $\dot{\boldsymbol{\epsilon}}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$ are independent of these rates. This follows as a result of the constitutive assumptions regarding the free energy, entropy, and stress. Assuming we can arbitrarily select $\dot{\boldsymbol{\epsilon}}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$, it can be shown that to satisfy the inequality we must

have

$$\sigma = \sigma^e = \frac{1}{2} \rho_o \{ \partial_{\epsilon^e}(\psi) + [\partial_{\epsilon^e}(\psi)]^T \}, \quad (7.101)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho_o} (\sigma^e - \sigma^\theta) : \alpha, \quad (7.102)$$

$$\partial_{\mathbf{G}}(\psi) = \mathbf{0}, \quad (7.103)$$

$$\frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0. \quad (7.104)$$

Obviously, the third relation states that the free energy, and by association through the first and second relations, the entropy and stress, do not depend on temperature gradient. Therefore, from this point, we restrict attention to free-energy functions of the form $\psi(\epsilon^e, \epsilon^p, \epsilon^\theta, \xi, \theta)$. The heat flux vector still can depend on $\mathbf{G}$.

We assume continuity in the response functions as we enter the range of plastic flow, and we also assume that during plastic flow we have attached to each loading point an elastic range that can be entered by changing the rates. From these assumptions, we argue that the forms of the expressions for stress and entropy hold even during plastic flow and, therefore, write the Clausius-Duhem inequality during plastic flow as

$$-(\sigma^e - \sigma^b) : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0, \quad (7.105)$$

where the missing terms become zero from the analysis of the attached elastic range. We further assume that the temperature gradient can be controlled independent of the plastic strain rate. This is assumed to be achievable through manipulation of boundary conditions for the heat flux vector and changing of the radiation term, so that we always can create conditions for which the temperature gradient is zero. From this we conclude that all plastic flow must occur in such a way as to satisfy the inequality

$$-(\sigma^e - \sigma^b) : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi} \leq 0. \quad (7.106)$$

7.5.4 Balance of work and energy

Let us now consider the balance of work and energy. Before we do this, from the discussion of the thermodynamic constraint we can conclude that for a thermoplastic material we have

$$\rho_o \dot{\psi} - \sigma : \dot{\epsilon} + \rho_o \eta \dot{\theta} = -(\sigma^e - \sigma^b) : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi}. \quad (7.107)$$

The equation for the balance of work and energy was given by equation (6.108) as

$$\rho_o \dot{\psi} + \rho_o \dot{\eta} \theta + \rho_o \eta \dot{\theta} = \rho_o r - \text{Div}(\mathbf{q}_o) + \sigma : \dot{\epsilon}. \quad (7.108)$$

Therefore, the balance of work and energy for a thermoplastic material can be written as

$$\rho_o \dot{h} = -\Delta \sigma : \dot{\epsilon}^p + \rho_o \partial_\xi(\psi) \dot{\xi} + \rho_o \dot{\eta} \theta, \quad (7.109)$$

where the reader recalls that $\rho_o \dot{h} = \rho_o r - \text{Div}(\mathbf{q}_o)$ represents the rate of heat flow into the point per unit volume by conduction and radiation, and where $\Delta \sigma \equiv \sigma^e - \sigma^b = \sigma - \sigma^b$

is the overstress. We also note that from the Clausius-Duhem inequality we have shown in equation (7.106) that for all processes we have

$$\boxed{\rho_o \dot{h} - \rho_o \dot{\eta} \theta = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o \partial_\xi(\psi) \dot{\xi} \leq 0.} \quad (7.110)$$

For an *isothermal process*, we can integrate $\rho_o \dot{h}$ to get

$$\rho_o \Delta h = \rho_o \theta (\eta - \eta_o) - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt + \rho_o \int_{t_o}^t \partial_\xi(\psi) \dot{\xi} dt. \quad (7.111)$$

As can be seen, for an isothermal process, we have the condition

$$\rho_o \Delta h - \rho_o \theta (\eta - \eta_o) \leq 0, \quad (7.112)$$

which is also similar to stating that

$$\boxed{- \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt + \rho_o \int_{t_o}^t \partial_\xi(\psi) : \dot{\xi} dt \leq 0.} \quad (7.113)$$

For an *isentropic process* (one that has constant entropy), $-\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o \partial_\xi(\psi) \dot{\xi}$ represents the energy that needs to be removed through radiation and conduction for the entropy to remain constant. From the thermodynamic constraints on these two terms, we can see that for an isentropic process we always have

$$\rho_o \dot{h} = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o \partial_\xi(\psi) \dot{\xi} \leq 0, \quad (7.114)$$

which states that there needs to be a net zero or positive heat flow out of the point to sustain conditions of constant entropy. Integrating the expression over time for an isentropic process yields

$$\boxed{\rho_o \Delta h = - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt + \rho_o \int_{t_o}^t \partial_\xi(\psi) \dot{\xi} dt \leq 0.} \quad (7.115)$$

For an *adiabatic process* (one that is thermally isolated), the rate of change of entropy is given by

$$\dot{\eta} = \frac{1}{\rho_o \theta} [\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - \rho_o \partial_\xi(\psi) \dot{\xi}] \geq 0. \quad (7.116)$$

As can be seen, plastic flow always results in zero change or an increase of entropy in an adiabatic system. The integration of this equation results in

$$\boxed{\eta - \eta_o = \frac{1}{\rho_o} \int_{t_o}^t \frac{1}{\theta} [\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - \rho_o \partial_\xi(\psi) \dot{\xi}] dt \geq 0.} \quad (7.117)$$

7.5.5 Heat flux

When considering temperature effects, it becomes important to model both the contributions of the heat flow and heat generation. One can imagine the heat flow as coming from the body's contact with objects at different temperatures, while heat generation is related to the heat that is generated locally at a point by the plastic flow of the material. Here, we focus on the heat flux vector, which is one component of the heat flow. To model the heat flux vector, we can take a model such as Fourier's law given by

$$\boxed{\mathbf{q}_o = -\mathbf{K} \mathbf{G},} \quad (7.118)$$

where $\mathbf{q}_o$ is the heat flux vector, $\mathbf{K}$ is the linear coefficient of thermal conduction, and $\mathbf{G} = \nabla_{\mathbf{x}}(\theta)$ is the temperature gradient. The linear coefficient of thermal conduction $\mathbf{K}$ is a second-order tensor. For an *isotropic material*, $\mathbf{K}$ has the form

$$\boxed{\mathbf{K} = k\mathbf{I}}, \quad (7.119)$$

where k is a scalar function of temperature. For a *transversely isotropic material*, we can take $\mathbf{K}$ as

$$\boxed{\mathbf{K} = k_1\mathbf{I} + k_2\hat{\mathbf{h}} \otimes \hat{\mathbf{h}}}, \quad (7.120)$$

where k_1 and k_2 are scalar functions of temperature and $\hat{\mathbf{h}}$ is the preferred axis of transverse isotropy. For an *orthotropic material*, $\mathbf{K}$ can be taken as

$$\boxed{\mathbf{K} = k_1\hat{\mathbf{h}}_1 \otimes \hat{\mathbf{h}}_1 + k_2\hat{\mathbf{h}}_2 \otimes \hat{\mathbf{h}}_2 + k_3\hat{\mathbf{h}}_3 \otimes \hat{\mathbf{h}}_3}, \quad (7.121)$$

where k_i are scalar functions of temperature and $\hat{\mathbf{h}}_i$ are an orthonormal base representing the preferred axes of orthotropy.

Section 7.5-Problems:

- 7.5-3:** Show that the volumetric coefficient of thermal expansion, denoted by β , is given by $\beta = \text{tr}(\boldsymbol{\alpha})$. From this conclude that for an isotropic material $\beta = 3\alpha$, where α is the linear coefficient of thermal expansion.
- 7.5-4:** Consider an orthonormal base $\mathbf{e}_i$ and a material that is orthotropic with preferred direction of orthotropy given by the orthonormal base $\hat{\mathbf{h}}_i$. Knowing that $\hat{\mathbf{h}}_i$ is given in terms of $\mathbf{e}_j$ through the transformation matrix l_{ij} such that $\hat{\mathbf{h}}_i = l_{ij}\mathbf{e}_j$, show that the coefficient of thermal expansion for this orthotropic material can be written as

$$\boldsymbol{\alpha} = (\alpha_1 l_{1i} l_{1j} + \alpha_2 l_{2i} l_{2j} + \alpha_3 l_{3i} l_{3j}) \mathbf{e}_i \otimes \mathbf{e}_j. \quad (7.122)$$

- 7.5-5:** Prove that $\mathbf{A}$ must be skew-symmetric if $\mathbf{A} : \mathbf{B} = 0$ for a symmetric tensor $\mathbf{B}$. Use this to show that for a symmetric tensor $\mathbf{B}$ we always have the identity

$$\mathbf{A} : \mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) : \mathbf{B} = \mathbf{A}_{sym} : \mathbf{B}. \quad (7.123)$$

7.6 Free energy of quadratic form

Probably the most common form for the free energy is an expression that is quadratic in the elastic and plastic strains. This normally is written in the form

$$\boxed{\psi = \frac{1}{2\rho_o} \boldsymbol{\epsilon}^e : \mathbf{E} : \boldsymbol{\epsilon}^e + \frac{1}{2\rho_o} \boldsymbol{\epsilon}^p : \mathbf{E}^b : \boldsymbol{\epsilon}^p + \psi^\theta}, \quad (7.124)$$

where there is a quadratic contribution from the elastic strain, a quadratic contribution from the plastic strain, and a term depending only on temperature. In this expression, $\mathbf{E}$ and $\mathbf{E}^b$ are fourth-order tensor moduli and ψ^θ is a scalar, all three only depending on temperature.

This form is motivated from the mechanical analog shown in Figure 7.2, which has a potential energy V given by

$$V = \frac{1}{2}k^e \delta e^2 + \frac{1}{2}k^b \delta p^2. \quad (7.125)$$

The moduli are assumed to have symmetry properties such that

$$\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{A}^T : \mathbf{E} : \mathbf{B} = \mathbf{A} : \mathbf{E} : \mathbf{B}^T = \mathbf{B} : \mathbf{E} : \mathbf{A}, \quad (7.126)$$

$$\mathbf{A} : \mathbf{E}^b : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^b : \mathbf{B} = \mathbf{A} : \mathbf{E}^b : \mathbf{B}^T = \mathbf{B} : \mathbf{E}^b : \mathbf{A}, \quad (7.127)$$

for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$. Even if we introduce moduli that do not show this symmetry, they can be replaced by ones constructed from the original tensors that have these noted properties. The discussion of these symmetries is provided later in Chapter 10. In essence, these symmetries require that when expressing the moduli in component form we have the relations

$$\boxed{E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}}, \quad (7.128)$$

$$\boxed{E_{ijkl}^b = E_{jikl}^b = E_{ijlk}^b = E_{klij}^b}. \quad (7.129)$$

7.6.1 Thermodynamic constraint

For this model, the expression for $\dot{\psi}$ is given by

$$\dot{\psi} = \frac{1}{\rho_o} \boldsymbol{\epsilon}^e : \mathbf{E} : \dot{\boldsymbol{\epsilon}}^e + \frac{1}{\rho_o} \boldsymbol{\epsilon}^p : \mathbf{E}^b : \dot{\boldsymbol{\epsilon}}^p + \left(\frac{1}{2\rho_o} \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e + \frac{1}{2\rho_o} \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \boldsymbol{\epsilon}^p + \frac{d\psi^\theta}{d\theta} \right) \dot{\theta}. \quad (7.130)$$

It follows from this that the partial derivatives are

$$\partial_{\boldsymbol{\epsilon}^e}(\psi) = \frac{1}{\rho_o} \boldsymbol{\epsilon}^e : \mathbf{E} + \mathbf{A}^e, \quad (7.131)$$

$$\partial_{\boldsymbol{\epsilon}^p}(\psi) = \frac{1}{\rho_o} \boldsymbol{\epsilon}^e : \mathbf{E} + \mathbf{A}^p, \quad (7.132)$$

$$\partial_{\boldsymbol{\epsilon}^\theta}(\psi) = \mathbf{A}^\theta, \quad (7.133)$$

$$\partial_\xi(\psi) = 0, \quad (7.134)$$

$$\partial_\theta(\psi) = \frac{1}{2\rho_o} \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e + \frac{1}{2\rho_o} \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \boldsymbol{\epsilon}^p + \frac{d\psi^\theta}{d\theta}, \quad (7.135)$$

for arbitrary skew-symmetric tensors $\mathbf{A}^e$, $\mathbf{A}^p$, $\mathbf{A}^\theta$. It follows from these that the thermodynamic stresses are given by

$$\boxed{\boldsymbol{\sigma}^e = \mathbf{E} : \boldsymbol{\epsilon}^e}, \quad (7.136)$$

$$\boxed{\boldsymbol{\sigma}^b = \mathbf{E}^b : \boldsymbol{\epsilon}^p}, \quad (7.137)$$

$$\boxed{\boldsymbol{\sigma}^\theta = \mathbf{0}}. \quad (7.138)$$

Using these, we can rewrite the free energy as

$$\psi = \frac{1}{2\rho_o} \boldsymbol{\sigma}^e : \boldsymbol{\epsilon}^e + \frac{1}{2\rho_o} \boldsymbol{\sigma}^b : \boldsymbol{\epsilon}^p + \psi^\theta. \quad (7.139)$$

From the partial derivatives and the thermodynamic stresses, we can see that we arrive at the expressions for stress and entropy as

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^e = \mathbf{E} : \boldsymbol{\epsilon}^e,} \quad (7.140)$$

$$\boxed{\eta = -\frac{1}{2\rho_o} \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e - \frac{1}{2\rho_o} \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \boldsymbol{\epsilon}^p - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho_o} \boldsymbol{\sigma} : \boldsymbol{\alpha}.} \quad (7.141)$$

7.6.2 Balance of work and energy

The balance of work and energy for this model can be written as

$$\boxed{\rho_o \dot{h} = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o \dot{\eta} \theta,} \quad (7.142)$$

for the overstress $\Delta \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^b$, and with the thermodynamic constraint given by

$$\boxed{\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p \geq 0.} \quad (7.143)$$

For an *isothermal process*, from equation (7.111) we know that in this case

$$\rho_o \Delta h = \rho_o \theta (\eta - \eta_o) - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt. \quad (7.144)$$

For an *isentropic process*, from equation (7.115) we have that

$$\rho_o \Delta h = - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt \leq 0. \quad (7.145)$$

For the isentropic process, the rate of change of temperature is calculated from $\dot{\eta} = 0$.

Assuming a constant coefficient of thermal expansion $\boldsymbol{\alpha}$ and linear dependence on temperature for $\mathbf{E}$ and $\mathbf{E}^b$, the expression for $\dot{\eta}$ can be calculated as

$$\dot{\eta} = -\frac{d^2\psi^\theta}{d\theta^2} \dot{\theta} - \frac{1}{\rho_o} \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \dot{\boldsymbol{\epsilon}}^e - \frac{1}{\rho_o} \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \dot{\boldsymbol{\epsilon}}^p + \frac{1}{\rho_o} \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha}, \quad (7.146)$$

where

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}}^e + \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e \dot{\theta}. \quad (7.147)$$

For an isentropic process, we can use this to calculate the temperature change by setting $\dot{\eta} = 0$ to get

$$\dot{\theta} = \frac{1}{\rho_o \frac{d^2\psi^\theta}{d\theta^2} - \boldsymbol{\alpha} : \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\epsilon}^e} \left(-\boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \dot{\boldsymbol{\epsilon}}^e - \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \dot{\boldsymbol{\epsilon}}^p + \dot{\boldsymbol{\epsilon}}^e : \mathbf{E} : \boldsymbol{\alpha} \right). \quad (7.148)$$

For this model, the balance of energy can be written as

$$\rho_o \dot{h} = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - \theta \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \dot{\boldsymbol{\epsilon}}^e - \theta \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \dot{\boldsymbol{\epsilon}}^p - \rho_o \theta \frac{d^2\psi^\theta}{d\theta^2} \dot{\theta} + \theta \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha}. \quad (7.149)$$

For an adiabatic process we, have $\dot{h} = 0$ so that temperature change is calculated from

$$0 = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - \theta \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \dot{\boldsymbol{\epsilon}}^e - \theta \boldsymbol{\epsilon}^p : \frac{d\mathbf{E}^b}{d\theta} : \dot{\boldsymbol{\epsilon}}^p - \rho_o \theta \frac{d^2\psi^\theta}{d\theta^2} \dot{\theta} + \theta \dot{\boldsymbol{\epsilon}}^e : \mathbf{E} : \boldsymbol{\alpha} + \theta \boldsymbol{\epsilon}^e : \frac{d\mathbf{E}}{d\theta} : \boldsymbol{\alpha} \dot{\theta}. \quad (7.150)$$

This results in

$$\dot{\theta} = \frac{1}{\rho_o \theta \frac{d^2 \psi^\theta}{d\theta^2} - \theta \epsilon^e : \frac{d\mathbf{E}}{d\theta} : \alpha} \left(-\Delta\sigma : \dot{\epsilon}^p - \theta \epsilon^e : \frac{d\mathbf{E}}{d\theta} : \dot{\epsilon}^e - \theta \epsilon^p : \frac{d\mathbf{E}^b}{d\theta} : \dot{\epsilon}^p + \theta \dot{\epsilon}^e : \mathbf{E} : \alpha \right). \quad (7.151)$$

7.6.3 Flow rule, hardening rule, and yield function

To complete a plasticity model, we need to add a yield function, flow rule, and possibly a hardening rule. There are many models to choose from. Here, we make a simple selection to demonstrate the process of imposing the consistency condition. The thermodynamic constraint $\Delta\sigma : \dot{\epsilon}^p \geq 0$ suggests that a flow rule that provides plastic flow along $\Delta\sigma$ or its deviatoric part would satisfy the required condition. A common assumption is that plastic flow is shear driven, suggesting a flow rule of the form

$$\dot{\epsilon}^p = \begin{cases} \beta \Delta\mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (7.152)$$

where $\Delta\mathbf{S}$ is the deviatoric part of the overstress $\Delta\sigma = \sigma - \sigma^b$, and β is a scalar that needs to be greater or equal to zero to satisfy the thermodynamic constraint. The thermodynamic constraint $\Delta\sigma : \dot{\epsilon}^p \geq 0$ also provides a means of introducing a monotonically increasing work hardening model for ξ given by

$$\dot{\xi} = \Delta\sigma : \dot{\epsilon}^p. \quad (7.153)$$

A fairly common yield function is given by the equation

$$f = \sqrt{\frac{3}{2} \Delta\mathbf{S} : \Delta\mathbf{S} - \Delta\sigma_y}, \quad (7.154)$$

where $\Delta\sigma_y$ represents the overstress at yield in tension and is assumed to be a function of temperature θ and hardening parameter ξ .

We need to calculate $\dot{f}$ and impose the consistency condition during yield. We do this by first calculating $\dot{f}$. This is given by

$$\dot{f} = \frac{3}{2\sqrt{\frac{3}{2} \Delta\mathbf{S} : \Delta\mathbf{S}}} \Delta\mathbf{S} : \Delta\dot{\mathbf{S}} - \Delta\dot{\sigma}_y. \quad (7.155)$$

Noting that $\Delta\mathbf{S} : \Delta\dot{\mathbf{S}} = \Delta\mathbf{S} : \Delta\dot{\sigma}$, let us first look at $\Delta\dot{\sigma}$. For this term, we have

$$\begin{aligned} \Delta\dot{\sigma} &= \dot{\sigma} - \dot{\sigma}^b \\ &= \frac{d\mathbf{E}}{d\theta} \dot{\theta} : \epsilon^e + \mathbf{E} : \dot{\epsilon}^e - \frac{d\mathbf{E}^b}{d\theta} \dot{\theta} : \epsilon^p - \mathbf{E}^b : \dot{\epsilon}^p \\ &= \frac{d\mathbf{E}}{d\theta} \dot{\theta} : \epsilon^e + \mathbf{E} : (\dot{\epsilon} - \dot{\epsilon}^p - \alpha \dot{\theta}) - \frac{d\mathbf{E}^b}{d\theta} \dot{\theta} : \epsilon^p - \mathbf{E}^b : \dot{\epsilon}^p \\ &= \mathbf{E} : \dot{\epsilon} - (\mathbf{E} + \mathbf{E}^b) : \dot{\epsilon}^p + \left(\frac{d\mathbf{E}}{d\theta} : \epsilon^e - \frac{d\mathbf{E}^b}{d\theta} : \epsilon^p - \mathbf{E} : \alpha \right) \dot{\theta}. \end{aligned} \quad (7.156)$$

We also note that

$$\Delta\dot{\sigma}_y = \frac{\partial \Delta\sigma_y}{\partial \theta} \dot{\theta} + \frac{\partial \Delta\sigma_y}{\partial \xi} \dot{\xi}. \quad (7.157)$$

Substituting these back into the expression for $\dot{f}$ results in

$$\dot{f} = \frac{3}{2\sqrt{\frac{3}{2}\Delta\mathbf{S}:\Delta\mathbf{S}}} \Delta\mathbf{S} : [\mathbf{E} : \dot{\epsilon} - (\mathbf{E} + \mathbf{E}^b) : \dot{\epsilon}^p + \left(\frac{d\mathbf{E}}{d\theta} : \epsilon^e - \frac{d\mathbf{E}^b}{d\theta} : \epsilon^p - \mathbf{E} : \alpha\right)\dot{\theta}] - \frac{\partial\Delta\sigma_y}{\partial\theta}\dot{\theta} - \frac{\partial\Delta\sigma_y}{\partial\xi}\dot{\xi}. \quad (7.158)$$

This allows us now to calculate $\dot{f}$ for a strain and temperature-controlled process as

$$\dot{f} = \frac{3}{2\sqrt{\frac{3}{2}\Delta\mathbf{S}:\Delta\mathbf{S}}} \Delta\mathbf{S} : \left[\mathbf{E} : \dot{\epsilon} + \left(\frac{d\mathbf{E}}{d\theta} : \epsilon^e - \frac{d\mathbf{E}^b}{d\theta} : \epsilon^p - \mathbf{E} : \alpha \right) \dot{\theta} \right] - \frac{\partial\Delta\sigma_y}{\partial\theta}\dot{\theta}. \quad (7.159)$$

In the case of stress and temperature control, $\dot{f}$ would be given as

$$\dot{f} = \frac{3}{2\sqrt{\frac{3}{2}\Delta\mathbf{S}:\Delta\mathbf{S}}} \Delta\mathbf{S} : \left[\dot{\sigma} - \frac{d\mathbf{E}^b}{d\theta} : \epsilon^p \dot{\theta} \right] - \frac{\partial\Delta\sigma_y}{\partial\theta}\dot{\theta}. \quad (7.160)$$

In general, $\dot{f}$ is calculated from $\dot{f}$ under the assumption that

$$\dot{\sigma} = \frac{d\mathbf{E}}{d\theta}\dot{\theta} : \epsilon^e + \mathbf{E} : \dot{\epsilon}^e = \frac{d\mathbf{E}}{d\theta}\dot{\theta} : \epsilon^e + \mathbf{E} : (\dot{\epsilon} - \alpha\dot{\theta}), \quad (7.161)$$

$$\dot{\sigma}^b = \frac{d\mathbf{E}^b}{d\theta}\dot{\theta} : \epsilon^p, \quad (7.162)$$

$$\Delta\dot{\sigma}_y = \frac{d\Delta\sigma_y}{d\theta}\dot{\theta}. \quad (7.163)$$

To impose the consistency condition, let us rewrite $\dot{f}$ as

$$\dot{f} = \dot{f} - \frac{3\Delta\mathbf{S} : (\mathbf{E} + \mathbf{E}^b) : \dot{\epsilon}^p}{2\sqrt{\frac{3}{2}\Delta\mathbf{S}:\Delta\mathbf{S}}} - \frac{\partial\Delta\sigma_y}{\partial\xi}\dot{\xi}. \quad (7.164)$$

The consistency condition is only imposed when $f = 0$ so that we have $\sqrt{\frac{3}{2}\Delta\mathbf{S}:\Delta\mathbf{S}} = \Delta\sigma_y$. Imposing the consistency condition $\dot{f} = 0$ on the flow rule results in the equation

$$0 = \dot{f} - \frac{3\Delta\mathbf{S} : (\mathbf{E} + \mathbf{E}^b) : \beta\Delta\mathbf{S}}{2\Delta\sigma_y} - \frac{d\Delta\sigma_y}{d\xi}\Delta\sigma : \beta\Delta\mathbf{S}. \quad (7.165)$$

Noting that $\Delta\sigma : \beta\Delta\mathbf{S} = \Delta\mathbf{S} : \beta\Delta\mathbf{S}$, reorganizing to get β yields

$$\beta = \frac{2\Delta\sigma_y \dot{f}}{3\Delta\mathbf{S} : (\mathbf{E} + \mathbf{E}^b) : \Delta\mathbf{S} + 2\Delta\sigma_y \frac{d\Delta\sigma_y}{d\xi} \Delta\mathbf{S} : \Delta\mathbf{S}}. \quad (7.166)$$

This concludes the process of calculating a β that fits our assumptions.

Section 7.6-Problems:

7.6-1: Start with a free energy of quadratic form given by

$$\psi = \frac{1}{2\rho_o} \epsilon^e : \mathbf{E} : \epsilon^e + \frac{1}{2\rho_o} \epsilon^p : \mathbf{E}^b : \epsilon^p + \psi^\theta, \quad (7.167)$$

where $\mathbf{E}$ and $\mathbf{E}^b$ are fourth-order tensor moduli and ψ^θ is a scalar, all three only depending on temperature θ and hardening parameter ξ . Assume a work hardening expression of the form

$$\dot{\xi} = \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p, \quad (7.168)$$

for $\Delta \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^b$. Assume a yield function of the form

$$f = \sqrt{\frac{3}{2} \Delta \mathbf{S} : \Delta \mathbf{S}} - \sigma_{iso}, \quad (7.169)$$

with $\Delta \mathbf{S} = \mathbf{S} - \mathbf{S}^b$ for $\mathbf{S}$ and $\mathbf{S}^b$ being the deviatoric parts, respectively, of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^b$, and with

$$\sigma_{iso} = \sigma_{yo} + \sigma'_{iso} \xi \quad (7.170)$$

for σ_{yo} and σ'_{iso} being functions of temperature. Calculate the expressions for the thermodynamic stresses, entropy, and a consistent flow rule of the form

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.171)$$

7.7 Scalar stress and hardening functions

For isotropic material response, authors frequently write the yield function in terms of scalar parameters since this process simplifies the form and its visualization. We also have done so in most cases without really mentioning the scalar function for stress. The basic idea in this case is that a graph, either analytically defined or a fit to a set of data points, similar to Figure 7.8, is used to define the yielding in the material. This method also allows us to develop and visualize much more complex models, as will be shown at the end of this section. To focus our discussion, we only look at flow rules that have the form

$$\dot{\boldsymbol{\epsilon}}^p = \beta \mathbf{S}, \quad (7.172)$$

or

$$\dot{\boldsymbol{\epsilon}}^p = \beta \Delta \mathbf{S}, \quad (7.173)$$

where the first assumes the plastic strain changes in the direction of the deviatoric stress and the second assumes that the plastic strain changes in the direction of the deviatoric overstress. The result of both of these is that the plastic strain has a zero trace so that

$$\text{tr}(\boldsymbol{\epsilon}^e) = \text{tr}(\boldsymbol{\epsilon}). \quad (7.174)$$

We will focus only on isotropic models in this section so that the elastic response is given by

$$\mathbf{S} = 2G[\boldsymbol{\epsilon}^e - \frac{1}{3}\epsilon_v \mathbf{I}], \quad (7.175)$$

$$\sigma_{ave} = K\epsilon_v, \quad (7.176)$$

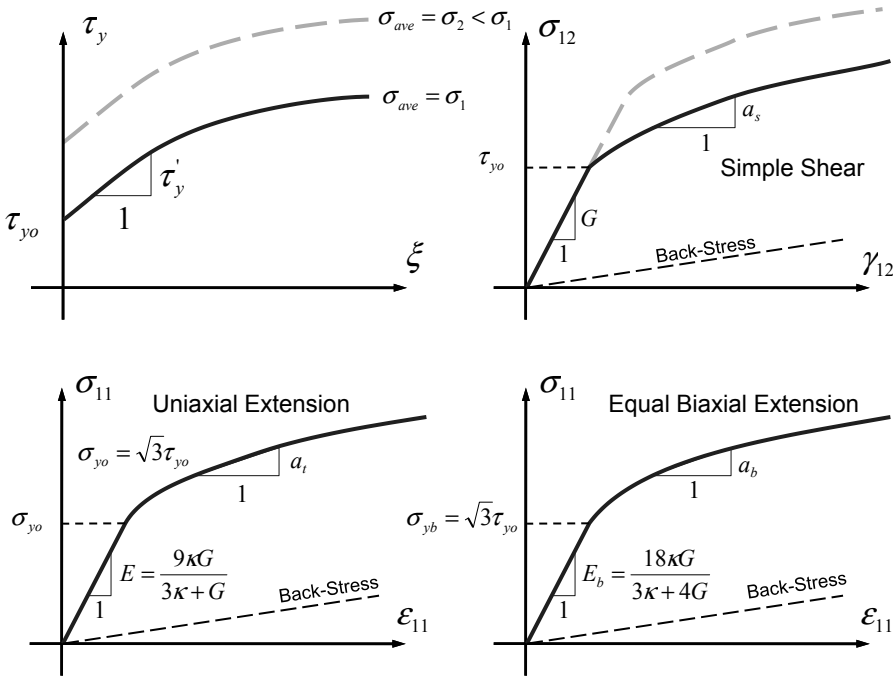


Figure 7.8: Schematic hardening rule in terms of scalar parameters $\Delta\tau$, σ_{ave} , and ξ , and associated response in simple shear, uniaxial extension, and equal biaxial extension.

where G is the elastic shear modulus, κ is the elastic bulk modulus, and $\epsilon_v = tr(\epsilon)$ is the volumetric strain. For the selected model for plastic flow, the back-stress only has a deviatoric part given by

$$\mathbf{S}^b = 2G^b \epsilon^p, \quad (7.177)$$

where G^b is the back-stress shear modulus. These are the basic assumptions on which the following modeling is based.

Plasticity in most materials is motivated through shear since shear is the most common mode of plastic deformation. As a result, the modeling structure separates the response into scalar parameters that measure the extent of shear and triaxial extension. In this modeling, the measure for shear stress is τ , the measure for average stress is σ_{ave} , the measure for overstress is $\Delta\tau$, the measure for plastic strain is ϵ^p , and the measure for hardening is ξ . It is commonly assumed that plastic flow is driven by shear so the most common assumption for the stress measure is

$$\tau \equiv \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}}. \quad (7.178)$$

The average stress is assumed to affect the yield point and is given in the standard way described above by a function of $\sigma_{ave} = \frac{1}{3} tr(\sigma)$. As expected, positive σ_{ave} should be associated with expansion and, therefore, reduction in the resistance to plastic flow, and negative values should be associated with compression and, therefore, increases in the resistance to

plastic flow. The scalar measure of overstress is normally only related to shear and given by

$$\Delta\tau \equiv \sqrt{\frac{1}{2}\Delta\mathbf{S} : \Delta\mathbf{S}}, \quad (7.179)$$

where $\Delta\mathbf{S} = \mathbf{S} - \mathbf{S}^b$; $\Delta\tau \neq \tau - \tau^b$. Both the scalar measures of shear stress and shear overstress are taken to be positive scalars, as is the case for all the scalar measures presented here. There are events that scale with the total amount of plastic strain, such as the back-stress, and may be related to the scalar measure of plastic strain γ^p given by

$$\gamma^p \equiv \sqrt{2\epsilon^p : \epsilon^p}. \quad (7.180)$$

There are also events associated with hardening in the material for which we define a hardening parameter ξ through a hardening rate $\dot{\xi}$ and initial conditions. There are two commonly used hardening rules. These are strain hardening and work hardening, respectively, given by

$$\dot{\xi} = \sqrt{2\dot{\epsilon}^p : \dot{\epsilon}^p}, \quad (7.181)$$

and

$$\dot{\xi} = \Delta\boldsymbol{\sigma} : \dot{\epsilon}^p. \quad (7.182)$$

It easily can be shown that by the deviatoric nature of the plastic flow, one can rewrite the work hardening rule as

$$\dot{\xi} = \Delta\mathbf{S} : \dot{\epsilon}^p. \quad (7.183)$$

Even though in some cases it can be shown that there exist proper models that have back-stress but which use a flow rule of the form $\dot{\epsilon}^p = \beta\mathbf{S}$, in most cases when there is a back-stress we use a plastic flow of the form $\dot{\epsilon}^p = \beta\Delta\mathbf{S}$. *To simplify the presentation, we will assume in this section that in all cases the flow rule is given by $\dot{\epsilon}^p = \beta\Delta\mathbf{S}$.* Readers may pursue other assumptions on their own.

Given these assumptions, the strain hardening rule can be written as

$$\dot{\xi} = 2\beta\Delta\tau, \quad (7.184)$$

and the work hardening rule can be written as

$$\dot{\xi} = 2\beta\Delta\tau^2, \quad (7.185)$$

both being nonzero only during plastic flow (i.e., they are nonzero only when $\dot{f} = 0$ and $\dot{f} > 0$). The reader may imagine that under the assumptions of this section the two hardening rules are “similar,” the conclusion being that the choice is based on the simplicity of the form of the yield function or the particular preference of the reader. We also will need the

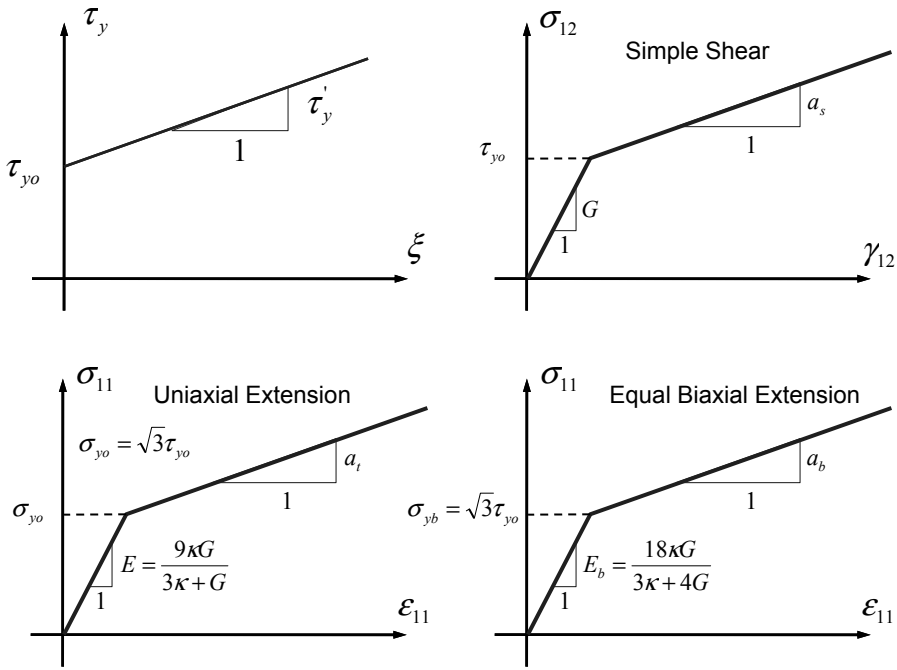


Figure 7.9: A model for a material with linear hardening.

rate of change of the scalar stresses that are given by

$$\dot{\tau} = \frac{1}{2\tau} \mathbf{S} : \dot{\mathbf{S}}, \quad (7.186)$$

$$\dot{\tau}^b = \frac{1}{2\tau^b} \mathbf{S}^b : \dot{\mathbf{S}}^b, \quad (7.187)$$

$$\Delta \dot{\tau} = \frac{1}{2\Delta\tau} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}}, \quad (7.188)$$

$$\dot{\sigma}_{ave} = \kappa \dot{\epsilon}_v, \quad (7.189)$$

$$\dot{\tau} = \frac{1}{2\tau} \mathbf{S} : \dot{\mathbf{S}} = \frac{G}{\tau} \mathbf{S} : \dot{\epsilon}, \quad (7.190)$$

$$\Delta \dot{\tau} = \frac{1}{2\Delta\tau} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}} = \frac{G}{\Delta\tau} \Delta \mathbf{S} : \dot{\epsilon}. \quad (7.191)$$

As the reader will note from the problem on simple shear, the scalar shear parameters are selected such that they reduce to the associate parameters in simple shear. This explains the constants selected, which are otherwise arbitrary and can be selected differently or altogether left out.

Now that we have defined all the parameters, we are in the position to discuss yield functions and their associated flow rules. The typical yield function in this way of viewing

the functions is given by the form

$$f(\Delta\tau, \sigma_{ave}, \gamma^p, \xi), \quad (7.192)$$

with the yield surface defined by the condition $f = 0$. There are many different yield functions that can be defined, but we will focus on a few that provide simple models with potential use in characterizing some materials.

The first function that we will consider is that shown in Figure 7.9, where the overstress is a linear function of the hardening parameter. From Figure 7.9, the expression for the yield function can be written as

$$f = \Delta\tau - \tau_y(\xi). \quad (7.193)$$

We will be on the yield surface when $\Delta\tau = \tau_y(\xi)$, for the current value of ξ . Also, being on the yield surface, plastic flow will occur when $\dot{f} > 0$. This can be written as

$$\dot{f} = \Delta \dot{\tau} = \frac{G}{\Delta\tau} \Delta \mathbf{S} : \dot{\epsilon}. \quad (7.194)$$

The consistency condition during plastic flow is $\dot{f} = 0$, which can be written as

$$\dot{f} = \frac{1}{2\Delta\tau} \Delta \mathbf{S} : \Delta \dot{\mathbf{S}} - \tau'_y \dot{\xi} = \frac{G}{\Delta\tau} \Delta \mathbf{S} : \dot{\epsilon}^e - \frac{G^b}{\Delta\tau} \Delta \mathbf{S} : \dot{\epsilon}^p - \tau'_y \dot{\xi} = 0, \quad (7.195)$$

where τ'_y is the derivative of τ_y with respect to ξ . This can be written for $\dot{\epsilon}^p = \beta \Delta \mathbf{S}$ and $\dot{\epsilon}^e = \dot{\epsilon} - \dot{\epsilon}^p$ as

$$\dot{f} = \frac{G}{\Delta\tau} \Delta \mathbf{S} : \dot{\epsilon} - 2\beta(G + G^b)\Delta\tau - \tau'_y \dot{\xi} = 0. \quad (7.196)$$

For strain hardening $\dot{\xi} = 2\beta\Delta\tau$, so that we get β as

$$\beta = \frac{G\Delta \mathbf{S} : \dot{\epsilon}}{2\Delta\tau^2(G + G^b + \tau'_y)}. \quad (7.197)$$

The same equation for work hardening, which has $\dot{\xi} = 2\beta\Delta\tau^2$, results in the expression for β given by

$$\beta = \frac{G\Delta \mathbf{S} : \dot{\epsilon}}{2\Delta\tau^2(G + G^b + \Delta\tau\tau'_y)}. \quad (7.198)$$

These expressions each can be followed in a particular deformation mode and expressions found for the model parameters in terms of the observed response. The kinematics and analysis of simple shear, uniaxial extension, and equal biaxial extension are presented in detail in the homework. From these one can show, for example, that under the assumption of strain hardening and during monotonic plastic flow, the expression for the hardening slope a_s in simple shear is given by

$$a_s = \frac{\dot{\sigma}_{12}}{\dot{\gamma}_{12}} = \frac{\dot{\sigma}_{12}}{2\dot{\epsilon}_{12}} = \frac{G(G^b + \tau'_y)}{G + G^b + \tau'_y}, \quad (7.199)$$

while the hardening slope a_t in uniaxial extension is given by

$$a_t = \frac{\dot{\sigma}_{11}}{\dot{\epsilon}_{11}} = \frac{9\kappa G(G^b + \tau'_y)}{3\kappa(G + G^b + \tau'_y) + G(G^b + \tau'_y)}, \quad (7.200)$$

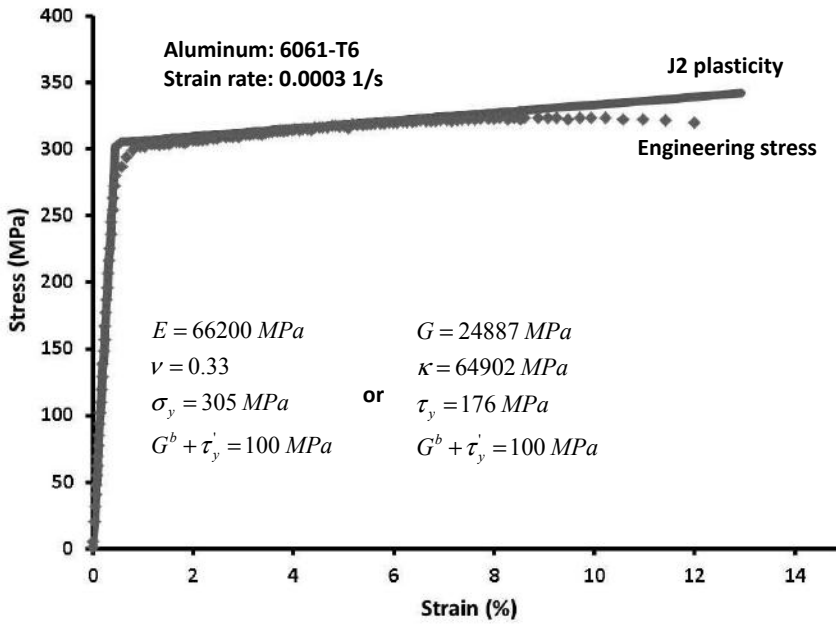


Figure 7.10: Comparison of a J2-plasticity model and measured response of a 6061-T6 aluminum sample in uniaxial extension.

and the hardening slope a_b in equal biaxial extension is given by

$$a_b = \frac{\dot{\sigma}_{11}}{\dot{\epsilon}_{11}} = \frac{18\kappa G(G^b + \tau_y')}{3\kappa(G + G^b + \tau_y') + 4G(G^b + \tau_y')}. \quad (7.201)$$

In this case, simple shear is in the 1-2 plane, uniaxial extension is along the 1-direction, and equal biaxial extension is in the 1-2 plane. An example of the modeling of the response of aluminum using this von Mises (J2-plasticity) model is shown in Figure 7.10. It can be shown that τ_y' is given by

$$\tau_y' = \frac{G(a_s - G^b) + a_s G^b}{G - a_s}, \quad (7.202)$$

$$\tau_y' = \frac{3a_t\kappa(G + G^b) + GG^b(a_t - 9\kappa)}{9\kappa G - a_t(3\kappa + G)}, \quad (7.203)$$

$$\tau_y' = \frac{3a_b\kappa(G + G^b) + 2GG^b(2a_b - 9\kappa)}{18\kappa G - a_b(3\kappa + 4G)}. \quad (7.204)$$

These can be used for obtaining the relation between measured parameters, such as a_t , and model parameters, such as τ_y' . For this yield function and the case of strain hardening, the relations obtained can be constructed in a uniform manner since, as shown in the homework, for each of the basic deformations the deviatoric overstress can be written as

$$\Delta S = \varpi M, \quad (7.205)$$

where ϖ is a scalar function and $\mathbf{M}$ is a constant second-order tensor. As shown in the problems, for simple shear we have

$$\varpi = \sigma_{12} - \sigma_{12}^b, \quad (7.206)$$

$$\mathbf{M} = (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (7.207)$$

for uniaxial extension we have

$$\varpi = \frac{1}{3}\sigma - \frac{1}{2}\sigma^b, \quad (7.208)$$

$$\mathbf{M} = 2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (7.209)$$

and for equal biaxial extension we have

$$\varpi = \frac{1}{3}\sigma - \sigma^b, \quad (7.210)$$

$$\mathbf{M} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (7.211)$$

If we recall that $\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}}^e$ and $\dot{\boldsymbol{\sigma}}^b = \mathbf{E}^b : \dot{\boldsymbol{\epsilon}}^p$, it can be shown from the consistency condition that the expression for β also can be written formally by the equation

$$\beta = \frac{\Delta \mathbf{S} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{\Delta \mathbf{S} : (\mathbf{E} + \mathbf{E}^b) : \Delta \mathbf{S} + 4\Delta\tau^2\tau_y'}, \quad (7.212)$$

where

$$\Delta\tau = \sqrt{\frac{1}{2}\Delta \mathbf{S} : \Delta \mathbf{S}} = |\varpi| \sqrt{\frac{1}{2}\mathbf{M} : \mathbf{M}}. \quad (7.213)$$

After substituting for $\Delta \mathbf{S} = \varpi \mathbf{M}$ into the expression for plastic flow we now find that

$$\dot{\boldsymbol{\epsilon}}^p = \beta \Delta \mathbf{S} = \frac{\mathbf{M} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{\mathbf{M} : (\mathbf{E} + \mathbf{E}^b) : \mathbf{M} + 4\tau_M^2\tau_y'} \mathbf{M}, \quad (7.214)$$

where $\tau_M = \sqrt{\frac{1}{2}\mathbf{M} : \mathbf{M}}$. We now can substitute this back into the expression for stress given by $\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}}^e = \mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p)$ to get

$$\dot{\boldsymbol{\sigma}} = \left[\mathbf{E} - \frac{(\mathbf{E} : \mathbf{M}) \otimes (\mathbf{M} : \mathbf{E})}{\mathbf{M} : (\mathbf{E} + \mathbf{E}^b) : \mathbf{M} + 4\tau_M^2\tau_y'} \right] : \dot{\boldsymbol{\epsilon}}. \quad (7.215)$$

We now solve this for the specific deformations and obtain the expressions shown above for the hardening slopes under the different deformation modes. For constant τ_y' , the coefficient of $\dot{\boldsymbol{\epsilon}}$ in this expression is constant, indicating a straight-line response in the stress-strain plot during hardening. As stated, these results are for strain hardening and, as the reader might assume, the form of the results will differ slightly if we assume work hardening.

The model we have developed for strain hardening results in a linear strain hardening curve when τ_y' is constant. We can extend this to more complex responses by assuming that the monotonic response considered is piecewise linear. This is easily achieved by defining the form of τ_y and its derivative for different values of the hardening parameter ξ . This extension to multiple linear segments is left as a problem.

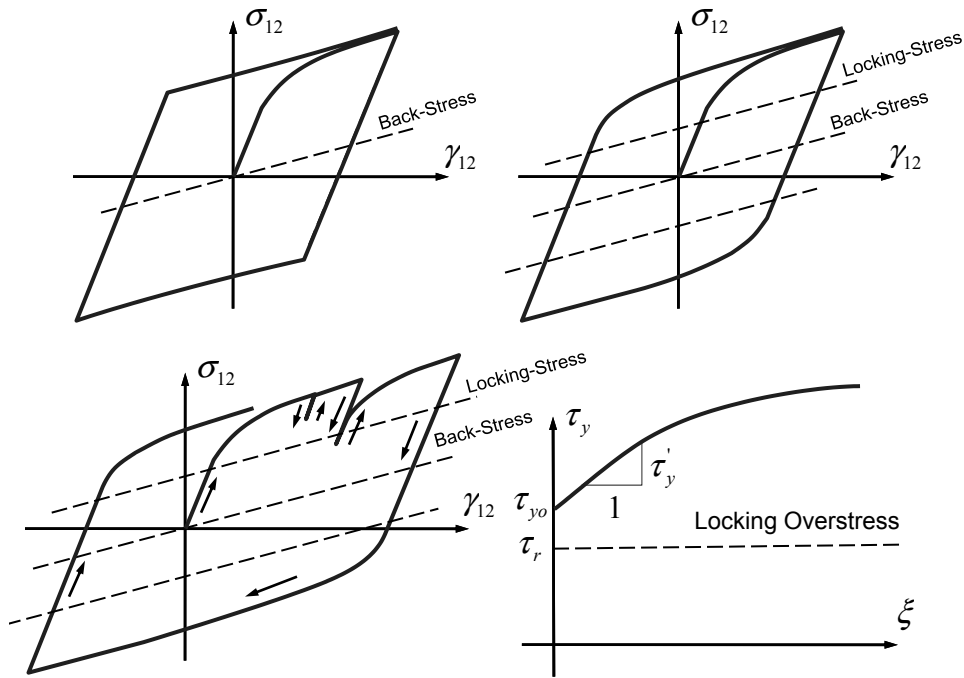


Figure 7.11: Examples of models with continuous hardening and models that have rejuvenation with unloading. Top left is a typical model that continuously hardens. The top right and bottom left show a model that includes rejuvenation after the load falls within a given distance from the back-stress. The bottom right shows the model characteristics for a rejuvenating model.

Another fairly common additional assumption is to assume that the initial yield point is actually a function of average stress. That is, we take the yield function to be

$$f = \Delta\tau - \tau_y(\sigma_{ave}, \xi), \quad (7.216)$$

where, for example, we can take

$$\tau_y(\sigma_{ave}, \xi) = \tau_{yo}(\sigma_{ave}) + \Delta\tau_y(\xi), \quad (7.217)$$

for appropriate functions τ_{yo} and $\Delta\tau_y$. For this function, we have

$$\dot{f} = \Delta\dot{\tau} - \tau'_{yo}\dot{\sigma}_{ave} - \Delta\tau'_y\dot{\xi}. \quad (7.218)$$

The final point to consider is that the hardening in all the models is progressive (monotonically increasing), but this need not be the case. For example, consider Figure 7.11. If the hardening is progressive, then what we get looks like the top left picture. In this case, the hardening parameter always increases, even in reverse yielding. On the other hand, the figure on the top right shows a material that returns to the initial hardening of zero once the applied load reaches a certain distance from the back-stress. This can in particular be

seen in the bottom left figure in which small unloading from the yield results in no change in the hardening, but large unloading returns the hardening parameter to zero, initiating a process resembling the initial yielding. This can be captured by the standard model shown in the bottom right corner with the added condition that if $\Delta\tau < \tau_r$, for a given locking stress τ_r , then ξ returns to its initial value of $\xi = 0$.

Section 7.7-Problems:

- 7.7-1:** A fairly general isotropic elastic material response is given by the expression for stress

$$\boldsymbol{\sigma} = \kappa \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2G \left[\boldsymbol{\epsilon}^e - \frac{1}{3} \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} \right], \quad (7.219)$$

the expression for back-stress given by

$$\boldsymbol{\sigma}^b = \kappa^b \text{tr}(\boldsymbol{\epsilon}^p) \mathbf{I} + 2G^b \left[\boldsymbol{\epsilon}^p - \frac{1}{3} \text{tr}(\boldsymbol{\epsilon}^p) \mathbf{I} \right], \quad (7.220)$$

and a flow rule

$$\dot{\boldsymbol{\epsilon}}^p = \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (7.221)$$

Without even knowing the yield function, show that the selection of the flow rule results in $\text{tr}(\boldsymbol{\epsilon}^p) = 0$ so that we always can write

$$\boldsymbol{\sigma}^b = 2G^b \boldsymbol{\epsilon}^p. \quad (7.222)$$

Also conclude that

$$\epsilon_v = \epsilon_v^e, \quad (7.223)$$

where $\epsilon_v = \text{tr}(\boldsymbol{\epsilon})$ is the volumetric strain and $\epsilon_v^e = \text{tr}(\boldsymbol{\epsilon}^e)$ is the elastic volumetric strain. As a corollary, for these models conclude that the plastic flow is volume preserving. Also, show that it will follow that

$$\boldsymbol{\sigma}^b = \mathbf{S}^b. \quad (7.224)$$

- 7.7-2:** Simple shear plays an important role in material characterization since it normally is understood that most plastic flow is a result of shear deformation. For this we consider a simple shear process. We know that such a process is characterized by a strain field of the form

$$\boldsymbol{\epsilon} = \frac{\gamma}{2} (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (7.225)$$

where γ is the total shear strain and $\mathbf{e}_i$ are orthonormal base vectors selected such that simple shear occurs in the first two directions. Consider the model given in the previous problem for an isotropic material and show that the expression for stress for simple shear always can be written as

$$\boldsymbol{\sigma} = 2G \boldsymbol{\epsilon}^e. \quad (7.226)$$

Conclude from this that the equation for stress before plastic flow always can be written by

$$\boldsymbol{\sigma} = \sigma_{12} (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (7.227)$$

Use this, the plastic flow rule, and the equation for back-stress to conclude that the diagonal terms of the plastic strain will remain constant so that in simple shear we always will have stress given by an expression of the form (7.227), the back-stress always is given by an equation of the form

$$\boldsymbol{\sigma}^b = \mathbf{S}^b = \sigma_{12}^b(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (7.228)$$

and that the plastic strain always is given by the form

$$\boldsymbol{\epsilon}^p = \epsilon_{12}^p(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (7.229)$$

For a constant second-order tensor $\mathbf{M}$ and scalar ϖ show that $\Delta\mathbf{S}$ can be written as $\Delta\mathbf{S} = \varpi\mathbf{M}$ for

$$\varpi = \sigma_{12} - \sigma_{12}^b, \quad (7.230)$$

$$\mathbf{M} = (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (7.231)$$

For this model and simple shear show that

$$\gamma^p = 2\epsilon_{12}^p, \quad (7.232)$$

$$\tau = \sigma_{12} = G(\gamma - \gamma^p), \quad (7.233)$$

$$\Delta\tau = G\gamma - (G + G^b)\gamma^p = |\varpi|, \quad (7.234)$$

$$\dot{\gamma}^p = 2\beta\Delta\tau. \quad (7.235)$$

7.7-3: Even though simple shear is an important deformation mode and motivates much of plastic flow, most material models are constructed based on tension and compression tests since they are “simpler.” In this problem, let us look at the same isotropic model considered in the first problem for the case of uniaxial extension. For an isotropic material, the basic assumption for uniaxial extension is that the axial strain ϵ is different from the transverse strain ϵ^* with the strain given by the expression

$$\boldsymbol{\epsilon} = \epsilon\mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon^*(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (7.236)$$

The stress $\boldsymbol{\sigma}$ is assumed to have only an axial component so that the stresses in the lateral directions are zero and the expression for stress can be written as

$$\boldsymbol{\sigma} = \sigma\mathbf{e}_1 \otimes \mathbf{e}_1. \quad (7.237)$$

It can be shown¹ that the form of the elastic and plastic strains must follow the form of the total strain so that

$$\boldsymbol{\epsilon}^e = \epsilon^e\mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon^{e*}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad (7.238)$$

$$\boldsymbol{\epsilon}^p = \epsilon^p\mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon^{p*}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (7.239)$$

Knowing the form of the plastic strain, from the flow rule show that

$$\epsilon^{p*} = -\frac{1}{2}\epsilon^p. \quad (7.240)$$

Use this to show that

$$\boldsymbol{\sigma}^b = \mathbf{S}^b = \sigma^b\mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{2}\sigma^b(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad (7.241)$$

¹This is based on representations of first gradient functionals.

where $\sigma^b = 2G^b\epsilon^p$. From this conclude that

$$\Delta \mathbf{S} = \left(\frac{2}{3}\sigma - \sigma^b\right)\mathbf{e}_1 \otimes \mathbf{e}_1 - \left(\frac{1}{3}\sigma - \frac{1}{2}\sigma^b\right)(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (7.242)$$

From this show that for a constant second-order tensor, $\mathbf{M}$ and scalar ϖ one can write $\Delta \mathbf{S} = \varpi \mathbf{M}$, where

$$\varpi = \frac{1}{3}\sigma - \frac{1}{2}\sigma^b, \quad (7.243)$$

$$\mathbf{M} = 2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (7.244)$$

For this model show that

$$\gamma^p = \sqrt{3}\epsilon^p, \quad (7.245)$$

$$\tau = \sqrt{\frac{1}{3}}\sigma, \quad (7.246)$$

$$\Delta \tau = \sqrt{3}\left|\frac{1}{3}\sigma - \frac{1}{2}\sigma^b\right| = \sqrt{3}|\varpi|, \quad (7.247)$$

$$\dot{\gamma}^p = 2\beta\Delta\tau. \quad (7.248)$$

Using the constitutive equation for stress and the assumptions on the state of stress and elastic strain, show that

$$\epsilon^e = \frac{3\kappa + G}{9\kappa G}\sigma, \quad (7.249)$$

$$\epsilon^{e*} = -\frac{3\kappa - 2G}{18\kappa G}\sigma. \quad (7.250)$$

From this show that the Poisson's ration ν is given by

$$\nu = -\frac{\epsilon^{e*}}{\epsilon^e} = \frac{3\kappa - 2G}{2(3\kappa + G)}. \quad (7.251)$$

Finally, show that the flow rule can be written as

$$\dot{\epsilon}^p = \beta\left(\frac{2}{3}\sigma - \sigma^b\right) = \beta\left(\frac{2}{3}\sigma - 2G^b\epsilon^p\right). \quad (7.252)$$

7.7-4: Biaxial extension is the next form of deformation that sometimes is used for characterizing material response. Let us look at this deformation in the context of the isotropic elastic model introduced in the first problem. The stress response in biaxial extension is taken to have the form

$$\boldsymbol{\sigma} = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2), \quad (7.253)$$

where σ is the stress applied on the material in the plane of biaxial extension, and $\mathbf{e}_i$ are orthonormal base vectors such that the plane of biaxial extension is in the plane of the first two directions. From this show that the deviatoric stress is given by

$$\mathbf{S} = \frac{1}{3}\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - \frac{2}{3}\sigma\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (7.254)$$

From this show that

$$\tau = \sqrt{\frac{1}{3}}\sigma. \quad (7.255)$$

It can be shown from representation theory that the form of the strain and plastic strain are given by

$$\boldsymbol{\epsilon} = \epsilon(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \epsilon^* \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (7.256)$$

$$\boldsymbol{\epsilon}^p = \epsilon^p(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \epsilon^{p*} \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (7.257)$$

where ϵ and ϵ^p are, respectively, the strain and plastic strain in the plane of biaxial extension and ϵ^* and ϵ^{p*} are, respectively, the strain and plastic strain perpendicular to the plane of biaxial extension. From this and the flow rule show that

$$\epsilon^{p*} = -2\epsilon^p. \quad (7.258)$$

Use the expression for back-stress to show that

$$\boldsymbol{\sigma}^b = \mathbf{S}^b = \sigma^b(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - 2\sigma^b \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (7.259)$$

where $\sigma^b = 2G^b \epsilon^p$. Use this to derive the expression

$$\Delta \mathbf{S} = \left(\frac{1}{3} \sigma - \sigma^b \right) (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3), \quad (7.260)$$

and from this show that for a constant second-order tensor $\mathbf{M}$ and scalar ϖ one can write $\Delta \mathbf{S} = \varpi \mathbf{M}$ where

$$\varpi = \frac{1}{3} \sigma - \sigma^b, \quad (7.261)$$

$$\mathbf{M} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (7.262)$$

From this obtain

$$\Delta \tau = \sqrt{3} \left| \frac{1}{3} \sigma - \sigma^b \right| = \sqrt{\frac{1}{3}} |\sigma - 3\sigma^b| = \sqrt{3} |\varpi|. \quad (7.263)$$

From the deviatoric overstress show that the expression for the flow rule is

$$\dot{\boldsymbol{\epsilon}}^p = \beta \left(\frac{1}{3} \sigma - \sigma^b \right) (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3) \quad (7.264)$$

when $f = 0$ and $\dot{f} > 0$. From this conclude that

$$\dot{\boldsymbol{\epsilon}}^p = \beta \left(\frac{1}{3} \sigma - \sigma^b \right) = \beta \left(\frac{1}{3} \sigma - 2G^b \epsilon^p \right). \quad (7.265)$$

Use the expression for the stress to show that

$$\boldsymbol{\epsilon}^e = \frac{3\kappa + 4G}{18\kappa G} \sigma, \quad (7.266)$$

$$\boldsymbol{\epsilon}^{e*} = -\frac{3\kappa - 2G}{9\kappa G} \sigma, \quad (7.267)$$

where $\boldsymbol{\epsilon}^e$ is the elastic strain in the plane of biaxial stretch and $\boldsymbol{\epsilon}^{e*}$ is the elastic strain perpendicular to this plane.

7.7-5: Use the expression

$$\dot{\boldsymbol{\sigma}} = [\mathbf{E} - \frac{(\mathbf{E} : \mathbf{M}) \otimes (\mathbf{M} : \mathbf{E})}{\mathbf{M} : (\mathbf{E} + \mathbf{E}^b) : \mathbf{M} + 4\tau_M^2 \tau_y'}] : \dot{\boldsymbol{\epsilon}} \quad (7.268)$$

to obtain the stress-strain response for each of the three common modes of deformation (simple shear, uniaxial extension, and equal biaxial extension).

7.7-6: Assume that the response of the material during monotonic flow in plastic loading is multilinear in the stress-strain response. Given the multilinear response in uniaxial stress-strain space, construct the associated model.

7.7-7: Consider the yield function given by

$$f = \Delta\tau - \tau_y, \quad (7.269)$$

where

$$\tau_y = \tau_{y\infty} - (\tau_{y\infty} - \tau_{yo})e^{-\frac{\xi}{\xi_o}} \quad (7.270)$$

for τ_{yo} representing the initial yield, $\tau_{y\infty}$ representing the steady state yield of the overstress, and where ξ_o is denoting the speed of conversion from τ_{yo} to $\tau_{y\infty}$. Draw the response of this model in shear, uniaxial extension, and equal biaxial extension. Extend this idea to multiple relaxation exponents. Discuss the use of Prony series for modeling complex response functions.

7.7-8: Consider a material with a yield function given by

$$f = \Delta\tau - \tau_y, \quad (7.271)$$

where

$$\tau_y = \tau_{yo} + (\tau_{y\infty} - \tau_{yo})\frac{A\xi}{1 + A\xi} \quad (7.272)$$

for τ_{yo} representing the initial yield, $\tau_{y\infty}$ representing the steady-state yield of the overstress, and where A denotes the speed of conversion from τ_{yo} to $\tau_{y\infty}$. Draw the response of this model in shear, uniaxial extension, and equal biaxial extension. Consider both monotonic and cyclic loading.

7.7-9: Consider a material that has a yield point that is linearly dependent on hydrostatic stress. Create a model for such a material and consider its response for a typical set of material parameters.

7.7-10: Consider a material that on unloading has a threshold equal to one-half the original yield function under which the hardening returns to zero hardening. Create such a model and consider its response for a typical set of material parameters.

7.8 Multiple elements in parallel

As one may imagine, the single element can be strung together either in parallel or in series to get multielement models. For example, Figure 7.12 shows a number of elements that are put in parallel. If we extend the number to infinity and use s as the variable that

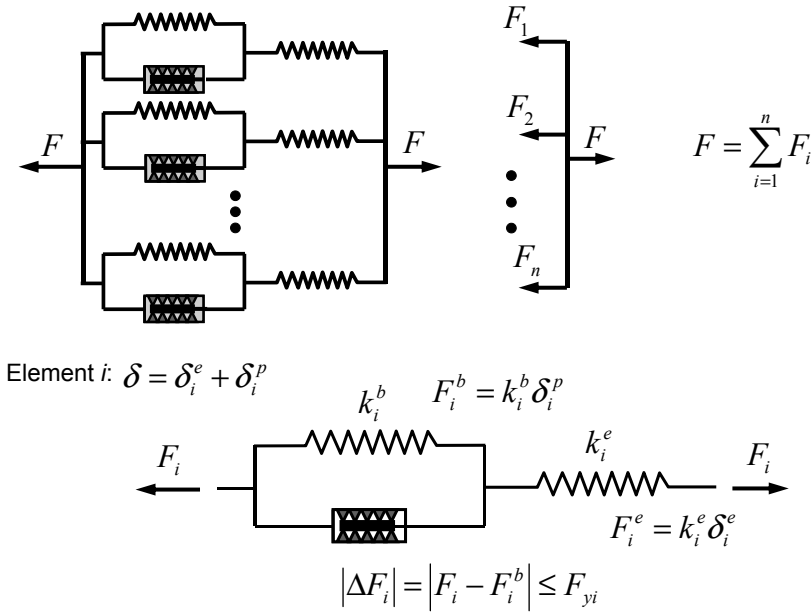


Figure 7.12: Mechanical analog for an elastic-plastic material with multiple hardening elements.

these are parameters by, we can assume the model to be of the form:

$$\begin{aligned}
 \epsilon &= \epsilon_s^e + \epsilon_s^p, \\
 \sigma &= \int_0^\infty \sigma_s ds, \\
 \sigma_s &= \mathbf{E}_s : \epsilon_s^e, \\
 \sigma_s^b &= \mathbf{E}_s^b : \epsilon_s^p, \\
 \mathbf{S}_s &= \sigma_s - \frac{\text{tr}(\sigma_s)}{3} \mathbf{I}, \\
 \mathbf{S}_s^b &= \sigma_s^b - \frac{\text{tr}(\sigma_s^b)}{3} \mathbf{I}, \\
 \Delta \mathbf{S}_s &= \mathbf{S}_s - \mathbf{S}_s^b, \\
 f_s &= \sqrt{\frac{3}{2} \Delta \mathbf{S}_s : \Delta \mathbf{S}_s} - \sigma_{ys}, \\
 \dot{\epsilon}_s^p &= \begin{cases} \beta_s \Delta \mathbf{S}_s & \text{if } f_s = 0 \text{ and } \dot{f}_s > 0, \\ 0 & \text{for all other cases.} \end{cases}
 \end{aligned}$$

As can be seen, we have assumed each element to have a stress-strain relation, back-stress plastic-strain relation, its own yield function, and plastic flow rule. As will be seen in the next section, an even simpler version of this can be used to describe hardening models that are approximately linear in each stage. In addition, we have the choice to take more complex yield functions, particularly for describing anisotropic materials, or to add isotropic hardening by making σ_{ys} be a function of a hardening parameter. Even though it is not required, the purpose of using multiple models suggests that one might want to construct the response from many simple elements so it seems reasonable to try and avoid more complex models. Here, we will take each σ_{ys} to be constant.

To complete such a model we have to provide an expression for β_s that follows the consistency condition. As all the derivatives of all the constants are zero, we can write $\dot{f}_s$ as

$$\dot{f}_s = \frac{3\Delta\mathbf{S}_s : \Delta\dot{\mathbf{S}}_s}{\sqrt{\frac{3}{2}\Delta\mathbf{S}_s : \Delta\mathbf{S}_s}}. \quad (7.273)$$

The consistency condition at $f_s = 0$ and $\dot{f}_s = 0$ during plastic flow requires that

$$\Delta\mathbf{S}_s : [\mathbf{E}_s : (\dot{\epsilon} - \dot{\epsilon}_s^p) - \mathbf{E}_s^b : \dot{\epsilon}_s^p] = 0, \quad (7.274)$$

where we have used the known substitution $\Delta\mathbf{S}_s : \Delta\mathbf{S}_s = \Delta\mathbf{S}_s : \Delta\boldsymbol{\sigma}_s$. During plastic flow, we can set $\dot{\epsilon}_s^p = \beta_s \Delta\mathbf{S}_s$ and solve for β_s to get

$$\beta_s = \frac{\Delta\mathbf{S}_s : \mathbf{E}_s : \dot{\epsilon}}{\Delta\mathbf{S}_s : (\mathbf{E}_s + \mathbf{E}_s^b) : \Delta\mathbf{S}_s}. \quad (7.275)$$

It can be seen from $\dot{f}_s$ that $\dot{f}_s^\diamond$ is given by

$$\dot{f}_s^\diamond = \frac{3\Delta\mathbf{S}_s : \mathbf{E}_s : \dot{\epsilon}}{\sqrt{\frac{3}{2}\Delta\mathbf{S}_s : \Delta\mathbf{S}_s}}. \quad (7.276)$$

The model now is complete. To use the model, the user must provide the spectrum of $\mathbf{E}_s$, $\mathbf{E}_s^b$, and f_s as a function of the parameter s . This normally is found from a microstructural model that separates the response into parts, depending on some variable s which signifies a relaxation in the material.

To obtain a temperature-dependent or thermodynamic model, we can make all the constants be temperature dependent and/or assume associated free energies. These are left as homework problems.

Section 7.8-Problems:

- 7.8-1:** Develop a temperature-dependent model similar to the model proposed by assuming $\mathbf{E}_s$, $\mathbf{E}_s^b$ and σ_{ys} are all functions of the temperature, and that $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^\theta + \boldsymbol{\epsilon}_s^e + \boldsymbol{\epsilon}_s^p$ where $\dot{\boldsymbol{\epsilon}}^\theta = \boldsymbol{\alpha}\dot{\theta}$, and where $\boldsymbol{\alpha}$ is a second-order tensor function of temperature.
- 7.8-2:** Develop a thermodynamic model that, in addition to the assumptions in the previous model, assumes a continuous array of free-energy functions ψ_s that depend on $\boldsymbol{\epsilon}_s^e$, $\boldsymbol{\epsilon}_s^p$, and θ .

7.8.1 A model made of perfectly plastic elements

Let us look at a system described with n elements, all in parallel, where each element is without a back-stress. This is shown in Figure 7.13. Let us also assume that the material is isotropic with yielding occurring only due to the deviatoric stress, and only resulting in deviatoric plastic strains. Essentially, we will use the model developed in the start of this section without back-stress and only using a finite number of elements. In this case, the

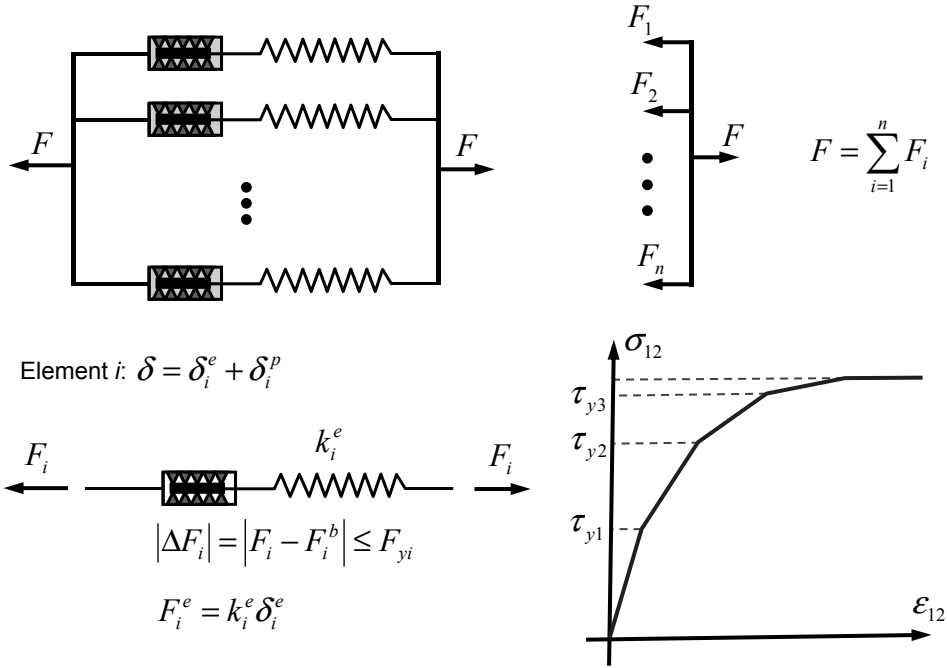


Figure 7.13: Mechanical analog for an elastic-plastic material with multiple perfectly plastic elements, with the resulting response in monotonic shear shown at the bottom right.

model for element m

$$\begin{aligned}
 \epsilon &= \epsilon_m^e + \epsilon_m^p, \\
 \sigma &= \sum_{m=1}^n \sigma_m \\
 \sigma_m &= \kappa^m \operatorname{tr}(\epsilon_m^e) \mathbf{I} + 2G^m [\epsilon_m^e - \frac{\operatorname{tr}(\epsilon_m^e)}{3} \mathbf{I}], \\
 \mathbf{S}_m &= \sigma_m - \frac{\operatorname{tr}(\sigma_m)}{3} \mathbf{I}, \\
 f_m &= \sqrt{\frac{3}{2} \mathbf{S}_m : \mathbf{S}_m} - \sigma_{ym}, \\
 \dot{\epsilon}_m^p &= \begin{cases} \beta_m \mathbf{S}_m & \text{if } f_m = 0 \text{ and } \dot{f}_m > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases}
 \end{aligned} \tag{7.277}$$

As the reader might imagine, as the load is increased, one by one the elements will reach their flow stress and then flow at constant stress, essentially removing their contribution from the tangent modulus. The result will be multiple linear segments, the first associated with the stiffness obtained by adding all the elastic stiffnesses together. If we assume the yield loads are organized, such that $\sigma_{y(i+1)} > \sigma_{y(i)}$, then during monotonic loading, the tangent modulus is reduced by the amount of the first modulus once the first element reaches its yield stress, and subsequently reduces by the modulus of the following elements once each element reaches its yield value. Finally, once all the elements have reached their yield values, the response becomes perfectly plastic. If one wants to avoid this case, it is possible to leave

mechanical analog for this material response is shown in Figure 7.14. Again, we assume that s represents a continuous parameter, normally associated with the relaxation time of the element used to parameterize the system. The general model will be

$$\begin{aligned}
 \epsilon &= \epsilon^e + \int_0^\infty \epsilon_s^p ds, \\
 \sigma &= \mathbf{E} : \epsilon^e, \\
 \sigma_s^b &= \mathbf{E}_s^b : \epsilon_s^p, \\
 \mathbf{S} &= \sigma - \frac{\text{tr}(\sigma)}{3} \mathbf{I}, \\
 \mathbf{S}_s^b &= \sigma_s^b - \frac{\text{tr}(\sigma_s^b)}{3} \mathbf{I}, \\
 \Delta \mathbf{S}_s &= \mathbf{S} - \mathbf{S}_s^b, \\
 f_s &= \sqrt{\frac{3}{2} \Delta \mathbf{S}_s : \Delta \mathbf{S}_s} - \sigma_{ys}, \\
 \dot{\epsilon}_s^p &= \begin{cases} \beta_s \Delta \mathbf{S}_s & \text{if } f_s = 0 \text{ and } \dot{f}_s > 0, \\ 0 & \text{for all other cases.} \end{cases}
 \end{aligned} \tag{7.278}$$

As for the case of multiple elements in parallel, there are several additions that can be done, including using a more complex yield function and flow rule. But, in general, these normally defeat the purpose. The idea is to have a simple plastic element and to obtain the complexity of the response through using many of them.

For this model, we need to calculate β_s and $\dot{f}_s$. The value of β_s is determined from the consistency condition. Before we calculate it, let us calculate $\dot{f}_s$ to get the equation

$$\dot{f}_s = \frac{3 \Delta \mathbf{S}_s : \Delta \dot{\mathbf{S}}_s}{\sqrt{\frac{3}{2} \Delta \mathbf{S}_s : \Delta \mathbf{S}_s}}. \tag{7.279}$$

The consistency condition is $\dot{f}_s = 0$, so we can write it as

$$\Delta \mathbf{S}_s : \Delta \dot{\sigma}_s = 0, \tag{7.280}$$

since $\Delta \mathbf{S}_s : \Delta \dot{\mathbf{S}}_s = \Delta \mathbf{S}_s : \Delta \dot{\sigma}_s$. At yield the denominator of $\dot{f}_s$ is equal to σ_{ys} , which is always greater than zero so that we need not consider it when we are looking at $\dot{f}_s = 0$. This consistency condition now can be written as

$$\Delta \mathbf{S}_s : (\dot{\sigma} - \mathbf{E}_s^b : \dot{\epsilon}_s^p) = 0. \tag{7.281}$$

We then use $\dot{\epsilon}_s^p = \beta_s \Delta \mathbf{S}_s$ and solve for β_s to get

$$\beta_s = \frac{\Delta \mathbf{S}_s : \dot{\sigma}}{\Delta \mathbf{S}_s : \mathbf{E}_s^b : \Delta \mathbf{S}_s} \tag{7.282}$$

for a given applied $\dot{\sigma}$. This is the same as

$$\beta_s = \frac{\Delta \mathbf{S}_s : \dot{\mathbf{S}}}{\Delta \mathbf{S}_s : \mathbf{E}_s^b : \Delta \mathbf{S}_s}. \tag{7.283}$$

This model is better suited for situations in which the applied stress is the controlling parameter. The total plastic strain rate $\dot{\epsilon}^p$ can be calculated from this model using the relation

$$\dot{\epsilon}^p = \int_0^\infty \dot{\epsilon}_s^p ds = \dot{\mathbf{S}} : \int_0^\infty \frac{\Delta \mathbf{S}_s \otimes \Delta \mathbf{S}_s}{\Delta \mathbf{S}_s : \mathbf{E}_s^b : \Delta \mathbf{S}_s} ds. \quad (7.284)$$

In calculating this integral, not all elements are active, so that the argument only should be integrated for the nonzero $\dot{\epsilon}_s^p$.

The user of this model will have to provide a spectra, in terms of s , of the back-stress moduli and yield functions. Extension to temperature-dependent plasticity can be achieved by adding a thermal strain and making all the constants temperature dependent. The thermodynamic theory is similar, but with free energies that provide as their derivatives the stress, back-stresses, and entropy. Details are left as homework problems.

Section 7.9-Problems:

- 7.9-1:** Construct a temperature-dependent mechanical theory assuming you have elements in series with the total strain given by

$$\epsilon = \epsilon^\theta + \epsilon^e + \epsilon^p, \quad (7.285)$$

where

$$\epsilon^\theta = \alpha \dot{\theta}, \quad (7.286)$$

and

$$\epsilon^p = \int_0^\infty \epsilon_s^p ds. \quad (7.287)$$

Assume a model as given above and assume that all constants, including α , depend on temperature.

- 7.9-2:** Construct a thermodynamically based model following the assumptions in the last problem, but starting with a free energy of the form $\psi = \psi^e + \psi^p$. In this expression, ψ^e depends on the elastic strain and temperature, and

$$\psi^p = \int_0^\infty \psi_s^p ds, \quad (7.288)$$

where ψ_s^p depends on ϵ_s^p and temperature.

7.9.1 A model made of a finite number of plastic elements

The simplest model one might imagine will have a finite number of plastic elements in series. The observed response is similar to that shown in the bottom right of Figure 7.13, even though the mechanical analog is different. As a result, one might imagine a different source for the observed response of the material. The model with n plastic elements in series and one elastic element is given by the model

$$\begin{aligned}
 \epsilon &= \epsilon^e + \epsilon^p, \\
 \epsilon^p &= \sum_{i=1}^n \epsilon_m^p, \\
 \sigma_m &= \sigma = \kappa \operatorname{tr}(\epsilon^e) \mathbf{I} + 2G[\epsilon^e - \frac{\operatorname{tr}(\epsilon^e)}{3} \mathbf{I}], \\
 \sigma_m^b &= 2G_m^b(\epsilon_m^p - \frac{\operatorname{tr}(\epsilon_m^p)}{3} \mathbf{I}), \\
 \mathbf{S}_m &= \mathbf{S} = \sigma - \frac{\operatorname{tr}(\sigma)}{3} \mathbf{I}, \\
 \mathbf{S}_m^b &= \sigma_m^b - \frac{\operatorname{tr}(\sigma_m^b)}{3} \mathbf{I}, \\
 \Delta \mathbf{S}_m &= \mathbf{S} - \mathbf{S}_m^b, \\
 f_m &= \sqrt{\frac{3}{2} \Delta \mathbf{S}_m : \Delta \mathbf{S}_m} - \sigma_{ym}, \\
 \dot{\epsilon}_m^p &= \begin{cases} \beta_m \Delta \mathbf{S}_m & \text{if } f_m = 0 \text{ and } \dot{f}_m > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases}
 \end{aligned} \tag{7.289}$$

As expected, the model is better suited for characterizing response that is controlled by the stress. The details of this model are left as homework.

Section 7.9-Problems:

- 7.9-3:** For the model made of n plastic elements in series and one elastic element, evaluate the model for β_m . Construct a model made of two plastic elements and one elastic element. Assume the elements are characterized by $\kappa = 20$, $G = 10$, $G_1^b = 5$, $G_2^b = 5$, $\sigma_{y1} = 2$, and $\sigma_{y2} = 5$. Consider both simple shear and uniaxial extension. Plot the response for monotonic loading and cyclic loading.
- 7.9-4:** For the previous problem, consider the response to equal biaxial loading.

7.10 Rate-dependent plasticity

Rate-dependent plasticity is actually simpler in construction than normal plasticity since we no longer enforce the consistency condition. In this case, the yield function is the boundary between conditions that have no change in plastic strain and conditions for which we have plastic flow. A typical model for rate-dependent plasticity is given as

$$\begin{aligned}
\boldsymbol{\epsilon} &= \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p, \\
\boldsymbol{\sigma} &= \kappa \operatorname{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2G[\boldsymbol{\epsilon}^e - \frac{\operatorname{tr}(\boldsymbol{\epsilon}^e)}{3} \mathbf{I}], \\
\boldsymbol{\sigma}^b &= 2G^b[\boldsymbol{\epsilon}^p - \frac{\operatorname{tr}(\boldsymbol{\epsilon}^p)}{3} \mathbf{I}], \\
\mathbf{S} &= \boldsymbol{\sigma} - \frac{\operatorname{tr}(\boldsymbol{\sigma})}{3} \mathbf{I}, \\
\mathbf{S}^b &= \boldsymbol{\sigma}^b - \frac{\operatorname{tr}(\boldsymbol{\sigma}^b)}{3} \mathbf{I}, \\
\Delta \mathbf{S} &= \mathbf{S} - \mathbf{S}^b, \\
f &= \sqrt{\frac{3}{2} \Delta \mathbf{S} : \Delta \mathbf{S}} - \sigma_y, \\
\dot{\boldsymbol{\epsilon}}^p &= \begin{cases} \beta \Delta \mathbf{S} & \text{if } f > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases}
\end{aligned}$$

where

$$\beta = \frac{1}{\mu} \left(\frac{f}{\sigma_y} \right)^n, \quad (7.290)$$

where as in the first chapter μ is an appropriate viscosity and n is an exponent that determines the strength of the rate dependence. As can be seen from the model, there is no plastic flow when the yield function is negative. The typical response for this model in shear is as shown in Figure 1.28. Plastic flow occurs when the yield function becomes positive, being in the direction of the deviatoric overstress, and proportional to the value of the yield function to a given power. In this expression, μ and n are material parameters that depend on the specific yield function selected. As with the other models, one can attach bells and whistles to make the yield function or β more complex. More details are provided in Chapter 14.

7.11 Deformation plasticity

As one might expect, there are ways to make an elasticity model not only take the shape of the initial elastic response in monotonic loading but also bend at the yield stress and follow the postyield hardening seen in monotonic loading. Such models are elastic response models so they will unload on the same loading path. As a result, they are not suited for problems that include unloading after plastic flow but can be very effective and efficient when solving problems that only include loading of material points. Most numerical methods will provide such an option for the users since the solutions take a fraction of the time needed to solve the full plasticity problems and are much more robust since an exact measure of error can be established. In computational analysis, these models normally are used to solve for plastic flow around cracks and for buckling analysis.

One of the most common plasticity-mimicking elastic models is the *deformation plasticity* model. The general form of this model is given by

$$\boldsymbol{\epsilon} = \frac{1-2\nu}{E} \sigma_{ave} \mathbf{I} + \frac{1+\nu}{E} \mathbf{S} + \frac{3\alpha}{2E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \mathbf{S}, \quad (7.291)$$

where E is the elastic Young's modulus, ν is Poisson's ratio, α is a scaling factor known as the yield offset, and n is the hardening exponent. The hardening exponent n is greater than 1, typically taken to be 5 or larger. For this model, when the equivalent shear stress $\tau = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}}$ is substantially below the yield stress τ_y , the last term on the right-hand side is

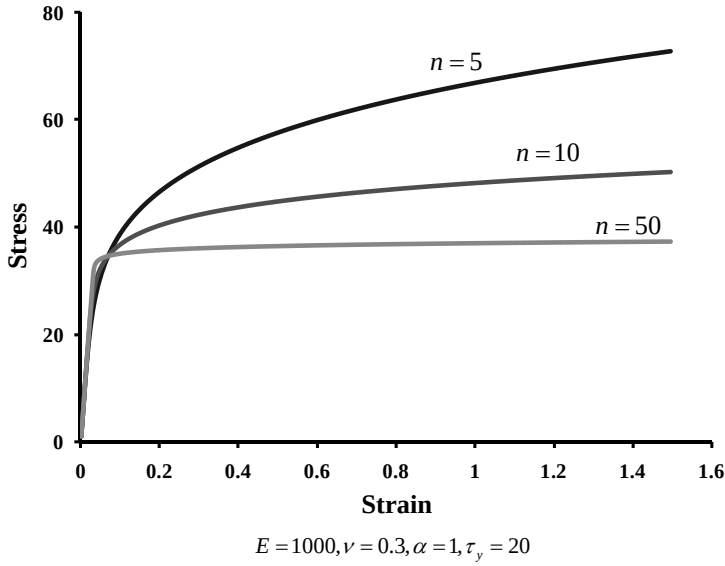


Figure 7.15: Uniaxial extension for a deformation plasticity model with the parameters $E = 1000$, $\nu = 0.3$, $\alpha = 1$, $n = 5, 10, 50$, and $\tau_y = 20$ ($\sigma_y = 35$).

approximately zero and one gets the expected isotropic elastic response. Once the equivalent shear stress τ gets close to and exceeds the yield stress τ_y in shear, the last term on the right-hand side becomes significant and produces large strains, and the model mimics the postyield hardening seen in plastically flowing materials.

Let us consider the response of the model to simple deformations. First, let us consider uniaxial extension along the $\hat{\mathbf{e}}_1$ direction, where $\hat{\mathbf{e}}_i$ are orthonormal base vectors. For this case, the stress is given by $\boldsymbol{\sigma} = \sigma \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1$, so that

$$\sigma_{ave} = \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) = \frac{1}{3} \sigma, \quad (7.292)$$

$$\mathbf{S} = \boldsymbol{\sigma} - \sigma_{ave} \mathbf{I} = \frac{2}{3} \sigma \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 - \frac{1}{3} \sigma (\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3), \quad (7.293)$$

$$\tau = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}} = \sqrt{\frac{1}{3}} \sigma. \quad (7.294)$$

As a result, the expression for strain will be given by

$$\boldsymbol{\epsilon} = \frac{1-2\nu}{3E} \sigma \mathbf{I} + \left[\frac{1+\nu}{E} + \frac{3\alpha}{2E} \left(\frac{\sigma}{\sqrt{3}\tau_y} \right)^{n-1} \right] \times \frac{1}{3} \sigma [2\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 - (\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3)]. \quad (7.295)$$

If we focus on the $\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2$ directions, we get the axial strain $\epsilon_a = \epsilon_{11}$ and the transverse strain $\epsilon_t = \epsilon_{22}$ given as

$$\epsilon_a = \frac{1}{E} \sigma + \frac{\alpha}{E} \left(\frac{\sigma}{\sqrt{3}\tau_y} \right)^{n-1} \sigma, \quad (7.296)$$

$$\epsilon_t = \frac{1-\nu}{E} \sigma + \frac{\alpha}{2E} \left(\frac{\sigma}{\sqrt{3}\tau_y} \right)^{n-1} \sigma. \quad (7.297)$$

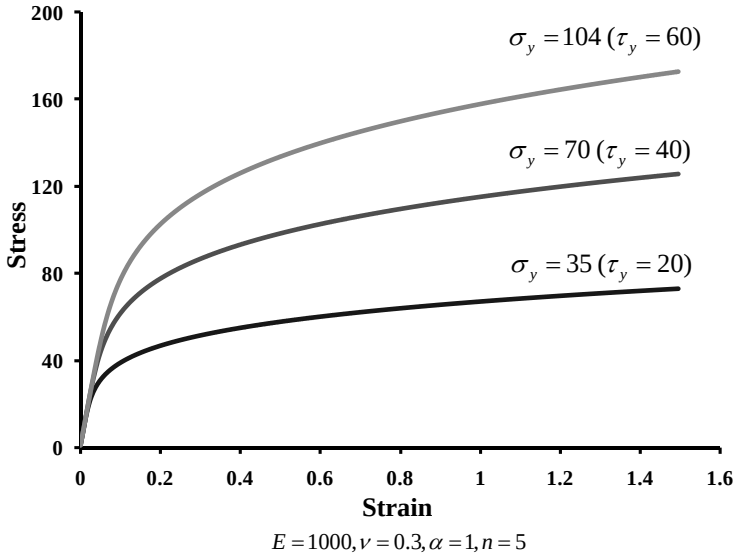


Figure 7.16: Uniaxial extension for a deformation plasticity model with the parameters $E = 1000$, $\nu = 0.3$, $\alpha = 1$, $n = 5$, and $\tau_y = 20, 40, 60$ ($\sigma_y = 35, 70, 104$).

This response is shown for different parameters in Figures 7.15 and 7.16.

For the case of pure shear, we have $\boldsymbol{\sigma} = \tau(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1)$, which yields the expressions

$$\sigma_{ave} = 0, \quad (7.298)$$

$$\mathbf{S} = \tau(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1). \quad (7.299)$$

The resulting strain is given by

$$\boldsymbol{\epsilon} = \left[\frac{1+\nu}{E} + \frac{3\alpha}{2E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \right] \times \tau(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1). \quad (7.300)$$

From this we can evaluate the shear $\gamma = 2\epsilon_{12}$ as

$$\begin{aligned} \gamma &= \left[\frac{2(1+\nu)}{E} + \frac{3\alpha}{E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \right] \tau \\ &= \left[\frac{1}{G} + \frac{3\alpha}{E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \right] \tau, \end{aligned} \quad (7.301)$$

since the shear modulus G is given by $G = \frac{E}{2(1+\nu)}$. The response will be similar to that of uniaxial extension shown in Figures 7.15 and 7.16..

An example of a deformation plasticity model constructed to capture the response of aluminum is shown in Figure 7.17. As can be seen, even though the model cannot reproduce unloading, the deformation plasticity model closely can model the monotonic extension of the material.

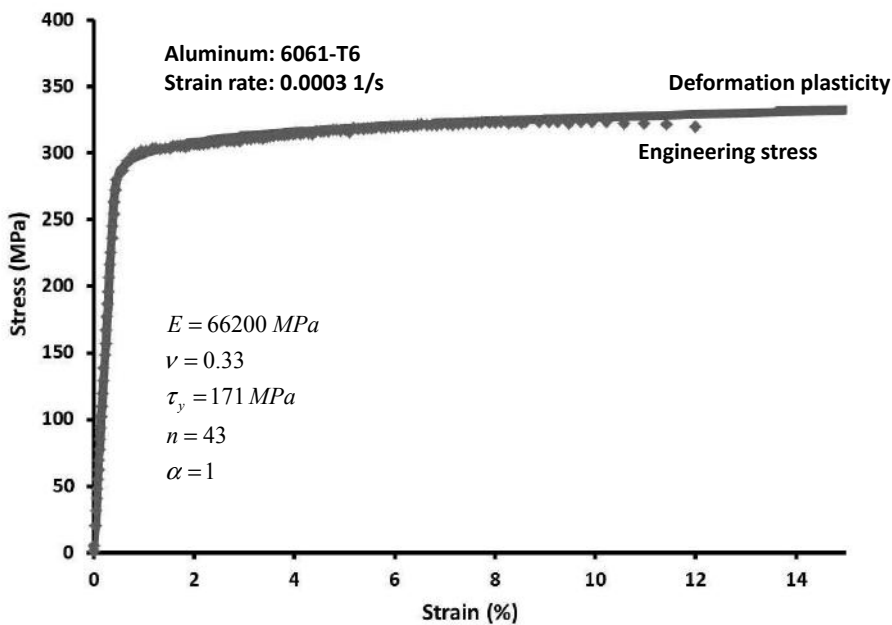


Figure 7.17: Comparison of a deformation plasticity model and measured response of a 6061-T6 aluminum sample in uniaxial extension.

7.12 Notation

The following table shows the symbols used in the text and their definitions.

" $\diamond$ "		trial time derivative taken holding plastic strain constant
a_b		hardening slope in biaxial extension
a_s		hardening slope in shear
a_t		hardening slope in tension
c_i		material parameters
$\mathbf{C}^p$		plastic right Cauchy stretch
$\mathbf{C}$		plastic flow material tensor
d		dissipation ($-\rho \dot{h}$)
ΔF		overload ($F - F^b$)
$\Delta \mathbf{S}$		deviatoric part of overstress
e		elastic (superscript)
E		elastic modulus
$\mathbf{E}_i$		base tensors for representing symmetry
$\mathbf{E}$		elastic modulus tensor
$\mathbf{E}^b$		back-stress modulus tensor
$\mathbf{E}_s$		elastic modulus tensor for element with parameter s

$\mathbf{E}_s^b$		back-stress modulus tensor for element with parameter s
f		yield function
f_s		yield function for element with parameter s
F		load
F^b		back-stress spring load
F_{yo}		initial yield load
G		shear modulus
G^b		shear back-stress modulus
$\mathbf{G}$		temperature gradient
$\dot{h}$		rate of heat flowing into a point ($r - \frac{1}{\rho} \text{div}(\mathbf{q})$)
$\dot{H}$		dissipation rate
$\hat{\mathbf{h}}_i$		material directions
J_{2S}		second invariant J_2 of $\mathbf{S}$
k^e		elastic spring modulus
k^{equiv}		equivalent quasi-equilibrium spring modulus
k^b		back-stress spring modulus
$\mathbf{K}$		coefficient of thermal conduction
$\mathbf{M}$		deviatoric overstress base tensor in proportional loading
n		flow nonlinearity exponent or hardening exponent
p		plastic (superscript)
$\mathbf{q}_o$		nominal heat flux vector
S_i		principal values of $\mathbf{S}$
$\mathbf{S}$		deviatoric stress ($\boldsymbol{\sigma} - \sigma_{ave} \mathbf{I}$)
$\mathbf{S}^b$		deviatoric back-stress ($\boldsymbol{\sigma}^b - \sigma_{ave}^b \mathbf{I}$)
$\mathbf{S}_s$		deviatoric “stress intensity” in element with parameter s
$\mathbf{S}_s^b$		deviatoric “back-stress intensity” in element with parameter s
t		time
$\mathbf{X}$		position in reference configuration
α		linear coefficient of thermal expansion, or scaling factor
α_i		thermal expansion parameters
$\boldsymbol{\alpha}$		coefficient of thermal expansion
β		plastic flow scaling parameter
δ		extension
δ^e		elastic extension
δ^p		plastic extension
δ_{ij}		Kronecker delta
$\Delta\boldsymbol{\sigma}$		overstress ($\boldsymbol{\sigma} - \boldsymbol{\sigma}_c$)
$\Delta\tau$		shear overstress or equivalent shear overstress
ϵ^p		equivalent plastic strain
ϵ_v		volumetric strain ($tr(\boldsymbol{\epsilon})$)
$\boldsymbol{\epsilon}$		infinitesimal strain
$\boldsymbol{\epsilon}^e$		infinitesimal elastic strain
$\boldsymbol{\epsilon}^p$		infinitesimal plastic strain
$\boldsymbol{\epsilon}_s^e$		infinitesimal elastic strain for element with parameter s
$\boldsymbol{\epsilon}_s^p$		infinitesimal plastic strain for element with parameter s
$\boldsymbol{\epsilon}^\theta$		infinitesimal thermal strain
η		specific entropy
γ		shear strain
γ^p		plastic shear strain
κ		bulk modulus
κ^b		back-stress bulk modulus

λ		first Lamé constant
μ		second Lamé constant (same as G)
μ_i		material parameters
ν		Poisson's ratio
ψ		specific free energy
ψ^θ		purely thermal part of specific free energy
ρ_o		nominal density
σ		axial stress
σ_{ave}		average stress ($\frac{tr(\sigma)}{3}$)
σ_i		principal values of stress
σ_{iso}		isotropic hardening yield stress
σ_y		yield stress in tension
σ_{yo}		initial yield stress in tension
σ		stress
σ^e		elastic stress
σ^b		back-stress
σ_c		back-stress
σ_s		"stress intensity" in element with parameter s
σ_s^b		"back-stress intensity" in element with parameter s
σ^θ		thermal back-stress
τ		shear stress or equivalent shear stress
τ^b		equivalent shear back-stress
τ_c		shear back-stress
τ_{max}		maximum shear stress
τ_y		yield stress in shear
τ_y'		hardening derivative of equivalent shear yield stress
τ_{yo}		initial yield stress in shear
θ		temperature
ϖ		scalar scaling factor for deviatoric overstress in proportional loading
ξ		hardening parameter

Chapter 8

Solutions for infinitesimal plasticity

In this chapter, we will look at some solutions associated with infinitesimal plasticity. The reader is directed to the books of Hill, Kachanov, and Skrzypek for more details and problems. These solutions are associated primarily with isotropic material response, even though at times we discuss more complex situations. The reader should be warned that there are only a limited number of such solutions and they are associated with very specific material responses. This is because, unlike linear elasticity, which has a simple and unique expression, linear plasticity comes in many flavors and, as a result, provides too complex a structure to obtain general solutions. In most cases, numerical methods are preferred for solving plasticity problems, particularly with the current computational methods available.

In the following, other than when specified, we consider a mechanical model that is isotropic with the stress given by

$$\boldsymbol{\sigma} = \left(\kappa - \frac{2}{3}G\right)tr(\boldsymbol{\epsilon}^e)\mathbf{I} + 2G\boldsymbol{\epsilon}^e, \quad (8.1)$$

where $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p$. The inverse of this equation can be shown to be given as

$$\begin{aligned} \boldsymbol{\epsilon}^e &= -\frac{3\kappa - 2G}{18\kappa G}tr(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\boldsymbol{\sigma} \\ &= \frac{1}{3\kappa}\sigma_{ave}\mathbf{I} + \frac{1}{2G}\mathbf{S} \\ &= \frac{1 - 2\nu}{E}\sigma_{ave}\mathbf{I} + \frac{1 + \nu}{E}\mathbf{S}. \end{aligned} \quad (8.2)$$

The yield function in general is taken to have the form of the von Mises yield function, and is given for isotropic hardening by

$$f = \tau - \tau_y, \quad (8.3)$$

where $\tau = \sqrt{\frac{1}{2}\mathbf{S} : \mathbf{S}}$, and for kinematic hardening or combined isotropic and kinematic hardening is given by

$$f = \Delta\tau - \tau_y, \quad (8.4)$$

for $\Delta\tau = \sqrt{\frac{1}{2}\Delta\mathbf{S} : \Delta\mathbf{S}}$, and where $\tau_y = \tau_{y0} + \tau'_y\xi$ for a strain hardening parameter ξ is defined as

$$\dot{\xi} = \sqrt{2\dot{\boldsymbol{\epsilon}}^p : \dot{\boldsymbol{\epsilon}}^p} = 2\beta\tau, \quad (8.5)$$

for isotropic hardening with

$$\beta = \frac{G\mathbf{S} : \dot{\boldsymbol{\epsilon}}}{2\tau^2(G + G^b + \tau'_y)}, \quad (8.6)$$

and for kinematic hardening or combined isotropic and kinematic hardening is defined as

$$\dot{\xi} = \sqrt{2\dot{\boldsymbol{\epsilon}}^p : \dot{\boldsymbol{\epsilon}}^p} = 2\beta\Delta\tau, \quad (8.7)$$

with

$$\beta = \frac{G\Delta\mathbf{S} : \dot{\boldsymbol{\epsilon}}}{2\Delta\tau^2(G + G^b + \tau'_y)}. \quad (8.8)$$

The flow rule used to consider only isotropic hardening is taken to be

$$\dot{\epsilon}^p = \begin{cases} \beta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (8.9)$$

where

$$\dot{f} = \dot{\tau} = \frac{G}{\tau} \mathbf{S} : \dot{\epsilon}. \quad (8.10)$$

When considering combined isotropic and kinematic hardening, we use the flow rule

$$\dot{\epsilon}^p = \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (8.11)$$

where

$$\dot{f} = \Delta \dot{\tau} = \frac{G}{\Delta \tau} \Delta \mathbf{S} : \dot{\epsilon}. \quad (8.12)$$

For all the mechanical simulations, other than when specified, the reader is to use the parameters

$$\begin{aligned} G &= 300, & \kappa &= 800, & G^b &= 30, \\ \tau_{yo} &= 30, & \tau_y' &= 10. \end{aligned} \quad (8.13)$$

If we consider temperature effects, we will assume

$$\dot{\epsilon}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (8.14)$$

where $\epsilon = \epsilon^e + \epsilon^p + \epsilon^\theta$. In this case, each of the elastic and plastic material “constants” can be considered temperature dependent. For simplicity, we will assume these constants to have linear dependence on temperature and to be given by

$$G = \bar{G} + G' \Delta \theta, \quad (8.15)$$

$$\kappa = \bar{\kappa} + \kappa' \Delta \theta, \quad (8.16)$$

$$G^b = \bar{G}^b + G^{b'} \Delta \theta, \quad (8.17)$$

$$\tau_y = \bar{\tau}_{yo} + \frac{\partial \tau_y}{\partial \theta} \Delta \theta + \frac{\partial \tau_y}{\partial \xi} \xi + \frac{\partial \tau_y}{\partial \sigma_{ave}} \sigma_{ave}, \quad (8.18)$$

where the bar over a character designates the value of the parameter evaluated at the reference temperature $\bar{\theta}$, and in this expression we have $\Delta \theta = \theta - \bar{\theta}$. The value of $\bar{\tau}_{yo}$ is evaluated at the reference temperature, at zero hardening and at zero average stress. The coefficients are assumed to be temperature independent. For all the simulations, other than when specified, the reader should use the parameters

$$\begin{aligned} \bar{G} &= 300, & G' &= -10, \\ \bar{\kappa} &= 800, & \kappa' &= -10, \\ \bar{G}^b &= 30, & G^{b'} &= -1, \\ \bar{\tau}_{yo} &= 30, & \frac{\partial \tau_y}{\partial \theta} &= -1, & \frac{\partial \tau_y}{\partial \xi} &= 10, & \frac{\partial \tau_y}{\partial \sigma_{ave}} &= -1, \\ \alpha &= 0.01, & \bar{\theta} &= 0. \end{aligned} \quad (8.19)$$

In each case, when not prescribed, we will use an orthonormal set of base vectors $\mathbf{e}_i$, and use the notation

$$\boldsymbol{\sigma} = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad (8.20)$$

$$\boldsymbol{\epsilon} = \epsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad (8.21)$$

$$\boldsymbol{\epsilon}^e = \epsilon_{ij}^e \mathbf{e}_i \otimes \mathbf{e}_j, \quad (8.22)$$

$$\boldsymbol{\epsilon}^p = \epsilon_{ij}^p \mathbf{e}_i \otimes \mathbf{e}_j, \quad (8.23)$$

$$\boldsymbol{\epsilon}^\theta = \epsilon_{ij}^\theta \mathbf{e}_i \otimes \mathbf{e}_j. \quad (8.24)$$

In general, we look at several classes of problems. First, we look at homogeneous problems, some of which we already have considered in the last chapter. We follow this with several inhomogeneous deformations, including unloading after plastic flow, which is a particular feature of plasticity. We also consider steady-state and dynamic problems, concluding with issues related to heat generation. We will avoid one-dimensional problems for the axial motion in a bar since we already have looked at similar problems in Chapter 1. These problems cover approximately all types of problems from homogeneous to nonhomogeneous, and static to dynamic deformations. The summary of the types of problems considered here is as follows:

1. Homogeneous problems
 - (a) Equal triaxial
 - (b) Pure shear
 - (c) Uniaxial tension/compression
 - (d) Equal biaxial extension
 - (e) Torsion-extension of a thin section (combined extension and shear)
 - (f) Compression in plane strain (confined compression)
2. Inhomogeneous problems defined by deformation and using simplified constitutive assumptions
 - (a) Bending (constructed from elements in uniaxial extension)
 - (b) Torsion (constructed from elements in pure shear)
 - (c) Unloading after loading (development of residual stresses)
 - (d) Torsion of prismatic sections (warping function and membrane and rooftop analogies)
 - (e) Solutions using perfect plasticity (approximate limit cases)
3. Nonuniform loading of bars along the length
 - (a) Equilibrium problems
 - (b) Problems with small rotational inertia terms
 - (c) Statically determinate terms (problems that separate in the solution process)
 - (d) A beam rotating around one end (steady-state dynamic problem)
4. Cylindrical and spherical symmetry
 - (a) Pressurized cylinder (statically indeterminate problem)
 - (b) Rotating disks (statically indeterminate problem)

5. Full two-dimensional problems
 - (a) Plane stress
 - (b) Plane strain
6. Heat generation

8.0.1 Simulation of response

We can use the simulation tools developed in Chapter 1 for following the response in one dimension for studying the response under homogeneous conditions. Any of the proposed integration methods, given sufficiently small steps, should result in fairly good simulations, even though they are not optimized to achieve the desired accuracy in the minimum number of steps.

In the case of nonhomogeneous deformations for prescribed deformation fields, such as seen in bending and torsion, the problem can be evaluated at a finite number of “nodes” (using the terminology of the finite element method) and interpolated between the nodes. A node is a point in space at which we are following the development of the material response. For each node, we can use separately the integration methods discussed for homogeneous conditions. The implementation requires that we separately keep track of the time/loading step and the location. If we use “ q ” to denote node number and “ i ” to denote loading step, then any variable U can be considered at node “ q ” to be a function $U(q, i)$.

The simplest integration method for such a variable is a one-term Taylor series expansion where

$$U(q, i + 1) = U(q, i) + \dot{U}(q, i)\Delta t, \quad (8.25)$$

which gives the new value of U in terms of the old value and its derivative at the old time, and the time increment Δt . In numerical methods, we have many more complicated ways of calculating this that also provide much more accuracy, but the proposed method will do for our purpose and will work as long as the time step is small enough, which is not an issue for the problems we consider. The reader is directed to books on numerical methods for more accurate and robust integration methods.

Once we have calculated the response at n nodal locations, we can use standard interpolations to get the values between nodes. For example, if we put node q at point X_q , for nodes 1 to n consecutively arranged such that $X_q > X_{q+1}$, we can interpolate the value of U at X between two nodes q and $q + 1$ using the shape functions of the two node line elements (see shape functions) so that the value of U at X at time t_i associated with load step i is denoted by $U(X, t_i)$ and given by

$$U(X, t_i) = U(q, i) \frac{X - X_{q+1}}{X_q - X_{q+1}} + U(q + 1, i) \frac{X - X_q}{X_{q+1} - X_q}. \quad (8.26)$$

Nonlinear interpolations also can be made using more complex methods; the easiest would be using Lagrange shape functions for multiple-node elements. For our use, this two-node interpolation is sufficient.

8.1 Homogeneous deformations

We will start by looking at several common homogeneous problems. These include equal triaxial stress, pure shear, uniaxial tension/compression, and equal biaxial tension/compression. These are shown in Figure 8.1. We also will consider in the two following

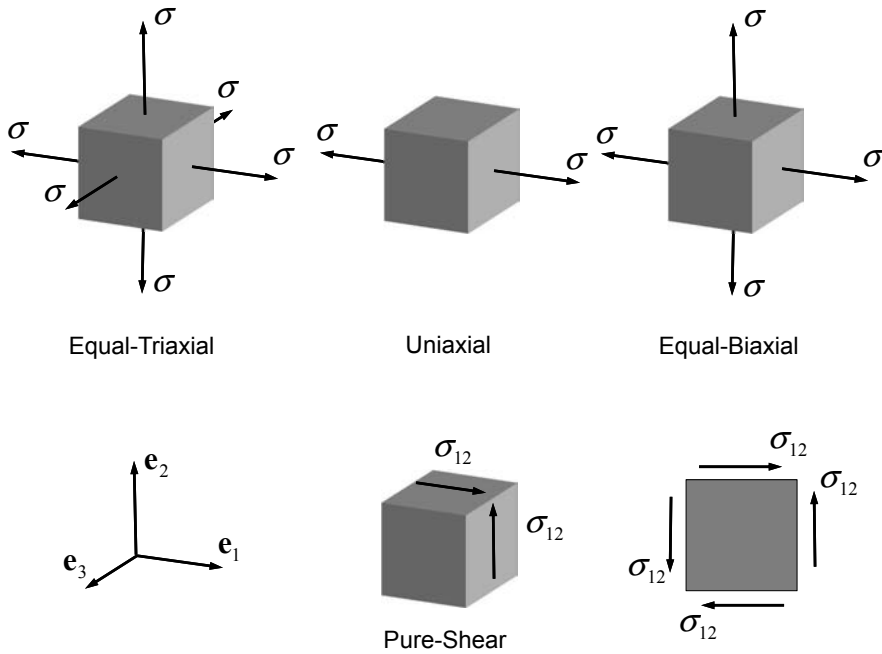


Figure 8.1: Basic homogeneous stress fields, including equal-triaxial, uniaxial, equal-biaxial, and pure-shear stresses.

sections homogeneous deformations of special interest. To make the presentation simple, *we will avoid kinematic hardening* by setting $G^b = 0$, but most results also can be shown with kinematic hardening. The user is expected to include kinematic hardening when doing simulations.

8.1.1 Equal triaxial stress

For the case of equal triaxial stress, we know that the expression for stress is given by

$$\boldsymbol{\sigma} = \sigma \mathbf{I}, \quad (8.27)$$

so that the average stress is given by

$$\sigma_{ave} = \sigma, \quad (8.28)$$

and the deviatoric stress is given by

$$\mathbf{S} = \mathbf{0}. \quad (8.29)$$

Combining this with the plastic flow rule of $\dot{\boldsymbol{\epsilon}}^p = \beta \mathbf{S}$, we conclude that there can be no plastic flow under this condition. Therefore, the response is given by $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^\theta$ where $\dot{\boldsymbol{\epsilon}}^\theta = \alpha \dot{\theta} \mathbf{I}$ and $\boldsymbol{\epsilon}^e$ is given by

$$\boldsymbol{\epsilon}^e = \frac{1}{3\kappa} \sigma \mathbf{I}. \quad (8.30)$$

Taking the trace of this equation, we see that $\sigma = \kappa \operatorname{tr}(\boldsymbol{\epsilon}^e) = \kappa \epsilon_v^e$.

8.1.2 Pure shear

As described in Chapter 5, pure shear is characterized by a stress field of the form

$$\boldsymbol{\sigma} = \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (8.31)$$

where the shear occurs in the 1-2 plane. As a result, for pure shear the average stress is zero, and, therefore, the elastic strain is given by

$$\boldsymbol{\epsilon}^e = \frac{1}{2G}\boldsymbol{\sigma}, \quad (8.32)$$

which results in

$$\dot{\epsilon}_{12}^e = \frac{\dot{\sigma}_{12}}{2G}. \quad (8.33)$$

It follows from the form of the stress that the initial plastic strain rate must be of the form

$$\dot{\epsilon}^p = \dot{\epsilon}_{12}^p(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (8.34)$$

It then follows that the back-stress will develop only terms similar to this, resulting in the fact that the plastic strain rate always will be of this form. From this we can conclude that the strain rate also has a similar form. Therefore, the only nonzero terms in the case of pure shear are the 1-2 and 2-1 terms. It follows from the equations that

$$\dot{\epsilon}_{12}^p = \frac{\dot{\sigma}_{12}}{2(G^b + \tau_y')}. \quad (8.35)$$

The total strain during plastic flow is then given by $\dot{\epsilon}_{12} = \dot{\epsilon}_{12}^e + \dot{\epsilon}_{12}^p$ to yield

$$\dot{\epsilon}_{12} = \frac{G + G^b + \tau_y'}{2G(G^b + \tau_y')} \dot{\sigma}_{12}. \quad (8.36)$$

Section 8.1-Problems:

- 8.1-1:** Show that for infinitesimal mechanical loading the cases of pure shear and simple shear become the same.
- 8.1-2:** Simulate the mechanical response of a material during pure shear to monotonic and cyclic loading.
- 8.1-3:** Study pure shear response during temperature change. Calculate the temperature rise that would result in yield for a given pure shear. Simulate temperature-dependent response due to monotonic rising of temperature and due to sinusoidal temperature change.

8.1.3 Uniaxial tension/compression

Uniaxial extension/compression is defined by the stress field

$$\boldsymbol{\sigma} = \sigma \mathbf{e}_1 \otimes \mathbf{e}_1, \quad (8.37)$$

where the extension/compression is along the 1-1 direction, and the material is free along the lateral directions. This results in an average stress given by

$$\sigma_{ave} = \frac{1}{3}\sigma, \quad (8.38)$$

and the deviatoric stress given by

$$\mathbf{S} = \frac{1}{3}\sigma(2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (8.39)$$

The expression for strain is given by

$$\boldsymbol{\epsilon} = \epsilon_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon_{22}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad (8.40)$$

where by symmetry it can be shown that the strains in the second and third directions are identical. The corresponding expression for elastic strain is given by

$$\boldsymbol{\epsilon}^e = -\frac{3\kappa - 2G}{18\kappa G}\sigma\mathbf{I} + \frac{1}{2G}\sigma\mathbf{e}_1 \otimes \mathbf{e}_1. \quad (8.41)$$

It is easy to show that the Young's modulus E and Poisson's ratio ν , which are defined through a uniaxial extension test, are given by

$$E = \frac{9\kappa G}{3\kappa + G}, \quad (8.42)$$

$$\nu = \frac{3\kappa - 2G}{2(3\kappa + G)}, \quad (8.43)$$

where $\sigma = E\epsilon_{11}^e$ and $\epsilon_{22}^e = -\nu\epsilon_{11}^e$.

For uniaxial stress, we can calculate τ as

$$\tau = \sqrt{\frac{1}{3}}|\sigma|, \quad (8.44)$$

and obtain

$$\dot{\tau} = \frac{1}{3}\frac{\sigma}{\tau}\dot{\sigma}. \quad (8.45)$$

We now can use the expressions

$$\dot{\sigma}_{ave} = \frac{1}{3}\dot{\sigma}, \quad (8.46)$$

to get

$$\begin{aligned} \dot{f} &= \dot{\tau} - \dot{\tau}_y = \frac{1}{3}\frac{\sigma}{\tau}\dot{\sigma} - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi} - \frac{\partial\tau_y}{\partial\sigma_{ave}}\dot{\sigma}_{ave} \\ &= \frac{1}{3}\left(\frac{\sigma}{\tau} - \frac{\partial\tau_y}{\partial\sigma_{ave}}\right)\dot{\sigma} - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi}. \end{aligned} \quad (8.47)$$

For *controlled axial strain history* ϵ_{11} , it follows that we can use $\dot{\sigma} = E(\dot{\epsilon}_{11} - \dot{\epsilon}_{11}^p) - dE/d\theta(\epsilon_{11} - \epsilon_{11}^p)\dot{\theta}$ to get

$$\dot{f} = \frac{1}{3}\left(\frac{\sigma}{\tau} - \frac{\partial\tau_y}{\partial\sigma_{ave}}\right)\left(E\dot{\epsilon}_{11} - \frac{1}{E}\frac{dE}{d\theta}\sigma\dot{\theta}\right) - \frac{\partial\tau_y}{\partial\theta}\dot{\theta}. \quad (8.48)$$

This gives $\dot{f}$ as

$$\dot{f} = \dot{f} - \frac{1}{3}\left(\frac{\sigma}{\tau} - \frac{\partial\tau_y}{\partial\sigma_{ave}}\right)E\dot{\epsilon}_{11}^p - \frac{\partial\tau_y}{\partial\xi}\dot{\xi}. \quad (8.49)$$

We can use $\dot{\epsilon}^p = \beta \mathbf{S}$ and thus $\dot{\xi} = 2\beta\tau$ to conclude that

$$\dot{f} = \overset{\diamond}{f} - \frac{1}{3} \left(\frac{\sigma}{\tau} - \frac{\partial \tau_y}{\partial \sigma_{ave}} \right) E \beta S_{11} - 2\beta\tau \frac{\partial \tau_y}{\partial \xi}. \quad (8.50)$$

During plastic flow ($f = 0$ and $\overset{\diamond}{f} > 0$), we have from the consistency condition $\dot{f} = 0$, and we know that $S_{11} = 2\sigma/3$ so that

$$\beta = \frac{\overset{\diamond}{f}}{\frac{2}{9} \left(\frac{\sigma}{\tau} - \frac{\partial \tau_y}{\partial \sigma_{ave}} \right) E \sigma + 2\tau \frac{\partial \tau_y}{\partial \xi}}. \quad (8.51)$$

For *controlled axial stress history* σ , it follows that we have

$$\overset{\diamond}{f} = \frac{1}{3} \left(\frac{\sigma}{\tau} - \frac{\partial \tau_y}{\partial \sigma_{ave}} \right) \dot{\sigma} - \frac{\partial \tau_y}{\partial \theta} \dot{\theta}. \quad (8.52)$$

We can use $\dot{\xi} = 2\beta\tau$ to conclude that

$$\dot{f} = \overset{\diamond}{f} - 2\beta\tau \frac{\partial \tau_y}{\partial \xi}. \quad (8.53)$$

During plastic flow ($f = 0$ and $\overset{\diamond}{f} > 0$), we have from the consistency condition $\dot{f} = 0$ that

$$\beta = \frac{\overset{\diamond}{f}}{2\tau \frac{\partial \tau_y}{\partial \xi}}. \quad (8.54)$$

The plastic strain rate then is given by

$$\dot{\epsilon}^p = \beta \mathbf{S} = \frac{1}{6} \frac{\sigma}{\tau} \frac{\overset{\diamond}{f}}{\frac{\partial \tau_y}{\partial \xi}} (2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3). \quad (8.55)$$

Section 8.1-Problems:

- 8.1-4:** Simulate the response of a bar undergoing uniaxial extension subject to both monotonic loading and cyclic loading. Consider both constant temperature and monotonically increasing temperature.
- 8.1-5:** Consider a bar held at constant length under homogeneous conditions similar to uniaxial extension, as shown in Figure 8.2a. Assume the temperature decreases uniformly. Calculate the temperature at which the bar first will reach its yield condition. After reducing the temperature below the point of initial yield, heat the sample back up to the original temperature and calculate the stress in the bar.
- 8.1-6:** Consider a bar hanging under its own weight, as shown in Figure 8.2b. Assume that the first direction is pointing down and the second and third directions are horizontal. Ignore end effects, assuming the bar is under uniaxial extension with stress σ along the first direction. Show that the equilibrium equation can be written as

$$\frac{\partial \sigma}{\partial X_1} + \rho_o g = 0, \quad (8.56)$$

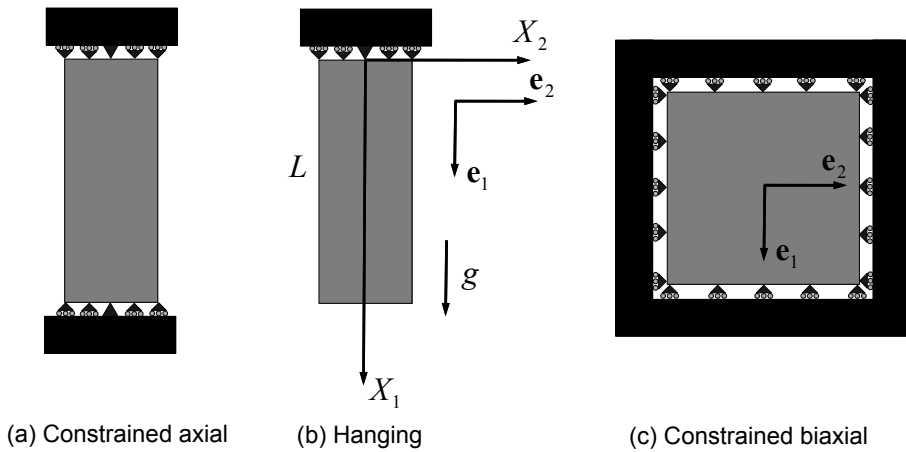


Figure 8.2: Constrained axial, hanging member, and constrained biaxial problems.

where g is the acceleration of gravity. Assuming that the bar is of length L , show that this equation can be integrated to give

$$\sigma = \rho_o g(L - X_1), \quad (8.57)$$

where X_1 is measured down from the point the bar is hanging, and we have introduced the boundary condition $\sigma = 0$ at $X_1 = L$. Locate the point in the bar that is at the yield point. Describe the elastic and plastic strain fields in the bar and calculate the total extension in the bar due to its own weight.

8.1.4 Equal biaxial extension

For equal biaxial extension, we assume the stress is given by

$$\boldsymbol{\sigma} = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2), \quad (8.58)$$

where the equal biaxial stress is in the 1-2 plane and it is free in the third direction. This results in an average stress given by

$$\sigma_{ave} = \frac{2}{3}\sigma, \quad (8.59)$$

and the deviatoric stress given by

$$\mathbf{S} = \frac{1}{3}\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3). \quad (8.60)$$

From this we can calculate τ as

$$\tau = \sqrt{\frac{1}{3}}|\sigma|, \quad (8.61)$$

and obtain

$$\dot{\tau} = \frac{1}{3} \frac{\sigma}{\tau} \dot{\sigma}. \quad (8.62)$$

We now can use the expressions

$$\dot{\sigma}_{ave} = \frac{2}{3}\dot{\sigma}, \quad (8.63)$$

to get

$$\begin{aligned} \dot{f} = \dot{\tau} - \dot{\tau}_y &= \frac{1}{3}\frac{\sigma}{\tau}\dot{\sigma} - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi} - \frac{\partial\tau_y}{\partial\sigma_{ave}}\dot{\sigma}_{ave} \\ &= \frac{1}{3}\left(\frac{\sigma}{\tau} - 2\frac{\partial\tau_y}{\partial\sigma_{ave}}\right)\dot{\sigma} - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi}. \end{aligned} \quad (8.64)$$

It follows that for a controlled applied stress history we have

$$\dot{f} = \frac{1}{3}\left(\frac{\sigma}{\tau} - 2\frac{\partial\tau_y}{\partial\sigma_{ave}}\right)\dot{\sigma} - \frac{\partial\tau_y}{\partial\theta}\dot{\theta}. \quad (8.65)$$

In this case we can use $\dot{\xi} = 2\beta\tau$ to conclude that

$$\dot{f} = \dot{f} - 2\beta\tau\frac{\partial\tau_y}{\partial\xi}. \quad (8.66)$$

During plastic flow ($f = 0$ and $\dot{f} > 0$), we have from the consistency condition $\dot{f} = 0$ that

$$\beta = \frac{\dot{f}}{2\tau\frac{\partial\tau_y}{\partial\xi}}. \quad (8.67)$$

The plastic strain rate then is given by

$$\dot{\epsilon}^p = \beta\mathbf{S} = \frac{1}{6}\frac{\sigma}{\tau}\frac{\dot{f}}{\frac{\partial\tau_y}{\partial\xi}}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3). \quad (8.68)$$

Section 8.1-Problems:

8.1-7: Show that for equal biaxial extension the elastic strain rate is given by

$$\dot{\epsilon}^e = -\frac{3\kappa - 2G}{9\kappa G}\dot{\sigma}\mathbf{I} + \frac{1}{2G}\dot{\sigma}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2), \quad (8.69)$$

to obtain

$$\dot{\epsilon}_{11}^e = \dot{\epsilon}_{22}^e = \frac{3\kappa + 4G}{18\kappa G}\dot{\sigma}, \quad (8.70)$$

$$\dot{\epsilon}_{33}^e = -\frac{3\kappa - 2G}{9\kappa G}\dot{\sigma}. \quad (8.71)$$

Combine this with the expression for thermal expansion $\dot{\epsilon}^\theta = \alpha\dot{\theta}\mathbf{I}$ and the expression for plastic strain rate to obtain the total strain rate during plastic flow.

8.1-8: Simulate the response of a material undergoing equal biaxial stress under monotonic and cyclic loading. Include both constant temperature and monotonically increasing temperature.

8.1-9: A material is held biaxially at zero total strain in the 1-2 plane and is free along the third direction, as shown in Figure 8.2c. The material is cooled uniformly. Assume uniform deformations. Calculate the temperature at which the material will reach the yield point. Evaluate the stress needed to keep the sample in this condition both before and after yielding.

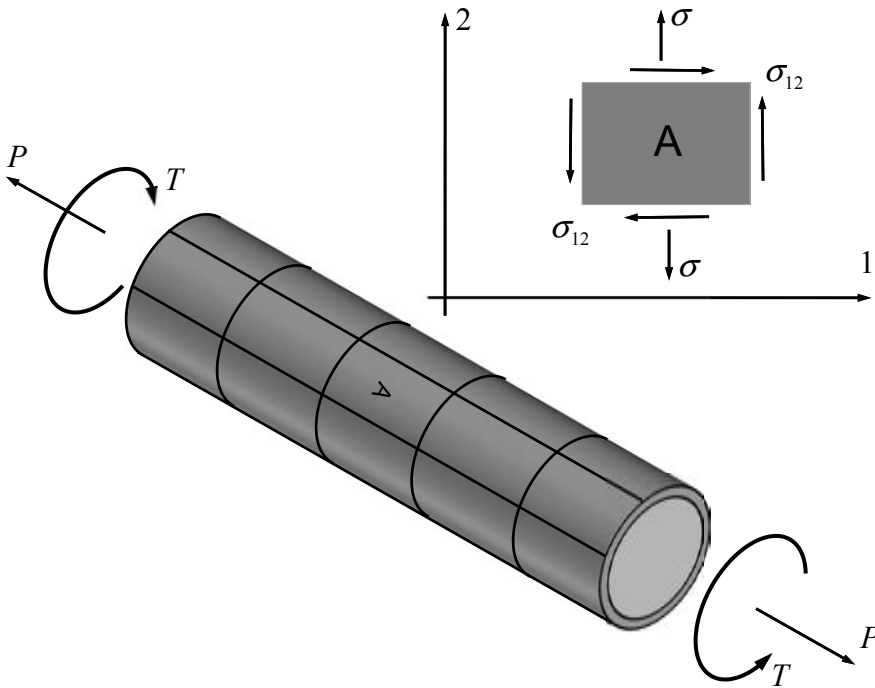


Figure 8.3: Torsion and tension of a thin cylindrical member.

8.2 Torsion-extension of a thin circular cylindrical tube

The problem we will consider here is a homogeneous deformation problem that is characterized by a hollow circular cylinder first twisted up to yield and then, holding the shear strain constant, is extended or compressed. Figure 8.3 provides a description of the deformation. The problem provides an interesting “paradox” since even though at the yield point there is no initial plastic flow when extending the tube. The paradox also can be seen when we first compress or extend the cylinder up to the point of yield and then start twisting it at constant compression/extension.

Let us start by discussing the assumption of constant shear strain during the extension. If the end-to-end twist of the cylinder is being held fixed, the approximation of a constant strain becomes better as the ratio of the length-to-mean radius increases. One obviously can obtain the constant shear deformation for any length cylinder through controlling the twist as we extend the cylinder. It turns out that the approximation is exact at the initial point of stretching.

We consider here only isotropic hardening so that as before we ignore the back-stress (i.e., $G^b = 0$). During the initial torsion, we assume that the shear in the thin-walled cylindrical tube is represented by the deformation

$$\epsilon = \epsilon^e = \frac{\gamma}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (8.72)$$

so that the stress response is given by

$$\boldsymbol{\sigma} = \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (8.73)$$

We first load the material until we reach the yield point. To do this we set $f = 0$ and obtain $\tau = \tau_y$. For the initial torsion, it follows that $\sigma_{12} = \tau_{yo}$.

Next, we start extending the system at constant γ along the two-direction, assuming the conditions require the stress in the first and third directions to be zero. We, therefore, assume the following response for the strain response

$$\boldsymbol{\epsilon} = \epsilon_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \epsilon_3 \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{\gamma}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (8.74)$$

with an associated stress response given by

$$\boldsymbol{\sigma} = \sigma \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (8.75)$$

where σ is the axial stress and starts from zero and σ_{12} is the shear stress and starts from the value $\sigma_{12} = \tau_{yo}$. For this case, the deviatoric stress is given by

$$\mathbf{S} = -\frac{1}{3}\sigma \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{2}{3}\sigma \mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{1}{3}\sigma \mathbf{e}_3 \otimes \mathbf{e}_3 + \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (8.76)$$

The expression for the strain rate for constant γ is given by

$$\dot{\boldsymbol{\epsilon}} = \dot{\epsilon}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \dot{\epsilon}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \dot{\epsilon}_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (8.77)$$

If we calculate the trial derivative of the yield function, we get

$$\overset{\diamond}{f} = \frac{G}{\tau} \mathbf{S} : \dot{\boldsymbol{\epsilon}} = \frac{G\sigma(-\dot{\epsilon}_1 + 2\dot{\epsilon}_2 - \dot{\epsilon}_3)}{3\tau}, \quad (8.78)$$

where $\tau = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{12}^2}$. At the start of the extension, we know that $\sigma = 0$ and $\sigma_{12} = \tau_{yo}$ so that we have

$$\overset{\diamond}{f} = \frac{G\sigma(-\dot{\epsilon}_1 + 2\dot{\epsilon}_2 - \dot{\epsilon}_3)}{3\tau_{yo}} = 0. \quad (8.79)$$

Therefore, even though the bar has been twisted until it has reached the yield surface in shear, the initial extension will be elastic. This is the paradox. Even though further shear would result in plastic flow, the initial extension does not. The source of the paradox is the fact that the subsequent flow has a zero trial derivative of the yield function. This means that infinitesimally small extensions can happen without changing the yield function from zero (the initial extension is a stationary direction for f).

This “paradox” will happen for each case where $\mathbf{S} : \dot{\boldsymbol{\epsilon}} = 0$ (i.e., $\mathbf{S}$ and $\dot{\boldsymbol{\epsilon}}$ are orthogonal). This same “paradox” will happen for models with kinematic hardening when $\Delta \mathbf{S} : \dot{\boldsymbol{\epsilon}} = 0$ (i.e., $\Delta \mathbf{S}$ and $\dot{\boldsymbol{\epsilon}}$ are orthogonal). A similar condition will happen when we compress or extend the cylinder to yield and then start to twist it at constant compression/extension.

Section 8.2-Problems:

8.2-1: Show that in general for an isotropic material and for volume preserving plastic flow we have

$$\dot{\boldsymbol{\sigma}} = \left(\kappa - \frac{2}{3}G \right) \text{tr}(\dot{\boldsymbol{\epsilon}}) \mathbf{I} + 2G(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p). \quad (8.80)$$

- 8.2-2:** Consider the problem of torsion followed by extension as given above. Consider the equation for stress rate given by

$$\dot{\boldsymbol{\sigma}} = \left(\kappa - \frac{2}{3}G \right) \text{tr}(\dot{\boldsymbol{\epsilon}}) \mathbf{I} + 2G(\dot{\boldsymbol{\epsilon}} - \beta \Delta \mathbf{S}). \quad (8.81)$$

Show that we have

$$\beta = \frac{G\sigma(2\dot{\epsilon}_2 - \dot{\epsilon}_1 - \dot{\epsilon}_3)}{6\tau_y^2(G + \tau_y')} = \mu\sigma(2\dot{\epsilon}_2 - \dot{\epsilon}_1 - \dot{\epsilon}_3), \quad (8.82)$$

for

$$\mu = \frac{G}{6\tau_y^2(G + \tau_y')}. \quad (8.83)$$

Write the four independent equations from the stress equation to find

$$0 = \left(\kappa - \frac{2}{3}G \right) (\dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3) + 2G\dot{\epsilon}_1 + \frac{2G\mu\sigma}{3}(2\dot{\epsilon}_2 - \dot{\epsilon}_1 - \dot{\epsilon}_3), \quad (8.84)$$

$$\dot{\sigma} = \left(\kappa - \frac{2}{3}G \right) (\dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3) + 2G\dot{\epsilon}_2 - \frac{4G\mu\sigma}{3}(2\dot{\epsilon}_2 - \dot{\epsilon}_1 - \dot{\epsilon}_3), \quad (8.85)$$

$$0 = \left(\kappa - \frac{2}{3}G \right) (\dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3) + 2G\dot{\epsilon}_3 + \frac{2G\mu\sigma}{3}(2\dot{\epsilon}_2 - \dot{\epsilon}_1 - \dot{\epsilon}_3), \quad (8.86)$$

$$\dot{\sigma}_{12} = -2G\mu\sigma\sigma_{12}. \quad (8.87)$$

From this show that $\dot{\epsilon}_1 = \dot{\epsilon}_3$. Show that initially we have

$$\dot{\epsilon}_1 = \dot{\epsilon}_3 = -\nu\dot{\epsilon}_2, \quad (8.88)$$

where the Poisson's ratio ν is given by

$$\nu = \frac{3\kappa - 2G}{2(3\kappa + G)}. \quad (8.89)$$

Show that in general we have

$$\beta = \frac{G\sigma(\dot{\epsilon}_2 - \dot{\epsilon}_1)}{3\tau_y^2(G + \tau_y')}. \quad (8.90)$$

Integrate this system for values for $G = 10$, $\kappa = 20$, $\tau_{yo} = 2$, and $\tau_y' = 2$ during the extension.

- 8.2-3:** Reconstruct the steps of the previous problem for the case of first compressing the bar to the point of yield and then twisting it.

8.3 Compression in plane strain

In this section, we consider another homogeneous deformation. The deformation considered is plane strain tension or compression, as shown in Figure 8.4. In this case, the body is pulled or compressed along the first direction and restricted along the second direction but is free along the third direction. As in the past sections, we will assume only isotropic

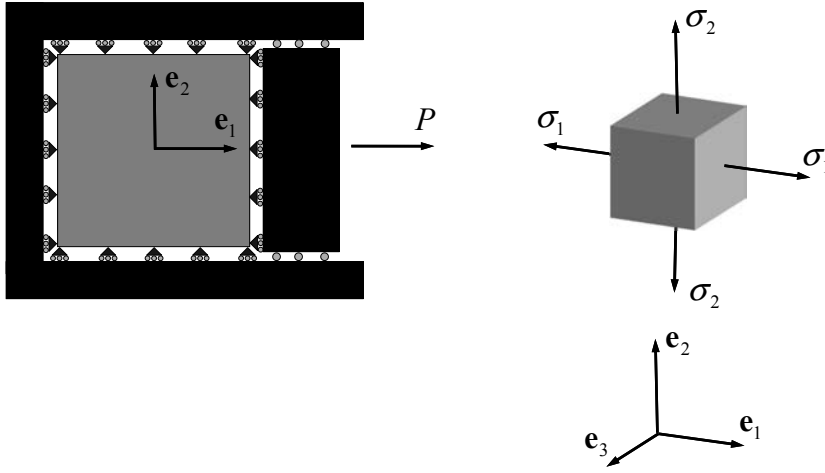


Figure 8.4: Plane strain tension/compression.

hardening, so we will ignore kinematic hardening by setting the back-stress equal to zero by taking $G^b = 0$.

The general expression for stress in this case is

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2, \quad (8.91)$$

while the expression for strain is given by

$$\boldsymbol{\epsilon} = \epsilon_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \epsilon_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (8.92)$$

As a result, the expression for average stress and deviatoric stress are given by

$$\sigma_{ave} = \frac{1}{3}(\sigma_1 + \sigma_2), \quad (8.93)$$

$$\mathbf{S} = \frac{1}{3}(2\sigma_1 - \sigma_2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{3}(2\sigma_2 - \sigma_1)\mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{1}{3}(\sigma_2 + \sigma_1)\mathbf{e}_3 \otimes \mathbf{e}_3. \quad (8.94)$$

Since the deviatoric stress only has three components, the plastic strain can develop only along these directions so that we can write plastic strain rate in the form

$$\dot{\boldsymbol{\epsilon}}^p = \beta \mathbf{S} = \dot{\epsilon}_1^p \mathbf{e}_1 \otimes \mathbf{e}_1 + \dot{\epsilon}_2^p \mathbf{e}_2 \otimes \mathbf{e}_2 + \dot{\epsilon}_3^p \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (8.95)$$

The thermal strain rate is given by

$$\dot{\boldsymbol{\epsilon}}^\theta = \alpha \dot{\theta} \mathbf{I}. \quad (8.96)$$

Since $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p + \dot{\boldsymbol{\epsilon}}^\theta$, from the form of the total, plastic, and thermal strains it follows that the elastic strain rate has the form

$$\dot{\boldsymbol{\epsilon}}^e = \dot{\epsilon}_1^e \mathbf{e}_1 \otimes \mathbf{e}_1 + \dot{\epsilon}_2^e \mathbf{e}_2 \otimes \mathbf{e}_2 + \dot{\epsilon}_3^e \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (8.97)$$

From the expression of deviatoric stress, we can calculate τ as

$$\tau = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}} = \sqrt{\frac{1}{3}(\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2)}. \quad (8.98)$$

This gives the expression for $\dot{\tau}$ as

$$\dot{\tau} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2]. \quad (8.99)$$

From the expression of the average stress, we have

$$\dot{\sigma}_{ave} = \frac{1}{3}(\dot{\sigma}_1 + \dot{\sigma}_2). \quad (8.100)$$

The expression for the yield function is given by the equation

$$f = \tau - \tau_y = \sqrt{\frac{1}{3}(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)} - \tau_y, \quad (8.101)$$

where

$$\tau_y = \tau_{yo} + \frac{\partial \tau_y}{\partial \theta} \Delta \theta + \frac{\partial \tau_y}{\partial \xi} \xi + \frac{\partial \tau_y}{\partial \sigma_{ave}} \sigma_{ave}. \quad (8.102)$$

This results in the expression for the rate of change of the yield function given by

$$\dot{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] - \frac{\partial \tau_y}{\partial \theta} \dot{\theta} - \frac{\partial \tau_y}{\partial \xi} \dot{\xi} - \frac{1}{3} \frac{\partial \tau_y}{\partial \sigma_{ave}} (\dot{\sigma}_1 + \dot{\sigma}_2). \quad (8.103)$$

8.3.1 During elastic flow

Let us focus on the elastic flow, where the plastic strain rate is zero. In this case $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^\theta$ so that

$$\dot{\epsilon}^e = \dot{\epsilon} - \dot{\epsilon}^\theta. \quad (8.104)$$

If we substitute this into the expression for stress rate given by

$$\dot{\sigma} = \left(\kappa - \frac{2}{3}G \right) tr(\dot{\epsilon}^e) \mathbf{I} + 2G\dot{\epsilon}^e, \quad (8.105)$$

one can obtain three independent equations given by

$$\dot{\sigma}_1 = \left(\kappa + \frac{4}{3}G \right) \dot{\epsilon}_1 + \left(\kappa - \frac{2}{3}G \right) \dot{\epsilon}_3 - 3\alpha\kappa\dot{\theta}, \quad (8.106)$$

$$\dot{\sigma}_2 = \left(\kappa - \frac{2}{3}G \right) \dot{\epsilon}_1 + \left(\kappa - \frac{2}{3}G \right) \dot{\epsilon}_3 - 3\alpha\kappa\dot{\theta}, \quad (8.107)$$

$$0 = \left(\kappa - \frac{2}{3}G \right) \dot{\epsilon}_1 + \left(\kappa + \frac{4}{3}G \right) \dot{\epsilon}_3 - 3\alpha\kappa\dot{\theta}. \quad (8.108)$$

We can solve the last equation for $\dot{\epsilon}_3$ to obtain

$$\dot{\epsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G} \dot{\epsilon}_1 + \frac{9\alpha\kappa}{3\kappa + 4G} \dot{\theta}. \quad (8.109)$$

Eliminating $\dot{\epsilon}_3$ from the other two equations results in

$$\dot{\sigma}_1 = \frac{4G(3\kappa + G)}{3\kappa + 4G} \dot{\epsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G} \dot{\theta}, \quad (8.110)$$

$$\dot{\sigma}_2 = \frac{2G(3\kappa - 2G)}{3\kappa + 4G} \dot{\epsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G} \dot{\theta}. \quad (8.111)$$

We then can get $\dot{f}$ by substituting into the expression for the rate of change of the yield function given by

$$\dot{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\dot{\sigma}_1 + \dot{\sigma}_2). \quad (8.112)$$

If the flow is not elastic, then the above solution becomes the steps taken to calculate the trial derivative of f . In this case, we can replace $\dot{f} = \dot{f}^\diamond$ from the above equation where $\dot{\sigma}_1 = \dot{\sigma}_1$ and $\dot{\sigma}_2 = \dot{\sigma}_2$ given in the above equation.

8.3.2 During plastic flow

Let us consider what will happen when the material is undergoing plastic flow (i.e., $f = 0$ and $\dot{f} > 0$). During plastic flow, we have $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p + \dot{\epsilon}^\theta$ so that we have

$$\dot{\epsilon}^e = \dot{\epsilon} - \dot{\epsilon}^p - \dot{\epsilon}^\theta = \dot{\epsilon} - \beta\mathbf{S} - \alpha\dot{\theta}\mathbf{I}. \quad (8.113)$$

If we substitute the elastic strain rate into the expression for stress rate, we get

$$\dot{\sigma} = (\kappa - \frac{2}{3}G)tr(\dot{\epsilon}^e)\mathbf{I} + 2G\dot{\epsilon}^e \quad (8.114)$$

$$= (\kappa - \frac{2}{3}G)tr(\dot{\epsilon})\mathbf{I} + 2G\dot{\epsilon} - 2G\beta\mathbf{S} - 3\alpha\kappa\dot{\theta}\mathbf{I}. \quad (8.115)$$

This results in the three independent equations

$$\dot{\sigma}_1 = \frac{3\kappa + 4G}{3}\dot{\epsilon}_1 + \frac{3\kappa - 2G}{3}\dot{\epsilon}_3 - \frac{2G}{3}(2\sigma_1 - \sigma_2)\beta - 3\alpha\kappa\dot{\theta}, \quad (8.116)$$

$$\dot{\sigma}_2 = \frac{3\kappa - 2G}{3}\dot{\epsilon}_1 + \frac{3\kappa - 2G}{3}\dot{\epsilon}_3 - \frac{2G}{3}(2\sigma_2 - \sigma_1)\beta - 3\alpha\kappa\dot{\theta}, \quad (8.117)$$

$$0 = \frac{3\kappa - 2G}{3}\dot{\epsilon}_1 + \frac{3\kappa + 4G}{3}\dot{\epsilon}_3 + \frac{2G}{3}(\sigma_1 + \sigma_2)\beta - 3\alpha\kappa\dot{\theta}. \quad (8.118)$$

We use the last equation to solve for $\dot{\epsilon}_3$ and obtain

$$\dot{\epsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\epsilon}_1 - \frac{2G}{3\kappa + 4G}(\sigma_1 + \sigma_2)\beta + \frac{9\alpha\kappa}{3\kappa + 4G}\dot{\theta}. \quad (8.119)$$

This can be used to eliminate $\dot{\epsilon}_3$ from the other two equations to get

$$\begin{aligned} \dot{\sigma}_1 &= \frac{4G(3\kappa + G)}{3\kappa + 4G}\dot{\epsilon}_1 - \frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \\ &= \dot{\sigma}_1^\diamond - \frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta, \end{aligned} \quad (8.120)$$

$$\begin{aligned} \dot{\sigma}_2 &= \frac{2G(3\kappa - 2G)}{3\kappa + 4G}\dot{\epsilon}_1 + \frac{2G}{3\kappa + 4G}[(2G\sigma_1 - (3\kappa + 2G)\sigma_2)]\beta - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \\ &= \dot{\sigma}_2^\diamond + \frac{2G}{3\kappa + 4G}[(2G\sigma_1 - (3\kappa + 2G)\sigma_2)]\beta. \end{aligned} \quad (8.121)$$

Considering the elastic solution and the definition of $\dot{f}^\diamond$, the expression for the rate of the

yield function during plastic flow can be written as

$$\begin{aligned}
 \dot{f} &= \frac{1}{6\tau} [(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\dot{\sigma}_1 + \dot{\sigma}_2) \\
 &= \left[\frac{1}{6\tau}(2\sigma_1 - \sigma_2) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}} \right] \dot{\sigma}_1 + \left[\frac{1}{6\tau}(2\sigma_2 - \sigma_1) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}} \right] \dot{\sigma}_2 - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - 2\frac{\partial\tau_y}{\partial\xi}\tau\beta \\
 &= \dot{f} + \left[\frac{1}{6\tau}(2\sigma_1 - \sigma_2) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}} \right] \left\{ -\frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta \right\} \\
 &\quad + \left[\frac{1}{6\tau}(2\sigma_2 - \sigma_1) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}} \right] \left\{ \frac{2G}{3\kappa + 4G}[(2G\sigma_1 - (3\kappa + 2G)\sigma_2)\beta] \right\} - 2\frac{\partial\tau_y}{\partial\xi}\tau\beta. \quad (8.122)
 \end{aligned}$$

The consistency condition $\dot{f} = 0$ during plastic flow now gives β as

$$\beta = \frac{\tau(3\kappa + 4G)\dot{f}}{2G[(\kappa + G)(\sigma_1^2 + \sigma_2^2) - (\kappa + 2G)\sigma_1\sigma_2 - \kappa\tau\frac{\partial\tau_y}{\partial\sigma_{ave}}(\sigma_1 + \sigma_2)] + 2\tau^2(3\kappa + 4G)\frac{\partial\tau_y}{\partial\xi}}. \quad (8.123)$$

Section 8.3-Problems:

- 8.3-1:** Consider isothermal plane strain compression. Evaluate the slope of the compressive stress-strain plot during a monotonic loading, including both the initial elastic and the following plastic response (i.e., the slopes of the σ_1 - ϵ_1 plot). Calculate the ratio of the transverse to axial strain rates in each range (i.e., the ratio of $\dot{\epsilon}_3$ to $\dot{\epsilon}_1$).
- 8.3-2:** Calculate the minimum temperature rise that will force the response to reach the yield surface. Calculate this for both the case of $\epsilon_1 = 0$ and also for the case of $\sigma_1 = 0$.
- 8.3-3:** Simulate the response under plane strain compression in monotonic compression and cyclic compression. Plot as a function of ϵ_1 the stresses σ_1 and σ_2 , and the strain ϵ_3 .

8.4 Bending

Bending is an inhomogeneous deformation. The most common model used for bending is for a long and straight bar, and assumes that the response is like that of fibers undergoing uniaxial extension. In doing so, we ignore any restriction offered by the lateral surfaces. This normally is justified by the smallness of the lateral dimensions compared to the axial dimension. The typical bending problem has the form shown in Figure 8.5, where we have two bending moments and an axial load. The typically assumed deformation field of the body is characterized by linearly varying bending strains resulting from bending about two centers C_1 and C_2 and a constant axial strain. The total axial strain ϵ is given by the expression

$$\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{\rho_1}X_2 - \frac{1}{\rho_2}X_1 + \epsilon_3, \quad (8.124)$$

where ρ_1 and ρ_2 , as shown in the figure, are radii of curvature associated with, respectively, bending about the first and second axes, and X_1 and X_2 are the positions from C along the

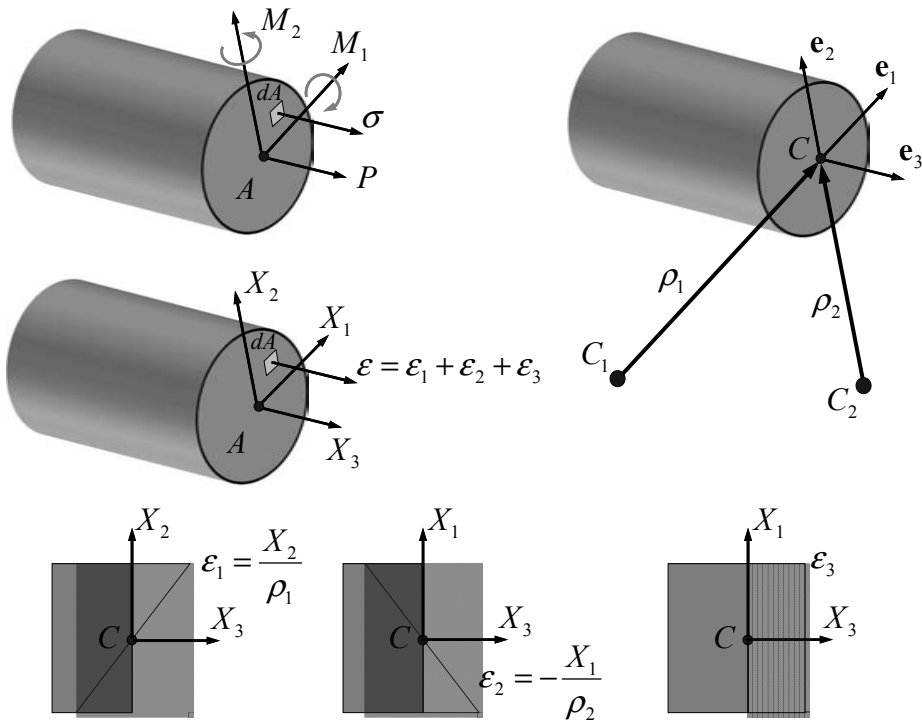


Figure 8.5: Bending moments M_1 and M_2 about two principal axes of the cross section and an axial load P applied along the bar.

two coordinate directions. The location of C is a priori unknown but becomes the centroid of the cross section for a linear elastic material.

The expressions for the axial load and bending moments can be calculated from

$$P = \int_A \sigma dA, \quad (8.125)$$

$$M_1 = \int_A \sigma X_2 dA, \quad (8.126)$$

$$M_2 = - \int_A \sigma X_1 dA, \quad (8.127)$$

where P is the axial load, M_1 and M_2 , respectively, are the bending moments about the X_1 and X_2 axis, and A is the cross-sectional area.

The neutral fiber is the axial line that neither extends nor contracts. The location of the neutral fiber can be found by setting the strain equal to zero. Therefore, the fibers satisfying the condition

$$\frac{1}{\rho_1} X_2 - \frac{1}{\rho_2} X_1 + \epsilon_3 = 0 \quad (8.128)$$

are neutral fibers. Since this represents a flat surface, the neutral fibers define a straight line of the cross section passing through points with coordinates (X_1, X_2) given by

$$(0, -\rho_1 \epsilon_3), \quad (\rho_2 \epsilon_3, 0). \quad (8.129)$$

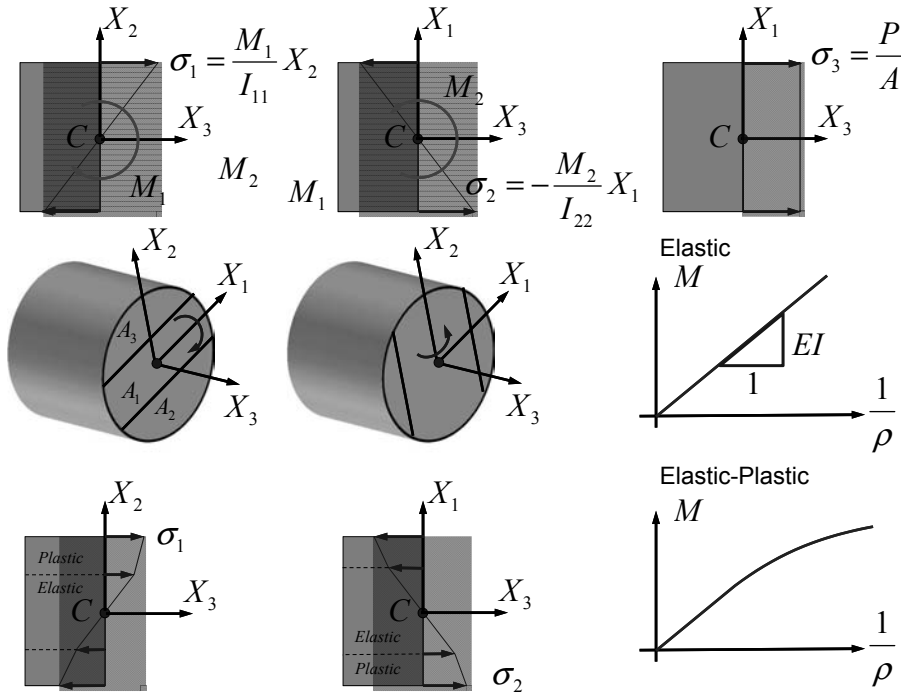


Figure 8.6: Elastic and plastic stress fields resulting from application of a single bending moment or force. The top shows the elastic stress fields, the bottom shows the elastic-plastic stress fields, the middle shows the point that yielding starts for only one applied bending moment, and the right shows the moment-curvature relations for a single applied moment.

When only one of the two bending moments exists, the line becomes parallel to the axis around which bending is occurring.

8.4.1 Elastic bending

Infinitesimal elasticity is linear in the sense that the response from the sum of two strain fields is the sum of the responses from each individual strain field. This is also true for linear viscoelasticity. As a result, for the elasticity problem we can study the response to each one of the three strain fields separately and then add them together. We thus have

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3, \quad (8.130)$$

where σ_i is the stress response to imposing the individual strain field ϵ_i . Since we assume the fibers are under uniaxial extension, the stress for each strain field is given by

$$\sigma_1 = E\epsilon_1 = \frac{E}{\rho_1} X_2, \quad (8.131)$$

$$\sigma_2 = E\epsilon_2 = -\frac{E}{\rho_2} X_1, \quad (8.132)$$

$$\sigma_3 = E\epsilon_3. \quad (8.133)$$

These are shown at the top of Figure 8.6.

We first consider only ϵ_1 so that $\epsilon_2 = \epsilon_3 = 0$. As a result we have $\sigma_2 = \sigma_3 = 0$, and we assume the deformation fully results from M_1 , so that M_2 and P are zero. From this we obtain

$$P = \int_A \sigma_1 dA = \int_A \frac{E}{\rho_1} X_2 dA = \frac{E}{\rho_1} \int_A X_2 dA = 0, \quad (8.134)$$

$$M_1 = \int_A \sigma_1 X_2 dA = \int_A \frac{E}{\rho_1} X_2^2 dA = \frac{E}{\rho_1} \int_A X_2^2 dA, \quad (8.135)$$

$$M_2 = - \int_A \sigma_1 X_1 dA = - \int_A \frac{E}{\rho_1} X_2 X_1 dA = - \frac{E}{\rho_1} \int_A X_2 X_1 dA = 0. \quad (8.136)$$

From $P = 0$ it follows that C is at the centroid of the section along the X_2 direction. From $M_2 = 0$ it follows that the area product of inertia I_{12} defined by

$$I_{12} = \int_A X_2 X_1 dA \quad (8.137)$$

is zero. Finally, from the equation for M_1 , we find that

$$M_1 = \frac{EI_{11}}{\rho_1}, \quad (8.138)$$

where I_{11} is the area moment of inertia about the X_1 axis defined by

$$I_{11} = \int_A X_2^2 dA. \quad (8.139)$$

This elastic response can be shown as a straight line in the moment-curvature relation, as shown in the right of Figure 8.6. If one calculates the radius of curvature ρ_1 for the given moment M_1 and then back substitutes it into the expression for the stress, we will get

$$\sigma_1 = \frac{M_1}{I_{11}} X_2, \quad (8.140)$$

showing the relation between the applied moment and the resulting stress.

In a similar fashion, we can show if we consider only ϵ_2 , assuming that P and M_1 are zero, that C is on the centroid of the section along the X_1 direction, and that the area product of inertia I_{12} must be zero. We also can show that

$$M_2 = \frac{EI_{22}}{\rho_2}, \quad (8.141)$$

where I_{22} is the area moment of inertia about the X_2 axis defined as

$$I_{22} = \int_A X_1^2 dA. \quad (8.142)$$

It also follows that

$$\sigma_2 = - \frac{M_2}{I_{22}} X_1. \quad (8.143)$$

It follows that if the area product of inertia is zero, then one can have a condition that M_1 produces ϵ_1 and M_2 produces ϵ_2 , without any interaction. Or, in other words, if X_1 and X_2 are in the principal directions of the cross section, then we can consider the two bending moments as being decoupled in their associated strain and stress fields. To simplify this problem, we assume this to be the case so that $I_{12} = 0$.

Considering only the axial strain ϵ_3 so that $\sigma = \sigma_3 = E\epsilon_3$, we will get

$$P = \int_A \sigma_3 dA = \int_A E\epsilon_3 dA = AE\epsilon_3, \quad (8.144)$$

$$M_1 = \int_A \sigma_3 X_2 dA = \int_A E\epsilon_3 X_2 dA = E\epsilon_3 \int_A X_2 dA = 0, \quad (8.145)$$

$$M_2 = - \int_A \sigma_3 X_1 dA = - \int_A E\epsilon_3 X_1 dA = -E\epsilon_3 \int_A X_1 dA = 0. \quad (8.146)$$

Therefore, as long as the force is applied at the centroid, the moments M_1 and M_2 are zero, and the equations are automatically satisfied by putting the coordinate axis at the centroid.

We only consider here isotropic hardening so that the yield function $f = \tau - \tau_y$ for uniaxial extension is given by

$$f = \sqrt{\frac{1}{3}}|\sigma| - \tau_y = \sqrt{\frac{1}{3}}|\sigma_1 + \sigma_2 + \sigma_3| - \tau_y. \quad (8.147)$$

Unfortunately, plastic flow is not additive so to consider this we need to consider the sum of the three responses together. For this we must write

$$f = \sqrt{\frac{1}{3}}\left|\frac{M_1}{I_{11}}X_2 - \frac{M_2}{I_{22}}X_1 + \frac{P}{A}\right| - \tau_y. \quad (8.148)$$

As can be seen, the value of the yield function is different at each point. In the case that we only have the bending moment M_1 , the yield function will be

$$f = \sqrt{\frac{1}{3}}\left|\frac{M_1}{I_{11}}X_2\right| - \tau_y, \quad (8.149)$$

so that the yield point $f = 0$ is given by

$$|X_2| = \frac{I_{11}}{M_1}\sqrt{3}\tau_y = \frac{I_{11}}{M_1}\sigma_y. \quad (8.150)$$

Any values of $|X_2|$ larger than this will result in stresses larger than allowed by the yield function. This can be seen in the middle left of Figure 8.6, where A_1 is elastic, A_2 and A_3 are elastic-plastic regions, and the two regions are separated by lines parallel to the X_1 axis. A similar equation exists if we only consider M_2 . In the case that we only consider P , we get that the stress is uniform and yield will occur only if the load P reaches the yield load given by $|P| = A\sigma_y$. Outside the elastic regions we must have plastic flow. Since plastic flow is path dependent, how we load the bar becomes important. If we only consider bending about one of the two axes, as we increase the moment monotonically from zero, the yield point will start from the points farthest from the axis of the moment and gradually come closer to the axis as the moment increases, the elastic region being between parallel lines of equal distance from the axis of the moment. If we consider both moments, the yield function is still a linear function so that the yielding will be defined by straight lines, no longer parallel to the coordinate directions.

Section 8.4-Problems:

- 8.4-1:** In this section, we have considered the material to be uniform, even though the strain field is not. Consider a bar that is constructed from two uniform rectangular bars of identical size welded together along one side to construct a composite bar. As a result, the material is no longer uniform, even though it is uniform in each part. Analyze the elastic response of this composite bar.

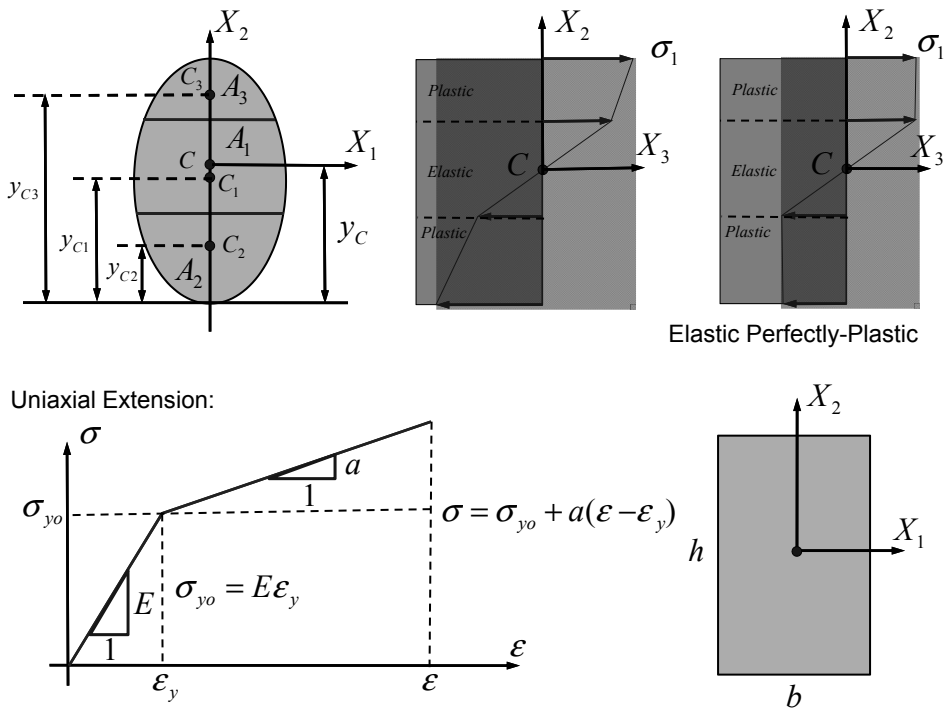


Figure 8.7: Elastic-plastic bending due to a single bending moment M_1 about axis X_1 .

8.4-2: A bar is constructed of a nonuniform material where the Young's modulus is given as a function of position by the function $E(X_1, X_2)$. Study the elastic response of this bar to extension and bending.

8.4.2 Plastic bending under a single monotonic moment

Here we will focus on the problem of monotonically bending a bar with only a single applied bending moment M_1 . This is described by the response shown in the bottom row of Figure 8.6. As described above, we will have three regions, the center being elastic and the outer regions being elastic-plastic. The plastic region will start at the farthest distance from the X_1 axis and move in. The three regions and the resulting response are shown in Figure 8.7.

As before, we will let C denote the neutral axis. The area A_1 is the part which is under elastic loading, and areas A_2 and A_3 undergo elastic-plastic loading. The stress response in A_1 is given by

$$\sigma = \frac{E}{\rho_1} X_2. \quad (8.151)$$

From the figure we can show that the stress response in A_2 is given by

$$\sigma = -\frac{E-a}{E} \sigma_{yo} + \frac{a}{\rho_1} X_2, \quad (8.152)$$

and in A_3 it is given by

$$\sigma = \frac{E-a}{E}\sigma_{yo} + \frac{a}{\rho_1}X_2. \quad (8.153)$$

Setting P equal to zero should provide us with the location of C which from the figure is given by y_C relative to the bottom of the cross-section. From this we can calculate the axial load in the bar as

$$P = \int_A \sigma dA = \int_{A_1} \frac{E}{\rho_1} X_2 dA + \int_{A_2} \left(-\frac{E-a}{E}\sigma_{yo} + \frac{a}{\rho_1}X_2 \right) dA + \int_{A_3} \left(\frac{E-a}{E}\sigma_{yo} + \frac{a}{\rho_1}X_2 \right) dA. \quad (8.154)$$

This can be written as

$$P = \frac{E}{\rho_1}Q_1 + \frac{a}{\rho_1}(Q_2 + Q_3) + \frac{E-a}{E}\sigma_{yo}(A_3 - A_2), \quad (8.155)$$

where Q_i is the area moment given by

$$Q_i = \int_{A_i} X_2 dA. \quad (8.156)$$

It easily can be shown that

$$Q_i = A_i(y_{Ci} - y_C), \quad (8.157)$$

where, as shown in Figure 8.7, $(y_{Ci} - y_C)$ is the X_2 value of point C_i . We now can rewrite the expression for the load to get

$$P = \frac{1}{\rho_1} \{ EA_1(y_{C1} - y_C) + a[A_2(y_{C2} - y_C) + A_3(y_{C3} - y_C)] \} + \frac{E-a}{E}\sigma_{yo}(A_3 - A_2). \quad (8.158)$$

Assuming no axial load, we can set $P = 0$ and solve for y_C to get

$$y_C = \frac{EA_1y_{C1} + a(A_2y_{C2} + A_3y_{C3}) + \rho_1 \frac{E-a}{E}\sigma_{yo}(A_3 - A_2)}{EA_1 + a(A_2 + A_3)}. \quad (8.159)$$

Knowing the location of C we now can calculate the bending moment using the equation

$$\begin{aligned} M_1 &= \int_A \sigma X_2 dA \\ &= \int_{A_1} \frac{E}{\rho_1} X_2^2 dA + \int_{A_2} \left(-\frac{E-a}{E}\sigma_{yo} + \frac{a}{\rho_1}X_2 \right) X_2 dA + \int_{A_3} \left(\frac{E-a}{E}\sigma_{yo} + \frac{a}{\rho_1}X_2 \right) X_2 dA. \end{aligned} \quad (8.160)$$

This can be written as

$$M_1 = \frac{1}{\rho_1} [EI_{A1} + a(I_{A2} + I_{A3})] + \frac{E-a}{E}\sigma_{yo}(Q_3 - Q_2), \quad (8.161)$$

where

$$I_{Ai} = \int_{A_i} X_2^2 dA. \quad (8.162)$$

Section 8.4-Problems:

- 8.4-3:** Consider a rectangular beam of height h and width b as shown in Figure 8.7. Given that $E = 20$, $a = 2$, $\sigma_{yo} = 2$, $h = 10$, and $b = 1$, draw the moment-curvature diagram for the beam as it is monotonically loaded by a bending moment parallel to the b side.

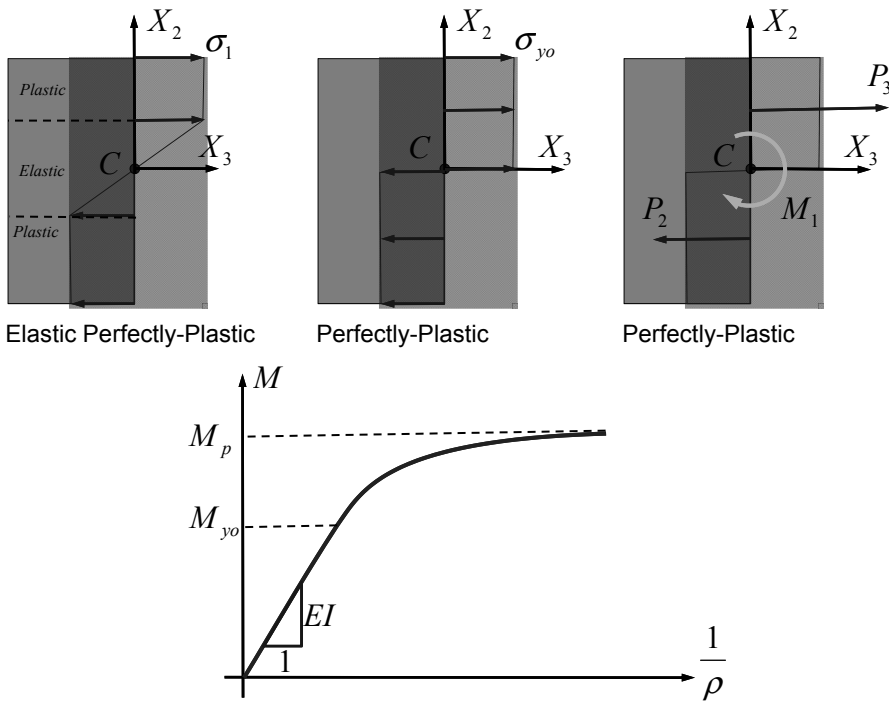


Figure 8.8: Perfectly plastic bending due to a single bending moment M_1 about axis X_1 .

- 8.4-4:** A simplification to the problem has the material following an elastic perfectly plastic loading. Construct the theory for such a material. The stress response for this beam is shown at the top right of Figure 8.7 and in Figure 8.8.
- 8.4-5:** For a material in pure bending modeled as an elastic perfectly plastic material, find the bending moment at which the elastic region disappears. Argue that the member cannot support any moment higher than this moment. Show that under this condition the surface is separated into two equal areas, one in tension and one in compression, each at the yield stress σ_{y0} . This moment is known as the plastic moment, or the plastic hinge moment. For each of the sections shown in Figure 8.9, calculate the plastic moment.

8.5 Torsion of circular members

Frequently, when we consider torsion we are considering torsion of circular sections. This is natural since most members designed to carry torsion are circular. A common example is the shape of the shafts used for power transmission in cars and power plants. One also might consider torsion in noncircular section. An obvious example is the torsion seen in

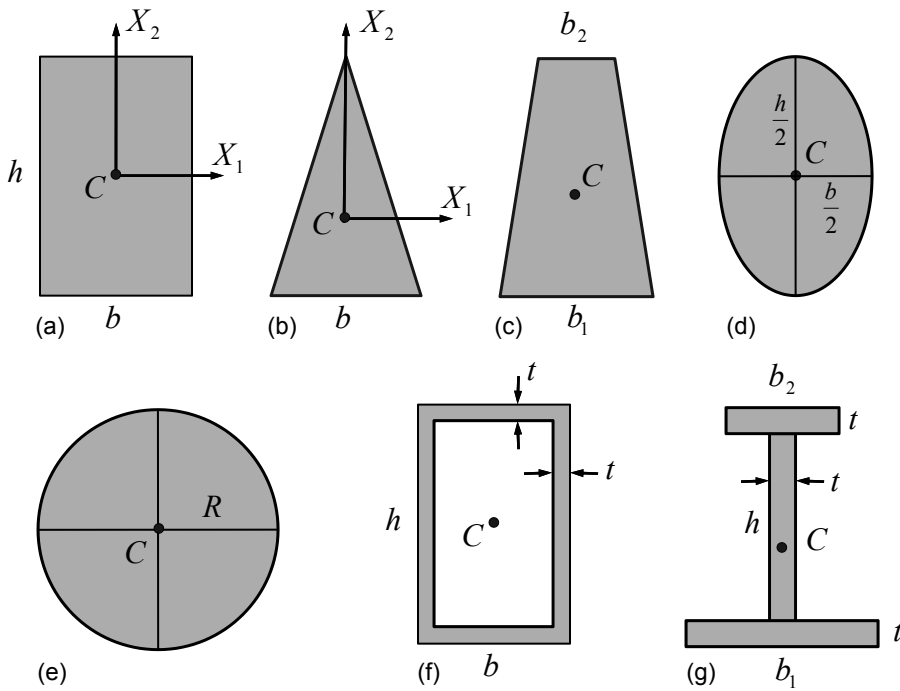


Figure 8.9: Cross sections of members.

hollow sections such as observed in the wing of an airplane or wind-power generators. The analysis of the two types of torsion is very different both for the elastic and for the plastic response. We will consider only circular members here since they are simpler to analyze. We also restrict our attention to monotonic loading at the points.

The basic loading considered here is shown in Figure 8.9, which includes both axial force and torsional moment. The basic kinematics of torsion in circular sections is that the circular cross sections remain circular and simply rotate as a rigid body. As has been shown in Chapter 3, torsion is characterized by a shear strain that is increasing linearly from zero at the center of the cross section to its maximum at the outer edge. The axial strain is assumed to be constant over the cross section. The basic kinematic assumptions are, therefore, given as

$$\epsilon = \text{Constant}, \quad (8.163)$$

$$\gamma_{r\theta} = r \frac{d\phi}{dX}, \quad (8.164)$$

where ϵ is the axial strain, $\gamma_{r\theta}$ is the circumferential shear strain, r is the radial distance from the center, $\frac{d\phi}{dX}$ is the twist (rotation) per unit axial length, and X is the position along the axis.

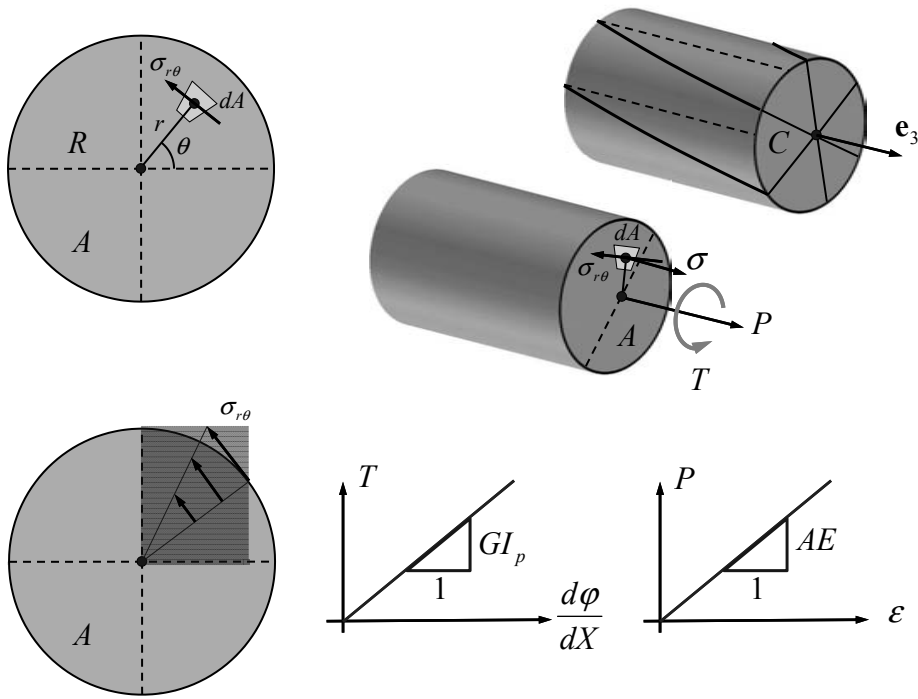


Figure 8.10: The elastic response of a circular bar subjected to an axial load and a torsional moment.

The corresponding equations for the axial load P and torsional moment T are given by

$$P = \int_A \sigma dA, \quad (8.165)$$

$$T = \int_A \sigma_{r\theta} r dA, \quad (8.166)$$

where A is the cross-sectional area, σ is the axial stress on the cross section, and $\sigma_{r\theta}$ is the circumferential torsional (shear) stress on the cross section. Typically, the axial response of the bar is given in terms of a plot of the load P as a function of the applied strain ϵ and the torsional response is given in terms of a plot of T as a function of $\frac{d\phi}{dX}$ shown in Figure 8.10.

8.5.1 Elastic torsion

For circular cross sections, we normally assume the response is a combination of simple shear and uniaxial extension. The typical loading and associated response for an elastic material is shown in Figure 8.9. The elastic response is linear so that torsion and extension can be decoupled. This is not true for the plastic response.

For uniform axial strain ϵ , assuming uniaxial extension, the axial stress for an elastic member is given by

$$\sigma = E\epsilon, \quad (8.167)$$

and the resulting axial load P can be obtained from the equation

$$P = \int_A \sigma dA = \int_A E\epsilon dA = EA\epsilon. \quad (8.168)$$

For a uniform axial load P , and given that the initial length of the bar is L , this equation gives the change in length ΔL as

$$\Delta L = \frac{PL}{EA}. \quad (8.169)$$

We also can calculate the axial stress σ from the load as

$$\sigma = \frac{P}{A}. \quad (8.170)$$

The shear stress for the elastic member is given by

$$\sigma_{r\theta} = Gr \frac{d\phi}{dX}, \quad (8.171)$$

and the torsional moment is evaluated from the equation

$$T = \int_A \sigma_{r\theta} r dA = \int_A Gr^2 \frac{d\phi}{dX} dA = GI_p \frac{d\phi}{dX}, \quad (8.172)$$

where I_p is the polar area moment of inertia of the cross section given by

$$I_p = \int_A r^2 dA. \quad (8.173)$$

For the elastic bar, the stress can be given by the applied torque as

$$\sigma_{r\theta} = \frac{Tr}{I_p}. \quad (8.174)$$

The corresponding expression for stress is given by

$$\boldsymbol{\sigma} = \sigma \mathbf{e}_3 \otimes \mathbf{e}_3 + \sigma_{r\theta} (\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r). \quad (8.175)$$

This results in a deviatoric stress given by

$$\mathbf{S} = \frac{1}{3} \sigma (-\mathbf{e}_r \otimes \mathbf{e}_r - \mathbf{e}_\theta \otimes \mathbf{e}_\theta + 2\mathbf{e}_3 \otimes \mathbf{e}_3) + \sigma_{r\theta} (\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r), \quad (8.176)$$

resulting in

$$\mathbf{S} : \mathbf{S} = \frac{2}{3} \sigma^2 + 2\sigma_{r\theta}^2. \quad (8.177)$$

We consider pure isotropic hardening. Therefore, the yield function can be written as

$$f = \tau - \tau_y = \sqrt{\frac{1}{3} \sigma^2 + \sigma_{r\theta}^2} - \tau_y. \quad (8.178)$$

For the fully elastic bar, we can write this as

$$f = \sqrt{\frac{1}{3} \frac{P^2}{A^2} + \frac{T^2 r^2}{I_p^2}} - \tau_y. \quad (8.179)$$

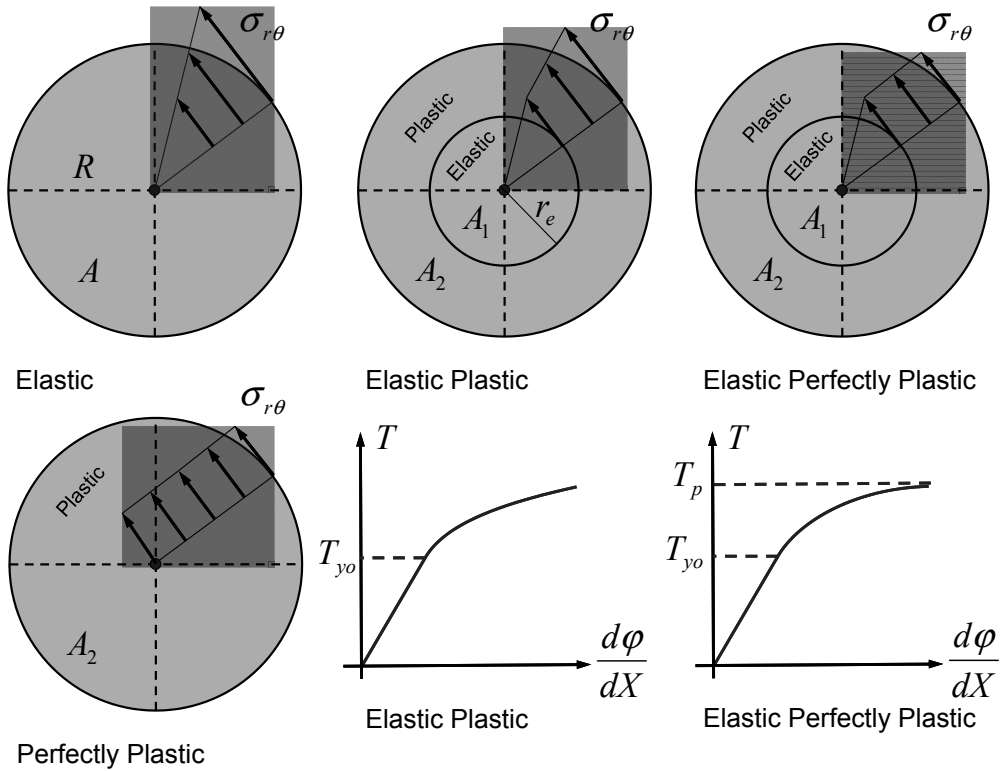


Figure 8.11: The elastic-plastic response of a circular bar subjected to an axial load and a torsional moment.

The maximum elastic load the bar can take then is given by reaching yield at the largest radius R . This condition can be written as

$$\frac{1}{3} \frac{P^2}{A^2} + \frac{T^2 R^2}{I_p^2} = \tau_{yo}^2. \quad (8.180)$$

As can be seen, as the axial load is increased, the largest torsional moment decreases. If there is no axial load, the maximum torsional moment that the bar can take elastically is $T = I_p \tau_{yo} / R$.

Section 8.5-Problems:

- 8.5-1:** Exactly the same equations are obtained for a hollow circular cylindrical bar. Derive the expressions and show that the only thing that needs to be done is to use the area of the cross section of the hollow cylinder as A , and the area polar moment of inertia of the cross section of the hollow cylinder as I_p .

8.5.2 Elastic-plastic torsion

Figure 8.11 shows the loading resulting for a bar that undergoes plastic flow. As can be seen, the center is undeformed by the torsion, as a result of the kinematic assumption, with

an elastic central section reaching the initial yield and plastic flow as the radius increases. As was shown by the yield function, both the axial load and the torsional moment play a role in yielding so the radius r_e of initial yield can be obtained from the equation $f = 0$ given by

$$\begin{aligned} f &= \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}^2 - \tau_{yo}} \\ &= \sqrt{\frac{1}{3}(E\epsilon)^2 + \left(Gr_e \frac{d\phi}{dX}\right)^2 - \tau_{yo}} = 0. \end{aligned} \quad (8.181)$$

For the remainder of the section, we will consider pure torsion. That is, we will set the axial load equal to zero so that the point of initial yield is given in terms of the twist by

$$r_e = \frac{\tau_{yo}}{G \frac{d\phi}{dX}}. \quad (8.182)$$

We use this to identify the two areas A_1 and A_2 , representing, respectively, the region of elastic flow and elastic-plastic flow. From pure shear, we know that the expression for stress in the region of plastic flow is given by

$$\sigma_{r\theta} = G\gamma_{yo} + \frac{G\tau_y'}{G + \tau_y'}(\gamma_{r\theta} - \gamma_{yo}) = \frac{G}{G + \tau_y'}(\tau_{yo} + \tau_y'\gamma_{r\theta}). \quad (8.183)$$

The torque now can be found from the equation

$$\begin{aligned} T &= \int_{A_1} \sigma_{r\theta} r dA + \int_{A_2} \sigma_{r\theta} r dA \\ &= \int_{A_1} G\gamma_{r\theta} r dA + \int_{A_2} \left[\frac{G}{G + \tau_y'}(\tau_{yo} + \tau_y'\gamma_{r\theta}) \right] r dA \\ &= \int_{A_1} G \frac{d\phi}{dX} r^2 dA + \int_{A_2} \left[\frac{G}{G + \tau_y'} \left(\tau_{yo} + \tau_y' \frac{d\phi}{dX} r \right) \right] r dA \\ &= G \frac{d\phi}{dX} \int_{A_1} r^2 dA + \frac{G}{G + \tau_y'} \left[\tau_{yo} \int_{A_2} r dA + \tau_y' \frac{d\phi}{dX} \int_{A_2} r^2 dA \right]. \end{aligned} \quad (8.184)$$

We can write this as

$$T = G \frac{d\phi}{dX} I_{p1} + \frac{G}{G + \tau_y'} \left(\tau_{yo} Q_{p2} + \tau_y' \frac{d\phi}{dX} I_{p2} \right), \quad (8.185)$$

where

$$Q_{pi} = \int_{A_i} r dA, \quad (8.186)$$

$$I_{pi} = \int_{A_i} r^2 dA. \quad (8.187)$$

In this case, it is easy to show that

$$Q_{p2} = \frac{2\pi}{3} (R^3 - r_e^3), \quad (8.188)$$

$$I_{p1} = \frac{\pi}{2} r_e^4, \quad (8.189)$$

$$I_{p2} = \frac{\pi}{2} (R^4 - r_e^4). \quad (8.190)$$

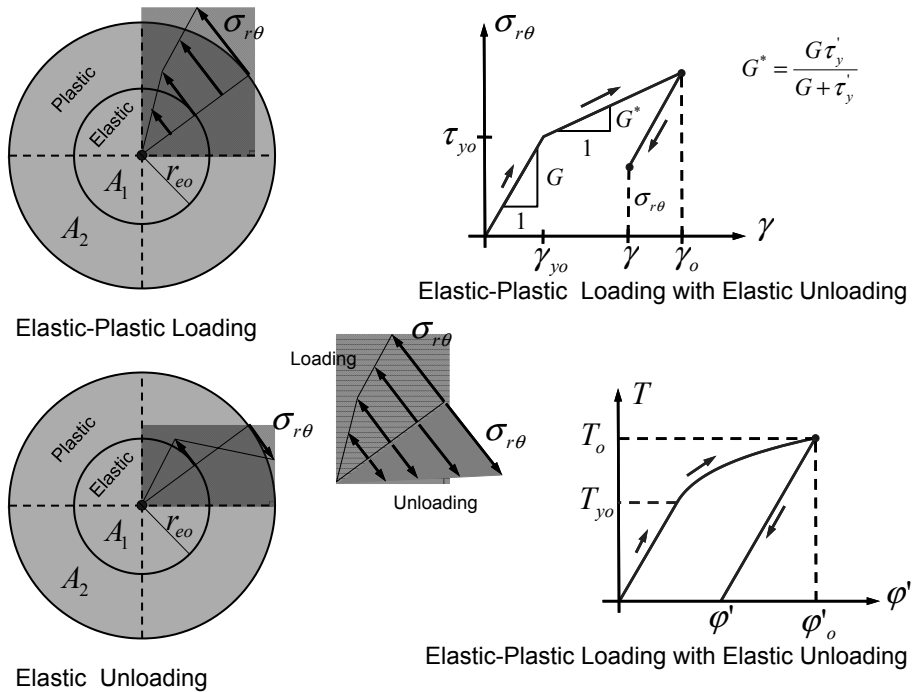


Figure 8.12: The elastic-plastic loading of a circular member in pure torsion followed by an elastic unloading of the member to zero torsional moment.

Section 8.5-Problems:

- 8.5-2:** Calculate the pure torsion moment T_p for the case of perfect plasticity.
- 8.5-3:** Develop the case of pure torsion for a hollow cylindrical bar.
- 8.5-4:** Simulate the simultaneous extension and torsion of a solid cylindrical bar.

8.6 Unloading

In homogeneous straining, unloading is always to zero stress. That is, a homogeneous body that is deformed homogeneously always can be fully unloaded by removing the boundary loads. This is not the case when considering inhomogeneous strain fields such as those seen for bending and torsion. In such cases, the system normally unloads to nonzero stresses even when the external loads are put to zero. For these cases, the internal stresses balance each other in a way that produces conditions of equilibrium between the parts of the body, making the applied load zero.

The two cases of bending and torsion that we studied were defined by prescribed forms of strain. When unloading these members, we assume also that the same conditions of strain

hold. Most more complicated problems are studied numerically since today's computational capabilities make this much easier to do than obtaining analytical solutions. In Chapter 1 we studied different methods to obtain numerical results for plasticity models, which can be used here for each specific problem, with slight modification to handle three-dimensional situations. The one drawback of numerical solutions is that they do not provide analytical equations that we can use to better understand the relation between the components of the problem.

We will focus here on pure torsion since the solution is simple and allows us to see the result of the unloading. Obviously, if the entire bar remains elastic then the unloading is elastic and results in a zero stress distribution. The more interesting case is that for which we have applied a large enough twist that it results in an elastic core section of radius r_e and an outer elastic-plastic section. To designate the initial monotonic loading, we consider the applied twist per unit length at the end of the monotonic loading to be ϕ'_o and the corresponding torque to be T_o . From the equations of torsion, we know that

$$\gamma_o = r\phi'_o, \quad (8.191)$$

$$r_{eo} = \frac{\tau_{yo}}{G\phi'_o}, \quad (8.192)$$

$$T_o = G\phi'_o I_{p1o} + \frac{G}{G + \tau'_y} (\tau_{yo} Q_{p2o} + \tau'_y \phi'_o I_{p2o}), \quad (8.193)$$

where I_{p1o} and Q_{p2o} are the corresponding values of I_{pi} and Q_{p2} for $r_e = r_{eo}$.

The controlling condition when unloaded is that the torsional moment needs to be zero, even though the twist will not be zero. For simplicity in the notation, we will take the twist after unloading to be ϕ' . We will assume elastic unloading so that in the elastic range we have the expression for the stress given by

$$\sigma_{r\theta} = G\gamma = G\phi' r. \quad (8.194)$$

In the plastic zone, as shown in Figure 8.12, the stress is given by

$$\sigma_{r\theta} = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}) - G(\gamma_o - \gamma), \quad (8.195)$$

where

$$G^* = \frac{G\tau'_y}{G + \tau'_y}. \quad (8.196)$$

Considering that $\tau_{yo} = G\gamma_{yo}$, $\gamma_o = r\phi'_o$, and $\gamma = r\phi'$, this becomes

$$\sigma_{r\theta} = \tau_{yo} + G^*(\phi'_o r - \frac{\tau_{yo}}{G}) - G(\phi'_o r - \phi' r). \quad (8.197)$$

The torsional moment can be evaluated by putting the stresses into equation for the moment to obtain

$$\begin{aligned} T &= \int_{A_1} \sigma_{r\theta} r dA + \int_{A_2} \sigma_{r\theta} r dA \\ &= \int_{A_1} G\phi' r^2 dA + \int_{A_2} [\tau_{yo} + G^*(\phi'_o r - \frac{\tau_{yo}}{G}) - G(\phi'_o r - \phi' r)] r dA \\ &= GI_{p1o}\phi' + \tau_{yo}Q_{p2o} + G^*\phi'_o I_{p2o} - \frac{G^*\tau_{yo}}{G}Q_{p2o} - GI_{p2o}\phi'_o + GI_{p2o}\phi'. \end{aligned} \quad (8.198)$$

As can be seen in Figure 8.12, ultimately the unloading is elastic and represents the subtraction of a linearly increasing stress with the radius. This can be written as

$$T = T_o - GI_p(\phi'_o - \phi'). \quad (8.199)$$

This shows that for elastic unloading the relation between the torsional moment and the angle of twist is linear during unloading. As a result of this equation, one also can calculate the twist as a function of the applied torsional moment from the equation

$$\phi' = \phi'_o + \frac{T - T_o}{GI_p}. \quad (8.200)$$

For total unloading, we will have $T = 0$ to get

$$\phi' = \phi'_o - \frac{T_o}{GI_p}. \quad (8.201)$$

The stress response can be obtained from the original stress response at T_o by subtracting the elastic unloading, which is a linearly varying stress along the radius for a negative twist of $\phi'_o - \phi'$. This is shown in the Figure 8.12.

The case of parts of the cross section undergoing reverse plastic flow also can occur. To identify this case, we need to check the stresses at the points to make sure they have not yielded. This easily can be done when numerically modeling the problem.

Section 8.6-Problems:

- 8.6-1:** Simulate the monotonic and cyclic loading of a circular bar subjected to pure torsion.
- 8.6-2:** Simulate the monotonic and cyclic torsional loading of a circular bar subjected to a constant axial load.
- 8.6-3:** Simulate the monotonic and cyclic bending of a rectangular bar in pure bending around one of its two principal axes.
- 8.6-4:** Simulate the monotonic and cyclic response of a square bar subjected to equal bending along the two principal axes of the section.
- 8.6-5:** Simulate the monotonic and cyclic response in bending of a rectangular bar subjected to bending along one of its principal axes, assuming a given constant axial load.

8.7 Torsion of prismatic sections

The solution for torsion of a prismatic section is the first nonhomogeneous problem we consider that actually uses the equilibrium equations to solve for the response. Figure 8.13 shows a schematic description of the problem. Again, as in the previous sections, we consider a given deformation field in terms of unknowns that then will be solved for through the equilibrium equation. Before we start, we will point out that the solution has some historic importance since the elastic solution has the same equations as that of a membrane and so one can understand the solution by observing the shape of a blown on soap bubble that is put on a wire in the shape of the cross section. The perfectly plastic solution also has special significance since the solution resembles the shape of a pile of sand put on a plate cut in the shape of the cross section.

The basic kinematic assumptions for torsion of prismatic sections builds on two deformations. One is of rigid-body twisting of cross sections and the other is described by uniform

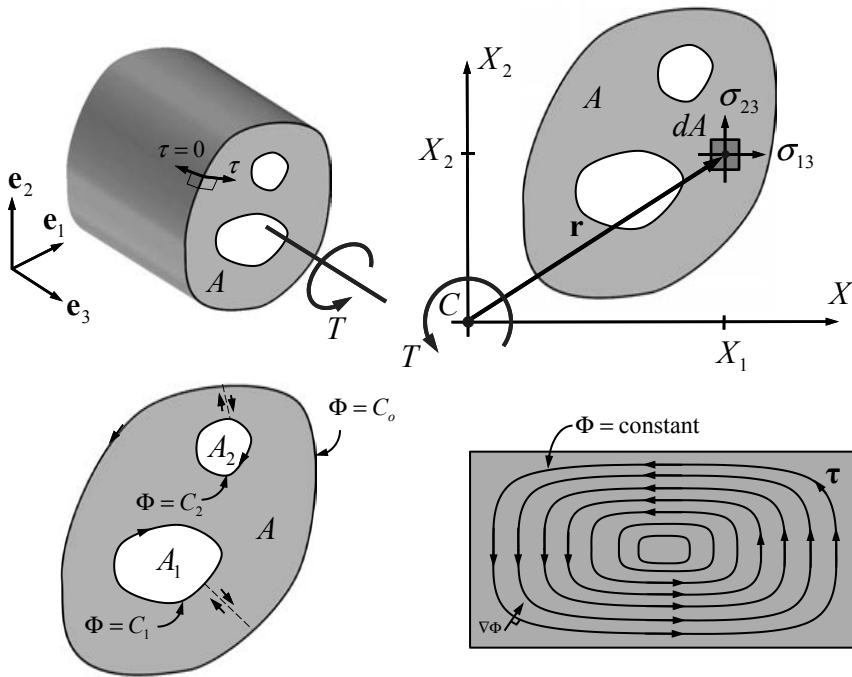


Figure 8.13: Elastic torsion of prismatic sections.

warping along the axis of the cross sections. If the bar is along the $\mathbf{e}_3$ direction, uniform rigid-body rotation is given by $\phi' \mathbf{e}_3 \times \mathbf{X}$, where ϕ' , just as above, is the twisting of the bar per unit length. Twisting only rotates points around the axis of the bar. On the other hand, warping describes motions along the bar of points on the surface of the cross section but is assumed to be the same for all the cross sections of the bar. Therefore, the current displacement of points is given by the equation

$$\mathbf{u} = (\phi' X_3 \mathbf{e}_3) \times \mathbf{X} + w(X_1, X_2, \phi') \mathbf{e}_3, \quad (8.202)$$

where $w(X_1, X_2, \phi')$ is the warping function. The result of this displacement field is a strain field that is given by

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{12} = 0, \quad (8.203)$$

$$\epsilon_{13} = \frac{1}{2} \left(\frac{\partial w}{\partial X_1} - \phi' X_2 \right), \quad (8.204)$$

$$\epsilon_{23} = \frac{1}{2} \left(\frac{\partial w}{\partial X_2} + \phi' X_1 \right). \quad (8.205)$$

The equilibrium equation without consideration of any body forces can be written as

$Div(\boldsymbol{\sigma}) = \mathbf{0}$. This has three components written as

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{21}}{\partial X_2} + \frac{\partial \sigma_{31}}{\partial X_3} = 0, \quad (8.206)$$

$$\frac{\partial \sigma_{12}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \sigma_{32}}{\partial X_3} = 0, \quad (8.207)$$

$$\frac{\partial \sigma_{13}}{\partial X_1} + \frac{\partial \sigma_{23}}{\partial X_2} + \frac{\partial \sigma_{33}}{\partial X_3} = 0. \quad (8.208)$$

Assuming the stress to be homogeneous along the bar (i.e., is the same for all cross sections), we have $\boldsymbol{\sigma}(X_1, X_2)$. Since the stress is independent of X_3 , the three equations become

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{21}}{\partial X_2} = 0, \quad (8.209)$$

$$\frac{\partial \sigma_{12}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} = 0, \quad (8.210)$$

$$\frac{\partial \sigma_{13}}{\partial X_1} + \frac{\partial \sigma_{23}}{\partial X_2} = 0. \quad (8.211)$$

The traction vector on the cross section has components given by

$$\mathbf{t}^{e_3} = \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3, \quad (8.212)$$

so that the moment on any cross section about the point $(X_1, X_2) = (0, 0)$ is given by

$$T = \int_A (X_1 \sigma_{23} - X_2 \sigma_{13}) dA. \quad (8.213)$$

8.7.1 Elastic torsion

For elastic torsion, as a result of the assumed strain field, all of the stresses other than σ_{13} and σ_{23} become zero. In this case, we have

$$\sigma_{13} = 2G\epsilon_{13} = G \left(\frac{\partial w}{\partial X_1} - \phi' X_2 \right), \quad (8.214)$$

$$\sigma_{23} = 2G\epsilon_{23} = G \left(\frac{\partial w}{\partial X_2} + \phi' X_1 \right). \quad (8.215)$$

The relation between the stresses and the twist can be obtained by eliminating the warping to get

$$\frac{\partial \sigma_{23}}{\partial X_1} - \frac{\partial \sigma_{13}}{\partial X_2} = 2G\phi', \quad (8.216)$$

which is the twisting equation, that when combined with the equilibrium equation can be solved for the shear stresses on the surface. Once we introduce the zero stresses, the only remaining equilibrium equation is given as

$$\frac{\partial \sigma_{13}}{\partial X_1} + \frac{\partial \sigma_{23}}{\partial X_2} = 0. \quad (8.217)$$

The equilibrium equation automatically can be satisfied by selecting a stress function, also known as a potential function, $\Phi(X_1, X_2)$, such that the nonzero stresses are defined as

$$\sigma_{13} = \frac{\partial \Phi}{\partial X_2}, \quad \sigma_{23} = -\frac{\partial \Phi}{\partial X_1}. \quad (8.218)$$

The twisting equation then becomes

$$\frac{\partial^2 \Phi}{\partial X_1^2} + \frac{\partial^2 \Phi}{\partial X_2^2} = -2G\phi'. \quad (8.219)$$

This is the same equation as that of the membrane under pressure, having a solution of contour lines that must be parallel to the cross-section boundaries, so that the boundary condition associated with zero shear stress on the lateral surfaces can be satisfied.

Let us now look at the relation between the direction of the shear stress vector and the gradient of the stress potential. The reader will note that the direction of the gradient of Φ is the direction of largest changes in Φ given by

$$\mathbf{Grad}(\Phi) = \frac{\partial \Phi}{\partial X_1} \mathbf{e}_1 + \frac{\partial \Phi}{\partial X_2} \mathbf{e}_2 = -\sigma_{23} \mathbf{e}_1 + \sigma_{13} \mathbf{e}_2. \quad (8.220)$$

These are the directions perpendicular to the contour lines of Φ (i.e., the contour lines are the lines of constant Φ). The shear traction on the surface is a vector denoted by $\boldsymbol{\tau}$ and given by

$$\boldsymbol{\tau} = \sigma_{13} \mathbf{e}_1 + \sigma_{23} \mathbf{e}_2. \quad (8.221)$$

It can be seen that the

$$\mathbf{Grad}(\Phi) \circ \boldsymbol{\tau} = (-\sigma_{23} \mathbf{e}_1 + \sigma_{13} \mathbf{e}_2) \circ (\sigma_{13} \mathbf{e}_1 + \sigma_{23} \mathbf{e}_2) = 0, \quad (8.222)$$

so that the component of $\boldsymbol{\tau}$ along the direction of the gradient of Φ is zero. Therefore, the shear stress vector $\boldsymbol{\tau}$ always is pointing along the contour lines of Φ . Also, since along the boundary the shear stress is zero, it follows that the gradient of Φ is perpendicular to the boundary. In other words, the contour lines of Φ must be parallel to the boundary. This also proves that Φ takes a constant value along each boundary, potentially being a different constant around each closed boundary. Since the constant of integration in this case is arbitrary, the value of Φ on one of the boundaries can be taken as zero.

Another point to note is that the magnitude of the gradient of Φ is equal to the magnitude of the stress vector. Therefore, the places with the largest gradient are also the places with the largest shear stress. For example, for the rectangular section shown in Figure 8.13, the maximum shear stresses are at the top and bottom centers where the contour lines are the closest.

The equation for the axial moment can be written in terms of the potential function Φ as

$$T = - \int_A \left(X_1 \frac{\partial \Phi}{\partial X_1} + X_2 \frac{\partial \Phi}{\partial X_2} \right) dA. \quad (8.223)$$

If we consider $\mathbf{Grad}(\Phi)$ as the two-dimensional gradient and $\mathbf{X}$ as the two-dimensional position vector, we can show that

$$\begin{aligned} T &= - \int_A \mathbf{X} \circ \mathbf{Grad}(\Phi) dA \\ &= - \int_A \text{Div}(\Phi \mathbf{X}) dA + \int_A \Phi \text{Div}(\mathbf{X}) dA \\ &= - \int_C \Phi \mathbf{X} \circ \mathbf{n} dC + 2 \int_A \Phi dA, \end{aligned} \quad (8.224)$$

where C is the contour of the line defining the boundary of A , where we have used the divergence theorem to move the area integral to a line integral, and where we have noted

that in two dimensions $\text{Div}(\mathbf{X}) = 2$. If the cross section is such that we have an external boundary $\mathcal{C}_o$ and internal boundaries $\mathcal{C}_i$ for i from 1 to n , we can rewrite this equation as

$$T = - \int_{\mathcal{C}_o} \Phi \mathbf{X} \circ \mathbf{n} d\mathcal{C} + \sum_{i=1}^n \int_{\mathcal{C}_i} \Phi \mathbf{X} \circ \mathbf{n} d\mathcal{C} + 2 \int_A \Phi dA, \quad (8.225)$$

where the line integral is taken counterclockwise on the outer contour and clockwise on each of the inner contours. Figure 8.13 shows the contour cuts used in the integration process. As will be shown in the following, the stress function is constant over each closed boundary. If the stress function takes the value of C_i on the section $\mathcal{C}_i$, we then can write the axial torque as

$$T = -2C_o A_o + \sum_{i=1}^n 2C_i A_i + 2 \int_A \Phi dA, \quad (8.226)$$

where A_i is the area inside the contour $\mathcal{C}_i$ as show in Figure 8.13 ($A = A_o - \sum_{i=1}^n A_i$). For a solid cross section, we will not have any internal contours, and since we can take the Φ on the external contour equal to zero we can conclude that in this case

$$T = 2 \int_A \Phi dA, \quad (8.227)$$

which shows that the torque is twice the volume under the plot of Φ .

Section 8.7-Problems:

- 8.7-1:** For thin hollow cross sections, the solution can be considered one dimensional, with variations only perpendicular to the boundary. We can write the equation in this case as

$$\frac{\partial^2 \Phi}{\partial s^2} = -2G\phi', \quad (8.228)$$

where s is a coordinate perpendicular to the surface of the section as shown in Figure 8.15, and we assume $\Phi(s)$. Solve this system for Φ and use it to estimate the torsional response of a thin hollow cylindrical section constructed from a plate of uniform cross-sectional thickness t and length L (i.e., L is the length of the plate in the cross section).

8.7.2 Perfect plasticity

As we apply load, at one point the elastic shear stress reaches its limit and parts of the cross section start going through plastic flow. If the material is perfectly plastic, in the regions that undergo plastic deformation the magnitude of the shear stress will be equal to the yield stress. That is, we get regions in which we have

$$\sigma_{13}^2 + \sigma_{23}^2 = \tau_{yo}^2. \quad (8.229)$$

In terms of the potential function Φ , this can be written as

$$\left(\frac{\partial \Phi}{\partial X_1} \right)^2 + \left(\frac{\partial \Phi}{\partial X_2} \right)^2 = \tau_{yo}^2, \quad (8.230)$$

or alternately as

$$|\mathbf{Grad}(\Phi)| = \tau_{yo}. \quad (8.231)$$

Perfectly Plastic

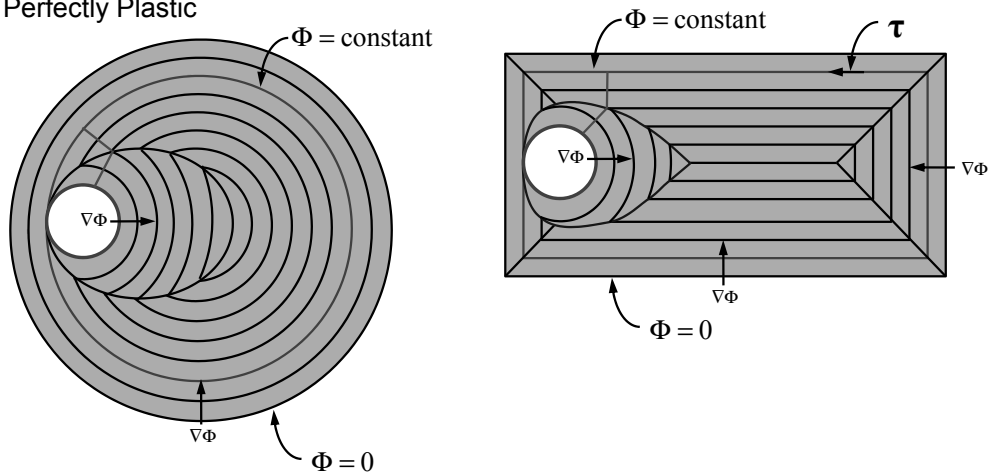


Figure 8.14: Perfectly plastic torsion with holes.

Therefore, the gradient of Φ has a constant magnitude equal to τ_{yo} inside the range that undergoes perfect plastic deformation. The solution in the plastic range resembles that of a sand hill or of rooftops so that Φ has a constant slope of τ_{yo} normal to the boundaries, both satisfying the zero shear stress boundary condition and the yield function. As in the case of elastic torsion, holes in the cross section represent lines of constant Φ , the constant value possibly being different from the values at other free surfaces. The case of torsion of a section with a circular hole is shown in Figure 8.14 both in the case of a circular and a rectangular bar. As can be seen, the value of Φ on the hole is constant, but different from zero. In this case, the value of Φ on the inside circle is shown in blue. Also, the red lines show that for perfect plasticity the magnitude of the gradient is constant so the associated perpendicular distances are the same for fixed changes of the contour line values.

For the elastic perfectly plastic response, the member has two ranges, one elastic following the membrane theory, and the other being perfectly plastic following the rooftop solution. The end of the elastic range is defined by yielding, so the transition is continuous. Figure 8.15 shows a rectangular cross section that has a solution that is a combination of the bubble analogy for the central elastic region and the rooftop analogy for the end regions that have undergone plastic flow.

Section 8.7-Problems:

- 8.7-2:** Discuss the difference between a thin hollow section under torsion with and without a slit that connects the inner and outer parts.
- 8.7-3:** For a thin hollow section with a slit and subjected to torsion, estimate the twist-moment relation for purely elastic and for perfectly-plastic response.
- 8.7-4:** Construct a diagram showing Φ for a rectangular bar under torsion, assuming perfect plastic response and a circular hole at its center.
- 8.7-5:** Assuming the kinematics of pure torsion, construct the theory that includes strain hardening and then simulate the torsion-twist relation for both monotonic and cyclic loading.

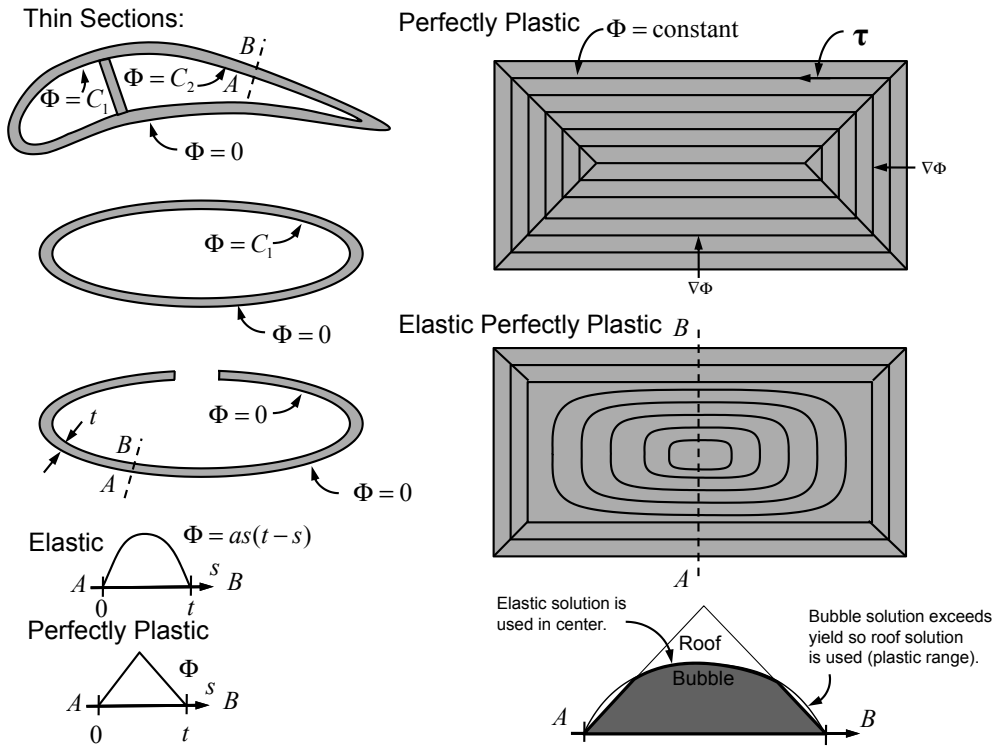


Figure 8.15: Elastic, perfectly plastic, and elastic perfectly plastic torsion of prismatic sections.

8.8 Nonuniform loading of bars

We now will consider the deformation of bars under multiple loading, possibly varying along the length of the bar. In doing so, we will assume small deformations and eliminate certain terms. We already have considered a bar in tension in Chapter 1, so we will focus more only on bending and torsion. In each case, we can study both equilibrium and dynamic problems, and will look for one-dimensional problems. These problems are important in that they result in one-dimensional equations with applications in civil and industrial structures.

The general description of this problem is given in Figure 8.16. The kinematic parameters that describe the simplest assumption about the deformation of a straight beam are the three components of displacement of the “centerline” $\mathbf{u}$ and the one component of axial rotation ϕ . We consider a bar along the X_3 direction, so that u_3 is the axial extension. For this beam u_1 and u_2 are the deflection, respectively, of the “center” line of the beam along the first and second directions. The rotation ϕ describes the twisting rotation around X_3 of cross sections of the bar.

The balance equations for a bar are calculated by looking at elements of the bar, assuming no point loads, moments, or torques. On a cross section, we assume there are two shear forces and one axial force applied. We denote the resultant force by $\mathbf{V}$ which has components

$$\mathbf{V} = V_1 \mathbf{e}_1 + V_2 \mathbf{e}_2 + V_3 \mathbf{e}_3, \quad (8.232)$$

where V_1 and V_2 are shear loads and V_3 is the axial load P described before. The moment

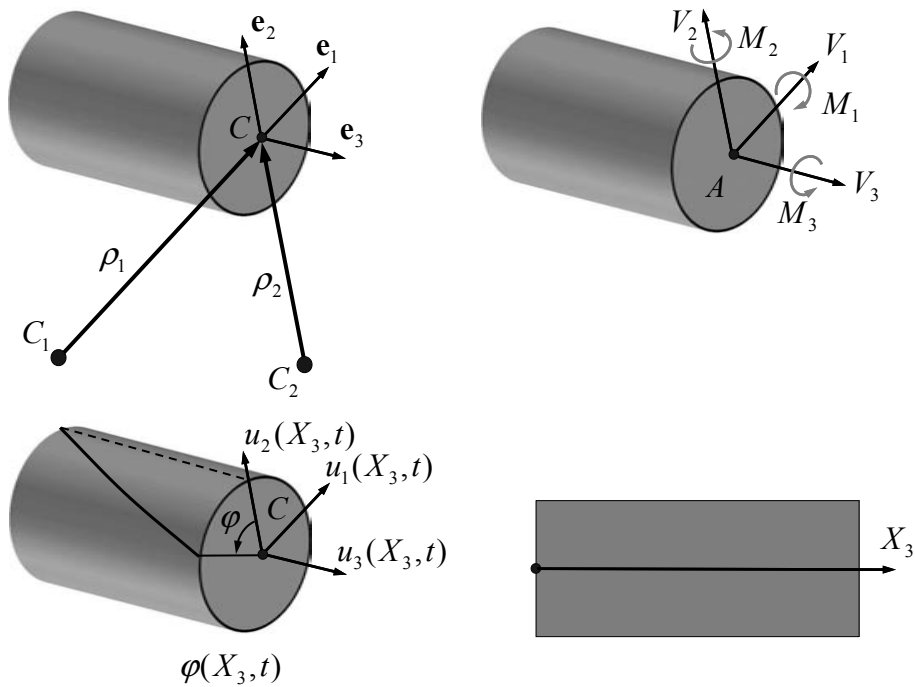


Figure 8.16: Nonuniform loading of a bar due to loads and moments.

on the cross-sectional area is given by the vector $\mathbf{M}$ given in component form

$$\mathbf{M} = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2 + M_3 \mathbf{e}_3, \quad (8.233)$$

where M_1 and M_2 are bending moments, as described above, and M_3 is the torsional moment described as T above. We will assume that a distributed load $\mathbf{w}$ is applied on the bar that has components

$$\mathbf{w} = w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2 + w_3 \mathbf{e}_3, \quad (8.234)$$

where each component of $\mathbf{w}$ is the force per unit length of X_3 applied on the lateral surfaces of the bar. We will use q to denote the torsional moment per unit length around X_3 applied to the lateral surface. In addition to the loading from the distributed loads, there can be body forces. We will denote the body force per unit mass as $\mathbf{b}$ so that we have

$$\mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3. \quad (8.235)$$

The drawings in Figure 8.17 are typical free-body diagrams for bending and torsional elements of the body. From these we can calculate the balance equations and jump conditions. The balance of linear momentum for the beam, with appropriate approximations, will

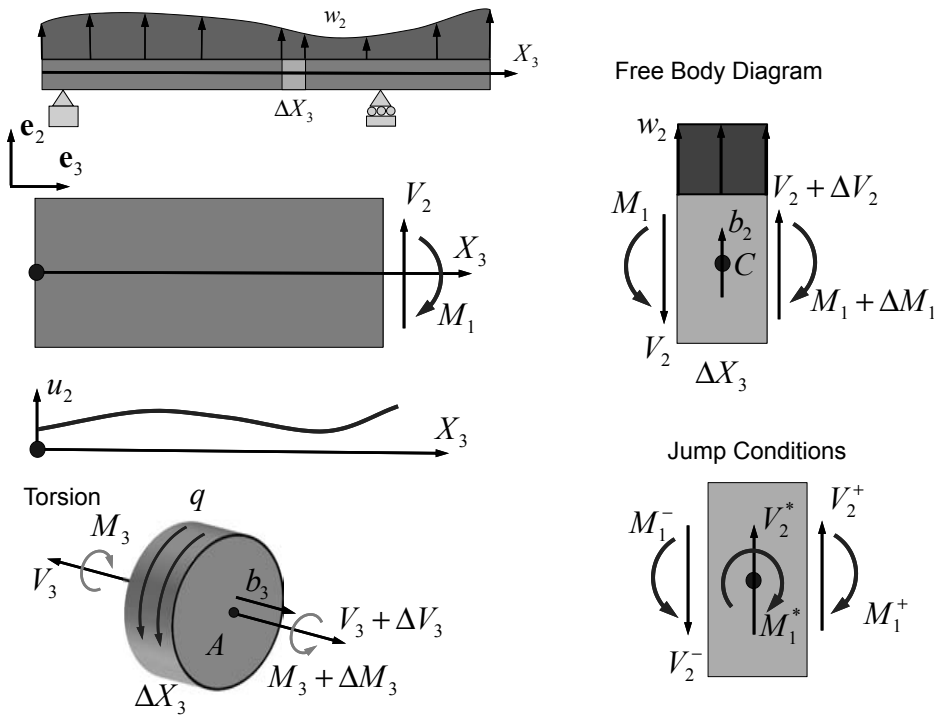


Figure 8.17: Diagrams to calculate balance laws and jump conditions for beams under distributed and point loads and moments.

result in the equations

$$\frac{\partial V_1}{\partial X_3} + w_1 + \rho A b_1 = \rho A \ddot{u}_1, \quad (8.236)$$

$$\frac{\partial V_2}{\partial X_3} + w_2 + \rho A b_2 = \rho A \ddot{u}_2, \quad (8.237)$$

$$\frac{\partial V_3}{\partial X_3} + w_3 + \rho A b_3 = \rho A \ddot{u}_3, \quad (8.238)$$

where ρ is the density, not to be confused with the radii of curvature ρ_i . The equations for the balance of angular momentum, again considering appropriate approximations, can be written for the bar as

$$-\frac{\partial M_1}{\partial X_3} + V_2 = \rho \bar{I}_{11} \frac{\partial \ddot{u}_2}{\partial X_3}, \quad (8.239)$$

$$\frac{\partial M_2}{\partial X_3} + V_1 = \rho \bar{I}_{22} \frac{\partial \ddot{u}_1}{\partial X_3}, \quad (8.240)$$

$$\frac{\partial M_3}{\partial X_3} + q = \rho \bar{I}_p \ddot{\phi}, \quad (8.241)$$

where $\bar{I}_{11}$ and $\bar{I}_{22}$ are, respectively, the area moments of inertia about the first and second axes, and $\bar{I}_p = \bar{I}_{11} + \bar{I}_{22}$ is the area polar moment of inertia around the third axis. This can

be calculated from

$$\bar{I}_{11} = \int_A X_2^2 dA, \quad (8.242)$$

$$\bar{I}_{22} = \int_A X_1^2 dA. \quad (8.243)$$

The balance equations always can be used to eliminate V_1 and V_2 to obtain four equations.

The balance laws as described in the differential forms above relate to continuous fields of $\mathbf{V}$ and $\mathbf{M}$ so that the derivatives are meaningful. Whenever there is a point load or moment, such as under a support or at the root of a cantilevered beam, we need to add jump conditions to relate the values of $\mathbf{V}$ and $\mathbf{M}$ on two sides of the point load and/or moment. As before, we will use “+” to denote the right and “-” to denote the left of the jump as shown in Figure 8.17. The jump conditions under static conditions or when the line of discontinuity moves with the material become

$$\mathbf{V}^+ = \mathbf{V}^- - \mathbf{V}^*, \quad (8.244)$$

$$\mathbf{M}^+ = \mathbf{M}^- - \mathbf{M}^*. \quad (8.245)$$

In general, the jump conditions are given by

$$|[\mathbf{V}]| + \mathbf{V}^* = |[\rho A \dot{\mathbf{u}}_3 - v^*]|, \quad (8.246)$$

$$|[\mathbf{M}]| + \mathbf{M}^* = |[\rho \bar{\mathbf{I}} \boldsymbol{\omega}(\dot{\mathbf{u}}_3 - v^*)]|, \quad (8.247)$$

where v^* is the velocity of the discontinuity along the bar and

$$\bar{\mathbf{I}} \boldsymbol{\omega} = \bar{I}_{11} \omega_1 \mathbf{e}_1 + \bar{I}_{22} \omega_2 \mathbf{e}_2 + \bar{I}_{33} \omega_3 \mathbf{e}_3, \quad (8.248)$$

for

$$\omega_1 = -\frac{\partial \dot{u}_2}{\partial X_3}, \quad (8.249)$$

$$\omega_2 = \frac{\partial \dot{u}_1}{\partial X_3}, \quad (8.250)$$

$$\omega_3 = \dot{\phi}. \quad (8.251)$$

In general, the three components of the moment and the axial shear load are given through material constitutive models that are given in terms of the histories of the two curvatures ρ_i , the axial strain ϵ , and the rate of twisting per unit length ϕ' . These are given by the approximations

$$\frac{1}{\rho_1} = -\frac{\partial^2 u_1}{\partial X_3^2}, \quad (8.252)$$

$$\frac{1}{\rho_2} = -\frac{\partial^2 u_2}{\partial X_3^2}, \quad (8.253)$$

$$\epsilon = \frac{\partial u_3}{\partial X_3}, \quad (8.254)$$

$$\phi' = \frac{\partial \phi}{\partial X_3}, \quad (8.255)$$

where $\mathbf{u}$ and ϕ are assumed to be functions of X_3 and time t . The radii of curvature relations and the axial strain are restricted to small slopes, so the first derivatives of u_i with respect to X_3 must be small.

Section 8.8-Problems:

- 8.8-1:** Show that for a differential element of length ΔX_3 we have the three components of the differential angular momentum ΔH given by

$$\Delta H_1 = [\alpha_1 \bar{I}_{11} + \omega_2 \omega_3 (\bar{I}_{33} - \bar{I}_{22})] \rho \Delta X_3, \quad (8.256)$$

$$\Delta H_2 = [\alpha_2 \bar{I}_{22} - \omega_1 \omega_3 (\bar{I}_{33} - \bar{I}_{11})] \rho \Delta X_3, \quad (8.257)$$

$$\Delta H_3 = [\alpha_3 \bar{I}_{33} + \omega_1 \omega_2 (\bar{I}_{22} - \bar{I}_{11})] \rho \Delta X_3, \quad (8.258)$$

where

$$\alpha_1 = -\frac{\partial \ddot{u}_2}{\partial X_3}, \quad (8.259)$$

$$\alpha_2 = \frac{\partial \ddot{u}_1}{\partial X_3}, \quad (8.260)$$

$$\alpha_3 = \ddot{\phi}. \quad (8.261)$$

- 8.8-2:** To construct the full derivation of the beam equations, we can start from the balance of mass, linear momentum and angular momentum. Before we do this, we make kinematic assumptions on how the bar is described. We take the bar to be a straight bar situated along the $\mathbf{e}_3$ direction and where points on the bar are given by

$$\mathbf{X} = \mathbf{X}_o + X_3 \mathbf{e}_3 + \mathbf{R}, \quad (8.262)$$

where $\mathbf{X}_o$ is on the axis of the bar and at location $X_3 = 0$, and $\mathbf{R} = R_1 \mathbf{e}_1 + R_2 \mathbf{e}_2$ is the relative position of the point to the centerline along a line perpendicular to the axis. The motion $\mathbf{x}$ of a point on the bar is assumed to be given by

$$\mathbf{x} = \mathbf{X} + \mathbf{u} + \mathbf{\Phi} \times \mathbf{R} = \mathbf{X}_o + X_3 \mathbf{e}_3 + \mathbf{R} + \mathbf{u} + \mathbf{\Phi} \times \mathbf{R}, \quad (8.263)$$

where $\mathbf{u}$ is the displacement of the centerline, and $\mathbf{\Phi}$ is the rotation of the cross section given by

$$\Phi_1 = -\frac{\partial u_2}{\partial X_3}, \quad (8.264)$$

$$\Phi_2 = \frac{\partial u_1}{\partial X_3}, \quad (8.265)$$

$$\Phi_3 = \phi. \quad (8.266)$$

Here we are assuming that $\mathbf{u}(X_3, t)$, $\mathbf{\Phi}(X_3, t)$, and $\phi(X_3, t)$ is the rotation of the cross section around its axis. From this we have the expression for the angular velocity $\boldsymbol{\omega} = \dot{\mathbf{\Phi}}$ and the angular acceleration $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$. The expression for the velocity and acceleration of a particle is given by

$$\mathbf{v} = \dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{R}, \quad (8.267)$$

$$\mathbf{a} = \ddot{\mathbf{u}} + \boldsymbol{\alpha} \times \mathbf{R}. \quad (8.268)$$

For the bar we assume that the density in the reference configuration ρ_o , the cross-sectional area in the reference configuration A_o and the mass per unit length in the reference configuration ϖ are each only a function of the position along the bar X_3 . Show that the linear momentum $\mathbf{L}$ of the body between points a_o and b_o in the reference configuration is given by

$$\mathbf{L} = \int_B \mathbf{v} dm = \int_{a_o}^{b_o} \rho_o A_o \dot{\mathbf{u}} dX_3 + \int_{a_o}^{b_o} \rho_o A_o \boldsymbol{\omega} \times \mathbf{R}_C dX_3, \quad (8.269)$$

where $\mathbf{R}_C$ is the location of the centroid of the cross section relative to the X_3 axis given by

$$A_o \mathbf{R}_C = \int_{A_o} \mathbf{R} dA_o. \quad (8.270)$$

Show that the rate of change of the linear momentum is given by

$$\dot{\mathbf{L}} = \frac{D}{Dt} \left(\int_{\mathcal{B}} \mathbf{v} dm \right) = \int_{a_o}^{b_o} \rho_o A_o \ddot{\mathbf{u}} dX_3 + \int_{a_o}^{b_o} \rho_o A_o \boldsymbol{\alpha} \times \mathbf{R}_C dX_3. \quad (8.271)$$

Assume the surface of the bar is separated into the end surfaces S_a and S_b and the lateral surface S_L and show that the total traction applied on the body is given by

$$\int_S \mathbf{t}^{(n)} dS = \int_{S_o} \mathbf{t}_o^{(N)} dS_o = \mathbf{V}_b - \mathbf{V}_a + \int_{a_o}^{b_o} \mathbf{w}_o dX_3, \quad (8.272)$$

where $\mathbf{V}_a$ and $\mathbf{V}_b$ are the loads, respectively, on S_a and S_b , and where $\mathbf{w}_o$ is the load on the lateral surface per unit length of the bar. Show that as a result of these, the equation for the balance of linear momentum can be written as

$$\mathbf{V}_b - \mathbf{V}_a + \int_{a_o}^{b_o} \mathbf{w}_o dX_3 + \int_{a_o}^{b_o} \rho_o A_o \mathbf{b} dX_3 + \sum \mathbf{V} = \int_{a_o}^{b_o} \rho_o A_o \ddot{\mathbf{u}} dX_3, \quad (8.273)$$

where we have assumed that X_3 passes through the centroid of the cross sections (i.e., $\mathbf{R}_C = \mathbf{0}$), and $\mathbf{b}$ is the body force per unit mass that is assumed to depend only on X_3 , and $\sum \mathbf{V}$ are all the applied point loads on the lateral surfaces. For the segments between point loads, show that the equation can be written in differential form as

$$\frac{\partial \mathbf{V}}{\partial X_3} + \mathbf{w}_o + \rho_o A_o \mathbf{b} = \rho_o A_o \ddot{\mathbf{u}}. \quad (8.274)$$

8.8-3: Following the development of the last problem, using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{a} \circ \mathbf{b})\mathbf{c}$, show that the expression for angular momentum can be written as

$$\begin{aligned} \mathbf{H}_O &= \int_{\mathcal{B}} (\mathbf{x} - \mathbf{X}_O) \times \mathbf{v} dm \\ &= \int_{\mathcal{B}} (X_3 \mathbf{e}_3 + \mathbf{R} + \mathbf{u} + \boldsymbol{\Phi} \times \mathbf{R}) \times (\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{R}) dm \\ &= \mathbf{e}_3 \times \int_{a_o}^{b_o} \rho_o A_o X_3 \dot{\mathbf{u}} dX_3 \\ &\quad + \int_{a_o}^{b_o} \rho_o [\bar{\mathbf{I}} \boldsymbol{\omega} + A_o \mathbf{u} \times \dot{\mathbf{u}} + (\boldsymbol{\omega} \times \boldsymbol{\Phi}) \mathbf{I}^*] dX_3, \end{aligned} \quad (8.275)$$

where we have assumed that the X_3 axes passes through the centroid of the cross sections in the reference configuration (i.e., we set $\mathbf{R}_C = \mathbf{O}$), and the two terms $\bar{\mathbf{I}}$ and $\mathbf{I}^*$ used in the equation are defined as

$$\bar{\mathbf{I}} = \int_{A_o} \mathbf{R} \circ \mathbf{R} dA_o \mathbf{I} - \mathbf{I}^*, \quad (8.276)$$

$$\mathbf{I}^* = \int_{A_o} \mathbf{R} \otimes \mathbf{R} dA_o, \quad (8.277)$$

where $\mathbf{I}$ here is the identity tensor. From this show that the angular acceleration is given by

$$\dot{\mathbf{H}}_O = \int_{a_o}^{b_o} \rho_o [A_o X_3 \mathbf{e}_3 \times \ddot{\mathbf{u}} + \bar{\mathbf{I}} \alpha + A_o \mathbf{u} \times \dot{\mathbf{u}} + (\boldsymbol{\omega} \times \boldsymbol{\Phi}) \mathbf{I}^*] dX_3. \quad (8.278)$$

Show that the moment of the point loads on the lateral surfaces about the point $\mathbf{X}_O$ is given by

$$\sum (\mathbf{x}_i - \mathbf{X}_O) \times \mathbf{V}_i = \sum (X_{3i} \mathbf{e}_3 + \mathbf{u}_i) \times \mathbf{V}_i, \quad (8.279)$$

where X_{3i} is the location of point load $\mathbf{V}_i$ and $\mathbf{u}_i$ is the associated displacement of the centerline. Show that the moment due to the traction forces can be calculated as

$$\begin{aligned} \int_S (\mathbf{x} - \mathbf{X}_O) \times \mathbf{t}^{(n)} dS &= \int_{S_o} (\mathbf{x} - \mathbf{X}_O) \times \mathbf{t}_o^{(N)} dS_o \\ &= \mathbf{e}_3 \times (X_b \mathbf{V}_b - X_a \mathbf{V}_a + \int_{a_o}^{b_o} X_3 \mathbf{w}_o dX_3) \\ &\quad + \mathbf{M}_b - \mathbf{M}_a + \int_{a_o}^{b_o} \mathbf{q}_o dX_3 \\ &\quad + \mathbf{u}_b \times \mathbf{V}_b - \mathbf{u}_a \times \mathbf{V}_a + \int_{a_o}^{b_o} \mathbf{u} \times \mathbf{w}_o dX_3 \\ &\quad + \int_{a_o}^{b_o} \boldsymbol{\Phi} \int_{\mathcal{C}_o} (\mathbf{t}_o^{(N)} \otimes \mathbf{R} - \mathbf{R} \otimes \mathbf{t}_o^{(N)}) d\mathcal{C}_o dX_3, \end{aligned} \quad (8.280)$$

where $\mathbf{M}_a$ and $\mathbf{M}_b$ are the moments on the end surfaces, $\mathbf{q}_o$ is the applied torsional moment per unit length in the reference configuration, and $\mathcal{C}_o$ is the arc around the circumference of the bar at each X_3 . Prove that

$$\begin{aligned} &\int_{a_o}^{b_o} \boldsymbol{\Phi} \int_{\mathcal{C}_o} (\mathbf{t}_o^{(N)} \otimes \mathbf{R} - \mathbf{R} \otimes \mathbf{t}_o^{(N)}) d\mathcal{C}_o dX_3 \\ &= \int_{a_o}^{b_o} (\Phi_2 \mathbf{e}_1 - \Phi_1 \mathbf{e}_2) \int_{\mathcal{C}_o} (t_{o2}^{(N)} R_1 - t_{o1}^{(N)} R_2) d\mathcal{C}_o dX_3. \end{aligned} \quad (8.281)$$

Show that the moment of the body force is given by

$$\int_B \rho (\mathbf{x} - \mathbf{X}_O) \times \mathbf{b} dm = \int_{a_o}^{b_o} \rho_o A_o (X_3 \mathbf{e}_3 + \mathbf{R}_C + \mathbf{u} + \boldsymbol{\Phi} \times \mathbf{R}_C) \times \mathbf{b} dX_3. \quad (8.282)$$

From this show that the balance of angular momentum in the integral form can be written as

$$\begin{aligned} \dot{\mathbf{H}}_O &= \mathbf{M}_b - \mathbf{M}_a + \int_{a_o}^{b_o} \mathbf{q}_o dX_3 \\ &\quad + \mathbf{u}_b \times \mathbf{V}_b - \mathbf{u}_a \times \mathbf{V}_a + \int_{a_o}^{b_o} \mathbf{u} \times \mathbf{w}_o dX_3 \\ &\quad + \int_{a_o}^{b_o} (\Phi_2 \mathbf{e}_1 - \Phi_1 \mathbf{e}_2) \int_{\mathcal{C}_o} (t_{o2}^{(N)} R_1 - t_{o1}^{(N)} R_2) d\mathcal{C}_o dX_3 \\ &\quad + \int_{a_o}^{b_o} \rho_o A_o (X_3 \mathbf{e}_3 + \mathbf{R}_C + \mathbf{u} + \boldsymbol{\Phi} \times \mathbf{R}_C) \times \mathbf{b} dX_3 \\ &\quad + \sum (X_{3i} \mathbf{e}_3 + \mathbf{u}_i) \times \mathbf{V}_i + \sum \mathbf{M}_O. \end{aligned} \quad (8.283)$$

If we assume the deformations are small so that the slopes can be ignored (i.e., $\Phi = \mathbf{0}$) and that the coordinate is selected such that all R_C go to zero, then show that the balance of angular momentum becomes

$$\int_{a_o}^{b_o} \rho_o \bar{\mathbf{I}} \alpha dX_3 = \int_{a_o}^{b_o} (\mathbf{e}_3 \times \mathbf{V} + \frac{\partial \mathbf{M}}{\partial X_3} + \mathbf{q}_o) dX_3 + \sum (X_{3i} \mathbf{e}_3 + \mathbf{u}_i) \times \mathbf{V}_i + \sum \mathbf{M}_O. \quad (8.284)$$

From this show that for segments along the length between the applied point loads and point moments, we can get a differential form for this equation given by

$$\rho_o \bar{\mathbf{I}} \alpha = \mathbf{e}_3 \times \mathbf{V} + \frac{\partial \mathbf{M}}{\partial X_3} + \mathbf{q}_o. \quad (8.285)$$

Show that this results in the three equations

$$\frac{\partial M_1}{\partial X_3} - V_2 = \rho_o \bar{I}_{1i} \alpha_i, \quad (8.286)$$

$$\frac{\partial M_2}{\partial X_3} + V_1 = \rho_o \bar{I}_{2i} \alpha_i, \quad (8.287)$$

$$\frac{\partial M_3}{\partial X_3} + q_{o3} = \rho_o \bar{I}_{3i} \alpha_i. \quad (8.288)$$

8.8.1 Equilibrium problems

For equilibrium problems, we can eliminate all the time derivatives to get the balance of linear momentum equations

$$\frac{\partial V_1}{\partial X_3} + w_1 + \rho A b_1 = 0, \quad (8.289)$$

$$\frac{\partial V_2}{\partial X_3} + w_2 + \rho A b_2 = 0, \quad (8.290)$$

$$\frac{\partial V_3}{\partial X_3} + w_3 + \rho A b_3 = 0, \quad (8.291)$$

and the balance of angular momentum equations

$$-\frac{\partial M_1}{\partial X_3} + V_2 = 0, \quad (8.292)$$

$$\frac{\partial M_2}{\partial X_3} + V_1 = 0, \quad (8.293)$$

$$\frac{\partial M_3}{\partial X_3} + q = 0. \quad (8.294)$$

In this case, the two bending moment equations can be eliminated by substituting into the

first two linear momentum equations to get the four equations

$$\frac{\partial^2 M_1}{\partial X_3^2} + w_2 + \rho A b_2 = 0, \quad (8.295)$$

$$-\frac{\partial^2 M_2}{\partial X_3^2} + w_1 + \rho A b_1 = 0, \quad (8.296)$$

$$\frac{\partial M_3}{\partial X_3} + q = 0, \quad (8.297)$$

$$\frac{\partial V_3}{\partial X_3} + w_3 + \rho A b_3 = 0. \quad (8.298)$$

The shear load equations still can be used to calculate V_1 and V_2 . We also always can combine $\mathbf{w} + \rho A \mathbf{b}$ into one term representing the applied distributed load.

8.8.2 Assuming small rotational inertia in bending

For the dynamics problems, it is fairly common to ignore rotational inertia. This additional approximation does not affect the equations for the balance of linear momentum that are given by

$$\frac{\partial V_1}{\partial X_3} + w_1 + \rho A b_1 = \rho A \ddot{u}_1, \quad (8.299)$$

$$\frac{\partial V_2}{\partial X_3} + w_2 + \rho A b_2 = \rho A \ddot{u}_2, \quad (8.300)$$

$$\frac{\partial V_3}{\partial X_3} + w_3 + \rho A b_3 = \rho A \ddot{u}_3, \quad (8.301)$$

where again we recall that ρ is the density, not to be confused with the radii of curvature ρ_i . The equations for the balance of angular momentum, considering the additional approximations, can be written for the bar as

$$-\frac{\partial M_1}{\partial X_3} + V_2 = 0, \quad (8.302)$$

$$\frac{\partial M_2}{\partial X_3} + V_1 = 0, \quad (8.303)$$

$$\frac{\partial M_3}{\partial X_3} + q = \rho \bar{I}_p \ddot{\phi}. \quad (8.304)$$

Solving for V_1 and V_2 in these equations and substituting them in the equations for the balance of linear momentum leave four equations

$$\frac{\partial^2 M_1}{\partial X_3^2} + w_2 + \rho A b_2 = \rho A \ddot{u}_2, \quad (8.305)$$

$$-\frac{\partial^2 M_2}{\partial X_3^2} + w_1 + \rho A b_1 = \rho A \ddot{u}_1, \quad (8.306)$$

$$\frac{\partial M_3}{\partial X_3} + q = \rho \bar{I}_p \ddot{\phi}, \quad (8.307)$$

$$\frac{\partial V_3}{\partial X_3} + w_3 + \rho A b_3 = \rho A \ddot{u}_3. \quad (8.308)$$

8.8.3 Statically determinate problems

There are a set of equilibrium problems that are statically determinate so that we can calculate the values of $\mathbf{V}$ and $\mathbf{M}$ directly from the static equilibrium equations. If so, solving

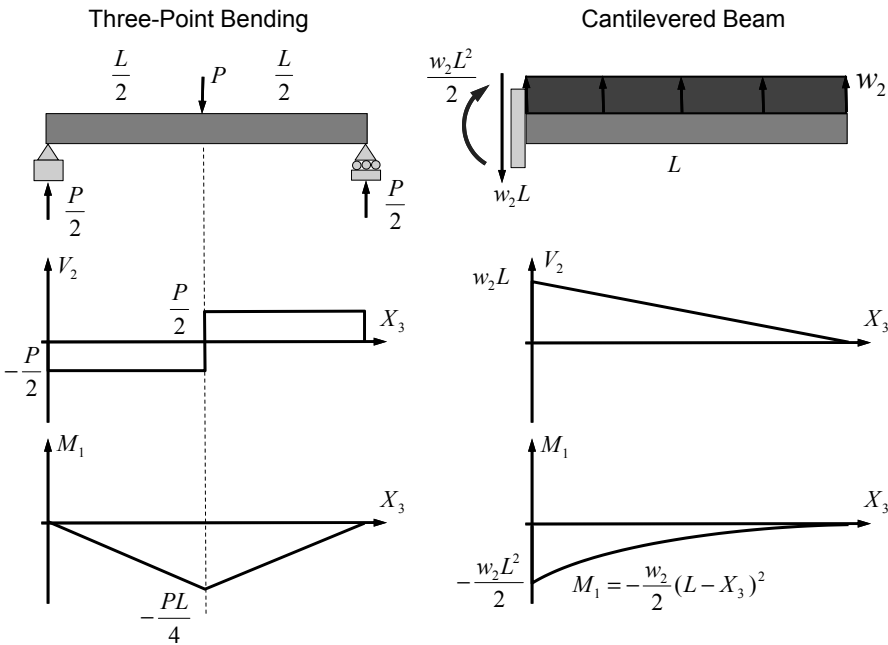


Figure 8.18: Diagrams to calculate balance laws and jump conditions for beams under distributed and point loads and moments.

the problem becomes substantially simpler since the part of the problem that is related to calculating $\mathbf{V}$ and $\mathbf{M}$ is separated from the part that is related to calculating $\mathbf{u}$ and ϕ . In such cases, knowing $\mathbf{M}$ and $\mathbf{V}$ we set up equations for $\mathbf{u}$ and ϕ through the constitutive equations.

Figure 8.18 shows two statically determinate problems and their associated shear and moment diagrams. The first problem is that of a simply supported beam with a central point load. The other is a cantilevered beam under a constant distributed load. In each case, the point of maximum moment can be determined from the moment diagram and compared to the maximum elastic moment. If it is below the maximum elastic moment then the beam is under elastic loading and its deflection can be studied with the standard relation, in this case, of

$$M_1 = \frac{EI_1}{\rho_1} = -EI_1 \frac{d^2 u_2}{dX_3^2}. \quad (8.309)$$

For example, for the cantilevered beam, we have

$$\frac{d^2 u_2}{dX_3^2} = \frac{w_2}{2EI_1} (L - X_3)^2, \quad (8.310)$$

so that

$$\frac{du_2}{dX_3} = \frac{w_2}{6EI_1} (L - X_3)^3 + D_1, \quad (8.311)$$

$$u_2 = \frac{w_2}{24EI_1} (L - X_3)^4 + D_1 X_3 + D_2, \quad (8.312)$$

where D_1 and D_2 are constants of integration that need to be fit to the boundary conditions given at the root of the beam as

$$u_2|_{X_3=0} = \frac{du_2}{dX_3}|_{X_3=0} = 0. \quad (8.313)$$

Another important fact for statically determinate problems is that, as long as we have the moment-curvature diagram for the beam under bending, we can determine the curvature of each point of the beam numerically and then integrate this to find the displacement. We also can approximate the response as elastic perfectly plastic and for each moment identify the range of the section that has undergone plastic deformation. Another important fact is that we know the maximum moment. If we set this equal to the limit moment M_p for perfectly plastic response, we can calculate the conditions under which we get a section that has become like a hinge (a plastic hinge) since it no longer can carry any higher moments. For example, for the three-point bending problem, the plastic hinge occurs when

$$M_p = \frac{PL}{4}, \quad (8.314)$$

which means the maximum load we can apply on the beam before failure is

$$P_{max} = \frac{4M_p}{L}. \quad (8.315)$$

When looking at simultaneous loadings in the case of elastic response, we know that the system is linear so that the solutions are additive and can be considered separately. If the problem is statically determinate, the load $\mathbf{V}$ and the moment $\mathbf{M}$ still can be calculated as before, but the constitutive equations are not linear so the solutions for the displacements no longer add. For this we need to consider all loadings simultaneously to calculate the point of yield and the response after that. Figure 8.19 shows a typical situation for a beam under three-point bending. As can be seen, the central portion has undergone both elastic and plastic deformation, while the ends are only elastic. Using the moment-curvature response, we can calculate the curvature for each point in the beam. We then can use this and the two boundary conditions of $u_2 = 0$ at the ends to obtain the curve for u_2 . The stress distribution for each moment also can be calculated so we know all the information about both the deflection and stress distribution in the beam.

Section 8.8-Problems:

- 8.8-4:** For the cantilevered beam subjected to a uniform distributed load, calculate the maximum distributed load w_2 that can be applied before formation of a plastic hinge at its root.
- 8.8-5:** For the cantilevered beam subjected to a point load P at its free end, calculate the maximum load P that can be applied before formation of a plastic hinge at its root.
- 8.8-6:** Simulate the response of a beam under conditions of three-point bending for the applied load monotonically increasing and for when it is cyclic. Assume the response is quasi-static.
- 8.8-7:** Simulate the response of a cantilevered beam under constant distributed load for an applied distributed load that is monotonically increasing and for when it is cyclic. Assume the response is quasi-static.

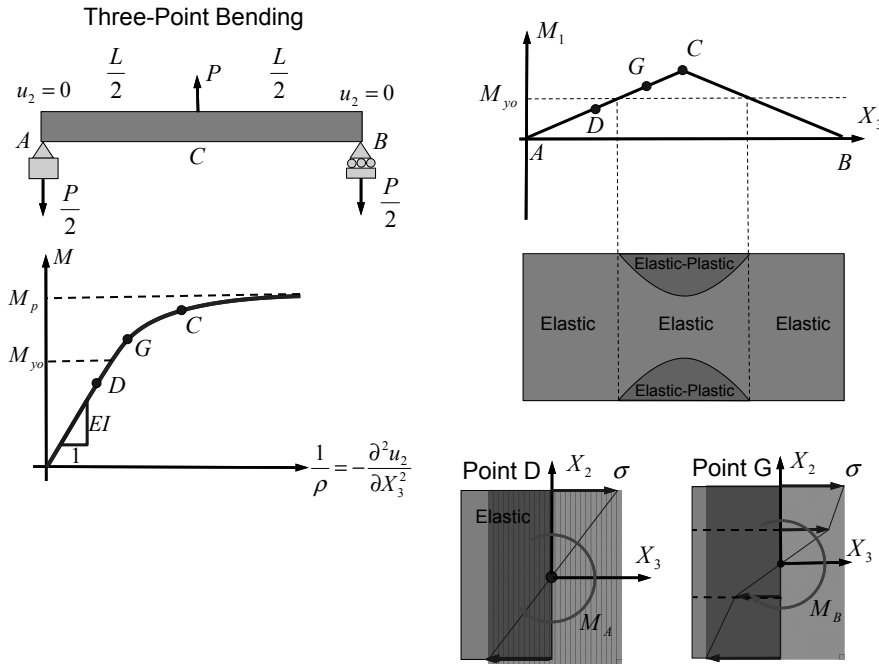


Figure 8.19: Example of calculating the curvature and stress distribution for a beam under three-point bending that has a part that is elastic and one that is elastic-plastic.

8.8.4 Rotating beam

Here, we will consider a beam rotating steadily about one end at a constant rate Ω . Figure 8.20 shows the details of the problem. This problem can be solved without considering the large rotations involved in the problem by setting the system in a frame that rotates with the bar. As a result, the deformations of the bar still remain small so the problem can be considered under the theory of infinitesimal deformations. We will ignore gravity and assume the rotation is around the X_1 axis (i.e., in the 2-3 plane). We will assume that the beam is, as before, situated along the X_3 direction and that all the applied distributed loads and moments are zero. All the moments and loads other than V_3 can be taken as zero, and for this we have the equation

$$\frac{\partial V_3}{\partial X_3} = -\rho A \Omega^2 X_3, \quad (8.316)$$

where the system is assumed to be rotating about end $X_3 = 0$ and the acceleration is therefore equal to $-\Omega^2 X_3$. We can integrate this to get

$$V_3 = -\rho A \Omega^2 \frac{X_3^2}{2} + C, \quad (8.317)$$

for a constant of integration C that we can fit to the zero value of V_3 at its free end $X_3 = L$. We, therefore, get

$$V_3 = \frac{\rho A \Omega^2}{2} (L^2 - X_3^2). \quad (8.318)$$

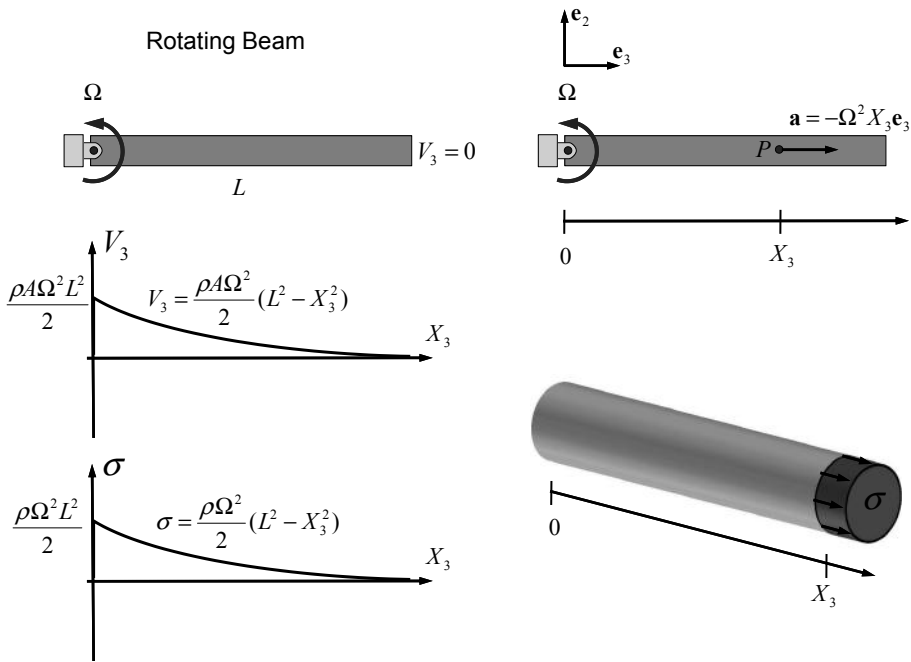


Figure 8.20: A bar rotating at a constant rate Ω about one end.

As can be seen, the problem is statically determinate. It corresponds to a bar with non-uniform axial load along the bar but uniform axial stress in each cross section. The axial stress σ is given by

$$\sigma = \frac{V_3}{A} = \frac{\rho\Omega^2}{2}(L^2 - X_3^2). \quad (8.319)$$

Section 8.8-Problems:

- 8.8-8:** For a bar rotating steadily around one end, calculate the rate of rotation Ω for which plastic flow will start in the beam.

8.9 Cylindrical and spherical symmetry

There are a number of important industrial problems that are either cylindrical in shape or spherical in shape so the subject of studying them is an important area. Examples of such problems are the flow and fracture of a pressurized cylinder, the explosion in a spherical cavity, stability and flow of a rapidly rotating disk, and the dynamics of solid bushings. A number of these problems are either “cylindrically” symmetric or “spherically” symmetric.

In each case, we assume the solution to be best represented in terms of the variables of either cylindrical or spherical coordinate systems, and we look at solutions of a simplifying

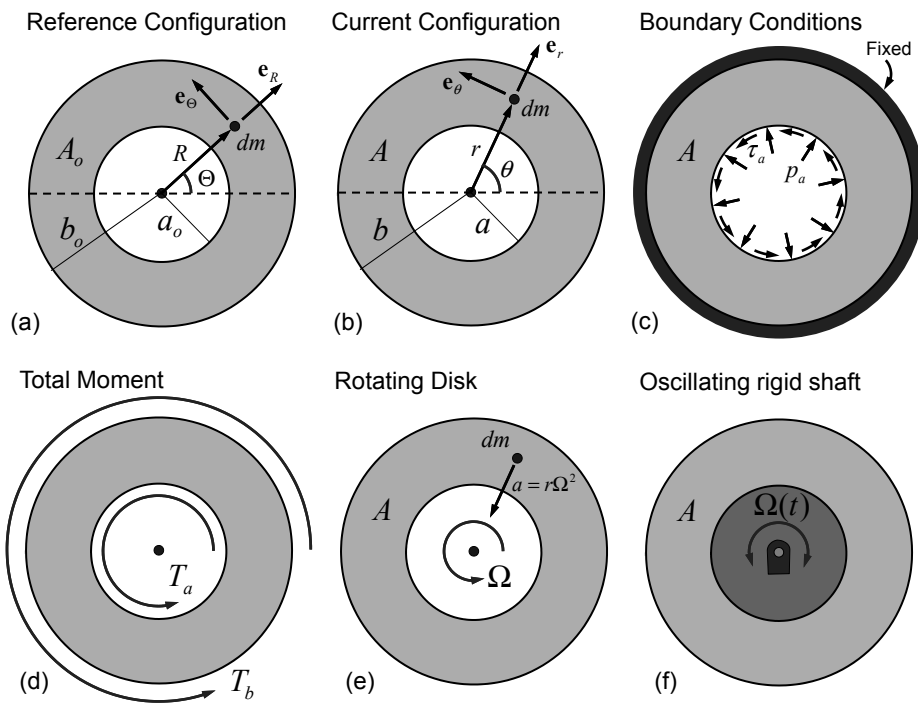


Figure 8.21: Cylinder shaped objects: (a) reference configuration, (b) current configuration, (c) boundary conditions, (d) total moment applied on surfaces, (e) rotating disk, and (f) oscillating shaft.

nature, which we will describe by several examples. All the equations written for either cylindrical or spherical coordinate systems are provided in Appendix C. We will focus primarily on cylindrically shaped problems, even though spherically shaped problems will result in similar equations, most of which are left as homework problems.

The basic cylindrical problem considered here is shown in Figure 8.21, which represents a cylinder transforming into a new cylinder. This shape is ideal for description in this coordinate system. This problem may represent a disk or a shaft-like object. A disk normally is considered as being plane-stress, while a shaft is considered plane-strain. We assume the reference cylinder has inside and outside radii of a_o and b_o and associated radii a and b in the current configuration. For the problems that we consider here, we will assume the variation along the length of the cylinder is zero, even though the reader can use the same system for solving problems that have axial variation, for example, the shearing of a cylindrical disk on a rotating table that can be modeled by a radius that changes as a function of the axial coordinate.

As shown in Appendix C, the expressions for the balance of linear momentum written in terms of cylindrical coordinates for both the reference and current configurations can be written as follows. In these expressions, the coordinates (R, Θ, Z) are the positions in the reference configuration and (r, θ, z) are the coordinates used in the current configuration. The difference is assumed to be small. The equations describing the balance of linear momentum

in cylindrical coordinates are given by

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\Theta R}}{\partial \Theta} + \frac{\partial \sigma_{ZR}}{\partial Z} + \frac{\sigma_{RR} - \sigma_{\Theta\Theta}}{R} + \rho_o b_R = \rho_o \ddot{u}_R, \quad (8.320)$$

$$\frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\Theta\Theta}}{\partial \Theta} + \frac{\partial \sigma_{Z\Theta}}{\partial Z} + \frac{2\sigma_{R\Theta}}{R} + \rho_o b_\Theta = \rho_o \ddot{u}_\Theta, \quad (8.321)$$

$$\frac{\partial \sigma_{RZ}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\Theta Z}}{\partial \Theta} + \frac{\partial \sigma_{ZZ}}{\partial Z} + \frac{\sigma_{RZ}}{R} + \rho_o b_Z = \rho_o \ddot{u}_Z. \quad (8.322)$$

Cylindrically symmetric problems are those that have solutions independent of Θ . We further will confine our attention to problems that do not depend on the axial coordinate Z , and we ignore the body force. As a result, stress will depend only on the radial variable R so that we can simplify this set of equations to

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{\sigma_{RR} - \sigma_{\Theta\Theta}}{R} = \rho_o \ddot{u}_R, \quad (8.323)$$

$$\frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{2\sigma_{R\Theta}}{R} = \rho_o \ddot{u}_\Theta, \quad (8.324)$$

$$\frac{\partial \sigma_{RZ}}{\partial R} + \frac{\sigma_{RZ}}{R} = \rho_o \ddot{u}_Z. \quad (8.325)$$

From Appendix C, we know that the expression for the deformation gradient is given by

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & r \frac{\partial \theta}{\partial \Theta} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}. \quad (8.326)$$

For the plane-strain types of problems, we assume that the displacement only depends on the radial location so that we can write

$$r = R + \Delta r(R, t), \quad (8.327)$$

$$\theta = \Theta + \Delta \theta(R, t), \quad (8.328)$$

$$z = Z + \Delta z(R, t), \quad (8.329)$$

where $\Delta r(R, t)$, $\Delta \theta(R, t)$, and $\Delta z(R, t)$ are each functions that are small and are small in their first derivative with respect to R . In this case, the expression for the deformation gradient is given by

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} 1 + \frac{\partial \Delta r}{\partial R} & 0 & 0 \\ (R + \Delta r) \frac{\partial \Delta \theta}{\partial R} & 1 + \frac{\Delta r}{R} & 0 \\ \frac{\partial \Delta z}{\partial R} & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}. \quad (8.330)$$

Consistent with the kinematic assumption that the changes are small, we can ignore the difference between the base vectors and can calculate the strain using the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{I}$ to obtain

$$\boldsymbol{\epsilon} = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_Z \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta r}{\partial R} & \frac{R}{2} \frac{\partial \Delta \theta}{\partial R} & \frac{1}{2} \frac{\partial \Delta z}{\partial R} \\ \frac{R}{2} \frac{\partial \Delta \theta}{\partial R} & \frac{\Delta r}{R} & 0 \\ \frac{1}{2} \frac{\partial \Delta z}{\partial R} & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}. \quad (8.331)$$

For the plane-stress problems, in place of the above assumption, we assume that $z = Z + \Delta z(R, Z, t)$ to allow an axial strain to accommodate the plane-stress assumption. We also make a similar assumption for some other problems that allow axial extension. In these cases, we get

$$\mathbf{F} = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} 1 + \frac{\partial \Delta r}{\partial R} & 0 & 0 \\ (R + \Delta r) \frac{\partial \Delta \theta}{\partial R} & 1 + \frac{\Delta r}{R} & 0 \\ \frac{\partial \Delta z}{\partial R} & 0 & 1 + \frac{\partial \Delta z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}, \quad (8.332)$$

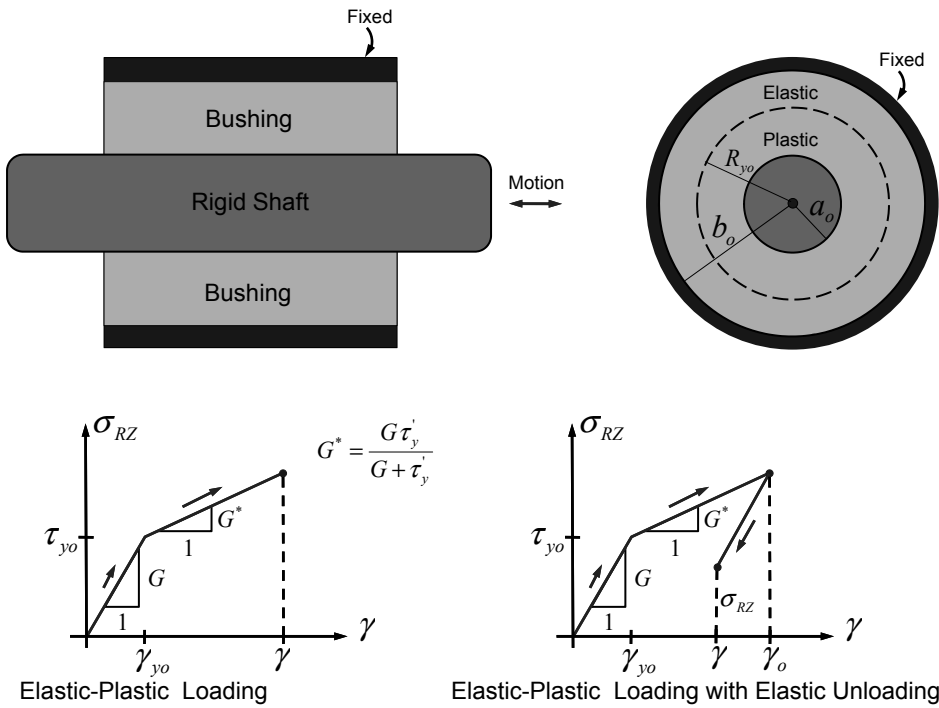


Figure 8.22: Antiplane motion of a cylindrical bushing connected to a rigid shaft on the inside and a fixed support on the outside.

so that the strain is given by

$$\epsilon = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta r}{\partial R} & \frac{R}{2} \frac{\partial \Delta \theta}{\partial R} & \frac{1}{2} \frac{\partial \Delta z}{\partial R} \\ \frac{R}{2} \frac{\partial \Delta \theta}{\partial R} & \frac{\Delta r}{R} & 0 \\ \frac{1}{2} \frac{\partial \Delta z}{\partial R} & 0 & \frac{\partial \Delta z}{\partial z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \quad (8.333)$$

There is a large number of problems that can be solved within this setting. We will focus on a few. These include antiplane shear, internal pressure, and in-plane shear, and that of a steadily rotating disk.

8.9.1 Antiplane shear

Let us consider antiplane shearing motions along the axis of the cylinder, such as shown in Figure 8.22 for the axial motion of a flexible bushing connected at the outer radius to a fixed support and at the inner radius to a rigid moving circular shaft.

In the case of antiplane shear, we assume that motion is defined only by motion along the axis of the cylinder so that $\Delta r = \Delta \theta = 0$. We only consider the plane strain case so that the problem is defined by the strain

$$\epsilon = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \frac{\partial \Delta z}{\partial R} \\ 0 & 0 \\ \frac{1}{2} \frac{\partial \Delta z}{\partial R} & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \quad (8.334)$$

This results in a stress field for the elastic response described by

$$\boldsymbol{\sigma} = \left(\kappa - \frac{2}{3}G\right)tr(\boldsymbol{\epsilon})\mathbf{I} + 2G\boldsymbol{\epsilon} = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_Z \end{bmatrix} \begin{bmatrix} 0 & G\frac{\partial\Delta z}{\partial R} \\ 0 & 0 & 0 \\ G\frac{\partial\Delta z}{\partial R} & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}. \quad (8.335)$$

The only nontrivial term in the stress is

$$\sigma_{RZ} = G\frac{\partial\Delta z}{\partial R}. \quad (8.336)$$

The only nontrivial term in this case for the balance of linear momentum will be the component along the axis of the cylinder given by

$$\frac{\partial\sigma_{RZ}}{\partial R} + \frac{\sigma_{RZ}}{R} = \rho_o \frac{\partial^2\Delta z}{\partial t^2}. \quad (8.337)$$

Substituting the expression for the elastic stress we arrive at the partial differential equation for the associated elastic response given by

$$\frac{\partial}{\partial R} \left(G\frac{\partial\Delta z}{\partial R} \right) + \frac{1}{R}G\frac{\partial\Delta z}{\partial R} = \rho_o \frac{\partial^2\Delta z}{\partial t^2}. \quad (8.338)$$

The static problem is statically determinate and given by the equation

$$\frac{d\sigma_{RZ}}{dR} + \frac{\sigma_{RZ}}{R} = 0. \quad (8.339)$$

We can reorganize and integrate this once to get

$$\ln(\sigma_{RZ}) = -\ln(R) + \ln(C), \quad (8.340)$$

where C is an integration constant. We can reorganize this to get

$$\sigma_{RZ} = \frac{C}{R}. \quad (8.341)$$

The corresponding equation for the axial load P on a cylindrical surface of radius R and length L is given by integrating this stress, which is constant, over the surface of a cylinder to obtain

$$P = \int_A \sigma_{RZ} dA = \int_0^{2\pi} \int_0^L \sigma_{RZ} dZ d\Theta = 2\pi RL\sigma_{RZ} = 2\pi LC. \quad (8.342)$$

As can be seen, the axial load on all cross sections will be the same. We can calculate the constant of integration in terms of P to get

$$C = \frac{P}{2\pi L}. \quad (8.343)$$

The shear stress can be written as a function of the axial load at any cross section as

$$\sigma_{RZ} = \frac{P}{2\pi LR}. \quad (8.344)$$

The maximum stress is at the minimum radius so that we can conclude that the first place to undergo plastic flow will be the inner surface. The elastic solution can be obtained by substituting the elastic constitutive expression for σ_{RZ} into this equation to obtain

$$G\frac{d\Delta z}{dR} = \frac{P}{2\pi LR}. \quad (8.345)$$

We can integrate this expression to obtain

$$\Delta z = \frac{P}{2\pi LG} \ln(R) + D, \quad (8.346)$$

where D is a constant of integration obtained from the boundary conditions. If we assume that the cylinder is fixed (i.e., $\Delta z = 0$) at a radius of $R = R_o$, then we have

$$\Delta z = \frac{P}{2\pi LG} \ln\left(\frac{R}{R_o}\right). \quad (8.347)$$

If the load is sufficient to induce plastic flow, as long as the applied load is not too large, the outer region still will be elastic, with an elastic plastic core. The boundary of the elastic and plastic parts is defined by the radius R_{yo} and can be obtained by setting the shear stress in the elastic solution to be equal to the initial yield stress so that we get

$$\tau_{yo} = \frac{P}{2\pi L R_{yo}}, \quad (8.348)$$

to obtain the radius inside of which you have plastic flow given by

$$R_{yo} = \frac{P}{2\pi L \tau_{yo}}. \quad (8.349)$$

The static problem is not only an example of a statically determinate problem, but also shows some very important relations. One of these is related to bushings and their use as an axial spring. For example, if we construct an axial spring from a bushing that is fixed on its outside at radius b_o and connected to a shaft at its inner radius of a_o , we know that the motion on the shaft, here given by x , is equal to the value of Δz at the radius a_o . This is given by

$$x = \frac{P}{2\pi LG} \ln\left(\frac{a_o}{b_o}\right). \quad (8.350)$$

In other words, the load P applied on the shaft is given by

$$P = kx, \quad (8.351)$$

where the spring constant k of the bushing can be calculated by

$$k = \frac{2\pi LG}{\ln\left(\frac{a_o}{b_o}\right)}. \quad (8.352)$$

For a shaft of mass M , the system can be considered as a linear spring-mass system that has a solution

$$x = A \sin(\omega t + \phi), \quad (8.353)$$

where $\omega = \sqrt{\frac{k}{M}}$ is the circular natural frequency, and the constant amplitude A and phase shift ϕ are evaluated from initial conditions. In addition, the maximum load P_{yo} that we can subject the bushing to before inducing plastic flow is given by the start of yielding at the inner surface given by

$$P_{yo} = 2\pi L a_o \tau_{yo}. \quad (8.354)$$

For the static problem, considering only loading, we know that in the plastic zone the response is given by

$$\sigma_{RZ} = \tau_{yo} + G^*(\gamma - \gamma_{yo}), \quad (8.355)$$

shown in Figure 8.22, where

$$G^* = \frac{G\tau_y'}{G + \tau_y'} \quad (8.356)$$

$$\gamma_{yo} = \frac{\tau_{yo}}{G}, \quad (8.357)$$

$$\gamma = \frac{\partial \Delta z}{\partial R}. \quad (8.358)$$

Since the problem is statically determinate and we have the solution, the response in the plastically deformed region can be written as

$$\frac{P}{2\pi LR} = \tau_{yo} + G^* \left(\frac{\partial \Delta z}{\partial R} - \gamma_{yo} \right). \quad (8.359)$$

This can be reorganized to get

$$\frac{\partial \Delta z}{\partial R} = \gamma_{yo} - \frac{\tau_{yo}}{G^*} + \frac{P}{2\pi G^* LR}. \quad (8.360)$$

Integrating this, we obtain the relation

$$\Delta z = \left(\gamma_{yo} - \frac{\tau_{yo}}{G^*} \right) R + \frac{P}{2\pi G^* L} \ln(R) + C, \quad (8.361)$$

where C is the associated integration constant. To calculate the integration constant, let us assume that the boundary is fixed at the outer surface so that $\Delta z = 0$ at $R = b_o$. In this case, the elastic solution is

$$\Delta z = \frac{P}{2\pi LG} \ln \left(\frac{R}{b_o} \right). \quad (8.362)$$

If we match the displacement for the elastic and plastic solutions at the radius R_{yo} at which plastic flow starts, we get the equation

$$\frac{P}{2\pi LG} \ln \left(\frac{R_{yo}}{b_o} \right) = \left(\gamma_{yo} - \frac{\tau_{yo}}{G^*} \right) R_{yo} + \frac{P}{2\pi G^* L} \ln(R_{yo}) + C, \quad (8.363)$$

where

$$R_{yo} = \frac{P}{2\pi L \tau_{yo}}. \quad (8.364)$$

Therefore, the response in the plastic region $a_o \leq R \leq R_{yo}$ is given by

$$\Delta z = \left(\gamma_{yo} - \frac{\tau_{yo}}{G^*} \right) (R - R_{yo}) + \frac{P}{2\pi G^* L} \ln \left(\frac{R}{R_{yo}} \right) - \frac{P}{2\pi LG} \ln \left(\frac{R_{yo}}{b_o} \right). \quad (8.365)$$

We next will look at unloading after plastic flow. Because of the statically determinate solution, the stress will go to zero when the load is reduced to zero, yet a residual strain will develop in the plastic portion. From equation (8.362), when we reduce the load to zero the displacement in the elastic region goes to zero so that $\Delta z = 0$ from radius R_{yo} to b_o . If the load before unloading is P_o , from the statically determinate solution, we can calculate the shear strain γ_o before unloading from the equation

$$\sigma_{RZ} = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}), \quad (8.366)$$

which yields

$$\frac{P_o}{2\pi LR} = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}). \quad (8.367)$$

Solving for the strain γ_o before unloading we obtain

$$\gamma_o = \frac{P}{2\pi G^* LR} + \gamma_{yo} - \frac{\tau_{yo}}{G^*}. \quad (8.368)$$

During unloading, the stress is given by the equation

$$\sigma_{RZ} = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}) + G(\gamma - \gamma_o). \quad (8.369)$$

The stress goes to zero with the removal of the applied load so we can write this as

$$0 = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}) + G\left(\frac{\partial \Delta z}{\partial R} - \gamma_o\right). \quad (8.370)$$

We now can substitute for γ_o and get

$$0 = \frac{P_o}{2\pi LR} + G\left(\frac{\partial \Delta z}{\partial R} - \frac{P}{2\pi G^* LR} - \gamma_{yo} + \frac{\tau_{yo}}{G^*}\right). \quad (8.371)$$

This provides the differential equation

$$\frac{\partial \Delta z}{\partial R} = \frac{P_o(G - G^*)}{2\pi LGG^* R} + \gamma_{yo} - \frac{\tau_{yo}}{G^*}. \quad (8.372)$$

Integrating this and forcing continuity at the elastic boundary (zero displacement at R_{yo}) yields the unloaded solution in the plastic zone from a_o to R_{yo} given by

$$\Delta z = \frac{P_o(G - G^*)}{2\pi LGG^*} \ln\left(\frac{R}{R_{yo}}\right) + \left(\gamma_{yo} - \frac{\tau_{yo}}{G^*}\right)(R - R_{yo}). \quad (8.373)$$

Section 8.9-Problems:

- 8.9-1:** Find the smallest monotonically increasing quasi-static load at which the entire bushing undergoes plastic flow.
- 8.9-2:** Consider the problem fixed at the inner radius a_o and connected to a rigid hollow cylinder of inner radius b_o . Obtain the expression for the stiffness of the resulting bushing under elastic deformations.
- 8.9-3:** Simulate the quasi-static loading and unloading of the bushing and calculate the residual displacement of the rigid central shaft after unloading.
- 8.9-4:** Simulate the quasi-static cyclic loading of the bushing.
- 8.9-5:** Simulate the dynamic response of the shaft assuming a constant load of $\bar{P}$ is applied to the central shaft of mass M and that the response of the bushing can be considered quasi-static (i.e., ignore inertia for the bushing, but not for the shaft).

8.9.2 Internal pressure under plane strain

Another interesting and important problem is that of a cylinder subjected to an internal pressure and either fixed or free at the outer radius. The resulting problem, even in its simplest form, is statically indeterminate. To simplify the problem, we ignore all axial and shear deformations so that we set $\Delta\theta = \Delta z = 0$. We also assume the shear stresses to be zero. In this case, the expression for strain is given by

$$\epsilon = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_Z \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta r}{\partial R} & 0 & 0 \\ 0 & \frac{\Delta r}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}, \quad (8.374)$$

so that its trace is given by

$$\text{tr}(\epsilon) = \frac{\partial \Delta r}{\partial R} + \frac{\Delta r}{R}. \quad (8.375)$$

The only nontrivial equation for the balance of linear momentum in this case is given by

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{\sigma_{RR} - \sigma_{\Theta\Theta}}{R} = \rho_o \frac{\partial^2 \Delta r}{\partial t^2}. \quad (8.376)$$

The problem is statically indeterminate so that we need to introduce the constitutive equation. We first look at the elastic response to obtain the elastic solution. Fortunately, the expression for strain is simple, and we obtain only three nontrivial expressions for the elastic response. These are

$$\sigma_{RR} = \frac{(3\kappa + 4G)}{3} \frac{\partial \Delta r}{\partial R} + \frac{(3\kappa - 2G)}{3} \frac{\Delta r}{R}, \quad (8.377)$$

$$\sigma_{\Theta\Theta} = \frac{(3\kappa - 2G)}{3} \frac{\partial \Delta r}{\partial R} + \frac{(3\kappa + 4G)}{3} \frac{\Delta r}{R}, \quad (8.378)$$

$$\sigma_{ZZ} = \frac{(3\kappa - 2G)}{3} \left(\frac{\partial \Delta r}{\partial R} + \frac{\Delta r}{R} \right). \quad (8.379)$$

Therefore, the balance law for elastic deformations is given by

$$\frac{\partial}{\partial R} \left[\frac{(3\kappa + 4G)}{3} \frac{\partial \Delta r}{\partial R} + \frac{(3\kappa - 2G)}{3} \frac{\Delta r}{R} \right] + \frac{2G}{R} \left[\frac{\partial \Delta r}{\partial R} - \frac{\Delta r}{R} \right] = \rho_o \frac{\partial^2 \Delta r}{\partial t^2}. \quad (8.380)$$

For a homogeneous body, the equilibrium problem takes the form

$$\frac{(3\kappa + 4G)}{3} \frac{d^2 \Delta r}{dR^2} + \frac{(3\kappa - 2G)}{3} \left[\frac{1}{R} \frac{d\Delta r}{dR} - \frac{\Delta r}{R^2} \right] + \frac{2G}{R} \left[\frac{d\Delta r}{dR} - \frac{\Delta r}{R} \right] = 0. \quad (8.381)$$

Reorganizing this equation we get

$$\frac{(3\kappa + 4G)}{3} \frac{d^2 \Delta r}{dR^2} + \frac{(3\kappa + 4G)}{3} \left[\frac{1}{R} \frac{d\Delta r}{dR} - \frac{\Delta r}{R^2} \right] = 0. \quad (8.382)$$

Since both terms have the same scalar coefficient, we can divide by it to eliminate the dependence on the material parameters and get

$$\frac{d^2 \Delta r}{dR^2} + \frac{1}{R} \frac{d\Delta r}{dR} - \frac{\Delta r}{R^2} = 0. \quad (8.383)$$

By inspection we can show that this is the same as

$$\frac{d}{dR} \left(\frac{d\Delta r}{dR} + \frac{\Delta r}{R} \right) = 0. \quad (8.384)$$

Therefore, we can integrate this to obtain

$$\frac{d\Delta r}{dR} + \frac{\Delta r}{R} = C, \quad (8.385)$$

where C is a constant of integration. We can write this as

$$\frac{1}{R} \frac{d}{dR} (R\Delta r) = C. \quad (8.386)$$

Reorganizing the terms gives

$$\frac{d}{dR} (R\Delta r) = CR, \quad (8.387)$$

which can be integrated to get

$$R\Delta r = \frac{1}{2}CR^2 + D, \quad (8.388)$$

for the integration constant D . We, therefore, get the solution for the radial displacement given by

$$\Delta r = \frac{1}{2}CR + \frac{D}{R}. \quad (8.389)$$

This results in the expressions

$$\frac{\Delta r}{R} = \frac{1}{2}C + \frac{D}{R^2}, \quad (8.390)$$

$$\frac{d\Delta r}{dR} = \frac{1}{2}C - \frac{D}{R^2}, \quad (8.391)$$

$$\frac{d\Delta r}{dR} + \frac{\Delta r}{R} = \frac{1}{2}C - \frac{D}{R^2} + \frac{1}{2}C + \frac{D}{R^2} = C. \quad (8.392)$$

One can substitute this back into the the expressions for the stresses to find

$$\begin{aligned} \sigma_{RR} &= \frac{(3\kappa + 4G)}{3} \left(\frac{1}{2}C - \frac{D}{R^2} \right) + \frac{(3\kappa - 2G)}{3} \left(\frac{1}{2}C + \frac{D}{R^2} \right) \\ &= \frac{(3\kappa + G)}{3}C - 2G \frac{D}{R^2} \end{aligned} \quad (8.393)$$

$$\begin{aligned} \sigma_{\Theta\Theta} &= \frac{(3\kappa - 2G)}{3} \left(\frac{1}{2}C - \frac{D}{R^2} \right) + \frac{(3\kappa + 4G)}{3} \left(\frac{1}{2}C + \frac{D}{R^2} \right) \\ &= \frac{(3\kappa + G)}{3}C + 2G \frac{D}{R^2} \end{aligned} \quad (8.394)$$

$$\sigma_{ZZ} = \frac{(3\kappa - 2G)}{3}C. \quad (8.395)$$

There are two integration constants in the solution that can be fit to two boundary conditions for the elastic problem. Let us consider first the solid bar. In this case, the displacement at the center needs to go to zero so that $D = 0$, and we can fit the pressure p_b at the outer radius b_o to get

$$-p_b = \frac{(3\kappa + G)}{3}C, \quad (8.396)$$

which gives

$$C = -\frac{3p_b}{(3\kappa + G)}. \quad (8.397)$$

This results in the stresses

$$\sigma_{RR} = \sigma_{\Theta\Theta} = -p_b, \quad (8.398)$$

$$\sigma_{ZZ} = -\frac{(3\kappa - 2G)}{(3\kappa + G)}p_b, \quad (8.399)$$

and the displacement

$$\Delta r = -\frac{3p_b}{2(3\kappa + G)}R. \quad (8.400)$$

This special solution is homogeneous in the stress and strain fields. As a result, yielding will occur at all points once the pressure p_b becomes large enough to induce plastic flow. In this case, the average stress is given by

$$\sigma_{ave} = \frac{1}{3}tr(\boldsymbol{\sigma}) = -\frac{3\kappa}{(3\kappa + G)}p_b. \quad (8.401)$$

The deviatoric stress only has diagonal terms and is given by

$$S_{RR} = S_{\Theta\Theta} = -\frac{G}{(3\kappa + G)}p_b, \quad (8.402)$$

$$S_{ZZ} = \frac{(2G)}{(3\kappa + G)}p_b. \quad (8.403)$$

The initial yield condition $\tau = \tau_{yo}$ gives

$$\sqrt{3}\frac{G}{(3\kappa + G)}p_b = \tau_{yo}, \quad (8.404)$$

which can be solved to obtain the pressure p_b for initial yield.

In the case of the hollow cylinder with pressures p_a and p_b at, respectively, the inner radius a_o and the outer radius b_o , we can force the radial stress to match the boundary conditions to obtain the conditions

$$-p_a = \frac{(3\kappa + G)}{3}C - 2G\frac{D}{a_o^2}, \quad (8.405)$$

$$-p_b = \frac{(3\kappa + G)}{3}C - 2G\frac{D}{b_o^2}. \quad (8.406)$$

Subtracting the two equations gives

$$-(p_b - p_a) = -2GD\left(\frac{1}{b_o^2} - \frac{1}{a_o^2}\right). \quad (8.407)$$

This yields

$$D = \frac{(p_a - p_b)b_o^2a_o^2}{2G(b_o^2 - a_o^2)}. \quad (8.408)$$

Back substitution of D gives C as

$$C = \frac{3(p_a a_o^2 - p_b b_o^2)}{(3\kappa + G)(b_o^2 - a_o^2)}. \quad (8.409)$$

8.9.3 Other problems

There are a number of other problems that can be considered here, some of which are shown in Figure 8.21. These could include the case of in-plane shearing, resulting from torsional moments put on the inside or outside surface, such as is shown in Figure 8.21c and 8.21d. Another interesting problem is the steady-state rotation of a disk as shown in Figure 8.21e. In this case, the steady-state motion provides the acceleration as shown in the figure so that the balance of linear momentum will include an acceleration in the radial direction. Figure 8.21f shows another interesting problem defined by a cylinder with an oscillating rigid

central shaft. All of these problems have analytical elastic solutions with different degrees of analytical or numerical elastic-plastic solutions.

Section 8.9-Problems:

- 8.9-6:** Consider the elastic problem discussed above for plane strain of a hollow cylinder with internal pressure p_a at radius $R = a_o$ and fixed outer boundary at the radius $R = b_o$. Evaluate the integration constants, the stresses, and the displacements. Evaluate the pressure p_a for which plastic flow starts. If the pressure p_a is larger than that which starts plastic flow in the cylinder, calculate the radius defining the boundary between elastic and plastic deformations. Under what minimum pressure p_a will the cylinder totally be undergoing plastic deformation?
- 8.9-7:** For the previous problem simulate the response of the cylinder (stresses and displacements) for monotonic and cyclic pressure p_a applied at the center.
- 8.9-8:** Consider the problem of the hollow cylinder under internal and external pressure and add axial extension. Solve this problem and show that the stress for the elastic case is given by

$$\sigma_{RR} = A + \frac{B}{R^2}, \quad (8.410)$$

$$\sigma_{\Theta\Theta} = A - \frac{B}{R^2}, \quad (8.411)$$

$$\sigma_{ZZ} = E\epsilon_{ZZ} + \nu(\sigma_{RR} + \sigma_{\Theta\Theta}) = E\epsilon_{ZZ} + 2\nu A. \quad (8.412)$$

An interesting problem is the pressurization of a rigidly capped cylindrical tube, where the axial stress P on the tube comes from the internal pressure pushing against the end plates to give $P = p_a \pi a_o^2$, resulting in an axial stress given by $\sigma_{ZZ} = p_a a_o^2 / (b_o^2 - a_o^2)$. The outside pressure is taken to be zero, and one can ignore the end effects if the tube is long. For this case calculate the internal pressure p_a at which initial plastic flow starts, and the pressure at which all the tube undergoes plastic deformation.

- 8.9-9:** For the previous problem, simulate the response of the cylinder under monotonically increasing internal pressure and also for cyclically changing internal pressure.
- 8.9-10:** Consider the problem described by Figure 8.21e for the steady-state rotation of a hollow disk with zero stress on the inside and outside surfaces. Write the equations of balance of linear momentum. Solve this system for an elastic material. Calculate the angular velocity Ω for which plastic flow starts in the disk. Calculate the minimum angular velocity for which the disk entirely undergoes plastic flow.
- 8.9-11:** The balance of linear momentum in spherical coordinates (ρ, ϕ, θ) using the

current density denoted by ρ^* is given by the equations

$$\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\phi\rho}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial \sigma_{\theta\rho}}{\partial \theta} + \frac{2\sigma_{\rho\rho} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{\phi\rho} \cot(\phi)}{\rho} + \rho^* b_\rho = \rho^* \ddot{u}_\rho, \quad (8.413)$$

$$\frac{\partial \sigma_{\rho\phi}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{(\sigma_{\phi\phi} - \sigma_{\theta\theta}) \cot(\phi) + 3\sigma_{\rho\phi}}{\rho} + \rho^* b_\phi = \rho^* \ddot{u}_\phi, \quad (8.414)$$

$$\frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{\theta\rho} + 2\sigma_{\phi\theta} \cot(\phi)}{\rho} + \rho^* b_\theta = \rho^* \ddot{u}_\theta. \quad (8.415)$$

Write this system for a spherically symmetric problem, assuming all variables only depend on (ρ, t) . Assume the domain of the body is given by a hollow spherical shape of inner radius a_o and outer radius b_o , where the inside is subjected to hydrostatic pressure p_a and the outside to p_b . Assume the shear stresses $\sigma_{\rho\phi} = \sigma_{\rho\theta} = \sigma_{\phi\theta} = 0$ and solve for the stress distribution as a function of ρ for an elastic body under equilibrium conditions without a body force. Assuming the outside pressure is zero, calculate the initial internal pressure p_a at which the body starts flowing plastically. Show that the elastic response of the sphere is characterized by the equations

$$\sigma_{RR} = A + \frac{B}{R^3}, \quad (8.416)$$

$$\sigma_{\Phi\Phi} = A - \frac{B}{R^3}. \quad (8.417)$$

Impose the boundary conditions to find the two constants A and B . Show that with the increase of internal pressure the plastic flow starts from the inside radius and moves outward. Find the initial pressure that induces plastic flow at the inner surface, and the smallest internal pressure that results in plastic flow in the entire sphere.

- 8.9-12:** For the previous problem, calculate the response for an elastic perfectly plastic material both on monotonic loading through an increase of internal pressure and during unloading to zero internal pressure.

8.10 Two-dimensional problems

The main two-dimensional problems we consider are plane-stress and plane-strain problems. Plane-stress normally refers to a problem on a shell-like object, where the surfaces of the shell are free of stress. For thin shell-like structures the assumption that the traction normal to the surface is zero through the thickness is not too large an approximation if the traction on the surfaces of the shell are zero and is the main source and application of this assumption. Plane-strain refers normally to long objects that are loaded “similarly” along the length so that, other than end effects along the length, the deflection and resulting strains are assumed to be uniform through the cross section. Even though these are the two main classes of two-dimensional problems, one can make others, but they will not be considered in this section.

8.10.1 Plane-stress problems

For plane-stress problems, we are considering problems that have zero traction on all surfaces perpendicular to a given direction. If we formulate the problem such that the stresses all fall within the 1-2 plane, then we can assume that the traction is zero on all surfaces perpendicular to the 3-direction. If $\mathbf{e}_i$ is an orthonormal base for this coordinate system and we have $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, then this assumption is equivalent to assuming $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The three scalar equations representing the balance of linear momentum for this case can be written as

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{21}}{\partial X_2} + \rho b_1 = \rho a_1, \quad (8.418)$$

$$\frac{\partial \sigma_{12}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \rho b_2 = \rho a_2, \quad (8.419)$$

$$\rho b_3 = \rho a_3. \quad (8.420)$$

For the elastic problem, the six constitutive equations are written as

$$\sigma_{11} = \left(\kappa - \frac{2}{3}G\right)(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2G\epsilon_{11}, \quad (8.421)$$

$$\sigma_{22} = \left(\kappa - \frac{2}{3}G\right)(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2G\epsilon_{22}, \quad (8.422)$$

$$\sigma_{33} = \left(\kappa - \frac{2}{3}G\right)(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2G\epsilon_{33}, \quad (8.423)$$

$$\sigma_{12} = 2G\epsilon_{12}, \quad (8.424)$$

$$\sigma_{13} = 2G\epsilon_{13}, \quad (8.425)$$

$$\sigma_{23} = 2G\epsilon_{23}. \quad (8.426)$$

Imposing the plane-stress assumptions $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ requires that $\epsilon_{13} = \epsilon_{23} = 0$ and that

$$\epsilon_{33} = -\frac{3\kappa - 2G}{3\kappa + 4G}(\epsilon_{11} + \epsilon_{22}). \quad (8.427)$$

Removing these from the constitutive equations results in the three equations

$$\sigma_{11} = \frac{2G}{(\kappa + 4G)}[2(3\kappa + G)\epsilon_{11} + (3\kappa - 2G)\epsilon_{22}] = \frac{E}{1 - \nu^2}(\epsilon_{11} + \nu\epsilon_{22}), \quad (8.428)$$

$$\sigma_{22} = \frac{2G}{(\kappa + 4G)}[(3\kappa - 2G)\epsilon_{11} + 2(3\kappa + G)\epsilon_{22}] = \frac{E}{1 - \nu^2}(\nu\epsilon_{11} + \epsilon_{22}), \quad (8.429)$$

$$\sigma_{12} = 2G\epsilon_{12}. \quad (8.430)$$

A variation of this will assume the shear stresses in the third direction are zero but that there is a known normal stress. As before, since the elasticity problem is linear, the solution can be separated, but this is not the case for plasticity.

8.10.2 Plane-strain problems

For the plane-strain problem in the 1-2 plane, we assume that displacements in the plane are independent of the position in the third direction and that displacements in the third direction are only a function of position in the third direction. We can write this as

$$u_1(X_1, X_2), \quad u_2(X_1, X_2), \quad u_3(X_3). \quad (8.431)$$

As a result of this assumption, we have that $\epsilon_{13} = \epsilon_{23} = 0$, and ϵ_{33} is constant for a given X_3 . For the plane-strain problem, we assume $\epsilon_{33} = 0$, indicating that there is no strain in the third direction, but a close problem also can be defined that takes it as a constant.

Considering the linear elastic problem given by the constitutive equations (8.421), it follows that for the elastic case and plane-strain we get $\sigma_{13} = \sigma_{23} = 0$, and we have the remaining expressions for stress given by

$$\sigma_{11} = \left(\kappa + \frac{4}{3}G \right) \epsilon_{11} + \left(\kappa - \frac{2}{3}G \right) \epsilon_{22}, \quad (8.432)$$

$$\sigma_{22} = \left(\kappa - \frac{2}{3}G \right) \epsilon_{11} + \left(\kappa + \frac{4}{3}G \right) \epsilon_{22}, \quad (8.433)$$

$$\sigma_{33} = \left(\kappa - \frac{2}{3}G \right) (\epsilon_{11} + \epsilon_{22}), \quad (8.434)$$

$$\sigma_{12} = 2G\epsilon_{12}. \quad (8.435)$$

An interesting special case occurs when both the material is assumed to respond as a perfectly plastic material (neglecting elastic deformation), and we assume that the response is incompressible. As a result, the stress is either zero or on the yield surface, and the total strain equals the plastic strain with the condition that volumetric strain $\epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0$ results in $\epsilon_{11} + \epsilon_{22} = 0$. Another slight variation is to take both the elastic and plastic deformations to be incompressible so that the total deformation has the same property, but the stress is given by the equation

$$\boldsymbol{\sigma} = p\mathbf{I} + 2G\boldsymbol{\epsilon}^e, \quad (8.436)$$

where p is an indeterminate function (see the chapter on thermoelastic response). In this case, we can write the balance of linear momentum as

$$\mathbf{Grad}(p) + 2G\mathbf{Div}(\boldsymbol{\epsilon}^e) + \rho_o\mathbf{b} = \rho_o\mathbf{a}, \quad (8.437)$$

with the condition $\epsilon_v^e = \epsilon_{11}^e + \epsilon_{22}^e + \epsilon_{33}^e = 0$ ($\boldsymbol{\epsilon}^e$ is deviatoric) and $\epsilon_{33} = \epsilon_{33}^e + \epsilon_{33}^p = 0$. Looking at the equilibrium problem and ignoring body forces, the equilibrium equations in principal directions are given by

$$\frac{\partial p}{\partial X_1} + 2G \frac{\partial \epsilon_{11}^e}{\partial X_1} = 0, \quad (8.438)$$

$$\frac{\partial p}{\partial X_2} + 2G \frac{\partial \epsilon_{22}^e}{\partial X_2} = 0, \quad (8.439)$$

which in view of the conditions become

$$\frac{\partial p}{\partial X_1} + 2G \frac{\partial \epsilon_{11}^e}{\partial X_1} = 0, \quad (8.440)$$

$$\frac{\partial p}{\partial X_2} - 2G \frac{\partial \epsilon_{11}^e}{\partial X_2} + 2G \frac{\partial \epsilon_{33}^p}{\partial X_2} = 0. \quad (8.441)$$

Taking the partial derivative of the first with respect to X_2 and the second with respect to X_1 and adding the two yields

$$2 \frac{\partial^2 (p + G\epsilon_{33}^p)}{\partial X_1 \partial X_2} = 0. \quad (8.442)$$

This gives

$$p = -G\epsilon_{33}^p + f(X_1) + g(X_2), \quad (8.443)$$

where f and g are functions of the given variables.

8.11 Heat and its generation

To each of the problems, we should add a heat component including temperature as an unknown in the problem. The simplest problems are those for which the material's properties are weakly connected to temperature or for which the temperature change is small enough so that this dependence can be ignored. This is probably the most commonly made assumption.

The balance of work and energy is given as

$$\rho_o \dot{h} = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o \dot{\eta} \theta, \quad (8.444)$$

for the overstress $\Delta \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^b$, and with the thermodynamic constraint given by

$$\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p \geq 0. \quad (8.445)$$

For an isothermal process, from equation (7.111) we know that we have

$$\rho_o \Delta h = \rho_o \theta (\eta - \eta_o) - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt. \quad (8.446)$$

For an isentropic process, from equation (7.115) we have that

$$\rho_o \Delta h = - \int_{t_o}^t \Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p dt \leq 0. \quad (8.447)$$

For the isentropic process the rate of change of temperature is calculated from $\dot{\eta} = 0$.

Assuming constant coefficients $\boldsymbol{\alpha}$, $\mathbf{E}$, and $\mathbf{E}^b$, the expression for $\dot{\eta}$ can be calculated as

$$\dot{\eta} = -\frac{d^2 \psi^\theta}{d\theta^2} \dot{\theta} + \frac{1}{\rho_o} \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha}, \quad (8.448)$$

where

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}}^e. \quad (8.449)$$

For purely thermal heating, we can assume the stresses are zero and the heat capacity at constant zero stress per unit mass is given by

$$c_p = \frac{\dot{h}}{\dot{\theta}} = -\frac{d^2 \psi^\theta}{d\theta^2} \theta. \quad (8.450)$$

We therefore can rewrite the general balance of work and energy using the heat capacity to read

$$\rho_o \dot{h} = -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + \rho_o c_p \dot{\theta} \dot{\eta} + \theta \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha}. \quad (8.451)$$

For an isentropic process, we can use this to calculate the temperature change by setting $\dot{\eta} = 0$ to get

$$\rho_o \frac{d^2 \psi^\theta}{d\theta^2} \dot{\theta} = \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha}. \quad (8.452)$$

For this model, the balance of energy can be written as

$$\begin{aligned} \rho_o \dot{h} &= -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - \rho_o \theta \frac{d^2 \psi^\theta}{d\theta^2} \dot{\theta} + \theta \dot{\boldsymbol{\sigma}} : \boldsymbol{\alpha} \\ &= -\Delta \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p. \end{aligned} \quad (8.453)$$

For an adiabatic process, we have $\dot{h} = 0$ so that temperature change is calculated from

$$0 = -\Delta\sigma : \dot{\epsilon}^p - \rho_o \theta \frac{d^2\psi^\theta}{d\theta^2} \dot{\theta} + \theta \dot{\sigma} : \alpha. \quad (8.454)$$

This results in

$$\rho_o \theta \frac{d^2\psi^\theta}{d\theta^2} \dot{\theta} = -\Delta\sigma : \dot{\epsilon}^p + \theta \dot{\sigma} : \alpha. \quad (8.455)$$

We can rewrite this to read

$$\rho_o c_p \dot{\theta} = \Delta\sigma : \dot{\epsilon}^p - \theta \dot{\sigma} : \alpha, \quad (8.456)$$

which shows that, for an isolated system, plastic working of the overstress increases the temperature since we always have $\Delta\sigma : \dot{\epsilon}^p \geq 0$, while increasing the stress can decrease the temperature.

Section 8.11-Problems:

- 8.11-1:** Knowing that the heat capacity c_p of a material is constant, calculate the expression for the thermal part ψ^θ of the free energy.
- 8.11-2:** Knowing that the heat capacity c_p of most polymers is linear (i.e., $c_p = c_{p0} + c'_p \theta$), calculate the expression for the thermal part ψ^θ of the free energy.
In the following problems assume that the thermal response of the material is fast enough so that it can be considered that this process is adiabatic. Assume the system starts at constant uniform temperature.
- 8.11-3:** Simulate the thermal response during monotonic and cyclic extension of a bar.
- 8.11-4:** Simulate the thermal response during monotonic and cyclic shear of a bar.
- 8.11-5:** Simulate the thermal response during monotonic and cyclic torsion of a bar.
- 8.11-6:** Simulate the thermal response during monotonic and cyclic bending of a bar.

Chapter 9

First-gradient thermomechanical materials

In this chapter, we look at first-gradient thermomechanical material models. These are models where the response of the material depends on the history of position, temperature, and their first gradients. Plasticity is one example of a first-gradient material model, and elasticity and viscoelasticity are other examples.

Since all of plasticity falls within the constitutive theory of modeling first-gradient materials, we will use this chapter to introduce the basic modeling structure and develop the needed background material before looking at elasticity and plasticity in the following chapters. Of particular interest here are the results associated with the superposition of rigid body motions, and the method of imposing material symmetry.

We also will consider state variable theories within the context of first-gradient materials, a starting point for developing thermoplasticity as a state variable theory within the framework of first-gradient material models.

9.1 First-gradient theories

A full first-gradient thermomechanical theory assumes the response of the material at any point is fully characterized through the histories of the position and temperature and their first gradients. We will use $\mathcal{U}$ to denote the value of these four variables at any given time so that

$$\mathcal{U}(\mathbf{X}, t) = \{\mathbf{x}(\mathbf{X}, t), \mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}. \quad (9.1)$$

For a thermomechanical material, the response is described by following the values of specific free energy, specific entropy, Cauchy stress, and the heat flux vector. We will use $\mathcal{R}$ to denote this response, and its value for the material point at position $\mathbf{X}$ in the reference configuration and at time t is

$$\mathcal{R}(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{T}(\mathbf{X}, t), \mathbf{q}(\mathbf{X}, t)\}. \quad (9.2)$$

For our general first-gradient thermo mechanical material, the response at the current time t for a material point $\mathbf{X}$ is assumed to be calculated from functions or functionals of the history of $\mathcal{U}$. This can be written symbolically as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^{\dagger}[\mathbf{X}, \mathcal{U}(\mathbf{X}, s)]_{s=t_o}^t, \quad (9.3)$$

where we are using the superscript $\dagger$ to designate the function or functional used to calculate the value of the response, and where t_o is the initial time from which the history of $\mathcal{U}$ needs to be provided to the specific constitutive equations to calculate the current response. The time t_o simply splits the past into two parts. The history before t_o is fixed and defines the initial conditions of the material. It can be shown that any time can be selected as t_o since

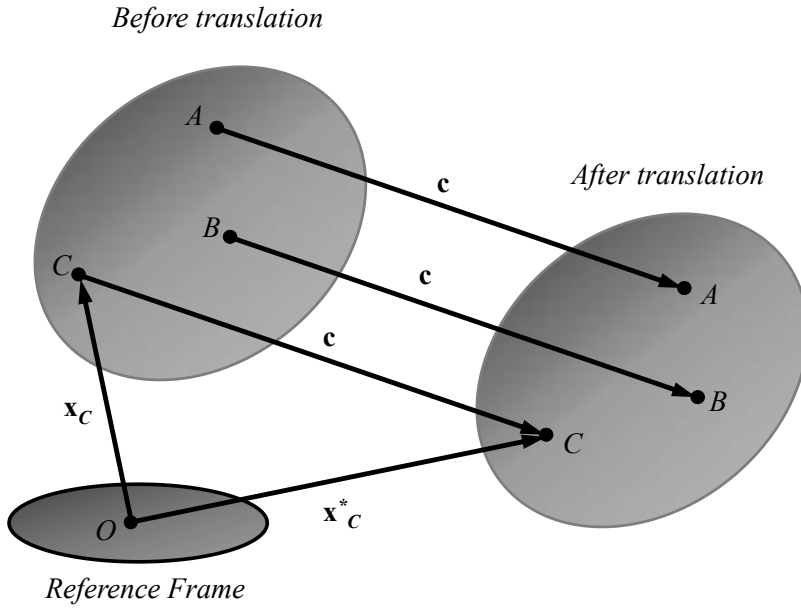


Figure 9.1: Pure rigid body translation of the current configuration.

it simply defines the initial part of the past history which we consider fixed, defining the initial conditions when developing the model.

9.2 Superposition of pure translations

We now will restrict our attention to first-gradient theories that behave in a particular way when superimposing pure rigid body translations on a given history. We assume that superimposing a pure translation onto any given motion of a body will not change the response. As shown in Figure 9.1, a pure translation moves all points of the body by the same amount and in the same direction. Given the motion of a body as $\mathbf{x}(\mathbf{X}, t)$, a motion derived from this by superimposing a pure translation at each time can be written $\mathbf{x}^*(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) + \mathbf{c}(t)$, where $\mathbf{c}(t)$ is an arbitrary function of time. The condition requires that $\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^*(\mathbf{X}, t)$, written explicitly as

$$\mathcal{R}^\dagger[\mathbf{X}, \mathbf{x}(\mathbf{X}, s), \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] = \mathcal{R}^\dagger[\mathbf{X}, \mathbf{x}(\mathbf{X}, s) + \mathbf{c}(s), \theta(\mathbf{X}, s), \mathbf{F}(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)]. \quad (9.4)$$

Since for each material point $\mathbf{X}$ we can select $\mathbf{c}$ to be negative the actual position, we can conclude that

$$\mathcal{R}^\dagger[\mathbf{X}, \mathbf{x}(\mathbf{X}, s), \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] = \mathcal{R}^\dagger[\mathbf{X}, 0, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad (9.5)$$

which suggests that none of the response functions depends on $\mathbf{x}$.

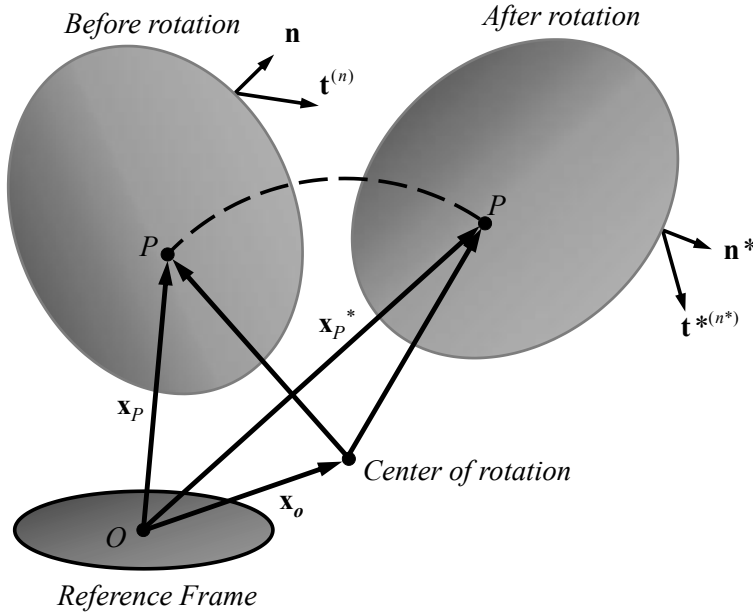


Figure 9.2: Rigid body rotation of the current configuration.

To simplify the presentation, from this point on we assume

$$\mathcal{U}(\mathbf{X}, t) = \{\mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}, \quad (9.6)$$

essentially removing $\mathbf{x}$ from the variable set used to calculate the response.

9.3 Superposition of rigid body rotations

We already have looked at the expected effect of imposing rigid body translations on the history, concluding that the response at a point should not depend on its current location. We now focus on superimposing rigid body rotations onto the history. As can be seen in Figure 9.2, pure rigid body rotations move points on the body on arcs of spheres defined by the particular point on the body, the center of rotation, and the specified rotation. Given the motion of a body as $\mathbf{x}(\mathbf{X}, t)$, a motion derived from this by superimposing rigid body rotations about $\mathbf{x}_o(s)$ can be written as

$$\mathbf{x}^*(\mathbf{X}, s) = \mathbf{Q}(s)[\mathbf{x}(\mathbf{X}, s) - \mathbf{x}_o(s)] + \mathbf{x}_o(s), \quad (9.7)$$

where $\mathbf{Q}(s)$ is an orthogonal tensor representing a rigid body rotation, and s takes values from the starting time t_o to the current time t . Consider a general first-gradient material having a response $\mathcal{R}(\mathbf{X}, t)$, given by

$$\mathcal{R}(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{T}(\mathbf{X}, t), \mathbf{q}(\mathbf{X}, t)\} \quad (9.8)$$

when subjected to the history described by $\mathbf{x}(\mathbf{X}, s)$, and the response $\mathcal{R}^*(\mathbf{X}, t)$, given by

$$\mathcal{R}^*(\mathbf{X}, t) = \{\psi^*(\mathbf{X}, t), \eta^*(\mathbf{X}, t), \mathbf{T}^*(\mathbf{X}, t), \mathbf{q}^*(\mathbf{X}, t)\} \quad (9.9)$$

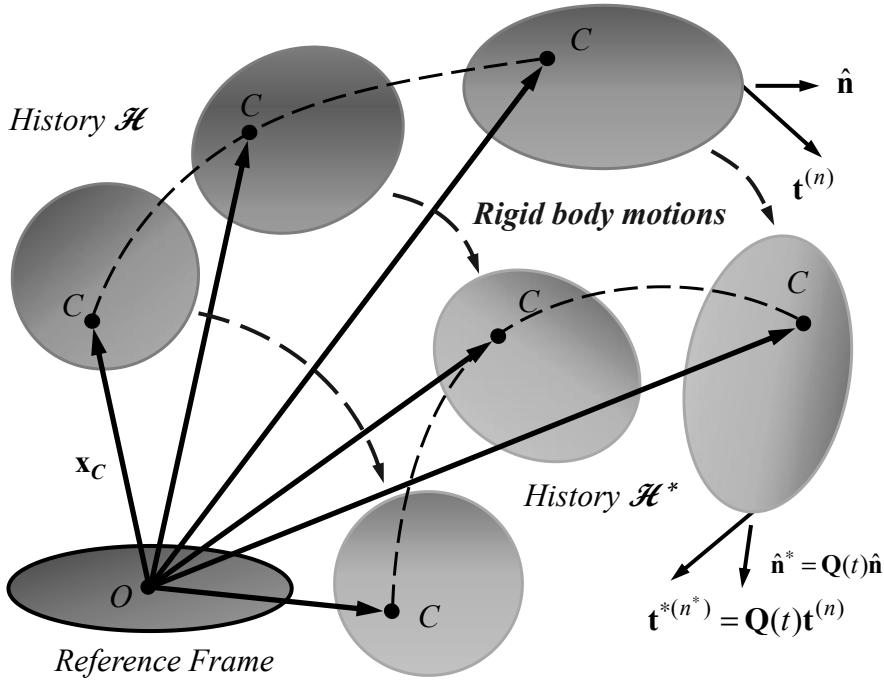


Figure 9.3: Two deformation histories that are identical except for the fact that each configuration in $\mathcal{H}^*$ is obtained by an arbitrary rigid body translation and rotation of the associated configuration in $\mathcal{H}$. It is assumed that this change rotates the traction vector by the amount of the associated rotation of the configuration.

when subjected to the history described by $\mathbf{x}^*(\mathbf{X}, s)$. It would be expected that the rigid body rotations should not alter the value of the specific free energy and specific entropy. The influence of rigid body motions on the stress normally is dictated by the influence of rigid body motions on the traction. Normally, a rigid body motion is assumed to reorient the traction by the amount of the rigid body rotation. As shown in Figure 9.3, if the body is rotated by a rigid body rotation given by the orthogonal transformation $\mathbf{Q}$, then any normal $\hat{\mathbf{n}}$ transforms to $\hat{\mathbf{n}}^* = \mathbf{Q}\hat{\mathbf{n}}$, and the traction on the surface with normal $\hat{\mathbf{n}}$ given by $\mathbf{t}^n$ changes to $\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}$. It can be demonstrated that the Cauchy stress $\mathbf{T}$ then changes to the Cauchy stress $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. As for the traction, the heat flux vector is assumed to rotate by the amount of the rigid body rotation so that we can write

$$\mathcal{R}^*(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{Q}(t)\mathbf{T}(\mathbf{X}, t)\mathbf{Q}^T(t), \mathbf{Q}(t)\mathbf{q}(\mathbf{X}, t)\}. \quad (9.10)$$

The general response functional $\mathcal{R}^\dagger$ for a first-gradient material, considering the already excluded current position vector, can be written as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.11)$$

This same response functional should give $\mathcal{R}^*(\mathbf{X}, t)$ when introducing into it the “*” history so that we should have

$$\mathcal{R}^*(\mathbf{X}, t) = \mathcal{R}^\dagger[\mathbf{X}, \mathbf{Q}(s)\mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.12)$$

Since $\mathbf{Q}(s)$ is an arbitrary rotation, we can select $\mathbf{Q}(s) = \mathbf{R}^T(s)$, where $\mathbf{R}(s)$ is the orthogonal part in the polar decomposition of $\mathbf{F}(s) = \mathbf{R}(s)\mathbf{U}(s)$. This selection requires that the individual response functions have the properties

$$\begin{aligned}\psi(\mathbf{X}, t) &= \psi^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \psi^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)],\end{aligned}\quad (9.13)$$

$$\begin{aligned}\eta(\mathbf{X}, t) &= \eta^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \eta^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)],\end{aligned}\quad (9.14)$$

$$\begin{aligned}\mathbf{T}(\mathbf{X}, t) &= \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \mathbf{R}(\mathbf{X}, t)\mathbf{T}^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)]\mathbf{R}^T(\mathbf{X}, t),\end{aligned}\quad (9.15)$$

$$\begin{aligned}\mathbf{q}(\mathbf{X}, t) &= \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \mathbf{R}(\mathbf{X}, t)\mathbf{q}^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)].\end{aligned}\quad (9.16)$$

In short, in terms of the dependence on the history of the deformation gradient $\mathbf{F}$, the assumption states that the specific free energy and entropy can be written in terms of the history of $\mathbf{U}$, the right symmetric part of $\mathbf{F}$ in its polar decomposition, and that the Cauchy stress and heat flux can be calculated from the history of $\mathbf{U}$ plus the current value of $\mathbf{R}$.

Possibly the most important observation is that the past history of $\mathbf{R}$ never contributes to the current response of the material, only the current value of $\mathbf{R}$ does. This suggests the introduction of a rotated stress $\mathbf{T}^R$ and a rotated heat flux vector $\mathbf{q}^R$ defined by

$$\mathbf{T}^R(\mathbf{X}, t) = \mathbf{R}^T(\mathbf{X}, t)\mathbf{T}(\mathbf{X}, t)\mathbf{R}(\mathbf{X}, t), \quad (9.17)$$

$$\mathbf{q}^R(\mathbf{X}, t) = \mathbf{R}^T(\mathbf{X}, t)\mathbf{q}(\mathbf{X}, t), \quad (9.18)$$

which can be evaluated by constitutive functionals that depend only on the history of the

symmetric part of $\mathbf{F}$. We now can summarize the results of the section by stating that

$$\psi(\mathbf{X}, t) = \psi^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.19)$$

$$\eta(\mathbf{X}, t) = \eta^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.20)$$

$$\mathbf{T}^R(\mathbf{X}, t) = \mathbf{T}^{R\dagger}[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.21)$$

$$\mathbf{q}^R(\mathbf{X}, t) = \mathbf{q}^{R\dagger}[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.22)$$

Since $\mathbf{U}$, the right Cauchy stretch $\mathbf{C}$, and the Green strain $\mathbf{E}$ are related uniquely, we can assume these functionals can be given in terms of either one of these alternate variables. We write this as

$$\psi(\mathbf{X}, t) = \psi^{\dagger\dagger}[\mathbf{X}, \mathbf{C}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.23)$$

$$\eta(\mathbf{X}, t) = \eta^{\dagger\dagger}[\mathbf{X}, \mathbf{C}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.24)$$

$$\mathbf{T}^R(\mathbf{X}, t) = \mathbf{T}^{R\dagger\dagger}[\mathbf{X}, \mathbf{C}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.25)$$

$$\mathbf{q}^R(\mathbf{X}, t) = \mathbf{q}^{R\dagger\dagger}[\mathbf{X}, \mathbf{C}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.26)$$

or

$$\psi(\mathbf{X}, t) = \psi^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.27)$$

$$\eta(\mathbf{X}, t) = \eta^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.28)$$

$$\mathbf{T}^R(\mathbf{X}, t) = \mathbf{T}^{R\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.29)$$

$$\mathbf{q}^R(\mathbf{X}, t) = \mathbf{q}^{R\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.30)$$

This also opens up the possibility of using any other strain measure derivable directly from $\mathbf{U}$ in constructing such functionals. One example might be the use of the logarithmic strain $\ln(\mathbf{U})$.

One additional point to note is that because the second Piola-Kirchhoff stress $\tilde{\mathbf{T}}$ behaves in a similar manner as the rotated stress one can show that

$$\begin{aligned}
 \tilde{\mathbf{T}}(\mathbf{X}, t) &= \tilde{\mathbf{T}}^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \overset{t}{\theta}(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \\
 &\quad \quad \quad \underset{s=t_o}{\overset{t}{\theta}} \\
 &= \tilde{\mathbf{T}}^{\dagger\dagger}[\mathbf{X}, \mathbf{C}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \\
 &\quad \quad \quad \underset{s=t_o}{\overset{t}{\theta}} \\
 &= \tilde{\mathbf{T}}^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)]. \\
 &\quad \quad \quad \underset{s=t_o}{\overset{t}{\theta}}
 \end{aligned} \tag{9.31}$$

Section 9.3-Problems:

9.3-1: Prove that if $\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}$ and $\hat{\mathbf{n}}^* = \mathbf{Q}\hat{\mathbf{n}}$, then one can conclude that

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T.$$

9.3-2: Prove relations (9.31) for the constitutive relation of the second Piola-Kirchhoff stress tensor.

9.4 Material symmetry

Material symmetry normally refers to existing symmetry in the structure of the material. This is the symmetry that exists before we start the thermomechanical loading process. If present, symmetry in the material structure means that certain reorganizations of the structure are thermomechanically indistinguishable from each other. For example, for an initially isotropic material, we can rotate the material by any amount before cutting a testing specimen and will observe the exact same response and obtain the exact same results, irrespective of the test.

Material symmetry is described by mathematical mappings that reorganize the neighborhood of a point in the reference configuration. Figure 9.4 shows a schematic of the idea of reorganizing a material's neighborhood. For first-gradient materials, these reorganizations can be represented by second-order transformation tensors, here denoted by $\mathbf{M}$ and shown in Figure 9.4. Each material symmetry is characterized by a group of transformations $\mathbf{M}$ that take the neighborhood of a material point and reorganize it to materially equivalent neighborhoods (i.e., ones that are indistinguishable in their response from the original neighborhood). This group will be denoted by $\mathcal{G}$. As might be expected, the description of the transformation $\mathbf{M}$ is related closely to the reference configuration since these transformations (the $\mathbf{M}$ s) represent reorganizations of the reference configuration. Let us look at how changing the reference configuration changes the representation of material symmetry. If $\mathbf{M}_\kappa$ is a symmetry described relative to configuration κ , and $\mathbf{M}_{\kappa^*}$ is the same symmetry described relative to configuration κ^* , then Noll's rule states that

$$\mathbf{M}_{\kappa^*} = \tilde{\mathbf{F}}\mathbf{M}_\kappa\tilde{\mathbf{F}}^{-1}, \tag{9.32}$$

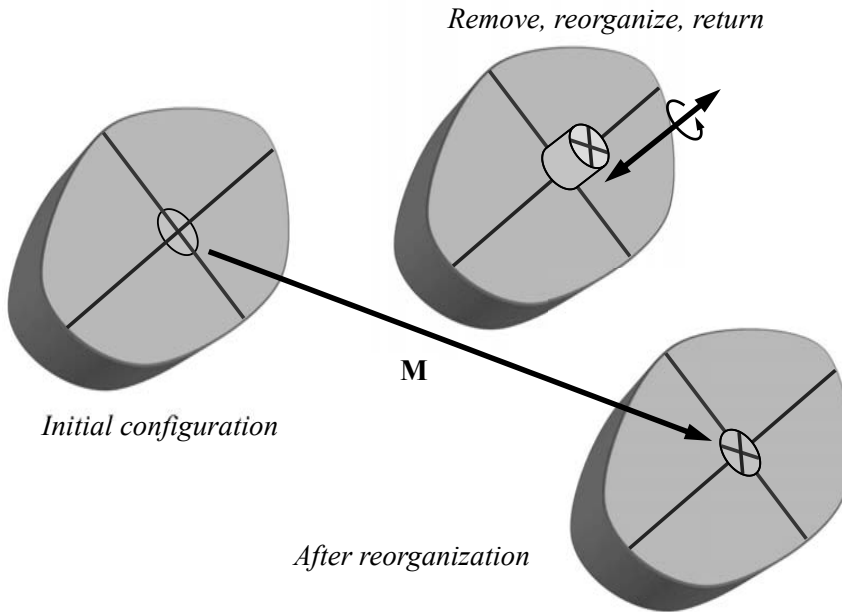


Figure 9.4: Schematic of a material symmetry reorganization of the neighborhood of a material point.

where $\tilde{\mathbf{F}}$, as shown in Figure 9.5, is the deformation gradient mapping the neighborhood in κ to the associated neighborhood in κ^* . Since each member of the material symmetry group follows Noll's rule, we then can write the relation between the groups as

$$\mathcal{G}_{\kappa^*} = \tilde{\mathbf{F}} \mathcal{G}_{\kappa} \tilde{\mathbf{F}}^{-1}. \quad (9.33)$$

The reader might note that Noll's rule refers to changing the reference configuration, but *Noll's rule does not apply to actually deforming the material.*

The transformations in the material symmetry group need not be orthogonal. For example, an isotropic material normally is represented by the set of orthogonal transformations. This is actually a set of transformations with respect to a given reference configuration. If we choose to use another reference configuration, through Noll's rule we get a new set of transformations representing the same isotropy, but possibly some of these new transformations may not be orthogonal. This is because Noll's rule does not preserve orthogonality of the transformations.

With thermomechanical loading the material's symmetry can change. For example, by melting a crystal we can increase its symmetry, or by drawing an isotropic bar we might make it anisotropic. These types of changes are not what Noll's rule applies to. By melting the crystal or drawing the bar, we actually are subjecting the material to thermomechanical loading that is altering the state of the material, in the process creating a new material with possibly new symmetries. When changing the reference configuration, we simply are using a different, but related, mathematical formulation, without subjecting the actual material to any deformation whatsoever. Since we do not intend to make changes to the reference configuration, we will not use the required configuration subscript on the transformations

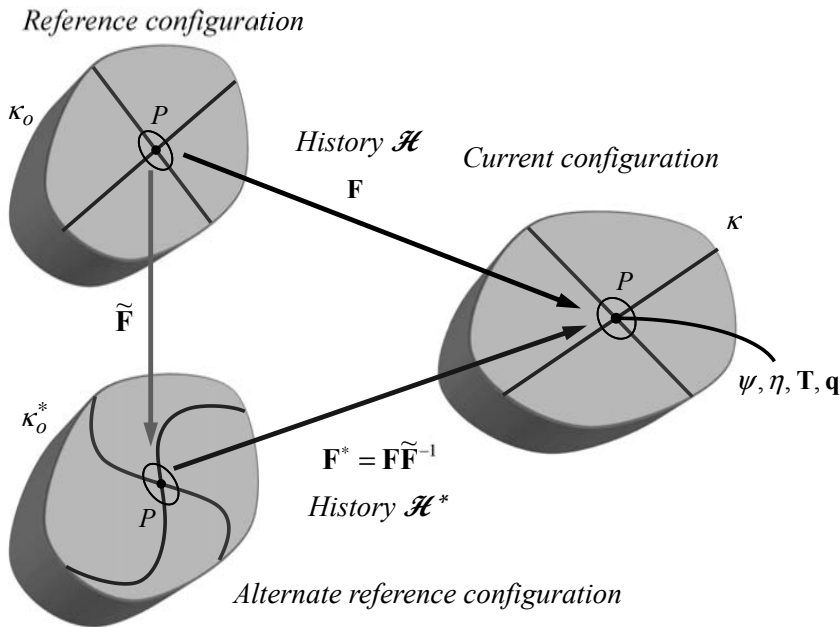


Figure 9.5: The effect of change of reference condition on the definition of the deformation gradient.

$\mathbf{M}$, always implied to be transformations of the reference configuration, which also normally will be the initial configuration.

Let us now consider the restriction imposed by material symmetry. Let us assume that t_o is the initial time from which we are to model the response of the material to further loading. For our first-gradient material, a loading history imposed on the state at t_o will be noted by $\mathcal{H}(t_o, t)$ and symbolically represented by

$$\mathcal{H}(t_o, t) = \mathcal{U}(s) = \{\mathbf{F}(s), \theta(s), \mathbf{G}(s)\}. \quad (9.34)$$

$s = t_o$ $s = t_o$

A history that is essentially identical to this, but for a reorganization of the neighborhood by material symmetry transformation $\mathbf{M}$ before the loading will be denoted by $\bar{\mathcal{H}}(t_o, t)$ and given by

$$\begin{aligned} \bar{\mathcal{H}}(t_o, t) = \bar{\mathcal{U}}(s) &= \{\bar{\mathbf{F}}(s), \bar{\theta}(s), \bar{\mathbf{G}}(s)\} \\ &= \{\mathbf{F}(s)\mathbf{M}, \theta(s), \mathbf{G}(s)\mathbf{M}\}. \end{aligned} \quad (9.35)$$

$s = t_o$ $s = t_o$

The existing material symmetries at time t_o can be represented by a material symmetry group $\mathcal{G}(t_o)$. Each transformation $\mathbf{M}$ in $\mathcal{G}(t_o)$ represents a reorganization at time t_o that leaves the response identical for the two histories $\mathcal{H}(t_o, t)$ and $\bar{\mathcal{H}}(t_o, t)$, irrespective of history

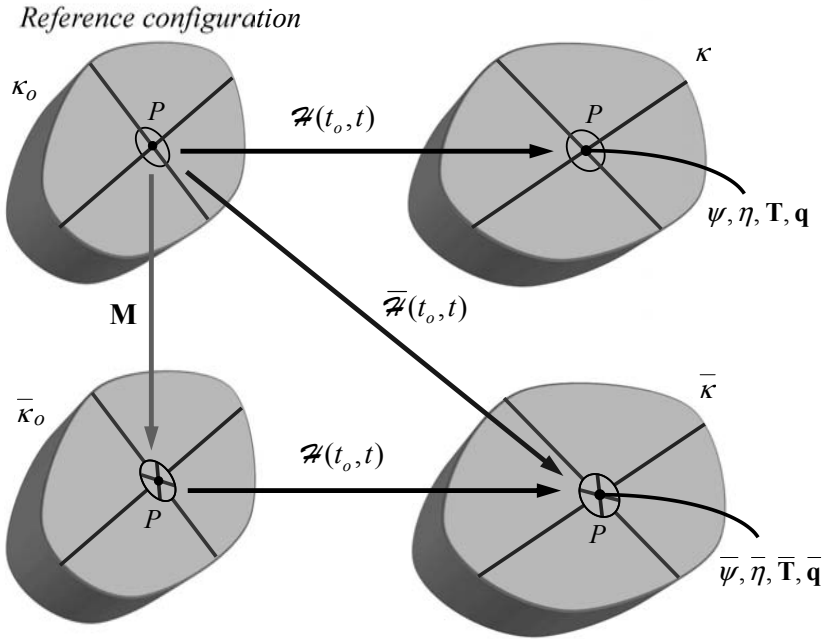


Figure 9.6: A deformation imposed before or after a material symmetry transformation $\mathbf{M}$.

$\mathcal{H}(t_o, t)$ and the current time t , as shown in Figure 9.6. Symbolically, we write this as

$$\boxed{\mathcal{R}(t) = \bar{\mathcal{R}}(t)} \quad (9.36)$$

for all pairs of histories $\mathcal{H}(t_o, t)$ and $\bar{\mathcal{H}}(t_o, t)$ related by each symmetry transformation $\mathbf{M}$ in the symmetry group of the material $\mathcal{G}(t_o)$.

Section 9.4-Problems:

9.4-1: Prove Noll's rule.

9.5 First-gradient state variable models

State variable models essentially condense the influence of the history into a number of variables. The current value of these variables and the current value of $\mathcal{U}$ defines the state of the material, from which it is assumed the response can be evaluated fully. These additional variables may be scalars, vectors, or tensors. Letting y_i , $\mathbf{y}_i$, and $\mathbf{Y}_i$, designate, respectively, scalar, vector, and second-order tensorial variables, the current state of the material $\mathcal{S}$ is assumed to be defined as

$$\boxed{\mathcal{S}(\mathbf{X}, t) = \{\mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t), y_1(\mathbf{X}, t), \dots, y_m(\mathbf{X}, t), \mathbf{y}_1(\mathbf{X}, t), \dots, \mathbf{y}_n(\mathbf{X}, t), \mathbf{Y}_1(\mathbf{X}, t), \dots, \mathbf{Y}_p(\mathbf{X}, t)\}} \quad (9.37)$$

for m scalar variables, n vector variables, and p tensor valued variables. For this to be a first-gradient theory, it has to be assumed that these variables are representing the history of $\mathcal{U}$ in the response functions, so that they each somehow should be calculated from this history. Therefore, each of these variables should be given by a constitutive function of the form

$$y_i(\mathbf{X}, t) = y_i^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.38)$$

$$\mathbf{y}_i(\mathbf{X}, t) = \mathbf{y}_i^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.39)$$

$$\mathbf{Y}_i(\mathbf{X}, t) = \mathbf{Y}_i^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad s = t_o \quad (9.40)$$

Obviously, there would be no particular interest in state variable formulations if these constitutive functionals were not simple and physically meaningful. Possibly, the most simple state variable theory is that of thermoelasticity. For a *thermoelastic material*, the state of the material is fully determined by

$$\mathcal{S}(\mathbf{X}, t) = \{\mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}, \quad (9.41)$$

with no additional variables defined. Another state variable theory is *plasticity*, in which the state is given by

$$\mathcal{S}(\mathbf{X}, t) = \{\mathbf{F}(\mathbf{X}, t), \mathbf{F}^p(\mathbf{X}, t), \mathbf{F}^\theta(\mathbf{X}, t), \xi(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}, \quad (9.42)$$

where the additional variables are the plastic deformation gradient $\mathbf{F}^p$, the thermal deformation gradient $\mathbf{F}^\theta$, and the hardening parameter ξ . In state variable theories, the response is fully determined by the state variables. That is, we can write the response as functions of the current state. This can be written symbolically as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^\dagger[\mathbf{X}, \mathcal{S}(\mathbf{X}, t)]. \quad (9.43)$$

The state variables inherit certain characteristics from their first-gradient parent material models. One of these has to do with the influence of rigid body motions. Essentially, if $\mathcal{S}(\mathbf{X}, t)$ is the value of the state variables evaluated for any given history, and if $\mathcal{S}^*(\mathbf{X}, t)$ is the value of the state variables for a history obtained from the original by superposition of rigid body motions, then the relation between $\mathcal{R}$ evaluated from $\mathcal{S}$ and $\mathcal{R}^*$ evaluated from $\mathcal{S}^*$ should be exactly the same as if we would evaluate directly $\mathcal{R}$ and $\mathcal{R}^*$ from the histories. In view of the above discussion for general first-gradient materials, this requires that

$$\psi(\mathbf{X}, t) = \psi^\dagger[\mathbf{X}, \mathcal{S}(\mathbf{X}, t)] = \psi^\dagger[\mathbf{X}, \mathcal{S}^*(\mathbf{X}, t)], \quad (9.44)$$

$$\eta(\mathbf{X}, t) = \eta^\dagger[\mathbf{X}, \mathcal{S}(\mathbf{X}, t)] = \eta^\dagger[\mathbf{X}, \mathcal{S}^*(\mathbf{X}, t)], \quad (9.45)$$

$$\mathbf{T}(\mathbf{X}, t) = \mathbf{T}^\dagger[\mathbf{X}, \mathcal{S}(\mathbf{X}, t)] = \mathbf{R}(\mathbf{X}, t) \mathbf{T}^\dagger[\mathbf{X}, \mathcal{S}^*(\mathbf{X}, t)] \mathbf{R}^T(\mathbf{X}, t), \quad (9.46)$$

$$\mathbf{q}(\mathbf{X}, t) = \mathbf{q}^\dagger[\mathbf{X}, \mathcal{S}(\mathbf{X}, t)] = \mathbf{R}(\mathbf{X}, t) \mathbf{q}^\dagger[\mathbf{X}, \mathcal{S}^*(\mathbf{X}, t)], \quad (9.47)$$

where in this case $\mathcal{S}^*$ is

$$\begin{aligned} \mathcal{S}^*(\mathbf{X}, t) = \{ & \mathbf{U}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t), y_1^*(\mathbf{X}, t), \dots, y_m^*(\mathbf{X}, t), \\ & \mathbf{y}_1^*(\mathbf{X}, t), \dots, \mathbf{y}_n^*(\mathbf{X}, t), \mathbf{Y}_1^*(\mathbf{X}, t), \dots, \mathbf{Y}_p^*(\mathbf{X}, t)\}, \end{aligned} \quad (9.48)$$

for

$$y_i^*(\mathbf{X}, t) = y_i^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad (9.49)$$

$$s = t_o$$

$$\mathbf{y}_i^*(\mathbf{X}, t) = \mathbf{y}_i^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)], \quad (9.50)$$

$$s = t_o$$

$$\mathbf{Y}_i^*(\mathbf{X}, t) = \mathbf{Y}_i^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)]. \quad (9.51)$$

$$s = t_o$$

Namely, there is no contribution from the past history of the orthogonal part $\mathbf{R}$ of the deformation gradient to the response of the material, and we always can construct models that provide the response from state variables unaffected by the history of $\mathbf{R}$. The latter is important in that it tells us we need not concern ourselves with determining the influence of the history of $\mathbf{R}$ on the state variables, since we always can write the response functions in a way that avoids such contributions. In short, if we can identify the values of the state variables for the history of $\mathbf{U}$, as opposed to $\mathbf{F}$, we have all we require to calculate the response, assuming we know the current value of $\mathbf{R}$.

We need to separately determine the influence of material symmetry on the state variables other than those in $\mathcal{U}$. These should be determined based on the role these variables play in the model. Essentially, using the same notation as above when discussing material symmetry, we need to establish the form of $\bar{y}_i$, $\bar{\mathbf{y}}_i$, and $\bar{\mathbf{Y}}_i$ in relation to y_i , $\mathbf{y}_i$, and $\mathbf{Y}_i$, for a material symmetry transformation $\mathbf{M}$.

9.6 Higher-gradient and nonlocal models

There are many models and model development strategies being used to characterize real materials. As long as the model evaluates the response of the material from the history of position and temperature, and the history of their first gradients, by our definition, the model is a first-gradient material model. As soon as higher gradients or nonlocal information is needed for evaluating the model, then the models fall into larger classes of models known, respectively, as higher-gradient and nonlocal models. For most of what is considered as plasticity, it seems unnecessary to use higher-gradient or nonlocal models. As a result, in this book we only focus on first-gradient models.

Different authors have selected different strategies to construct plasticity models. Most can be put into the state variable form described above. Many are a special subclass that assumes the stress depends only on the elastic deformation and temperature. A few have generalized this model to allow the symmetry of the elastic response to change with the deformation. Some even have viewed plasticity as a theory of changing reference configurations. Yet, ultimately, all seem to calculate the response based on the history of the motion and its first gradient, and the history of temperature and its first gradient so that one safely can conclude that the models are first-gradient models. Since it is easy to get lost in the underlying assumptions made to construct a theory, particularly if the model is not described clearly or the model is highly interconnected, it sometimes helps in realizing the underlying assumptions of a model if one unfolds it into the form of a general first-gradient model or the above described state variable form.

9.7 Notation

The following table shows the symbols used in the text and their definitions.

C		right Cauchy stretch tensor ($\mathbf{F}^T \mathbf{F}$)
E		Green strain
F		deformation gradient
G		temperature gradient with respect to reference configuration
$\mathcal{H}$		history of $\mathcal{U}$
M		transformation tensor representing material symmetry
q		heat flux vector
Q		orthogonal tensor representing rigid body motion
R		orthogonal factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathcal{R}$		response variables $\{\psi, \eta, \mathbf{T}, \mathbf{q}\}$
$\mathcal{S}$		state of a material point
t		time
$\mathbf{t}^{(\mathbf{n})}$		traction vector on surface with normal $\mathbf{n}$
T		Cauchy stress
U		right symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathcal{U}$		variable set $\{\mathbf{x}, \mathbf{F}, \theta, \mathbf{G}\}$ or $\{\mathbf{F}, \theta, \mathbf{G}\}$
V		left symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
x		current position
X		reference position
y_i		scalar internal state variable i
$\mathbf{y}_i$		vector internal state variable i
$\mathbf{Y}_i$		tensor internal state variable i
η		specific entropy
ψ		specific free energy
θ		temperature

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Chapter 10

Elastic and thermoelastic solids

We will focus on thermoelastic response in this chapter, looking primarily at finite deformation theory developed in the context of continuum thermodynamics. The theory of elasticity is a much simpler theory than plasticity, and it provides a means to develop and understand the concepts needed later in plasticity without having to deal with the complications that come with plasticity. A reader familiar with nonlinear elasticity might wish to skip this chapter and go directly to the chapter on thermoplasticity.

A characteristic of the elastic response is that the unloading response is very close to the loading response, sufficiently so that the difference can be ignored. A typical nonlinear elastic response is shown for a poly(dimethyl siloxane)(PDMS) at room temperature in Figure 10.1. The figure shows both the engineering and true stresses, which are noticeably different for this close to incompressible polymer. This is the first indication that when looking at large deformations we need to take note of the shape changes and even revisit our understanding of well-defined material parameters such as Poisson's ratio shown in Figure 10.2 for an incompressible material. As can be seen in the figure, even for an incompressible material, the Poisson's ratio can be significantly different from its well-known value of $\nu = 0.5$ under infinitesimal deformations, depending on how we define it. One potentially can construct a definition for Poisson's ratio based on the ratio of the strains or the ratio of the change of the strains or the ratio of velocity gradients, all of which reduce to the same parameter under infinitesimal deformations.

We will start this chapter by defining a thermoelastic solid. As is the case in all of this book, the development will be confined to first-gradient material models. That is, we focus on models that assume the responses at a material point are characterized by functions or functionals of position, temperature, and their first spatial gradients. We will start by focusing on making the constitutive models behave as expected when superimposing rigid body motions. This is followed by a discussion of material symmetry and then a change of the reference configuration. These are included in this chapter, in spite of their discussion in the chapter on first-gradient material models, since they help in seeing the specific arguments in the simpler setting of elasticity.

Only after these preliminaries do we study the constraint of thermodynamics on the constitutive functions of thermoelasticity. We then show how one can develop general representations for thermodynamically consistent isotropic, transversely isotropic, and orthotropic materials. As a special example, we will consider a material that has a specific free energy that is quadratic in the Green strain and show the corresponding finite deformation expressions for Cauchy stress and entropy that result from this assumption. To show the connection between finite deformation theory and infinitesimal modeling, the results obtained in this example are approximated to obtain directly the relations normally used in infinitesimal models.

Since heat and heat flow are important considerations, we then look at the balance of energy and show how this relation can be used to understand the relation between heat and mechanical working.

As an introduction to the decomposition of the deformation into elastic, plastic, and thermal deformations, in this chapter we look at the separation of the deformation gradient

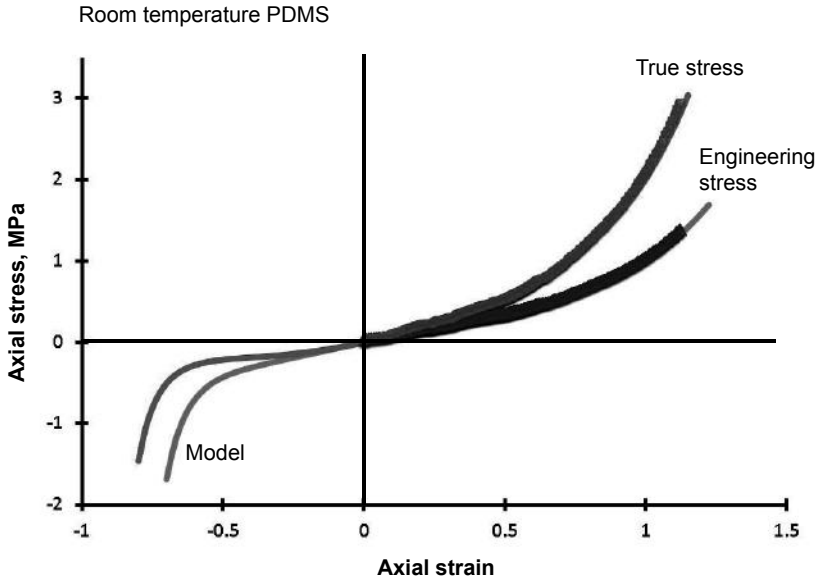


Figure 10.1: The true and engineering stresses as a function of uniaxial strain for PDMS at room temperature.

into elastic and thermal parts and examine how this decomposition affects the equations.

We also look at constraints. Constraints frequently are used to simplify modeling, particularly when some types of deformations are relatively more difficult to initiate than others, such as extension along a fiber in a fiber-reinforced composite. We consider both single and multiple constraints.

We conclude the chapter with the study of superposition of deformations and consider the example of wave propagation in a preloaded body.

10.1 The thermoelastic solid

We will consider the response of a thermodynamic body to be the values of its specific free energy ψ , specific entropy η , Cauchy stress $\mathbf{T}$, and heat flux vector $\mathbf{q}$. We will denote the response by $\mathcal{R}$ and consider it to be a function of the material point, given by the position in the reference configuration $\mathbf{X}$, and current time t . We write this as

$$\mathcal{R}(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{T}(\mathbf{X}, t), \mathbf{q}(\mathbf{X}, t)\}. \quad (10.1)$$

We will select the independent variables to be the current position $\mathbf{x}$, temperature θ , and their first gradients with respect to space, $\mathbf{F}$ and $\mathbf{G}$. We use $\mathcal{U}$ to denote this variable set and write it for a given material point and at current time as

$$\mathcal{U}(\mathbf{X}, t) = \{\mathbf{x}(\mathbf{X}, t), \mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}. \quad (10.2)$$

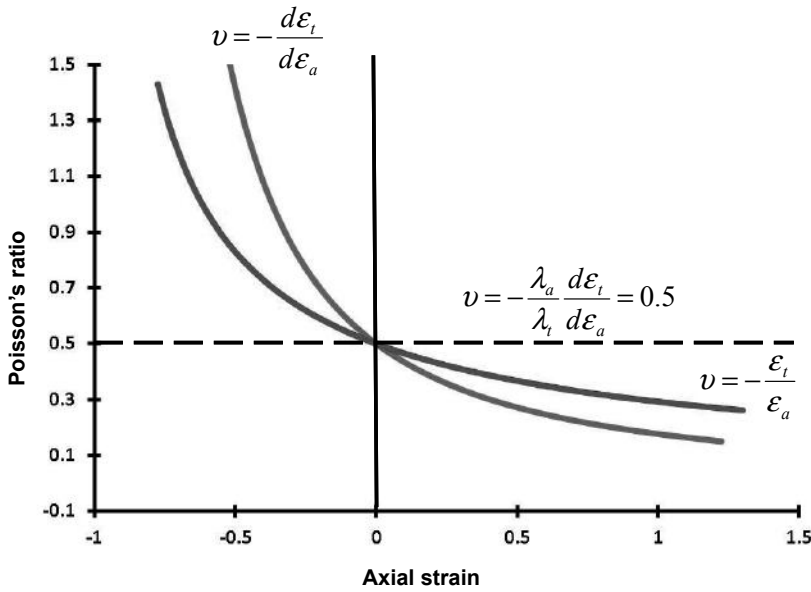


Figure 10.2: The Poisson's ratio for an incompressible material under uniaxial extension defined in terms of the ratio of strains or the ratio of their changes.

A thermoelastic material will be defined as one for which its thermodynamic response at each material point $\mathbf{X}$ can be written in terms of the current value of this variable set. We symbolically write this assumption as

$$\boxed{\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^\dagger[\mathbf{X}, \mathcal{U}(\mathbf{X}, t)]}, \quad (10.3)$$

where $\mathcal{R}^\dagger$ denotes the function that is used to evaluate $\mathcal{R}$. This assumption must be modified in the presence of material constraints such as incompressibility or inextensibility. The influence of such constraints will be considered later in the chapter. The absence of time t as an explicit variable in the constitutive response function $\mathcal{R}^\dagger$ implies that time does not independently influence the response. This excludes all possibilities of explicit changes in the constitutive response with time such as aging or curing.

To simplify the presentation, frequently we will omit in the notation the dependence on $(\mathbf{X}, t)$. In most cases, this should not create any confusion since the response functions in this chapter are assumed always to depend on the current values of $\mathcal{U}$, and they should be evaluated for the $\mathbf{X}$ already explicitly noted in the response function. Therefore, we consider a thermoelastic material to be one that is characterized by the following explicit constitutive assumptions

$$\boxed{\psi = \psi^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G})}, \quad (10.4)$$

$$\boxed{\eta = \eta^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G})}, \quad (10.5)$$

$$\boxed{\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G})}, \quad (10.6)$$

$$\boxed{\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G})}, \quad (10.7)$$

where, as above, the superscript “†” is used to distinguish between the material response on the left of the equality sign and its constitutive model on the right. The assumption that all four constitutive response functions depend on the same arguments, $\mathcal{U}$ in this case, is not mandatory. This assumption sometimes is called the Axiom of Equal Presence.

We now will lay out a set of assumptions on how the material responds under specific conditions, and follow through to see what simplifications in the form of the constitutive response functions are implied by these assumptions.

10.2 The influence of pure rigid body translation on the constitutive response

The first assumption we will make is that a pure rigid body translation will not alter the response. That is, it will leave unchanged the values of free energy, entropy, Cauchy stress, and heat flux. This assumption imposes the hypothesis that the response of each material point will remain unaltered if the distortion in the body is not altered, and that if the relative orientation and temperature of points in a body are not altered.

As shown in Figure 9.1, if points in a body undergo a pure rigid body translation, then the current position of each point $\mathbf{x}$ is shifted to new position $\mathbf{x}^*$ given by

$$\mathbf{x}^*(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) + \mathbf{c}(t), \quad (10.8)$$

where the translation $\mathbf{c}$ is the same for all points of the body. Since $\mathbf{c}$ is a constant over space (is not a function of $\mathbf{X}$), then its gradient will be zero. It is clear that the deformation gradients for these two deformations are the same. If we use $\mathbf{F}$ to denote the deformation gradient associated with the motion $\mathbf{x}$ (i.e., $d\mathbf{x} = \mathbf{F}d\mathbf{X}$) and we use $\mathbf{F}^*$ to denote the deformation gradient associated with the motion $\mathbf{x}^*$ (i.e., $d\mathbf{x}^* = \mathbf{F}^*d\mathbf{X}$), then we have

$$\mathbf{F}^* = \mathbf{F}. \quad (10.9)$$

Since the deformation gradient is a measure of the distortion and rotation in a body, there is no further distortion of the body as a result of this pure translation. The assumption that each material point retains its temperature and that there is no further distortion in the body implies that $\theta^*(\mathbf{X}, t) = \theta(\mathbf{X}, t)$ and, therefore, it follows that

$$\mathbf{G}^* = \mathbf{G}. \quad (10.10)$$

Thus, such a pure translation would change the variable set $\mathcal{U}$ to the new set $\mathcal{U}^*$ given by

$$\mathcal{U}^* = \{\mathbf{x} + \mathbf{c}, \mathbf{F}, \theta, \mathbf{G}\}. \quad (10.11)$$

If such a pure translation is to leave the response of the material unaltered at each material point, then the constitutive model (function) used for each material response should be such that it remains unaffected by changing the arguments from $\mathcal{U}$ to $\mathcal{U}^*$. This requires that

$$\boxed{\mathcal{R}^\dagger[\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G}] = \mathcal{R}^\dagger[\mathbf{X}, \mathbf{x} + \mathbf{c}, \mathbf{F}, \theta, \mathbf{G}]} \quad (10.12)$$

for all $\mathbf{c}$. It is easy to show that a function $f(x)$ does not depend on x if it has this property $f(x) = f(x + c)$ for all c (i.e., f is a constant). Therefore, if the response is to be unaffected by a pure rigid body translation as described above, then the response functions must not depend on the current position $\mathbf{x}$. The result of imposing this restriction on the response will be reflected in the response functions by removing $\mathbf{x}$ from the argument set and writing

$$\boxed{\begin{aligned} \psi &= \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \\ \eta &= \eta^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \\ \mathbf{T} &= \mathbf{T}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \\ \mathbf{q} &= \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}). \end{aligned}} \quad (10.13)$$

We will do the same to $\mathcal{U}$ and write it as

$$\mathcal{U} = \{\mathbf{F}, \theta, \mathbf{G}\}. \quad (10.14)$$

10.3 The influence of pure rigid body rotation on the constitutive response

We now will consider the alteration of the current distortion of the body by adding a pure rigid body rotation. It will be assumed that the temperature of each material particle stays the same (i.e., the temperature field rigidly rotates the same as the body is rotated). Let $\mathcal{U}$ designate the argument set before the rigid body rotation and let $\mathcal{U}^*$ designate the argument set after such a rigid body rotation is imposed on the current conditions. As seen in Figure 9.2, any rigid body rotation can be written as

$$\mathbf{x}^*(\mathbf{X}, t) = \mathbf{Q}(t)[\mathbf{x}(\mathbf{X}, t) - \mathbf{x}_o(t)] + \mathbf{x}_o(t), \quad (10.15)$$

where $\mathbf{Q}$ is an orthogonal second-order tensor describing the rotation and $\mathbf{x}_o$ is the point about which the body is rotated. For the motion to be a rigid body motion, both $\mathbf{Q}$ and $\mathbf{x}_o$ must only be functions of time (i.e., they are the same for all material points). Taking the gradient of both sides, it is clear that

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \quad (10.16)$$

where $\mathbf{F}$ is the deformation gradient associated with the motion $\mathbf{x}$ and $\mathbf{F}^*$ is the deformation gradient associated with the motion $\mathbf{x}^*$. Since the temperature field is assumed to rotate rigidly with the body, the temperature of the material points will not change. That is,

$$\theta^*(\mathbf{X}, t) = \theta(\mathbf{X}, t). \quad (10.17)$$

It therefore follows that

$$\mathbf{G}^* = \mathbf{G}, \quad (10.18)$$

where $\mathbf{G}$ is the temperature gradient associated with the temperature field θ (i.e., $d\theta = \mathbf{G} \circ d\mathbf{X}$), and $\mathbf{G}^*$ is the temperature gradient associated with the temperature field θ^* (i.e., $d\theta^* = \mathbf{G}^* \circ d\mathbf{X}$). One then must conclude that if

$$\mathcal{U} = \{\mathbf{F}, \theta, \mathbf{G}\},$$

then

$$\mathcal{U}^* = \{\mathbf{Q}\mathbf{F}, \theta, \mathbf{G}\}. \quad (10.19)$$

A plausible assumption is that such a rigid body motion will leave the free energy and entropy unchanged and will result in a rotation of the traction vector and the heat flux. That is, since the relative conditions of the material particles have only changed by a rigid body rotation, the free energy and entropy are unchanged, and the heat flux and traction vector reorient to reflect the new relative location of the material particles. Any vector $\mathbf{u}$ connected to material points of the body will be transformed to $\mathbf{u}^* = \mathbf{Q}\mathbf{u}$ by such a rigid body rotation. Therefore, we will assume that both the traction vector and heat flux rotate in a similar manner. The combination of these assumptions makes us conclude that

$$\psi^* = \psi, \quad (10.20)$$

$$\eta^* = \eta, \quad (10.21)$$

$$\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}, \quad (10.22)$$

$$\mathbf{q}^* = \mathbf{Q}\mathbf{q}. \quad (10.23)$$

As in the case of pure translation, these assumptions restrict the form of the constitutive functions. Since we have not proposed a constitutive equation for the traction vector, let us look at how the restriction on the traction vector results in a restriction on the Cauchy stress, for which we wish to prescribe a constitutive function. First, we note that the normal vector $\mathbf{n}$ to a material surface before the rigid body rotation transforms to $\mathbf{n}^* = \mathbf{Q}\mathbf{n}$, that the traction $\mathbf{t}^{(n)}$ on the material surface with normal $\mathbf{n}$ before rotation is given by $\mathbf{t}^{(n)} = \mathbf{T}^T \mathbf{n}$, and that traction on the same material surface after the rotation is given by $\mathbf{t}^{*(n^*)} = \mathbf{T}^{*T} \mathbf{n}^*$. Combining these relations with 10.22 results in

$$\boxed{\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T.} \quad (10.24)$$

Therefore, given the initial response

$$\mathcal{R} = \{\psi, \eta, \mathbf{T}, \mathbf{q}\}, \quad (10.25)$$

then the response after superimposing a pure rigid body rotation is given by

$$\boxed{\mathcal{R}^* = \{\psi, \eta, \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \mathbf{Q}\mathbf{q}\}.} \quad (10.26)$$

Let us now impose these assumptions of how the material should respond onto the response functions. First, we start with the specific free energy. The results will be similar for specific entropy. Next, we will look at Cauchy stress and then the heat flux vector. If $\psi^* = \psi$ for every rigid body rotation, then the constitutive function for the specific free energy must be of such form that

$$\psi^\dagger(\mathbf{X}, \mathbf{Q}\mathbf{F}, \theta, \mathbf{G}) = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) \quad (10.27)$$

for every orthogonal $\mathbf{Q}$. If $\mathbf{R}$ and $\mathbf{U}$ are, respectively, the orthogonal and right symmetric factors in the polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U}$, and if we take $\mathbf{Q} = \mathbf{R}^T$, then we can rewrite this as

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \psi^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}).$$

Therefore, the constitutive function for the specific free energy will yield the same value for every deformation gradient $\mathbf{F}$ as it would give for its associated $\mathbf{U}$. We now can conclude that the constitutive function for free energy cannot depend on the orthogonal part $\mathbf{R}$ of the deformation gradient. Since both the specific free energy and specific entropy have the same values after rigid body rotation, we can rewrite their constitutive functions in terms of functions of $\mathbf{U}$, as opposed to $\mathbf{F}$. That is,

$$\boxed{\psi = \bar{\psi}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}),} \quad (10.28)$$

$$\boxed{\eta = \bar{\eta}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}).} \quad (10.29)$$

Since $\mathbf{U}$ is unaffected by rigid body rotations, these new forms for the constitutive functions automatically satisfy our assumptions, behaving as expected under rigid body rotations.

Now let us consider the implications of the restrictions on the constitutive function for Cauchy stress. The assumption $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ imposes a constraint on the constitutive function for Cauchy stress. This can be written as

$$\mathbf{T}^\dagger(\mathbf{X}, \mathbf{Q}\mathbf{F}, \theta, \mathbf{G}) = \mathbf{Q}\mathbf{T}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G})\mathbf{Q}^T, \quad (10.30)$$

which must hold for all orthogonal tensors $\mathbf{Q}$. As above, taking $\mathbf{Q} = \mathbf{R}^T$ and rearranging the expressions give

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathbf{R}\mathbf{T}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G})\mathbf{R}^T.$$

Defining the rotated stress $\mathbf{T}^R = \mathbf{R}^T \mathbf{T} \mathbf{R}$, it follows from the above relation that

$$\mathbf{T}^R = \mathbf{T}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}), \quad (10.31)$$

where $\mathbf{T}^{R\dagger}$ is the function used to calculate the rotated stress. This $\mathbf{T}^{R\dagger}$ is actually the same function as the one used to calculate the Cauchy stress, but for inputting $\mathbf{U}$ in place of $\mathbf{F}$. Therefore, the rotated stress does not depend on the rigid body rotation. The Cauchy stress is obtained from the rotated stress by the relation $\mathbf{T} = \mathbf{R} \mathbf{T}^R \mathbf{R}^T$. It also can be shown that the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{T}}$, that is given by the relation

$$\mathbf{T} = \frac{1}{J} \mathbf{F} \tilde{\mathbf{T}} \mathbf{F}^T,$$

is given by a constitutive function of the form

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \quad (10.32)$$

Finally, let us look at the simplification imposed on the constitutive equation for the heat flux. The condition $\mathbf{q}^* = \mathbf{Q} \mathbf{q}$ requires that

$$\mathbf{q}^\dagger(\mathbf{X}, \mathbf{Q} \mathbf{F}, \theta, \mathbf{G}) = \mathbf{Q} \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \quad (10.33)$$

which must hold for all orthogonal $\mathbf{Q}$. Again, selecting $\mathbf{Q}$ equal to $\mathbf{R}^T$ yields

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathbf{R} \mathbf{q}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \quad (10.34)$$

Therefore, we can introduce the rotated heat flux $\mathbf{q}^R = \mathbf{R}^T \mathbf{q}$ that has a constitutive representation given by

$$\mathbf{q}^R = \mathbf{q}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \quad (10.35)$$

The heat flux then can be calculated by first calculating $\mathbf{q}^R$ using $\mathbf{U}$, and then using $\mathbf{q} = \mathbf{R} \mathbf{q}^R$.

As the reader recalls, the right Cauchy stretch is given by $\mathbf{C} = \mathbf{U}^2$ and can uniquely be inverted, because $\mathbf{U}$ is positive definite, to get $\mathbf{U} = \sqrt{\mathbf{C}}$. Similarly, the Green strain is $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ so that $\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$ and $\mathbf{U} = \sqrt{\mathbf{I} + 2\mathbf{E}}$, again uniquely defined. As a result, we may replace for $\mathbf{U}$ in any of the above constitutive functions and obtain new versions of these functions that depend on $\mathbf{C}$ or $\mathbf{E}$. We then conclude that we always can write

$$\psi = \bar{\psi}^{\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G}), \quad (10.36)$$

$$\eta = \bar{\eta}^{\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G}), \quad (10.37)$$

$$\mathbf{T}^R = \mathbf{T}^{R\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G}), \quad (10.38)$$

$$\mathbf{q}^R = \mathbf{q}^{R\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G}), \quad (10.39)$$

where $\dagger\dagger$ indicates that the functions are in terms of the new argument sets. Similarly, we can write these as

$$\psi = \bar{\psi}^{\dagger\dagger\dagger}(\mathbf{X}, \mathbf{E}, \theta, \mathbf{G}), \quad (10.40)$$

$$\eta = \bar{\eta}^{\dagger\dagger\dagger}(\mathbf{X}, \mathbf{E}, \theta, \mathbf{G}), \quad (10.41)$$

$$\mathbf{T}^R = \mathbf{T}^{R\dagger\dagger\dagger}(\mathbf{X}, \mathbf{E}, \theta, \mathbf{G}), \quad (10.42)$$

$$\mathbf{q}^R = \mathbf{q}^{R\dagger\dagger\dagger}(\mathbf{X}, \mathbf{E}, \theta, \mathbf{G}). \quad (10.43)$$

Also, the same is true for the expression for the second Piola-Kirchhoff stress. That is, we can write it as

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}^{\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G}) = \tilde{\mathbf{T}}^{\dagger\dagger\dagger}(\mathbf{X}, \mathbf{E}, \theta, \mathbf{G}). \quad (10.44)$$

Section 10.3-Problems:

- 10.3-1:** Show that if a given traction vector transforms by a pure rigid body rotation to $\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}$ then one can conclude that the associated Cauchy stress transforms to $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$.
- 10.3-2:** Show that the second Piola-Kirchhoff stress $\tilde{\mathbf{T}}$ does not depend on the orthogonal part $\mathbf{R}$ of the polar decomposition of the deformation gradient.

10.4 Material symmetry

Symmetry in the material's response can limit substantially the possible forms of the constitutive equations and, therefore, simplify the process of characterizing the material response. The basic idea behind material symmetry is as follows. Let us assume that we have the capability to remove material at any given material point, reorganize it, and then return it back to the neighborhood of the point. Even though this is not possible in practice without damaging the structure of the material, theoretically one can assume it to be possible. For example, as shown in Figure 9.4, one might imagine removing a segment of the material, rotating it, and then inserting it back. It is possible that some of these reorganizations will be such that after the reorganization the material responds exactly the same as it did before (i.e., there are no distinguishing changes in the characteristics of the material response as a result of the reorganization). The constitutive models, being mathematical relations describing the material response, also should mimic this behavior and, therefore, should be indifferent to such reorganizations of the material. Since all reorganizations are mathematical mappings from one configuration to a new configuration, part of the characteristics of the material's response can be a set of mappings of the neighborhood of each given point into configurations that for all practical purposes are indistinguishable from the original neighborhood in the reference configuration. The set of all such mappings describes the symmetry of the material.

In general, the symmetry at a material point is described by the set of mappings of its neighborhood that leave the response of the material the same. Let us first look a little closer at the mapping. A mapping is a mathematical expression that takes one set of points and reorganizes them into a new configuration. Examples of such mappings are rotations and reflections, even though more complex mappings also can exist. In Figure 10.3, we have the initial configuration κ_o and a configuration $\bar{\kappa}_o$ constructed from κ_o by the removal of the neighborhood of point P and its reorganization according to the transformation described by the deformation gradient $\mathbf{M}$. If $\mathbf{M}$ represents a transformation of the neighborhood of P that leaves the response of κ_o and $\bar{\kappa}_o$ indistinguishable to all possible loadings, then $\mathbf{M}$ represents a symmetry of the material. For a given material, the set of all such deformation gradients is known as the *material symmetry group* of the material and will be designated by $\mathcal{G}_{\kappa_o}$.

How do we know that two configurations are indistinguishable and, therefore, represent a material symmetry at a particular point? Either we know of the existence of the symmetry

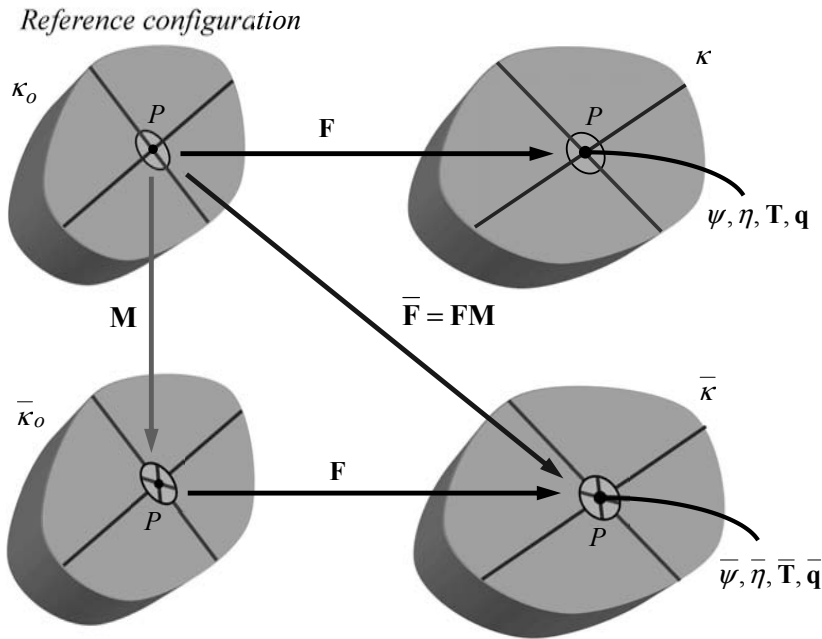


Figure 10.3: The reorganization described by the deformation gradient $\mathbf{M}$ takes the configuration κ_o to $\bar{\kappa}_o$. If we impose the same deformation gradient $\mathbf{F}$ on $\bar{\kappa}_o$ as was imposed on κ_o , the deformation gradient $\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$ maps the neighborhood of P in κ_o to the neighborhood of P in $\bar{\kappa}$.

(for example, from knowledge of the microstructure), or we have to conduct tests to establish it. Let us say that we have configurations κ_o and $\bar{\kappa}_o$ as shown in Figure 10.3 and we want to determine by experimentation whether they are indistinguishable from each other and, therefore, establish whether the mapping $\mathbf{M}$ represents a material symmetry at point P . To do this, we need to subject both configurations to all possible loadings and compare the responses at the point. In our case, the response is characterized by the values of the specific free energy, specific entropy, Cauchy stress, and the heat flux vector. Each loading is characterized by the deformation gradient, temperature, and temperature gradient. Let $\mathcal{L}$ denote the loading. It is clear that for an elastic material $\mathcal{L} = \mathcal{U}$. For a general first-gradient material, the loading is given by the entire history of $\mathcal{U}$. To check to see whether $\mathbf{M}$ represents a symmetry, we need to subject κ_o and $\bar{\kappa}_o$ to all possible $\mathcal{L}$ and verify that the response is identical. As can be seen in Figure 10.3, imposing the deformation described by the deformation gradient $\mathbf{F}$ on $\bar{\kappa}_o$ is equivalent to imposing the deformation described by the deformation gradient $\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$ on κ_o . A similar relation holds for temperature gradient, in that a temperature field described by $\mathbf{G}$ on $\bar{\kappa}_o$ is equivalent to imposing a temperature field described by the temperature gradient $\bar{\mathbf{G}} = \mathbf{G}\mathbf{M}$ on κ_o . In this sense, imposing the loading $\mathcal{L} = \{\mathbf{F}, \theta, \mathbf{G}\}$ on $\bar{\kappa}_o$ is equivalent to imposing the loading $\bar{\mathcal{L}} = \{\mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M}\}$ directly on κ_o . Let the response resulting from imposing $\mathcal{L}$ on κ_o be $\mathcal{R} = \{\psi, \eta, \mathbf{T}, \mathbf{q}\}$ and the response resulting from imposing $\bar{\mathcal{L}}$ on κ_o be $\bar{\mathcal{R}} = \{\bar{\psi}, \bar{\eta}, \bar{\mathbf{T}}, \bar{\mathbf{q}}\}$, then if $\mathbf{M}$ describes a material symmetry at P , we will have $\mathcal{R} = \bar{\mathcal{R}}$ for all $\mathcal{L}$. Conversely, if for all possible pairs of loading $\mathcal{L}$ and $\bar{\mathcal{L}}$ we find that $\mathcal{R} = \bar{\mathcal{R}}$, then $\mathbf{M}$ must represent a symmetry of the material.

Table 10.1: Transformations used to create common symmetry groups.

Symmetry	Transformations constructing the symmetry groups
Isotropic	All orthogonal transformations ($\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M} = \mathbf{I}$)
Triclinic	$\pm\mathbf{I}$
Monoclinic	$\pm\mathbf{I}, \pm\mathbf{R}_1$
Orthotropic	$\pm\mathbf{I}, \pm\mathbf{R}_1, \pm\mathbf{R}_2, \pm\mathbf{R}_3$
Transversely isotropic	$\pm\mathbf{I}, \pm\mathbf{R}_1, \pm\mathbf{M}_\omega$ for all ω

The next task is to force our constitutive equations to recognize symmetries in the material. That is, for every transformation $\mathbf{M}$ that represents a symmetry of the material, we have to force the constitutive equations to be such that $\boxed{\psi = \bar{\psi}}$, $\boxed{\eta = \bar{\eta}}$, $\boxed{\mathbf{T} = \bar{\mathbf{T}}}$, and $\boxed{\mathbf{q} = \bar{\mathbf{q}}}$ for all possible pairs of $\mathcal{L}$ and $\bar{\mathcal{L}}$. In terms of our response functions $\mathcal{R}^\dagger$, this can be written as

$$\boxed{\mathcal{R}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathcal{R}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M})} \quad (10.45)$$

for every $\mathbf{M}$ in the symmetry group of the material. The reader might note that we added a subscript κ_o to $\mathcal{R}_{\kappa_o}^\dagger$ to remind us that the response functions are written with respect to a specific configuration (in this case κ_o), and that the symmetries also are written in terms of this same configuration. This dependence will be explored later in the chapter. The explicit forms of these equations are therefore written as the following four conditions:

$$\begin{aligned} \boxed{\psi_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) &= \psi_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M}),} \\ \boxed{\eta_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) &= \eta_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M}),} \\ \boxed{\mathbf{T}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) &= \mathbf{T}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M}),} \\ \boxed{\mathbf{q}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) &= \mathbf{q}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}\mathbf{M}, \theta, \mathbf{G}\mathbf{M}),} \end{aligned} \quad (10.46)$$

which must hold for all $\mathcal{L}$. These four constraint conditions, known as the material symmetry constraints, must be imposed on the constitutive models for each symmetry of the material. In some cases, it is possible to have symmetry in one response function, for example, in the Cauchy stress, which does not exist in another response function, for example, heat flux vector. In such cases, one could separate out the material symmetry and talk about the symmetry for each response individually. As will be shown later, for a thermodynamically consistent model, we will get stress and entropy related to derivatives of the free energy so it will follow that for thermodynamically based models the symmetries of the free energy will dictate the symmetries of the stress and of entropy.

Typically, a set of transformations $\mathbf{M}$ defines a material's symmetry. This set forms a mathematical group and is known as the material symmetry group. As indicated, this group contains a number of transformations of the reference configuration κ_o , each mapping the neighborhood under consideration to one indistinguishable in its response from the original neighborhood. As noted above, the material symmetry group is denoted by $\mathcal{G}_{\kappa_o}$, each member of which is a transformation $\mathbf{M}$. As is the characteristic of a mathematical group, the product of any two members of the group is also within the group, so that if $\mathbf{M}_1$ and $\mathbf{M}_2$ are two members of $\mathcal{G}_{\kappa_o}$ then $\mathbf{M}_3 = \mathbf{M}_1\mathbf{M}_2$ is also in $\mathcal{G}_{\kappa_o}$. It is typical that one describes the material symmetry group by its characteristic members. These are a set of members from which one can generate the entire group by evaluation of all the possible products of the members in the characteristic set.

Table 10.1 describes some of the transformations used to construct the common mate-

rial symmetry groups. The transformations noted in this table are given in terms of the orthonormal base $\mathbf{e}_i$ as

$$\begin{aligned}\mathbf{I} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_1 &= -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_2 &= \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_3 &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{M}_\omega &= \cos \omega (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin \omega (\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3.\end{aligned}\tag{10.47}$$

As can be seen, $\mathbf{I}$ is the identity, $\mathbf{R}_1$ is a reflection about the $\mathbf{e}_2 - \mathbf{e}_3$ plane, $\mathbf{R}_2$ is a reflection about the $\mathbf{e}_3 - \mathbf{e}_1$ plane, $\mathbf{R}_3$ is a reflection about the $\mathbf{e}_1 - \mathbf{e}_2$ plane, and $\mathbf{M}_\omega$ is a rotation of ω around the $\mathbf{e}_3$ axis. If $\tau = \mathbf{e}_3$ defines the preferred axis of a transversely isotropic material, then transverse isotropy can be represented by five groups of tensors given by

$$T_1 : \mathbf{R}_\tau, \quad T_2 : \mathbf{R}_\tau, \mathbf{R}_2, \quad T_3 : \mathbf{R}_\tau, \mathbf{R}_1, \quad T_4 : \mathbf{R}_\tau, \mathbf{R}_2, \mathbf{R}_1, \quad T_5 : \mathbf{R}_\tau, -\mathbf{R}_3, \tag{10.48}$$

where $\mathbf{R}_\tau$ represents all rotations about τ , and, as described above, $\mathbf{R}_i$ represent the reflection about the plane perpendicular to $\mathbf{e}_i$. The most common representation of transverse isotropy is T_2 , which is in the appendix to construct functional representations. There are three representations for orthotropy given by

$$O_1 : \mathbf{R}_2, \mathbf{R}_3, \quad O_2 : -\mathbf{R}_2, -\mathbf{R}_3, \quad O_3 : \mathbf{R}_2, \mathbf{R}_3, -\mathbf{I}. \tag{10.49}$$

The most commonly used representation for orthotropy is O_3 .

As will be seen in the next section, the representation of symmetry is directly related to the selection of the reference configuration. Therefore, even though the symmetry of a material may be described with respect to one reference configuration in one of the forms given in Table 10.1, the same material symmetry's description with respect to another reference configuration might look very different. More on this can be found in the following section. Here, we look at developing general representations for the response functions under the assumption of isotropy, transverse isotropy, and orthotropy.

In the following representation, we combine the material symmetry condition and the assumed dependence on superimposed rigid body motions. As such, we will use the representations that use the right Cauchy stretch tensor $\mathbf{C}$ in place of $\mathbf{U}$. For these, we must note that for *orthogonal* $\mathbf{M}$ we can show that

$$\boxed{\bar{\mathbf{T}}^R = \mathbf{M}^T \mathbf{T}^R \mathbf{M}}, \quad \boxed{\bar{\mathbf{q}}^R = \mathbf{M}^T \mathbf{q}^R}, \tag{10.50}$$

and $\boxed{\bar{\mathbf{C}} = \mathbf{M}^T \mathbf{C} \mathbf{M}}$. The constraint of material symmetry now can be written for our alternate constitutive functions as

$$\begin{aligned}\boxed{\psi_{\kappa_o}^{\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G})} &= \boxed{\psi_{\kappa_o}^{\dagger\dagger}(\mathbf{X}, \mathbf{M}^T \mathbf{C} \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G})}, \\ \boxed{\eta_{\kappa_o}^{\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G})} &= \boxed{\eta_{\kappa_o}^{\dagger\dagger}(\mathbf{X}, \mathbf{M}^T \mathbf{C} \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G})}, \\ \boxed{\mathbf{T}_{\kappa_o}^{R\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G})} &= \boxed{\mathbf{M} \mathbf{T}_{\kappa_o}^{R\dagger\dagger}(\mathbf{X}, \mathbf{M}^T \mathbf{C} \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G}) \mathbf{M}^T}, \\ \boxed{\mathbf{q}_{\kappa_o}^{R\dagger\dagger}(\mathbf{X}, \mathbf{C}, \theta, \mathbf{G})} &= \boxed{\mathbf{M} \mathbf{q}_{\kappa_o}^{R\dagger\dagger}(\mathbf{X}, \mathbf{M}^T \mathbf{C} \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G})},\end{aligned}\tag{10.51}$$

where we have used the identity $\boxed{\mathbf{G} \mathbf{M} = \mathbf{M}^T \mathbf{G}}$, which follows from the fact that a vector is identical to its transpose. The mathematical subject of functional representations directly

addresses how to construct general representations for functions with the type of restrictions described in (10.51). In general, the representations turn out to be of the form

$$\psi = \psi^\dagger(\mathbf{X}, \theta, I_1, I_2, \dots, I_n), \quad (10.52)$$

$$\eta = \eta^\dagger(\mathbf{X}, \theta, I_1, I_2, \dots, I_n), \quad (10.53)$$

$$\mathbf{T}^R = \sum_{i=1}^m \mu_i(\mathbf{X}, \theta, I_1, I_2, \dots, I_n) \mathbf{H}_i, \quad (10.54)$$

$$\mathbf{q}^R = \sum_{i=1}^p k_i(\mathbf{X}, \theta, I_1, I_2, \dots, I_n) \mathbf{f}_i, \quad (10.55)$$

where for the particular symmetry $I_1, I_2, \dots, I_n$ are n scalar invariants, $\mathbf{f}_i$ are p vector invariants, and $\mathbf{H}_i$ are m symmetric tensor invariants. These functions are tabulated in Appendix A. The reader should note that $\mathbf{H}_i$ is not the displacement gradient. The associated expressions for the Cauchy stress and the heat flux vector are given by

$$\mathbf{T} = \sum_{i=1}^m \mu_i(\mathbf{X}, \theta, I_1, I_2, \dots, I_n) \mathbf{R} \mathbf{H}_i \mathbf{R}^T, \quad (10.56)$$

$$\mathbf{q} = \sum_{i=1}^p k_i(\mathbf{X}, \theta, I_1, I_2, \dots, I_n) \mathbf{R} \mathbf{f}_i. \quad (10.57)$$

10.4.1 Isotropic elastic materials

For an isotropic material, we can take a reference configuration in which the material symmetry group is the full group of orthogonal transformations. For such a case, from the appendix we note that isotropic scalar invariants of a symmetric tensor $\mathbf{C}$ and a vector $\mathbf{G}$ are given by

$$\begin{array}{lll} I_1 = \text{tr}(\mathbf{C}), & I_2 = \text{tr}(\mathbf{C}^2), & I_3 = \text{tr}(\mathbf{C}^3), \\ I_4 = \mathbf{G} \circ \mathbf{G}, & I_5 = \mathbf{G} \circ (\mathbf{C}\mathbf{G}), & I_6 = \mathbf{G} \circ (\mathbf{C}^2\mathbf{G}). \end{array} \quad (10.58)$$

The isotropic vector invariant functions of one symmetric tensor $\mathbf{C}$ and one vector $\mathbf{G}$ are given by

$$\mathbf{f}_1 = \mathbf{G}, \quad \mathbf{f}_2 = \mathbf{C}\mathbf{G}, \quad \mathbf{f}_3 = \mathbf{C}^2\mathbf{G}. \quad (10.59)$$

Since we later will conclude that, because of the entropy production inequality, the Cauchy stress cannot depend on the temperature gradient, we will give only the isotropic symmetric tensor functions of a single symmetric tensor $\mathbf{C}$. These are given by

$$\mathbf{H}_1 = \mathbf{I}, \quad \mathbf{H}_2 = \mathbf{C}, \quad \mathbf{H}_3 = \mathbf{C}^2. \quad (10.60)$$

It will be shown in a future section that specific free energy, specific entropy, and Cauchy stress do not depend on the temperature gradient. As a result, we omit the terms with $\mathbf{G}$ for these representations. Therefore, the general representations for isotropy can be written

as

$$\boxed{\psi = \psi^\dagger(\mathbf{X}, \theta, I_1, I_2, I_3),} \quad (10.61)$$

$$\boxed{\eta = \eta^\dagger(\mathbf{X}, \theta, I_1, I_2, I_3),} \quad (10.62)$$

$$\mathbf{T}^R = \mu_1(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{I} + \mu_2(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{C} + \mu_3(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{C}^2, \quad (10.63)$$

$$\boxed{\mathbf{T} = \mu_1(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{I} + \mu_2(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{B} + \mu_3(\mathbf{X}, \theta, I_1, I_2, I_3)\mathbf{B}^2,} \quad (10.64)$$

$$\mathbf{q}^R = k_1(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{G} + k_2(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{C}\mathbf{G} + k_3(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{C}^2\mathbf{G}, \quad (10.65)$$

$$\mathbf{q} = k_1(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{R}\mathbf{G} + k_2(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{B}\mathbf{R}\mathbf{G} + k_3(\mathbf{X}, \theta, I_1, \dots, I_6)\mathbf{B}^2\mathbf{R}\mathbf{G}. \quad (10.66)$$

Even though the equations are mathematically correct representations, the expressions for the heat fluxes are not exactly what one normally would use. For example, in either of these representations, Fourier's law of heat conduction (i.e., $\mathbf{q} = -k\mathbf{g}$) is not a trivially obtained special case! This points out a real issue in representation. That is, even though the representations are general and constructed from irreducible functional bases, they still might not be in the most convenient form for fitting a material's response.

For the case of the heat flux vector, let us look at a slightly different development of the general representation. This development will result in expressions that automatically provide Fourier's law of heat conduction as a special case. Since the temperature gradients $\mathbf{g}$ and $\mathbf{G}$ are related through the expression $\mathbf{g}\mathbf{F} = \mathbf{G}$, we can replace the variable set $\mathcal{U} = \{\mathbf{F}, \theta, \mathbf{G}\}$ with the variable set $\{\mathbf{F}, \theta, \mathbf{g}\}$ without any loss in generality. Therefore, let us assume that the heat flux vector is given by the constitutive expression

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{g}). \quad (10.67)$$

Since superimposing a rigid body rotation represented by $\mathbf{Q}$ on the motion would make the variable set change to $\{\mathbf{Q}\mathbf{F}, \theta, \mathbf{Q}\mathbf{g}\}$, imposing the expected response due to rigid body rotations requires that

$$\mathbf{Q}\mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{g}) = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{Q}\mathbf{F}, \theta, \mathbf{Q}\mathbf{g}). \quad (10.68)$$

As before, selecting $\mathbf{Q} = \mathbf{R}^T$ results in

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{g}) = \mathbf{R}\mathbf{q}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{R}^T\mathbf{g}). \quad (10.69)$$

Defining, as before, $\mathbf{q}^R = \mathbf{R}^T\mathbf{q}$, and also defining the rotated temperature gradient as $\mathbf{g}^R = \mathbf{R}^T\mathbf{g}$, we can conclude that

$$\mathbf{q}^R = \mathbf{q}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{g}^R). \quad (10.70)$$

That is, the rotated heat flux vector is a function of the rotated temperature gradient. Noting that $\bar{\mathbf{g}}^R = \mathbf{M}^T\mathbf{g}^R$, the requirement of material symmetry now can be written as

$$\mathbf{q}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{g}^R) = \mathbf{M}\mathbf{q}^{R\dagger}(\mathbf{X}, \mathbf{M}^T\mathbf{U}\mathbf{M}, \theta, \mathbf{M}^T\mathbf{g}^R). \quad (10.71)$$

Therefore, for this case we have the scalar invariants

$$\boxed{\begin{array}{lll} I_1 = \text{tr}(\mathbf{U}), & I_2 = \text{tr}(\mathbf{U}^2), & I_3 = \text{tr}(\mathbf{U}^3), \\ I_4 = \mathbf{g}^R \circ \mathbf{g}^R, & I_5 = \mathbf{g}^R \circ (\mathbf{U}\mathbf{g}^R), & I_6 = \mathbf{g}^R \circ (\mathbf{U}^2\mathbf{g}^R). \end{array}} \quad (10.72)$$

The isotropic vector invariant functions of one symmetric tensor $\mathbf{U}$ and one vector $\mathbf{g}^R$ are given by

$$\boxed{\mathbf{f}_1 = \mathbf{g}^R, \quad \mathbf{f}_2 = \mathbf{U}\mathbf{g}^R, \quad \mathbf{f}_3 = \mathbf{U}^2\mathbf{g}^R.} \quad (10.73)$$

Therefore, we can represent $\mathbf{q}^R$ as

$$\mathbf{q}^R = \tilde{k}_1(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{g}^R + \tilde{k}_2(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{U} \mathbf{g}^R + \tilde{k}_3(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{U}^2 \mathbf{g}^R, \quad (10.74)$$

where $\tilde{k}_i$ are a new set of scalar functions. We now can rewrite this in terms of the heat flux vector $\mathbf{q} = \mathbf{R} \mathbf{q}^R$ and the temperature gradient $\mathbf{g} = \mathbf{R} \mathbf{g}^R$ to obtain

$$\mathbf{q} = \tilde{k}_1(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{g} + \tilde{k}_2(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{V} \mathbf{g} + \tilde{k}_3(\mathbf{X}, \theta, I_1, \dots, I_6) \mathbf{B} \mathbf{g}. \quad (10.75)$$

The first term with a constant coefficient $\tilde{k}_1$ is Fourier's law. This shows that we can modify Fourier's law in two ways. One way is by letting $\tilde{k}_1$ vary with the invariants, yet retaining the fact that the direction of the heat flux is the same as the temperature gradient (i.e., setting the last two terms equal to zero). The second way is to let the direction of the heat flux move away from the direction of the temperature gradient (i.e., heat moving partially along the directions $\mathbf{V} \mathbf{g}$ and $\mathbf{B} \mathbf{g}$). Finally, in a following section, we will introduce the second law of thermodynamics, and it will impose additional restrictions on the form of the heat flux that will not be satisfied by every representation. For example, if we take only the first term in this representation, the thermodynamic constraint will require that the scalar function $\tilde{k}_1$ be negative.

Section 10.4-Problems:

10.4-1: For an isotropic material, we can consider material parameters that are polynomial functions of the scalar invariants and from these select the terms that are at most linear, as a result getting fairly simple models. For example, if we take polynomials for μ_i to construct a mechanical theory and select stress terms that are at most linear in $\mathbf{C}$, we get

$$\mathbf{T}^R = [\mu_{11} + \mu_{12} \text{tr}(\mathbf{C})] \mathbf{I} + \mu_2 \mathbf{C}, \quad (10.76)$$

where all the coefficients are constants. Show that the condition that the rotated stress is zero for $\mathbf{C} = \mathbf{I}$ requires that

$$\mu_{11} + 3\mu_{12} + \mu_2 = 0. \quad (10.77)$$

From this conclude that a model made in this way will be given by

$$\mathbf{T} = 2\mu_{12} \text{tr}(\mathbf{E}) \mathbf{I} + \mu_2 (\mathbf{B} - \mathbf{I}). \quad (10.78)$$

Knowing that pure thermal expansion is given by $\mathbf{F} = \lambda_\theta \mathbf{I}$, where λ_θ is a given function of temperature, reconsider this model with temperature included in the coefficients and show that

$$\mathbf{T} = \mu_{12} [\text{tr}(\mathbf{C}) - 3\lambda_\theta^2] \mathbf{I} + \mu_2 (\mathbf{B} - \lambda_\theta^2 \mathbf{I}). \quad (10.79)$$

Using this idea, start from the heat flux representation

$$\mathbf{q}^R = k_1 \mathbf{G} + k_2 \mathbf{C} \mathbf{G} + k_3 \mathbf{C}^2 \mathbf{G}, \quad (10.80)$$

and construct a model that is at most bilinear in $\mathbf{C}$ and $\mathbf{G}$. On this model, impose the condition that heat flux is zero at zero temperature gradient and show that the model becomes

$$\mathbf{q}^R = [k_{11} + k_{12} \text{tr}(\mathbf{C})] \mathbf{G} + k_2 \mathbf{C} \mathbf{G}, \quad (10.81)$$

where the coefficients are constants, or depend on temperature. Show that the heat flow before deformation is given by

$$\mathbf{q}^R = k_o \mathbf{G}, \quad (10.82)$$

where $k_o = k_{11} + 3k_{12} + k_2$ is the coefficient of thermal conduction without strain. Rewrite the heat flux equation as

$$\mathbf{q}^R = [k_o + 2k_{12}tr(\mathbf{E})]\mathbf{G} + 2k_2\mathbf{E}\mathbf{G}. \quad (10.83)$$

10.4.2 Transversely isotropic elastic materials

For a transversely isotropic material, we can take a reference configuration in which the material symmetry group is given by rotations about the axis of transverse isotropy $\mathbf{m}$ in the reference configuration and also includes one reflection, essentially making all $\mathbf{M}$ orthogonal transformations. To obtain representations more commonly seen, we take the representations that use $\mathbf{C}$ and $\mathbf{g}^R$, as was described in the development on isotropic elastic materials. From the appendix, we note that the transversely isotropic scalar invariants of a symmetric tensor $\mathbf{C}$ and a vector $\mathbf{g}^R$ are given by

$I_1 = tr(\mathbf{C}),$	$I_2 = tr(\mathbf{C}^2),$	$I_3 = tr(\mathbf{C}^3),$	(10.84)
$I_4 = \mathbf{m} \circ (\mathbf{C}\mathbf{m}),$	$I_5 = \mathbf{m} \circ (\mathbf{C}^2\mathbf{m}),$		
$I_6 = \mathbf{g}^R \circ \mathbf{g}^R,$	$I_7 = \mathbf{g}^R \circ \mathbf{m},$	$I_8 = \mathbf{g}^R \circ (\mathbf{C}\mathbf{g}^R),$	
$I_9 = \mathbf{m} \circ (\mathbf{C}\mathbf{g}^R),$	$I_{10} = \mathbf{m} \circ (\mathbf{C}^2\mathbf{g}^R).$		

The transversely isotropic vector invariant functions of one symmetric tensor $\mathbf{C}$ and one vector $\mathbf{g}^R$ are given by

$\mathbf{f}_1 = \mathbf{g}^R,$	$\mathbf{f}_2 = \mathbf{m},$	$\mathbf{f}_3 = \mathbf{C}\mathbf{g}^R,$	(10.85)
$\mathbf{f}_4 = \mathbf{C}\mathbf{m},$	$\mathbf{f}_5 = \mathbf{C}^2\mathbf{m}.$		

Since we later will conclude that, because of the entropy production inequality, the Cauchy stress cannot depend on the temperature gradient, we will give only the transversely isotropic symmetric tensor functions of a single symmetric tensor $\mathbf{C}$. These are given by

$\mathbf{H}_1 = \mathbf{I},$	$\mathbf{H}_2 = \mathbf{C},$	$\mathbf{H}_3 = \mathbf{C}^2,$	(10.86)
$\mathbf{H}_4 = \mathbf{m} \otimes \mathbf{m},$			
$\mathbf{H}_5 = \mathbf{m} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{m},$			
$\mathbf{H}_6 = \mathbf{m} \otimes (\mathbf{C}^2\mathbf{m}) + (\mathbf{C}^2\mathbf{m}) \otimes \mathbf{m}.$			

Following the same discussion as given for an isotropic material, we now can provide general representations for the response functions as

$\psi = \psi^\dagger(\mathbf{X}, \theta, I_1, \dots, I_{10}),$	(10.87)

$\eta = \eta^\dagger(\mathbf{X}, \theta, I_1, \dots, I_{10}),$	(10.88)

$$\mathbf{T}^R = \mu_1 \mathbf{I} + \mu_2 \mathbf{C} + \mu_3 \mathbf{C}^2 + \mu_4 \mathbf{m} \otimes \mathbf{m} + \mu_5 [\mathbf{m} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{m}] + \mu_6 [\mathbf{m} \otimes (\mathbf{C}^2\mathbf{m}) + (\mathbf{C}^2\mathbf{m}) \otimes \mathbf{m}], \quad (10.89)$$

$$\mathbf{q}^R = k_1 \mathbf{g}^R + k_2 \mathbf{m} + k_3 \mathbf{C}\mathbf{g}^R + k_4 \mathbf{C}\mathbf{m} + k_5 \mathbf{C}^2\mathbf{m}, \quad (10.90)$$

where μ_i are scalar functions of $(\mathbf{X}, \theta, I_1, \dots, I_{10})$ and k_i are scalar functions of $(\mathbf{X}, \theta, I_1, \dots, I_{10})$. This yields the expressions

$$\begin{aligned} \mathbf{T} &= \mu_1 \mathbf{I} + \mu_2 \mathbf{B} + \mu_3 \mathbf{B}^2 + \mu_4 \check{\mathbf{m}} \otimes \check{\mathbf{m}} + \mu_5 [\check{\mathbf{m}} \otimes (\mathbf{B}\check{\mathbf{m}}) + (\mathbf{B}\check{\mathbf{m}}) \otimes \check{\mathbf{m}}] \\ &\quad + \mu_6 [\check{\mathbf{m}} \otimes (\mathbf{B}^2 \check{\mathbf{m}}) + (\mathbf{B}^2 \check{\mathbf{m}}) \otimes \check{\mathbf{m}}], \\ \mathbf{q} &= k_1 \mathbf{g} + k_2 \check{\mathbf{m}} + k_3 \mathbf{B}\mathbf{g} + k_4 \mathbf{B}\check{\mathbf{m}} + k_5 \mathbf{B}^2 \check{\mathbf{m}}, \end{aligned} \quad (10.91)$$

where $\check{\mathbf{m}} = \mathbf{R}\mathbf{m}$ represents the rigid body rotation of the axis of transverse isotropy $\mathbf{m}$ by the amount $\mathbf{R}$, for $\mathbf{R}$ being the orthogonal factor in the polar decomposition of $\mathbf{F}$.

Section 10.4-Problems:

- 10.4-2:** For a transversely isotropic material, we can consider material parameters μ_i that are polynomial functions of the scalar invariants and from these select the terms that are at most linear in $\mathbf{C}$ to get a fairly simple model. For a mechanical theory this will result in

$$\begin{aligned} \mathbf{T}^R &= [\mu_{11} + \mu_{12} \text{tr}(\mathbf{C}) + \mu_{13} \mathbf{m} \circ (\mathbf{C}\mathbf{m})] \mathbf{I} + \mu_2 \mathbf{C} \\ &\quad + [\mu_{41} + \mu_{42} \text{tr}(\mathbf{C}) + \mu_{43} \mathbf{m} \circ (\mathbf{C}\mathbf{m})] \mathbf{m} \otimes \mathbf{m} \\ &\quad + \mu_5 [\mathbf{m} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{m}], \end{aligned} \quad (10.92)$$

where $\mathbf{m}$ is assumed to be a unit vector, and all the coefficients are constants. Show that the condition that stress is zero for $\mathbf{C} = \mathbf{I}$ requires that

$$\mu_{11} + 3\mu_{12} + \mu_{13} + \mu_2 = 0, \quad (10.93)$$

$$\mu_{41} + 3\mu_{42} + \mu_{43} + 2\mu_5 = 0. \quad (10.94)$$

From this conclude that the rotated stress is given by

$$\begin{aligned} \mathbf{T}^R &= [2\mu_{12} \text{tr}(\mathbf{E}) + 2\mu_{13} \mathbf{m} \circ (\mathbf{E}\mathbf{m})] \mathbf{I} + 2\mu_2 \mathbf{E} \\ &\quad + [2\mu_{42} \text{tr}(\mathbf{E}) + 2\mu_{43} \mathbf{m} \circ (\mathbf{E}\mathbf{m})] \mathbf{m} \otimes \mathbf{m} \\ &\quad + 2\mu_5 [\mathbf{m} \otimes (\mathbf{E}\mathbf{m}) + (\mathbf{E}\mathbf{m}) \otimes \mathbf{m}]. \end{aligned} \quad (10.95)$$

Using the expressions for left Green strain $\mathbf{E}_L$ and $\check{\mathbf{m}} = \mathbf{R}\mathbf{m}$, calculate the expression for Cauchy stress.

- 10.4-3:** Following the idea given in the last problem, start with the model for the rotated heat flux

$$\mathbf{q}^R = k_1 \mathbf{g}^R + k_2 \mathbf{m} + k_3 \mathbf{C}\mathbf{g}^R + k_4 \mathbf{C}\mathbf{m} + k_5 \mathbf{C}^2 \mathbf{m}, \quad (10.96)$$

and expand k_i in terms of polynomials and select the terms that result in at most bilinear terms in $\mathbf{C}$ and $\mathbf{g}^R$ to obtain

$$\begin{aligned} \mathbf{q}^R &= [k_{11} + k_{12} \text{tr}(\mathbf{C}) + k_{13} \mathbf{m} \circ (\mathbf{C}\mathbf{m})] \mathbf{g}^R \\ &\quad + [k_{21} + k_{22} \text{tr}(\mathbf{C}) + k_{23} \mathbf{m} \circ (\mathbf{C}\mathbf{m}) + k_{24} \mathbf{g}^R \circ \mathbf{m} + k_{25} \mathbf{m} \circ (\mathbf{C}\mathbf{g}^R)] \mathbf{m} \\ &\quad + k_3 \mathbf{C}\mathbf{g}^R + [k_{41} + k_{42} \mathbf{g}^R \circ \mathbf{m}] \mathbf{C}\mathbf{m}. \end{aligned} \quad (10.97)$$

Assume the heat flux is zero for a zero temperature gradient and conclude that this requires that

$$\begin{aligned} \mathbf{q}^R &= [k_{11} + k_{12} \text{tr}(\mathbf{C}) + k_{13} \mathbf{m} \circ (\mathbf{C}\mathbf{m})] \mathbf{g}^R \\ &\quad + [k_{24} \mathbf{g}^R \circ \mathbf{m} + k_{25} \mathbf{m} \circ (\mathbf{C}\mathbf{g}^R)] \mathbf{m} + k_3 \mathbf{C}\mathbf{g}^R + k_{42} (\mathbf{g}^R \circ \mathbf{m}) \mathbf{C}\mathbf{m}. \end{aligned} \quad (10.98)$$

10.4.3 Orthotropic elastic materials

To represent orthotropy by a group of orthogonal tensors, we select a reference configuration in which the three preferred directions of orthotropy are orthogonal and given by the three vectors $\mathbf{e}_i$. In this case, we take

$$\mathbf{m} = \mathbf{e}_1, \quad \mathbf{Y} = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (10.99)$$

As in the case for transverse isotropy, we build the representations based on $\mathbf{C}$ and $\mathbf{g}^R$. From the appendix, we note that the orthotropic scalar invariants for a symmetric tensor $\mathbf{C}$ and a vector $\mathbf{g}^R$ are given by

$$\begin{array}{lll} I_1 = \text{tr}(\mathbf{C}), & I_2 = \text{tr}(\mathbf{C}^2), & I_3 = \text{tr}(\mathbf{C}^3), \\ I_4 = \text{tr}(\mathbf{Y}\mathbf{C}), & I_5 = \text{tr}(\mathbf{Y}\mathbf{C}^2), & I_6 = \mathbf{m} \circ (\mathbf{C}\mathbf{m}), \\ I_7 = \mathbf{m} \circ (\mathbf{C}^2\mathbf{m}), & & \\ I_8 = \mathbf{g}^R \circ \mathbf{g}^R, & I_9 = \mathbf{g}^R \circ (\mathbf{Y}\mathbf{g}^R), & I_{10} = (\mathbf{g}^R \circ \mathbf{m})^2, \\ I_{11} = \mathbf{g}^R \circ (\mathbf{C}\mathbf{g}^R), & I_{12} = \mathbf{g}^R \circ (\mathbf{C}^2\mathbf{g}^R), & I_{13} = \mathbf{g}^R \circ (\mathbf{Y}\mathbf{C}\mathbf{g}^R). \end{array} \quad (10.100)$$

The orthotropic vector invariant functions of one symmetric tensor $\mathbf{C}$ and one vector $\mathbf{g}^R$ are given by

$$\begin{array}{lll} \mathbf{f}_1 = \mathbf{g}^R, & \mathbf{f}_2 = \mathbf{Y}\mathbf{g}^R, & \mathbf{f}_3 = (\mathbf{g}^R \circ \mathbf{m})\mathbf{m}, \\ \mathbf{f}_4 = \mathbf{C}\mathbf{g}^R, & \mathbf{f}_5 = \mathbf{C}^2\mathbf{g}^R, & \mathbf{f}_6 = (\mathbf{Y}\mathbf{C} - \mathbf{C}\mathbf{Y})\mathbf{g}^R. \end{array} \quad (10.101)$$

Since we later will conclude that, because of the entropy production inequality, the Cauchy stress cannot depend on the temperature gradient, we will give only the orthotropic symmetric tensor functions of a single symmetric tensor $\mathbf{C}$. These are given by

$$\begin{array}{lll} \mathbf{H}_1 = \mathbf{I}, & \mathbf{H}_2 = \mathbf{C}, & \mathbf{H}_3 = \mathbf{C}^2, \\ \mathbf{H}_4 = \mathbf{Y}, & \mathbf{H}_5 = \mathbf{m} \otimes \mathbf{m}, & \\ \mathbf{H}_6 = \mathbf{Y}\mathbf{C} + \mathbf{C}\mathbf{Y}, & & \mathbf{H}_7 = \mathbf{m} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{m}. \end{array} \quad (10.102)$$

Following the same discussion as given for an isotropic material, we now can provide general representations for the response functions as

$$\psi = \psi^\dagger(\mathbf{X}, \theta, I_1, \dots, I_{13}), \quad (10.103)$$

$$\eta = \eta^\dagger(\mathbf{X}, \theta, I_1, \dots, I_{13}), \quad (10.104)$$

$$\begin{aligned} \mathbf{T}^R = & \mu_1 \mathbf{I} + \mu_2 \mathbf{C} + \mu_3 \mathbf{C}^2 + \mu_4 \mathbf{Y} + \mu_5 \mathbf{m} \otimes \mathbf{m} \\ & + \mu_6 (\mathbf{Y}\mathbf{C} + \mathbf{C}\mathbf{Y}) + \mu_7 [\mathbf{m} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{m}], \end{aligned} \quad (10.105)$$

$$\mathbf{q}^R = k_1 \mathbf{g}^R + k_2 \mathbf{Y}\mathbf{g}^R + k_3 (\mathbf{g}^R \circ \mathbf{m})\mathbf{m} + k_4 \mathbf{C}\mathbf{g}^R + k_5 \mathbf{C}^2\mathbf{g}^R + k_6 (\mathbf{Y}\mathbf{C} - \mathbf{C}\mathbf{Y})\mathbf{g}^R, \quad (10.106)$$

where μ_i are scalar functions of $(\mathbf{X}, \theta, I_1, \dots, I_{13})$ and k_i are scalar functions of $(\mathbf{X}, \theta, I_1, \dots, I_{13})$. This yields the expressions

$$\begin{aligned} \mathbf{T} = & \mu_1 \mathbf{I} + \mu_2 \mathbf{B} + \mu_3 \mathbf{B}^2 + \mu_4 \check{\mathbf{Y}} + \mu_5 \check{\mathbf{m}} \otimes \check{\mathbf{m}} \\ & + \mu_6 (\check{\mathbf{Y}}\mathbf{B} + \mathbf{B}\check{\mathbf{Y}}) + \mu_7 [\check{\mathbf{m}} \otimes (\mathbf{B}\check{\mathbf{m}}) + (\mathbf{B}\check{\mathbf{m}}) \otimes \check{\mathbf{m}}], \\ \mathbf{q} = & k_1 \mathbf{g} + k_2 \check{\mathbf{Y}}\mathbf{g} + k_3 (\mathbf{g} \circ \check{\mathbf{m}})\check{\mathbf{m}} + k_4 \mathbf{B}\mathbf{g} + k_5 \mathbf{B}^2\mathbf{g} + k_6 (\check{\mathbf{Y}}\mathbf{B} - \mathbf{B}\check{\mathbf{Y}})\mathbf{g}, \end{aligned} \quad (10.107)$$

where $\check{\mathbf{m}} = \mathbf{R}\mathbf{m}$ and $\check{\mathbf{Y}} = \mathbf{R}\mathbf{Y}\mathbf{R}^T$, each representing the rigid body rotation of the associated material tensor by the orthogonal part of $\mathbf{F}$.

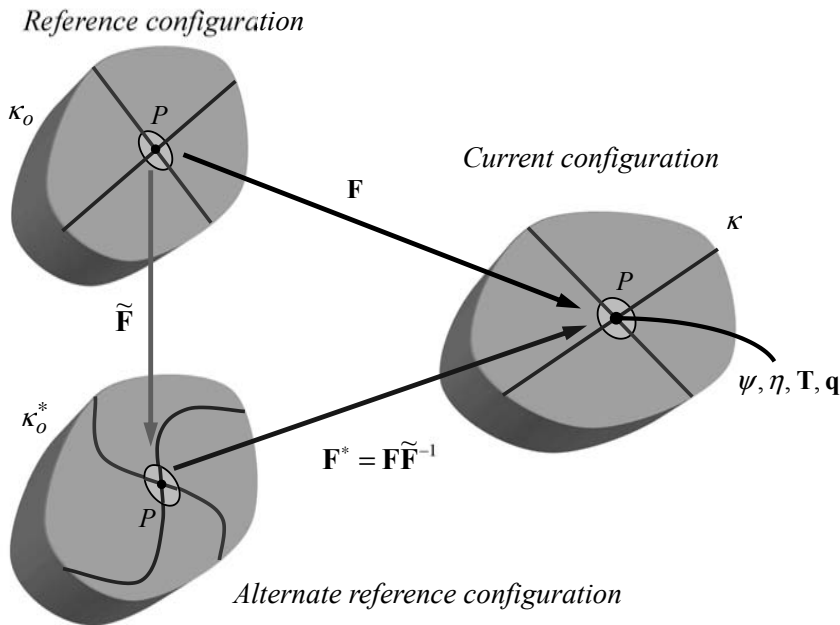


Figure 10.4: The relation between deformation gradient $\mathbf{F}$ with respect to the reference configuration κ_o and the deformation gradient $\mathbf{F}^*$ with respect to the reference configuration κ_o^* is given by $\mathbf{F}^* = \tilde{\mathbf{F}}\mathbf{F}^{-1}$, where $\tilde{\mathbf{F}}$ is the deformation gradient comparing the configuration κ_o^* to κ_o .

Section 10.4-Problems:

- 10.4-4:** Show that for an orthogonal tensor $\mathbf{M}$, if we replace $\mathcal{L}$ by $\bar{\mathcal{L}}$, as described above, then $\bar{\mathbf{T}}^R = \mathbf{M}^T \mathbf{T}^R \mathbf{M}$, $\bar{\mathbf{q}}^R = \mathbf{M}^T \mathbf{q}^R$, and $\bar{\mathbf{g}}^R = \mathbf{M}^T \mathbf{g}^R$.

10.5 Change of reference configuration

Frequently, the first configuration in which one encounters a material body is used as the reference configuration, even though this need not always be the case. Sometimes it becomes necessary to compare information referenced to two different reference configurations. For example, consider the case where the symmetry of a material is characterized relative to its as-received initial configuration, subsequent to which the material is formed into a part and its new material symmetries are evaluated relative to the configuration of the completed part. If the change in symmetry resulting from the processing of the material into a part is to be evaluated, then the two symmetries need to be referenced to the same configuration, so that one can recognize which symmetries represent the same reorganizations.

Consider the two reference configurations κ_o and κ_o^* and the current configuration κ shown in Figure 10.4. The deformation gradient $\mathbf{F}$ maps the reference configuration κ_o

into the current configuration κ , the deformation gradient $\mathbf{F}^*$ maps the alternate reference configuration κ_o^* into the same current configuration κ , and the deformation gradient $\tilde{\mathbf{F}}$ maps the reference configuration κ_o into an alternate reference configuration κ_o^* . It is easy to see that

$$\boxed{\mathbf{F} = \mathbf{F}^* \tilde{\mathbf{F}}.} \quad (10.108)$$

One also can show that the temperature gradients $\mathbf{G}$ and $\mathbf{G}^*$ are related by a similar relation given by

$$\boxed{\mathbf{G} = \mathbf{G}^* \tilde{\mathbf{F}},} \quad (10.109)$$

where $\mathbf{G}$ is the temperature gradient with respect to changes of position in κ_o , and $\mathbf{G}^*$ is the temperature gradient with respect to changes of position in κ_o^* . Changing the reference configuration does not affect the description of temperature since it does not depend on the relative location of different material points. Therefore,

$$\boxed{\theta = \theta^*.} \quad (10.110)$$

When a constitutive equation is written, it is written with respect to a particular reference configuration. Yet, the response of the material is independent of the reference configuration we select in the development of the constitutive models. Therefore, the relation between constitutive models developed based on two different reference configurations must be such that the two models predict the exact same response. We can write this as

$$\mathcal{R} = \mathcal{R}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathcal{R}_{\kappa_o^*}^\dagger(\mathbf{X}^*, \mathbf{F}^*, \theta^*, \mathbf{G}^*), \quad (10.111)$$

where $\mathbf{X}^*$ is the position in κ_o^* of the same material point which is at location $\mathbf{X}$ in κ_o . The reader will note that each function takes variables written in regard to its own reference configuration, here denoted by κ_o and κ_o^* . Also note that a subscript has been added to the functions for $\mathcal{R}$ to distinguish between the representations with respect to different configurations. From this relation, it is easy to see that $\mathcal{R}_{\kappa_o^*}^\dagger$ can be written in terms of $\mathcal{R}_{\kappa_o}^\dagger$ by the following substitution:

$$\mathcal{R}_{\kappa_o^*}^\dagger(\mathbf{X}^*, \mathbf{F}^*, \theta^*, \mathbf{G}^*) = \mathcal{R}_{\kappa_o}^\dagger(\mathbf{X}, \mathbf{F}^* \tilde{\mathbf{F}}, \theta^*, \mathbf{G}^* \tilde{\mathbf{F}}). \quad (10.112)$$

To obtain these results, we simply have replaced in the constitutive equations written with respect to κ_o the values $\mathbf{F} = \mathbf{F}^* \tilde{\mathbf{F}}$, $\theta = \theta^*$, and $\mathbf{G} = \mathbf{G}^* \tilde{\mathbf{F}}$. The idea is as simple as it looks, and will be illustrated in the following example. Consider a constitutive equation for an isotropic material given by

$$\mathbf{T} = \kappa(J - 1)\mathbf{I} + G(\mathbf{B} - J^{2/3}\mathbf{I}).$$

Let us assume that this equation is written with respect to the reference configuration κ_o (i.e., it represents $\mathbf{T}_{\kappa_o}^\dagger$), and that we want to know its representation with respect to κ_o^* (i.e., $\mathbf{T}_{\kappa_o^*}^\dagger$). Since $\mathbf{F} = \mathbf{F}^* \tilde{\mathbf{F}}$, we have $J = \det(\mathbf{F}) = \det(\mathbf{F}^* \tilde{\mathbf{F}}) = \det(\mathbf{F}^*) \det(\tilde{\mathbf{F}}) = J^* \tilde{J}$, where $J^* = \det(\mathbf{F}^*)$, and we have $\mathbf{B} = \mathbf{F} \mathbf{F}^T = \mathbf{F}^* \tilde{\mathbf{F}} \tilde{\mathbf{F}}^T \mathbf{F}^{*T}$. Therefore, the constitutive equation for Cauchy stress written with respect to the reference configuration κ_o^* (i.e., $\mathbf{T}_{\kappa_o^*}^\dagger$) is given as

$$\mathbf{T} = \kappa(\tilde{J} J^* - 1)\mathbf{I} + G[\mathbf{F}^* \tilde{\mathbf{B}} \mathbf{F}^{*T} - (\tilde{J} J^*)^{2/3} \mathbf{I}].$$

The next relation that we would like to establish is between the transformations that describe the same material symmetry, but with respect to two different reference configurations. Let the deformation gradient $\mathbf{M}$ be a transformation that takes the neighborhood of a material point in κ_o to a materially equivalent neighborhood. Let the deformation gradient $\mathbf{M}^*$ represent the same reorganization as $\mathbf{M}$, but let it be the transformation that maps

the neighborhood of the point in κ_o^* into the materially equivalent neighborhood. We want to find the relation between deformation gradients $\mathbf{M}$ and $\mathbf{M}^*$, which represent the same material symmetry from the perspective of different reference configurations. As was shown in the section on material symmetry, in the constitutive equations written with respect to the reference configuration κ_o , we can replace $\mathbf{F}$ by $\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$ and $\mathbf{G}$ by $\bar{\mathbf{G}} = \mathbf{G}\mathbf{M}$. If $\mathbf{M}$ represents a material symmetry, then the response functions should be indifferent to this change of arguments. In a similar way, if in the constitutive equations written with respect to the reference configuration κ_o^* we replace $\mathbf{F}^*$ with $\bar{\mathbf{F}}^* = \mathbf{F}^*\mathbf{M}^*$ and $\mathbf{G}^*$ with $\bar{\mathbf{G}}^* = \mathbf{G}^*\mathbf{M}^*$ and if $\mathbf{M}^*$ represents a material symmetry, then these response functions should be indifferent to this change of arguments. Since by the transformation rule we have $\mathbf{F}^* = \mathbf{F}\tilde{\mathbf{F}}^{-1}$ and $\bar{\mathbf{F}}^* = \bar{\mathbf{F}}\tilde{\mathbf{F}}^{-1}$, we then can write $\bar{\mathbf{F}}^* = \mathbf{F}^*\mathbf{M}^*$ as $\bar{\mathbf{F}}\tilde{\mathbf{F}}^{-1} = \mathbf{F}\tilde{\mathbf{F}}^{-1}\mathbf{M}^*$. Reorganizing this yields

$$\bar{\mathbf{F}} = \mathbf{F}\tilde{\mathbf{F}}^{-1}\mathbf{M}^*\tilde{\mathbf{F}}.$$

If we compare this relation to $\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$, and recall that this relation must hold for all $\mathbf{F}$, we arrive at the conclusion that $\mathbf{M} = \tilde{\mathbf{F}}^{-1}\mathbf{M}^*\tilde{\mathbf{F}}$ or

$$\boxed{\mathbf{M}^* = \tilde{\mathbf{F}}\mathbf{M}\tilde{\mathbf{F}}^{-1}.} \quad (10.113)$$

This relation between the description of the same material symmetry with respect to two different configurations related by the deformation gradient $\tilde{\mathbf{F}}$ as shown in Figure 10.4 is known as *Noll's rule*. Since each member of the material symmetry group, when represented in the new configuration, can be obtained by Noll's rule, we can write

$$\boxed{\mathcal{G}_{\kappa_o^*} = \tilde{\mathbf{F}}\mathcal{G}_{\kappa_o}\tilde{\mathbf{F}}^{-1}.} \quad (10.114)$$

Section 10.5-Problems:

- 10.5-1:** Calculate the elements that generate the material symmetry group $\mathcal{G}_{\kappa_o^*}$ given that the elements that construct the material symmetry group $\mathcal{G}_{\kappa_o}$ are one of the sets presented in Table 10.1. Do this for each symmetry represented in the table and for the reference configuration κ_o^* given by the uniaxial extension $\tilde{\mathbf{F}} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)$ with respect to κ_o .

10.6 A thermodynamically consistent model

It is very desirable to construct constitutive models that never violate the entropy production inequality. Otherwise, we would need to check continuously to make sure the predicted response did not violate this law. The entropy production inequality, in the form of the Clausius-Duhem inequality, was given in (6.69) as

$$\rho\dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0.$$

This inequality ensures that entropy at each material point increases at least as fast as the entropy being added to it through heat flow. Therefore, our task in this section will be to see what implications the Clausius-Duhem inequality has on the possible forms of our constitutive models. In particular, we shall show that specific free energy, specific entropy, and Cauchy stress become related, and the form of the heat flux vector is restricted.

The reader recalls that the free energy is given by a constitutive equation of the form

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}),$$

so that its material time derivative can be expressed by

$$\dot{\psi} = \partial_{\mathbf{F}}(\psi) : \dot{\mathbf{F}} + \partial_{\theta}(\psi)\dot{\theta} + \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}. \quad (10.115)$$

The reader will note that since $\mathbf{X}$ denotes the position in the reference configuration of the particle under consideration, and that the particle under consideration does not change with time in a material time derivative, $\mathbf{X}$ is treated as a constant when taking the time derivative for obtaining (10.115). Therefore, we have no contribution to $\dot{\psi}$ from $\mathbf{X}$. Using the relation $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\text{tr}(\mathbf{AB}) = \mathbf{A}^T : \mathbf{B}$, we can show that

$$\text{tr}(\mathbf{TL}) = \text{tr}(\mathbf{T}\dot{\mathbf{F}}\mathbf{F}^{-1}) = \text{tr}(\mathbf{F}^{-1}\mathbf{T}\dot{\mathbf{F}}) = (\mathbf{F}^{-1}\mathbf{T})^T : \dot{\mathbf{F}}.$$

Substituting for $\dot{\psi}$ and $\text{tr}(\mathbf{TL})$ in the Clausius-Duhem inequality and reorganizing terms result in the equation

$$[\rho\partial_{\mathbf{F}}(\psi) - (\mathbf{F}^{-1}\mathbf{T})^T] : \dot{\mathbf{F}} + \rho[\partial_{\theta}(\psi) + \eta]\dot{\theta} + \rho[\partial_{\mathbf{G}}(\psi)] \circ \dot{\mathbf{G}} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0. \quad (10.116)$$

This inequality should hold for all possible loading processes. Since the material is in no way constrained, at each loading point described by $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$, the loading rate described by $\dot{\mathcal{L}} = (\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}})$ can take any value independent of the current value of $\mathcal{L}$. Since the terms in the three square brackets and the last term are each only a function of $\mathcal{L}$, and since $\dot{\mathcal{L}}$ can take any arbitrary values, then the three terms in the square brackets must each be equal to zero and the last term must be less than or equal to zero. If any one of the three terms within the square brackets is not equal to zero, then one always can find values of the loading rate $\dot{\mathcal{L}}$ that will result in a violation of the inequality. Therefore, one must conclude that, if the constitutive models are not to violate the second law of thermodynamics, we must have the following four conditions satisfied

$$\mathbf{T}^T = \rho\partial_{\mathbf{F}}(\psi)\mathbf{F}^T, \quad (10.117)$$

$$\eta = -\partial_{\theta}(\psi), \quad (10.118)$$

$$\partial_{\mathbf{G}}(\psi) = \mathbf{0}, \quad (10.119)$$

$$\frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0. \quad (10.120)$$

The first and second conditions relate the Cauchy stress and specific entropy to derivatives of free energy. The third of these conditions excludes the possibility of specific free energy, depending on the temperature gradient, and in view of the first two conditions we must conclude that Cauchy stress and specific entropy cannot depend on the temperature gradient either. The last condition requires that the heat flux vector $\mathbf{q}$ and temperature gradient $\mathbf{g}$ be more than 90° away from each other. One might interpret the last equation to say that none of the heat flux vector can be along the direction of increasing temperature.

Let us now proceed to study the behavior of the Cauchy stress now constructed from the free energy through expression (10.117). In particular let us show that (10.117) is consistent with the symmetry of the Cauchy stress imposed by the balance of angular momentum and is consistent with the assumptions made about how the Cauchy stress changes with the imposition of rigid body motions and how it changes with transformations associated with

material symmetry. First, let us consider imposing rigid body motions by considering the two piecewise continuous histories $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t) = \mathbf{Q}(t)\mathbf{F}_1(t)$ for $\mathbf{Q}(t)$ being a continuous history of orthogonal second-order tensors. Let $\psi_1(t)$ and $\psi_2(t)$ be the histories of the free energy associated, respectively, with $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ with all other parameters kept the same so that

$$\psi_1(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}_1(t), \theta(t)], \quad (10.121)$$

$$\psi_2(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}_2(t), \theta(t)]. \quad (10.122)$$

Since $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ differ only by rigid body motions, the assumption we have made that superimposing rigid body motions does not alter the free energy required that $\psi_1(t) = \psi_2(t)$ for all time. Therefore, we can conclude that $\dot{\psi}_1(t) = \dot{\psi}_2(t)$. Now that it is clear which variables depend on time, let us eliminate explicit mention of time in the notation. From equation (10.115), we can conclude that

$$\dot{\psi}_1 = [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_1} : \dot{\mathbf{F}}_1 + [\partial_\theta(\psi)]_{\mathbf{F}=\mathbf{F}_1} \dot{\theta}, \quad (10.123)$$

$$\begin{aligned} \dot{\psi}_2 &= [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_2} : \dot{\mathbf{F}}_2 + [\partial_\theta(\psi)]_{\mathbf{F}=\mathbf{F}_2} \dot{\theta} \\ &= [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_2} : (\dot{\mathbf{Q}}\mathbf{F}_1 + \mathbf{Q}\dot{\mathbf{F}}_1) + [\partial_\theta(\psi)]_{\mathbf{F}=\mathbf{F}_2} \dot{\theta}. \end{aligned} \quad (10.124)$$

Equating $\dot{\psi}_1$ and $\dot{\psi}_2$, and considering that this equality should hold for all histories of $\mathbf{F}_1$ and θ , and for all admissible $\mathbf{Q}$ and $\dot{\mathbf{Q}}$, then we must conclude that

$$\{[\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_2} \mathbf{F}_1^T\} : \dot{\mathbf{Q}} = 0, \quad (10.125)$$

$$[\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_1} = \mathbf{Q}^T [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_2}. \quad (10.126)$$

Using the latter result, for the thermodynamically consistent model, we know from (10.117) that

$$\mathbf{T}_1^T = \rho [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_1} \mathbf{F}_1^T, \quad (10.127)$$

$$\begin{aligned} \mathbf{T}_2^T &= \rho [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_2} \mathbf{F}_2^T \\ &= \rho \mathbf{Q} [\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_1} \mathbf{F}_1^T \mathbf{Q}^T \\ &= \mathbf{Q} \mathbf{T}_1^T \mathbf{Q}^T. \end{aligned} \quad (10.128)$$

Therefore, the thermodynamically constructed Cauchy stress changes in the same way with the superposition of rigid body motions as originally was postulated. Next, let us show that the thermodynamically constructed stress is symmetric and, therefore, consistent with the result of the balance of angular momentum. Since $\mathbf{Q}$ is orthogonal we know that $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$ so that we have $\dot{\mathbf{Q}}^T \mathbf{Q} + \mathbf{Q}^T \dot{\mathbf{Q}} = \mathbf{0}$. It therefore follows that $\dot{\mathbf{Q}}^T \mathbf{Q} = -\mathbf{Q}^T \dot{\mathbf{Q}}$, which shows that $\dot{\mathbf{Q}}^T \mathbf{Q}$ is skew symmetric. Substituting from (10.126) into (10.125) we obtain, after using the identity $\mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^T \mathbf{A}) : \mathbf{C}$, the relation

$$\{[\partial_{\mathbf{F}}(\psi)]_{\mathbf{F}=\mathbf{F}_1} \mathbf{F}_1^T\} : (\mathbf{Q}^T \dot{\mathbf{Q}}) = 0. \quad (10.129)$$

This can be written as

$$\mathbf{T}_1^T : (\mathbf{Q}^T \dot{\mathbf{Q}}) = 0, \quad (10.130)$$

which since $\mathbf{Q}^T \dot{\mathbf{Q}}$ is skew symmetric requires that $\mathbf{T}_1$ be symmetric. This proves that the Cauchy stress constructed from equation (10.117) will be symmetric once we impose the condition that free energy does not change with the superposition of rigid body motions.

So far we have shown that the thermodynamically consistent Cauchy stress given in (10.117) is symmetric and behaves as expected when imposing rigid body motions on the deformation if we postulate that the free energy is invariant to the superposition of these

rigid body motions. We also can look at the material symmetry restriction and show that if the free energy is unchanged by transformations of the history associated with material symmetry, then the Cauchy stress obtained from equation (10.117) also is unchanged by such changes. The proof of this is left as a homework problem.

Section 10.6-Problems:

10.6-1: Consider two deformation histories given by $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t) = \mathbf{F}_1(t)\mathbf{M}$ where $\mathbf{M}$ is a transformation that is constant in time and that represents a symmetry of the material. Show that the Cauchy stress obtained using (10.117) is the same for these two deformation histories if the free energy is the same for these two deformation histories (i.e., if $\psi_1(t) = \psi_2(t)$ for all t , then show that $\mathbf{T}_1(t) = \mathbf{T}_2(t)$). From this conclude that once we impose the postulated condition on free energy, then the thermodynamically constructed Cauchy stress also behaves as originally was postulated in (10.51).

10.6.1 Isotropic materials

For the thermodynamically consistent elastic material model, the specific entropy and Cauchy stress become related to derivatives of the specific free energy, so we need only provide representations for the specific free energy and heat flux vector. We recall that the general representation for the specific free energy for an isotropic material is given by

$$\psi = \psi^\dagger(\mathbf{X}, \theta, I_1, I_2, I_3), \quad (10.131)$$

where

$$I_1 = \text{tr}(\mathbf{C}), \quad I_2 = \text{tr}(\mathbf{C}^2), \quad I_3 = \text{tr}(\mathbf{C}^3). \quad (10.132)$$

The general expressions for Cauchy stress and specific entropy, in view of this representation of specific free energy, are given by

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T = \rho \sum_{i=1}^3 \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}}(I_i) \mathbf{F}^T, \quad (10.133)$$

$$\eta = -\frac{\partial \psi}{\partial \theta}. \quad (10.134)$$

The expression for Cauchy stress further can be simplified by calculating the expressions for $\partial_{\mathbf{F}}(I_i)$. For I_1 we have $I_1 = \text{tr}(\mathbf{F}^T \mathbf{F})$ so that

$$\dot{I}_1 = \overline{\text{tr}(\dot{\mathbf{F}}^T \mathbf{F})} = \text{tr}(\dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}}) = \text{tr}(\dot{\mathbf{F}}^T \mathbf{F}) + \text{tr}(\mathbf{F}^T \dot{\mathbf{F}}) = 2\mathbf{F} : \dot{\mathbf{F}}. \quad (10.135)$$

In a similar manner, using $I_2 = \text{tr}(\mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F})$ and $I_3 = \text{tr}(\mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F})$, it can be shown that

$$\dot{I}_2 = 4(\mathbf{F} \mathbf{F}^T \mathbf{F}) : \dot{\mathbf{F}} = 4(\mathbf{F} \mathbf{C}) : \dot{\mathbf{F}}, \quad (10.136)$$

$$\dot{I}_3 = 6(\mathbf{F} \mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F}) : \dot{\mathbf{F}} = 6(\mathbf{F} \mathbf{C}^2) : \dot{\mathbf{F}}. \quad (10.137)$$

Therefore, we have

$$\partial_{\mathbf{F}}(I_1) = 2\mathbf{F}, \quad \partial_{\mathbf{F}}(I_2) = 4\mathbf{F} \mathbf{C}, \quad \partial_{\mathbf{F}}(I_3) = 6\mathbf{F} \mathbf{C}^2. \quad (10.138)$$

It follows that the general representation for the Cauchy stress of an isotropic elastic material can be written as

$$\mathbf{T}^T = \rho \left(2 \frac{\partial \psi}{\partial I_1} \mathbf{B} + 4 \frac{\partial \psi}{\partial I_2} \mathbf{B}^2 + 6 \frac{\partial \psi}{\partial I_3} \mathbf{B}^3 \right). \quad (10.139)$$

As discussed earlier, the representations are not unique in form. We will look at two alternate representations that are as general, but expose the different possible forms, each potentially useful in fitting actual experimental data. First, we know that many materials have a vastly different response to volumetric strains as opposed to shear. It seems desirable to use an equivalent set of isotropic invariants that includes the volume ratio. We already have looked at such a set and denoted them by J_i defined by

$$J_1 = \text{tr}(\mathbf{C}), \quad J_2 = \frac{1}{2}[\text{tr}^2(\mathbf{C}) - \text{tr}(\mathbf{C}^2)], \quad J_3 = \det(\mathbf{F}) = J. \quad (10.140)$$

This invariant set is equivalent to I_i , so that we can write

$$\psi = \psi^{\dagger\dagger}(\mathbf{X}, \theta, J_1, J_2, J_3). \quad (10.141)$$

We also note that

$$\dot{J}_1 = 2\mathbf{F} : \dot{\mathbf{F}}, \quad \dot{J}_2 = 2J_1\mathbf{F} : \dot{\mathbf{F}} - 2(\mathbf{F}\mathbf{C}) : \dot{\mathbf{F}}, \quad \dot{J}_3 = \dot{J} = J\mathbf{F}^{-T} : \dot{\mathbf{F}}, \quad (10.142)$$

so that

$$\partial_{\mathbf{F}}(J_1) = 2\mathbf{F}, \quad \partial_{\mathbf{F}}(J_2) = 2J_1\mathbf{F} - 2\mathbf{F}\mathbf{C}, \quad \partial_{\mathbf{F}}(J_3) = J_3\mathbf{F}^{-T}. \quad (10.143)$$

The representation for Cauchy stress in this case can be written as

$$\mathbf{T}^T = \rho \left[\frac{\partial \psi}{\partial J} J\mathbf{I} + 2 \left(\frac{\partial \psi}{\partial J_1} + J_1 \frac{\partial \psi}{\partial J_2} \right) \mathbf{B} - 2 \frac{\partial \psi}{\partial J_2} \mathbf{B}^2 \right]. \quad (10.144)$$

Even though the invariants J_i explicitly introduce volume ratio, they are not decoupled. That is, changing volume ratio results in changes of both J_1 and J_2 . A possible alternate set of invariants for isotropy might be

$$K_1 = J, \quad K_2 = \frac{I_1}{J^{2/3}}, \quad K_3 = \frac{I_2}{I_1^2}. \quad (10.145)$$

The rates of these invariants are given by

$$\dot{K}_1 = J\mathbf{F}^{-T} : \dot{\mathbf{F}}, \quad \dot{K}_2 = \frac{2}{J^{2/3}} \left(\mathbf{F} - \frac{I_1}{3}\mathbf{F}^{-T} \right) : \dot{\mathbf{F}}, \quad \dot{K}_3 = \frac{4}{I_1^2} \left(\mathbf{F}\mathbf{C} - \frac{I_2}{I_1}\mathbf{F} \right) : \dot{\mathbf{F}}, \quad (10.146)$$

so that

$$\partial_{\mathbf{F}}(K_1) = J\mathbf{F}^{-T}, \quad \partial_{\mathbf{F}}(K_2) = \frac{2}{J^{2/3}} \left(\mathbf{B} - \frac{I_1}{3}\mathbf{I} \right) \mathbf{F}^{-T}, \quad \partial_{\mathbf{F}}(K_3) = \frac{4}{I_1^2} \left(\mathbf{B} - \frac{I_2}{I_1}\mathbf{I} \right) \mathbf{F}. \quad (10.147)$$

The expression for Cauchy stress in this case is given by

$$\mathbf{T}^T = \rho \left[\frac{\partial \psi}{\partial K_1} J\mathbf{I} + \frac{2}{J^{2/3}} \frac{\partial \psi}{\partial K_2} \left(\mathbf{B} - \frac{I_1}{3}\mathbf{I} \right) + \frac{4}{I_1^2} \frac{\partial \psi}{\partial K_3} \left(\mathbf{B}^2 - \frac{I_2}{I_1}\mathbf{B} \right) \right]. \quad (10.148)$$

A particularly attractive characteristic of this set of invariants is that for equal-triaxial extension, K_2 and K_3 are constant ($K_2 = 3$ and $K_3 = 1/3$). In addition, if one superimposes equal-triaxial extension on any deformation, during the superimposed equal-triaxial extension one gets $\dot{K}_2 = \dot{K}_3 = 0$. Another attractive characteristic of this invariant set is seen when taking the trace of the Cauchy stress. In this case one gets

$$T_{ave} = \frac{tr(\mathbf{T})}{3} = \rho_o \frac{\partial \psi}{\partial K_1}. \quad (10.149)$$

Therefore, the average normal stress is obtained from the derivative of the free energy with respect to K_1 , completely decoupling the average normal stress from the other two derivatives of the free energy.

Section 10.6-Problems:

10.6-2: Consider the inequality

$$f(x)y + g(x) \leq 0$$

for bounded functions $f(x)$ and $g(x)$ and variable y . Assuming that this inequality should be satisfied for all y , and that y is independent of x , show that this condition requires that

$$f(x) = 0, \quad g(x) \leq 0.$$

10.6-3: Show that superimposing equal-triaxial extension on any fixed deformation gradient $\mathbf{F}$ results in $\dot{K}_2 = \dot{K}_3 = 0$.

10.6-4: Show that for the deformation gradient $\mathbf{F}$, the right Cauchy stretch tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, and any *constant second-order tensors* $\mathbf{A}$ and $\mathbf{B}$ we have

$$f = tr(\mathbf{C}\mathbf{A}) \Rightarrow \dot{f} = [\mathbf{F}(\mathbf{A} + \mathbf{A}^T)] : \dot{\mathbf{F}} \Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{F}(\mathbf{A} + \mathbf{A}^T), \quad (10.150)$$

$$\begin{aligned} f = tr(\mathbf{C}^2 \mathbf{A}) &\Rightarrow \dot{f} = \{\mathbf{F}[\mathbf{C}(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)\mathbf{C}]\} : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{F}[\mathbf{C}(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)\mathbf{C}], \end{aligned} \quad (10.151)$$

$$\begin{aligned} f = tr(\mathbf{C}^2 \mathbf{A} \mathbf{C} \mathbf{B}) \\ &\Rightarrow \dot{f} = \{\mathbf{F}[\mathbf{C}(\mathbf{A} \mathbf{C} \mathbf{B} + \mathbf{B}^T \mathbf{C} \mathbf{A}^T) + (\mathbf{A} \mathbf{C} \mathbf{B} + \mathbf{B}^T \mathbf{C} \mathbf{A}^T)\mathbf{C} \\ &\quad + \mathbf{B} \mathbf{C}^2 \mathbf{A} + \mathbf{A}^T \mathbf{C}^2 \mathbf{B}^T]\} : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{F}[\mathbf{C}(\mathbf{A} \mathbf{C} \mathbf{B} + \mathbf{B}^T \mathbf{C} \mathbf{A}^T) + (\mathbf{A} \mathbf{C} \mathbf{B} + \mathbf{B}^T \mathbf{C} \mathbf{A}^T)\mathbf{C} \\ &\quad + \mathbf{B} \mathbf{C}^2 \mathbf{A} + \mathbf{A}^T \mathbf{C}^2 \mathbf{B}^T], \end{aligned} \quad (10.152)$$

$$f = tr[(\mathbf{F} + \mathbf{F}^T)\mathbf{A}] \Rightarrow \dot{f} = (\mathbf{A} + \mathbf{A}^T) : \dot{\mathbf{F}} \Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{A} + \mathbf{A}^T, \quad (10.153)$$

$$f = tr[(\mathbf{F} - \mathbf{F}^T)\mathbf{A}] \Rightarrow \dot{f} = (\mathbf{A}^T - \mathbf{A}) : \dot{\mathbf{F}} \Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{A}^T - \mathbf{A}, \quad (10.154)$$

$$\begin{aligned} f = tr[(\mathbf{F} + \mathbf{F}^T)^2 \mathbf{A}] &\Rightarrow \dot{f} = [(\mathbf{F} + \mathbf{F}^T)(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F} + \mathbf{F}^T)] : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = (\mathbf{F} + \mathbf{F}^T)(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F} + \mathbf{F}^T), \end{aligned} \quad (10.155)$$

$$\begin{aligned} f = tr[(\mathbf{F} - \mathbf{F}^T)^2 \mathbf{A}] &\Rightarrow \dot{f} = -[(\mathbf{F} - \mathbf{F}^T)(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F} - \mathbf{F}^T)] : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = -[(\mathbf{F} - \mathbf{F}^T)(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F} - \mathbf{F}^T)]. \end{aligned} \quad (10.156)$$

10.6-5: Calculate the expression for $\partial_{\mathbf{F}}(I)$ for *constant second-order tensors* $\mathbf{A}$ and $\mathbf{B}$ for each of the following scalar functions:

$$I = tr[(\mathbf{F} + \mathbf{F}^T)^2 \mathbf{A}(\mathbf{F} + \mathbf{F}^T)\mathbf{B}], \quad I = tr[(\mathbf{F} - \mathbf{F}^T)^2 \mathbf{A}(\mathbf{F} - \mathbf{F}^T)\mathbf{B}]. \quad (10.157)$$

10.6-6: Show that for the deformation gradient $\mathbf{F}$, the right Cauchy stretch tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, and any *constant vectors* $\mathbf{m}$ and $\mathbf{n}$ we have

$$\begin{aligned} f = \mathbf{m} \circ (\mathbf{C}\mathbf{n}) &= (\mathbf{m} \otimes \mathbf{n}) : \mathbf{C} \Rightarrow \dot{f} = [\mathbf{F}(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n})] : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{F}(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}), \end{aligned} \quad (10.158)$$

$$\begin{aligned} f = \mathbf{m} \circ (\mathbf{C}^2 \mathbf{n}) &= (\mathbf{m} \otimes \mathbf{n}) : \mathbf{C}^2 \\ &\Rightarrow \dot{f} = \{\mathbf{F}[(\mathbf{C}\mathbf{n}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{C}\mathbf{n}) + \mathbf{n} \otimes (\mathbf{C}\mathbf{m}) + (\mathbf{C}\mathbf{m}) \otimes \mathbf{n}]\} : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{F}[\mathbf{C}(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}) + (\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n})\mathbf{C}], \end{aligned} \quad (10.159)$$

$$\begin{aligned} f = \mathbf{m} \circ [(\mathbf{F} + \mathbf{F}^T)\mathbf{n}] &= (\mathbf{m} \otimes \mathbf{n}) : (\mathbf{F} + \mathbf{F}^T) \Rightarrow \dot{f} = (\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}) : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}, \end{aligned} \quad (10.160)$$

$$\begin{aligned} f = \mathbf{m} \circ [(\mathbf{F} - \mathbf{F}^T)\mathbf{n}] &= (\mathbf{m} \otimes \mathbf{n}) : (\mathbf{F} - \mathbf{F}^T) \Rightarrow \dot{f} = (\mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m}) : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = \mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m}, \end{aligned} \quad (10.161)$$

$$\begin{aligned} f = \mathbf{m} \circ [(\mathbf{F} + \mathbf{F}^T)^2 \mathbf{n}] &= (\mathbf{m} \otimes \mathbf{n}) : (\mathbf{F} + \mathbf{F}^T)^2 \\ &\Rightarrow \dot{f} = [(\mathbf{F} + \mathbf{F}^T)(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}) + (\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n})(\mathbf{F} + \mathbf{F}^T)] : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = (\mathbf{F} + \mathbf{F}^T)(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}) + (\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n})(\mathbf{F} + \mathbf{F}^T), \end{aligned} \quad (10.162)$$

$$\begin{aligned} f = \mathbf{m} \circ [(\mathbf{F} - \mathbf{F}^T)^2 \mathbf{n}] &= (\mathbf{m} \otimes \mathbf{n}) : (\mathbf{F} - \mathbf{F}^T)^2 \\ &\Rightarrow \dot{f} = -[(\mathbf{F} - \mathbf{F}^T)(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})(\mathbf{F} - \mathbf{F}^T)] : \dot{\mathbf{F}} \\ &\Rightarrow \partial_{\mathbf{F}}(f) = -[(\mathbf{F} - \mathbf{F}^T)(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})(\mathbf{F} - \mathbf{F}^T)]. \end{aligned} \quad (10.163)$$

10.6.2 Isotropic rubber-elasticity-based model

Rubber elasticity is based on statistical mechanics of disordered long-chain molecules. The theory looks at the probability of chains taking different configurations. The outcome of this is a representation for the response of the polymer system in the rubbery range (at temperatures between the glass-transition temperature and the melting temperature). The method is based on equilibrium, so it does not capture viscoelastic characteristics, but does provide a commonly used model for equilibrium elastic response and is motivated strongly by the underlying molecular structure. The most common form of this model uses Gaussian distribution functions and results in an isotropic model that has a shear stress response

$$\tau = G\gamma, \quad (10.164)$$

where τ is the shear stress, γ is the shear strain, and G is the shear modulus, which turns out to be linearly propositional to the temperature and inversely propositional to the chain length. For the Gaussian distribution, we have $G = \rho R\theta/M_c$, where ρ is the density, R is the gas constant, θ is temperature, and M_c is the average molecular weight between cross-linking points.¹ Typically, this model goes with the assumption of incompressibility of the material, arriving at a model of the form

$$\mathbf{T} = -p\mathbf{I} + G\mathbf{B}, \quad (10.165)$$

where p is the indeterminant pressure resulting from the incompressibility constraint (see the following section on material constraints).

¹See *Mechanical Properties of Solid Polymers* by I. M. Ward.

Let us construct a model like this but include compressibility. For this purpose, let us assume we have isotropic thermal expansion governed by

$$\frac{j^\theta}{J^\theta} = \beta_V \dot{\theta}, \quad (10.166)$$

where J^θ is the thermal volume ratio, assuming $J^\theta = 1$ at the reference temperature θ_o , and β_V is the volumetric coefficient of thermal expansion. Next, let us remove the thermal expansion from the deformation gradient to get the deformation gradient $\mathbf{F}^*$ given by

$$\mathbf{F}^* = \frac{1}{J^{\theta \frac{1}{3}}} \mathbf{F}. \quad (10.167)$$

Since the material is isotropic, we assume the free energy is a function of the three isotropic invariants of $\mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{*T}$ and of temperature. This will be written as

$$\psi = \psi^\dagger(K_1^*, K_2^*, K_3^*, \theta), \quad (10.168)$$

where

$$K_1^* = \det(\mathbf{F}^*) = \frac{K_1}{J^\theta} = J^*, \quad K_2^* = \frac{\text{tr}(\mathbf{B}^*)}{J^{*2/3}} = K_2, \quad K_3^* = \frac{\text{tr}(\mathbf{B}^{*2})}{\text{tr}^2(\mathbf{B}^*)} = K_3. \quad (10.169)$$

The rate of change of the first invariant is given by

$$\dot{K}_1^* = \frac{J}{J^\theta} \mathbf{F}^{-T} : \dot{\mathbf{F}} - \frac{J}{J^\theta} \beta_V \dot{\theta}, \quad (10.170)$$

so that

$$\partial_{\mathbf{F}}(K_1^*) = \frac{J}{J^\theta} \mathbf{F}^{-T}, \quad \partial_\theta(K_1^*) = -\beta_V \frac{J}{J^\theta}. \quad (10.171)$$

The expressions for Cauchy stress and entropy in this case are given by

$$\begin{aligned} \mathbf{T} &= \rho \left[\frac{\partial \psi}{\partial K_1^*} \frac{J}{J^\theta} \mathbf{I} + \frac{2}{J^{2/3}} \frac{\partial \psi}{\partial K_2^*} (\mathbf{B} - \frac{I_1}{3} \mathbf{I}) + \frac{4}{I_1^2} \frac{\partial \psi}{\partial K_3^*} (\mathbf{B}^2 - \frac{I_2}{I_1} \mathbf{B}) \right], \\ \eta &= -\frac{\partial \psi}{\partial \theta} + \beta_V \frac{J}{J^\theta} \frac{\partial \psi}{\partial K_1^*}. \end{aligned} \quad (10.172)$$

As a specific example to reproduce rubber-elasticity type response, we consider a simple model given by a free energy of the form

$$\begin{aligned} \psi &= \frac{\kappa_o J^\theta}{2m\rho_o} K_1^* \left(\frac{K_1^{*m}}{m+1} + \frac{K_1^{*-m}}{m-1} \right) + \frac{G}{2\rho_o} (K_2^* - 3) + \psi^\theta \\ &= \frac{\kappa_o J^\theta}{2m\rho_o} J^* \left(\frac{J^{*m}}{m+1} + \frac{J^{*-m}}{m-1} \right) + \frac{G}{2\rho_o} \left[\frac{\text{tr}(\mathbf{B}^*)}{J^{*2/3}} - 3 \right] + \psi^\theta, \end{aligned} \quad (10.173)$$

where the material parameters κ_o , G and ψ^θ are functions of temperature only, and m is a nonzero constant. The expressions for Cauchy stress and entropy in this case become

$$\mathbf{T} = \sigma_{ave} \mathbf{I} + \mathbf{S}, \quad (10.174)$$

$$\eta = -\frac{J^\theta}{2m\rho_o} \left(\frac{J^{*(m+1)}}{m+1} + \frac{J^{*(-m+1)}}{m-1} \right) \left(\frac{d\kappa_o}{d\theta} + \kappa_o \beta_V \right) + \frac{\sigma_{ave} \beta_V}{\rho} - \frac{1}{2\rho_o} \left[\frac{\text{tr}(\mathbf{B}^*)}{J^{*2/3}} - 3 \right] \frac{dG}{d\theta} - \frac{d\psi^\theta}{d\theta}, \quad (10.175)$$

where the average stress σ_{ave} and the deviatoric stress $\mathbf{S}$ are given by

$$\sigma_{ave} = \frac{\kappa_o}{2m} \left(J^{*m} - \frac{1}{J^{*m}} \right), \quad (10.176)$$

$$\mathbf{S} = \frac{G}{J^{5/3}} \left[\mathbf{B} - \frac{tr(\mathbf{B})}{3} \mathbf{I} \right]. \quad (10.177)$$

For this model, the deformation in simple shear in the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane of the orthonormal base $\hat{\mathbf{e}}_i$ is given by $\mathbf{F} = \mathbf{I} + \gamma \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2$ for a shear strain γ with $J = 1$ and $\mathbf{B} = \mathbf{I} + \gamma(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1) + \gamma^2 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1$. This results in the stress response

$$T_{11} = \frac{\kappa_o}{2m} \left(\frac{1}{J^{\theta m}} - J^{\theta m} \right) + \frac{2G}{3} \gamma^2, \quad (10.178)$$

$$T_{22} = T_{33} = \frac{\kappa_o}{2m} \left(\frac{1}{J^{\theta m}} - J^{\theta m} \right) - \frac{G}{3} \gamma^2, \quad (10.179)$$

$$T_{12} = G\gamma, \quad (10.180)$$

$$T_{13} = T_{23} = 0. \quad (10.181)$$

To match this with traditional rubber elasticity, we can take $G = \rho_o R\theta/M_c$. We note that the response for the shear stress in terms of the shear strain is linear, yet there are normal stresses developed as a result of shearing and these are quadratic functions of the shear strain. This suggests that one may ignore the normal stresses at small shear strains but should consider them once one reaches shears of about $\gamma = 0.5$ (about 26°). The response of this model in simple shear is shown in Figure 10.5.

The response in uniaxial extension can be written in terms of the axial stretch λ_a and the transverse stretch λ_t , with $J = \lambda_a \lambda_t^2$ and $tr(\mathbf{B}) = \lambda_a^2 + 2\lambda_t^2$. One obtains two equations, one for the axial stress σ and one for the zero transverse stress, given by

$$\sigma = \sigma_{ave} + \frac{G}{J^{5/3}} \left[\lambda_a^2 - \frac{tr(\mathbf{B})}{3} \right], \quad (10.182)$$

$$0 = \sigma_{ave} + \frac{G}{J^{5/3}} \left[\lambda_t^2 - \frac{tr(\mathbf{B})}{3} \right]. \quad (10.183)$$

The second equation can be used to solve for the transverse stretch in terms of the axial stretch. This is a nonlinear equation that can be solved numerically. Subtracting the two equations gives a simpler relation for the axial stress given by

$$\sigma = \frac{G}{J^{5/3}} (\lambda_a^2 - \lambda_t^2). \quad (10.184)$$

In the limit of incompressibility, the transverse stretch is given by the simpler relation $J = 1$ that yields $\lambda_t = \sqrt{1/\lambda_a}$. As a result, in the case of incompressibility one gets

$$\sigma = G \left(\lambda_a^2 - \frac{1}{\lambda_a} \right). \quad (10.185)$$

The response of this model under uniaxial extension is shown in Figures 10.5 and 10.6. As expected, as the bulk modulus increases, the volumetric strains become smaller and the response more closely resembles that of the incompressible model.

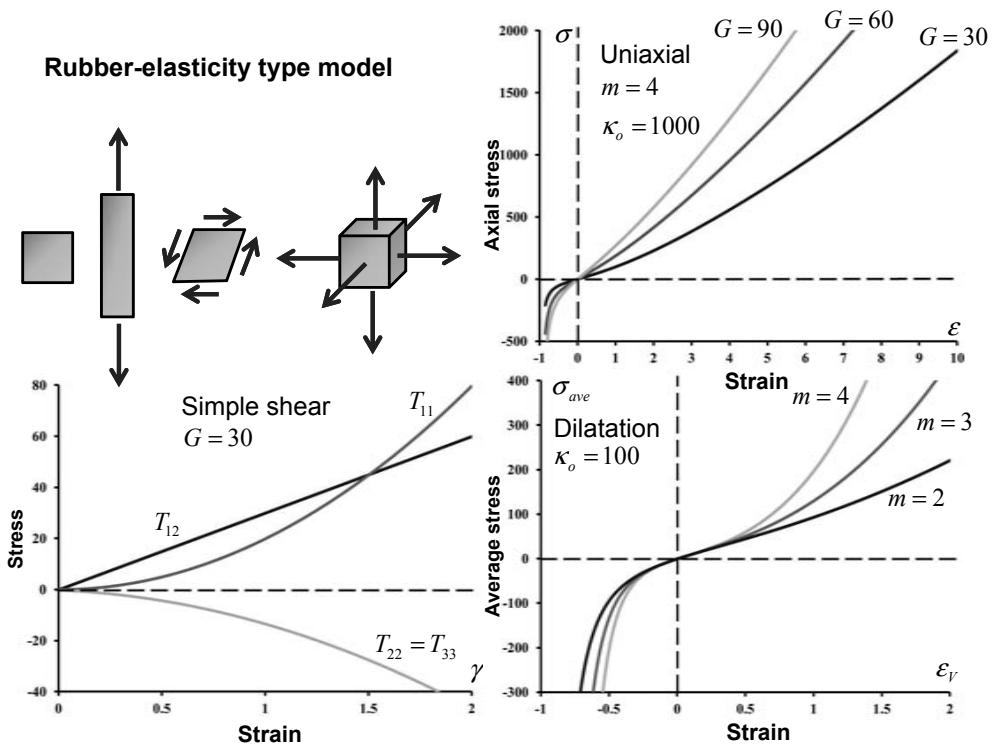


Figure 10.5: Rubber-elasticity type model: Response in extension (top right), shear (bottom left), and dilatation (bottom right).

Section 10.6-Problems:

- 10.6-7:** Simulate the response in uniaxial extension for the rubber-elasticity type model and compare your results to those shown in Figures 10.5 and 10.6. Assume isothermal response at the reference temperature and use the parameters given in the figures.
- 10.6-8:** Calculate the bulk modulus under volumetric deformations and plot your results for different m .
- 10.6-9:** Consider the response of the model in simple shear and uniaxial extension for the limit of incompressibility by taking κ_o very large. Show that even though the simple shear response is linear at all shear strains, uniaxial extension behaves as a quadratic function of the axial stretch for large stretches.

10.6.3 Isotropic Ogden type models

Another isotropic material model that has found use for polymers and, particularly, biological materials, is the Ogden model. This model, with some modification, has a free energy of the form

$$\psi = \sum_{i=1}^m \frac{\kappa_{oi} J^\theta}{2\rho_o \beta_i} J^* \left(\frac{J^{*\beta_i}}{\beta_i + 1} + \frac{J^{*-\beta_i}}{\beta_i - 1} \right) + \sum_{i=1}^n \frac{G_i}{2\rho_o \alpha_i} \left[\frac{\text{tr}(\mathbf{B}^{\alpha_i})}{J^{\frac{2}{3}\alpha_i}} - 3 \right] + \psi^\theta, \quad (10.186)$$

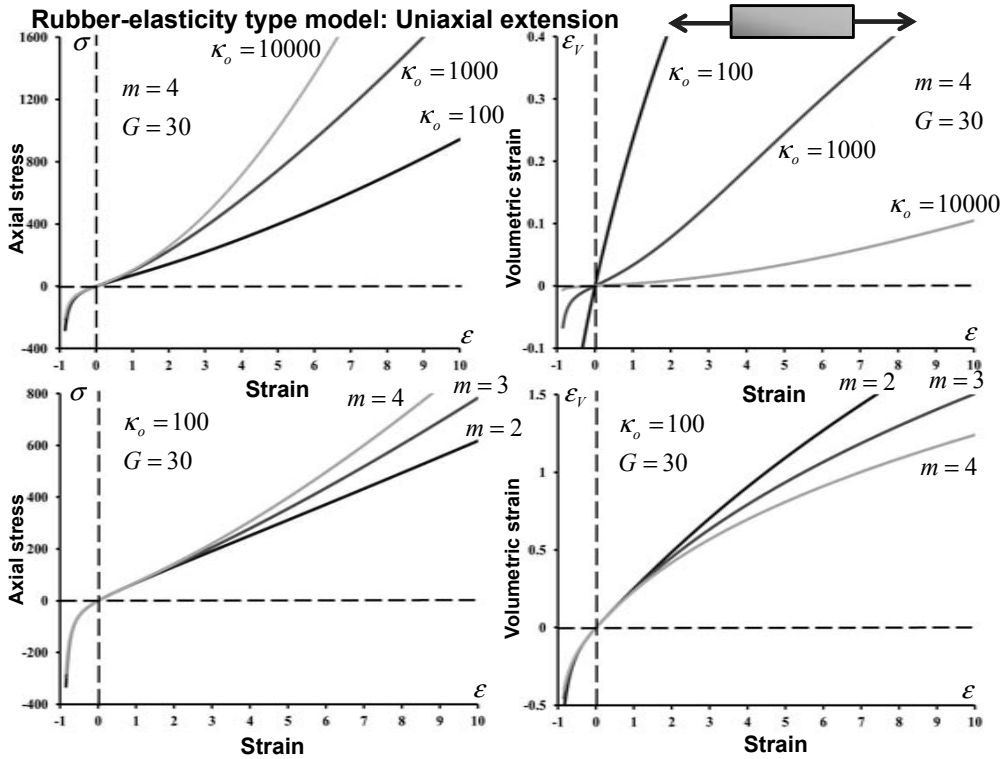


Figure 10.6: Rubber-elasticity type model: Response in uniaxial extension for different bulk moduli κ_o (top) and exponent m (bottom).

where, as in the last section, $J^* = J/J^\theta$ is the volume ratio normalized to the thermal expansion. The thermal expansion is calculated by integrating $\dot{J}^\theta = \beta_V J^\theta \dot{\theta}$ for β_V being the coefficient of volumetric thermal expansion. The material parameters κ_{oi} , β_V , G_i , and ψ^θ are assumed to be only functions of temperature, and α_i and β_i are assumed to be nonzero constants. In particular, α_i and β_i are real numbers, not necessarily integers, selected to best fit the data, and $\beta_i > 1$.

To calculate the Cauchy stress and entropy for this model we need to know how to calculate the time derivative of $\text{tr}(\mathbf{B}^{\alpha_i})$, where α_i is not necessarily an integer power. For this we need to recall that $\mathbf{B}^{\alpha_i} = \mathbf{P}^T \mathbf{B}_d^{\alpha_i} \mathbf{P}$, where $\mathbf{B} = \mathbf{P}^T \mathbf{B}_d \mathbf{P}$ for $\mathbf{B}_d$ being diagonal and composed of the eigenvalues of $\mathbf{B}$, and $\mathbf{P}$ being orthogonal and constructed from its eigenvectors. From this we can show that $\overline{\text{tr}(\mathbf{B}^{\alpha_i})} = 2\alpha_i(\mathbf{B}^{\alpha_i-1}\mathbf{F}) : \dot{\mathbf{F}}$, the proof of which is left as homework. Using this we can show that

$$\mathbf{T} = \sigma_{ave} \mathbf{I} + \mathbf{S}, \quad (10.187)$$

$$\begin{aligned} \eta = & - \sum_{i=1}^m \frac{J^\theta}{2\rho_o\beta_i} J^* \left(\frac{J^{*\beta_i}}{\beta_i + 1} + \frac{J^{*-\beta_i}}{\beta_i - 1} \right) \left(\frac{d\kappa_{oi}}{d\theta} + \kappa_{oi}\beta_V \right) + \frac{\beta_V \sigma_{ave}}{\rho} \\ & - \sum_{i=1}^n \frac{1}{2\rho_o\alpha_i} \left[\frac{\text{tr}(\mathbf{B}^{\alpha_i})}{J^{2/3\alpha_i}} - 3 \right] \frac{dG_i}{d\theta} - \frac{d\psi^\theta}{d\theta}, \end{aligned} \quad (10.188)$$

where the average stress σ_{ave} and the deviatoric stress $\mathbf{S}$ are given by

$$\sigma_{ave} = \sum_{i=1}^m \frac{\kappa_{oi}}{2\beta_i} \left(J^{*\beta_i} - \frac{1}{J^{*\beta_i}} \right), \quad (10.189)$$

$$\mathbf{S} = \sum_{i=1}^n \frac{G_i}{J^{\frac{2}{3}\alpha_i+1}} \left[\mathbf{B}^{\alpha_i} - \frac{tr(\mathbf{B}^{\alpha_i})}{3} \mathbf{I} \right]. \quad (10.190)$$

When considering pure volumetric changes the expressions simplify by dropping the Ogden terms (terms with α_i and G_i). This is since we have a nicely decoupled system where the hydrostatic and deviatoric terms are separated. In this case, we get

$$\mathbf{T} = \sigma_{ave} \mathbf{I}, \quad (10.191)$$

$$\eta = - \sum_{i=1}^m \frac{J^\theta}{2\rho_o\beta_i} J^* \left(\frac{J^{*\beta_i}}{\beta_i+1} + \frac{J^{*-\beta_i}}{\beta_i-1} \right) \left(\frac{d\kappa_{oi}}{d\theta} + \kappa_{oi}\beta_V \right) + \frac{\beta_V \sigma_{ave}}{\rho} - \frac{d\psi^\theta}{d\theta}. \quad (10.192)$$

At any temperature, we will define the bulk modulus at constant temperature, calculated relative to the base volume at that temperature, as the rate of change of σ_{ave} with respect to changes in J^* . This gives

$$\kappa = \frac{d\sigma_{ave}}{dJ^*} \Big|_\theta = \sum_{i=1}^m \frac{\kappa_{oi}}{2J^*} \left(J^{*\beta_i} + \frac{1}{J^{*\beta_i}} \right). \quad (10.193)$$

We note that for positive κ_{oi} the model always produces positive bulk moduli. Also, since $\beta_i > 1$, this is true even when the volume ratio goes to infinity. Evaluating this at $J^* = 1$ we get the bulk modulus at zero stress κ_o as

$$\kappa_o = \sum_{i=1}^m \kappa_{oi}, \quad (10.194)$$

which provides the interpretation of the temperature-dependent moduli κ_{oi} . The response in dilatation is similar to that of the rubber-elasticity type model, but with the option of having multiple terms.

In uniaxial extension along the $\hat{\mathbf{e}}_1$ direction of the orthonormal base directions $\hat{\mathbf{e}}_i$, the stress components are all zero other than that $T_{11} = \sigma$. For the axial stretch λ_a and transverse stretch λ_t , this gives

$$\sigma = \sigma_{ave} + \sum_{i=1}^n \frac{G_i}{(\lambda_a \lambda_t^2)^{\frac{2}{3}\alpha_i+1}} \left[\lambda_a^{2\alpha_i} - \frac{1}{3}(\lambda_a^{2\alpha_i} + 2\lambda_t^{2\alpha_i}) \right], \quad (10.195)$$

$$0 = \sigma_{ave} + \sum_{i=1}^n \frac{G_i}{(\lambda_a \lambda_t^2)^{\frac{2}{3}\alpha_i+1}} \left[\lambda_t^{2\alpha_i} - \frac{1}{3}(\lambda_a^{2\alpha_i} + 2\lambda_t^{2\alpha_i}) \right]. \quad (10.196)$$

The second equation can be used to solve for the transverse stretch λ_t in terms of λ_a . Subtracting the two equations gives

$$\sigma = \sum_{i=1}^n \frac{G_i}{(\lambda_a \lambda_t^2)^{\frac{2}{3}\alpha_i+1}} (\lambda_a^{2\alpha_i} - \lambda_t^{2\alpha_i}). \quad (10.197)$$

As will be discussed later, in the limit of incompressibility under isothermal conditions, the average stress becomes indeterminate, but the condition $J = J^\theta$ gives $\lambda_t = \sqrt{J^\theta/\lambda_a}$ to yield

$$\sigma = \sum_{i=1}^n G_i \left[\left(\frac{\lambda_a}{J^{\theta\frac{1}{3}}} \right)^{2\alpha_i} - \left(\frac{J^{\theta\frac{1}{3}}}{\lambda_a} \right)^{2\alpha_i} \right]. \quad (10.198)$$

The response of this model for both uniaxial extension and simple shear is shown in Figure 10.7 for a single Ogden element.

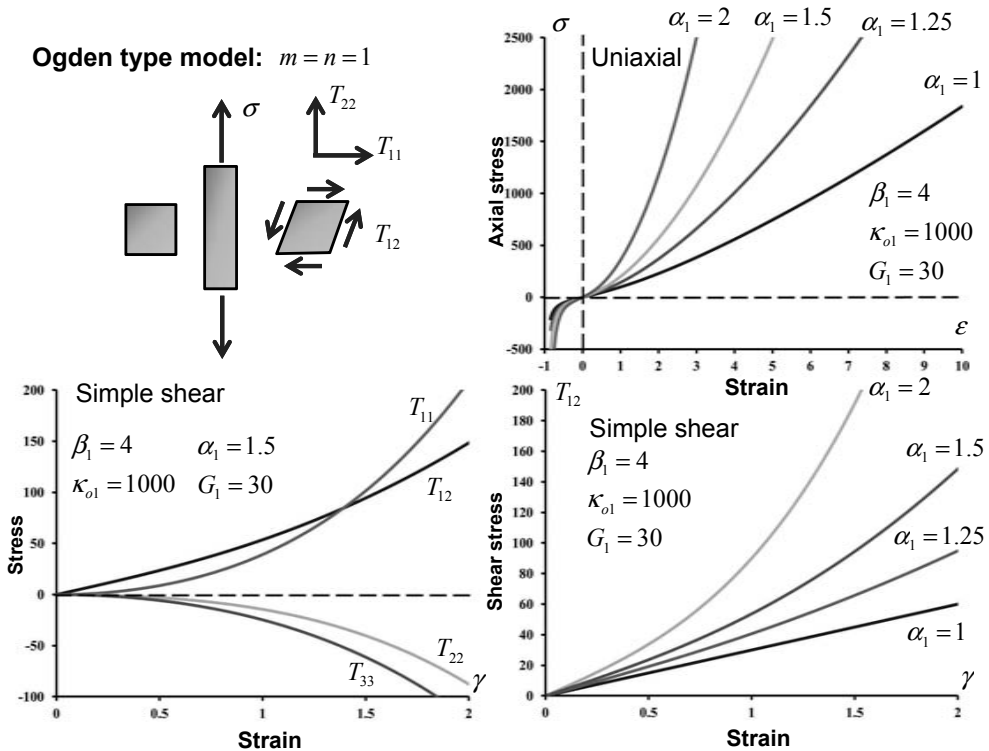


Figure 10.7: Ogden type model: Response in uniaxial extension (top right) and simple shear (bottom).

Section 10.6-Problems:

- 10.6-10:** Prove that the time derivative of $\text{tr}(\mathbf{B}^\alpha)$ is given by $2\alpha(\mathbf{B}^{\alpha-1}\mathbf{F}) : \dot{\mathbf{F}}$ for any real value α . Recall that for the real symmetric tensor $\mathbf{B}$ we define $\mathbf{B}^\alpha = \mathbf{P}^T \mathbf{B}_d^\alpha \mathbf{P}$, where $\mathbf{B} = \mathbf{P}^T \mathbf{B}_d \mathbf{P}$ for $\mathbf{B}_d$ being diagonal and composed of the eigenvalues of $\mathbf{B}$, and $\mathbf{P}$ is orthogonal and constructed from its eigenvectors. Construct your proof by noting that $\text{tr}(\mathbf{B}^\alpha) = \text{tr}(\mathbf{B}_d^\alpha)$ and taking its derivative to find $\frac{d}{dt} \text{tr}(\mathbf{B}^\alpha) = \alpha \text{tr}(\mathbf{B}^{\alpha-1} \mathbf{P}^T \dot{\mathbf{B}}_d \mathbf{P})$. Next, show that $\dot{\mathbf{B}} = \dot{\mathbf{P}}^T \mathbf{P} \mathbf{B} + \mathbf{P}^T \dot{\mathbf{B}}_d \mathbf{P} - \mathbf{B} \dot{\mathbf{P}}^T \mathbf{P}$, and use this to substitute for $\mathbf{P}^T \dot{\mathbf{B}}_d \mathbf{P}$ and complete the proof.
- 10.6-11:** The proposed Ogden model provides substantial flexibility so one can model many different types of response in shear and in dilatation. One way to generalize this model is to change the first term to a more complicated form. For example, consider the model

$$\psi = \psi^h \left(\frac{J}{J^\theta}, \theta \right) + \sum_{i=1}^n \frac{G_i}{2\alpha_i \rho_o} \left[\frac{\text{tr}(\mathbf{B}^{\alpha_i})}{J^{\frac{2}{3}\alpha_i}} - 3 \right] + \psi^\theta, \quad (10.199)$$

where $\psi^h(J^*, \theta)$ is selected to fit the pure hydrostatic response (note that

$J^* = \frac{J}{J^\theta}$). The stress and entropy for such a model are given by

$$\mathbf{T} = \frac{1}{J} \left\{ \frac{\rho_o}{J^\theta} \frac{\partial \psi^h}{\partial J^*} \mathbf{I} + \sum_{i=1}^n \frac{G_i}{J^{\frac{2}{3}\alpha_i}} \left[\mathbf{B}^{\alpha_i} - \frac{\text{tr}(\mathbf{B}^{\alpha_i})}{3} \mathbf{I} \right] \right\}, \quad (10.200)$$

$$\eta = -\frac{\partial \psi^h}{\partial \theta} + \beta_V \frac{\sigma_{ave}}{\rho} - \frac{1}{2\rho_o \alpha_i} \left[\frac{\text{tr}(\mathbf{B}^{\alpha_i})}{J^{2/3\alpha_i}} - 3 \right] \frac{dG_i}{d\theta} - \frac{d\psi^\theta}{d\theta}. \quad (10.201)$$

Derive these expressions for the cases

$$\psi^h(J^*, \theta) = \frac{\kappa_o J^\theta}{2\rho_o} (J^* - 1)^2, \quad (10.202)$$

$$\psi^h(J^*, \theta) = \frac{\kappa_o J^\theta}{2\rho_o} \ln^2(J^*), \quad (10.203)$$

$$\psi^h(J^*, \theta) = \frac{\kappa_o J^\theta}{(n+1)\rho_o} \left[\frac{J^{*(n+1)}}{n+1} - \ln(J^*) \right], \quad (10.204)$$

$$\psi^h(J^*, \theta) = \frac{\kappa_o J^\theta}{8\rho_o} \{ J^{*2} + 2 \ln(J^*) [\ln(J^*) - 1] \}, \quad (10.205)$$

$$\psi^h(J^*, \theta) = \frac{\kappa_o J^\theta}{\rho_o} J^* [\ln(J^*) - 1], \quad (10.206)$$

where κ_o is a function of temperature and n is a constant. In each case, derive the expressions for the average stress and the bulk modulus, and discuss the behavior of the material.

10.6-12: Simulate the response of the Ogden type model for uniaxial extension and simple shear and compare your simulations to those shown in Figure 10.7.

10.7 Models based on $\mathbf{F}^e$ and $\mathbf{F}^\theta$

It is a good idea to separate the deformation into the part that is due to mechanical loading and that is due to thermal expansion. The easiest way to do this is to assume that the deformation gradient is given by

$$\boxed{\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta}, \quad (10.207)$$

where $\mathbf{F}^e$ is the elastic part, and $\mathbf{F}^\theta$ is the thermal part. For this separation to have a meaning, we need to define how to do it. We will do this by giving a method to calculate the thermal deformation gradient. We could provide a function for $\mathbf{F}^\theta$, but we will select to follow its derivative by giving a differential evolution equation to calculate it. We will assume the current value of $\mathbf{F}^\theta$ can be calculated from an initial value $\mathbf{F}_o^\theta$ and an evolution equation given as

$$\boxed{\dot{\mathbf{L}}^\theta = \boldsymbol{\alpha} \dot{\theta}}, \quad (10.208)$$

where $\boldsymbol{\alpha}$ is the coefficient of thermal expansion, a second-order tensor function of $\mathbf{F}^e$, $\mathbf{F}^\theta$, and θ . We can use this evolution equation to integrate $\dot{\mathbf{F}}^\theta = \mathbf{L}^\theta \mathbf{F}^\theta$ to get $\mathbf{F}^\theta$ from $\mathbf{F}_o^\theta$. If one expects reversibility of thermal expansion, which is a natural expectation, then the

dependence of α on $\mathbf{F}^e$ should be removed, which results in the thermal deformation gradient being fully evaluated from the temperature history.

The next assumption that we will make is that the free energy can be written as

$$\boxed{\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta, \mathbf{G}),} \quad (10.209)$$

so that its derivative can be written as

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi^\dagger) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^\theta}(\psi^\dagger) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi^\dagger)\dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \quad (10.210)$$

We now rewrite this in terms of $\dot{\mathbf{F}}$ by using $\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta$. We know that the derivative of $\mathbf{F}$ is

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^\theta + \mathbf{F}^e \dot{\mathbf{F}}^\theta, \quad (10.211)$$

so that

$$\dot{\mathbf{F}}^e = \dot{\mathbf{F}} \mathbf{F}^{\theta-1} - \mathbf{F}^e \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1}. \quad (10.212)$$

If we substitute this into $\dot{\psi}$, we get

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi^\dagger) : (\dot{\mathbf{F}} \mathbf{F}^{\theta-1} - \mathbf{F}^e \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1}) + \partial_{\mathbf{F}^\theta}(\psi^\dagger) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi^\dagger)\dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \quad (10.213)$$

We now can substitute this into the Clausius-Duhem inequality (6.69) to get

$$\begin{aligned} & [\rho \partial_{\mathbf{F}^e}(\psi^\dagger) \mathbf{F}^{eT} - \mathbf{T}^T] : \mathbf{L} + \{\rho \eta + \rho \partial_\theta(\psi^\dagger) + [\rho \partial_{\mathbf{F}^\theta}(\psi^\dagger) \mathbf{F}^{\theta T} - \rho \mathbf{F}^e \partial_{\mathbf{F}^e}(\psi^\dagger)] : \alpha\} \dot{\theta} \\ & + \rho \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (10.214)$$

where we have used $\dot{\mathbf{F}}^\theta = \mathbf{L}^\theta \mathbf{F}^\theta = \alpha \mathbf{F}^\theta \dot{\theta}$. Let us now define, respectively, the thermodynamic elastic and thermal stresses as

$$\boxed{\mathbf{T}^{eT} \equiv \rho \partial_{\mathbf{F}^e}(\psi^\dagger) \mathbf{F}^{eT},} \quad (10.215)$$

$$\boxed{\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi^\dagger) \mathbf{F}^{\theta T},} \quad (10.216)$$

and the thermal overstress as

$$\boxed{\Delta \mathbf{T}^\theta \equiv \mathbf{F}^e \mathbf{T}^e \mathbf{F}^{e-1} - \mathbf{T}^\theta.} \quad (10.217)$$

Using these abbreviations, the Clausius-Duhem inequality becomes

$$[\mathbf{T}^{eT} - \mathbf{T}^T] : \mathbf{L} + \rho \left[\eta + \partial_\theta(\psi^\dagger) - \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \alpha \right] \dot{\theta} + [\rho \partial_{\mathbf{G}}(\psi^\dagger)] \circ \dot{\mathbf{G}} + \left[\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \right] \leq 0. \quad (10.218)$$

We assume that the entropy, Cauchy stress, and heat flux depend on the same argument set as we selected for the free energy. As a result, the terms in the square brackets are independent of the rates $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$, which, given the assumption that this equation must hold for all rates, results in the relations

$$\mathbf{T}^{eT} - \mathbf{T}^T = \mathbf{0}, \quad (10.219)$$

$$\eta + \partial_\theta(\psi^\dagger) - \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \alpha = 0, \quad (10.220)$$

$$\rho \partial_{\mathbf{G}}(\psi^\dagger) = \mathbf{0}, \quad (10.221)$$

$$\boxed{\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.} \quad (10.222)$$

It follows that the free energy does not depend on the temperature gradient, so that

$$\boxed{\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta),} \quad (10.223)$$

and that

$$\boxed{\mathbf{T}^T = \mathbf{T}^{eT} = \rho \partial_{\mathbf{F}^e}(\psi^\dagger) \mathbf{F}^{eT},} \quad (10.224)$$

$$\boxed{\eta = -\partial_\theta(\psi^\dagger) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha},} \quad (10.225)$$

which indicate that the stress and the entropy do not depend on the temperature gradient.

Before we can impose the effect of rigid body motions, we need to assume how the elastic and thermal parts of the deformation gradient change with rigid body motions. We know that as a result of a rigid body motion the deformation gradient $\mathbf{F}$ becomes $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$. We will assume that the thermal deformation gradient does not change with rigid body motions, so that we have

$$\mathbf{F}^{e*} = \mathbf{Q}\mathbf{F}^e, \quad \mathbf{F}^{\theta*} = \mathbf{F}^\theta, \quad (10.226)$$

so that $\mathbf{F}^* = \mathbf{F}^{e*} \mathbf{F}^{\theta*} = \mathbf{Q}\mathbf{F}^e \mathbf{F}^\theta = \mathbf{Q}\mathbf{F}$. The assumption that the free energy does not change with rigid body motions requires that

$$\psi^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^\theta, \theta) = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta) \quad (10.227)$$

for every orthogonal $\mathbf{Q}$. Since $\mathbf{G}$ does not change with rigid body motions, the assumption that the heat flux rotates by $\mathbf{Q}$ results in

$$\mathbf{q}^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^\theta, \theta, \mathbf{G}) = \mathbf{Q}\mathbf{q}^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta, \mathbf{G}) \quad (10.228)$$

for every orthogonal $\mathbf{Q}$. By the selection of $\mathbf{Q} = \mathbf{R}^{eT}$, we conclude that

$$\boxed{\psi = \psi^\dagger(\mathbf{U}^e, \mathbf{F}^\theta, \theta),} \quad (10.229)$$

and

$$\boxed{\mathbf{q} = \mathbf{R}^e \mathbf{q}^{R^e}} \quad (10.230)$$

for the rotated heat flux $\mathbf{q}^{R^e}$ with the model

$$\boxed{\mathbf{q}^{R^e} = \mathbf{q}^{R^e \dagger}(\mathbf{U}^e, \mathbf{F}^\theta, \theta, \mathbf{G}).} \quad (10.231)$$

This allows us to rewrite ψ and $\mathbf{q}^{R^e}$ in one of the alternate forms

$$\boxed{\psi = \psi^{\dagger\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta) = \psi^{\dagger\dagger\dagger}(\mathbf{E}^e, \mathbf{F}^\theta, \theta),} \quad (10.232)$$

$$\boxed{\mathbf{q}^{R^e} = \mathbf{q}^{R^e \dagger\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta, \mathbf{G}) = \mathbf{q}^{R^e \dagger\dagger\dagger}(\mathbf{E}^e, \mathbf{F}^\theta, \theta, \mathbf{G}).} \quad (10.233)$$

The material symmetry constraint can be imposed once we assume how the elastic and thermal deformation gradients change. We assume that $\mathbf{F}$ changing to $\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$ under the material symmetry transformation $\mathbf{M}$ results in the elastic and thermal parts transforming to

$$\boxed{\bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M},} \quad \boxed{\bar{\mathbf{F}}^\theta = \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M},} \quad (10.234)$$

which gives $\bar{\mathbf{F}} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta = \mathbf{F}^e \mathbf{M} \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M} = \mathbf{F} \mathbf{M}$. This results in the condition on the coefficient of thermal expansion being

$$\boxed{\bar{\boldsymbol{\alpha}} = \mathbf{M}^{-1} \boldsymbol{\alpha} \mathbf{M}.} \quad (10.235)$$

Assuming that the free energy and heat flux do not change because of a material symmetry reorganization before the deformation, we must have

$$\psi^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M}, \theta) = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta), \quad (10.236)$$

$$\mathbf{q}^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G}) = \mathbf{q}^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta, \mathbf{G}). \quad (10.237)$$

The other two response functions, for the stress and entropy, are given in terms of the free energy, so we need not consider them independently. The condition on the constitutive model for the coefficient of thermal expansion is

$$\boldsymbol{\alpha}^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M}, \theta) = \mathbf{M}^{-1} \boldsymbol{\alpha}^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta) \mathbf{M}. \quad (10.238)$$

Combining the constraint imposed by rigid body motion and the constraint imposed by material symmetry, for the case of orthogonal $\mathbf{M}$, results in requirements such as

$$\psi^{\dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \theta) = \psi^{\dagger\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta), \quad (10.239)$$

$$\mathbf{M} \mathbf{q}^{R^e \dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \theta, \mathbf{M}^T \mathbf{G}) = \mathbf{q}^{R^e \dagger\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta, \mathbf{G}), \quad (10.240)$$

$$\mathbf{M} \boldsymbol{\alpha}^{\dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \theta) \mathbf{M}^T = \boldsymbol{\alpha}^{\dagger\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta). \quad (10.241)$$

10.7.1 Models depending on $(\mathbf{F}^e, \theta, \mathbf{g})$

The most common models do not assume an explicit dependence of the free energy on the thermal deformation gradient. As such, we start with all material response functions, depending on $(\mathbf{F}^e, \theta, \mathbf{g})$, and conclude from Clausius-Duhem inequality that

$$\psi = \psi^\dagger(\mathbf{F}^e, \theta), \quad (10.242)$$

$$\eta = \eta^\dagger(\mathbf{F}^e, \theta), \quad (10.243)$$

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \theta), \quad (10.244)$$

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{F}^e, \theta, \mathbf{g}), \quad (10.245)$$

and that

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e} (\psi^\dagger) \mathbf{F}^{eT},$$

$$\mathbf{T}^\theta = \mathbf{0},$$

$$\Delta \mathbf{T}^\theta = \mathbf{F}^e \mathbf{T} \mathbf{F}^{e-1},$$

$$\eta = -\partial_\theta (\psi^\dagger) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha},$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

The starting assumption seems to be reasonable and fairly general since $\mathbf{F}^e$ provides the means of calculating the contribution of mechanical loading, θ provides the method of introducing the contribution of temperature changes, and $\mathbf{g}$ provides the needed temperature gradient in calculating the heat flow. As shown above, the model is fully determined by providing the models for the thermal expansion, the specific free energy, and the heat flux vector. The introduction of the constraint we imposed for rigid body motions allows us to write

$$\psi = \psi^{\dagger\dagger}(\mathbf{C}^e, \theta) = \psi^{\dagger\dagger\dagger}(\mathbf{E}^e, \theta), \quad (10.246)$$

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^{\dagger\dagger}(\mathbf{C}^e, \theta) = \boldsymbol{\alpha}^{\dagger\dagger\dagger}(\mathbf{E}^e, \theta), \quad (10.247)$$

$$\mathbf{q}^{R^e} = \mathbf{q}^{R^e \dagger\dagger}(\mathbf{C}^e, \theta, \mathbf{g}^{R^e}) = \mathbf{q}^{R^e \dagger\dagger\dagger}(\mathbf{E}^e, \theta, \mathbf{g}^{R^e}), \quad (10.248)$$

where $\mathbf{g}^{R^e} = \mathbf{R}^{eT} \mathbf{g}$ is a rotated temperature gradient. For the purpose of constructing representations, let us focus on orthogonal symmetry transformations so that the symmetry constrains can be written, for example, as

$$\psi^{\dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \theta) = \psi^{\dagger\dagger}(\mathbf{C}^e, \theta), \quad (10.249)$$

$$\mathbf{M} \boldsymbol{\alpha}^{\dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \theta) \mathbf{M}^T = \boldsymbol{\alpha}^{\dagger\dagger}(\mathbf{C}^e, \theta), \quad (10.250)$$

$$\mathbf{M} \mathbf{q}^{R^e \dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \theta, \mathbf{M}^T \mathbf{g}^{R^e}) = \mathbf{q}^{R^e \dagger\dagger}(\mathbf{C}^e, \theta, \mathbf{g}^{R^e}), \quad (10.251)$$

or

$$\psi^{\dagger\dagger\dagger}(\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \theta) = \psi^{\dagger\dagger\dagger}(\mathbf{E}^e, \theta), \quad (10.252)$$

$$\mathbf{M} \boldsymbol{\alpha}^{\dagger\dagger\dagger}(\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \theta) \mathbf{M}^T = \boldsymbol{\alpha}^{\dagger\dagger\dagger}(\mathbf{E}^e, \theta), \quad (10.253)$$

$$\mathbf{M} \mathbf{q}^{R^e \dagger\dagger\dagger}(\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \theta, \mathbf{M}^T \mathbf{g}^{R^e}) = \mathbf{q}^{R^e \dagger\dagger\dagger}(\mathbf{E}^e, \theta, \mathbf{g}^{R^e}). \quad (10.254)$$

We can now construct the general representations using Appendix A.

Isotropic materials

For an isotropic material with the symmetry represented as the full group of orthogonal transformations, the representation for free energy can be written as

$$\psi = \tilde{\psi}^{\dagger\dagger}(I_1^e, I_2^e, I_3^e, \theta), \quad (10.255)$$

where $I_1^e = \text{tr}(\mathbf{C}^e)$, $I_2^e = \text{tr}(\mathbf{C}^{e2})$, $I_3^e = \text{tr}(\mathbf{C}^{e3})$, or any other set of isotropic invariants of $\mathbf{C}^e$. This gives the expression for stress as

$$\begin{aligned} \mathbf{T}^T &= \rho \sum_{i=1}^3 \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_i^e} \partial_{\mathbf{F}^e}(I_i^e) \mathbf{F}^{eT} \\ &= 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_1^e} \mathbf{B}^e + 4\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_2^e} \mathbf{B}^{e2} + 6\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_3^e} \mathbf{B}^{e3}. \end{aligned} \quad (10.256)$$

Possibly a more useful representation comes from using the isotropic invariants

$$K_1^e = J^e, \quad K_2^e = \frac{\text{tr}(\mathbf{B}^e)}{J^{e2/3}}, \quad K_3^e = \frac{\text{tr}(\mathbf{B}^{e2})}{\text{tr}^2(\mathbf{B}^e)}, \quad (10.257)$$

which help decouple the volumetric deformation from the others. In this case, we take

$$\psi = \tilde{\psi}^{\dagger\dagger}(K_1^e, K_2^e, K_3^e, \theta), \quad (10.258)$$

and obtain the Cauchy stress as

$$\begin{aligned} \mathbf{T}^T &= \rho \sum_{i=1}^3 \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial K_i^e} \partial_{\mathbf{F}^e}(K_i^e) \mathbf{F}^{eT} \\ &= \rho \left[\frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial K_1^e} J^e \mathbf{I} + \frac{2}{J^{e2/3}} \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial K_2^e} \left(\mathbf{B}^e - \frac{I_1^e}{3} \mathbf{I} \right) + \frac{4}{I_1^{e2}} \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial K_3^e} \left(\mathbf{B}^{e2} - \frac{I_2^e}{I_1^e} \mathbf{B}^e \right) \right]. \end{aligned} \quad (10.259)$$

The representation for the coefficient of thermal expansion can be written as

$$\boldsymbol{\alpha} = \alpha_o \mathbf{I} + \alpha_1 \mathbf{C}^e + \alpha_2 \mathbf{C}^{e2}, \quad (10.260)$$

where α_i are scalar valued functions of temperature and the three isotropic invariants of $\mathbf{C}^e$. The form suggests that we could, when appropriate, use the rotated stress directly to get, for example, models such as

$$\boldsymbol{\alpha} = \hat{\alpha}_o \mathbf{I} + \hat{\alpha}_1 \mathbf{T}^{R^e},$$

where $\mathbf{T}^{R^e} = \mathbf{R}^{eT} \mathbf{T} \mathbf{R}^e$. If the thermal expansion is to be recoverable in a closed thermo-mechanical process, then we want to use a model of the form

$$\boldsymbol{\alpha} = \alpha_o \mathbf{I}, \quad (10.261)$$

where α_o only depends on temperature. The representation for the heat flux can be written as

$$\begin{aligned} \mathbf{q} &= \mathbf{R}^e (k_o \mathbf{I} + k_1 \mathbf{C}^e + k_2 \mathbf{C}^{e2}) \mathbf{g}^{R^e} \\ &= (k_o \mathbf{I} + k_1 \mathbf{B}^e + k_2 \mathbf{B}^{e2}) \mathbf{g}, \end{aligned} \quad (10.262)$$

where each k_i is a function of temperature and the six isotropic scalar invariants of $\mathbf{C}^e$ and $\mathbf{g}^{R^e}$ given as the three isotropic invariants of $\mathbf{C}^e$ plus the additional invariants $I_4^e = \mathbf{g}^{R^e} \circ \mathbf{g}^{R^e} = \mathbf{g} \circ \mathbf{g}$, $I_5^e = \mathbf{g}^{R^e} \circ (\mathbf{C}^e \mathbf{g}^{R^e}) = \mathbf{g} \circ (\mathbf{B}^e \mathbf{g})$, and $I_6^e = \mathbf{g}^{R^e} \circ (\mathbf{C}^{e2} \mathbf{g}^{R^e}) = \mathbf{g} \circ (\mathbf{B}^{e2} \mathbf{g})$. This suggests that one might directly use stress when it is appropriate or convenient for the modeling and come up, for example, with models such as

$$\mathbf{q} = (\hat{k}_o \mathbf{I} + \hat{k}_1 \mathbf{T}^2) \mathbf{g}.$$

The model for $\mathbf{q}$ still needs to satisfy the constraint imposed by the Clausius-Duhem inequality. In this case, we must have

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} = \frac{1}{\theta} [\hat{k}_o \mathbf{g} \circ \mathbf{g} + \hat{k}_1 (\mathbf{T} \mathbf{g}) \circ (\mathbf{T} \mathbf{g})] \leq 0,$$

which can be satisfied by assuming the $\hat{k}_i$ are positive functions (assuming temperature is positive valued).

Section 10.7-Problems:

10.7-1: Construct a rubber-elasticity type model based on a free energy of the form

$$\psi = \frac{\kappa_o J^\theta}{2m\rho_o} K_1^e \left(\frac{K_1^{em}}{m+1} + \frac{K_1^{e-m}}{m-1} \right) + \frac{G}{2\rho_o} (K_2^e - 3) + \psi^\theta, \quad (10.263)$$

where κ_o , G , and ψ^θ are functions of temperature and m and ρ_o are constants.

10.7-2: Construct an Ogden type model based on the free energy of the form

$$\sum_{i=1}^m \frac{\kappa_{oi} J^\theta}{2\rho_o \beta_i} J^e \left(\frac{J^{e\beta_i}}{\beta_i + 1} + \frac{J^{e-\beta_i}}{\beta_i - 1} \right) + \sum_{i=1}^n \frac{G_i}{2\rho_o \alpha_i} \left[\frac{\text{tr}(\mathbf{B}^{e\alpha_i})}{J^{e\frac{2}{3}\alpha_i}} - 3 \right] + \psi^\theta, \quad (10.264)$$

where m , n , and ρ_o are constants, and κ_{oi} , G_i , and ψ^θ are functions of temperature.

Transversely isotropic materials

For a transversely isotropic material with axis of transverse isotropy given by $\mathbf{m}$, we can replace $\mathbf{C}$ with $\mathbf{C}^e$ in (10.84) to get the 10 transversely isotropic scalar invariants of $\mathbf{C}^e$ and $\mathbf{g}^{R^e}$ as

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}^e), & I_2 &= \text{tr}(\mathbf{C}^{e2}), & I_3 &= \text{tr}(\mathbf{C}^{e3}), \\ I_4 &= \mathbf{m} \circ (\mathbf{C}^e \mathbf{m}), & I_5 &= \mathbf{m} \circ (\mathbf{C}^{e2} \mathbf{m}), & & \\ I_6 &= \mathbf{g}^{R^e} \circ \mathbf{g}^{R^e}, & I_7 &= \mathbf{g}^{R^e} \circ \mathbf{m}, & I_8 &= \mathbf{g}^{R^e} \circ (\mathbf{C}^e \mathbf{g}^{R^e}), \\ I_9 &= \mathbf{m} \circ (\mathbf{C}^e \mathbf{g}^{R^e}), & I_{10} &= \mathbf{m} \circ (\mathbf{C}^{e2} \mathbf{g}^{R^e}), & & \end{aligned} \quad (10.265)$$

the vector invariants from (10.85) as

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{g}^{R^e}, & \mathbf{f}_2 &= \mathbf{m}, & \mathbf{f}_3 &= \mathbf{C}^e \mathbf{g}^{R^e}, \\ \mathbf{f}_4 &= \mathbf{C}^e \mathbf{m}, & \mathbf{f}_5 &= \mathbf{C}^{e2} \mathbf{m}, \end{aligned} \quad (10.266)$$

the second-order symmetric tensor invariants from (10.86) as

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I}, & \mathbf{H}_2 &= \mathbf{C}^e, & \mathbf{H}_3 &= \mathbf{C}^{e2}, \\ \mathbf{H}_4 &= \mathbf{m} \otimes \mathbf{m}, \\ \mathbf{H}_5 &= \mathbf{m} \otimes (\mathbf{C}^e \mathbf{m}) + (\mathbf{C}^e \mathbf{m}) \otimes \mathbf{m}, \\ \mathbf{H}_6 &= \mathbf{m} \otimes (\mathbf{C}^{e2} \mathbf{m}) + (\mathbf{C}^{e2} \mathbf{m}) \otimes \mathbf{m}, \end{aligned} \quad (10.267)$$

and from the appendix the skew-symmetric tensor invariants as

$$\begin{aligned} \mathbf{Z}_1 &= \mathbf{m} \otimes (\mathbf{C}^e \mathbf{m}) - (\mathbf{C}^e \mathbf{m}) \otimes \mathbf{m}, \\ \mathbf{Z}_2 &= \mathbf{m} \otimes (\mathbf{C}^{e2} \mathbf{m}) - (\mathbf{C}^{e2} \mathbf{m}) \otimes \mathbf{m}, \\ \mathbf{Z}_3 &= (\mathbf{C}^e \mathbf{m}) \otimes (\mathbf{C}^{e2} \mathbf{m}) - (\mathbf{C}^{e2} \mathbf{m}) \otimes (\mathbf{C}^e \mathbf{m}). \end{aligned} \quad (10.268)$$

From these we can construct the representation of the free energy as

$$\psi = \tilde{\psi}^{\dagger\dagger}(I_1, I_2, I_3, I_4, I_5, \theta), \quad (10.269)$$

which gives an expression for the Cauchy stress as

$$\begin{aligned} \mathbf{T}^T &= \rho \sum_{i=1}^5 \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT} \\ &= 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_1} \mathbf{B}^e + 4\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_2} \mathbf{B}^{e2} + 6\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_3} \mathbf{B}^{e3} \\ &\quad + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_4} \mathbf{F}^e (\mathbf{m} \otimes \mathbf{m}) \mathbf{F}^{eT} + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_5} \mathbf{F}^e [\mathbf{C}^e (\mathbf{m} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{m}) \mathbf{C}^e] \mathbf{F}^{eT}. \end{aligned} \quad (10.270)$$

The representation of the coefficient of thermal expansion can be written as

$$\boldsymbol{\alpha} = \sum_{i=1}^6 \alpha_{1i} \mathbf{H}_i + \sum_{i=1}^3 \alpha_{2i} \mathbf{Z}_i, \quad (10.271)$$

where α_i are scalar functions of temperature and the five invariants I_1 – I_5 . If we assume thermal expansion is reversible, then we can write it as

$$\boldsymbol{\alpha} = \hat{\alpha}_\theta \mathbf{I} + \hat{\alpha}_1 \mathbf{m} \otimes \mathbf{m}, \quad (10.272)$$

where the scalar coefficients are functions of temperature. Finally, the heat flux has a representation

$$\mathbf{q} = \mathbf{R}^e \sum_{i=1}^5 k_i \mathbf{f}_i, \quad (10.273)$$

where the coefficients k_i are scalar functions of temperature and the 10 scalar invariants I_1 – I_{10} given above. The heat flux model needs to be selected such that it satisfies the general condition $\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0$.

Section 10.7-Problems:

10.7-3: A transversely isotropic material is assumed to have a free energy of the form

$$\begin{aligned}\psi = & \frac{\kappa J^\theta}{2\rho_o} (J^e - 1)^2 + \frac{G_1 J^\theta}{2\rho_o} \left[\frac{\text{tr}(\mathbf{C}^e)}{J^{e2/3}} - 3 \right] + \frac{G_2 J^\theta}{2\rho_o} \mathbf{m} \circ (\mathbf{C}^e \mathbf{m}) \\ & + \frac{G_3 J^\theta}{2\rho_o} \mathbf{m} \circ (\mathbf{C}^{e-1} \mathbf{m}) + \psi^\theta,\end{aligned}\quad (10.274)$$

where κ , G_1 , G_2 , G_3 , and ψ^θ are functions of temperature. Calculate the expression for Cauchy stress showing that

$$\begin{aligned}\mathbf{T} = & \kappa(J^e - 1)\mathbf{I} + \frac{G_1}{J^{e5/3}} \left[\mathbf{B}^e - \frac{\text{tr}(\mathbf{B}^e)}{3} \mathbf{I} \right] + \frac{G_2}{J^e} \mathbf{F}^e (\mathbf{m} \otimes \mathbf{m}) \mathbf{F}^{eT} \\ & - \frac{G_3}{J^e} \mathbf{F}^e [(\mathbf{C}^{e-1} \mathbf{m}) \otimes (\mathbf{C}^{e-1} \mathbf{m})] \mathbf{F}^{eT}.\end{aligned}\quad (10.275)$$

Impose the condition that for pure thermal expansion ($\mathbf{F}^e = \mathbf{R}^e$) we should have zero stress and conclude that

$$\begin{aligned}\mathbf{T} = & \kappa(J^e - 1)\mathbf{I} + \frac{G_1}{J^{e5/3}} \left[\mathbf{B}^e - \frac{\text{tr}(\mathbf{B}^e)}{3} \mathbf{I} \right] \\ & + \frac{G_2}{J^e} \mathbf{F}^e [\mathbf{m} \otimes \mathbf{m} - (\mathbf{C}^{e-1} \mathbf{m}) \otimes (\mathbf{C}^{e-1} \mathbf{m})] \mathbf{F}^{eT}.\end{aligned}\quad (10.276)$$

10.7-4: Consider a model for the coefficient of thermal expansion that does not depend on elastic deformation. Show that for a transversely isotropic material we will have

$$\boldsymbol{\alpha} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{m} \otimes \mathbf{m}, \quad (10.277)$$

where α_1 and α_2 are functions of temperature. If the coefficient of thermal expansion is a linear function of $\mathbf{C}^e$, show that it will be given for a transversely isotropic material as

$$\begin{aligned}\boldsymbol{\alpha} = & [\alpha_{11} + \alpha_{12} \text{tr}(\mathbf{C}^e) + \alpha_{13} \mathbf{m} \circ (\mathbf{C}^e \mathbf{m})] \mathbf{I} \\ & + [\alpha_{21} + \alpha_{22} \text{tr}(\mathbf{C}^e) + \alpha_{23} \mathbf{m} \circ (\mathbf{C}^e \mathbf{m})] \mathbf{m} \otimes \mathbf{m} \\ & + \alpha_3 \mathbf{C}^e + \alpha_4 \mathbf{m} \otimes (\mathbf{C}^e \mathbf{m}) + \alpha_5 (\mathbf{C}^e \mathbf{m}) \otimes \mathbf{m}.\end{aligned}\quad (10.278)$$

10.7-5: A first-order model for heat flow may assume that it is independent of the elastic deformation. Show that the general representation of such a model is

$$\mathbf{q}^{R^e} = k_1 \mathbf{g}^{R^e} + k_2 \mathbf{m}, \quad (10.279)$$

where k_1 and k_2 are functions of temperature and the invariants $\mathbf{g}^{R^e} \circ \mathbf{g}^{R^e}$ and $\mathbf{m} \circ \mathbf{g}^{R^e}$. Assume the coefficients are polynomial functions and the heat flux is at most linear in the temperature gradient and show that

$$\mathbf{q}^{R^e} = k_1 \mathbf{g}^{R^e} + [k_{21} + k_{22} \mathbf{m} \circ \mathbf{g}^{R^e}] \mathbf{m}. \quad (10.280)$$

Assume that the rotated heat flow at zero temperature gradient is zero and show that

$$\mathbf{q}^{R^e} = k_1 \mathbf{g}^{R^e} + k_{22} (\mathbf{m} \circ \mathbf{g}^{R^e}) \mathbf{m}. \quad (10.281)$$

In a similar manner construct a model for which the material is transversely isotropic and the heat flux is linear in the temperature gradient and at most linear in $\mathbf{C}^e$. Show that this results in

$$\begin{aligned}\mathbf{q}^{R^e} = & [k_{11} + k_{12} \text{tr}(\mathbf{C}^e) + k_{13} \mathbf{m} \circ (\mathbf{C}^e \mathbf{m})] \mathbf{g}^{R^e} \\ & + [k_{21} \mathbf{g}^{R^e} \circ \mathbf{m} + k_{22} (\mathbf{C}^e \mathbf{g}^{R^e}) \circ \mathbf{m}] \mathbf{m} \\ & + k_3 \mathbf{C}^e \mathbf{g}^{R^e} + k_4 (\mathbf{g}^{R^e} \circ \mathbf{m}) \mathbf{C}^e \mathbf{m}.\end{aligned}\quad (10.282)$$

Orthotropic materials

For an orthotropic material, we can do a similar substitution and get the invariants. Assuming that the three preferred directions of orthotropy are orthogonal and given by the three vectors $\mathbf{e}_i$ and that

$$\mathbf{m} = \mathbf{e}_1, \quad \mathbf{Y} = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (10.283)$$

From (10.100) we can get the scalar invariants of $\mathbf{C}^e$ and $\mathbf{g}^{R^e}$ as

$$\begin{aligned}I_1 &= \text{tr}(\mathbf{C}^e), & I_2 &= \text{tr}(\mathbf{C}^{e2}), & I_3 &= \text{tr}(\mathbf{C}^{e3}), \\ I_4 &= \text{tr}(\mathbf{Y} \mathbf{C}^e), & I_5 &= \text{tr}(\mathbf{Y} \mathbf{C}^{e2}), & I_6 &= \mathbf{m} \circ (\mathbf{C}^e \mathbf{m}), \\ I_7 &= \mathbf{m} \circ (\mathbf{C}^{e2} \mathbf{m}), & & & & \\ I_8 &= \mathbf{g}^{R^e} \circ \mathbf{g}^{R^e}, & I_9 &= \mathbf{g}^{R^e} \circ (\mathbf{Y} \mathbf{g}^{R^e}), & I_{10} &= (\mathbf{g}^{R^e} \circ \mathbf{m})^2, \\ I_{11} &= \mathbf{g}^{R^e} \circ (\mathbf{C}^e \mathbf{g}^{R^e}), & I_{12} &= \mathbf{g}^{R^e} \circ (\mathbf{C}^{e2} \mathbf{g}^{R^e}), & I_{13} &= \mathbf{g}^{R^e} \circ (\mathbf{Y} \mathbf{C}^e \mathbf{g}^{R^e}),\end{aligned}\quad (10.284)$$

from (10.101) we can get the vector invariant of $\mathbf{C}^e$ and $\mathbf{g}^{R^e}$ as

$$\begin{aligned}\mathbf{f}_1 &= \mathbf{g}^{R^e}, & \mathbf{f}_2 &= \mathbf{Y} \mathbf{g}^{R^e}, & \mathbf{f}_3 &= (\mathbf{g}^{R^e} \circ \mathbf{m}) \mathbf{m}, \\ \mathbf{f}_4 &= \mathbf{C}^e \mathbf{g}^{R^e}, & \mathbf{f}_5 &= \mathbf{C}^{e2} \mathbf{g}^{R^e}, & \mathbf{f}_6 &= (\mathbf{Y} \mathbf{C}^e - \mathbf{C}^e \mathbf{Y}) \mathbf{g}^{R^e},\end{aligned}\quad (10.285)$$

from (10.102) we can get symmetric tensor invariants of $\mathbf{C}^e$ as

$$\begin{aligned}\mathbf{H}_1 &= \mathbf{I}, & \mathbf{H}_2 &= \mathbf{C}^e, & \mathbf{H}_3 &= \mathbf{C}^{e2}, \\ \mathbf{H}_4 &= \mathbf{Y}, & \mathbf{H}_5 &= \mathbf{m} \otimes \mathbf{m}, & & \\ \mathbf{H}_6 &= \mathbf{Y} \mathbf{C}^e + \mathbf{C}^e \mathbf{Y}, & & & \mathbf{H}_7 &= \mathbf{m} \otimes (\mathbf{C}^e \mathbf{m}) + (\mathbf{C}^e \mathbf{m}) \otimes \mathbf{m},\end{aligned}\quad (10.286)$$

and from the appendix the skew-symmetric tensors invariants as

$$\begin{aligned}\mathbf{Z}_1 &= \mathbf{Y} \mathbf{C}^e - \mathbf{C}^e \mathbf{Y}, \\ \mathbf{Z}_2 &= \mathbf{Y} \mathbf{C}^{e2} - \mathbf{C}^{e2} \mathbf{Y}, \\ \mathbf{Z}_3 &= \mathbf{m} \otimes (\mathbf{C}^e \mathbf{m}) - (\mathbf{C}^e \mathbf{m}) \otimes \mathbf{m}, \\ \mathbf{Z}_4 &= (\mathbf{Y} \mathbf{C}^e \mathbf{m}) \otimes \mathbf{m} + (\mathbf{C}^e \mathbf{m}) \otimes (\mathbf{m} \mathbf{Y}) + \mathbf{m} \otimes (\mathbf{m} \mathbf{Y} \mathbf{C}^e) \\ &\quad - \mathbf{m} \otimes (\mathbf{m} \mathbf{C}^e \mathbf{Y}) - (\mathbf{C}^e \mathbf{Y} \mathbf{m}) \otimes \mathbf{m} - (\mathbf{Y} \mathbf{m}) \otimes (\mathbf{m} \mathbf{C}^e).\end{aligned}\quad (10.287)$$

From these we can construct the representation of the free energy as

$$\psi = \tilde{\psi}^{\dagger\dagger}(I_1, \dots, I_7, \theta), \quad (10.288)$$

which gives an expression for stress as

$$\begin{aligned}\mathbf{T}^T &= \rho \sum_{i=1}^7 \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT} \\ &= 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_1} \mathbf{B}^e + 4\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_2} \mathbf{B}^{e2} + 6\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_3} \mathbf{B}^{e3} \\ &\quad + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_4} \mathbf{F}^e \mathbf{Y} \mathbf{F}^{eT} + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_5} \mathbf{F}^e (\mathbf{C}^e \mathbf{Y} + \mathbf{Y} \mathbf{C}^e) \mathbf{F}^{eT} \\ &\quad + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_6} \mathbf{F}^e (\mathbf{m} \otimes \mathbf{m}) \mathbf{F}^{eT} + 2\rho \frac{\partial \tilde{\psi}^{\dagger\dagger}}{\partial I_7} \mathbf{F}^e [\mathbf{C}^e (\mathbf{m} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{m}) \mathbf{C}^e] \mathbf{F}^{eT}.\end{aligned}\quad (10.289)$$

The representation of the coefficient of thermal expansion can be written as

$$\boldsymbol{\alpha} = \sum_{i=1}^7 \alpha_{1i} \mathbf{H}_i + \sum_{i=1}^4 \alpha_{2i} \mathbf{Z}_i, \quad (10.290)$$

where α_i are scalar functions of temperature and the invariants I_1 – I_7 . If we assume thermal expansion is reversible, then we can write it as

$$\boldsymbol{\alpha} = \hat{\alpha}_o \mathbf{I} + \hat{\alpha}_1 \mathbf{Y} + \hat{\alpha}_2 \mathbf{m} \otimes \mathbf{m}, \quad (10.291)$$

where the scalar coefficients are functions of temperature. This can also be written as

$$\boldsymbol{\alpha} = \tilde{\alpha}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \tilde{\alpha}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \tilde{\alpha}_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (10.292)$$

Finally, the heat flux has a representation

$$\mathbf{q} = \mathbf{R}^e \sum_{i=1}^6 k_i \mathbf{f}_i, \quad (10.293)$$

where the coefficients k_i are scalar functions of temperature and the scalar invariants I_1 – I_{13} given above. Again, it should be noted that the heat flux model needs to be selected such that it satisfies the general condition $\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0$.

10.8 Specific free energy of quadratic form in strain

As a special example, let us consider a model constructed on a specific free energy that is quadratic in Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$. A general form of such a material model can be written as

$$\psi = \frac{1}{2\rho_o} \mathbf{E} : \mathbf{E} : \mathbf{E} : \mathbf{E} + \psi^\theta, \quad (10.294)$$

where $\mathbf{E}$ is a fourth-order tensor modulus and ψ^θ is a scalar, and both depend on temperature. Our goal is to calculate the expressions (10.117) and (10.118) for Cauchy stress and entropy for this model, and to explore the condition of material symmetry.

Let us first reorganize this quadratic free energy to take advantage of the fact that the Green strain is symmetric. In component form, this equation is given as

$$\begin{aligned} \psi &= \frac{1}{2\rho_o} (E_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{E}_{klmn} \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n) : (E_{op} \mathbf{e}_o \otimes \mathbf{e}_p) + \psi^\theta \\ &= \frac{1}{2\rho_o} E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{mo} g_{np} E_{op} + \psi^\theta \\ &= \frac{1}{2\rho_o} E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{mo} g_{np} E_{op} + \psi^\theta \\ &= \frac{1}{2\rho_o} E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{mo} g_{np} E_{op} + \psi^\theta \\ &= \frac{1}{2\rho_o} E_{ij} g_{ik} g_{jl} \mathbf{E}_{lknm} g_{mo} g_{np} E_{op} + \psi^\theta, \end{aligned} \quad (10.295)$$

where the last three relations follow from the symmetry of the Green strain. For example:

$$\begin{aligned}\mathbf{E} : \mathbf{E} : \mathbf{E} &= \mathbf{E} : \mathbf{E} : \mathbf{E}^T = (E_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{E}_{klmn} \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n) : (E_{op} \mathbf{e}_p \otimes \mathbf{e}_o) \\ &= E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{mp} g_{no} E_{op} = E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{np} g_{mo} E_{op} \\ &= E_{ij} g_{ik} g_{jl} \mathbf{E}_{klmn} g_{mo} g_{np} E_{op}.\end{aligned}\quad (10.296)$$

We, therefore, conclude that the expression for the specific free energy can be rewritten in terms of the new modulus $\bar{\mathbf{E}}$ in the form

$$\psi = \frac{1}{2\rho_o} \mathbf{E} : \bar{\mathbf{E}} : \mathbf{E} + \psi^\theta, \quad (10.297)$$

where

$$\bar{\mathbf{E}} = \frac{1}{4} (\mathbf{E}_{klmn} + \mathbf{E}_{lkmn} + \mathbf{E}_{klnm} + \mathbf{E}_{lknm}) \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \quad (10.298)$$

with the property that

$$\bar{\mathbf{E}}_{klmn} = \bar{\mathbf{E}}_{lkmn} = \bar{\mathbf{E}}_{klnm} = \bar{\mathbf{E}}_{lknm}. \quad (10.299)$$

The modulus tensor $\bar{\mathbf{E}}$ has the property that

$$\mathbf{A} : \bar{\mathbf{E}} : \mathbf{B} = \mathbf{A}^T : \bar{\mathbf{E}} : \mathbf{B} = \mathbf{A} : \bar{\mathbf{E}} : \mathbf{B}^T = \mathbf{A}^T : \bar{\mathbf{E}} : \mathbf{B}^T \quad (10.300)$$

for all second-order tensors $\mathbf{A}$ and $\mathbf{B}$. The proof of this is left as a homework problem.

The time derivative of the specific free energy is given by the expression

$$\dot{\psi} = \frac{1}{2\rho_o} \dot{\mathbf{E}} : \bar{\mathbf{E}} : \mathbf{E} + \frac{1}{2\rho_o} \mathbf{E} : \bar{\mathbf{E}} : \dot{\mathbf{E}} + \frac{1}{2\rho_o} \mathbf{E} : \dot{\bar{\mathbf{E}}} : \mathbf{E} + \dot{\psi}^\theta. \quad (10.301)$$

The rate of the Green strain is given by the expression

$$\dot{\mathbf{E}} = \frac{1}{2} (\dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}}) = \frac{1}{2} [(\mathbf{F}^T \dot{\mathbf{F}})^T + \mathbf{F}^T \dot{\mathbf{F}}], \quad (10.302)$$

so that

$$\begin{aligned}\dot{\mathbf{E}} : \bar{\mathbf{E}} : \mathbf{E} &= \frac{1}{2} [(\mathbf{F}^T \dot{\mathbf{F}})^T + \mathbf{F}^T \dot{\mathbf{F}}] : \bar{\mathbf{E}} : \mathbf{E} \\ &= \frac{1}{2} (\mathbf{F}^T \dot{\mathbf{F}})^T : \bar{\mathbf{E}} : \mathbf{E} + \frac{1}{2} (\mathbf{F}^T \dot{\mathbf{F}}) : \bar{\mathbf{E}} : \mathbf{E} \\ &= \frac{1}{2} (\mathbf{F}^T \dot{\mathbf{F}}) : \bar{\mathbf{E}} : \mathbf{E} + \frac{1}{2} (\mathbf{F}^T \dot{\mathbf{F}}) : \bar{\mathbf{E}} : \mathbf{E} \\ &= (\bar{\mathbf{E}} : \mathbf{E}) : (\mathbf{F}^T \dot{\mathbf{F}}) = \text{tr}[(\bar{\mathbf{E}} : \mathbf{E}) \dot{\mathbf{F}}^T \mathbf{F}] = \text{tr}[\mathbf{F}(\bar{\mathbf{E}} : \mathbf{E}) \dot{\mathbf{F}}^T] \\ &= [\mathbf{F}(\bar{\mathbf{E}} : \mathbf{E})] : \dot{\mathbf{F}}.\end{aligned}\quad (10.303)$$

In a similar way, it can be shown that

$$\mathbf{E} : \bar{\mathbf{E}} : \dot{\mathbf{E}} = [\mathbf{F}(\mathbf{E} : \bar{\mathbf{E}})] : \dot{\mathbf{F}}. \quad (10.304)$$

Therefore, the expression for the rate of change of the specific free energy can be written as

$$\dot{\psi} = \frac{1}{2\rho_o} [\mathbf{F}(\bar{\mathbf{E}} : \mathbf{E} + \mathbf{E} : \bar{\mathbf{E}})] : \dot{\mathbf{F}} + \left(\frac{1}{2\rho_o} \mathbf{E} : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E} + \frac{d\psi^\theta}{d\theta} \right) \dot{\theta}. \quad (10.305)$$

Examination of the terms yields the expressions

$$\partial_{\mathbf{F}}(\psi) = \frac{1}{2\rho_o}[\mathbf{F}(\bar{\mathbf{E}} : \mathbf{E} + \mathbf{E} : \bar{\mathbf{E}})], \quad \partial_{\theta}(\psi) = \frac{1}{2\rho_o}\mathbf{E} : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E} + \frac{d\psi^{\theta}}{d\theta}. \quad (10.306)$$

It now follows that the expressions for Cauchy stress and entropy are given by

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T = \frac{1}{2J} \mathbf{F}(\bar{\mathbf{E}} : \mathbf{E} + \mathbf{E} : \bar{\mathbf{E}}) \mathbf{F}^T, \quad (10.307)$$

$$\eta = -\partial_{\theta}(\psi) = -\frac{1}{2\rho_o} \mathbf{E} : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E} - \frac{d\psi^{\theta}}{d\theta}. \quad (10.308)$$

The expression for Cauchy stress can be simplified to obtain

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}(\tilde{\mathbf{E}} : \mathbf{E}) \mathbf{F}^T, \quad (10.309)$$

where $\tilde{\mathbf{E}}$ is a new material modulus defined by

$$\tilde{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{E}} + \bar{\mathbf{E}}^T), \quad (10.310)$$

so its components have the property

$$\tilde{E}_{ijkl} = \tilde{E}_{jikl} = \tilde{E}_{ijlk} = \tilde{E}_{klij}. \quad (10.311)$$

The Cauchy stress should be unaffected by the replacement of $\mathbf{F}$ by $\mathbf{FM}$ if $\mathbf{M}$ is material symmetry transformation, representing a reorganization of the reference configuration to a materially equivalent configuration. As can be shown, for the Green strain, the replacement of $\mathbf{F}$ by $\mathbf{FM}$ is equivalent to replacing $\mathbf{E}$ with $\mathbf{M}^T \mathbf{E} \mathbf{M}$. Since the modulus tensor is not a function of the deformation gradient, the constraint of material symmetry on the Cauchy stress can be written as

$$\frac{1}{J} \mathbf{F}(\tilde{\mathbf{E}} : \mathbf{E}) \mathbf{F}^T = \frac{1}{J} \mathbf{FM}[\tilde{\mathbf{E}} : (\mathbf{M}^T \mathbf{E} \mathbf{M})] \mathbf{M}^T \mathbf{F}^T. \quad (10.312)$$

Therefore, one must have

$$\tilde{\mathbf{E}} : \mathbf{E} = \mathbf{M}[\tilde{\mathbf{E}} : (\mathbf{M}^T \mathbf{E} \mathbf{M})] \mathbf{M}^T, \quad (10.313)$$

which must be satisfied for each $\mathbf{M}$ in the material symmetry group and all symmetric tensors $\mathbf{E}$. This can be written in component form, with respect to an orthogonal base of unit vectors, as

$$\tilde{E}_{ijkl} E_{kl} = M_{im} \tilde{E}_{mnop} M_{ko} E_{kl} M_{lp} M_{jn} = (M_{im} M_{jn} M_{ko} M_{lp} \tilde{E}_{mnop}) E_{kl}, \quad (10.314)$$

for every E_{kl} . This is satisfied by requiring that

$$\tilde{E}_{ijkl} = M_{im} M_{jn} M_{ko} M_{lp} \tilde{E}_{mnop}. \quad (10.315)$$

For example, this states that for an isotropic material $\tilde{\mathbf{E}}$ is a fourth-order isotropic tensor given by the general form, considering the properties of $\tilde{\mathbf{E}}$, of

$$\tilde{\mathbf{E}} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{II}, \quad (10.316)$$

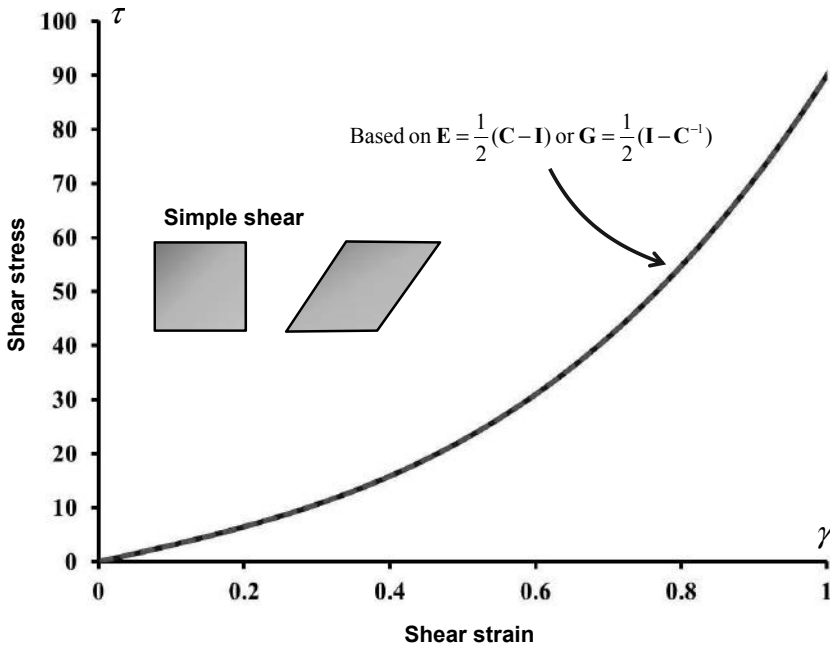


Figure 10.8: Shear response of an isotropic elastic material constructed from a free energy that is quadratic in Green strain or quadratic in the strain $\mathbf{G} = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$.

where $\mathbf{II}$ is the fourth-order tensor identity having the property $\mathbf{II} : \mathbf{A} = \mathbf{A}$ for every second-order tensor $\mathbf{A}$. The expression for the Cauchy stress for an isotropic material with quadratic specific free energy now can be written as

$$\begin{aligned}\mathbf{T}^T &= \frac{1}{J} \mathbf{F} [(\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{II}) : \mathbf{E}] \mathbf{F}^T \\ &= \frac{1}{J} \mathbf{F} [\lambda \text{tr}(\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}] \mathbf{F}^T,\end{aligned}\quad (10.317)$$

to give

$$\boxed{\mathbf{T}^T = \frac{1}{J} [\lambda \text{tr}(\mathbf{E}) \mathbf{B} + 2\mu \mathbf{F} \mathbf{E} \mathbf{F}^T]} \quad (10.318)$$

10.8.1 Models based on $\mathbf{E}^e$

We could assume that the free energy has a quadratic form in terms of the elastic strain $\mathbf{E}^e$ and write

$$\boxed{\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \mathbf{E} : \mathbf{E}^e + \psi^\theta}, \quad (10.319)$$

where $\mathbf{E}$ is a fourth-order tensor function of temperature as above with the same general characteristics, and ψ^θ is a function of temperature. To this we would also add the assumption for thermal expansion as $\mathbf{L}^\theta = \boldsymbol{\alpha} \dot{\theta}$ with an initial condition $\mathbf{F}^\theta = \mathbf{F}_o^\theta$. Following similar arguments as in the last section, we would get an expression for stress and entropy given

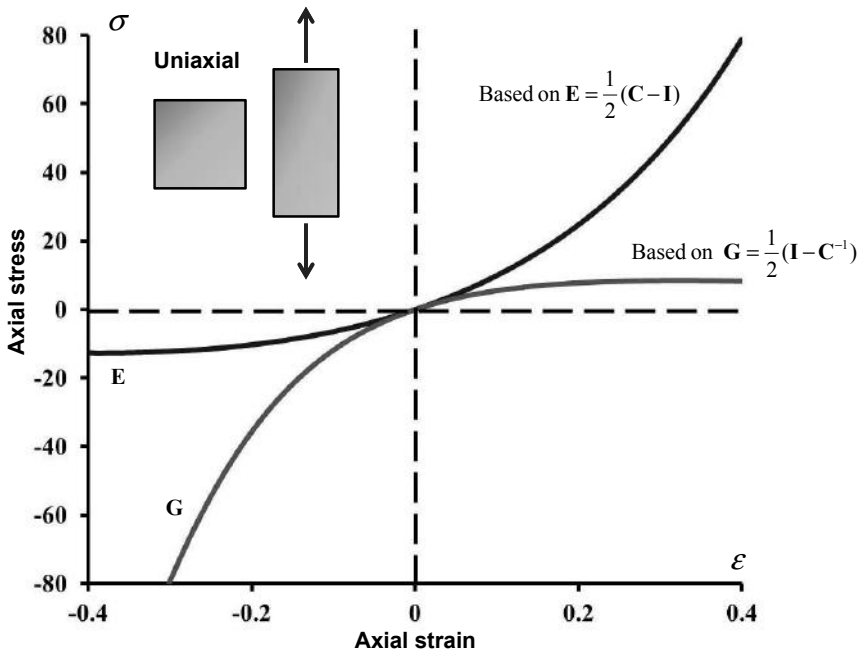


Figure 10.9: Uniaxial response of an isotropic elastic material constructed from a free energy that is quadratic in Green strain or quadratic in the strain $\mathbf{G} = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$.

by

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\tilde{\mathbf{E}} : \mathbf{E}^e) \mathbf{F}^{eT} \quad (10.320)$$

$$\eta = -\frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\tilde{\mathbf{E}}}{d\theta} : \mathbf{E}^e - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} (\mathbf{F}^{e-T} \mathbf{T}^T \mathbf{F}^{eT}) : \boldsymbol{\alpha}, \quad (10.321)$$

where, as above, $\tilde{\mathbf{E}}$ is a new material modulus defined by

$$\tilde{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{E}} + \bar{\mathbf{E}}^T). \quad (10.322)$$

The symmetry conditions will be similar resulting in similar representations. For example, an isotropic material will have an expression for Cauchy stress given by

$$\mathbf{T}^T = \frac{1}{J} [\lambda \text{tr}(\mathbf{E}^e) \mathbf{B}^e + 2\mu \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT}]. \quad (10.323)$$

Section 10.8-Problems:

10.8-1: Show that $\tilde{\mathbf{E}}$ has the property that

$$\mathbf{A} : \tilde{\mathbf{E}} : \mathbf{B} = \mathbf{A}^T : \tilde{\mathbf{E}} : \mathbf{B} = \mathbf{A} : \tilde{\mathbf{E}} : \mathbf{B}^T = \mathbf{A}^T : \tilde{\mathbf{E}} : \mathbf{B}^T, \quad (10.324)$$

for all second-order tensors $\mathbf{A}$ and $\mathbf{B}$.

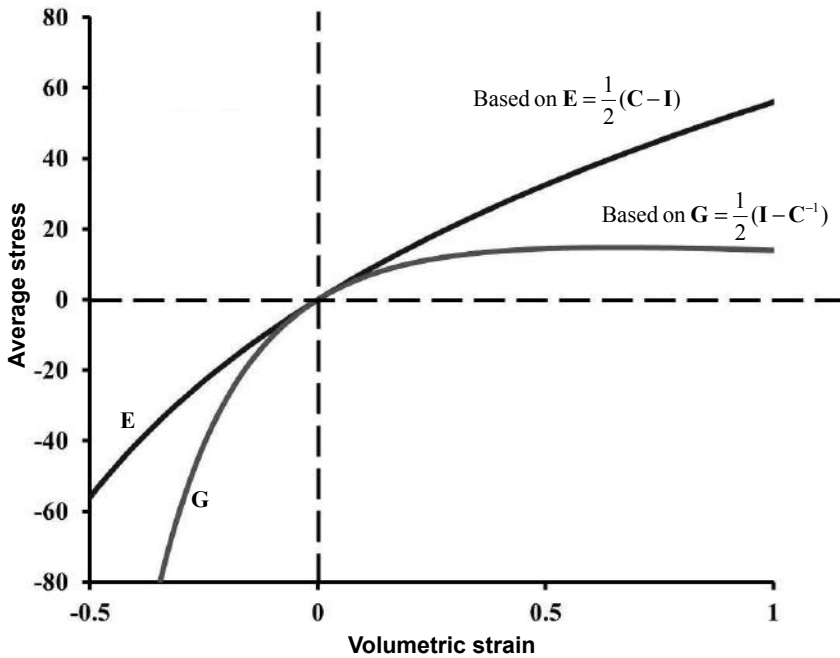


Figure 10.10: Volumetric response of an isotropic elastic material constructed from a free energy that is quadratic in Green strain or quadratic in the strain $\mathbf{G} = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$.

10.8-2: Show that the expression for Cauchy stress given by

$$\mathbf{T}^T = \frac{1}{2J} \mathbf{F}(\tilde{\mathbf{E}} : \mathbf{E} + \mathbf{E} : \tilde{\mathbf{E}}) \mathbf{F}^T$$

can be written as

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}(\tilde{\mathbf{E}} : \mathbf{E}) \mathbf{F}^T,$$

where $\tilde{\mathbf{E}}$ is a new material modulus defined by $\tilde{\mathbf{E}} = \frac{1}{2}(\tilde{\mathbf{E}} + \tilde{\mathbf{E}}^T)$. Show that the components of $\tilde{\mathbf{E}}$ have the property

$$\tilde{E}_{ijkl} = \tilde{E}_{jikl} = \tilde{E}_{ijlk} = \tilde{E}_{klij}.$$

10.8-3: Simulate the stress response of an isotropic thermoelastic material constructed using a free energy of quadratic form in Green strain as given in (10.318) and show that the response in simple shear, uniaxial extension, and pure dilatation are as given in Figures 10.8–10.10 for the material parameters $\kappa = 80$ and $\mu = 30$.

10.8-4: Construct a thermoelastic response model based on assuming that the free energy is quadratic in the strain $\mathbf{G} = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$. In constructing your model, start from the free energy

$$\psi = \frac{1}{2\rho_o} \mathbf{G} : \mathbf{E} : \mathbf{G} + \psi^\theta, \quad (10.325)$$

where ρ_o is the reference density, $\mathbf{E}$ is a fourth-order tensor of moduli only depending on temperature, and ψ^θ is a scalar function of temperature. Follow a similar procedure as developed in this section to show that the Cauchy stress is given by the expression

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^{-T} (\tilde{\mathbf{E}} : \mathbf{G}) \mathbf{F}^{-1}, \quad (10.326)$$

where $\tilde{\mathbf{E}}$ is a new material modulus defined by

$$\tilde{\mathbf{E}} = \frac{1}{2} (\bar{\mathbf{E}} + \bar{\mathbf{E}}^T). \quad (10.327)$$

Show that for an isotropic material this can be written as

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^{-T} [(\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{II}) : \mathbf{G}] \mathbf{F}^{-1}. \quad (10.328)$$

Use this model to simulate the mechanical response for this material to simple shear, uniaxial extension, and pure dilatation and show that the response is as given in Figures 10.8–10.10 for the material parameters $\kappa = 80$ and $\mu = 30$.

10.8.2 Infinitesimal strains

Here we will look at modeling for conditions characterized by small distortions of the body, either with or without large rigid body rotations. We can show that when the displacement gradient is small, such that we can assume $\|\mathbf{H}\| \ll 1$, the expression for Cauchy stress can be approximated by

$$\mathbf{T}^T \approx \tilde{\mathbf{E}} : \boldsymbol{\epsilon}, \quad (10.329)$$

where $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor given by $\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$. We normally write this as

$$\boxed{\boldsymbol{\sigma} = \tilde{\mathbf{E}} : \boldsymbol{\epsilon}}, \quad (10.330)$$

where we use the standard symbol for stress $\boldsymbol{\sigma}$ that normally is used when we do not need to distinguish between Cauchy stress and the nominal stress and the first Piola-Kirchhoff stress. For the case of an isotropic material, this can be written as

$$\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\epsilon}) \mathbf{I} + 2\mu \boldsymbol{\epsilon}. \quad (10.331)$$

When including temperature variations, it is convenient to separate the strain into the elastic and thermal parts. This becomes $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^\theta$. For the thermal part, we have an evolution equation $\dot{\boldsymbol{\epsilon}}^\theta = \boldsymbol{\alpha} \dot{\theta}$, where $\boldsymbol{\alpha}$ is assumed to be symmetric, and for the stress we have

$$\boxed{\boldsymbol{\sigma} = \tilde{\mathbf{E}} : \boldsymbol{\epsilon}^e}. \quad (10.332)$$

For the isotropic case, this gives us

$$\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2\mu \boldsymbol{\epsilon}^e. \quad (10.333)$$

When we have small deformations superimposed on large rigidbody rotation, the assumption is that $\|\mathbf{H}_U\| \ll 1$, where $\mathbf{U}$ is the right symmetric part in the polar decomposition of deformation gradient $\mathbf{F}$, and $\mathbf{H}_U = \mathbf{U} - \mathbf{I}$. In this case, we can approximate the expression for Cauchy stress by

$$\boxed{\mathbf{T}^T \approx \mathbf{R}(\tilde{\mathbf{E}} : \boldsymbol{\epsilon}_U) \mathbf{R}^T}, \quad (10.334)$$

where $\mathbf{R}$ is the orthogonal factor in the polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U}$ and

$$\epsilon_{\mathbf{U}} = \frac{1}{2}(\mathbf{H}_{\mathbf{U}} + \mathbf{H}_{\mathbf{U}}^T) = \mathbf{H}_{\mathbf{U}}. \quad (10.335)$$

If we separate the deformation into a thermal and a mechanical part, both of which are assumed to be small, then we can assume $\mathbf{R}^e = \mathbf{R}$ and take $\mathbf{U} = \mathbf{U}^e \mathbf{F}^\theta \approx \epsilon^e + \epsilon^\theta$ and obtain

$$\mathbf{T}^T \approx \mathbf{R}(\tilde{\mathbf{E}} : \epsilon^e) \mathbf{R}^T. \quad (10.336)$$

Section 10.8-Problems:

10.8-5: For infinitesimal strains, starting from the assumption $\|\mathbf{H}\| \ll 1$, show that for a material with quadratic specific free energy the expression for stress is given by

$$\mathbf{T}^T \approx \tilde{\mathbf{E}} : \epsilon.$$

10.9 Heat generation and heat capacity

We now will look at different thermodynamic processes and study the implications of having a thermoelastic material. From (6.78) we know that the rate of heat flow to a point is given by the equation

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta} - \frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}). \quad (10.337)$$

For the thermodynamically consistent thermoelastic model, we know that

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - \text{tr}(\mathbf{T}\mathbf{L}) = [\rho\partial_{\mathbf{F}}(\psi) - (\mathbf{F}^{-1}\mathbf{T})^T] : \dot{\mathbf{F}} + \rho[\partial_{\theta}(\psi) + \eta]\dot{\theta} = 0. \quad (10.338)$$

Therefore, the expression for the rate of the heat flowing into a point for a thermoelastic material can be written as

$$\dot{h} = \dot{\eta}\theta. \quad (10.339)$$

For a thermoelastic material, the equation for $\dot{\eta}$ can be written as

$$\dot{\eta} = \dot{\eta}_{\mathbf{L}} : \mathbf{L} + \dot{\eta}_{\dot{\theta}}\dot{\theta}, \quad (10.340)$$

where

$$\dot{\eta}_{\mathbf{L}} = -\frac{1}{\rho} \partial_{\theta}(\mathbf{T}^T), \quad (10.341)$$

$$\dot{\eta}_{\dot{\theta}} = -\partial_{\theta}(\eta). \quad (10.342)$$

The reader will note that the partial derivatives are defined through the assumption that $\eta^{\dagger}(\mathbf{X}, \mathbf{F}, \theta)$, consistent with the assumption that $\psi^{\dagger}(\mathbf{X}, \mathbf{F}, \theta)$. The expression for $\dot{h}$ is, therefore, given as

$$\dot{h} = \theta \dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta \dot{\eta}_{\dot{\theta}} \dot{\theta}. \quad (10.343)$$

Deformation can contribute to both the rising of temperature and the flowing of heat from the point. If we consider a process that is adiabatic (i.e., we thermally isolate the point), then $\dot{h}$ must be set to zero, and we should see the rising of the temperature with deformation. In the adiabatic case, we have

$$0 = \theta \dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta \dot{\eta}_{\dot{\theta}} \dot{\theta}.$$

Therefore, the rate of temperature rise due to deformation under adiabatic conditions is given by

$$\dot{\theta} = -\frac{1}{\dot{\eta}_{\dot{\theta}}} \dot{\eta}_{\mathbf{L}} : \mathbf{L}. \quad (10.344)$$

If, on the other hand, the process is happening under isothermal conditions, one can calculate the heat that needs to be added to the point. This is given by setting $\dot{\theta} = 0$ to get

$$\dot{h} = \theta \dot{\eta}_{\mathbf{L}} : \mathbf{L}. \quad (10.345)$$

Obviously, even for an elastic material we need to remove heat during deformation to get isothermal conditions if $\theta \dot{\eta}_{\mathbf{L}} \neq \mathbf{0}$.

The expression (6.89) can be used to calculate specific heat capacity c to get

$$c = \frac{\dot{h}}{\dot{\theta}} = \theta \frac{\dot{\eta}}{\dot{\theta}} = \theta \dot{\eta}_{\mathbf{L}} : \frac{1}{\dot{\theta}} \mathbf{L} + \theta \dot{\eta}_{\dot{\theta}}. \quad (10.346)$$

For this to be meaningful, one must provide additional conditions to specify the relation between $\mathbf{L}$ and $\dot{\theta}$. Obviously, the heat capacity under rigid body motions can be calculated by setting $\mathbf{L} = \dot{\mathbf{Q}}\mathbf{Q}^T$, where $\mathbf{Q}$ is orthogonal. Since $\dot{\eta}_{\mathbf{L}}$ inherits the symmetry of the stress, and since $\dot{\mathbf{Q}}\mathbf{Q}^T$ is skew symmetric, following the development preceding (6.90), the heat capacity under a rigid body process, denoted by c_{rb} , will be given by

$$c_{rb} = \theta \dot{\eta}_{\dot{\theta}} = -\theta \partial_{\theta}(\eta) = \theta \frac{\partial^2 \psi}{\partial \theta^2}. \quad (10.347)$$

10.10 Material constraints

As we have indicated, material constraints change the results for a thermodynamically consistent elastic model. Material constraints refer to internal restrictions in the material such as incompressibility, where the volume of the material cannot change, or inextensibility, where material lines along a particular direction cannot change their length. In general, material constraints are idealizations, reflecting the difficulty of initiating one mode of deformation, for example, relative to other modes. In general, at each material point, the material constraint is a relation between components of the loading $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$. For example, Figure 10.11 shows that the constraint of incompressibility allows deformations to occur but restrains them to be volume preserving. The equation describing incompressibility is

$$\det(\mathbf{F}) = \text{constant}, \quad (10.348)$$

since $J = \det(\mathbf{F})$ is the volume ratio relative to the volume in reference configuration. Obviously, the constant is unity if the reference configuration is one that the material actually takes, such as the initial configuration. Another example of a constraint is inextensibility shown in Figure 10.12. Consider, for example, that the reference configuration is the initial unloaded configuration and that the material along the unit vector $\mathbf{h}$ in the reference configuration is inextensible. If l is the current length of a line segment of length l_o along $\mathbf{h}$,

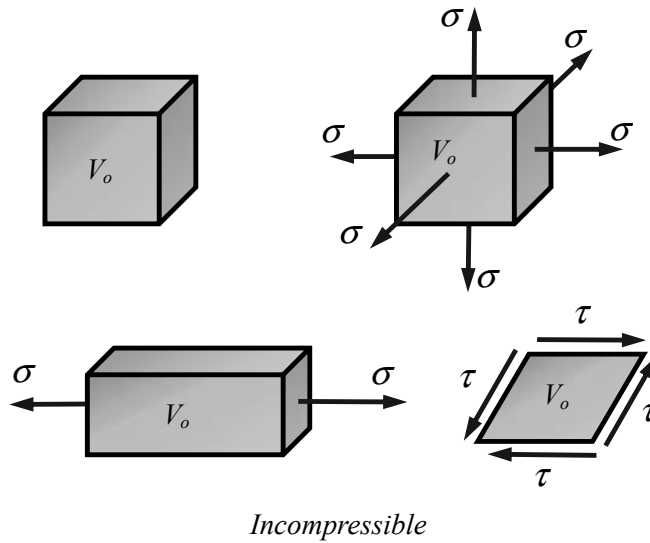


Figure 10.11: The volume of an incompressible material does not change with the application of load.

the constraint will be $l = l_o$, which also can be written as $l^2 = l_o^2$. We know from our study of kinematics of deformation that $l^2 = l_o^2 \mathbf{h} \circ (\mathbf{Ch})$, so the constraint can be written as

$$\mathbf{h} \circ (\mathbf{Ch}) = 1. \quad (10.349)$$

Letting C denote the constraint equation, the generic constraint equation can be written as

$$C(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0. \quad (10.350)$$

As can be seen, for each material point, C represents a relation between the deformation gradient, temperature, and temperature gradient. As we already have introduced the loading

$\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$, this constraint also can be written as

$$C(\mathbf{X}, \mathcal{L}) = 0.$$

Even though one can have several simultaneous constraint conditions, we will focus first on a single scalar constraint. If more than one constraint condition exists, they need to be compatible in the sense that satisfying one constraint will not exclude the possibility of satisfying the others.

The existence of a material constraint also changes the characteristics of the constitutive response functions. As shown in Figure 10.12, for example, the addition of load along the direction of inextensibility will not result in any changes in shape and, therefore, the deformation gradient, even though it will increase the traction. The stress, defined in terms of the traction, therefore can change without any change in the deformation. This, therefore, forces us to conclude that the stress is no longer fully determined by the knowledge of $(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G})$. For example, in the case of an inextensible direction in the material, to get the stress, in addition to $\mathcal{L}$, we need to know something about the loading (for example, the normal traction on the surface perpendicular to the direction of inextensibility). Since

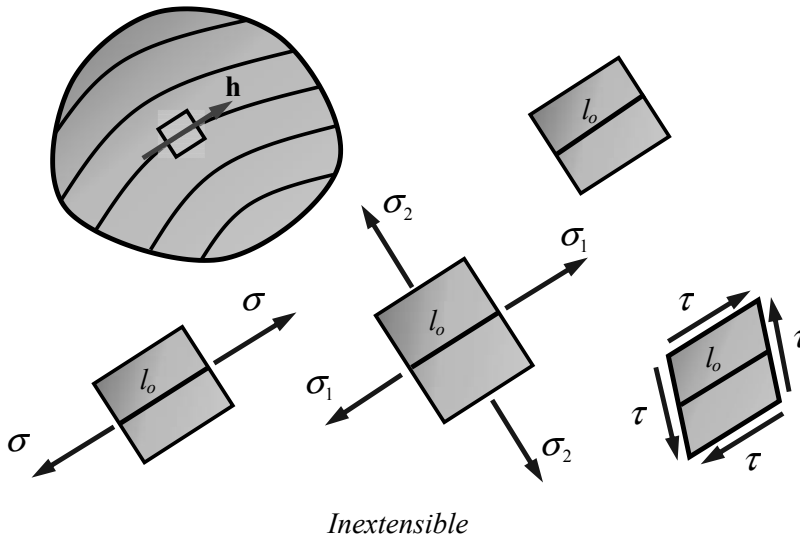


Figure 10.12: The length of line elements along the particular direction $\mathbf{h}$ of an inextensible material does not change with loading along the direction of inextensibility.

in such cases we cannot calculate the stress from just the knowledge of $(\mathbf{X}, \mathcal{L})$, we add an additional dependence to the constitutive functions to take care of this indeterminacy. Let p be a scalar that denotes the additional information needed to calculate the constitutive response functions. It will be assumed that all constitutive functions must depend on this additional variable. That is,

$$\begin{aligned} \psi &= \psi^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \mathbf{T} &= \mathbf{T}^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \eta &= \eta^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \mathbf{q} &= \mathbf{q}^\dagger(\mathbf{X}, \mathcal{L}, p). \end{aligned}$$

The constraint restricts how the components of $\mathcal{L}$ can change. The relation between the rates of change of these components with respect to time can be obtained by taking the time derivative of the constraint to obtain $\dot{\mathcal{C}} = 0$. Expanding this gives

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + \partial_{\theta}(C)\dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} = 0. \quad (10.351)$$

As can be seen, this is a scalar relation between the 13 components of $\dot{\mathcal{L}} = (\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}})$, and thus reduces the degrees of freedom from 13 to 12. As a result, we no longer arbitrarily can assign values to all 13 components of $\dot{\mathcal{L}}$. To simplify the presentation, we will write this equation as

$$\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}} = 0. \quad (10.352)$$

The constraint condition $\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}} = 0$ states that all admissible loading rates $\dot{\mathcal{L}}$ are “orthogonal” to $\partial_{\mathcal{L}}(C)$. That is, the projection of $\dot{\mathcal{L}}$ onto $\partial_{\mathcal{L}}(C)$ is zero. Therefore, from every arbitrary loading rate $\dot{\mathcal{L}}^*$, we can construct an admissible loading rate $\dot{\mathcal{L}}$ by removing

the portion of $\dot{\mathcal{L}}^*$ that provides a nonzero projection onto $\partial_{\mathcal{L}}(C)$. On the other hand, we can construct every arbitrary loading rate $\dot{\mathcal{L}}^*$ by adding to an admissible loading rate $\dot{\mathcal{L}}$ an appropriate loading rate along $\partial_{\mathcal{L}}(C)$. The latter can be written as

$$\dot{\mathcal{L}}^* = \dot{\mathcal{L}} + \alpha \partial_{\mathcal{L}}(C), \quad (10.353)$$

where α is a scalar factor, which may be changed as needed. Using this relation, one can construct an admissible loading rate $\dot{\mathcal{L}}$ from any arbitrary loading rate $\dot{\mathcal{L}}^*$ by selecting α such that $\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha \partial_{\mathcal{L}}(C)$ satisfies the constraint condition. To have $\dot{\mathcal{L}}$ satisfy the constraint condition $\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}} = 0$, we must, therefore, have

$$\partial_{\mathcal{L}}(C) \circ [\dot{\mathcal{L}}^* - \alpha \partial_{\mathcal{L}}(C)] = 0,$$

which results in an expression for α given by

$$\alpha = \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)}.$$

In conclusion, we will have

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C), \quad (10.354)$$

which yields the relations

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathbf{F}}(C), \quad (10.355)$$

$$\dot{\theta} = \dot{\theta}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\theta}(C), \quad (10.356)$$

$$\dot{\mathbf{G}} = \dot{\mathbf{G}}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathbf{G}}(C). \quad (10.357)$$

The material time derivative of the free energy is given by

$$\dot{\psi} = \partial_{\mathcal{L}}(\psi) \circ \dot{\mathcal{L}} + \partial_p(\psi) \dot{p}, \quad (10.358)$$

where $\dot{\mathcal{L}}$ is constrained to loading paths consistent with the constraint condition. One can write $\dot{\psi}$ in terms of an arbitrary loading rate $\dot{\mathcal{L}}^*$ using the above relation to obtain

$$\begin{aligned} \dot{\psi} &= \partial_{\mathcal{L}}(\psi) \circ [\dot{\mathcal{L}}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C)] + \partial_p(\psi) \dot{p} \\ &= [\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C)] \circ \dot{\mathcal{L}}^* + \partial_p(\psi) \dot{p}. \end{aligned}$$

The reader recalls the Clausius-Duhem inequality that was given in (6.69) as

$$\rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0,$$

which must hold for admissible $\mathcal{L}$ and $\dot{\mathcal{L}}$. Let us introduce into this expression the relations given above, recalling that $\text{tr}(\mathbf{T}\mathbf{L}) = (\mathbf{T}^T \mathbf{F}^{-T}) : \dot{\mathbf{F}}$, to obtain

$$\rho \left[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C) \right] \circ \dot{\mathcal{L}}^* + \rho \partial_p(\psi) \dot{p} - (\mathbf{T}^T \mathbf{F}^{-T}) : \left[\dot{\mathbf{F}}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathbf{F}}(C) \right]$$

$$+\rho\eta\left[\dot{\theta}^* - \frac{\partial_{\mathcal{L}}(C) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\theta}(C)\right] + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

Reorganization of the terms yields

$$\left\{ \rho \left[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C) \right] + \frac{(\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C) \right. \\ \left. - \rho\eta \frac{\partial_{\theta}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \partial_{\mathcal{L}}(C) \right\} \circ \dot{\mathcal{L}}^* - (\mathbf{T}^T \mathbf{F}^{-T}) : \dot{\mathbf{F}}^* + \rho\eta\dot{\theta}^* + \rho\partial_p(\psi)\dot{p} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0,$$

which must hold for every arbitrary $\dot{\mathcal{L}}^* = (\dot{\mathbf{F}}^*, \dot{\theta}^*, \dot{\mathbf{G}}^*)$, and any arbitrary $\dot{p}$. The reader will note that the system is linear in $\dot{\mathcal{L}}^*$ and $\dot{p}$, so that one can organize the equation into five terms, where the first term only contains $\dot{\mathbf{F}}^*$, the second term only contains $\dot{\theta}^*$, the third term only contains $\dot{\mathbf{G}}^*$, the fourth term only contains $\dot{p}$, and the fifth term is $(\mathbf{q} \circ \mathbf{g})/\theta$. The factor multiplying each rate is a function of $\mathbf{X}$, $\mathcal{L}$, and p , and independent of $\dot{\mathcal{L}}^*$ and $\dot{p}$. For the equation to hold for all values of $\dot{\mathcal{L}}^*$ and $\dot{p}$, the factors multiplying $\dot{\mathbf{F}}^*$, $\dot{\theta}^*$, $\dot{\mathbf{G}}^*$, and $\dot{p}$ each must be zero. The result of this process is the following five relations:

$$\rho\partial_{\mathbf{F}}(\psi) - \left[\frac{\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C) + \rho\eta\partial_{\theta}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \right] \partial_{\mathbf{F}}(C) - \mathbf{T}^T \mathbf{F}^{-T} = \mathbf{0},$$

$$\rho\partial_{\theta}(\psi) - \left[\frac{\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C) + \rho\eta\partial_{\theta}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \right] \partial_{\theta}(C) + \rho\eta = 0,$$

$$\rho\partial_{\mathbf{G}}(\psi) - \left[\frac{\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C) + \rho\eta\partial_{\theta}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)} \right] \partial_{\mathbf{G}}(C) = \mathbf{0},$$

$$\partial_p(\psi) = 0,$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

The fourth relation excludes the dependence of free energy on p , so that only the other constitutive functions may depend on it. Since we have not given any particular form or physical interpretation to p , other than it is the quantity needed to fully determine the values of the responses, let us take p to be

$$p = - \frac{\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C) + \rho\eta\partial_{\theta}(C)}{\partial_{\mathcal{L}}(C) \circ \partial_{\mathcal{L}}(C)}.$$

This will yield the following five relations

$$\boxed{\mathbf{T}^T = \rho\partial_{\mathbf{F}}(\psi)\mathbf{F}^T + p\partial_{\mathbf{F}}(C)\mathbf{F}^T}, \quad (10.359)$$

$$\boxed{\eta = -\partial_{\theta}(\psi) - \frac{p}{\rho}\partial_{\theta}(C)}, \quad (10.360)$$

$$\boxed{\partial_{\mathbf{G}}(\psi) = -\frac{p}{\rho}\partial_{\mathbf{G}}(C)}, \quad (10.361)$$

$$\boxed{\partial_p(\psi) = 0}, \quad (10.362)$$

$$\boxed{\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0}. \quad (10.363)$$

Also, one must add to this set the constraint condition

$$C(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0,$$

and note the fact that equation (10.362) states that *free energy does not depend on p* .

The relation of p to the physical constraint comes from the study of the response of the material. This is best illustrated by an example. For example, for the case of an absolutely incompressible material given by the constraint $C = \det(\mathbf{F}) - 1 = 0$, we have

$$\partial_{\mathbf{F}}(C) = \det(\mathbf{F})\mathbf{F}^{-T} = \mathbf{F}^{-T}, \quad \partial_{\theta}(C) = 0, \quad \partial_{\mathbf{G}}(C) = \mathbf{0}.$$

The response of such a material is given by

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T + p \mathbf{I},$$

$$\eta = -\partial_{\theta}(\psi),$$

$$\partial_{\mathbf{G}}(\psi) = \mathbf{0},$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

The first equation states that the Cauchy stress can be determined from the free energy only up to an unknown hydrostatic stress $p\mathbf{I}$. We physically interpret this to say that for an incompressible material the constraint makes it such that we can add any hydrostatic stress $p\mathbf{I}$ onto any state of stress without altering its shape. The second equation states that for this material the entropy is fully determined from the free energy, and is independent of p . The third equation states that the free energy is independent of the temperature gradient and, therefore, so are the Cauchy stress and entropy. The fourth equation has not changed and, therefore, has the same interpretation as before. For our absolutely incompressible material, the volume cannot change even with temperature. Therefore, this constraint does not accommodate for thermal expansion. One can construct an isothermally incompressible material that accommodates thermal expansion by making the volume change be fully determined by the temperature change, remaining constant if there is no temperature change.

Section 10.10-Problems:

10.10-1: Consider an isothermally incompressible material given by the condition

$$\det(\mathbf{F}) = h(\theta),$$

where $h(\theta)$ is a function of temperature giving the volume ratio at each temperature. An example of this function would be $h(\theta) = J_o + \beta(\theta - \theta_o)$, where J_o is the volume ratio at reference temperature θ_o , and β is the volumetric coefficient of thermal expansion.

1. For this constraint, write the general expressions for Cauchy stress and entropy.
2. For a free energy given by $\psi = A\theta(I_1 - 3)$, where $I_1 = \text{tr}(\mathbf{C})$, calculate the expressions for Cauchy stress and entropy.
3. For this model, relate the material constant A to the elastic and shear moduli.

10.10-2: Consider an isothermally inextensible material in which the material line element along the direction given by the unit vector $\mathbf{h}$ in the reference configuration at temperature θ_o has a length l_o , and the same line element in the current configuration at temperature θ has the length $l = l_o[1 + \alpha(\theta - \theta_o)]$, where α is the linear coefficient of thermal expansion.

1. Write the constraint equation.

2. For this constraint, calculate the general expressions for Cauchy stress and entropy.
3. For a free energy given by $\psi = A\theta(I_1 - 3)$, where $I_1 = \text{tr}(\mathbf{C})$, calculate the expressions for Cauchy stress and entropy.

10.11 Multiple material constraints

In the last section, we considered the existence of a single material constraint. We now will explore how to deal with multiple material constraints. Let us have n constraints, each given by a constraint equation of the form

$$\boxed{C_i(\mathbf{X}, \mathcal{L}) = 0}, \quad (10.364)$$

where i is an integer between 1 and n . For each constraint, we add an additional scalar variable p_i to the constitutive equations so that we have

$$\psi = \psi^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (10.365)$$

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (10.366)$$

$$\eta = \eta^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (10.367)$$

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n). \quad (10.368)$$

The differential form of the constraint equations can be written by setting each $\dot{C}_i = 0$ to obtain

$$\partial_{\mathcal{L}}(C_i) \circ \dot{\mathcal{L}} = 0. \quad (10.369)$$

As in the case of a single constraint, one can introduce an arbitrary loading rate $\dot{\mathcal{L}}^*$ that can be used to extract an admissible loading rate $\dot{\mathcal{L}}$ compatible with all constraints and is written as

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha_j \partial_{\mathcal{L}}(C_j). \quad (10.370)$$

The n scalar coefficients α_i are selected such that $\dot{\mathcal{L}}$ is forced to satisfy all the constraints. Substitution of this into the n constraints $\partial_{\mathcal{L}}(C_i) \circ \dot{\mathcal{L}} = 0$ on the loading rate results in the n equations

$$\partial_{\mathcal{L}}(C_i) \circ [\dot{\mathcal{L}}^* - \alpha_j \partial_{\mathcal{L}}(C_j)] = 0.$$

Reorganizing this results in the n equations

$$\partial_{\mathcal{L}}(C_i) \circ \partial_{\mathcal{L}}(C_j) \alpha_j = \partial_{\mathcal{L}}(C_i) \circ \dot{\mathcal{L}}^*.$$

Letting $\boxed{A_{ij} = \partial_{\mathcal{L}}(C_i) \circ \partial_{\mathcal{L}}(C_j)}$ and $\boxed{c_i = \partial_{\mathcal{L}}(C_i) \circ \dot{\mathcal{L}}^*}$, we obtain a linear system of n equations described by the equation

$$\boxed{A_{ij} \alpha_j = c_i}, \quad (10.371)$$

where A_{ij} are the components of a symmetric matrix of coefficients $[A]$. If $[A]$ is invertible, we can write

$$\alpha_i = A_{ij}^{-1} c_j.$$

The reader notes that the invertibility of the coefficient matrix $[A]$ is the key to the existence of α_i . If $[A]$ is not invertible, then there can only be nontrivial solutions if the loading is

such that all the c_j are zero. In general, there is always the possibility that the constraints are inconsistent, or that the constraints “lock” all or part of the loading $\mathcal{L}$ so that loading rate $\dot{\mathcal{L}}$ or part of it is forced to be zero. If the constraint conditions are totally incompatible, a loading $\mathcal{L}$ cannot be found to satisfy all the constraints simultaneously, and, therefore, further consideration of this set of constraints is impractical. Yet, the locking of part or all of the response might still result in a material of practical interest.

For the current discussion, let us assume that the constraints are consistent and $[A]$ is invertible so that we always can calculate the α_i . Under these conditions, we get

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha_i \partial_{\mathcal{L}}(C_i) = \dot{\mathcal{L}}^* - A_{ij}^{-1} c_j \partial_{\mathcal{L}}(C_i) = \dot{\mathcal{L}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathcal{L}}(C_i). \quad (10.372)$$

The specific rates for $\mathbf{F}$, θ , and $\mathbf{G}$ can be written as

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{F}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{F}}(C_i), \\ \dot{\theta} &= \dot{\theta}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\theta}(C_i), \\ \dot{\mathbf{G}} &= \dot{\mathbf{G}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{G}}(C_i). \end{aligned} \quad (10.373)$$

The material time derivative of the free energy under this condition is given by

$$\dot{\psi} = \partial_{\mathcal{L}}(\psi) \circ \dot{\mathcal{L}} + \partial_{p_i}(\psi) \dot{p}_i, \quad (10.374)$$

which can be reorganized as

$$\dot{\psi} = \{\partial_{\mathcal{L}}(\psi) - [\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i)] A_{ij}^{-1} \partial_{\mathcal{L}}(C_j)\} \circ \dot{\mathcal{L}}^* + \partial_{p_i}(\psi) \dot{p}_i. \quad (10.375)$$

Substitution of this and the above relations into the Clausius-Duhem inequality that was given in (6.69) as

$$\rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0,$$

will give

$$\begin{aligned} &\rho \{\partial_{\mathcal{L}}(\psi) - [\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i)] A_{ij}^{-1} \partial_{\mathcal{L}}(C_j)\} \circ \dot{\mathcal{L}}^* + \rho \partial_{p_i}(\psi) \dot{p}_i \\ &\quad - (\mathbf{T}^T \mathbf{F}^{-T}) : \{\dot{\mathbf{F}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{F}}(C_i)\} \\ &\quad + \rho \eta \{\dot{\theta}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(C_j) \circ \dot{\mathcal{L}}^*] \partial_{\theta}(C_i)\} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \end{aligned}$$

As can be seen, this equation is linear in $\dot{\mathcal{L}}^*$ and $\dot{p}_i$ so that we can organize it in the form of $14 + n$ terms, where the first $13 + n$ terms each only contain one of the $13 + n$ components of $\dot{\mathcal{L}}^*$ and $\dot{p}_i$, and the last term is $(\mathbf{q} \circ \mathbf{g})/\theta$. As before, the next thing to note is that the multiplying factor for each of the $13 + n$ rates is independent of rate, so that the only way for the equation to be true for all $\dot{\mathcal{L}}^*$ and $\dot{p}_i$ is if these multiplying factors are each equal to zero. The result of this process is the following equations

$$\rho \partial_{\mathbf{F}}(\psi) - \{\rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C_i) + \rho \eta \partial_{\theta}(C_i)\} A_{ij}^{-1} \partial_{\mathbf{F}}(C_j) - \mathbf{T}^T \mathbf{F}^{-T} = \mathbf{0},$$

$$\rho \partial_{\theta}(\psi) - \{\rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C_i) + \rho \eta \partial_{\theta}(C_i)\} A_{ij}^{-1} \partial_{\theta}(C_j) + \rho \eta = 0,$$

$$\rho \partial_{\mathbf{G}}(\psi) - \{\rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C_i) + \rho \eta \partial_{\theta}(C_i)\} A_{ij}^{-1} \partial_{\mathbf{G}}(C_j) = \mathbf{0},$$

$$\partial_{p_i}(\psi) = 0,$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

The fourth equation excludes the dependence of free energy on any of the n values of p_i . The form of each p_i at this point is still to be determined. We simply know that each p_i is a

variable needed to remove the indeterminacy introduced into the calculation of the response functions by one of the constraints. Without any loss of generality, we can take p_j to be

$$p_j = -\{\rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(C_i) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(C_i) + \rho \eta \partial_{\theta}(C_i)\} A_{ij}^{-1}. \quad (10.376)$$

Reorganization of the equations yields

$$\boxed{\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T + p_j \partial_{\mathbf{F}}(C_j) \mathbf{F}^T,} \quad (10.377)$$

$$\boxed{\eta = -\partial_{\theta}(\psi) - \frac{p_j}{\rho} \partial_{\theta}(C_j),} \quad (10.378)$$

$$\boxed{\partial_{\mathbf{G}}(\psi) = -\frac{p_j}{\rho} \partial_{\mathbf{G}}(C_j),} \quad (10.379)$$

$$\boxed{\partial_{p_i}(\psi) = 0,} \quad (10.380)$$

$$\boxed{\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.} \quad (10.381)$$

As in the case of a single constraint, the third and fourth equations can be taken as a constraint on the form of the free energy. In particular, *free energy can never depend on any of the p_i* . Also, if the constraint conditions do not depend on the temperature gradient $\mathbf{G}$, then the free energy cannot depend on temperature gradient. The Cauchy stress and entropy partially inherit such characteristics from the free energy through the first two equations, with the explicit dependence on p_i as given. The last equation restricts the possible directions of heat flux in relation to the direction of the temperature gradient. The reader is reminded that these equations still must be augmented by the n constraint equations

$$C_i(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0.$$

Section 10.11-Problems:

10.11-1: Consider a material that is simultaneously inextensible along the direction given by the unit vector $\mathbf{h}$ in the reference configuration and also incompressible.

1. Find the general expressions for Cauchy stress and entropy for this material.
2. Use the free energy $\psi = A\theta(I_1 - 3)$, where $I_1 = \text{tr}(\mathbf{C})$, and calculate the expressions for Cauchy stress and entropy.
3. Study the possible deformations in such a material.

10.11-2: Consider a material that is incompressible and simultaneously constrained by the restriction $\text{tr}(\mathbf{V}) = 3$ for $\mathbf{V}$ being the left symmetric factor in the polar decomposition of $\mathbf{F}$ which is written in terms of a reference configuration that is the same as the initial unloaded configuration. This later constraint is known as Bell's constraint.

1. Show that in the initial reference configuration these two constraints are compatible (i.e., both constraints can be satisfied simultaneously).
2. Using the relation

$$\partial_{\mathbf{F}}[\text{tr}(\mathbf{V})] = \mathbf{R},$$

show that these two constraints result in a singular matrix $[A]$.

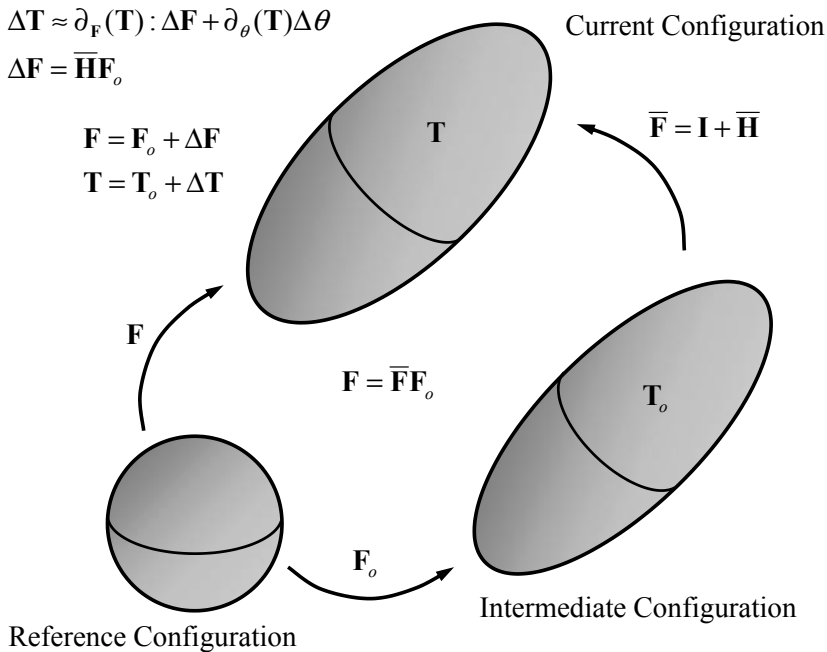


Figure 10.13: Superposition of two deformations on each other.

3. Show that for $\dot{\mathbf{F}}^*$ describing rigid body motions one obtains $c_i = 0$. For this case calculate the nontrivial solution for α_i and determine $\dot{\mathcal{L}}$. Discuss your result.

10.11-3: Construct expressions for the Cauchy stress and entropy for material simultaneously isothermally incompressible and isothermally inextensible.

10.11-4: Construct expressions for the Cauchy stress and entropy for a material simultaneously isothermally inextensible along two different directions.

10.12 Superposition of deformations

Many problems can be studied as the superposition of two deformations, frequently a small deformation superimposed on a large one. For example, in the study of wave motion in a prestressed body, we consider the wave as a perturbation superimposed on the deformation associated with the prestressing. Another example is seen when numerically solving the equations of nonlinear elasticity we frequently construct the solution through a sequence of linear approximations each superimposed on the previous total deformation. Evaluating the stability of a nonlinear deformation also frequently is done through studying the behavior of a superimposed small deformation. In any event, in most cases we superimpose a small

deformation on top of another so that a Taylor series expansion of some sort normally results in the desired equations.

Before we look at the expression for stress, we will consider the deformation gradient. Consider two deformations described by the deformation gradients $\mathbf{F}_o$ and $\mathbf{F}$. We can consider $\mathbf{F}$ to be constructed by superimposing a deformation with deformation gradient $\bar{\mathbf{F}}$ on the deformation described by the deformation gradient $\mathbf{F}_o$ so that we have

$$\boxed{\mathbf{F} = \bar{\mathbf{F}}\mathbf{F}_o.} \quad (10.382)$$

The difference between $\mathbf{F}$ and $\mathbf{F}_o$ will be denoted by $\Delta\mathbf{F}$ and is given by

$$\Delta\mathbf{F} = \mathbf{F} - \mathbf{F}_o = \bar{\mathbf{F}}\mathbf{F}_o - \mathbf{F}_o = (\bar{\mathbf{F}} - \mathbf{I})\mathbf{F}_o. \quad (10.383)$$

We always can write $\bar{\mathbf{F}} = \mathbf{I} + \bar{\mathbf{H}}$, where $\bar{\mathbf{H}}$ is the associated displacement gradient. We now can see that

$$\boxed{\Delta\mathbf{F} = \bar{\mathbf{H}}\mathbf{F}_o} \quad (10.384)$$

is the exact relation between the change in deformation gradient from $\mathbf{F}_o$ to $\mathbf{F}$ and for which the displacement gradient $\bar{\mathbf{H}}$ is associated with the superimposed deformation. As a result, if we consider $\mathbf{F}_o$ to be a fixed deformation on which we are imposing a variable deformation, we also can write

$$\boxed{\dot{\mathbf{F}} = \dot{\bar{\mathbf{H}}}\mathbf{F}_o,} \quad (10.385)$$

$$\boxed{\dot{\mathbf{F}} = \dot{\bar{\mathbf{H}}}\mathbf{F}_o.} \quad (10.386)$$

If the superimposed deformation results in small superimposed deformation gradients, then we can do the following. Let us assume that $(\mathbf{F}_o, \theta_o)$ describe, respectively, the values at time t_o of the deformation gradient and temperature, and let $\mathbf{T}_o$ be the associated value of the Cauchy stress and $\mathbf{T}^\dagger(\mathbf{F}, \theta)$ be the constitutive equation for calculating Cauchy stress. Note, that *in this section* $\mathbf{T}_o$ *DOES NOT* denote the nominal stress. The change from t_o to $t_o + \Delta t$ of the deformation gradient, temperature, and Cauchy stress are given through the expressions

$$\boxed{\mathbf{F} = \mathbf{F}_o + \Delta\mathbf{F},} \quad (10.387)$$

$$\boxed{\theta = \theta_o + \Delta\theta,} \quad (10.388)$$

$$\boxed{\mathbf{T} = \mathbf{T}_o + \Delta\mathbf{T}.} \quad (10.389)$$

Using a one-term Taylor series expansion of the deformation gradient, temperature, and Cauchy stress around the loading point $(\mathbf{F}_o, \theta_o)$, we can write these approximately

$$\mathbf{F}(t_o + \Delta t) \approx \mathbf{F}(t_o) + \dot{\mathbf{F}}(t)\Delta t, \quad (10.390)$$

$$\theta(t_o + \Delta t) \approx \theta(t_o) + \dot{\theta}(t)\Delta t, \quad (10.391)$$

$$\mathbf{T}(t_o + \Delta t) \approx \mathbf{T}(t_o) + \dot{\mathbf{T}}(t)\Delta t. \quad (10.392)$$

Even though we have used time to parameterize the change in deformation gradient, temperature, and stress, t can be any appropriate scalar parameter. In terms of the constitutive equation for Cauchy stress, the last equation can be written as

$$\mathbf{T}^\dagger[\mathbf{F}(t_o + \Delta t), \theta(t_o + \Delta t)] \approx \mathbf{T}^\dagger[\mathbf{F}(t_o), \theta(t_o)] + \partial_{\mathbf{F}}(\mathbf{T}) : \dot{\mathbf{F}}\Delta t + \partial_{\theta}(\mathbf{T})\dot{\theta}\Delta t. \quad (10.393)$$

This will be written as

$$\boxed{\mathbf{T} \approx \mathbf{T}_o + \partial_{\mathbf{F}}(\mathbf{T}) : \Delta\mathbf{F} + \partial_{\theta}(\mathbf{T})\Delta\theta.} \quad (10.394)$$

In this expression $\partial_{\mathbf{F}}(\mathbf{T})$ is the fourth-order tangent modulus associated with the Cauchy stress, representing the stiffness of the material subjected to a small superimposed deformation, when the material is held under isothermal conditions. The term $\partial_{\theta}(\mathbf{T})$ represents the sensitivity of the stress to changes in temperature, when holding the deformation gradient constant. The thermal expansion at constant stress can be evaluated from this equation by setting $\mathbf{T} = \mathbf{T}_o$. If we use $\mathbf{E}_{\mathbf{F}}$ to denote $\partial_{\mathbf{F}}(\mathbf{T})$ and $\mathbf{E}_{\theta}$ to denote $\partial_{\theta}(\mathbf{T})$, we can write this approximation by

$$\mathbf{T} \approx \mathbf{T}_o + \mathbf{E}_{\mathbf{F}} : \Delta \mathbf{F} + \mathbf{E}_{\theta} \Delta \theta. \quad (10.395)$$

A real, and not so academic, question arises when one asks what reference configuration is used for the measurement of the change in deformation gradient. In simple terms, we are using the initial undeformed configuration as the reference or the intermediate configuration on which we imposed a deformation. From above, we know that $\Delta \mathbf{F} = \bar{\mathbf{H}} \mathbf{F}_o$ so that

$$\mathbf{T} \approx \mathbf{T}_o + \mathbf{E}_{\mathbf{F}} : (\bar{\mathbf{H}} \mathbf{F}_o) + \mathbf{E}_{\theta} \Delta \theta. \quad (10.396)$$

This can be rewritten as

$$\boxed{\mathbf{T} \approx \mathbf{T}_o + (\mathbf{E}_{\mathbf{F}} \mathbf{F}_o^T) : \bar{\mathbf{H}} + \mathbf{E}_{\theta} \Delta \theta.} \quad (10.397)$$

As can be seen, the tangent modulus is scaled by the factor $\mathbf{F}_o^T$ when the expression is written in terms of $\bar{\mathbf{H}}$. In practice this means that there can be a large difference in the value of the moduli, depending on which deformation gradient is used in evaluating the stress. It turns out that when considering wave propagation in a preloaded material, the wave speeds are associated with the tangent moduli relative to the loaded state (i.e., we need to use $\mathbf{E}_{\mathbf{F}} \mathbf{F}_o^T$ as the tangent modulus).

Let us consider the following simple example. Let the stress be given by the constitutive relation

$$\mathbf{T} = \frac{1}{J} \mathbf{F} \left\{ \kappa \text{tr}(\mathbf{E}) \mathbf{I} + 2G \left[\mathbf{E} - \frac{\text{tr}(\mathbf{E})}{3} \mathbf{I} \right] \right\} \mathbf{F}^T, \quad (10.398)$$

where κ and G are, respectively, the bulk and shear moduli at small deformations, both depending on temperature. The time derivative of this stress can be obtained though using the following known relations:

$$\dot{J} = J \mathbf{F}^{-T} : \dot{\mathbf{F}}, \quad \text{tr}(\dot{\mathbf{E}}) = \mathbf{F} : \dot{\mathbf{F}}, \quad \dot{\mathbf{E}} = \frac{1}{2} (\dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}}). \quad (10.399)$$

Using these relations we can write

$$\begin{aligned} \dot{\mathbf{T}} = & -\frac{\dot{J}}{J} \mathbf{T} + \dot{\mathbf{F}} \mathbf{F}^{-1} \mathbf{T} + \mathbf{T} \mathbf{F}^{-T} \dot{\mathbf{F}}^T + \frac{1}{J} \mathbf{F} \left\{ \kappa (\mathbf{F} : \dot{\mathbf{F}}) \mathbf{I} + 2G \left[\frac{1}{2} (\dot{\mathbf{F}}^T \mathbf{F} + \mathbf{F}^T \dot{\mathbf{F}}) - \frac{\text{tr}(\mathbf{F} : \dot{\mathbf{F}})}{3} \mathbf{I} \right] \right\} \mathbf{F}^T \\ & + \frac{1}{J} \mathbf{F} \left\{ \frac{d\kappa}{d\theta} \dot{\theta} \text{tr}(\mathbf{E}) \mathbf{I} + 2 \frac{dG}{d\theta} \dot{\theta} \left[\mathbf{E} - \frac{\text{tr}(\mathbf{E})}{3} \mathbf{I} \right] \right\} \mathbf{F}^T. \end{aligned} \quad (10.400)$$

This can be put in the form $\mathbf{E}_{\mathbf{F}} : \dot{\mathbf{F}} + \mathbf{E}_{\theta} \dot{\theta}$ by using the identity

$$\mathbf{BAC} = (B_{ik} C_{lj} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : \mathbf{A}, \quad (10.401)$$

where $\mathbf{e}_i$ is any set of base vectors and B_{ik} and C_{lj} are, respectively, the components of $\mathbf{B}$ and $\mathbf{C}$ in this base (i.e., $\mathbf{B} = B_{ik} \mathbf{e}_i \otimes \mathbf{e}_k$ and $\mathbf{C} = C_{lj} \mathbf{e}_l \otimes \mathbf{e}_j$). Proof of this relation and the conversion of $\dot{\mathbf{T}}$ into the form $\dot{\mathbf{T}} = \mathbf{E}_{\mathbf{F}} : \dot{\mathbf{F}} + \mathbf{E}_{\theta} \dot{\theta}$ is left as a homework problem.

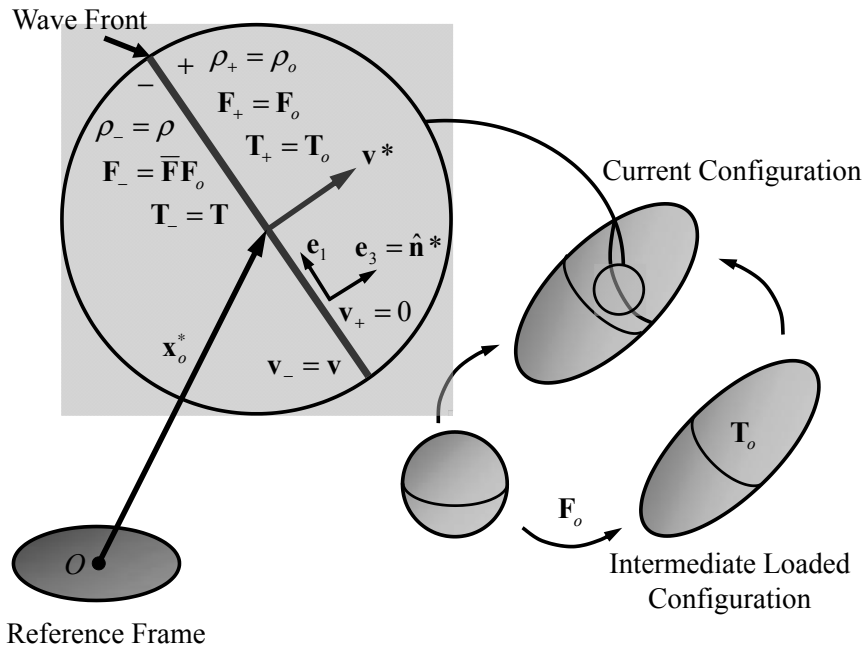


Figure 10.14: Motion of a discontinuity into a preloaded elastic body.

Section 10.12-Problems:

10.12-1: Prove the identity $\mathbf{BAC} = (B_{ik}C_{lj}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : \mathbf{A}$. Use this identity to obtain the relations

$$\mathbf{BA} = (B_{ik}g_{lj}^{-1}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : \mathbf{A}, \quad (10.402)$$

$$\mathbf{AC} = (g_{ik}^{-1}C_{lj}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : \mathbf{A}, \quad (10.403)$$

where $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ is the metric tensor for the selected base $\mathbf{e}_i$, and g_{ij}^{-1} is its inverse. Use these relations to convert equation (10.400) to the form $\dot{\mathbf{T}} = \mathbf{E}_F : \dot{\mathbf{F}} + \mathbf{E}_\theta \dot{\theta}$.

10.12.1 Wave propagation in preloaded bodies

We will focus on mechanical aspects of wave propagation in a preloaded body as shown in Figure 10.14. The equations that govern the motion of the wave front are given by the kinematic conditions, the jump conditions, and the constitutive equations. The focus will be on small compression and shear waves superimposed on an existing homogeneous stress, with the body at static equilibrium prior to introducing the wave.

As was introduced in Chapter 6, a subscript “+” will denote the value of a parameter just in front of the wave, and a subscript “-” will denote the parameter just behind the wave. Since in front of the wave the body is under load in static equilibrium, the velocity of particles will be zero so that $\mathbf{v}_+ = \mathbf{0}$. As above, we assume the material is under an initial

load described by the deformation gradient $\mathbf{F}_o$ and Cauchy stress $\mathbf{T}_o$ before the arrival of the wave. Therefore, we set $\mathbf{F}_+ = \mathbf{F}_o$ and $\mathbf{T}_+ = \mathbf{T}_o$. The wave results in a superimposed deformation given by the deformation gradient $\bar{\mathbf{F}}$ so that the total deformation gradient after the wave has passed is $\mathbf{F} = \bar{\mathbf{F}}\mathbf{F}_o$. To put the notation in a form consistent with above, behind the wave we will assume $\mathbf{v}_- = \mathbf{v}$, $\mathbf{F}_- = \mathbf{F}$ and $\mathbf{T}_- = \mathbf{T}$, with the understanding that $\mathbf{F} = \bar{\mathbf{F}}\mathbf{F}_o$ and $\Delta\mathbf{F} = \bar{\mathbf{H}}\mathbf{F}_o$.

To simplify the presentation, we will work in an orthonormal base $\mathbf{e}_i$ setup so that $\mathbf{e}_1$ and $\mathbf{e}_2$ are in the plane of the wave front and $\mathbf{e}_3$ is along $\hat{\mathbf{n}}^*$, which is normal to the surface of the discontinuity. The location of points in the initially loaded body will be denoted by $\mathbf{x}_o$ and the location of the wave front will be denoted by $\mathbf{x}_o^*$. The velocity of the wave front will be denoted by $\mathbf{v}^*$ and taken as $\mathbf{v}^* = v^*\hat{\mathbf{n}}^* = v^*\mathbf{e}_3$.

It will be assumed that the wave does not damage the integrity of the continuum so that there is no discontinuity in the displacement. Therefore, the superimposed displacement is continuous and at the wave front will be zero so that it matches the zero superimposed displacement in front of the wave. The superimposed displacement will be denoted by $\bar{\mathbf{u}}(\mathbf{x}_o, t)$ in terms of position $\mathbf{x}_o$ after the initial applied load, and time t . The condition of continuity at the wave front can be expressed as $\bar{\mathbf{u}}[\mathbf{x}_o^*(t), t] = \mathbf{0}$ for all time t . Taking the derivative of this condition with respect to time yields

$$\text{grad}_{\mathbf{x}_o}(\bar{\mathbf{u}})|_{\mathbf{x}_o=\mathbf{x}_o^*} \frac{d\mathbf{x}_o^*}{dt} + \frac{\partial \bar{\mathbf{u}}}{\partial t} = \mathbf{0}, \quad (10.404)$$

which can be written as

$$\bar{\mathbf{H}}\mathbf{v}^* + \mathbf{v} = \mathbf{0}, \quad (10.405)$$

where $\mathbf{v}$ and $\bar{\mathbf{H}}$ are, respectively, the particle velocity and the superimposed displacement gradient behind the wave. In component form, taking into consideration $\mathbf{v}^* = v^*\mathbf{e}_3$, this can be written as

$$\bar{H}_{i3}v^* + v_i = 0. \quad (10.406)$$

The conservation of mass on either side of the discontinuity is given by $\rho_- J_- = \rho_+ J_+ = \rho_i$, where ρ_i is the initial density before loading. In terms of the variables used in this section, this relation can be written as $\rho J = \rho_o J_o = \rho_i$. Introducing $J = \bar{J}J_o$ for $\bar{J} = \det(\bar{\mathbf{F}})$, we can write this as

$$\boxed{\rho\bar{J} = \rho_o}. \quad (10.407)$$

Across the discontinuity we must satisfy the jump condition $[[\rho(\mathbf{v} - \mathbf{v}^*)]] \circ \hat{\mathbf{n}}^* = 0$ associated with the conservation of mass. In terms of the notation used here, this equation can be written as

$$\rho(\mathbf{v} - \mathbf{v}^*) \circ \mathbf{e}_3 = -\rho_o \mathbf{v}^* \circ \mathbf{e}_3. \quad (10.408)$$

Introducing $\mathbf{v}^* = v^*\mathbf{e}_3$ results in

$$\boxed{\rho(v_3 - v^*) = -\rho_o v^*}. \quad (10.409)$$

By introducing (10.407) into this equation, we arrive at the equation

$$\boxed{\bar{J} = 1 - \frac{v_3}{v^*}}. \quad (10.410)$$

The jump condition for the balance of linear momentum in the absence of superimposed tractions on the jump surface is $[[\rho\mathbf{v} \otimes (\mathbf{v} - \mathbf{v}^*)]] \circ \hat{\mathbf{n}}^* = [[\mathbf{T}^T]]\hat{\mathbf{n}}^*$. This can be written in terms of the notation of this section as

$$\rho(v_3 - v^*)\mathbf{v} = \Delta\mathbf{T}^T\mathbf{e}_3, \quad (10.411)$$

where $\Delta \mathbf{T} = \mathbf{T} - \mathbf{T}_o$. We will assume a small superimposed deformation to characterize the difference between conditions in front of and behind the wave front so that we can write $\Delta \mathbf{T} = \bar{\mathbf{E}} : \bar{\mathbf{H}} + \mathbf{E}_\theta \Delta \theta$, where $\bar{\mathbf{E}} = \mathbf{E}_F \mathbf{F}_o^T$, and $\mathbf{E}_\theta$ and $\bar{\mathbf{E}}$ are evaluated at $(\mathbf{F}_o, \theta_o)$. In view of this assumption, the jump condition can be written as

$$\rho(v_3 - v^*)\mathbf{v} = \bar{\mathbf{H}}^T : \bar{\mathbf{E}}^T \mathbf{e}_3 + \mathbf{E}_\theta^T \mathbf{e}_3 \Delta \theta. \quad (10.412)$$

In component form this can be written as

$$\rho(v_3 - v^*)v_i = \bar{E}_{3ilk} \bar{H}_{lk} + E_{\theta,3i} \Delta \theta. \quad (10.413)$$

We now will constrain further the deformation to assume the superimposed displacement does not change with changes of position parallel to the wave surface. That is, we assume that the derivatives of the displacement with respect to the first and second components of $\mathbf{x}_o$ in the given coordinate $\mathbf{e}_i$ are zero so that $\bar{\mathbf{H}} = \gamma_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 + \gamma_{23} \mathbf{e}_2 \otimes \mathbf{e}_3 + \epsilon_{33} \mathbf{e}_3 \otimes \mathbf{e}_3$, or otherwise written as

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \gamma_{13} \\ 0 & 0 & \gamma_{23} \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}. \quad (10.414)$$

In view of the kinematic condition (10.406), we can write $\bar{\mathbf{H}}_{i3} = -v_i/v^*$ or otherwise giving

$$\begin{aligned} \gamma_{13} &= -\frac{v_1}{v^*}, & \gamma_{23} &= -\frac{v_2}{v^*}, & \epsilon_{33} &= -\frac{v_3}{v^*}. \end{aligned} \quad (10.415)$$

We also can see that $\bar{J} = \det(\bar{\mathbf{F}}) = \det(\mathbf{I} + \bar{\mathbf{H}}) = 1 + \epsilon_{33}$, which when compared to (10.410) also gives

$$\epsilon_{33} = -\frac{v_3}{v^*}, \quad (10.416)$$

which is consistent with the kinematic assumption resulting in (10.406). Since $\bar{H}_{i1} = \bar{H}_{i2} = 0$, equation (10.413) for the jump condition associated with the balance of linear momentum can be written as

$$\rho(v_3 - v^*)v_i = \bar{E}_{3il3} \bar{H}_{l3} + E_{\theta,3i} \Delta \theta. \quad (10.417)$$

In view of (10.406) and (10.409), this can be written as

$$\rho_o v^* v_i = \bar{E}_{3il3} \frac{v_l}{v^*} - E_{\theta,3i} \Delta \theta. \quad (10.418)$$

This can be reorganized into the following system

$$\begin{bmatrix} (\bar{E}_{3113} - \rho_o v^{*2}) & \bar{E}_{3123} & \bar{E}_{3133} \\ \bar{E}_{3213} & (\bar{E}_{3223} - \rho_o v^{*2}) & \bar{E}_{3233} \\ \bar{E}_{3313} & \bar{E}_{3323} & (\bar{E}_{3333} - \rho_o v^{*2}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} E_{\theta,31} v^* \Delta \theta \\ E_{\theta,32} v^* \Delta \theta \\ E_{\theta,33} v^* \Delta \theta \end{bmatrix}. \quad (10.419)$$

For an isothermal process, the system is homogeneous so that there will be a solution for the particle velocities only if the determinant of the coefficient matrix is zero. That is, we need to have $\det(\bar{E}_{3ij3} - \rho_o v^{*2} \delta_{ij}) = 0$, which represents a cubic equation in v^{*2} . The existence of a single wave speed v^* that accommodates the given deformation is contingent on the existence of a solution to this equation. In the event that the off-diagonal terms are zero (i.e., $\bar{E}_{3123} = \bar{E}_{3133} = \bar{E}_{3213} = \bar{E}_{3233} = \bar{E}_{3313} = \bar{E}_{3323} = 0$), then there will be three solutions given by

$$\begin{aligned} v^* &= \sqrt{\frac{\bar{E}_{3113}}{\rho_o}}, & v^* &= \sqrt{\frac{\bar{E}_{3223}}{\rho_o}}, & v^* &= \sqrt{\frac{\bar{E}_{3333}}{\rho_o}}. \end{aligned} \quad (10.420)$$

If the three moduli $\bar{E}_{3113}$, $\bar{E}_{3223}$, and $\bar{E}_{3333}$ are distinct, then depending on the wave speeds the solutions for the particle velocities will be

$$v^* = \sqrt{\frac{\bar{E}_{3113}}{\rho_o}} \rightarrow v_1 = -v^* \gamma_{13}, \quad v_2 = \gamma_{23} = 0, \quad v_3 = \epsilon_{33} = 0, \quad (10.421)$$

$$v^* = \sqrt{\frac{\bar{E}_{3223}}{\rho_o}} \rightarrow v_1 = \gamma_{13} = 0, \quad v_2 = -v^* \gamma_{23}, \quad v_3 = 0, \quad (10.422)$$

$$v^* = \sqrt{\frac{\bar{E}_{3333}}{\rho_o}} \rightarrow v_1 = \gamma_{13} = 0, \quad v_2 = \gamma_{23} = 0, \quad v_3 = -v^* \epsilon_{33}. \quad (10.423)$$

As can be seen, each wave speed is associated with a type of strain. The first two are associated with shear, respectively, in the 1-3 and 2-3 planes, and the third is associated with tension/compression along the 3-direction.

Section 10.12-Problems:

10.12-2: Consider the constitutive equation for Cauchy stress given as

$$\mathbf{T} = \kappa(J - 1)\mathbf{I} + 2G \left[\mathbf{B} - \frac{1}{3} \text{tr}(\mathbf{B})\mathbf{I} \right], \quad (10.424)$$

where κ and G are, respectively, the bulk and shear modulus at small strains, and both are only functions of temperature. Calculate $\mathbf{E}_{\mathbf{F}} = \partial_{\mathbf{F}}(\mathbf{T})$. From this calculate $\bar{\mathbf{E}} = \mathbf{E}_{\mathbf{F}} \mathbf{F}_o^T$, and use this to study the nature of wave propagation in a body that is subjected to an initial simple shear characterized by the shear strain γ_o .

10.13 Notation

The following table shows the symbols used in the text and their definitions.

B		left Cauchy stretch tensor ($\mathbf{F}\mathbf{F}^T$)
c		specific heat capacity
c_p		specific heat capacity at constant pressure
c_{rb}		specific heat capacity under rigid body processes
c		rigid body translation
C		right Cauchy stretch tensor ($\mathbf{F}^T\mathbf{F}$)
E		Green strain $[\frac{1}{2}(\mathbf{C} - \mathbf{I})]$
E		fourth-order tensor modulus
C		scalar constraint
$\mathbf{f}_i$		vector invariants
F		deformation gradient ($d\mathbf{x} = \mathbf{F}d\mathbf{X}$)
$\tilde{\mathbf{F}}$		deformation gradient between two reference configurations
g		temperature gradient with respect to current configuration $d\theta = \mathbf{g}d\mathbf{x}$
G		shear modulus
G		temperature gradient with respect to reference configuration

	$d\theta = \mathbf{G}d\mathbf{X}$ or $\frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$
$\mathcal{G}_{\kappa_o}$	material symmetry group represented with respect to κ_o
grad	gradient
$\dot{h}$	specific heat flow density into a point $[r - \frac{1}{\rho}div(\mathbf{q})]$
h	direction of constraint
H	displacement gradient ($d\mathbf{u} = \mathbf{H}d\mathbf{X}$)
$\mathbf{H}_i$	symmetric tensor invariants
$\mathbf{H}_U$	displacement gradient of $\mathbf{U}$
$\mathcal{H}$	history of $\mathcal{U}$
I_i	scalar invariants
I	identity
II	fourth-order tensor identity
J	volume ratio $[det(\mathbf{F})]$
J_i	scalar invariants
J_θ	volume ratio in isothermally incompressible material
K_i	scalar invariants
L	velocity gradient ($\dot{\mathbf{F}}\mathbf{F}^{-1}$)
$\mathcal{L}$	loading $\{\mathbf{F}, \theta, \mathbf{G}\}$, in elasticity it is the same as $\mathcal{U}$
m	axis of transverse isotropy
$\check{\mathbf{m}}$	Rm
M_c	average molecular weight between cross-linkings
M	transformation tensor representing material symmetry
$\mathbf{M}_\omega$	rotation tensor associated with representing material symmetry
p	indeterminate parameter associated with constraint f
q	heat flux vector
$\mathbf{q}^R$	rotated heat flux vector ($\mathbf{R}^T\mathbf{q}$)
Q	orthogonal tensor representing rigid body rotation
r	specific radiation or heat generation
R	gas constant
R	orthogonal factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathcal{R}$	response variables $\{\psi, \eta, \mathbf{T}, \mathbf{q}\}$
$\mathbf{R}_i$	reflection tensors associated with representing material symmetry
S	deviatoric stress
$\mathcal{S}$	state of a material point
t	time
$\mathbf{t}^{(n)}$	traction vector on surface with normal $\mathbf{n}$
T	Cauchy stress
$\tilde{\mathbf{T}}$	second Piola-Kirchhoff stress ($J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$)
$\mathbf{T}^R$	rotated stress ($\mathbf{R}^T\mathbf{T}\mathbf{R}$)
T_{ave}	average stress $[\frac{tr(\mathbf{T})}{3}]$
u	displacement from reference configuration ($\mathbf{x} = \mathbf{X} + \mathbf{u}$)
U	right symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathcal{U}$	variables $\{\mathbf{x}, \mathbf{F}, \theta, \mathbf{G}\}$ or $\{\mathbf{F}, \theta, \mathbf{G}\}$
v	velocity
$\mathbf{v}^*$	velocity of discontinuity
V	left symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
x	current position
X	reference position
Y	tensor base for representing symmetry ($\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3$)
$\check{\mathbf{Y}}$	RYR^T

ϵ		infinitesimal strain [$\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$]
η		specific entropy
γ		shear strain
κ		current configuration or bulk modulus
κ_o		reference configuration or initial bulk modulus
κ_o^*		alternate reference configuration
λ		stretch or Lamé constant
μ		shear modulus/Lamé constant
ν		Poisson's ratio
ψ		specific free energy
ψ^θ		purely thermal part of specific free energy
ρ		density ($\rho J = \rho_o$)
ρ_o		reference density
σ		stress
σ_{ave}		average stress
τ		shear stress
θ		temperature

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Chapter 11

Finite deformation mechanical theory of plasticity

In Chapter 7, we looked at plasticity at infinitesimal strains, restricting our attention to small deformations and small rigid body rotations. Some materials, such as the mild steel shown in Figure 11.1, undergo substantial deformations before failure, and in this process show substantial differences between the engineering and true stresses. When the material is undergoing large deformations, we need to use the kinematics developed for such motions and also distinguish between, for example, the nominal and Cauchy stresses and their associated tractions. In particular, in this chapter we will study plasticity at large deformations, including large rigid body motions. We will do this in the context of the multiplicative decomposition of the deformation gradient into an elastic part and a plastic part given by

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p. \quad (11.1)$$

As described before and shown in Figure 11.2, this decomposition introduces an intermediate configuration. This configuration is related to the reference configuration by the deformation gradient $\mathbf{F}^p$, which is the plastic deformation gradient. The current configuration is related to the intermediate configuration by the deformation gradient $\mathbf{F}^e$, which is the elastic deformation gradient. As the reader recalls, the intermediate configuration is not, in general, a continuous configuration of the body. Normally, to get the intermediate configuration, the current configuration of the neighborhood of each point is unloaded until the stress goes to zero. As was explained, the result may be a set of disjointed configurations that do not combine to construct a continuous configuration. Also, since each rigid body rotation of a stress-free configuration is itself stress free, the unloading process does not result in a unique intermediate configuration for the neighborhood. Given the elastic and plastic deformation gradients $\mathbf{F}^e$ and $\mathbf{F}^p$, respectively, it is easy to see that for any orthogonal transformation $\mathbf{M}$ the deformation gradients $\bar{\mathbf{F}}^p = \mathbf{M} \mathbf{F}^p$ and $\bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M}^T$ also satisfy the relation $\mathbf{F} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^p$ since

$$\mathbf{F} = \mathbf{F}^e \mathbf{M}^T \mathbf{M} \mathbf{F}^p = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^p.$$

A similar fact is true for a non orthogonal deformation gradient $\mathbf{M}$, but in this case $\bar{\mathbf{F}}^p = \mathbf{M} \mathbf{F}^p$ and $\bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M}^{-1}$, which also satisfies the relation $\mathbf{F} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^p$, yet $\mathbf{M}$ in this case does not represent a rigid body motion of the intermediate configuration. This flexibility in selecting the intermediate configuration may be used in many ways. One possibility is to rotate the unloaded intermediate configuration until the plastic deformation gradient becomes symmetric. Considering the polar decomposition of the plastic deformation gradient $\mathbf{F}^p$ given by $\mathbf{F}^p = \mathbf{R}^p \mathbf{U}^p = \mathbf{V}^p \mathbf{R}^p$, this assumption is consistent with

$$\mathbf{R}^p = \mathbf{I}, \quad \mathbf{F}^p = \mathbf{U}^p = \mathbf{V}^p.$$

Another possibility is to assume that the unloading is done without any rotation (i.e., by a pure deformation). If the polar decomposition of the elastic deformation gradient $\mathbf{F}^e$ is given by $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e$, this assumption is consistent with

$$\mathbf{R}^e = \mathbf{I}, \quad \mathbf{F}^e = \mathbf{U}^e = \mathbf{V}^e.$$

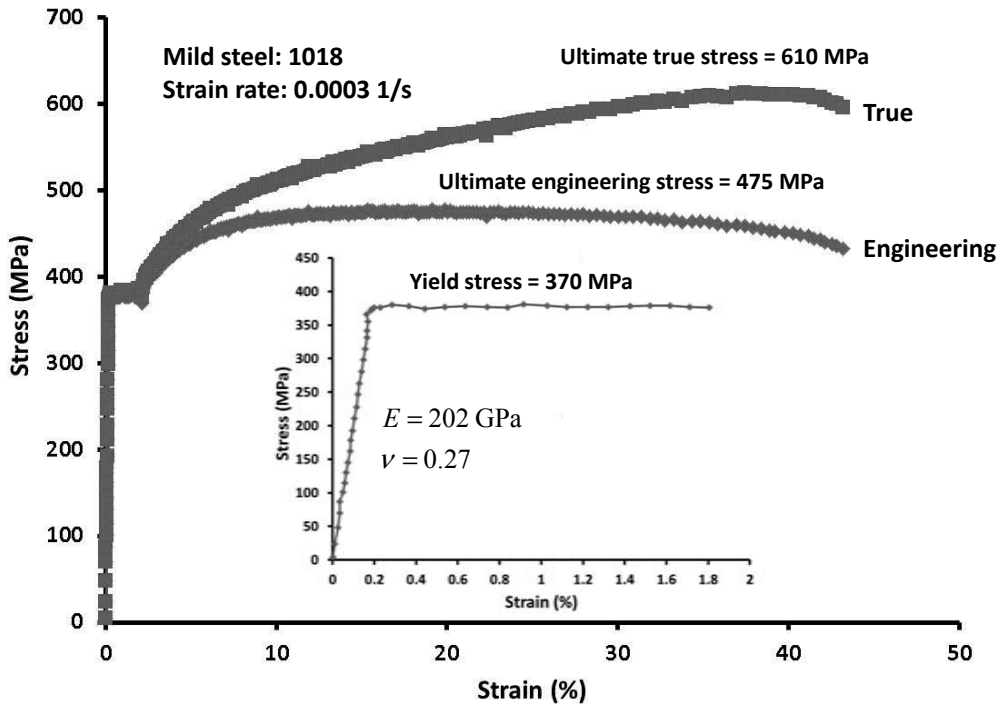


Figure 11.1: Monotonic extension of a mild steel.

Unfortunately, these two assumptions are not consistent with each other. That is, one may assume either the elastic deformation gradient to be symmetric or the plastic deformation gradient to be symmetric but not both. In what follows we avoid the case that assumes the elastic deformation gradient is symmetric since this forces the plastic deformation to change with rigid body motions of the body. We also will avoid forcing the plastic deformation gradient to be symmetric since that is also not required. Yet, we assume the plastic deformation gradient is calculated from the loading history and that the elastic deformation gradient is the tensor that is obtained from $\mathbf{F}^e = \mathbf{F}\mathbf{F}^p{}^{-1}$.

11.1 General mechanical theory of plasticity

Let us consider the components of a general mechanical theory of plasticity. In addition to the kinematic assumption that the deformation gradient $\mathbf{F}$ can be decomposed into elastic and plastic parts, to construct a theory of plasticity we need to specify how the stress is calculated and how the plastic deformation evolves with time. As we have described, the first assumption is that we can decompose $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, but this statement in itself does not provide the rule for how this can be done. We will assume that the plastic deformation gradient is calculated by an initial condition $\mathbf{F}_0^p$ on $\mathbf{F}^p$, and its change is calculated using a

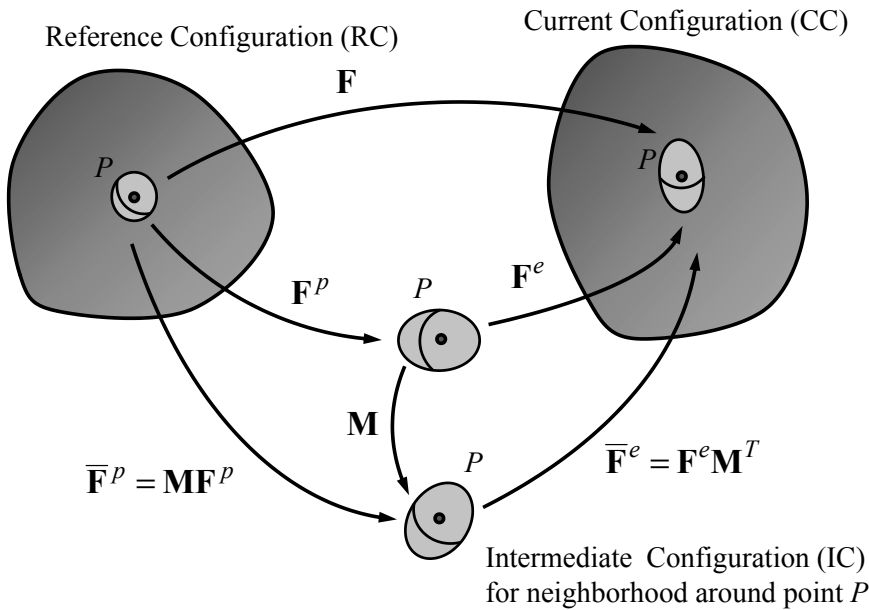


Figure 11.2: Decomposition of the deformation gradient $\mathbf{F}$ into a plastic deformation gradient $\mathbf{F}^p$ and an elastic deformation gradient $\mathbf{F}^e$.

flow rule of the form

$$\mathbf{L}^p \equiv \begin{cases} \mathbf{D} : \mathbf{L} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (11.2)$$

where $\mathbf{D}$ is a fourth-order tensor that is any function of the elastic and plastic deformation gradients, and a *hardening parameter* ξ . This allows us to calculate the future values of the plastic deformation gradient from the current conditions through integrating the rate equation $\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p$. The flow rule assumes knowledge of how to switch the plastic flow on and off. As before, this switch is constructed from a *yield function* f that always satisfies the condition $f \leq 0$, and with the assumption that plastic flow occurs on the yield surface, which is defined as $f = 0$. We will assume the yield function is given in terms of the elastic and plastic deformation gradients and the hardening parameter, so it is written as

$$f(\mathbf{F}^e, \mathbf{F}^p, \xi). \quad (11.3)$$

Since we assume that the plastic flow occurs on the yield surface, during plastic flow we force $f = 0$. This provides the *consistency condition* for plastic flow as

$$\dot{f} = 0. \quad (11.4)$$

Once at the yield surface $f = 0$, this condition forces the plastic flow to occur on the yield surface. We will have to consider this as a condition on the flow rule. As before, $\dot{f}$ is the time derivative of the yield function assuming the increment of loading has occurred without

plastic flow. The one condition that will result in the violation of $f \leq 0$ during an elastic process is when $f = 0$ and $\dot{f} > 0$, which is the switch selected to initiate plastic flow. We also need to follow the changes in the hardening parameter ξ . In a similar manner, we assume the hardening parameter is given by an initial condition ξ_o and a *hardening rule* of the form

$$\dot{\xi} \equiv \mathbf{A} : \mathbf{L}^p, \quad (11.5)$$

where $\mathbf{A}$ is a second-order tensor function of the elastic and plastic deformation gradients and the hardening parameter. We also would like to calculate the current stress, which we will assume is given by the current values of the elastic and plastic deformation gradients and the hardening parameter. We will write this by assuming the existence of a constitutive function $\mathbf{T}^\dagger$ for the Cauchy stress of the form

$$\mathbf{T} \equiv \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi). \quad (11.6)$$

In theory, once we know these constitutive functions, we can follow the stress response of a material point given the deformation history.

There are several common alterations to this structure. First, the yield function is frequently assumed to be a function of stress or an overstress. The case of the dependence on stress is simply a simplification of this model since stress is a function of the same variables as the yield function depends on. One may simply rewrite the yield function in this special case as

$$f(\mathbf{T}, \mathbf{F}^p, \xi), \quad (11.7)$$

replacing the dependence on the elastic deformation gradient with the stress. The case of dependence on the overstress is similar but requires that we first define a back-stress $\mathbf{T}^b$, which we assume is given by

$$\mathbf{T}^b \equiv \mathbf{T}^{b\dagger}(\mathbf{F}^p, \xi). \quad (11.8)$$

As will become clear in the next chapter, an appropriate overstress $\Delta\mathbf{T}$ is defined as

$$\Delta\mathbf{T} \equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b. \quad (11.9)$$

Again, one can construct a special case of the theory with a yield function that depends on this overstress to get

$$f(\Delta\mathbf{T}, \mathbf{F}^p, \xi). \quad (11.10)$$

Further simplifications might be to leave out the plastic deformation gradient or only use the deviatoric part of the stress. Removing the plastic deformation gradient would remove the possibility of distorting the yield surface with plastic flow. Using the deviatoric part of stress is frequently associated with materials for which their yield is not strongly influenced by the hydrostatic stress. Finally, one might select a different form of the hardening rule. The proposed form closely resembles the form that work hardening would result from.

11.1.1 Consistency condition

As we mentioned, the flow rule is constrained by the consistency condition, which forces the plastic flow to occur on a path that keeps the response on the yield surface. The condition is simple, $\dot{f} = 0$ if undergoing plastic flow, which is defined as any conditions that result in $f = 0$ and $\dot{f} > 0$. The manipulation of the flow rule to satisfy the consistency condition depends on the form of the yield function. For our general yield function, f depends on the elastic and plastic deformation gradients, and the hardening parameter so that

$$f(\mathbf{F}^e, \mathbf{F}^p, \xi).$$

In this case, its time derivative is

$$\dot{f} = \partial_{\mathbf{F}^e}(f) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^p}(f) : \dot{\mathbf{F}}^p + \partial_{\xi}(f)\dot{\xi}. \quad (11.11)$$

We know that $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ so that

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p, \quad (11.12)$$

which can be reorganized to give

$$\begin{aligned} \dot{\mathbf{F}}^e &= \dot{\mathbf{F}} \mathbf{F}^{p-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1} \\ &= \mathbf{L} \mathbf{F}^e - \mathbf{F}^e \mathbf{L}^p. \end{aligned} \quad (11.13)$$

Substituting this and the expression $\dot{\xi} = \mathbf{A} : \mathbf{L}^p$ into $\dot{f}$ results in

$$\dot{f} = \partial_{\mathbf{F}^e}(f) : (\mathbf{L} \mathbf{F}^e - \mathbf{F}^e \mathbf{L}^p) + \partial_{\mathbf{F}^p}(f) : \dot{\mathbf{F}}^p + \partial_{\xi}(f) \mathbf{A} : \mathbf{L}^p, \quad (11.14)$$

which can be reorganized to

$$\dot{f} = [\partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT}] : \mathbf{L} - [\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) - \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT} - \partial_{\xi}(f) \mathbf{A}] : \mathbf{L}^p. \quad (11.15)$$

Since the trial derivative is taken assuming no plastic flow, we have

$$\boxed{\dot{f} = [\partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT}] : \mathbf{L}.} \quad (11.16)$$

During plastic flow $\mathbf{L}^p = \mathbf{D} : \mathbf{L}$, so the consistency condition $\dot{f} = 0$ will become

$$[\partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT}] : \mathbf{L} - [\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) - \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT} - \partial_{\xi}(f) \mathbf{A}] : \mathbf{D} : \mathbf{L} = 0, \quad (11.17)$$

which can be written as

$$\boxed{\{\partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT} - [\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) - \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT} - \partial_{\xi}(f) \mathbf{A}] : \mathbf{D}\} : \mathbf{L} = 0.} \quad (11.18)$$

This must hold for all $\mathbf{L}$ that satisfy the condition $\dot{f} > 0$. The condition $\dot{f} > 0$ requires that we have

$$[\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) - \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT} - \partial_{\xi}(f) \mathbf{A}] : \mathbf{D} : \mathbf{L} > 0. \quad (11.19)$$

The terms in the square brackets are obtained from the derivatives of the yield function, which, having the insight from the thermodynamic theory, we will call associated stresslike terms defined by

$$\boxed{\mathbf{T}_f^{eT} \equiv \partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT},} \quad (11.20)$$

$$\boxed{\mathbf{T}_f^{bT} \equiv \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT},} \quad (11.21)$$

and an overstress associated with f defined as

$$\boxed{\Delta \mathbf{T}_f \equiv \mathbf{F}^{e-1} \mathbf{T}_f^e \mathbf{F}^e - \mathbf{T}_f^b - \partial_{\xi}(f) \mathbf{A}^T.} \quad (11.22)$$

This results in the *consistency condition* during plastic flow taking the expression

$$\boxed{(\mathbf{T}_f^{eT} - \Delta \mathbf{T}_f^T : \mathbf{D}) : \mathbf{L} = 0,} \quad (11.23)$$

and the trial derivative taking the expression

$$\boxed{\overset{\diamond}{f} = \mathbf{T}_f^{eT} : \mathbf{L}.} \quad (11.24)$$

For example, let us assume a flow rule of the Reuss-type

$$\mathbf{L}^p \equiv \begin{cases} \beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (11.25)$$

Since for the Reuss-type flow rule we have $\mathbf{D} : \mathbf{L} = \beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e$, the consistency condition can be written as

$$\mathbf{T}_f^{eT} : \mathbf{L} - \Delta \mathbf{T}_f^T : (\beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e) = 0, \quad (11.26)$$

to obtain the known result

$$\beta = \frac{\overset{\diamond}{f}}{\Delta \mathbf{T}_f^T : (\mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e)} \quad (11.27)$$

during plastic flow. Obviously, we need to know the form of the yield function before we can provide further details of β in the flow rule. For example, let us now assume a von Mises-type yield function given as

$$f = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}} - \tau_{yo} \quad (11.28)$$

for a constant yield stress τ_{yo} in shear. At yield we have $f = 0$ so that $\sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}} = \tau_{yo}$ and we get

$$\dot{f} = \frac{1}{2\tau_{yo}} \mathbf{S} : \dot{\mathbf{S}}. \quad (11.29)$$

Since $\mathbf{S} : \dot{\mathbf{S}} = \mathbf{S} : \dot{\mathbf{T}}$, we can write this for a general expression of stress as

$$\dot{f} = \frac{1}{2\tau_{yo}} \mathbf{S} : [\partial_{\mathbf{F}^e}(\mathbf{T}) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^p}(\mathbf{T}) : \dot{\mathbf{F}}^p + \partial_{\xi}(\mathbf{T}) \dot{\xi}], \quad (11.30)$$

giving the partial derivatives of the yield function as

$$\partial_{\mathbf{F}^e}(f) = \frac{1}{2\tau_{yo}} \mathbf{S} : \partial_{\mathbf{F}^e}(\mathbf{T}), \quad (11.31)$$

$$\partial_{\mathbf{F}^p}(f) = \frac{1}{2\tau_{yo}} \mathbf{S} : \partial_{\mathbf{F}^p}(\mathbf{T}), \quad (11.32)$$

$$\partial_{\xi}(f) = \frac{1}{2\tau_{yo}} \mathbf{S} : \partial_{\xi}(\mathbf{T}). \quad (11.33)$$

This results in

$$\mathbf{T}_f^{eT} = \frac{1}{2\tau_{yo}} [\mathbf{S} : \partial_{\mathbf{F}^e}(\mathbf{T})] \mathbf{F}^{eT}, \quad (11.34)$$

$$\mathbf{T}_f^{bT} = \frac{1}{2\tau_{yo}} [\mathbf{S} : \partial_{\mathbf{F}^p}(\mathbf{T})] \mathbf{F}^{pT}, \quad (11.35)$$

and

$$\Delta \mathbf{T}_f = \mathbf{F}^{e-1} \mathbf{T}_f^e \mathbf{F}^e - \mathbf{T}_f^b - \frac{1}{2\tau_{yo}} [\mathbf{S} : \partial_{\xi}(\mathbf{T})] \mathbf{A}^T. \quad (11.36)$$

Most simple models assume the stress only depends on the elastic deformation gradient and result in $\partial_{\mathbf{F}^p}(\mathbf{T}) = \mathbf{0}$ and $\partial_\xi(\mathbf{T}) = \mathbf{0}$, so that $\mathbf{T}_f^b = \mathbf{0}$ and $\Delta\mathbf{T}_f = \mathbf{F}^{e-1}\mathbf{T}_f^e\mathbf{F}^e$. This would give the expression for β as

$$\beta = \frac{\dot{f}}{(\mathbf{F}^{eT}\mathbf{T}_f^{eT}\mathbf{F}^{e-T}) : (\mathbf{F}^{e-1}\mathbf{S}\mathbf{F}^e)} = \frac{\dot{f}}{\mathbf{T}_f^{eT} : \mathbf{S}}. \quad (11.37)$$

As can be seen, to construct nonlinear mechanical theories of plasticity we need to provide constitutive expressions for the stress $\mathbf{T}$, the yield function f , the flow rule by providing $\mathbf{D}$, and the hardening rule by providing $\mathbf{A}$. Once suitable functions of $(\mathbf{F}^e, \mathbf{F}^p, \xi)$ are selected for these constitutive parameters, the mechanism provided embeds the selection to construct an appropriate flow rule with a switch to turn on and off plastic flow, and to keep it consistent with the consistency condition that $\dot{f} = 0$ during plastic flow.

Section 11.1-Problems:

11.1-1: Assume you are given the constitutive equation for the stress as

$$\mathbf{T} = \kappa^e \ln(J^e)\mathbf{I} + G^e \left[\mathbf{B}^e - \frac{\text{tr}(\mathbf{B}^e)}{3}\mathbf{I} \right], \quad (11.38)$$

where κ^e and G^e are constants representing, respectively, the initial elastic bulk and shear moduli. Construct a model for plastic flow with this expression for stress and assuming a Reuss-type flow rule and a von Mises-type yield function as described in the section.

11.1-2: Assume you are given the constitutive equation for the stress as

$$\mathbf{T} = \kappa^e \ln(J^e)\mathbf{I} + G^e \left[\mathbf{B}^e - \frac{\text{tr}(\mathbf{B}^e)}{3}\mathbf{I} \right], \quad (11.39)$$

and the back-stress as

$$\mathbf{T}^b = \kappa^b \ln(J^p)\mathbf{I} + G^b \left[\mathbf{C}^p - \frac{\text{tr}(\mathbf{C}^p)}{3}\mathbf{I} \right], \quad (11.40)$$

where κ^e and G^e are constants representing, respectively, the initial elastic bulk and shear moduli, and κ^b and G^b are constants representing, respectively, the initial back-stress bulk and shear moduli. Construct a model for plastic flow using these expressions for stress and back-stress, assuming a Reuss-type flow rule given by

$$\mathbf{L}^p \equiv \begin{cases} \beta \Delta \mathbf{S} & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (11.41)$$

and a von Mises-type of yield function given as

$$f = \sqrt{\frac{1}{2}\Delta\mathbf{S} : \Delta\mathbf{S}} - \tau_{yo} \quad (11.42)$$

for a constant initial yield stress τ_{yo} in shear. Show that the plastic flow rule does not induce change in the plastic volume ratio, and, thus, the hydrostatic back-stress will remain at zero even after substantial plastic flow.

11.2 Rigid body motions

The influence of rigid body motions on the stress is calculated through its influence on the traction. Normally, a rigid body motion is assumed to reorient the traction by the amount of the rigid body rotation. As shown in Figure 9.3, if the body is rotated by a rigid body rotation given by the orthogonal transformation $\mathbf{Q}$ such that the current deformation gradient changes from $\mathbf{F}$ to

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \quad (11.43)$$

then any normal $\hat{\mathbf{n}}$ transforms to $\hat{\mathbf{n}}^* = \mathbf{Q}\hat{\mathbf{n}}$ and the traction on the surface with the normal $\hat{\mathbf{n}}$, given by $\mathbf{t}^{(n)}$, changes to $\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}$. This can be shown to require that the Cauchy stress $\mathbf{T}$ changes to $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. Consider the given constitutive model for Cauchy stress

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi),$$

that now is required to satisfy the condition $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. It will have to have the characteristic that

$$\mathbf{T}^\dagger(\mathbf{F}^{e*}, \mathbf{F}^{p*}, \xi^*) = \mathbf{Q}\mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi)\mathbf{Q}^T \quad (11.44)$$

for every orthogonal history of $\mathbf{Q}$. To impose this restriction, we need to know how rigid body motions affect $\mathbf{F}^e$, $\mathbf{F}^p$, and ξ . Since $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$, we have

$$\mathbf{F}^* = \mathbf{F}^{e*}\mathbf{F}^{p*} = \mathbf{Q}\mathbf{F}^e\mathbf{F}^p.$$

Let us first consider the plastic deformation gradient. It seems rational to take the plastic deformation gradient to be unaffected by rigid body rotations since this is consistent with our idea that plastic deformation does not change with pure elastic deformation. Therefore, we will assume that

$$\mathbf{F}^{p*} = \mathbf{F}^p. \quad (11.45)$$

This then requires that

$$\mathbf{F}^{e*} = \mathbf{Q}\mathbf{F}^e, \quad (11.46)$$

so that the relation $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ is always satisfied. In the same spirit, we expect no hardening to occur due to rigid body motions, so we take

$$\xi^* = \xi. \quad (11.47)$$

Once imposed on the constitutive model for Cauchy stress, these assumptions result in

$$\mathbf{T}^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{Q}\mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi)\mathbf{Q}^T. \quad (11.48)$$

Selection of $\mathbf{Q} = \mathbf{R}^{eT}$ and reorganization yield

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{R}^e\mathbf{T}^\dagger(\mathbf{U}^e, \mathbf{F}^p, \xi)\mathbf{R}^{eT}. \quad (11.49)$$

By defining a rotated stress $\mathbf{T}^{R^e}$ for the rotation $\mathbf{R}^e$ through the relation

$$\mathbf{T} = \mathbf{R}^e\mathbf{T}^{R^e}\mathbf{R}^{eT}, \quad (11.50)$$

it becomes clear that we only need to provide a model for this rotated stress. Since there is a one-to-one relation between $\mathbf{U}^e$, $\mathbf{C}^e$, and $\mathbf{E}^e$, one can create models with the arguments

$$\begin{aligned} \mathbf{T}^{R^e} &= \mathbf{T}^{R^e\dagger}(\mathbf{U}^e, \mathbf{F}^p, \xi) \\ &= \mathbf{T}^{R^e\dagger\dagger}(\mathbf{C}^e, \mathbf{F}^p, \xi) \\ &= \mathbf{T}^{R^e\dagger\dagger\dagger}(\mathbf{E}^e, \mathbf{F}^p, \xi). \end{aligned} \quad (11.51)$$

The yield function f , the back-stress $\mathbf{T}^b$, the plastic velocity gradient $\mathbf{L}^p$, and the hardening parameter rate $\dot{\xi}$ remain unaffected by rigid body motions. This requires that

$$f(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^p, \xi) = f(\mathbf{F}^e, \mathbf{F}^p, \xi), \quad (11.52)$$

$$\mathbf{A}^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{A}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi) \quad (11.53)$$

for every orthogonal $\mathbf{Q}$, so that we can conclude that f and $\mathbf{A}$ can be rewritten in terms of the variables $(\mathbf{U}^e, \mathbf{F}^p, \xi)$, $(\mathbf{C}^e, \mathbf{F}^p, \xi)$, or $(\mathbf{E}^e, \mathbf{F}^p, \xi)$. The back-stress automatically satisfies this condition since neither of its arguments are affected by rigid body rotations. For the condition on $\mathbf{D}$, we need to recall that $\mathbf{L}^* = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{L}\mathbf{Q}^T$ so that the condition $\mathbf{L}^{p*} = \mathbf{L}^p$ requires that

$$\mathbf{D}^* : [\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{L}\mathbf{Q}^T] = \mathbf{D} : \mathbf{L} \quad (11.54)$$

for every history of $\mathbf{Q}$. It is easy to show from $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ that $\dot{\mathbf{Q}}\mathbf{Q}^T$ is skew-symmetric. Since this condition must hold for every history, we can take histories that have the current value of $\mathbf{Q} = \mathbf{I}$ and arbitrary skew-symmetric values of $\dot{\mathbf{Q}}$. Substituting this back into the equation requires that

$$\mathbf{D} : \dot{\mathbf{Q}} = \mathbf{0} \quad (11.55)$$

for every skew tensor $\dot{\mathbf{Q}}$. This suggests that we should write

$$\mathbf{D} = \mathbf{D}_1 \otimes \mathbf{D}_2, \quad (11.56)$$

or as a sum of such terms, where $\mathbf{D}_2$ is symmetric. The remaining conditions can be satisfied by taking

$$\mathbf{D}_1^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{D}_1^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi), \quad (11.57)$$

$$\mathbf{D}_2^\dagger(\mathbf{Q}\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{Q}\mathbf{D}_2^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi)\mathbf{Q}^T. \quad (11.58)$$

The condition on $\mathbf{D}_1$ allows us to rewrite its function in terms of the variables $(\mathbf{U}^e, \mathbf{F}^p, \xi)$, $(\mathbf{C}^e, \mathbf{F}^p, \xi)$, or $(\mathbf{E}^e, \mathbf{F}^p, \xi)$. The condition on $\mathbf{D}_2$ is similar to that on the Cauchy stress, which allows us to write it in the form

$$\mathbf{D}_2 = \mathbf{R}^e \mathbf{D}_2^{R^e} \mathbf{R}^{eT}, \quad (11.59)$$

where $\mathbf{D}_2^{R^e}$ is symmetric and can be written in terms of the variables $(\mathbf{U}^e, \mathbf{F}^p, \xi)$, $(\mathbf{C}^e, \mathbf{F}^p, \xi)$, or $(\mathbf{E}^e, \mathbf{F}^p, \xi)$. Since $\mathbf{D}_2$ is symmetric, if desired, one can use the property $\mathbf{D}_2 : \mathbf{L} = \mathbf{D}_2 : \mathbf{D}$ and write the flow rule as

$$\mathbf{L}^p = \begin{cases} (\mathbf{D}_2 : \mathbf{D})\mathbf{D}_1 & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (11.60)$$

where $\mathbf{D}$ is the deformation rate. The separation of $\mathbf{D}$ into a single term of the form $\mathbf{D}_1 \otimes \mathbf{D}_2$ actually allows us to eliminate $\mathbf{D}_2$ from the flow rule, and consequently from the model, by using the consistency condition. This results in the flow rule

$$\mathbf{L}^p = \begin{cases} \frac{\mathbf{T}_f^{eT} : \mathbf{L}}{\Delta \mathbf{T}_f^T : \mathbf{D}_1} \mathbf{D}_1 & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (11.61)$$

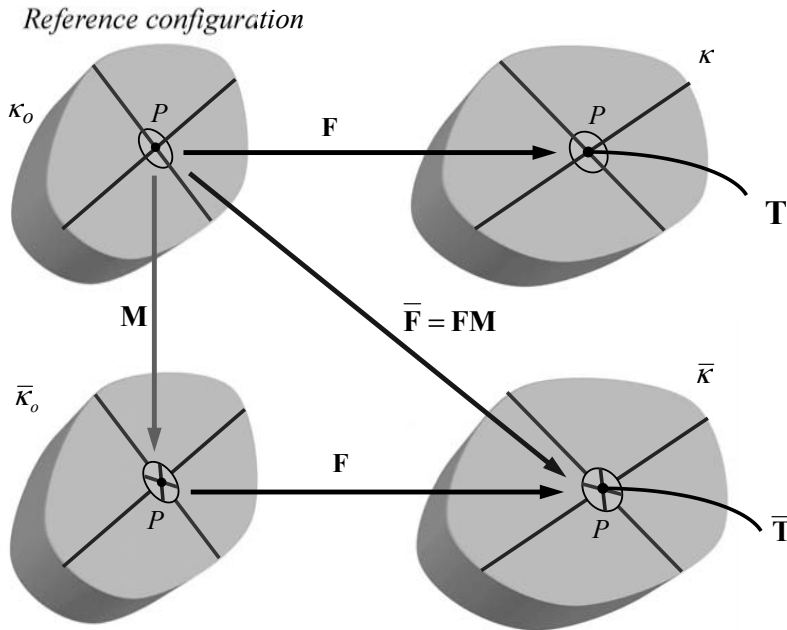


Figure 11.3: A reorganization of the reference configuration associated with the material's symmetry leaves the new material indistinguishable from the original material so that the response to any history would result in identical stress for the original and reorganized configurations.

which now *automatically satisfies the consistency condition*.

As was mentioned, one might also decide to separate $\mathbf{D}$ into a sum of terms, so that

$$\mathbf{D} = \sum_{i=1}^n \mathbf{D}_{i1} \otimes \mathbf{D}_{i2}, \quad (11.62)$$

where each $\mathbf{D}_{i2}$ is symmetric. The conditions on $\mathbf{D}_{i1}$ and $\mathbf{D}_{i2}$ are identical to $\mathbf{D}_1$ and $\mathbf{D}_2$.

11.3 Material symmetry

Material symmetry requires that the stress be the same for any two histories that are identical up to reorganizations of the reference configuration in ways that reflect symmetries of the material. This idea is shown schematically in Figure 9.4, where an alternate reference configuration is constructed using a symmetry of the material. This new configuration is supposed to be identical in response to the original one. Figure 11.3 shows this idea again, but with transformations shown. The two configurations κ_o and $\bar{\kappa}_o$ are related through the material symmetry transformation $\mathbf{M}$, so they should be “identical.” That is, if on each one of these two configurations we impose the same deformation history, then the resulting responses should be the same. As can be seen in the figure, imposing the same deformation $\mathbf{F}$ on both configurations κ_o and $\bar{\kappa}_o$ is the same as imposing the two deformation gradients

$\mathbf{F}$ and $\bar{\mathbf{F}}$ on κ_o . It is easy to see that one can write

$$\boxed{\bar{\mathbf{F}} = \mathbf{F}\mathbf{M}.} \quad (11.63)$$

If the deformation gradient $\mathbf{F}$ is decomposed into elastic and plastic parts given by $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$, and the deformation gradient $\bar{\mathbf{F}}$ is decomposed into elastic and plastic parts given by $\bar{\mathbf{F}} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^p$, then we will have

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^p = \mathbf{F}\mathbf{M} = \mathbf{F}^e \mathbf{F}^p \mathbf{M} = \mathbf{F}^e \mathbf{M} \mathbf{M}^{-1} \mathbf{F}^p \mathbf{M}.$$

As a rule, we will assume that

$$\boxed{\bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M}, \quad \bar{\mathbf{F}}^p = \mathbf{M}^{-1} \mathbf{F}^p \mathbf{M}.} \quad (11.64)$$

Consistent with this, we will have $\bar{\mathbf{L}}^p = \mathbf{M}^{-1} \mathbf{L}^p \mathbf{M}$ and $\bar{\mathbf{L}} = \mathbf{L}$. This requires that

$$\boxed{\bar{\mathbf{D}} : \mathbf{L} = \mathbf{M}^{-1} (\mathbf{D} : \mathbf{L}) \mathbf{M}.} \quad (11.65)$$

This suggests again that we should take

$$\mathbf{D} = \mathbf{D}_1 \otimes \mathbf{D}_2,$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are second-order tensor functions that satisfy the conditions

$$\boxed{\bar{\mathbf{D}}_1 = \mathbf{M}^{-1} \mathbf{D}_1 \mathbf{M},} \quad (11.66)$$

$$\boxed{\bar{\mathbf{D}}_2 = \mathbf{D}_2.} \quad (11.67)$$

As a scalar function, it will be assumed that the yield function will remain unchanged with a material symmetry transformation so that we have $\bar{f} = f$. We assume the hardening parameter is also unchanged by a symmetry transformation so that $\bar{\mathbf{A}} : \bar{\mathbf{L}}^p = \mathbf{A} : \mathbf{L}^p$. To satisfy this, we take

$$\boxed{\bar{\mathbf{A}} = \mathbf{M}^T \mathbf{A} \mathbf{M}^{-T}.} \quad (11.68)$$

The back-stress, due to its inability to change with rigid body motions, was put with the term $\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e$ in the expression for the overstress. This suggests that it should follow the symmetry condition

$$\boxed{\bar{\mathbf{T}}^b = \mathbf{M}^{-1} \mathbf{T}^b \mathbf{M}.} \quad (11.69)$$

The reader concerned with this and other selections is referred to the thermodynamic theory in which some of these expressions appear naturally.

We will concentrate on *symmetries described by orthogonal transformations* and, as such, use

$$\bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M}, \quad \bar{\mathbf{F}}^p = \mathbf{M}^T \mathbf{F}^p \mathbf{M}. \quad (11.70)$$

The assumption that the stress remains the same under such changes can be written as $\bar{\mathbf{T}} = \mathbf{T}$ and imposes the following condition on the constitutive model for Cauchy stress written as

$$\boxed{\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi) = \mathbf{T}^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi).} \quad (11.71)$$

The same condition written in terms of the rotated stress results in the restriction

$$\mathbf{T}^{Re\dagger}(\mathbf{U}^e, \mathbf{F}^p, \xi) = \mathbf{M} \mathbf{T}^{Re\dagger}(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) \mathbf{M}^T. \quad (11.72)$$

In a similar manner, we can write

$$\mathbf{D}_1^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{D}_1^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi) \mathbf{M}, \quad (11.73)$$

$$\mathbf{D}_2^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{D}_2^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi), \quad (11.74)$$

$$\mathbf{A}^\dagger(\mathbf{F}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{A}^\dagger(\mathbf{F}^e, \mathbf{F}^p, \xi) \mathbf{M}, \quad (11.75)$$

$$f(\mathbf{F}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = f(\mathbf{F}^e, \mathbf{F}^p, \xi), \quad (11.76)$$

$$\mathbf{T}^{b\dagger}(\mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{T}^{b\dagger}(\mathbf{F}^p, \xi) \mathbf{M}. \quad (11.77)$$

For the first four, we can combine the conditions for material symmetry and that imposed by rigid body motions to get

$$\mathbf{D}_1^\dagger(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{D}_1^\dagger(\mathbf{U}^e, \mathbf{F}^p, \xi) \mathbf{M}, \quad (11.78)$$

$$\mathbf{D}_2^{R^e \dagger}(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{D}_2^{R^e \dagger}(\mathbf{U}^e, \mathbf{F}^p, \xi) \mathbf{M}, \quad (11.79)$$

$$\mathbf{A}^\dagger(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = \mathbf{M}^T \mathbf{A}^\dagger(\mathbf{U}^e, \mathbf{F}^p, \xi) \mathbf{M}, \quad (11.80)$$

$$f(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = f(\mathbf{U}^e, \mathbf{F}^p, \xi). \quad (11.81)$$

As can be seen, the rotated stress $\mathbf{T}^{R^e}$ and the material functions for $\mathbf{D}_1$, $\mathbf{D}_2^{R^e}$, and $\mathbf{A}$ are all second-order tensors that have the same material symmetry restriction on them and, as a result, have similar general representations, even though two are symmetric and two are general.

For the special case where $f(\Delta \mathbf{T}, \mathbf{F}^p, \xi)$, the condition of material symmetry can be written as

$$f(\mathbf{M}^T \Delta \mathbf{T} \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \xi) = f(\Delta \mathbf{T}, \mathbf{F}^p, \xi). \quad (11.82)$$

Table 11.1 describes some of the common material symmetry groups. The transformations noted in this table are given in terms of the orthonormal base $\mathbf{e}_i$ as

$$\begin{aligned} \mathbf{I} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_1 &= -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_2 &= \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{R}_3 &= \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \mathbf{M}_\omega &= \cos \omega (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin \omega (\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3. \end{aligned} \quad (11.83)$$

As can be seen, $\mathbf{I}$ is the identity, $\mathbf{R}_1$ is a reflection about the $\mathbf{e}_2 - \mathbf{e}_3$ plane, $\mathbf{R}_2$ is a reflection about the $\mathbf{e}_3 - \mathbf{e}_1$ plane, $\mathbf{R}_3$ is a reflection about the $\mathbf{e}_1 - \mathbf{e}_2$ plane, and $\mathbf{M}_\omega$ is a rotation of ω around the $\mathbf{e}_3$ axis. The representation of symmetry is directly related to the selection of the reference configuration. As a result, even though the symmetry of a material may be described with respect to one reference configuration in one of the forms given in Table 11.1, the same material symmetry's description with respect to another reference configuration might look very different. We can represent the different material functions using representations developed for the different symmetries, many of which are provided in the appendices.

Table 11.1: Common symmetry groups.

Symmetry	Transformations constructing symmetry group
Isotropic	All orthogonal transformations ($\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M} = \mathbf{I}$)
Triclinic	$\pm\mathbf{I}$
Monoclinic	$\pm\mathbf{I}, \pm\mathbf{R}_1$
Orthotropic	$\pm\mathbf{I}, \pm\mathbf{R}_1, \pm\mathbf{R}_2, \pm\mathbf{R}_3$
Transversely isotropic	$\pm\mathbf{I}, \pm\mathbf{R}_1, \pm\mathbf{M}_\omega$ for all ω

11.4 Stress depending only on elastic deformation gradient

In this section, we will concentrate on the case of an inhomogeneous body where the Cauchy stress is modeled by a function of position in the reference configuration and of the elastic deformation gradient. Therefore, the general model we will consider can be written as

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathbf{F}^e). \quad (11.84)$$

As before, the inclusion of $\mathbf{X}$ is to allow characterization of an inhomogeneous material body. To simplify the presentation, from this point on we will omit $\mathbf{X}$ from the argument list of $\mathbf{T}^\dagger$, even though this dependence is implied throughout. Therefore, we will write our model as

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e).$$

As was shown above, to capture the proper dependence on rigid body motions we know that

$$\mathbf{T} = \mathbf{R}^e \mathbf{T}^{\mathbf{R}^e} \mathbf{R}^{eT}, \quad (11.85)$$

where the rotated stress $\mathbf{T}^{\mathbf{R}^e}$ is given by the model

$$\mathbf{T}^{\mathbf{R}^e} = \mathbf{T}^{\mathbf{R}^e\dagger}(\mathbf{U}^e). \quad (11.86)$$

For each transformation $\mathbf{M}$ representing an initial symmetry of the material, we know we must have the relation

$$\bar{\mathbf{F}}^e = \bar{\mathbf{R}}^e \bar{\mathbf{U}}^e = \mathbf{F}^e \mathbf{M} = \mathbf{R}^e \mathbf{U}^e \mathbf{M} = \mathbf{R}^e \mathbf{M} \mathbf{M}^T \mathbf{U}^e \mathbf{M}.$$

Since the polar decomposition is unique, and since $\mathbf{R}^e \mathbf{M}$ is orthogonal and $\mathbf{M}^T \mathbf{U}^e \mathbf{M}$ is symmetric positive definite, we must conclude that

$$\bar{\mathbf{R}}^e = \mathbf{R}^e \mathbf{M}, \quad \bar{\mathbf{U}}^e = \mathbf{M}^T \mathbf{U}^e \mathbf{M}.$$

Since material symmetry requires that $\mathbf{T} = \bar{\mathbf{T}}$ for each transformation $\mathbf{M}$, which represents a symmetry of the material, it is easy to show that $\bar{\mathbf{T}}^{\mathbf{R}^e} = \mathbf{M}^T \mathbf{T}^{\mathbf{R}^e} \mathbf{M}$ to give the condition

$$\mathbf{T}^{\mathbf{R}^e\dagger}(\mathbf{U}^e) = \mathbf{M} \mathbf{T}^{\mathbf{R}^e\dagger}(\mathbf{M}^T \mathbf{U}^e \mathbf{M}) \mathbf{M}^T. \quad (11.87)$$

Before we continue to express the form of the model for Cauchy stress for a specific material symmetry, we should point out that a model for Cauchy stress that only depends on the elastic deformation gradient has no way of reducing its symmetry. That is, whatever

symmetry the material has at the start of the process remains in the model. This might or might not be desirable. From the crystal point of view, as the material deforms plastically, the crystal lattice continuously comes back to its original structure. Since the stress is associated with small lattice distortions, this point of view suggests that the symmetry of the stress should stay the same since the structure of the lattice has not changed. On the other hand, experimental results show substantial changes in the symmetry of materials that have been plastically deformed, therefore suggesting that these models are oversimplistic and unable to capture the real characteristics of the response. As with most issues that arise when modeling reality, there are probably cases in which the current assumptions are valid and others in which they are not.

Let us proceed to examine the form of such models for an isotropic material.

11.4.1 Initially isotropic materials

For the case of an isotropic material, $\mathbf{M}$ can be any orthogonal transformation. The most general function that satisfies the symmetry conditions for an isotropic material can be written as

$$\mathbf{T}^{\mathbf{R}^e} = \mu_o \mathbf{I} + \mu_1 \mathbf{U}^e + \mu_2 \mathbf{U}^{e2}, \quad (11.88)$$

where μ_o, μ_1, μ_2 may each be a function of the three isotropic invariants of $\mathbf{U}^e$ given by

$$I_{1\mathbf{U}}^e = \text{tr}(\mathbf{U}^e), \quad I_{2\mathbf{U}}^e = \text{tr}(\mathbf{U}^{e2}), \quad I_{3\mathbf{U}}^e = \text{tr}(\mathbf{U}^{e3}),$$

or

$$J_{1\mathbf{U}}^e = \text{tr}(\mathbf{U}^e), \quad J_{2\mathbf{U}}^e = \frac{1}{2}[\text{tr}^2(\mathbf{U}^e) - \text{tr}(\mathbf{U}^{e2})], \quad J_{3\mathbf{U}}^e = \det(\mathbf{U}^e),$$

or any other representation of the isotropic invariants of $\mathbf{U}^e$. The Cayley-Hamilton theorem can be used to change the powers of $\mathbf{U}^e$ in the expression for $\mathbf{T}^{\mathbf{R}^e}$. An example of such a case would be to write this same expression as

$$\mathbf{T}^{\mathbf{R}^e} = \bar{\mu}_{-1} \mathbf{U}^{e-1} + \bar{\mu}_o \mathbf{I} + \bar{\mu}_1 \mathbf{U}^e, \quad (11.89)$$

where each $\bar{\mu}_i$ is a function of the same isotropic invariants. As we know that $\mathbf{V}^e = \mathbf{R}^e \mathbf{U}^e \mathbf{R}^{eT}$, the expression for Cauchy stress can be written as

$$\mathbf{T} = \mu_o \mathbf{I} + \mu_1 \mathbf{V}^e + \mu_2 \mathbf{V}^{e2}, \quad (11.90)$$

or

$$\mathbf{T} = \bar{\mu}_{-1} \mathbf{V}^{e-1} + \bar{\mu}_o \mathbf{I} + \bar{\mu}_1 \mathbf{V}^e. \quad (11.91)$$

Without any loss of generality, one can use $\mathbf{C}^e$ in place of $\mathbf{U}^e$ to construct a representation for the stress. This results in the general model

$$\mathbf{T}^{\mathbf{R}^e} = \beta_o \mathbf{I} + \beta_1 \mathbf{C}^e + \beta_2 \mathbf{C}^{e2}, \quad (11.92)$$

where each β_i is a function of any three isotropic invariants of $\mathbf{C}^e$, such as

$$I_{1\mathbf{C}}^e = \text{tr}(\mathbf{C}^e), \quad I_{2\mathbf{C}}^e = \text{tr}(\mathbf{C}^{e2}), \quad I_{3\mathbf{C}}^e = \text{tr}(\mathbf{C}^{e3}),$$

or

$$J_{1\mathbf{C}}^e = \text{tr}(\mathbf{C}^e), \quad J_{2\mathbf{C}}^e = \frac{1}{2}[\text{tr}^2(\mathbf{C}^e) - \text{tr}(\mathbf{C}^{e2})], \quad J_{3\mathbf{C}}^e = \det(\mathbf{C}^e).$$

As we have the relation $\mathbf{B}^e = \mathbf{R}^e \mathbf{C}^e \mathbf{R}^{eT}$, one concludes that we can write the model for Cauchy stress as

$$\mathbf{T} = \beta_o \mathbf{I} + \beta_1 \mathbf{B}^e + \beta_2 \mathbf{B}^{e2}. \quad (11.93)$$

An alternate form, based on changing the powers using the Cayley-Hamilton theorem, can be written as

$$\mathbf{T}^{\mathbf{R}^e} = \bar{\beta}_{-1} \mathbf{C}^{e-1} + \bar{\beta}_o \mathbf{I} + \bar{\beta}_1 \mathbf{C}^e, \quad (11.94)$$

or

$$\mathbf{T} = \bar{\beta}_{-1} \mathbf{B}^{e-1} + \bar{\beta}_o \mathbf{I} + \bar{\beta}_1 \mathbf{B}^e. \quad (11.95)$$

A general polynomial representation of the rotated stress for an isotropic material can be written as

$$\mathbf{T}^{\mathbf{R}^e} = a_o \mathbf{I} + a_1 \mathbf{U}^e + a_2 \mathbf{U}^{e2} + a_3 \mathbf{U}^{e3} + \dots, \quad (11.96)$$

for constant coefficients a_i , which would result in

$$\mathbf{T} = a_o \mathbf{I} + a_1 \mathbf{V}^e + a_2 \mathbf{V}^{e2} + a_3 \mathbf{V}^{e3} + \dots \quad (11.97)$$

An alternate form of this may be

$$\mathbf{T}^{\mathbf{R}^e} = b_o \mathbf{I} + b_1 \mathbf{C}^e + b_2 \mathbf{C}^{e2} + b_3 \mathbf{C}^{e3} + \dots \quad (11.98)$$

for constant coefficients b_i , which would result in

$$\mathbf{T} = b_o \mathbf{I} + b_1 \mathbf{B}^e + b_2 \mathbf{B}^{e2} + b_3 \mathbf{B}^{e3} + \dots \quad (11.99)$$

The representations for $\mathbf{D}_1$, $\mathbf{D}_2^{R^e}$, and $\mathbf{A}$ would be similar to the representations of the rotated stress if we also assumed that these functions only depend on the elastic deformation gradient.

Section 11.4-Problems:

11.4-1: Using the representations described in Appendix A, develop the general mathematical expressions for Cauchy stress in the case of

1. An initially transversely isotropic material.
2. An initially orthotropic material.

11.5 Stress depending on both elastic deformation and plastic strain

One way to introduce deformation-induced change of symmetry in the elastic response is to assume the Cauchy stress depends not only on the elastic deformation gradient but also on the plastic strain. Other changes in the elastic moduli, besides those that result from symmetry, sometimes can be related to the hardening parameter so that we also introduce this into the expression to model stress and write

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{U}^p, \xi), \quad (11.100)$$

with the symmetry condition of

$$\mathbf{T}^{R^e \dagger}(\mathbf{U}^e, \mathbf{U}^p, \xi) = \mathbf{M} \mathbf{T}^{R^e \dagger}(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{U}^p \mathbf{M}, \xi) \mathbf{M}^T \quad (11.101)$$

for every orthogonal transformation $\mathbf{M}$ associated with a symmetry of the initial state of the material. For this type of model, we can use the representations given in Appendix A for two symmetric tensors. For example, for an isotropic material we can write

$$\begin{aligned}\mathbf{T}^{R^e} = & \mu_1 \mathbf{I} + \mu_2 \mathbf{U}^e + \mu_3 \mathbf{U}^{e2} + \mu_4 \mathbf{U}^p + \mu_5 \mathbf{U}^{p2} + \mu_6 (\mathbf{U}^e \mathbf{U}^p + \mathbf{U}^p \mathbf{U}^e) \\ & + \mu_7 (\mathbf{U}^{e2} \mathbf{U}^p + \mathbf{U}^p \mathbf{U}^{e2}) + \mu_8 (\mathbf{U}^e \mathbf{U}^{p2} + \mathbf{U}^{p2} \mathbf{U}^e),\end{aligned}\quad (11.102)$$

where μ_i are functions of ξ and the 10 isotropic invariants of $\mathbf{U}^e$ and $\mathbf{U}^p$ given by $I_1 = \text{tr}(\mathbf{U}^e)$, $I_2 = \text{tr}(\mathbf{U}^{e2})$, $I_3 = \text{tr}(\mathbf{U}^{e3})$, $I_4 = \text{tr}(\mathbf{U}^p)$, $I_5 = \text{tr}(\mathbf{U}^{p2})$, $I_6 = \text{tr}(\mathbf{U}^{p3})$, $I_7 = \text{tr}(\mathbf{U}^e \mathbf{U}^p)$, $I_8 = \text{tr}(\mathbf{U}^{e2} \mathbf{U}^p)$, $I_9 = \text{tr}(\mathbf{U}^e \mathbf{U}^{p2})$, and $I_{10} = \text{tr}(\mathbf{U}^{e2} \mathbf{U}^{p2})$.

A possibly more convenient representation uses $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger}(\mathbf{C}^e, \mathbf{C}^p, \xi)$. For example, for an initially isotropic material this gives

$$\begin{aligned}\mathbf{T}^{R^e} = & \mu_1 \mathbf{I} + \mu_2 \mathbf{C}^e + \mu_3 \mathbf{C}^{e2} + \mu_4 \mathbf{C}^p + \mu_5 \mathbf{C}^{p2} + \mu_6 (\mathbf{C}^e \mathbf{C}^p + \mathbf{C}^p \mathbf{C}^e) \\ & + \mu_7 (\mathbf{C}^{e2} \mathbf{C}^p + \mathbf{C}^p \mathbf{C}^{e2}) + \mu_8 (\mathbf{C}^e \mathbf{C}^{p2} + \mathbf{C}^{p2} \mathbf{C}^e),\end{aligned}\quad (11.103)$$

where μ_i are functions of ξ and the 10 isotropic invariants of $\mathbf{C}^e$ and $\mathbf{C}^p$ given by $I_1 = \text{tr}(\mathbf{C}^e)$, $I_2 = \text{tr}(\mathbf{C}^{e2})$, $I_3 = \text{tr}(\mathbf{C}^{e3})$, $I_4 = \text{tr}(\mathbf{C}^p)$, $I_5 = \text{tr}(\mathbf{C}^{p2})$, $I_6 = \text{tr}(\mathbf{C}^{p3})$, $I_7 = \text{tr}(\mathbf{C}^e \mathbf{C}^p)$, $I_8 = \text{tr}(\mathbf{C}^{e2} \mathbf{C}^p)$, $I_9 = \text{tr}(\mathbf{C}^e \mathbf{C}^{p2})$, and $I_{10} = \text{tr}(\mathbf{C}^{e2} \mathbf{C}^{p2})$.

Another possible form might be based on $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger \dagger}(\mathbf{E}^e, \mathbf{E}^p, \xi)$. In this case the representation for an initially isotropic material is given by

$$\begin{aligned}\mathbf{T}^{R^e} = & \mu_1 \mathbf{I} + \mu_2 \mathbf{E}^e + \mu_3 \mathbf{E}^{e2} + \mu_4 \mathbf{E}^p + \mu_5 \mathbf{E}^{p2} + \mu_6 (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \\ & + \mu_7 (\mathbf{E}^{e2} \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^{e2}) + \mu_8 (\mathbf{E}^e \mathbf{E}^{p2} + \mathbf{E}^{p2} \mathbf{E}^e),\end{aligned}\quad (11.104)$$

where μ_i are functions of ξ and the 10 isotropic invariants of $\mathbf{E}^e$ and $\mathbf{E}^p$ given by $I_1 = \text{tr}(\mathbf{E}^e)$, $I_2 = \text{tr}(\mathbf{E}^{e2})$, $I_3 = \text{tr}(\mathbf{E}^{e3})$, $I_4 = \text{tr}(\mathbf{E}^p)$, $I_5 = \text{tr}(\mathbf{E}^{p2})$, $I_6 = \text{tr}(\mathbf{E}^{p3})$, $I_7 = \text{tr}(\mathbf{E}^e \mathbf{E}^p)$, $I_8 = \text{tr}(\mathbf{E}^{e2} \mathbf{E}^p)$, $I_9 = \text{tr}(\mathbf{E}^e \mathbf{E}^{p2})$, and $I_{10} = \text{tr}(\mathbf{E}^{e2} \mathbf{E}^{p2})$.

In each case we may need to impose additional conditions. For example, we could require that the stress be zero for $\mathbf{U}^e = \mathbf{I}$ so that the intermediate configuration represents one of a family of unloaded configurations.

The model for $\mathbf{D}_2^{R^e}$ would be similar to the rotated stress $\mathbf{T}^{R^e}$, but the representations for $\mathbf{D}_1$ and $\mathbf{A}$ will not be restricted to symmetric tensor representations. For example, $\mathbf{A}$ might take the form

$$\begin{aligned}\mathbf{A} = & a_1 \mathbf{I} + a_2 \mathbf{E}^e + a_3 \mathbf{E}^{e2} + a_4 \mathbf{E}^p + a_5 \mathbf{E}^{p2} + a_6 \mathbf{E}^e \mathbf{E}^p + a_7 \mathbf{E}^p \mathbf{E}^e \\ & + a_8 \mathbf{E}^{e2} \mathbf{E}^p + a_9 \mathbf{E}^p \mathbf{E}^{e2} + a_{10} \mathbf{E}^e \mathbf{E}^{p2} + a_{11} \mathbf{E}^{p2} \mathbf{E}^e \\ & + a_{12} (\mathbf{E}^e \mathbf{E}^p \mathbf{E}^{e2} - \mathbf{E}^{e2} \mathbf{E}^p \mathbf{E}^e) + a_{13} (\mathbf{E}^p \mathbf{E}^e \mathbf{E}^{p2} - \mathbf{E}^{p2} \mathbf{E}^e \mathbf{E}^p).\end{aligned}\quad (11.105)$$

Section 11.5-Problems:

11.5-1: Consider the model for rotated stress during plastic deformation given by

$$\mathbf{T}^{R^e} = \frac{\kappa^e}{3} \left(J^{e2} - \frac{1}{J^e} \right) \mathbf{I} + G^e \left[\mathbf{C}^e - \frac{\text{tr}(\mathbf{C}^e)}{3} \mathbf{I} \right] + A^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e), \quad (11.106)$$

where κ^e and G^e are the initial elastic bulk and shear moduli, and A^e is the coefficient of evolving anisotropy, all constants.

- Show that this gives the expression for the Cauchy stress as

$$\mathbf{T} = \frac{\kappa^e}{3} \left(J^{e2} - \frac{1}{J^e} \right) \mathbf{I} + G^e \left[\mathbf{B}^e - \frac{\text{tr}(\mathbf{B}^e)}{3} \mathbf{I} \right] + A^e \mathbf{R}^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{R}^{eT}. \quad (11.107)$$

- What is the stress when there is no elastic deformation represents a pure rigid body translation ($\mathbf{F}^e = \mathbf{I}$), or represents a general rigid body motion (i.e., $\mathbf{F}^e$ is orthogonal)?
- 11.5-2:** Construct the representation of the stress response of an initially transversely isotropic material for (a) $\mathbf{T}^{R^e \dagger}(\mathbf{U}^e, \mathbf{U}^p, \xi)$, (b) $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger}(\mathbf{C}^e, \mathbf{C}^p, \xi)$, and (c) $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger \dagger}(\mathbf{E}^e, \mathbf{E}^p, \xi)$.
- 11.5-3:** Construct the representation of the stress response for an initially orthotropic material for (a) $\mathbf{T}^{R^e \dagger}(\mathbf{U}^e, \mathbf{U}^p, \xi)$, (b) $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger}(\mathbf{C}^e, \mathbf{C}^p, \xi)$, and (c) $\mathbf{T}^{R^e} = \mathbf{T}^{R^e \dagger \dagger \dagger}(\mathbf{E}^e, \mathbf{E}^p, \xi)$.

11.6 General comments

In this chapter, we presented and somewhat developed a mechanical theory for finite deformation plasticity. The presentation focused primarily on the basic ideas and the possible representations, without providing much detail, other than several example models. This is, to some extent, due to the following chapters on thermoplasticity and crystal plasticity, which are much more extensive.

The study of inhomogeneous bodies only requires us to add a dependence in each constitutive equation on material particle. That is, we only need to add $\mathbf{X}$ to each argument set. Otherwise, everything remains the same.

A possibility that we did not pursue is the existence of multiple plastic components either set in parallel or in series. If they are in parallel, for each element i we would have the rule $\mathbf{F} = \mathbf{F}_i^e \mathbf{F}_i^p$, and have associated flow rules, hardening rules, and yield functions. In this case we would probably assume the applied stress is the sum of the elastic stresses in the individual elements. On the other hand, if they were in parallel, we might take the assumption that $\mathbf{F} = \mathbf{F}^e \mathbf{F}_1^p \mathbf{F}_2^p \dots \mathbf{F}_n^p$, with associated additional assumptions. The discussion of such models can be seen in the following chapters.

Finally, there is a tradition to use elasticity models to represent plastic flow when there will be no unloading. These models normally provide simpler or faster solutions. Their use is entirely dependent on the appropriateness of the loading conditions. The following section provides one such model that has already been considered in the infinitesimal setting.

11.7 Deformation plasticity

As for infinitesimal deformations, we can develop a model for plasticity that mimics the shape of elastic-plastic flow but does not show the unloading seen in plasticity. As expected, the model is elastic and the unloading will follow the same loading curve. For this, we start from the same model used for deformation plasticity under infinitesimal deformation. As the reader recalls, this model was given by

$$\boldsymbol{\epsilon} = \frac{1-2\nu}{E} \sigma_{ave} \mathbf{I} + \frac{1+\nu}{E} \mathbf{S} + \frac{3\alpha}{2E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \mathbf{S}, \quad (11.108)$$

where τ is the equivalent shear stress given by $\tau = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}}$, E is the elastic Young's modulus, ν is Poisson's ratio, τ_y is the yield stress in shear, α is a scaling factor known as

the yield offset, and n is the hardening exponent. We have to correct this so that all terms have the same characteristics when subjecting the body to rigid body motions. We know the Cauchy stress $\mathbf{T}$ changes to $\mathbf{Q}\mathbf{T}\mathbf{Q}^T$ when subjecting the motion to a rigid body rotation described by the orthogonal transformation $\mathbf{Q}$. The same will be true for $\mathbf{S}$, so that $\mathbf{S}$ goes to $\mathbf{Q}\mathbf{S}\mathbf{Q}^T$. All terms of the right-hand side change in the same manner as the stress so that the left-hand side also should change in this same manner. Therefore, the model will have to include a strain that will change in a similar manner as $\mathbf{T}$ or $\mathbf{S}$. One possible model is

$$\mathbf{E}_L = \frac{1-2\nu}{E} \sigma_{ave} \mathbf{I} + \frac{1+\nu}{E} \mathbf{S} + \frac{3\alpha}{2E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \mathbf{S}, \quad (11.109)$$

where $\mathbf{E}_L$ is the left Green strain given by

$$\mathbf{E}_L = \frac{1}{2}(\mathbf{B} - \mathbf{I}). \quad (11.110)$$

The Almansi strain $\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$ also has the same characteristics so one may use it to construct the deformation plasticity model and write

$$\mathbf{A} = \frac{1-2\nu}{E} \sigma_{ave} \mathbf{I} + \frac{1+\nu}{E} \mathbf{S} + \frac{3\alpha}{2E} \left(\frac{\tau}{\tau_y} \right)^{n-1} \mathbf{S}. \quad (11.111)$$

Any number of other strains also might be used to make the model. Since at small deformations both the left Green strain $\mathbf{E}_L$ and the Almansi strain $\mathbf{A}$ reduce to the infinitesimal strain, the response of both these models will mimic closely the infinitesimal deformation plasticity model response described in Figures 7.15 and 7.16.

11.8 Notation

The following table shows the symbols used in the text and their definitions.

" $\diamond$ "		trial time derivative taken holding plastic strain constant
$\mathbf{A}$		Almansi strain
$\mathbf{A}$		constitutive function for hardening ($\dot{\xi} = \mathbf{A} : \mathbf{L}^p$)
ave		average (subscript)
$\mathbf{B}^e$		elastic left Cauchy stretch tensor ($\mathbf{F}^e \mathbf{F}^{eT}$)
$\mathbf{B}^p$		plastic left Cauchy stretch tensor ($\mathbf{F}^p \mathbf{F}^{pT}$)
$\mathbf{C}^e$		elastic right Cauchy stretch tensor ($\mathbf{F}^{eT} \mathbf{F}^e$)
$\mathbf{C}^p$		plastic right Cauchy stretch tensor ($\mathbf{F}^{pT} \mathbf{F}^p$)
$\mathbf{D}$		constitutive function in flow rule ($\mathbf{L}^p = \mathbf{D} : \mathbf{L}$)
$\mathbf{D}_1$		constitutive function in flow rule ($\mathbf{D} = \mathbf{D}_1 \otimes \mathbf{D}_2$)
$\mathbf{D}_2$		constitutive function in flow rule ($\mathbf{D} = \mathbf{D}_1 \otimes \mathbf{D}_2$)
$\Delta \mathbf{S}$		deviatoric part of overstress $\Delta \mathbf{T}$
$\Delta \mathbf{T}$		overstress ($\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b$)
$\Delta \mathbf{T}_f$		overstress-like term resulting from yield function
E		elastic modulus
$\mathbf{E}^e$		elastic Green strain
$\mathbf{E}^p$		plastic Green strain
$\mathbf{E}_L$		left Green strain [$\frac{1}{2}(\mathbf{B} - \mathbf{I})$]
$\mathbf{F}$		deformation gradient
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}^p$		plastic deformation gradient

G^e		elastic shear modulus
G^b		back-stress shear modulus
$\mathbf{I}$		identity
I_i		scalar invariant
J^e		elastic volume ratio [$\det(\mathbf{F}^e)$]
J_i		scalar invariant
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}}\mathbf{F}^{-1}$)
$\mathbf{L}^e$		elastic velocity gradient ($\dot{\mathbf{F}}^e\mathbf{F}^{e-1}$)
$\mathbf{L}^p$		plastic velocity gradient ($\dot{\mathbf{F}}^p\mathbf{F}^{p-1}$)
$\mathbf{M}$		transformation tensor representing material symmetry
$\mathbf{Q}$		orthogonal tensor representing rigid body motion
$\mathbf{R}$		orthogonal factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{R}^e$		orthogonal factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e$
$\mathbf{R}^p$		orthogonal factor in polar decomposition of $\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p = \mathbf{V}^p\mathbf{R}^p$
$\mathbf{S}$		deviatoric stress
$\mathbf{T}$		Cauchy stress
$\mathbf{T}^b$		back-stress
$\mathbf{T}^e$		elastic stress
T_{ave}		average stress
$\mathbf{T}^{R^e}$		elastic rotated Cauchy stress($\mathbf{R}^{eT}\mathbf{T}\mathbf{R}^e$)
$\mathbf{T}_f^e$		stresslike term resulting from yield function
$\mathbf{T}_f^b$		back-stress-like term resulting from yield function
$\mathbf{U}$		right symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{U}^e$		right symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e$
$\mathbf{U}^p$		right symmetric factor in polar decomposition of $\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p$
$\mathbf{V}$		left symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{V}^e$		left symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e$
$\mathbf{V}^p$		left symmetric factor in polar decomposition of $\mathbf{F}^p = \mathbf{V}^p\mathbf{R}^p$
β		scalar factor in flow rule
κ^e		elastic bulk modulus
κ^b		back-stress bulk modulus
ν		Poisson's ratio
τ		equivalent shear stress
τ_y		yield stress in shear
ξ		hardening parameter

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Chapter 12

Thermoplastic solids

In this chapter, we will study the way to include thermal effects in the modeling of plastic deformations. We will do this in the context of the thermodynamic theory of plasticity, paralleling the development in the chapter on thermoelasticity. One notable difference here will be the explicit introduction of the thermal deformation gradient from the start of the development.

Since all of plasticity falls within the constitutive theory of modeling first gradient materials, the reader is directed to the chapter on first gradient models for background material. We set plasticity as a state variable theory, and introduce the effect of the past flow through a number of state variables that include the plastic deformation gradient, among others. The result then is subjected to the thermodynamic constraint introduced by the Clausius-Duhem inequality, which, as expected, provides explicit expressions for Cauchy stress and entropy in terms of the specific free energy, among other restrictions.

As will be seen, the number of possible variations in the modeling is too numerous for us to follow all possible variations. Special examples are presented that capture the basic models introduced in the one-dimensional and infinitesimal cases. These include models that resemble the models previously provided for no hardening, isotropic hardening, kinematic hardening, and combined isotropic and kinematic hardening.

A particular point of interest in plasticity is heat generation, which also will be considered in this chapter.

12.1 A simple thermomechanical analog

We will start by looking at the thermomechanical analog shown in Figure 12.1 for a simple one-dimensional thermoplastic model. Before we proceed, we should note that even though analogs provide a valuable aid for us to view what happens in the different elements, they contain only part of the picture and normally are simplified substantially so that we should not take the assumptions too literally. To simplify the presentation, we will assume that the material parameters are constants that do not depend on temperature, leaving the case in which they depend on temperature as a homework problem.

The thermomechanical analog that we are considering is constructed from elastic, plastic, and thermal elements put into series as shown in Figure 12.1. The total displacement of the system, denoted by δ , is assumed to be the sum of the extension of the three components so that

$$\delta = \delta^e + \delta^p + \delta^\theta, \quad (12.1)$$

where δ^e , δ^p , and δ^θ are, respectively, the extension of the elastic, plastic, and thermal elements. The new element here is the thermal element and its contribution is assumed to be given by

$$\dot{\delta}^\theta = \alpha L \dot{\theta}, \quad (12.2)$$

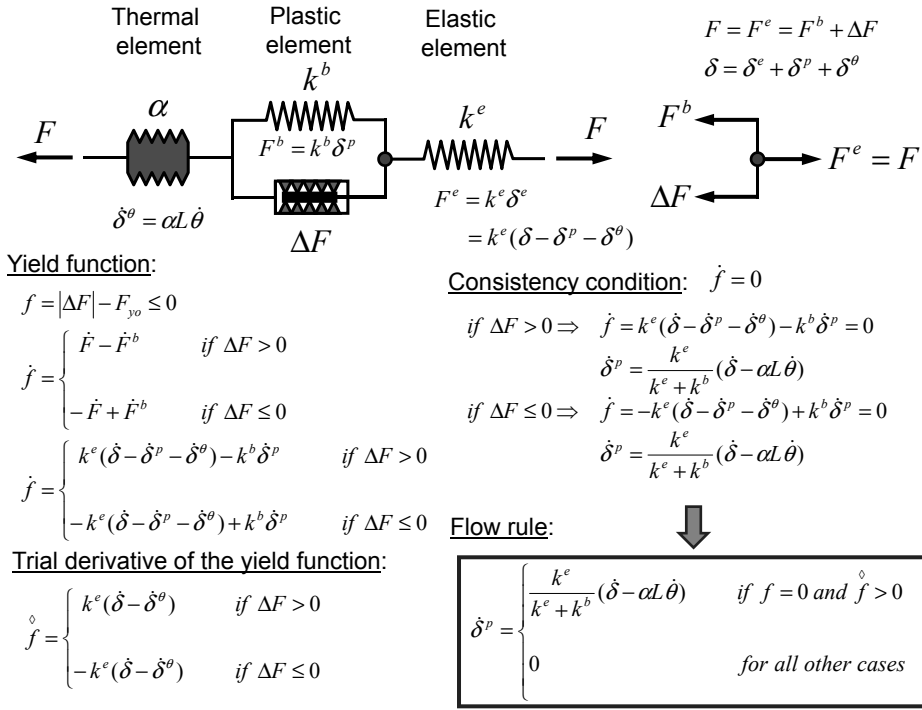


Figure 12.1: A simple thermomechanical analog for a simple one-dimensional thermoplasticity model.

where L is the total initial length, α is the linear coefficient of thermal expansion, and $\dot{\theta}$ is the rate of change of temperature. The elastic and plastic elements each have a spring. The spring in the elastic element has a stiffness of k^e . The spring in the plastic element is called the “back-stress” spring and has a stiffness k^b . The load in these two elements are given by

$$F^e = k^e \delta^e, \quad (12.3)$$

$$F^b = k^b \delta^p. \quad (12.4)$$

The free-body diagram shown in the figure suggests that the load in the elastic spring is balanced by the sum of the load in the two components of the plastic element. We use this and the suggestion that the applied load F is the same as the load in the elastic spring and write it as

$$F^e = F^b + \Delta F, \quad (12.5)$$

$$F^e = F, \quad (12.6)$$

where ΔF is the load in the “plastic flow” component of the plastic element, as shown in Figure 12.1.

The key concept that makes this a thermoplastic response analog, as opposed to viscoelastic or viscoplastic, is the way we characterize the “plastic flow” component of the plastic element. For thermoplastic response, we assume this element is controlled by a yield function f that always must be less than or equal to zero. The simplest such model is characterized by the assumption that the load in this element would be less than some upper

bound value. We can write this as $|\Delta F| \leq F_{yo}$, where F_{yo} is the yield load for the element. This idea results in a yield function for this component written as

$$f = |\Delta F| - F_{yo}. \quad (12.7)$$

The element will stay locked when $f < 0$, and under some conditions, it may stretch or contract when $f = 0$. The key is that the unlocking of the element only happens when the condition on f cannot be satisfied with changes only in the elastic and thermal elements. To formalize this idea, we introduce the trial derivative of f that assumes only changes in the thermal and elastic elements, and we denote it as $\overset{\diamond}{f}$. The reader will note that the unlocking of the system will happen only when $f = 0$ and $\overset{\diamond}{f} > 0$. Once the “plastic flow” condition is satisfied, the plastic element will flow under the condition that the yield function remains equal to zero. That is, plastic flow happens such that $\dot{f} = 0$, which is known as the *consistency condition*. To formalize the rule for the change in δ^p , known as the flow rule, we can consider the condition $\dot{f} = 0$. From the assumptions, we have

$$f = \begin{cases} \Delta F - F_{yo} & \text{if } \Delta F > 0, \\ -\Delta F - F_{yo} & \text{if } \Delta F \leq 0. \end{cases} \quad (12.8)$$

For this example, we assume all the material parameters are constant so their derivatives are zero. The equation for $\dot{f}$ in this case becomes

$$\dot{f} = \begin{cases} \Delta \dot{F} & \text{if } \Delta F > 0, \\ -\Delta \dot{F} & \text{if } \Delta F \leq 0. \end{cases} \quad (12.9)$$

We can use the above relation to write

$$\Delta F = F^e - F^b = k^e \delta^e - k^b \delta^p = k^e (\delta - \delta^p - \delta^\theta) - k^b \delta^p. \quad (12.10)$$

This gives the expression for the rate of ΔF as

$$\Delta \dot{F} = k^e (\dot{\delta} - \dot{\delta}^p - \dot{\delta}^\theta) - k^b \dot{\delta}^p = k^e (\dot{\delta} - \dot{\delta}^\theta) - (k^e + k^b) \dot{\delta}^p. \quad (12.11)$$

It follows that the yield function changes through the relation

$$\dot{f} = \begin{cases} k^e (\dot{\delta} - \dot{\delta}^\theta) - (k^e + k^b) \dot{\delta}^p & \text{if } \Delta F > 0, \\ -k^e (\dot{\delta} - \dot{\delta}^\theta) + (k^e + k^b) \dot{\delta}^p & \text{if } \Delta F \leq 0. \end{cases} \quad (12.12)$$

The trial derivative of the yield function simply replaces a zero for $\dot{\delta}^p$ so that

$$\overset{\diamond}{f} = \begin{cases} k^e (\dot{\delta} - \dot{\delta}^\theta) & \text{if } \Delta F > 0, \\ -k^e (\dot{\delta} - \dot{\delta}^\theta) & \text{if } \Delta F \leq 0. \end{cases} \quad (12.13)$$

If $f = 0$ and $\overset{\diamond}{f} > 0$, we have plastic flow governed by the rule $\dot{f} = 0$. For $\Delta F > 0$, this requires that

$$k^e (\dot{\delta} - \dot{\delta}^\theta) - (k^e + k^b) \dot{\delta}^p = 0. \quad (12.14)$$

Solving for $\dot{\delta}^p$ we get

$$\dot{\delta}^p = \frac{k^e}{k^e + k^b} (\dot{\delta} - \dot{\delta}^\theta). \quad (12.15)$$

The same results if we consider the case $\Delta F \leq 0$. The flow rule is therefore written as

$$\dot{\delta}^p = \begin{cases} \frac{k^e}{k^e + k^b} (\dot{\delta} - \alpha L \dot{\theta}) & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (12.16)$$

As can be noted from the development, the addition of the thermal element has not changed the condition for plastic flow (i.e., $f = 0$ and $\dot{f} > 0$), but both the trial derivative of the yield function and the flow rule now have terms that depend on $\dot{\theta}$. Also, with the addition of the thermal element, there follows no change in the consistency condition (i.e., $\dot{f} = 0$), even though the result of this condition has changed. This is the spirit in which we develop the full three-dimensional theory below. The problems for this section consider variations of the above development.

Section 12.1-Problems:

- 12.1-1:** For the model developed in this section, assume the heat generated is given by $\Delta F \dot{\delta}^p$ and calculate the heat needed to be removed from the system to maintain isothermal conditions during monotonic extension of the system. For this case, plot the heat per unit increment of strain needed to be removed as a function of strain. Does this plot depend on the rate of loading?
- 12.1-2:** In the thermomechanical analog discussed above add the dependence of F_{yo} on a hardening parameter ξ that is given by the hardening rule $\dot{\xi} = \Delta F \dot{\delta}^p$, with the initial condition that $\xi = 0$ before deformation. Calculate $\dot{f}$ and the associated flow rule (check both branches of the yield function).
- 12.1-3:** For the thermomechanical analog discussed in this section assume k^e , k^b , and F_{yo} are all functions of temperature. Derive $\dot{f}$ and the flow rule (check both branches of the yield function).
- 12.1-4:** For the thermomechanical analog developed here change the yield function to $f = [(\Delta F)^2 - F_{yo}^2]/2$. Derive $\dot{f}$ and the flow rule.
- 12.1-5:** Write the model developed in this section in a form appropriate for a system that runs under load control.

12.2 Thermoplasticity

The model we propose to develop here for plasticity is a special case of the general first-gradient thermomechanical material model. To construct this model, we will decompose the deformation gradient $\mathbf{F}$ into three parts. These parts will be the elastic deformation gradient $\mathbf{F}^e$, the plastic deformation gradient $\mathbf{F}^p$, and the thermal deformation gradient $\mathbf{F}^\theta$, and we assume they combine to give the deformation gradient through the equation¹

$$\mathbf{F}(t) = \mathbf{F}^e(t) \mathbf{F}^p(t) \mathbf{F}^\theta(t). \quad (12.17)$$

¹As discussed before, one may develop a model based on the decomposition $\mathbf{F}(t) = \mathbf{F}^e(t) \bar{\mathbf{F}}^\theta(t) \bar{\mathbf{F}}^p(t)$. Details of this are given in the last section.

This decomposition is not unique and becomes meaningful only after providing constitutive assumptions and expressions for calculating $\mathbf{F}^p$ and $\mathbf{F}^\theta$ from the history of $\mathcal{U} = (\mathbf{F}, \theta, \mathbf{G})$ at the point. One may combine $\mathbf{F}^\theta$ into $\mathbf{F}^p$, but this will require that the plastic deformation gradient change with temperature, even during thermoelastic response, which we would like to avoid. We also add a scalar hardening parameter ξ , which also will need to be evaluated from the history of $\mathcal{U}$. We assume that the total contribution of the history of $\mathcal{U}$ to the response of the material is given through knowing the current state of the material, and we denote this state by $\mathcal{S}$. For our plasticity model, we will assume that the state of the material is given by the following set of variables

$$\mathcal{S}(t) = \{\mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)\}. \quad (12.18)$$

In an abstract form, the assumption that the response depends on the state of the material can be written as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^\dagger[\mathbf{X}, \mathcal{S}(t)]. \quad (12.19)$$

This symbolic statement translates into the following specific assumptions

$$\psi(\mathbf{X}, t) = \psi^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)], \quad (12.20)$$

$$\eta(\mathbf{X}, t) = \eta^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)], \quad (12.21)$$

$$\mathbf{T}(\mathbf{X}, t) = \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)], \quad (12.22)$$

$$\mathbf{q}(\mathbf{X}, t) = \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)]. \quad (12.23)$$

We now will lay down specific assumptions that will change these dependencies.

12.3 Thermodynamic constraint

The entropy production inequality given in (6.69) introduces constraints on the constitutive response functions. This law, in the form of the Clausius-Duhem inequality, is written as

$$\rho\dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0,$$

and must be satisfied for all possible processes. The current assumptions on the constitutive dependence of the specific free energy (12.20) result in the expression for $\dot{\psi}$ as

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^p}(\psi) : \dot{\mathbf{F}}^p + \partial_{\mathbf{F}^\theta}(\psi) : \dot{\mathbf{F}}^\theta + \partial_\xi(\psi)\dot{\xi} + \partial_\theta(\psi)\dot{\theta} + \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}. \quad (12.24)$$

The rate of the deformation gradient can be written as

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p \mathbf{F}^\theta + \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^\theta + \mathbf{F}^e \mathbf{F}^p \dot{\mathbf{F}}^\theta$$

to yield an expression for the rate of elastic deformation gradient as

$$\dot{\mathbf{F}}^e = \dot{\mathbf{F}} \mathbf{F}^{\theta-1} \mathbf{F}^{p-1} - \mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1} - \mathbf{F}^e \mathbf{F}^p \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1} \mathbf{F}^{p-1}. \quad (12.25)$$

We now can introduce this into the equation for $\dot{\psi}$ to obtain

$$\begin{aligned} \dot{\psi} = & [\partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{p-T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}} - [\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{p-T}] : \dot{\mathbf{F}}^p - [\mathbf{F}^{pT} \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{p-T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}}^\theta \\ & + \partial_{\mathbf{F}^p}(\psi) : \dot{\mathbf{F}}^p + \partial_{\mathbf{F}^\theta}(\psi) : \dot{\mathbf{F}}^\theta + \partial_\xi(\psi)\dot{\xi} + \partial_\theta(\psi)\dot{\theta} + \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}. \end{aligned} \quad (12.26)$$

To simplify the presentation, we introduce, respectively, the thermodynamic elastic, back-, and thermal stresses

$$\boxed{\mathbf{T}^{eT} \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}}, \quad (12.27)$$

$$\boxed{\mathbf{T}^{bT} \equiv \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT}}, \quad (12.28)$$

$$\boxed{\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}}, \quad (12.29)$$

$$(12.30)$$

which results in the expression $\rho \dot{\psi}$ given by

$$\begin{aligned} \rho \dot{\psi} = & [\mathbf{T}^{eT} \mathbf{F}^{-T}] : \dot{\mathbf{F}} - [\mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T}] : \dot{\mathbf{F}}^p - [\mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}}^\theta \\ & + [\mathbf{T}^{bT} \mathbf{F}^{p-T}] : \dot{\mathbf{F}}^p + [\mathbf{T}^{\theta T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}}^\theta + \rho \partial_\xi(\psi) \dot{\xi} + \rho \partial_\theta(\psi) \dot{\theta} + \rho \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}. \end{aligned} \quad (12.31)$$

The Clausius-Duhem inequality now can be written as

$$\begin{aligned} & [(\mathbf{T}^{eT} - \mathbf{T}^T) \mathbf{F}^{-T}] : \dot{\mathbf{F}} + \rho [\eta + \partial_\theta(\psi)] \dot{\theta} + [(\mathbf{T}^{\theta T} - \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T}) \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}}^\theta \\ & + [(\mathbf{T}^{bT} - \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T}) \mathbf{F}^{p-T}] : \dot{\mathbf{F}}^p + \rho \partial_\xi(\psi) \dot{\xi} + \rho \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (12.32)$$

which can be reorganized as

$$\begin{aligned} & [\mathbf{T}^{eT} - \mathbf{T}^T] : \mathbf{L} + \rho [\eta + \partial_\theta(\psi)] \dot{\theta} + [\mathbf{T}^{\theta T} - \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T}] : \mathbf{L}^\theta \\ & + [\mathbf{T}^{bT} - \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T}] : \mathbf{L}^p + \rho \partial_\xi(\psi) \dot{\xi} + \rho \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (12.33)$$

and which must be satisfied for all admissible processes. For an unconstrained material, we consider a process to be defined by the history of $\mathcal{U}$ and assume all continuous processes to be admissible. As such, from each point of an admissible process, we can branch off along another admissible process by moving along a branch with a different $\dot{\mathcal{U}}$, where

$$\boxed{\dot{\mathcal{U}} = \left\{ \dot{\mathbf{F}}(t), \dot{\theta}(t), \dot{\mathbf{G}}(t) \right\}}. \quad (12.34)$$

Since in this development each point in a process represents a state $\mathcal{S}$ of the material, we will interpret this to require that the Clausius-Duhem inequality must be satisfied for all admissible states $\mathcal{S}$ and all loading directions $\dot{\mathcal{U}}$.

The next step will be to make further constitutive assumptions to simplify the expressions and extract information.

12.3.1 Thermal deformation gradient

We will consider only thermal deformation gradients that can be calculated from a constitutively prescribed rate form given by

$$\boxed{\mathbf{L}^\theta(t) = \boldsymbol{\alpha}(t) \dot{\theta}(t)}, \quad (12.35)$$

where $\boldsymbol{\alpha}$ is a second-order tensor coefficient of thermal expansion. This results in the relation $\dot{\mathbf{F}}^\theta = \boldsymbol{\alpha} \mathbf{F}^\theta \dot{\theta}$. We will assume that the coefficient of thermal expansion can be a function of all the state variables other than the temperature gradient. We, therefore, are assuming that

$$\boxed{\boldsymbol{\alpha}(\mathbf{X}, t) = \boldsymbol{\alpha}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t)]}. \quad (12.36)$$

12.3.2 Thermoelastic range

We will assume that for each material state there is an attached thermoelastic range. Within this range the material is assumed to behave as a thermoelastic material. In particular, the plastic deformation gradient and hardening parameter are assumed to be constant for loading paths that remain within this range. We will define the thermoelastic range through a yield function of the form

$$f[\mathbf{X}, \mathcal{S}(t)], \quad (12.37)$$

where $f = 0$ indicates states that are on the yield surface, and $f < 0$ indicates states that are within the yield surface. As is commonly assumed, *we will leave out the dependence of the yield function on the temperature gradient.*² If we are within the thermoelastic range, we know that $\mathbf{L}^p = \mathbf{0}$ so that the Clausius-Duhem inequality can be written as

$$\begin{aligned} & [\mathbf{T}^{eT} - \mathbf{T}^T] : \mathbf{L} + [\rho\eta + \rho\partial_\theta(\psi) + (\mathbf{T}^{\theta T} - \mathbf{F}^{pT}\mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}\mathbf{F}^{p-T}) : \boldsymbol{\alpha}]\dot{\theta} \\ & + \rho\partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (12.38)$$

which must hold for every state $\mathcal{S}$ inside the current thermoelastic range and for all $\dot{\mathcal{U}}$. An examination of the terms shows that the only thermodynamically admissible constitutive functions are those that satisfy the following conditions:

$$\mathbf{T}^{eT} - \mathbf{T}^T = \mathbf{0}, \quad (12.39)$$

$$\rho\eta + \rho\partial_\theta(\psi) + (\mathbf{T}^{\theta T} - \mathbf{F}^{pT}\mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}\mathbf{F}^{p-T}) : \boldsymbol{\alpha} = 0, \quad (12.40)$$

$$\rho\partial_{\mathbf{G}}(\psi) = \mathbf{0}, \quad (12.41)$$

$$\frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0. \quad (12.42)$$

One therefore may conclude that the specific free energy is not a function of $\mathbf{G}$, and that

$$\mathbf{T}^T = \mathbf{T}^{eT} = \rho\partial_{\mathbf{F}^e}(\psi)\mathbf{F}^{eT}, \quad (12.43)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho}\Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha}, \quad (12.44)$$

where the thermal overstress $\Delta\mathbf{T}^\theta$ is given by

$$\Delta\mathbf{T}^\theta = \mathbf{F}^{p-1}\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^e\mathbf{F}^p - \mathbf{T}^\theta. \quad (12.45)$$

Examination of these expressions reveals that the Cauchy stress and the specific entropy cannot be functions of $\mathbf{G}$ either, leaving only the heat flux vector with a possible dependence on the temperature gradient. Also, we will assume these are continuous functions of their arguments, allowing us to argue that they will continue to hold during plastic flow, and, therefore, under all conditions. The reader is urged to construct the proof, recalling that we have assumed the existence of a thermoelastic range attached to each state of the material, including those during plastic flow.

²Inclusion of the dependence of the yield function on temperature gradient $\mathbf{G}$ will alter the form of the flow rule but will not alter the expressions for stress and entropy.

12.3.3 Flow rule and hardening parameter

Before providing a flow rule, let us look at what conditions need to be satisfied by this rule. The part of the Clausius-Duhem inequality that has not been satisfied requires that during plastic flow we have

$$[\mathbf{T}^{bT} - \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T}] : \mathbf{L}^p + \rho \partial_\xi(\psi) \dot{\xi} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (12.46)$$

which must hold for all $\dot{\mathbf{U}}$. Since histories with zero temperature gradient are assumed to be admissible, we must require that

$$[\mathbf{T}^{bT} - \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T}] : \mathbf{L}^p + \rho \partial_\xi(\psi) \dot{\xi} \leq 0. \quad (12.47)$$

For the hardening rule, we take any function of the form

$$\dot{\xi}(t) = \mathbf{A} : \mathbf{L}^p, \quad (12.48)$$

where $\mathbf{A}$ is a second-order tensor function of the state variables, excluding the temperature gradient. We write this as

$$\mathbf{A}(\mathbf{X}, t) = \mathbf{A}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t)]. \quad (12.49)$$

This rule also can be written as $\dot{\xi}(t) = (\mathbf{A} \mathbf{F}^{p-T}) : \dot{\mathbf{F}}^p$.

Substituting the hardening rule into equation (12.47), after some reorganization, will result in

$$[\mathbf{T}^{bT} - \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} + \rho \partial_\xi(\psi) \mathbf{A}] : \mathbf{L}^p \leq 0. \quad (12.50)$$

This expression suggests the introduction of an overstress $\Delta \mathbf{T}$ defined by

$$\Delta \mathbf{T} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T. \quad (12.51)$$

The restriction imposed by the Clausius-Duhem inequality on the plastic flow now can be written as

$$-\Delta \mathbf{T}^T : \mathbf{L}^p \leq 0. \quad (12.52)$$

The heat flow per unit volume, $\rho \dot{h}$, is the sum of heat flowing to the point by radiation and heat conduction ($\rho r - \text{div}(\mathbf{q})$). This can be calculated from the equation

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T} \mathbf{L}) + \rho \eta \dot{\theta} + \rho \dot{\eta} \theta.$$

Since the first three terms directly appear in the Clausius-Duhem inequality, it is simple to show that we get

$$\rho \dot{h} = -\Delta \mathbf{T}^T : \mathbf{L}^p + \rho \theta \dot{\eta}. \quad (12.53)$$

The constraint imposed by the Clausius-Duhem inequality on $\mathbf{L}^p$ suggests using flow rules of the form

$$\mathbf{L}^p(t) = \begin{cases} \mathbf{C}^p : \Delta \mathbf{T}^T & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (12.54)$$

where $\mathbf{C}^p$ is any fourth-order tensor function with the property $\Delta \mathbf{T}^T : \mathbf{C}^p : \Delta \mathbf{T}^T \geq 0$ for all admissible processes. As will be shown in the examples, there are other possible selections, particularly for the simpler models.

In addition to satisfying the Clausius-Duhem inequality, we need to satisfy the consistency condition $\dot{f} = 0$ during plastic flow. For our yield function, the rate is given by

$$\dot{f} = \partial_{\mathbf{F}^e}(f) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^p}(f) : \dot{\mathbf{F}}^p + \partial_{\mathbf{F}^\theta}(f) : \dot{\mathbf{F}}^\theta + \partial_\xi(f)\dot{\xi} + \partial_\theta(f)\dot{\theta}. \quad (12.55)$$

We can replace the rate of change of the elastic deformation gradient in favor of using the rate of the total deformation gradient, which then can be considered to change as an independent variable. The result of this operation is the equation

$$\begin{aligned} \dot{f} = & [\partial_{\mathbf{F}^e}(f) \mathbf{F}^{p-T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}} - [\mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) \mathbf{F}^{p-T}] : \dot{\mathbf{F}}^p - [\mathbf{F}^{pT} \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(f) \mathbf{F}^{p-T} \mathbf{F}^{\theta-T}] : \dot{\mathbf{F}}^\theta \\ & + \partial_{\mathbf{F}^p}(f) : \dot{\mathbf{F}}^p + \partial_{\mathbf{F}^\theta}(f) : \dot{\mathbf{F}}^\theta + \partial_\xi(f)\dot{\xi} + \partial_\theta(f)\dot{\theta}. \end{aligned} \quad (12.56)$$

Before we proceed, let us define the following thermodynamic “stresses” associated with the yield function. These are inspired by the thermodynamic stresses defined for the free energy, and are defined as

$$\mathbf{T}_f^{eT} \equiv \partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT}, \quad (12.57)$$

$$\mathbf{T}_f^{bT} \equiv \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT}, \quad (12.58)$$

$$\mathbf{T}_f^{\theta T} \equiv \partial_{\mathbf{F}^\theta}(f) \mathbf{F}^{\theta T}. \quad (12.59)$$

These allow us to write the expression for $\dot{f}$ as

$$\begin{aligned} \dot{f} = & \mathbf{T}_f^{eT} : \mathbf{L} - [\mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T}] : \mathbf{L}^p - [\mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T}] : \mathbf{L}^\theta \\ & + \mathbf{T}_f^{bT} : \mathbf{L}^p + \mathbf{T}_f^{\theta T} : \mathbf{L}^\theta + \partial_\xi(f)\dot{\xi} + \partial_\theta(f)\dot{\theta}. \end{aligned} \quad (12.60)$$

We reorganize this expression to get

$$\begin{aligned} \dot{f} = & \mathbf{T}_f^{eT} : \mathbf{L} + [\mathbf{T}_f^{bT} - \mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T}] : \mathbf{L}^p + [\mathbf{T}_f^{\theta T} - \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T}] : \mathbf{L}^\theta \\ & + \partial_\xi(f)\dot{\xi} + \partial_\theta(f)\dot{\theta}. \end{aligned} \quad (12.61)$$

Finally, we introduce the expression for thermal expansion $\mathbf{L}^\theta = \alpha \dot{\theta}$ and the hardening rule $\dot{\xi} = \mathbf{A} : \mathbf{L}^p$ into this expression to get

$$\dot{f} = \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \alpha] \dot{\theta} - \Delta \mathbf{T}_f^{pT} : \mathbf{L}^p, \quad (12.62)$$

for the “thermodynamic overstresses of f ” defined as

$$\Delta \mathbf{T}_f^p \equiv \mathbf{F}^{e-1} \mathbf{T}_f^e \mathbf{F}^e - \mathbf{T}_f^b - \partial_\xi(f) \mathbf{A}^T, \quad (12.63)$$

$$\Delta \mathbf{T}_f^\theta \equiv \mathbf{F}^{p-1} \mathbf{F}^{e-1} \mathbf{T}_f^e \mathbf{F}^e \mathbf{F}^p - \mathbf{T}_f^\theta. \quad (12.64)$$

From this it is immediately clear that the trial derivative $\overset{\diamond}{f}$, which is calculated from $\dot{f}$ by setting the plastic flow equal to zero, is given by

$$\overset{\diamond}{f} \equiv \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \alpha] \dot{\theta}, \quad (12.65)$$

and the consistency condition $\dot{f} = 0$ can be written as

$$\boxed{\overset{\diamond}{f} - \Delta \mathbf{T}_f^{pT} : \mathbf{L}^p = 0.} \quad (12.66)$$

For the flow rule $\mathbf{L}^p = \mathbf{C}^p : \Delta \mathbf{T}^T$, this becomes a constraint on $\mathbf{C}^p$ that is written as

$$\overset{\diamond}{f} - \Delta \mathbf{T}_f^{pT} : \mathbf{C}^p : \Delta \mathbf{T}^T = 0. \quad (12.67)$$

12.4 Isotropic examples with J_2 type yield functions

In the following sections, we look at several special examples that mimic the expected behaviors for perfect plasticity, plasticity with isotropic hardening, plasticity with kinematic hardening, and plasticity with combined isotropic and kinematic hardening. These all use the von Mises-type yield function and may be considered to be thermodynamic models from the class of J_2 theories. To show the richness of the possibilities, we select on purpose different assumptions in the different models. As will be seen, in the simpler models we can construct forms that closely resemble the assumptions made for the infinitesimal case, even though the effort to get these results for a nonlinear thermodynamic model seems to be much greater.

12.4.1 Perfectly plastic material

As a first example, we consider the finite deformation thermodynamically based version of an isotropic model for perfectly plastic response. This is possibly the simplest plasticity model. To construct the model, we need to select expressions for the specific free energy, yield function, and hardening parameter. We will use a specific free energy function that represents isotropic elastic response. The form we will select is

$$\psi = \frac{1}{2\rho_o} \left[\kappa (J^e - 1)^2 + G \left(\frac{I^e}{J^{e2/3}} - 3 \right) \right] + \psi^\theta, \quad (12.68)$$

where $J^e = \det(\mathbf{F}^e)$ and $I^e = \text{tr}(\mathbf{B}^e)$, and κ , G and ψ^θ are functions of temperature only. This model is one of many such models that reduce to isotropic linear elastic response at small elastic strains, and which at the small strain limit have κ and G representing, respectively, the bulk and shear moduli. For the yield function, we will consider a von Mises-type yield function written as

$$f = \mathbf{S} : \mathbf{S} - \frac{2}{3} \sigma_{y_o}^2, \quad (12.69)$$

where σ_{y_o} is the yield stress in uniaxial tension and is assumed to be a function of temperature. There is no hardening parameter used here so there is no need of introducing a hardening rule. For the flow rule, let us take a Reuss-type flow rule given by

$$\mathbf{L}^p(t) = \begin{cases} \beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e & \text{if } f = 0 \text{ and } \overset{\diamond}{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (12.70)$$

where use of $\mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e$ in place of $\mathbf{S}$ is suggested by the thermodynamic requirement that $\Delta \mathbf{T}^T : \mathbf{L}^p \geq 0$, which will be shown later.

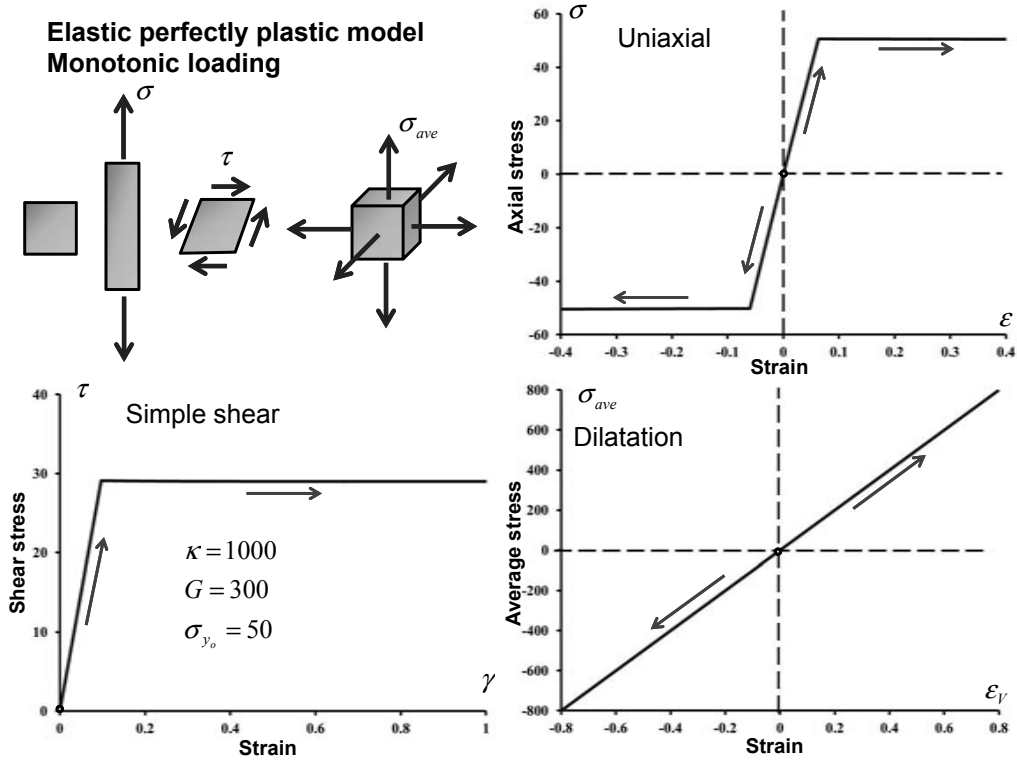


Figure 12.2: Nonlinear elastic perfectly plastic model shown under monotonic loading conditions.

From the specific free energy, we can calculate the expression for the Cauchy stress given by

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{J^p J^\theta} \left[\kappa (J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) \right], \quad (12.71)$$

where use has been made of the conservation of mass $\rho J = \rho_o$, and the relation $J = J^e J^p J^\theta$. The plastic and thermal back-stresses in this case are

$$\mathbf{T}^{bT} = \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \mathbf{0}, \quad (12.72)$$

$$\mathbf{T}^{\theta T} = \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T} = \mathbf{0}, \quad (12.73)$$

since the specific free energy is not a function of either plastic deformation gradient or the thermal deformation gradient. Also, noting that the specific free energy is not a function of ξ , it is clear that the overstress is given by

$$\Delta \mathbf{T} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e = \frac{1}{J^p J^\theta} \left[\kappa (J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) \right]. \quad (12.74)$$

The average stress in this case is given by

$$T_{ave} = \frac{tr(\mathbf{T})}{3} = \frac{1}{J^p J^\theta} \left[\kappa (J^e - 1) + \frac{G}{3J^{e5/3}} (tr(\mathbf{B}^e) - I^e) \right] = \frac{\kappa}{J^p J^\theta} (J^e - 1). \quad (12.75)$$

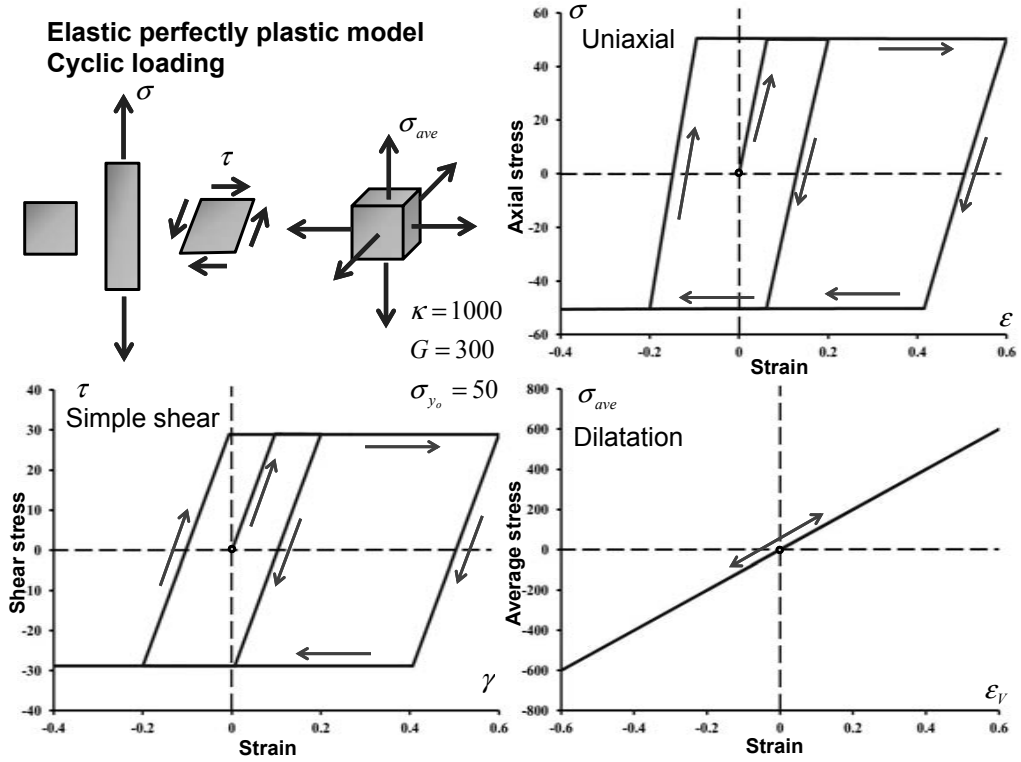


Figure 12.3: Nonlinear elastic perfectly plastic model shown under cyclic loading conditions.

Therefore, the deviatoric stress is given by

$$\mathbf{S} = \mathbf{T} - T_{ave} \mathbf{I} = \frac{G}{J J^{e2/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right). \quad (12.76)$$

The expression for specific free energy is not a function of $\mathbf{F}^\theta$, so, in this case, the specific entropy is given by

$$\begin{aligned} \eta &= -\partial_\theta(\psi) + \frac{1}{\rho} (\mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} \mathbf{F}^{p-T}) : \boldsymbol{\alpha} \\ &= -\frac{1}{2\rho_o} \left[\frac{d\kappa}{d\theta} (J^e - 1)^2 + \frac{dG}{d\theta} \left(\frac{I^e}{J^{e2/3}} - 3 \right) \right] - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} (\mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} \mathbf{F}^{p-T}) : \boldsymbol{\alpha}. \end{aligned} \quad (12.77)$$

During plastic flow the Clausius-Duhem inequality requires that $-\Delta \mathbf{T}^T : \mathbf{L}^p \leq 0$. To check this we recall that $\mathbf{T}$ and $\mathbf{S}$ are symmetric so we can write

$$\begin{aligned} -\Delta \mathbf{T}^T : \mathbf{L}^p &= -(\mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T}) : (\beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e) = -\beta \text{tr}(\mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} \mathbf{F}^e \mathbf{S} \mathbf{F}^{e-T}) \\ &= -\beta \text{tr}(\mathbf{T}^T \mathbf{S}) = -\beta \mathbf{S} : \mathbf{S}. \end{aligned} \quad (12.78)$$

It is now clear that the condition $-\beta \mathbf{S} : \mathbf{S} \leq 0$ is satisfied when β is a positive function since at yield we have $\mathbf{S} : \mathbf{S} = \frac{2}{3} \sigma_{yo}^2 > 0$. Therefore, we will require that

$$\beta \geq 0. \quad (12.79)$$

We now will look at the consistency condition. We first calculate $\mathbf{S} : \mathbf{S}$, which becomes

$$\mathbf{S} : \mathbf{S} = \frac{G}{JJ^{e2/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) : \frac{G}{JJ^{e2/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) = \frac{G^2}{J^2 J^{e4/3}} \left[tr(\mathbf{B}^{e2}) - \frac{1}{3} tr^2(\mathbf{B}^e) \right], \quad (12.80)$$

so that the yield function is given by

$$f = \frac{G^2}{J^2 J^{e4/3}} \left[tr(\mathbf{B}^{e2}) - \frac{1}{3} tr^2(\mathbf{B}^e) \right] - \frac{2}{3} \sigma_{y_o}^2. \quad (12.81)$$

Substituting $J = J^e J^p J^\theta$ and taking the time derivative result in

$$\begin{aligned} \dot{f} = & 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S} \mathbf{F}^e - \frac{10}{9} \sigma_{y_o}^2 \mathbf{F}^{e-T} \right) : \dot{\mathbf{F}}^e - \frac{4}{3} \sigma_{y_o}^2 \mathbf{F}^{p-T} : \dot{\mathbf{F}}^p - \frac{4}{3} \sigma_{y_o}^2 \mathbf{F}^{\theta-T} : \dot{\mathbf{F}}^\theta \\ & + \frac{4}{3} \left(\frac{\sigma_{y_o}^2}{G} \frac{dG}{d\theta} - \sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \right) \dot{\theta}, \end{aligned} \quad (12.82)$$

where we have used $\mathbf{S} : \mathbf{S} = \frac{2}{3} \sigma_{y_o}^2$, and which yields

$$\partial_{\mathbf{F}^e}(f) = 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S} \mathbf{F}^e - \frac{10}{9} \sigma_{y_o}^2 \mathbf{F}^{e-T} \right), \quad (12.83)$$

$$\partial_{\mathbf{F}^p}(f) = -\frac{4}{3} \sigma_{y_o}^2 \mathbf{F}^{p-T}, \quad (12.84)$$

$$\partial_{\mathbf{F}^\theta}(f) = -\frac{4}{3} \sigma_{y_o}^2 \mathbf{F}^{\theta-T}, \quad (12.85)$$

$$\partial_\xi(f) = 0, \quad (12.86)$$

$$\partial_\theta(f) = \frac{4}{3} \left(\frac{\sigma_{y_o}^2}{G} \frac{dG}{d\theta} - \sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \right). \quad (12.87)$$

These give the thermodynamic stresses of f as

$$\mathbf{T}_f^{eT} = 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S} \mathbf{B}^e - \frac{10}{9} \sigma_{y_o}^2 \mathbf{I} \right), \quad (12.88)$$

$$\mathbf{T}_f^{bT} = -\frac{4}{3} \sigma_{y_o}^2 \mathbf{I}, \quad (12.89)$$

$$\mathbf{T}_f^{\theta T} = -\frac{4}{3} \sigma_{y_o}^2 \mathbf{I}, \quad (12.90)$$

with the over stresses of f given as

$$\Delta \mathbf{T}_f^{pT} = \frac{4G}{JJ^{e2/3}} \mathbf{F}^{eT} \mathbf{S} \mathbf{F}^e - \frac{8}{9} \sigma_{y_o}^2 \mathbf{I}, \quad (12.91)$$

$$\Delta \mathbf{T}_f^{\theta T} = \frac{4G}{JJ^{e2/3}} \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{S} \mathbf{F}^e \mathbf{F}^{p-T} - \frac{8}{9} \sigma_{y_o}^2 \mathbf{I}. \quad (12.92)$$

The consistency condition requires that

$$\Delta \mathbf{T}_f^{pT} : \mathbf{L}^p = \dot{f}^\diamond. \quad (12.93)$$

For the model we can write this relation as

$$\left[\frac{4G}{JJ^{e2/3}} \mathbf{F}^{eT} \mathbf{S} \mathbf{F}^e - \frac{8}{9} \sigma_{y_o}^2 \mathbf{I} \right] : (\beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e) = \dot{f}^\diamond. \quad (12.94)$$

Solving for β we obtain

$$\beta = \frac{JJ^{e2/3}}{4G(\mathbf{S}\mathbf{F}^e) : (\mathbf{S}\mathbf{F}^e)} \dot{f}, \quad (12.95)$$

which automatically satisfies the Clausius-Duhem inequality since $\dot{f} > 0$ during plastic flow. The expression for $\dot{f}$ can be calculated from

$$\dot{f} = \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \boldsymbol{\alpha}] \dot{\theta}.$$

This can be reorganized to get

$$\begin{aligned} \dot{f} = & 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S}\mathbf{B}^e - \frac{10}{9} \sigma_{y_o}^2 \mathbf{I} \right) : \mathbf{L} \\ & + \left\{ \frac{4}{3} \left(\frac{\sigma_{y_o}^2}{G} \frac{dG}{d\theta} - \sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \right) - \left(\frac{4G}{JJ^{e2/3}} \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{S}\mathbf{F}^e \mathbf{F}^{p-T} - \frac{8}{9} \sigma_{y_o}^2 \mathbf{I} \right) : \boldsymbol{\alpha} \right\} \dot{\theta}. \end{aligned} \quad (12.96)$$

With all its complexity, the model we have developed is a finite deformation thermodynamically based version of an elastic perfectly plastic material response.

12.4.2 Isotropic hardening

Let us now look at a model constructed to include isotropic hardening. We will build this model based on the elastic perfectly plastic model described in the previous section. This will be done by introducing a hardening model for the yield stress given by

$$\sigma_y = \sigma_{y_o} + \frac{d\sigma_y}{d\xi} \xi, \quad (12.97)$$

where σ_{y_o} and $d\sigma_y/d\xi$ are functions of temperature only, and the hardening parameter ξ is calculated from the relation

$$\dot{\xi} = \Delta \mathbf{T}^T : \mathbf{L}^p, \quad (12.98)$$

with the initial condition $\xi_o = 0$. In this case it is clear that $\mathbf{A} = \Delta \mathbf{T}^T$. We will take the yield function to be of the von Mises form, modified from above to be given as

$$f = \mathbf{S} : \mathbf{S} - \frac{2}{3} \sigma_y^2. \quad (12.99)$$

We take the specific free energy and flow rule as were given in the previous section.

Since there is no change in the free energy, the expressions for the Cauchy stress, over-stress, average stress, and deviatoric stress remain identical to those given, respectively, in (12.71), (12.74), (12.75), and (12.76). Also, we have the same expression for entropy given in (12.77). As a result of (12.78), we can write the hardening rule as

$$\dot{\xi} = \Delta \mathbf{T}^T : \mathbf{L}^p = \beta \mathbf{S} : \mathbf{S} = \frac{2}{3} \sigma_y^2 \beta. \quad (12.100)$$

The expression for $\dot{f}$ in this case is given by

$$\begin{aligned} \dot{f} = & 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S}\mathbf{F}^e - \frac{10}{9} \sigma_y^2 \mathbf{F}^{e-T} \right) : \dot{\mathbf{F}}^e - \frac{4}{3} \sigma_y^2 \mathbf{F}^{p-T} : \dot{\mathbf{F}}^p - \frac{4}{3} \sigma_y^2 \mathbf{F}^{\theta-T} : \dot{\mathbf{F}}^\theta \\ & - \frac{4}{3} \sigma_y \frac{d\sigma_y}{d\xi} \dot{\xi} + \frac{4}{3} \left\{ \frac{\sigma_y^2}{G} \frac{dG}{d\theta} - \sigma_y \left[\frac{d\sigma_{y_o}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_y}{d\xi} \right) \xi \right] \right\} \dot{\theta}, \end{aligned} \quad (12.101)$$

which yields

$$\partial_{\mathbf{F}^e}(f) = 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S}\mathbf{F}^e - \frac{10}{9} \sigma_y^2 \mathbf{F}^{e-T} \right), \quad (12.102)$$

$$\partial_{\mathbf{F}^p}(f) = -\frac{4}{3} \sigma_y^2 \mathbf{F}^{p-T}, \quad (12.103)$$

$$\partial_{\mathbf{F}^\theta}(f) = -\frac{4}{3} \sigma_y^2 \mathbf{F}^{\theta-T}, \quad (12.104)$$

$$\partial_\xi(f) = -\frac{4}{3} \sigma_y \frac{d\sigma_y}{d\xi}, \quad (12.105)$$

$$\partial_\theta(f) = \frac{4}{3} \left\{ \frac{\sigma_y^2}{G} \frac{dG}{d\theta} - \sigma_y \left[\frac{d\sigma_{y_o}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_y}{d\xi} \right) \xi \right] \right\}. \quad (12.106)$$

These give the thermodynamic stresses of f as

$$\mathbf{T}_f^{eT} = 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S}\mathbf{B}^e - \frac{10}{9} \sigma_y^2 \mathbf{I} \right), \quad (12.107)$$

$$\mathbf{T}_f^{bT} = -\frac{4}{3} \sigma_y^2 \mathbf{I}, \quad (12.108)$$

$$\mathbf{T}_f^{\theta T} = -\frac{4}{3} \sigma_y^2 \mathbf{I}, \quad (12.109)$$

with the overstresses of f given by

$$\Delta \mathbf{T}_f^{pT} = \frac{4G}{JJ^{e2/3}} \mathbf{F}^{eT} \mathbf{S}\mathbf{F}^e - \frac{8}{9} \sigma_y^2 \mathbf{I} + \frac{4}{3} \sigma_y \frac{d\sigma_y}{d\xi} \Delta \mathbf{T}^T, \quad (12.110)$$

$$\Delta \mathbf{T}_f^{\theta T} = \frac{4G}{JJ^{e2/3}} \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{S}\mathbf{F}^e \mathbf{F}^{p-T} - \frac{8}{9} \sigma_y^2 \mathbf{I}. \quad (12.111)$$

In this case $\overset{\diamond}{f}$ can be written as

$$\begin{aligned} \overset{\diamond}{f} &= \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} \\ &= 2 \left(\frac{2G}{JJ^{e2/3}} \mathbf{S}\mathbf{B}^e - \frac{10}{9} \sigma_y^2 \mathbf{I} \right) : \mathbf{L} + \left\{ \frac{4}{3} \left[\frac{\sigma_y^2}{G} \frac{dG}{d\theta} - \sigma_y \left(\frac{d\sigma_{y_o}}{d\theta} + \frac{d}{d\theta} \left(\frac{d\sigma_y}{d\xi} \right) \xi \right) \right] \right. \\ &\quad \left. - 2 \left[\frac{2G}{JJ^{e2/3}} \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{S}\mathbf{F}^e \mathbf{F}^{p-T} - \frac{4}{9} \sigma_y^2 \mathbf{I} \right] : \boldsymbol{\alpha} \right\} \dot{\theta}. \end{aligned} \quad (12.112)$$

The consistency condition requires that

$$\left\{ 2 \left[\frac{2G}{JJ^{e2/3}} \mathbf{F}^{eT} \mathbf{S}\mathbf{F}^e - \frac{10}{9} \sigma_y^2 \mathbf{I} \right] + \frac{4}{3} \sigma_y^2 \mathbf{I} + \frac{4}{3} \sigma_y \frac{d\sigma_y}{d\xi} \Delta \mathbf{T}^T \right\} : (\beta \mathbf{F}^{e-1} \mathbf{S}\mathbf{F}^e) = \overset{\diamond}{f}. \quad (12.113)$$

Solving for β we obtain

$$\beta = \frac{1}{\frac{4G}{JJ^{e2/3}} (\mathbf{S}\mathbf{F}^e) : (\mathbf{S}\mathbf{F}^e) + \frac{8}{9} \sigma_y^3 \frac{d\sigma_y}{d\xi}} \overset{\diamond}{f}. \quad (12.114)$$

12.4.3 Kinematic hardening

For our next example, we construct a model that incorporates a back-stress, and we show how this mimics the idea of kinematic hardening. To construct the model, we select

a specific free energy function that represents isotropic elastic response for both the stress and back-stress. This model is

$$\psi = \frac{1}{2\rho_o} \left[\kappa(J^e - 1)^2 + G \left(\frac{I^e}{J^{e2/3}} - 3 \right) \right] + \frac{1}{2\rho_o} \left[\kappa^b(J^p - 1)^2 + G^b \left(\frac{I^p}{J^{p2/3}} - 3 \right) \right] + \psi^\theta, \quad (12.115)$$

where $J^p = \det(\mathbf{F}^p)$ and $I^p = \text{tr}(\mathbf{B}^p)$. In this equation κ^b and G^b are, respectively, the bulk and shear moduli associated with the back-stress and are both functions of temperature only. For the yield function, we will consider a von Mises-type yield function written as

$$f = \Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3} \sigma_{y_o}^2, \quad (12.116)$$

where σ_{y_o} is the initial yield stress in tension and is given as a function of temperature only, and $\Delta \mathbf{S}$ is the deviatoric part of the overstress $\Delta \mathbf{T}$ given by

$$\Delta \mathbf{S} = \Delta \mathbf{T} - \frac{\text{tr}(\Delta \mathbf{T})}{3} \mathbf{I}. \quad (12.117)$$

There is no hardening parameter used here so there is no need to introduce a hardening rule. Let us take a Reuss type flow rule given by

$$\mathbf{L}^p(t) = \begin{cases} \beta \Delta \mathbf{S}^T & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (12.118)$$

where $\Delta \mathbf{T}^T : \mathbf{L}^p = \beta \Delta \mathbf{T}^T : \Delta \mathbf{S}^T = \beta \Delta \mathbf{S}^T : \Delta \mathbf{S}^T$, which automatically will satisfy the Clausius-Duhem inequality if $\beta \geq 0$.

From the specific free energy, we can calculate the expressions for the Cauchy stress and back-stress as

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{J^p J^\theta} \left[\kappa(J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) \right], \quad (12.119)$$

$$\mathbf{T}^{bT} = \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \frac{1}{J^e J^\theta} \left[\kappa^b(J^p - 1) \mathbf{I} + \frac{G^b}{J^{p5/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right) \right]. \quad (12.120)$$

Also, noting that the specific free energy is not a function of ξ , it is clear that the overstress is given by

$$\begin{aligned} \Delta \mathbf{T} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b &= \frac{1}{J^p J^\theta} \left[\kappa(J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) \right] \\ &\quad - \frac{1}{J^e J^\theta} \left[\kappa^b(J^p - 1) \mathbf{I} + \frac{G^b}{J^{p5/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right) \right]. \end{aligned} \quad (12.121)$$

The average overstress in this case becomes

$$\Delta T_{ave} = \frac{\text{tr}(\Delta \mathbf{T})}{3} = \frac{\kappa}{J^p J^\theta} (J^e - 1) - \frac{\kappa^b}{J^e J^\theta} (J^p - 1). \quad (12.122)$$

Therefore, the deviatoric overstress is

$$\Delta \mathbf{S} = \Delta \mathbf{T} - \Delta T_{ave} \mathbf{I} = \frac{G}{J J^{e2/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) - \frac{G^b}{J J^{p2/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right). \quad (12.123)$$

The expression for specific free energy is not a function of $\mathbf{F}^\theta$, so, in this case, the specific entropy is given by

$$\begin{aligned}\eta &= -\partial_\theta(\psi) + \frac{1}{\rho}(\mathbf{F}^{pT}\mathbf{F}^{eT}\mathbf{T}^T\mathbf{F}^{e-T}\mathbf{F}^{p-T}) : \boldsymbol{\alpha} \\ &= -\frac{1}{2\rho_o} \left[\frac{d\kappa}{d\theta}(J^e - 1)^2 + \frac{dG}{d\theta} \left(\frac{I^e}{J^{e2/3}} - 3 \right) + \frac{d\kappa^b}{d\theta}(J^p - 1)^2 + \frac{dG^b}{d\theta} \left(\frac{I^p}{J^{p2/3}} - 3 \right) \right] \\ &\quad - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho}(\mathbf{F}^{pT}\mathbf{F}^{eT}\mathbf{T}^T\mathbf{F}^{e-T}\mathbf{F}^{p-T}) : \boldsymbol{\alpha}.\end{aligned}\quad (12.124)$$

Let us now calculate the rate of change of the yield function. For the selected yield function, this can be written as

$$\dot{f} = 2\Delta\mathbf{S} : \Delta\dot{\mathbf{S}} - \frac{4}{3}\sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \dot{\theta}.\quad (12.125)$$

The expression for $\Delta\dot{\mathbf{S}}$ can be written as

$$\begin{aligned}\Delta\dot{\mathbf{S}} &= -\frac{1}{J^2}(\mathbf{J}\mathbf{F}^{-T} : \dot{\mathbf{F}}) \left[\frac{G}{J^{e2/3}} \left(\mathbf{C}^e - \frac{I^e}{3}\mathbf{I} \right) - \frac{G^b}{J^{p2/3}} \left(\mathbf{B}^p - \frac{I^p}{3}\mathbf{I} \right) \right] \\ &\quad - \frac{2}{3J} \left[\frac{G}{J^{e5/3}}(\mathbf{J}^e\mathbf{F}^{e-T} : \dot{\mathbf{F}}^e) \left(\mathbf{C}^e - \frac{I^e}{3}\mathbf{I} \right) - \frac{G^b}{J^{p5/3}}(\mathbf{J}^p\mathbf{F}^{p-T} : \dot{\mathbf{F}}^p) \left(\mathbf{B}^p - \frac{I^p}{3}\mathbf{I} \right) \right] \\ &\quad + \frac{1}{J} \left[\frac{G}{J^{e2/3}} \left(\dot{\mathbf{C}}^e - \frac{\dot{I}^e}{3}\mathbf{I} \right) - \frac{G^b}{J^{p2/3}} \left(\dot{\mathbf{B}}^p - \frac{\dot{I}^p}{3}\mathbf{I} \right) \right] \\ &\quad + \frac{1}{J} \left[\frac{1}{J^{e2/3}} \frac{dG}{d\theta} \left(\mathbf{C}^e - \frac{I^e}{3}\mathbf{I} \right) - \frac{1}{J^{p2/3}} \frac{dG^b}{d\theta} \left(\mathbf{B}^p - \frac{I^p}{3}\mathbf{I} \right) \right] \dot{\theta}.\end{aligned}\quad (12.126)$$

Substitution of this relation back into the expression for $\dot{f}$, use of the relations

$$\Delta\mathbf{S} : \mathbf{I} = 0,\quad (12.127)$$

$$\Delta\mathbf{S} : \dot{\mathbf{C}}^e = [\mathbf{F}^e(\Delta\mathbf{S} + \Delta\mathbf{S}^T)] : \dot{\mathbf{F}}^e,\quad (12.128)$$

$$\Delta\mathbf{S} : \dot{\mathbf{B}}^p = [(\Delta\mathbf{S} + \Delta\mathbf{S}^T)\mathbf{F}^p] : \dot{\mathbf{F}}^p,\quad (12.129)$$

$$\mathbf{F}^{-T} : \dot{\mathbf{F}} = \mathbf{F}^{e-T} : \dot{\mathbf{F}}^e + \mathbf{F}^{p-T} : \dot{\mathbf{F}}^p + \mathbf{F}^{\theta-T} : \dot{\mathbf{F}}^\theta,\quad (12.130)$$

and noting that for this model $\Delta\mathbf{S}$ is symmetric result in

$$\begin{aligned}\dot{f} &= -2\Delta\mathbf{S} : \Delta\mathbf{S}(\mathbf{F}^{e-T} : \dot{\mathbf{F}}^e + \mathbf{F}^{p-T} : \dot{\mathbf{F}}^p + \mathbf{F}^{\theta-T} : \dot{\mathbf{F}}^\theta) \\ &\quad - \frac{4}{3J} \left[\frac{G}{J^{e2/3}}(\Delta\mathbf{S} : \mathbf{C}^e)\mathbf{F}^{e-T} : \dot{\mathbf{F}}^e - \frac{G^b}{J^{p2/3}}(\Delta\mathbf{S} : \mathbf{B}^p)\mathbf{F}^{p-T} : \dot{\mathbf{F}}^p \right] \\ &\quad + \frac{4}{J} \left[\frac{G}{J^{e2/3}}(\mathbf{F}^e\Delta\mathbf{S}) : \dot{\mathbf{F}}^e - \frac{G^b}{J^{p2/3}}(\Delta\mathbf{S}\mathbf{F}^p) : \dot{\mathbf{F}}^p \right] \\ &\quad + \frac{2}{J} \left[\frac{1}{J^{e2/3}} \frac{dG}{d\theta}(\Delta\mathbf{S} : \mathbf{C}^e) - \frac{1}{J^{p2/3}} \frac{dG^b}{d\theta}(\Delta\mathbf{S} : \mathbf{B}^p) \right] \dot{\theta} \\ &\quad - \frac{4}{3}\sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \dot{\theta}.\end{aligned}\quad (12.131)$$

It, therefore, follows that

$$\partial_{\mathbf{F}^e}(f) = -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{F}^{e-T} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{F}^e \Delta \mathbf{S}, \quad (12.132)$$

$$\partial_{\mathbf{F}^p}(f) = -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \right) \mathbf{F}^{p-T} - \frac{4}{J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} \mathbf{F}^p, \quad (12.133)$$

$$\partial_{\mathbf{F}^\theta}(f) = -2(\Delta \mathbf{S} : \Delta \mathbf{S}) \mathbf{F}^{\theta-T}, \quad (12.134)$$

$$\partial_\xi(f) = 0, \quad (12.135)$$

$$\partial_\theta(f) = \frac{2}{J} \left[\frac{1}{J^{e2/3}} \frac{dG}{d\theta} (\Delta \mathbf{S} : \mathbf{C}^e) - \frac{1}{J^{p2/3}} \frac{dG^b}{d\theta} (\Delta \mathbf{S} : \mathbf{B}^p) \right] - \frac{4}{3} \sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta}. \quad (12.136)$$

These give the thermodynamic stresses of f as

$$\mathbf{T}_f^{eT} = -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{F}^e \Delta \mathbf{S} \mathbf{F}^{eT}, \quad (12.137)$$

$$\mathbf{T}_f^{bT} = -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \right) \mathbf{I} - \frac{4}{J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} \mathbf{B}^p, \quad (12.138)$$

$$\mathbf{T}_f^{\theta T} = -2(\Delta \mathbf{S} : \Delta \mathbf{S}) \mathbf{I}, \quad (12.139)$$

with the over stresses of f given by

$$\begin{aligned} \Delta \mathbf{T}_f^{pT} &= -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{C}^e \Delta \mathbf{S} \\ &+ 2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \right) \mathbf{I} + \frac{4}{J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} \mathbf{B}^p, \end{aligned} \quad (12.140)$$

$$\begin{aligned} \Delta \mathbf{T}_f^{\theta T} &= -2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{F}^{pT} \mathbf{C}^e \Delta \mathbf{S} \mathbf{F}^{p-T} \\ &+ 2(\Delta \mathbf{S} : \Delta \mathbf{S}) \mathbf{I}. \end{aligned} \quad (12.141)$$

From this we get $\overset{\diamond}{f}$ given by

$$\begin{aligned} \overset{\diamond}{f} &= \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} \\ &= \left[-2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{F}^e \Delta \mathbf{S} \mathbf{F}^{eT} \right] : \mathbf{L} \\ &+ \left\{ \frac{2}{J} \left[\frac{1}{J^{e2/3}} \frac{dG}{d\theta} (\Delta \mathbf{S} : \mathbf{C}^e) - \frac{1}{J^{p2/3}} \frac{dG^b}{d\theta} (\Delta \mathbf{S} : \mathbf{B}^p) \right] - \frac{4}{3} \sigma_{y_o} \frac{d\sigma_{y_o}}{d\theta} \right. \\ &- \left[-2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} \right. \\ &\left. \left. + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{F}^{pT} \mathbf{C}^e \Delta \mathbf{S} \mathbf{F}^{p-T} + 2(\Delta \mathbf{S} : \Delta \mathbf{S}) \mathbf{I} \right] : \boldsymbol{\alpha} \right\} \dot{\theta}. \end{aligned} \quad (12.142)$$

The consistency condition for this model can be written as

$$\Delta \mathbf{T}_f^{pT} : \mathbf{L}^p = \overset{\diamond}{f}. \quad (12.143)$$

Substituting from the expressions above yields

$$\begin{aligned} &\left[-2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{2}{3J} \frac{G}{J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4}{J} \frac{G}{J^{e2/3}} \mathbf{C}^e \Delta \mathbf{S} \right. \\ &\left. + 2 \left(\Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \right) \mathbf{I} + \frac{4}{J} \frac{G^b}{J^{p2/3}} \Delta \mathbf{S} \mathbf{B}^p \right] : (\beta \Delta \mathbf{S}^T) = \overset{\diamond}{f}. \end{aligned} \quad (12.144)$$

Elastic-plastic model with combined isotropic and kinematic hardening
Monotonic loading

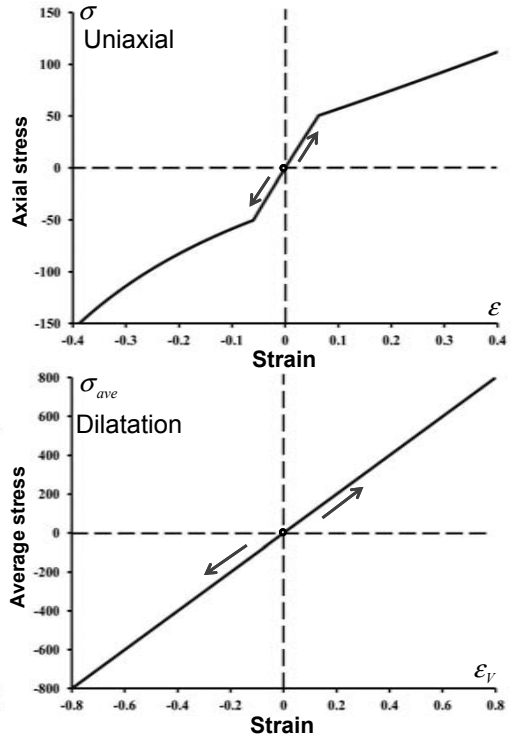
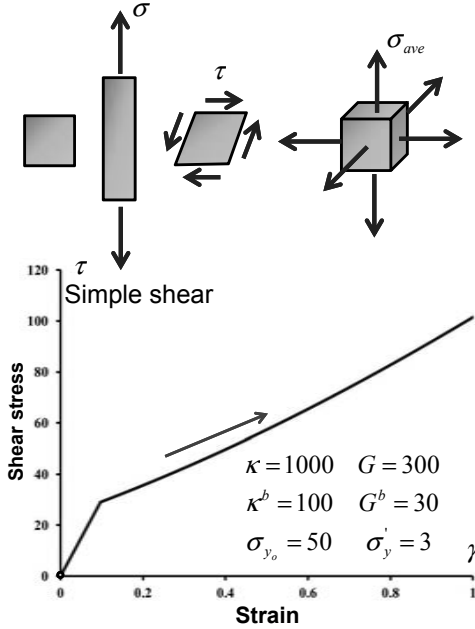


Figure 12.4: Nonlinear elastic-plastic model for combined isotropic and kinematic hardening shown under monotonic loading conditions.

Noting that $\Delta S^T : \mathbf{I} = 0$, $(\mathbf{C}^e \Delta \mathbf{S}) : \Delta \mathbf{S}^T = (\mathbf{F}^e \Delta \mathbf{S}) : (\mathbf{F}^e \Delta \mathbf{S})$, $(\Delta \mathbf{S}^b)^p : \Delta \mathbf{S}^T = (\Delta \mathbf{S}^b)^p : (\Delta \mathbf{S}^b)^p$, and solving for β yield

$$\beta = \frac{J}{\frac{4G}{J^{e2/3}} (\mathbf{F}^e \Delta \mathbf{S}) : (\mathbf{F}^e \Delta \mathbf{S}) + \frac{4G^b}{J^{p2/3}} (\Delta \mathbf{S}^b)^p : (\Delta \mathbf{S}^b)^p} \dot{f}. \quad (12.145)$$

12.4.4 Combined isotropic and kinematic hardening

Here we develop a model that includes both kinematic and isotropic hardening. The details are left as a homework problem. The model is developed as a combination of the two

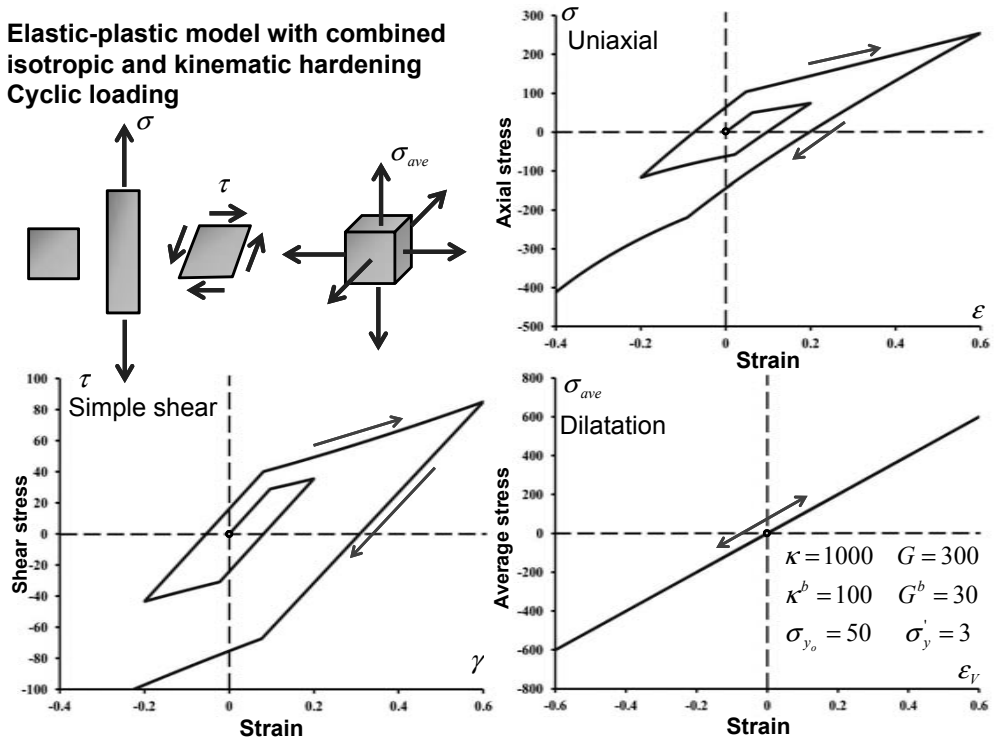


Figure 12.5: Nonlinear elastic-plastic model for combined isotropic and kinematic hardening shown under cyclic loading conditions.

previous examples. We will select for the model the following expressions:

$$\psi = \frac{1}{2\rho_o} \left[\kappa(J^e - 1)^2 + G \left(\frac{I^e}{Je^{2/3}} - 3 \right) \right] + \frac{1}{2\rho_o} \left[\kappa^b(J^p - 1)^2 + G^b \left(\frac{I^p}{Jp^{2/3}} - 3 \right) \right] + \psi^\theta, \quad (12.146)$$

$$f = \Delta \mathbf{S} : \Delta \mathbf{S} - \frac{2}{3} \sigma_y^2, \quad (12.147)$$

$$\sigma_y = \sigma_{y_o} + \sigma_y' \xi, \quad (12.148)$$

$$\mathbf{L}^p(t) = \begin{cases} \beta \Delta \mathbf{S}^T & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (12.149)$$

$$\dot{\xi} = \Delta \mathbf{T}^T : \mathbf{L}^p, \quad \xi_o = 0, \quad (12.150)$$

$$\mathbf{L}^\theta = \alpha \dot{\theta}. \quad (12.151)$$

In this model, we assume that κ , G , κ^b , G^b , ψ^θ , σ_{y_o} , and σ_y' are all functions only of

temperature. This model results in the expressions of thermodynamic stresses given by

$$\mathbf{T}^T \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{J^p J^\theta} \left[\kappa(J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) \right], \quad (12.152)$$

$$\mathbf{T}^{bT} \equiv \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \frac{1}{J^e J^\theta} \left[\kappa^b(J^p - 1) \mathbf{I} + \frac{G^b}{J^{p5/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right) \right], \quad (12.153)$$

$$\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T} = \mathbf{0}. \quad (12.154)$$

The overstress, in this case, is given by $\Delta \mathbf{T} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b$ since the specific free energy does not depend on the hardening parameter and results in

$$\begin{aligned} \Delta \mathbf{T} &\equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T \\ &= \frac{1}{J^p J^\theta} \left[\kappa(J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) \right] \\ &\quad - \frac{1}{J^e J^\theta} \left[\kappa^b(J^p - 1) \mathbf{I} + \frac{G^b}{J^{p5/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right) \right], \end{aligned} \quad (12.155)$$

$$\begin{aligned} \Delta \mathbf{T}^\theta &\equiv \mathbf{F}^{p-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^p - \mathbf{T}^\theta \\ &= \frac{1}{J^p J^\theta} \left[\kappa(J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{F}^{p-1} \mathbf{C}^e \mathbf{F}^p - \frac{I^e}{3} \mathbf{I} \right) \right]. \end{aligned} \quad (12.156)$$

The average and deviatoric overstress is given by

$$\Delta T_{ave} = \frac{\kappa}{J^p J^\theta} (J^e - 1) - \frac{\kappa^b}{J^e J^\theta} (J^p - 1), \quad (12.157)$$

$$\Delta \mathbf{S} = \frac{G}{J J^{e2/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) - \frac{G^b}{J J^{p2/3}} \left(\mathbf{B}^p - \frac{I^p}{3} \mathbf{I} \right). \quad (12.158)$$

From the yield function, we get the thermodynamic stresses associated with f as

$$\begin{aligned} \mathbf{T}_f^{eT} &\equiv \partial_{\mathbf{F}^e}(f) \mathbf{F}^{eT} \\ &= - \left(2\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{4G}{3J J^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} + \frac{4G}{J J^{e2/3}} \mathbf{F}^e \Delta \mathbf{S} \mathbf{F}^{eT}, \end{aligned} \quad (12.159)$$

$$\begin{aligned} \mathbf{T}_f^{bT} &\equiv \partial_{\mathbf{F}^p}(f) \mathbf{F}^{pT} \\ &= - \left(2\Delta \mathbf{S} : \Delta \mathbf{S} - \frac{4G^b}{3J J^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \right) \mathbf{I} - \frac{4G^b}{J J^{p2/3}} \Delta \mathbf{S} \mathbf{B}^p, \end{aligned} \quad (12.160)$$

$$\begin{aligned} \mathbf{T}_f^{\theta T} &\equiv \partial_{\mathbf{F}^\theta}(f) \mathbf{F}^{\theta T} \\ &= - 2(\Delta \mathbf{S} : \Delta \mathbf{S}) \mathbf{I}. \end{aligned} \quad (12.161)$$

The resulting overstresses associated with f are

$$\begin{aligned} \Delta \mathbf{T}_f^{pT} &\equiv \mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T} - \mathbf{T}_f^{bT} - \partial_\xi(f) \mathbf{A} \\ &= - \frac{4G}{3J J^{e2/3}} (\Delta \mathbf{S} : \mathbf{C}^e) \mathbf{I} + \frac{4G}{J J^{e2/3}} \mathbf{C}^e \Delta \mathbf{S} - \frac{4G^b}{3J J^{p2/3}} (\Delta \mathbf{S} : \mathbf{B}^p) \mathbf{I} + \frac{4G^b}{J J^{p2/3}} \Delta \mathbf{S} \mathbf{B}^p \\ &\quad + \frac{4}{3} \sigma_y \sigma'_y \Delta \mathbf{T}^T, \end{aligned} \quad (12.162)$$

$$\begin{aligned} \Delta \mathbf{T}_f^{\theta T} &\equiv \mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{T}_f^{eT} \mathbf{F}^{e-T} \mathbf{F}^{p-T} - \mathbf{T}_f^{\theta T} \\ &= - \frac{4G}{3J J^{e2/3}} (\Delta \mathbf{S} : \mathbf{C}^e) \mathbf{I} + \frac{4G}{J J^{e2/3}} \mathbf{F}^{pT} \mathbf{C}^e \Delta \mathbf{S} \mathbf{F}^{p-T}. \end{aligned} \quad (12.163)$$

From these expressions, we can calculate the trial derivative $\overset{\diamond}{f}$ as

$$\begin{aligned}\overset{\diamond}{f} &\equiv \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} \\ &= - \left(2\Delta \mathbf{S} : \Delta \mathbf{S} + \frac{4G}{3JJe^{2/3}} \Delta \mathbf{S} : \mathbf{C}^e \right) \mathbf{I} : \mathbf{L} + \frac{4G}{JJ^{e2/3}} (\mathbf{F}^e \Delta \mathbf{S} \mathbf{F}^{eT}) : \mathbf{L} \\ &\quad + \left\{ \frac{2}{JJ^{e2/3}} \Delta \mathbf{S} : \mathbf{C}^e \frac{dG}{d\theta} - \frac{2}{JJ^{p2/3}} \Delta \mathbf{S} : \mathbf{B}^p \frac{dG^b}{d\theta} - \frac{4}{3} \sigma_y \left(\frac{d\sigma_{yo}}{d\theta} + \frac{d\sigma'_y}{d\theta} \xi \right) \right. \\ &\quad \left. - \left[-\frac{4G}{3JJ^{e2/3}} (\Delta \mathbf{S} : \mathbf{C}^e) \mathbf{I} + \frac{4G}{JJ^{e2/3}} \mathbf{F}^{pT} \mathbf{C}^e \Delta \mathbf{S} \mathbf{F}^{p-T} \right] : \boldsymbol{\alpha} \right\} \dot{\theta}. \quad (12.164)\end{aligned}$$

Finally, using the consistency condition $\dot{f} = 0$ during plastic flow, we get $\overset{\diamond}{f} = \Delta \mathbf{T}_f^{pT} : \mathbf{L}^p = \Delta \mathbf{T}_f^{pT} : (\beta \Delta \mathbf{S}^T)$, which we can solve for β to get

$$\begin{aligned}\beta &= \frac{\overset{\diamond}{f}}{\Delta \mathbf{T}_f^{pT} : \Delta \mathbf{S}^T} \\ &= \frac{\overset{\diamond}{f}}{\frac{4G}{JJ^{e2/3}} (\mathbf{F}^e \Delta \mathbf{S}) : (\mathbf{F}^e \Delta \mathbf{S}) + \frac{4G^b}{JJ^{p2/3}} (\Delta \mathbf{S} \mathbf{F}^p) : (\Delta \mathbf{S} \mathbf{F}^p) + \frac{8}{9} \sigma_y^3 \sigma'_y}. \quad (12.165)\end{aligned}$$

Section 12.4-Problems:

- 12.4-1:** Develop a program to calculate the elastic perfectly plastic response for a given deformation and temperature history. Test your program using monotonic and cyclic simple shear loading under isothermal, constant temperature rate, and cyclic temperature heating. Compare your monotonic and cyclic loading to the responses shown in, respectively, Figures 12.2 and 12.3.
- 12.4-2:** Develop a program to calculate the elastic-plastic response with isotropic hardening for a given deformation and temperature history. Test your program using monotonic and cyclic simple shear loading under isothermal, constant temperature rate, and cyclic temperature heating.
- 12.4-3:** Develop a program to calculate the elastic-plastic response with kinematic hardening for a given deformation and temperature history. Test your program using monotonic and cyclic simple shear loading under isothermal, constant temperature rate, and cyclic temperature heating.
- 12.4-4:** Develop a program to calculate the elastic-plastic response with combined isotropic and kinematic hardening for a given deformation and temperature history. Test your program using monotonic and cyclic simple shear loading under isothermal, constant temperature rate, and cyclic temperature heating. For your demonstration simulations, use the simulation material parameters $\kappa = 1000(1 - 0.01\theta)$, $G = 300(1 - 0.01\theta)$, $\kappa^b = 100(1 - 0.01\theta)$, $G^b = 30(1 - 0.01\theta)$, $\sigma_{yo} = 50(1 - 0.01\theta)$, and $\sigma'_y = 10(1 - 0.01\theta)$. Compare your isothermal results to those given in Figures 12.4 and 12.5.
- 12.4-5:** Develop a program to calculate the elastic-plastic response with combined isotropic and kinematic hardening for a given deformation and temperature history. Assume the temperature rise is due entirely to the plastic work and is given through the equation

$$c\dot{\theta} = \Delta \mathbf{T} : \mathbf{L}^p.$$

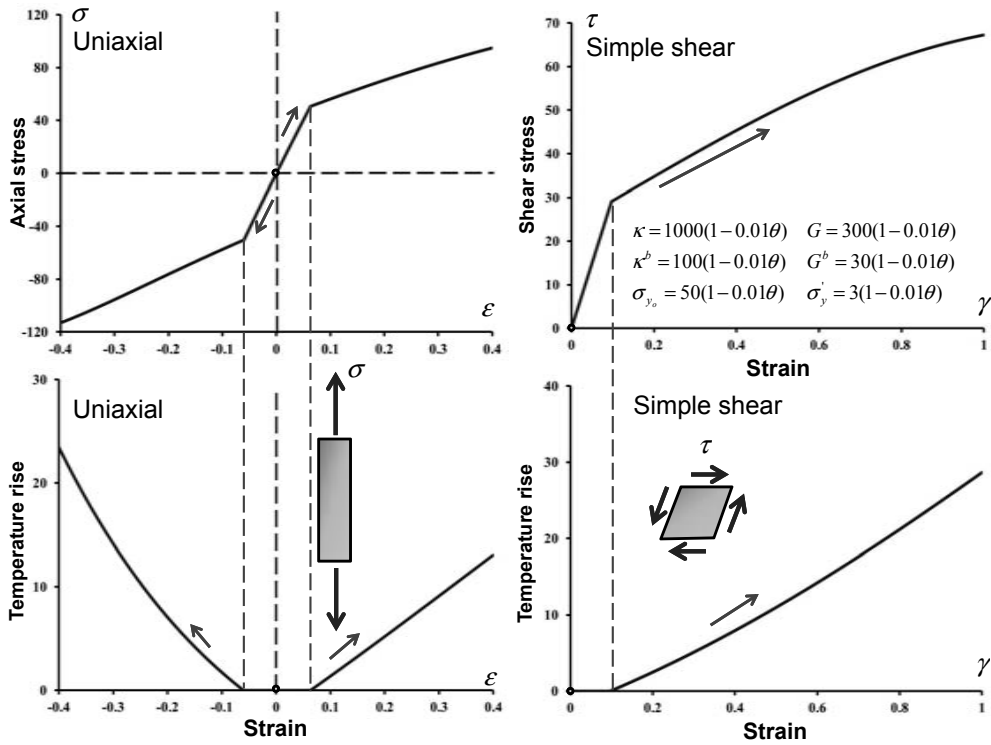


Figure 12.6: Softening due to heating during plastic flow for nonlinear elastic-plastic model for combined isotropic and kinematic hardening shown under monotonic loading conditions.

Select $c = 1$, $\kappa = 1000(1 - 0.01\theta)$, $G = 300(1 - 0.01\theta)$, $\kappa^b = 100(1 - 0.01\theta)$, $G^b = 30(1 - 0.01\theta)$, $\sigma_{y0} = 50(1 - 0.01\theta)$, and $\sigma'_y = 3(1 - 0.01\theta)$, and compare your results to those presented in Figures 12.6, 12.7 and 12.8.

- 12.4-6:** Derive the expressions for $\dot{f}$, $\hat{f}$, and β for combined kinematic and isotropic hardening.

12.5 Superposition of rigid body motions

We already have studied the constraints imposed by rigid body motions on first-gradient material formulations. Essentially, in the chapter on first gradient material models it was shown that given the motion of a body as $\mathbf{x}(\mathbf{X}, t)$, a motion derived from this by superimposing rigid body rotation about $\mathbf{x}_o(s)$ can be written as

$$\mathbf{x}^*(\mathbf{X}, s) = \mathbf{Q}(s)[\mathbf{x}(\mathbf{X}, s) - \mathbf{x}_o(s)] + \mathbf{x}_o(s), \quad (12.166)$$

where $\mathbf{Q}(s)$ is an orthogonal tensor, and s takes values from the starting time t_o to the current time t . For this change we have shown that using $\mathbf{Q}(s) = \mathbf{R}^T(s)$, where $\mathbf{R}(s)$ is the

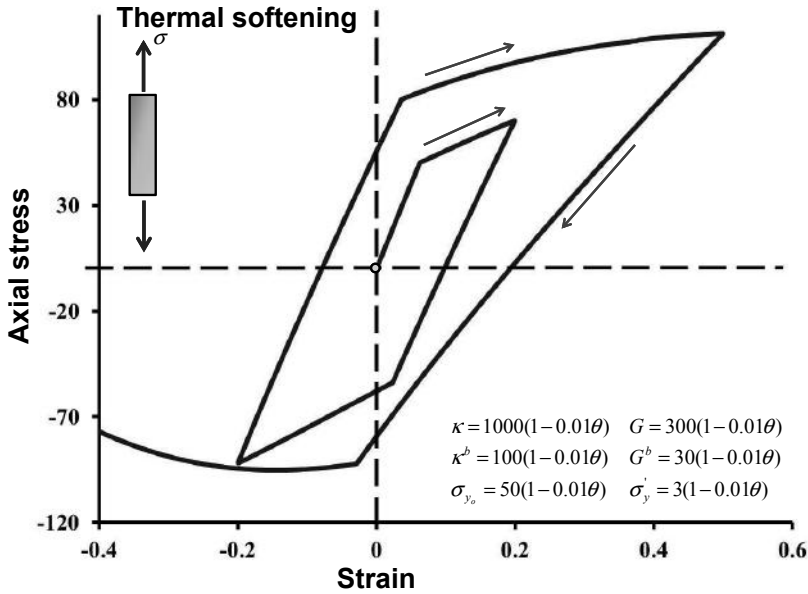


Figure 12.7: Softening due to heating during plastic flow for nonlinear elastic-plastic model for combined isotropic and kinematic hardening shown under uniaxial cyclic loading condition.

orthogonal part in the polar decomposition of $\mathbf{F}(s) = \mathbf{R}(s)\mathbf{U}(s)$, we need to require that

$$\begin{aligned}\psi(\mathbf{X}, t) &= \psi^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &= \psi^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)],\end{aligned}\quad (12.167)$$

$$\begin{aligned}\eta(\mathbf{X}, t) &= \eta^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &= \eta^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)],\end{aligned}\quad (12.168)$$

$$\begin{aligned}\mathbf{T}(\mathbf{X}, t) &= \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &= \mathbf{R}(\mathbf{X}, t)\mathbf{T}^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)]\mathbf{R}^T(\mathbf{X}, t),\end{aligned}\quad (12.169)$$

$$\begin{aligned}\mathbf{q}(\mathbf{X}, t) &= \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &= \mathbf{R}(\mathbf{X}, t)\mathbf{q}^\dagger[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)].\end{aligned}\quad (12.170)$$

In short, in terms of the dependence on the history of the deformation gradient $\mathbf{F}$, the

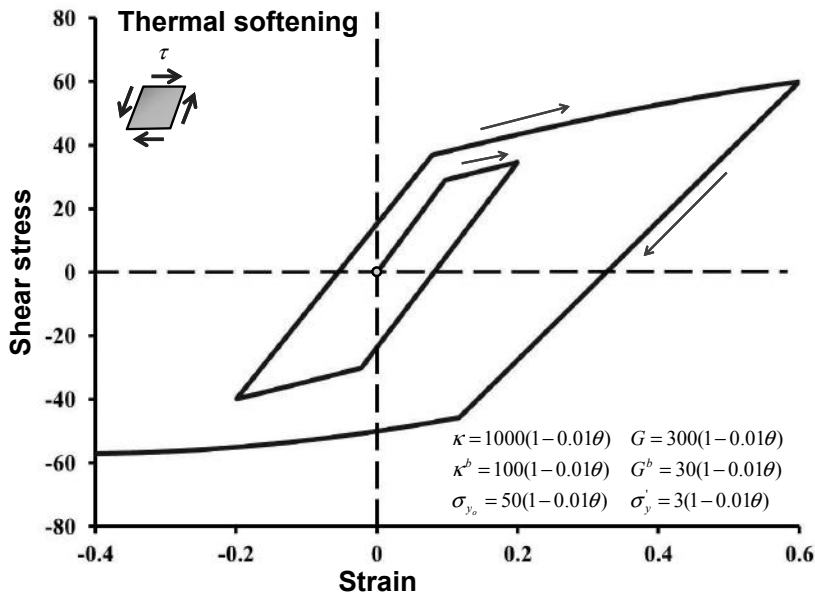


Figure 12.8: Softening due to heating during plastic flow for nonlinear elastic-plastic model for combined isotropic and kinematic hardening shown under shear cyclic loading condition.

assumption states that the specific free energy and entropy can be written in terms of the history of $\mathbf{U}$, the right symmetric part of $\mathbf{F}$ in its polar decomposition. Also, the Cauchy stress and heat flux vector can be calculated from the history of $\mathbf{U}$ plus the current value of $\mathbf{R}$.

Possibly the most important observation is that the past history of $\mathbf{R}$ does not contribute to the response of the material, only its current value does. In particular, if we calculate the response based on the pure deformation, leaving out the history of rigid body motion, we then can calculate the response using the current rigid body rotation through one of the above stated relations. In terms of constructing our plasticity theory in terms of state variables, this states that we can evaluate all state variables from the history of pure deformation, ignoring both the current value and past history of rigid body rotations. Using these state variables provides the value of the response functions up to the known correction given above, needing only the current value of the rigid body rotation. Arguably, the most controversial part of constructing theories of plasticity has been the modeling of the plastic deformation gradient $\mathbf{F}^p$. It seems desirable, as we already have done in the development of the examples, to have the plastic deformation gradient be constant during pure thermoelastic motions, which includes rigid body motions. The same assumption seems reasonable for the thermal deformation gradient $\mathbf{F}^\theta$. Therefore, without any loss of generality, we will assume $\mathbf{F}^p$ and $\mathbf{F}^\theta$ do not change when superimposing rigid body motions on the history. By analogy, we then must conclude that $\mathbf{F}^p$ and $\mathbf{F}^\theta$, such as the case of the specific free

energy and entropy, can be modeled by functionals of the form

$$\begin{aligned}\mathbf{F}^p(\mathbf{X}, t) &= \mathbf{F}^{p\dagger}[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \mathbf{F}^{p\dagger}[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)],\end{aligned}\quad (12.171)$$

$$\begin{aligned}\mathbf{F}^\theta(\mathbf{X}, t) &= \mathbf{F}^{\theta\dagger}[\mathbf{X}, \mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)] \\ &\quad s = t_o \\ &= \mathbf{F}^{\theta\dagger}[\mathbf{X}, \mathbf{U}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s)].\end{aligned}\quad (12.172)$$

In reality, for plasticity we give these functionals as differential equations, their integrals together with initial conditions being the functionals discussed in these equations. Yet, if $\mathbf{F}^p$ and $\mathbf{F}^\theta$ are of this form, then their rates also should be of this form and therefore unaffected by superposition of rigid body motions. The same seems a reasonable assumption for the hardening parameter ξ .

One can define $\mathbf{F}^p$ to depend on rigid body motion, but this will not be a plastic deformation that will be constant in thermoelastic deformations, therefore requiring us to make slightly different arguments in deriving the equations of the thermoelastic range. Also, given such a definition, this $\mathbf{F}^p$ will not necessarily be the internal parameter we can use in the expressions for free energy, entropy, Cauchy stress, and heat flux. We will need to derive from this $\mathbf{F}^p$, and possibly other information from the history, a variable that is consistent with the above.

Returning to the state variable construction of our theory, if $\mathcal{S}(t)$ is given by

$$\mathcal{S}(t) = \{\mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)\}, \quad (12.173)$$

and represents the value of the state of the point of the body due to the initial history, and if $\mathcal{S}^*(t)$ represents the value of the state of the same point for the “*” history and is given by

$$\mathcal{S}^*(t) = \{\mathbf{F}^{e*}(t), \mathbf{F}^{p*}(t), \mathbf{F}^{\theta*}(t), \xi^*(t), \theta^*(t), \mathbf{G}^*(t)\}, \quad (12.174)$$

we have argued our way to assuming

$$\mathcal{S}(t) = \{\mathbf{Q}(t)\mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)\}. \quad (12.175)$$

The relation $\mathbf{F}^{e*}(t) = \mathbf{Q}(t)\mathbf{F}^e(t)$ directly follows from the relation $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p\mathbf{F}^\theta$, which defines $\mathbf{F}^e$. We now conclude that in our state variable formulation we can write

$$\begin{aligned}\psi(\mathbf{X}, t) &= \psi^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \psi^\dagger[\mathbf{X}, \mathbf{U}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)],\end{aligned}\quad (12.176)$$

$$\begin{aligned}\eta(\mathbf{X}, t) &= \eta^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \eta^\dagger[\mathbf{X}, \mathbf{U}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)],\end{aligned}\quad (12.177)$$

$$\begin{aligned}\mathbf{T}(\mathbf{X}, t) &= \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \mathbf{R}^e(t)\mathbf{T}^\dagger[\mathbf{X}, \mathbf{U}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)]\mathbf{R}^{eT}(t),\end{aligned}\quad (12.178)$$

$$\begin{aligned}\mathbf{q}(\mathbf{X}, t) &= \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \mathbf{R}^e(t)\mathbf{q}^\dagger[\mathbf{X}, \mathbf{U}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)].\end{aligned}\quad (12.179)$$

12.6 Material symmetry

Material symmetry normally refers to existing symmetry in the structure of the material before we start the thermomechanical loading process. If present, symmetry in the material structure allows us to make certain rearrangements in the material structure before thermomechanical loading, which will leave the response of the material after the reorganization indistinguishable from the original material irrespective of the loading. We already have studied how to define and impose material symmetry on first-gradient models. The issue here is to look at the particular state variables for plasticity and create appropriate assumptions to impose constraints.

We need to introduce the effect of symmetry transformations on the internal state variables. A material symmetry transformation $\mathbf{M}$ results in the changes $\bar{\mathbf{F}}(t) = \mathbf{F}(t)\mathbf{M}$, $\bar{\theta}(t) = \theta(t)$, and $\bar{\mathbf{G}}(t) = \mathbf{G}(t)\mathbf{M}$, which can be calculated directly based on their definitions. What remains is to express the relations for $\bar{\mathbf{F}}^p$, $\bar{\mathbf{F}}^\theta$, and $\bar{\xi}$. A consistent set of assumptions is

$$\bar{\mathbf{F}}^p(t) = \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \quad (12.180)$$

$$\bar{\mathbf{F}}^\theta(t) = \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \quad (12.181)$$

$$\bar{\xi}(t) = \xi(t), \quad (12.182)$$

which result in the relation

$$\bar{\mathbf{F}}^e(t) = \mathbf{F}^e(t)\mathbf{M}. \quad (12.183)$$

Other assumptions exist but frequently result in anomalous response behavior. We now define the two states

$$\mathcal{S}(t) = \{\mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)\}, \quad (12.184)$$

$$\begin{aligned} \bar{\mathcal{S}}(t) &= \{\bar{\mathbf{F}}^e(t), \bar{\mathbf{F}}^p(t), \bar{\mathbf{F}}^\theta(t), \bar{\xi}(t), \bar{\theta}(t), \bar{\mathbf{G}}(t)\} \\ &= \{\mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t), \mathbf{G}(t)\mathbf{M}\}, \end{aligned} \quad (12.185)$$

which are derived from the two histories $\mathcal{H}(t_o, t)$ and $\bar{\mathcal{H}}(t_o, t)$ for a given $\mathbf{M}$ in the material symmetry group $\mathcal{G}(t_o)$ (see chapter on first-gradient models). The assumption that the response should be the same for these two histories requires that the response functions give the same values for the two state variables. This can be written as

$$\mathcal{R}^\dagger[\mathbf{X}, \mathcal{S}(t)] = \mathcal{R}^\dagger[\mathbf{X}, \bar{\mathcal{S}}(t)]. \quad (12.186)$$

More specifically, this can be written as

$$\begin{aligned} \psi(\mathbf{X}, t) &= \psi^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \psi^\dagger[\mathbf{X}, \mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t), \mathbf{G}(t)\mathbf{M}], \end{aligned} \quad (12.187)$$

$$\begin{aligned} \eta(\mathbf{X}, t) &= \eta^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \eta^\dagger[\mathbf{X}, \mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t), \mathbf{G}(t)\mathbf{M}], \end{aligned} \quad (12.188)$$

$$\begin{aligned} \mathbf{T}(\mathbf{X}, t) &= \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \mathbf{T}^\dagger[\mathbf{X}, \mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t), \mathbf{G}(t)\mathbf{M}], \end{aligned} \quad (12.189)$$

$$\begin{aligned} \mathbf{q}(\mathbf{X}, t) &= \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t), \mathbf{G}(t)] \\ &= \mathbf{q}^\dagger[\mathbf{X}, \mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t), \mathbf{G}(t)\mathbf{M}], \end{aligned} \quad (12.190)$$

for each $\mathbf{M}$ in $\mathcal{G}(t_o)$.

We consider the yield function as part of the response function of the material and so require that $\bar{f}(t) = f(t)$. This gives

$$\begin{aligned} f(\mathbf{X}, t) &= f^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^p(t), \mathbf{F}^\theta(t), \xi(t), \theta(t)] \\ &= f^\dagger[\mathbf{X}, \mathbf{F}^e(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^p(t)\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta(t)\mathbf{M}, \xi(t), \theta(t)]. \end{aligned} \quad (12.191)$$

Also, since each transformation $\mathbf{M}$ is a constant, we will obtain

$$\dot{\bar{\mathbf{F}}}^p(t) = \mathbf{M}^{-1}\dot{\mathbf{F}}^p(t)\mathbf{M}, \quad (12.192)$$

$$\dot{\bar{\mathbf{F}}}^\theta(t) = \mathbf{M}^{-1}\dot{\mathbf{F}}^\theta(t)\mathbf{M}, \quad (12.193)$$

for any time t greater than t_o , and for each $\mathbf{M}$ in $\mathcal{G}(t_o)$. It also follows that

$$\bar{\mathbf{L}}^p(t) = \mathbf{M}^{-1}\mathbf{L}^p(t)\mathbf{M}, \quad (12.194)$$

$$\bar{\mathbf{L}}^\theta(t) = \mathbf{M}^{-1}\mathbf{L}^\theta(t)\mathbf{M}. \quad (12.195)$$

A number of other relations can be derived from these expressions. For each of the following relations, it will go without mention that the relations must hold for every $\mathbf{M}$ in $\mathcal{G}(t_o)$ and for all time t greater than t_o . For thermal expansion given by $\mathbf{L}^\theta = \boldsymbol{\alpha}\dot{\theta}$, it can be shown that

$$\bar{\boldsymbol{\alpha}}(t) = \mathbf{M}^{-1}\boldsymbol{\alpha}(t)\mathbf{M}. \quad (12.196)$$

For the back-stress $\mathbf{T}^{bT}(t) = \rho\partial_{\mathbf{F}^p}(\psi)\mathbf{F}^{pT}$, it can be shown that

$$\bar{\mathbf{T}}^b(t) = \mathbf{M}^{-1}\mathbf{T}^b(t)\mathbf{M}. \quad (12.197)$$

For a hardening rule of the form $\dot{\xi}(t) = \mathbf{A}(t) : \mathbf{L}^p(t)$, it can be shown that the material symmetry constraint is to require

$$\bar{\mathbf{A}}(t) = \mathbf{M}^T\mathbf{A}(t)\mathbf{M}^{-T}. \quad (12.198)$$

The material restriction on the overstress $\Delta\mathbf{T}$ will be

$$\Delta\bar{\mathbf{T}}(t) = \mathbf{M}^{-1}\Delta\mathbf{T}(t)\mathbf{M}. \quad (12.199)$$

For a flow rule of the form $\mathbf{L}^p(t) = \mathbf{C}^p(t) : \Delta\mathbf{T}^T(t)$, the symmetry requirement is given by

$$\bar{\mathbf{C}}^p(t) : [\mathbf{M}^T\Delta\mathbf{T}^T(t)\mathbf{M}^{-T}] = \mathbf{M}^{-1}[\mathbf{C}^p(t) : \Delta\mathbf{T}^T(t)]\mathbf{M} \quad (12.200)$$

for each admissible $\Delta\mathbf{T}^T(t)$.

Even though technically the yield function is a function of the state variables, in practice, we normally write the yield function in terms of stress, since yielding is envisioned to occur when the stress in the material has reached a sufficiently large amount. There is no contradiction here since the stress can be calculated from the state variables. Therefore, yield functions written in terms of stress are a subset of the more general expression for the yield function directly given in terms of the state variables. In any case, as we have seen in the examples, the overstress seems to be an appropriate variable to use as a measure of stress, and, one possibly may add the plastic deformation gradient to introduce additional structure imposed by the “permanent” part of the deformation. Therefore, we consider a special case of the general yield function written as

$$f(\mathbf{X}, t) = f^{\dagger\dagger}[\mathbf{X}, \Delta\mathbf{T}(t), \mathbf{F}^p(t), \xi(t), \theta(t)]. \quad (12.201)$$

Even though $\mathbf{F}^\theta$ does not explicitly appear in the equation, it influences $\mathbf{F}^e$ through the expression $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$, which in turn influences the value of $\Delta \mathbf{T}(t)$. The material symmetry condition on this special form of the yield function can be written as

$$\begin{aligned} f(\mathbf{X}, t) &= f^{\dagger\dagger}[\mathbf{X}, \Delta \mathbf{T}(t), \mathbf{F}^p(t), \xi(t), \theta(t)] \\ &= f^{\dagger\dagger}[\mathbf{X}, \mathbf{M}^{-1} \Delta \mathbf{T}(t) \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}^p(t) \mathbf{M}, \xi(t), \theta(t)]. \end{aligned} \quad (12.202)$$

Section 12.6-Problems:

12.6-1: Prove the relations

$$\begin{aligned} \bar{\alpha}(t) &= \mathbf{M}^{-1} \alpha(t) \mathbf{M}, \\ \bar{\mathbf{T}}^b(t) &= \mathbf{M}^{-1} \mathbf{T}^b(t) \mathbf{M}. \end{aligned}$$

12.6-2: For the overstress defined by

$$\Delta \mathbf{T} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T,$$

show that the material symmetry restriction can be written as

$$\Delta \bar{\mathbf{T}} = \mathbf{M}^{-1} \Delta \mathbf{T} \mathbf{M}.$$

12.6-3: For the hardening rule $\dot{\xi} = \mathbf{A} : \mathbf{L}^p$, derive the relation

$$\bar{\mathbf{A}} = \mathbf{M}^T \mathbf{A} \mathbf{M}^{-T}.$$

12.6-4: Show that in component form, written in an orthonormal system, the restriction of material symmetry on $\bar{\mathbf{C}}^p$ can be written as

$$C_{ijkl}^p = M_{im} M_{nj}^{-1} M_{ko} M_{pl}^{-1} \bar{C}_{mnop}^p.$$

12.7 An initially isotropic material

Here we will consider the material symmetry restrictions on a fairly general, yet not the most general, isotropic thermoplastic material. The reason we will not look at the most general model is the number of possible terms in the representations, which makes the representations fairly large. A reader wishing to develop more general representations is directed for more information on representations and related key reference articles to the chapter by A. J. M. Spencer in *Continuum Physics Volume 1-Mathematics*, edited by A. C. Eringen, Academic Press, 1971, and to the representations given in Appendix A.

We focus here on an initially isotropic material at time t_o that has its symmetry represented by the full group of orthogonal transformations for the selected reference configuration. That is, $\mathcal{G} = \mathcal{O}$, where $\mathcal{O}$ is the group of all orthogonal transformations $\mathbf{M}$ so that $\mathbf{M}^{-1} = \mathbf{M}^T$. We, therefore, can write

$$\bar{\mathbf{F}}^p = \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \quad \bar{\mathbf{F}}^\theta = \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}. \quad (12.203)$$

Given the polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e$ for orthogonal factor $\mathbf{R}^e$ and right symmetric factor $\mathbf{U}^e$, we also can write

$$\bar{\mathbf{R}}^e = \mathbf{R}^e \mathbf{M}, \quad \bar{\mathbf{U}}^e = \mathbf{M}^T \mathbf{U}^e \mathbf{M}, \quad \bar{\mathbf{C}}^e = \mathbf{M}^T \mathbf{C}^e \mathbf{M}. \quad (12.204)$$

Introducing the thermodynamic restrictions and those restrictions imposed by the expected response to superimposed rigid body motions, we select a specific free energy given by the function

$$\psi(t) = \psi^\dagger[\mathbf{C}^e(t), \mathbf{F}^p(t), \xi(t), \theta(t)], \quad (12.205)$$

where the dependence on $\mathbf{X}$ is not explicitly written, even though assumed. The thermal deformation gradient $\mathbf{F}^\theta$ does not directly contribute to the specific free energy, even though its effects can be felt indirectly through the relation $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$. For example, if we constrain the material during thermal expansion, the effect will be manifested in development of elastic strains, that, in turn, would result in a change in the specific free energy. The constraint of initial isotropy on this function can be written as

$$\psi^\dagger[\mathbf{C}^e(t), \mathbf{F}^p(t), \xi(t), \theta(t)] = \psi^\dagger[\mathbf{M}^T \mathbf{C}^e(t) \mathbf{M}, \mathbf{M}^T \mathbf{F}^p(t) \mathbf{M}, \xi(t), \theta(t)], \quad (12.206)$$

for every orthogonal $\mathbf{M}$. We may consider ψ to be given in terms of two symmetric tensors $\mathbf{C}^e$ and $\frac{1}{2}(\mathbf{F}^p + \mathbf{F}^{pT})$, one skew-symmetric tensor $\frac{1}{2}(\mathbf{F}^p - \mathbf{F}^{pT})$ and two scalars ξ and θ . From Appendix A, we can construct a general representation for a scalar valued isotropic function of two symmetric tensors and one skew-symmetric tensor. In this case, the isotropic scalar invariants are

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}^e), & I_2 &= \text{tr}(\mathbf{C}^{e2}), \\ I_3 &= \text{tr}(\mathbf{C}^{e3}), & I_4 &= \frac{1}{2} \text{tr}(\mathbf{F}^p + \mathbf{F}^{pT}), \\ I_5 &= \frac{1}{4} \text{tr}[(\mathbf{F}^p + \mathbf{F}^{pT})^2], & I_6 &= \frac{1}{8} \text{tr}[(\mathbf{F}^p + \mathbf{F}^{pT})^3], \\ I_7 &= \frac{1}{2} \text{tr}[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})], & I_8 &= \frac{1}{2} \text{tr}[\mathbf{C}^{e2}(\mathbf{F}^p + \mathbf{F}^{pT})], \\ I_9 &= \frac{1}{4} \text{tr}[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2], & I_{10} &= \frac{1}{4} \text{tr}[\mathbf{C}^{e2}(\mathbf{F}^p + \mathbf{F}^{pT})^2], \\ I_{11} &= \frac{1}{4} \text{tr}[(\mathbf{F}^p - \mathbf{F}^{pT})^2], & I_{12} &= \frac{1}{4} \text{tr}[\mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2], \end{aligned} \quad (12.207)$$

and

$$\begin{aligned} I_{13} &= \frac{1}{4} \text{tr}[\mathbf{C}^{e2}(\mathbf{F}^p - \mathbf{F}^{pT})^2], \\ I_{14} &= \frac{1}{8} \text{tr}[\mathbf{C}^{e2}(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})], \\ I_{15} &= \frac{1}{8} \text{tr}[(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})^2], \\ I_{16} &= \frac{1}{16} \text{tr}[(\mathbf{F}^p + \mathbf{F}^{pT})^2(\mathbf{F}^p - \mathbf{F}^{pT})^2], \\ I_{17} &= \frac{1}{64} \text{tr}[(\mathbf{F}^p + \mathbf{F}^{pT})^2(\mathbf{F}^p - \mathbf{F}^{pT})^2(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})], \\ I_{18} &= \frac{1}{4} \text{tr}[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})], \\ I_{19} &= \frac{1}{8} \text{tr}[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2(\mathbf{F}^p - \mathbf{F}^{pT})], \\ I_{20} &= \frac{1}{4} \text{tr}[\mathbf{C}^{e2}(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})], \\ I_{21} &= \frac{1}{16} \text{tr}[\mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})]. \end{aligned} \quad (12.208)$$

Therefore, we can write the specific free energy in terms of these invariants, the hardening parameter and temperature. This gives the representation for ψ as

$$\psi = \psi^{\dagger\dagger}(I_1, \dots, I_{21}, \xi, \theta). \quad (12.209)$$

We can calculate the Cauchy stress through the relation

$$\mathbf{T}^T \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \rho \sum_{i=1}^{21} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT}. \quad (12.210)$$

To complete this representation, we need to calculate the derivative of I_i with respect to $\mathbf{F}^e$. These can be calculated directly from the expressions for each invariant. For example,

$$\begin{aligned} \dot{I}_1 &= \overline{\dot{\mathbf{C}}^e} = \text{tr}(\dot{\mathbf{C}}^e) = \text{tr}(\dot{\mathbf{F}}^{eT} \mathbf{F}^e + \mathbf{F}^{eT} \dot{\mathbf{F}}^e) = \text{tr}(\dot{\mathbf{F}}^{eT} \mathbf{F}^e) + \text{tr}(\mathbf{F}^{eT} \dot{\mathbf{F}}^e) \\ &= 2 \text{tr}(\mathbf{F}^{eT} \dot{\mathbf{F}}^e) = 2 \mathbf{F}^e : \dot{\mathbf{F}}^e, \end{aligned} \quad (12.211)$$

so that $\partial_{\mathbf{F}^e}(I_1) = 2\mathbf{F}^e$. In general, for any constant second-order tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\partial_{\mathbf{F}^e}[tr(\mathbf{C}^e \mathbf{A})] = \mathbf{F}^e(\mathbf{A} + \mathbf{A}^T), \quad (12.212)$$

$$\partial_{\mathbf{F}^e}[tr(\mathbf{C}^{e2} \mathbf{A})] = \mathbf{F}^e[\mathbf{C}^e(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)\mathbf{C}^e], \quad (12.213)$$

$$\begin{aligned} \partial_{\mathbf{F}^e}[tr(\mathbf{C}^{e2} \mathbf{A} \mathbf{C}^e \mathbf{B})] &= \mathbf{F}^e[\mathbf{C}^e(\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T) \\ &\quad + (\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T)\mathbf{C}^e + \mathbf{B} \mathbf{C}^{e2} \mathbf{A} + \mathbf{A}^T \mathbf{C}^{e2} \mathbf{B}^T]. \end{aligned} \quad (12.214)$$

These can be used to calculate the following partial derivatives of the invariants:

$$\begin{aligned} \partial_{\mathbf{F}^e}(I_1) &= 2\mathbf{F}^e, & \partial_{\mathbf{F}^e}(I_2) &= 4\mathbf{F}^e \mathbf{C}^e, \\ \partial_{\mathbf{F}^e}(I_3) &= 6\mathbf{F}^e \mathbf{C}^{e2}, & \partial_{\mathbf{F}^e}(I_4) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_5) &= \mathbf{0}, & \partial_{\mathbf{F}^e}(I_6) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_7) &= \mathbf{F}^e(\mathbf{F}^p + \mathbf{F}^{pT}), & \partial_{\mathbf{F}^e}(I_8) &= \mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT}) + (\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_9) &= \frac{1}{2}\mathbf{F}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2, & \partial_{\mathbf{F}^e}(I_{10}) &= \frac{1}{2}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2 + (\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{11}) &= \mathbf{0}, & \partial_{\mathbf{F}^e}(I_{12}) &= \frac{1}{2}\mathbf{F}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2, \\ \partial_{\mathbf{F}^e}(I_{13}) &= \frac{1}{2}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2 + (\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{14}) &= \frac{1}{8}\mathbf{F}^e\{\mathbf{C}^e[(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT}) - (\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2] \\ &\quad + [(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT}) - (\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{C}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2]\mathbf{C}^e \\ &\quad + (\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{C}^{e2}(\mathbf{F}^p - \mathbf{F}^{pT})^2 - (\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{C}^{e2}(\mathbf{F}^p - \mathbf{F}^{pT})\}, \\ \partial_{\mathbf{F}^e}(I_{15}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{16}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{17}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{18}) &= \frac{1}{4}\mathbf{F}^e[(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT}) - (\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})], \\ \partial_{\mathbf{F}^e}(I_{19}) &= \frac{1}{8}\mathbf{F}^e[(\mathbf{F}^p + \mathbf{F}^{pT})^2(\mathbf{F}^p - \mathbf{F}^{pT}) - (\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})^2], \\ \partial_{\mathbf{F}^e}(I_{20}) &= \frac{1}{4}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT}) - (\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{21}) &= \frac{1}{16}\mathbf{F}^e[(\mathbf{F}^p - \mathbf{F}^{pT})^2(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT}) \\ &\quad - (\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{F}^p - \mathbf{F}^{pT})^2]. \end{aligned} \quad (12.215)$$

The general expression for Cauchy stress is then given by

$$\begin{aligned} \mathbf{T}^T &= \rho\{2\frac{\partial\psi}{\partial I_1}\mathbf{B}^e + 4\frac{\partial\psi}{\partial I_2}\mathbf{B}^{e2} + 6\frac{\partial\psi}{\partial I_3}\mathbf{B}^{e3} + \frac{\partial\psi}{\partial I_7}\mathbf{F}^e(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{F}^{eT} \\ &\quad + \frac{\partial\psi}{\partial I_8}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT}) + (\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{C}^e]\mathbf{F}^{eT} + \frac{1}{2}\frac{\partial\psi}{\partial I_9}\mathbf{F}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2\mathbf{F}^{eT} \\ &\quad + \frac{1}{2}\frac{\partial\psi}{\partial I_{10}}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^p + \mathbf{F}^{pT})^2 + (\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{C}^e]\mathbf{F}^{eT} \\ &\quad + \frac{1}{2}\frac{\partial\psi}{\partial I_{12}}\mathbf{F}^e(\mathbf{F}^p - \mathbf{F}^{pT})^2\mathbf{F}^{eT} + \dots\}. \end{aligned} \quad (12.216)$$

The calculation of the back-stress is similar and uses the expression

$$\mathbf{T}^{bT} \equiv \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \rho \sum_{i=1}^{21} \frac{\partial\psi}{\partial I_i} \partial_{\mathbf{F}^p}(I_i) \mathbf{F}^{pT}. \quad (12.217)$$

The partial derivatives of the invariants are as simple to calculate. For example,

$$2\dot{I}_4 = \overline{\dot{tr}(\mathbf{F}^p + \mathbf{F}^{pT})} = tr(\dot{\mathbf{F}}^p) + tr(\dot{\mathbf{F}}^{pT}) = 2tr(\dot{\mathbf{F}}^p) = 2\mathbf{I} : \dot{\mathbf{F}}^p, \quad (12.218)$$

so that

$$\partial_{\mathbf{F}^p}(I_4) = \mathbf{I}. \quad (12.219)$$

Or, for example,

$$\begin{aligned} 4\dot{I}_5 &= \overline{tr[(\mathbf{F}^p + \mathbf{F}^{pT})^2]} = tr[(\dot{\mathbf{F}}^p + \dot{\mathbf{F}}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})] + tr[(\mathbf{F}^p + \mathbf{F}^{pT})(\dot{\mathbf{F}}^p + \dot{\mathbf{F}}^{pT})] \\ &= 2tr[(\dot{\mathbf{F}}^p + \dot{\mathbf{F}}^{pT})(\mathbf{F}^p + \mathbf{F}^{pT})] = 4(\mathbf{F}^p + \mathbf{F}^{pT}) : \dot{\mathbf{F}}^p, \end{aligned} \quad (12.220)$$

so that

$$\partial_{\mathbf{F}^p}(I_5) = \mathbf{F}^p + \mathbf{F}^{pT}. \quad (12.221)$$

The section problems include a set of general tools for calculation of the partial derivatives. The resulting expression for the back-stress is given as

$$\mathbf{T}^{bT} = \rho \left\{ 2 \frac{\partial \psi}{\partial I_4} \mathbf{F}^{pT} + 4 \frac{\partial \psi}{\partial I_5} (\mathbf{B}^p + \mathbf{F}^{p2T}) + \dots \right\}. \quad (12.222)$$

As can be seen, if the plastic deformation gradient is not symmetric then the expression for the back-stress will not be symmetric.

In addition to the specific free energy, we need to provide models for the yield function and for the heat flux. In general, the yield function is a function of the same variables as the specific free energy and is invariant to symmetry transformations in the same manner so that a general expression for the yield function of an initially isotropic material can be written as

$$f(\mathbf{X}, t) = f^\dagger(I_1, \dots, I_{21}, \xi, \theta), \quad (12.223)$$

where, as in the case of the specific free energy, we have left out the dependence on $\mathbf{F}^\theta$, assuming it will contribute to the yielding only through its influence on $\mathbf{F}^e$ through $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$. Though very general, this is probably the least desirable form for a yield function. It is a tradition, and with good reason, to write the yield function for an isotropic material in terms of expressions involving the stress. This is because the “level” of stress is what induces yielding. We introduced a specialized form of this yield function which was a function of the overstress, the plastic deformation gradient, the hardening parameter, and temperature that seems to be general enough to capture most commonly used yield functions. This alternate form of the yield function can be written in terms of temperature, the hardening parameter and the isotropic scalar invariants of the two symmetric tensors $(\Delta \mathbf{T} + \Delta \mathbf{T}^T)$ and $(\mathbf{F}^p + \mathbf{F}^{pT})$, and the two skew-symmetric tensors $(\Delta \mathbf{T} - \Delta \mathbf{T}^T)$ and $(\mathbf{F}^p - \mathbf{F}^{pT})$. These can be extracted from Appendix A.

Section 12.7-Problems:

12.7-1: Show that for any constant second-order tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\partial_{\mathbf{F}^e} [tr(\mathbf{C}^e \mathbf{A})] = \mathbf{F}^e (\mathbf{A} + \mathbf{A}^T), \quad (12.224)$$

$$\partial_{\mathbf{F}^e} [tr(\mathbf{C}^{e2} \mathbf{A})] = \mathbf{F}^e [\mathbf{C}^e (\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T) \mathbf{C}^e], \quad (12.225)$$

$$\begin{aligned} \partial_{\mathbf{F}^e} [tr(\mathbf{C}^{e2} \mathbf{A} \mathbf{C}^e \mathbf{B})] &= \mathbf{F}^e [\mathbf{C}^e (\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T) \\ &\quad + (\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T) \mathbf{C}^e + \mathbf{B} \mathbf{C}^{e2} \mathbf{A} + \mathbf{A}^T \mathbf{C}^{e2} \mathbf{B}^T]. \end{aligned} \quad (12.226)$$

12.7-2: Show that for any constant second-order tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{A}]\} = \mathbf{A} + \mathbf{A}^T, \quad (12.227)$$

$$\partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{A}]\} = \mathbf{A}^T - \mathbf{A}, \quad (12.228)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{A}]\} = \\ (\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F}^p + \mathbf{F}^{pT}), \end{aligned} \quad (12.229)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{A}]\} = \\ - [(\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)(\mathbf{F}^p - \mathbf{F}^{pT})], \end{aligned} \quad (12.230)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{A}(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{B}]\} = \\ (\mathbf{F}^p + \mathbf{F}^{pT})[\mathbf{A}(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{B} + \mathbf{B}^T(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{A}^T] \\ + [\mathbf{A}(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{B} + \mathbf{B}^T(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{A}^T](\mathbf{F}^p + \mathbf{F}^{pT}) \\ + \mathbf{B}(\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{A} + \mathbf{A}^T(\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{B}^T, \end{aligned} \quad (12.231)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{tr[(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{A}(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{B}]\} = \\ - \{(\mathbf{F}^p - \mathbf{F}^{pT})[\mathbf{A}(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{B} - \mathbf{B}^T(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{A}^T] \\ + [\mathbf{A}(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{B} - \mathbf{B}^T(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{A}^T](\mathbf{F}^p - \mathbf{F}^{pT}) \\ + \mathbf{B}(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{A} - \mathbf{A}^T(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{B}^T\}. \end{aligned} \quad (12.232)$$

12.7-3: Show that for any constant vectors $\mathbf{m}$ and $\mathbf{n}$ we have

$$\partial_{\mathbf{F}^e} [\mathbf{m} \circ (\mathbf{C}^e \mathbf{n})] = \mathbf{F}^e (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}), \quad (12.233)$$

$$\partial_{\mathbf{F}^e} [\mathbf{m} \circ (\mathbf{C}^{e2} \mathbf{n})] = \mathbf{F}^e [\mathbf{C}^e (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})\mathbf{C}^e], \quad (12.234)$$

$$\partial_{\mathbf{F}^p} \{\mathbf{m} \circ [(\mathbf{F}^p + \mathbf{F}^{pT})\mathbf{n}]\} = \mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}, \quad (12.235)$$

$$\partial_{\mathbf{F}^p} \{\mathbf{m} \circ [(\mathbf{F}^p - \mathbf{F}^{pT})\mathbf{n}]\} = \mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m}, \quad (12.236)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{\mathbf{m} \circ [(\mathbf{F}^p + \mathbf{F}^{pT})^2 \mathbf{n}]\} = \\ (\mathbf{F}^p + \mathbf{F}^{pT})(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})(\mathbf{F}^p + \mathbf{F}^{pT}), \end{aligned} \quad (12.237)$$

$$\begin{aligned} \partial_{\mathbf{F}^p} \{\mathbf{m} \circ [(\mathbf{F}^p - \mathbf{F}^{pT})^2 \mathbf{n}]\} = \\ - [(\mathbf{F}^p - \mathbf{F}^{pT})(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})(\mathbf{F}^p - \mathbf{F}^{pT})]. \end{aligned} \quad (12.238)$$

12.8 Models depending on $\mathbf{C}^p$

In the development of constitutive models, there was no requirement that the plastic deformation gradient must appear directly in the response functions. An interesting and possibly useful simplification is to assume that only the plastic strain, not the full deformation gradient, appears in the response functions. In particular, let us look at a simple material response characterized by a free energy given by the assumption

$$\boxed{\psi = \psi^\dagger(\mathbf{C}^e, \mathbf{C}^p, \theta)}, \quad (12.239)$$

and a yield function described by a function of the form

$$\boxed{f(\Delta \mathbf{T}, \mathbf{C}^p, \xi, \theta)}. \quad (12.240)$$

This somewhat simplified set of assumptions provides a general enough base to construct models that capture evolving anisotropy both in the elastic range and in the yield function, yet are simple enough to be manageable. One should note that we still assume that $\mathbf{F}^p$ is possibly nonsymmetric, only assuming that its symmetric part contributes to the free energy and yield function. Similar assumptions can be made for the other response functions, such as the heat flux vector.

The conditions of material symmetry on the free energy and on the yield function are given by

$$\psi^\dagger(\mathbf{C}^e, \mathbf{C}^p, \theta) = \psi^\dagger(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{C}^p \mathbf{M}, \theta), \quad (12.241)$$

$$f(\Delta \mathbf{T}, \mathbf{C}^p, \xi, \theta) = f(\mathbf{M}^{-1} \Delta \mathbf{T} \mathbf{M}, \mathbf{M}^T \mathbf{C}^p \mathbf{M}, \xi, \theta), \quad (12.242)$$

for every symmetry transformation $\mathbf{M}$. In general, we focus on orthogonal symmetry transformations for which we have scalar invariants. In this case we can write

$$\psi = \psi^\dagger(I_1, \dots, I_m, \theta), \quad (12.243)$$

$$f(\bar{I}_1, \dots, \bar{I}_n, \xi, \theta), \quad (12.244)$$

where I_i are the m scalar invariants associated with the two symmetric tensors $\mathbf{C}^e$ and $\mathbf{C}^p$, and $\bar{I}_i$ are the n scalar invariants associated with the two symmetric tensors $(\Delta \mathbf{T} + \Delta \mathbf{T}^T)$ and $\mathbf{C}^p$, and one skew-symmetric tensor $(\Delta \mathbf{T} - \Delta \mathbf{T}^T)$. In this case, the expressions for the thermodynamic stresses become

$$\mathbf{T}^T = \rho \sum_{i=1}^m \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e(I_i)} \mathbf{F}^{eT}, \quad (12.245)$$

$$\mathbf{T}^{bT} = \rho \sum_{i=1}^m \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^p(I_i)} \mathbf{F}^{pT}, \quad (12.246)$$

$$\mathbf{T}^{\theta T} = \mathbf{0}. \quad (12.247)$$

The associated thermodynamic stresses of f are given by

$$\mathbf{T}_f^{eT} = \sum_{i=1}^n \frac{\partial f}{\partial \bar{I}_i} \partial_{\mathbf{F}^e(\bar{I}_i)} \mathbf{F}^{eT}, \quad (12.248)$$

$$\mathbf{T}_f^{bT} = \sum_{i=1}^n \frac{\partial f}{\partial \bar{I}_i} \partial_{\mathbf{F}^p(\bar{I}_i)} \mathbf{F}^{pT}, \quad (12.249)$$

$$\mathbf{T}_f^{\theta T} = \sum_{i=1}^n \frac{\partial f}{\partial \bar{I}_i} \partial_{\mathbf{F}^\theta(\bar{I}_i)} \mathbf{F}^{\theta T}. \quad (12.250)$$

12.8.1 Isotropic materials

For an isotropic material, a selection for the isotropic invariants I_i is given in the appendix as

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}^e), & I_2 &= \text{tr}(\mathbf{C}^{e2}), & I_3 &= \text{tr}(\mathbf{C}^{e3}), \\ I_4 &= \text{tr}(\mathbf{C}^p), & I_5 &= \text{tr}(\mathbf{C}^{p2}), & I_6 &= \text{tr}(\mathbf{C}^{p3}), \\ I_7 &= \text{tr}(\mathbf{C}^e \mathbf{C}^p), & I_8 &= \text{tr}(\mathbf{C}^{e2} \mathbf{C}^p), \\ I_9 &= \text{tr}(\mathbf{C}^e \mathbf{C}^{p2}), & I_{10} &= \text{tr}(\mathbf{C}^{e2} \mathbf{C}^{p2}). \end{aligned} \quad (12.251)$$

A possibly better set to work with that somewhat decouples the invariants can be

$$\begin{aligned} I_1^* &= \frac{I_1}{J^{e2/3}}, & I_2^* &= \frac{I_2}{I_1^2}, & I_3^* &= J^e, \\ I_4^* &= \frac{I_4}{J^{p2/3}}, & I_5^* &= \frac{I_5}{I_4^2}, & I_6^* &= J^p, \\ I_7^* &= I_7 - I_1 - I_4 + 3, & I_8^* &= I_8 - I_2 - I_4 + 3, \\ I_9^* &= I_9 - I_1 - I_5 + 3, & I_{10}^* &= I_{10} - I_2 - I_5 + 3. \end{aligned} \quad (12.252)$$

For this second set, the stress is given by

$$\begin{aligned} \mathbf{T} = \rho \left\{ \frac{2}{J^{e2/3}} \frac{\partial \psi}{\partial I_1^*} \left(\mathbf{B}^e - \frac{I_1}{3} \mathbf{I} \right) + \frac{4}{I_1^2} \frac{\partial \psi}{\partial I_2^*} \mathbf{B}^e \left(\mathbf{B}^e - \frac{I_2}{I_1} \mathbf{I} \right) + J^e \frac{\partial \psi}{\partial I_3^*} \mathbf{I} \right. \\ \left. + 2 \frac{\partial \psi}{\partial I_7^*} \mathbf{F}^e (\mathbf{C}^p - \mathbf{I}) \mathbf{F}^{eT} + 2 \frac{\partial \psi}{\partial I_8^*} \mathbf{F}^e (\mathbf{C}^e \mathbf{C}^p + \mathbf{C}^p \mathbf{C}^e - 2 \mathbf{C}^e) \mathbf{F}^{eT} \right. \\ \left. + 2 \frac{\partial \psi}{\partial I_9^*} \mathbf{F}^e (\mathbf{C}^{p2} - \mathbf{I}) \mathbf{F}^{eT} + 2 \frac{\partial \psi}{\partial I_{10}^*} \mathbf{F}^e (\mathbf{C}^e \mathbf{C}^{p2} + \mathbf{C}^{p2} \mathbf{C}^e - 2 \mathbf{C}^e) \mathbf{F}^{eT} \right\}, \quad (12.253) \end{aligned}$$

and the back-stress is given by

$$\begin{aligned} \mathbf{T}^b = \rho \left\{ \frac{2}{J^{p2/3}} \frac{\partial \psi}{\partial I_4^*} \left(\mathbf{B}^p - \frac{I_4}{3} \mathbf{I} \right) + \frac{4}{I_4^2} \frac{\partial \psi}{\partial I_5^*} \mathbf{B}^p \left(\mathbf{B}^p - \frac{I_5}{I_4} \mathbf{I} \right) + J^p \frac{\partial \psi}{\partial I_6^*} \mathbf{I} \right. \\ \left. + 2 \frac{\partial \psi}{\partial I_7^*} \mathbf{F}^p (\mathbf{C}^e - \mathbf{I}) \mathbf{F}^{pT} + 2 \frac{\partial \psi}{\partial I_8^*} \mathbf{F}^p (\mathbf{C}^{e2} - \mathbf{I}) \mathbf{F}^{pT} + 2 \frac{\partial \psi}{\partial I_9^*} \mathbf{F}^p (\mathbf{C}^e \mathbf{C}^p + \mathbf{C}^p \mathbf{C}^e - 2 \mathbf{C}^p) \mathbf{F}^{pT} \right. \\ \left. + 2 \frac{\partial \psi}{\partial I_{10}^*} \mathbf{F}^p (\mathbf{C}^p \mathbf{C}^{e2} + \mathbf{C}^{e2} \mathbf{C}^p - 2 \mathbf{C}^p) \mathbf{F}^{pT} \right\}. \quad (12.254) \end{aligned}$$

In this case, there are 21 isotropic invariants of $\mathbf{C}^p$ and $\Delta \mathbf{T}$ given by

$$\begin{aligned} \bar{I}_1 &= tr(\mathbf{C}^p), & \bar{I}_2 &= tr(\mathbf{C}^{p2}), \\ \bar{I}_3 &= tr(\mathbf{C}^{p3}), & \bar{I}_4 &= \frac{1}{2} tr(\Delta \mathbf{T} + \Delta \mathbf{T}^T), \\ \bar{I}_5 &= \frac{1}{4} tr[(\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2], & \bar{I}_6 &= \frac{1}{8} tr[(\Delta \mathbf{T} + \Delta \mathbf{T}^T)^3], \\ \bar{I}_7 &= \frac{1}{2} tr[\mathbf{C}^p (\Delta \mathbf{T} + \Delta \mathbf{T}^T)], & \bar{I}_8 &= \frac{1}{2} tr[\mathbf{C}^{p2} (\Delta \mathbf{T} + \Delta \mathbf{T}^T)], \\ \bar{I}_9 &= \frac{1}{4} tr[\mathbf{C}^p (\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2], & \bar{I}_{10} &= \frac{1}{4} tr[\mathbf{C}^{p2} (\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2], \\ \bar{I}_{11} &= \frac{1}{4} tr[(\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2], & \bar{I}_{12} &= \frac{1}{4} tr[\mathbf{C}^p (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2], \end{aligned} \quad (12.255)$$

and

$$\begin{aligned} \bar{I}_{13} &= \frac{1}{4} tr[\mathbf{C}^{p2} (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2], \\ \bar{I}_{14} &= \frac{1}{8} tr[\mathbf{C}^{p2} (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2 \mathbf{C}^p (\Delta \mathbf{T} - \Delta \mathbf{T}^T)], \\ \bar{I}_{15} &= \frac{1}{8} tr[(\Delta \mathbf{T} + \Delta \mathbf{T}^T) (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2], \\ \bar{I}_{16} &= \frac{1}{16} tr[(\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2 (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2], \\ \bar{I}_{17} &= \frac{1}{64} tr[(\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2 (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2 (\Delta \mathbf{T} + \Delta \mathbf{T}^T) (\Delta \mathbf{T} - \Delta \mathbf{T}^T)], \\ \bar{I}_{18} &= \frac{1}{4} tr[\mathbf{C}^p (\Delta \mathbf{T} + \Delta \mathbf{T}^T) (\Delta \mathbf{T} - \Delta \mathbf{T}^T)], \\ \bar{I}_{19} &= \frac{1}{8} tr[\mathbf{C}^p (\Delta \mathbf{T} + \Delta \mathbf{T}^T)^2 (\Delta \mathbf{T} - \Delta \mathbf{T}^T)], \\ \bar{I}_{20} &= \frac{1}{4} tr[\mathbf{C}^{p2} (\Delta \mathbf{T} + \Delta \mathbf{T}^T) (\Delta \mathbf{T} - \Delta \mathbf{T}^T)], \\ \bar{I}_{21} &= \frac{1}{16} tr[\mathbf{C}^p (\Delta \mathbf{T} - \Delta \mathbf{T}^T)^2 (\Delta \mathbf{T} + \Delta \mathbf{T}^T) (\Delta \mathbf{T} - \Delta \mathbf{T}^T)]. \end{aligned} \quad (12.256)$$

Section 12.8-Problems:

- 12.8-1:** Show that $\Delta \mathbf{T} : \Delta \mathbf{T}$ is an isotropic invariant of $\Delta \mathbf{T}$ and that it can be constructed from the invariants $\bar{I}_5$ and $\bar{I}_{11}$.
- 12.8-2:** Construct a yield function linear in the invariants $\bar{I}_5$, $\bar{I}_9$, $\bar{I}_{11}$. Calculate the thermodynamic stresses associated with f .

12.9 Heat generation and heat flow

As was shown in Chapter 6, the general form of the balance of work and energy can be written as

$$\dot{h} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta} - \frac{1}{\rho}tr(\mathbf{T}\mathbf{L}), \quad (12.257)$$

where $\dot{h}$ is the rate of heat added to the material point through conduction and radiation. To calculate $\dot{h}$ let us first consider the terms we already have calculated. Namely, we can show that for a thermodynamically consistent thermoplastic material we can write

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - tr(\mathbf{T}\mathbf{L}) = -\Delta\mathbf{T}^T : \mathbf{L}^p. \quad (12.258)$$

Therefore, the expression for the rate of change of the heat added can be written as

$$\dot{h} = \begin{cases} \dot{\eta}\theta - \frac{1}{\rho}\Delta\mathbf{T}^T : \mathbf{L}^p & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \dot{\eta}\theta & \text{for all other cases.} \end{cases} \quad (12.259)$$

The equation for $\dot{\eta}$ can be written as

$$\dot{\eta} = \dot{\eta}_{\mathbf{L}} : \mathbf{L} + \dot{\eta}_{\dot{\theta}}\dot{\theta} + \dot{\eta}_{\mathbf{L}^p} : \mathbf{L}^p, \quad (12.260)$$

where

$$\dot{\eta}_{\mathbf{L}} \equiv \partial_{\mathbf{F}^e}(\eta)\mathbf{F}^{eT}, \quad (12.261)$$

$$\dot{\eta}_{\dot{\theta}} \equiv \partial_{\theta}(\eta) + [\partial_{\mathbf{F}^e}(\eta)\mathbf{F}^{\theta T} - \mathbf{F}^{pT}\mathbf{F}^{eT}\partial_{\mathbf{F}^e}(\eta)\mathbf{F}^{p-T}] : \boldsymbol{\alpha}, \quad (12.262)$$

$$\dot{\eta}_{\mathbf{L}^p} \equiv \partial_{\mathbf{F}^p}(\eta)\mathbf{F}^{pT} - \mathbf{F}^{eT}\partial_{\mathbf{F}^e}(\eta) + \partial_{\xi}(\eta)\mathbf{A}. \quad (12.263)$$

The reader will note that the partial derivatives are defined through the assumption that $\eta^\dagger(\mathbf{X}, \mathbf{F}^e, \mathbf{F}^p, \mathbf{F}^\theta, \xi, \theta)$, consistent with the assumption that $\psi^\dagger(\mathbf{X}, \mathbf{F}^e, \mathbf{F}^p, \mathbf{F}^\theta, \xi, \theta)$. As can be seen, $\dot{\eta}$ contains terms affected by the plastic flow. Therefore, with the plastic flow terms exposed, the expression for $\dot{h}$ is given as

$$\dot{h} = \begin{cases} \theta\dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta\dot{\eta}_{\dot{\theta}}\dot{\theta} + \theta\dot{\eta}_{\mathbf{L}^p} : \mathbf{L}^p - \frac{1}{\rho}\Delta\mathbf{T}^T : \mathbf{L}^p & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \theta\dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta\dot{\eta}_{\dot{\theta}}\dot{\theta} & \text{for all other cases.} \end{cases} \quad (12.264)$$

Plastic flow normally would contribute to both the rising of temperature and the flowing of heat from the point. If we consider a process that is adiabatic (i.e., we thermally isolate the point), then $\dot{h}$ must be set to zero, and we should see an elevation of temperature. In the adiabatic case, during plastic flow we have

$$0 = \theta\dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta\dot{\eta}_{\dot{\theta}}\dot{\theta} + \theta\dot{\eta}_{\mathbf{L}^p} : \mathbf{L}^p - \frac{1}{\rho}\Delta\mathbf{T}^T : \mathbf{L}^p. \quad (12.265)$$

Therefore, the rate of temperature increases due to plastic flow under *adiabatic* conditions is given by

$$\dot{\theta} = \frac{1}{\theta\dot{\eta}_{\dot{\theta}}}\left[\frac{1}{\rho}\Delta\mathbf{T}^T : \mathbf{L}^p - \theta\dot{\eta}_{\mathbf{L}} : \mathbf{L} - \theta\dot{\eta}_{\mathbf{L}^p} : \mathbf{L}^p\right]. \quad (12.266)$$

If, on the other hand, the process is occurring under *isothermal* conditions, one can calculate the heat that must be removed from the point. This is given by setting $\dot{\theta} = 0$ to obtain

$$\dot{h} = \begin{cases} \theta \dot{\eta}_{\mathbf{L}} : \mathbf{L} + \theta \dot{\eta}_{\mathbf{L}^p} : \mathbf{L}^p - \frac{1}{\rho} \Delta \mathbf{T}^T : \mathbf{L}^p & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \theta \dot{\eta}_{\mathbf{L}} : \mathbf{L} & \text{for all other cases.} \end{cases} \quad (12.267)$$

A process that is *isentropic* would have constant entropy so the amount of heat that must move into the point is

$$\dot{h} = \begin{cases} -\frac{1}{\rho} \Delta \mathbf{T}^T : \mathbf{L}^p & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ 0 & \text{for all other cases.} \end{cases} \quad (12.268)$$

Section 12.9-Problems:

12.9-1: Show that for a thermoplastic material we have

$$\rho \dot{\psi} + \rho \eta \dot{\theta} - \text{tr}(\mathbf{T}\mathbf{L}) = -\Delta \mathbf{T}^T : \mathbf{L}^p.$$

12.9-2: Show that

$$\begin{aligned} \dot{\eta}_{\mathbf{L}} &= \partial_{\mathbf{F}^e}(\eta) \mathbf{F}^{eT}, \\ \dot{\eta}_{\dot{\theta}} &= \partial_{\theta}(\eta) + [\partial_{\mathbf{F}^e}(\eta) \mathbf{F}^{\theta T} - \mathbf{F}^{pT} \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\eta) \mathbf{F}^{p-T}] : \boldsymbol{\alpha}, \\ \dot{\eta}_{\mathbf{L}^p} &= \partial_{\mathbf{F}^p}(\eta) \mathbf{F}^{pT} - \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\eta) + \partial_{\xi}(\eta) \mathbf{A}. \end{aligned}$$

12.9-3: For an elastic perfectly plastic material, we have

$$\begin{aligned} \psi &= \frac{1}{2\rho_o} \left[\kappa (J^e - 1)^2 + G \left(\frac{I^e}{J^{e2/3}} - 3 \right) \right] + \psi^{\theta}, \\ \mathbf{L}^{\theta} &= \boldsymbol{\alpha} \dot{\theta}, \\ f &= \mathbf{S} : \mathbf{S} - \frac{2}{3} \sigma_{y_o}^2, \\ \mathbf{L}^p(t) &= \begin{cases} \beta \mathbf{F}^{e-1} \mathbf{S} \mathbf{F}^e & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \end{aligned}$$

where

$$\beta = \frac{J J^{e2/3}}{4G(\mathbf{S}\mathbf{F}^e) : (\mathbf{S}\mathbf{F}^e)} \dot{f}.$$

For this model, we have shown that

$$\begin{aligned}
 \mathbf{T} &= \frac{1}{J^p J^\theta} \left[\kappa (J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right) \right], \\
 \mathbf{S} &= \frac{G}{J J^{e2/3}} \left(\mathbf{B}^e - \frac{I^e}{3} \mathbf{I} \right), \\
 \Delta \mathbf{T} &= \frac{1}{J^p J^\theta} \left[\kappa (J^e - 1) \mathbf{I} + \frac{G}{J^{e5/3}} \left(\mathbf{C}^e - \frac{I^e}{3} \mathbf{I} \right) \right], \\
 \eta &= -\frac{1}{2\rho_o} \left[\frac{d\kappa}{d\theta} (J^e - 1)^2 + \frac{dG}{d\theta} \left(\frac{I^e}{J^{e2/3}} - 3 \right) \right] - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} [\mathbf{F}^{\theta-T} \mathbf{F}^T \mathbf{T}^T \mathbf{F}^{-T} \mathbf{F}^{\theta T}] : \boldsymbol{\alpha}, \\
 &\quad - \Delta \mathbf{T}^T : \mathbf{L}^p = -\beta \mathbf{S} : \mathbf{S}, \\
 \mathbf{A} &= \mathbf{0}, \\
 \partial_\xi(\psi) &= 0.
 \end{aligned}$$

Assume $\boldsymbol{\alpha}$ is constant, and κ , G , ψ^θ and σ_{y_o} are each linear in temperature. For this model, calculate the expression for the rate of temperature rise during adiabatic conditions and the rate of heat flow during isothermal and isentropic conditions.

12.10 Specific free energy of quadratic form in strain

Here we will consider a model for specific free energy of quadratic form in the elastic and plastic Green strains. The general form of such a model can be written as

$$\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \mathbf{E} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^p : \mathbf{E}^b : \mathbf{E}^p + \psi^\theta, \quad (12.269)$$

where $\mathbf{E}$ and $\mathbf{E}^b$ are fourth-order tensor moduli, ψ^θ is a scalar, and all three are assumed to be functions of temperature only. As was shown for the elastic solid, because of the symmetry of the strains $\mathbf{E}^e$ and $\mathbf{E}^p$, one always can rewrite this equation as

$$\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \bar{\mathbf{E}} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^p : \bar{\mathbf{E}}^b : \mathbf{E}^p + \psi^\theta, \quad (12.270)$$

where the moduli $\bar{\mathbf{E}}$ and $\bar{\mathbf{E}}^b$ have the properties

$$\mathbf{A} : \bar{\mathbf{E}} : \mathbf{B} = \mathbf{A}^T : \bar{\mathbf{E}} : \mathbf{B} = \mathbf{A} : \bar{\mathbf{E}} : \mathbf{B}^T = \mathbf{A}^T : \bar{\mathbf{E}} : \mathbf{B}^T, \quad (12.271)$$

$$\mathbf{A} : \bar{\mathbf{E}}^b : \mathbf{B} = \mathbf{A}^T : \bar{\mathbf{E}}^b : \mathbf{B} = \mathbf{A} : \bar{\mathbf{E}}^b : \mathbf{B}^T = \mathbf{A}^T : \bar{\mathbf{E}}^b : \mathbf{B}^T, \quad (12.272)$$

for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$.

The time derivative of the specific free energy can be written as

$$\begin{aligned}
 \dot{\psi} &= \frac{1}{2\rho_o} \dot{\mathbf{E}}^e : \bar{\mathbf{E}} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^e : \bar{\mathbf{E}} : \dot{\mathbf{E}}^e + \frac{1}{2\rho_o} \mathbf{E}^e : \dot{\bar{\mathbf{E}}} : \mathbf{E}^e \\
 &\quad + \frac{1}{2\rho_o} \dot{\mathbf{E}}^p : \bar{\mathbf{E}}^b : \mathbf{E}^p + \frac{1}{2\rho_o} \mathbf{E}^p : \bar{\mathbf{E}}^b : \dot{\mathbf{E}}^p + \frac{1}{2\rho_o} \mathbf{E}^p : \dot{\bar{\mathbf{E}}^b} : \mathbf{E}^p + \dot{\psi}^\theta.
 \end{aligned} \quad (12.273)$$

It can be shown, in the same way as was done for the thermoelastic solid, that this can be expressed as

$$\begin{aligned} \dot{\psi} = & \frac{1}{2\rho_o} [\mathbf{F}^e (\bar{\mathbf{E}} : \mathbf{E}^e + \mathbf{E}^e : \bar{\mathbf{E}})] : \dot{\mathbf{F}}^e + \frac{1}{2\rho_o} [\mathbf{F}^p (\bar{\mathbf{E}}^b : \mathbf{E}^p + \mathbf{E}^p : \bar{\mathbf{E}}^b)] : \dot{\mathbf{F}}^p \\ & + \left[\frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^p : \frac{d\bar{\mathbf{E}}^b}{d\theta} : \mathbf{E}^p + \frac{d\psi^\theta}{d\theta} \right] \dot{\theta}. \end{aligned} \quad (12.274)$$

Examination of the terms reveals that

$$\partial_{\mathbf{F}^e}(\psi) = \frac{1}{2\rho_o} \mathbf{F}^e (\bar{\mathbf{E}} : \mathbf{E}^e + \mathbf{E}^e : \bar{\mathbf{E}}), \quad (12.275)$$

$$\partial_{\mathbf{F}^p}(\psi) = \frac{1}{2\rho_o} \mathbf{F}^p (\bar{\mathbf{E}}^b : \mathbf{E}^p + \mathbf{E}^p : \bar{\mathbf{E}}^b), \quad (12.276)$$

$$\partial_\theta(\psi) = \frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^p : \frac{d\bar{\mathbf{E}}^b}{d\theta} : \mathbf{E}^p + \frac{d\psi^\theta}{d\theta}. \quad (12.277)$$

The expressions for Cauchy stress, back-stress, and specific entropy can be written as

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{2J} \mathbf{F}^e (\bar{\mathbf{E}} : \mathbf{E}^e + \mathbf{E}^e : \bar{\mathbf{E}}) \mathbf{F}^{eT}, \quad (12.278)$$

$$\mathbf{T}^{bT} = \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \frac{1}{2J} \mathbf{F}^p (\bar{\mathbf{E}}^b : \mathbf{E}^p + \mathbf{E}^p : \bar{\mathbf{E}}^b) \mathbf{F}^{pT}, \quad (12.279)$$

$$\begin{aligned} \eta = & -\partial_\theta(\psi) + \frac{1}{\rho} (\mathbf{F}^{\theta-T} \mathbf{F}^T \mathbf{T}^T \mathbf{F}^{-T} \mathbf{F}^{\theta T}) : \boldsymbol{\alpha} \\ = & -\frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\bar{\mathbf{E}}}{d\theta} : \mathbf{E}^e - \frac{1}{2\rho_o} \mathbf{E}^p : \frac{d\bar{\mathbf{E}}^b}{d\theta} : \mathbf{E}^p - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} (\mathbf{F}^{\theta-T} \mathbf{F}^T \mathbf{T}^T \mathbf{F}^{-T} \mathbf{F}^{\theta T}) : \boldsymbol{\alpha}. \end{aligned} \quad (12.280)$$

The Cauchy stress and back-stress can be recast in the following forms

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\tilde{\mathbf{E}} : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (12.281)$$

$$\mathbf{T}^{bT} = \frac{1}{J} \mathbf{F}^p (\tilde{\mathbf{E}}^b : \mathbf{E}^p) \mathbf{F}^{pT}, \quad (12.282)$$

where

$$\tilde{\mathbf{E}} = \frac{1}{2} (\bar{\mathbf{E}} + \bar{\mathbf{E}}^T), \quad \tilde{\mathbf{E}}^b = \frac{1}{2} (\bar{\mathbf{E}}^b + \bar{\mathbf{E}}^{bT}). \quad (12.283)$$

For an orthogonal material symmetry transformation $\mathbf{M}$, we know that $\mathbf{F}^e$ changes to $\mathbf{F}^e \mathbf{M}$, $\mathbf{F}^p$ transforms to $\mathbf{M}^T \mathbf{F}^p \mathbf{M}$, $\mathbf{T}$ transforms to $\mathbf{T}$, and $\mathbf{T}^b$ transforms to $\mathbf{M}^T \mathbf{T}^b \mathbf{M}$. Therefore, the condition of material symmetry on the Cauchy stress and the back-stress, in this case, can be written as

$$\frac{1}{J} \mathbf{F}^e (\tilde{\mathbf{E}} : \mathbf{E}^e) \mathbf{F}^{eT} = \frac{1}{J} \mathbf{F}^e \mathbf{M} [\tilde{\mathbf{E}} : (\mathbf{M}^T \mathbf{E}^e \mathbf{M})] \mathbf{M}^T \mathbf{F}^{eT}, \quad (12.284)$$

$$\frac{1}{J} \mathbf{M}^T \mathbf{F}^p (\tilde{\mathbf{E}}^b : \mathbf{E}^p) \mathbf{F}^{pT} \mathbf{M} = \frac{1}{J} \mathbf{M}^T \mathbf{F}^p \mathbf{M} [\tilde{\mathbf{E}}^b : (\mathbf{M}^T \mathbf{E}^p \mathbf{M})] \mathbf{M}^T \mathbf{F}^{pT} \mathbf{M}. \quad (12.285)$$

We thus obtain

$$\tilde{\mathbf{E}} : \mathbf{E}^e = \mathbf{M} [\tilde{\mathbf{E}} : (\mathbf{M}^T \mathbf{E}^e \mathbf{M})] \mathbf{M}^T, \quad (12.286)$$

$$\tilde{\mathbf{E}}^b : \mathbf{E}^p = \mathbf{M} [\tilde{\mathbf{E}}^b : (\mathbf{M}^T \mathbf{E}^p \mathbf{M})] \mathbf{M}^T, \quad (12.287)$$

for every $\mathbf{E}^e$ and $\mathbf{E}^p$, and for all $\mathbf{M}$ in the symmetry group of the material. Therefore, in an orthogonal base of unit vectors the components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}^b$ need to satisfy the relations

$$\tilde{E}_{ijkl} = M_{im}M_{jn}M_{ko}M_{lp}\tilde{E}_{mnop}, \quad (12.288)$$

$$\tilde{E}_{ijkl}^b = M_{im}M_{jn}M_{ko}M_{lp}\tilde{E}_{mnop}^b. \quad (12.289)$$

For an initially isotropic material, given $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}^b$ have the properties given above, the material symmetry restriction requires that

$$\tilde{\mathbf{E}} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{II}, \quad (12.290)$$

$$\tilde{\mathbf{E}}^b = \lambda^b \mathbf{I} \otimes \mathbf{I} + 2\mu^b \mathbf{II}, \quad (12.291)$$

resulting in the expressions for Cauchy stress and back-stress given as

$$\begin{aligned} \mathbf{T}^T &= \frac{1}{J} \mathbf{F}^e [(\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{II}) : \mathbf{E}^e] \mathbf{F}^{eT} = \frac{1}{J} \mathbf{F}^e [\lambda \text{tr}(\mathbf{E}^e) \mathbf{I} + 2\mu \mathbf{E}^e] \mathbf{F}^{eT} \\ &= \frac{1}{J} [\lambda \text{tr}(\mathbf{E}^e) \mathbf{B}^e + 2\mu \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT}], \end{aligned} \quad (12.292)$$

$$\begin{aligned} \mathbf{T}^{bT} &= \frac{1}{J} \mathbf{F}^p [(\lambda^b \mathbf{I} \otimes \mathbf{I} + 2\mu^b \mathbf{II}) : \mathbf{E}^p] \mathbf{F}^{pT} = \frac{1}{J} \mathbf{F}^p [\lambda^b \text{tr}(\mathbf{E}^p) \mathbf{I} + 2\mu^b \mathbf{E}^p] \mathbf{F}^{pT} \\ &= \frac{1}{J} [\lambda^b \text{tr}(\mathbf{E}^p) \mathbf{B}^p + 2\mu^b \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT}]. \end{aligned} \quad (12.293)$$

Section 12.10-Problems:

- 12.10-1:** The elastic moduli frequently depend on the temperature and on the hardening parameter. Discuss what changes will result if you make the elastic and back-stress moduli depend on temperature and the hardening parameter ξ .

12.10.1 Small strain approximation

The quadratic form seems ideal for developing infinitesimal strain approximation models. For example, if we assume the elastic, plastic, and thermal displacement gradients are small, then we can form the linear approximations for the Cauchy stress and the back-stress in the following manner. We can start by recalling that we have

$$\begin{aligned} \mathbf{F}^e &= \mathbf{I} + \mathbf{H}^e, \\ \mathbf{F}^p &= \mathbf{I} + \mathbf{H}^p, \\ \mathbf{E}^e &= \boldsymbol{\epsilon}^e + \mathbf{H}^{eT} \mathbf{H}^e, \\ \mathbf{E}^p &= \boldsymbol{\epsilon}^p + \mathbf{H}^{pT} \mathbf{H}^p, \\ J &= 1 + \epsilon_V. \end{aligned}$$

Substituting these into the expression for Cauchy stress and back-stress, noting that both stresses are symmetric, and eliminating second-order terms results in

$$\mathbf{T} \approx \tilde{\mathbf{E}} : \boldsymbol{\epsilon}^e, \quad (12.294)$$

$$\mathbf{T}^b \approx \tilde{\mathbf{E}}^b : \boldsymbol{\epsilon}^p. \quad (12.295)$$

Note that if we assume $\|\mathbf{H}^e\| \ll 1$, $\|\mathbf{H}^p\| \ll 1$, and $\|\mathbf{H}^\theta\| \ll 1$, this requires that $\|\mathbf{H}\| \ll 1$, so that $|\epsilon_V|$ is small.

In the case that the rigid body motions are large, yet the strains are small, we can construct the approximation as follows. The only term in the expressions for stress and back-stress that is influenced by rigid body motions is $\mathbf{F}^e$, which only appears in the expression for Cauchy stress. Therefore, the expression for the back-stress remains the same, but the expression for the stress changes. In this case, we can use the polar decomposition $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e$ with the relation

$$\mathbf{U}^e = \mathbf{I} + \mathbf{H}_{U^e},$$

for which we assume $\|\mathbf{H}_{U^e}\|$ is small. In this case, the approximations become

$$\mathbf{T} \approx \mathbf{R}^e (\tilde{\mathbf{E}} : \epsilon^e) \mathbf{R}^{eT}, \quad (12.296)$$

$$\mathbf{T}^b \approx \tilde{\mathbf{E}}^b : \epsilon^p. \quad (12.297)$$

12.11 Plasticity models based on Green strains

Here we focus on special classes of models that have a prominent role in the development of plasticity. Both classes of models use the Green strain, as opposed to the deformation gradients, in the expression of the specific free energy. As the reader recalls, for the decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$ we get the Green strains related by

$$\mathbf{E} = \mathbf{E}^\theta + \mathbf{F}^{\theta T} \mathbf{E}^p \mathbf{F}^\theta + \mathbf{F}^{\theta T} \mathbf{F}^{pT} \mathbf{E}^e \mathbf{F}^p \mathbf{F}^\theta.$$

The first class of models assumes a specific free energy of the form

$$\psi = \psi^\dagger(\mathbf{E}^e, \mathbf{E}^p, \xi, \theta). \quad (12.298)$$

The second class of models assumes this form to be separable into additive parts given by

$$\psi = \psi^e(\mathbf{E}^e, \xi, \theta) + \psi^b(\mathbf{E}^p, \xi, \theta) + \psi^\theta(\theta). \quad (12.299)$$

There is one subtle difference between these two assumptions. Models of the former type can exhibit changing material symmetry in their elastic response, but models of the latter type are too simplified to show any change in elastic material symmetry. That is, the decoupling of the parts essentially eliminates any change in the elastic stress response due to plastic flow.

To each of these two assumptions, we add the assumption that the yield function is given by a function of the form

$$f = f^\dagger(\Delta \mathbf{T}, \mathbf{E}^p, \theta), \quad (12.300)$$

and a hardening rule of the form

$$\dot{\xi} = \mathbf{A} : \mathbf{L}^p, \quad (12.301)$$

and a flow rule of the form

$$\mathbf{L}^p(t) = \begin{cases} \beta \Delta \mathbf{S}^T & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases.} \end{cases} \quad (12.302)$$

12.11.1 Models using elastic and plastic Green strains

First, we consider the case that the specific free energy is assumed to depend on the elastic and plastic Green strains and temperature as given in (12.298). For orthogonal $\mathbf{M}$, the constraint of material symmetry can be written as

$$\psi^\dagger(\mathbf{E}^e, \mathbf{E}^p, \xi, \theta) = \psi^\dagger(\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \mathbf{M}^T \mathbf{E}^p \mathbf{M}, \xi, \theta) \quad (12.303)$$

for each $\mathbf{M}$ in the initial material symmetry group. We can construct general representations for this function for initially isotropic, initially transversely isotropic, and initially orthotropic materials.

Initially isotropic materials

For initially isotropic materials, we can express the specific free energy as

$$\psi = \psi^{\dagger\dagger}(I_1, \dots, I_{10}, \xi, \theta), \quad (12.304)$$

where

i	I_i	$\partial_{\mathbf{F}^e}(I_i)$	$\partial_{\mathbf{F}^p}(I_i)$
1	$\det(\mathbf{F}^e)$	$\det(\mathbf{F}^e) \mathbf{F}^{e-T}$	$\mathbf{0}$
2	$\text{tr}(\mathbf{E}^e)$	$\mathbf{F}^e$	$\mathbf{0}$
3	$\text{tr}(\mathbf{E}^{e2})$	$2\mathbf{F}^e \mathbf{E}^e$	$\mathbf{0}$
4	$\det(\mathbf{F}^p)$	$\mathbf{0}$	$\det(\mathbf{F}^p) \mathbf{F}^{p-T}$
5	$\text{tr}(\mathbf{E}^p)$	$\mathbf{0}$	$\mathbf{F}^p$
6	$\text{tr}(\mathbf{E}^{p2})$	$\mathbf{0}$	$2\mathbf{F}^p \mathbf{E}^p$
7	$\text{tr}(\mathbf{E}^e \mathbf{E}^p)$	$\mathbf{F}^e \mathbf{E}^p$	$\mathbf{F}^p \mathbf{E}^e$
8	$\text{tr}(\mathbf{E}^e \mathbf{E}^{p2})$	$\mathbf{F}^e \mathbf{E}^{p2}$	$\mathbf{F}^p (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e)$
9	$\text{tr}(\mathbf{E}^{e2} \mathbf{E}^p)$	$\mathbf{F}^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e)$	$\mathbf{F}^p \mathbf{E}^{e2}$
10	$\text{tr}(\mathbf{E}^{e2} \mathbf{E}^{p2})$	$\mathbf{F}^e (\mathbf{E}^e \mathbf{E}^{p2} + \mathbf{E}^{p2} \mathbf{E}^e)$	$\mathbf{F}^p (\mathbf{E}^{e2} \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^{e2})$

From these we can calculate the expressions for the Cauchy stress as

$$\begin{aligned} \mathbf{T}^T &= \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \rho \sum_{i=1}^{10} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT} \\ &= \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right. \\ &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^e \mathbf{E}^p \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^e \mathbf{E}^{p2} \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{eT} \\ &\quad \left. + \frac{\partial \psi}{\partial I_{10}} \mathbf{F}^e (\mathbf{E}^e \mathbf{E}^{p2} + \mathbf{E}^{p2} \mathbf{E}^e) \mathbf{F}^{eT} \right\}. \end{aligned} \quad (12.306)$$

The expression for the back-stress is given by

$$\begin{aligned} \mathbf{T}^{bT} &= \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT} = \rho \sum_{i=1}^{10} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^p}(I_i) \mathbf{F}^{pT} \\ &= \rho \left\{ \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right. \\ &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^p \mathbf{E}^e \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^p (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^p \mathbf{E}^{e2} \mathbf{F}^{pT} \\ &\quad \left. + \frac{\partial \psi}{\partial I_{10}} \mathbf{F}^p (\mathbf{E}^{e2} \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^{e2}) \mathbf{F}^{pT} \right\}. \end{aligned} \quad (12.307)$$

The reader will note that in these models both the expression for the Cauchy stress and the expression for the back-stress are symmetric. The expression for the overstress is given by

$$\begin{aligned}
 \Delta \mathbf{T} &= \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T \\
 &= \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{C}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{E}^e \mathbf{C}^e \right. \\
 &\quad - \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} - \frac{\partial \psi}{\partial I_5} \mathbf{B}^p - 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \\
 &\quad + \frac{\partial \psi}{\partial I_7} (\mathbf{E}^p \mathbf{C}^e - \mathbf{F}^p \mathbf{E}^e \mathbf{F}^{pT}) + \frac{\partial \psi}{\partial I_8} [\mathbf{E}^{p2} \mathbf{C}^e - \mathbf{F}^p (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{pT}] \\
 &\quad + \frac{\partial \psi}{\partial I_9} [(\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{C}^e - \mathbf{F}^p \mathbf{E}^{e2} \mathbf{F}^{pT}] \\
 &\quad \left. + \frac{\partial \psi}{\partial I_{10}} [(\mathbf{E}^e \mathbf{E}^{p2} + \mathbf{E}^{p2} \mathbf{E}^e) \mathbf{C}^e - \mathbf{F}^p (\mathbf{E}^{e2} \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^{e2}) \mathbf{F}^{pT}] - \partial_\xi(\psi) \mathbf{A}^T \right\}. \quad (12.308)
 \end{aligned}$$

If the reader examines this expression, they will note that the overstress is not symmetric in the terms associated with I_7 to I_{10} , even though both the Cauchy stress and back-stress are symmetric.

Initially transversely isotropic materials

For an initially transversely isotropic material with a preferred axis of transverse isotropy given by $\mathbf{m}$, the expression for the specific free energy can be written as

$$\psi = \psi^{\dagger\dagger}(I_1, \dots, I_{14}, \xi, \theta), \quad (12.309)$$

where I_1 to I_9 are the same as that given in (12.305) and I_{10} to I_{14} are given as

i	I_i	$\partial_{\mathbf{F}^e}(I_i)$	$\partial_{\mathbf{F}^p}(I_i)$
10	$\mathbf{m} \circ (\mathbf{E}^e \mathbf{m})$	$(\mathbf{F}^e \mathbf{m}) \otimes \mathbf{m}$	$\mathbf{0}$
11	$\mathbf{m} \circ (\mathbf{E}^{e2} \mathbf{m})$	$\mathbf{F}^e [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})]$	$\mathbf{0}$
12	$\mathbf{m} \circ (\mathbf{E}^p \mathbf{m})$	$\mathbf{0}$	$(\mathbf{F}^p \mathbf{m}) \otimes \mathbf{m}$
13	$\mathbf{m} \circ (\mathbf{E}^{p2} \mathbf{m})$	$\mathbf{0}$	$\mathbf{F}^p [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})]$
14	$2\mathbf{m} \circ (\mathbf{E}^e \mathbf{E}^p \mathbf{m})$	$\mathbf{F}^e [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})]$	$\mathbf{F}^p [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})]$

(12.310)

The expression for the Cauchy stress is given by

$$\begin{aligned}
 \mathbf{T}^T &= \rho \sum_{i=1}^{14} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT} \\
 &= \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right. \\
 &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^e \mathbf{E}^p \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^e \mathbf{E}^{p2} \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{eT} \\
 &\quad + \frac{\partial \psi}{\partial I_{10}} (\mathbf{F}^e \mathbf{m}) \otimes (\mathbf{F}^e \mathbf{m}) + \frac{\partial \psi}{\partial I_{11}} \mathbf{F}^e [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})] \mathbf{F}^{eT} \\
 &\quad \left. + \frac{\partial \psi}{\partial I_{14}} \mathbf{F}^e [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})] \mathbf{F}^{eT} \right\}. \quad (12.311)
 \end{aligned}$$

The expression for the back-stress is given by

$$\begin{aligned}
 \mathbf{T}^{bT} &= \rho \sum_{i=1}^{14} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^p}(I_i) \mathbf{F}^{pT} \\
 &= \rho \left\{ \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right. \\
 &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^p \mathbf{E}^e \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^p (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^p \mathbf{E}^{e2} \mathbf{F}^{pT} \\
 &\quad + \frac{\partial \psi}{\partial I_{12}} (\mathbf{F}^p \mathbf{m}) \otimes (\mathbf{F}^p \mathbf{m}) + \frac{\partial \psi}{\partial I_{13}} \mathbf{F}^p [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})] \mathbf{F}^{pT} \\
 &\quad \left. + \frac{\partial \psi}{\partial I_{14}} \mathbf{F}^p [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})] \mathbf{F}^{pT} \right\}. \quad (12.312)
 \end{aligned}$$

Initially orthotropic materials

For an initially orthotropic material with a preferred axis of orthotropy given by the three orthogonal directions $\mathbf{e}_i$, we define

$$\mathbf{m} = \mathbf{e}_1, \quad \mathbf{Y} = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (12.313)$$

The invariants in this case are the invariants I_1 to I_{13} for transversely isotropic materials given in (12.310) plus invariants I_{14} to I_{17} given by

i	I_i	$\partial_{\mathbf{F}^e}(I_i)$	$\partial_{\mathbf{F}^p}(I_i)$
14	$tr(\mathbf{Y}\mathbf{E}^e)$	$\mathbf{F}^e \mathbf{Y}$	$\mathbf{0}$
15	$tr(\mathbf{Y}\mathbf{E}^{e2})$	$\mathbf{F}^e (\mathbf{Y}\mathbf{E}^e + \mathbf{E}^e \mathbf{Y})$	$\mathbf{0}$
16	$tr(\mathbf{Y}\mathbf{E}^p)$	$\mathbf{0}$	$\mathbf{F}^p \mathbf{Y}$
17	$tr(\mathbf{Y}\mathbf{E}^{p2})$	$\mathbf{0}$	$\mathbf{F}^p (\mathbf{Y}\mathbf{E}^p + \mathbf{E}^p \mathbf{Y})$

(12.314)

The expression for the Cauchy stress is given by

$$\begin{aligned}
 \mathbf{T}^T &= \rho \sum_{i=1}^{17} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT} \\
 &= \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right. \\
 &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^e \mathbf{E}^p \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^e \mathbf{E}^{p2} \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^e (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{eT} \\
 &\quad + \frac{\partial \psi}{\partial I_{10}} (\mathbf{F}^e \mathbf{m}) \otimes (\mathbf{F}^e \mathbf{m}) + \frac{\partial \psi}{\partial I_{11}} \mathbf{F}^e [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})] \mathbf{F}^{eT} \\
 &\quad \left. + \frac{\partial \psi}{\partial I_{14}} \mathbf{F}^e \mathbf{Y} \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_{15}} \mathbf{F}^e [\mathbf{Y}\mathbf{E}^e + \mathbf{E}^e \mathbf{Y}] \mathbf{F}^{eT} \right\}. \quad (12.315)
 \end{aligned}$$

The expression for the back-stress is given by

$$\begin{aligned}
 \mathbf{T}^{bT} &= \rho \sum_{i=1}^{17} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^p} (I_i) \mathbf{F}^{pT} \\
 &= \rho \left\{ \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right. \\
 &\quad + \frac{\partial \psi}{\partial I_7} \mathbf{F}^p \mathbf{E}^e \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_8} \mathbf{F}^p (\mathbf{E}^e \mathbf{E}^p + \mathbf{E}^p \mathbf{E}^e) \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_9} \mathbf{F}^p \mathbf{E}^{e2} \mathbf{F}^{pT} \\
 &\quad + \frac{\partial \psi}{\partial I_{12}} (\mathbf{F}^p \mathbf{m}) \otimes (\mathbf{F}^p \mathbf{m}) + \frac{\partial \psi}{\partial I_{13}} \mathbf{F}^p [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})] \mathbf{F}^{pT} \\
 &\quad \left. + \frac{\partial \psi}{\partial I_{16}} \mathbf{F}^p \mathbf{Y} \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_{17}} \mathbf{F}^p [\mathbf{Y} \mathbf{E}^p + \mathbf{E}^p \mathbf{Y}] \mathbf{F}^{pT} \right\}. \quad (12.316)
 \end{aligned}$$

12.11.2 Separable free energy

When the specific free energy is separable, as given in (12.299), the constraint of material symmetry can be written as

$$\psi^e(\mathbf{E}^e, \xi, \theta) = \psi^e(\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \xi, \theta), \quad (12.317)$$

$$\psi^b(\mathbf{E}^p, \xi, \theta) = \psi^b(\mathbf{M}^T \mathbf{E}^p \mathbf{M}, \xi, \theta), \quad (12.318)$$

for every $\mathbf{M}$ in the initial material symmetry group.

Initially isotropic materials

For an initially isotropic material, the material symmetry restrictions require that

$$\psi^e = \psi^e(I_1, I_2, I_3, \xi, \theta), \quad (12.319)$$

$$\psi^b = \psi^b(I_4, I_5, I_6, \xi, \theta), \quad (12.320)$$

where I_i are as given in (12.305). In this case the expression for the Cauchy stress and back-stress are given by

$$\mathbf{T}^T = \rho \left\{ \frac{\partial \psi^e}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi^e}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi^e}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right\}, \quad (12.321)$$

$$\mathbf{T}^{bT} = \rho \left\{ \frac{\partial \psi^b}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi^b}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi^b}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right\}. \quad (12.322)$$

As the reader will note, all the terms involving both the elastic and plastic deformation gradient are missing. As a result, the expression for the Cauchy stress is independent of plastic strain, so there can be no change in the elastic response due to plastic flow. The overstress is given by

$$\begin{aligned}
 \Delta \mathbf{T} &= \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{C}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{E}^e \mathbf{C}^e \right. \\
 &\quad \left. - \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} - \frac{\partial \psi}{\partial I_5} \mathbf{B}^p - 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} - \partial_\xi(\psi) \mathbf{A}^T \right\}. \quad (12.323)
 \end{aligned}$$

The reader will note that the overstress is symmetric in this case if $\mathbf{A}$ is symmetric.

Initially transversely isotropic materials

For an initially transversely isotropic material, the material symmetry restrictions require that

$$\psi^e = \psi^e(I_1, I_2, I_3, I_{10}, I_{11}, \xi, \theta), \quad (12.324)$$

$$\psi^b = \psi^b(I_4, I_5, I_6, I_{12}, I_{13}, \xi, \theta), \quad (12.325)$$

where I_i are the invariants for transverse isotropy given in (12.310). The expressions for Cauchy stress and back-stress are given as

$$\begin{aligned} \mathbf{T}^T = \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right. \\ \left. + \frac{\partial \psi}{\partial I_{10}} (\mathbf{F}^e \mathbf{m}) \otimes (\mathbf{F}^e \mathbf{m}) + \frac{\partial \psi}{\partial I_{11}} \mathbf{F}^e [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})] \mathbf{F}^{eT} \right\}, \end{aligned} \quad (12.326)$$

$$\begin{aligned} \mathbf{T}^{bT} = \rho \left\{ \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right. \\ \left. + \frac{\partial \psi}{\partial I_{12}} (\mathbf{F}^p \mathbf{m}) \otimes (\mathbf{F}^p \mathbf{m}) + \frac{\partial \psi}{\partial I_{13}} \mathbf{F}^p [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})] \mathbf{F}^{pT} \right\}. \end{aligned} \quad (12.327)$$

Initially orthotropic materials

For an initially orthotropic material, the material symmetry restrictions require that

$$\psi^e = \psi^e(I_1, I_2, I_3, I_{10}, I_{11}, I_{14}, I_{15}, \xi, \theta), \quad (12.328)$$

$$\psi^b = \psi^b(I_4, I_5, I_6, I_{12}, I_{13}, I_{16}, I_{17}, \xi, \theta), \quad (12.329)$$

where I_i are the invariants for orthotropy given in (12.314). The expressions for Cauchy stress and back-stress are given as

$$\begin{aligned} \mathbf{T}^T = \rho \left\{ \frac{\partial \psi}{\partial I_1} J^e \mathbf{I} + \frac{\partial \psi}{\partial I_2} \mathbf{B}^e + 2 \frac{\partial \psi}{\partial I_3} \mathbf{F}^e \mathbf{E}^e \mathbf{F}^{eT} \right. \\ + \frac{\partial \psi}{\partial I_{10}} (\mathbf{F}^e \mathbf{m}) \otimes (\mathbf{F}^e \mathbf{m}) + \frac{\partial \psi}{\partial I_{11}} \mathbf{F}^e [(\mathbf{E}^e \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^e \mathbf{m})] \mathbf{F}^{eT} \\ \left. + \frac{\partial \psi}{\partial I_{14}} \mathbf{F}^e \mathbf{Y} \mathbf{F}^{eT} + \frac{\partial \psi}{\partial I_{15}} \mathbf{F}^e (\mathbf{Y} \mathbf{E}^e + \mathbf{E}^e \mathbf{Y}) \mathbf{F}^{eT} \right\}. \end{aligned} \quad (12.330)$$

$$\begin{aligned} \mathbf{T}^{bT} = \rho \left\{ \frac{\partial \psi}{\partial I_4} J^p \mathbf{I} + \frac{\partial \psi}{\partial I_5} \mathbf{B}^p + 2 \frac{\partial \psi}{\partial I_6} \mathbf{F}^p \mathbf{E}^p \mathbf{F}^{pT} \right. \\ + \frac{\partial \psi}{\partial I_{12}} (\mathbf{F}^p \mathbf{m}) \otimes (\mathbf{F}^p \mathbf{m}) + \frac{\partial \psi}{\partial I_{13}} \mathbf{F}^p [(\mathbf{E}^p \mathbf{m}) \otimes \mathbf{m} + \mathbf{m} \otimes (\mathbf{E}^p \mathbf{m})] \mathbf{F}^{pT} \\ \left. + \frac{\partial \psi}{\partial I_{16}} \mathbf{F}^p \mathbf{Y} \mathbf{F}^{pT} + \frac{\partial \psi}{\partial I_{17}} \mathbf{F}^p (\mathbf{Y} \mathbf{E}^p + \mathbf{E}^p \mathbf{Y}) \mathbf{F}^{pT} \right\}. \end{aligned} \quad (12.331)$$

12.12 Heat flux vector

The representations for the heat flux are similar to the representations described for elasticity in Chapter 10 with the addition of invariants accounting for the introduction of

the plastic deformation parameter. In each case, the representations need to satisfy the general requirement

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

For example, consider the representation for $\mathbf{q}$ given by

$$\mathbf{q} = \mathbf{R}^e \mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{X}, \mathbf{C}^e, \mathbf{F}^p, \mathbf{F}^\theta, \xi, \theta, \mathbf{g}^{\mathbf{R}^e}], \quad (12.332)$$

where $\mathbf{g}^{\mathbf{R}^e}$ is the rotated temperature gradient given by

$$\mathbf{g}^{\mathbf{R}^e} = \mathbf{R}^{eT} \mathbf{g}. \quad (12.333)$$

For this we can show that the material symmetry condition for an orthogonal $\mathbf{M}$ is

$$\mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{C}^e, \mathbf{F}^p, \mathbf{F}^\theta, \xi, \theta, \mathbf{g}^{\mathbf{R}^e}] = \mathbf{M} \mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^p \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \xi, \theta, \mathbf{M}^T \mathbf{g}^{\mathbf{R}^e}]. \quad (12.334)$$

As a result, $\mathbf{q}^{\mathbf{R}^e}$ can be represented in terms of vector invariants for the given symmetry. For example, if the material is isotropic represented by the full group of orthogonal transformations, then $\mathbf{q}^{\mathbf{R}^e}$ can be written in terms of the isotropic vector invariants of three symmetric and two skew-symmetric tensors and one vector.

As a special case, let us consider a much simpler model. The model we select is given by

$$\mathbf{q}^{\mathbf{R}^e} = \mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{E}^e, \mathbf{E}^p, \xi, \theta, \mathbf{g}^{\mathbf{R}^e}], \quad (12.335)$$

which has the symmetry requirement

$$\mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{E}^e, \mathbf{E}^p, \xi, \theta, \mathbf{g}^{\mathbf{R}^e}] = \mathbf{M} \mathbf{q}^{\mathbf{R}^e \dagger\dagger} [\mathbf{M}^T \mathbf{E}^e \mathbf{M}, \mathbf{M}^T \mathbf{E}^p \mathbf{M}, \xi, \theta, \mathbf{M}^T \mathbf{g}^{\mathbf{R}^e}]. \quad (12.336)$$

For an isotropic material, we can extract the scalar and vector invariants associated with functions of two symmetric tensors $\mathbf{E}^e$ and $\mathbf{E}^p$ and one vector $\mathbf{g}^{\mathbf{R}^e}$. In this case there are 16 scalar invariants and six vector invariants. This gives the expression

$$\begin{aligned} \mathbf{q}^{\mathbf{R}^e} = & k_1 \mathbf{g}^{\mathbf{R}^e} + k_2 \mathbf{E}^e \mathbf{g}^{\mathbf{R}^e} + k_3 \mathbf{E}^{e2} \mathbf{g}^{\mathbf{R}^e} \\ & + k_4 \mathbf{E}^p \mathbf{g}^{\mathbf{R}^e} + k_5 \mathbf{E}^{p2} \mathbf{g}^{\mathbf{R}^e} + k_6 (\mathbf{E}^e \mathbf{E}^p - \mathbf{E}^p \mathbf{E}^e) \mathbf{g}^{\mathbf{R}^e}, \end{aligned} \quad (12.337)$$

where k_i are scalar functions of ξ, θ and the sixteen scalar isotropic invariants consisting of the set given in (12.305) and the six additional ones given by

$$\begin{aligned} & \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{g}^{\mathbf{R}^e}, \quad \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{E}^e \mathbf{g}^{\mathbf{R}^e}, \quad \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{E}^{e2} \mathbf{g}^{\mathbf{R}^e}, \\ & \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{E}^p \mathbf{g}^{\mathbf{R}^e}, \quad \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{E}^{p2} \mathbf{g}^{\mathbf{R}^e}, \quad \mathbf{g}^{\mathbf{R}^e} \circ \mathbf{E}^e \mathbf{E}^p \mathbf{g}^{\mathbf{R}^e}. \end{aligned} \quad (12.338)$$

This results in the expression of the heat flux being given as

$$\begin{aligned} \mathbf{q} = & k_1 \mathbf{g} + k_2 \mathbf{R}^e \mathbf{E}^e \mathbf{R}^{eT} \mathbf{g} + k_3 \mathbf{R}^e \mathbf{E}^{e2} \mathbf{R}^{eT} \mathbf{g} \\ & + k_4 \mathbf{R}^e \mathbf{E}^p \mathbf{R}^{eT} \mathbf{g} + k_5 \mathbf{R}^e \mathbf{E}^{p2} \mathbf{R}^{eT} \mathbf{g} + k_6 \mathbf{R}^e (\mathbf{E}^e \mathbf{E}^p - \mathbf{E}^p \mathbf{E}^e) \mathbf{R}^{eT} \mathbf{g}. \end{aligned} \quad (12.339)$$

As the reader notes, the first term resembles Fourier's law, while the additional terms are new and represent distortions in conductivity coming from the elastic and plastic strains. One also might use the rotated stress in place of the elastic strain. That is, if one assumes the representation

$$\mathbf{q}^{\mathbf{R}^e} = \mathbf{q}^{\mathbf{R}^e \dagger\dagger\dagger} [\mathbf{T}^{\mathbf{R}^e}, \mathbf{E}^p, \xi, \theta, \mathbf{g}^{\mathbf{R}^e}], \quad (12.340)$$

where $\mathbf{T} = \mathbf{R}^e \mathbf{T}^{\mathbf{R}^e} \mathbf{R}^{eT}$, then the isotropic representation of this will be

$$\begin{aligned} \mathbf{q} = & k'_1 \mathbf{g} + k'_2 \mathbf{T} \mathbf{g} + k'_3 \mathbf{T}^2 \mathbf{g} \\ & + k'_4 \mathbf{R}^e \mathbf{E}^p \mathbf{R}^{eT} \mathbf{g} + k'_5 \mathbf{R}^e \mathbf{E}^{p2} \mathbf{R}^{eT} \mathbf{g} + k'_6 (\mathbf{T} \mathbf{R}^e \mathbf{E}^p \mathbf{R}^{eT} - \mathbf{R}^e \mathbf{E}^p \mathbf{R}^{eT} \mathbf{T}) \mathbf{g}, \end{aligned} \quad (12.341)$$

where k'_i are scalar functions of ξ , θ and the sixteen isotropic invariants of $\mathbf{T}^{\mathbf{R}^e}$, $\mathbf{E}^p$, and $\mathbf{g}^{\mathbf{R}^e}$.

The representations for other symmetries can be constructed in very much the same way. Finally, let us note that the representations are not unique.

12.13 Material constraints

Let us conclude the chapter by looking at constraints such as inextensibility or incompressibility. The general constraint is assumed to be between the deformation gradient, temperature, and temperature gradient and given by a scalar constraint function C written as

$$C(\mathbf{F}, \theta, \mathbf{G}) = 0. \quad (12.342)$$

This requires that

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + \partial_{\theta}(C)\dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} = 0. \quad (12.343)$$

Using $\mathcal{U} \equiv \{\mathbf{F}, \theta, \mathbf{G}\}$ and $\dot{\mathcal{U}} \equiv \{\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}}\}$, we will write this as

$$\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}} = 0. \quad (12.344)$$

Since $\dot{\mathcal{U}}$ represents a loading rate, the constraint requires that an admissible loading rate be such that its projection along $\partial_{\mathcal{U}}(C)$ is zero. This being the only constraint, we can construct any arbitrary loading rate $\dot{\mathcal{U}}^*$ from an admissible loading rate $\dot{\mathcal{U}}$ by adding an arbitrary amount along $\partial_{\mathcal{U}}(C)$. We will write this as

$$\dot{\mathcal{U}}^* = \dot{\mathcal{U}} + \alpha \partial_{\mathcal{U}}(C), \quad (12.345)$$

where α is an arbitrary scalar. Introducing this into the constraint equation requires that

$$\partial_{\mathcal{U}}(C) \circ [\dot{\mathcal{U}}^* - \alpha \partial_{\mathcal{U}}(C)] = 0. \quad (12.346)$$

We can solve this for α to get

$$\alpha = \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)}. \quad (12.347)$$

Therefore, we can write

$$\dot{\mathcal{U}}^* = \dot{\mathcal{U}} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathcal{U}}(C). \quad (12.348)$$

This can be written in “component form,” with use of $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, as

$$\mathbf{L}^* = \mathbf{L} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathbf{F}}(C) \mathbf{F}^{-1}, \quad (12.349)$$

$$\dot{\theta}^* = \dot{\theta} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\theta}(C), \quad (12.350)$$

$$\dot{\mathbf{G}}^* = \dot{\mathbf{G}} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathbf{G}}(C). \quad (12.351)$$

As was argued before, in a constrained material the response is not obtained totally from the “state.” For example, we know the stress in an incompressible material only can be calculated up to a hydrostatic component, which is then matched to the applied stress,

or selected such that it satisfies the balance laws and boundary conditions. As before, not knowing what the form of the indeterminacy is going to be, we add a scalar p to the response functions to represent the added information needed to calculate the response, and later show that its form can be evaluated knowing the constraint. Therefore, for our thermoplastic material, we will assume the free energy is given by

$$\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^p, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}, p), \quad (12.352)$$

and will assume all the other response functions also follow the same dependence. Putting this into the Clausius-Duhem inequality we get

$$\begin{aligned} & -\Delta \mathbf{T}^{eT} : \mathbf{L} + [\rho\eta + \rho\partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} + \rho\partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} \\ & + \rho\partial_p(\psi)\dot{p} - \Delta \mathbf{T}^{pT} : \mathbf{L}^p + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (12.353)$$

where we have defined the thermodynamic over stresses as

$$\Delta \mathbf{T}^e \equiv \mathbf{T} - \mathbf{T}^e, \quad (12.354)$$

$$\Delta \mathbf{T}^p \equiv \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e - \mathbf{T}^p - \rho\partial_\xi(\psi) \mathbf{A}^T, \quad (12.355)$$

$$\Delta \mathbf{T}^\theta \equiv \mathbf{F}^{p-1} \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e \mathbf{F}^p - \mathbf{T}^\theta. \quad (12.356)$$

Before we proceed, let us note that without a constraint $\Delta \mathbf{T}^e = \mathbf{0}$ since $\mathbf{T} = \mathbf{T}^e$. If we now substitute for $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$ from, respectively, (12.349), (12.350), (12.351), we get

$$\begin{aligned} & [-\Delta \mathbf{T}^{eT} + \varpi \partial_{\mathbf{F}}(C) \mathbf{F}^T] : \mathbf{L}^* \\ & + [\rho\eta + \rho\partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \varpi \partial_\theta(C)] \dot{\theta}^* \\ & + [\rho\partial_{\mathbf{G}}(\psi) + \varpi \partial_{\mathbf{G}}(C)] \circ \dot{\mathbf{G}}^* \\ & + \rho\partial_p(\psi)\dot{p} - \Delta \mathbf{T}^{pT} : \mathbf{L}^p + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (12.357)$$

where

$$\begin{aligned} \varpi = \frac{1}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \{ & \Delta \mathbf{T}^{eT} : [\partial_{\mathbf{F}}(C) \mathbf{F}^{-1}] \\ & - [\rho\eta + \rho\partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \partial_\theta(C) - \rho\partial_{\mathbf{G}}\psi \circ \partial_{\mathbf{G}}(C) \}. \end{aligned} \quad (12.358)$$

Since we have set it up so that $\dot{\mathcal{U}}^*$ is arbitrary and we assume that $\dot{p}$ can be selected as arbitrary, following similar arguments as before, we can conclude that in the thermoelastic range we must have the quantities in the square brackets go to zero. This gives

$$\mathbf{T}^T = \rho\partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} + \varpi \partial_{\mathbf{F}}(C) \mathbf{F}^T, \quad (12.359)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} - \frac{\varpi}{\rho} \partial_\theta(C), \quad (12.360)$$

$$\partial_{\mathbf{G}}(\psi) = -\frac{\varpi}{\rho} \partial_{\mathbf{G}}(C), \quad (12.361)$$

$$\partial_p(\psi) = 0, \quad (12.362)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (12.363)$$

Examining the equations one immediately notes that the free energy cannot depend on the indeterminant parameter p . Further examination of the terms, noting that ψ does not depend on p , reveals that ϖ is the only term in the expression for stress and for entropy that can depend on p , which it inherits through the possible dependence of Cauchy stress and entropy on p . Also, one notes that the free energy cannot depend on the temperature gradient if the constraint does not, which represents the most common form of the constraint. It seems reasonable to assume the relations hold at yield and have the plastic flow occur such that

$$\boxed{-\Delta \mathbf{T}^{pT} : \mathbf{L}^p \leq 0.} \quad (12.364)$$

For example, for an isothermally incompressible material, one may write the constraint

$$\det(\mathbf{F}) = J_o(\theta), \quad (12.365)$$

where $C = \det(\mathbf{F}) - J_o(\theta)$ and $J_o(\theta)$ is a function of temperature that gives the imposed value of the volume ratio. For this constraint, we have

$$\partial_{\mathbf{F}}(C) = J\mathbf{F}^{-T}, \quad (12.366)$$

$$\partial_{\theta}(C) = -\frac{dJ_o(\theta)}{d\theta}, \quad (12.367)$$

$$\partial_{\mathbf{G}}(C) = \mathbf{0}. \quad (12.368)$$

Obviously, for this case, the free energy does not depend on $\mathbf{G}$, and we have the relations

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} + \varpi J \mathbf{I}, \quad (12.369)$$

$$\eta = -\partial_{\theta}(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \frac{\varpi}{\rho} \frac{dJ_o(\theta)}{d\theta}. \quad (12.370)$$

Since ϖ is the only term on the right-hand side of these equations that can depend on p and it is a scalar, with little loss of generality we can replace it with p or combine it with other terms and call it p . For example, we could write

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} + p \mathbf{I}, \quad (12.371)$$

$$\eta = -\partial_{\theta}(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \frac{p}{\rho_o} \frac{dJ_o(\theta)}{d\theta}. \quad (12.372)$$

Section 12.13-Problems:

- 12.13-1:** Materials, such as composites, that have directions of large rigidity are sometimes modeled as inextensible along these directions. A material that is inextensible along a material direction during isothermal processes is termed an isothermally inextensible material. Consider a material that is isothermally inextensible along a fiber with direction $\hat{\mathbf{h}}$ in the current configuration and with corresponding direction $\hat{\mathbf{h}}_o$ in the reference configuration. The constraint for this material is described by the relation

$$\hat{\mathbf{h}}_o \circ (\mathbf{C} \hat{\mathbf{h}}_o) = \lambda_o^2(\theta), \quad (12.373)$$

where $\lambda_o(\theta)$ is the temperature-dependent stretch ratio along the direction of isothermal inextensibility. For this constraint, show that

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} + 2\varpi \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad (12.374)$$

$$\eta = -\partial_{\theta}(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \frac{2\varpi}{\rho} \lambda_o \frac{d\lambda_o}{d\theta}. \quad (12.375)$$

12.14 Models based on $\mathbf{F} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$

If the reader would like to construct a plasticity model based on the decomposition $\mathbf{F} = \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$ as opposed to $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$, the process is fairly similar. Most of the assumptions and arguments remain the same with a change of the kinematic relation between the velocity gradients given by

$$\mathbf{L} = \bar{\mathbf{L}}^e + \bar{\mathbf{F}}^e \bar{\mathbf{L}}^\theta \bar{\mathbf{F}}^{e-1} + \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^p \bar{\mathbf{F}}^{\theta-1} \bar{\mathbf{F}}^{e-1}. \quad (12.376)$$

In this case, the thermodynamic stresses are defined as

$$\mathbf{T}^{eT} \equiv \rho \partial_{\bar{\mathbf{F}}^e}(\psi) \bar{\mathbf{F}}^{eT}, \quad (12.377)$$

$$\mathbf{T}^{bT} \equiv \rho \partial_{\bar{\mathbf{F}}^p}(\psi) \bar{\mathbf{F}}^{pT}, \quad (12.378)$$

$$\mathbf{T}^{\theta T} \equiv \rho \partial_{\bar{\mathbf{F}}^\theta}(\psi) \bar{\mathbf{F}}^{\theta T}, \quad (12.379)$$

and the over stresses become

$$\Delta \mathbf{T} \equiv \bar{\mathbf{F}}^{\theta-1} \bar{\mathbf{F}}^{e-1} \mathbf{T}^e \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta - \mathbf{T}^b - \rho \partial_{\bar{\xi}}(\psi) \bar{\mathbf{A}}^T, \quad (12.380)$$

$$\Delta \mathbf{T}^\theta \equiv \bar{\mathbf{F}}^{e-1} \mathbf{T}^e \bar{\mathbf{F}}^e - \mathbf{T}^\theta, \quad (12.381)$$

for the evolution rules

$$\dot{\bar{\xi}} = \bar{\mathbf{A}} : \bar{\mathbf{L}}^p, \quad (12.382)$$

$$\bar{\mathbf{L}}^\theta = \bar{\alpha} \dot{\theta}. \quad (12.383)$$

In this case, we assume the response is a function of this alternate decomposition and arrive at similar results that include

$$\mathbf{T}^T = \mathbf{T}^{eT} = \rho \partial_{\bar{\mathbf{F}}^e}(\psi) \bar{\mathbf{F}}^{eT}, \quad (12.384)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \bar{\alpha}, \quad (12.385)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (12.386)$$

during all deformations, and

$$-\Delta \mathbf{T}^T : \bar{\mathbf{L}}^p + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0 \quad (12.387)$$

during plastic flow, and with a heat flow given by

$$\rho \dot{h} = -\Delta \mathbf{T}^T : \bar{\mathbf{L}}^p + \rho \theta \dot{\eta}. \quad (12.388)$$

This suggests, as one option, a plastic flow rule of the form

$$\bar{\mathbf{L}}^p(t) = \begin{cases} \bar{\mathbf{C}}^p : \Delta \mathbf{T}^T & \text{if } f = 0 \text{ and } \dot{f} > 0, \\ \mathbf{0} & \text{for all other cases,} \end{cases} \quad (12.389)$$

with the condition $\Delta \mathbf{T}^T : \bar{\mathbf{C}}^p : \Delta \mathbf{T}^T \geq 0$ for all admissible processes.

In this case, we can define the “thermodynamic stresses” associated with the yield function as

$$\mathbf{T}_f^{eT} \equiv \partial_{\bar{\mathbf{F}}^e}(f) \bar{\mathbf{F}}^{eT}, \quad (12.390)$$

$$\mathbf{T}_f^{bT} \equiv \partial_{\bar{\mathbf{F}}^p}(f) \bar{\mathbf{F}}^{pT}, \quad (12.391)$$

$$\mathbf{T}_f^{\theta T} \equiv \partial_{\bar{\mathbf{F}}^\theta}(f) \bar{\mathbf{F}}^{\theta T}, \quad (12.392)$$

and the associated over stresses as

$$\Delta \mathbf{T}_f^p \equiv \bar{\mathbf{F}}^{\theta-1} \bar{\mathbf{F}}^{e-1} \mathbf{T}_f^e \bar{\mathbf{F}}^e \bar{\mathbf{F}}^\theta - \mathbf{T}_f^b - \partial_{\bar{\xi}}(f) \bar{\mathbf{A}}^T, \quad (12.393)$$

$$\Delta \mathbf{T}_f^\theta \equiv \bar{\mathbf{F}}^{e-1} \mathbf{T}_f^e \bar{\mathbf{F}}^e - \mathbf{T}_f^\theta. \quad (12.394)$$

The associate values for $\dot{f}$ and $\dot{f}^\diamond$ are given by

$$\dot{f} = \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \bar{\boldsymbol{\alpha}}] \dot{\theta} - \Delta \mathbf{T}_f^{pT} : \bar{\mathbf{L}}^p, \quad (12.395)$$

$$\dot{f}^\diamond \equiv \mathbf{T}_f^{eT} : \mathbf{L} + [\partial_\theta(f) - \Delta \mathbf{T}_f^{\theta T} : \bar{\boldsymbol{\alpha}}] \dot{\theta}. \quad (12.396)$$

The consistency condition $\dot{f} = 0$ is given by

$$\dot{f}^\diamond - \Delta \mathbf{T}_f^{pT} : \bar{\mathbf{L}}^p = 0. \quad (12.397)$$

For the flow rule $\bar{\mathbf{L}}^p = \bar{\mathbf{C}}^p : \Delta \mathbf{T}^T$, the consistency condition becomes a constraint on $\bar{\mathbf{C}}^p$ given by

$$\dot{f}^\diamond - \Delta \mathbf{T}_f^{pT} : \bar{\mathbf{C}}^p : \Delta \mathbf{T}^T = 0. \quad (12.398)$$

The reader will note that the two decompositions provide similar results with slight differences. For the most part, the affects of rigid body motion and material symmetry are the same. As a result, the resulting representations for the different symmetries will be similar.

If we assume the elastic deformation gradient does not change for the two types of decompositions, we can write $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta = \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$ which gives

$$\mathbf{F}^p \mathbf{F}^\theta = \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p. \quad (12.399)$$

Taking the time derivative of both sides gives

$$\dot{\mathbf{F}}^p \mathbf{F}^\theta + \mathbf{F}^p \dot{\mathbf{F}}^\theta = \dot{\bar{\mathbf{F}}}^\theta \bar{\mathbf{F}}^p + \bar{\mathbf{F}}^\theta \dot{\bar{\mathbf{F}}}^p. \quad (12.400)$$

This can be written as

$$\mathbf{L}^p \mathbf{F}^p \mathbf{F}^\theta + \mathbf{F}^p \mathbf{L}^\theta \mathbf{F}^\theta = \bar{\mathbf{L}}^\theta \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p + \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^p \bar{\mathbf{F}}^p. \quad (12.401)$$

In the thermoelastic range, the plastic velocity gradients should be zero so we get $\mathbf{F}^p \mathbf{L}^\theta \mathbf{F}^\theta = \bar{\mathbf{L}}^\theta \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$ that can be written as

$$\bar{\mathbf{L}}^\theta = \mathbf{F}^p \mathbf{L}^\theta \mathbf{F}^{p-1}. \quad (12.402)$$

Since we have assumed that $\mathbf{L}^\theta = \boldsymbol{\alpha} \dot{\theta}$ and $\bar{\mathbf{L}}^\theta = \bar{\boldsymbol{\alpha}} \dot{\theta}$, that each point in the loading is attached to an elastic range, and that $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ depend only on the state (not the rates), we can conclude that this relation holds for all points of the loading and gives the general relation

$$\bar{\boldsymbol{\alpha}} = \mathbf{F}^p \boldsymbol{\alpha} \mathbf{F}^{p-1}, \quad (12.403)$$

where $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ are the associated thermal expansions for theories based on these two decompositions. Since this must hold for all deformations, we can conclude that $\mathbf{L}^p \mathbf{F}^p \mathbf{F}^\theta = \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^p \bar{\mathbf{F}}^p$, which can be written as

$$\bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^p \bar{\mathbf{F}}^{\theta-1} = \mathbf{L}^p. \quad (12.404)$$

Therefore, given the thermal expansion and flow rule for one decomposition, one can uniquely construct them for the other decomposition.

Section 12.14-Problems:

- 12.14-1:** Show that the two decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$ and $\mathbf{F} = \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^p$ will result in exactly the same volumetric part of the respective thermal and plastic deformation gradients.

12.15 Notation

The following table shows the symbols used in the text and their definitions.

$\diamond$		trial time derivative taken holding plastic strain constant
ave		average (subscript)
$\mathbf{B}^e$		elastic left Cauchy stretch tensor ($\mathbf{F}^e \mathbf{F}^{eT}$)
$\mathbf{B}^p$		plastic left Cauchy stretch tensor ($\mathbf{F}^p \mathbf{F}^{pT}$)
$\mathbf{C}^e$		elastic right Cauchy stretch tensor ($\mathbf{F}^{eT} \mathbf{F}^e$)
$\mathbf{C}^p$		plastic right Cauchy stretch tensor ($\mathbf{F}^{pT} \mathbf{F}^p$)
$\mathbf{C}^p$		plastic flow coefficient
dev		deviatoric (subscript)
$\Delta \mathbf{T}$		plastic overstress (same as $\Delta \mathbf{T}^p$)
$\Delta \mathbf{T}^e$		elastic overstress ($\mathbf{T} - \mathbf{T}^e$)
$\Delta \mathbf{T}^p$		plastic overstress ($\mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e - \mathbf{T}^b$)
$\Delta \mathbf{T}^\theta$		thermal overstress ($\mathbf{F}^{p-1} \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e \mathbf{F}^p - \mathbf{T}^\theta$)
e		elastic (superscript)
$\mathbf{E}^e$		elastic Green strain
$\mathbf{E}^p$		plastic Green strain
$\mathbf{E}$		elastic modulus tensor
$\mathbf{E}^b$		plastic back-stress modulus
$\mathbf{F}$		deformation gradient
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}^\theta$		thermal deformation gradient
$\mathbf{F}^p$		plastic deformation gradient
$\mathbf{g}$		temperature gradient with respect to current configuration
$\mathbf{g}^{R^e}$		“elastic” rotated temperature gradient ($\mathbf{R}^{eT} \mathbf{g}$)
$\mathbf{G}$		temperature gradient with respect to reference configuration
G		elastic shear modulus
G^b		back-stress shear modulus
G^e		elastic shear modulus
I_i		scalar invariant i
J		volume ratio ($\det(\mathbf{F})$)
$\mathbf{J}_i$		base tensors for the representation of a tensor function
k		thermal conductivity
K_i		scalar invariant i
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}} \mathbf{F}^{-1}$)
$\mathbf{L}^\theta$		thermal velocity gradient ($\dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1}$)
$\mathbf{L}^p$		plastic velocity gradient ($\dot{\mathbf{F}}^p \mathbf{F}^{p-1}$)
$\mathbf{M}$		transformation tensor representing material symmetry
p		plastic (superscript), or indeterminate parameter
$\mathbf{P}$		transformation between reference frames
$\mathbf{q}$		heat flux vector
$\mathbf{q}^{R^e}$		“elastic” rotated heat flux vector ($\mathbf{R}^{eT} \mathbf{q}$)
$\mathbf{Q}$		orthogonal tensor representing rigid body motion
$\mathbf{R}$		orthogonal factor in polar decomposition of $\mathbf{F} = \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{R}$
$\mathbf{R}^e$		orthogonal factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e$
$\mathbf{S}$		deviatoric stress
$\mathbf{T}$		Cauchy stress
$\mathbf{T}^e$		thermodynamic elastic stress ($\rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}$)
$\mathbf{T}^p$		thermodynamic plastic stress ($\rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT}$)

$\mathbf{T}^\theta$		thermodynamic thermal stress ($\rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}$)
T_{ave}		average stress
$\mathbf{U}$		right symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{U}^e$		right symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e$
$\mathcal{U}$		variable set $\{\mathbf{F}, \theta, \mathbf{G}\}$
$\mathbf{V}$		left symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
α		coefficient of thermal expansion
ϵ		infinitesimal strain
ϵ^e		infinitesimal elastic strain
ϵ^θ		infinitesimal thermal strain
ϵ^p		infinitesimal plastic strain
η		specific entropy
κ^b		back-stress bulk modulus
κ^e		elastic bulk modulus
λ		stretch or Lamé constant
μ		viscosity
μ^{p-d}		plastic deviatoric viscosity
μ^{p-h}		plastic hydrostatic viscosity
ψ		specific free energy
ψ^θ		purely thermal component of specific free energy
ρ		density
σ_{ave}		average stress
$\boldsymbol{\sigma}$		stress
$\boldsymbol{\sigma}^b$		plastic back-stress
τ_{yo}		yield stress in shear
θ		temperature
ξ_i		hardening parameter i

Chapter 13

Viscoelastic solids

Viscoelastic materials are those that show stress relaxation at all magnitudes of loading, even at very small loads. Models normally describing viscoelasticity show rate-dependent response and even yielding like behavior without the use of a yield function. As such, they normally are characterized as a different class of material response.

The typical characteristic of the response of a nonlinear viscoelastic material is shown in Figure 13.1 for the flow and relaxation of poly(methyl methacrylate) (PMMA) at 140°C , which is a temperature in the rubbery response range of this material. The figure shows constant strain rate loading, starting from an initially rigid response followed by what looks like yielding and then flow. For a viscoelastic material, the point of yielding depends on the loading rate as shown in Figure 13.2 for PMMA at 120°C . For the test shown in Figure 13.1, at the end of the constant strain rate loading, the sample is held at constant strain, which results in stress relaxation to an equilibrium stress other than zero. This nonzero stress is called the back-stress and is shown in Figure 13.3 for PMMA at 140°C . As can be seen, not only does the stress relax to the the back-stress, but when it is at values below it one will see a stress buildup to the equilibrium. Mechanical analogs to describe the phenomenon seen in viscoelasticity are constructed from springs and viscous dampers, such as is shown for a standard linear solid in Figure 13.4. For a nonlinear response, the components of these analogs can be nonlinear.

We start the chapter by looking at one-dimensional linear and nonlinear viscoelastic response. The most common model for linear viscoelasticity is called the “standard linear solid,” which will be discussed first; its mechanical analog is shown in Figure 13.4. This is then generalized to create a viscoelastic model constructed from a continuous series of standard linear solids put in parallel. We use the same standard linear solid analog to construct a standard nonlinear solid that is made with components that behave nonlinearly, and then generalize the idea by looking at the response of a series of nonlinear elements in parallel. These ideas are then again generalized to multidimensions for linear and nonlinear response functions, including thermal effects.

Since the purpose of this chapter is to support the development of the viscoelastic-viscoplastic response presented in the next chapter, the development of viscoelasticity is done in a way similar in construction to that of plasticity so that it will fit seamlessly into that development. As such, it may differ in places with what normally is done in classical presentations of the subject.

13.1 One-dimensional linear viscoelasticity

Linear viscoelasticity refers to a particular material response that exhibits the feature of linearity. That is, the material response to the superposition of two stimuli is the sum of the response it would have to the individual stimuli. In a model, the response is characterized by the stress history and the stimulating agent is taken to be the strain history. In this

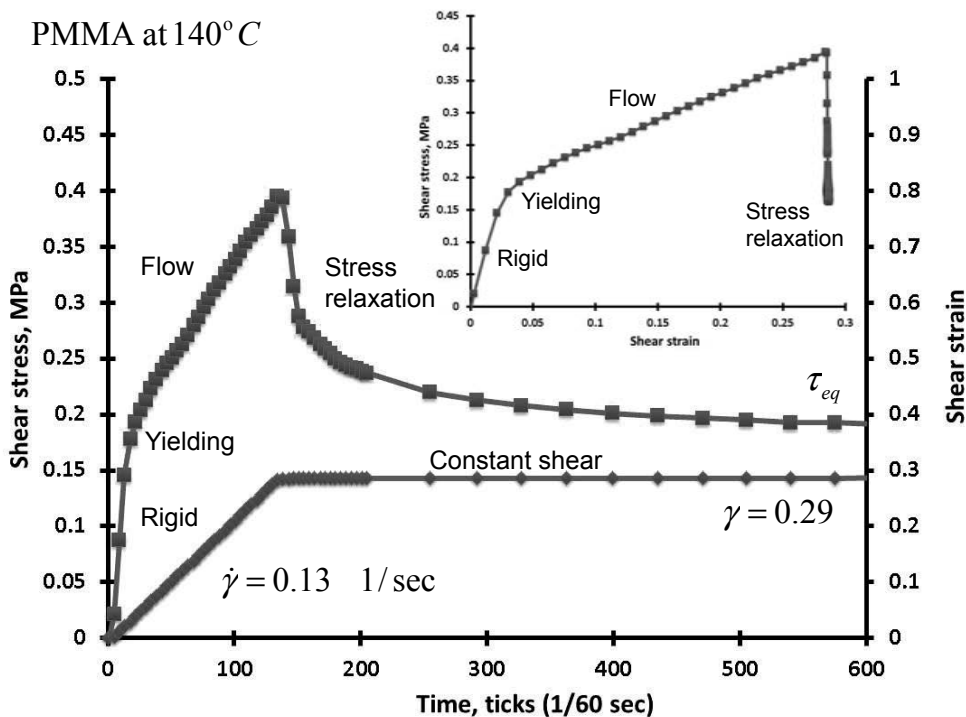


Figure 13.1: The shear response of PMMA at 140°C under a constant strain rate loading followed by a constant strain stress relaxation.

case, we can say that, if $\sigma_1(t)$ is the stress in response to the strain history $\epsilon_1(t)$ and $\sigma_2(t)$ is the stress in response to the strain history $\epsilon_2(t)$, then the response to the strain history $\epsilon(t) = \alpha_1\epsilon_1(t) + \alpha_2\epsilon_2(t)$ is the stress history $\sigma(t) = \alpha_1\sigma_1(t) + \alpha_2\sigma_2(t)$. All the models presented in this section will have this characteristic, even though we will not explore in much detail the implications of linearity from the modeling point of view. The reader is directed to other texts for such discussions.¹ We first will develop the basic ideas using what is known as the “standard linear solid” that has one relaxation time associated with it and then will generalize this to models with multiple relaxation times.

13.1.1 Standard linear solid

The standard linear solid is one of the simplest models for viscoelastic behavior. The mechanical analog for this model is shown in Figure 13.4. As can be seen, the mechanical analog for this material is constructed using two linear springs and a viscous damper. For the linear springs the load in each spring is proportional to its extension, and for the viscous damper the load in the damper is proportional to the rate of its extension. In the figure the mechanical analog is constructed of two units in series which have been denoted as the elastic and viscoelastic elements. The extension of the elastic element is denoted by δ^e and the extension of the viscoelastic element is denoted by δ^{ve} . The load in the spring of the

¹ *Creep and Relaxation of Nonlinear Viscoelastic Materials with an Introduction to Linear Viscoelasticity*, by Findly, Lai, and Onaran, North-Holland Publishing Company, Amsterdam, 1976 (Dover edition, 1989).

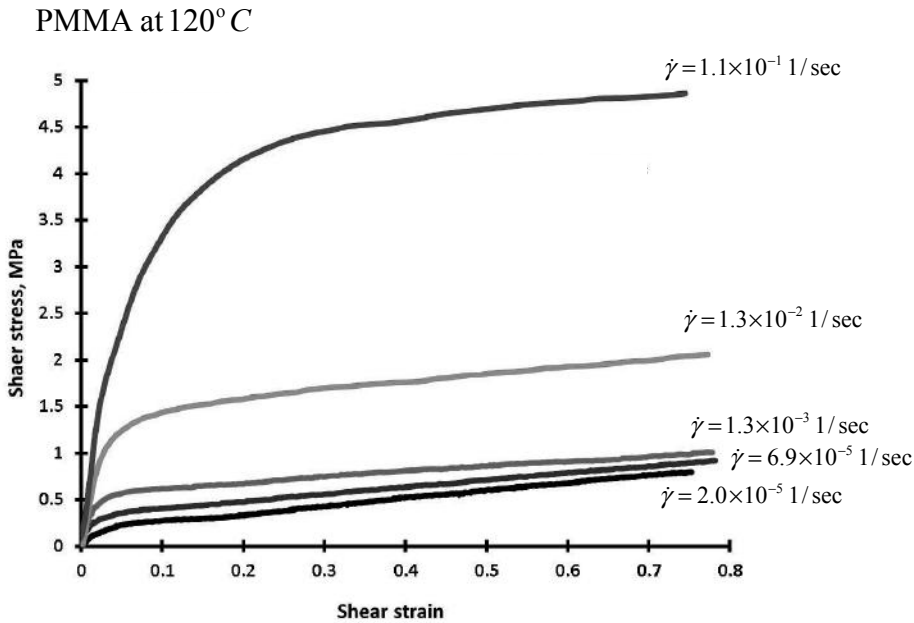


Figure 13.2: The shear response of PMMA at 120°C under constant strain rates from 2.0×10^{-5} to 1.1×10^{-1} .

elastic element, which is identical to the applied load, is denoted by F^e , the load in the spring of the viscoelastic element is denoted by $F^{k^{ve}}$, and the load in the viscous damper of the viscoelastic element is denoted by $F^{\mu^{ve}}$. As shown in the figure, the extension of the mechanical analog is equal to the extension of the elastic plus the viscoelastic elements (i.e., $\delta = \delta^e + \delta^{ve}$), and the load in each element is equal to the applied external load (i.e., $F = F^e = F^{k^{ve}} + F^{\mu^{ve}}$). The load in each spring is proportional to its extension (i.e., $F^e = k^e \delta^e$ and $F^{k^{ve}} = k^{ve} \delta^{ve}$), and the load in the viscous damper is proportional to its rate of extension (i.e., $F^{\mu^{ve}} = \mu^{ve} \dot{\delta}^{ve}$).

The response of this system is very simple, yet provides significant physical interpretation for what happens in real materials. For example, under very rapid loading, the viscous damper “locks” so the response is fully described by the extension of the spring in the elastic element. At the other extreme, when subjected to very slow loading, the viscous damper extends with little resistance so the response is controlled by the extension of the two springs in series. Therefore, at the two limits of high and low extension rates, the response looks linear elastic, but with different spring stiffnesses. At intermediate loading rates the extension of the viscoelastic element is controlled by the extension of the viscous damper given by the equation

$$\dot{\delta}^{ve} = \frac{1}{\mu^{ve}} \Delta F, \quad (13.1)$$

where ΔF is the overload given by $\Delta F = F - F^{k^{ve}}$. The overload is simply the load in the viscous damper. The overload drives the extension of the viscous damper and is the difference between the applied load and the load in the spring of the viscoelastic element. The load in the spring of the viscoelastic element also is known as the back-load so the load driving the extension of the viscoelastic element is the difference between the applied

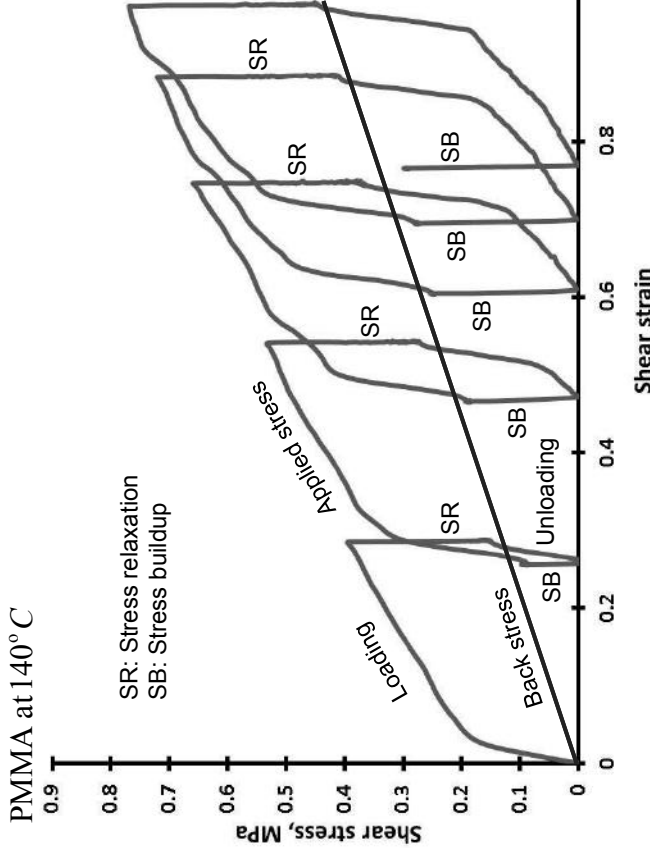


Figure 13.3: The back-stress or equilibrium stress in PMMA at 140°C evaluated by subjecting the material to cycles of loading that include segments of loading, stress relaxation at constant strain, unloading to zero stress, and stress buildup at constant strain.

load and the back-load. As a result, the extension of the viscoelastic element is not directly driven by the applied load.

Making the analogy between loads and stresses, and between extensions and strains, we can make the following one-dimensional model. As for the extensions, we assume the strain to be an addition of the strain in the elastic and viscoelastic elements and we write this as

$$\epsilon = \epsilon^e + \epsilon^{ve}, \quad (13.2)$$

where ϵ^{ve} denotes the strain in the viscoelastic element, and ϵ^e is the strain in the elastic element. The stress in the “spring” of the elastic element is assumed to be equal to the applied stress so we write

$$\sigma = E\epsilon^e, \quad (13.3)$$

where E is the modulus associated with the elastic element. The stress in the spring of the viscoelastic element is the back-stress, and we will denote it by σ^b . This stress is given in terms of the strain in the viscoelastic unit as

$$\sigma^b = E^b\epsilon^{ve}, \quad (13.4)$$

where E^b is the modulus associated with the spring in the viscoelastic element. The rate of strain of the viscoelastic unit is controlled through the viscous damper by the equation

$$\dot{\epsilon}^{ve} = \frac{1}{\mu}(\sigma - \sigma^b), \quad (13.5)$$

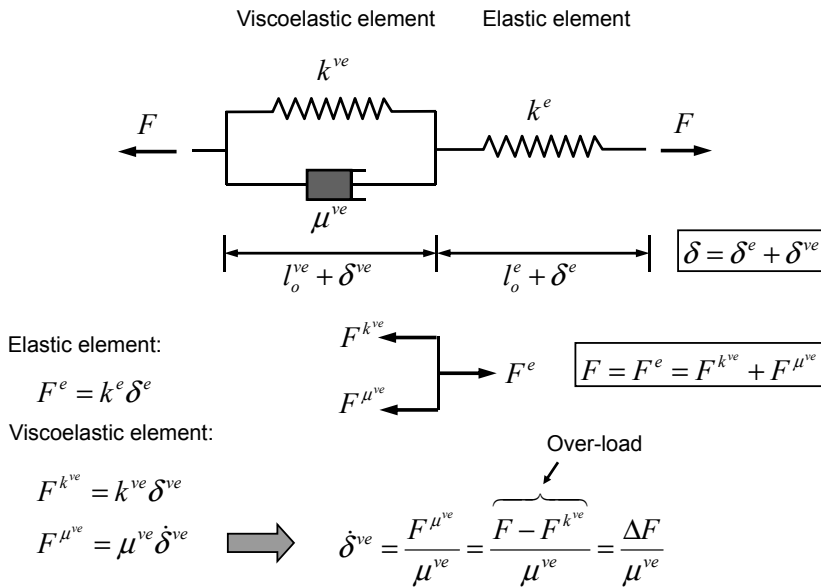


Figure 13.4: A mechanical analog for the “standard linear solid.”

where μ is the viscosity associated with the viscous damping. As can be seen from this equation, the strain in the viscoelastic element is driven by the overstress $\Delta\sigma = \sigma - \sigma^b$, which is the difference between the applied stress and the back-stress in the viscoelastic element. In the terminology of plasticity, one may consider this equation to be the flow rule for the internal parameter ϵ^{ve} .

The response of the standard linear solid to arbitrary applied strains can be evaluated through the following procedure. One can reorganize the flow rule (13.5), after the substitution for the stress and back-stress from equations (13.3) and (13.4), to obtain the differential equation

$$\dot{\epsilon}^{ve} + \frac{E + E^b}{\mu} \epsilon^{ve} = \frac{E}{\mu} \epsilon. \quad (13.6)$$

This equation is a linear ordinary differential equation for ϵ^{ve} , which for piecewise continuous strain histories has a general solution of the form

$$\epsilon^{ve}(t) = \int_{t_o}^t e^{\frac{u-t}{s}} \frac{E}{s(E + E^b)} \epsilon(u) du + \epsilon^{ve}(t_o) e^{-\frac{t-t_o}{s}}, \quad (13.7)$$

where s is the characteristic relaxation time for this model given by

$$s = \frac{\mu}{E + E^b}. \quad (13.8)$$

One can get the general constitutive equation associated with this model by substituting the expression for $\epsilon^{ve}(t)$ into (13.3) and using $\epsilon^e = \epsilon - \epsilon^{ve}$ to get

$$\sigma(t) = E\epsilon(t) - E \int_{t_o}^t e^{\frac{u-t}{s}} \frac{E}{s(E + E^b)} \epsilon(u) du - E\epsilon^{ve}(t_o) e^{-\frac{t-t_o}{s}}. \quad (13.9)$$

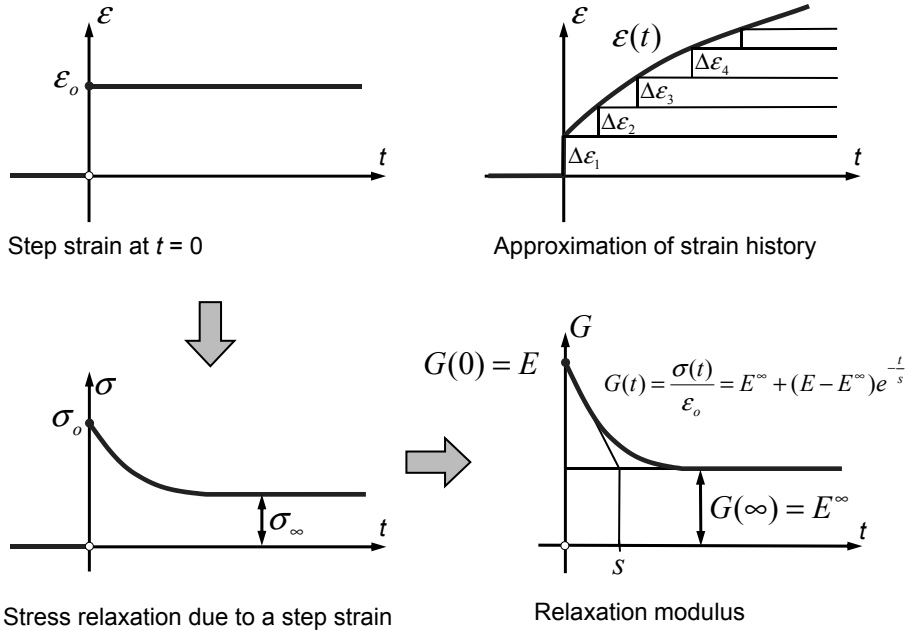


Figure 13.5: Schematic of the response of a standard linear solid to a step in strain applied at time zero.

The response of a linear viscoelastic material best can be understood by studying the response to a step strain such as is shown in Figure 13.5. The reason for this is that any piecewise continuous strain history, as shown in the figure, can be approximated by a series of superimposed step strains. As a result of the linearity of the response, one then can understand the response to an arbitrary strain history as the response to a series of step strain histories superimposed on one another. We now will proceed to look at the response of the standard linear solid to a step strain of magnitude ϵ_o applied at time zero. We will assume that the stress and the back-stress are both zero prior to the application of the step strain so that both the elastic and viscoelastic strains are zero prior to the application of the step strain. It can be shown from equation (13.9) that the response of the standard linear solid model to a step strain of magnitude ϵ_o applied at time zero is given by

$$\sigma(t) = E\epsilon_o - \frac{E^2}{E + E^b}(1 - e^{-\frac{t}{s}})\epsilon_o \quad (13.10)$$

for times $t \geq 0$. As would be expected, the initial strain extends the elastic component only since the viscoelastic component cannot extend instantaneously because of the viscous damper. This is confirmed by setting time equal to zero in the expression for stress to obtain

$$\sigma(0) = E\epsilon_o. \quad (13.11)$$

At long times the changes in the system will tend to an equilibrium characterized by constant elastic and viscoelastic strains. As a result, the viscous damper will stop carrying load, and all the load will be carried by the two springs. This can be seen by substitution of infinity for the time in the expression for stress to obtain

$$\sigma(\infty) = E^\infty \epsilon_o, \quad (13.12)$$

where E^∞ is the equivalent modulus for the two springs in series (assuming no viscous damper). This modulus can be calculated from the equation for two springs in series given by

$$\frac{1}{E^\infty} = \frac{1}{E^b} + \frac{1}{E}. \quad (13.13)$$

It is common to study this response in terms of the stress per unit strain since the response scales linearly with the strain. The relaxation modulus $G(t)$ is defined as the response to a unit step strain applied at time zero so that we have

$$G(t) = \frac{\sigma(t)}{\epsilon_o}. \quad (13.14)$$

For the standard linear solid, the relaxation modulus is given by

$$G(t) = E - \frac{E^2}{E + E^b}(1 - e^{-\frac{t}{s}}). \quad (13.15)$$

As shown in Figure 13.5, the initial modulus $G(0)$ equals the modulus of the spring in the elastic element and the long-term modulus $G(\infty)$ equals E^∞ . Reorganizing the expression for the relaxation modulus in terms of E and E^∞ , one also can write the relaxation modulus as

$$G(t) = E^\infty + (E - E^\infty)e^{-\frac{t}{s}}, \quad (13.16)$$

which shows that the transient portion, which has initial magnitude $(E - E^\infty)$, decays exponentially. For the standard linear solid, the relaxation time s provides the time it would take for the linear approximation of $G(t)$ at time zero to reach the equilibrium response. This interpretation of s is shown in Figure 13.5, indicating that the effect of a step strain on the stress response decays toward the long-term equilibrium within a time interval equal to several relaxation times. If one calculates the response at times $t = s$, $t = 2s$, and $t = 3s$, it will be seen that the magnitude of the transient part of the response reduces, respectively, to 37%, 14%, and 5% of its initial value.

Before we conclude, we should look at creep, which is the response of the material to a suddenly applied constant stress as shown in Figure 13.6. In particular, creep has a special place in the development of a general linear theory since relaxation and creep represent similar starting points to construct a general theory. Let us consider a step stress of magnitude σ_o applied at time zero. Equation (13.5) for the evolution of ϵ^{ve} for a constant stress can be written as

$$\dot{\epsilon}^{ve} + \frac{E^b}{\mu}\epsilon^{ve} = \frac{\sigma_o}{\mu}. \quad (13.17)$$

This is a first-order linear ordinary differential equation with a characteristic time t_C known as the creep time and given by

$$t_C = \frac{\mu}{E_b}. \quad (13.18)$$

If we solve the equation and impose the initial condition of $\sigma_o = E\epsilon_o$, we obtain the solution for creep under a constant stress as

$$\epsilon(t) = \frac{\sigma_o}{E} + \frac{\sigma_o}{E_b}(1 - e^{-\frac{t}{t_C}}). \quad (13.19)$$

Normalizing this solution to the applied stress, we get the creep compliance $J(t)$ given by

$$J(t) = \frac{\epsilon(t)}{\sigma_o} = \frac{1}{E} + \frac{1}{E_b}(1 - e^{-\frac{t}{t_C}}). \quad (13.20)$$

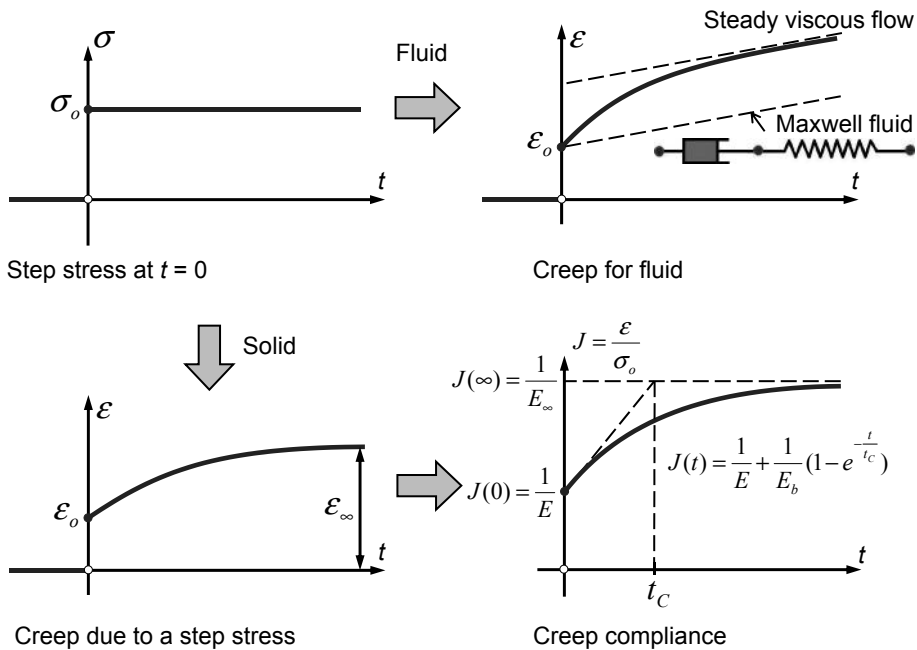


Figure 13.6: Schematic of the creep response under a constant stress for a standard linear solid and the expected response of a fluid.

A point to note is that the creep time normally is orders of magnitude larger than the relaxation time since the back-stress modulus is normally much smaller than the elastic modulus.

Section 13.1-Problems:

- 13.1-1:** Calculate the stress response of the standard linear solid to a constant strain rate history starting at time zero as shown in Figure 13.7b. Assume both the stress and back-stress are zero prior to the application of the constant strain rate given by $\epsilon(t) = \dot{\epsilon}_0 t$ for $t \geq 0$.
- 13.1-2:** Calculate the response of the standard linear solid to a sinusoidal strain history starting at time zero as described in Figure 13.7d. Assume both the stress and back-stress are zero prior to the application of the sinusoidal strain given by $\epsilon(t) = \epsilon_0 \sin(\omega t)$ for $t \geq 0$.
- 13.1-3:** Calculate the creep response of a standard linear solid to a constant stress and calculate the creep compliance. Remove the back-stress spring to obtain a Maxwell fluid, see Figure 13.6, and recalculate the creep response and creep compliance.

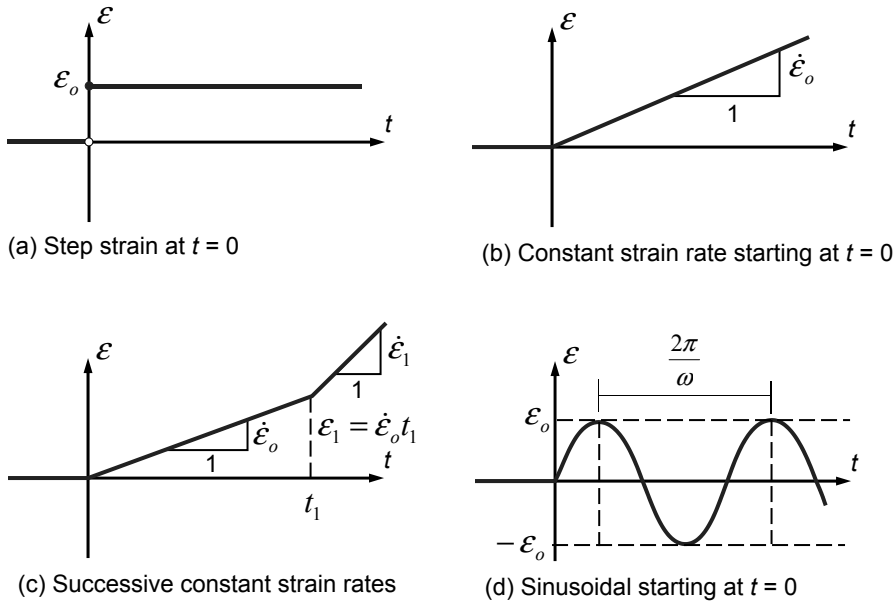


Figure 13.7: Strain histories.

13.1.2 General linear viscoelastic material

A general viscoelastic model can be constructed by combining a continuum of different standard linear solid elements in parallel. Figure 13.8 shows the mechanical analog for a finite number of standard linear solid elements assembled in parallel. One can imagine extending this idea to an infinite number of elements with the stress in the elements adding to give the stress in the entire system. We will use a continuous parameter s to characterize the elements, where s takes values from zero to infinity. The stress in each element can be denoted by σ_s , and we will assume that the total stress is given by

$$\sigma = \int_0^\infty \sigma_s ds, \quad (13.21)$$

where s is clearly a parameter identifying the different elements, not to be confused with time. Each element is assumed to behave like the standard linear solid. The response of each element will be given by the assumption that the strain of each element is decomposed into an elastic and a viscoelastic part written as

$$\epsilon = \epsilon_s^e + \epsilon_s^{ve}. \quad (13.22)$$

As in the standard linear solid, we assume that the stress in the element is given by

$$\sigma_s = E_s \epsilon_s^e, \quad (13.23)$$

combined with the flow rule

$$\dot{\epsilon}_s^{ve} = \frac{1}{\mu_s} (\sigma_s - \sigma_s^b), \quad (13.24)$$

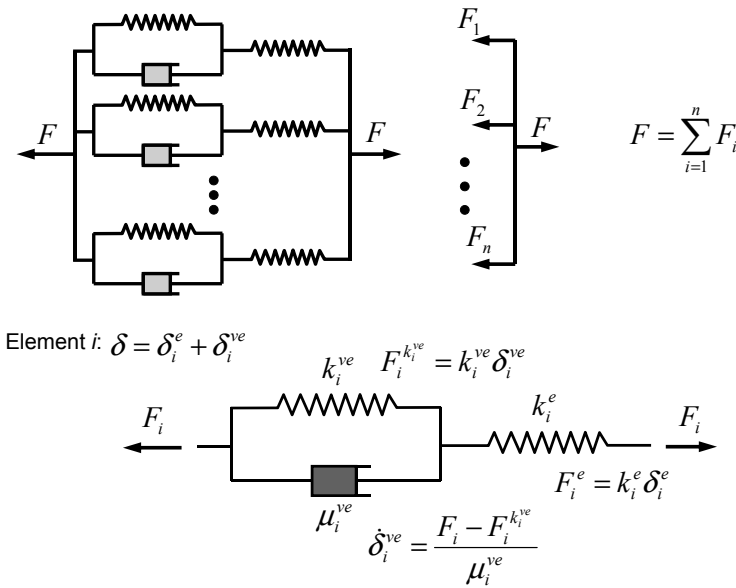


Figure 13.8: A mechanical analog for a general viscoelastic solid constructed from standard linear solid elements in parallel.

where the back-stress σ_s^b for each element is given by

$$\sigma_s^b = E_s^b \epsilon_s^{ve}. \quad (13.25)$$

In terms of an externally applied strain history, the internal parameter ϵ_s^{ve} can be calculated from the differential equation

$$\dot{\epsilon}_s^{ve} + \frac{E_s + E_s^b}{\mu_s} \epsilon_s^{ve} = \frac{E_s}{\mu_s} \epsilon, \quad (13.26)$$

which is obtained directly from the substitution $\sigma_s = E_s \epsilon_s^e$ and $\sigma_s^b = E_s^b \epsilon_s^{ve}$ into the flow rule. This ordinary differential equation has a characteristic decay time t_s given by

$$t_s = \frac{\mu_s}{E_s + E_s^b}. \quad (13.27)$$

For the one-dimensional response, one can select the parameter s to be equal to the characteristic relaxation time of the elements so that $t_s = s$. In this case, the general solution of the differential equation can be written as

$$\epsilon_s^{ve}(t) = \int_{t_o}^t e^{\frac{u-t}{s}} \frac{E_s}{s(E_s + E_s^b)} \epsilon(u) du + \epsilon_s^{ve}(t_o) e^{-\frac{t-t_o}{s}}. \quad (13.28)$$

The response of the material is given by the expression

$$\sigma(t) = \int_0^\infty E_s [\epsilon(t) - \epsilon_s^{ve}(t)] ds = \int_0^\infty E_s ds \epsilon(t) - \int_0^\infty E_s \epsilon_s^{ve}(t) ds. \quad (13.29)$$

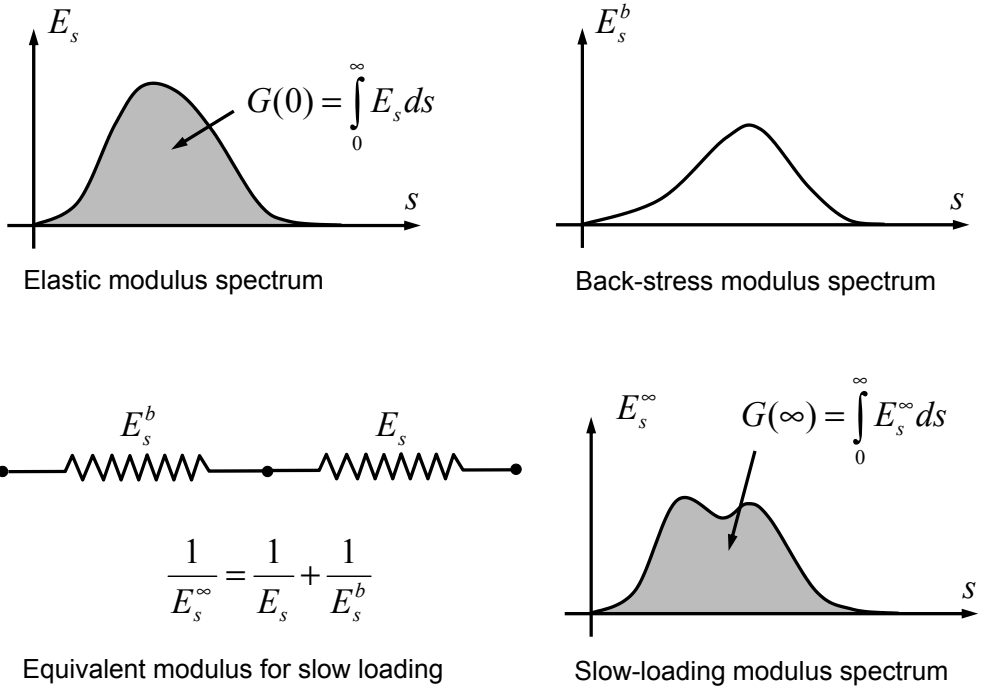


Figure 13.9: Schematic of the elastic element modulus and the back-stress response modulus as a function of the relaxation time s and their relation to the instantaneous modulus $G(0)$ and the long-term modulus $G(\infty)$.

Substituting the general solution for $\epsilon_s^{ve}(t)$ into this equation and reorganization yield the expression for stress given by

$$\sigma(t) = \int_0^{\infty} E_s ds \epsilon(t) - \int_0^{\infty} E_s \left[\int_{t_0}^t e^{-\frac{u-t}{s}} \frac{E_s}{s(E_s + E_s^b)} \epsilon(u) du \right] ds, \quad (13.30)$$

for initial values of $\epsilon_s^{ve}(t_0) = 0$. Assuming the order of the integrals can be changed and reorganizing the terms allows us to write the general expression for stress subjected to a piecewise continuous strain history as

$$\sigma(t) = G(0)\epsilon(t) + \int_{t_0}^t G'(t-u)\epsilon(u)du, \quad (13.31)$$

where

$$G(0) = \int_0^{\infty} E_s ds, \quad (13.32)$$

$$G(\infty) = \int_0^{\infty} \frac{E_s E_s^b}{E_s + E_s^b} ds, \quad (13.33)$$

$$G(t) = G(\infty) + \int_0^{\infty} \frac{E_s^2}{E_s + E_s^b} e^{-\frac{t}{s}} ds. \quad (13.34)$$

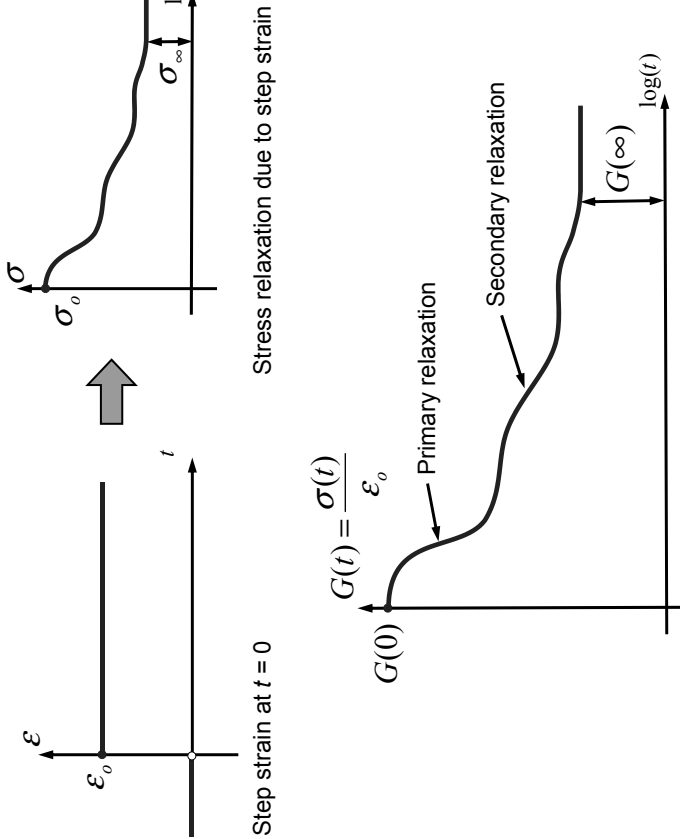


Figure 13.10: Schematic of the response of a general linear viscoelastic material to a step strain applied at time zero.

In this expression, G' denotes the derivative of G with respect to its argument so that

$$G'(t) = \frac{dG(t)}{dt}. \quad (13.35)$$

Figure 13.9 shows that the initial relaxation modulus $G(0)$ is the area under the plot of E_s as a function of the relaxation time s . If we introduce the equivalent long-term modulus E_s^∞ for each relaxation time s , as was done for a standard linear solid, then this modulus can be calculated (see Figure 13.9) for each relaxation time s by the equation

$$\frac{1}{E_s^\infty} = \frac{1}{E_s^b} + \frac{1}{E_s}. \quad (13.36)$$

As a result, the expression for calculating $G(\infty)$ is equivalent to calculating the area under the plot of E_s^∞ as a function of relaxation time s , as shown in Figure 13.9.

The response of a linear viscoelastic material best can be understood by studying the response to a step strain such as was shown for the standard linear solid in Figure 13.5. Figure 13.10 shows the schematic of such a response for a general linear viscoelastic material. As can be seen, in a general linear viscoelastic material model, one has different relaxation events occurring that produce a much more complex response than that seen for the standard linear solid which only had one relaxation time.

Section 13.1-Problems:

- 13.1-4:** Figure 13.7a shows a step strain history starting at time zero. For the general linear viscoelastic model, show that the expression for the viscoelastic strain

ϵ_s^{ve} is given by

$$\epsilon_s^{ve}(t) = \frac{E_s}{E_s + E_s^b} (1 - e^{-\frac{t}{s}}) \epsilon_o, \quad (13.37)$$

and that the expression for stress is given by

$$\sigma(t) = \int_0^\infty E_s \left[1 - \frac{E_s}{E_s + E_s^b} (1 - e^{-\frac{t}{s}}) \right] ds \epsilon_o. \quad (13.38)$$

From this show that

$$G(t) = \frac{\sigma(t)}{\epsilon_o}. \quad (13.39)$$

- 13.1-5:** Figure 13.7b shows a constant strain rate history starting at time zero. For the general linear viscoelastic model, show that the expression for the viscoelastic strain ϵ_s^{ve} is given by

$$\epsilon_s^{ve}(t) = \frac{E_s}{E_s + E_s^b} [t - s(1 - e^{-\frac{t}{s}})] \dot{\epsilon}_o \quad (13.40)$$

and that the expression for stress is given by

$$\sigma(t) = \int_0^\infty E_s \left[\frac{E_s^b}{E_s + E_s^b} t + \frac{s E_s}{E_s + E_s^b} (1 - e^{-\frac{t}{s}}) \right] ds \dot{\epsilon}_o. \quad (13.41)$$

For this case, calculate the tangent modulus $G_T(t)$ given by

$$G_T(t) = \frac{\dot{\sigma}(t)}{\dot{\epsilon}} \quad (13.42)$$

and show that the tangent modulus is the same as the relaxation modulus $G(t)$.

- 13.1-6:** Figure 13.7c shows consecutive constant strain rate histories imposed one after the other. Consider the case that the strain rate is reversed (i.e., $\dot{\epsilon}_1 = -\dot{\epsilon}_o$). For this case, show that

$$G_T(0) = \frac{1}{2} [G_T(t_1-) + G_T(t_1+)], \quad (13.43)$$

where $G_T(t_1-)$ and $G_T(t_1+)$ are, respectively, the tangent moduli just before and just after the time t_1 at which the strain rate is reversed.

- 13.1-7:** Figure 13.7d shows the case of a sinusoidal strain history applied at time zero. For the general linear viscoelastic model, show that the viscoelastic strain ϵ_s^{ve} is given by

$$\epsilon_s^{ve}(t) = \frac{E_s \epsilon_o}{(E_s + E_s^b)(1 + s^2 \omega^2)} \{ \sin(\omega t) + s \omega [e^{-\frac{t}{s}} - \cos(\omega t)] \}, \quad (13.44)$$

and the stress is given by

$$\frac{\sigma(t)}{\epsilon_o} = G'(\omega) \sin(\omega t) + G''(\omega) \cos(\omega t) - \int_0^\infty \frac{s \omega E_s^2 e^{-\frac{t}{s}}}{(E_s + E_s^b)(1 + s^2 \omega^2)} ds, \quad (13.45)$$

where $G'(\omega)$ and $G''(\omega)$ are, respectively, the storage and loss moduli and are given by

$$G'(\omega) = \int_0^\infty E_s \left[1 - \frac{E_s}{(E_s + E_s^b)(1 + s^2\omega^2)} \right] ds, \quad (13.46)$$

$$G''(\omega) = \int_0^\infty \frac{s\omega E_s^2}{(E_s + E_s^b)(1 + s^2\omega^2)} ds. \quad (13.47)$$

Show that $G(0) = G'(\infty)$ and $G(\infty) = G'(0)$.

13.1-8: For infinitesimal strains, write the jump conditions and show that a disturbance will travel into a stationary viscoelastic body with a speed of $\sqrt{\frac{G(0)}{\rho}}$.

13.2 One-dimensional nonlinear viscoelasticity

Let us start the development of a nonlinear viscoelasticity model by constructing nonlinear versions of the standard linear solid. First, we simply replace the linear elements in the standard linear solid analog with nonlinear elements. The result is a model that has the same physical interpretation for the role of the different elements as we saw in the standard linear solid, but responds nonlinearly due to the nonlinear elements. We call this the quasi-nonlinear standard solid since we keep the linear kinematic relation $\epsilon = \epsilon^e + \epsilon^{ve}$ between the elements. We next generalize the kinematics also to construct the (large deformation) nonlinear standard solid. Both these models provide structures that can be generalized in the same way as we did for the linear response.

13.2.1 A standard quasi-nonlinear viscoelastic solid

Let us look at making a nonlinear viscoelastic response by selecting nonlinear elements for the standard linear solid analog shown in Figure 13.4. There are many different ways of constructing nonlinear models, many based, like this one, on taking a linear model and adding nonlinearity to it. For the quasi-nonlinear standard solid, we keep the linear kinematics, so the result is not fully nonlinear. The particular attraction of the idea of replacing the linear elements with nonlinear elements in a mechanical analog is that it keeps the way in which each element contributes to the total response (i.e., it keeps the physics that defines the analog), even though it changes the behavior of the elements.

The relation between the elements of the mechanical analog for a standard linear solid shown in Figure 13.4 can be described by the kinematic relation between the strains in each of the elements and balance of their stresses. We can write these two conditions as

$$\epsilon = \epsilon^e + \epsilon^{ve}, \quad (13.48)$$

$$\sigma^e = \sigma^b + \sigma^v, \quad (13.49)$$

where σ^e , σ^b , and σ^v are the stresses, respectively, in the elastic, back-stress, and viscous elements. We also add to this the fact that the applied stress is equal to the stress in the elastic element and write

$$\sigma = \sigma^e. \quad (13.50)$$

Stress relaxation in a standard quasi-nonlinear solid defined by the element models:

$$\sigma^e = \frac{E}{2m-1} \left[(1+\epsilon^e)^m - \frac{1}{(1+\epsilon^e)^{m-1}} \right]$$

$$\sigma^b = \frac{E^b}{2m^b-1} \left[(1+\epsilon^{ve})^{m^b} - \frac{1}{(1+\epsilon^{ve})^{m^b-1}} \right]$$

$$\dot{\epsilon}^{ve} = \left(\frac{|\sigma^v|}{\sigma_o^v} \right)^n \frac{\sigma^v}{\mu}$$

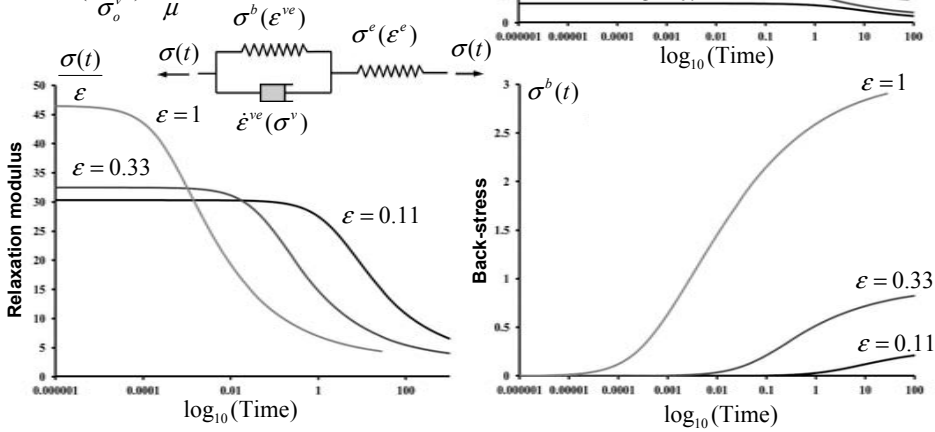


Figure 13.11: The stress relaxation for a standard quasi-nonlinear solid with elements with models as indicated.

For us, these three equations provide the physics behind the analog and its relation to the applied/observed stress and strain.

We now proceed to construct the nonlinear model by assuming each element of the analog is nonlinear. The simplest way to do this would be to assume the elements, although nonlinear, are noninteracting. In this case we can write the constitutive models for each element as

$$\sigma^e = \sigma^{e\dagger}(\epsilon^e), \quad (13.51)$$

$$\sigma^b = \sigma^{b\dagger}(\epsilon^{ve}), \quad (13.52)$$

$$\dot{\epsilon}^{ve} = \dot{\epsilon}^{ve\dagger}(\sigma^v), \quad (13.53)$$

where the “ $\dagger$ ” denotes the function used to calculate the variable. The first equation states that the stress in the elastic element is a function of the strain in the elastic element, the second equation states that the stress in the back-stress element is a function of the strain in the back-stress element, and the third equation states that the strain rate of the viscous element is given by a function of the stress in the viscous element. Unlike the linear system, here we have many options for the constitutive function of each element. For example, we

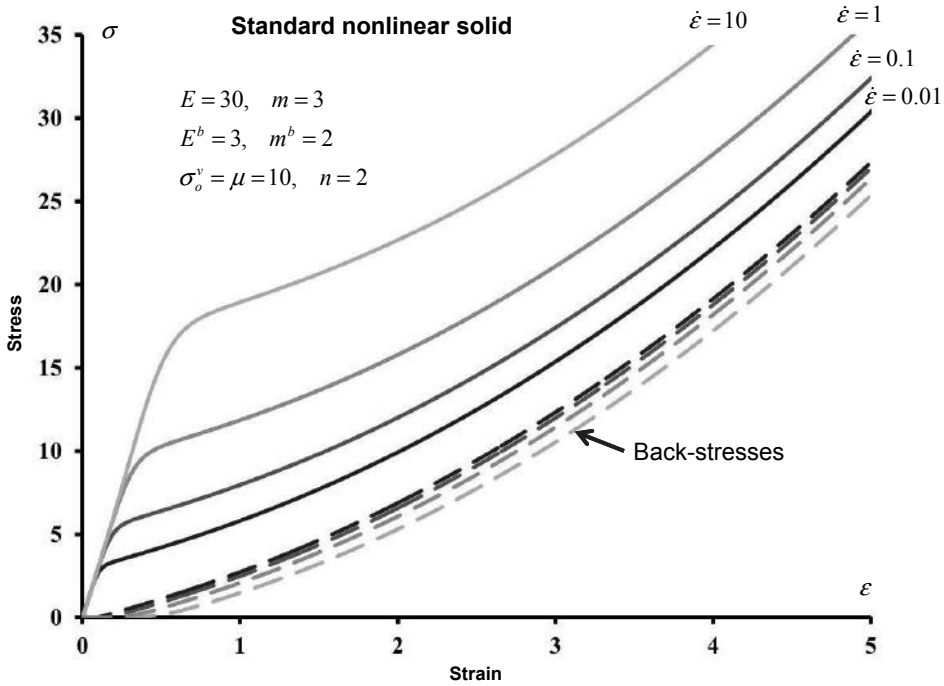


Figure 13.12: The stress relaxation for a standard quasi-nonlinear solid under constant strain rate extension.

might take for the elastic element one of the constitutive functions

$$\sigma^e = E_1 \epsilon^e + \frac{E_3}{3} \epsilon^{e3} + \frac{E_5}{5} \epsilon^{e5} \dots, \quad (13.54)$$

$$\sigma^e = E \ln(1 + \epsilon^e), \quad (13.55)$$

$$\sigma^e = \frac{E}{3} \left[(1 + \epsilon^e)^2 - \frac{1}{1 + \epsilon^e} \right], \quad (13.56)$$

$$\sigma^e = \frac{E}{2m-1} \left[(1 + \epsilon^e)^m - \frac{1}{(1 + \epsilon^e)^{m-1}} \right], \quad (13.57)$$

$$\sigma^e = \sum_{i=1}^n \frac{E_i}{3\alpha_i} \left[(1 + \epsilon^e)^{2\alpha_i} - \frac{1}{(1 + \epsilon^e)^{\alpha_i}} \right], \quad (13.58)$$

$$\sigma^e = E_\infty \epsilon^e + \sigma_o \tanh \left(\frac{E_o}{\sigma_o} \epsilon^e \right). \quad (13.59)$$

Even though there are an endless number of models one can select from, these represent some of the most common/useful ones. Equation (13.54) is simply a nonlinear polynomial model, with E_1 being the standard elastic modulus and the remaining E_i being higher-order moduli. Equation (13.55) uses the logarithmic strain, which forces the stress to go infinity as the sample's strain goes to -1 . Equation (13.56) is inspired from rubber elasticity, provides a quadratic response in tension for large strains and, like the logarithmic strain model, provides an infinite stress when the strain goes to -1 . Equation (13.57) is inspired by generalization of rubber elasticity with Gaussian distribution, which is the special case when $m = 2$.

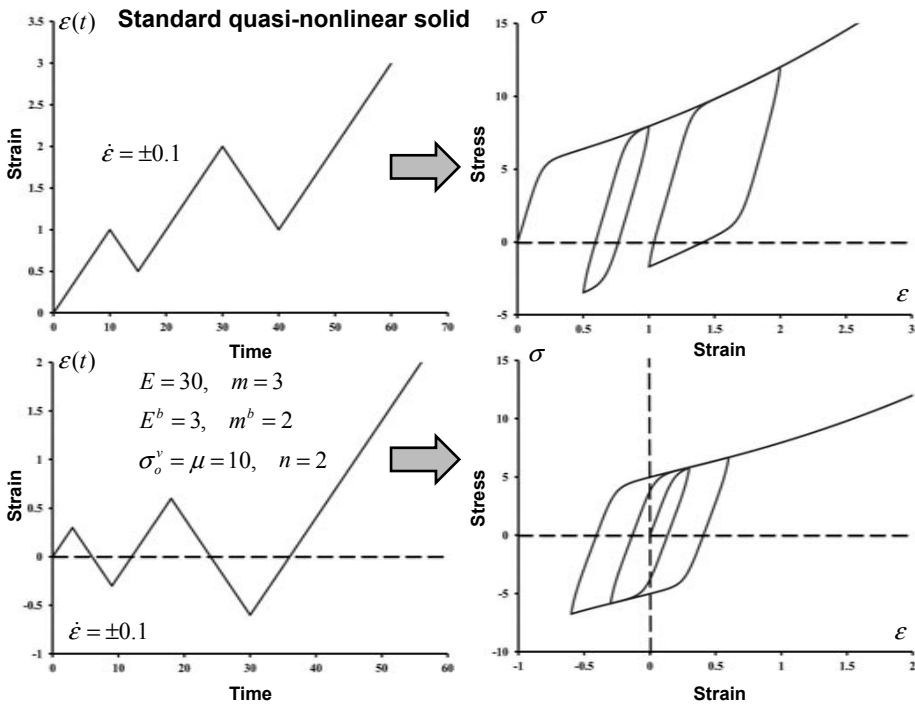


Figure 13.13: Response of standard quasi-nonlinear solid to constant strain rate cyclic loading as indicated.

Equation (13.58) is inspired by an n -element Ogden model, where the exponents α_i need not be integers. Equation (13.59) is inspired by the four-parameter plasticity model, where, compared to a J_2 plasticity model, E_o is the initial elastic modulus, E_∞ is the hardening slope, and σ_o is the intercept with the vertical axis of the hardening asymptote (in value close to the yield stress). One can select from the similar equations for the back-stress model. An example nonlinear model for the viscous element would be

$$\dot{\epsilon}^{ve} = \left(\frac{|\sigma^v|}{\sigma_o^v} \right)^n \frac{\sigma^v}{\mu}, \quad (13.60)$$

where $\sigma^v = \sigma - \sigma^b$ is the same as the overstress $\Delta\sigma$, and σ_o^v is a scaling factor selected such that when the stress σ^v in the element is equal to σ_o^v the element will extend like a linear element with a viscosity of μ . This can be generalized to include multiple terms and noninteger exponents. We can write this for n terms with each term “ i ” having different parameters as

$$\dot{\epsilon}^{ve} = \left[\sum_{i=1}^n \frac{1}{\mu_i} \left(\frac{|\sigma^v|}{\sigma_{oi}^v} \right)^{\zeta_i} \right] \sigma^v. \quad (13.61)$$

One example of a standard nonlinear solid model is noted in Figure 13.11 and the response during stress relaxation is shown. As can be seen, possibly best by looking at the relaxation modulus (which is defined as the stress divided by the constant applied strain), the figure shows that the response is nonlinear both in the initial elastic jump and in the

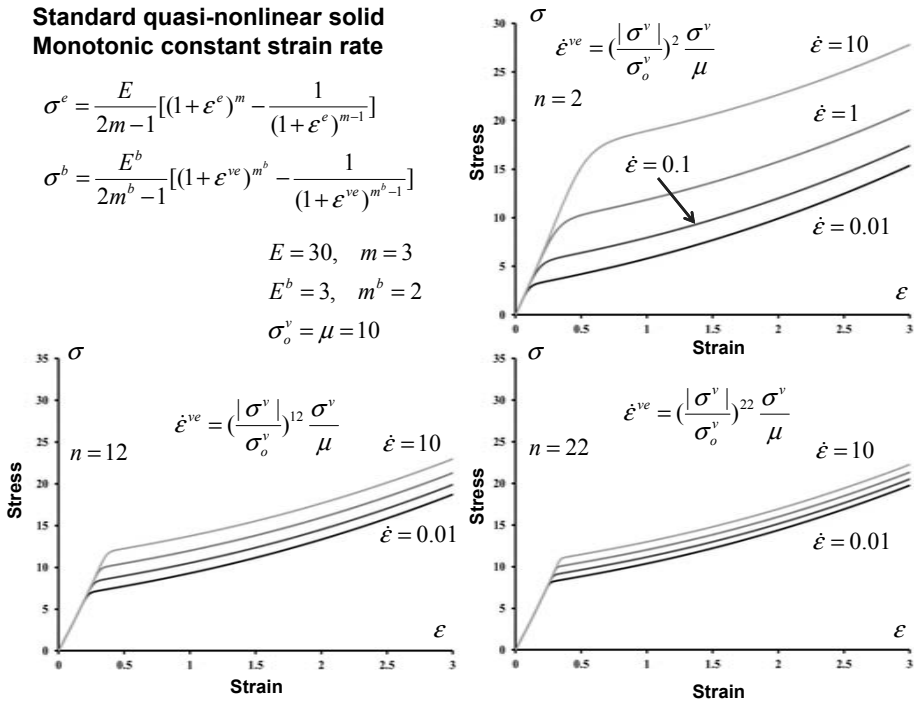


Figure 13.14: Influence of the flow exponent n on the response of the standard quasi-nonlinear solid during monotonic constant strain rate loading.

relaxation time. In this case, the response at higher strains relaxes faster than the response at lower strains. The response of the same model for constant strain rate monotonic loading is shown in Figure 13.12, and for constant strain rate cyclic loading is shown in Figure 13.13.

A slightly more complicated model arises when we allow the models for the elements to interact. This type of model normally can be written in the form

$$\sigma^e = \sigma^{e\dagger}(\epsilon^e, \epsilon^{ve}), \quad (13.62)$$

$$\sigma^b = \sigma^{b\dagger}(\epsilon^e, \epsilon^{ve}), \quad (13.63)$$

$$\dot{\epsilon}^{ve} = \dot{\epsilon}^{ve\dagger}(\sigma^b, \sigma^v). \quad (13.64)$$

In developing such models, one has to take particular care since the interactions can produce unexpected, possibly wrong or impossible, responses under certain conditions.

There are other ways to create nonlinear viscoelastic systems. One other way is to write the response of the linear system in terms of a linear differential equation with constant coefficients and then processed to nonlinearize the system by changing the system, for example, by making the constant coefficients into functions. The common characteristic of such methods is that one frequently loses the physics that originally motivated the analog, and with it the relation between the contribution of the different parts.

Section 13.2-Problems:

- 13.2-1:** Consider the standard nonlinear solid model indicated in Figure 13.11. Simulate the response of this material model under stress relaxation and compare

Stress relaxation in a standard nonlinear solid defined by the element models:

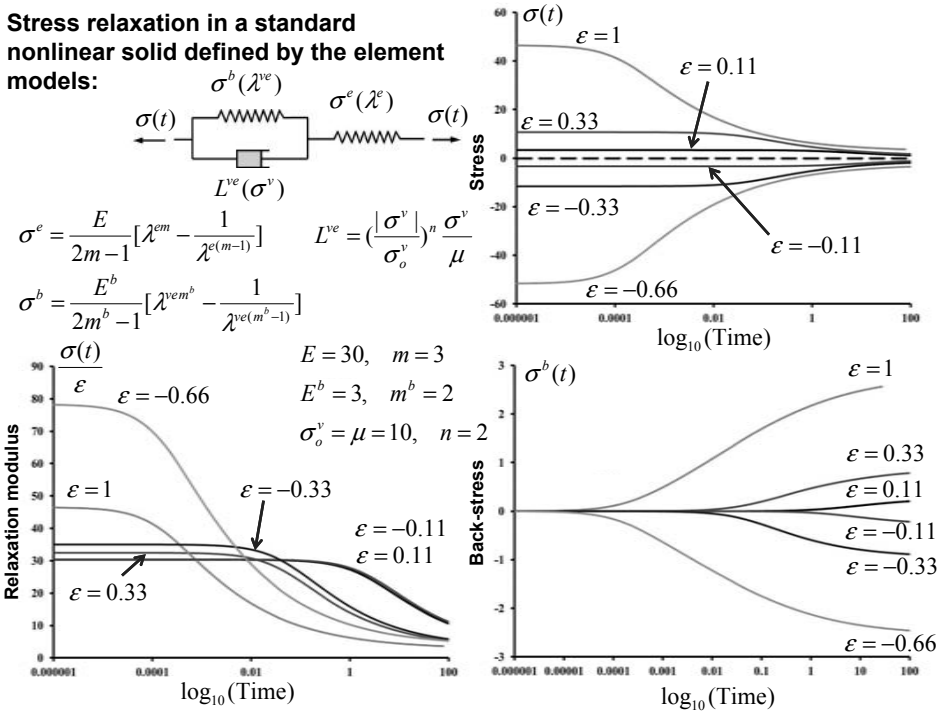


Figure 13.15: The stress relaxation for a standard nonlinear solid with elements with models as indicated.

your results to those provided in the figure.

- 13.2-2:** Consider the standard nonlinear solid model indicated in Figure 13.11. Study the response of this model by simulating the response for different initial step strains ϵ_o followed at different intervals by a second step of either $\Delta\epsilon$ or $-\Delta\epsilon$.
- 13.2-3:** Consider the standard nonlinear solid model indicated in Figure 13.11. Study the effect of the initial start-up time in a stress relaxation experiment by simulating the response under constant strain rate $\dot{\epsilon} = \dot{\epsilon}_o$ followed by a constant strain $\epsilon = \epsilon_o$. Compare the stress obtained to that in a stress relaxation test. In particular, for each strain calculate how soon after the completion of the ramp one can ignore the startup effects.

13.2.2 A standard nonlinear viscoelastic solid

In the quasi-nonlinear standard solid, we kept the linear kinematics between the elements. Leaving all else the same, here we replace the linear relation $\epsilon = \epsilon^e + \epsilon^{ve}$ with its large deformation counterpart

$$\lambda = \lambda^e \lambda^{ve}, \quad (13.65)$$

where the λ s are the associated stretch ratios given by $\lambda = 1 + \epsilon$, $\lambda^e = 1 + \epsilon^e/4$, and $\lambda^{ve} = 1 + \epsilon^{ve}$. The result is a model that works not only for nonlinear elements, but also

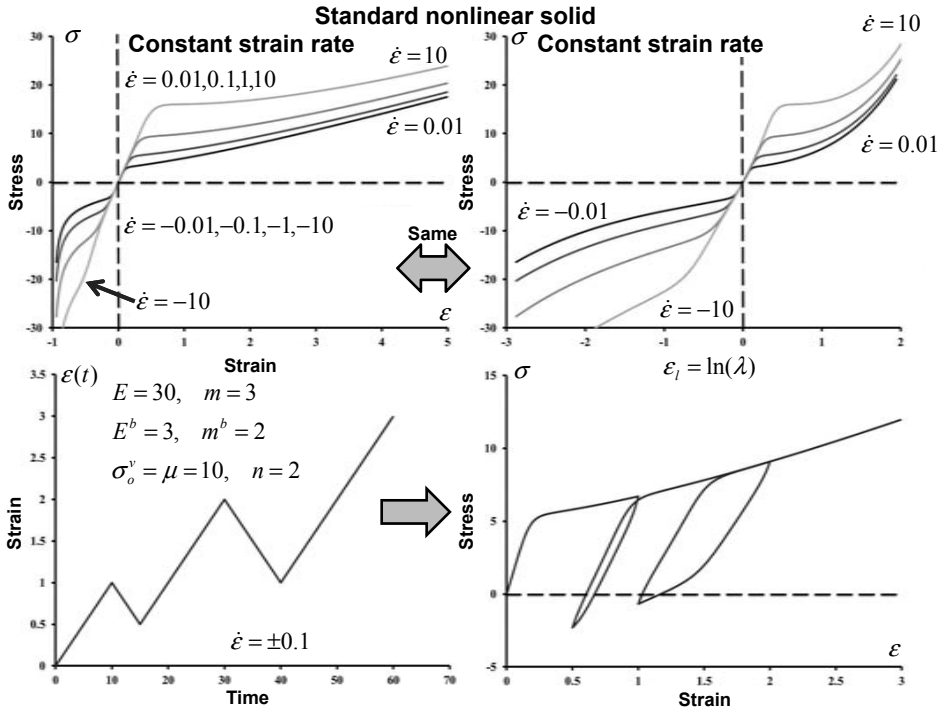


Figure 13.16: The stress relaxation for a standard nonlinear solid under monotonic tension and compression strain rates (top), and under cyclic strain rate reversal (bottom). Note the effect of plotting the same response with respect to strain (top left) or to the logarithmic strain (top right).

for large deformation. The reader notes that we can write the large deformation kinematic relation as $\epsilon_l = \epsilon_l^e + \epsilon_l^{ve}$ using the logarithmic strains ($\epsilon_l = \ln(\lambda)$), but as pointed out earlier, we avoid this since the logarithmic strain does not provide a convenient counterpart in three-dimensional deformation. Yet, the reader could replace the logarithmic strain, in an appropriate manner, in the expressions of the previous section and obtain the correct structure of the model.

In addition to the kinematic relation, we take the relations between the loads in the elements as for the standard quasi-nonlinear solid. That is, we take

$$\sigma = \sigma^e, \quad (13.66)$$

$$\sigma^e = \sigma^b + \sigma^v. \quad (13.67)$$

For the springs, as for the quasi-nonlinear standard solid, we take constitutive models that provide the stress in terms of the stretch in the element and write

$$\sigma^e = \sigma^{e\dagger}(\lambda^e), \quad (13.68)$$

$$\sigma^b = \sigma^{b\dagger}(\lambda^{ve}). \quad (13.69)$$

Finally, we need to provide a model for the extension of the viscous damper. For reasons that are best left for the discussion of thermodynamics, we take the extension of the viscous

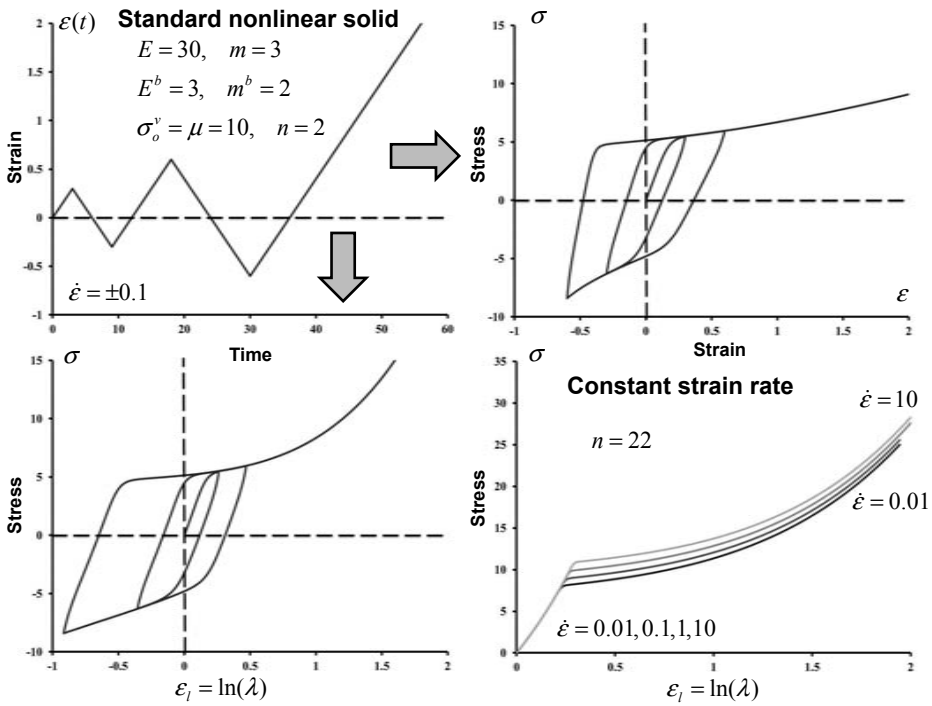


Figure 13.17: Response of standard nonlinear solid to growing cycles of constant strain rate loading and unloading, and the influence of the viscous flow exponent n .

element given by a model of the form

$$\frac{\dot{\lambda}^{ve}}{\lambda^{ve}} = L^{ve\ddagger}(\sigma^v), \quad (13.70)$$

where $L^{ve\ddagger}$ is the model used for the velocity gradient of the viscous element.

Example models for the elastic response are

$$\sigma^e = E_1 \ln(\lambda^e) + \frac{E_3}{3} \ln^3(\lambda^e) + \frac{E_5}{5} \ln^5(\lambda^e) \dots, \quad (13.71)$$

$$\sigma^e = \frac{E}{3} \left[\lambda^{e2} - \frac{1}{\lambda^e} \right], \quad (13.72)$$

$$\sigma^e = \frac{E}{2m-1} \left[\lambda^{em} - \frac{1}{\lambda^{e(m-1)}} \right], \quad (13.73)$$

$$\sigma^e = \sum_{i=1}^n \frac{E_i}{3\alpha_i} \left(\lambda^{e2\alpha_i} - \frac{1}{\lambda^{e\alpha_i}} \right), \quad (13.74)$$

$$\sigma^e = \frac{E_\infty}{2m-1} \left[\lambda^{em} - \frac{1}{\lambda^{e(m-1)}} \right] + \sigma_o \tanh \left[\frac{E_o}{\sigma_o} (\lambda^e - 1) \right]. \quad (13.75)$$

Similar models can be used for the back-stress. As in the quasi-nonlinear case, we can take

a model for the extension of the viscous element given by

$$L^{ve} = \left[\sum_{i=1}^n \frac{1}{\mu_i} \left(\frac{|\sigma^v|}{\sigma_{oi}^v} \right)^{\zeta_i} \right] \sigma^v. \quad (13.76)$$

Many of the features of the response of the standard nonlinear solid are the same as for the standard quasi-linear solid. The differences come, for the most part, from the difference of the kinematic assumption between the total, elastic, and viscoelastic strains. Figures 13.15 to 13.17 show the response of the standard nonlinear solid. Note that even though we do not explicitly use the logarithmic strain, when plotting one can use this strain, and also note that the responses look markedly different when plotted against the different strains.

Section 13.2-Problems:

- 13.2-4:** Simulate the responses shown in Figures 13.15 to 13.17 for the standard nonlinear solid indicated.

13.2.3 Generalization to multiple elements

There are many ways to generalize the single nonlinear element. Possibly, the simplest way is to assume that there are a number of such elements in parallel, therefore all undergoing the same strain, and the stresses in the elements add together to give the applied stress. This provides a structure similar to what we did for the linear elements, even though most of the niceties that come with linear differential equations are lost with the nonlinear models. Yet, the numerical integration of the resulting systems is fairly straightforward. Another advantage of this method is that it always can be linearized to obtain a fairly general linear model. As before, if we assume a continuous spectrum of elements, the response can be parameterized and written in an integral. This is the model we develop here. The case of a finite number of elements can be considered as a special case.

As in the case of linear viscoelasticity, we will use a continuous parameter s to characterize the elements, where s takes values from zero to infinity. The way we parameterize is not unique, and so the range is also open to change. The parameter simply provides a method for distinctly labeling an infinite number of distinct elements. The stress in each element is denoted by σ_s and we will assume that the total stress is given by

$$\sigma = \int_0^\infty \sigma_s ds, \quad (13.77)$$

where s is clearly a parameter identifying the different elements, not to be confused with time. Each element is assumed to behave like either a standard linear solid, a standard quasi-nonlinear solid, or a standard nonlinear solid. In the limit of small deformations, the nonlinear elements become linear, providing a smooth transition between a full (read multi-element) nonlinear model and its full linear counterpart. The response of each element will be given by the assumption that the strain of each element is decomposed into an elastic and a viscoelastic part written as

$$\lambda = \lambda_s^e \lambda_s^{ve}, \quad (13.78)$$

where for the quasi-nonlinear and linear elements it is approximated as

$$\epsilon = \epsilon_s^e + \epsilon_s^{ve}. \quad (13.79)$$

For each element, we assume that the applied stress is the same as the stress in the elastic element and therefore write

$$\sigma_s = \sigma_s^e. \quad (13.80)$$

Each s represents a standard element given by two springs and viscous damper. The constitutive equations for these elements are assumed to be noninteracting between the different s and also between the different elements in each s . We, thus, provide equations for the material response by providing functions of the form

$$\sigma_s^e = \sigma_s^{e\dagger}(\lambda_s^e), \quad (13.81)$$

$$\sigma_s^b = \sigma_s^{b\dagger}(\lambda_s^{ve}), \quad (13.82)$$

$$\dot{\lambda}_s^{ve} = \lambda_s^{ve} L_s^{ve\dagger}(\lambda_s^{ve}, \sigma_s^v), \quad (13.83)$$

where σ_s^v is the overstress given by $\Delta\sigma_s = \sigma_s^e - \sigma_s^b$.

The simulation of the response is fairly straightforward if the range of s that has nonzero σ_s^e is finite. In this case, the integral can be approximated using a method mimicking the meshing done for solving problems with the finite-element method, and the approximation can be improved by taking smaller and smaller or higher-order elements. If this method of approximation is selected, then the internal parameter λ_s^{ve} needs to be integrated only at a finite number of points s_i (at the nodes). If the strain history is given, then the integral for the stress can be constructed at each time step from the known values of $\lambda_{s_i}^e = \lambda/\lambda_{s_i}^{ve}$ at the nodes. On the other hand, if the stress history is given, then a root finder can be used to find λ given the stress and the known values of $\lambda_{s_i}^{ve}$. Like most numerical problems, many different strategies can be used to solve for the response. The bigger problem of finding the best integration method is outside our scope, but we will look at a simple process for solving for the response for a finite range for s .

Let us assume (s_a, s_b) defines the range of s for which σ_s^e is nonzero, and assume that this interval is separated into nn nodes and ne elements, where the nn nodes start with $s_1 = s_a$, and sequentially increase in s until we get to the last node that is $s_{nn} = s_b$. For simplicity, we take two node elements, where each element e sits between nodes $i = e$ and $i = e + 1$ so that there are $nn - 1$ elements. As in finite-element methods, for element e the node $i = e$ will be given the local node number 1 and the node $i = e + 1$ is given the local node 2, so that points s_e and s_{e+1} locally will be referred to as s_1 and s_2 . The integral for the stress then can be written as

$$\sigma = \int_0^\infty \sigma_s ds = \sum_e^{ne} \int_{s_e}^{s_{e+1}} \sigma_s^e ds. \quad (13.84)$$

Over each element e the stress σ_s can be interpolated linearly as

$$\sigma_s^e \approx \sigma_{s_1}^e \frac{s - s_2}{s_1 - s_2} + \sigma_{s_2}^e \frac{s - s_1}{s_2 - s_1}, \quad (13.85)$$

where local numbering is being used so that $\sigma_{s_1}^e$ is actually $\sigma_{s_e}^e$ and $\sigma_{s_2}^e$ is $\sigma_{s_{e+1}}^e$, etc. Substituting this into the integral over the element results in

$$\int_{s_1}^{s_2} \sigma_s^e ds \approx \int_1^{s_2} \left(\sigma_{s_1}^e \frac{s - s_2}{s_1 - s_2} + \sigma_{s_2}^e \frac{s - s_1}{s_2 - s_1} \right) ds. \quad (13.86)$$

Since $\sigma_{s_1}^e$ and $\sigma_{s_2}^e$ are constants over the integration, it can be shown that this results in

$$\int_{s_1}^{s_2} \sigma_s^e ds \approx \frac{1}{s_2 - s_1} \left(\frac{\sigma_{s_1}^e}{2} + \frac{\sigma_{s_2}^e}{2} \right). \quad (13.87)$$

Introducing this into the original integral we get

$$\sigma = \int_0^\infty \sigma_s ds \approx \sum_e^{ne} \frac{\sigma_{s_e}^e + \sigma_{s_{e+1}}^e}{2(s_{e+1} - s_e)}. \quad (13.88)$$

This provides a simple approximation of the applied stress in terms of the values of the stress of only nn values of s . Even though it is outside the scope of this book, one can improve the solution by adaptively increasing the number of points or increasing the order of the interpolation in each element by increasing the number of nodes per element. The evaluation of nn values of σ_s^e at the nodes requires the knowledge of λ_s^e at each node. We can calculate this from the corresponding λ_s^{ve} . We thus need to calculate $\lambda_{s_i}^{ve}$ for each of the nn nodes. To do this, we select the simplest integration method, which is the Euler method. Be forewarned that the method is only conditionally stable so you will need to take very small time steps (see Chapter 1) or replace the integration with a better process. In any case, for each node s_i we write the equation

$$\lambda_{s_i}^{ve}|_{new} = \lambda_{s_i}^{ve}[1 + L_s^{ve\dagger}(\lambda_{s_i}^{ve}, \sigma_{s_i}^v)\Delta t]|_{old}, \quad (13.89)$$

where the new value of each $\lambda_{s_i}^{ve}$ after the time step Δt is calculated from its old value before the time step, and the value of $\sigma_{s_i}^v$ before the step. Once the new values of $\lambda_{s_i}^{ve}$ are calculated, the new values of $\lambda_{s_i}^e$ are calculated using

$$\lambda_{s_i}^e|_{new} = \frac{\lambda|_{new}}{\lambda_{s_i}^{ve}|_{new}}. \quad (13.90)$$

This completes the process, as we can calculate the new stresses $\sigma_{s_i}^e$, assuming that all the old values are known, and calculate the total stress σ . As stated, if the new stress is known, then the new strains are unknowns, and we need to add a root-finding method to iteratively solve for the $\lambda|_{new}$ given the $\sigma|_{new}$.

Again, the reader is cautioned to use very small time steps to avoid stability issues. For inclusion in computationally intensive operations, it is suggested that at least the time integrator be replaced by a unconditionally stable integrator.

13.3 Three-dimensional linear viscoelasticity

The linear theory of viscoelasticity in three dimensions closely resembles the one-dimensional theory. As in the previous section, we first will develop the three-dimensional model that resembles the standard linear solid. Even though this results in a model that has more than one relaxation time, the number of distinct relaxation times is finite. As in the one-dimensional case, we then will generalize this to a model with a continuum of standard linear solids put in parallel.

13.3.1 Standard linear solid

The three-dimensional counterpart to our one-dimensional model for the standard linear solid can be constructed as follows. We can assume that the strain ϵ is the sum of elastic and viscoelastic parts given by the expression

$$\epsilon = \epsilon^e + \epsilon^{ve}. \quad (13.91)$$

The applied stress σ can be taken to be given in terms of the elastic strain by the expression

$$\sigma = \mathbf{E} : \epsilon^e, \quad (13.92)$$

where $\mathbf{E}$ is a fourth-order tensor modulus associated with the “spring” of the elastic element. The back-stress is taken to be given by the expression

$$\sigma^b = \mathbf{E}^b : \epsilon^{ve}, \quad (13.93)$$

where $\mathbf{E}^b$ is a fourth-order tensor modulus associated with the spring in the “viscoelastic element.” The flow rule for the internal parameter ϵ^{ve} is assumed to be given by

$$\dot{\epsilon}^{ve} = \mathbf{C} : \Delta\sigma, \quad (13.94)$$

where $\Delta\sigma$ is the overstress given by the relation $\Delta\sigma = \sigma - \sigma^b$, and $\mathbf{C}$ is a fourth-order tensor. The tensor $\mathbf{C}$ needs to be such that the system always has a dissipative nature characterized by $\Delta\sigma : \dot{\epsilon}^{ve} \geq 0$ for all processes. As will be shown later, this can be achieved by several different methods, one of which is to take $\mathbf{C} = \mathbf{Z} \otimes \mathbf{Z}$ for any real second-order tensor $\mathbf{Z}$. Another method is to postulate that the flow occurs along $\Delta\sigma$ or along the deviatoric part of $\Delta\sigma$. The reader will find a more detailed discussion of this and the conditions imposed by material symmetry in the following sections as part of discussing the thermodynamic theory.

The general solution for the three-dimensional version of the model for a standard linear solid follows a similar development to that of the one-dimensional version. Substituting (13.92) and (13.93) in the flow rule (13.94) and reorganization result in an ordinary differential equation for ϵ^{ve} given as

$$\dot{\epsilon}^{ve} + \mathbf{C} : (\mathbf{E} + \mathbf{E}^b) : \epsilon^{ve} = \mathbf{C} : \mathbf{E} : \epsilon. \quad (13.95)$$

This ordinary differential equation with constant coefficients can be solved in general form for a piecewise continuous applied strain history. The component form of this equation in an orthonormal base can be written as

$$\dot{\epsilon}_{ij}^{ve} + C_{ijkl}(\mathbf{E}_{klmn} + \mathbf{E}_{klmn}^b)\epsilon_{mn}^{ve} = C_{ijkl}\mathbf{E}_{klmn}\epsilon_{mn}. \quad (13.96)$$

We can introduce the transformation T_{ijk} from the nine components of the second-order tensor ϵ_{ij} to the six components of its vector representation ϵ_i through the relation $\epsilon_i = T_{ijk}\epsilon_{jk}$ and its inverse relation given as $\epsilon_{ij} = T_{ijk}^{-1}\epsilon_k$. The reader will note that in these expressions ϵ_i represents the six-dimensional vector of components $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{23}, \epsilon_{31}, \epsilon_{12})$. Using this transformation, we can obtain the general solution²

$$\epsilon_{ij}^{ve}(t) = \int_{t_o}^t T_{ijk}^{-1}[e^{(t-u)\mathbf{A}}]_{kl}f_l(u)du + T_{ijk}^{-1}[e^{(t-t_o)\mathbf{A}}]_{kl}T_{lmn}\epsilon_{mn}^{ve}(t_o), \quad (13.97)$$

where

$$A_{po} = -T_{pij}C_{ijkl}(\mathbf{E}_{klmn} + \mathbf{E}_{klmn}^b)T_{mno}^{-1}, \quad f_p(u) = T_{pij}C_{ijkl}\mathbf{E}_{klmn}\epsilon_{mn}(u). \quad (13.98)$$

Substitution of $\epsilon^{ve}(t)$ back into (13.92) results in a general expression for stress in terms of the strain history given by

$$\begin{aligned} \sigma_{op}(t) &= \mathbf{E}_{opij}[\epsilon_{ij}(t) - \epsilon_{ij}^{ve}(t)] \\ &= \mathbf{E}_{opij}\epsilon_{ij}(t) - \mathbf{E}_{opij} \int_{t_o}^t T_{ijk}^{-1}[e^{(t-u)\mathbf{A}}]_{kl}f_l(u)du - \mathbf{E}_{opij}T_{ijk}^{-1}[e^{(t-t_o)\mathbf{A}}]_{kl}T_{lmn}\epsilon_{mn}^{ve}(t_o). \end{aligned} \quad (13.99)$$

As will be shown later, for an isotropic material one can show that there are two distinct relaxation times, one associated with the bulk response and one associated with the deviatoric response. A detailed description of material symmetry and the linear model will be presented in the section on quadratic free energy functions. This is presented as a special example in the thermodynamic theory of viscoelasticity.

²See, for example, Myskis, A. D., *Advanced Mathematics for Engineers*, Mir Publishers, Moscow, 1975.

13.3.2 General linear viscoelastic model

A general three-dimensional model for linear viscoelasticity can be constructed as in the one-dimensional case from considering an infinite number of standard linear solid models put in parallel. As in the one-dimensional case, the parameter s will be used to identify the elements, and it will be assumed that s is continuous and going from zero to infinity. Since the elements are in parallel, each element has the same applied strain, and this strain is partitioned into elastic and viscoelastic parts. We thus assume that for each element the strain ϵ is the sum of elastic and viscoelastic parts given by the expression

$$\epsilon = \epsilon_s^e + \epsilon_s^{ve}, \quad (13.100)$$

where ϵ_s^e and ϵ_s^{ve} are the elastic and viscoelastic strains in the element characterized by the value of s . Since the elements are assumed to be in parallel, the stresses in the elements should add to equal the applied stress. Therefore, we postulate the applied stress can be evaluated from the stresses in the individual elements by integrating them over s going from zero to infinity. If the stress in element s is given by σ_s , this will be written as

$$\sigma = \int_0^\infty \sigma_s ds. \quad (13.101)$$

As in the case of the standard linear solid, the stress in each element is taken as equal to the stress in the elastic element and can be taken as given in terms of the elastic strain by the expression

$$\sigma_s = \mathbf{E}_s : \epsilon_s^e, \quad (13.102)$$

where $\mathbf{E}_s$ is a fourth-order tensor modulus associated with the “spring” of the elastic element of s . The back-stress is taken to be given by the expression

$$\sigma_s^b = \mathbf{E}_s^b : \epsilon_s^{ve}, \quad (13.103)$$

where $\mathbf{E}_s^b$ is a fourth-order tensor modulus associated with the spring in the “viscoelastic element” of s . The flow rule for the internal parameter ϵ_s^{ve} is assumed to be given by

$$\dot{\epsilon}_s^{ve} = \mathbf{C}_s : \Delta\sigma_s, \quad (13.104)$$

where $\Delta\sigma_s$ is the overstress given by the relation $\Delta\sigma_s = \sigma_s - \sigma_s^b$, and $\mathbf{C}_s$ is a fourth-order tensor. The tensor $\mathbf{C}_s$ needs to be such that the system always has a dissipative nature. This can be satisfied by setting $\Delta\sigma_s : \dot{\epsilon}_s^{ve} \geq 0$ for all processes. As will be shown later, this can be achieved by several different methods, one of which is to take $\mathbf{C}_s = \mathbf{Z}_s \otimes \mathbf{Z}_s$ for any real second-order tensor $\mathbf{Z}_s$. Another method is to postulate that the flow occurs along $\Delta\sigma_s$ or along the deviatoric part of $\Delta\sigma_s$. The reader will find a more detailed discussion of this and the conditions imposed by material symmetry in the following sections that discuss the thermodynamic theory.

The general solution for the three-dimensional version of the model follows the one-dimensional version and the three-dimensional standard linear solid model. Substituting (13.102) and (13.103) into the flow rule (13.104) and reorganization result in an ordinary differential equation for ϵ_s^{ve} given as

$$\dot{\epsilon}_s^{ve} + \mathbf{C}_s : (\mathbf{E}_s + \mathbf{E}_s^b) : \epsilon_s^{ve} = \mathbf{C}_s : \mathbf{E}_s : \epsilon. \quad (13.105)$$

This ordinary differential equation can be solved in general form for a piecewise continuous applied strain history. The component form of this equation in an orthonormal base can be written as

$$\dot{\epsilon}_{s\_ij}^{ve} + \mathbf{C}_{s\_ijkl}(\mathbf{E}_{s\_klmn} + \mathbf{E}_{s\_klmn}^b)\epsilon_{s\_mn}^{ve} = \mathbf{C}_{s\_ijkl}\mathbf{E}_{s\_klmn}\epsilon_{mn}, \quad (13.106)$$

where “_” in the notation is used to separate the parameter s and the indices of the tensors. As for the standard linear solid, we can introduce the transformation T_{ijk} from the nine components of the second-order tensor ϵ_{ij} to the six components of its vector representation ϵ_i through the relation $\epsilon_i = T_{ijk}\epsilon_{jk}$ and its inverse relation $\epsilon_{ij} = T_{ijk}^{-1}\epsilon_k$. As mentioned before, in these expressions ϵ_i represents the six-dimensional vector of components $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{23}, \epsilon_{31}, \epsilon_{12})$. Using this transformation, we can obtain the general solution

$$\epsilon_{s\_ij}^{ve}(t) = \int_{t_o}^t T_{ijk}^{-1}[e^{(t-u)\mathbf{A}_s}]_{kl}f_{s\_l}(u)du + T_{ijk}^{-1}[e^{(t-t_o)\mathbf{A}_s}]_{kl}T_{lmn}\epsilon_{s\_mn}^{ve}(t_o), \quad (13.107)$$

where

$$A_{s\_po} = -T_{pij}\mathbf{C}_{s\_ijkl}(\mathbf{E}_{s\_klmn} + \mathbf{E}_{s\_klmn}^b)T_{mno}^{-1}, \quad f_{s\_p}(u) = T_{pij}\mathbf{C}_{s\_ijkl}\mathbf{E}_{s\_klmn}\epsilon_{mn}(u). \quad (13.108)$$

Substitution of $\epsilon_s^{ve}(t)$ back into (13.102) results in an expression for stress in element s in terms of the strain history given by

$$\begin{aligned} \sigma_{s\_op}(t) &= \mathbf{E}_{s\_opij}[\epsilon_{ij}(t) - \epsilon_{s\_ij}^{ve}(t)] \\ &= \mathbf{E}_{s\_opij}\epsilon_{ij}(t) - \mathbf{E}_{s\_opij} \int_{t_o}^t T_{ijk}^{-1}[e^{(t-u)\mathbf{A}_s}]_{kl}f_{s\_l}(u)du \\ &\quad - \mathbf{E}_{s\_opij}T_{ijk}^{-1}[e^{(t-t_o)\mathbf{A}_s}]_{kl}T_{lmn}\epsilon_{s\_mn}^{ve}(t_o). \end{aligned} \quad (13.109)$$

Substitution of this back into (13.101) results in the constitutive equation for the response of a general linear viscoelastic model constructed from a continuous array of standard linear solid elements put in parallel. This process results in the expression for stress given by

$$\begin{aligned} \sigma_{op}(t) &= \int_0^\infty \mathbf{E}_{s\_opij}ds\epsilon_{ij}(t) - \int_0^\infty \mathbf{E}_{s\_opij} \int_{t_o}^t T_{ijk}^{-1}[e^{(t-u)\mathbf{A}_s}]_{kl}f_{s\_l}(u)duds \\ &\quad - \int_0^\infty \mathbf{E}_{s\_opij}T_{ijk}^{-1}[e^{(t-t_o)\mathbf{A}_s}]_{kl}T_{lmn}\epsilon_{s\_mn}^{ve}(t_o)ds. \end{aligned} \quad (13.110)$$

This expression for stress can be reorganized for the case that $\epsilon_s^{ve}(t_o) = \mathbf{0}$ and put in the form

$$\sigma(t) = \mathbf{G}(0) : \epsilon(t) + \int_0^t \mathbf{G}'(t-u) : \epsilon(u)du, \quad (13.111)$$

where

$$\mathbf{G}_{ijst}(t) = \int_0^\infty \mathbf{E}_{s\_ijkl}T_{klm}^{-1}[e^{-t\mathbf{A}_s}]_{mv}A_{s\_vn}^{-1}T_{nop}\mathbf{C}_{s\_opqr}\mathbf{E}_{s\_qrst}ds + \mathbf{G}_{ijst}(\infty), \quad (13.112)$$

$$\mathbf{G}_{ijst}(0) = \int_0^\infty \mathbf{E}_{s\_ijst}ds, \quad (13.113)$$

$$\mathbf{G}_{ijst}(\infty) = \int_0^\infty \mathbf{E}_{s\_ijkl}(\delta_{ks}\delta_{lt} - T_{klm}^{-1}A_{s\_mn}^{-1}T_{nop}\mathbf{C}_{s\_opqr}\mathbf{E}_{s\_qrst})ds. \quad (13.114)$$

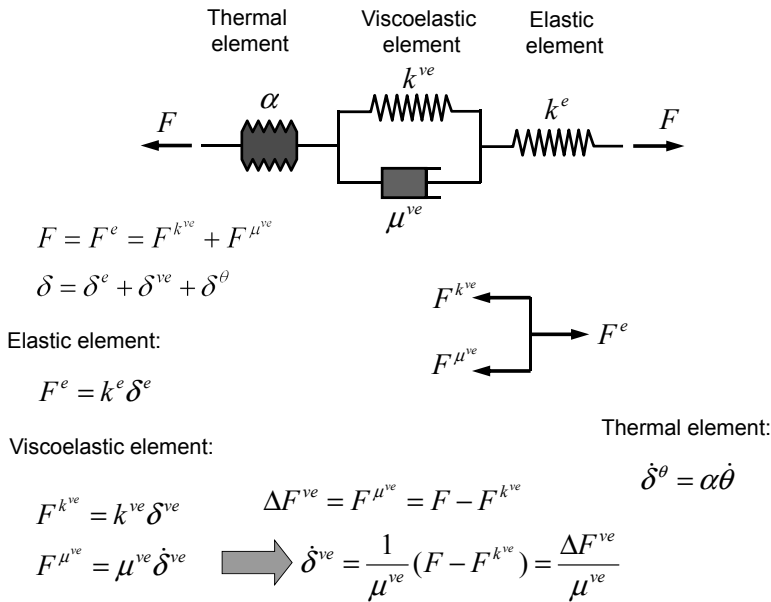


Figure 13.18: A mechanical analog for a viscoelastic material constructed from elastic, viscoelastic and thermal elements.

13.4 A one-element thermoviscoelastic model

One of the simplest models one can construct that includes thermal strains and viscoelastic response is represented by the mechanical analog shown in Figure 13.18. This mechanical analog simply adds a thermal element in series to the standard linear solid studied earlier in this chapter. The equations that describe the response of the mechanical analog are shown in the figure. In this section, we will construct models that are motivated by this simple mechanical analog, first considering a one-dimensional model and then presenting a fully consistent nonlinear three-dimensional model. As was done for the mechanical theories presented in the previous sections, we then generalize the model by considering an infinite number of elements similar to that given in Figure 13.18 put in parallel.

13.4.1 One-dimensional model

We can use the equations given in Figure 13.18 as a template for constructing a model for the one-dimensional response of a linear thermoviscoelastic material. As the elements are in series, the basic assumption for strain in this case will be that the strain of the material is the sum of the elastic, viscoelastic, and thermal parts given by the equation

$$\epsilon = \epsilon^e + \epsilon^{ve} + \epsilon^\theta, \quad (13.115)$$

where, as before, ϵ^e and ϵ^{ve} denote, respectively, the strain in the elastic and viscoelastic elements and ϵ^θ denotes the strain in the thermal element. The stress in the “springs” of

the elastic and viscoelastic elements are given by the linear expressions

$$\sigma^e = E^e \epsilon^e, \quad (13.116)$$

$$\sigma^{ve} = E^{ve} \epsilon^{ve}, \quad (13.117)$$

where E^e and E^{ve} are, respectively, the elastic moduli for the “springs” of the elastic and viscoelastic elements. The total stress in each element is the same as the applied stress σ so that the stress in the elastic element is equal to the applied stress and gives

$$\sigma = E^e \epsilon^e. \quad (13.118)$$

The extension of the viscoelastic element is given by the viscoelastic flow rule

$$\dot{\epsilon}^{ve} = \frac{1}{\mu^{ve}} (\sigma - \sigma^{ve}), \quad (13.119)$$

where μ^{ve} is the “viscosity” for the viscous damping in the viscoelastic element. The thermal strain is given by integrating the thermal expansion rule

$$\dot{\epsilon}^\theta = \alpha \dot{\theta}, \quad (13.120)$$

where α is the linear coefficient of thermal expansion. This simple model, which is constructed with many linear ideas in mind, provides a response that shows many of the features that are observed in the response of materials that show both viscoelastic and thermal response. The model has four material constants given by $(E^e, E^{ve}, \mu^{ve}, \alpha)$. To extend the applicability of the theory, we can make these material parameters depend on temperature.

13.4.2 A one-element three-dimensional thermodynamically consistent model

One can construct a three-dimensional model based on the same ideas. We will do this in a thermodynamically consistent form for large deformations. First, we assume that the deformations can be separated into elastic, viscoelastic, and thermal parts such that we can write

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^\theta, \quad (13.121)$$

where we have assumed that there are, as in the one-dimensional case, three “elements” set up in series represented by, respectively, the elastic, viscoelastic, and thermal deformation gradients $\mathbf{F}^e$, $\mathbf{F}^{ve}$, and $\mathbf{F}^\theta$. We assume that the free energy depends on all these deformation gradients plus temperature so that we have

$$\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta). \quad (13.122)$$

We need to calculate the rate of change of the free energy to include in the Clausius-Duhem inequality. This rate is given by

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^{ve}}(\psi) : \dot{\mathbf{F}}^{ve} + \partial_{\mathbf{F}^\theta}(\psi) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi) \dot{\theta}. \quad (13.123)$$

The rate of change of the deformation gradient is given by

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^{ve} \mathbf{F}^\theta + \mathbf{F}^e \dot{\mathbf{F}}^{ve} \mathbf{F}^\theta + \mathbf{F}^e \mathbf{F}^{ve} \dot{\mathbf{F}}^\theta. \quad (13.124)$$

Solving for $\dot{\mathbf{F}}^e$ yields

$$\dot{\mathbf{F}}^e = \dot{\mathbf{F}} \mathbf{F}^{\theta-1} \mathbf{F}^{ve-1} - \mathbf{F}^e \dot{\mathbf{F}}^{ve} \mathbf{F}^{ve-1} - \mathbf{F}^e \mathbf{F}^{ve} \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1} \mathbf{F}^{ve-1}. \quad (13.125)$$

This can be written in terms of the velocity gradients as

$$\dot{\mathbf{F}}^e = \mathbf{L}\mathbf{F}^e - \mathbf{F}^e\mathbf{L}^{ve} - \mathbf{F}^e\mathbf{F}^{ve}\mathbf{L}^\theta\mathbf{F}^{ve-1}. \quad (13.126)$$

Using this relation and the identity $\mathbf{C} : (\mathbf{A}\mathbf{F}\mathbf{B}) = (\mathbf{A}^T\mathbf{C}\mathbf{B}^T) : \mathbf{F}$, any expression of the form $\mathbf{A} : \dot{\mathbf{F}}^e + \mathbf{B} : \dot{\mathbf{F}}^{ve} + \mathbf{C} : \dot{\mathbf{F}}^\theta$ can be written as

$$\begin{aligned} & \mathbf{A} : \dot{\mathbf{F}}^e + \mathbf{B} : \dot{\mathbf{F}}^{ve} + \mathbf{C} : \dot{\mathbf{F}}^\theta \\ &= \mathbf{A} : (\mathbf{L}\mathbf{F}^e - \mathbf{F}^e\mathbf{L}^{ve} - \mathbf{F}^e\mathbf{F}^{ve}\mathbf{L}^\theta\mathbf{F}^{ve-1}) + (\mathbf{B}\mathbf{F}^{veT}) : \mathbf{L}^{ve} + (\mathbf{C}\mathbf{F}^{\theta T}) : \mathbf{L}^\theta \\ &= (\mathbf{A}\mathbf{F}^{eT}) : \mathbf{L} + (\mathbf{B}\mathbf{F}^{veT} - \mathbf{F}^{eT}\mathbf{A}) : \mathbf{L}^{ve} + (\mathbf{C}\mathbf{F}^{\theta T} - \mathbf{F}^{veT}\mathbf{F}^{eT}\mathbf{A}\mathbf{F}^{ve-T}) : \mathbf{L}^\theta. \end{aligned} \quad (13.127)$$

Using this relation, the time derivative of the free energy can be written as

$$\rho\dot{\psi} = \mathbf{T}^{eT} : \mathbf{L} + (\mathbf{T}^{veT} - \mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}) : \mathbf{L}^{ve} + (\mathbf{T}^{\theta T} - \mathbf{F}^{veT}\mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}\mathbf{F}^{ve-T}) : \mathbf{L}^\theta + \rho\partial_\theta(\psi)\dot{\theta} \quad (13.128)$$

for the “thermodynamic stresses” in the elastic, viscoelastic, and thermal elements defined, respectively, as

$$\boxed{\mathbf{T}^{eT} \equiv \rho\partial_{\mathbf{F}^e}(\psi)\mathbf{F}^{eT}}, \quad (13.129)$$

$$\boxed{\mathbf{T}^{veT} \equiv \rho\partial_{\mathbf{F}^{ve}}(\psi)\mathbf{F}^{veT}}, \quad (13.130)$$

$$\boxed{\mathbf{T}^{\theta T} \equiv \rho\partial_{\mathbf{F}^\theta}(\psi)\mathbf{F}^{\theta T}}. \quad (13.131)$$

Introducing the rate of change of free energy into the Clausius-Duhem inequality (6.69) and reorganizing result in

$$\begin{aligned} & (\mathbf{T}^{eT} - \mathbf{T}^T) : \mathbf{L} + (\mathbf{T}^{veT} - \mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}) : \mathbf{L}^{ve} \\ & + [\rho\eta + (\mathbf{T}^{\theta T} - \mathbf{F}^{veT}\mathbf{F}^{eT}\mathbf{T}^{eT}\mathbf{F}^{e-T}\mathbf{F}^{ve-T}) : \boldsymbol{\alpha} + \rho\partial_\theta(\psi)]\dot{\theta} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (13.132)$$

where we have assumed that the thermal expansion is given by the rule

$$\boxed{\mathbf{L}^\theta = \boldsymbol{\alpha}\dot{\theta}}, \quad (13.133)$$

where $\boldsymbol{\alpha}$ is the second-order tensor of the coefficients of linear thermal expansion. We now assume that the viscoelastic velocity gradients and the coefficient of thermal expansion depend on current values of $(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta)$ but not on their rates. Then, by altering $\mathbf{L}$ and $\dot{\theta}$, one can conclude that the Clausius-Duhem inequality only can be satisfied for all processes if

$$\boxed{\mathbf{T}^T = \mathbf{T}^{eT} = \rho\partial_{\mathbf{F}^e}(\psi)\mathbf{F}^{eT}}, \quad (13.134)$$

$$\boxed{\eta = -\partial_\theta(\psi) + \frac{1}{\rho}\Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha}}, \quad (13.135)$$

where $\Delta\mathbf{T}^\theta$ is the thermal overstress given by

$$\boxed{\Delta\mathbf{T}^\theta \equiv \mathbf{F}^{ve-1}\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^e\mathbf{F}^{ve} - \mathbf{T}^\theta}. \quad (13.136)$$

Through similar arguments as presented before we can conclude that we always must have

$$\boxed{-\Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0}, \quad (13.137)$$

for a viscoelastic overstress $\Delta \mathbf{T}^{ve}$ defined as

$$\Delta \mathbf{T}^{ve} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}. \quad (13.138)$$

We also have remaining the condition that

$$-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (13.139)$$

As the reader recalls, $\rho \dot{h}$ is the rate of heat entering a point (per unit volume) through radiation and conduction, so that $\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q})$. Most features about heating of a point can be studied through knowing $\rho \dot{h}$, so let us calculate it for this model. We recall from Chapter 6 that one can calculate it from

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T} \mathbf{L}) + \rho \eta \dot{\theta} + \rho \dot{\eta} \theta.$$

The first three terms on the right-hand side directly appear in the Clausius-Duhem inequality, so we easily can calculate $\rho \dot{h}$ as

$$\rho \dot{h} = -\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \rho \dot{\eta} \theta. \quad (13.140)$$

A possible selection for the flow rule might be

$$\mathbf{L}^{ve} = \mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}, \quad (13.141)$$

where $\mathbf{C}^{ve}$ is a fourth-order tensor function with the property $\Delta \mathbf{T}^{veT} : \mathbf{C}^{ve} : \Delta \mathbf{T}^{veT} \geq 0$.

13.4.3 Rigid body motions and the symmetry of the Cauchy stress

How the “internal parameters” $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$ are influenced by rigid body motions strongly influences how the resulting thermodynamic model behaves. In particular, it can be shown that certain selections result in expressions that are consistent with other imposed conditions, such as the symmetry of Cauchy stress that results from the balance of angular momentum. This section will look at one such selection and its influence on the resulting expressions.

We will assume, with good motivation, that $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$ are unaffected by rigid body motions superimposed on the deformation history.³ We will consider two deformation histories that only differ by rigid body motions. Let $\mathbf{F}(t)$ represent a deformation history and let $\mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t)$ be constructed from it by superimposing rigid body motions described by a history of orthogonal transformations $\mathbf{Q}(t)$. We will describe this transformation of $\mathbf{F}(t)$ by $\mathbf{F}(t) \rightarrow \mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t)$, or simply $\mathbf{F}(t) \rightarrow \mathbf{Q}(t)\mathbf{F}(t)$. We now postulate that

$$\mathbf{F}^e(t) \rightarrow \mathbf{F}^{e*}(t) = \mathbf{Q}(t)\mathbf{F}^e(t), \quad (13.142)$$

$$\mathbf{F}^{ve}(t) \rightarrow \mathbf{F}^{ve*}(t) = \mathbf{F}^{ve}(t), \quad (13.143)$$

$$\mathbf{F}^\theta(t) \rightarrow \mathbf{F}^{\theta*}(t) = \mathbf{F}^\theta(t), \quad (13.144)$$

which is consistent with the assumption $\mathbf{F}^* = \mathbf{F}^{e*}\mathbf{F}^{ve*}\mathbf{F}^{\theta*} = \mathbf{Q}\mathbf{F}$. It will be shown below that this set of assumptions and the assumption that free energy is not affected by rigid

³It can be shown that to calculate the response of the material one needs only to evaluate the internal parameters for the history of the pure deformations, where pure deformations refer to the symmetry part in the polar decomposition of the deformation gradient. This is true for all first-gradient models.

body motions result in

$$\psi(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}^e(t), \mathbf{F}^{ve}(t), \mathbf{F}^\theta(t), \theta(t)] \quad (13.145)$$

$$= \psi^\dagger[\mathbf{X}, \mathbf{U}^e(t), \mathbf{F}^{ve}(t), \mathbf{F}^\theta(t), \theta(t)] \quad (13.146)$$

$$= \psi^{\dagger\dagger}[\mathbf{X}, \mathbf{C}^e(t), \mathbf{F}^{ve}(t), \mathbf{F}^\theta(t), \theta(t)] \quad (13.147)$$

$$= \psi^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}^e(t), \mathbf{F}^{ve}(t), \mathbf{F}^\theta(t), \theta(t)], \quad (13.148)$$

where $\mathbf{U}^e$ is the right symmetric factor in the polar decomposition of $\mathbf{F}^e$ and $\mathbf{C}^e$ is the right Cauchy stretch tensor constructed from $\mathbf{F}^e$. It further will be shown that Cauchy stress is symmetric and the thermodynamic stresses have the property

$$\mathbf{T}^e(t) \rightarrow \mathbf{T}^{e*}(t) = \mathbf{Q}^T(t) \mathbf{T}^e(t) \mathbf{Q}(t), \quad (13.149)$$

$$\mathbf{T}^{ve}(t) \rightarrow \mathbf{T}^{ve*}(t) = \mathbf{T}^{ve}(t), \quad (13.150)$$

$$\mathbf{T}^\theta(t) \rightarrow \mathbf{T}^{\theta*}(t) = \mathbf{T}^\theta(t), \quad (13.151)$$

which result in the over stresses having the properties

$$\Delta \mathbf{T}^{ve}(t) \rightarrow \Delta \mathbf{T}^{ve*}(t) = \Delta \mathbf{T}^{ve}(t), \quad (13.152)$$

$$\Delta \mathbf{T}^\theta(t) \rightarrow \Delta \mathbf{T}^{\theta*}(t) = \Delta \mathbf{T}^\theta(t). \quad (13.153)$$

To prove the above, let $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ be two deformation gradient histories such that one is obtained from the other by superimposing rigid body motions. We always can write $\mathbf{F}_2(t) = \mathbf{Q}(t) \mathbf{F}_1(t)$ for some history $\mathbf{Q}(t)$ of orthogonal transformations. As stated, we will postulate that both $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$ are indifferent to rigid body motions so that

$$\mathbf{F}_2^{ve}(t) = \mathbf{F}_1^{ve}(t), \quad (13.154)$$

$$\mathbf{F}_2^\theta(t) = \mathbf{F}_1^\theta(t). \quad (13.155)$$

To keep the relation $\mathbf{F} = \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^\theta$ true for both deformation histories, given the assumptions regarding $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$, we will have to have

$$\mathbf{F}_2^e(t) = \mathbf{Q}(t) \mathbf{F}_1^e(t). \quad (13.156)$$

We have postulated that superimposing rigid body motions does not change the value of the free energy. Given the model for free energy described above we will have

$$\psi_1(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)], \quad (13.157)$$

$$\psi_2(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}_2^e(t), \mathbf{F}_2^{ve}(t), \mathbf{F}_2^\theta(t), \theta(t)], \quad (13.158)$$

which yields

$$\psi^\dagger[\mathbf{X}, \mathbf{F}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)] = \psi^\dagger[\mathbf{X}, \mathbf{Q}(t) \mathbf{F}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)] \quad (13.159)$$

for every orthogonal $\mathbf{Q}(t)$. Selecting $\mathbf{Q}(t) = \mathbf{R}_1^e(t)$, for $\mathbf{R}_1^e$ being the orthogonal factor in the polar decomposition of $\mathbf{F}_1^e$, it follows that for every $\mathbf{F}_1^e(t)$ we have

$$\psi_1(t) = \psi^\dagger[\mathbf{X}, \mathbf{F}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)] = \psi^\dagger[\mathbf{X}, \mathbf{U}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)], \quad (13.160)$$

which proves (13.146). Because of the uniqueness and invertability of the relation between $\mathbf{U}^e$ and $\mathbf{C}^e = \mathbf{U}^{e2}$, any function of $\mathbf{U}^e$ can be rewritten as a function of $\mathbf{C}^e$, which proves the relation (13.147). In a similar manner the uniqueness of the relation $\mathbf{C}^e$ and $\mathbf{E}^e$ proves the relation (13.148). Since the free energy does not change with rigid body motions, we

should have $\psi_1(t) = \psi_2(t)$ for all time. Therefore, we have $\dot{\psi}_1(t) = \dot{\psi}_2(t)$. These rates can be written as

$$\dot{\psi}_1 = [\partial_{\mathbf{F}^e}(\psi)]_1 : \dot{\mathbf{F}}_1^e + [\partial_{\mathbf{F}^{ve}}(\psi)]_1 : \dot{\mathbf{F}}_1^{ve} + [\partial_{\mathbf{F}^\theta}(\psi)]_1 : \dot{\mathbf{F}}_1^\theta + [\partial_\theta(\psi)]_1 \dot{\theta}, \quad (13.161)$$

$$\begin{aligned} \dot{\psi}_2 &= [\partial_{\mathbf{F}^e}(\psi)]_2 : \dot{\mathbf{F}}_2^e + [\partial_{\mathbf{F}^{ve}}(\psi)]_2 : \dot{\mathbf{F}}_2^{ve} + [\partial_{\mathbf{F}^\theta}(\psi)]_2 : \dot{\mathbf{F}}_2^\theta + [\partial_\theta(\psi)]_2 \dot{\theta} \\ &= [\partial_{\mathbf{F}^e}(\psi)]_2 : (\dot{\mathbf{Q}}\mathbf{F}_1^e + \mathbf{Q}\dot{\mathbf{F}}_1^e) + [\partial_{\mathbf{F}^{ve}}(\psi)]_2 : \dot{\mathbf{F}}_1^{ve} + [\partial_{\mathbf{F}^\theta}(\psi)]_2 : \dot{\mathbf{F}}_1^\theta + [\partial_\theta(\psi)]_2 \dot{\theta}, \end{aligned} \quad (13.162)$$

where the explicit inclusion of “(t)” is omitted from the notation, and the subscripts “1” and “2” denote the value evaluated, respectively, for the history of $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$. Since the superimposed rigid body motion can be selected arbitrarily, the values of $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ can be selected independently and arbitrarily. In view of this, the requirement that $\dot{\psi}_1 = \dot{\psi}_2$ results, after using the identity $\mathbf{A} : (\mathbf{BCD}) = (\mathbf{B}^T \mathbf{A} \mathbf{D}^T) : \mathbf{C}$ and reorganization, in the relations

$$[\partial_{\mathbf{F}^e}(\psi)]_1 = \mathbf{Q}^T [\partial_{\mathbf{F}^e}(\psi)]_2, \quad (13.163)$$

$$([\partial_{\mathbf{F}^e}(\psi)]_2 \mathbf{F}_1^{eT}) : \dot{\mathbf{Q}} = 0, \quad (13.164)$$

$$[\partial_{\mathbf{F}^{ve}}(\psi)]_1 = [\partial_{\mathbf{F}^{ve}}(\psi)]_2, \quad (13.165)$$

$$[\partial_{\mathbf{F}^\theta}(\psi)]_1 = [\partial_{\mathbf{F}^\theta}(\psi)]_2, \quad (13.166)$$

$$[\partial_\theta(\psi)]_1 = [\partial_\theta(\psi)]_2. \quad (13.167)$$

The second relation, in view of the first relation, can be rewritten as

$$([\partial_{\mathbf{F}^e}(\psi)]_1 \mathbf{F}_1^{eT}) : (\mathbf{Q}^T \dot{\mathbf{Q}}) = 0, \quad (13.168)$$

which in view of (13.129) and the fact that $\mathbf{Q}^T \dot{\mathbf{Q}}$ is skew-symmetric requires that the $\mathbf{T}_1^e$ be symmetric (note that $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$ yields $\dot{\mathbf{Q}}^T \mathbf{Q} + \mathbf{Q}^T \dot{\mathbf{Q}} = \mathbf{0}$, which shows that $\mathbf{Q}^T \dot{\mathbf{Q}} = -(\mathbf{Q}^T \dot{\mathbf{Q}})^T$). Since the Cauchy stress and $\mathbf{T}^e$ are equal, it follows from the arbitrary nature of the two deformation histories that the Cauchy stress must be symmetric so the thermodynamic model, given the assumptions made regarding $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$, is consistent with the result of the balance of angular momentum. Another result of these relations is that

$$\mathbf{T}_2^{eT} = \rho [\partial_{\mathbf{F}^e}(\psi)]_2 \mathbf{F}_2^{eT} = \rho \mathbf{Q} [\partial_{\mathbf{F}^e}(\psi)]_1 \mathbf{F}_1^{eT} \mathbf{Q}^T = \mathbf{Q} \mathbf{T}_1^{eT} \mathbf{Q}^T, \quad (13.169)$$

showing that the thermodynamic model, given the assumptions made regarding $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$, produces a Cauchy stress that behaves with rigid body rotations by simply rotating the traction vectors an amount equal to the rotation, consistent with what was postulated in Chapter 9. Other direct results of the assumptions made for $\mathbf{F}^{ve}$ and $\mathbf{F}^\theta$, are that

$$\mathbf{T}_2^{veT} = \mathbf{T}_1^{veT}, \quad (13.170)$$

$$\mathbf{T}_2^{\theta T} = \mathbf{T}_1^{\theta T}, \quad (13.171)$$

which in addition to stating that the thermodynamic stresses for $\mathbf{T}^{ve}$ and $\mathbf{T}^\theta$ are unaffected by rigid body motions, also, once combined with the results for $\mathbf{T}^e$, states that the over stresses $\Delta \mathbf{T}^{ve}$ and $\Delta \mathbf{T}^\theta$ also are unchanged by superimposed rigid body motions.

Section 13.4-Problems:

13.4-1: There are theories that satisfy the relation $\mathbf{F}_2(t) = \mathbf{Q}(t)\mathbf{F}_1(t)$ by postulating

$$\mathbf{F}_2^e(t) = \mathbf{Q}(t)\mathbf{F}_1^e(t)\mathbf{Q}^T(t), \quad (13.172)$$

$$\mathbf{F}_2^{ve}(t) = \mathbf{Q}(t)\mathbf{F}_1^{ve}(t)\mathbf{Q}^T(t), \quad (13.173)$$

$$\mathbf{F}_2^\theta(t) = \mathbf{Q}(t)\mathbf{F}_1^\theta(t). \quad (13.174)$$

Study the results of such a selection.

13.4.4 Material symmetry constraints

We next will look at imposing material symmetry on the constitutive functions. We will let $\mathbf{M}$ denote a general transformation representing a symmetry of the material at a material point of consideration. As was discussed before, $\mathbf{M}$ is a transformation that represents a reorganization of the neighborhood of the given point in the reference configuration, and this transformation changes the neighborhood in a way that leaves it thermodynamically indistinguishable from before reorganization. For details on material symmetry, the reader should consult Chapters 9 and 10. A symmetry of the material represented by the transformation $\mathbf{M}$ allows us to change the history of deformation gradient $\mathbf{F}(t)$ to $\mathbf{F}(t)\mathbf{M}$ for any deformation gradient history without seeing a change in the response of the material at that point. We denote this change of history by a bar over the symbol and write it as $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$. To construct a consistent theory, we need to have the kinematic variables $\mathbf{F}^e$, $\mathbf{F}^{ve}$, and $\mathbf{F}^\theta$ change in a way that is consistent with $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$. To create such a consistent relation, we will assume

$$\mathbf{F}^e \rightarrow \bar{\mathbf{F}}^e = \mathbf{F}^e\mathbf{M}, \quad (13.175)$$

$$\mathbf{F}^{ve} \rightarrow \bar{\mathbf{F}}^{ve} = \mathbf{M}^{-1}\mathbf{F}^{ve}\mathbf{M}, \quad (13.176)$$

$$\mathbf{F}^\theta \rightarrow \bar{\mathbf{F}}^\theta = \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \quad (13.177)$$

which is consistent with the condition $\bar{\mathbf{F}} = \bar{\mathbf{F}}^e\bar{\mathbf{F}}^{ve}\bar{\mathbf{F}}^\theta = \mathbf{F}\mathbf{M}$. The assumption that the free energy is the same in value for a change in the deformation history described by $\mathbf{F} \rightarrow \mathbf{F}\mathbf{M}$ can be written in terms of the different models for the free energy as one of the following:

$$\psi^\dagger[\mathbf{X}, \mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta] = \psi^\dagger[\mathbf{X}, \mathbf{F}^e\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^{ve}\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta], \quad (13.178)$$

$$\psi^\dagger[\mathbf{X}, \mathbf{U}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta] = \psi^\dagger[\mathbf{X}, \bar{\mathbf{U}}^e, \mathbf{M}^{-1}\mathbf{F}^{ve}\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta], \quad (13.179)$$

$$\psi^{\dagger\dagger}[\mathbf{X}, \mathbf{C}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta] = \psi^{\dagger\dagger}[\mathbf{X}, \mathbf{M}^T\mathbf{C}^e\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^{ve}\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta], \quad (13.180)$$

$$\psi^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta] = \psi^{\dagger\dagger\dagger}[\mathbf{X}, \bar{\mathbf{E}}^e, \mathbf{M}^{-1}\mathbf{F}^{ve}\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta], \quad (13.181)$$

where $\bar{\mathbf{U}}^e$ is the right symmetric factor in the polar decomposition of $\bar{\mathbf{F}}^e$ and $\bar{\mathbf{E}}^e = \frac{1}{2}(\mathbf{M}^T\mathbf{C}^e\mathbf{M} - \mathbf{I})$.⁴ In the case of an orthogonal $\mathbf{M}$, we will have $\bar{\mathbf{U}}^e = \mathbf{M}^T\mathbf{U}^e\mathbf{M}$ and $\bar{\mathbf{E}}^e = \mathbf{M}^T\mathbf{E}^e\mathbf{M}$. A consistent set of assumptions on how the thermodynamic stresses change are given by the rules

$$\mathbf{T}^e \rightarrow \bar{\mathbf{T}}^e = \mathbf{T}^e, \quad (13.182)$$

$$\mathbf{T}^{ve} \rightarrow \bar{\mathbf{T}}^{ve} = \mathbf{M}^{-1}\mathbf{T}^{ve}\mathbf{M}, \quad (13.183)$$

$$\mathbf{T}^\theta \rightarrow \bar{\mathbf{T}}^\theta = \mathbf{M}^{-1}\mathbf{T}^\theta\mathbf{M}. \quad (13.184)$$

The proof of the consistency of these assumptions with those given for $\mathbf{F}^e$, $\mathbf{F}^{ve}$, and $\mathbf{F}^\theta$ is left as a homework problem. One can motivate these assumptions by the following observations. We know that the Cauchy stress should be unchanged by a change in the history associated with the material symmetry (i.e., $\mathbf{T} \rightarrow \bar{\mathbf{T}} = \mathbf{T}$). Since $\mathbf{T}^e$ is equal to the Cauchy stress, it should follow the same rule, motivating the assumption that $\mathbf{T}^e \rightarrow \bar{\mathbf{T}}^e = \mathbf{T}^e$. The assumptions on $\mathbf{T}^{ve}$ and $\mathbf{T}^\theta$ can be motivated directly by looking at the terms in the expressions for the overstresses $\Delta\mathbf{T}^{ve}$ and $\Delta\mathbf{T}^\theta$. For example, if we expect both terms constructing the overstress $\Delta\mathbf{T}^{ve} = \mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^e - \mathbf{T}^{ve}$ to change in a similar way, then the assumptions on how $\mathbf{T}$ and $\mathbf{F}^e$ change bring us to the assumption that $\mathbf{T}^{ve} \rightarrow \mathbf{M}^{-1}\mathbf{T}^{ve}\mathbf{M}$. The result is that overstresses follow the same rules as the thermodynamic stresses. The constraints of

⁴Note that $\bar{\mathbf{E}}^e = \frac{1}{2}(\mathbf{M}^T\mathbf{C}^e\mathbf{M} - \mathbf{I}) = \frac{1}{2}\mathbf{M}^T(\mathbf{C}^e - \mathbf{M}^{-T}\mathbf{M}^{-1})\mathbf{M} \neq \mathbf{M}^T\mathbf{E}^e\mathbf{M}$ for a general nonorthogonal $\mathbf{M}$.

material symmetry on the viscoelastic flow then becomes

$$\mathbf{M}^{-1}(\mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}) \mathbf{M} = \bar{\mathbf{C}}^{ve} : (\mathbf{M}^T \Delta \mathbf{T}^{veT} \mathbf{M}^{-T}), \quad (13.185)$$

and the constraint on the coefficient of thermal expansion becomes

$$\bar{\alpha} = \mathbf{M}^{-1} \alpha \mathbf{M}. \quad (13.186)$$

A full thermodynamic multidimensional model would be obtained once we specify a function for the specific free energy, the viscoelastic velocity gradient, the thermal expansion coefficient, and the heat flux, keeping in mind that in all processes $\mathbf{L}^{ve}$ and $\mathbf{q}$ need to satisfy the thermodynamic constraints. As an example, we will look at a material that is characterized by quadratic expressions for the free energy.

Section 13.4-Problems:

13.4-2: Prove that imposing the material symmetry condition on the function for the free energy using the “kinematic” assumptions (13.175)–(13.177) results in the relations (13.182)–(13.184). Construct your proof as follows. Following the results of the Clausius-Duhem inequality, assume the free energy is given by a function of the form

$$\psi(t) = \psi^\dagger[\mathbf{F}^e(t), \mathbf{F}^{ve}(t), \mathbf{F}^\theta(t), \theta(t)]. \quad (13.187)$$

Define two deformation histories described by the deformation gradient histories $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ where $\mathbf{F}_2(t)$ is constructed from $\mathbf{F}_1(t)$ using the relation $\mathbf{F}_2(t) = \mathbf{F}_1(t) \mathbf{M}$ for $\mathbf{M}$ being a second-order tensor representing a symmetry of the material. Denote the free energy associated with each of these deformation histories, respectively, as $\psi_1(t)$ and $\psi_2(t)$, so that

$$\psi_1(t) = \psi^\dagger[\mathbf{F}_1^e(t), \mathbf{F}_1^{ve}(t), \mathbf{F}_1^\theta(t), \theta(t)], \quad (13.188)$$

$$\psi_2(t) = \psi^\dagger[\mathbf{F}_2^e(t), \mathbf{F}_2^{ve}(t), \mathbf{F}_2^\theta(t), \theta(t)]. \quad (13.189)$$

Use the expression

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^{ve}}(\psi) : \dot{\mathbf{F}}^{ve} + \partial_{\mathbf{F}^\theta}(\psi) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi) \dot{\theta} \quad (13.190)$$

for each history to arrive at the expressions

$$\dot{\psi}_1 = [\partial_{\mathbf{F}^e}(\psi)]_1 : \dot{\mathbf{F}}_1^e + [\partial_{\mathbf{F}^{ve}}(\psi)]_1 : \dot{\mathbf{F}}_1^{ve} + [\partial_{\mathbf{F}^\theta}(\psi)]_1 : \dot{\mathbf{F}}_1^\theta + [\partial_\theta(\psi)]_1 \dot{\theta}, \quad (13.191)$$

$$\dot{\psi}_2 = [\partial_{\mathbf{F}^e}(\psi)]_2 : \dot{\mathbf{F}}_2^e + [\partial_{\mathbf{F}^{ve}}(\psi)]_2 : \dot{\mathbf{F}}_2^{ve} + [\partial_{\mathbf{F}^\theta}(\psi)]_2 : \dot{\mathbf{F}}_2^\theta + [\partial_\theta(\psi)]_2 \dot{\theta}, \quad (13.192)$$

where the subscripts “1” and “2” indicate the history for which the variable is evaluated. Use the material symmetry condition that the free energy will be the same at all times for the two histories to conclude that $\dot{\psi}_1 = \dot{\psi}_2$. Use the relations (13.175)–(13.177) to evaluate the relation between the rates for the two histories and then use $\dot{\psi}_1 = \dot{\psi}_2$ to evaluate the relation between the partial derivatives for the two histories. Introduce the relation between the partial derivatives in the expression for the thermodynamic stresses (13.129)–(13.131) to obtain the relations (13.182)–(13.184).

13.4.5 Free energies of quadratic form in Green strains

As a special example, we will consider a material that has a specific free energy that is quadratic in elastic and viscoelastic Green strains. A general expression for such a material can be written as

$$\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \mathbf{E}^e : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^{ve} : \mathbf{E}^{ve} : \mathbf{E}^{ve} + \psi^\theta, \quad (13.193)$$

where $\mathbf{E}^e$, $\mathbf{E}^{ve}$, and ψ^θ are assumed to be only a function of temperature. In this expression $\mathbf{E}^e$ and $\mathbf{E}^{ve}$ are the fourth-order elastic moduli of, respectively, the elastic and viscoelastic elements and the associated terms in the free energy characterize the energy stored in the “springs” of these components, and ψ^θ represents the purely thermal component of the free energy.

As was shown in the chapter on thermoelastic response, the symmetry of the Green strain allows us to select the fourth-order moduli such that they have, for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$, the properties

$$\mathbf{A} : \mathbf{E}^e : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^e : \mathbf{B} = \mathbf{A} : \mathbf{E}^e : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^e : \mathbf{B}^T, \quad (13.194)$$

$$\mathbf{A} : \mathbf{E}^{ve} : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^{ve} : \mathbf{B} = \mathbf{A} : \mathbf{E}^{ve} : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^{ve} : \mathbf{B}^T. \quad (13.195)$$

The expressions for the thermodynamic stresses for this model are

$$\mathbf{T}^{eT} = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{J} \mathbf{F}^e (\tilde{\mathbf{E}}^e : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (13.196)$$

$$\mathbf{T}^{veT} = \rho \partial_{\mathbf{F}^{ve}}(\psi) \mathbf{F}^{veT} = \frac{1}{J} \mathbf{F}^{ve} (\tilde{\mathbf{E}}^{ve} : \mathbf{E}^{ve}) \mathbf{F}^{veT}, \quad (13.197)$$

$$\mathbf{T}^{\theta T} = \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T} = \mathbf{0}, \quad (13.198)$$

where

$$\tilde{\mathbf{E}}^e = \frac{1}{2}(\mathbf{E}^e + \mathbf{E}^{eT}), \quad (13.199)$$

$$\tilde{\mathbf{E}}^{ve} = \frac{1}{2}(\mathbf{E}^{ve} + \mathbf{E}^{veT}). \quad (13.200)$$

A point to note is that all the thermodynamic stresses constructed from this process are symmetric, even though the over stresses may not be.

If the material has some form of symmetry, then this symmetry imposes constraints on the form of the moduli $\tilde{\mathbf{E}}^e$ and $\tilde{\mathbf{E}}^{ve}$. As was shown in Chapter 10, the restriction of material symmetry results in

$$\tilde{\mathbf{E}}^e : \mathbf{E}^e = \mathbf{M}[\tilde{\mathbf{E}}^e : (\mathbf{M}^T \mathbf{E}^e \mathbf{M})] \mathbf{M}^T, \quad (13.201)$$

$$\tilde{\mathbf{E}}^{ve} : \mathbf{E}^{ve} = \mathbf{M}[\tilde{\mathbf{E}}^{ve} : (\mathbf{M}^T \mathbf{E}^{ve} \mathbf{M})] \mathbf{M}^T, \quad (13.202)$$

for every orthogonal transformation $\mathbf{M}$ representing a symmetry of the material, and for all $\mathbf{E}^e$ and $\mathbf{E}^{ve}$. The specific forms of these fourth-order tensors for different material symmetries can be found in Appendix B. For an isotropic material, they take the familiar forms

$$\tilde{\mathbf{E}}^e = \lambda^e \mathbf{I} \otimes \mathbf{I} + G^e \mathbf{e}_i \otimes \mathbf{e}_j \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad (13.203)$$

$$\tilde{\mathbf{E}}^{ve} = \lambda^{ve} \mathbf{I} \otimes \mathbf{I} + G^{ve} \mathbf{e}_i \otimes \mathbf{e}_j \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad (13.204)$$

where λ^e , G^e , λ^{ve} , and G^{ve} are the associated Lamé constants and $\mathbf{e}_i$ is any orthonormal base. The associated expressions for thermodynamic stress for an isotropic material becomes

$$\mathbf{T}^{eT} = \frac{1}{J} \mathbf{F}^e [\lambda^e \text{tr}(\mathbf{E}^e) \mathbf{I} + 2G^e \mathbf{E}^e] \mathbf{F}^{eT}, \quad (13.205)$$

$$\mathbf{T}^{veT} = \frac{1}{J} \mathbf{F}^{ve} [\lambda^{ve} \text{tr}(\mathbf{E}^{ve}) \mathbf{I} + 2G^{ve} \mathbf{E}^{ve}] \mathbf{F}^{veT}. \quad (13.206)$$

Section 13.4-Problems:

- 13.4-3:** Derive the expressions for the thermodynamic stresses $\mathbf{T}^e$, $\mathbf{T}^{ve}$, and $\mathbf{T}^\theta$ given in equations (13.196)–(13.198).
- 13.4-4:** Show that the form of the equations (13.205) and (13.206) also can be written as

$$\mathbf{T}^{eT} = \frac{1}{J} \mathbf{F}^e \{ \kappa^e \text{tr}(\mathbf{E}^e) \mathbf{I} + 2G^e [\mathbf{E}^e - \frac{1}{3} \text{tr}(\mathbf{E}^e) \mathbf{I}] \} \mathbf{F}^{eT}, \quad (13.207)$$

$$\mathbf{T}^{veT} = \frac{1}{J} \mathbf{F}^{ve} \{ \kappa^{ve} \text{tr}(\mathbf{E}^{ve}) \mathbf{I} + 2G^{ve} [\mathbf{E}^{ve} - \frac{1}{3} \text{tr}(\mathbf{E}^{ve}) \mathbf{I}] \} \mathbf{F}^{veT}, \quad (13.208)$$

where κ^e and κ^{ve} are associated bulk moduli at infinitesimal strains. The bulk moduli are given by the standard relation

$$\kappa = \lambda + \frac{2}{3}G = \frac{2G(1+\nu)}{3(1-2\nu)} = \frac{E}{3(1-2\nu)} = \frac{GE}{3(3G-E)} = \frac{\lambda(1+\nu)}{3\nu}, \quad (13.209)$$

where G is the shear modulus, E is the elastic modulus (Young's modulus), ν is Poisson's ratio, and λ is the first Lamé constant.

13.4.6 Linear flow rules

If we take $\mathbf{C}^{ve}$ to be only a function of temperature, we will get linear relations between the associated velocity gradients and over stresses. This provides one of the simplest possible flow models. The constraints of material symmetry in this case reduce to

$$\mathbf{M}^{-1}(\mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}) \mathbf{M} = \mathbf{C}^{ve} : (\mathbf{M}^T \Delta \mathbf{T}^{veT} \mathbf{M}^{-T}), \quad (13.210)$$

for every transformation $\mathbf{M}$ representing a symmetry of the material, and for any admissible over stresses. Writing these in component form in an orthonormal base, assuming that the symmetries are represented by orthogonal transformations, and assuming that the over stresses can take the form of any second-order tensor, these constraints result in the relations

$$C_{ijkl}^{ve} = M_{im} M_{jn} M_{ko} M_{lp} C_{mnop}^{ve}. \quad (13.211)$$

For an isotropic material, one can show that the isotropy requires that the flow rules can be written as

$$\mathbf{L}^{ve} = c_1^{ve} \text{tr}(\Delta \mathbf{T}^{ve}) \mathbf{I} + c_2^{ve} \Delta \mathbf{T}^{ve} + c_3^{ve} \Delta \mathbf{T}^{veT}. \quad (13.212)$$

Yet, we still need to satisfy the thermodynamic restrictions $-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0$. This constraint can be written as

$$-c_1^{ve} \text{tr}^2(\Delta \mathbf{T}^{ve}) - (c_2^{ve} + c_3^{ve}) |\Delta \mathbf{T}_{sym}^{veT}|^2 - (c_3^{ve} - c_2^{ve}) |\Delta \mathbf{T}_{skew}^{veT}|^2 \leq 0. \quad (13.213)$$

A way to satisfy this is to require that $c_1^{ve} \geq 0$, $(c_2^{ve} + c_3^{ve}) \geq 0$, and $(c_3^{ve} - c_2^{ve}) \geq 0$.

Section 13.4-Problems:

- 13.4-5:** Starting from the condition $-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0$, show that the conditions $c_1^{ve} \geq 0$, $(c_2^{ve} + c_3^{ve}) \geq 0$, and $(c_3^{ve} - c_2^{ve}) \geq 0$ are sufficient to satisfy the thermodynamic constraint for the linear isotropic model for $\mathbf{L}^{ve}$.

13.4.7 Infinitesimal strains

The model with quadratic free energy can be used as a base to create an infinitesimal response model. The infinitesimal counterpart to this model will be given by the kinematic assumption

$$\epsilon = \epsilon^e + \epsilon^{ve} + \epsilon^\theta, \quad (13.214)$$

where it is assumed that the displacement gradients associated with the elastic, viscoelastic, and thermal deformation gradients are all small. The thermodynamic stresses will be given by

$$\sigma^e = \mathbf{E}^e : \epsilon^e, \quad (13.215)$$

$$\sigma^{ve} = \mathbf{E}^{ve} : \epsilon^{ve}, \quad (13.216)$$

with the assumption that the applied stress $\sigma = \sigma^e$, and for fourth-order elastic moduli that depend only on temperature and which have the symmetry conditions, for components expressed in an orthonormal base, as

$$\mathbf{E}_{ijkl}^e = M_{im}M_{jn}M_{ko}M_{lp}\mathbf{E}_{mnop}^e, \quad (13.217)$$

$$\mathbf{E}_{ijkl}^{ve} = M_{im}M_{jn}M_{ko}M_{lp}\mathbf{E}_{mnop}^{ve}, \quad (13.218)$$

for each orthogonal material symmetry transformation $\mathbf{M}$. Each of these fourth-order moduli has the index symmetries described by $\mathbf{E}_{ijkl} = \mathbf{E}_{jikl} = \mathbf{E}_{ijlk} = \mathbf{E}_{klij}$ when expressed in an orthonormal base. The flow rule for the viscoelastic element is given by

$$\dot{\epsilon}^{ve} = \mathbf{C}^{ve} : \Delta\sigma^{ve}, \quad (13.219)$$

where the viscoelastic overstress is given by

$$\Delta\sigma^{ve} = \sigma - \sigma^{ve}. \quad (13.220)$$

The fourth-order modulus $\mathbf{C}^{ve}$ has the same index symmetry and material symmetry conditions as the fourth-order elastic moduli. The thermal expansion is given by the expression

$$\dot{\epsilon}^\theta = \alpha \dot{\theta}, \quad (13.221)$$

where α is symmetric and has the constraint

$$\alpha = \mathbf{M}^T \alpha \mathbf{M} \quad (13.222)$$

for every orthogonal transformation $\mathbf{M}$ associated with the material symmetry.

The heat flux for infinitesimal strains can be expressed in the form

$$\mathbf{q} = \mathbf{k}\mathbf{g}, \quad (13.223)$$

where $\mathbf{g}$ is the temperature gradient and the coefficient of thermal conduction $\mathbf{k}$ has the property

$$\mathbf{q} \circ \mathbf{g} = \mathbf{g} \circ (\mathbf{k}\mathbf{g}) \leq 0 \quad (13.224)$$

for every vector $\mathbf{g}$. The material symmetry constraint on $\mathbf{k}$ is

$$\mathbf{k} = \mathbf{M}^T \mathbf{k} \mathbf{M} \quad (13.225)$$

for every orthogonal transformation $\mathbf{M}$ associated with the material symmetry.

In the case of an initially isotropic material, we have the following. The thermodynamic stresses can be expressed as

$$\boldsymbol{\sigma}^e = \lambda^e \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2G^e \boldsymbol{\epsilon}^e, \quad (13.226)$$

$$\boldsymbol{\sigma}^{ve} = \lambda^{ve} \text{tr}(\boldsymbol{\epsilon}^{ve}) \mathbf{I} + 2G^{ve} \boldsymbol{\epsilon}^{ve}. \quad (13.227)$$

The expressions for the viscoelastic flow rule can be written as

$$\dot{\boldsymbol{\epsilon}}^{ve} = \frac{1}{3\mu^{ve,h}} \Delta\boldsymbol{\sigma}_{ave}^{ve} \mathbf{I} + \frac{1}{\mu^{ve,d}} \Delta\mathbf{S}^{ve}, \quad (13.228)$$

where $\Delta\boldsymbol{\sigma}_{ave}^{ve} = \text{tr}(\Delta\boldsymbol{\sigma}^{ve})/3$ is the average part of the overstress $\Delta\boldsymbol{\sigma}^{ve}$, and $\Delta\mathbf{S}^{ve}$ is the deviatoric part of $\Delta\boldsymbol{\sigma}^{ve}$, and where $\mu^{ve,h}$ and $\mu^{ve,d}$ are viscosities associated, respectively, with the hydrostatic and deviatoric overstresses.

13.4.8 An example model with quadratic free energy

Now that we have developed the details of a simple model, let us proceed to explore the response of an isotropic material with a free energy of quadratic form in the Green strains, with a linear flow rule, and linear thermal conductivity. The response for such a material is given as

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^\theta, \quad (13.229)$$

$$\mathbf{L}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (13.230)$$

$$\mathbf{L}^{ve} = \frac{1}{9\mu^{ve,h}} \text{tr}(\Delta\mathbf{T}^{ve}) \mathbf{I} + \frac{1}{\mu^{ve,d}} \Delta\mathbf{S}^{ve}, \quad (13.231)$$

$$\Delta\mathbf{T}^{ve} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}, \quad (13.232)$$

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e \{ \kappa \text{tr}(\mathbf{E}^e) \mathbf{I} + 2G[\mathbf{E}^e - \frac{1}{3} \text{tr}(\mathbf{E}^e) \mathbf{I}] \} \mathbf{F}^{eT}, \quad (13.233)$$

$$\mathbf{T}^{veT} = \frac{1}{J} \mathbf{F}^{ve} \{ \kappa^{ve} \text{tr}(\mathbf{E}^{ve}) \mathbf{I} + 2G^{ve} [\mathbf{E}^{ve} - \frac{1}{3} \text{tr}(\mathbf{E}^{ve}) \mathbf{I}] \} \mathbf{F}^{veT}, \quad (13.234)$$

$$\mathbf{q} = -k\mathbf{g}, \quad (13.235)$$

where α is the linear coefficient of thermal expansion, $\mu^{ve,h}$ and $\mu^{ve,d}$ are, respectively, the viscosities associated with the hydrostatic and deviatoric parts of the viscoelastic response, $\Delta\mathbf{S}^{ve}$ is the deviatoric part of the overstress $\Delta\mathbf{T}^{ve}$, κ and G are the bulk and shear moduli associated with the elastic response, κ^{ve} and G^{ve} are the bulk and shear moduli associated with the viscoelastic back-stress, and k is the linear coefficient of thermal conduction. The material parameters all are assumed to be temperature dependent.

This model has two characteristic times, one associated with volume changes (dilatation) and one associated with shear. In the range of small strains, the equations associated with pure volume changes decouple from that associated with shear, and we get two characteristic relaxation times, one associated with each. If we denote the relaxation time associated with pure volumetric deformations as t_R^h (h for hydrostatic) and the relaxation time associated with shear as t_R^d (d for deviatoric), then we have for infinitesimal strains

$$t_R^h \approx \frac{\mu^{ve,h}}{\kappa + \kappa^{ve}}, \quad t_R^d \approx \frac{\mu^{ve,d}}{G + G^{ve}}. \quad (13.236)$$

Obviously, as the deformations get large, the model becomes nonlinear and the characteristic times change in association with this change.

In the following problems, we will demonstrate the characteristics of this model by selecting model parameters that provide characteristic relaxation times of approximately 1

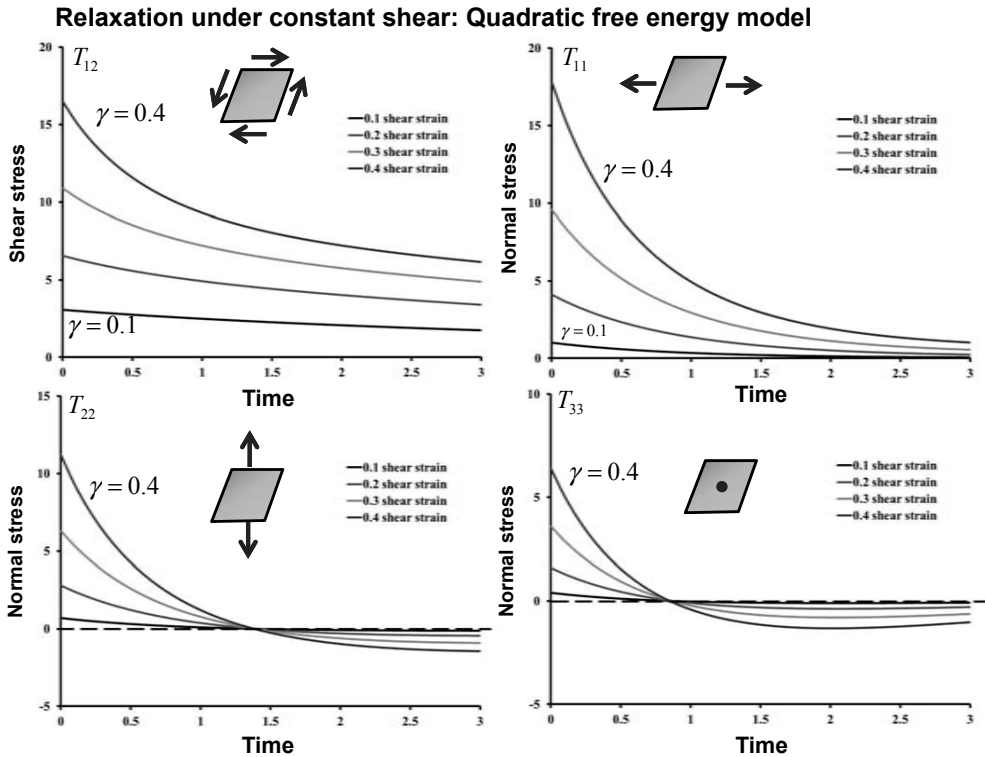


Figure 13.19: Model with quadratic free energy: Stress relaxation for shear strains of $\gamma = 0.1, 0.2, 0.3, 0.4$: (a) shear stress T_{12} , (b) in-plane normal stress T_{11} , (c) in-plane normal stress T_{22} , and (d) out-of-plane normal stress T_{33} . Material parameters for the model are described in Section 13.

for infinitesimal volumetric deformation and 10 for infinitesimal shear. Figures 13.19–13.21 show the response seen under different loading conditions using the model and parameters: $\kappa = 100$, $\kappa^{ve} = 10$, $\mu^{ve,h} = 110$, $G = 30$, $G^{ve} = 3$, and $\mu^{ve,d} = 330$. Some of the details can be found in the problem statements.

Section 13.4-Problems:

- 13.4-6:** Consider stress relaxation in pure volumetric deformation (dilatation). Define the relaxation time as the approximate time it would take the material to reach zero stress in a linear approximation to the response so that

$$t_R^h = -\frac{\sigma_{ave}}{\dot{\sigma}_{ave}}. \quad (13.237)$$

This is an approximation to the actual relaxation since the stress actually tends toward the equilibrium stress, not zero stress. Show that for the quadratic model developed in this section the initial relaxation time t_{Ro}^h is given by

$$t_{Ro}^h = \frac{\mu^{ve,h}}{\kappa(2J^{1/3} - J^{-1/3})}, \quad (13.238)$$

where J is the applied constant volume ratio so that applied volumetric strain is $\epsilon_v = J - 1$. Plot the initial relaxation time as a function of the applied

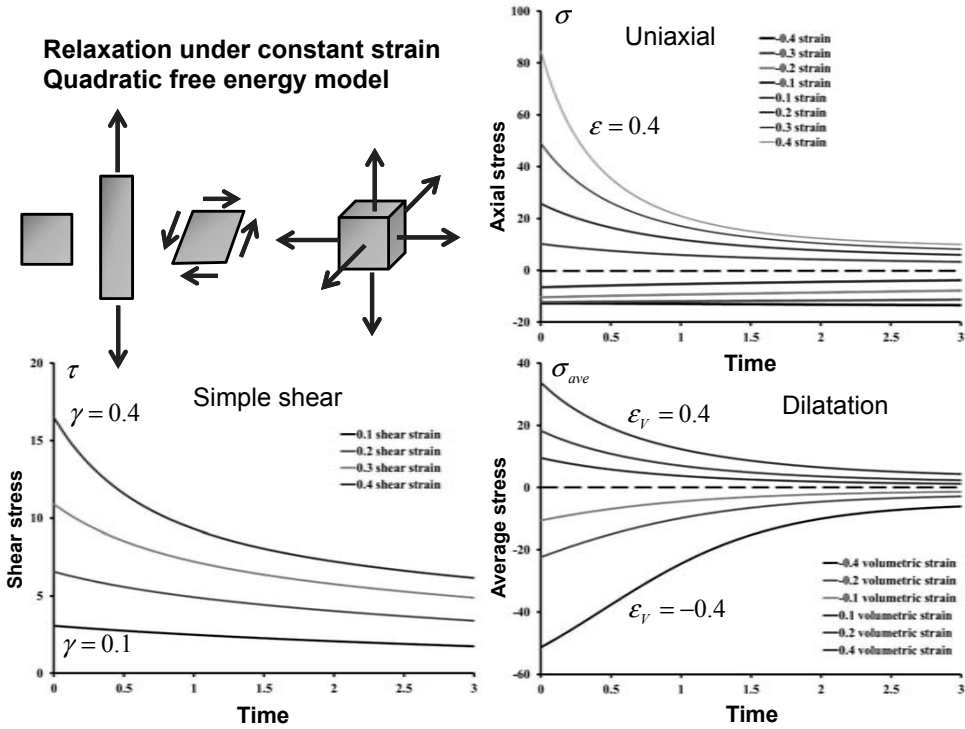


Figure 13.20: Model with quadratic free energy: Stress relaxation at constant strain for (a) uniaxial extension/compression, (b) simple shear, and (c) volumetric extension/compression (dilatation). Material parameters for the model are described in Section 13.

volumetric strain. Note that for this model the initial relaxation time goes to infinity for $J = (1/2)^{3/2} \approx 0.354$.

- 13.4-7:** Consider pure dilatation deformations under isothermal conditions so that $\mathbf{F} = \lambda \mathbf{I}$ and $\mathbf{F}^\theta = \mathbf{I}$. Assume both the elastic and viscoelastic deformations are described by pure dilatation so that $\mathbf{F}^e = \lambda^e \mathbf{I}$ and $\mathbf{F}^{ve} = \lambda^{ve} \mathbf{I}$. For the quadratic model developed in this section, calculate the average stress σ_{ave} . Show that the average stress rate $\dot{\sigma}_{ave}$ for constant applied dilatation $J = J_o$ has a zero for $\lambda^e = \sqrt{1/2} \approx 0.707$. Study the behavior of the model during constant dilatation and show that the relation between the rate of average stress and the rate of elastic stretch $\dot{\lambda}^e$ changes sign at $\lambda^e = \sqrt{1/2}$. Conclude that the response shows stress relaxation for $\lambda^e > \sqrt{1/2}$ and stress increase for $\lambda^e < \sqrt{1/2}$. Evaluate $tr(\mathbf{E}^e) = \frac{3}{2}(\lambda^{e2} - 1)$, and conclude that the model should not be used for conditions that $tr(\mathbf{E}^e) \leq -3/4$. Show that this condition occurs at a volumetric strain $\epsilon_v^e = J^e - 1 = \sqrt{1/8} - 1 \approx -0.646$.

- 13.4-8:** To alleviate the potential problem described in the previous questions, one might consider changing the models for stress given above to the models

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e \left\{ \kappa \ln(J^e) \mathbf{I} + 2G[\mathbf{E}^e - \frac{1}{3} tr(\mathbf{E}^e) \mathbf{I}] \right\} \mathbf{F}^{eT}, \quad (13.239)$$

$$\mathbf{T}^{veT} = \frac{1}{J} \mathbf{F}^{ve} \left\{ \kappa^{ve} \ln(J^{ve}) \mathbf{I} + 2G^{ve}[\mathbf{E}^{ve} - \frac{1}{3} tr(\mathbf{E}^{ve}) \mathbf{I}] \right\} \mathbf{F}^{veT}. \quad (13.240)$$

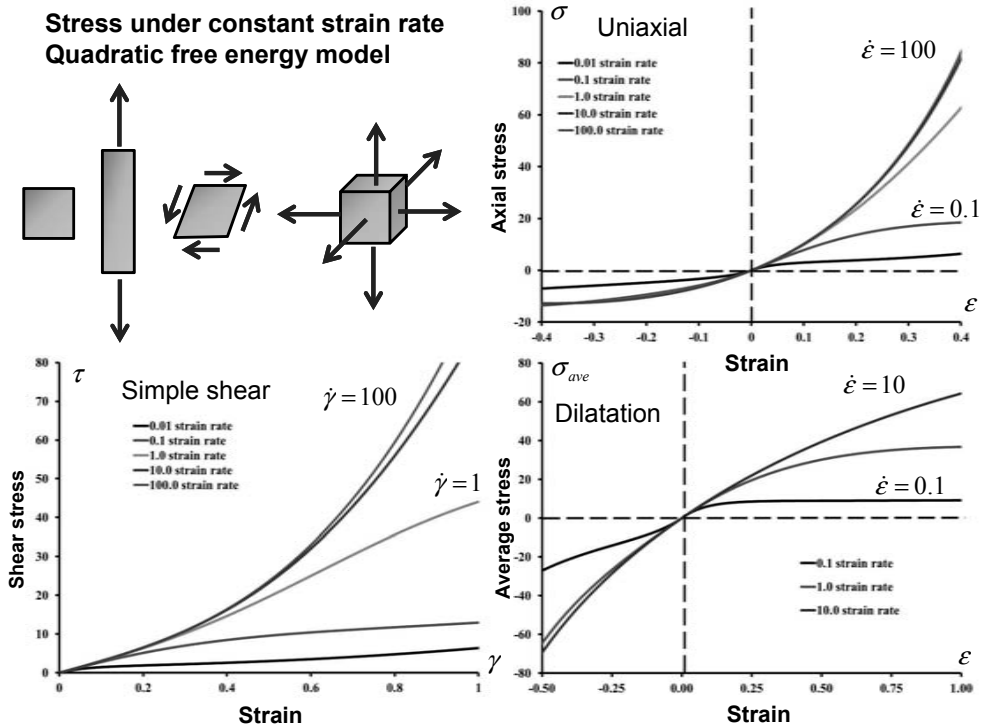


Figure 13.21: Model with quadratic free energy: Stress response under monotonic constant strain rate for (a) uniaxial extension/compression, (b) simple shear, and (c) pure dilatation. Material parameters for the model are described in Section 13.

Show that this change results in models that exhibit the same condition, but that this condition occurs at $J^e = e^{-3/2} \approx 0.223$ (this is equivalent to $\lambda^e = e^{-1/2} \approx 0.607$ and $\epsilon_v^e = J^e - 1 \approx -0.777$).

- 13.4-9:** Simulate the response of the quadratic model developed in this section for stress relaxation under simple shear. Create plots for (a) shear stress T_{12} , (b) in-plane normal stress T_{11} , (c) in-plane normal stress T_{22} , (d) out-of-plane normal stress T_{33} , and (e) relaxation modulus (shear stress over shear strain). Plot the response for shear strains of $\gamma = 0.1, 0.2, 0.4, 0.8$. Compare your results to Figure 13.19.
- 13.4-10:** Simulate the response of the quadratic model developed in this section for stress relaxation under pure volumetric deformation (dilatation) for volumetric strains of $\epsilon_v = J - 1 = -0.4, -0.2, -0.1, 0.1, 0.2, 0.4, 0.8$. Compare your results to those given in Figure 13.20.
- 13.4-11:** Simulate the response of the quadratic model developed in this section for stress relaxation under uniaxial extension for axial strains of $\epsilon = -0.4, -0.2, -0.1, 0.1, 0.2, 0.4, 0.8$. Compare your results to those given in Figure 13.20.
- 13.4-12:** Simulate the response of the quadratic model developed in this section for constant strain rate loading in simple shear. Use the strain rates $\dot{\gamma} = 0.01, 0.1, 1.0, 10, 100$. Compare your results to those given in Figure 13.21.

- 13.4-13:** Simulate the response of the quadratic model developed in this section for constant strain rate loading in pure dilatation. Use the strain rates $\dot{\epsilon}_v = \pm 0.1, \pm 1.0, \pm 10$. Compare your results to those given in Figure 13.21.
- 13.4-14:** Simulate the response of the quadratic model developed in this section for constant strain rate loading in uniaxial extension. Use the strain rates $\dot{\epsilon} = \pm 0.01, \pm 0.1, \pm 1.0, \pm 10, \pm 100$. Compare your results to those given in Figure 13.21.
- 13.4-15:** Simulate the response of the quadratic model developed in this section for cyclic loading consisting of constant strain rate segments of loading and unloading in (a) simple shear, (b) uniaxial tension, and (c) volumetric strain. In each case start from zero strain and cycle to associated strains of plus and minus 0.2. Use associated strain rates of 0.01, 0.1, 1.0, 10, 100.
- 13.4-16:** In this section, we developed a model that had a free energy that was quadratic in the Green strain. An alternate, but similar, model would use the strain $\mathbf{G} = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$. At small deformations, this strain and the Green strain become similar so similar models will produce similar responses. Construct a model for free energy that is quadratic in $\mathbf{G}$ and calculate the expressions for the thermodynamic stresses. Replace this free energy for the free energy used in the example developed in this section and simulate the stress relaxation response for pure dilatation, uniaxial extension, and simple shear. Compare your response to that for the model using the Green strain and shown in Figure 13.20.

13.4.9 An isotropic coupled elastic-viscoelastic free energy

Let us look at a material model that has a free energy that is initially isotropic and selected to be given by a free energy of the form

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{C}^e, \mathbf{F}^{ve}, \theta). \quad (13.241)$$

From Appendix A, we know that condition of initial isotropy allows us to write the free energy in the form

$$\psi = \psi^\dagger(\mathbf{X}, I_1, \dots, I_{21}, \theta), \quad (13.242)$$

where I_1 – I_{21} are the 21 isotropic invariants of $\mathbf{C}^e$ and $\mathbf{F}^{ve}$ given by

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}^e), & I_2 &= \text{tr}(\mathbf{C}^{e2}), \\ I_3 &= \text{tr}(\mathbf{C}^{e3}), & I_4 &= \frac{1}{2}\text{tr}(\mathbf{F}^{ve} + \mathbf{F}^{veT}), \\ I_5 &= \frac{1}{4}\text{tr}[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2], & I_6 &= \frac{1}{8}\text{tr}[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^3], \\ I_7 &= \frac{1}{2}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})], & I_8 &= \frac{1}{2}\text{tr}[\mathbf{C}^{e2}(\mathbf{F}^{ve} + \mathbf{F}^{veT})], \\ I_9 &= \frac{1}{4}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2], & I_{10} &= \frac{1}{4}\text{tr}[\mathbf{C}^{e2}(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2], \\ I_{11} &= \frac{1}{4}\text{tr}[(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2], & I_{12} &= \frac{1}{4}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2], \\ I_{13} &= \frac{1}{4}\text{tr}[\mathbf{C}^{e2}(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2], & & \\ I_{14} &= \frac{1}{8}\text{tr}[\mathbf{C}^{e2}(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})], & & \\ I_{15} &= \frac{1}{8}\text{tr}[(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2], & & \\ I_{16} &= \frac{1}{16}\text{tr}[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2], & & \\ I_{17} &= \frac{1}{64}\text{tr}[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})], & & \\ I_{18} &= \frac{1}{4}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})], & & \\ I_{19} &= \frac{1}{8}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2(\mathbf{F}^{ve} - \mathbf{F}^{veT})], & & \\ I_{20} &= \frac{1}{4}\text{tr}[\mathbf{C}^{e2}(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})], & & \\ I_{21} &= \frac{1}{16}\text{tr}[\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})]. & & \end{aligned} \quad (13.243)$$

We can calculate the Cauchy stress through the relation

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \rho \sum_{i=1}^{21} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT}. \quad (13.244)$$

To complete this representation, we need to calculate the derivative of I_i with respect to $\mathbf{F}^e$. These can be calculated directly from the expressions for each invariant. For example,

$$\begin{aligned} \dot{I}_1 &= \overline{\dot{(\mathbf{C}^e)}} = \text{tr}(\dot{\mathbf{C}}^e) = \text{tr}(\dot{\mathbf{F}}^{eT} \mathbf{F}^e + \mathbf{F}^{eT} \dot{\mathbf{F}}^e) = \text{tr}(\dot{\mathbf{F}}^{eT} \mathbf{F}^e) + \text{tr}(\mathbf{F}^{eT} \dot{\mathbf{F}}^e) \\ &= 2\text{tr}(\mathbf{F}^{eT} \dot{\mathbf{F}}^e) = 2\mathbf{F}^e : \dot{\mathbf{F}}^e, \end{aligned} \quad (13.245)$$

so that $\partial_{\mathbf{F}^e}(I_1) = 2\mathbf{F}^e$. In general, for any constant tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\partial_{\mathbf{F}^e}[\text{tr}(\mathbf{C}^e \mathbf{A})] = \mathbf{F}^e(\mathbf{A} + \mathbf{A}^T), \quad (13.246)$$

$$\partial_{\mathbf{F}^e}[\text{tr}(\mathbf{C}^{e2} \mathbf{A})] = \mathbf{F}^e[\mathbf{C}^e(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} + \mathbf{A}^T)\mathbf{C}^e], \quad (13.247)$$

$$\begin{aligned} \partial_{\mathbf{F}^e}[\text{tr}(\mathbf{C}^{e2} \mathbf{A} \mathbf{C}^e \mathbf{B})] &= \mathbf{F}^e[\mathbf{C}^e(\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T) \\ &\quad + (\mathbf{A} \mathbf{C}^e \mathbf{B} + \mathbf{B}^T \mathbf{C}^e \mathbf{A}^T)\mathbf{C}^e + \mathbf{B} \mathbf{C}^{e2} \mathbf{A} + \mathbf{A}^T \mathbf{C}^{e2} \mathbf{B}^T]. \end{aligned} \quad (13.248)$$

These can be used to calculate the following partial derivatives of the invariants:

$$\begin{aligned} \partial_{\mathbf{F}^e}(I_1) &= 2\mathbf{F}^e, & \partial_{\mathbf{F}^e}(I_2) &= 4\mathbf{F}^e \mathbf{C}^e, \\ \partial_{\mathbf{F}^e}(I_3) &= 6\mathbf{F}^e \mathbf{C}^{e2}, & \partial_{\mathbf{F}^e}(I_4) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_5) &= \mathbf{0}, & \partial_{\mathbf{F}^e}(I_6) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_7) &= \mathbf{F}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT}), & \partial_{\mathbf{F}^e}(I_8) &= \mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT}) + (\mathbf{F}^{ve} + \mathbf{F}^{veT})\mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_9) &= \frac{1}{2}\mathbf{F}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2, & \partial_{\mathbf{F}^e}(I_{10}) &= \frac{1}{2}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2 + (\mathbf{F}^{ve} + \mathbf{F}^{veT})^2 \mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{11}) &= \mathbf{0}, & \partial_{\mathbf{F}^e}(I_{12}) &= \frac{1}{2}\mathbf{F}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2, \\ \partial_{\mathbf{F}^e}(I_{13}) &= \frac{1}{2}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 + (\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 \mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{14}) &= \frac{1}{8}\mathbf{F}^e\{\mathbf{C}^e[(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 \mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT}) - (\mathbf{F}^{ve} - \mathbf{F}^{veT})\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2] \\ &\quad + [(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 \mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT}) - (\mathbf{F}^{ve} - \mathbf{F}^{veT})\mathbf{C}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2] \mathbf{C}^e \\ &\quad + (\mathbf{F}^{ve} - \mathbf{F}^{veT})\mathbf{C}^{e2}(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 - (\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 \mathbf{C}^{e2}(\mathbf{F}^{ve} - \mathbf{F}^{veT})\}, \\ \partial_{\mathbf{F}^e}(I_{15}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{16}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{17}) &= \mathbf{0}, \\ \partial_{\mathbf{F}^e}(I_{18}) &= \frac{1}{4}\mathbf{F}^e[(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT}) - (\mathbf{F}^{ve} - \mathbf{F}^{veT})(\mathbf{F}^{ve} + \mathbf{F}^{veT})], \\ \partial_{\mathbf{F}^e}(I_{19}) &= \frac{1}{8}\mathbf{F}^e[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2(\mathbf{F}^{ve} - \mathbf{F}^{veT}) - (\mathbf{F}^{ve} - \mathbf{F}^{veT})(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2], \\ \partial_{\mathbf{F}^e}(I_{20}) &= \frac{1}{8}\mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{veT} \mathbf{F}^{ve} - \mathbf{F}^{ve} \mathbf{F}^{veT}) + (\mathbf{F}^{veT} \mathbf{F}^{ve} - \mathbf{F}^{ve} \mathbf{F}^{veT})\mathbf{C}^e], \\ \partial_{\mathbf{F}^e}(I_{21}) &= \frac{1}{16}\mathbf{F}^e[(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT}) \\ &\quad - (\mathbf{F}^{ve} - \mathbf{F}^{veT})(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2]. \end{aligned} \quad (13.249)$$

The general expression for Cauchy stress then is given by

$$\begin{aligned} \mathbf{T} &= \rho \left\{ 2 \frac{\partial \psi}{\partial I_1} \mathbf{B}^e + 4 \frac{\partial \psi}{\partial I_2} \mathbf{B}^{e2} + 6 \frac{\partial \psi}{\partial I_3} \mathbf{B}^{e3} + \frac{\partial \psi}{\partial I_7} \mathbf{F}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT}) \mathbf{F}^{eT} \right. \\ &\quad + \frac{\partial \psi}{\partial I_8} \mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT}) + (\mathbf{F}^{ve} + \mathbf{F}^{veT})\mathbf{C}^e] \mathbf{F}^{eT} + \frac{1}{2} \frac{\partial \psi}{\partial I_9} \mathbf{F}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2 \mathbf{F}^{eT} \\ &\quad + \frac{1}{2} \frac{\partial \psi}{\partial I_{10}} \mathbf{F}^e[\mathbf{C}^e(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2 + (\mathbf{F}^{ve} + \mathbf{F}^{veT})^2 \mathbf{C}^e] \mathbf{F}^{eT} \\ &\quad \left. + \frac{1}{2} \frac{\partial \psi}{\partial I_{12}} \mathbf{F}^e(\mathbf{F}^{ve} - \mathbf{F}^{veT})^2 \mathbf{F}^{eT} + \dots \right\}. \end{aligned} \quad (13.250)$$

The calculation of the back-stress is similar and uses the expression

$$\mathbf{T}^{veT} = \rho \partial_{\mathbf{F}^{ve}}(\psi) \mathbf{F}^{veT} = \rho \sum_{i=1}^{21} \frac{\partial \psi}{\partial I_i} \partial_{\mathbf{F}^{ve}}(I_i) \mathbf{F}^{veT}. \quad (13.251)$$

The partial derivatives of the invariants are as simple to calculate. For example,

$$2\dot{I}_4 = \overline{tr(\mathbf{F}^{ve} + \mathbf{F}^{veT})} = tr(\dot{\mathbf{F}}^{ve}) + tr(\dot{\mathbf{F}}^{veT}) = 2tr(\dot{\mathbf{F}}^{ve}) = 2\mathbf{I} : \dot{\mathbf{F}}^{ve}, \quad (13.252)$$

so that

$$\partial_{\mathbf{F}^{ve}}(I_4) = \mathbf{I}. \quad (13.253)$$

Or, for example,

$$\begin{aligned} 4\dot{I}_5 &= \overline{tr[(\mathbf{F}^{ve} + \mathbf{F}^{veT})^2]} = tr[(\overline{(\mathbf{F}^{ve} + \mathbf{F}^{veT})}(\mathbf{F}^{ve} + \mathbf{F}^{veT})) + tr[(\mathbf{F}^{ve} + \mathbf{F}^{veT})(\overline{(\mathbf{F}^{ve} + \mathbf{F}^{veT})})] \\ &= 2tr[(\overline{(\mathbf{F}^{ve} + \mathbf{F}^{veT})}(\mathbf{F}^{ve} + \mathbf{F}^{veT}))] = 4(\mathbf{F}^{ve} + \mathbf{F}^{veT}) : \dot{\mathbf{F}}^{ve}, \end{aligned} \quad (13.254)$$

so that

$$\partial_{\mathbf{F}^{ve}}(I_5) = \mathbf{F}^{ve} + \mathbf{F}^{veT}. \quad (13.255)$$

The resulting expression for the back-stress is given as

$$\mathbf{T}^{veT} = \rho \left[\frac{\partial \psi}{\partial I_4} \mathbf{F}^{veT} + \frac{\partial \psi}{\partial I_5} (\mathbf{B}^{ve} + \mathbf{F}^{ve2T}) + \dots \right]. \quad (13.256)$$

Obviously, if the viscoelastic deformation gradient is not symmetric, then the expression for the back-stress will not be symmetric.

A simplification, which can be even more useful, with the same ability to produce strong anisotropy in the elastic response with large deformation, can be achieved by assuming the free energy is affected only by $\mathbf{F}^{ve}$ through $\mathbf{C}^{ve}$. That is, the free energy is written as

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{C}^e, \mathbf{C}^{ve}, \theta). \quad (13.257)$$

This provides a consistent way to allow $\mathbf{F}^{ve}$ to be nonsymmetric, yet its influence on the free energy is only through the strains it generates, here measured represented through $\mathbf{C}^{ve}$. For an initially isotropic material, assuming the symmetry is represented by the set of all orthogonal transformations of this configuration, using the results in Appendix A, we can reduce this to

$$\psi = \psi^\dagger(\mathbf{X}, I_1, \dots, I_{10}, \theta), \quad (13.258)$$

where I_1 – I_{10} are the 10 isotropic invariants of the two symmetric tensors $\mathbf{C}^e$ and $\mathbf{C}^{ve}$. These are given as

$$\begin{aligned} I_1 &= tr(\mathbf{C}^e), & I_2 &= tr(\mathbf{C}^{e2}), \\ I_3 &= tr(\mathbf{C}^{e3}), & I_4 &= tr(\mathbf{C}^{ve}), \\ I_5 &= tr(\mathbf{C}^{ve2}), & I_6 &= tr(\mathbf{C}^{ve3}), \\ I_7 &= tr(\mathbf{C}^e \mathbf{C}^{ve}), & I_8 &= tr(\mathbf{C}^{e2} \mathbf{C}^{ve}), \\ I_9 &= tr(\mathbf{C}^e \mathbf{C}^{ve2}), & I_{10} &= tr(\mathbf{C}^{e2} \mathbf{C}^{ve2}). \end{aligned} \quad (13.259)$$

Section 13.4-Problems:

- 13.4-17:** Consider the special model for free energy of an isotropic material described by equation (13.258) and construct the expression for the thermodynamic stresses.

13.4.10 An example model motivated by rubber elasticity

When considering rubber-like materials, much of the modeling motivation comes from statistical mechanics of randomly distributed long-chain structures. This normally results in isotropic elastic constitutive equations for the response, and their form is the motivation for the model presented here.

Before we proceed, we consider the separation of volumetric effects from the rest since the volumetric response in these materials is much stiffer than the shear response. For this reason, the models are constructed either assuming an incompressible response or separated as follows. In particular, to separate the variables, we will remove the volumetric deformation from the deformation gradient by introducing the “gradient” $\mathbf{F}^*$ defined as

$$\mathbf{F}^* = \frac{1}{J^{1/3}} \mathbf{F}, \quad (13.260)$$

for $J = \det(\mathbf{F})$ so that $\det(\mathbf{F}^*) = 1$. One may consider $\mathbf{F}^*$ to be a volume-preserving counterpart of $\mathbf{F}$. For $\mathbf{F}^*$ we can define an associated left Cauchy stretch tensor $\mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{*T}$. One set of the isotropic invariance of $\mathbf{B}^*$ is given as

$$J_1^* = \text{tr}(\mathbf{B}^*) = \frac{1}{J^{2/3}} \text{tr}(\mathbf{B}) = \frac{1}{J^{2/3}} J_1, \quad (13.261)$$

$$J_2^* = \frac{1}{2} [\text{tr}^2(\mathbf{B}^*) - \text{tr}(\mathbf{B}^{*2})] = \frac{1}{2J^{4/3}} [\text{tr}^2(\mathbf{B}) - \text{tr}(\mathbf{B}^2)] = \frac{1}{J^{4/3}} J_2, \quad (13.262)$$

$$J_3^* = \det(\mathbf{B}^*) = 1. \quad (13.263)$$

Comparing these to the invariants J_i of $\mathbf{B}$, we can see that there is a simple algebraic relation between J_i and the three invariants J_1^* , J_2^* , and J . We can do the same for $\mathbf{B}^e$ and $\mathbf{B}^{ve}$ to obtain the associated isotropic invariants J_1^{e*} , J_2^{e*} , J^e , J_1^{ve*} , J_2^{ve*} , and J^{ve} . When taking derivatives, we will need to know the partial derivatives of the invariants with respect to the deformation gradients. These are

$$\partial_{\mathbf{F}}(J) = J\mathbf{F}^{-T}, \quad (13.264)$$

$$\partial_{\mathbf{F}}(J_1^*) = 2(\mathbf{B}^* - \frac{J_1^*}{3}\mathbf{I})\mathbf{F}^{-T}, \quad (13.265)$$

$$\partial_{\mathbf{F}}(J_2^*) = 2(J_1^*\mathbf{B}^* - \mathbf{B}^{*2} - \frac{2}{3}J_2^*\mathbf{I})\mathbf{F}^{-T}. \quad (13.266)$$

For an isotropic elastic response, we would select a free energy of the form $\psi = \psi^\dagger(J, J_1^*, J_2^*, \theta)$ and calculate the stress as

$$\begin{aligned} \mathbf{T}^T &= \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T = \rho \left[\frac{\partial \psi}{\partial J} \partial_{\mathbf{F}}(J) + \frac{\partial \psi}{\partial J_1^*} \partial_{\mathbf{F}}(J_1^*) + \frac{\partial \psi}{\partial J_2^*} \partial_{\mathbf{F}}(J_2^*) \right] \mathbf{F}^T \\ &= \rho_o \frac{\partial \psi}{\partial J} \mathbf{I} + \frac{2\rho_o}{J} \frac{\partial \psi}{\partial J_1^*} \left(\mathbf{B}^* - \frac{J_1^*}{3}\mathbf{I} \right) + \frac{2\rho_o}{J} \frac{\partial \psi}{\partial J_2^*} \left(J_1^*\mathbf{B}^* - \mathbf{B}^{*2} - \frac{2}{3}J_2^*\mathbf{I} \right). \end{aligned} \quad (13.267)$$

For a model consistent with statistical mechanics, we select a free energy of the form

$$\psi = \frac{\kappa_o}{2\rho_o} \ln^2(J) + \frac{G_o}{2\rho_o} (J_1^* - 3) + \psi^\theta, \quad (13.268)$$

where the initial bulk modulus κ_o , the initial shear modulus G_o , and the pure thermal parameter ψ^θ are functions only of temperature. This gives an expression for the Cauchy stress as

$$\mathbf{T} = \kappa_o \frac{\ln(J)}{J} \mathbf{I} + \frac{G_o}{J} \left(\mathbf{B}^* - \frac{J_1^*}{3}\mathbf{I} \right), \quad (13.269)$$

which is similar to the model predicted from statistical mechanics and is the base for the viscoelastic model that follows. One characteristic of this model is that the response in simple shear is linear, even for large deformations.

As a base for viscoelasticity, a fairly general isotropic model for free energy might take the form $\psi = \psi^\dagger(J^e, J_1^{e*}, J_2^{e*}, J^{ve}, J_1^{ve*}, J_2^{ve*}, \theta)$. Even though this form is missing the cross invariants between $\mathbf{F}^e$ and $\mathbf{F}^{ve}$, and therefore will not propagate changes in the local anisotropy, it still allows for capturing effects from the elastic stress and viscoelastic back-stress. The model we propose, which is motivated by rubber elasticity, has the form

$$\psi = \frac{\kappa}{2\rho_o} \ln^2(J^e) + \frac{G}{2\rho_o} (J_1^{e*} - 3) + \frac{\kappa^{ve}}{2\rho_o} \ln^2(J^{ve}) + \frac{G^{ve}}{2\rho_o} (J_1^{ve*} - 3) + \psi^\theta, \quad (13.270)$$

where κ , G , κ^{ve} , G^{ve} , and ψ^θ are only functions of temperature. This model results in expressions for $\mathbf{T}$ and $\mathbf{T}^{ve}$ given by

$$\mathbf{T} = \frac{1}{J} \left[\kappa \ln(J^e) \mathbf{I} + G \left(\mathbf{B}^{e*} - \frac{J_1^{e*}}{3} \mathbf{I} \right) \right], \quad (13.271)$$

$$\mathbf{T}^{ve} = \frac{1}{J} \left[\kappa^{ve} \ln(J^{ve}) \mathbf{I} + G^{ve} \left(\mathbf{B}^{ve*} - \frac{J_1^{ve*}}{3} \mathbf{I} \right) \right], \quad (13.272)$$

where κ and G are, respectively, the initial bulk and shear modulus of the elastic response, and κ^{ve} and G^{ve} are, respectively, the initial bulk and shear modulus of the viscoelastic back-stress response. In this case, the thermodynamic stress $\mathbf{T}^\theta = \mathbf{0}$.

For the thermal expansion and viscoelastic flow parameters, we will select the evolution rules

$$\mathbf{L}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (13.273)$$

$$\mathbf{L}^{ve} = \frac{\zeta_h}{9\mu^{ve,h}} \text{tr}(\Delta \mathbf{T}^{ve}) \mathbf{I} + \frac{\zeta_d}{\mu^{ve,d}} \Delta \mathbf{S}^{ve}, \quad (13.274)$$

where $\Delta \mathbf{T}^{ve} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}$, $\Delta \mathbf{S}^{ve}$ is its deviatoric part, α is the linear coefficient of thermal expansion, $\mu^{ve,h}$ is the infinitesimal dilatation viscosity, $\mu^{ve,d}$ is the infinitesimal shear viscosity, and ζ_h and ζ_d are scaling functions added to accommodate the typically highly nonlinear flow characteristics. The scaling factors are to be unit for infinitesimal conditions and grow with the overstress. Examples might be functions of the form

$$1 + \left[\frac{\Delta \tau}{A(1 - B\epsilon_V)} \right]^n, \quad e^{\left[\frac{\Delta \tau}{A(1 - B\epsilon_V)} \right]^n}, \quad 1 + A_1 \frac{\Delta \tau}{A(1 - B\sigma_{ave})} + A_2 \left[\frac{\Delta \tau}{A(1 - B\sigma_{ave})} \right]^2 + \dots, \quad (13.275)$$

where $\Delta \tau = \sqrt{\frac{1}{2} \Delta \mathbf{S}^{ve} : \Delta \mathbf{S}^{ve}}$. The scaling factors frequently include, in addition to the sensitivity of the flow to overstress, the influence of pressure or volume on the flow behavior. In addition, we need to provide an expression for heat flow, which we take as

$$\mathbf{q} = -k \mathbf{g}, \quad (13.276)$$

where k is the coefficient of thermal conduction. For the most part, all the parameters can be temperature dependent and we still have the general relation $\mathbf{F} = \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^\theta$.

This model has two characteristic times, one associated with volume changes (dilatation) and one associated with shear. In the range of small strains, the scaling factors are unity and the equations associated with pure volume changes decouple from that associated with shear, and we get two characteristic relaxation times, one associated with each. If we denote the relaxation time associated with pure volumetric deformations as t_R^h (h for hydrostatic)

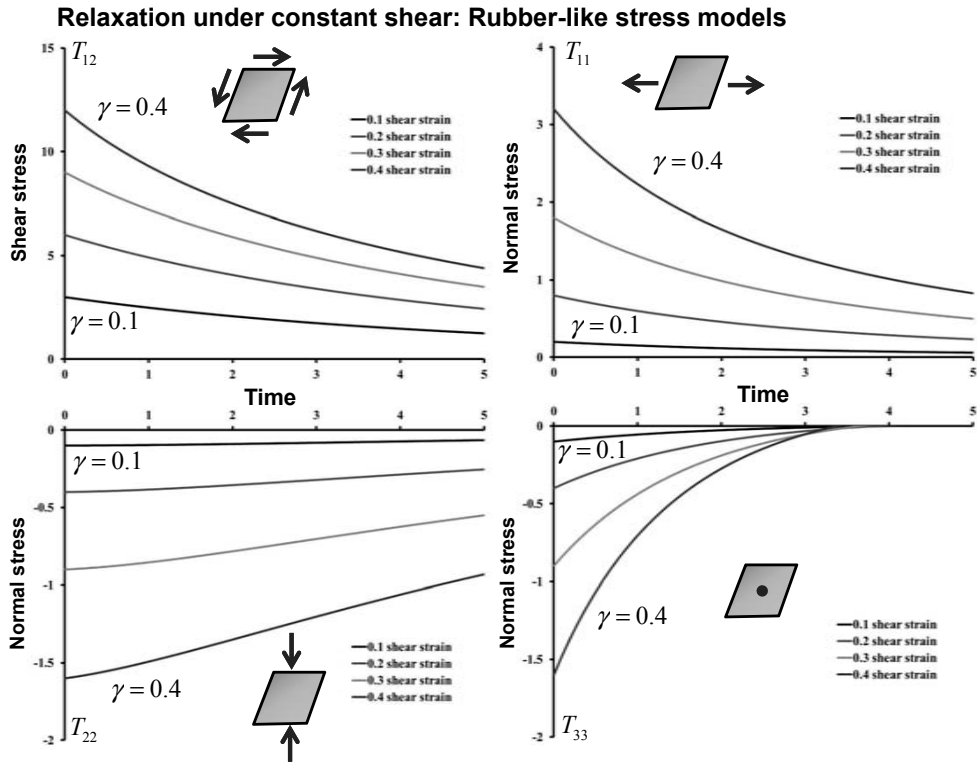


Figure 13.22: Rubber-elasticity-based nonlinear model: Stress relaxation under simple shear strains of $\gamma = 0.1, 0.2, 0.3, 0.4$: (a) shear stress T_{12} , (b) in-plane normal stress T_{11} , (c) in-plane normal stress T_{22} , and (d) out-of-plane normal stress T_{33} . Material parameters for the model are described in Section 13.

and the relaxation time associated with shear as t_R^d (d for deviatoric), then we have for infinitesimal strains

$$t_R^h \approx \frac{\mu^{ve,h}}{\kappa + \kappa^{ve}}, \quad t_R^d \approx \frac{\mu^{ve,d}}{G + G^{ve}}. \quad (13.277)$$

Obviously, as the deformations get large, the model becomes nonlinear and the characteristic times change.

In the following problems, we will demonstrate the characteristics of this model by selecting model parameters that provide characteristic relaxation times of approximately 1 for infinitesimal volumetric deformation and 10 for infinitesimal shear. Figures 13.22–13.24 show the response seen under different loading conditions using the model and parameters: $\kappa = 100$, $\kappa^{ve} = 10$, $\mu^{ve,h} = 110$, $G = 30$, $G^{ve} = 3$, and $\mu^{ve,d} = 330$. For showing the characteristics of the response function, we take scaling factors of the form

$$\zeta_h = J^2, \quad \zeta_d = 1 + \left(J \frac{\Delta \tau}{20} \right)^2. \quad (13.278)$$

Some of the details can be found in the problem statements.

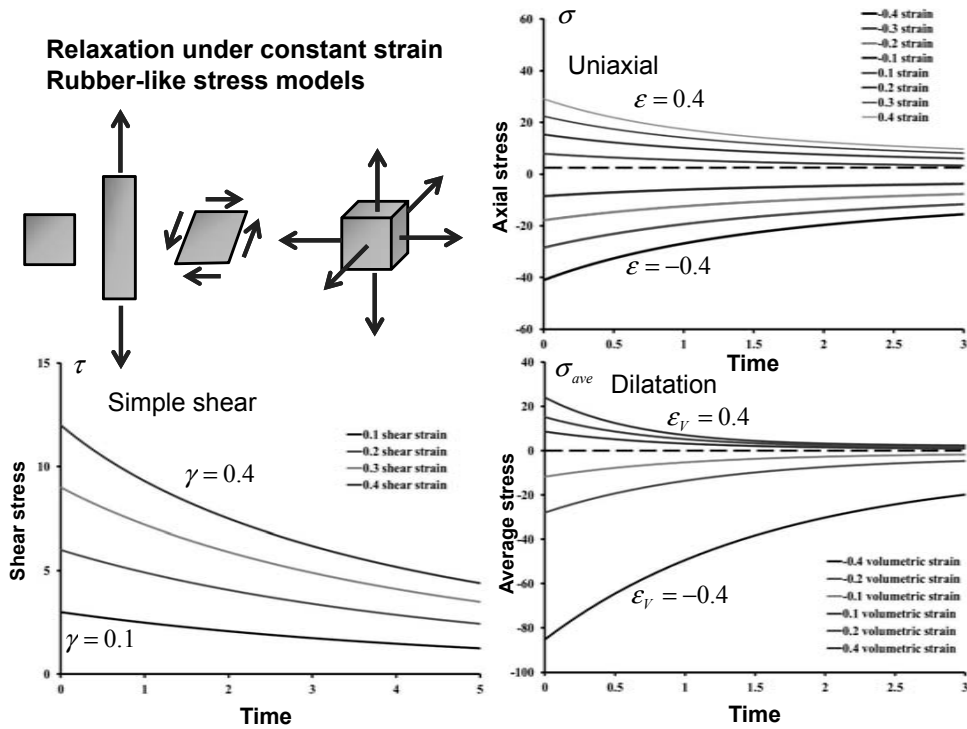


Figure 13.23: Rubber-elasticity-based nonlinear model: Stress relaxation at constant strain for (a) uniaxial extension/compression, (b) simple shear, and (c) volumetric extension/compression (dilatation). Material parameters for the model are described in Section 13.

Section 13.4-Problems:

- 13.4-18:** Consider stress relaxation in pure volumetric deformation (dilatation). Ignoring the equilibrium stress, define the relaxation time as the approximate time it would take the material to reach zero stress in a linear approximation to the response so that

$$t_R^h = -\frac{\sigma_{ave}}{\dot{\sigma}_{ave}}. \quad (13.279)$$

Show that for this model the initial relaxation time t_{Ro}^h is given by

$$t_{Ro}^h = \frac{\mu^{ve,h} J}{\zeta_h \kappa}, \quad (13.280)$$

where J is the applied constant volume ratio so that the applied volumetric strain is $\epsilon_v = J - 1$. Plot the initial relaxation time as a function of the applied volumetric strain. Note that the relaxation time approaches zero for extreme compressions and approaches infinity for extreme expansions. If the relaxation becomes more rapid both with compression and expansion, one way would be to select ζ_h , for example, as $\zeta_h = (1 + J^2)/2$ or by changing the model for free

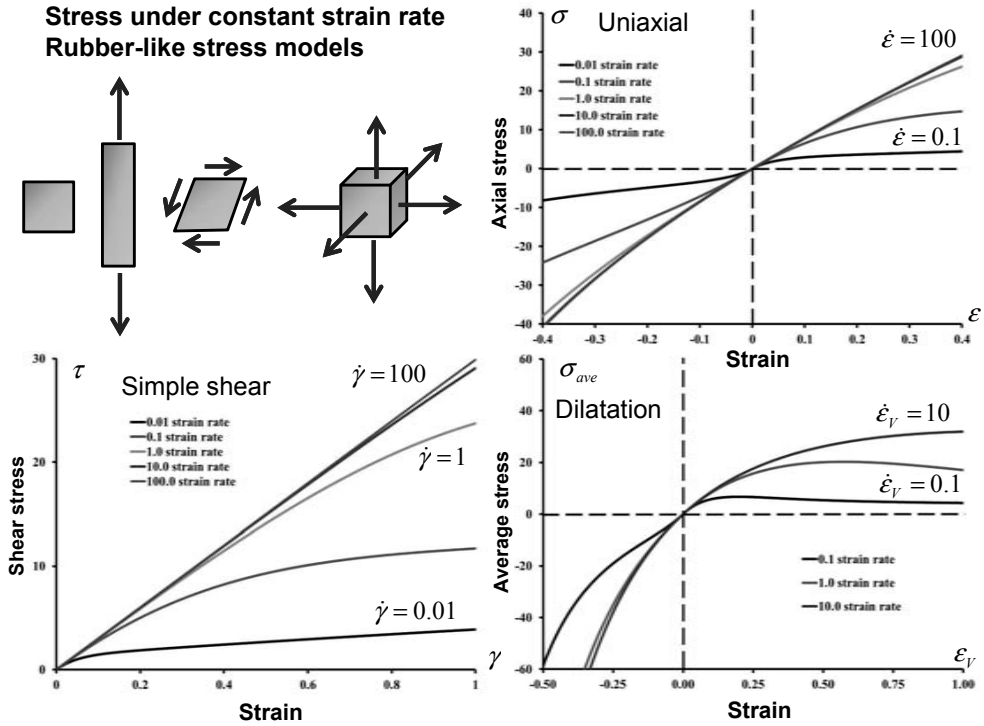


Figure 13.24: Rubber-elasticity-based nonlinear model: Stress response under monotonic constant strain rate for (a) uniaxial extension/compression, (b) simple shear, and (c) pure dilatation. Material parameters for the model are described in Section 13.

energy. An example of a change of the free energy would be taking

$$\begin{aligned} \psi = & \frac{\kappa}{8\rho_o} \{ J^{e2} + 2 \ln(J^e) [\ln(J^e) - 1] \} + \frac{G}{2\rho_o} (J_1^{e*} - 3) \\ & + \frac{\kappa^{ve}}{8\rho_o} \{ J^{ve2} + 2 \ln(J^{ve}) [\ln(J^{ve}) - 1] \} + \frac{G^{ve}}{2\rho_o} (J_1^{ve*} - 3) + \psi^\theta, \end{aligned} \quad (13.281)$$

where κ , G , κ^{ve} , G^{ve} , and ψ^θ are, as above, only functions of temperature and with the same interpretations. This model results in expressions for $\mathbf{T}$ and $\mathbf{T}^{ve}$ given by

$$\mathbf{T} = \frac{\kappa}{4J^{ve}} \left[J^e - \frac{1}{J^e} + 2 \frac{\ln(J^e)}{J^e} \right] \mathbf{I} + \frac{G}{J} \left(\mathbf{B}^{e*} - \frac{J_1^{e*}}{3} \mathbf{I} \right), \quad (13.282)$$

$$\mathbf{T}^{ve} = \frac{\kappa^{ve}}{4J^e} \left[J^{ve} - \frac{1}{J^{ve}} + 2 \frac{\ln(J^{ve})}{J^{ve}} \right] \mathbf{I} + \frac{G^{ve}}{J} \left(\mathbf{B}^{ve*} - \frac{J_1^{ve*}}{3} \mathbf{I} \right). \quad (13.283)$$

Show that for this model the approximate initial relaxation time is

$$t_{Ro}^h = \frac{4\mu^{ve,h}}{\zeta_h \kappa \left(J + \frac{3-2 \ln(J^e)}{J} \right)}. \quad (13.284)$$

Plot this relaxation time and show that for constant $\zeta_h = 1$ it goes to zero for both extreme compression and extreme expansion. Plot the initial stress under

sudden hydrostatic expansion or compressions as a function of the applied J and verify that the response is reasonable.

- 13.4-19:** One expects that the relaxations slow down with compression and increase with expansion. Consider a free energy of the form

$$\psi = \frac{\kappa}{(n+1)\rho_o} \left[\frac{J^{e(n+1)}}{n+1} - \ln(J^e) \right] + \frac{G}{2\rho_o} (J_1^{e*} - 3) + \frac{\kappa^{ve}}{(m+1)\rho_o} \left[\frac{J^{ve(m+1)}}{m+1} - \ln(J^{ve}) \right] + \frac{G^{ve}}{2\rho_o} (J_1^{ve*} - 3) + \psi^\theta, \quad (13.285)$$

where m and n are material factors expressing the nonlinearity. Show that for pure dilatation the average stress is given by

$$\sigma_{ave} = \frac{\kappa}{(n+1)J^{ve}} \left(J^{en} - \frac{1}{J^e} \right). \quad (13.286)$$

Plot this for different n to verify the form of the response and the influence of the parameter n . From this conclude that under constant dilatation ($\dot{J} = 0$) the rate of change of average stress is given by

$$\dot{\sigma}_{ave} = -\frac{\kappa J^{en} \dot{J}^{ve}}{J^{ve2}}. \quad (13.287)$$

Using this, calculate the approximate initial relaxation time described in the previous problem. For $\zeta_h = 1$, from this conclude that the relaxation of stress under sudden compressions slows down, while under sudden expansions it accelerates for positive values of n . Plot the relaxation time as a function of a sudden volumetric expansion or contraction (sudden imposed J). Using the model originally constructed in this section, develop a function for ζ_h that would result in the same general relaxation.

- 13.4-20:** Considering the problems described with stress relaxation under large volumetric compression for the isotropic quadratic form of the free energy, one might consider creating a hybrid model that mixes the newly proposed dilatation part with the quadratic model. One such model might have a free energy of the form

$$\psi = \frac{\kappa}{8\rho_o} \{ J^{e2} + 2 \ln(J^e) [\ln(J^e) - 1] \} + \frac{G}{\rho_o} \left[\mathbf{E}^e : \mathbf{E}^e - \frac{1}{3} tr^2(\mathbf{E}^e) \right] + \frac{\kappa^{ve}}{8\rho_o} \{ J^{ve2} + 2 \ln(J^{ve}) [\ln(J^{ve}) - 1] \} + \frac{G^{ve}}{\rho_o} \left[\mathbf{E}^{ve} : \mathbf{E}^{ve} - \frac{1}{3} tr^2(\mathbf{E}^{ve}) \right]. \quad (13.288)$$

For this model calculate the expressions for the thermodynamic stresses.

- 13.4-21:** Simulate the response of the proposed model in this section for stress relaxation under simple shear. Create plots for (a) shear stress T_{12} , (b) in-plane normal stress T_{11} , (c) in-plane normal stress T_{22} , (d) out-of-plane normal stress T_{33} , and (e) relaxation modulus (shear stress over shear strain). Plot the response for shear strains of $\gamma = 0.1, 0.2, 0.4, 0.8$. Compare your results to Figure 13.22.
- 13.4-22:** Simulate the response of the proposed model in this section for stress relaxation under pure volumetric deformation (dilatation) for volumetric strains of $\epsilon_v = J - 1 = -0.4, -0.2, -0.1, 0.1, 0.2, 0.4$. Compare your results to those given in Figure 13.23.

- 13.4-23:** Simulate the response of the proposed model in this section for stress relaxation under uniaxial extension for axial strains of $\epsilon = -0.4, -0.2, -0.1, 0.1, 0.2, 0.4$. Compare your results to those given in Figure 13.23.
- 13.4-24:** Simulate the response of the proposed model in this section for constant strain rate loading in simple shear. Use the strain rates $\dot{\gamma} = 0.01, 0.1, 1.0, 10, 100$. Compare your results to those given in Figure 13.24.
- 13.4-25:** Simulate the response of the proposed model in this section for constant strain rate loading in pure dilatation. Use the strain rates $\dot{\epsilon}_v = \pm 0.1, \pm 1.0, \pm 10$. Compare your results to those given in Figure 13.24.
- 13.4-26:** Simulate the response of the proposed model in this section for constant strain rate loading in uniaxial extension. Use the strain rates $\dot{\epsilon} = \pm 0.01, \pm 0.1, \pm 1.0, \pm 10, \pm 100$. Compare your results to those given in Figure 13.24.
- 13.4-27:** Simulate the response of the proposed model in this section for cyclic loading consisting of constant strain rate segments of loading and unloading in (a) simple shear, (b) uniaxial tension, and (c) volumetric strain. In each case start from zero strain and cycle to associated strains of plus and minus 0.2. Use associated strain rates of 0.01, 0.1, 1.0, 10, 100.

13.5 Multielement thermodynamic viscoelastic model

As we did for the mechanical theory, we will generalize the simple model we presented in the previous section by assuming there is a continuous set of relaxation events each represented by a mechanical analog similar to that presented in Figure 13.18, but each with different material parameters. Again, we use “ s ” to denote a continuous real parameter characterizing the different relaxation events and introduce the assumption that the total free energy of the material at a point can be calculated from the expression

$$\psi(t) = \int_0^\infty \phi_s(t) ds, \quad (13.289)$$

where $\phi_s(t)$ represents the contribution from the relaxation event s to the total free energy $\psi(t)$. Obviously, $\phi_s(t)$ represents the intensity of the free energy characterized by relaxation event s . In the following we will use a subscript “ s ” to denote parameters associated with the relaxation event characterized by parameter s . The deformation gradient will be assumed to be decomposed differently for each s . This decomposition of the deformation gradient into elastic, viscoelastic, and thermal parts will be written as

$$\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^\theta, \quad (13.290)$$

and it will be assumed that ϕ_s is given in terms of the current values of the component of this decomposition and of temperature. This will be written as

$$\phi_s = \phi_s^\dagger(\mathbf{F}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta). \quad (13.291)$$

To introduce these assumptions into the Clausius-Duhem inequality (6.69), we need to calculate the time derivative of the free energy. The time derivative of ϕ_s can be written as

$$\dot{\phi}_s = \partial_{\mathbf{F}_s^e}(\phi_s) : \dot{\mathbf{F}}_s^e + \partial_{\mathbf{F}_s^{ve}}(\phi_s) : \dot{\mathbf{F}}_s^{ve} + \partial_{\mathbf{F}_s^\theta}(\phi_s) : \dot{\mathbf{F}}_s^\theta + \partial_\theta(\phi_s)\dot{\theta}. \quad (13.292)$$

We can rewrite this by eliminating $\dot{\mathbf{F}}_s^e$ as follows. Taking the material time derivative of the deformation gradient results in the expression

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_s^e \mathbf{F}_s^e \mathbf{F}_s^\theta + \mathbf{F}_s^e \dot{\mathbf{F}}_s^{ve} \mathbf{F}_s^\theta + \mathbf{F}_s^e \mathbf{F}_s^{ve} \dot{\mathbf{F}}_s^\theta \quad (13.293)$$

that can be reorganized to give $\dot{\mathbf{F}}_s^e$ as

$$\dot{\mathbf{F}}_s^e = \dot{\mathbf{F}} \mathbf{F}_s^{\theta-1} \mathbf{F}_s^{ve-1} - \mathbf{F}_s^e \dot{\mathbf{F}}_s^{ve} \mathbf{F}_s^{ve-1} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \dot{\mathbf{F}}_s^\theta \mathbf{F}_s^{\theta-1} \mathbf{F}_s^{ve-1}. \quad (13.294)$$

This can be organized by using the velocity gradients as

$$\dot{\mathbf{F}}_s^e = \mathbf{L} \mathbf{F}_s^e - \mathbf{F}_s^e \mathbf{L}_s^{ve} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{L}_s^\theta \mathbf{F}_s^{ve-1}. \quad (13.295)$$

Using procedures similar to those presented in the previous section, we can eliminate $\dot{\mathbf{F}}_s^e$ from the expression for $\dot{\phi}_s$ to get

$$\rho \dot{\phi}_s = \boldsymbol{\tau}_s^{eT} : \mathbf{L} - \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} - \Delta \boldsymbol{\tau}_s^{\theta T} : \mathbf{L}_s^\theta + \rho \partial_\theta(\phi_s) \dot{\theta}, \quad (13.296)$$

where the overstresses $\Delta \boldsymbol{\tau}_s^{ve}$ and $\Delta \boldsymbol{\tau}_s^\theta$ are given by

$$\Delta \boldsymbol{\tau}_s^{ve} \equiv \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e - \boldsymbol{\tau}_s^{ve}, \quad (13.297)$$

$$\Delta \boldsymbol{\tau}_s^\theta \equiv \mathbf{F}_s^{ve-1} \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e \mathbf{F}_s^{ve} - \boldsymbol{\tau}_s^\theta, \quad (13.298)$$

in terms of the thermodynamic stress intensities

$$\boldsymbol{\tau}_s^{eT} \equiv \rho \partial_{\mathbf{F}_s^e}(\phi_s) \mathbf{F}_s^{eT}, \quad (13.299)$$

$$\boldsymbol{\tau}_s^{veT} \equiv \rho \partial_{\mathbf{F}_s^{ve}}(\phi_s) \mathbf{F}_s^{veT}, \quad (13.300)$$

$$\boldsymbol{\tau}_s^{\theta T} \equiv \rho \partial_{\mathbf{F}_s^\theta}(\phi_s) \mathbf{F}_s^{\theta T}. \quad (13.301)$$

We now can introduce

$$\dot{\psi} = \int_0^\infty \dot{\phi}_s ds \quad (13.302)$$

into the Clausius-Duhem inequality (6.69) to get, after some reorganization, the equation

$$\begin{aligned} & \left[\int_0^\infty \boldsymbol{\tau}_s^{eT} ds - \mathbf{T}^T \right] : \mathbf{L} - \int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds \\ & + \left[\rho \eta + \rho \int_0^\infty \partial_\theta(\phi_s) ds - \int_0^\infty \Delta \boldsymbol{\tau}_s^{\theta T} : \boldsymbol{\alpha}_s ds \right] \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (13.303)$$

where we have assumed that each relaxation event is characterized by a thermal expansion given by

$$\mathbf{L}_s^\theta = \boldsymbol{\alpha}_s \dot{\theta} \quad (13.304)$$

for a coefficient of thermal expansion α_s that can be a function of $(\mathbf{F}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta)$. Standard arguments require that

$$\mathbf{T}^T = \int_o^\infty \boldsymbol{\tau}_s^{eT} ds, \quad (13.305)$$

$$\eta = - \int_o^\infty \partial_\theta(\phi_s) ds + \frac{1}{\rho} \int_o^\infty \Delta \boldsymbol{\tau}_s^{\theta T} : \boldsymbol{\alpha}_s ds. \quad (13.306)$$

What remains to be satisfied is

$$- \int_o^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (13.307)$$

It seems plausible to assume that each relaxation event should satisfy the Clausius-Duhem inequality independent of the other relaxation events. This assumption could be interpreted to require that

$$-\Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} \leq 0 \quad (13.308)$$

for every relaxation event s . The reader will note that this is a sufficient condition to satisfy the second law of thermodynamics and has not been established as a necessary condition. Nonetheless, we will postulate this to be the way through which the Clausius-Duhem inequality is satisfied. In terms of the mechanical analog, this postulate is equivalent to assuming that the viscous damping in each viscoelastic element is such that the viscous damper always resists the changing of its length (i.e., it always dissipates energy).

As the reader recalls, $\rho \dot{h}$ is the rate of heat entering a point (per unit volume) through radiation and conduction, so that $\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q})$. We recall from Chapter 6 that one can calculate it from

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta}.$$

The first three terms on the right-hand side directly appear in the Clausius-Duhem inequality so we easily can calculate $\rho \dot{h}$ as

$$\rho \dot{h} = - \int_o^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds + \rho \eta \dot{\theta}. \quad (13.309)$$

13.5.1 Rigid body motions and the symmetry of the Cauchy stress

Motivated by the simple model, we will assume that the “internal parameters” $\mathbf{F}_s^{ve}$ and $\mathbf{F}_s^\theta$ are not affected by the superimposition of rigid body motions. As before we let $\mathbf{F}(t)$ represent a deformation history, and we let $\mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t)$ be constructed from it by superimposing rigid body motions described by a history of orthogonal transformations $\mathbf{Q}(t)$. We denote this transformation of $\mathbf{F}(t)$ as $\mathbf{F}(t) \rightarrow \mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t)$ or simply $\mathbf{F}(t) \rightarrow \mathbf{Q}(t)\mathbf{F}(t)$. The effect of rigid body motions on the “kinematic” variables then will be summarized as

$$\mathbf{F}_s^e(t) \rightarrow \mathbf{F}_s^{e*}(t) = \mathbf{Q}(t)\mathbf{F}_s^e(t), \quad (13.310)$$

$$\mathbf{F}_s^{ve}(t) \rightarrow \mathbf{F}_s^{ve*}(t) = \mathbf{F}_s^{ve}(t), \quad (13.311)$$

$$\mathbf{F}_s^\theta(t) \rightarrow \mathbf{F}_s^{\theta*}(t) = \mathbf{F}_s^\theta(t), \quad (13.312)$$

to be consistent with the assumption $\mathbf{F}^* = \mathbf{F}_s^{e*}\mathbf{F}_s^{ve*}\mathbf{F}_s^{\theta*} = \mathbf{Q}\mathbf{F}$. This set of assumptions and the assumption that free energy ϕ_s in each element is not affected by rigid body motions

result in

$$\phi_s(t) = \phi_s^\dagger[\mathbf{X}, \mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] \quad (13.313)$$

$$= \phi_s^\dagger[\mathbf{X}, \mathbf{U}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] \quad (13.314)$$

$$= \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] \quad (13.315)$$

$$= \phi_s^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)]. \quad (13.316)$$

It can be shown, through similar arguments as given for the simple model, that these assumptions when introduced into the thermodynamic model will result in the relations

$$\boldsymbol{\tau}_s^e(t) \rightarrow \boldsymbol{\tau}_s^{e*}(t) = \mathbf{Q}^T(t) \boldsymbol{\tau}_s^e(t) \mathbf{Q}(t), \quad (13.317)$$

$$\boldsymbol{\tau}_s^{ve}(t) \rightarrow \boldsymbol{\tau}_s^{ve*}(t) = \boldsymbol{\tau}_s^{ve}(t), \quad (13.318)$$

$$\boldsymbol{\tau}_s^\theta(t) \rightarrow \boldsymbol{\tau}_s^{\theta*}(t) = \boldsymbol{\tau}_s^\theta(t), \quad (13.319)$$

which result in the overstresses having the properties

$$\Delta \boldsymbol{\tau}_s^{ve}(t) \rightarrow \Delta \boldsymbol{\tau}_s^{ve*}(t) = \Delta \boldsymbol{\tau}_s^{ve}(t), \quad (13.320)$$

$$\Delta \boldsymbol{\tau}_s^\theta(t) \rightarrow \Delta \boldsymbol{\tau}_s^{\theta*}(t) = \Delta \boldsymbol{\tau}_s^\theta(t). \quad (13.321)$$

It also follows that $\boldsymbol{\tau}_s^e(t)$ is symmetric, which makes the resulting model for the Cauchy stress symmetric. Finally, it also can be shown that the resulting model for the Cauchy stress has the expected transformation $\mathbf{T}(t) \rightarrow \mathbf{T}^*(t) = \mathbf{Q}^T(t) \mathbf{T}(t) \mathbf{Q}(t)$.

13.5.2 Material symmetry

As before, we will let $\mathbf{M}$ denote a transformation representing a symmetry of the material at the initial time for a given material point. If we have a symmetry, we will see no change in the response of the material at the point if we change the history of the deformation gradient $\mathbf{F}(t)$ to $\mathbf{F}(t)\mathbf{M}$ for any deformation gradient history $\mathbf{F}(t)$. We denote this change of history by a bar over the symbol and write it as $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$. To construct a consistent theory, we need to have the kinematic variables $\mathbf{F}_s^e$, $\mathbf{F}_s^{ve}$, and $\mathbf{F}_s^\theta$ change in a way that is consistent with $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$. Motivated by the simple model, we will assume

$$\mathbf{F}_s^e \rightarrow \bar{\mathbf{F}}_s^e = \mathbf{F}_s^e \mathbf{M}, \quad (13.322)$$

$$\mathbf{F}_s^{ve} \rightarrow \bar{\mathbf{F}}_s^{ve} = \mathbf{M}^{-1} \mathbf{F}_s^{ve} \mathbf{M}, \quad (13.323)$$

$$\mathbf{F}_s^\theta \rightarrow \bar{\mathbf{F}}_s^\theta = \mathbf{M}^{-1} \mathbf{F}_s^\theta \mathbf{M}. \quad (13.324)$$

The assumption that the free energy in each element s is the same in value for a change in the deformation history described by $\mathbf{F} \rightarrow \mathbf{F}\mathbf{M}$ can be written in terms of the different models for the free energy as one of the following:

$$\phi_s^\dagger[\mathbf{X}, \mathbf{F}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta] = \phi_s^\dagger[\mathbf{X}, \mathbf{F}_s^e \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^{ve} \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^\theta \mathbf{M}, \theta], \quad (13.325)$$

$$\phi_s^\dagger[\mathbf{X}, \mathbf{U}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta] = \phi_s^\dagger[\mathbf{X}, \bar{\mathbf{U}}_s^e, \mathbf{M}^{-1} \mathbf{F}_s^{ve} \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^\theta \mathbf{M}, \theta], \quad (13.326)$$

$$\phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta] = \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{M}^T \mathbf{C}_s^e \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^{ve} \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^\theta \mathbf{M}, \theta], \quad (13.327)$$

$$\phi_s^{\dagger\dagger\dagger}[\mathbf{X}, \mathbf{E}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta] = \phi_s^{\dagger\dagger\dagger}[\mathbf{X}, \bar{\mathbf{E}}_s^e, \mathbf{M}^{-1} \mathbf{F}_s^{ve} \mathbf{M}, \mathbf{M}^{-1} \mathbf{F}_s^\theta \mathbf{M}, \theta], \quad (13.328)$$

where $\bar{\mathbf{U}}_s^e$ is the right symmetric factor in the polar decomposition of $\bar{\mathbf{F}}_s^e$ and $\bar{\mathbf{E}}_s^e = \frac{1}{2}(\mathbf{M}^T \mathbf{C}_s^e \mathbf{M} - \mathbf{I})$. These can be shown to result in the thermodynamic stresses in element s transforming through the relations

$$\boldsymbol{\tau}_s^e \rightarrow \bar{\boldsymbol{\tau}}_s^e = \boldsymbol{\tau}_s^e, \quad (13.329)$$

$$\boldsymbol{\tau}_s^{ve} \rightarrow \bar{\boldsymbol{\tau}}_s^{ve} = \mathbf{M}^{-1} \boldsymbol{\tau}_s^{ve} \mathbf{M}, \quad (13.330)$$

$$\boldsymbol{\tau}_s^\theta \rightarrow \bar{\boldsymbol{\tau}}_s^\theta = \mathbf{M}^{-1} \boldsymbol{\tau}_s^\theta \mathbf{M}, \quad (13.331)$$

which will in turn result in the transformations of the over stresses given by

$$\Delta \tau_s^{ve} \rightarrow \Delta \bar{\tau}_s^{ve} = \mathbf{M}^{-1} \Delta \tau_s^{ve} \mathbf{M}, \quad (13.332)$$

$$\Delta \tau_s^\theta \rightarrow \Delta \bar{\tau}_s^\theta = \mathbf{M}^{-1} \Delta \tau_s^\theta \mathbf{M}. \quad (13.333)$$

If we assume that $\mathbf{L}_s^{ve}$ and $\mathbf{L}_s^\theta$ are given by the models

$$\mathbf{L}_s^{ve} = \mathbf{C}_s^{ve} : \Delta \tau_s^{veT}, \quad (13.334)$$

$$\mathbf{L}_s^\theta = \alpha_s \dot{\theta}, \quad (13.335)$$

for, respectively, the fourth- and second-order moduli $\mathbf{C}_s^{ve}$ and α_s , then symmetry restrictions can be written as

$$\mathbf{M}^{-1}(\mathbf{C}_s^{ve} : \Delta \tau_s^{veT})\mathbf{M} = \bar{\mathbf{C}}_s^{ve} : (\mathbf{M}^T \Delta \tau_s^{veT} \mathbf{M}^{-T}), \quad (13.336)$$

$$\bar{\alpha}_s = \mathbf{M}^{-1} \alpha_s \mathbf{M}. \quad (13.337)$$

13.5.3 Change of reference configuration

For the modeling to be consistent, the way in which the internal variables change with a change in the reference configuration needs to be consistent with how they change with transformations associated with material symmetry since the two are similar. Consider changing the reference configuration from $\mathcal{B}_o$ to $\check{\mathcal{B}}_o$ so that $\check{\mathbf{F}} = \mathbf{F}\mathbf{P}$, where $\mathbf{F}$ is the deformation gradient calculated relative to the reference configuration $\mathcal{B}_o$, $\check{\mathbf{F}}$ is the deformation gradient calculated relative to the reference configuration $\check{\mathcal{B}}_o$, and $\mathbf{P}$ is the deformation gradient of $\mathcal{B}_o$ relative to $\check{\mathcal{B}}_o$. If one assumes that

$$\check{\mathbf{F}}_s^e = \mathbf{F}_s^e \mathbf{P}, \quad \check{\mathbf{F}}_s^{ve} = \mathbf{P}^{-1} \mathbf{F}_s^{ve} \mathbf{P}, \quad \check{\mathbf{F}}_s^\theta = \mathbf{P}^{-1} \mathbf{F}_s^\theta \mathbf{P}, \quad (13.338)$$

then one arrives at a consistent set of assumptions. Under these conditions Noll's rule for the evaluation of material symmetry transformation also holds, yielding the new representation of $\check{\mathbf{M}}$ with respect to $\check{\mathcal{B}}_o$ of the symmetry represented by $\mathbf{M}$ with respect to $\mathcal{B}_o$ given by

$$\check{\mathbf{M}} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P}. \quad (13.339)$$

13.6 Initially isotropic models: Free energy and thermodynamic stresses

The construction of a model first starts from selecting a function for the free energy. We will select the form of the function for free energy given by (13.315) so that

$$\phi_s(t) = \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)]. \quad (13.340)$$

For an initially isotropic material, the condition of material symmetry requires that

$$\phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^\theta, \theta] = \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{M}^T \mathbf{C}_s^e \mathbf{M}, \mathbf{M}^T \mathbf{F}_s^{ve} \mathbf{M}, \mathbf{M}^T \mathbf{F}_s^\theta \mathbf{M}, \theta] \quad (13.341)$$

for every orthogonal $\mathbf{M}$. Therefore, $\phi_s^{\dagger\dagger}$ is an isotropic scalar function of the three symmetric tensors $\mathbf{C}_s^e$, $\frac{1}{2}(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})$ and $\frac{1}{2}(\mathbf{F}_s^\theta + \mathbf{F}_s^{\theta T})$, and the two skew-symmetric tensors $\frac{1}{2}(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})$ and $\frac{1}{2}(\mathbf{F}_s^\theta - \mathbf{F}_s^{\theta T})$, plus it also will be a function of $\mathbf{X}$ and θ . Appendix A provides the solution to this in the form of 76 scalar invariants to select from. In this case, we can write

$$\phi_s(t) = \check{\phi}_s^{\dagger\dagger}[\mathbf{X}, I_{s1}, \dots, I_{s76}, \theta]. \quad (13.342)$$

Most probably no experimental effort can provide sufficient information to select the form of a function that is written in terms of all of these invariants. If we leave out $\mathbf{F}_s^\theta$ by assuming that there is no effect on free energy from the thermal expansion, or assuming that all the energy associated with it can be calculated with the knowledge of θ , then we can reduce the number of invariants to the 21 invariants of the two symmetric tensors $\mathbf{C}_s^e$, $\frac{1}{2}(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})$, and one skew-symmetric tensor $\frac{1}{2}(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})$. If we further assume that only $\mathbf{C}_s^e$ and the symmetric part of $\mathbf{F}_s^{ve}$, either in the form of $\frac{1}{2}(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})$ or its polar decomposition, appears in the function for the free energy, then the number of invariants is reduced to the 10 invariants of two symmetric tensors. Finally, if we assume the free energy is only a function of $\mathbf{C}_s^e$, then the invariants are reduced to the three isotropic $\mathbf{C}_s^e$. For the general case, the expressions for the thermodynamic stresses become

$$\boldsymbol{\tau}_s^{eT} = \rho \sum_{i=1}^{76} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^e(I_{si})} \mathbf{F}_s^{eT}, \quad (13.343)$$

$$\boldsymbol{\tau}_s^{veT} = \rho \sum_{i=1}^{76} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^{ve}(I_{si})} \mathbf{F}_s^{veT}, \quad (13.344)$$

$$\boldsymbol{\tau}_s^{\theta T} = \rho \sum_{i=1}^{76} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^\theta(I_{si})} \mathbf{F}_s^{\theta T}. \quad (13.345)$$

13.6.1 Separable free energies

We first will consider free energy functions that can be separated into additive elastic, viscoelastic, and thermal parts of the free energy. We will write this as

$$\phi_s(t) = \phi_s^e(t) + \phi_s^{ve}(t) + \phi_s^\theta(t), \quad (13.346)$$

where the three parts are given by functions described by

$$\phi_s^e = \phi_s^{e\dagger}[\mathbf{X}, \mathbf{C}_s^e, \theta], \quad (13.347)$$

$$\phi_s^{ve} = \phi_s^{ve\dagger}[\mathbf{X}, \mathbf{F}_s^{ve}, \theta], \quad (13.348)$$

$$\phi_s^\theta = \phi_s^{\theta\dagger}[\mathbf{X}, \mathbf{F}_s^\theta, \theta]. \quad (13.349)$$

As a result of isotropy of the initial material, the elastic part of the free energy can be written in terms of the isotropic invariants of $\mathbf{C}_s^e$. As in the case of an isotropic thermoelastic material, one can select among sets of isotropic invariants. If we use $I_{s1}^e = \text{tr}(\mathbf{C}_s^e)$, $I_{s2}^e = \text{tr}(\mathbf{C}_s^{e2})$, and $I_{s3}^e = \text{tr}(\mathbf{C}_s^{e3})$, then

$$\phi_s^e = \phi_s^{e\dagger\dagger}[\mathbf{X}, I_{s1}^e, I_{s2}^e, I_{s3}^e, \theta]. \quad (13.350)$$

The expression for the thermodynamic elastic stress then will become

$$\boldsymbol{\tau}_s^{eT} = \rho \sum_{i=1}^3 \frac{\partial \phi_s^e}{\partial I_{si}^e} \partial_{\mathbf{F}_s^e(I_{si}^e)} \mathbf{F}_s^{eT}. \quad (13.351)$$

Using the partial derivatives

$$\partial_{\mathbf{F}_s^e(I_{s1}^e)} = 2\mathbf{F}_s^e, \quad \partial_{\mathbf{F}_s^e(I_{s2}^e)} = 4\mathbf{F}_s^e \mathbf{C}_s^e, \quad \partial_{\mathbf{F}_s^e(I_{s3}^e)} = 6\mathbf{F}_s^e \mathbf{C}_s^{e2}, \quad (13.352)$$

we can conclude that

$$\boldsymbol{\tau}_s^{eT} = \rho \left(2 \frac{\partial \phi_s^e}{\partial I_{s1}^e} \mathbf{B}_s^e + 4 \frac{\partial \phi_s^e}{\partial I_{s2}^e} \mathbf{B}_s^{e2} + 6 \frac{\partial \phi_s^e}{\partial I_{s3}^e} \mathbf{B}_s^{e3} \right). \quad (13.353)$$

It seems plausible to assume that the τ_s^e becomes zero for pure rigid body motions so that we would require that

$$\left[\frac{\partial \phi_s^e}{\partial I_{s1}^e} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 2 \left[\frac{\partial \phi_s^e}{\partial I_{s2}^e} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 3 \left[\frac{\partial \phi_s^e}{\partial I_{s3}^e} \right]_{\mathbf{F}_s^e = \mathbf{Q}} = 0, \quad (13.354)$$

for all orthogonal $\mathbf{Q}$. The expression for Cauchy stress is given by integrating τ_s^e over s to obtain

$$\mathbf{T}^T(t) = \rho \int_0^\infty \left(2 \frac{\partial \phi_s^e}{\partial I_{s1}^e} \mathbf{B}_s^e + 4 \frac{\partial \phi_s^e}{\partial I_{s2}^e} \mathbf{B}_s^{e2} + 6 \frac{\partial \phi_s^e}{\partial I_{s3}^e} \mathbf{B}_s^{e3} \right) ds. \quad (13.355)$$

Clearly, as in the elastic case, we can use in place of $I_{s1}^e = \text{tr}(\mathbf{C}_s^e)$, $I_{s2}^e = \text{tr}(\mathbf{C}_s^{e2})$, and $I_{s3}^e = \text{tr}(\mathbf{C}_s^{e3})$ one of the other equivalent sets such as $J_{s1}^e = \text{tr}(\mathbf{C}_s^e)$, $J_{s2}^e = \frac{1}{2}[\text{tr}^2(\mathbf{C}_s^e) - \text{tr}(\mathbf{C}_s^{e2})]$, and $J_{s3}^e = J_s^e = \det(\mathbf{F}_s^e)$, or the set $K_{s1}^e = J_s^e$, $K_{s2}^e = \frac{I_{s1}^e}{J_s^{e2/3}}$, and $K_{s3}^e = \frac{I_{s2}^e}{I_{s1}^e}$. The results of these selections will be alternate representations given by

$$\tau_s^{eT} = \rho \left[\frac{\partial \phi_s^e}{\partial J_s^e} J_s^e \mathbf{I} + 2 \left(\frac{\partial \phi_s^e}{\partial J_{s1}^e} + J_{s1}^e \frac{\partial \phi_s^e}{\partial J_{s2}^e} \right) \mathbf{B}_s^e - 2 \frac{\partial \phi_s^e}{\partial J_{s2}^e} \mathbf{B}_s^{e2} \right], \quad (13.356)$$

$$= \rho \left[\frac{\partial \phi_s^e}{\partial J_s^e} J_s^e \mathbf{I} + \frac{2}{J_s^{e2/3}} \frac{\partial \phi_s^e}{\partial K_{s2}^e} \left(\mathbf{B}_s^e - \frac{I_{s1}^e}{3} \mathbf{I} \right) + \frac{4}{I_{s1}^{e2}} \frac{\partial \phi_s^e}{\partial K_{s3}^e} \left(\mathbf{B}_s^{e2} - \frac{I_{s2}^e}{I_{s1}^e} \mathbf{B}_s^e \right) \right]. \quad (13.357)$$

As a result of the initial isotropy, we can write the ϕ_s^{ve} in terms of the seven isotropic invariants of the symmetric and skew-symmetric parts of $\mathbf{F}_s^{ve}$. These isotropic invariants are described in Appendix A as

$$\begin{aligned} I_{s1}^{ve} &= \frac{1}{2} \text{tr}(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT}), & I_{s2}^{ve} &= \frac{1}{4} \text{tr}[(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2], & I_{s3}^{ve} &= \frac{1}{8} \text{tr}[(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^3], \\ I_{s4}^{ve} &= \frac{1}{4} \text{tr}[(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2], & I_{s5}^{ve} &= \frac{1}{8} \text{tr}[(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2], \\ I_{s6}^{ve} &= \frac{1}{16} \text{tr}[(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2], \\ I_{s7}^{ve} &= \frac{1}{64} \text{tr}[(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})]. \end{aligned} \quad (13.358)$$

The expression for the thermodynamic viscoelastic stress then will become

$$\tau_s^{veT} = \rho \sum_{i=1}^7 \frac{\partial \phi_s^{ve}}{\partial I_{si}^{ve}} \partial_{\mathbf{F}_s^{ve}} (I_{si}^{ve}) \mathbf{F}_s^{veT}. \quad (13.359)$$

The partial derivatives of the invariants with respect to $\mathbf{F}_s^{ve}$ are given by

$$\begin{aligned} \partial_{\mathbf{F}_s^{ve}} (I_{s1}^{ve}) &= \mathbf{I}, & \partial_{\mathbf{F}_s^{ve}} (I_{s2}^{ve}) &= \mathbf{F}_s^{ve} + \mathbf{F}_s^{veT}, & \partial_{\mathbf{F}_s^{ve}} (I_{s3}^{ve}) &= \frac{3}{4} (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2, \\ \partial_{\mathbf{F}_s^{ve}} (I_{s4}^{ve}) &= -(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT}), \\ \partial_{\mathbf{F}_s^{ve}} (I_{s5}^{ve}) &= \frac{1}{4} (\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2 - \frac{1}{4} [(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT}) + (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})], \\ \partial_{\mathbf{F}_s^{ve}} (I_{s6}^{ve}) &= \frac{1}{8} [(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT}) + (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})^2 \\ &\quad - (\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})(\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2 - (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})^2(\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})], \end{aligned} \quad (13.360)$$

and using $\mathbf{A} = (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT})$ and $\mathbf{W} = (\mathbf{F}_s^{ve} - \mathbf{F}_s^{veT})$, then

$$\begin{aligned} \partial_{\mathbf{F}_s^{ve}} (I_{s7}^{ve}) &= \frac{1}{64} [\mathbf{A}(\mathbf{W}^2 \mathbf{A} \mathbf{W} - \mathbf{W} \mathbf{A} \mathbf{W}^2) + (\mathbf{W}^2 \mathbf{A} \mathbf{W} - \mathbf{W} \mathbf{A} \mathbf{W}^2) \mathbf{A} - \mathbf{W}(\mathbf{A} \mathbf{W} \mathbf{A}^2 - \mathbf{A}^2 \mathbf{W} \mathbf{A}) \\ &\quad - (\mathbf{A} \mathbf{W} \mathbf{A}^2 - \mathbf{A}^2 \mathbf{W} \mathbf{A}) \mathbf{W} + \mathbf{W} \mathbf{A}^2 \mathbf{W}^2 - \mathbf{W}^2 \mathbf{A}^2 \mathbf{W} + \mathbf{A} \mathbf{W}^2 \mathbf{A}^2 - \mathbf{A}^2 \mathbf{W}^2 \mathbf{A}]. \end{aligned} \quad (13.361)$$

In this case the back-stress for each element is given by an expression of the form

$$\boldsymbol{\tau}_s^{veT} = \rho \left\{ \frac{\partial \phi_s^{ve}}{\partial I_{s1}^{ve}} \mathbf{F}_s^{veT} + \frac{\partial \phi_s^{ve}}{\partial I_{s2}^{ve}} (\mathbf{F}_s^{ve} + \mathbf{F}_s^{veT}) \mathbf{F}_s^{veT} + \dots \right\}. \quad (13.362)$$

13.6.2 Coupled elastic and viscoelastic model

A general coupled initially isotropic model for free energy will be given by an isotropic function as described at the start of this section. A simpler coupled model can be constructed by assuming the free energy to be a function of the form

$$\phi_s(t) = \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e(t), \mathbf{C}_s^{ve}(t), \theta(t)], \quad (13.363)$$

where $\mathbf{F}_s^\theta$ is absent from the function and $\mathbf{F}_s^{ve}$ only contributes through its symmetric factor. One might argue that this is the desired form of the free energy since the energy stored in the elastic and back-stress “springs” should contribute to the free energy through the associated measures of strain, not the associated deformation gradients.

As a result of the constraint imposed by the initial isotropy, we can rewrite this as

$$\phi_s(t) = \check{\phi}_s^{\dagger\dagger}[\mathbf{X}, I_{s1}, \dots, I_{s10}, \theta], \quad (13.364)$$

where $I_{s1} = \text{tr}(\mathbf{C}_s^e)$, $I_{s2} = \text{tr}(\mathbf{C}_s^{e2})$, $I_{s3} = \text{tr}(\mathbf{C}_s^{e3})$, $I_{s4} = \text{tr}(\mathbf{C}_s^{ve})$, $I_{s5} = \text{tr}(\mathbf{C}_s^{ve2})$, $I_{s6} = \text{tr}(\mathbf{C}_s^{ve3})$, $I_{s7} = \text{tr}(\mathbf{C}_s^e \mathbf{C}_s^{ve})$, $I_{s8} = \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}_s^{ve})$, $I_{s9} = \text{tr}(\mathbf{C}_s^e \mathbf{C}_s^{ve2})$, and $I_{s10} = \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}_s^{ve2})$. The expressions for the thermodynamic stresses then can be written as

$$\boldsymbol{\tau}_s^{eT} = \rho \sum_{i=1}^{10} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^e} (I_{si}) \mathbf{F}_s^{eT}, \quad (13.365)$$

$$\boldsymbol{\tau}_s^{veT} = \rho \sum_{i=1}^{10} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^{ve}} (I_{si}) \mathbf{F}_s^{veT}, \quad (13.366)$$

$$\boldsymbol{\tau}_s^{\theta T} = \rho \sum_{i=1}^{10} \frac{\partial \phi_s}{\partial I_{si}} \partial_{\mathbf{F}_s^\theta} (I_{si}) \mathbf{F}_s^{\theta T} = \mathbf{0}. \quad (13.367)$$

Using the partial derivatives

$$\begin{aligned} \partial_{\mathbf{F}_s^e} (I_{s1}) &= 2\mathbf{F}_s^e, & \partial_{\mathbf{F}_s^e} (I_{s2}) &= 4\mathbf{F}_s^e \mathbf{C}_s^e, & \partial_{\mathbf{F}_s^e} (I_{s3}) &= 6\mathbf{F}_s^e \mathbf{C}_s^{e2}, \\ \partial_{\mathbf{F}_s^e} (I_{s4}) &= \partial_{\mathbf{F}_s^e} (I_{s5}) = \partial_{\mathbf{F}_s^e} (I_{s6}) = 0, \\ \partial_{\mathbf{F}_s^e} (I_{s7}) &= 2\mathbf{F}_s^e \mathbf{C}_s^{ve} \mathbf{F}_s^{eT}, & \partial_{\mathbf{F}_s^e} (I_{s8}) &= 2\mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve} + \mathbf{C}_s^{ve} \mathbf{C}_s^e) \mathbf{F}_s^{eT}, \\ \partial_{\mathbf{F}_s^e} (I_{s9}) &= 2\mathbf{F}_s^e \mathbf{C}_s^{ve2} \mathbf{F}_s^{eT}, & \partial_{\mathbf{F}_s^e} (I_{s10}) &= 2\mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve2} + \mathbf{C}_s^{ve2} \mathbf{C}_s^e) \mathbf{F}_s^{eT}, \end{aligned} \quad (13.368)$$

we can conclude that

$$\begin{aligned} \boldsymbol{\tau}_s^{eT} = \rho \left[2 \frac{\partial \phi_s}{\partial I_{s1}} \mathbf{B}_s^e + 4 \frac{\partial \phi_s}{\partial I_{s2}} \mathbf{B}_s^{e2} + 6 \frac{\partial \phi_s}{\partial I_{s3}} \mathbf{B}_s^{e3} \right. \\ + 2 \frac{\partial \phi_s}{\partial I_{s7}} \mathbf{F}_s^e \mathbf{C}_s^{ve} \mathbf{F}_s^{eT} + 2 \frac{\partial \phi_s}{\partial I_{s8}} \mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve} + \mathbf{C}_s^{ve} \mathbf{C}_s^e) \mathbf{F}_s^{eT} \\ \left. + 2 \frac{\partial \phi_s}{\partial I_{s9}} \mathbf{F}_s^e \mathbf{C}_s^{ve2} \mathbf{F}_s^{eT} + 2 \frac{\partial \phi_s}{\partial I_{s10}} \mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve2} + \mathbf{C}_s^{ve2} \mathbf{C}_s^e) \mathbf{F}_s^{eT} \right]. \end{aligned} \quad (13.369)$$

It seems plausible to assume that the $\boldsymbol{\tau}_s^e$ becomes zero for pure rigid body motions so that we would require that

$$\left\{ \left[\frac{\partial \phi_s}{\partial I_{s1}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 2 \left[\frac{\partial \phi_s}{\partial I_{s2}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 3 \left[\frac{\partial \phi_s}{\partial I_{s3}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} \right\} \mathbf{I} + \left\{ \left[\frac{\partial \phi_s}{\partial I_{s7}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 2 \left[\frac{\partial \phi_s}{\partial I_{s8}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} \right\} \mathbf{C}^{ve} + \left\{ \left[\frac{\partial \phi_s}{\partial I_{s9}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} + 2 \left[\frac{\partial \phi_s}{\partial I_{s10}} \right]_{\mathbf{F}_s^e = \mathbf{Q}} \right\} \mathbf{C}_s^{ve2} = 0, \quad (13.370)$$

for all orthogonal $\mathbf{Q}$. One way to satisfy this is to have each of the expressions in the curly brackets be equal to zero. The expression for Cauchy stress is given by integrating $\boldsymbol{\tau}_s^e$ over s to obtain

$$\begin{aligned} \mathbf{T}^T(t) = \rho \int_0^\infty & \left[2 \frac{\partial \phi_s}{\partial I_{s1}} \mathbf{B}_s^e + 4 \frac{\partial \phi_s}{\partial I_{s2}} \mathbf{B}_s^{e2} + 6 \frac{\partial \phi_s}{\partial I_{s3}} \mathbf{B}_s^{e3} \right. \\ & + 2 \frac{\partial \phi_s}{\partial I_{s7}} \mathbf{F}_s^e \mathbf{C}_s^{ve} \mathbf{F}_s^{eT} + 2 \frac{\partial \phi_s}{\partial I_{s8}} \mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve} + \mathbf{C}_s^{ve} \mathbf{C}_s^e) \mathbf{F}_s^{eT} \\ & \left. + 2 \frac{\partial \phi_s}{\partial I_{s9}} \mathbf{F}_s^e \mathbf{C}_s^{ve2} \mathbf{F}_s^{eT} + 2 \frac{\partial \phi_s}{\partial I_{s10}} \mathbf{F}_s^e (\mathbf{C}_s^e \mathbf{C}_s^{ve2} + \mathbf{C}_s^{ve2} \mathbf{C}_s^e) \mathbf{F}_s^{eT} \right] ds. \quad (13.371) \end{aligned}$$

In this case the back-stress in the viscoelastic element can be calculated using the partial derivatives

$$\begin{aligned} \partial_{\mathbf{F}_s^{ve}}(I_{s1}) &= \partial_{\mathbf{F}_s^{ve}}(I_{s2}) = \partial_{\mathbf{F}_s^{ve}}(I_{s3}) = 0, \\ \partial_{\mathbf{F}_s^{ve}}(I_{s4}) &= 2\mathbf{F}_s^{ve}, \quad \partial_{\mathbf{F}_s^{ve}}(I_{s5}) = 4\mathbf{F}_s^{ve} \mathbf{C}_s^{ve}, \quad \partial_{\mathbf{F}_s^{ve}}(I_{s6}) = 6\mathbf{F}_s^{ve} \mathbf{C}_s^{ve2}, \\ \partial_{\mathbf{F}_s^{ve}}(I_{s7}) &= 2\mathbf{F}_s^{ve} \mathbf{C}_s^e \mathbf{F}_s^{veT}, \quad \partial_{\mathbf{F}_s^{ve}}(I_{s8}) = 2\mathbf{F}_s^{ve} \mathbf{C}_s^{e2} \mathbf{F}_s^{ve}, \\ \partial_{\mathbf{F}_s^{ve}}(I_{s9}) &= 2\mathbf{F}_s^{ve} (\mathbf{C}_s^e \mathbf{C}_s^{ve} + \mathbf{C}_s^{ve} \mathbf{C}_s^e) \mathbf{F}_s^{veT}, \quad \partial_{\mathbf{F}_s^{ve}}(I_{s10}) = 2\mathbf{F}_s^{ve} (\mathbf{C}_s^e \mathbf{C}_s^{ve2} + \mathbf{C}_s^{ve2} \mathbf{C}_s^e) \mathbf{F}_s^{veT}, \end{aligned} \quad (13.372)$$

to get

$$\begin{aligned} \boldsymbol{\tau}_s^{veT} = \rho & \left[2 \frac{\partial \phi_s}{\partial I_{s4}} \mathbf{B}_s^{ve} + 4 \frac{\partial \phi_s}{\partial I_{s5}} \mathbf{B}_s^{ve2} + 6 \frac{\partial \phi_s}{\partial I_{s6}} \mathbf{B}_s^{ve3} + 2 \frac{\partial \phi_s}{\partial I_{s7}} \mathbf{F}_s^{ve} \mathbf{C}_s^e \mathbf{F}_s^{veT} \right. \\ & + 2 \frac{\partial \phi_s}{\partial I_{s8}} \mathbf{F}_s^{ve} \mathbf{C}_s^{e2} \mathbf{F}_s^{veT} + 2 \frac{\partial \phi_s}{\partial I_{s9}} \mathbf{F}_s^{ve} (\mathbf{C}_s^e \mathbf{C}_s^{ve} + \mathbf{C}_s^{ve} \mathbf{C}_s^e) \mathbf{F}_s^{veT} \\ & \left. + 2 \frac{\partial \phi_s}{\partial I_{s10}} \mathbf{F}_s^{ve} (\mathbf{C}_s^e \mathbf{C}_s^{ve2} + \mathbf{C}_s^{ve2} \mathbf{C}_s^e) \mathbf{F}_s^{veT} \right]. \quad (13.373) \end{aligned}$$

One also might add the requirement that the back-stress be zero at zero viscoelastic strain. This provides a set of relations between the partial derivatives of the free energy for every $\mathbf{F}^{ve} = \mathbf{Q}$.

Section 13.6-Problems:

13.6-1: Construct the representation of the thermodynamic stresses for an initially isotropic material given by a free energy of the form

$$\phi_s(t) = \phi_s^{\dagger\dagger}[\mathbf{X}, \mathbf{C}_s^e(t), \mathbf{F}_s^{ve}(t), \theta(t)]. \quad (13.374)$$

13.6-2: As has been noted, the invariants are not unique. It might be desirable to select an invariant set that better separates the different effects. One such set

is described by the then isotropic invariants of $\mathbf{C}_s^e$ and $\mathbf{C}_s^p$ given by

$$\begin{aligned} K_{s1} &= J_s^e, & K_{s2} &= \frac{\text{tr}(\mathbf{C}_s^e)}{J_s^{e2/3}}, & K_{s3} &= \frac{\text{tr}(\mathbf{C}_s^{e2})}{\text{tr}^2(\mathbf{C}_s^e)}, \\ K_{s4} &= J_s^p, & K_{s5} &= \frac{\text{tr}(\mathbf{C}_s^p)}{J_s^{p2/3}}, & K_{s6} &= \frac{\text{tr}(\mathbf{C}_s^{p2})}{\text{tr}^2(\mathbf{C}_s^p)}, \\ K_{s7} &= I_{s7} - I_{s1} - I_{s4} + 3, & K_{s8} &= I_{s8} - I_{s2} - I_{s4} + 3, \\ K_{s9} &= I_{s9} - I_{s1} - I_{s5} + 3, & K_{s10} &= I_{s10} - I_{s2} - I_{s5} + 3. \end{aligned} \quad (13.375)$$

Use this set of invariants to construct a model for an initially isotropic material that assumes the free energy is given by

$$\phi_s(t) = \phi_s^{\dagger}[\mathbf{X}, K_s(t), \dots, K_{s10}(t), \theta(t)]. \quad (13.376)$$

13.7 Quasi-linear viscoelastic model

We conclude the models in this chapter by looking at the quasi-linear model that even though is not directly related to the theme of the book, never the less has substantial use for modeling biological materials and lends itself to using some of the tools that we have developed. In addition, it provides us with a different point of view for applying some of the methods and tools.

Let us start the process by considering, for a general material, the response during relaxation after a constant thermomechanical step. We assume the material before the step is in a static base state, that is, undeformed, stress free, etc., and that a step of deformation and temperature is applied at time zero. For a fairly large class of materials, we can write the free energy of the material, after the application of the step, as a function of the applied constant deformation $\mathbf{F}$, constant $\mathbf{G}$, constant temperature θ , and varying time t as

$$\psi(t) = \psi^{\dagger}(\mathbf{F}, \mathbf{G}, \theta, t). \quad (13.377)$$

Note that $\psi^{\dagger}$ is not the general response function for ψ , it is only the response to a single step history. A similar relation can be assumed for entropy, Cauchy stress, and heat flux. Now, since the deformation gradient, temperature gradient, and temperature are constants after the step, we have $\dot{\mathbf{L}} = \mathbf{0}$, $\dot{\mathbf{G}} = \mathbf{0}$, and $\dot{\theta} = 0$ at all times after the step. Assuming uniform temperature fields are always an option, from the entropy production inequality (6.69) we have to conclude that for all physically possible thermomechanical relaxations we will have

$$\dot{\psi}(t) = \frac{\partial \psi^{\dagger}(\mathbf{F}, \mathbf{G}, \theta, t)}{\partial t} \leq 0 \quad (13.378)$$

for all times after the step. Therefore, all thermomechanical relaxations are free-energy reducing after the application of the step. On the other hand, the step itself introduces an instantaneous jump in the free energy at time zero that we can write as

$$\psi(0+) = \psi^{\dagger}(\mathbf{F}, \mathbf{G}, \theta, 0). \quad (13.379)$$

The instantaneous response due to the jump is given by a function that resembles the model for a nonlinear thermoelastic material. We will call it the instantaneous elastic response. Therefore, we have argued that the entropy production inequality states that the

instantaneous elastic jump in free energy always drops during the following relaxation. Yet, we have not said anything about the jump itself. If we assume that the instantaneous elastic response functions are continuous in their arguments, and that the jump can be considered an elastic extension, albeit over a very small time interval, one could force oneself to accept that the step loading results in relations between the instantaneous elastic response functions that are identical to what we got for thermodynamically based elasticity. If we note the instantaneous elastic response functions with superscript “e” and subscript “o,” we have argued ourselves into accepting that

$$\psi(0+) = \psi_o^{e\dagger}(\mathbf{F}, \mathbf{G}, \theta) = \psi^\dagger(\mathbf{F}, \mathbf{G}, \theta, 0), \quad (13.380)$$

$$\eta(0+) = \eta_o^{e\dagger}(\mathbf{F}, \mathbf{G}, \theta) = \eta^\dagger(\mathbf{F}, \mathbf{G}, \theta, 0), \quad (13.381)$$

$$\mathbf{T}(0+) = \mathbf{T}_o^{e\dagger}(\mathbf{F}, \mathbf{G}, \theta) = \mathbf{T}^\dagger(\mathbf{F}, \mathbf{G}, \theta, 0), \quad (13.382)$$

$$\mathbf{q}(0+) = \mathbf{q}_o^{e\dagger}(\mathbf{F}, \mathbf{G}, \theta) = \mathbf{q}^\dagger(\mathbf{F}, \mathbf{G}, \theta, 0), \quad (13.383)$$

with the relations

$$\mathbf{T}^T(0+) = \rho \partial_{\mathbf{F}}[\psi(0+)] \mathbf{F}^T, \quad (13.384)$$

$$\eta(0+) = -\partial_\theta[\psi(0+)], \quad (13.385)$$

$$\partial_{\mathbf{G}}[\psi(0+)] = \mathbf{0}, \quad (13.386)$$

$$\frac{1}{\theta} \mathbf{q}(0+) \circ \mathbf{g}(0+) \leq 0. \quad (13.387)$$

Therefore, the inclusion of the temperature gradient $\mathbf{G}$ in the free energy, entropy, and stress is not needed.

Independently, we could argue that deforming the body in a way that rigidly rotates the material before applying the same strains along the same material directions should not influence the free energy. Standard arguments, as we have made before, bring us to the realization that we could write the free energy in terms of the right Cauchy stretch tensor as

$$\psi(t) = \psi^{\dagger\dagger}(\mathbf{C}, \theta, \mathbf{G}, t). \quad (13.388)$$

The same argument could be made for the entropy. A slight variation, with similar arguments as used before, provides us with the results that

$$\mathbf{T}(t) = \mathbf{R} \mathbf{T}^{R\dagger\dagger}(\mathbf{C}, \theta, \mathbf{G}, t) \mathbf{R}^T, \quad (13.389)$$

$$\mathbf{q}(t) = \mathbf{R} \mathbf{q}^{R\dagger\dagger}(\mathbf{C}, \theta, \mathbf{G}, t), \quad (13.390)$$

where the superscript “R,” as before, denotes the rigid body motion removed version of the variable.

We also could argue that if we have any symmetry in the material response, and if the symmetries are characterized by transformations $\mathbf{M}$ of the reference configuration, the free energy should be the same for two-step deformations that are only different because for one we have first done a symmetry transformation before the deformation. Without providing the details, this suggests that the function for the free energy should have the property

$$\psi^{\dagger\dagger}(\mathbf{C}, \theta, \mathbf{G}, t) = \psi^{\dagger\dagger}(\mathbf{M}^T \mathbf{C} \mathbf{M}, \theta, \mathbf{G} \mathbf{M}, t). \quad (13.391)$$

So, without much restriction, we can state that, for a fairly general material, the free energy during a constant thermomechanical step (let us call it a thermomechanical stress relaxation) always will have a free energy of the form

$$\psi(t) = \psi^{\dagger\dagger\dagger}(I_1, \dots, I_n, \theta, t), \quad (13.392)$$

where I_i are an appropriate set of scalar invariants, associated with the material symmetry restriction, of the tensor $\mathbf{C}$ and the vector $\mathbf{G}$. For example, if the material was isotropic, and the symmetries were characterized by all the orthogonal transformation of the reference configuration, then the free energy could be written in terms of the isotropic invariants of $\mathbf{C}$ and $\mathbf{G}$, temperature, and time. One could make the similar argument for entropy, rotated stress, and rotated heat flux.

In particular, let us ignore the dependence of free energy, stress, and entropy on the temperature gradient, as suggested by the entropy production inequality. This simplifies the representations, and possibly is justifiable. Let us look at an isotropic material, and let us assume that the reference configuration is selected appropriately such that the symmetry is given by all orthogonal transformation of this configuration. The representations are not unique, but they tell us that we can write the response functions as

$$\psi(t) = \psi^{\dagger\dagger\dagger}(I_1, I_2, I_3, \theta, t), \quad (13.393)$$

$$\eta(t) = \eta^{\dagger\dagger\dagger}(I_1, I_2, I_3, \theta, t), \quad (13.394)$$

$$\mathbf{T}(t) = \mu_1 \mathbf{I} + \mu_2 \mathbf{B} + \mu_3 \mathbf{B}^2, \quad (13.395)$$

$$\mathbf{q}(t) = k_1 \mathbf{g} + k_2 \mathbf{Vg} + k_3 \mathbf{Bg}, \quad (13.396)$$

where μ_i and k_i are scalar functions, respectively, of “ $(I_1, I_2, I_3, \theta, t)$ ” and “ $(I_1, \dots, I_6, \theta, t)$,” for the isotropic invariants

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}), & I_2 &= \text{tr}(\mathbf{C}^2), & I_3 &= \text{tr}(\mathbf{C}^3), \\ I_4 &= \mathbf{G} \circ \mathbf{G}, & I_5 &= \mathbf{G} \circ (\mathbf{CG}), & I_6 &= \mathbf{G} \circ (\mathbf{C}^2 \mathbf{G}). \end{aligned} \quad (13.397)$$

Similar expressions can be obtained for different material symmetry conditions. Since $\mathbf{F}$ and $\mathbf{G}$ are fixed after the step, what we have concluded is that for fairly general material types, the stress and heat flux are given in terms of fixed directions, but with coefficients that are varying with time. We also have argued that the initial values of the scalar function for the entropy, and the scalar functions for the stress are given by the instantaneous elastic response.

Here is where we take one additional leap of faith. We assume that the time dependence in the μ_i can be separated from the dependence on the step deformation and temperature. That is, we assume

$$\mu_i(I_1, I_2, I_3, \theta, t) = \mu_i^e(I_1, I_2, I_3, \theta) G_i(t). \quad (\text{no sum on } i) \quad (13.398)$$

That is, we assume the size of the step deformation and temperature provides the initial value of the coefficients, but that the size of the deformation and temperature provides no influence on the relaxation process, which is given by the relaxation functions $G_i(t)$. For this to work, the relaxation functions have unit value at the initial time. In general, for a given symmetry, we can write the Cauchy stress as

$$\mathbf{T}(t) = \sum_{i=1}^n \mu_i^e(I_1, \dots, I_n, \theta) G_i(t) \mathbf{J}_i, \quad (\text{no sum on } i) \quad (13.399)$$

where $\mathbf{J}_i$ are the known tensor functions of the step deformation. For example, for the isotropic material, we have $\mathbf{J}_1 = \mathbf{I}$, $\mathbf{J}_2 = \mathbf{B}$, $\mathbf{J}_3 = \mathbf{B}^2$.

To get the final result, we need to take another leap of faith. We assume that the response for multiple steps adds quasi-linearly. This also gives the justification for the model's name. The assumption here is that the new step produces an instantaneous response equal to the difference between the instantaneous elastic response of the first and subsequent step, with relaxation functions that are shifted to start at the time of the application of the new step.

This guarantees that even multiple nonlinear steps taken at the same time produce the same total result. We then extend this to continuous functions by allowing a limit to small steps when needed and obtain the equation

$$\mathbf{T}(t) = \sum_{i=1}^n \int_{t_o}^t G_i(t-s) d\mathbf{S}_i(s), \quad (13.400)$$

where

$$\mathbf{S}_i(\mathbf{F}, \theta) = \mu_i^e(I_1, \dots, I_n, \theta) \mathbf{J}_i. \quad (\text{no sum on } i) \quad (13.401)$$

In the event that the loading is continuous, up to m step jumps $\Delta \mathbf{S}_{ij}$ in $\mathbf{S}_i$, the model can be written as

$$\mathbf{T}(t) = \sum_{i=1}^n \int_{t_o}^t G_i(t-s) \dot{\mathbf{S}}_i(s) ds + \sum_{j=1}^m \sum_{i=1}^n G_i(t-t_j) \Delta \mathbf{S}_{ij}. \quad (13.402)$$

Change of variables and integration by parts over the continuous intervals can be used to rewrite this as

$$\mathbf{T}(t) = \sum_{i=1}^n \mathbf{S}_i(t) + \sum_{i=1}^n \int_{t_o}^t \dot{G}_i(t-s) \mathbf{S}_i(s) ds. \quad (13.403)$$

Even though one can find minimal bases for the isotropic invariants, they are not unique. So, for example, a different representation of the isotropic response would result in $\mathbf{J}_1 = \mathbf{I}$, $\mathbf{J}_2 = \mathbf{B}$, and $\mathbf{J}_3 = \mathbf{B}^{-1}$, or we could have $\mathbf{J}_1 = \mathbf{I}$, $\mathbf{J}_2 = (\mathbf{B} - \frac{I_2}{3} \mathbf{I})$, $\mathbf{J}_3 = (\mathbf{B}^2 - \frac{I_2}{I_1} \mathbf{B})$. Even if the deformation and temperature part of $\mathbf{S}_i$ may be separable from the time-dependent part G_i in one representation, this fact could be used to show that the separation could not be obtained for another representation. That is, the assumption of the separation in one representation precludes the existence of a separation in another representation. Therefore, the application of the quasi-linear model for a particular material response might work for one representation, but not for another. For example, if the volumetric relaxations are very different from the deviatoric relaxations, then possibly the choice $\mathbf{J}_1 = \mathbf{I}$, $\mathbf{J}_2 = (\mathbf{B} - \frac{I_2}{3} \mathbf{I})$, $\mathbf{J}_3 = (\mathbf{B}^2 - \frac{I_2}{I_1} \mathbf{B})$ is a better one since it separates the hydrostatic and deviatoric parts.

Getting back to the instantaneous elastic response, since we take all the relaxation functions to be unity at time zero, then the instantaneous elastic response is given by

$$\mathbf{T}_o^e = \sum_{i=1}^n \mathbf{S}_i \quad (13.404)$$

for the current values of $\mathbf{S}_i$, which are calculated from the current values of the deformation gradient and temperature. Note that the arguments led us to a thermodynamic representation for the instantaneous elastic response as

$$\mathbf{T}_o^e = \sum_{i=1}^n \rho \frac{\partial \psi(0+)}{\partial I_i} \partial_{\mathbf{F}}(I_i) \mathbf{F}^T, \quad (13.405)$$

which suggest taking $\mathbf{S}_i$ as the terms in the summation. Also, we have not assumed any particular way to satisfy thermal expansion, and consequently the point of zero stress for the thermoelastic response. As before, we might separate the deformation into an elastic and a thermal deformation gradient with the property $\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta$ and let the free energy and the stress only depend on the elastic part, and on temperature. And for convenience, we can call $\psi(0+)$ the instantaneous elastic free energy, and denote it as ψ^e . This would provide a natural way to accommodate for this and would result in the modified expression

$$\mathbf{T}_o^e = \sum_{i=1}^n \rho \frac{\partial \psi^e}{\partial I_i^e} \partial_{\mathbf{F}^e}(I_i^e) \mathbf{F}^{eT}. \quad (13.406)$$

The quasi-linear viscoelastic response represented in equation (13.403) now can be written as

$$\mathbf{T}(t) = \mathbf{T}_o^e(t) + \sum_{i=1}^n \int_{t_o}^t \dot{G}_i(t-s) \mathbf{S}_i(s) ds. \quad (13.407)$$

Section 13.7-Problems:

13.7-1: Consider an isotropic quasi-linear viscoelastic material defined by an elastic free energy of

$$\psi^e = \frac{\kappa_o J^\theta}{3\rho_o} \left[\frac{J^{e3} - 1}{3} - \ln(J^e) \right] + \frac{G_o J^\theta}{2\rho_o} \left(\frac{I_1^e}{J^{e2/3}} - 3 \right) + \psi^\theta, \quad (13.408)$$

where κ_o , G_o , and ψ^θ are only a function of temperature, $\mathbf{F}^e = \frac{\mathbf{F}}{J^{e1/3}}$, $\mathbf{C}^e = \mathbf{F}^{eT} \mathbf{F}^e$, $J^e = \det(\mathbf{F}^e)$, $I_1^e = \text{tr}(\mathbf{C}^e)$, and $j^\theta = \beta J^\theta \dot{\theta}$, where β is the volumetric coefficient of thermal expansion, that only depends on temperature. Show that this results in the expression for stress given by

$$\mathbf{T}_o^e = \frac{\kappa_o}{3} \left(J^{e2} - \frac{1}{J^e} \right) \mathbf{I} + \frac{G_o}{J^{\theta5/3}} \left(\mathbf{B}^e - \frac{I_1^e}{3} \mathbf{I} \right), \quad (13.409)$$

where κ_o is the bulk modulus at $J^e = 1$ calculated relative to changes in J^e at the given temperature, and G_o is the shear modulus at the given temperature. Assume that

$$\mathbf{S}_1 = \frac{\kappa_o}{3} \left(J^{e2} - \frac{1}{J^e} \right) \mathbf{I}, \quad (13.410)$$

$$\mathbf{S}_2 = \frac{G_o}{J^{e5/3}} \left(\mathbf{B}^e - \frac{I_1^e}{3} \mathbf{I} \right), \quad (13.411)$$

and construct a quasi-linear viscoelastic model with relaxation moduli

$$G_1 = e^{-\frac{t}{t_\kappa}}, \quad (13.412)$$

$$G_2 = e^{-\frac{t}{t_G}}, \quad (13.413)$$

where t_κ is the relaxation time associated with the bulk relaxation and t_G is the relaxation time associated with the shear relaxation.

13.8 Material constraints

The process of constructing the influence of the material constraints, such as inextensibility, follows similar paths as described in the chapters for thermoelasticity and thermoplasticity. As such, the reader is directed to these chapters for specific details. We consider a constraint to be described by a general constraint equation of the form

$$\boxed{C(\mathbf{F}, \theta, \mathbf{G}) = 0}, \quad (13.414)$$

which results in

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + \partial_\theta(C) \dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} = 0, \quad (13.415)$$

that using $\mathcal{U} \equiv \{\mathbf{F}, \theta, \mathbf{G}\}$ and $\dot{\mathcal{U}} \equiv \{\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}}\}$ can be written as

$$\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}} = 0. \quad (13.416)$$

Since $\dot{\mathcal{U}}$ represents a loading rate, the constraint requires that an admissible loading rate be such that its projection along $\partial_{\mathcal{U}}(C)$ be zero. We can construct any arbitrary loading rate $\dot{\mathcal{U}}^*$ from an admissible loading rate $\dot{\mathcal{U}}$ by adding an arbitrary amount along $\partial_{\mathcal{U}}(C)$ so that

$$\dot{\mathcal{U}}^* = \dot{\mathcal{U}} + \alpha \partial_{\mathcal{U}}(C), \quad (13.417)$$

where α is an arbitrary scalar. Introducing this into the constraint equation, we can solve for α to get

$$\dot{\mathcal{U}}^* \equiv \dot{\mathcal{U}} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathcal{U}}(C), \quad (13.418)$$

which in “component form” can be written as

$$\mathbf{L}^* = \mathbf{L} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathbf{F}}(C) \mathbf{F}^{-1}, \quad (13.419)$$

$$\dot{\theta}^* = \dot{\theta} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\theta}(C), \quad (13.420)$$

$$\dot{\mathbf{G}}^* = \dot{\mathbf{G}} + \frac{\partial_{\mathcal{U}}(C) \circ \dot{\mathcal{U}}^*}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \partial_{\mathbf{G}}(C). \quad (13.421)$$

Let us construct the arguments for a single-element viscoelastic model. The results will be similar for the integral model. In a constrained material, we need more information to calculate the response than just the history of the deformation gradient and temperature. As before, not knowing what the form of the indeterminacy is going to be, we add a scalar p to the response functions to represent the added information needed to calculate the response, and later show that its form can be evaluated knowing the constraint. For the single-element viscoelastic model we will assume the free energy is given by

$$\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta, \mathbf{G}, p), \quad (13.422)$$

and will assume similar forms for the other response functions, but taking the viscoelastic flow rule given by

$$\mathbf{L}^{ve} = \mathbf{L}^{ve\dagger}(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta, p). \quad (13.423)$$

Putting this into the Clausius-Duhem inequality we get

$$\begin{aligned} & -\Delta \mathbf{T}^{eT} : \mathbf{L} + [\rho \eta + \rho \partial_{\theta}(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} + \rho \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} \\ & + \rho \partial_p(\psi) \dot{p} - \Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (13.424)$$

where we have defined the thermodynamic stresses as

$$\mathbf{T}^{eT} \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}, \quad (13.425)$$

$$\mathbf{T}^{veT} \equiv \rho \partial_{\mathbf{F}^{ve}}(\psi) \mathbf{F}^{veT}, \quad (13.426)$$

$$\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}, \quad (13.427)$$

and where the thermodynamic over stresses are defined as

$$\Delta \mathbf{T}^e \equiv \mathbf{T} - \mathbf{T}^e, \quad (13.428)$$

$$\Delta \mathbf{T}^{ve} \equiv \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e - \mathbf{T}^{ve}, \quad (13.429)$$

$$\Delta \mathbf{T}^\theta \equiv \mathbf{F}^{p-1} \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e \mathbf{F}^p - \mathbf{T}^\theta. \quad (13.430)$$

Before we proceed, let us note that without a constraint $\Delta \mathbf{T}^e = \mathbf{0}$ since $\mathbf{T}$ becomes equal to $\mathbf{T}^e$, which is not the case here. If we now substitute for $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$ from, respectively, (13.419), (13.420), and (13.421), we get

$$\begin{aligned} & [-\Delta \mathbf{T}^{eT} + \varpi \partial_{\mathbf{F}}(C) \mathbf{F}^T] : \mathbf{L}^* \\ & + [\rho \eta + \rho \partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \varpi \partial_\theta(C)] \dot{\theta}^* \\ & + [\rho \partial_{\mathbf{G}}(\psi) + \varpi \partial_{\mathbf{G}}(C)] \circ \dot{\mathbf{G}}^* \\ & + \rho \partial_p(\psi) \dot{p} - \Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \end{aligned} \quad (13.431)$$

where

$$\begin{aligned} \varpi = \frac{1}{\partial_{\mathcal{U}}(C) \circ \partial_{\mathcal{U}}(C)} \{ & \Delta \mathbf{T}^{eT} : [\partial_{\mathbf{F}}(C) \mathbf{F}^{-1}] \\ & - [\rho \eta + \rho \partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \partial_\theta(C) - \rho \partial_{\mathbf{G}} \psi \circ \partial_{\mathbf{G}}(C) \}. \end{aligned} \quad (13.432)$$

The development is very much identical to those given in thermoelasticity and thermoplasticity. To extract the equations, we note that the term $-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g}$ does not depend on the loading rate $\dot{\mathcal{U}}$, $\dot{\mathcal{U}}^*$, or $\dot{p}$. Since $\dot{\mathcal{U}}^*$ and $\dot{p}$ can be selected arbitrarily, we have to conclude, through arguments that are similar to those given before, that we must have

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} + \varpi \partial_{\mathbf{F}}(C) \mathbf{F}^T, \quad (13.433)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} - \frac{\varpi}{\rho} \partial_\theta(C), \quad (13.434)$$

$$\partial_{\mathbf{G}}(\psi) = -\frac{\varpi}{\rho} \partial_{\mathbf{G}}(C), \quad (13.435)$$

$$\partial_p(\psi) = 0, \quad (13.436)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (13.437)$$

Examining the equations one immediately notes that the free energy cannot depend on the indeterminant parameter p . Further examination of the terms, noting that ψ does not depend on p , reveals that ϖ is the only term on the right-hand side in the expression for stress and for entropy that can depend on p , which it inherits through the possible dependence of Cauchy stress and entropy on p . Also, one notes that the free energy cannot depend on the temperature gradient if the constraint does not, which represents the most common form of the constraint. We next assume that homogeneous temperature fields always are admissible and, therefore, conclude that

$$-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0. \quad (13.438)$$

One may note that in this model the parameter p can influence the viscoelastic flow. For example, the viscosity of an incompressible material can depend on pressure.

The results for a general model with a continuous series of elements in parallel are similar.

13.9 Models based on $\mathbf{F} = \mathbf{F}_s^e \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve}$

The assumptions we used to create the viscoelasticity theory have a special structure that relates the models formulated based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^\theta$ and models based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve}$. To see this relation consider having two theories that start from similar assumptions, but one based on the former decomposition and the other is based on the latter. For each we would have a flow rule and thermal expansion rule. The flow rules for $\mathbf{L}_s^{ve}$ and $\bar{\mathbf{L}}_s^{ve}$ would each be given by the associated variables defining the state, but would not depend on the rates. The thermal expansions would be given by

$$\mathbf{L}_s^\theta = \alpha_s \dot{\theta}, \quad (13.439)$$

$$\bar{\mathbf{L}}_s^\theta = \bar{\alpha}_s \dot{\theta}, \quad (13.440)$$

where, again, α_s and $\bar{\alpha}_s$ would each be given by the associated variables defining the state, but would not depend on the rates. If we now consider the relation $\mathbf{F}_s^{ve} \mathbf{F}_s^\theta = \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve}$ and take its time derivative, we would get

$$\dot{\mathbf{F}}_s^{ve} \mathbf{F}_s^\theta + \mathbf{F}_s^{ve} \dot{\mathbf{F}}_s^\theta = \dot{\bar{\mathbf{F}}}_s^\theta \bar{\mathbf{F}}_s^{ve} + \bar{\mathbf{F}}_s^\theta \dot{\bar{\mathbf{F}}}_s^{ve}, \quad (13.441)$$

which could be rewritten as

$$\mathbf{L}_s^{ve} \mathbf{F}_s^{ve} \mathbf{F}_s^\theta + \mathbf{F}_s^{ve} \mathbf{L}_s^\theta \mathbf{F}_s^\theta = \bar{\mathbf{L}}_s^\theta \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve} + \bar{\mathbf{F}}_s^\theta \bar{\mathbf{L}}_s^{ve} \bar{\mathbf{F}}_s^{ve}. \quad (13.442)$$

We can now introduce the thermal expansion rules to get

$$\mathbf{L}_s^{ve} \mathbf{F}_s^{ve} \mathbf{F}_s^\theta + \mathbf{F}_s^{ve} \alpha_s \dot{\theta} \mathbf{F}_s^\theta = \bar{\alpha}_s \dot{\theta} \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve} + \bar{\mathbf{F}}_s^\theta \bar{\mathbf{L}}_s^{ve} \bar{\mathbf{F}}_s^{ve}. \quad (13.443)$$

If we examine the terms, only the second term on the left side of the equality and the first term on the right side of the equality depend on the rate of loading. These terms only depend on the temperature rate. If we assume the temperature rate can be selected arbitrarily, then the relation can only be satisfied if the first term on the left equals the second term on the right, and the second term on the left equals the first term on the right. This gives

$$\mathbf{L}_s^{ve} \mathbf{F}_s^{ve} \mathbf{F}_s^\theta = \bar{\mathbf{F}}_s^\theta \bar{\mathbf{L}}_s^{ve} \bar{\mathbf{F}}_s^{ve}, \quad (13.444)$$

$$\mathbf{F}_s^{ve} \alpha_s \mathbf{F}_s^\theta = \bar{\alpha}_s \bar{\mathbf{F}}_s^\theta \bar{\mathbf{F}}_s^{ve} \quad (13.445)$$

and can be reorganized to get the relations

$$\bar{\alpha}_s = \mathbf{F}_s^{ve} \alpha_s \mathbf{F}_s^{ve-1}, \quad (13.446)$$

$$\bar{\mathbf{F}}_s^\theta \bar{\mathbf{L}}_s^{ve} \bar{\mathbf{F}}_s^{\theta-1} = \mathbf{L}_s^{ve}. \quad (13.447)$$

Examination of the relations brings us to the conclusion that if one constructs the thermal expansion and flow rule for one decomposition, then the values of the thermal expansion and flow rule for the other decomposition are uniquely defined. Since we can integrate these equations for each given loading history to get the corresponding thermal and viscoplastic deformation gradients, this result states that for each loading history the two decompositions are uniquely related. Therefore, models based on one decomposition will be closely related to models based on the other and would use state parameters that could be calculated, even though not algebraically, from each other.

13.10 Notation

The following table shows the symbols used in the text and their definitions.

ave		average (subscript)
$\mathbf{B}$		left Cauchy stretch tensor ($\mathbf{F}\mathbf{F}^T$)
$\mathbf{B}^e$		elastic left Cauchy stretch tensor ($\mathbf{F}^e\mathbf{F}^{eT}$)
$\mathbf{C}$		right Cauchy stretch tensor ($\mathbf{F}^T\mathbf{F}$)
$\mathbf{C}^e$		elastic right Cauchy stretch tensor ($\mathbf{F}^{eT}\mathbf{F}^e$)
$\mathbf{C}^{ve}$		viscoelastic flow coefficient
$\mathbf{C}_s^{ve}$		viscoelastic flow coefficient for element parameter s
dev		deviatoric (subscript)
$\Delta\mathbf{T}^\theta$		thermal overstress ($\mathbf{F}^{ve-1}\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^e\mathbf{F}^{ve} - \mathbf{T}^\theta$)
$\Delta\mathbf{T}^{ve}$		viscoelastic overstress ($\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^e - \mathbf{T}^{ve}$)
e		elastic (superscript)
$\mathbf{E}^e$		elastic Green strain
$\mathbf{E}^{ve}$		viscoelastic Green strain
$\mathbf{E}$		elastic modulus tensor
$\mathbf{E}_s$		elastic modulus tensor for element parameter s
$\mathbf{E}^{ve}$		viscoelastic back-stress modulus tensor
$\mathbf{E}_s^b$		viscoelastic back-stress modulus for element parameter s
$\mathbf{F}$		deformation gradient
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}_s^e$		elastic deformation gradient for element parameter s
$\mathbf{F}^\theta$		thermal deformation gradient
$\mathbf{F}_s^\theta$		thermal deformation gradient for element parameter s
$\mathbf{F}^{ve}$		viscoelastic deformation gradient
$\mathbf{F}_s^{ve}$		viscoelastic deformation gradient for element parameter s
$\mathbf{g}$		temperature gradient with respect to current configuration
$\mathbf{g}^{R^e}$		“elastic” rotated temperature gradient ($\mathbf{R}^{eT}\mathbf{g}$)
$\mathbf{G}$		temperature gradient with respect to reference configuration
$\mathbf{G}$		fourth-order viscoelastic relaxation modulus
G		elastic shear modulus
G^{ve}		viscoelastic back-stress shear modulus
G^e		elastic shear modulus
I_i		scalar invariant i
J		volume ratio ($det(\mathbf{F})$)
$\mathbf{J}_i$		base tensors for the representation of a tensor function
k		thermal conductivity
K_i		scalar invariant i
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}}\mathbf{F}^{-1}$)
$\mathbf{L}^\theta$		thermal velocity gradient ($\dot{\mathbf{F}}^\theta\mathbf{F}^{\theta-1}$)
$\mathbf{L}^{ve}$		viscoelastic velocity gradient ($\dot{\mathbf{F}}^{ve}\mathbf{F}^{ve-1}$)
$\mathbf{M}$		transformation tensor representing material symmetry
$\mathbf{P}$		transformation between reference frames
$\mathbf{q}$		heat flux vector
$\mathbf{q}^{R^e}$		“elastic” rotated heat flux vector ($\mathbf{R}^{eT}\mathbf{q}$)
$\mathbf{Q}$		orthogonal tensor representing rigid body motion
$\mathbf{R}$		orthogonal factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{R}^e$		orthogonal factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e$

s		parameter for representing continuous series of viscoelastic elements (subscript)
$\mathbf{S}$		deviatoric stress
$\mathbf{S}_i$		base deformation functions for quasi-linear viscoelastic model
t_R		relaxation time
$\mathbf{T}$		Cauchy stress
$\mathbf{T}_o^e$		instantaneous elastic response in a quasi-linear viscoelastic model
$\mathbf{T}^e$		elastic stress
T_{ave}		average stress
$\mathbf{U}$		right symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
$\mathbf{U}^e$		right symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e$
$\mathbf{V}$		left symmetric factor in polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$
ve		viscoelastic (superscript)
α		coefficient of thermal expansion
$\Delta\sigma_s^{ve}$		viscoelastic overstress for element parameter s
ϵ		infinitesimal strain
ϵ^e		infinitesimal elastic strain
ϵ_s^e		infinitesimal elastic strain for element parameter s
ϵ^θ		infinitesimal thermal strain
ϵ^{ve}		infinitesimal viscoelastic strain
ϵ_s^{ve}		infinitesimal viscoelastic strain for element parameter s
η		specific entropy
κ^{b-ve}		viscoelastic back-stress bulk modulus
κ^e		elastic bulk modulus
λ		stretch or Lamé constant
μ		viscosity
μ^{ve-d}		viscoelastic deviatoric viscosity
μ^{ve-h}		viscoelastic hydrostatic viscosity
ϕ_s		free energy “intensity” for element s
ψ		specific free energy
ψ^θ		purely thermal component of specific free energy
ρ		density
σ		stress
σ_{ave}		average stress
σ^b		viscoelastic back-stress
σ^{ve}		viscoelastic back-stress
σ_s		stress in viscoelastic system with parameter s
τ_s^e		thermodynamic elastic stress for element with parameter s
τ_s^θ		thermodynamic thermal stress for element with parameter s
τ_s^{ve}		thermodynamic viscoelastic stress for element with parameter s
θ		temperature
ζ		nonlinearity factor in flow rule

Chapter 14

Rate-dependent plasticity

In this chapter, we will look at modeling material response that shows rate sensitivity in the plastic flow. Most materials show some form of rate sensitivity during deformation. That is, they respond differently to loading applied slowly as opposed to rapidly. The classical theories of plasticity do not include this rate dependence. In the classical theories, this is because time appears in the equations in such a way that it can be scaled without altering the response in the stress-strain space.

The extent of the rate dependence is what determines whether one should include rate dependence in the model or whether one should ignore it. In this chapter, we consider methods to include rate-dependent response both before the yield point and during plastic flow. A material that exhibits rate dependence below the yield point is considered to be viscoelastic. If the plastic flow shows rate dependence, then it can be termed viscoplastic. As a result, one can imagine four main response types given as elastic-plastic, elastic-viscoplastic, viscoelastic-plastic, and viscoelastic-viscoplastic, depending on which part exhibits the rate-dependent response.

We will consider the cases of a mechanical or thermomechanical response that is constructed from an elastic or viscoelastic base response that is enhanced by one or more viscoplastic elements. To start we consider infinitesimal models constructed based on thermomechanical analogs. We then consider models that are geometrically nonlinear and thermodynamically based.

14.1 Infinitesimal mechanical and thermomechanical models with viscoplastic flow

Figure 14.1 shows a typical mechanical analog for a viscoelastic-viscoplastic material. It is constructed of three elements termed in the figure as elastic, viscoelastic, and viscoplastic. A reader familiar with viscoelasticity will recognize that the combination of the elastic and viscoelastic elements is the mechanical analog for what is known in viscoelasticity as the standard linear solid, which is characterized by three material parameters, in this case k^e , k^{ve} , and μ^{ve} . The viscoplastic element in this case is characterized by three material parameters, a back-stress stiffness k^{vp} , a yield load F_y , and a viscosity μ^{vp} . The response of this mechanical analog is characterized by a viscoelastic response that shows viscoplastic flow at high loads that terminates when the load is reduced. This combination allows for permanent flow under large loads, with time-dependent recoverable viscoelastic response superimposed on it.

The mechanical analog in Figure 14.1 shows all the basic characteristics of a viscoelastic-viscoplastic response, even though both the viscoelastic and the viscoplastic responses are idealized. In particular, the viscoelastic response has only a single relaxation time associated with that of a standard linear solid. More general models can be constructed both for the

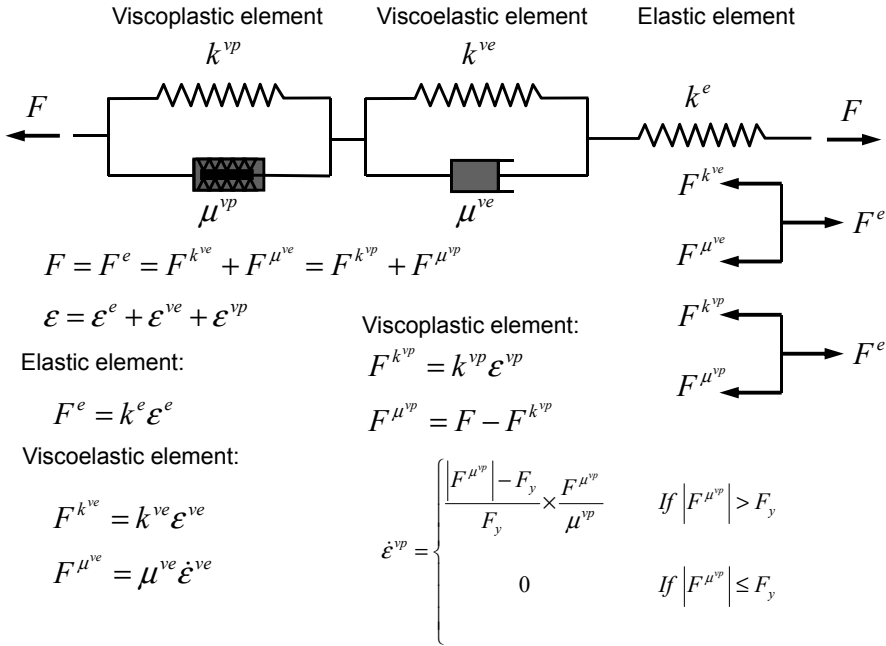


Figure 14.1: A mechanical analog for viscoelastic-viscoplastic response.

viscoelastic part and for the viscoplastic part.

Before we consider a general thermodynamic model for large deformations, let us construct several multidimensional infinitesimal models. These models can be directly motivated by analogs similar to those shown in Figures 14.1 to 14.3. In each case, the model has an initial range of response that is elastic, thermoelastic, viscoelastic, or thermoviscoelastic. This base response type is augmented by a viscoplastic element initiated once we reach some limit load. For the viscoplastic element, the element is rigid when the given yield function f is negative or zero, and the element exhibits viscous flow with a back-stress once the yield function becomes positive. A general linear form for the viscoplastic element can take the form

$$\dot{\epsilon}^{vp} = \begin{cases} \mathbf{C}^{vp} : \Delta \sigma^{vp} & \text{if } f > 0, \\ \mathbf{0} & \text{if } f \leq 0, \end{cases} \quad (14.1)$$

where $\mathbf{C}^{vp}$ is a fourth-order tensor describing the viscoplastic flow characteristics and $\Delta \sigma^{vp}$ is the viscoplastic overstress given by

$$\Delta \sigma^{vp} = \sigma - \sigma^{b-vp} \quad (14.2)$$

for the viscoplastic back-stress given by $\sigma^{b-vp} = \mathbf{E}^{vp} : \epsilon^{vp}$. Expecting the viscoplastic flow to be dissipative, we restrict the models to have $\Delta \sigma^{vp} : \dot{\epsilon}^{vp} > 0$. This requires that the viscoplastic flow tensor $\mathbf{C}^{vp}$ satisfies the relation

$$\Delta \sigma^{vp} : \mathbf{C}^{vp} : \Delta \sigma^{vp} > 0 \quad (14.3)$$

for all admissible viscoplastic overstresses.

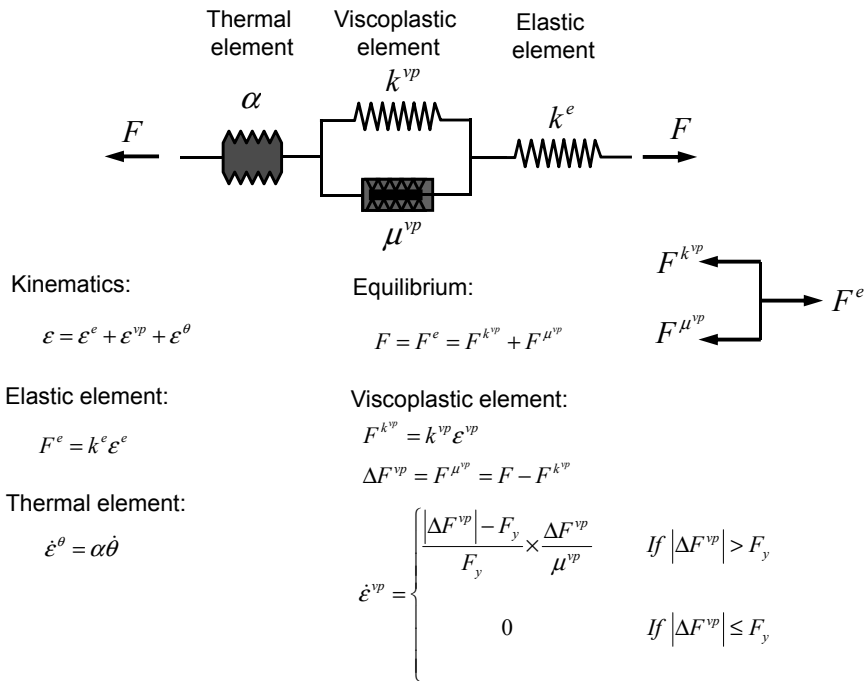


Figure 14.2: Mechanical analog for a thermoelastic solid with a viscoplastic element in series.

14.1.1 Thermoelastic base

The first example we consider is a material that starts as a thermoelastic material, but starts to flow viscously once it reaches a limit load. The thermomechanical analog for this is shown in Figure 14.2. Essentially, the infinitesimal response of the material is characterized by an additive decomposition of the strain into an elastic strain ϵ^e , thermal strain ϵ^θ , and viscoplastic strain ϵ^{vp} . For the multidimensional case, this will be written as

$$\epsilon = \epsilon^e + \epsilon^\theta + \epsilon^{vp}. \quad (14.4)$$

The analog suggests that the elastic response carries all the applied load so the applied stress σ is given in terms of the elastic strain by a relation

$$\sigma = \mathbf{E}^e : \epsilon^e, \quad (14.5)$$

where $\mathbf{E}^e$ is a fourth-order tensor that represents the elastic stiffness and can be assumed to be a function of temperature. The form of this modulus is exactly as described before, so depending on the material symmetries, it will take different forms. The thermal response is assumed to be given by the thermal expansion rule

$$\dot{\epsilon}^\theta = \alpha \dot{\theta}, \quad (14.6)$$

where α is the second-order coefficient of thermal expansion.

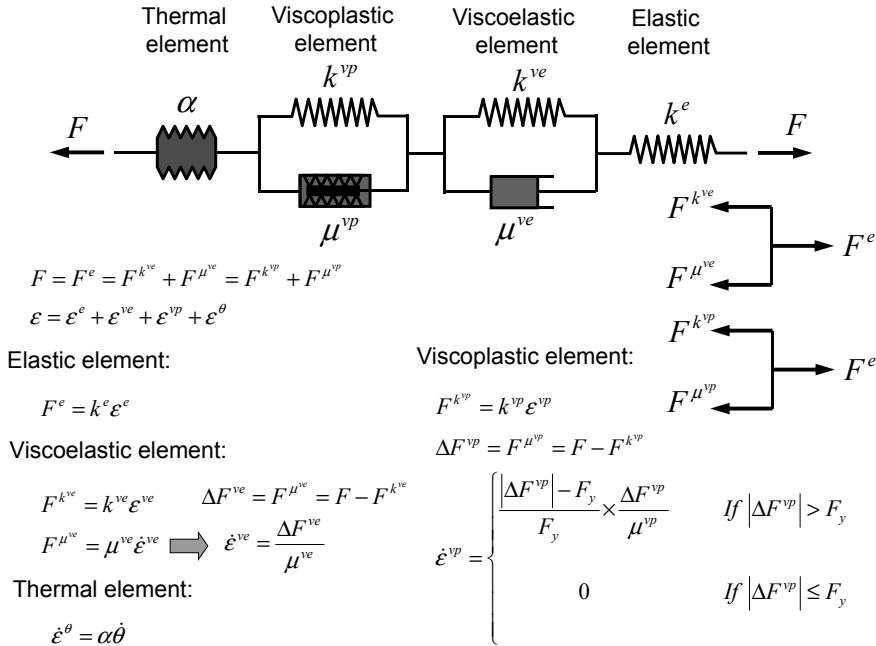


Figure 14.3: A mechanical analog for a viscoelastic-viscoplastic material constructed from elastic, viscoelastic, viscoplastic, and thermal elements.

An isotropic model with a von Mises yield function might take the form

$$\boldsymbol{\sigma} = \left(\kappa^e - \frac{2}{3} G^e \right) \text{tr}(\boldsymbol{\epsilon}^e) \mathbf{I} + 2G^e \boldsymbol{\epsilon}^e, \quad (14.7)$$

$$\boldsymbol{\sigma}^{b-vp} = \left(\kappa^{vp} - \frac{2}{3} G^{vp} \right) \text{tr}(\boldsymbol{\epsilon}^{vp}) \mathbf{I} + 2G^{vp} \boldsymbol{\epsilon}^{vp}, \quad (14.8)$$

$$\dot{\boldsymbol{\epsilon}}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (14.9)$$

$$f = \sqrt{\frac{1}{2} \Delta \mathbf{S}^{vp} : \Delta \mathbf{S}^{vp} - \tau_{y_o}}, \quad (14.10)$$

$$\dot{\boldsymbol{\epsilon}}^{vp} = \begin{cases} \frac{1}{\mu^{vp}} \left(\frac{f}{\tau_{y_o}} \right)^n \Delta \mathbf{S}^{vp} & \text{if } f > 0, \\ 0 & \text{if } f \leq 0. \end{cases} \quad (14.11)$$

As can be seen, the model is described in terms of the bulk and shear moduli of the elastic spring (κ^e , G^e), and the viscoplastic back-stress spring (κ^{vp} , G^{vp}), the linear coefficient of thermal expansion (α), the initial yield stress in shear (τ_{y_o}), the nonlinearity exponent for the viscoplastic flow (n), and the viscosity of viscoplastic flow (μ^{vp}).

14.1.2 Viscoelastic base: Standard linear solid

Let us now construct a model that resembles the one given in Figure 14.3, that is, a response characterized by a standard linear solid viscoelastic response that has a viscoplastic element augmented to it. As in the figure, we assume the strain is a summation of an elastic

strain ϵ^e , a thermal strain ϵ^θ , a viscoelastic strain ϵ^{ve} , and a viscoplastic strain ϵ^{vp} so that

$$\epsilon = \epsilon^e + \epsilon^\theta + \epsilon^{ve} + \epsilon^{vp}. \quad (14.12)$$

The stresses in the “springs” of the elastic, viscoelastic, and viscoplastic elements are assumed to be given, respectively, by

$$\sigma^e = \mathbf{E}^e : \epsilon^e, \quad (14.13)$$

$$\sigma^{b-ve} = \mathbf{E}^{ve} : \epsilon^{ve}, \quad (14.14)$$

$$\sigma^{b-vp} = \mathbf{E}^{vp} : \epsilon^{vp}, \quad (14.15)$$

with the assumption that the applied stress is the same as the stress in the elastic element so that $\sigma = \sigma^e$. The flow of the viscoelastic element is controlled by the viscoelastic overstress $\Delta\sigma^{ve} = \sigma - \sigma^{b-ve}$ and given by the model

$$\dot{\epsilon}^{ve} = \mathbf{C}^{ve} : \Delta\sigma^{ve}. \quad (14.16)$$

We use the exact same viscoplastic element as before to describe the additional flow after reaching the yield surface.

In this case, an isotropic model might take the form

$$\sigma = \left(\kappa^e - \frac{2}{3} G^e \right) \text{tr}(\epsilon^e) \mathbf{I} + 2G^e \epsilon^e, \quad (14.17)$$

$$\sigma^{b-ve} = \left(\kappa^{ve} - \frac{2}{3} G^{ve} \right) \text{tr}(\epsilon^{ve}) \mathbf{I} + 2G^{ve} \epsilon^{ve}, \quad (14.18)$$

$$\sigma^{b-vp} = \left(\kappa^{vp} - \frac{2}{3} G^{vp} \right) \text{tr}(\epsilon^{vp}) \mathbf{I} + 2G^{vp} \epsilon^{vp}, \quad (14.19)$$

$$\dot{\epsilon}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (14.20)$$

$$\dot{\epsilon}^{ve} = \frac{1}{3\mu^{ve,h}} \Delta\sigma_{ave}^{ve} \mathbf{I} + \frac{1}{\mu^{ve,d}} \Delta\mathbf{S}^{ve}, \quad (14.21)$$

$$f = \sqrt{\frac{1}{2} \Delta\mathbf{S}^{vp} : \Delta\mathbf{S}^{vp} - \tau_{y_o}}, \quad (14.22)$$

$$\dot{\epsilon}^{vp} = \begin{cases} \frac{1}{\mu^{vp}} \left(\frac{f}{\tau_{y_o}} \right)^n \Delta\mathbf{S}^{vp} & \text{if } f > 0, \\ \mathbf{0} & \text{if } f \leq 0. \end{cases} \quad (14.23)$$

As can be seen, the model is described in terms of the bulk and shear moduli of the elastic spring (κ^e , G^e), the viscoelastic back-stress spring (κ^{ve} , G^{ve}), and the viscoplastic back-stress spring (κ^{vp} , G^{vp}), the linear coefficient of thermal expansion (α), the viscoelastic hydrostatic and deviatoric viscosities ($\mu^{ve,h}$, $\mu^{ve,d}$), the initial yield stress in shear (τ_{y_o}), the nonlinearity exponent for the viscoplastic flow (n), and the viscosity of the viscoplastic flow (μ^{vp}).

14.1.3 Viscoelastic base: General linear model

We have shown that we can construct a general linear viscoelastic solid of integral form by putting an infinite number of standard linear solid elements in parallel. We did this by introducing a continuous parameter s that spans the real numbers and to each s we assigned a “stress” σ_s such that the applied stress is given by

$$\sigma = \int_0^\infty \sigma_s ds. \quad (14.24)$$

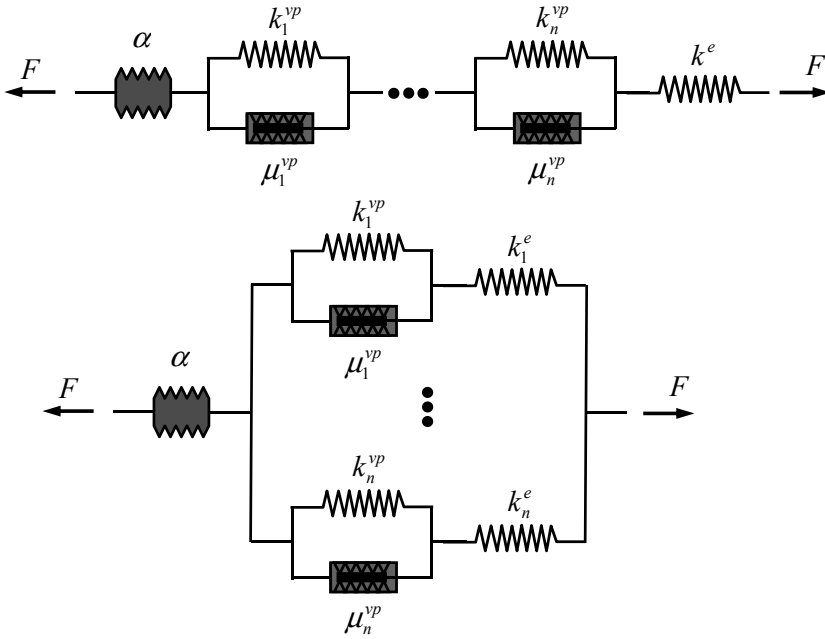


Figure 14.4: Mechanical analog for a thermoelastic solid with n viscoplastic elements in series (top) and in parallel (bottom).

We also assumed that the response in each “element” s was given by an elastic and viscoelastic response characterized by the “springs”

$$\sigma_s = \mathbf{E}_s^e : \epsilon_s^e, \quad (14.25)$$

$$\sigma_s^{b,ve} = \mathbf{E}_s^{ve} : \epsilon_s^{ve}, \quad (14.26)$$

where now the strain is separated into

$$\epsilon = \epsilon^\theta + \epsilon_s^e + \epsilon_s^{ve} + \epsilon^{vp}. \quad (14.27)$$

That is, there is a single thermal element and a single viscoplastic element, but an infinite number of standard linear solid elements in parallel, so that the sum of the $\epsilon_s^e + \epsilon_s^{ve}$ is the same for all the elements. The flow in each standard linear solid element is given by a viscous flow relation of the form

$$\dot{\epsilon}_s^{ve} = \mathbf{C}_s^{ve} : \Delta \sigma_s^{ve}, \quad (14.28)$$

where $\mathbf{C}_s^{ve}$ is a fourth-order tensor characterizing the flow, and $\Delta \sigma_s^{ve} = \sigma_s - \sigma_s^{b,ve}$ is the element’s overstress. To this we add a model for thermal expansion and one for viscoplastic flow and obtain a general viscoelastic response within the yield surface and a combined viscoelastic/viscoplastic response once we penetrate the yield surface.

14.1.4 Thermoelastic base: Multiple viscoplastic elements

Another interesting model can be constructed with two or more viscoplastic elements in series with an elastic element. Probably the most natural idea here would be to start new

viscoplastic elements as the load reaches thresholds for activating new microscopic events. There are two possible configurations to consider that are shown in Figure 14.4. First, let us look at the case of multiple viscoplastic elements in series as shown in the top of the figure. In this case, let us assume there are multiple yield functions f_i , each associated with a viscoplastic flow event that is characterized by a back-stress of $\sigma_i^{b,vp} = \mathbf{E}_i^{vp} : \epsilon_i^{vp}$, and each with an associated flow rule of

$$\dot{\epsilon}_i^{vp} = \begin{cases} \mathbf{C}_i^{vp} : \Delta\sigma_i^{vp} & \text{if } f_i > 0, \\ \mathbf{0} & \text{if } f_i \leq 0, \end{cases} \quad (14.29)$$

where each $\mathbf{C}_i^{vp}$ is a fourth-order tensor describing the viscoplastic flow characteristics of the element and $\Delta\sigma_i^{vp}$ is the viscoplastic overstress given by

$$\Delta\sigma_i^{vp} = \sigma - \sigma_i^{b,vp}. \quad (14.30)$$

For the n elements, the strain will be given by

$$\epsilon = \epsilon^\theta + \epsilon^e + \sum_{i=1}^n \epsilon_i^{vp}. \quad (14.31)$$

An isotropic model with von Mises yield functions might take the form

$$\sigma = \left(\kappa^e - \frac{2}{3} G^e \right) \text{tr}(\epsilon^e) \mathbf{I} + 2G^e \epsilon^e, \quad (14.32)$$

$$\sigma_i^{b,vp} = \left(\kappa_i^{vp} - \frac{2}{3} G_i^{vp} \right) \text{tr}(\epsilon_i^{vp}) \mathbf{I} + 2G_i^{vp} \epsilon_i^{vp}, \quad (14.33)$$

$$\dot{\epsilon}^\theta = \alpha \dot{\theta} \mathbf{I}, \quad (14.34)$$

$$f_i = \sqrt{\frac{1}{2} \Delta\mathbf{S}_i^{vp} : \Delta\mathbf{S}_i^{vp} - \tau_{y_{o-i}}}, \quad (14.35)$$

$$\dot{\epsilon}_i^{vp} = \begin{cases} \frac{1}{\mu_i^{vp}} \left(\frac{f_i}{\tau_{y_{o-i}}} \right)^{n_i} \Delta\mathbf{S}_i^{vp} & \text{if } f_i > 0, \\ \mathbf{0} & \text{if } f_i \leq 0, \end{cases} \quad (14.36)$$

where $\Delta\mathbf{S}_i^{vp}$ is the deviatoric part of the overstress for the element.

As shown in the bottom of Figure 14.4, another method to get a similar effect is to put in parallel the basic elastic-viscoplastic elements. For example, consider n elastic-viscoplastic elements in parallel. Each will be characterized by an elastic stress $\sigma_i = \mathbf{E}_i^e : \epsilon_i^e$, a back-stress $\sigma_i^{b,vp} = \mathbf{E}_i^{vp} : \epsilon_i^{vp}$, a yield function f_i , and a flow rule

$$\dot{\epsilon}_i^{vp} = \begin{cases} \mathbf{C}_i^{vp} : \Delta\sigma_i^{vp} & \text{if } f_i > 0, \\ \mathbf{0} & \text{if } f_i \leq 0, \end{cases}$$

where $\Delta\sigma_i^{vp} = \sigma_i - \sigma_i^{b,vp}$ is the overstress for the element. Since all the elements extend the same amount, we will have the relation

$$\epsilon_i^e + \epsilon_i^{vp} = \epsilon - \epsilon^\theta. \quad (14.37)$$

14.2 Nonlinear thermoelastic-viscoplastic model

The most common type of rate-dependent plasticity is one that has an elastic base response that shows viscoplastic flow after reaching the yield point. As in the example infinitesimal models developed, we build the model based on the modification of the thermoelastic model, yet we allow for full geometric nonlinearity. In the analog shown in Figure 14.2, the strain is the sum of the strains in the elastic, thermal, and viscoplastic elements. For the nonlinear case, we assume the deformation gradient $\mathbf{F}$ is the product of the elastic, viscoplastic, and thermal deformation gradients so that

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^{vp} \mathbf{F}^\theta. \quad (14.38)$$

One might consider the state of the material to be given by the variable set $\{\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \theta, \mathbf{G}, \xi_i\}$, where ξ_i are scalar, vector, and/or tensor hardening parameters.

As an example, let us consider a model that only depends on one hardening parameter, and that hardening parameter is a scalar given by the initial value ξ_o and hardening rule

$$\dot{\xi} = \mathbf{A} : \mathbf{L}^{vp}, \quad (14.39)$$

where $\mathbf{A}$ is a second-order tensor function of the variables $\{\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}\}$. Let us assume that the response functions are given by the state as

$$\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}), \quad (14.40)$$

$$\eta = \eta^\dagger(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}), \quad (14.41)$$

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}), \quad (14.42)$$

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}). \quad (14.43)$$

To this we add the assumption of an existing yield function $f(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta)$, and a viscoplastic flow rule given by

$$\mathbf{L}^{vp} = \mathbf{L}^{vp\dagger}(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta), \quad (14.44)$$

which is only activated when $f > 0$. Finally, we assume the thermal deformation gradient changes based on the rule

$$\mathbf{L}^\theta = \boldsymbol{\alpha} \dot{\theta}, \quad (14.45)$$

where $\boldsymbol{\alpha}$ is a second-order tensor coefficient of thermal expansion, given as a function of $(\mathbf{F}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta)$.

One can take a time derivative of the kinematic decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^{vp} \mathbf{F}^\theta$ to obtain the relation

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}^e \dot{\mathbf{F}}^{vp} \mathbf{F}^\theta + \mathbf{F}^e \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta. \quad (14.46)$$

One can use this to calculate $\dot{\mathbf{F}}^e$ to obtain

$$\dot{\mathbf{F}}^e = \dot{\mathbf{F}} \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1} - \mathbf{F}^e \dot{\mathbf{F}}^{vp} \mathbf{F}^{vp-1} - \mathbf{F}^e \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1}. \quad (14.47)$$

This can be written in terms of the velocity gradients as

$$\dot{\mathbf{F}}^e = \mathbf{L} \mathbf{F}^e - \mathbf{F}^e \mathbf{L}^{vp} - \mathbf{F}^e \mathbf{F}^{vp} \mathbf{L}^\theta \mathbf{F}^{vp-1}. \quad (14.48)$$

Let us now consider the second law of thermodynamics, given in the form of the Clausius-Duhem inequality in (6.69) as

$$\rho \dot{\psi} - \mathbf{T}^T : \mathbf{L} + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0.$$

To use this, we need to calculate $\dot{\psi}$, which from the assumed dependence of the free energy is given by

$$\begin{aligned} \dot{\psi} = & \partial_{\mathbf{F}^e}(\psi^\dagger) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^{vp}}(\psi^\dagger) : \dot{\mathbf{F}}^{vp} + \partial_{\mathbf{F}^\theta}(\psi^\dagger) : \dot{\mathbf{F}}^\theta + \partial_\xi(\psi^\dagger) \dot{\xi} \\ & + \partial_\theta(\psi^\dagger) \dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \end{aligned} \quad (14.49)$$

Substituting for $\dot{\mathbf{F}}^e$ from the kinematic relation gives

$$\begin{aligned} \dot{\psi} = & [\partial_{\mathbf{F}^e}(\psi^\dagger) \mathbf{F}^{eT}] : \mathbf{L} + [\partial_{\mathbf{F}^{vp}}(\psi^\dagger) \mathbf{F}^{vpT} - \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\psi^\dagger) + \partial_\xi(\psi^\dagger) \mathbf{A}] : \mathbf{L}^{vp} \\ & + [\partial_{\mathbf{F}^\theta}(\psi^\dagger) \mathbf{F}^{\theta T} - \mathbf{F}^{vpT} \mathbf{F}^{eT} \partial_{\mathbf{F}^e}(\psi^\dagger) \mathbf{F}^{vp-T}] : \mathbf{L}^\theta + \partial_\theta(\psi^\dagger) \dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \end{aligned} \quad (14.50)$$

We define the “thermodynamic stresses” for the elastic, viscoplastic and thermal elements, respectively, as

$$\boxed{\mathbf{T}^{eT} \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}}, \quad (14.51)$$

$$\boxed{\mathbf{T}^{vpT} \equiv \rho \partial_{\mathbf{F}^{vp}}(\psi) \mathbf{F}^{vpT}}, \quad (14.52)$$

$$\boxed{\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}}, \quad (14.53)$$

and the thermodynamic overstresses, respectively, as

$$\boxed{\Delta \mathbf{T}^e \equiv \mathbf{T}^e - \mathbf{T}}, \quad (14.54)$$

$$\boxed{\Delta \mathbf{T}^{vp} \equiv \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e - \mathbf{T}^{vp} - \rho \partial_\xi(\psi^\dagger) \mathbf{A}^T}, \quad (14.55)$$

$$\boxed{\Delta \mathbf{T}^\theta \equiv \mathbf{F}^{vp-1} \mathbf{F}^{e-1} \mathbf{T}^e \mathbf{F}^e \mathbf{F}^{vp} - \mathbf{T}^\theta}. \quad (14.56)$$

With these definitions, we can write $\rho \dot{\psi}$ as

$$\rho \dot{\psi} = \mathbf{T}^{eT} : \mathbf{L} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} \dot{\theta} + \rho \partial_\theta(\psi^\dagger) \dot{\theta} + \rho \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \quad (14.57)$$

Substituting for $\rho \dot{\psi}$ in the Clausius-Duhem inequality and rearranging yield

$$\Delta \mathbf{T}^{eT} : \mathbf{L} + [\rho \eta + \rho \partial_\theta(\psi^\dagger) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \rho \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (14.58)$$

This relation must hold for all admissible processes, including purely elastic processes that occur when $f < 0$ so that $\dot{\mathbf{F}}^{vp} = \mathbf{0}$. Examining the structure of this equation in the purely elastic case, we note, as was shown before, that the only way to satisfy the equation is for

$$\begin{aligned} \mathbf{T} &= \mathbf{T}^e, \\ \eta &= -\partial_\theta(\psi^\dagger) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}, \\ \partial_{\mathbf{G}}(\psi^\dagger) &= \mathbf{0}, \\ \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} &\leq 0. \end{aligned}$$

Following standard arguments, we conclude that the free energy does not depend on the temperature gradient and, as a result, neither does stress nor entropy. We now assume these relations hold into the viscoplastic range ($f > 0$) arriving at the relations

$$\mathbf{T}^T = \mathbf{T}^{eT} = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}, \quad (14.59)$$

$$\eta = -\partial_\theta(\psi^\dagger) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}, \quad (14.60)$$

$$-\Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \leq 0, \quad (14.61)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (14.62)$$

We can add more structure by requiring that the viscoplastic flow take the form

$$\mathbf{L}^{vp} = \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} \quad (14.63)$$

during viscoplastic flow. In this case, the “dissipation” inequality $-\Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \leq 0$ requires that during viscoplastic flow the fourth-order tensor $\mathbf{C}^{vp}$ be such that

$$-\Delta \mathbf{T}^{vpT} : \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} \leq 0 \quad (14.64)$$

for every admissible viscoplastic overstress. This can be specialized further if we select $\mathbf{C}^{vp} = \sum_{i=1}^n \beta_i \mathbf{Z}_i \otimes \mathbf{Z}_i$ for any n positive-valued scalar functions β_i and second-order tensor functions $\mathbf{Z}_i$ of the stated variables.

As the reader recalls, $\rho \dot{h}$ is the rate of heat entering a point (per unit volume) through radiation and conduction, so that $\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q})$. Most information about heating of a point can be studied through knowing $\rho \dot{h}$ so let us calculate it for this model. We recall from Chapter 6 that one can calculate it from

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T} \mathbf{L}) + \rho \eta \dot{\theta} + \rho \dot{\eta} \theta.$$

The first three terms on the right-hand side directly appear in the Clausius-Duhem inequality so we easily can calculate $\rho \dot{h}$ as

$$\rho \dot{h} = -\Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \rho \dot{\eta} \theta. \quad (14.65)$$

14.2.1 Invariance to rigid body motions

As before, let us assume that rigid body motions do not change the free energy and entropy, but rigidly rotate the stress and heat flux vector by the current rotation. If $\mathcal{H}$ is the history of $(\mathbf{F}, \theta)$, and $\mathcal{H}^*$ is the history of $(\mathbf{Q}\mathbf{F}, \theta)$, for any time varying $\mathbf{Q}$, then we are assuming

$$\psi(t) \rightarrow \psi^*(t) = \psi(t), \quad (14.66)$$

$$\eta(t) \rightarrow \eta^*(t) = \eta(t), \quad (14.67)$$

$$\mathbf{T}(t) \rightarrow \mathbf{T}^*(t) = \mathbf{Q}(t) \mathbf{T}(t) \mathbf{Q}^T(t), \quad (14.68)$$

$$\mathbf{q}(t) \rightarrow \mathbf{q}^*(t) = \mathbf{Q}(t) \mathbf{q}(t). \quad (14.69)$$

In addition, we assume that since $\mathbf{F}(t) \rightarrow \mathbf{F}^*(t) = \mathbf{Q}(t) \mathbf{F}(t)$, its constitutive partition will take the form

$$\mathbf{F}^e(t) \rightarrow \mathbf{F}^{e*}(t) = \mathbf{Q}(t) \mathbf{F}^e(t), \quad (14.70)$$

$$\mathbf{F}^{vp}(t) \rightarrow \mathbf{F}^{vp*}(t) = \mathbf{F}^{vp}(t), \quad (14.71)$$

$$\mathbf{F}^\theta(t) \rightarrow \mathbf{F}^{\theta*}(t) = \mathbf{F}^\theta(t). \quad (14.72)$$

Without getting into the details, from this restriction one can conclude that the free energy can be written in terms of the right symmetric factor of the deformation gradient in its polar decomposition, and the viscoplastic deformation gradient calculated for the history of this symmetric factor. But since we have assumed that rigid body motions do not change the plastic deformation gradient, this assumption requires that

$$\psi = \psi^\dagger(\mathbf{U}^e, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta). \quad (14.73)$$

A simplification of this, which we will use, is to assume the free energy is influenced only by the viscoplastic strain, so that we only include the symmetric part of $\mathbf{F}^{vp}$ in the expression for the free energy. In addition, we leave out the dependence on the thermal deformation gradient. Since $\mathbf{U}^e$ is symmetric positive-definite, we can replace $\mathbf{U}^e$ by the right Cauchy stretch tensor $\mathbf{C}^e$ without any loss of generality. We can do the same for the $\mathbf{U}^{vp}$ and write

$$\psi = \psi^{\dagger\dagger}(\mathbf{C}^e, \mathbf{C}^{vp}, \xi, \theta). \quad (14.74)$$

Since the Cauchy stress and the entropy are derived from the free energy, the influence of these assumptions on them can be evaluated directly. For the heat flux, we can write

$$\mathbf{q} = \mathbf{R}^e \mathbf{q}^{R^e\dagger}(\mathbf{C}^e, \mathbf{C}^{vp}, \xi, \theta, \mathbf{g}^{R^e}), \quad (14.75)$$

where $\mathbf{q}^{R^e} = \mathbf{R}^{eT} \mathbf{q}$ is a rotated heat flux vector and $\mathbf{g}^{R^e} = \mathbf{R}^{eT} \mathbf{g}$ is a rotated temperature gradient.

14.2.2 Material symmetry

We assume that the free energy, entropy, Cauchy stress, and heat flux vector remain the same when altering the response by a symmetry transformation. Let us assume the internal parameters behave as described for a thermoelastic-plastic case. That is, for a symmetry transformation $\mathbf{M}$, that transforms $\mathbf{F} \rightarrow \mathbf{F}\mathbf{M}$, we assume its parts transform as

$$\mathbf{F}^\theta \rightarrow \mathbf{M}^{-1} \mathbf{F}^\theta \mathbf{M}, \quad (14.76)$$

$$\mathbf{F}^{vp} \rightarrow \mathbf{M}^{-1} \mathbf{F}^{vp} \mathbf{M}, \quad (14.77)$$

$$\mathbf{F}^e \rightarrow \mathbf{F}^e \mathbf{M}. \quad (14.78)$$

This requires that $\mathbf{L}^{vp} \rightarrow \mathbf{M}^{-1} \mathbf{L}^{vp} \mathbf{M}$ and $\Delta \mathbf{T}^{vp} \rightarrow \mathbf{M}^{-1} \Delta \mathbf{T}^{vp} \mathbf{M}$. If we impose these on the assumption $\mathbf{L}^{vp} = \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT}$, we get

$$\mathbf{M}^{-1}(\mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT})\mathbf{M} = \bar{\mathbf{C}}^{vp} : (\mathbf{M}^T \Delta \mathbf{T}^{vpT} \mathbf{M}^{-T}), \quad (14.79)$$

for all admissible $\Delta \mathbf{T}^{vpT}$ and where $\mathbf{C}^{vp} \rightarrow \bar{\mathbf{C}}^{vp}$.

A simple theory can be constructed if we assume the symmetry transformation does not change $\mathbf{C}^{vp}$, that is, if $\bar{\mathbf{C}}^{vp} = \mathbf{C}^{vp}$ for all symmetry transformations $\mathbf{M}$. In this case, we can use the representations for fourth-order tensors given in the appendix, but using function coefficients. This will be our assumption for the special examples.

As described, one way to automatically satisfy the entropy production inequality is to assume $\mathbf{C}^{vp} = \sum_{i=1}^n \beta_i \mathbf{Z}_i \otimes \mathbf{Z}_i$ for any n positive-valued scalar functions β_i and second-order tensor functions $\mathbf{Z}_i$. Examination of the symmetry requirement suggests that $\mathbf{Z}_i = \mathbf{M}^{-1} \mathbf{Z}_i \mathbf{M}$.

14.2.3 Isotropic material

For the isotropic case, let us assume the reference configuration is selected such that the symmetry of the material is described by the group of orthogonal transformations. In

Table 14.1: Isotropic invariants and their derivatives for the two symmetric tensors $\mathbf{C}^e$ and $\mathbf{C}^{vp}$.

i	I_i	$\partial_{\mathbf{F}^e}(I_i)$	$\partial_{\mathbf{F}^{vp}}(I_i)$
1	$tr(\mathbf{C}^e)$	$2\mathbf{F}^e$	$\mathbf{0}$
2	$tr(\mathbf{C}^{e2})$	$4\mathbf{F}^e\mathbf{C}^e$	$\mathbf{0}$
3	$tr(\mathbf{C}^{e3})$	$6\mathbf{F}^e\mathbf{C}^{e2}$	$\mathbf{0}$
4	$tr(\mathbf{C}^{vp})$	$\mathbf{0}$	$2\mathbf{F}^{vp}$
5	$tr(\mathbf{C}^{vp2})$	$\mathbf{0}$	$4\mathbf{F}^{vp}\mathbf{C}^{vp}$
6	$tr(\mathbf{C}^{vp3})$	$\mathbf{0}$	$6\mathbf{F}^{vp}\mathbf{C}^{vp2}$
7	$tr(\mathbf{C}^e\mathbf{C}^{vp})$	$2\mathbf{F}^e\mathbf{C}^{vp}$	$2\mathbf{F}^{vp}\mathbf{C}^e$
8	$tr(\mathbf{C}^{e2}\mathbf{C}^{vp})$	$2\mathbf{F}^e(\mathbf{C}^e\mathbf{C}^{vp} + \mathbf{C}^{vp}\mathbf{C}^e)$	$2\mathbf{F}^{vp}\mathbf{C}^{e2}$
9	$tr(\mathbf{C}^e\mathbf{C}^{vp2})$	$2\mathbf{F}^e\mathbf{C}^{vp2}$	$2\mathbf{F}^{vp}(\mathbf{C}^e\mathbf{C}^{vp} + \mathbf{C}^{vp}\mathbf{C}^e)$
10	$tr(\mathbf{C}^{e2}\mathbf{C}^{vp2})$	$2\mathbf{F}^e(\mathbf{C}^e\mathbf{C}^{vp2} + \mathbf{C}^{vp2}\mathbf{C}^e)$	$2\mathbf{F}^{vp}(\mathbf{C}^{e2}\mathbf{C}^{vp} + \mathbf{C}^{vp}\mathbf{C}^{e2})$

general, the requirement of material symmetry on the function for the free energy is given by

$$\psi^{\dagger\dagger}(\mathbf{C}^e, \mathbf{C}^{vp}, \xi, \theta) = \psi^{\dagger\dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{C}^{vp} \mathbf{M}, \xi, \theta) \quad (14.80)$$

for every $\mathbf{M}$ representing a symmetry in the material. From the representations given in the appendix, the free energy is given by an isotropic scalar function of two symmetric tensors $\mathbf{C}^e$ and $\mathbf{C}^{vp}$. Therefore, the free energy is given by a function of the form

$$\psi = \psi^{\dagger\dagger\dagger}(I_1, \dots, I_{10}, \xi, \theta), \quad (14.81)$$

where the I_i are the 10 isotropic scalar invariants of $\mathbf{C}^e$ and $\mathbf{C}^{vp}$, given by $I_1 = tr(\mathbf{C}^e)$, $I_2 = tr(\mathbf{C}^{e2})$, $I_3 = tr(\mathbf{C}^{e3})$, $I_4 = tr(\mathbf{C}^{vp})$, $I_5 = tr(\mathbf{C}^{vp2})$, $I_6 = tr(\mathbf{C}^{vp3})$, $I_7 = tr(\mathbf{C}^e\mathbf{C}^{vp})$, $I_8 = tr(\mathbf{C}^{e2}\mathbf{C}^{vp})$, $I_9 = tr(\mathbf{C}^e\mathbf{C}^{vp2})$, $I_{10} = tr(\mathbf{C}^{e2}\mathbf{C}^{vp2})$. The expression for stress is calculated from this free energy using the relation

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \rho \sum_{i=1}^{10} \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_i} \partial_{\mathbf{F}^e}(I_i) \mathbf{F}^{eT}, \quad (14.82)$$

where $\partial_{\mathbf{F}^e}(I_i)$ are given in Table 14.1. Therefore, the expression for stress is given by

$$\begin{aligned} \mathbf{T}^T = \rho \left\{ 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_1} \mathbf{B}^e + 4 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_2} \mathbf{B}^{e2} + 6 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_3} \mathbf{B}^{e3} \right. \\ + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_7} \mathbf{F}^e \mathbf{C}^{vp} \mathbf{F}^{eT} + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_8} \mathbf{F}^e (\mathbf{C}^e \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{C}^e) \mathbf{F}^{eT} \\ \left. + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_9} \mathbf{F}^e \mathbf{C}^{vp2} \mathbf{F}^{eT} + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_{10}} \mathbf{F}^e (\mathbf{C}^e \mathbf{C}^{vp2} + \mathbf{C}^{vp2} \mathbf{C}^e) \mathbf{F}^{eT} \right\}. \end{aligned} \quad (14.83)$$

In a similar way, we can calculate the viscoplastic back-stress as

$$\mathbf{T}^{vpT} = \rho \partial_{\mathbf{F}^{vp}}(\psi) \mathbf{F}^{vpT} = \rho \sum_{i=1}^{10} \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_i} \partial_{\mathbf{F}^{vp}}(I_i) \mathbf{F}^{vpT}, \quad (14.84)$$

which results in

$$\begin{aligned} \mathbf{T}^{vpT} = \rho \left\{ 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_4} \mathbf{B}^{vp} + 4 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_5} \mathbf{B}^{vp2} + 6 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_6} \mathbf{B}^{vp3} \right. \\ + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_7} \mathbf{F}^{vp} \mathbf{C}^e \mathbf{F}^{vpT} + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_8} \mathbf{F}^{vp} \mathbf{C}^{e2} \mathbf{F}^{vpT} \\ + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_9} \mathbf{F}^{vp} (\mathbf{C}^e \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{C}^e) \mathbf{F}^{vpT} \\ \left. + 2 \frac{\partial \psi^{\dagger\dagger\dagger}}{\partial I_{10}} \mathbf{F}^{vp} (\mathbf{C}^{e2} \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{C}^{e2}) \mathbf{F}^{vpT} \right\}. \end{aligned} \quad (14.85)$$

The expression for entropy does not change, but the expression for the heat flux vector must satisfy the requirement that

$$\mathbf{q}^{R^e \dagger}(\mathbf{C}^e, \mathbf{C}^{vp}, \xi, \theta, \mathbf{R}^{eT} \mathbf{g}) = \mathbf{M} \mathbf{q}^{R^e \dagger}(\mathbf{M}^T \mathbf{C}^e \mathbf{M}, \mathbf{M}^T \mathbf{C}^{vp} \mathbf{M}, \xi, \theta, \mathbf{M}^T \mathbf{R}^{eT} \mathbf{g}) \quad (14.86)$$

for each orthogonal $\mathbf{M}$ representing a material symmetry. Using the representations for an isotropic vector function of two symmetric second-order tensors and one vector, we can write the representation of the heat flux vector as

$$\begin{aligned} \mathbf{q} = q_1 \mathbf{g} + q_2 \mathbf{B}^e \mathbf{g} + q_3 \mathbf{B}^{e2} \mathbf{g} + q_4 \mathbf{R}^e \mathbf{C}^{vp} \mathbf{R}^{eT} \mathbf{g} + q_5 \mathbf{R}^e \mathbf{C}^{vp2} \mathbf{R}^{eT} \mathbf{g} \\ + q_6 \mathbf{R}^e (\mathbf{C}^e \mathbf{C}^{vp} - \mathbf{C}^{vp} \mathbf{C}^e) \mathbf{R}^{eT} \mathbf{g}, \end{aligned} \quad (14.87)$$

where q_i are scalar functions of the 10 isotropic invariants of $\mathbf{C}^e$ and $\mathbf{C}^{vp}$, temperature, the hardening parameter, and the additional invariants $I_{11} = \mathbf{g} \circ \mathbf{g}$, $I_{12} = \mathbf{g} \circ (\mathbf{B}^e \mathbf{g})$, $I_{13} = \mathbf{g} \circ (\mathbf{B}^{e2} \mathbf{g})$, $I_{14} = \mathbf{g} \circ (\mathbf{R}^e \mathbf{C}^{vp} \mathbf{R}^{eT} \mathbf{g})$, $I_{15} = \mathbf{g} \circ (\mathbf{R}^e \mathbf{C}^{vp2} \mathbf{R}^{eT} \mathbf{g})$, $I_{16} = \mathbf{g} \circ (\mathbf{B}^e \mathbf{R}^e \mathbf{C}^{vp} \mathbf{R}^{eT} \mathbf{g})$.

If we assume $\bar{\mathbf{C}}^{vp} = \mathbf{C}^{vp}$, the expression for the viscoplastic flow is given through the fourth-order tensor representations as

$$\mathbf{L}^{vp} = \begin{cases} \mathbf{C}_1^{vp} \text{tr}(\Delta \mathbf{T}^{vp}) \mathbf{I} + \mathbf{C}_2^{vp} \Delta \mathbf{T}_{sym}^{vp} + \mathbf{C}_3^{vp} \Delta \mathbf{T}_{skew}^{vpT} & \text{if } f > 0, \\ \mathbf{0} & \text{if } f \leq 0, \end{cases} \quad (14.88)$$

where $\mathbf{C}_i^{vp}$ are scalar functions of temperature, the hardening parameter, and the 10 isotropic invariants of $\mathbf{C}^e$ and $\mathbf{C}^{vp}$. The dissipation inequality can be satisfied if these are positive. It probably is prudent to remind the reader that representations are not unique so that we easily can construct alternate forms for the representations. In the case of isotropic materials, since the volumetric and deviatoric parts of the response normally are vastly different and there is no directional preference, one might want to somehow separate these two parts in the construction of the representations. For the model proposed, the volumetric viscoplastic flow is given by

$$\frac{j^{vp}}{J^{vp}} = (3\mathbf{C}_1^{vp} + \mathbf{C}_2^{vp}) \text{tr}(\Delta \mathbf{T}^{vp}), \quad (14.89)$$

and the deviatoric part is given by

$$\mathbf{L}_{dev}^{vp} = \mathbf{C}_2^{vp} \Delta \mathbf{S}_{sym}^{vp} + \mathbf{C}_3^{vp} \Delta \mathbf{T}_{skew}^{vpT}, \quad (14.90)$$

where $\Delta \mathbf{S}_{sym}^{vp}$ is the deviatoric part of the symmetric part of the overstress given by

$$\Delta \mathbf{S}_{sym}^{vp} = \Delta \mathbf{T}_{sym}^{vp} - \frac{\text{tr}(\Delta \mathbf{T}^{vp})}{3} \mathbf{I}. \quad (14.91)$$

Further separating the $\mathbf{L}_{dev}^{vp}$ into its symmetric and skew-symmetric parts (deformation rate and spin tensor parts), we can write

$$\mathbf{D}_{dev}^{vp} = \mathbf{C}_2^{vp} \Delta \mathbf{S}_{sym}^{vp}, \quad (14.92)$$

$$\mathbf{W}_{dev}^{vp} = \mathbf{C}_3^{vp} \Delta \mathbf{T}_{skew}^{vpT}. \quad (14.93)$$

One may make many further alterations to this form, one of which is to select two different yield functions for the volumetric and deviatoric parts. The number of alternate representations are endless so we will not pursue other possibilities and instead will present a special example.

If we assume $\mathbf{C}^{vp} = \sum_{i=1}^n \beta_i \mathbf{Z}_i \otimes \mathbf{Z}_i$ for any n positive-valued scalar functions β_i and second-order tensor functions $\mathbf{Z}_i$, then we have each $\bar{\mathbf{Z}}_i = \mathbf{M}^T \mathbf{Z}_i \mathbf{M}$ for every orthogonal tensor $\mathbf{M}$. Therefore, each $\mathbf{Z}_i$ is an isotropic second-order tensor function of $\mathbf{C}^e$ and $\mathbf{C}^{vp}$, with dependence also on ξ and θ . Each $\mathbf{Z}_i$ can be written as

$$\begin{aligned} \mathbf{Z}_i = & \zeta_{i1} \mathbf{I} + \zeta_{i2} \mathbf{C}^e + \zeta_{i3} \mathbf{C}^{e2} + \zeta_{i4} \mathbf{C}^{vp} + \zeta_{i5} \mathbf{C}^{vp2} \\ & + \zeta_{i6} (\mathbf{C}^e \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{C}^e) + \zeta_{i7} (\mathbf{C}^e \mathbf{C}^{vp} - \mathbf{C}^{vp} \mathbf{C}^e) \\ & + \zeta_{i8} (\mathbf{C}^{e2} \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{C}^{e2}) + \zeta_{i9} (\mathbf{C}^{e2} \mathbf{C}^{vp} - \mathbf{C}^{vp} \mathbf{C}^{e2}) \\ & + \zeta_{i10} (\mathbf{C}^e \mathbf{C}^{vp2} + \mathbf{C}^{vp2} \mathbf{C}^e) + \zeta_{i11} (\mathbf{C}^e \mathbf{C}^{vp2} - \mathbf{C}^{vp2} \mathbf{C}^e) + \dots \end{aligned} \quad (14.94)$$

One might select to have each $\mathbf{Z}_i$ represent only one nonzero term from the right-hand side. Another possibility is to select $\mathbf{Z}_i$ to be dependent on any combinations of the functions of $\mathbf{T}^{Re}$, $\Delta \mathbf{T}^{vp}$, $\mathbf{C}^e$, $\mathbf{C}^{vp}$, ξ , and θ . For example, if one assumes $\mathbf{C}^{vp}$ is a function of stress and the extent of plastic strain, one can assume it depends on the variables $\mathbf{T}^{Re}$, $\mathbf{C}^{vp}$, ξ , and θ , and arrive at the representation

$$\begin{aligned} \mathbf{Z}_i = & \zeta_{i1} \mathbf{I} + \zeta_{i2} \mathbf{T}^{Re} + \zeta_{i3} \mathbf{T}^{Re2} + \zeta_{i4} \mathbf{C}^{vp} + \zeta_{i5} \mathbf{C}^{vp2} \\ & + \zeta_{i6} (\mathbf{T}^{Re} \mathbf{C}^{vp} + \mathbf{C}^{vp} \mathbf{T}^{Re}) + \zeta_{i7} (\mathbf{T}^{Re} \mathbf{C}^{vp} - \mathbf{C}^{vp} \mathbf{T}^{Re}) + \dots \end{aligned} \quad (14.95)$$

where the coefficients ζ_{ij} are isotropic scalar functions of $\mathbf{T}^{Re}$, $\mathbf{C}^{vp}$, ξ , and θ .

14.2.4 Special model: Rubber-elasticity type energy

As a simple model, let us consider one with a free energy given by the simple form

$$\begin{aligned} \psi = & \frac{\kappa^e}{3\rho_o} \left[\frac{J^{e3} - 1}{3} - \ln(J^e) \right] + \frac{G^e}{2\rho_o} \left[\frac{\text{tr}(\mathbf{C}^e)}{J^{e\frac{2}{3}}} - 3 \right] \\ & + \frac{\kappa^{vp}}{3\rho_o} \left[\frac{J^{vp3} - 1}{3} - \ln(J^{vp}) \right] + \frac{G^{vp}}{2\rho_o} \left[\frac{\text{tr}(\mathbf{C}^{vp})}{J^{vp\frac{2}{3}}} - 3 \right] + \psi^\theta, \end{aligned} \quad (14.96)$$

where the material functions κ^e , κ^{vp} , G^e , G^{vp} , and ψ^θ only depend on temperature. The viscoplastic flow rule is separated into two parts and given by

$$\frac{j^{vp}}{J^{vp}} = \begin{cases} \frac{\beta_h}{\mu_h} \Delta T_{ave}^{vp} & \text{if } f_h > 0, \\ 0 & \text{if } f_h \leq 0, \end{cases} \quad (14.97)$$

and

$$\mathbf{L}_{dev}^{vp} = \begin{cases} \frac{\beta_d}{\mu_d} \Delta \mathbf{S}^{vp} & \text{if } f_d > 0, \\ \mathbf{0} & \text{if } f_d \leq 0, \end{cases} \quad (14.98)$$

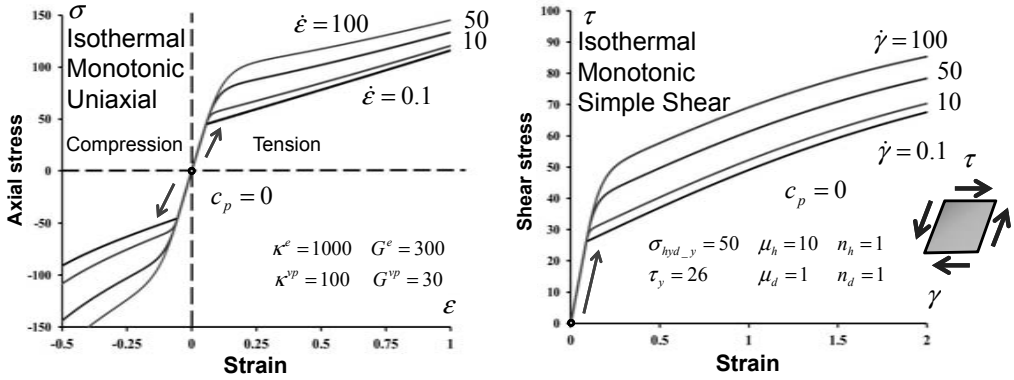


Figure 14.5: Isothermal viscoplastic responses during constant strain rate uniaxial extension and simple shear for a rubber-like free-energy function.

for the viscoplastic yield functions

$$f_h = |\Delta T_{ave}^{vp}| - \sigma_{hy d-y}, \quad (14.99)$$

$$f_d = \sqrt{\frac{1}{2} \Delta \mathbf{S}^{vp} : \Delta \mathbf{S}^{vp}} - \tau_y, \quad (14.100)$$

and for

$$\beta_h = \left(\frac{f_h}{\sigma_{hy d-y}} \right)^{n_h}, \quad (14.101)$$

$$\beta_d = \left(\frac{f_d}{\tau_y} \right)^{n_d}. \quad (14.102)$$

The model presented has a separable free energy, and two, in place of one, yield functions to allow different, yet simple, models to control the yielding in dilatation and shear of the viscoplastic flow. The stress and viscoplastic back-stress for this model are given as

$$\mathbf{T} = \frac{\kappa^e}{3J^{vp}J^\theta} \left(J^{e2} - \frac{1}{J^e} \right) \mathbf{I} + \frac{G^e}{JJ^{e\frac{2}{3}}} \left[\mathbf{B}^e - \frac{tr(\mathbf{B}^e)}{3} \mathbf{I} \right], \quad (14.103)$$

$$\mathbf{T}^{vp} = \frac{\kappa^{vp}}{3J^eJ^\theta} \left(J^{vp2} - \frac{1}{J^{vp}} \right) \mathbf{I} + \frac{G^{vp}}{JJ^{vp\frac{2}{3}}} \left[\mathbf{B}^{vp} - \frac{tr(\mathbf{B}^{vp})}{3} \mathbf{I} \right]. \quad (14.104)$$

The expression for entropy is given by

$$\begin{aligned} \eta = & -\frac{1}{3\rho_o} \frac{d\kappa^e}{d\theta} \left[\frac{J^{e3} - 1}{3} - \ln(J^e) \right] - \frac{1}{2\rho_o} \frac{dG^e}{d\theta} \left[\frac{tr(\mathbf{C}^e)}{J^{e\frac{2}{3}}} - 3 \right] \\ & - \frac{1}{3\rho_o} \frac{d\kappa^{vp}}{d\theta} \left[\frac{J^{vp3} - 1}{3} - \ln(J^{vp}) \right] - \frac{1}{2\rho_o} \frac{dG^{vp}}{d\theta} \left[\frac{tr(\mathbf{C}^{vp})}{J^{vp\frac{2}{3}}} - 3 \right] \\ & - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} (\mathbf{F}^{vpT} \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} \mathbf{F}^{vp-T}) : \boldsymbol{\alpha}. \end{aligned} \quad (14.105)$$

Finally, let us take a heat flux rule that takes the form

$$\mathbf{q} = \left[k_o \mathbf{I} + k^e \mathbf{S} + k^{vp} \mathbf{R}^e \left(\frac{\mathbf{C}^{vp}}{J^{vp\frac{1}{3}}} - \mathbf{I} \right) \mathbf{R}^{eT} \right] \mathbf{g}. \quad (14.106)$$

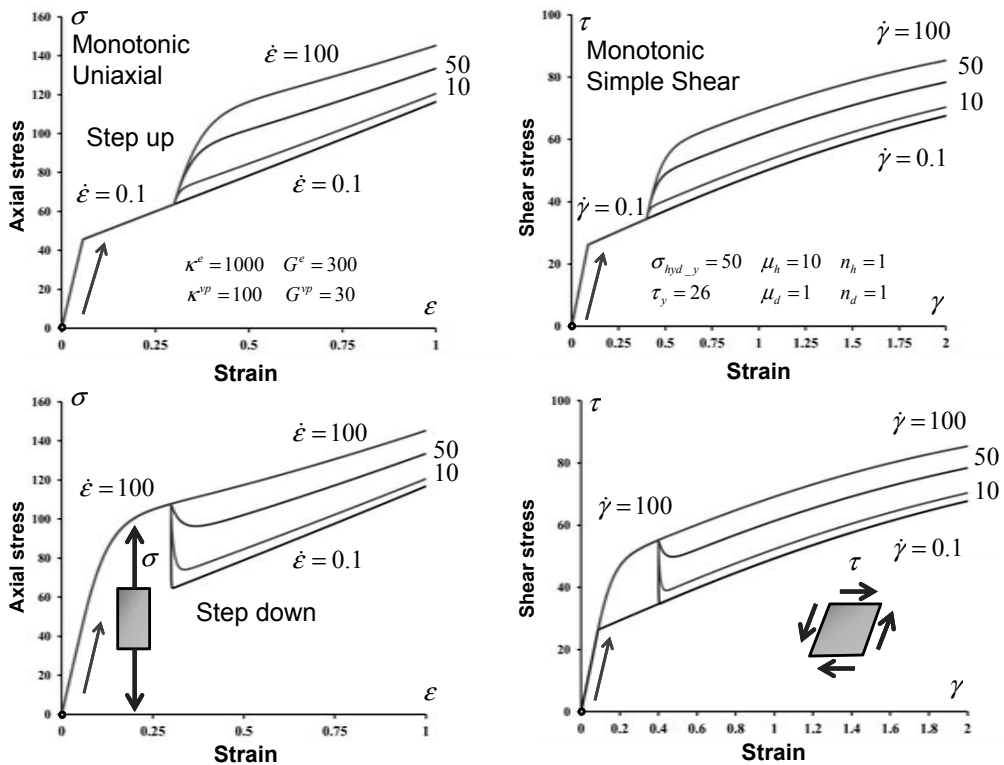


Figure 14.6: Isothermal viscoplastic responses to a change in strain rate under uniaxial extension or simple shear for a rubber-like free-energy function.

The form selected will provide conductivity changes with stress and with viscoplastic flow. This might be motivated by the experimentally observed development of anisotropy in the conductivity with viscoplastic flow as is shown in Figure 14.8 for two glassy polymers.

The model presented is a simple nonlinear model that has a free energy that is separable between the elastic and viscoplastic stored energies. As a result, the model does not evolve anisotropy in the elastic response. Many materials, such as the polymers shown in Figure 14.9, actually do change the symmetry of their elastic response. To capture such changes, we need to include an evolving structure variable. The easiest way to accomplish this is to use the viscoplastic deformation by including some of the mixed invariants (I_7 , I_8 , I_9 , and I_{10}) in the free energy.

Section 14.2-Problems:

- 14.2-1:** One way to take advantage of variable selection to simplify fitting is to select a set of invariants that separates the variables associated with volumetric deformations from those associated with the other deformations. One such set of

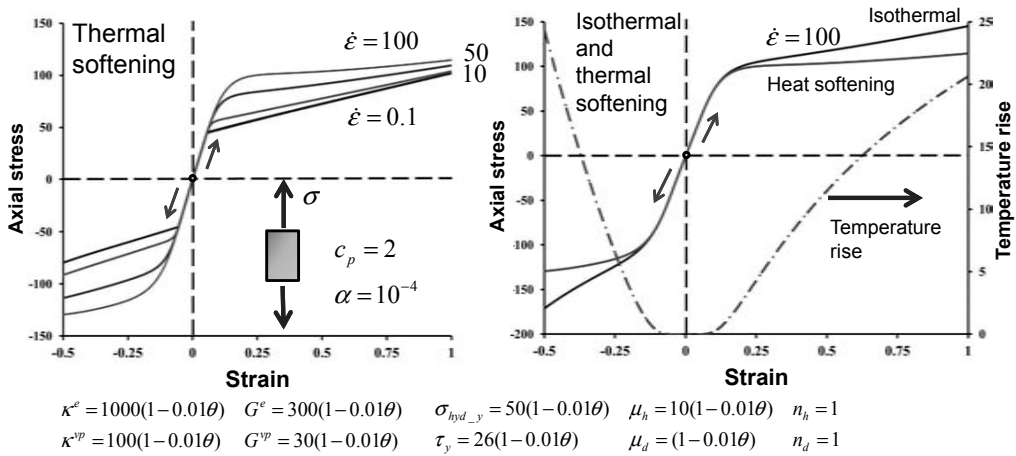


Figure 14.7: Thermal softening viscoplastic response resulting from the heating of the sample due to viscoplastic flow during monotonic uniaxial extension shown for a model with the rubber-like free-energy function.

invariants is

$$\begin{aligned}
 I_1^* &= \frac{I_1}{J^{\frac{e}{3}}}, & I_2^* &= \frac{I_2}{I_1^2}, & I_3^* &= J^e, \\
 I_4^* &= \frac{I_4}{J^{vp \frac{2}{3}}}, & I_5^* &= \frac{I_5}{I_4^2}, & I_6^* &= J^{vp}, \\
 I_7^* &= I_7 - I_1 - I_4 + 3, & I_8^* &= I_8 - I_2 - I_4 + 3, \\
 I_9^* &= I_9 - I_1 - I_5 + 3, & I_{10}^* &= I_{10} - I_2 - I_5 + 3.
 \end{aligned} \tag{14.107}$$

Construct the expressions for Cauchy stress, and viscoplastic back-stress for this set of invariants.

14.2-2: Using the representations given in the appendix, construct a thermoelastic-viscoplastic model for (a) an initially transversely isotropic material, and (b) an initially orthotropic material.

14.2-3: Develop a program to simulate the response for the simple model. For your material parameters, use $\kappa^e = 1000(1 - 0.01\theta)$, $G^e = 300(1 - 0.01\theta)$, $\kappa^{vp} = 100(1 - 0.01\theta)$, $G^{vp} = 30(1 - 0.01\theta)$, $\mu_d = (1 - 0.01\theta)$, $\mu_h = 10(1 - 0.01\theta)$, $\sigma_{hyd-y} = 50(1 - 0.01\theta)$, $\tau_y = 26(1 - 0.01\theta)$, and $n_h = n_d = 1$. Use this to simulate the response of the material under isothermal uniaxial and shear loadings for monotonic rates of 0.1, 10, and 100, and compare your results to those shown in Figures 14.5 and 14.6 for isothermal loading. Conduct the same simulations assuming that the temperature rise is entirely due to the viscoplastic work and given through the equation

$$c_p \dot{\theta} = \Delta \mathbf{T}^{vp} : \mathbf{L}^{vp}.$$

Select $c_p = 1, 2, 5$, and $\alpha = 10^{-4} \mathbf{I}$ and compare your results to those shown in Figure 14.7.

14.2-4: Complete the last problem for a strain rate of 1 for the first 0.4 units of time, followed by a rate of either 0.1, 1, or 10. Repeat this problem for a strain rate of 1 for 0.3 units of time, -1 for 0.1 units of time, 1 for 0.3 units of time, etc.

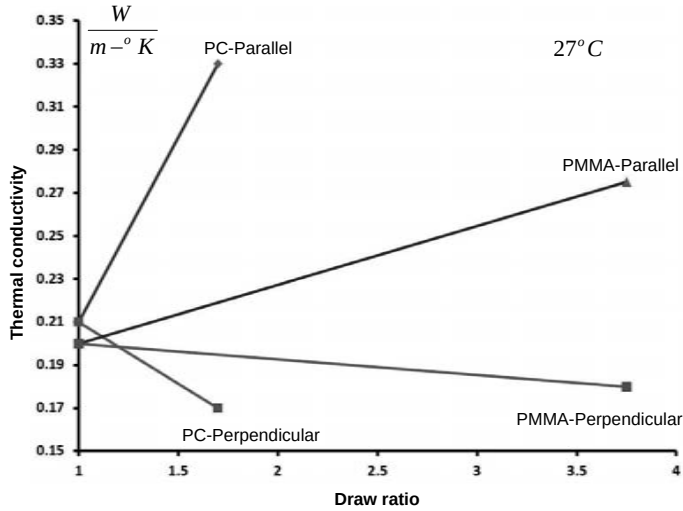


Figure 14.8: The development of thermal anisotropy with drawing in poly(methyl methacrylate) (PMMA) and polycarbonate (PC) (from K. Kurabayashi, *International Journal of Thermophysics*, Vol. 22, 2001).

14.3 Single-element viscoelastic-viscoplastic

One of the simplest models one can construct that has thermal strains, a viscoelastic range, and viscoplastic flow above yield is characterized by the mechanical analog shown in Figure 14.3. The basic assumption for an infinitesimal one-dimensional strain in this case would be

$$\epsilon = \epsilon^e + \epsilon^{ve} + \epsilon^{vp} + \epsilon^\theta. \quad (14.108)$$

The stress in the elastic components in each of the three elements can be obtained from the relations

$$\sigma^e = E^e \epsilon^e, \quad (14.109)$$

$$\sigma^{ve} = E^{ve} \epsilon^{ve}, \quad (14.110)$$

$$\sigma^{vp} = E^{vp} \epsilon^{vp}, \quad (14.111)$$

where E^e , E^{ve} , and E^{vp} are, respectively, the elastic moduli for the “springs” of the elastic, viscoelastic, and viscoplastic elements. The total stress in each element is the same as the applied stress σ . For the elastic element, this states that

$$\sigma = \sigma^e = E^e \epsilon^e. \quad (14.112)$$

The extension of the viscoelastic element is given by the viscoelastic flow rule

$$\dot{\epsilon}^{ve} = \frac{1}{\mu^{ve}} (\sigma - \sigma^{ve}), \quad (14.113)$$

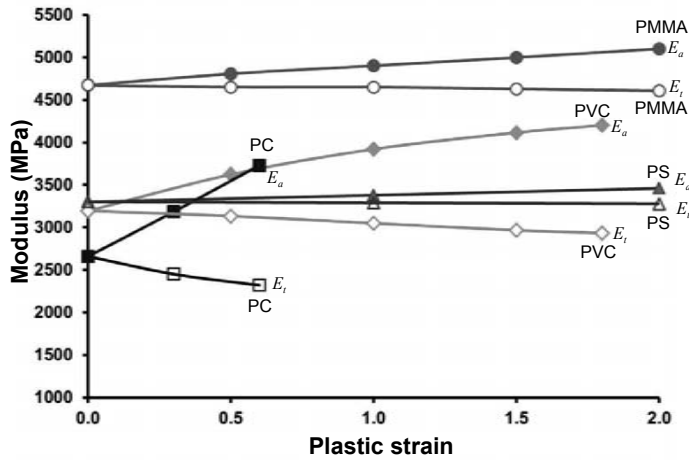


Figure 14.9: The development of elastic anisotropy with drawing in poly(methyl methacrylate) (PMMA), polycarbonate (PC), polystyrene (PS) and polyvinylchloride (PVC) (from I. M. Ward, *Mechanical Properties of Solid Polymers*, Wiley, 1979).

where μ^{ve} is the “viscosity” of the viscous damping in the viscoelastic element. The extension of the viscoplastic element is controlled by the viscoplastic flow rule

$$\dot{\epsilon}^{vp} = \begin{cases} \frac{1}{\mu^{vp}} \text{sign}(\sigma - \sigma^{vp}) f & \text{for } f > 0, \\ 0 & \text{for } f \leq 0, \end{cases} \quad (14.114)$$

where $f = |\sigma - \sigma^{vp}| - \sigma_{y_o}$ is the yield function, σ_{y_o} is the initial yield stress, and μ^{vp} is the viscosity associated with the viscous flow in the viscoplastic element. The thermal strain is given by integrating the thermal expansion rule

$$\dot{\epsilon}^{\theta} = \alpha \dot{\theta}, \quad (14.115)$$

where α is the linear coefficient of thermal expansion. This simple model, which is constructed with many linear ideas in mind, provides a response that shows many of the features observed in the response of materials that show both viscoelastic and viscoplastic response. The model has seven material constants given by $(E^e, E^{ve}, E^{vp}, \mu^{ve}, \mu^{vp}, \sigma_{y_o}, \alpha)$.

One can construct a three-dimensional model based on the same ideas. We will do this in a thermodynamically consistent form for large deformations. First, we assume that the deformations can be separated into elastic, viscoelastic, viscoplastic, and thermal parts such that we can write

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \mathbf{F}^{\theta}, \quad (14.116)$$

where we have assumed that there is, as in the one-dimensional case, four “elements” set up in series represented by, respectively, the elastic, viscoelastic, viscoplastic, and thermal deformation gradients. We assume that the free energy depends on all these deformation gradients plus temperature and hardening parameter so that we have

$$\psi = \psi^{\dagger}(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^{\theta}, \xi, \theta), \quad (14.117)$$

where we will assume the hardening parameter is given by the initial value ξ_o and the hardening rule

$$\dot{\xi} = \mathbf{A} : \mathbf{L}^{vp}, \quad (14.118)$$

and the thermal expansion is given by the rule

$$\mathbf{L}^\theta = \alpha \dot{\theta} \quad (14.119)$$

for $\mathbf{A}$ and α given by second-order tensor functions of $\mathbf{F}^e$, $\mathbf{F}^{ve}$, $\mathbf{F}^{vp}$, $\mathbf{F}^\theta$, ξ , and θ .

We need to calculate the rate of change of the free energy to include in the Clausius-Duhem inequality. This rate is given by

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^{ve}}(\psi) : \dot{\mathbf{F}}^{ve} + \partial_{\mathbf{F}^{vp}}(\psi) : \dot{\mathbf{F}}^{vp} + \partial_{\mathbf{F}^\theta}(\psi) : \dot{\mathbf{F}}^\theta + \partial_\xi(\psi)\dot{\xi} + \partial_\theta(\psi)\dot{\theta}. \quad (14.120)$$

The rate of change of the deformation gradient is given by

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}^e \dot{\mathbf{F}}^{ve} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}^e \mathbf{F}^{ve} \dot{\mathbf{F}}^{vp} \mathbf{F}^\theta + \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta. \quad (14.121)$$

Solving for $\dot{\mathbf{F}}^e$ yields

$$\begin{aligned} \dot{\mathbf{F}}^e = & \dot{\mathbf{F}} \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1} \mathbf{F}^{ve-1} - \mathbf{F}^e \dot{\mathbf{F}}^{ve} \mathbf{F}^{vp-1} - \mathbf{F}^e \mathbf{F}^{ve} \dot{\mathbf{F}}^{vp} \mathbf{F}^{vp-1} \mathbf{F}^{ve-1} \\ & - \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1} \mathbf{F}^{ve-1}. \end{aligned} \quad (14.122)$$

This can be written in terms of the velocity gradients as

$$\dot{\mathbf{F}}^e = \mathbf{L} \mathbf{F}^e - \mathbf{F}^e \mathbf{L}^{ve} - \mathbf{F}^e \mathbf{F}^{ve} \mathbf{L}^{vp} \mathbf{F}^{ve-1} - \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \mathbf{L}^\theta \mathbf{F}^{vp-1} \mathbf{F}^{ve-1}. \quad (14.123)$$

As a result, using this relation and the identity $\mathbf{C} : (\mathbf{A} \mathbf{F} \mathbf{B}) = (\mathbf{A}^T \mathbf{C} \mathbf{B}^T) : \mathbf{F}$, any expression of the form $\mathbf{A} : \dot{\mathbf{F}}^e + \mathbf{B} : \dot{\mathbf{F}}^{ve} + \mathbf{C} : \dot{\mathbf{F}}^{vp} + \mathbf{D} : \dot{\mathbf{F}}^\theta$ can be written as

$$\begin{aligned} & \mathbf{A} : \dot{\mathbf{F}}^e + \mathbf{B} : \dot{\mathbf{F}}^{ve} + \mathbf{C} : \dot{\mathbf{F}}^{vp} + \mathbf{D} : \dot{\mathbf{F}}^\theta \\ & = \mathbf{A} : (\mathbf{L} \mathbf{F}^e - \mathbf{F}^e \mathbf{L}^{ve} - \mathbf{F}^e \mathbf{F}^{ve} \mathbf{L}^{vp} \mathbf{F}^{ve-1} - \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} \mathbf{L}^\theta \mathbf{F}^{vp-1} \mathbf{F}^{ve-1}) \\ & + (\mathbf{B} \mathbf{F}^{veT}) : \mathbf{L}^{ve} + (\mathbf{C} \mathbf{F}^{vpT}) : \mathbf{L}^{vp} + (\mathbf{D} \mathbf{F}^{\theta T}) : \mathbf{L}^\theta \\ & = (\mathbf{A} \mathbf{F}^{eT}) : \mathbf{L} + (\mathbf{B} \mathbf{F}^{veT} - \mathbf{F}^{eT} \mathbf{A}) : \mathbf{L}^{ve} + (\mathbf{C} \mathbf{F}^{vpT} - \mathbf{F}^{veT} \mathbf{F}^{eT} \mathbf{A} \mathbf{F}^{ve-T}) : \mathbf{L}^{vp} \\ & + (\mathbf{D} \mathbf{F}^{\theta T} - \mathbf{F}^{vpT} \mathbf{F}^{veT} \mathbf{F}^{eT} \mathbf{A} \mathbf{F}^{ve-T} \mathbf{F}^{vp-T}) : \mathbf{L}^\theta. \end{aligned} \quad (14.124)$$

Using this relation, the time derivative of the free energy can be written as

$$\begin{aligned} \rho \dot{\psi} = & \mathbf{T}^{eT} : \mathbf{L} + (\mathbf{T}^{veT} - \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T}) : \mathbf{L}^{ve} + (\mathbf{T}^{vpT} - \mathbf{F}^{veT} \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{ve-T}) : \mathbf{L}^{vp} \\ & + (\mathbf{T}^{\theta T} - \mathbf{F}^{vpT} \mathbf{F}^{veT} \mathbf{F}^{eT} \mathbf{T}^{eT} \mathbf{F}^{e-T} \mathbf{F}^{ve-T} \mathbf{F}^{vp-T}) : \mathbf{L}^\theta + \rho \partial_\xi(\psi) \dot{\xi} + \rho \partial_\theta(\psi) \dot{\theta} \end{aligned} \quad (14.125)$$

for the “thermodynamic stresses” in the elastic, viscoelastic, viscoplastic and thermal elements defined, respectively, as

$$\mathbf{T}^{eT} \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}, \quad (14.126)$$

$$\mathbf{T}^{veT} \equiv \rho \partial_{\mathbf{F}^{ve}}(\psi) \mathbf{F}^{veT}, \quad (14.127)$$

$$\mathbf{T}^{vpT} \equiv \rho \partial_{\mathbf{F}^{vp}}(\psi) \mathbf{F}^{vpT}, \quad (14.128)$$

$$\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}. \quad (14.129)$$

Introducing the rate of change of free energy into the Clausius-Duhem inequality (6.69) and reorganization result in

$$\Delta \mathbf{T}^{eT} : \mathbf{L} + [\rho \eta + \rho \partial_\theta(\psi) - \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}] \dot{\theta} - \Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (14.130)$$

where we have used the detentions for the thermodynamic over stresses given by

$$\Delta \mathbf{T}^e \equiv \mathbf{T}^e - \mathbf{T}, \quad (14.131)$$

$$\Delta \mathbf{T}^{ve} \equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}, \quad (14.132)$$

$$\Delta \mathbf{T}^{vp} \equiv \mathbf{F}^{ve-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^{ve} - \mathbf{T}^{vp} - \rho \partial_\xi(\psi) \mathbf{A}^T, \quad (14.133)$$

$$\Delta \mathbf{T}^\theta \equiv \mathbf{F}^{vp-1} \mathbf{F}^{ve-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} - \mathbf{T}^\theta. \quad (14.134)$$

We now assume that the viscoelastic and viscoplastic velocity gradients and the coefficient of thermal expansion depend on current values of $(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta)$, but not on their rates. The Clausius-Duhem inequality must be satisfied for all possible processes, which we interpret as meaning for all $\dot{\mathbf{F}}$ and $\dot{\theta}$. Following standard arguments we conclude that we always should have

$$\mathbf{T}^T = \mathbf{T}^{eT} = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT}, \quad (14.135)$$

$$\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}. \quad (14.136)$$

Next, we assume that within the yield surface the viscoplastic velocity gradient is zero, and therefore conclude, through similar arguments as presented before, that we must have

$$-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0. \quad (14.137)$$

We take a sufficient condition to satisfy the Clausius-Duhem inequality in regards to the viscoplastic term. That is, we assume

$$-\Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \leq 0. \quad (14.138)$$

We also have the remaining condition

$$-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (14.139)$$

A possible selection for the flow rules might be

$$\mathbf{L}^{ve} = \mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}, \quad (14.140)$$

$$\mathbf{L}^{vp} = \begin{cases} \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0, \end{cases} \quad (14.141)$$

where f is an appropriate yield function and $\mathbf{C}^{ve}$ and $\mathbf{C}^{vp}$ are fourth-order tensor functions with the properties $\Delta \mathbf{T}^{veT} : \mathbf{C}^{ve} : \Delta \mathbf{T}^{veT} \geq 0$ and $\Delta \mathbf{T}^{vpT} : \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} \geq 0$. This can

be simplified further, and we can automatically satisfy the Clausius-Duhem inequality, by selecting

$$\mathbf{C}^{ve} = \sum_{i=1}^n \beta_i^{ve} \mathbf{Z}_i^{ve} \otimes \mathbf{Z}_i^{ve}, \quad (14.142)$$

$$\mathbf{C}^{vp} = \sum_{i=1}^m \beta_i^{vp} \mathbf{Z}_i^{vp} \otimes \mathbf{Z}_i^{vp}, \quad (14.143)$$

for positive scalar-valued functions β_i^{ve} and β_i^{vp} , and where $\mathbf{Z}_i^{ve}$ and $\mathbf{Z}_i^{vp}$ are second-order tensor-valued functions of the stated variables.

As the reader recalls, $\rho \dot{h}$ is the rate of heat entering a point (per unit volume) through radiation and conduction, so that $\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q})$. Most information about heating of a point can be studied through knowing $\rho \dot{h}$. We recall from Chapter 6 that one can calculate it from

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \rho \dot{\eta} \theta.$$

The first three terms on the right-hand side directly appear in the Clausius-Duhem inequality so we easily can calculate $\rho \dot{h}$ as

$$\boxed{\rho \dot{h} = -\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} + \rho \dot{\eta} \theta,} \quad (14.144)$$

during the viscoelastic range ($f \leq 0$), and

$$\boxed{\rho \dot{h} = -\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \rho \dot{\eta} \theta,} \quad (14.145)$$

during the viscoplastic range ($f > 0$).

We assume here that as a result of an *orthogonal material symmetry transformation* $\mathbf{M}$ (i.e., $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$), the internal kinematic parameters follow the rules

$$\mathbf{F}^e \rightarrow \bar{\mathbf{F}}^e = \mathbf{F}^e \mathbf{M}, \quad (14.146)$$

$$\mathbf{F}^{ve} \rightarrow \bar{\mathbf{F}}^{ve} = \mathbf{M}^T \mathbf{F}^{ve} \mathbf{M}, \quad (14.147)$$

$$\mathbf{F}^{vp} \rightarrow \bar{\mathbf{F}}^{vp} = \mathbf{M}^T \mathbf{F}^{vp} \mathbf{M}, \quad (14.148)$$

$$\mathbf{F}^\theta \rightarrow \bar{\mathbf{F}}^\theta = \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \quad (14.149)$$

and the thermodynamic stresses follow the rules

$$\mathbf{T}^e \rightarrow \bar{\mathbf{T}}^e = \mathbf{T}^e, \quad (14.150)$$

$$\mathbf{T}^{ve} \rightarrow \bar{\mathbf{T}}^{ve} = \mathbf{M}^T \mathbf{T}^{ve} \mathbf{M}, \quad (14.151)$$

$$\mathbf{T}^{vp} \rightarrow \bar{\mathbf{T}}^{vp} = \mathbf{M}^T \mathbf{T}^{vp} \mathbf{M}, \quad (14.152)$$

$$\mathbf{T}^\theta \rightarrow \bar{\mathbf{T}}^\theta = \mathbf{M}^T \mathbf{T}^\theta \mathbf{M}. \quad (14.153)$$

This set of kinematic assumptions is consistent with the fact that $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$, and the assumptions for the thermodynamic stresses are consistent with the fact that $\mathbf{T} \rightarrow \bar{\mathbf{T}} = \mathbf{T}$ when looking at the expressions obtained for the overstress in each case (for example, in $\Delta \mathbf{T}^{ve} = \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}$ we know $\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \rightarrow \mathbf{M}^T \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{M}$ so that we should have $\Delta \mathbf{T}^{ve} \rightarrow \mathbf{M}^T \Delta \mathbf{T}^{ve} \mathbf{M}$). After imposing the condition imposed by rigid body motions, for every material symmetry, it follows that the expression for the free energy must satisfy the condition

$$\boxed{\psi^\dagger(\mathbf{U}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta) = \psi^\dagger(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^{ve} \mathbf{M}, \mathbf{M}^T \mathbf{F}^{vp} \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \xi, \theta),} \quad (14.154)$$

and the expression for the heat flux vector must satisfy the requirement that

$$\begin{aligned} \mathbf{q}^{Re\uparrow}(\mathbf{U}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{R}^{eT} \mathbf{g}) \\ = \mathbf{M} \mathbf{q}^{Re\uparrow}(\mathbf{M}^T \mathbf{U}^e \mathbf{M}, \mathbf{M}^T \mathbf{F}^{ve} \mathbf{M}, \mathbf{M}^T \mathbf{F}^{vp} \mathbf{M}, \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \xi, \theta, \mathbf{M}^T \mathbf{R}^{eT} \mathbf{g}) \end{aligned} \quad (14.155)$$

for each orthogonal $\mathbf{M}$ representing a material symmetry. As a result, the over stresses follow the same rules as the thermodynamic stresses. The constraints on the models for the viscoelastic and viscoplastic flows then are given by

$$\mathbf{M}^T (\mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}) \mathbf{M} = \bar{\mathbf{C}}^{ve} : (\mathbf{M}^T \Delta \mathbf{T}^{veT} \mathbf{M}), \quad (14.156)$$

$$\mathbf{M}^T (\mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT}) \mathbf{M} = \bar{\mathbf{C}}^{vp} : (\mathbf{M}^T \Delta \mathbf{T}^{vpT} \mathbf{M}), \quad (14.157)$$

and the constraint on the coefficient of thermal expansion is

$$\bar{\alpha} = \mathbf{M}^T \alpha \mathbf{M}. \quad (14.158)$$

If we select to use the representation $\mathbf{C}^{ve} = \sum_{i=1}^n \beta_i^{ve} \mathbf{Z}_i^{ve} \otimes \mathbf{Z}_i^{ve}$ and $\mathbf{C}^{vp} = \sum_{i=1}^m \beta_i^{vp} \mathbf{Z}_i^{vp} \otimes \mathbf{Z}_i^{vp}$, then the symmetry restrictions are

$$\bar{\mathbf{Z}}_i^{ve} = \mathbf{M}^T \mathbf{Z}_i^{ve} \mathbf{M}, \quad (14.159)$$

$$\bar{\mathbf{Z}}_i^{vp} = \mathbf{M}^T \mathbf{Z}_i^{vp} \mathbf{M}. \quad (14.160)$$

A full thermodynamic multidimensional model would be obtained once we specify a function for the specific free energy, the viscoelastic velocity gradient, the yield function, the viscoplastic velocity gradient, the thermal expansion coefficient, and the heat flux, keeping in mind that in all processes $\mathbf{L}^{ve}$, $\mathbf{L}^{vp}$, and $\mathbf{q}$ need to satisfy the thermodynamic constraints. As an example, we will look at a material characterized by quadratic expressions for the free energy.

14.3.1 Quadratic free energy in Green strains

As a special example we will consider a material that has a specific free energy that is quadratic in elastic, viscoelastic, and viscoplastic Green strains. A general expression for such a material can be written as

$$\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \mathbf{E}^e : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^{ve} : \mathbf{E}^{ve} : \mathbf{E}^{ve} + \frac{1}{2\rho_o} \mathbf{E}^{vp} : \mathbf{E}^{vp} : \mathbf{E}^{vp} + \psi^\theta, \quad (14.161)$$

where $\mathbf{E}^e$, $\mathbf{E}^{ve}$, $\mathbf{E}^{vp}$, and ψ^θ are assumed to be functions only of temperature. In this expression, $\mathbf{E}^e$, $\mathbf{E}^{ve}$, and $\mathbf{E}^{vp}$ are the fourth-order elastic moduli of, respectively, the elastic, viscoelastic, and viscoplastic elements, and the associated terms in the free energy characterize the energy stored in the “elastic springs” of these components and ψ^θ represents the purely thermal component of the free energy.

As was shown in the chapter on thermoelastic response, the symmetry of the Green strain allows us to select the fourth-order moduli such that they have, for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$, the properties

$$\mathbf{A} : \mathbf{E}^e : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^e : \mathbf{B} = \mathbf{A} : \mathbf{E}^e : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^e : \mathbf{B}^T, \quad (14.162)$$

$$\mathbf{A} : \mathbf{E}^{ve} : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^{ve} : \mathbf{B} = \mathbf{A} : \mathbf{E}^{ve} : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^{ve} : \mathbf{B}^T, \quad (14.163)$$

$$\mathbf{A} : \mathbf{E}^{vp} : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^{vp} : \mathbf{B} = \mathbf{A} : \mathbf{E}^{vp} : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^{vp} : \mathbf{B}^T. \quad (14.164)$$

The expressions for the thermodynamic stresses for this model are

$$\mathbf{T}^{eT} = \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT} = \frac{1}{J} \mathbf{F}^e (\tilde{\mathbf{E}}^e : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (14.165)$$

$$\mathbf{T}^{veT} = \rho \partial_{\mathbf{F}^{ve}}(\psi) \mathbf{F}^{veT} = \frac{1}{J} \mathbf{F}^{ve} (\tilde{\mathbf{E}}^{ve} : \mathbf{E}^{ve}) \mathbf{F}^{veT}, \quad (14.166)$$

$$\mathbf{T}^{vpT} = \rho \partial_{\mathbf{F}^{vp}}(\psi) \mathbf{F}^{vpT} = \frac{1}{J} \mathbf{F}^{vp} (\tilde{\mathbf{E}}^{vp} : \mathbf{E}^{vp}) \mathbf{F}^{vpT}, \quad (14.167)$$

$$\mathbf{T}^{\theta T} = \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T} = \mathbf{0}, \quad (14.168)$$

where

$$\tilde{\mathbf{E}}^e = \frac{1}{2}(\mathbf{E}^e + \mathbf{E}^{eT}), \quad (14.169)$$

$$\tilde{\mathbf{E}}^{ve} = \frac{1}{2}(\mathbf{E}^{ve} + \mathbf{E}^{veT}), \quad (14.170)$$

$$\tilde{\mathbf{E}}^{vp} = \frac{1}{2}(\mathbf{E}^{vp} + \mathbf{E}^{vpT}). \quad (14.171)$$

A point to note is that all the thermodynamic stresses constructed from this process are symmetric, even though the over stresses may not be.

The expression for entropy in this case becomes

$$\begin{aligned} \eta = & -\frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\mathbf{E}^e}{d\theta} : \mathbf{E}^e - \frac{1}{2\rho_o} \mathbf{E}^{ve} : \frac{d\mathbf{E}^{ve}}{d\theta} : \mathbf{E}^{ve} - \frac{1}{2\rho_o} \mathbf{E}^{vp} : \frac{d\mathbf{E}^{vp}}{d\theta} : \mathbf{E}^{vp} \\ & - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} (\mathbf{F}^{vpT} \mathbf{F}^{veT} \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} \mathbf{F}^{ve-T} \mathbf{F}^{vp-T}) : \boldsymbol{\alpha}. \end{aligned} \quad (14.172)$$

14.3.2 Linear flow rules

If we take $\mathbf{C}^{ve}$ and $\mathbf{C}^{vp}$ as constants, we will get linear relations between the associated velocity gradients and over stresses. This provides one of the simplest possible flow models. The constraints of material symmetry in this case reduce to

$$\mathbf{M}^T (\mathbf{C}^{ve} : \Delta \mathbf{T}^{veT}) \mathbf{M} = \mathbf{C}^{ve} : (\mathbf{M}^T \Delta \mathbf{T}^{veT} \mathbf{M}), \quad (14.173)$$

$$\mathbf{M}^T (\mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT}) \mathbf{M} = \mathbf{C}^{vp} : (\mathbf{M}^T \Delta \mathbf{T}^{vpT} \mathbf{M}), \quad (14.174)$$

for every transformation $\mathbf{M}$ representing a symmetry of the material, and for any admissible over stress. Writing these in component form in an orthonormal base, and assuming that the over stresses can take the form of any second-order tensor, these constraints result in the relations

$$\mathbf{C}_{ijkl}^{ve} = M_{im} M_{jn} M_{ko} M_{lp} \mathbf{C}_{mnop}^{ve}, \quad (14.175)$$

$$\mathbf{C}_{ijkl}^{vp} = M_{im} M_{jn} M_{ko} M_{lp} \mathbf{C}_{mnop}^{vp}. \quad (14.176)$$

For an isotropic material, one can show that this material symmetry requires that the flow rules can be written as

$$\mathbf{L}^{ve} = \frac{1}{3\mu_{ve-h}} \text{tr}(\Delta \mathbf{T}^{ve}) \mathbf{I} + \frac{1}{\mu_1^{ve-d}} \Delta \mathbf{T}^{ve} + \frac{1}{\mu_2^{ve-d}} \Delta \mathbf{T}^{veT}, \quad (14.177)$$

$$\mathbf{L}^{vp} = \begin{cases} \frac{1}{3\mu_{vp-h}} \text{tr}(\Delta \mathbf{T}^{vp}) \mathbf{I} + \frac{1}{\mu_1^{vp-d}} \Delta \mathbf{T}^{vp} + \frac{1}{\mu_2^{vp-d}} \Delta \mathbf{T}^{vpT} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0. \end{cases} \quad (14.178)$$

Yet, we still need to satisfy the thermodynamic restrictions. For example, $-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} \leq 0$ now will become

$$-\frac{1}{3\mu_{ve,h}} \text{tr}^2(\Delta \mathbf{T}^{ve}) - \frac{1}{\mu_{1}^{ve,d}} \Delta \mathbf{T}^{veT} : \Delta \mathbf{T}^{ve} - \frac{1}{\mu_{2}^{ve,d}} \Delta \mathbf{T}^{veT} : \Delta \mathbf{T}^{veT} \leq 0. \quad (14.179)$$

An alternate, but similar, form for the flow rules of the isotropic material is given in terms of the average and deviatoric over stresses by

$$\mathbf{L}^{ve} = \frac{1}{3\mu_{ve,h}^*} \Delta T_{ave}^{ve} \mathbf{I} + \frac{1}{\mu_{1}^{ve,d*}} \Delta \mathbf{S}^{ve} + \frac{1}{\mu_{2}^{ve,d*}} \Delta \mathbf{S}^{veT}, \quad (14.180)$$

$$\mathbf{L}^{vp} = \begin{cases} \frac{1}{3\mu_{vp,h}^*} \Delta T_{ave}^{vp} \mathbf{I} + \frac{1}{\mu_{1}^{vp,d*}} \Delta \mathbf{S}^{vp} + \frac{1}{\mu_{2}^{vp,d*}} \Delta \mathbf{S}^{vpT} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0. \end{cases} \quad (14.181)$$

A similar result can be obtained for the case that $\mathbf{C}^{ve}$ and $\mathbf{C}^{vp}$ are not constants, but functions of $(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta)$ with the properties $\bar{\mathbf{C}}^{ve} = \mathbf{C}^{ve}$ and $\bar{\mathbf{C}}^{vp} = \mathbf{C}^{vp}$. In this case, the coefficients $\mu_{ve,h}^*, \mu_{1}^{ve,d*}, \mu_{2}^{ve,d*}, \mu_{vp,h}^*, \mu_{1}^{vp,d*}, \mu_{2}^{vp,d*}$ become scalar functions of the appropriate isotropic invariants.

14.3.3 Infinitesimal strains

The model with quadratic free energy can be used as a base to create an infinitesimal response model. The infinitesimal counterpart to this model will be given by the kinematic assumption

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^{ve} + \boldsymbol{\epsilon}^{vp} + \boldsymbol{\epsilon}^\theta, \quad (14.182)$$

where it is assumed that the displacement gradients associated with the elastic, viscoelastic, viscoplastic, and thermal deformation gradients are all small. The thermodynamic stresses will be given by

$$\boldsymbol{\sigma}^e = \mathbf{E}^e : \boldsymbol{\epsilon}^e, \quad (14.183)$$

$$\boldsymbol{\sigma}^{ve} = \mathbf{E}^{ve} : \boldsymbol{\epsilon}^{ve}, \quad (14.184)$$

$$\boldsymbol{\sigma}^{vp} = \mathbf{E}^{vp} : \boldsymbol{\epsilon}^{vp}, \quad (14.185)$$

with the assumption that the applied stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^e$, and for fourth-order elastic moduli that depend only on temperature and which have the symmetry conditions, for components expressed in an orthonormal base, as

$$\mathbf{E}_{ijkl}^e = M_{im} M_{jn} M_{ko} M_{lp} \mathbf{E}_{mnop}^e, \quad (14.186)$$

$$\mathbf{E}_{ijkl}^{ve} = M_{im} M_{jn} M_{ko} M_{lp} \mathbf{E}_{mnop}^{ve}, \quad (14.187)$$

$$\mathbf{E}_{ijkl}^{vp} = M_{im} M_{jn} M_{ko} M_{lp} \mathbf{E}_{mnop}^{vp}, \quad (14.188)$$

for each orthogonal material symmetry transformation $\mathbf{M}$. Each of these fourth-order moduli has the index symmetries described by $\mathbf{E}_{ijkl} = \mathbf{E}_{jikl} = \mathbf{E}_{ijlk} = \mathbf{E}_{klij}$ when expressed in an orthonormal base. The flow rules for the viscoelastic and viscoplastic elements are given by

$$\dot{\boldsymbol{\epsilon}}^{ve} = \mathbf{C}^{ve} : \Delta \boldsymbol{\sigma}^{ve}, \quad (14.189)$$

$$\dot{\boldsymbol{\epsilon}}^{vp} = \begin{cases} \mathbf{C}^{vp} : \Delta \boldsymbol{\sigma}^{vp} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0, \end{cases} \quad (14.190)$$

where the viscoelastic and viscoplastic overstresses are given by

$$\Delta\sigma^{ve} = \sigma - \sigma^{ve}, \quad (14.191)$$

$$\Delta\sigma^{vp} = \sigma - \sigma^{vp}. \quad (14.192)$$

The fourth-order moduli $\mathbf{C}^{ve}$ and $\mathbf{C}^{vp}$ have the same index symmetry and material symmetry conditions as the fourth-order elastic moduli. The thermal expansion is given by the expression

$$\dot{\epsilon}^\theta = \alpha \dot{\theta}, \quad (14.193)$$

where α is symmetric and has the constraint

$$\alpha = \mathbf{M}^T \alpha \mathbf{M} \quad (14.194)$$

for every orthogonal transformation $\mathbf{M}$ associated with the material symmetry.

The heat flux for infinitesimal strains can be expressed in the form

$$\mathbf{q} = \mathbf{k} \mathbf{g}, \quad (14.195)$$

where the coefficient of thermal conduction $\mathbf{k}$ has the property

$$\mathbf{q} \circ \mathbf{g} = \mathbf{g} \circ (\mathbf{k} \mathbf{g}) \leq 0 \quad (14.196)$$

for every vector $\mathbf{g}$. The material symmetry constraint on $\mathbf{k}$ is

$$\mathbf{k} = \mathbf{M}^T \mathbf{k} \mathbf{M}, \quad (14.197)$$

for every orthogonal transformation $\mathbf{M}$ associated with the material symmetry.

In the case of an initially isotropic material, we have the following. The thermodynamic stresses can be expressed as

$$\sigma^e = \lambda^e \text{tr}(\epsilon^e) \mathbf{I} + 2G^e \epsilon^e, \quad (14.198)$$

$$\sigma^{ve} = \lambda^{ve} \text{tr}(\epsilon^{ve}) \mathbf{I} + 2G^{ve} \epsilon^{ve}, \quad (14.199)$$

$$\sigma^{vp} = \lambda^{vp} \text{tr}(\epsilon^{vp}) \mathbf{I} + 2G^{vp} \epsilon^{vp}. \quad (14.200)$$

The expressions for the viscoelastic and viscoplastic flow rules can be written as

$$\dot{\epsilon}^{ve} = \frac{1}{3\mu^{ve-h}} \Delta\sigma_{ave}^{ve} \mathbf{I} + \frac{1}{\mu^{ve-d}} \Delta\mathbf{S}^{ve}, \quad (14.201)$$

$$\dot{\epsilon}^{vp} = \begin{cases} \frac{f}{3\mu^{vp-h}} \Delta\sigma_{ave}^{vp} \mathbf{I} + \frac{f}{\mu^{vp-d}} \Delta\mathbf{S}^{vp} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0, \end{cases} \quad (14.202)$$

where $\Delta\sigma_{ave}^{ve}$ and $\Delta\sigma_{ave}^{vp}$ are the average parts, and $\Delta\mathbf{S}^{ve}$ and $\Delta\mathbf{S}^{vp}$ are the deviatoric parts, respectively, of $\Delta\sigma^{ve}$ and $\Delta\sigma^{vp}$, and where the yield function f is normalized to the associated yield stress so it starts from -1 in the initial unloaded case.

14.4 Full viscoelastic-viscoplastic model

A thermodynamic model for a general viscoelastic material that also exhibits viscoplastic behavior after yielding can be constructed as follows. In this model, we will assume there is only one viscoplastic flow mechanism and that the free energy is given by an expression of the form

$$\psi = \int_0^\infty \phi_s ds + \psi^{b-vp} + \psi^\theta, \quad (14.203)$$

where s is a parameter used to distinguish between the different viscoelastic relaxation mechanisms, ϕ_s characterizes the contribution to the free energy of the viscoelastic relaxation mechanism s , ψ^{b-vp} characterizes the stored energy due to the back-stress resulting from the viscoplastic flow, and ψ^θ characterizes the pure thermal contribution to the free energy. The functional forms of ϕ_s , ψ^{b-vp} , and ψ^θ will be taken as

$$\phi_s = \phi_s^\dagger(\mathbf{F}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}^{vp}, \xi, \mathbf{F}^\theta, \theta), \quad (14.204)$$

$$\psi^{b-vp} = \psi^{b-vp\dagger}(\mathbf{F}^{vp}, \xi, \theta), \quad (14.205)$$

$$\psi^\theta = \psi^{\theta\dagger}(\theta), \quad (14.206)$$

where we assume that the deformations can be separated into elastic, viscoelastic, viscoplastic, and thermal parts such that we can write

$$\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} \mathbf{F}^\theta, \quad (14.207)$$

where s , as before, is a parameter used to distinguish between the different relaxation mechanisms of the viscoelastic response. In addition, we assume that the thermal deformation gradient and the hardening rule are given by initial conditions and rules of the form

$$\mathbf{L}^\theta = \boldsymbol{\alpha} \dot{\theta}, \quad (14.208)$$

$$\dot{\xi} = \mathbf{A} : \mathbf{L}^{vp}, \quad (14.209)$$

where $\boldsymbol{\alpha}$ and $\mathbf{A}$ are second-order tensor functions of $\{\mathbf{F}^{vp}, \xi, \theta\}$. The rate of change of the deformation gradient then is given by

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}_s^e \dot{\mathbf{F}}_s^{ve} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}_s^e \mathbf{F}_s^{ve} \dot{\mathbf{F}}^{vp} \mathbf{F}^\theta + \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta. \quad (14.210)$$

Solving for $\dot{\mathbf{F}}_s^e$ yields

$$\begin{aligned} \dot{\mathbf{F}}_s^e = & \dot{\mathbf{F}} \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1} \mathbf{F}_s^{ve-1} - \mathbf{F}_s^e \dot{\mathbf{F}}_s^{ve} \mathbf{F}_s^{ve-1} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \dot{\mathbf{F}}^{vp} \mathbf{F}^{vp-1} \mathbf{F}_s^{ve-1} \\ & - \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1} \mathbf{F}^{vp-1} \mathbf{F}_s^{ve-1}. \end{aligned} \quad (14.211)$$

This can be written in terms of the velocity gradients as

$$\dot{\mathbf{F}}_s^e = \mathbf{L} \mathbf{F}_s^e - \mathbf{F}_s^e \mathbf{L}_s^{ve} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{L}^{vp} \mathbf{F}_s^{ve-1} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} \mathbf{L}^\theta \mathbf{F}^{vp-1} \mathbf{F}_s^{ve-1}. \quad (14.212)$$

We can calculate $\dot{\phi}_s$ as

$$\dot{\phi}_s = \partial_{\mathbf{F}_s^e}(\phi_s) : \dot{\mathbf{F}}_s^e + \partial_{\mathbf{F}_s^{ve}}(\phi_s) : \dot{\mathbf{F}}_s^{ve} + \partial_{\mathbf{F}^{vp}}(\phi_s) : \dot{\mathbf{F}}^{vp} + \partial_{\mathbf{F}^\theta}(\phi_s) : \dot{\mathbf{F}}^\theta + \partial_\xi(\phi_s) \dot{\xi} + \partial_\theta(\phi_s) \dot{\theta}. \quad (14.213)$$

We can eliminate $\dot{\mathbf{F}}_s^e$ from this equation in favor of $\dot{\mathbf{F}}$ and rewrite it as

$$\begin{aligned} \rho \dot{\phi}_s = & \boldsymbol{\tau}_s^{eT} : \mathbf{L} + (\boldsymbol{\tau}_s^{veT} - \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T}) : \mathbf{L}_s^{ve} \\ & + [\boldsymbol{\tau}_s^{vpT} - \mathbf{F}_s^{veT} \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T} \mathbf{F}_s^{ve-T} + \rho \partial_\xi(\phi_s) \mathbf{A}] : \mathbf{L}^{vp} \\ & + [(\boldsymbol{\tau}_s^{\theta T} - \mathbf{F}^{vpT} \mathbf{F}_s^{veT} \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T} \mathbf{F}_s^{ve-T} \mathbf{F}^{vp-T}) : \boldsymbol{\alpha} + \rho \partial_\theta(\phi_s)] \dot{\theta}, \end{aligned} \quad (14.214)$$

for the thermodynamic viscoelastic stress “intensities” given by

$$\boldsymbol{\tau}_s^{eT} \equiv \rho \partial_{\mathbf{F}_s^e}(\phi_s) \mathbf{F}_s^{eT}, \quad (14.215)$$

$$\boldsymbol{\tau}_s^{veT} \equiv \rho \partial_{\mathbf{F}_s^{ve}}(\phi_s) \mathbf{F}_s^{veT}, \quad (14.216)$$

$$\boldsymbol{\tau}_s^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\phi_s) \mathbf{F}^{\theta T}, \quad (14.217)$$

$$\boldsymbol{\tau}_s^{vpT} \equiv \rho \partial_{\mathbf{F}^{vp}}(\phi_s) \mathbf{F}^{vpT}. \quad (14.218)$$

We also can calculate $\dot{\psi}^{b-vp}$ as

$$\rho \dot{\psi}^{b-vp} = [\mathbf{T}^{b-vpT} + \rho \partial_\xi(\psi^{b-vp}) \mathbf{A}] : \mathbf{L}^{vp} + \rho \partial_\theta(\psi^{b-vp}) \dot{\theta}, \quad (14.219)$$

where the thermodynamic viscoplastic back-stress is defined as

$$\boxed{\mathbf{T}^{b-vpT} \equiv \rho \partial_{\mathbf{F}^{vp}}(\psi^{b-vp}) \mathbf{F}^{vpT}}. \quad (14.220)$$

Organizing these into the Clausius-Duhem inequality results in

$$\begin{aligned} & \left[\int_0^\infty \boldsymbol{\tau}_s^{eT} ds - \mathbf{T}^T \right] : \mathbf{L} + \int_0^\infty [(\boldsymbol{\tau}_s^{veT} - \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T}) : \mathbf{L}_s^{ve}] ds \\ & + \left\{ \int_0^\infty [\boldsymbol{\tau}_s^{vpT} - \mathbf{F}_s^{veT} \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T} \mathbf{F}_s^{ve-T} + \rho \partial_\xi(\phi_s) \mathbf{A}] ds + \mathbf{T}^{b-vpT} + \rho \partial_\xi(\psi^{b-vp}) \mathbf{A} \right\} : \mathbf{L}^{vp} \\ & + \left[\int_0^\infty (\boldsymbol{\tau}_s^{\theta T} - \mathbf{F}^{vpT} \mathbf{F}_s^{veT} \mathbf{F}_s^{eT} \boldsymbol{\tau}_s^{eT} \mathbf{F}_s^{e-T} \mathbf{F}_s^{ve-T} \mathbf{F}^{vp-T}) ds \right] : \boldsymbol{\alpha} \dot{\theta} \\ & + \rho \left\{ \eta + \int_0^\infty [\partial_\theta(\phi_s)] ds + \partial_\theta(\psi^{b-vp}) + \partial_\theta(\psi^\theta) \right\} \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \end{aligned} \quad (14.221)$$

Before we proceed, let us introduce the thermodynamic overstresses

$$\boxed{\Delta \boldsymbol{\tau}_s^{ve} \equiv \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e - \boldsymbol{\tau}_s^{ve}}, \quad (14.222)$$

$$\boxed{\Delta \mathbf{T}^e \equiv \int_0^\infty \boldsymbol{\tau}_s^e ds - \mathbf{T}}, \quad (14.223)$$

$$\boxed{\Delta \mathbf{T}^{vp} \equiv \int_0^\infty [\mathbf{F}_s^{ve-1} \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e \mathbf{F}_s^{ve} - \boldsymbol{\tau}_s^{vp} - \rho \partial_\xi(\phi_s) \mathbf{A}^T] ds - \mathbf{T}^{b-vp} - \rho \partial_\xi(\psi^{b-vp}) \mathbf{A}^T}, \quad (14.224)$$

$$\boxed{\Delta \mathbf{T}^\theta \equiv \int_0^\infty (\mathbf{F}^{vp-1} \mathbf{F}_s^{ve-1} \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}^{vp} - \boldsymbol{\tau}_s^\theta) ds}, \quad (14.225)$$

and rewrite the Clausius-Duhem inequality as

$$\begin{aligned} & \Delta \mathbf{T}^{eT} : \mathbf{L} - \int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \\ & + \rho \left\{ \eta + \int_0^\infty [\partial_\theta(\phi_s)] ds + \partial_\theta(\psi^{b-vp}) + \partial_\theta(\psi^\theta) - \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} \right\} \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \end{aligned} \quad (14.226)$$

Let us consider viscoelastic flow inside the yield surface. In this range, we assume $\mathbf{L}^{vp} = \mathbf{0}$, and the equation is satisfied through the assumptions

$$\boxed{\mathbf{T} = \int_0^\infty \boldsymbol{\tau}_s^e ds}, \quad (14.227)$$

$$\boxed{\eta = - \int_0^\infty [\partial_\theta(\phi_s)] ds - \partial_\theta(\psi^{b-vp}) - \partial_\theta(\psi^\theta) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha}}, \quad (14.228)$$

and the inequalities

$$- \int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds \leq 0, \quad (14.229)$$

$$- \int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (14.230)$$

During viscoplastic flow, we assume the additional requirements

$$-\Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \leq 0, \quad (14.231)$$

$$-\int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (14.232)$$

One way to satisfy the viscoelastic dissipation rule is to assume the argument of the integral is dissipative for each s so that

$$-\Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} \leq 0. \quad (14.233)$$

One can achieve this with a viscoelastic flow of the form

$$\boxed{\mathbf{L}_s^{ve} = \mathbf{C}_s^{ve} : \Delta \boldsymbol{\tau}_s^{veT}}, \quad (14.234)$$

with a dissipation condition on $\mathbf{C}_s^{ve}$ given by

$$\boxed{-\Delta \boldsymbol{\tau}_s^{veT} : \mathbf{C}_s^{ve} : \Delta \boldsymbol{\tau}_s^{veT} \leq 0} \quad (14.235)$$

for all admissible $\Delta \boldsymbol{\tau}_s^{ve}$. In a similar manner, we can satisfy the viscoplastic dissipation inequality by requiring that

$$\boxed{\mathbf{L}^{vp} = \begin{cases} \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0, \end{cases}} \quad (14.236)$$

where f is a suitably selected yield function, and with the dissipation condition on $\mathbf{C}^{vp}$ given by

$$\boxed{-\Delta \mathbf{T}^{vpT} : \mathbf{C}^{vp} : \Delta \mathbf{T}^{vpT} \leq 0} \quad (14.237)$$

for all admissible $\Delta \mathbf{T}^{vpT}$.

As the reader recalls, $\rho \dot{h}$ is the rate of heat entering a point (per unit volume) through radiation and conduction, so that $\rho \dot{h} \equiv \rho r - \text{div}(\mathbf{q})$. Most information about heating a point can be studied through knowing $\rho \dot{h}$, so let us calculate it for this model. We recall from Chapter 6 that one can calculate it from

$$\rho \dot{h} = \rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \rho \dot{\eta} \theta.$$

The first three terms on the right-hand side directly appear in the Clausius-Duhem inequality, so we easily can calculate $\rho \dot{h}$ as

$$\boxed{\rho \dot{h} = -\int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds + \rho \dot{\eta} \theta,} \quad (14.238)$$

during the viscoelastic range ($f \leq 0$), and

$$\boxed{\rho \dot{h} = -\int_0^\infty \Delta \boldsymbol{\tau}_s^{veT} : \mathbf{L}_s^{ve} ds - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} + \rho \dot{\eta} \theta,} \quad (14.239)$$

during the viscoplastic range ($f > 0$).

We assume here that, as a result of an orthogonal material symmetry transformation $\mathbf{M}$ (i.e., $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$), the internal kinematic parameters follow the rules

$$\mathbf{F}_s^e \rightarrow \bar{\mathbf{F}}_s^e = \mathbf{F}_s^e \mathbf{M}, \quad (14.240)$$

$$\mathbf{F}_s^{ve} \rightarrow \bar{\mathbf{F}}_s^{ve} = \mathbf{M}^T \mathbf{F}_s^{ve} \mathbf{M}, \quad (14.241)$$

$$\mathbf{F}^{vp} \rightarrow \bar{\mathbf{F}}^{vp} = \mathbf{M}^T \mathbf{F}^{vp} \mathbf{M}, \quad (14.242)$$

$$\mathbf{F}^\theta \rightarrow \bar{\mathbf{F}}^\theta = \mathbf{M}^T \mathbf{F}^\theta \mathbf{M}, \quad (14.243)$$

and the thermodynamic stress intensities for the different s follow the rules

$$\boldsymbol{\tau}_s^e \rightarrow \bar{\boldsymbol{\tau}}_s^e = \boldsymbol{\tau}_s^e, \quad (14.244)$$

$$\boldsymbol{\tau}_s^{ve} \rightarrow \bar{\boldsymbol{\tau}}_s^{ve} = \mathbf{M}^T \boldsymbol{\tau}_s^{ve} \mathbf{M}, \quad (14.245)$$

$$\boldsymbol{\tau}_s^{vp} \rightarrow \bar{\boldsymbol{\tau}}_s^{vp} = \mathbf{M}^T \boldsymbol{\tau}_s^{vp} \mathbf{M}, \quad (14.246)$$

$$\boldsymbol{\tau}_s^\theta \rightarrow \bar{\boldsymbol{\tau}}_s^\theta = \mathbf{M}^T \boldsymbol{\tau}_s^\theta \mathbf{M}, \quad (14.247)$$

and the back-stress of the viscoplastic part follows the rule

$$\mathbf{T}^{b-vp} \rightarrow \bar{\mathbf{T}}^{b-vp} = \mathbf{M}^T \mathbf{T}^{b-vp} \mathbf{M}. \quad (14.248)$$

This set of kinematic assumptions is consistent with the fact that $\mathbf{F} \rightarrow \bar{\mathbf{F}} = \mathbf{F}\mathbf{M}$, and the assumptions for the thermodynamic stresses are consistent with the fact that $\mathbf{T} \rightarrow \bar{\mathbf{T}} = \mathbf{T}$ when looking at the expressions obtained for the overstress in each case (for example, in $\Delta \boldsymbol{\tau}_s^{ve} = \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e - \boldsymbol{\tau}_s^{ve}$ we know $\mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e \rightarrow \mathbf{M}^T \mathbf{F}_s^{e-1} \boldsymbol{\tau}_s^e \mathbf{F}_s^e \mathbf{M}$ so that we should have $\Delta \boldsymbol{\tau}_s^{ve} \rightarrow \mathbf{M}^T \Delta \boldsymbol{\tau}_s^{ve} \mathbf{M}$).

As a result, the overstresses follow with the same rules as the thermodynamic stresses. The constraints on the models for the viscoelastic and viscoplastic flow are then given by

$$\mathbf{M}^T (\mathbf{C}_s^{ve} : \Delta \boldsymbol{\tau}_s^{veT}) \mathbf{M} = \bar{\mathbf{C}}_s^{ve} : (\mathbf{M}^T \Delta \boldsymbol{\tau}_s^{veT} \mathbf{M}), \quad (14.249)$$

$$\mathbf{M}^T (\mathbf{C}^{vp} : \Delta \boldsymbol{\tau}_s^{vpT}) \mathbf{M} = \bar{\mathbf{C}}^{vp} : (\mathbf{M}^T \Delta \boldsymbol{\tau}_s^{vpT} \mathbf{M}), \quad (14.250)$$

and the constraint on the coefficient of thermal expansion is

$$\bar{\boldsymbol{\alpha}} = \mathbf{M}^T \boldsymbol{\alpha} \mathbf{M}. \quad (14.251)$$

In spite of all the assumptions going into the model, it is fairly general in structure and can capture both viscoelastic and viscoplastic flows that become anisotropic with deformation. In the current form, the model might be too general to be of practical use. We probably need to make simplifying assumptions by selecting special forms for the different constitutive terms. This normally is done based on either experimental evidence, molecular motivation, or expected outcome. What follows is an example of one simple model.

14.4.1 An isotropic model based on $\mathbf{C}_s^e$, $\mathbf{C}_s^{ve}$, and $\mathbf{C}^{vp}$

The model we present here is simplified substantially from the more general one presented. First, we assume the response to be initially isotropic with the material symmetry given, for the reference configuration selected, by the full group of orthogonal transformations. Next, let us assume that the free energy is a function of the viscoelastic and viscoplastic strains, as opposed to the full deformation gradients, so that we have

$$\psi = \int_0^\infty \phi_s^\dagger(\mathbf{C}_s^e, \mathbf{C}_s^{ve}, \mathbf{C}^{vp}, \theta) ds + \psi^{b-vp\dagger}(\mathbf{C}^{vp}, \theta) + \psi^{\theta\dagger}(\theta). \quad (14.252)$$

We further assume that the symmetry conditions are satisfied for the free energy by taking

$$\phi_s^\dagger(\mathbf{C}_s^e, \mathbf{C}_s^{ve}, \mathbf{C}^{vp}, \theta) = \phi_s^{\dagger\dagger}(I_1, \dots, I_{22}, \theta), \quad (14.253)$$

$$\psi^{b\cdot vp\dagger}(\mathbf{C}^{vp}, \theta) = \psi^{b\cdot vp\dagger\dagger}(I_7, I_8, I_9, \theta), \quad (14.254)$$

where I_i are the 22 isotropic scalar invariants of the three symmetric tensors $\mathbf{C}_s^e$, $\mathbf{C}_s^{ve}$, and $\mathbf{C}^{vp}$ given by

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}_s^e), & I_2 &= \text{tr}(\mathbf{C}_s^{e2}), & I_3 &= \text{tr}(\mathbf{C}_s^{e3}), \\ I_4 &= \text{tr}(\mathbf{C}_s^{ve}), & I_5 &= \text{tr}(\mathbf{C}_s^{ve2}), & I_6 &= \text{tr}(\mathbf{C}_s^{ve3}), \\ I_7 &= \text{tr}(\mathbf{C}^{vp}), & I_8 &= \text{tr}(\mathbf{C}^{vp2}), & I_9 &= \text{tr}(\mathbf{C}^{vp3}), \end{aligned} \quad (14.255)$$

and

$$\begin{aligned} I_{10} &= \text{tr}(\mathbf{C}_s^e \mathbf{C}_s^{ve}), & I_{11} &= \text{tr}(\mathbf{C}_s^e \mathbf{C}_s^{ve2}), & I_{12} &= \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}_s^{ve}), & I_{13} &= \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}_s^{ve2}), \\ I_{14} &= \text{tr}(\mathbf{C}_s^e \mathbf{C}^{vp}), & I_{15} &= \text{tr}(\mathbf{C}_s^e \mathbf{C}^{vp2}), & I_{16} &= \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}^{vp}), & I_{17} &= \text{tr}(\mathbf{C}_s^{e2} \mathbf{C}^{vp2}), \\ I_{18} &= \text{tr}(\mathbf{C}_s^{ve} \mathbf{C}^{vp}), & I_{19} &= \text{tr}(\mathbf{C}_s^{ve} \mathbf{C}^{vp2}), & I_{20} &= \text{tr}(\mathbf{C}_s^{ve2} \mathbf{C}^{vp}), & I_{21} &= \text{tr}(\mathbf{C}_s^{ve2} \mathbf{C}^{vp2}), \\ I_{22} &= \text{tr}(\mathbf{C}_s^e \mathbf{C}_s^{ve} \mathbf{C}^{vp}). \end{aligned} \quad (14.256)$$

As a result, we can write the “stress intensities” as

$$\boldsymbol{\tau}_s^{eT} = \rho \sum_{i=1}^{22} \frac{\partial \phi_s}{\partial I_i} \partial_{\mathbf{F}_s^e} (I_i) \mathbf{F}_s^{eT}, \quad (14.257)$$

$$\boldsymbol{\tau}_s^{veT} = \rho \sum_{i=1}^{22} \frac{\partial \phi_s}{\partial I_i} \partial_{\mathbf{F}_s^{ve}} (I_i) \mathbf{F}_s^{veT}, \quad (14.258)$$

$$\boldsymbol{\tau}_s^{\theta T} = \rho \sum_{i=1}^{22} \frac{\partial \phi_s}{\partial I_i} \partial_{\mathbf{F}^\theta} (I_i) \mathbf{F}^{\theta T}, \quad (14.259)$$

$$\boldsymbol{\tau}_s^{vpT} = \rho \sum_{i=1}^{22} \frac{\partial \phi_s}{\partial I_i} \partial_{\mathbf{F}^{vp}} (I_i) \mathbf{F}^{vpT}, \quad (14.260)$$

and

$$\mathbf{T}^{b\cdot vpT} = \rho \sum_{i=7}^9 \frac{\partial \psi^{b\cdot vp}}{\partial I_i} \partial_{\mathbf{F}^{vp}} (I_i) \mathbf{F}^{vpT}. \quad (14.261)$$

We can augment this by viscoelastic and viscoplastic flow rules of the form

$$\mathbf{L}_s^{ve} = \frac{1}{3\mu_{s\_ave}^{ve}} \Delta \tau_{s\_ave}^{ve} \mathbf{I} + \frac{1}{\mu_{s\_1}^{ve,d}} \Delta \mathbf{S}_s^{ve} + \frac{1}{\mu_{s\_2}^{ve,d}} \Delta \mathbf{S}_s^{veT}, \quad (14.262)$$

$$\mathbf{L}^{vp} = \begin{cases} \frac{f}{3\mu_{vp,h}^{vp}} \Delta T_{ave}^{vp} \mathbf{I} + \frac{f}{\mu_1^{vp,d}} \Delta \mathbf{S}^{vp} + \frac{f}{\mu_2^{vp,d}} \Delta \mathbf{S}^{vpT} & \text{for } f > 0, \\ \mathbf{0} & \text{for } f \leq 0, \end{cases} \quad (14.263)$$

where $\Delta \tau_{s\_ave}^{ve}$ and $\Delta \mathbf{S}_s^{ve}$ are, respectively, the average and the deviatoric part of $\Delta \boldsymbol{\tau}_s^{ve}$, and the yield function f is normalized by the appropriate initial yield stress so that its value is -1 for the initial unloaded material. The scalar viscosities are subject to the dissipation laws. We also need a yield function that we take in the form

$$f(\Delta \mathbf{T}^{vp}, \mathbf{C}^{vp}, \theta), \quad (14.264)$$

or on the total and viscoplastic strains and temperature. As a result, the yield function can be written as appropriate invariants of the tensor variables and of temperature. The

expression for α can be taken from the isotropic tensor invariants of $\mathbf{C}$ and $\mathbf{C}^{vp}$, and the heat flux can be constructed from $\mathbf{q}^{Re}$, which is selected from the appropriate vector form constructed from vector isotropic invariants of $\mathbf{C}$, $\mathbf{C}^{vp}$, and $\mathbf{g}^{Re}$. An alternate is to have these variables depend on the rotated stress, viscoplastic strain, the hardening parameter and, in the case of $\mathbf{q}^{Re}$, on $\mathbf{g}^{Re}$.

14.5 Material constraints

The process of constructing the influence of the material constraints, such as incompressibility, follows similar paths as described in the chapters on viscoelasticity, and the results also are similar. The reader is directed to this chapter to see the development and how it is obtained. In general, the result of a constraint is to partially disconnect the applied quantities from what may be possible to evaluate from the history of deformation and temperature. For a constraint of the form $C(\mathbf{F}, \theta, \mathbf{G}) = 0$, a free energy of the form

$$\psi = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, \mathbf{G}, p), \quad (14.265)$$

with similar dependence for the other response functions, and viscoelastic and viscoplastic flows of the form

$$\mathbf{L}^{ve} = \mathbf{L}^{ve\dagger}(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, p), \quad (14.266)$$

$$\mathbf{L}^{vp} = \mathbf{L}^{vp\dagger}(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^{vp}, \mathbf{F}^\theta, \xi, \theta, p), \quad (14.267)$$

and a hardening rule of the form $\dot{\xi} = \mathbf{A} : \mathbf{L}^{vp}$, we have

$$\boxed{\mathbf{T}^T = \mathbf{T}^{eT} + \varpi \partial_{\mathbf{F}}(C) \mathbf{F}^T}, \quad (14.268)$$

$$\boxed{\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \boldsymbol{\alpha} - \frac{\varpi}{\rho} \partial_\theta(C)}, \quad (14.269)$$

$$\boxed{\partial_{\mathbf{G}}(\psi) = -\frac{\varpi}{\rho} \partial_{\mathbf{G}}(C)}, \quad (14.270)$$

$$\boxed{\partial_p(\psi) = 0}, \quad (14.271)$$

$$\boxed{\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0}, \quad (14.272)$$

where ϖ is a scalar that can be assumed to be the indeterminant parameter. One would still require that

$$\boxed{-\Delta \mathbf{T}^{veT} : \mathbf{L}^{ve} - \Delta \mathbf{T}^{vpT} : \mathbf{L}^{vp} \leq 0}. \quad (14.273)$$

Similar results follow for the full viscoelastic model.

14.6 Models based on $\mathbf{F} = \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^{vp}$

The assumptions we used to create the standard elastic-viscoplastic theory have a special structure that relates the models formulated based on the decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^{vp} \mathbf{F}^\theta$ and models based on the decomposition $\mathbf{F} = \mathbf{F}^e \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^{vp}$. These are similar to those given in viscoelasticity. To see the relations, consider having two theories that start from similar assumptions, with one based on the former decomposition and the other based on the latter. For each we would have a flow rule and a thermal expansion rule. The flow rules $\mathbf{L}^{vp}$ and $\bar{\mathbf{L}}^{vp}$

would each be given by the associated variables defining the state, but in rate-dependent plasticity, they would not depend on the rates. The thermal expansions would be given by

$$\mathbf{L}^\theta = \alpha \dot{\theta}, \quad (14.274)$$

$$\bar{\mathbf{L}}^\theta = \bar{\alpha} \dot{\theta}, \quad (14.275)$$

where, again, α and $\bar{\alpha}$ would each be given by the associated variables defining the state but would not depend on the rates. If we now consider the relation $\mathbf{F}^{vp} \mathbf{F}^\theta = \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^{vp}$ and take its time derivative, we would get

$$\dot{\mathbf{F}}^{vp} \mathbf{F}^\theta + \mathbf{F}^{vp} \dot{\mathbf{F}}^\theta = \dot{\bar{\mathbf{F}}}^\theta \bar{\mathbf{F}}^{vp} + \bar{\mathbf{F}}^\theta \dot{\bar{\mathbf{F}}}^{vp}, \quad (14.276)$$

which could be rewritten as

$$\mathbf{L}^{vp} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}^{vp} \mathbf{L}^\theta \mathbf{F}^\theta = \bar{\mathbf{L}}^\theta \bar{\mathbf{F}}^\theta \bar{\mathbf{F}}^{vp} + \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^{vp} \bar{\mathbf{F}}^{vp}. \quad (14.277)$$

We can now introduce the thermal expansion rules to get

$$\mathbf{L}^{vp} \mathbf{F}^{vp} \mathbf{F}^\theta + \mathbf{F}^{vp} \alpha \dot{\mathbf{F}}^\theta = \bar{\alpha} \dot{\bar{\mathbf{F}}}^\theta \bar{\mathbf{F}}^{vp} + \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^{vp} \bar{\mathbf{F}}^{vp}. \quad (14.278)$$

If we examine the terms, only the second term on the left side of the equality and the first term on the right side of the equality depend on the rate of loading. These terms only depend on the temperature rate. If we assume the temperature rate can be selected arbitrarily, then the relation can only be satisfied if the first term on the left equals the second term on the right and the second term on the left equals the first term on the right. This gives

$$\mathbf{L}^{vp} \mathbf{F}^{vp} \mathbf{F}^\theta = \bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^{vp} \bar{\mathbf{F}}^{vp}, \quad (14.279)$$

$$\mathbf{F}^{vp} \alpha \dot{\mathbf{F}}^\theta = \bar{\alpha} \dot{\bar{\mathbf{F}}}^\theta \bar{\mathbf{F}}^{vp}. \quad (14.280)$$

This can be reorganized to get the relations

$$\bar{\alpha} = \mathbf{F}^{vp} \alpha \mathbf{F}^{vp-1}, \quad (14.281)$$

$$\bar{\mathbf{F}}^\theta \bar{\mathbf{L}}^{vp} \bar{\mathbf{F}}^{\theta-1} = \mathbf{L}^{vp}. \quad (14.282)$$

Examination of the relations brings us to the conclusion that if one constructs the thermal expansion and flow rule for one decomposition, then the values of the thermal expansion and flow rule for the other decomposition are uniquely defined. Since we can integrate these equations for each given loading history to get the corresponding thermal and viscoplastic deformation gradients, this result states that for each loading history the two decompositions are uniquely related. Therefore, models based on one decomposition will be closely related to models based on the other and would use state parameters that could be calculated, even though not algebraically, from each other. In addition, since the viscoplastic flow for one will have to occur when there is viscoplastic flow in the other, the yield surfaces for both cases must be identical points in the loading space. Since we have a thermoelastic range, the current isothermal yield surface should be expressible in terms of either decomposition. This will be true for the initial loading, which states that both models will have exactly the same initial yield surface.

14.7 Notation

The following table shows the symbols used in the text and their definitions.

ave		average
$\mathbf{B}^e$		elastic left Cauchy stretch tensor ($\mathbf{F}^e \mathbf{F}^{eT}$)
$\mathbf{B}^{vp}$		viscoplastic left Cauchy stretch tensor ($\mathbf{F}^{vp} \mathbf{F}^{vpT}$)
$\mathbf{C}^e$		elastic right Cauchy stretch tensor ($\mathbf{F}^{eT} \mathbf{F}^e$)
$\mathbf{C}^{vp}$		viscoplastic right Cauchy stretch tensor ($\mathbf{F}^{vpT} \mathbf{F}^{vp}$)
$\mathbf{C}^{ve}$		viscoelastic flow coefficient
$\mathbf{C}_s^{ve}$		viscoelastic flow coefficient for element parameter s
$\mathbf{C}^{vp}$		viscoplastic flow coefficient
$\mathbf{C}_i^{vp}$		viscoplastic flow coefficient for element i
$\mathbf{D}$		deformation rate
dev		deviatoric
$\Delta \mathbf{T}^{ve}$		viscoelastic overstress ($\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{ve}$)
$\Delta \mathbf{T}^{vp}$		viscoplastic overstress ($\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^{vp} - \rho \partial_\xi(\psi) \mathbf{A}^T$) = $\mathbf{F}^{ve-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^{ve} - \mathbf{T}^{vp} - \rho \partial_\xi(\psi) \mathbf{A}^T$ if viscoelastic-viscoplastic
$\Delta \mathbf{T}^\theta$		thermal overstress ($\mathbf{F}^{ve-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^{ve} - \mathbf{T}^\theta$) = $\mathbf{F}^{vp-1} \mathbf{F}^{ve-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^{ve} \mathbf{F}^{vp} - \mathbf{T}^\theta$ if viscoelastic-viscoplastic
e		elastic
$\mathbf{E}^e$		elastic modulus tensor
$\mathbf{E}_s^e$		elastic modulus tensor for element parameter s
$\mathbf{E}^{ve}$		viscoelastic back-stress modulus tensor
$\mathbf{E}_s^{ve}$		viscoelastic back-stress modulus for element parameter s
$\mathbf{E}^{vp}$		viscoplastic back-stress modulus tensor
f		yield function
$\mathbf{F}$		deformation gradient
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}^\theta$		thermal deformation gradient
$\mathbf{F}^{ve}$		viscoelastic deformation gradient
$\mathbf{F}^{vp}$		viscoplastic deformation gradient
$\mathbf{g}$		temperature gradient with respect to current configuration
$\mathbf{g}^{R^e}$		“elastic” rotated temperature gradient ($\mathbf{R}^{eT} \mathbf{g}$)
$\mathbf{G}$		temperature gradient with respect to reference configuration
G^e		elastic shear modulus
G^{ve}		viscoelastic back-stress shear modulus
G^{vp}		viscoplastic back-stress shear modulus
I_i		scalar invariant i
J		volume ratio ($\det(\mathbf{F})$)
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}} \mathbf{F}^{-1}$)
$\mathbf{L}^\theta$		thermal velocity gradient ($\dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1}$)
$\mathbf{L}^{ve}$		viscoelastic velocity gradient ($\dot{\mathbf{F}}^{ve} \mathbf{F}^{ve-1}$)
$\mathbf{L}^{vp}$		viscoplastic velocity gradient ($\dot{\mathbf{F}}^{vp} \mathbf{F}^{vp-1}$)
$\mathbf{M}$		transformation tensor representing material symmetry
$\mathbf{q}$		heat flux vector
$\mathbf{q}^{R^e}$		“elastic” rotated heat flux vector ($\mathbf{R}^{eT} \mathbf{q}$)

Q		orthogonal tensor representing rigid body motion
R^e		orthogonal factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e$
<i>s</i>		parameter for representing continuous series of viscoelastic elements
S		deviatoric stress
T		Cauchy stress
T^e		thermodynamic elastic stress
T^{ve}		thermodynamic viscoelastic back-stress
T^{vp}		thermodynamic viscoplastic back-stress
<i>T_{ave}</i>		average stress
U^e		right symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e$
<i>ve</i>		viscoelastic
<i>vp</i>		viscoplastic
W		spin tensor
α		coefficient of thermal expansion
$\Delta \sigma_s^{ve}$		viscoelastic overstress for element parameter <i>s</i>
$\Delta \sigma^{vp}$		viscoplastic overstress ($\sigma - \sigma^{b, vp}$)
$\Delta \sigma_s^{vp}$		viscoplastic overstress for element parameter <i>s</i>
$\Delta \sigma_i^{vp}$		viscoplastic overstress for element <i>i</i> ($\sigma - \sigma_i^{b, vp}$)
ε		infinitesimal strain
ε^e		infinitesimal elastic strain
ε_s^e		infinitesimal elastic strain for element parameter <i>s</i>
ε^θ		infinitesimal thermal strain
ε^{ve}		infinitesimal viscoelastic strain
ε_s^{ve}		infinitesimal viscoelastic strain for element parameter <i>s</i>
ε^{vp}		infinitesimal viscoplastic strain
ε_i^{vp}		infinitesimal viscoplastic strain for element <i>i</i>
η		specific entropy
κ^{ve}		viscoelastic back-stress bulk modulus
κ^{vp}		viscoplastic back-stress bulk modulus
κ^e		elastic bulk modulus
μ^{ve,d}		viscoelastic deviatoric viscosity
μ^{ve,h}		viscoelastic hydrostatic viscosity
μ^{vp,d}		viscoplastic deviatoric viscosity
μ^{vp,h}		viscoplastic hydrostatic viscosity
μ^{vp}		viscoplastic viscosity
φ_s		free energy “intensity” for element <i>s</i>
ψ		specific free energy
ψ^{b, vp}		viscoplastic back-stress component of specific free energy
ψ^θ		purely thermal component of specific free energy
ρ		density
σ		stress
σ^{b, ve}		viscoelastic back-stress
σ^{b, vp}		viscoplastic back-stress
σ_i^{b, vp}		viscoplastic back-stress for element <i>i</i>
σ_s		stress in viscoelastic system with parameter <i>s</i>
σ^{ve}		viscoelastic back-stress (same as $\sigma^{b, ve}$)
σ^{vp}		viscoplastic back-stress (same as $\sigma^{b, vp}$)
τ_{yo}		yield stress in shear
τ_s^e		thermodynamic elastic stress for element with parameter <i>s</i>
τ_s^θ		thermodynamic thermal stress for element with parameter <i>s</i>
τ_s^{ve}		thermodynamic viscoelastic stress for element with parameter <i>s</i>

τ_s^{vp}		thermodynamic viscoplastic stress for element with parameter s
θ		temperature
ξ_i		hardening parameter i

Chapter 15

Crystal plasticity

In this chapter, we will look at crystal plasticity, which is based on the observation that there are preferred planes and directions on which dislocations will move in a single crystal. The motion of dislocations and the process of twinning normally are identified as the primary sources of plastic flow in single crystals. Even though we will not focus on dislocations and their specific motion, the theory of crystal plasticity we develop in this chapter is directly motivated by these motions in a single crystal.

The preferred planes of motion in a crystal are called slip planes, and the preferred directions of slip in these planes are called slip directions. Each pair of one slip plane and one slip direction is called a slip system. A crystal normally would have many slip systems, each of which would be designated by a plane and a direction in the reference configuration. Since planes are characterized by their normals, the α th slip system is identified by a vector $\hat{\mathbf{n}}_o^\alpha$ normal to the slip plane and a vector $\hat{\mathbf{s}}_o^\alpha$ in the slip plane pointing along the slip direction. If there were m slip systems in a crystal, for each slip system we would have a pair

$$(\hat{\mathbf{n}}_o^\alpha, \hat{\mathbf{s}}_o^\alpha), \quad (15.1)$$

where α takes values from 1 to m . Figure 15.1 shows a single slip system in the deformed configuration where $\hat{\mathbf{n}}^\alpha$ designates the normal to the slip plane and $\hat{\mathbf{s}}^\alpha$ designates the slip direction.

At any point in the loading, a number of slip systems may be activated, resulting in slip along the designated slip directions. A point of caution is that for each slip direction we select two distinct slip systems, one along the slip direction and the other in the opposite direction. As a result of this decision, at each instant not more than half the slip systems can be activated.

The combination of slip in the slip systems and crystal distortion accounts for the total deformation of the crystal. Slip is assumed to be the microstructural mechanism associated with plastic deformation in the crystal, and lattice distortion is assumed to be the underlying microstructural element associated with thermal expansion and elastic deformation. Twinning is another possible source of plastic deformation, but it is outside the scope of our discussion in this chapter.

Each slip system can be described in the current configuration by the unit normal vector $\hat{\mathbf{n}}^\alpha$ to the slip plane and the unit vector $\hat{\mathbf{s}}^\alpha$ along the slip direction, as is shown in Figure 15.1. The simplest models for crystal plasticity assume the slip system is driven by the resolved shear stress, which is the shear component of traction on the slip plane resolved along the slip direction. The resolved shear stress τ^α is given in terms of the traction vector by the relation

$$\tau^\alpha \equiv \mathbf{t}^{(n^\alpha)} \circ \hat{\mathbf{s}}^\alpha,$$

and in terms of the Cauchy stress it is given by the relation

$$\tau^\alpha = (\mathbf{T}^T \hat{\mathbf{n}}^\alpha) \circ \hat{\mathbf{s}}^\alpha = (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{T}.$$

According to Schmid's law, slip (yield and plastic flow) starts along the slip direction when the resolved shear stress reaches a critical value given by τ_c^α . This will result in one yield

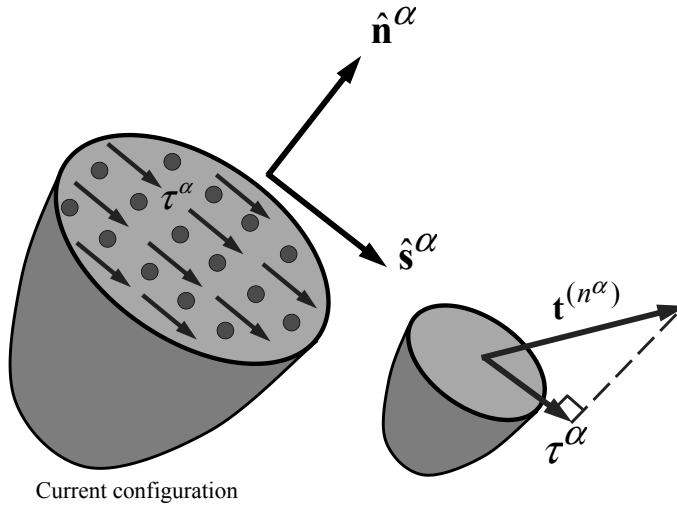


Figure 15.1: Schematic of the slip system α consisting of the slip plane with $\hat{\mathbf{n}}^\alpha$ normal to the slip plane and $\hat{\mathbf{s}}^\alpha$ tangent to the slip direction. The resolved shear stress τ^α is the shear stress on the slip plane in the direction of slip.

function for each slip system. The yield functions associated with Schmid's law for the α th slip system can be written as

$$\boxed{f^\alpha = \tau^\alpha - \tau_c^\alpha} \quad (15.2)$$

Normally, not all slip systems in a crystal are activated at the same time. Once a slip system is activated, plastic flow proceeds along the slip direction, increasing the resolved shear strain $\gamma^{p\alpha}$ for that slip system and, normally, resulting in hardening of the slip system through increasing the critical resolved shear stress τ_c^α . A simple model provided by Taylor and Elam to describe this hardening of the slip system is given by

$$\tau_c^\alpha = \beta \sqrt{\gamma^{p\alpha}},$$

where β is a material constant. As will be shown later, this simple model is generalized when considering combined self-hardening of the slip system, and cross-hardening in one slip system due to slip in other slip systems.

One may construct both rate-independent and rate-dependent crystal plasticity models. As before, for rate-independent models the condition of plastic flow in each slip system is given by $f^\alpha = 0$ and $\dot{f}^\alpha > 0$, and for rate-dependent models it is simply $f^\alpha \geq 0$. For rate-independent models, we also need to satisfy the consistency condition $\dot{f}^\alpha = 0$ when a slip system is active, but there is no such condition for rate-dependent models. Typical examples of flow rules for rate-independent crystal plasticity, including thermal effects, will be of the form

$$\dot{\gamma}^{p\alpha} = \begin{cases} \beta_{\mathbf{L}} : \mathbf{L} + \beta_{\dot{\theta}} \dot{\theta} & \text{for } f^\alpha = 0 \text{ and } \dot{f}^\alpha > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (15.3)$$

where $\beta_{\mathbf{L}}$ and $\beta_{\dot{\theta}}$ are appropriate material functions, selected to both characterize the response and satisfy the consistency condition. An example of a model for rate-dependent

crystal plasticity will be of the form

$$\dot{\gamma}^{p\alpha} = \begin{cases} (\frac{f^\alpha}{\mu^\alpha})^{r^\alpha} & \text{for } f^\alpha \geq 0, \\ 0 & \text{for } f^\alpha < 0, \end{cases} \quad (15.4)$$

where μ^α and r^α are positive valued material functions characterizing the slip.

We will generalize most of these ideas in the following sections, but first we will look at the description of crystal structures and how information from the crystallographic description of crystals can be used to identify the slip systems. This will be followed by a study of the mechanical theory of crystal plasticity, looking at elasticity of crystal distortion, the kinematics of crystal slip, the description of the resolved shear stress and overstress, yield functions, and flow rules. We finally conclude the chapter by including thermal effects.

15.1 Crystal structures and slip systems

Crystal structures are structures created by repeating a unit cell in space, creating regular harmonic structure. Figure 15.2 shows the unit cell for the 14 Bravais lattices, commonly used for the description of crystal structures. Each unit cell is characterized by three lattice vectors, here denoted by $\mathbf{e}_i$. These lattice vectors define the smallest repeating unit of the crystal. The description of vectors in crystallography is given either through components in the base $\mathbf{e}_i$ or its dual base $\mathbf{e}_i^*$ described in Chapter 2 as

$$\mathbf{e}_1^* = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{V}, \quad (15.5)$$

$$\mathbf{e}_2^* = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{V}, \quad (15.6)$$

$$\mathbf{e}_3^* = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{V}, \quad (15.7)$$

where $V = \mathbf{e}_1 \circ (\mathbf{e}_2 \times \mathbf{e}_3)$ is the volume of the unit cell. The dual base defines the unit cell of what is known as the reciprocal lattice. As such, it is common to refer to the dual base also as the reciprocal base. Figure 15.3 shows the unit cell for the cubic crystals, showing both the unit cell created by the lattice vectors $\mathbf{e}_i$ and the reciprocal unit cell created by the dual base. The same is provided for the hexagonal system in Figure 15.5. In general, the volume of the original unit cell and the one created by the dual base are reciprocals of each other. The two bases have the relation:

$$\mathbf{e}_i \circ \mathbf{e}_j^* = \delta_{ij}. \quad (15.8)$$

The reader will note that the dual to the dual base is the original lattice base. A particularly important use of the dual base is to extract components, especially in nonorthogonal systems. If a vector $\mathbf{a}$ has the description $\mathbf{a} = a_i \mathbf{e}_i$ in the lattice base and the description $\mathbf{a} = a_i^* \mathbf{e}_i^*$ in the dual base, then it is easy to show that

$$a_i = \mathbf{e}_i^* \circ \mathbf{a}, \quad (15.9)$$

$$a_i^* = \mathbf{e}_i \circ \mathbf{a}. \quad (15.10)$$

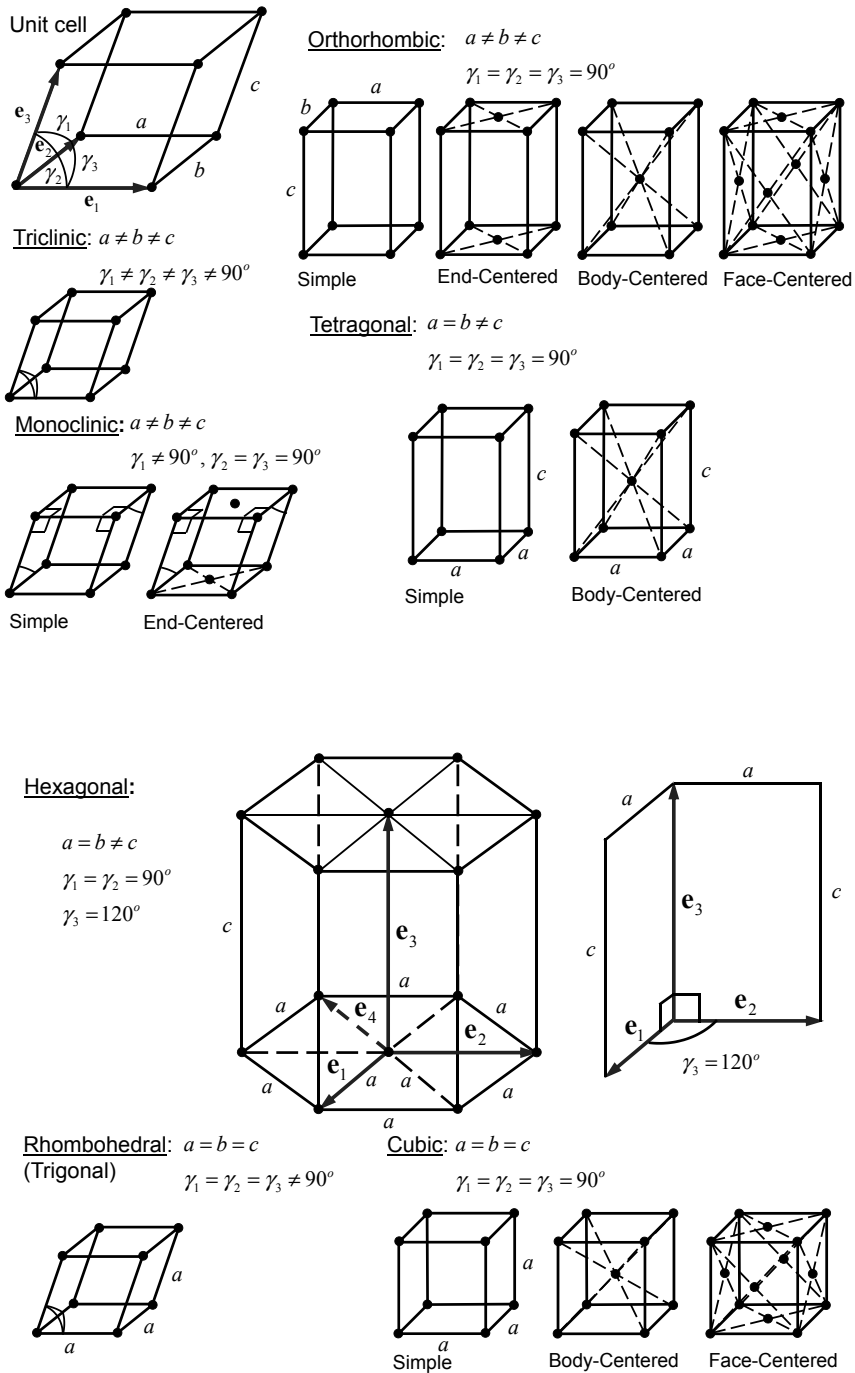


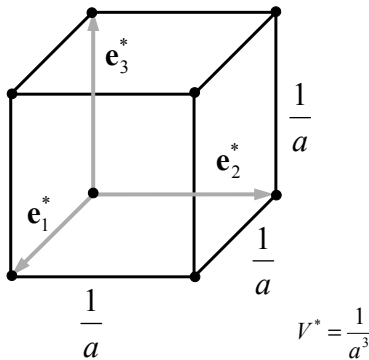
Figure 15.2: Unit cells of the 14 Bravais lattices.

Cubic:

$$a = b = c$$

$$\gamma_1 = \gamma_2 = \gamma_3 = 90^\circ$$

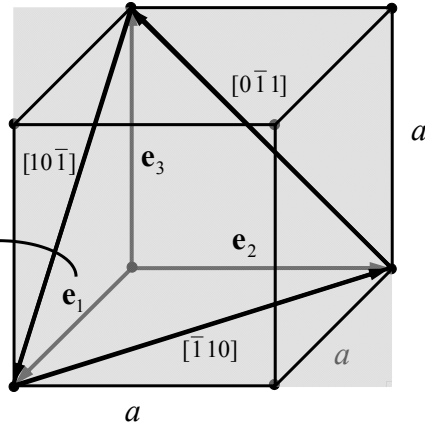
Dual (reciprocal) cell:



Unit cell:

$$V = a^3$$

(111) plane



$$|\mathbf{e}_1^*| = \frac{|\mathbf{e}_2 \times \mathbf{e}_3|}{V} = \frac{a^2}{a^3} = \frac{1}{a}$$

$$|\mathbf{e}_2^*| = \frac{|\mathbf{e}_3 \times \mathbf{e}_1|}{V} = \frac{a^2}{a^3} = \frac{1}{a}$$

$$|\mathbf{e}_3^*| = \frac{|\mathbf{e}_1 \times \mathbf{e}_2|}{V} = \frac{a^2}{a^3} = \frac{1}{a}$$

Figure 15.3: Unit cell and dual (reciprocal) lattice cell for cubic crystals.

Miller indices normally are used in crystallographic descriptions of directions and planes. The Miller indices are triplets of integer numbers designating either components of vectors in the lattice base $\mathbf{e}_i$ or the dual base $\mathbf{e}_i^*$. Triplets with square brackets, such as $[110]$, are used to designate components in the lattice base $\mathbf{e}_i$, while triplets in parentheses, such as (110) , represent components in the dual base $\mathbf{e}_i^*$. A bar over an index such as $\bar{k}$ denotes negative k . We, therefore, have the identifications:

$$[hkl] \leftrightarrow h\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3, \quad (15.11)$$

$$(hkl) \leftrightarrow h\mathbf{e}_1^* + k\mathbf{e}_2^* + l\mathbf{e}_3^*. \quad (15.12)$$

For example, if vector $\mathbf{a}$ is described by $[\bar{1}21]$, it means that $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$, and if vector $\mathbf{b}$ is described by $(\bar{1}21)$, it means that $\mathbf{b} = \mathbf{e}_1^* - 2\mathbf{e}_2^* + \mathbf{e}_3^*$. Obviously, the reader will note that $[\bar{1}21] \neq (\bar{1}21)$.

Planes normally are described in the reciprocal base by giving the Miller indices (hkl) of their normal, while directions normally are given in the lattice base by providing the Miller indices $[hkl]$. The reason for doing this is related primarily to the ease of calculation of the components. For example, the surface ABC in Figure 15.4 is defined by its unit normal vector $\mathbf{n}$ through the relation $d = \mathbf{r} \circ \mathbf{n}$, where d is a constant representing the orthogonal distance of the plane from the origin, and $\mathbf{r}$ is any vector going from the origin to a point on the plane. If $\mathbf{a} = [a00]$, $\mathbf{b} = [0b0]$, and $\mathbf{c} = [00c]$ are the three vectors designating the intercept of the plane with, respectively, the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ axes as shown in the figure, then

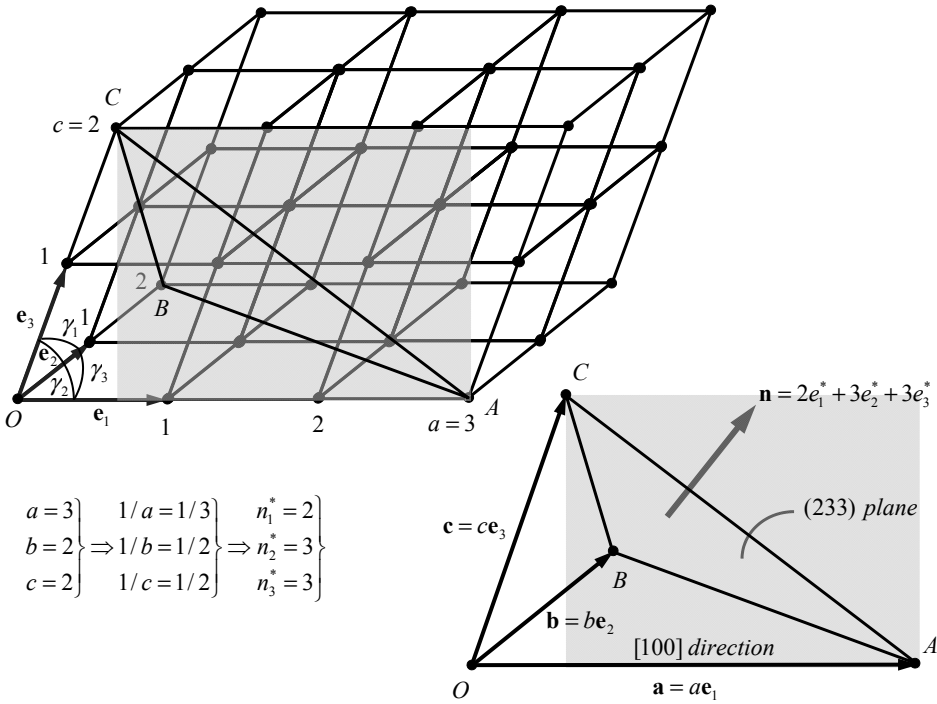


Figure 15.4: Crystal unit cell, crystal directions, crystal planes, and Miller indices.

we have

$$d = \mathbf{a} \circ \mathbf{n} = a\mathbf{e}_1 \circ \mathbf{n},$$

$$d = \mathbf{b} \circ \mathbf{n} = b\mathbf{e}_2 \circ \mathbf{n},$$

$$d = \mathbf{c} \circ \mathbf{n} = c\mathbf{e}_3 \circ \mathbf{n},$$

so that

$$n_1^* = \mathbf{e}_1 \circ \mathbf{n} = \frac{d}{a}, \quad (15.13)$$

$$n_2^* = \mathbf{e}_2 \circ \mathbf{n} = \frac{d}{b}, \quad (15.14)$$

$$n_3^* = \mathbf{e}_3 \circ \mathbf{n} = \frac{d}{c}. \quad (15.15)$$

In other words, if we know the intercepts of the plane with the lattice axes (given in fractions of the corresponding lattice vectors), we easily can write the normal to the plane as a vector in the dual base described as

$$\mathbf{n} = \frac{d}{a}\mathbf{e}_1^* + \frac{d}{b}\mathbf{e}_2^* + \frac{d}{c}\mathbf{e}_3^*. \quad (15.16)$$

To convert this description to Miller indices, we need to clear the fractions by multiplying the vector by the smallest number that will make all three components integers. Therefore, knowledge of d is not necessary, and we simply follow the steps:

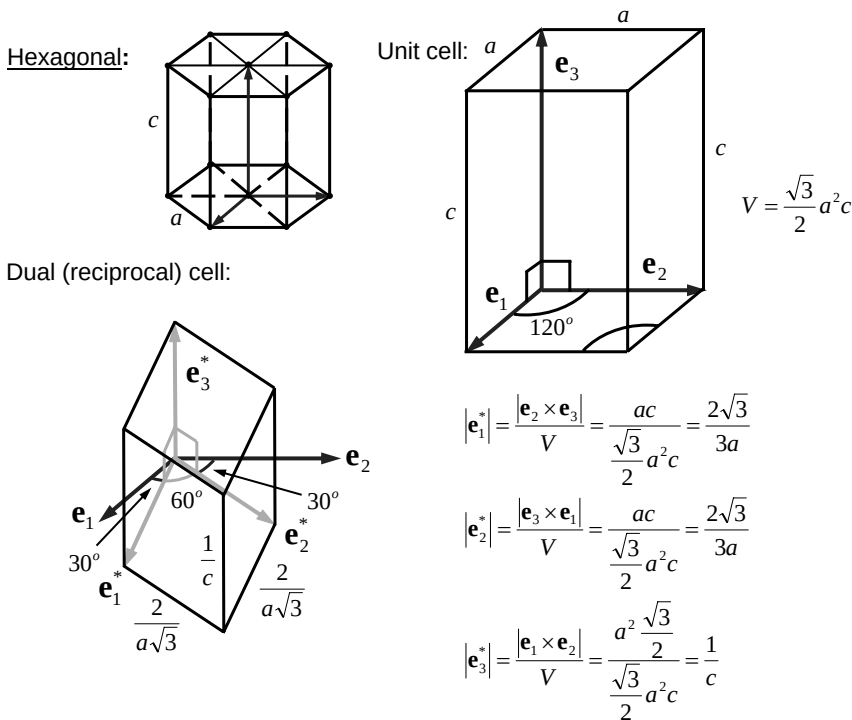


Figure 15.5: Unit cell and dual (reciprocal) lattice cell for cubic crystals.

1. Find the intercepts $\mathbf{a} = [a00]$, $\mathbf{b} = [0b0]$, and $\mathbf{c} = [00c]$ of the plane with the lattice directions, noting that a , b , and c are described as fractions of the associated lattice vectors.
2. Calculate the reciprocals $1/a$, $1/b$, and $1/c$.
3. Clear the fractions by multiplying the reciprocals by a number p that is the smallest number to make all reciprocals integers to describe the Miller indices of the plane (hkl) as

$$(hkl) = \left(\frac{p}{a} \frac{p}{b} \frac{p}{c} \right).$$

There is normally a special method that uses four indices to describe directions and planes for hexagonal and rhombohedral lattices. Planes of hexagonal lattices normally are given by the description $(h'k'l'i')$ where h' , k' , and l' are the same as the h , k , and l for the Miller indices representing the plane (hkl) and $i = -h' - k'$ is redundant and can be ignored. For example, the plane (123) is the same as $(12\bar{3}3)$. The index i in this description is associated with lattice direction $\mathbf{e}_4$ shown in Figure 15.2. For describing directions as $[h'k'l'i']$ using a four-index system we can convert the normal Miller indices $[hkl]$ using the

relations:

$$h' = \frac{1}{3}(2h - k), \quad (15.17)$$

$$k' = \frac{1}{3}(2k - h), \quad (15.18)$$

$$h' = -h' - k', \quad (15.19)$$

$$l' = l. \quad (15.20)$$

For example, the direction $[123]$ is the same as $[0\bar{3}\bar{3}]$. The inverse relation is given as

$$h = 2h' + k', \quad (15.21)$$

$$k = 2k' + h', \quad (15.22)$$

$$l = l'. \quad (15.23)$$

For example, the direction $[1\bar{1}\bar{2}1]$ is the same as $[331]$.

Slip in crystals is described by providing the crystal's slip systems, which are pairs containing one slip plane and a slip direction in this plane. Normally, similar slip systems are organized into families of similar planes and similar directions. The notation $\{hkl\}$ is used to denote the family of similar planes constructed from taking the plane (hkl) and transforming it using the symmetries of the crystal. Similarly, the notation $\langle hkl \rangle$ is used to denote the family of similar directions obtained by taking the direction $[hkl]$ and transforming it using the symmetry transformations of the crystal. For example, a family of slip systems could be written as $\{111\} \langle 100 \rangle$, which denotes the family of planes similar to (111) and the directions similar to $[100]$ that fall within them. The reader will note that not all directions in a family $\langle hkl \rangle$ will fall in each plane of the family $\{mno\}$. As a result, for each plane we need to check the directions to see which are in the plane. The direction $\mathbf{a}$ is in the plane with normal $\mathbf{n}$ if $\mathbf{a} \circ \mathbf{n} = 0$, so that if $\mathbf{n}$ is represented by $\mathbf{n} = n_i^* \mathbf{e}_i^*$ in the dual base and $\mathbf{a}$ is represented as $\mathbf{a} = a_i \mathbf{e}_i$ in the lattice base, then we will have

$$\mathbf{a} \circ \mathbf{n} = (a_i \mathbf{e}_i) \circ (n_j^* \mathbf{e}_j^*) = a_i n_j^* (\mathbf{e}_i \circ \mathbf{e}_j^*) = a_i n_j^* \delta_{ij} = a_i n_i^* = 0. \quad (15.24)$$

This exposes the advantage of using the lattice base for the description of vectors and the dual base for description of planes, since checking for normality is transformed into simple multiplication of components when describing one vector in the lattice base and the other in the dual base. For example, it easily can be seen that the direction $[1\bar{1}0]$ is in the plane (111) by calculating $(1)(1) + (-1)(1) + (0)(1) = 0$. Table 15.1 shows the common slip systems in cubic and hexagonal systems.

Not all the slip systems in a crystal are activated as easily, so we need not always consider all the slip systems of a crystal when considering plastic flow. Normally, there are preferred families of slip system for each material. Most of the metals are face-centered cubic (FCC), body-centered cubic (BCC), or hexagonal close-packed (HCP). The preferred slip systems for some of the FCC and BCC crystals are given in Table 15.2, and for some of the HCP crystals in Table 15.4. As the reader may have noted, to calculate the directions and planes in the cubic crystals we need no information of the crystal size so that no information is provided in Table 15.2 in regards to the lattice size. On the other hand, for hexagonal crystals we need to know the ratio of a and c lattice dimension to obtain physical orientation of vectors and planes. As a result, in Table 15.4 the c/a ratio also is given.

For example, let us consider a material such as copper which has a FCC crystal structure. As indicated in Table 15.2, the preferred slip system for copper is $\{111\} \langle 110 \rangle$. The cubic symmetry (see Appendix B) is given by the transformations

$$\mathbf{I}, \quad \mathbf{D}_1, \quad \mathbf{D}_2, \quad \mathbf{D}_3, \quad \mathbf{M}_1, \quad \mathbf{M}_2 \quad (15.25)$$

$$\mathbf{D}_1 \mathbf{M}_1, \quad \mathbf{D}_2 \mathbf{M}_1, \quad \mathbf{D}_3 \mathbf{M}_1, \quad \mathbf{D}_1 \mathbf{M}_2, \quad \mathbf{D}_2 \mathbf{M}_2, \quad \mathbf{D}_3 \mathbf{M}_2, \quad (15.26)$$

Table 15.1: Common slip system for different crystal types.

Crystal type	Slip plane	Slip direction	Number of planes	Number of direction in each plane	Number of slip systems
FCC	{111}	$\langle 110 \rangle$	4	3	$3 \times 4 \times 2 = 24$
BCC	{110}	$\langle 111 \rangle$	6	2	$6 \times 2 \times 2 = 24$
	{112}	$\langle 111 \rangle$	12	1	$12 \times 1 \times 2 = 24$
	{123}	$\langle 111 \rangle$	24	1	$24 \times 1 \times 2 = 48$
HCP	{0001}	$\langle 11\bar{2}0 \rangle$	1	3	$1 \times 3 \times 2 = 6$
	$\{10\bar{1}0\}^1$	$\langle 11\bar{2}0 \rangle$	3	1	$3 \times 1 \times 2 = 6$
	$\{10\bar{1}0\}^1$	$\langle 11\bar{2}0 \rangle$	3	1	$3 \times 1 \times 2 = 6$
	$\{11\bar{2}0\}^1$	$\langle 11\bar{2}0 \rangle$	3	1	$3 \times 1 \times 2 = 6$

1 Active in special crystals or high temperatures.

where

$$\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i, \quad (15.27)$$

$$\mathbf{D}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (15.28)$$

$$\mathbf{D}_2 = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (15.29)$$

$$\mathbf{D}_3 = -\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (15.30)$$

$$\mathbf{M}_1 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, \quad (15.31)$$

$$\mathbf{M}_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_2. \quad (15.32)$$

Fortunately, for a cubic crystal the lattice directions and the dual directions are parallel so we need not worry about their difference. Since the magnitude of the lattice base vectors are equal for a cubic system, and since the directions do not get altered by scaling, for the cubic crystals we can treat the base vectors like unit vectors without any loss of generality. Doing this, and taking $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ to represent the (111) plane, we get

$$\begin{aligned}
\mathbf{In} &\rightarrow (111), & \mathbf{D}_1\mathbf{n} &\rightarrow (1\bar{1}\bar{1}), \\
\mathbf{D}_2\mathbf{n} &\rightarrow (\bar{1}1\bar{1}), & \mathbf{D}_3\mathbf{n} &\rightarrow (\bar{1}\bar{1}1), \\
\mathbf{M}_1\mathbf{n} &\rightarrow (111), & \mathbf{M}_2\mathbf{n} &\rightarrow (111), \\
\mathbf{D}_1\mathbf{M}_1\mathbf{n} &\rightarrow (1\bar{1}\bar{1}), & \mathbf{D}_1\mathbf{M}_2\mathbf{n} &\rightarrow (1\bar{1}\bar{1}), \\
\mathbf{D}_2\mathbf{M}_1\mathbf{n} &\rightarrow (\bar{1}1\bar{1}), & \mathbf{D}_2\mathbf{M}_2\mathbf{n} &\rightarrow (\bar{1}\bar{1}1), \\
\mathbf{D}_3\mathbf{M}_1\mathbf{n} &\rightarrow (\bar{1}\bar{1}1), & \mathbf{D}_3\mathbf{M}_2\mathbf{n} &\rightarrow (\bar{1}\bar{1}1).
\end{aligned} \quad (15.33)$$

Therefore, the family {111} of planes in the cubic crystals are given by

$$(111), \quad (\bar{1}1\bar{1}), \quad (1\bar{1}\bar{1}), \quad (\bar{1}\bar{1}1). \quad (15.34)$$

We have ignored planes that are essentially the same, but with a negative normal. The family of $\langle 110 \rangle$ directions can be obtained, for $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$, as

$$\begin{aligned}
\mathbf{Ia} &\rightarrow [110], & \mathbf{D}_1\mathbf{a} &\rightarrow [1\bar{1}0], \\
\mathbf{D}_2\mathbf{a} &\rightarrow [\bar{1}10], & \mathbf{D}_3\mathbf{a} &\rightarrow [\bar{1}\bar{1}0], \\
\mathbf{M}_1\mathbf{a} &\rightarrow [101], & \mathbf{M}_2\mathbf{a} &\rightarrow [011], \\
\mathbf{D}_1\mathbf{M}_1\mathbf{a} &\rightarrow [10\bar{1}], & \mathbf{D}_1\mathbf{M}_2\mathbf{a} &\rightarrow [0\bar{1}\bar{1}], \\
\mathbf{D}_2\mathbf{M}_1\mathbf{a} &\rightarrow [\bar{1}0\bar{1}], & \mathbf{D}_2\mathbf{M}_2\mathbf{a} &\rightarrow [01\bar{1}], \\
\mathbf{D}_3\mathbf{M}_1\mathbf{a} &\rightarrow [\bar{1}01], & \mathbf{D}_3\mathbf{M}_2\mathbf{a} &\rightarrow [0\bar{1}1].
\end{aligned} \quad (15.35)$$

Table 15.2: Preferred slip systems and their associated critical resolved shear stress for cubic crystals. When more than one system is given, they are provided in the order of preference.

Metal	Preferred slip systems		τ_c^α (MPa) Room temperature	Purity %
	Room temperature	High temperature		
FCC lattices				
Cu	{111} < 110 >	{100} < 110 >	0.35, ¹ 0.65 ² 0.94 ²	99.999 99.98
Au	{111} < 110 >	{100} < 110 >	0.50 ¹	99.999
Ag	{111} < 110 >	{100} < 110 >	0.40–0.70 ¹ 0.48 ² 0.73 ² 1.31 ²	99.999 99.99 99.97 99.3
Ni	{111} < 110 >	{100} < 110 >	3.30–7.50 ¹	99.98
Al	{111} < 110 > {100} < 110 >	{100} < 110 >	0.55–1.00 ¹	99.994
BCC lattices				
Cr	{123} < 111 >	{123} <i>and</i> {hkl} < 111 >	15.00, ¹ 28 ²	99.96
Fe	{110} < 111 > {112} < 111 > {123} < 111 >			
Fe+3% Si	{110} < 111 > {112} < 111 >			
Mo	{110} < 111 > {112} < 111 >	50 ³	50 ²	99.8
Nb	{110} < 111 >			
Ni	{110} < 111 >			
Ta	{110} < 111 > {112} < 111 > {123} < 111 >	{112} < 111 >	34.00 ¹ 5.8 ²	
V, Mo-Re	{110} < 111 > {112} < 111 >			
W	{110} < 111 > {112} < 111 >			

1 *Physical Principles of Plastic Deformation*, Polukhin, Gorelik, and Vorontsov, Mir Publications, Moscow, 1983.

2 *Structure, Deformation, and Integrity of Materials: Volume I. Fundamental*, With, Wiley-VCH, 2006.

Therefore, the family of directions $\langle 110 \rangle$ for the cubic crystal, after eliminating negatives of existing terms, consists of

$$[110], \quad [101], \quad [011], \quad [\bar{1}10], \quad [\bar{1}01], \quad [0\bar{1}1]. \quad (15.36)$$

Not all the directions in $\langle 110 \rangle$ are in all the planes in $\{111\}$. It is easy to see, for example, that $[110]$ is not a direction in plane (111) since $(1)(1) + (1)(1) + (1)(0) = 2 \neq 0$. As shown in Figure 15.3, only three of these vectors are in the (111) plane. As a result, we will have the 12 slip systems (24 when we count each reverse slip as a distinct slip system) given in Table 15.3.

Table 15.3: Slip system $\{111\} \langle 110 \rangle$ for a cubic-tetartoidal crystal class.

Plane	$\pm$ Directions		
(111)	$[110]$,	$[101]$,	$[011]$
$(\bar{1}\bar{1}\bar{1})$	$[110]$,	$[011]$,	$[\bar{1}01]$
$(1\bar{1}\bar{1})$	$[110]$,	$[101]$,	$[0\bar{1}1]$
$(\bar{1}\bar{1}1)$	$[101]$,	$[011]$,	$[\bar{1}10]$

Table 15.4: Preferred slip systems and their associated critical resolved shear stress for hexagonal crystals. When more than one system is given, they are provided in the order of preference.

Metal	Preferred slip systems		τ_c^α (MPa) Room temperature	Purity %
	Room temperature	High temperature		
HCP lattices				
Cd ($c/a = 1.886$)	$\{0001\} \langle 11\bar{2}0 \rangle$ $\{1101\} \langle 11\bar{2}0 \rangle$	$\{10\bar{1}0\} \langle 11\bar{2}0 \rangle$ $\{0001\} \langle 11\bar{2}0 \rangle$	0.13^1	99.999
Co ($c/a = 1.621$)	$\{0001\} \langle 11\bar{2}0 \rangle$		6.50^1	99.999
Mg ($c/a = 1.624$)	$\{0001\} \langle 11\bar{2}0 \rangle$ $\{1\bar{1}00\} \langle 11\bar{2}0 \rangle$	$\{1011\} \langle 11\bar{2}0 \rangle$ $\{0001\} \langle 11\bar{2}0 \rangle$	0.50^1	99.99
Ti ($c/a = 1.59$)	$\{1010\} \langle 11\bar{2}0 \rangle$	$\{0001\} \langle 11\bar{2}0 \rangle$	$14.0^{1,2}$ 92^2	99.99 99.9
Zn ($c/a = 1.856$)	$\{0001\} \langle 11\bar{2}0 \rangle$ $\{11\bar{2}2\} \langle 11\bar{2}0 \rangle$	$\{10\bar{1}0\} \langle 11\bar{2}0 \rangle$ $\{11\bar{2}2\} \langle 11\bar{2}3 \rangle$	$0.30, ^1 0.18^2$	99.999
Zr ($c/a = 1.59$)	$\{10\bar{1}0\} \langle 11\bar{2}0 \rangle$	$\{0001\} \langle 11\bar{2}0 \rangle$	7.0^1	99.0

1 *Physical Principles of Plastic Deformation*, Polukhin, Gorelik, and Vorontsov, Mir Publications, Moscow, 1983.

2 *Structure, Deformation, and Integrity of Materials: Volume I. Fundamental*, With, Wiley-VCH, 2006.

Section 15.1-Problems:

15.1-1: Find all the planes in the family $\{112\}$ for a cubic crystal.

15.1-2: Find all the planes in the family $\{123\}$ for a cubic crystal.

15.1-3: Calculate all the slip systems (planes and slip directions) associated with $\{110\} \langle 111 \rangle$, assuming a cubic crystal.

- 15.1-4:** Transform to the form using three Miller indices the description of the slip system given by $\{0001\} \langle 11\bar{2}0 \rangle$ for a hexagonal crystal.
- 15.1-5:** Use the trigonal-dipyramidal symmetry described in Appendix B to find all the directions in the family $\langle 11\bar{2}0 \rangle$.

15.2 Elastic crystal distortion

The distortion of the crystal is assumed to be elastic and described by anisotropic elastic laws consistent with the symmetry of the crystal lattice. Even though one can use any general model for stress consistent with the symmetry of the crystal, we will focus here on a model for which the material parameters are readily available yet that works for finite deformations. The model that we will be considering is one that has a specific free energy that is quadratic in the elastic and plastic Green strains. This model, which was described in Chapter 12, when restricted to small strains reduced to the classical infinitesimal theory of elasticity. The specific free energy for this model was given by

$$\psi = \frac{1}{2\rho_o} \mathbf{E}^e : \mathbf{E} : \mathbf{E}^e + \frac{1}{2\rho_o} \mathbf{E}^p : \mathbf{E}^b : \mathbf{E}^p + \psi^\theta, \quad (15.37)$$

where $\mathbf{E}$ and $\mathbf{E}^b$ are fourth-order moduli tensors and ψ^θ is a scalar, and all three are assumed to be functions of only temperature, even though in most of this chapter we only consider mechanical models that ignore the thermal effects. It was shown that one can select the moduli tensors such that they have the properties

$$\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{A}^T : \mathbf{E} : \mathbf{B} = \mathbf{A} : \mathbf{E} : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E} : \mathbf{B}^T, \quad (15.38)$$

$$\mathbf{A} : \mathbf{E}^b : \mathbf{B} = \mathbf{A}^T : \mathbf{E}^b : \mathbf{B} = \mathbf{A} : \mathbf{E}^b : \mathbf{B}^T = \mathbf{A}^T : \mathbf{E}^b : \mathbf{B}^T, \quad (15.39)$$

for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$, and

$$\mathbf{E} = \mathbf{E}^T, \quad \mathbf{E}^b = \mathbf{E}^{bT}. \quad (15.40)$$

In this case, the expressions for Cauchy stress, back-stress, and entropy are given by

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\mathbf{E} : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (15.41)$$

$$\mathbf{T}^{bT} = \frac{1}{J} \mathbf{F}^p (\mathbf{E}^b : \mathbf{E}^p) \mathbf{F}^{pT}, \quad (15.42)$$

$$\eta = -\frac{1}{2\rho_o} \mathbf{E}^e : \frac{d\mathbf{E}}{d\theta} : \mathbf{E}^e - \frac{1}{2\rho_o} \mathbf{E}^p : \frac{d\mathbf{E}^b}{d\theta} : \mathbf{E}^p - \frac{d\psi^\theta}{d\theta} + \frac{1}{\rho} \mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T} : \boldsymbol{\alpha}, \quad (15.43)$$

where we are using the decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta \mathbf{F}^p$ with the model for thermal expansion given by $\mathbf{L}^\theta = \boldsymbol{\alpha} \dot{\theta}$. In the case of infinitesimal elastic deformations ($\|\mathbf{H}^e\| \ll 1$), the stress can be written as

$$\mathbf{T} = \mathbf{E} : \mathbf{E}^e = \mathbf{E} : \boldsymbol{\epsilon}^e. \quad (15.44)$$

For large rigid body motions imposed on small elastic strains ($\|\mathbf{H}_U^e\| \ll 1$), this can be written as

$$\mathbf{T} = \mathbf{R}^e(\mathbf{E} : \mathbf{E}^e)\mathbf{R}^{eT} = \mathbf{R}^e(\mathbf{E} : \boldsymbol{\epsilon}_{IJ}^e)\mathbf{R}^{eT}. \quad (15.45)$$

The result of the conditions (15.38) to (15.40) is that each modulus tensor only contains 21 independent components (as opposed to $3 \times 3 \times 3 \times 3 = 81$ components). As shown in Appendix B, each one of these moduli tensors can be written as

$$\begin{aligned} \mathbf{E} = & \mu_{11}\mathbf{A}_{11} + \mu_{12}\mathbf{A}_{12} + \mu_{13}\mathbf{A}_{13} + \mu_{14}\mathbf{A}_{14} + \mu_{15}\mathbf{A}_{15} + \mu_{16}\mathbf{A}_{16} \\ & + \mu_{22}\mathbf{A}_{22} + \mu_{23}\mathbf{A}_{23} + \mu_{24}\mathbf{A}_{24} + \mu_{25}\mathbf{A}_{25} + \mu_{26}\mathbf{A}_{26} \\ & + \mu_{33}\mathbf{A}_{33} + \mu_{34}\mathbf{A}_{34} + \mu_{35}\mathbf{A}_{35} + \mu_{36}\mathbf{A}_{36} \\ & + \mu_{44}\mathbf{A}_{44} + \mu_{45}\mathbf{A}_{45} + \mu_{46}\mathbf{A}_{46} \\ & + \mu_{55}\mathbf{A}_{55} + \mu_{56}\mathbf{A}_{56} \\ & + \mu_{66}\mathbf{A}_{66}, \end{aligned} \quad (15.46)$$

where $\mathbf{A}_{ij}$ are 21 base tensors for representing a fourth-order tensor with the characteristics given by (15.38) to (15.40). These base vectors are given by

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{E}_i \otimes \mathbf{E}_j & \text{for } j = i, \\ \mathbf{E}_i \otimes \mathbf{E}_j + \mathbf{E}_j \otimes \mathbf{E}_i & \text{for } j > i, \end{cases} \quad (15.47)$$

where

$$\mathbf{E}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1, \quad (15.48)$$

$$\mathbf{E}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2, \quad (15.49)$$

$$\mathbf{E}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (15.50)$$

$$\mathbf{E}_4 = \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2, \quad (15.51)$$

$$\mathbf{E}_5 = \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, \quad (15.52)$$

$$\mathbf{E}_6 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \quad (15.53)$$

for the orthonormal vector base $\mathbf{e}_i$. The components of a general modulus tensor can be arranged as follows

$$\begin{array}{cccccc} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} \\ & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} \\ & & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36} \\ & & & \mu_{44} & \mu_{45} & \mu_{46} \\ & & & & \mu_{55} & \mu_{56} \\ & & & & & \mu_{66} \end{array} \quad (15.54)$$

For cubic crystals, as shown in Appendix B for crystals of types 6 and 7, there are only three independent components with all the components given as

$$\begin{array}{cccccc} \mu_{11} & \mu_{12} & \mu_{12} & 0 & 0 & 0 \\ & \mu_{11} & \mu_{12} & 0 & 0 & 0 \\ & & \mu_{11} & 0 & 0 & 0 \\ & & & \mu_{44} & 0 & 0 \\ & & & & \mu_{44} & 0 \\ & & & & & \mu_{44} \end{array} \quad (15.55)$$

Table 15.5 shows typical elastic moduli for cubic crystals for a number of elements.

For hexagonal close-packed (HCP) crystals, as shown in Appendix B for crystals of types 10 and 11, there are five independent components of the modulus tensor, with all the

Table 15.5: Elastic moduli for cubic crystals (GPa).¹

Material	μ_{11}	μ_{12}	μ_{44}
FCC lattices			
Ag	120.0	90.0	43.0
Al	108.2	62.3	28.5
Au	186.0	157.0	42.0
Cu	108.4	121.4	75.4
Ni	246.5	147.3	124.7
Pb	46.6	39.2	14.4
BCC lattices			
Fe	287.0	141.0	116.0
K	4.6	3.7	3.6
Mo	460.0	176.0	110.0
Na	6.0	4.6	5.9
W	501.0	198.0	151.4

¹ *Physical Principles of Plastic Deformation*, Polukhin, Gorelik, and Vorontsov, Mir Publications, Moscow, 1983.

components given as

$$\begin{array}{ccccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & 0 & \\
 & \mu_{11} & \mu_{13} & 0 & 0 & 0 & \\
 & & \mu_{33} & 0 & 0 & 0 & \\
 & & & \mu_{44} & 0 & 0 & \\
 & & & & \mu_{44} & 0 & \\
 & & & & & \frac{1}{2}(\mu_{11} - \mu_{12}) &
 \end{array} \quad (15.56)$$

Table 15.6 shows typical elastic moduli for HPC crystals for several elements.

Table 15.6: Elastic moduli for hexagonal close-packed crystals (GPa).

Material	c/a	μ_{11}	μ_{12}	μ_{13}	μ_{33}	μ_{44}
Cd ¹	1.886	121.0	48.1	44.2	51.3	18.5
Co ²	1.621	306.0	165.0	102.0	357.0	75.0
Mg ¹	1.623	59.7	26.2	21.7	61.7	16.4
Re ²		616.0	273.0	206.0	683.0	161.0
Zn ¹	1.856	161.0	34.2	50.1	61.0	38.3

¹ *Physical Principles of Plastic Deformation*, Polukhin, Gorelik, and Vorontsov, Mir Publications, Moscow, 1983.

² First-Principle elastic constants for hpc transition metals Fe, Co, Re at high pressure, Steinle-Neumann, Stixrude, Cohen, *Physical Review B*, 60, 791–99, 1999.

Section 15.2-Problems:

15.2-1: For infinitesimal deformations and assuming a cubic crystal, calculate the expression for stress in terms of strain.

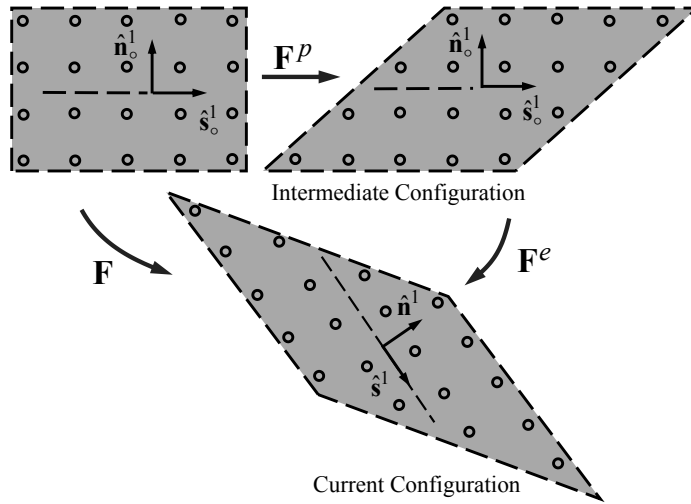


Figure 15.6: In the mechanical theory of crystal plasticity, it is assumed that the deformation occurs by slip along slip planes to yield plastic flow and then elastic distortion superimposed on top of that, which results in distortion of the crystal structure.

- 15.2-2:** For infinitesimal deformations and assuming a hexagonal crystal, calculate the expression for stress in terms of strain.

15.3 Kinematics of single-crystal deformation

We will focus on the mechanical theory of crystal deformation in this and the following sections, ignoring any thermal effects. As shown in Figure 15.6, deformation of a body can be resolved into two parts. First, we have slip along each slip system resulting in plastic flow in the form of simple shear. As shown in the figure, the shear of the body due to slip does not result in any change of the slip plane or the slip direction. Therefore, if the slip direction in the initial configuration is given by $\hat{s}_o^\alpha$ and the slip plane is given by the unit normal $\hat{n}_o^\alpha$, these same vectors represent the slip direction and the slip plane after plastic flow due to initiation of the slip system. The second part of the deformation is an elastic deformation imposed on top of the plastic flow. As shown in Figure 15.6, the elastic deformation rotates and distorts the crystal so both the direction of the slip and the slip plane change. After the elastic deformation, the slip system will have a slip plane with normal $\hat{n}^\alpha$ and slip direction $\hat{s}^\alpha$. If the plastic deformation gradient is given by $\mathbf{F}^p$ and the elastic deformation gradient is given by $\mathbf{F}^e$, there are the relations

$$\mathbf{s}^\alpha = \mathbf{F}^e \hat{s}_o^\alpha, \quad \mathbf{n}^\alpha = \hat{n}_o^\alpha \mathbf{F}^{e-1}.$$

The first relation is a result of the slip direction being a material direction, which is extended and rotated by the deformation gradient $\mathbf{F}^e$. On the other hand, since in the current

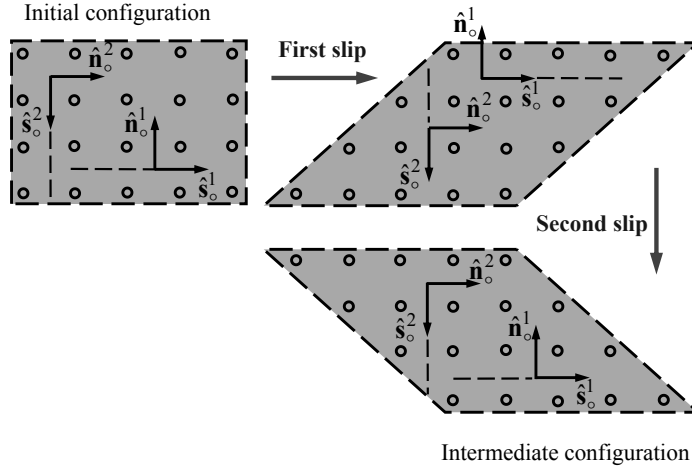


Figure 15.7: Consecutive slips, first along the first slip system and then along the second slip system.

configuration the two material lines along $\hat{\mathbf{n}}_o^\alpha$ and $\hat{\mathbf{s}}_o^\alpha$ in the intermediate configuration do not remain orthogonal, we cannot use the same relation for obtaining $\mathbf{n}^\alpha$. The proposed relation results in a vector $\mathbf{n}^\alpha$ that is orthogonal to $\mathbf{s}^\alpha$ since

$$\mathbf{n}^\alpha \circ \mathbf{s}^\alpha = (\hat{\mathbf{n}}_o^\alpha \mathbf{F}^{e-1}) \circ (\mathbf{F}^e \hat{\mathbf{s}}_o^\alpha) = \hat{\mathbf{n}}_o^\alpha \circ \hat{\mathbf{s}}_o^\alpha = 0.$$

The unit vectors describing the slip system easily are obtained by normalizing $\mathbf{n}^\alpha$ and $\mathbf{s}^\alpha$ using the relations

$$\hat{\mathbf{n}}^\alpha = \frac{1}{\sqrt{\mathbf{n}^\alpha \circ \mathbf{n}^\alpha}} \mathbf{n}^\alpha, \quad \hat{\mathbf{s}}^\alpha = \frac{1}{\sqrt{\mathbf{s}^\alpha \circ \mathbf{s}^\alpha}} \mathbf{s}^\alpha.$$

The plastic deformation associated with the motion of each slip system results in simple shear along the direction of slip. This deformation can be written as

$$\mathbf{F}^{p\alpha} = \mathbf{I} + \gamma^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha, \quad (15.57)$$

where $\gamma^{p\alpha}$ is known as the resolved shear strain for the slip system. As shown in Figure 15.6, even though the material lines undergo simple shear during the plastic deformation, the orientation of the slip system does not change, as is indicated by using the same vectors $\hat{\mathbf{n}}_o^\alpha$ and $\hat{\mathbf{s}}_o^\alpha$ to describe the slip system in the initial and intermediate configurations. Since $\hat{\mathbf{n}}_o^\alpha$ and $\hat{\mathbf{s}}_o^\alpha$ are constant, the rate of change of the plastic deformation gradient is given by

$$\dot{\mathbf{F}}^{p\alpha} = \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha. \quad (15.58)$$

The velocity gradient associated with the plastic deformation gradient now can be calculated for this slip system. This calculation can be written as

$$\mathbf{L}^{p\alpha} = \dot{\mathbf{F}}^{p\alpha} \mathbf{F}^{p-1\alpha} = \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha (\mathbf{I} - \gamma^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha). \quad (15.59)$$

It follows that

$$\mathbf{L}^{p\alpha} = \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha. \quad (15.60)$$

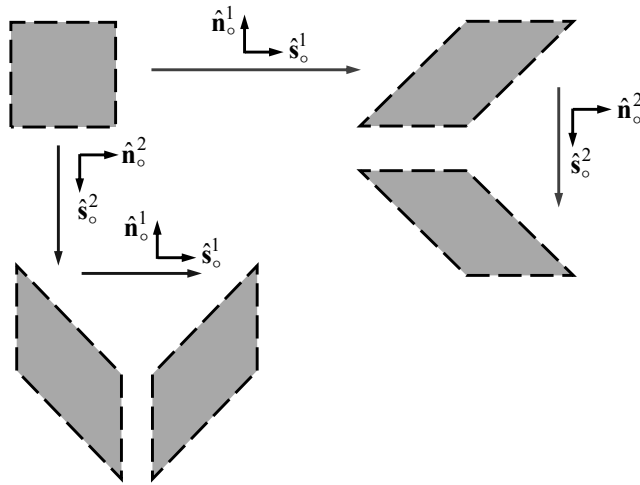


Figure 15.8: A schematic showing the effect of changing the order of slip.

This results in the expressions for the rate of deformation and spin tensors given by

$$\mathbf{D}^{p\alpha} = \frac{1}{2}(\mathbf{L}^{p\alpha} + \mathbf{L}^{pT\alpha}) = \frac{\dot{\gamma}^{p\alpha}}{2}(\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha + \hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha), \quad (15.61)$$

$$\mathbf{W}^{p\alpha} = \frac{1}{2}(\mathbf{L}^{p\alpha} - \mathbf{L}^{pT\alpha}) = \frac{\dot{\gamma}^{p\alpha}}{2}(\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha - \hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha). \quad (15.62)$$

Since the velocity gradients add, the total velocity gradient associated with the plastic flow given by m slip systems can be written as

$$\mathbf{L}^p = \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha. \quad (15.63)$$

The associated plastic deformation rate and spin tensors are

$$\mathbf{D}^p = \sum_{\alpha=1}^m \frac{\dot{\gamma}^{p\alpha}}{2}(\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha + \hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha), \quad (15.64)$$

$$\mathbf{W}^p = \sum_{\alpha=1}^m \frac{\dot{\gamma}^{p\alpha}}{2}(\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha - \hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha). \quad (15.65)$$

As is shown in Figure 15.7, motion along each slip system results in shear along the direction associated with that particular slip system. As indicated in Figure 15.8, the sequence in which slip systems are activated does affect the final configuration of the intermediate configuration. Yet, all the needed information is contained in the $\mathbf{L}^p$, which can be used directly to calculate $\mathbf{F}^p$ through the relation $\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p$.

As was shown in Chapter 4, for a purely mechanical decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$, we have the total velocity gradient given by

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1}.$$

Introducing the expression for plastic velocity gradient, we get

$$\mathbf{L} = \mathbf{L}^e + \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \mathbf{F}^e \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha \mathbf{F}^{e-1},$$

which results in

$$\boxed{\mathbf{L} = \mathbf{L}^e + \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \mathbf{s}^\alpha \otimes \mathbf{n}^\alpha.} \quad (15.66)$$

To calculate $\mathbf{n}^\alpha = \hat{\mathbf{n}}_o^\alpha \mathbf{F}^{e-1}$ and $\mathbf{s}^\alpha = \mathbf{F}^e \hat{\mathbf{s}}_o^\alpha$ from the velocity gradients, one can differentiate these expressions to get

$$\dot{\mathbf{s}}^\alpha = \dot{\mathbf{F}}^e \mathbf{s}_o^\alpha = \dot{\mathbf{F}}^e \mathbf{F}^{e-1} \mathbf{F}^e \mathbf{s}_o^\alpha, \quad \dot{\mathbf{n}}^\alpha = \dot{\hat{\mathbf{n}}_o^\alpha \overline{\mathbf{F}^{e-1}}} = -\hat{\mathbf{n}}_o^\alpha \mathbf{F}^{e-1} \dot{\mathbf{F}}^e \mathbf{F}^{e-1}.$$

This results in

$$\boxed{\dot{\mathbf{s}}^\alpha = \mathbf{L}^e \mathbf{s}^\alpha,} \quad \boxed{\dot{\mathbf{n}}^\alpha = -\mathbf{n}^\alpha \mathbf{L}^e.} \quad (15.67)$$

The rates of change of the normalized vectors $\hat{\mathbf{n}}^\alpha$ and $\hat{\mathbf{s}}^\alpha$ easily can be shown to be given by

$$\boxed{\dot{\hat{\mathbf{n}}}^\alpha = -\frac{1}{\sqrt{\mathbf{n}^\alpha \circ \mathbf{n}^\alpha}} [\mathbf{n}^\alpha \mathbf{L}^e - \frac{\mathbf{n}^\alpha \circ (\mathbf{n}^\alpha \mathbf{L}^e)}{\mathbf{n}^\alpha \circ \mathbf{n}^\alpha} \mathbf{n}^\alpha],} \quad (15.68)$$

$$\boxed{\dot{\hat{\mathbf{s}}}^\alpha = \frac{1}{\sqrt{\mathbf{s}^\alpha \circ \mathbf{s}^\alpha}} [\mathbf{L}^e \mathbf{s}^\alpha - \frac{\mathbf{s}^\alpha \circ (\mathbf{L}^e \mathbf{s}^\alpha)}{\mathbf{s}^\alpha \circ \mathbf{s}^\alpha} \mathbf{s}^\alpha].} \quad (15.69)$$

Both the plastic deformation gradient and the elastic deformation gradient can be calculated by integration using the standard expressions

$$\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p, \quad \dot{\mathbf{F}}^e = \mathbf{L}^e \mathbf{F}^e. \quad (15.70)$$

Section 15.3-Problems:

15.3-1: Prove the relations

$$\frac{\dot{|\mathbf{s}^\alpha|}}{|\mathbf{s}^\alpha|} = \frac{1}{|\mathbf{s}^\alpha|} (\mathbf{L}^e \mathbf{s}^\alpha) \circ \mathbf{s}^\alpha = \frac{1}{|\mathbf{s}^\alpha|} [(\mathbf{F}^e \hat{\mathbf{s}}_o^\alpha) \otimes \hat{\mathbf{s}}_o^\alpha] : \dot{\mathbf{F}}^e, \quad (15.71)$$

$$\frac{\dot{|\mathbf{n}^\alpha|}}{|\mathbf{n}^\alpha|} = -\frac{1}{|\mathbf{n}^\alpha|} (\mathbf{n}^\alpha \mathbf{L}^e) \circ \mathbf{n}^\alpha = -\frac{1}{|\mathbf{n}^\alpha|} [\mathbf{n}^\alpha \otimes (\mathbf{n}^\alpha \mathbf{F}^{e-T})] : \dot{\mathbf{F}}^e, \quad (15.72)$$

and use them to derive the relations given for the rates of change of $\hat{\mathbf{n}}^\alpha$ and $\hat{\mathbf{s}}^\alpha$.

15.4 Resolved shear stress and overstress

As described in the introduction, the resolved shear stress is the component of the traction vector on the slip plane that is along the slip direction. For the α th slip system the resolved shear stress is given by

$$\boxed{\tau^\alpha = (\mathbf{T}^T \hat{\mathbf{n}}^\alpha) \circ \hat{\mathbf{s}}^\alpha = (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{T}.} \quad (15.73)$$

In terms of the slip plane and slip directions in the initial undeformed configuration, this can be written as

$$\tau^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} \hat{\mathbf{s}}_o^\alpha \circ [(\mathbf{F}^{eT} \mathbf{T}^T \mathbf{F}^{e-T}) \hat{\mathbf{n}}_o^\alpha] = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : (\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e), \quad (15.74)$$

which suggests the introduction of the stress $\tilde{\mathbf{T}}$ defined as

$$\boxed{\tilde{\mathbf{T}} \equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e}, \quad (15.75)$$

so that we have

$$\boxed{\tau^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} \hat{\mathbf{s}}_o^\alpha \circ (\tilde{\mathbf{T}}^T \hat{\mathbf{n}}_o^\alpha) = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \tilde{\mathbf{T}}}. \quad (15.76)$$

$\tilde{\mathbf{T}}$ is not influenced by rigid body motions, and neither are $|\hat{\mathbf{s}}^\alpha|$ and $|\hat{\mathbf{n}}^\alpha|$. The rate of change of the resolved shear stress can be calculated from the expression

$$\boxed{\dot{\tau}^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\tilde{\mathbf{T}}} - \tau^\alpha \left(\frac{\dot{|\mathbf{s}^\alpha|}}{|\mathbf{s}^\alpha|} + \frac{\dot{|\mathbf{n}^\alpha|}}{|\mathbf{n}^\alpha|} \right)}. \quad (15.77)$$

In general, for a thermomechanical theory we can assume that $\tilde{\mathbf{T}}$ is a function of $(\mathbf{F}^e, \mathbf{F}^p, \theta)$ so that

$$\dot{\tilde{\mathbf{T}}} = \partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^p}(\tilde{\mathbf{T}}) : \dot{\mathbf{F}}^p + \partial_\theta(\tilde{\mathbf{T}}) \dot{\theta}. \quad (15.78)$$

We can replace $\dot{\mathbf{F}}^e$ with $\mathbf{L}^e \mathbf{F}^e$ and $\dot{\mathbf{F}}^p$ with $\mathbf{L}^p \mathbf{F}^p$ in this expression to get

$$\dot{\tilde{\mathbf{T}}} = [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : \mathbf{L}^e + [\partial_{\mathbf{F}^p}(\tilde{\mathbf{T}}) \mathbf{F}^{pT}] : \mathbf{L}^p + \partial_\theta(\tilde{\mathbf{T}}) \dot{\theta}. \quad (15.79)$$

Now, we can introduce the relations $\mathbf{L}^e = \mathbf{L} - \mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1}$, $\mathbf{L}^p = \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha$, and $\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} = \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \mathbf{s}^\alpha \otimes \mathbf{n}^\alpha$ to get

$$\begin{aligned} \dot{\tilde{\mathbf{T}}} &= [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : \mathbf{L} - \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : (\mathbf{s}^\alpha \otimes \mathbf{n}^\alpha) \\ &\quad + \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} [\partial_{\mathbf{F}^p}(\tilde{\mathbf{T}}) \mathbf{F}^{pT}] : (\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha) + \partial_\theta(\tilde{\mathbf{T}}) \dot{\theta}. \end{aligned} \quad (15.80)$$

This can be reorganized to give

$$\dot{\tilde{\mathbf{T}}} = [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : \mathbf{L} + \partial_\theta(\tilde{\mathbf{T}}) \dot{\theta} + \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \{ [\partial_{\mathbf{F}^p}(\tilde{\mathbf{T}}) \mathbf{F}^{pT}] : (\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha) - [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : (\mathbf{s}^\alpha \otimes \mathbf{n}^\alpha) \}. \quad (15.81)$$

We can introduce this into the expression for the rate of change of the resolved shear stress (see section problems) to get

$$\begin{aligned} \dot{\tau}^\alpha &= \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \{ [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : \mathbf{L} + \partial_\theta(\tilde{\mathbf{T}}) \dot{\theta} \\ &\quad + \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \{ [\partial_{\mathbf{F}^p}(\tilde{\mathbf{T}}) \mathbf{F}^{pT}] : (\hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha) - [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}}) \mathbf{F}^{eT}] : (\mathbf{s}^\alpha \otimes \mathbf{n}^\alpha) \} \} \\ &\quad + \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L} - \tau^\alpha \sum_{\beta=1}^m \dot{\gamma}^{p\beta} (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) : (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha). \end{aligned} \quad (15.82)$$

This can be reorganized into the form

$$\dot{\tau}^\alpha = \dot{\tau}_L^\alpha : \mathbf{L} + \dot{\tau}_\theta^\alpha \dot{\theta} + \sum_{\beta=1}^m \dot{\tau}_{\dot{\gamma}^{p\beta}}^\alpha \dot{\gamma}^{p\beta}. \quad (15.83)$$

The yield function for kinematic hardening normally contains the overstress $\Delta \mathbf{T}$ as opposed to the stress. An expression for overstress naturally appeared in the entropy production inequality and was given in Chapter 12 as

$$\Delta \mathbf{T} \equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T = \tilde{\mathbf{T}} - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T, \quad (15.84)$$

where $\mathbf{T}^b$ was the back-stress given by

$$\mathbf{T}^{bT} \equiv \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT}, \quad (15.85)$$

and ξ was a hardening parameter given by the rule $\dot{\xi} = \mathbf{A} : \mathbf{L}^p$. We can use the overstress to define a resolved shear overstress by an expression similar to that given for the resolved shear stress. We will take the resolved shear overstress $\Delta \tau^\alpha$ to be given as

$$\Delta \tau^\alpha = \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} \hat{\mathbf{s}}_o^\alpha \circ (\Delta \mathbf{T}^T \hat{\mathbf{n}}_o^\alpha) = \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \Delta \mathbf{T}. \quad (15.86)$$

If we select a work hardening relation of the form $\dot{\xi} = \Delta \mathbf{T}^T : \mathbf{L}^p \geq 0$ (from the entropy production inequality), then we can introduce $\mathbf{A} = \Delta \mathbf{T}^T$ into (15.84) and solve for the overstress to obtain

$$\Delta \mathbf{T} = \frac{1}{1 + \rho \partial_\xi(\psi)} (\tilde{\mathbf{T}} - \mathbf{T}^b). \quad (15.87)$$

In this special case, the resolved shear overstress can be written as

$$\Delta \tau^\alpha = \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha| [1 + \rho \partial_\xi(\psi)]} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : (\tilde{\mathbf{T}} - \mathbf{T}^b). \quad (15.88)$$

This naturally can be written as

$$\Delta \tau^\alpha = \frac{1}{[1 + \rho \partial_\xi(\psi)]} (\tau^\alpha - \tau^{b\alpha}), \quad (15.89)$$

where the resolved shear back-stress $\tau^{b\alpha}$ is defined as

$$\tau^{b\alpha} \equiv \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \mathbf{T}^b. \quad (15.90)$$

The rate of change of $\tau^{b\alpha}$ can be written as

$$\dot{\tau}^{b\alpha} = \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}}^b - \tau^\alpha [(\mathbf{L}^e \hat{\mathbf{s}}^\alpha) \circ \hat{\mathbf{s}}^\alpha - (\hat{\mathbf{n}}^\alpha \mathbf{L}^e) \circ \hat{\mathbf{n}}^\alpha]. \quad (15.91)$$

Section 15.4-Problems:

15.4-1: Prove that the expression for rate of change of the resolved shear stress given by

$$\dot{\tau}^\alpha = \frac{1}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} \hat{\mathbf{s}}_o^\alpha \circ (\dot{\mathbf{T}} \hat{\mathbf{n}}_o^\alpha) - \tau^\alpha \left(\frac{\dot{|\mathbf{s}^\alpha|}}{|\mathbf{s}^\alpha|} + \frac{\dot{|\mathbf{n}^\alpha|}}{|\mathbf{n}^\alpha|} \right),$$

can be written as

$$\dot{\tau}^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}} - \tau^\alpha [(\mathbf{L}^e \hat{\mathbf{s}}^\alpha) \circ \hat{\mathbf{s}}^\alpha - (\hat{\mathbf{n}}^\alpha \mathbf{L}^e) \circ \hat{\mathbf{n}}^\alpha]. \quad (15.92)$$

Use this to show that

$$\dot{\tau}^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}} - \tau^\alpha (\hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha - \hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha) : \mathbf{L}^e, \quad (15.93)$$

which will result in the relation

$$\begin{aligned} \dot{\tau}^\alpha = & \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}} + \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L} \\ & - \tau^\alpha \sum_{\beta=1}^m \dot{\gamma}^{p\beta} (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) : (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha). \end{aligned} \quad (15.94)$$

15.4-2: Evaluate the explicit expressions for $\dot{\tau}_L^\alpha$, $\dot{\tau}_\theta^\alpha$, and $\dot{\tau}_{\dot{\gamma}^{p\beta}}^\alpha$.

15.5 Yield function

For each slip system, we need a yield function to describe when it is inactive and when it may become active. We will introduce a yield function that has the characteristics of both kinematic hardening and isotropic hardening. A typical yield function of this type for slip system α will be written as

$$f^\alpha = \Delta\tau^\alpha - \tau_c^\alpha, \quad (15.95)$$

where τ_c^α is the critical resolved shear stress and $\Delta\tau^\alpha$ is the resolved shear overstress given in (15.86). The critical resolved shear stress τ_c^α is a material characteristic parameter given by a constant, or a function or functional of the slip in the different systems. The role of the critical resolved shear stress is identical to that of isotropic hardening, but of isotropic hardening on the individual slip systems. If τ_c^α is constant, we get a model for each slip system that behaves like pure kinematic hardening. If τ_c^α changes, we get a model that on the slip system level behaves like a combined kinematic and isotropic hardening.

One point to note is that in a slip plane we consider a slip direction and its negative as two distinct slip systems, each given by its own yield function. As such, our construction is not exactly the same as in previous chapters when we assumed yield in the reverse direction was managed by the same yield function. We now will discuss two special examples, one for pure isotropic hardening and the other for pure kinematic hardening.

15.5.1 Isotropic hardening

Possibly the simplest form of the yield function is given by a specific free energy that does not depend on the plastic deformation gradient or hardening parameter. As a result, we will have no back-stress, and the resolved shear overstress will reduce to the resolved shear stress. This assumption results in a model that either resembles what is known as elastic perfect-plastic response or resembles plasticity with isotropic hardening. The yield function in this case can be written as

$$f^\alpha = \tau^\alpha - \tau_c^\alpha. \quad (15.96)$$

Arguably the simplest version of this form is given when the critical resolved shear stress τ_c^α is constant. The special case of a constant τ_c^α would be equivalent to an elastic perfectly plastic response. A slightly more general isotropic hardening rule is given by a generalization of the simple hardening rule introduced by Taylor and Elam and shown in the introduction. This slightly more general rule accounts for hardening in one slip system as a result of slip in other slip systems. If we have m slip systems, to capture such effects one may introduce a hardening rule for slip on each slip system α as

$$\dot{\tau}_c^\alpha = \sum_{\beta=1}^m h_{\alpha\beta} \dot{\gamma}^{p\beta}, \quad (15.97)$$

where $h_{\alpha\beta}$ is a material characteristic matrix and where $\alpha = \beta$ represents self hardening and $\alpha \neq \beta$ represents cross (or latent) hardening from slip on other slip systems. It seems plausible to assume $h_{\alpha\beta}$ as a constant for linear hardening or a function of $\gamma^{p\alpha}$ for nonlinear hardening. The expression for the rate of change of the yield functions is given by

$$\begin{aligned} \dot{f}^\alpha = \dot{\tau}^\alpha - \dot{\tau}_c^\alpha = & \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}} + \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L} \\ & - \tau^\alpha \sum_{\beta=1}^m \dot{\gamma}^{p\beta} (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) : (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) - \dot{\tau}_c^\alpha, \end{aligned} \quad (15.98)$$

where $\dot{\mathbf{T}}$ for the cases that the Cauchy stress depends on $(\mathbf{F}^e, \mathbf{F}^p, \theta)$ is given in (15.81). The trial derivative $\dot{f}^\alpha$ is calculated as before assuming no plastic flow or hardening so that

$$\dot{f}^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\mathbf{T}} + \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L}, \quad (15.99)$$

where in calculating $\dot{\mathbf{T}}$ we set the plastic flow equal to zero so that $\mathbf{L}^e = \mathbf{L}$. Since the plastic velocity gradient and all hardening rates are set equal to zero, for a general expression for Cauchy stress as a function of $(\mathbf{F}^e, \mathbf{F}^p, \theta)$, the following expression can be written

$$\dot{\mathbf{T}} = [\partial_{\mathbf{F}^e}(\tilde{\mathbf{T}})\mathbf{F}^{eT}] : \mathbf{L} + \partial_\theta(\tilde{\mathbf{T}})\dot{\theta}. \quad (15.100)$$

15.5.2 Kinematic hardening

To get the effect of pure kinematic hardening, we need to include the back-stress, but we assume the critical resolved shear stress to be constant. As a result, the rate of change of the yield function for each slip system in this case is given by

$$\begin{aligned} \dot{f}^\alpha = \Delta \dot{\tau}^\alpha = & \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \Delta \dot{\mathbf{T}} + \Delta \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L} \\ & - \Delta \tau^\alpha \sum_{\beta=1}^m \dot{\gamma}^{p\beta} (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) : (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha). \end{aligned} \quad (15.101)$$

The expression for the trial derivative $\dot{f}^\alpha$ is given by

$$\dot{f}^\alpha = \frac{1}{|\mathbf{s}^\alpha||\mathbf{n}^\alpha|} (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \Delta \dot{\mathbf{T}} + \Delta \tau^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{L}. \quad (15.102)$$

Assuming no hardening parameters in the specific free energy (i.e., $\partial_\xi(\psi) = 0$), we have $\Delta \mathbf{T} = \tilde{\mathbf{T}} - \mathbf{T}^b$ so that $\Delta \dot{\mathbf{T}} = \dot{\tilde{\mathbf{T}}} - \dot{\mathbf{T}}^b$ and $\Delta \overset{\circ}{\mathbf{T}} = \overset{\circ}{\tilde{\mathbf{T}}} - \overset{\circ}{\mathbf{T}}^b$, where $\overset{\circ}{\tilde{\mathbf{T}}}$ is given in (15.100) and $\overset{\circ}{\mathbf{T}}^b$ is given by a similar expression.

Section 15.5-Problems:

15.5-1: Assuming Cauchy stress only depends on $\mathbf{F}^e$ and θ , derive the relation

$$\begin{aligned} \hat{\mathbf{s}}_o^\alpha \circ (\dot{\tilde{\mathbf{T}}}^T \hat{\mathbf{n}}_o^\alpha) &= (\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \dot{\tilde{\mathbf{T}}} \\ &= [\mathbf{T}^T \mathbf{n}^\alpha \otimes \mathbf{s}^\alpha - \mathbf{n}^\alpha \otimes \mathbf{s}^\alpha \mathbf{T}^T + (\mathbf{n}^\alpha \otimes \mathbf{s}^\alpha) : \partial_{\mathbf{F}^e}(\mathbf{T}) \mathbf{F}^{eT}] : \mathbf{L}^e \\ &\quad + (\mathbf{n}^\alpha \otimes \mathbf{s}^\alpha) : \partial_\theta(\mathbf{T}) \dot{\theta}. \end{aligned}$$

15.6 Thermomechanical models

To include thermal effects, we normally modify the decomposition of the deformation gradient from $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ to either $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \mathbf{F}^\theta$ or $\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta \mathbf{F}^p$. The former assumes the crystal to thermally expand, plastically deform, and then elastically distort, while the latter assumes the crystal to plastically deform, thermally expand, and elastically distort. Since the crystal structure seems to be unaffected by plastic deformation when we impose the deformation on the initial configuration, we will preserve this characteristic and use the latter decomposition given by

$$\boxed{\mathbf{F} = \mathbf{F}^e \mathbf{F}^\theta \mathbf{F}^p.} \quad (15.103)$$

The reader will note that *this assumption is different from that made in all the preceding chapters*. This decomposition is consistent with the decomposition of the velocity gradient given by

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1} + \mathbf{F}^e \mathbf{F}^\theta \mathbf{L}^p \mathbf{F}^{\theta-1} \mathbf{F}^{e-1}. \quad (15.104)$$

We assume that the thermal expansion is given by

$$\boxed{\mathbf{L}^\theta = \alpha \dot{\theta}.} \quad (15.105)$$

In this context, we can define the Cauchy stress $\mathbf{T}$, the back-stress $\mathbf{T}^b$, and the thermal stress $\mathbf{T}^\theta$ by the relations

$$\boxed{\mathbf{T}^T \equiv \rho \partial_{\mathbf{F}^e}(\psi) \mathbf{F}^{eT},} \quad (15.106)$$

$$\boxed{\mathbf{T}^{bT} \equiv \rho \partial_{\mathbf{F}^p}(\psi) \mathbf{F}^{pT},} \quad (15.107)$$

$$\boxed{\mathbf{T}^{\theta T} \equiv \rho \partial_{\mathbf{F}^\theta}(\psi) \mathbf{F}^{\theta T}.} \quad (15.108)$$

The expression for Cauchy stress is a direct result of the entropy production inequality, where as the other two are defined in a similar fashion to simplify the presentation. The entropy production inequality also provides the expression for the specific entropy as

$$\boxed{\eta = -\partial_\theta(\psi) + \frac{1}{\rho} \Delta \mathbf{T}^{\theta T} : \alpha,} \quad (15.109)$$

where the thermal overstress, *for this decomposition of the deformation gradient*, is given by

$$\Delta \mathbf{T}^\theta \equiv \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e - \mathbf{T}^\theta. \quad (15.110)$$

We will focus on hardening parameters that are given by

$$\dot{\xi} = \mathbf{A} : \mathbf{L}^p. \quad (15.111)$$

The entropy production inequality requires that during plastic flow we have the relation

$$-\Delta \mathbf{T}^T : \mathbf{L}^p \leq 0, \quad (15.112)$$

where the overstress $\Delta \mathbf{T}$ is defined *for this decomposition* as

$$\Delta \mathbf{T} \equiv \mathbf{F}^{\theta-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^\theta - \mathbf{T}^b - \rho \partial_\xi(\psi) \mathbf{A}^T. \quad (15.113)$$

For the special case where the hardening rule is given by $\dot{\xi} = \Delta \mathbf{T}^T : \mathbf{L}^p$, it can be shown that the overstress is given by

$$\Delta \mathbf{T} = \frac{1}{1 + \rho \partial_\xi(\psi)} (\tilde{\mathbf{T}} - \mathbf{T}^b), \quad (15.114)$$

where

$$\tilde{\mathbf{T}} \equiv \mathbf{F}^{\theta-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^\theta. \quad (15.115)$$

Assuming the same role for plastic flow as we assumed for the pure mechanical theory, the slip planes and slip directions are unchanged by the plastic flow so any change in them is a result of the thermal expansion and elastic deformation. As such, in most of our development we can replace $\mathbf{F}^e$ for $\mathbf{F}^e \mathbf{F}^\theta$ to obtain the correct relations. Therefore, we have

$$\mathbf{s}^\alpha = \mathbf{F}^e \mathbf{F}^\theta \hat{\mathbf{s}}_o^\alpha, \quad \mathbf{n}^\alpha = \hat{\mathbf{n}}_o^\alpha \mathbf{F}^{\theta-1} \mathbf{F}^{e-1}. \quad (15.116)$$

As before, the plastic velocity gradient is given by

$$\mathbf{L}^p = \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \hat{\mathbf{s}}_o^\alpha \otimes \hat{\mathbf{n}}_o^\alpha, \quad (15.117)$$

and the total velocity gradient can be decomposed as

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1} + \tilde{\mathbf{L}}^p, \quad (15.118)$$

where

$$\tilde{\mathbf{L}}^p = \mathbf{F}^e \mathbf{F}^\theta \mathbf{L}^p \mathbf{F}^{\theta-1} \mathbf{F}^{e-1} = \sum_{\alpha=1}^m \dot{\gamma}^{p\alpha} \mathbf{s}^\alpha \otimes \mathbf{n}^\alpha. \quad (15.119)$$

The rate of change of $\mathbf{s}^\alpha$ and $\mathbf{n}^\alpha$ are given by

$$\dot{\mathbf{s}}^\alpha = (\mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1}) \mathbf{s}^\alpha = (\mathbf{L} - \tilde{\mathbf{L}}^p) \mathbf{s}^\alpha, \quad (15.120)$$

$$\dot{\mathbf{n}}^\alpha = -\mathbf{n}^\alpha (\mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1}) = -\mathbf{n}^\alpha (\mathbf{L} - \tilde{\mathbf{L}}^p), \quad (15.121)$$

and result in the relations

$$\dot{\mathbf{s}}^\alpha = (\mathbf{I} - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) (\mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1}) \hat{\mathbf{s}}^\alpha = (\mathbf{I} - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) (\mathbf{L} - \tilde{\mathbf{L}}^p) \hat{\mathbf{s}}^\alpha, \quad (15.122)$$

$$\dot{\mathbf{n}}^\alpha = -\hat{\mathbf{n}}^\alpha (\mathbf{L}^e + \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1}) (\mathbf{I} - \hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha) = -\hat{\mathbf{n}}^\alpha (\mathbf{L} - \tilde{\mathbf{L}}^p) (\mathbf{I} - \hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha), \quad (15.123)$$

for the unit vectors $\hat{\mathbf{s}}^\alpha$ and $\hat{\mathbf{n}}^\alpha$ along, respectively, the directions $\mathbf{s}^\alpha$ and $\mathbf{n}^\alpha$.

The expression for the resolved shear stress for slip system α is given by

$$\tau^\alpha = \hat{\mathbf{s}}^\alpha \circ (\mathbf{T}^T \hat{\mathbf{n}}^\alpha) = (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \mathbf{T}, \quad (15.124)$$

so that its rate is given by

$$\dot{\tau}^\alpha = (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \dot{\mathbf{T}} + \mathbf{T} : (\dot{\hat{\mathbf{n}}}^\alpha \otimes \hat{\mathbf{s}}^\alpha + \hat{\mathbf{n}}^\alpha \otimes \dot{\hat{\mathbf{s}}}^\alpha). \quad (15.125)$$

The resolved shear overstress is given by

$$\Delta\tau^\alpha = \frac{\hat{\mathbf{s}}_o^\alpha \circ (\Delta\mathbf{T}^T \hat{\mathbf{n}}_o^\alpha)}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|} = \frac{(\hat{\mathbf{n}}_o^\alpha \otimes \hat{\mathbf{s}}_o^\alpha) : \Delta\mathbf{T}}{|\mathbf{s}^\alpha| |\mathbf{n}^\alpha|}, \quad (15.126)$$

$$= (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : (\mathbf{F}^e \mathbf{F}^\theta \Delta\mathbf{T} \mathbf{F}^{\theta-1} \mathbf{F}^{e-1}), \quad (15.127)$$

$$= (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : (\mathbf{F} \mathbf{F}^{p-1} \Delta\mathbf{T} \mathbf{F}^p \mathbf{F}^{-1}). \quad (15.128)$$

The rate of change of the resolved shear overstress is given by

$$\begin{aligned} \Delta\dot{\tau}^\alpha &= (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \overline{(\mathbf{F}^e \mathbf{F}^\theta \Delta\mathbf{T} \mathbf{F}^{\theta-1} \mathbf{F}^{e-1})} + (\mathbf{F}^e \mathbf{F}^\theta \Delta\mathbf{T} \mathbf{F}^{\theta-1} \mathbf{F}^{e-1}) : (\dot{\hat{\mathbf{n}}}^\alpha \otimes \hat{\mathbf{s}}^\alpha + \hat{\mathbf{n}}^\alpha \otimes \dot{\hat{\mathbf{s}}}^\alpha) \\ &= (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \overline{(\mathbf{F} \mathbf{F}^{p-1} \Delta\mathbf{T} \mathbf{F}^p \mathbf{F}^{-1})} + (\mathbf{F} \mathbf{F}^{p-1} \Delta\mathbf{T} \mathbf{F}^p \mathbf{F}^{-1}) : (\dot{\hat{\mathbf{n}}}^\alpha \otimes \hat{\mathbf{s}}^\alpha + \hat{\mathbf{n}}^\alpha \otimes \dot{\hat{\mathbf{s}}}^\alpha). \end{aligned} \quad (15.129)$$

15.6.1 A rate-independent model using Schmid's law

As a special example, let us construct a thermomechanical crystal plasticity model based on a rate-independent flow rule using Schmid's law to construct the yield function, and using the stress obtained from using a quadratic specific free energy in Green strains. From Schmid's law, the yield law for each slip system α can be written as

$$f^\alpha = \tau^\alpha - \tau_c^\alpha, \quad (15.130)$$

with its rate given by

$$\dot{f}^\alpha = \dot{\tau}^\alpha - \dot{\tau}_c^\alpha = (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \dot{\mathbf{T}} + \mathbf{T} : (\dot{\hat{\mathbf{n}}}^\alpha \otimes \hat{\mathbf{s}}^\alpha + \hat{\mathbf{n}}^\alpha \otimes \dot{\hat{\mathbf{s}}}^\alpha) - \dot{\tau}_c^\alpha. \quad (15.131)$$

For the stress given by the model

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\mathbf{E} : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (15.132)$$

we can show that

$$\dot{f}^\alpha = \mathbf{Z}_1^\alpha : \mathbf{L} + \mathbf{Z}_2^\alpha : (\mathbf{L} - \mathbf{F}^e \mathbf{L}^\theta \mathbf{F}^{e-1} - \tilde{\mathbf{L}}^p) + \mathbf{Z}_3^\alpha : (\mathbf{L} - \tilde{\mathbf{L}}^p) + \mathbf{Z}_4^\alpha \dot{\theta} - \dot{\tau}_c^\alpha, \quad (15.133)$$

where

$$\mathbf{Z}_1^\alpha \equiv -\tau^\alpha \mathbf{I}, \quad (15.134)$$

$$\mathbf{Z}_2^\alpha \equiv \hat{\mathbf{s}}^\alpha \otimes (\hat{\mathbf{n}}^\alpha \mathbf{T}) + \hat{\mathbf{n}}^\alpha \otimes (\mathbf{T} \hat{\mathbf{s}}^\alpha) + \frac{1}{J} \mathbf{F}^e ([(\hat{\mathbf{n}}^\alpha \mathbf{F}^e) \otimes (\mathbf{F}^{eT} \hat{\mathbf{s}}^\alpha)] : \mathbf{E}) \mathbf{F}^{eT}, \quad (15.135)$$

$$\mathbf{Z}_3^\alpha \equiv (\hat{\mathbf{n}}^\alpha \mathbf{T}) \otimes \hat{\mathbf{s}}^\alpha - \hat{\mathbf{n}}^\alpha \otimes (\mathbf{T} \hat{\mathbf{s}}^\alpha) + (\mathbf{T} \hat{\mathbf{s}}^\alpha) \circ \hat{\mathbf{n}}^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha), \quad (15.136)$$

$$\mathbf{Z}_4^\alpha \equiv \frac{1}{J} [(\hat{\mathbf{n}}^\alpha \mathbf{F}^e) \otimes (\mathbf{F}^{eT} \hat{\mathbf{s}}^\alpha)] : \frac{d\mathbf{E}}{d\theta} : \mathbf{E}^e. \quad (15.137)$$

The proof of this follows from the solution to the section problems. This expression can be reorganized to give

$$\dot{f}^\alpha = (\mathbf{Z}_1^\alpha + \mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \mathbf{L} + [\mathbf{Z}_4^\alpha - \mathbf{Z}_2^\alpha : (\mathbf{F}^e \boldsymbol{\alpha} \mathbf{F}^{e-1})] \dot{\theta} - (\mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \tilde{\mathbf{L}}^p - \dot{\tau}_c^\alpha. \quad (15.138)$$

The trial derivative $\dot{f}^\alpha$ is, therefore, given by

$$\dot{f}^\alpha = (\mathbf{Z}_1^\alpha + \mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \mathbf{L} + [\mathbf{Z}_4^\alpha - \mathbf{Z}_2^\alpha : (\mathbf{F}^e \boldsymbol{\alpha} \mathbf{F}^{e-1})] \dot{\theta}. \quad (15.139)$$

Taking the hardening rule given by

$$\dot{\tau}_c^\alpha = \sum_{\beta=1}^m h_{\alpha\beta} \dot{\gamma}^{p\beta}, \quad (15.140)$$

the consistency condition during plastic flow can be written as

$$(\mathbf{Z}_1^\alpha + \mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \mathbf{L} + [\mathbf{Z}_4^\alpha - \mathbf{Z}_2^\alpha : (\mathbf{F}^e \boldsymbol{\alpha} \mathbf{F}^{e-1})] \dot{\theta} - (\mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \sum_{\beta=1}^m \dot{\gamma}^{p\beta} \mathbf{s}^\beta \otimes \mathbf{n}^\beta - \sum_{\alpha=1}^m h_{\alpha\beta} \dot{\gamma}^{p\beta} = 0. \quad (15.141)$$

Reorganizing this yields

$$\sum_{\beta=1}^m [(\mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) + h_{\alpha\beta}] \dot{\gamma}^{p\beta} = (\mathbf{Z}_1^\alpha + \mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : \mathbf{L} + [\mathbf{Z}_4^\alpha - \mathbf{Z}_2^\alpha : (\mathbf{F}^e \boldsymbol{\alpha} \mathbf{F}^{e-1})] \dot{\theta}. \quad (15.142)$$

We will have one consistency condition for each slip system that is active. Assuming n of the m slip systems are active, then only n of m values of $\dot{\gamma}^{p\beta}$ are nonzero. The consistency conditions for the n active slip systems can be organized into a system of equation written as

$$\sum_{\beta=1}^n A_{\alpha\beta} \dot{\gamma}^{p\beta} = \dot{f}^\alpha, \quad (15.143)$$

where

$$A_{\alpha\beta} = (\mathbf{Z}_2^\alpha + \mathbf{Z}_3^\alpha) : (\mathbf{s}^\beta \otimes \mathbf{n}^\beta) + h_{\alpha\beta}. \quad (15.144)$$

This system will have a solution if $A_{\alpha\beta}$ is invertible yielding

$$\dot{\gamma}^{p\alpha} = A_{\alpha\beta}^{-1} \dot{f}^\beta. \quad (15.145)$$

As can be seen, in the event that a solution exists it will provide the flow rule

$$\dot{\gamma}^{p\alpha} = \begin{cases} \beta_{\mathbf{L}}^\alpha : \mathbf{L} + \beta_\theta^\alpha \dot{\theta} & \text{for } f^\alpha = 0 \text{ and } \dot{f}^\alpha > 0, \\ 0 & \text{for all other cases,} \end{cases} \quad (15.146)$$

where

$$\beta_{\mathbf{L}}^\alpha = \sum_{\beta=1}^n A_{\alpha\beta}^{-1} (\mathbf{Z}_1^\beta + \mathbf{Z}_2^\beta + \mathbf{Z}_3^\beta), \quad (15.147)$$

$$\beta_\theta^\alpha = \sum_{\beta=1}^n A_{\alpha\beta}^{-1} [\mathbf{Z}_4^\beta - \mathbf{Z}_2^\beta : (\mathbf{F}^e \boldsymbol{\alpha} \mathbf{F}^{e-1})]. \quad (15.148)$$

Section 15.6-Problems:

15.6-1: Show that for any second-order tensor $\mathbf{Z}$ we can write

$$\begin{aligned}\mathbf{Z} : (\dot{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha + \hat{\mathbf{n}}^\alpha \otimes \dot{\hat{\mathbf{s}}}^\alpha) \\ = [(\hat{\mathbf{n}}^\alpha \mathbf{Z}) \otimes \hat{\mathbf{s}}^\alpha - \hat{\mathbf{n}}^\alpha \otimes (\mathbf{Z} \hat{\mathbf{s}}^\alpha) + (\mathbf{Z} \hat{\mathbf{s}}^\alpha) \circ \hat{\mathbf{n}}^\alpha (\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{n}}^\alpha - \hat{\mathbf{s}}^\alpha \otimes \hat{\mathbf{s}}^\alpha)] : (\mathbf{L} - \tilde{\mathbf{L}}^p).\end{aligned}\quad (15.149)$$

15.6-2: Show that for the case that the Cauchy stress is given by the expression

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\mathbf{E} : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (15.150)$$

we have

$$\begin{aligned}(\hat{\mathbf{n}}^\alpha \otimes \hat{\mathbf{s}}^\alpha) : \dot{\mathbf{T}} &= -\tau^\alpha \mathbf{I} : \mathbf{L} + \{\hat{\mathbf{s}}^\alpha \otimes (\hat{\mathbf{n}}^\alpha \mathbf{T}) + \hat{\mathbf{n}}^\alpha \otimes (\mathbf{T} \hat{\mathbf{s}}^\alpha) \\ &\quad + \frac{1}{J} \mathbf{F}^e [(\hat{\mathbf{n}}^\alpha \mathbf{F}^e) \otimes (\mathbf{F}^{eT} \hat{\mathbf{s}}^\alpha)] : \mathbf{E} \mathbf{F}^{eT}\} : \mathbf{L}^e \\ &\quad + \frac{1}{J} [(\hat{\mathbf{n}}^\alpha \mathbf{F}^e) \otimes (\mathbf{F}^{eT} \hat{\mathbf{s}}^\alpha)] : \frac{d\mathbf{E}}{d\theta} : \mathbf{E}^e \dot{\theta}.\end{aligned}\quad (15.151)$$

15.6-3: Construct a rate-independent model based on the overstress based yield function

$$f^\alpha = \Delta \tau^\alpha - \tau_c^\alpha, \quad (15.152)$$

where the critical resolved shear stress τ_c^α is taken to be constant. Assume the quadratic form specific free energy, assuming no dependence on a hardening parameter so that

$$\mathbf{T}^T = \frac{1}{J} \mathbf{F}^e (\mathbf{E} : \mathbf{E}^e) \mathbf{F}^{eT}, \quad (15.153)$$

and

$$\mathbf{T}^{bT} = \frac{1}{J} \mathbf{F}^p (\mathbf{E}^b : \mathbf{E}^p) \mathbf{F}^{pT}. \quad (15.154)$$

15.7 Rate-dependent models

We can construct rate-dependent crystal plasticity models just as we did rate-dependent plasticity models. As before, we control the slip through the yield function and assume the yield function either through Schmid's law given by

$$f^\alpha = \tau^\alpha - \tau_c^\alpha, \quad (15.155)$$

or by an overstress law such as

$$f^\alpha = \Delta \tau^\alpha - \tau_c^\alpha. \quad (15.156)$$

Since we do not have a consistency condition for rate-dependent models, the flow rule can be written directly in a form such as

$$\dot{\gamma}^{p\alpha} = \begin{cases} \dot{\gamma}_o^{p\alpha} \left(\frac{f^\alpha}{\tau_c^\alpha}\right)^{r^\alpha} & \text{if } f^\alpha > 0, \\ 0 & \text{if } f^\alpha \leq 0, \end{cases} \quad (15.157)$$

where $\dot{\gamma}_o^{p\alpha}$ is an arbitrary scaling factor normally taken to be the same for all slip systems (10^{-3} 1/s), and τ_c^α and r^α are the material parameters characterizing slip for the particular slip system. In particular, r^α characterizes the rate sensitivity of the slip system.

Section 15.7-Problems:

- 15.7-1:** Construct a program to calculate the rate-dependent response of a crystal undergoing simple shear given information about the slip systems, the critical resolved shear stress τ_c^α , and the exponent n^α . Assume your crystal is cubic with the preferred slip system given as the $\{111\}\langle 110 \rangle$ system. Evaluate the response for different values of τ_c^α and n^α .
- 15.7-2:** It has been observed that the response for many metals during plastic flow can be characterized by a power-law model given by

$$\sigma = K\epsilon_l^n, \quad (15.158)$$

where σ is the true stress in tension, ϵ_l is the associated logarithmic strain, and K and n are material parameters shown in Table 15.7 for several metals. Assume that the critical resolved shear stress is characterized by a relation of the type

$$\tau_c^\alpha = \tau_{c0}^\alpha (1 + \beta^\alpha \gamma^{p^\alpha})^{r^\alpha}, \quad (15.159)$$

and that the yield function is given by Schmid's rule. Develop a model for such a hardening and study what parameters β^α and r^α result in equivalent response as provided in Table 15.7 for one of the materials given.

Table 15.7: Strain hardening coefficients for post yield response.¹

Metal	Structure	n	K (MPa)
Ti	HCP	0.05	1,210
Annealed alloy steel	BCC	0.15	641
Quenched and tempered medium-carbon steel	BCC	0.10	1,570
Mo	BCC	0.13	724
Cu	FCC	0.54	317
Cu-30%Zn	FCC	0.50	896
Austenitic stainless steel	FCC	0.52	1,520

¹ *The Science and Engineering of Materials*, Askeland, PWS Publishing Company, 1994.

15.8 Notation

The following table shows the symbols used in the text and their definitions.

" $\diamond$ "		trial time derivative taken holding plastic strain constant
$[hkl]$		Miller index for $h\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3$
(hkl)		Miller index for $h\mathbf{e}_1^* + k\mathbf{e}_2^* + l\mathbf{e}_3^*$
$\{hkl\}$		family of materially similar planes to (hkl)
$\langle hkl \rangle$		family of materially similar directions to $[hkl]$
$\{abc\}$		
$\langle hkl \rangle$		family of slip systems defined by planes $\{abc\}$ and directions $\langle hkl \rangle$
A		hardening coefficient ($\dot{\xi} = \mathbf{A} : \mathbf{L}^p$)

$\mathbf{A}_{ij}$		base tensors for representation of fourth-order tensors
BCC		body-centered cubic
$\mathbf{D}^p$		plastic deformation rate
e		elastic (superscript)
$\mathbf{e}_i$		base vectors
$\mathbf{E}_i$		base tensors for representing second-order tensors
$\mathbf{e}_i^*$		dual or reciprocal base vectors
$\mathbf{E}^e$		elastic Green strain
$\mathbf{E}^p$		plastic Green strain
$\mathbf{E}$		elastic modulus tensor
$\mathbf{E}^b$		plastic back-stress modulus
f^α		yield function for slip system α
$\mathbf{F}$		deformation gradient ($\mathbf{F}^e \mathbf{F}^p$ or $\mathbf{F}^e \mathbf{F}^\theta \mathbf{F}^p$)
$\mathbf{F}^e$		elastic deformation gradient
$\mathbf{F}^p$		plastic deformation gradient
$\mathbf{F}^\theta$		thermal deformation gradient
FCC		face-centered cubic
$h_{\alpha\beta}$		hardening parameter between slip systems
HCP		hexagonal close-packed
$\mathbf{H}$		displacement gradient
$\mathbf{H}^e$		elastic displacement gradient
$\mathbf{H}^p$		plastic displacement gradient
$\mathbf{H}_U^e$		displacement gradient of $\mathbf{U}^e$
$\mathbf{I}$		identity
$\mathbf{L}$		velocity gradient ($\dot{\mathbf{F}} \mathbf{F}^{-1}$)
$\mathbf{L}^\theta$		thermal velocity gradient ($\dot{\mathbf{F}}^\theta \mathbf{F}^{\theta-1}$)
$\mathbf{L}^p$		plastic velocity gradient ($\dot{\mathbf{F}}^p \mathbf{F}^{p-1}$)
$\hat{\mathbf{n}}$		unit normal
$\hat{\mathbf{n}}_o^\alpha$		initial unit normal to surface of slip system α
$\hat{\mathbf{n}}^\alpha$		current unit normal to surface of slip system α
r^α		nonlinearity of flow viscosity of slip system α
$\mathbf{R}^e$		orthogonal factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e = \mathbf{V}^e \mathbf{R}^e$
$\hat{\mathbf{s}}_o^\alpha$		initial direction of slip for slip system α
$\hat{\mathbf{s}}^\alpha$		current direction of slip for slip system α
$\mathbf{t}^{(\mathbf{n})}$		traction vector on surface with normal $\mathbf{n}$
$\mathbf{T}$		Cauchy stress
$\mathbf{T}^b$		plastic back-stress
$\mathbf{T}^\theta$		thermodynamic thermal stress
$\Delta \mathbf{T}$		overstress
$\Delta \mathbf{T}^\theta$		thermodynamic overstress
$\tilde{\mathbf{T}}$		$\mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e$ in mechanical and $\mathbf{F}^{\theta-1} \mathbf{F}^{e-1} \mathbf{T} \mathbf{F}^e \mathbf{F}^\theta$ in thermodynamic theory
$\mathbf{U}^e$		right symmetric factor in polar decomposition of $\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e$
V		volume
$\mathbf{W}^p$		plastic spin tensor
α		slip system
α		coefficient of thermal expansion
β		slip system
ϵ		infinitesimal strain
ϵ^e		infinitesimal elastic strain
ϵ^θ		infinitesimal thermal strain

ϵ^p		infinitesimal plastic strain
$\gamma^{p\alpha}$		resolved plastic shear strain for slip system α
μ^α		viscosity of slip system α
ψ		specific free energy
ψ^θ		purely thermal component of specific free energy
ρ		density
ρ_o		initial density
τ^α		resolved shear stress for slip system α
τ_c^α		critical resolved shear stress for slip system α
θ		temperature

Appendix A

Representation of functions

The following are functional representations for isotropy, transverse isotropy, and orthotropy for scalar, vector, symmetric tensors, and skew symmetric tensors. The representations are given for functions of at most three symmetric tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$, and one vector $\mathbf{v}$. A scalar invariant function ψ , a vector invariant function $\mathbf{q}$, and a tensor invariant function $\mathbf{T}$ of these variables are an invariant function under the tensor transformation $\mathbf{M}$ if they have the property

$$\psi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{W}, \mathbf{X}, \mathbf{v}) = \psi(\mathbf{MAM}^T, \mathbf{MBM}^T, \mathbf{MCM}^T, \mathbf{MWM}^T, \mathbf{MXM}^T, \mathbf{Mv}), \quad (\text{A.1})$$

$$\mathbf{q}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{W}, \mathbf{X}, \mathbf{v}) = \mathbf{M}^T \mathbf{q}(\mathbf{MAM}^T, \mathbf{MBM}^T, \mathbf{MCM}^T, \mathbf{MWM}^T, \mathbf{MXM}^T, \mathbf{Mv}), \quad (\text{A.2})$$

$$\mathbf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{W}, \mathbf{X}, \mathbf{v}) = \mathbf{M}^T \mathbf{T}(\mathbf{MAM}^T, \mathbf{MBM}^T, \mathbf{MCM}^T, \mathbf{MWM}^T, \mathbf{MXM}^T, \mathbf{Mv}) \mathbf{M}. \quad (\text{A.3})$$

In general there is a group of transformations under which a function is invariant. Here we consider the groups associated with isotropy, transverse isotropy, and orthotropy. We have selected to use the full orthogonal group of tensors, as opposed to the proper group of orthogonal tensors, to represent the symmetry transformations for isotropy. If $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are three orthogonal directions, and $\boldsymbol{\tau} = \mathbf{e}_3$, then transverse isotropy can be represented by five groups of tensors given by

$$T_1 : \mathbf{R}_{\boldsymbol{\tau}}, \quad T_2 : \mathbf{R}_{\boldsymbol{\tau}}, \mathbf{R}_2, \quad T_3 : \mathbf{R}_{\boldsymbol{\tau}}, \mathbf{R}_1, \quad T_4 : \mathbf{R}_{\boldsymbol{\tau}}, \mathbf{R}_2, \mathbf{R}_1, \quad T_5 : \mathbf{R}_{\boldsymbol{\tau}}, -\mathbf{R}_3, \quad (\text{A.4})$$

where $\boldsymbol{\tau}$ is the preferred axis of transverse isotropy, $\mathbf{R}_{\boldsymbol{\tau}}$ represents all rotations about $\boldsymbol{\tau}$, and $\mathbf{R}_i$ represent the reflection about the plane perpendicular to $\mathbf{e}_i$. The most common representation of transverse isotropy is T_2 , which is used for the following representations. There are three representations for orthotropy given by

$$O_1 : \mathbf{R}_2, \mathbf{R}_3, \quad O_2 : -\mathbf{R}_2, -\mathbf{R}_3, \quad O_3 : \mathbf{R}_2, \mathbf{R}_3, -\mathbf{I}. \quad (\text{A.5})$$

The most commonly used representation for orthotropy is O_3 , which is used in the following representations.

Vector functions are constructed by expressions of the form

$$\mathbf{q} = \sum_{i=1}^n a_i \mathbf{f}_i, \quad (\text{A.6})$$

where a_i are scalar functions of the scalar invariants and $\mathbf{f}_i$ are the given vector base. Symmetric tensor functions are constructed by expressions of the form

$$\mathbf{T}_{sym} = \sum_{i=1}^n b_i \mathbf{H}_i, \quad (\text{A.7})$$

where b_i are scalar functions of the scalar invariants and $\mathbf{H}_i$ are the given symmetric tensor base. Skew symmetric tensor functions are constructed by expressions of the form

$$\mathbf{T}_{skew} = \sum_{i=1}^n c_i \mathbf{Z}_i, \quad (\text{A.8})$$

where c_i are scalar functions of the scalar invariants and $\mathbf{Z}_i$ are the given skew symmetric tensor base.

For each symmetry, the invariants are organized in groups, including one, two, and three symmetric tensor arguments. In each case, as arguments are added, only the new terms are given, so you will need to add the previous terms. The previous terms are always numbered sequentially from 1 to one less than the first new term.

A.1 Isotropic

The following complete and irreducible invariants for the full orthogonal group have been extracted from the article by Zheng.¹

A.1.1 Scalar invariants

Including one symmetric tensor

Scalar invariants of one symmetric tensor $\mathbf{A}$:

$$I_1 = \text{tr}(\mathbf{A}) \quad I_2 = \text{tr}(\mathbf{A}^2) \quad I_3 = \text{tr}(\mathbf{A}^3) \quad (\text{A.9})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$ and a skew symmetric tensor $\mathbf{W}$: Add

$$I_4 = \text{tr}(\mathbf{W}^2) \quad I_5 = \text{tr}(\mathbf{A}\mathbf{W}^2) \quad I_6 = \text{tr}(\mathbf{A}^2\mathbf{W}^2) \quad I_7 = \text{tr}(\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}) \quad (\text{A.10})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned} I_8 &= \mathbf{v} \circ \mathbf{v} \\ I_9 &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{10} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{v} \\ I_{11} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{v} \\ I_{12} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} & I_{13} &= \mathbf{v} \circ \mathbf{W}\mathbf{A}\mathbf{W}^2\mathbf{v} & I_{14} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{W}\mathbf{v} \end{aligned} \quad (\text{A.11})$$

Including two symmetric tensors

Scalar invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$:

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{A}) & I_2 &= \text{tr}(\mathbf{A}^2) & I_3 &= \text{tr}(\mathbf{A}^3) \\ I_4 &= \text{tr}(\mathbf{B}) & I_5 &= \text{tr}(\mathbf{B}^2) & I_6 &= \text{tr}(\mathbf{B}^3) \\ I_7 &= \text{tr}(\mathbf{A}\mathbf{B}) & I_8 &= \text{tr}(\mathbf{A}\mathbf{B}^2) & I_9 &= \text{tr}(\mathbf{A}^2\mathbf{B}) & I_{10} &= \text{tr}(\mathbf{A}^2\mathbf{B}^2) \end{aligned} \quad (\text{A.12})$$

Scalar invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, and one skew symmetric tensor $\mathbf{W}$: Add

$$\begin{aligned} I_{11} &= \text{tr}(\mathbf{W}^2) \\ I_{12} &= \text{tr}(\mathbf{A}\mathbf{W}^2) & I_{13} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2) & I_{14} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}) \\ I_{15} &= \text{tr}(\mathbf{B}\mathbf{W}^2) & I_{16} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2) & I_{17} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2\mathbf{B}\mathbf{W}) \\ I_{18} &= \text{tr}(\mathbf{A}\mathbf{B}\mathbf{W}) & I_{19} &= \text{tr}(\mathbf{A}\mathbf{B}^2\mathbf{W}) & I_{20} &= \text{tr}(\mathbf{A}^2\mathbf{B}\mathbf{W}) & I_{21} &= \text{tr}(\mathbf{A}\mathbf{W}^2\mathbf{B}\mathbf{W}) \end{aligned} \quad (\text{A.13})$$

¹Zheng, Q.-S., "On transversely isotropic, orthotropic and relative isotropic functions of symmetric tensors, skew symmetric tensors and vectors. Part I: Two dimensional orthotropic and relative isotropic functions and three dimensional relative isotropic functions," *International Journal of Engineering Science*, vol. 31, no. 10, pages 1399–1409, 1993.

Scalar invariants of two symmetric tensors **A** and **B**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned}
 I_{22} &= \mathbf{v} \circ \mathbf{v} \\
 I_{23} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{24} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{v} \\
 I_{25} &= \mathbf{v} \circ \mathbf{B}\mathbf{v} & I_{26} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{v} \\
 I_{27} &= \mathbf{v} \circ \mathbf{A}\mathbf{B}\mathbf{v} \\
 I_{28} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{v} \\
 I_{29} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} & I_{30} &= \mathbf{v} \circ \mathbf{W}\mathbf{A}\mathbf{W}^2\mathbf{v} & I_{31} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{W}\mathbf{v} \\
 I_{32} &= \mathbf{v} \circ \mathbf{B}\mathbf{W}\mathbf{v} & I_{33} &= \mathbf{v} \circ \mathbf{W}\mathbf{B}\mathbf{W}^2\mathbf{v} & I_{34} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{W}\mathbf{v}
 \end{aligned} \tag{A.14}$$

Including three symmetric tensors

Scalar invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 I_1 &= tr(\mathbf{A}) & I_2 &= tr(\mathbf{A}^2) & I_3 &= tr(\mathbf{A}^3) \\
 I_4 &= tr(\mathbf{B}) & I_5 &= tr(\mathbf{B}^2) & I_6 &= tr(\mathbf{B}^3) \\
 I_7 &= tr(\mathbf{C}) & I_8 &= tr(\mathbf{C}^2) & I_9 &= tr(\mathbf{C}^3) \\
 I_{10} &= tr(\mathbf{A}\mathbf{B}) & I_{11} &= tr(\mathbf{A}\mathbf{B}^2) & I_{12} &= tr(\mathbf{A}^2\mathbf{B}) & I_{13} &= tr(\mathbf{A}^2\mathbf{B}^2) \\
 I_{14} &= tr(\mathbf{A}\mathbf{C}) & I_{15} &= tr(\mathbf{A}\mathbf{C}^2) & I_{16} &= tr(\mathbf{A}^2\mathbf{C}) & I_{17} &= tr(\mathbf{A}^2\mathbf{C}^2) \\
 I_{18} &= tr(\mathbf{B}\mathbf{C}) & I_{19} &= tr(\mathbf{B}\mathbf{C}^2) & I_{20} &= tr(\mathbf{B}^2\mathbf{C}) & I_{21} &= tr(\mathbf{B}^2\mathbf{C}^2) \\
 I_{22} &= tr(\mathbf{A}\mathbf{B}\mathbf{C})
 \end{aligned} \tag{A.15}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 I_{23} &= tr(\mathbf{W}^2) \\
 I_{24} &= tr(\mathbf{A}\mathbf{W}^2) & I_{25} &= tr(\mathbf{A}^2\mathbf{W}^2) & I_{26} &= tr(\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}) \\
 I_{27} &= tr(\mathbf{B}\mathbf{W}^2) & I_{28} &= tr(\mathbf{B}^2\mathbf{W}^2) & I_{29} &= tr(\mathbf{B}^2\mathbf{W}^2\mathbf{B}\mathbf{W}) \\
 I_{30} &= tr(\mathbf{C}\mathbf{W}^2) & I_{31} &= tr(\mathbf{C}^2\mathbf{W}^2) & I_{32} &= tr(\mathbf{C}^2\mathbf{W}^2\mathbf{C}\mathbf{W}) \\
 I_{33} &= tr(\mathbf{A}\mathbf{B}\mathbf{W}) & I_{34} &= tr(\mathbf{A}\mathbf{B}^2\mathbf{W}) & I_{35} &= tr(\mathbf{A}^2\mathbf{B}\mathbf{W}) & I_{36} &= tr(\mathbf{A}\mathbf{W}^2\mathbf{B}\mathbf{W}) \\
 I_{37} &= tr(\mathbf{A}\mathbf{C}\mathbf{W}) & I_{38} &= tr(\mathbf{A}\mathbf{C}^2\mathbf{W}) & I_{39} &= tr(\mathbf{A}^2\mathbf{C}\mathbf{W}) & I_{40} &= tr(\mathbf{A}\mathbf{W}^2\mathbf{C}\mathbf{W}) \\
 I_{41} &= tr(\mathbf{B}\mathbf{C}\mathbf{W}) & I_{42} &= tr(\mathbf{B}\mathbf{C}^2\mathbf{W}) & I_{43} &= tr(\mathbf{B}^2\mathbf{C}\mathbf{W}) & I_{44} &= tr(\mathbf{B}\mathbf{W}^2\mathbf{C}\mathbf{W})
 \end{aligned} \tag{A.16}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 I_{45} &= tr(\mathbf{X}^2) \\
 I_{46} &= tr(\mathbf{A}\mathbf{X}^2) & I_{47} &= tr(\mathbf{A}^2\mathbf{X}^2) & I_{48} &= tr(\mathbf{A}^2\mathbf{X}^2\mathbf{A}\mathbf{X}) \\
 I_{49} &= tr(\mathbf{B}\mathbf{X}^2) & I_{50} &= tr(\mathbf{B}^2\mathbf{X}^2) & I_{51} &= tr(\mathbf{B}^2\mathbf{X}^2\mathbf{B}\mathbf{X}) \\
 I_{52} &= tr(\mathbf{C}\mathbf{X}^2) & I_{53} &= tr(\mathbf{C}^2\mathbf{X}^2) & I_{54} &= tr(\mathbf{C}^2\mathbf{X}^2\mathbf{C}\mathbf{X}) \\
 I_{55} &= tr(\mathbf{A}\mathbf{B}\mathbf{X}) & I_{56} &= tr(\mathbf{A}\mathbf{B}^2\mathbf{X}) & I_{57} &= tr(\mathbf{A}^2\mathbf{B}\mathbf{X}) & I_{58} &= tr(\mathbf{A}\mathbf{X}^2\mathbf{B}\mathbf{X}) \\
 I_{59} &= tr(\mathbf{A}\mathbf{C}\mathbf{X}) & I_{60} &= tr(\mathbf{A}\mathbf{C}^2\mathbf{X}) & I_{61} &= tr(\mathbf{A}^2\mathbf{C}\mathbf{X}) & I_{62} &= tr(\mathbf{A}\mathbf{X}^2\mathbf{C}\mathbf{X}) \\
 I_{63} &= tr(\mathbf{B}\mathbf{C}\mathbf{X}) & I_{64} &= tr(\mathbf{B}\mathbf{C}^2\mathbf{X}) & I_{65} &= tr(\mathbf{B}^2\mathbf{C}\mathbf{X}) & I_{66} &= tr(\mathbf{B}\mathbf{X}^2\mathbf{C}\mathbf{X}) \\
 I_{67} &= tr(\mathbf{W}\mathbf{X}) \\
 I_{68} &= tr(\mathbf{A}\mathbf{W}\mathbf{X}) & I_{69} &= tr(\mathbf{A}\mathbf{W}\mathbf{X}^2) & I_{70} &= tr(\mathbf{A}\mathbf{W}^2\mathbf{X}) \\
 I_{71} &= tr(\mathbf{B}\mathbf{W}\mathbf{X}) & I_{72} &= tr(\mathbf{B}\mathbf{W}\mathbf{X}^2) & I_{73} &= tr(\mathbf{B}\mathbf{W}^2\mathbf{X}) \\
 I_{74} &= tr(\mathbf{C}\mathbf{W}\mathbf{X}) & I_{75} &= tr(\mathbf{C}\mathbf{W}\mathbf{X}^2) & I_{76} &= tr(\mathbf{C}\mathbf{W}^2\mathbf{X})
 \end{aligned} \tag{A.17}$$

Scalar invariants of three symmetric tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 I_{77} &= \mathbf{v} \circ \mathbf{v} \\
 I_{78} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{79} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{v} \\
 I_{80} &= \mathbf{v} \circ \mathbf{B}\mathbf{v} & I_{81} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{v} \\
 I_{82} &= \mathbf{v} \circ \mathbf{C}\mathbf{v} & I_{83} &= \mathbf{v} \circ \mathbf{C}^2\mathbf{v} \\
 I_{84} &= \mathbf{v} \circ \mathbf{A}\mathbf{B}\mathbf{v} & I_{85} &= \mathbf{v} \circ \mathbf{A}\mathbf{C}\mathbf{v} & I_{86} &= \mathbf{v} \circ \mathbf{B}\mathbf{C}\mathbf{v} \\
 I_{87} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{v} & I_{88} &= \mathbf{v} \circ \mathbf{X}^2\mathbf{v} \\
 I_{89} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} & I_{90} &= \mathbf{v} \circ \mathbf{W}\mathbf{A}\mathbf{W}^2\mathbf{v} & I_{91} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{W}\mathbf{v} \\
 I_{92} &= \mathbf{v} \circ \mathbf{B}\mathbf{W}\mathbf{v} & I_{93} &= \mathbf{v} \circ \mathbf{W}\mathbf{B}\mathbf{W}^2\mathbf{v} & I_{94} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{W}\mathbf{v} \\
 I_{95} &= \mathbf{v} \circ \mathbf{C}\mathbf{W}\mathbf{v} & I_{96} &= \mathbf{v} \circ \mathbf{W}\mathbf{C}\mathbf{W}^2\mathbf{v} & I_{97} &= \mathbf{v} \circ \mathbf{C}^2\mathbf{W}\mathbf{v} \\
 I_{98} &= \mathbf{v} \circ \mathbf{A}\mathbf{X}\mathbf{v} & I_{99} &= \mathbf{v} \circ \mathbf{X}\mathbf{A}\mathbf{X}^2\mathbf{v} & I_{100} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{X}\mathbf{v} \\
 I_{101} &= \mathbf{v} \circ \mathbf{B}\mathbf{X}\mathbf{v} & I_{102} &= \mathbf{v} \circ \mathbf{X}\mathbf{B}\mathbf{X}^2\mathbf{v} & I_{103} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{X}\mathbf{v} \\
 I_{104} &= \mathbf{v} \circ \mathbf{C}\mathbf{X}\mathbf{v} & I_{105} &= \mathbf{v} \circ \mathbf{X}\mathbf{C}\mathbf{X}^2\mathbf{v} & I_{106} &= \mathbf{v} \circ \mathbf{C}^2\mathbf{X}\mathbf{v} \\
 I_{107} &= \mathbf{v} \circ \mathbf{W}\mathbf{X}\mathbf{v} & I_{108} &= \mathbf{v} \circ \mathbf{W}\mathbf{X}^2\mathbf{v} & I_{109} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{X}\mathbf{v}
 \end{aligned} \tag{A.18}$$

A.1.2 Vector invariants

Including one symmetric tensor

Vector invariants of one symmetric tensor $\mathbf{A}$ and one vector $\mathbf{v}$:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} \\
 \mathbf{f}_2 &= \mathbf{A}\mathbf{v} & \mathbf{f}_3 &= \mathbf{A}^2\mathbf{v}
 \end{aligned} \tag{A.19}$$

Vector invariants of one symmetric tensor $\mathbf{A}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_4 &= \mathbf{W}\mathbf{v} & \mathbf{f}_5 &= \mathbf{W}^2\mathbf{v} \\
 \mathbf{f}_6 &= (\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A})\mathbf{v}
 \end{aligned} \tag{A.20}$$

Including two symmetric tensors

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, and one vector $\mathbf{v}$:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} \\
 \mathbf{f}_2 &= \mathbf{A}\mathbf{v} & \mathbf{f}_3 &= \mathbf{A}^2\mathbf{v} \\
 \mathbf{f}_4 &= \mathbf{B}\mathbf{v} & \mathbf{f}_5 &= \mathbf{B}^2\mathbf{v} \\
 \mathbf{f}_6 &= (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})\mathbf{v}
 \end{aligned} \tag{A.21}$$

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_7 &= \mathbf{W}\mathbf{v} & \mathbf{f}_8 &= \mathbf{W}^2\mathbf{v} \\
 \mathbf{f}_9 &= (\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A})\mathbf{v} & \mathbf{f}_{10} &= (\mathbf{B}\mathbf{W} + \mathbf{W}\mathbf{B})\mathbf{v}
 \end{aligned} \tag{A.22}$$

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_{11} &= \mathbf{X}\mathbf{v} & \mathbf{f}_{12} &= \mathbf{X}^2\mathbf{v} \\
 \mathbf{f}_{13} &= (\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A})\mathbf{v} & \mathbf{f}_{14} &= (\mathbf{B}\mathbf{X} + \mathbf{X}\mathbf{B})\mathbf{v} \\
 \mathbf{f}_{15} &= (\mathbf{W}\mathbf{X} - \mathbf{X}\mathbf{W})\mathbf{v}
 \end{aligned} \tag{A.23}$$

Including three symmetric tensors

Vector invariants of three symmetric tensors **A**, **B**, and **C**, and one vector **v**:

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{v} \\ \mathbf{f}_2 &= \mathbf{A}\mathbf{v} & \mathbf{f}_3 &= \mathbf{A}^2\mathbf{v} \\ \mathbf{f}_4 &= \mathbf{B}\mathbf{v} & \mathbf{f}_5 &= \mathbf{B}^2\mathbf{v} \\ \mathbf{f}_6 &= \mathbf{C}\mathbf{v} & \mathbf{f}_7 &= \mathbf{C}^2\mathbf{v} \\ \mathbf{f}_8 &= (\mathbf{AB} - \mathbf{BA})\mathbf{v} & \mathbf{f}_9 &= (\mathbf{AC} - \mathbf{CA})\mathbf{v} & \mathbf{f}_{10} &= (\mathbf{BC} - \mathbf{CB})\mathbf{v} \end{aligned} \quad (\text{A.24})$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned} \mathbf{f}_{11} &= \mathbf{W}\mathbf{v} & \mathbf{f}_{12} &= \mathbf{W}^2\mathbf{v} \\ \mathbf{f}_{13} &= (\mathbf{AW} + \mathbf{WA})\mathbf{v} & \mathbf{f}_{14} &= (\mathbf{BW} + \mathbf{WB})\mathbf{v} & \mathbf{f}_{15} &= (\mathbf{CW} + \mathbf{WC})\mathbf{v} \end{aligned} \quad (\text{A.25})$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, two skew symmetric tensors **W** and **X**, and one vector **v**: Add

$$\begin{aligned} \mathbf{f}_{16} &= \mathbf{X}\mathbf{v} & \mathbf{f}_{17} &= \mathbf{X}^2\mathbf{v} \\ \mathbf{f}_{18} &= (\mathbf{AX} + \mathbf{XA})\mathbf{v} & \mathbf{f}_{19} &= (\mathbf{BX} + \mathbf{XB})\mathbf{v} & \mathbf{f}_{20} &= (\mathbf{CX} + \mathbf{XC})\mathbf{v} \\ \mathbf{f}_{21} &= (\mathbf{WX} - \mathbf{XW})\mathbf{v} \end{aligned} \quad (\text{A.26})$$

A.1.3 Symmetric tensor invariants**Including one symmetric tensor**

Symmetric tensor invariants of one symmetric tensor **A**:

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I} \\ \mathbf{H}_2 &= \mathbf{A} & \mathbf{H}_3 &= \mathbf{A}^2 \end{aligned} \quad (\text{A.27})$$

Symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{H}_4 &= \mathbf{W}^2 \\ \mathbf{H}_5 &= \mathbf{AW} - \mathbf{WA} & \mathbf{H}_6 &= \mathbf{WAW} \\ \mathbf{H}_7 &= \mathbf{WAW}^2 - \mathbf{W}^2\mathbf{AW} & \mathbf{H}_8 &= \mathbf{A}^2\mathbf{W} - \mathbf{WA}^2 \end{aligned} \quad (\text{A.28})$$

Including two symmetric tensors

Symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I} \\ \mathbf{H}_2 &= \mathbf{A} & \mathbf{H}_3 &= \mathbf{A}^2 \\ \mathbf{H}_4 &= \mathbf{B} & \mathbf{H}_5 &= \mathbf{B}^2 \\ \mathbf{H}_6 &= \mathbf{AB} + \mathbf{BA} & \mathbf{H}_7 &= \mathbf{AB}^2 + \mathbf{B}^2\mathbf{A} & \mathbf{H}_8 &= \mathbf{A}^2\mathbf{B} + \mathbf{BA}^2 \end{aligned} \quad (\text{A.29})$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{H}_9 &= \mathbf{W}^2 \\ \mathbf{H}_{10} &= \mathbf{AW} - \mathbf{WA} & \mathbf{H}_{11} &= \mathbf{WAW} & \mathbf{H}_{12} &= \mathbf{WAW}^2 - \mathbf{W}^2\mathbf{AW} & \mathbf{H}_{13} &= \mathbf{A}^2\mathbf{W} - \mathbf{WA}^2 \\ \mathbf{H}_{14} &= \mathbf{BW} - \mathbf{WB} & \mathbf{H}_{15} &= \mathbf{WBW} & \mathbf{H}_{16} &= \mathbf{WBW}^2 - \mathbf{W}^2\mathbf{BW} & \mathbf{H}_{17} &= \mathbf{B}^2\mathbf{W} - \mathbf{WB}^2 \end{aligned} \quad (\text{A.30})$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 H_{18} &= X^2 \\
 H_{19} &= AX - XA & H_{20} &= XAX & H_{21} &= XAX^2 - X^2AX & H_{22} &= A^2X - XA^2 \\
 H_{23} &= BX - XB & H_{24} &= XBX & H_{25} &= XBX^2 - X^2BX & H_{26} &= B^2X - XB^2 \\
 H_{27} &= XW + WX & H_{28} &= XW^2 - W^2X & H_{29} &= X^2W - WX^2
 \end{aligned} \tag{A.31}$$

Including three symmetric tensors

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 H_1 &= I \\
 H_2 &= A & H_3 &= A^2 \\
 H_4 &= B & H_5 &= B^2 \\
 H_6 &= C & H_7 &= C^2 \\
 H_8 &= AB + BA & H_9 &= AB^2 + B^2A & H_{10} &= A^2B + BA^2 \\
 H_{11} &= AC + CA & H_{12} &= AC^2 + C^2A & H_{13} &= A^2C + CA^2 \\
 H_{14} &= BC + CB & H_{15} &= BC^2 + C^2B & H_{16} &= B^2C + CB^2
 \end{aligned} \tag{A.32}$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 H_{17} &= W^2 \\
 H_{18} &= AW - WA & H_{19} &= WAW & H_{20} &= WAW^2 - W^2AW & H_{21} &= A^2W - WA^2 \\
 H_{22} &= BW - WB & H_{23} &= WBW & H_{24} &= WBW^2 - W^2BW & H_{25} &= B^2W - WB^2 \\
 H_{26} &= CW - WC & H_{27} &= WCW & H_{28} &= WCW^2 - W^2CW & H_{29} &= C^2W - WC^2
 \end{aligned} \tag{A.33}$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 H_{30} &= X^2 \\
 H_{31} &= AX - XA & H_{32} &= XAX & H_{33} &= XAX^2 - X^2AX & H_{34} &= A^2X - XA^2 \\
 H_{35} &= BX - XB & H_{36} &= XBX & H_{37} &= XBX^2 - X^2BX & H_{38} &= B^2X - XB^2 \\
 H_{39} &= CX - XC & H_{40} &= XCX & H_{41} &= XCX^2 - X^2CX & H_{42} &= C^2X - XC^2 \\
 H_{43} &= XW + WX & H_{44} &= XW^2 - W^2X & H_{45} &= X^2W - WX^2
 \end{aligned} \tag{A.34}$$

A.1.4 Skew symmetric tensor invariants

Including one symmetric tensor

Skew symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**:

$$\begin{aligned}
 Z_1 &= W \\
 Z_2 &= AW + WA & Z_3 &= AW^2 - W^2A
 \end{aligned} \tag{A.35}$$

Including two symmetric tensors

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned}
 Z_1 &= AB - BA & Z_2 &= AB^2 - B^2A & Z_3 &= A^2B - BA^2 \\
 Z_4 &= ABA^2 - A^2BA & Z_5 &= BAB^2 - B^2AB
 \end{aligned} \tag{A.36}$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{Z}_6 &= \mathbf{W} \\ \mathbf{Z}_7 &= \mathbf{AW} + \mathbf{WA} & \mathbf{Z}_8 &= \mathbf{AW}^2 - \mathbf{W}^2\mathbf{A} \\ \mathbf{Z}_9 &= \mathbf{BW} + \mathbf{WB} & \mathbf{Z}_{10} &= \mathbf{BW}^2 - \mathbf{W}^2\mathbf{B} \end{aligned} \quad (\text{A.37})$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned} \mathbf{Z}_{11} &= \mathbf{X} \\ \mathbf{Z}_{12} &= \mathbf{AX} + \mathbf{XA} & \mathbf{Z}_{13} &= \mathbf{AX}^2 - \mathbf{X}^2\mathbf{A} \\ \mathbf{Z}_{14} &= \mathbf{BX} + \mathbf{XB} & \mathbf{Z}_{15} &= \mathbf{BX}^2 - \mathbf{X}^2\mathbf{B} \\ \mathbf{Z}_{16} &= \mathbf{WX} - \mathbf{XW} \end{aligned} \quad (\text{A.38})$$

Including three symmetric tensors

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned} \mathbf{Z}_1 &= \mathbf{AB} - \mathbf{BA} & \mathbf{Z}_2 &= \mathbf{AB}^2 - \mathbf{B}^2\mathbf{A} & \mathbf{Z}_3 &= \mathbf{A}^2\mathbf{B} - \mathbf{BA}^2 \\ \mathbf{Z}_4 &= \mathbf{AC} - \mathbf{CA} & \mathbf{Z}_5 &= \mathbf{AC}^2 - \mathbf{C}^2\mathbf{A} & \mathbf{Z}_6 &= \mathbf{A}^2\mathbf{C} - \mathbf{CA}^2 \\ \mathbf{Z}_7 &= \mathbf{BC} - \mathbf{CB} & \mathbf{Z}_8 &= \mathbf{BC}^2 - \mathbf{C}^2\mathbf{B} & \mathbf{Z}_9 &= \mathbf{B}^2\mathbf{C} - \mathbf{CB}^2 \\ \mathbf{Z}_{10} &= \mathbf{ABA}^2 - \mathbf{A}^2\mathbf{BA} & \mathbf{Z}_{11} &= \mathbf{BAB}^2 - \mathbf{B}^2\mathbf{AB} \\ \mathbf{Z}_{12} &= \mathbf{ACA}^2 - \mathbf{A}^2\mathbf{CA} & \mathbf{Z}_{13} &= \mathbf{CAC}^2 - \mathbf{C}^2\mathbf{AC} \\ \mathbf{Z}_{14} &= \mathbf{BCB}^2 - \mathbf{B}^2\mathbf{CB} & \mathbf{Z}_{15} &= \mathbf{CBC}^2 - \mathbf{C}^2\mathbf{BC} \\ \mathbf{Z}_{16} &= \mathbf{ABC} + \mathbf{BCA} + \mathbf{CAB} - \mathbf{CBA} - \mathbf{BAC} - \mathbf{ACB} \end{aligned} \quad (\text{A.39})$$

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{Z}_{17} &= \mathbf{W} \\ \mathbf{Z}_{18} &= \mathbf{AW} + \mathbf{WA} & \mathbf{Z}_{19} &= \mathbf{AW}^2 - \mathbf{W}^2\mathbf{A} \\ \mathbf{Z}_{20} &= \mathbf{BW} + \mathbf{WB} & \mathbf{Z}_{21} &= \mathbf{BW}^2 - \mathbf{W}^2\mathbf{B} \\ \mathbf{Z}_{22} &= \mathbf{CW} + \mathbf{WC} & \mathbf{Z}_{23} &= \mathbf{CW}^2 - \mathbf{W}^2\mathbf{C} \end{aligned} \quad (\text{A.40})$$

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned} \mathbf{Z}_{24} &= \mathbf{X} \\ \mathbf{Z}_{25} &= \mathbf{AX} + \mathbf{XA} & \mathbf{Z}_{26} &= \mathbf{AX}^2 - \mathbf{X}^2\mathbf{A} \\ \mathbf{Z}_{27} &= \mathbf{BX} + \mathbf{XB} & \mathbf{Z}_{28} &= \mathbf{BX}^2 - \mathbf{X}^2\mathbf{B} \\ \mathbf{Z}_{29} &= \mathbf{CX} + \mathbf{XC} & \mathbf{Z}_{30} &= \mathbf{CX}^2 - \mathbf{X}^2\mathbf{C} \\ \mathbf{Z}_{31} &= \mathbf{WX} - \mathbf{XW} \end{aligned} \quad (\text{A.41})$$

A.2 Transversely isotropic

The following complete and irreducible invariants for the group associated with transverse isotropy (T_2) have been extracted from the following article by Zheng.² The group

²Zheng, Q.-S., "On transversely isotropic, orthotropic and relative isotropic functions of symmetric tensors, skew symmetric tensors and vectors. Part II: The representation for three dimensional transversely isotropic functions," *International Journal of Engineering Science*, vol. 31, no. 10, pages 1411–1423, 1993.

associated with transverse isotropy (T_2) is given by $\mathbf{R}_\tau$ and $\mathbf{R}_2$, where $\mathbf{R}_\tau$ is all rotations about the preferred axis τ and $\mathbf{R}_2$ is the reflection about the plane orthogonal to $\mathbf{e}_2$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are three orthogonal directions with

$$\tau = \mathbf{e}_3. \quad (\text{A.42})$$

A.2.1 Scalar invariants

Including one symmetric tensor

Scalar invariants of one symmetric tensor $\mathbf{A}$:

$$I_1 = \text{tr}(\mathbf{A}) \quad I_2 = \text{tr}(\mathbf{A}^2) \quad I_3 = \text{tr}(\mathbf{A}^3) \quad I_4 = \tau \circ \mathbf{A}\tau \quad I_5 = \tau \circ \mathbf{A}^2\tau \quad (\text{A.43})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$ and a skew symmetric tensor $\mathbf{W}$: Add

$$\begin{aligned} I_6 &= \text{tr}(\mathbf{W}^2) & I_7 &= \tau \circ \mathbf{W}^2\tau \\ I_8 &= \text{tr}(\mathbf{A}\mathbf{W}^2) & I_9 &= \text{tr}(\mathbf{A}^2\mathbf{W}^2) & I_{10} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}) \\ I_{11} &= \tau \circ \mathbf{A}\mathbf{W}\tau & I_{12} &= \tau \circ \mathbf{A}^2\mathbf{W}\tau & I_{13} &= \tau \circ \mathbf{W}\mathbf{A}\mathbf{W}^2\tau \end{aligned} \quad (\text{A.44})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$, a skew symmetric tensor $\mathbf{W}$, and a vector $\mathbf{v}$: Add

$$\begin{aligned} I_{14} &= \mathbf{v} \circ \mathbf{v} & I_{15} &= \tau \circ \mathbf{v} \\ I_{16} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{17} &= \tau \circ \mathbf{A}\mathbf{v} & I_{18} &= \tau \circ \mathbf{A}^2\mathbf{v} \\ I_{19} &= \tau \circ \mathbf{W}\mathbf{v} & I_{20} &= \tau \circ \mathbf{W}^2\mathbf{v} \\ I_{21} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} & I_{22} &= \tau \circ (\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A})\mathbf{v} \end{aligned} \quad (\text{A.45})$$

Including two symmetric tensors

Scalar invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$:

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{A}) & I_2 &= \text{tr}(\mathbf{A}^2) & I_3 &= \text{tr}(\mathbf{A}^3) & I_4 &= \tau \circ \mathbf{A}\tau & I_5 &= \tau \circ \mathbf{A}^2\tau \\ I_6 &= \text{tr}(\mathbf{B}) & I_7 &= \text{tr}(\mathbf{B}^2) & I_8 &= \text{tr}(\mathbf{B}^3) & I_9 &= \tau \circ \mathbf{B}\tau & I_{10} &= \tau \circ \mathbf{B}^2\tau \\ I_{11} &= \text{tr}(\mathbf{A}\mathbf{B}) & I_{12} &= \text{tr}(\mathbf{A}\mathbf{B}^2) & I_{13} &= \text{tr}(\mathbf{A}^2\mathbf{B}) & I_{14} &= \tau \circ \mathbf{A}\mathbf{B}\tau \end{aligned} \quad (\text{A.46})$$

Scalar invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, and one skew symmetric tensor $\mathbf{W}$: Add

$$\begin{aligned} I_{15} &= \text{tr}(\mathbf{W}^2) & I_{16} &= \tau \circ \mathbf{W}^2\tau \\ I_{17} &= \text{tr}(\mathbf{A}\mathbf{W}^2) & I_{18} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2) & I_{19} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}) \\ I_{20} &= \tau \circ \mathbf{A}\mathbf{W}\tau & I_{21} &= \tau \circ \mathbf{A}^2\mathbf{W}\tau & I_{22} &= \tau \circ \mathbf{W}\mathbf{A}\mathbf{W}^2\tau \\ I_{23} &= \text{tr}(\mathbf{B}\mathbf{W}^2) & I_{24} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2) & I_{25} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2\mathbf{B}\mathbf{W}) \\ I_{26} &= \tau \circ \mathbf{B}\mathbf{W}\tau & I_{27} &= \tau \circ \mathbf{B}^2\mathbf{W}\tau & I_{28} &= \tau \circ \mathbf{W}\mathbf{B}\mathbf{W}^2\tau \\ I_{29} &= \text{tr}(\mathbf{A}\mathbf{B}\mathbf{W}) & I_{30} &= \text{tr}(\mathbf{A}\mathbf{B}^2\mathbf{W}) & I_{31} &= \text{tr}(\mathbf{A}^2\mathbf{B}\mathbf{W}) & I_{32} &= \text{tr}(\mathbf{A}\mathbf{W}^2\mathbf{B}\mathbf{W}) \end{aligned} \quad (\text{A.47})$$

Scalar invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned} I_{33} &= \mathbf{v} \circ \mathbf{v} & I_{34} &= \tau \circ \mathbf{v} \\ I_{35} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{36} &= \tau \circ \mathbf{A}\mathbf{v} & I_{37} &= \tau \circ \mathbf{A}^2\mathbf{v} \\ I_{38} &= \mathbf{v} \circ \mathbf{B}\mathbf{v} & I_{39} &= \tau \circ \mathbf{B}\mathbf{v} & I_{40} &= \tau \circ \mathbf{B}^2\mathbf{v} \\ I_{41} &= \tau \circ (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})\mathbf{v} \\ I_{42} &= \tau \circ \mathbf{W}\mathbf{v} & I_{43} &= \tau \circ \mathbf{W}^2\mathbf{v} \\ I_{44} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} & I_{45} &= \tau \circ (\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A})\mathbf{v} \\ I_{46} &= \mathbf{v} \circ \mathbf{B}\mathbf{W}\mathbf{v} & I_{47} &= \tau \circ (\mathbf{B}\mathbf{W} + \mathbf{W}\mathbf{B})\mathbf{v} \end{aligned} \quad (\text{A.48})$$

Including three symmetric tensors

Scalar invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 I_1 &= \text{tr}(\mathbf{A}) & I_2 &= \text{tr}(\mathbf{A}^2) & I_3 &= \text{tr}(\mathbf{A}^3) & I_4 &= \boldsymbol{\tau} \circ \mathbf{A}\boldsymbol{\tau} & I_5 &= \boldsymbol{\tau} \circ \mathbf{A}^2\boldsymbol{\tau} \\
 I_6 &= \text{tr}(\mathbf{B}) & I_7 &= \text{tr}(\mathbf{B}^2) & I_8 &= \text{tr}(\mathbf{B}^3) & I_9 &= \boldsymbol{\tau} \circ \mathbf{B}\boldsymbol{\tau} & I_{10} &= \boldsymbol{\tau} \circ \mathbf{B}^2\boldsymbol{\tau} \\
 I_{11} &= \text{tr}(\mathbf{C}) & I_{12} &= \text{tr}(\mathbf{C}^2) & I_{13} &= \text{tr}(\mathbf{C}^3) & I_{14} &= \boldsymbol{\tau} \circ \mathbf{C}\boldsymbol{\tau} & I_{15} &= \boldsymbol{\tau} \circ \mathbf{C}^2\boldsymbol{\tau} \\
 I_{16} &= \text{tr}(\mathbf{AB}) & I_{17} &= \text{tr}(\mathbf{AB}^2) & I_{18} &= \text{tr}(\mathbf{A}^2\mathbf{B}) & I_{19} &= \boldsymbol{\tau} \circ \mathbf{AB}\boldsymbol{\tau} \\
 I_{20} &= \text{tr}(\mathbf{AC}) & I_{21} &= \text{tr}(\mathbf{AC}^2) & I_{22} &= \text{tr}(\mathbf{A}^2\mathbf{C}) & I_{23} &= \boldsymbol{\tau} \circ \mathbf{AC}\boldsymbol{\tau} \\
 I_{24} &= \text{tr}(\mathbf{BC}) & I_{25} &= \text{tr}(\mathbf{BC}^2) & I_{26} &= \text{tr}(\mathbf{B}^2\mathbf{C}) & I_{27} &= \boldsymbol{\tau} \circ \mathbf{BC}\boldsymbol{\tau} \\
 I_{28} &= \text{tr}(\mathbf{ABC})
 \end{aligned}
 \tag{A.49}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 I_{29} &= \text{tr}(\mathbf{W}^2) & I_{30} &= \boldsymbol{\tau} \circ \mathbf{W}^2\boldsymbol{\tau} \\
 I_{31} &= \text{tr}(\mathbf{AW}^2) & I_{32} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2) & I_{33} &= \text{tr}(\mathbf{A}^2\mathbf{W}^2\mathbf{AW}) \\
 I_{34} &= \boldsymbol{\tau} \circ \mathbf{AW}\boldsymbol{\tau} & I_{35} &= \boldsymbol{\tau} \circ \mathbf{A}^2\mathbf{W}\boldsymbol{\tau} & I_{36} &= \boldsymbol{\tau} \circ \mathbf{WAW}^2\boldsymbol{\tau} \\
 I_{37} &= \text{tr}(\mathbf{BW}^2) & I_{38} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2) & I_{39} &= \text{tr}(\mathbf{B}^2\mathbf{W}^2\mathbf{BW}) \\
 I_{40} &= \boldsymbol{\tau} \circ \mathbf{BW}\boldsymbol{\tau} & I_{41} &= \boldsymbol{\tau} \circ \mathbf{B}^2\mathbf{W}\boldsymbol{\tau} & I_{42} &= \boldsymbol{\tau} \circ \mathbf{WBW}^2\boldsymbol{\tau} \\
 I_{43} &= \text{tr}(\mathbf{CW}^2) & I_{44} &= \text{tr}(\mathbf{C}^2\mathbf{W}^2) & I_{45} &= \text{tr}(\mathbf{C}^2\mathbf{W}^2\mathbf{CW}) \\
 I_{46} &= \boldsymbol{\tau} \circ \mathbf{CW}\boldsymbol{\tau} & I_{47} &= \boldsymbol{\tau} \circ \mathbf{C}^2\mathbf{W}\boldsymbol{\tau} & I_{48} &= \boldsymbol{\tau} \circ \mathbf{WCW}^2\boldsymbol{\tau} \\
 I_{49} &= \text{tr}(\mathbf{ABW}) & I_{50} &= \text{tr}(\mathbf{AB}^2\mathbf{W}) & I_{51} &= \text{tr}(\mathbf{A}^2\mathbf{BW}) & I_{52} &= \text{tr}(\mathbf{AW}^2\mathbf{BW}) \\
 I_{53} &= \text{tr}(\mathbf{ACW}) & I_{54} &= \text{tr}(\mathbf{AC}^2\mathbf{W}) & I_{55} &= \text{tr}(\mathbf{A}^2\mathbf{CW}) & I_{56} &= \text{tr}(\mathbf{AW}^2\mathbf{CW}) \\
 I_{57} &= \text{tr}(\mathbf{BCW}) & I_{58} &= \text{tr}(\mathbf{BC}^2\mathbf{W}) & I_{59} &= \text{tr}(\mathbf{B}^2\mathbf{CW}) & I_{60} &= \text{tr}(\mathbf{BW}^2\mathbf{CW})
 \end{aligned}
 \tag{A.50}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 I_{61} &= \text{tr}(\mathbf{X}^2) & I_{62} &= \boldsymbol{\tau} \circ \mathbf{X}^2\boldsymbol{\tau} \\
 I_{63} &= \text{tr}(\mathbf{AX}^2) & I_{64} &= \text{tr}(\mathbf{A}^2\mathbf{X}^2) & I_{65} &= \text{tr}(\mathbf{A}^2\mathbf{X}^2\mathbf{AX}) \\
 I_{66} &= \boldsymbol{\tau} \circ \mathbf{AX}\boldsymbol{\tau} & I_{67} &= \boldsymbol{\tau} \circ \mathbf{A}^2\mathbf{X}\boldsymbol{\tau} & I_{68} &= \boldsymbol{\tau} \circ \mathbf{XAX}^2\boldsymbol{\tau} \\
 I_{69} &= \text{tr}(\mathbf{BX}^2) & I_{70} &= \text{tr}(\mathbf{B}^2\mathbf{X}^2) & I_{71} &= \text{tr}(\mathbf{B}^2\mathbf{X}^2\mathbf{BX}) \\
 I_{72} &= \boldsymbol{\tau} \circ \mathbf{BX}\boldsymbol{\tau} & I_{73} &= \boldsymbol{\tau} \circ \mathbf{B}^2\mathbf{X}\boldsymbol{\tau} & I_{74} &= \boldsymbol{\tau} \circ \mathbf{XBX}^2\boldsymbol{\tau} \\
 I_{75} &= \text{tr}(\mathbf{CX}^2) & I_{76} &= \text{tr}(\mathbf{C}^2\mathbf{X}^2) & I_{77} &= \text{tr}(\mathbf{C}^2\mathbf{X}^2\mathbf{CX}) \\
 I_{78} &= \boldsymbol{\tau} \circ \mathbf{CX}\boldsymbol{\tau} & I_{79} &= \boldsymbol{\tau} \circ \mathbf{C}^2\mathbf{X}\boldsymbol{\tau} & I_{80} &= \boldsymbol{\tau} \circ \mathbf{XCX}^2\boldsymbol{\tau} \\
 I_{81} &= \text{tr}(\mathbf{ABX}) & I_{82} &= \text{tr}(\mathbf{AB}^2\mathbf{X}) & I_{83} &= \text{tr}(\mathbf{A}^2\mathbf{BX}) & I_{84} &= \text{tr}(\mathbf{AX}^2\mathbf{BX}) \\
 I_{85} &= \text{tr}(\mathbf{ACX}) & I_{86} &= \text{tr}(\mathbf{AC}^2\mathbf{X}) & I_{87} &= \text{tr}(\mathbf{A}^2\mathbf{CX}) & I_{88} &= \text{tr}(\mathbf{AX}^2\mathbf{CX}) \\
 I_{89} &= \text{tr}(\mathbf{BCX}) & I_{90} &= \text{tr}(\mathbf{BC}^2\mathbf{X}) & I_{91} &= \text{tr}(\mathbf{B}^2\mathbf{CX}) & I_{92} &= \text{tr}(\mathbf{BX}^2\mathbf{CX}) \\
 I_{93} &= \text{tr}(\mathbf{WX}) & I_{94} &= \boldsymbol{\tau} \circ \mathbf{WX}\boldsymbol{\tau} & I_{95} &= \boldsymbol{\tau} \circ \mathbf{WX}^2\boldsymbol{\tau} & I_{96} &= \boldsymbol{\tau} \circ \mathbf{W}^2\mathbf{X}\boldsymbol{\tau} \\
 I_{97} &= \text{tr}(\mathbf{AWX}) & I_{98} &= \text{tr}(\mathbf{AWX}^2) & I_{99} &= \text{tr}(\mathbf{AW}^2\mathbf{X}) \\
 I_{100} &= \text{tr}(\mathbf{BWX}) & I_{101} &= \text{tr}(\mathbf{BWX}^2) & I_{102} &= \text{tr}(\mathbf{BW}^2\mathbf{X}) \\
 I_{103} &= \text{tr}(\mathbf{CWX}) & I_{104} &= \text{tr}(\mathbf{CWX}^2) & I_{105} &= \text{tr}(\mathbf{CW}^2\mathbf{X})
 \end{aligned}
 \tag{A.51}$$

Scalar invariants of three symmetric tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 I_{106} &= \mathbf{v} \circ \mathbf{v} & I_{107} &= \boldsymbol{\tau} \circ \mathbf{v} & I_{110} &= \boldsymbol{\tau} \circ \mathbf{A}^2 \mathbf{v} \\
 I_{108} &= \mathbf{v} \circ \mathbf{A} \mathbf{v} & I_{109} &= \boldsymbol{\tau} \circ \mathbf{A} \mathbf{v} & I_{113} &= \boldsymbol{\tau} \circ \mathbf{B}^2 \mathbf{v} \\
 I_{111} &= \mathbf{v} \circ \mathbf{B} \mathbf{v} & I_{112} &= \boldsymbol{\tau} \circ \mathbf{B} \mathbf{v} & I_{116} &= \boldsymbol{\tau} \circ \mathbf{C}^2 \mathbf{v} \\
 I_{114} &= \mathbf{v} \circ \mathbf{C} \mathbf{v} & I_{115} &= \boldsymbol{\tau} \circ \mathbf{C} \mathbf{v} & I_{119} &= \boldsymbol{\tau} \circ (\mathbf{B} \mathbf{C} - \mathbf{C} \mathbf{B}) \mathbf{v} \\
 I_{117} &= \boldsymbol{\tau} \circ (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A}) \mathbf{v} & I_{118} &= \boldsymbol{\tau} \circ (\mathbf{A} \mathbf{C} - \mathbf{C} \mathbf{A}) \mathbf{v} & & \\
 I_{120} &= \boldsymbol{\tau} \circ \mathbf{W} \mathbf{v} & I_{121} &= \boldsymbol{\tau} \circ \mathbf{W}^2 \mathbf{v} & & \\
 I_{122} &= \mathbf{v} \circ \mathbf{A} \mathbf{W} \mathbf{v} & I_{123} &= \boldsymbol{\tau} \circ (\mathbf{A} \mathbf{W} + \mathbf{W} \mathbf{A}) \mathbf{v} & & \\
 I_{124} &= \mathbf{v} \circ \mathbf{B} \mathbf{W} \mathbf{v} & I_{125} &= \boldsymbol{\tau} \circ (\mathbf{B} \mathbf{W} + \mathbf{W} \mathbf{B}) \mathbf{v} & & \\
 I_{126} &= \mathbf{v} \circ \mathbf{C} \mathbf{W} \mathbf{v} & I_{127} &= \boldsymbol{\tau} \circ (\mathbf{C} \mathbf{W} + \mathbf{W} \mathbf{C}) \mathbf{v} & & \\
 I_{128} &= \boldsymbol{\tau} \circ \mathbf{X} \mathbf{v} & I_{129} &= \boldsymbol{\tau} \circ \mathbf{X}^2 \mathbf{v} & & \\
 I_{130} &= \mathbf{v} \circ \mathbf{A} \mathbf{X} \mathbf{v} & I_{131} &= \boldsymbol{\tau} \circ (\mathbf{A} \mathbf{X} + \mathbf{X} \mathbf{A}) \mathbf{v} & & \\
 I_{132} &= \mathbf{v} \circ \mathbf{B} \mathbf{X} \mathbf{v} & I_{133} &= \boldsymbol{\tau} \circ (\mathbf{B} \mathbf{X} + \mathbf{X} \mathbf{B}) \mathbf{v} & & \\
 I_{134} &= \mathbf{v} \circ \mathbf{C} \mathbf{X} \mathbf{v} & I_{135} &= \boldsymbol{\tau} \circ (\mathbf{C} \mathbf{X} + \mathbf{X} \mathbf{C}) \mathbf{v} & & \\
 I_{136} &= \boldsymbol{\tau} \circ (\mathbf{W} \mathbf{X} - \mathbf{X} \mathbf{W}) \mathbf{v} & & & &
 \end{aligned} \tag{A.52}$$

A.2.2 Vector invariants

Including one symmetric tensor

Vector invariants of one symmetric tensor $\mathbf{A}$ and one vector $\mathbf{v}$:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \boldsymbol{\tau} \\
 \mathbf{f}_3 &= \mathbf{A} \mathbf{v} & \mathbf{f}_4 &= \mathbf{A} \boldsymbol{\tau} & \mathbf{f}_5 &= \mathbf{A}^2 \boldsymbol{\tau}
 \end{aligned} \tag{A.53}$$

Vector invariants of one symmetric tensor $\mathbf{A}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_6 &= \mathbf{W} \mathbf{v} & \mathbf{f}_7 &= \mathbf{W} \boldsymbol{\tau} & \mathbf{f}_8 &= \mathbf{W}^2 \boldsymbol{\tau} \\
 \mathbf{f}_9 &= (\mathbf{A} \mathbf{W} + \mathbf{W} \mathbf{A}) \boldsymbol{\tau}
 \end{aligned} \tag{A.54}$$

Including two symmetric tensors

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, and one vector $\mathbf{v}$:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \boldsymbol{\tau} \\
 \mathbf{f}_3 &= \mathbf{A} \mathbf{v} & \mathbf{f}_4 &= \mathbf{A} \boldsymbol{\tau} & \mathbf{f}_5 &= \mathbf{A}^2 \boldsymbol{\tau} \\
 \mathbf{f}_6 &= \mathbf{B} \mathbf{v} & \mathbf{f}_7 &= \mathbf{B} \boldsymbol{\tau} & \mathbf{f}_8 &= \mathbf{B}^2 \boldsymbol{\tau} \\
 \mathbf{f}_9 &= (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A}) \boldsymbol{\tau}
 \end{aligned} \tag{A.55}$$

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, one skew symmetric tensor $\mathbf{W}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_{10} &= \mathbf{W} \mathbf{v} & \mathbf{f}_{11} &= \mathbf{W} \boldsymbol{\tau} & \mathbf{f}_{12} &= \mathbf{W}^2 \boldsymbol{\tau} \\
 \mathbf{f}_{13} &= (\mathbf{A} \mathbf{W} + \mathbf{W} \mathbf{A}) \boldsymbol{\tau} & \mathbf{f}_{14} &= (\mathbf{B} \mathbf{W} + \mathbf{W} \mathbf{B}) \boldsymbol{\tau}
 \end{aligned} \tag{A.56}$$

Vector invariants of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$, two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$, and one vector $\mathbf{v}$: Add

$$\begin{aligned}
 \mathbf{f}_{15} &= \mathbf{X} \mathbf{v} & \mathbf{f}_{16} &= \mathbf{X} \boldsymbol{\tau} & \mathbf{f}_{17} &= \mathbf{X}^2 \boldsymbol{\tau} \\
 \mathbf{f}_{18} &= (\mathbf{A} \mathbf{X} + \mathbf{X} \mathbf{A}) \boldsymbol{\tau} & \mathbf{f}_{19} &= (\mathbf{B} \mathbf{X} + \mathbf{X} \mathbf{B}) \boldsymbol{\tau} \\
 \mathbf{f}_{20} &= (\mathbf{W} \mathbf{X} - \mathbf{X} \mathbf{W}) \boldsymbol{\tau}
 \end{aligned} \tag{A.57}$$

Including three symmetric tensors

Vector invariants of three symmetric tensors **A**, **B**, and **C**, and one vector **v**:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \boldsymbol{\tau} & \mathbf{f}_5 &= \mathbf{A}^2 \boldsymbol{\tau} \\
 \mathbf{f}_3 &= \mathbf{A} \mathbf{v} & \mathbf{f}_4 &= \mathbf{A} \boldsymbol{\tau} & \mathbf{f}_8 &= \mathbf{B}^2 \boldsymbol{\tau} \\
 \mathbf{f}_6 &= \mathbf{B} \mathbf{v} & \mathbf{f}_7 &= \mathbf{B} \boldsymbol{\tau} & \mathbf{f}_{11} &= \mathbf{C}^2 \boldsymbol{\tau} \\
 \mathbf{f}_9 &= \mathbf{C} \mathbf{v} & \mathbf{f}_{10} &= \mathbf{C} \boldsymbol{\tau} & \mathbf{f}_{14} &= (\mathbf{B} \mathbf{C} - \mathbf{C} \mathbf{B}) \boldsymbol{\tau} \\
 \mathbf{f}_{12} &= (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A}) \boldsymbol{\tau} & \mathbf{f}_{13} &= (\mathbf{A} \mathbf{C} - \mathbf{C} \mathbf{A}) \boldsymbol{\tau} & &
 \end{aligned} \tag{A.58}$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned}
 \mathbf{f}_{15} &= \mathbf{W} \mathbf{v} & \mathbf{f}_{16} &= \mathbf{W} \boldsymbol{\tau} & \mathbf{f}_{17} &= \mathbf{W}^2 \boldsymbol{\tau} \\
 \mathbf{f}_{18} &= (\mathbf{A} \mathbf{W} + \mathbf{W} \mathbf{A}) \boldsymbol{\tau} & \mathbf{f}_{19} &= (\mathbf{B} \mathbf{W} + \mathbf{W} \mathbf{B}) \boldsymbol{\tau} & \mathbf{f}_{20} &= (\mathbf{C} \mathbf{W} + \mathbf{W} \mathbf{C}) \boldsymbol{\tau}
 \end{aligned} \tag{A.59}$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, two skew symmetric tensors **W** and **X**, and one vector **v**: Add

$$\begin{aligned}
 \mathbf{f}_{21} &= \mathbf{X} \mathbf{v} & \mathbf{f}_{22} &= \mathbf{X} \boldsymbol{\tau} & \mathbf{f}_{23} &= \mathbf{X}^2 \boldsymbol{\tau} \\
 \mathbf{f}_{24} &= (\mathbf{A} \mathbf{X} + \mathbf{X} \mathbf{A}) \boldsymbol{\tau} & \mathbf{f}_{25} &= (\mathbf{B} \mathbf{X} + \mathbf{X} \mathbf{B}) \boldsymbol{\tau} & \mathbf{f}_{26} &= (\mathbf{C} \mathbf{X} + \mathbf{X} \mathbf{C}) \boldsymbol{\tau} \\
 \mathbf{f}_{27} &= (\mathbf{W} \mathbf{X} - \mathbf{X} \mathbf{W}) \boldsymbol{\tau} & & & &
 \end{aligned} \tag{A.60}$$

A.2.3 Symmetric tensor invariants**Including one symmetric tensor**

Symmetric tensor invariants of one symmetric tensor **A**:

$$\begin{aligned}
 \mathbf{H}_1 &= \mathbf{I} & \mathbf{H}_2 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{H}_3 &= \mathbf{A} & \mathbf{H}_4 &= \mathbf{A}^2 \\
 \mathbf{H}_5 &= \boldsymbol{\tau} \otimes \mathbf{A} \boldsymbol{\tau} + \mathbf{A} \boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_6 &= \boldsymbol{\tau} \otimes \mathbf{A}^2 \boldsymbol{\tau} + \mathbf{A}^2 \boldsymbol{\tau} \otimes \boldsymbol{\tau}
 \end{aligned} \tag{A.61}$$

Symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 \mathbf{H}_7 &= \mathbf{W}^2 & \mathbf{H}_8 &= \boldsymbol{\tau} \otimes \mathbf{W} \boldsymbol{\tau} + \mathbf{W} \boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_9 &= \mathbf{W} \boldsymbol{\tau} \otimes \mathbf{W} \boldsymbol{\tau} \\
 \mathbf{H}_{10} &= \mathbf{W} \boldsymbol{\tau} \otimes \mathbf{W}^2 \boldsymbol{\tau} + \mathbf{W}^2 \boldsymbol{\tau} \otimes \mathbf{W} \boldsymbol{\tau} \\
 \mathbf{H}_{11} &= \mathbf{A} \mathbf{W} - \mathbf{W} \mathbf{A} & \mathbf{H}_{12} &= \mathbf{W} \mathbf{A} \mathbf{W}^2 - \mathbf{W}^2 \mathbf{A} \mathbf{W} & \mathbf{H}_{13} &= \mathbf{A}^2 \mathbf{W} - \mathbf{W} \mathbf{A}^2
 \end{aligned} \tag{A.62}$$

Including two symmetric tensors

Symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned}
 \mathbf{H}_1 &= \mathbf{I} & \mathbf{H}_2 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{H}_3 &= \mathbf{A} & \mathbf{H}_4 &= \mathbf{A}^2 \\
 \mathbf{H}_5 &= \boldsymbol{\tau} \otimes \mathbf{A} \boldsymbol{\tau} + \mathbf{A} \boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_6 &= \boldsymbol{\tau} \otimes \mathbf{A}^2 \boldsymbol{\tau} + \mathbf{A}^2 \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{H}_7 &= \mathbf{B} & \mathbf{H}_8 &= \mathbf{B}^2 \\
 \mathbf{H}_9 &= \boldsymbol{\tau} \otimes \mathbf{B} \boldsymbol{\tau} + \mathbf{B} \boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_{10} &= \boldsymbol{\tau} \otimes \mathbf{B}^2 \boldsymbol{\tau} + \mathbf{B}^2 \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{H}_{11} &= \mathbf{A} \mathbf{B} + \mathbf{B} \mathbf{A}
 \end{aligned} \tag{A.63}$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 H_{12} &= \mathbf{W}^2 & H_{13} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} + \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{14} &= \mathbf{W}\boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} \\
 H_{15} &= \mathbf{W}\boldsymbol{\tau} \otimes \mathbf{W}^2\boldsymbol{\tau} + \mathbf{W}^2\boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} & H_{17} &= \mathbf{W}\mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A}\mathbf{W} & H_{18} &= \mathbf{A}^2\mathbf{W} - \mathbf{W}\mathbf{A}^2 \\
 H_{16} &= \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} & H_{20} &= \mathbf{W}\mathbf{B}\mathbf{W}^2 - \mathbf{W}^2\mathbf{B}\mathbf{W} & H_{21} &= \mathbf{B}^2\mathbf{W} - \mathbf{W}\mathbf{B}^2 \\
 H_{19} &= \mathbf{B}\mathbf{W} - \mathbf{W}\mathbf{B} & & & &
 \end{aligned} \tag{A.64}$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 H_{22} &= \mathbf{X}^2 & H_{23} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} + \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{24} &= \mathbf{X}\boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} \\
 H_{25} &= \mathbf{X}\boldsymbol{\tau} \otimes \mathbf{X}^2\boldsymbol{\tau} + \mathbf{X}^2\boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} & H_{27} &= \mathbf{X}\mathbf{A}\mathbf{X}^2 - \mathbf{X}^2\mathbf{A}\mathbf{X} & H_{28} &= \mathbf{A}^2\mathbf{X} - \mathbf{X}\mathbf{A}^2 \\
 H_{26} &= \mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{A} & H_{30} &= \mathbf{X}\mathbf{B}\mathbf{X}^2 - \mathbf{X}^2\mathbf{B}\mathbf{X} & H_{31} &= \mathbf{B}^2\mathbf{X} - \mathbf{X}\mathbf{B}^2 \\
 H_{29} &= \mathbf{B}\mathbf{X} - \mathbf{X}\mathbf{B} & H_{33} &= \mathbf{W}\mathbf{X}^2 - \mathbf{X}^2\mathbf{W} & H_{34} &= \mathbf{W}^2\mathbf{X} - \mathbf{X}\mathbf{W}^2 \\
 H_{32} &= \mathbf{W}\mathbf{X} + \mathbf{X}\mathbf{W} & & & &
 \end{aligned} \tag{A.65}$$

Including three symmetric tensors

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 H_1 &= \mathbf{I} & H_2 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 H_3 &= \mathbf{A} & H_4 &= \mathbf{A}^2 \\
 H_5 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_6 &= \boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} + \mathbf{A}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 H_7 &= \mathbf{B} & H_8 &= \mathbf{B}^2 \\
 H_9 &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{10} &= \boldsymbol{\tau} \otimes \mathbf{B}^2\boldsymbol{\tau} + \mathbf{B}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 H_{11} &= \mathbf{C} & H_{12} &= \mathbf{C}^2 \\
 H_{13} &= \boldsymbol{\tau} \otimes \mathbf{C}\boldsymbol{\tau} + \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{14} &= \boldsymbol{\tau} \otimes \mathbf{C}^2\boldsymbol{\tau} + \mathbf{C}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 H_{15} &= \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} & H_{16} &= \mathbf{A}\mathbf{C} + \mathbf{C}\mathbf{A} & H_{17} &= \mathbf{B}\mathbf{C} + \mathbf{C}\mathbf{B}
 \end{aligned} \tag{A.66}$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 H_{18} &= \mathbf{W}^2 & H_{19} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} + \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{20} &= \mathbf{W}\boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} \\
 H_{21} &= \mathbf{W}\boldsymbol{\tau} \otimes \mathbf{W}^2\boldsymbol{\tau} + \mathbf{W}^2\boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} & H_{23} &= \mathbf{W}\mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A}\mathbf{W} & H_{24} &= \mathbf{A}^2\mathbf{W} - \mathbf{W}\mathbf{A}^2 \\
 H_{22} &= \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} & H_{26} &= \mathbf{W}\mathbf{B}\mathbf{W}^2 - \mathbf{W}^2\mathbf{B}\mathbf{W} & H_{27} &= \mathbf{B}^2\mathbf{W} - \mathbf{W}\mathbf{B}^2 \\
 H_{25} &= \mathbf{B}\mathbf{W} - \mathbf{W}\mathbf{B} & H_{29} &= \mathbf{W}\mathbf{C}\mathbf{W}^2 - \mathbf{W}^2\mathbf{C}\mathbf{W} & H_{30} &= \mathbf{C}^2\mathbf{W} - \mathbf{W}\mathbf{C}^2 \\
 H_{28} &= \mathbf{C}\mathbf{W} - \mathbf{W}\mathbf{C} & & & &
 \end{aligned} \tag{A.67}$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 H_{31} &= \mathbf{X}^2 & H_{32} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} + \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & H_{33} &= \mathbf{X}\boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} \\
 H_{34} &= \mathbf{X}\boldsymbol{\tau} \otimes \mathbf{X}^2\boldsymbol{\tau} + \mathbf{X}^2\boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} & H_{36} &= \mathbf{X}\mathbf{A}\mathbf{X}^2 - \mathbf{X}^2\mathbf{A}\mathbf{X} & H_{37} &= \mathbf{A}^2\mathbf{X} - \mathbf{X}\mathbf{A}^2 \\
 H_{35} &= \mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{A} & H_{39} &= \mathbf{X}\mathbf{B}\mathbf{X}^2 - \mathbf{X}^2\mathbf{B}\mathbf{X} & H_{40} &= \mathbf{B}^2\mathbf{X} - \mathbf{X}\mathbf{B}^2 \\
 H_{38} &= \mathbf{B}\mathbf{X} - \mathbf{X}\mathbf{B} & H_{42} &= \mathbf{X}\mathbf{C}\mathbf{X}^2 - \mathbf{X}^2\mathbf{C}\mathbf{X} & H_{43} &= \mathbf{C}^2\mathbf{X} - \mathbf{X}\mathbf{C}^2 \\
 H_{41} &= \mathbf{C}\mathbf{X} - \mathbf{X}\mathbf{C} & H_{45} &= \mathbf{W}\mathbf{X}^2 - \mathbf{X}^2\mathbf{W} & H_{46} &= \mathbf{W}^2\mathbf{X} - \mathbf{X}\mathbf{W}^2 \\
 H_{44} &= \mathbf{W}\mathbf{X} + \mathbf{X}\mathbf{W} & & & &
 \end{aligned} \tag{A.68}$$

A.2.4 Skew symmetric tensor invariants

Including one symmetric tensor

Skew symmetric tensor invariants of one symmetric tensor **A**:

$$\mathbf{Z}_1 = \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \quad \mathbf{Z}_2 = \boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \quad \mathbf{Z}_3 = \mathbf{A}\boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} \quad (\text{A.69})$$

Skew symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{Z}_4 &= \mathbf{W} \\ \mathbf{Z}_5 &= \mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} & \mathbf{Z}_6 &= \mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A} \\ \mathbf{Z}_7 &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} - \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_8 &= \boldsymbol{\tau} \otimes \mathbf{W}^2\boldsymbol{\tau} - \mathbf{W}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \end{aligned} \quad (\text{A.70})$$

Including two symmetric tensors

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned} \mathbf{Z}_1 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_2 &= \boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_3 &= \mathbf{A}\boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} \\ \mathbf{Z}_4 &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} - \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_5 &= \boldsymbol{\tau} \otimes \mathbf{B}^2\boldsymbol{\tau} - \mathbf{B}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_6 &= \mathbf{B}\boldsymbol{\tau} \otimes \mathbf{B}^2\boldsymbol{\tau} - \mathbf{B}^2\boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} \\ \mathbf{Z}_7 &= \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} & \mathbf{Z}_8 &= \mathbf{A}\mathbf{B}^2 - \mathbf{B}^2\mathbf{A} & \mathbf{Z}_9 &= \mathbf{A}^2\mathbf{B} - \mathbf{B}\mathbf{A}^2 \\ \mathbf{Z}_{10} &= \mathbf{A}\mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A}\mathbf{B} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B}\mathbf{A} - \mathbf{B}\mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B} \end{aligned} \quad (\text{A.71})$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{Z}_{11} &= \mathbf{W} & \mathbf{Z}_{12} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} - \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_{13} &= \boldsymbol{\tau} \otimes \mathbf{W}^2\boldsymbol{\tau} - \mathbf{W}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{Z}_{14} &= \mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} & \mathbf{Z}_{15} &= \mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A} \\ \mathbf{Z}_{16} &= \mathbf{B}\mathbf{W} + \mathbf{W}\mathbf{B} & \mathbf{Z}_{17} &= \mathbf{B}\mathbf{W}^2 - \mathbf{W}^2\mathbf{B} \end{aligned} \quad (\text{A.72})$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned} \mathbf{Z}_{18} &= \mathbf{X} & \mathbf{Z}_{19} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} - \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_{20} &= \boldsymbol{\tau} \otimes \mathbf{X}^2\boldsymbol{\tau} - \mathbf{X}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{Z}_{21} &= \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} & \mathbf{Z}_{22} &= \mathbf{A}\mathbf{X}^2 - \mathbf{X}^2\mathbf{A} \\ \mathbf{Z}_{23} &= \mathbf{B}\mathbf{X} + \mathbf{X}\mathbf{B} & \mathbf{Z}_{24} &= \mathbf{B}\mathbf{X}^2 - \mathbf{X}^2\mathbf{B} \\ \mathbf{Z}_{25} &= \mathbf{W}\mathbf{X} - \mathbf{X}\mathbf{W} \end{aligned} \quad (\text{A.73})$$

Including three symmetric tensors

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned} \mathbf{Z}_1 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_2 &= \boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_3 &= \mathbf{A}\boldsymbol{\tau} \otimes \mathbf{A}^2\boldsymbol{\tau} - \mathbf{A}^2\boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} \\ \mathbf{Z}_4 &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} - \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_5 &= \boldsymbol{\tau} \otimes \mathbf{B}^2\boldsymbol{\tau} - \mathbf{B}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_6 &= \mathbf{B}\boldsymbol{\tau} \otimes \mathbf{B}^2\boldsymbol{\tau} - \mathbf{B}^2\boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} \\ \mathbf{Z}_7 &= \boldsymbol{\tau} \otimes \mathbf{C}\boldsymbol{\tau} - \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_8 &= \boldsymbol{\tau} \otimes \mathbf{C}^2\boldsymbol{\tau} - \mathbf{C}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_9 &= \mathbf{C}\boldsymbol{\tau} \otimes \mathbf{C}^2\boldsymbol{\tau} - \mathbf{C}^2\boldsymbol{\tau} \otimes \mathbf{C}\boldsymbol{\tau} \\ \mathbf{Z}_{10} &= \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} & \mathbf{Z}_{11} &= \mathbf{A}\mathbf{B}^2 - \mathbf{B}^2\mathbf{A} & \mathbf{Z}_{12} &= \mathbf{A}^2\mathbf{B} - \mathbf{B}\mathbf{A}^2 \\ \mathbf{Z}_{13} &= \mathbf{A}\mathbf{C} - \mathbf{C}\mathbf{A} & \mathbf{Z}_{14} &= \mathbf{A}\mathbf{C}^2 - \mathbf{C}^2\mathbf{A} & \mathbf{Z}_{15} &= \mathbf{A}^2\mathbf{C} - \mathbf{C}\mathbf{A}^2 \\ \mathbf{Z}_{16} &= \mathbf{B}\mathbf{C} - \mathbf{C}\mathbf{B} & \mathbf{Z}_{17} &= \mathbf{B}\mathbf{C}^2 - \mathbf{C}^2\mathbf{B} & \mathbf{Z}_{18} &= \mathbf{B}^2\mathbf{C} - \mathbf{C}\mathbf{B}^2 \\ \mathbf{Z}_{19} &= \mathbf{A}\mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A}\mathbf{B} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B}\mathbf{A} - \mathbf{B}\mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B} \\ \mathbf{Z}_{20} &= \mathbf{A}\mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A}\mathbf{C} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{C}\mathbf{A} - \mathbf{C}\mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{C} \\ \mathbf{Z}_{21} &= \mathbf{B}\mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B}\mathbf{C} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{C}\mathbf{B} - \mathbf{C}\mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{C} \end{aligned} \quad (\text{A.74})$$

Skew symmetric tensor invariants of three symmetric tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and one skew symmetric tensor $\mathbf{W}$: Add

$$\begin{aligned} \mathbf{Z}_{22} &= \mathbf{W} & \mathbf{Z}_{23} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} - \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_{24} &= \boldsymbol{\tau} \otimes \mathbf{W}^2\boldsymbol{\tau} - \mathbf{W}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{Z}_{25} &= \mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} & \mathbf{Z}_{26} &= \mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A} \\ \mathbf{Z}_{27} &= \mathbf{B}\mathbf{W} + \mathbf{W}\mathbf{B} & \mathbf{Z}_{28} &= \mathbf{B}\mathbf{W}^2 - \mathbf{W}^2\mathbf{B} \\ \mathbf{Z}_{29} &= \mathbf{C}\mathbf{W} + \mathbf{W}\mathbf{C} & \mathbf{Z}_{30} &= \mathbf{C}\mathbf{W}^2 - \mathbf{W}^2\mathbf{C} \end{aligned} \quad (\text{A.75})$$

Skew symmetric tensor invariants of three symmetric tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and two skew symmetric tensors $\mathbf{W}$ and $\mathbf{X}$: Add

$$\begin{aligned} \mathbf{Z}_{31} &= \mathbf{X} & \mathbf{Z}_{32} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} - \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{Z}_{33} &= \boldsymbol{\tau} \otimes \mathbf{X}^2\boldsymbol{\tau} - \mathbf{X}^2\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{Z}_{34} &= \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} & \mathbf{Z}_{35} &= \mathbf{A}\mathbf{X}^2 - \mathbf{X}^2\mathbf{A} \\ \mathbf{Z}_{36} &= \mathbf{B}\mathbf{X} + \mathbf{X}\mathbf{B} & \mathbf{Z}_{37} &= \mathbf{B}\mathbf{X}^2 - \mathbf{X}^2\mathbf{B} \\ \mathbf{Z}_{38} &= \mathbf{C}\mathbf{X} + \mathbf{X}\mathbf{C} & \mathbf{Z}_{39} &= \mathbf{C}\mathbf{X}^2 - \mathbf{X}^2\mathbf{C} \\ \mathbf{Z}_{40} &= \mathbf{W}\mathbf{X} - \mathbf{X}\mathbf{W} \end{aligned} \quad (\text{A.76})$$

A.3 Orthotropic

The following complete and irreducible invariants for the orthotropic symmetry (O_3) have been extracted from the following article by Zheng.³ Given that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are three orthogonal directions representing the preferred directions of the orthotropy in the material, in the following we take

$$\boldsymbol{\tau} = \mathbf{e}_1, \quad \mathbf{M} = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3. \quad (\text{A.77})$$

A.3.1 Scalar invariants

Including one symmetric tensor

Scalar invariants of one symmetric tensor $\mathbf{A}$:

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{A}) & I_2 &= \text{tr}(\mathbf{A}^2) & I_3 &= \text{tr}(\mathbf{A}^3) \\ I_4 &= \text{tr}(\mathbf{M}\mathbf{A}) & I_5 &= \text{tr}(\mathbf{M}\mathbf{A}^2) & I_6 &= \boldsymbol{\tau} \circ \mathbf{A}\boldsymbol{\tau} & I_7 &= \boldsymbol{\tau} \circ \mathbf{A}^2\boldsymbol{\tau} \end{aligned} \quad (\text{A.78})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$ and a skew symmetric tensor $\mathbf{W}$: Add

$$\begin{aligned} I_8 &= \text{tr}(\mathbf{W}^2) & I_9 &= \text{tr}(\mathbf{M}\mathbf{W}^2) & I_{10} &= \boldsymbol{\tau} \circ \mathbf{W}^2\boldsymbol{\tau} & I_{11} &= \boldsymbol{\tau} \circ \mathbf{W}\mathbf{M}\mathbf{W}^2\boldsymbol{\tau} \\ I_{12} &= \text{tr}(\mathbf{A}\mathbf{W}^2) & I_{13} &= \text{tr}(\mathbf{M}\mathbf{A}\mathbf{W}) & I_{14} &= \text{tr}(\mathbf{M}\mathbf{A}^2\mathbf{W}) & I_{15} &= \boldsymbol{\tau} \circ \mathbf{A}\mathbf{W}\boldsymbol{\tau} \end{aligned} \quad (\text{A.79})$$

Scalar invariants of a symmetric tensor $\mathbf{A}$, a skew symmetric tensor $\mathbf{W}$, and a vector $\mathbf{v}$: Add

$$\begin{aligned} I_{16} &= \mathbf{v} \circ \mathbf{v} & I_{17} &= \mathbf{v} \circ \mathbf{M}\mathbf{v} & I_{18} &= (\mathbf{v} \circ \boldsymbol{\tau})^2 \\ I_{19} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{20} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{v} & I_{21} &= \mathbf{v} \circ \mathbf{M}\mathbf{A}\mathbf{v} \\ I_{22} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{v} & I_{23} &= \mathbf{v} \circ \mathbf{M}\mathbf{W}\mathbf{v} & I_{24} &= (\mathbf{v} \circ \boldsymbol{\tau})\mathbf{v} \circ \mathbf{W}\boldsymbol{\tau} \\ I_{25} &= \mathbf{v} \circ \mathbf{A}\mathbf{W}\mathbf{v} \end{aligned} \quad (\text{A.80})$$

³Zheng, Q.-S., "On transversely isotropic, orthotropic and relative isotropic functions of symmetric tensors, skew symmetric tensors and vectors. Part IV: Two dimensional orthotropic and relative isotropic functions and three dimensional relative isotropic functions," *International Journal of Engineering Science*, vol. 31, no. 10, pages 1399–1409, 1993.

Including two symmetric tensors

Scalar invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned}
 I_1 &= tr(\mathbf{A}) & I_2 &= tr(\mathbf{A}^2) & I_3 &= tr(\mathbf{A}^3) \\
 I_4 &= tr(\mathbf{MA}) & I_5 &= tr(\mathbf{MA}^2) & I_6 &= \boldsymbol{\tau} \circ \mathbf{A}\boldsymbol{\tau} & I_7 &= \boldsymbol{\tau} \circ \mathbf{A}^2\boldsymbol{\tau} \\
 I_8 &= tr(\mathbf{B}) & I_9 &= tr(\mathbf{B}^2) & I_{10} &= tr(\mathbf{B}^3) \\
 I_{11} &= tr(\mathbf{MB}) & I_{12} &= tr(\mathbf{MB}^2) & I_{13} &= \boldsymbol{\tau} \circ \mathbf{B}\boldsymbol{\tau} & I_{14} &= \boldsymbol{\tau} \circ \mathbf{B}^2\boldsymbol{\tau} \\
 I_{15} &= tr(\mathbf{AB}) & I_{16} &= tr(\mathbf{AB}^2) & I_{17} &= tr(\mathbf{A}^2\mathbf{B}) & I_{18} &= tr(\mathbf{MAB})
 \end{aligned} \tag{A.81}$$

Scalar invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**:
Add

$$\begin{aligned}
 I_{19} &= tr(\mathbf{W}^2) & I_{20} &= tr(\mathbf{MW}^2) & I_{21} &= \boldsymbol{\tau} \circ \mathbf{W}^2\boldsymbol{\tau} & I_{22} &= \boldsymbol{\tau} \circ \mathbf{WMW}^2\boldsymbol{\tau} \\
 I_{23} &= tr(\mathbf{AW}^2) & I_{24} &= tr(\mathbf{MAW}) & I_{25} &= tr(\mathbf{MA}^2\mathbf{W}) & I_{26} &= \boldsymbol{\tau} \circ \mathbf{AW}\boldsymbol{\tau} \\
 I_{27} &= tr(\mathbf{BW}^2) & I_{28} &= tr(\mathbf{MBW}) & I_{29} &= tr(\mathbf{MB}^2\mathbf{W}) & I_{30} &= \boldsymbol{\tau} \circ \mathbf{BW}\boldsymbol{\tau} \\
 I_{31} &= tr(\mathbf{ABW})
 \end{aligned} \tag{A.82}$$

Scalar invariants of two symmetric tensors **A** and **B**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned}
 I_{32} &= \mathbf{v} \circ \mathbf{v} & I_{33} &= \mathbf{v} \circ \mathbf{M}\mathbf{v} & I_{34} &= (\mathbf{v} \circ \boldsymbol{\tau})^2 \\
 I_{35} &= \mathbf{v} \circ \mathbf{A}\mathbf{v} & I_{36} &= \mathbf{v} \circ \mathbf{A}^2\mathbf{v} & I_{37} &= \mathbf{v} \circ \mathbf{MA}\mathbf{v} \\
 I_{38} &= \mathbf{v} \circ \mathbf{B}\mathbf{v} & I_{39} &= \mathbf{v} \circ \mathbf{B}^2\mathbf{v} & I_{40} &= \mathbf{v} \circ \mathbf{MB}\mathbf{v} \\
 I_{41} &= \mathbf{v} \circ \mathbf{AB}\mathbf{v} \\
 I_{42} &= \mathbf{v} \circ \mathbf{W}^2\mathbf{v} & I_{43} &= \mathbf{v} \circ \mathbf{MW}\mathbf{v} & I_{44} &= (\mathbf{v} \circ \boldsymbol{\tau})\mathbf{v} \circ \mathbf{W}\boldsymbol{\tau} \\
 I_{45} &= \mathbf{v} \circ \mathbf{AW}\mathbf{v} & I_{46} &= \mathbf{v} \circ \mathbf{BW}\mathbf{v}
 \end{aligned} \tag{A.83}$$

Including three symmetric tensors

Scalar invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 I_1 &= tr(\mathbf{A}) & I_2 &= tr(\mathbf{A}^2) & I_3 &= tr(\mathbf{A}^3) \\
 I_4 &= tr(\mathbf{MA}) & I_5 &= tr(\mathbf{MA}^2) & I_6 &= \boldsymbol{\tau} \circ \mathbf{A}\boldsymbol{\tau} & I_7 &= \boldsymbol{\tau} \circ \mathbf{A}^2\boldsymbol{\tau} \\
 I_8 &= tr(\mathbf{B}) & I_9 &= tr(\mathbf{B}^2) & I_{10} &= tr(\mathbf{B}^3) \\
 I_{11} &= tr(\mathbf{MB}) & I_{12} &= tr(\mathbf{MB}^2) & I_{13} &= \boldsymbol{\tau} \circ \mathbf{B}\boldsymbol{\tau} & I_{14} &= \boldsymbol{\tau} \circ \mathbf{B}^2\boldsymbol{\tau} \\
 I_{15} &= tr(\mathbf{C}) & I_{16} &= tr(\mathbf{C}^2) & I_{17} &= tr(\mathbf{C}^3) \\
 I_{18} &= tr(\mathbf{MC}) & I_{19} &= tr(\mathbf{MC}^2) & I_{20} &= \boldsymbol{\tau} \circ \mathbf{C}\boldsymbol{\tau} & I_{21} &= \boldsymbol{\tau} \circ \mathbf{C}^2\boldsymbol{\tau} \\
 I_{22} &= tr(\mathbf{AB}) & I_{23} &= tr(\mathbf{AB}^2) & I_{24} &= tr(\mathbf{A}^2\mathbf{B}) \\
 I_{25} &= tr(\mathbf{AC}) & I_{26} &= tr(\mathbf{AC}^2) & I_{27} &= tr(\mathbf{A}^2\mathbf{C}) \\
 I_{28} &= tr(\mathbf{BC}) & I_{29} &= tr(\mathbf{BC}^2) & I_{30} &= tr(\mathbf{B}^2\mathbf{C}) \\
 I_{31} &= tr(\mathbf{MAB}) & I_{32} &= tr(\mathbf{MAC}) & I_{33} &= tr(\mathbf{MBC}) \\
 I_{34} &= tr(\mathbf{ABC})
 \end{aligned} \tag{A.84}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 I_{35} &= tr(\mathbf{W}^2) & I_{36} &= tr(\mathbf{MW}^2) & I_{37} &= \boldsymbol{\tau} \circ \mathbf{W}^2\boldsymbol{\tau} & I_{38} &= \boldsymbol{\tau} \circ \mathbf{WMW}^2\boldsymbol{\tau} \\
 I_{39} &= tr(\mathbf{AW}^2) & I_{40} &= tr(\mathbf{MAW}) & I_{41} &= tr(\mathbf{MA}^2\mathbf{W}) & I_{42} &= \boldsymbol{\tau} \circ \mathbf{AW}\boldsymbol{\tau} \\
 I_{43} &= tr(\mathbf{BW}^2) & I_{44} &= tr(\mathbf{MBW}) & I_{45} &= tr(\mathbf{MB}^2\mathbf{W}) & I_{46} &= \boldsymbol{\tau} \circ \mathbf{BW}\boldsymbol{\tau} \\
 I_{47} &= tr(\mathbf{CW}^2) & I_{48} &= tr(\mathbf{MCW}) & I_{49} &= tr(\mathbf{MC}^2\mathbf{W}) & I_{50} &= \boldsymbol{\tau} \circ \mathbf{CW}\boldsymbol{\tau} \\
 I_{51} &= tr(\mathbf{ABW}) & I_{52} &= tr(\mathbf{ACW}) & I_{53} &= tr(\mathbf{BCW})
 \end{aligned} \tag{A.85}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 I_{54} &= tr(\mathbf{X}^2) & I_{55} &= tr(\mathbf{MX}^2) & I_{56} &= \boldsymbol{\tau} \circ \mathbf{X}^2 \boldsymbol{\tau} & I_{57} &= \boldsymbol{\tau} \circ \mathbf{MX}^2 \boldsymbol{\tau} \\
 I_{58} &= tr(\mathbf{AX}^2) & I_{59} &= tr(\mathbf{MAX}) & I_{60} &= tr(\mathbf{MA}^2 \mathbf{X}) & I_{61} &= \boldsymbol{\tau} \circ \mathbf{AX} \boldsymbol{\tau} \\
 I_{62} &= tr(\mathbf{BX}^2) & I_{63} &= tr(\mathbf{MBX}) & I_{64} &= tr(\mathbf{MB}^2 \mathbf{X}) & I_{65} &= \boldsymbol{\tau} \circ \mathbf{BX} \boldsymbol{\tau} \\
 I_{66} &= tr(\mathbf{CX}^2) & I_{67} &= tr(\mathbf{MCX}) & I_{68} &= tr(\mathbf{MC}^2 \mathbf{X}) & I_{69} &= \boldsymbol{\tau} \circ \mathbf{CX} \boldsymbol{\tau} \\
 I_{70} &= tr(\mathbf{ABX}) & I_{71} &= tr(\mathbf{ACX}) & I_{72} &= tr(\mathbf{BCX}) & & \\
 I_{73} &= tr(\mathbf{WX}) & I_{74} &= tr(\mathbf{MWX}) & I_{75} &= tr(\mathbf{MWX}^2) & I_{76} &= tr(\mathbf{MW}^2 \mathbf{X}) \\
 I_{77} &= tr(\mathbf{AWX}) & I_{78} &= tr(\mathbf{BWX}) & I_{79} &= tr(\mathbf{CWX}) & &
 \end{aligned} \tag{A.86}$$

Scalar invariants of three symmetric tensors **A**, **B**, and **C**, two skew symmetric tensors **W** and **X**, and one vector **v**: Add

$$\begin{aligned}
 I_{80} &= \mathbf{v} \circ \mathbf{v} & I_{81} &= \mathbf{v} \circ \mathbf{Mv} & I_{82} &= (\mathbf{v} \circ \boldsymbol{\tau})^2 \\
 I_{83} &= \mathbf{v} \circ \mathbf{Av} & I_{84} &= \mathbf{v} \circ \mathbf{A}^2 \mathbf{v} & I_{85} &= \mathbf{v} \circ \mathbf{MAv} \\
 I_{86} &= \mathbf{v} \circ \mathbf{Bv} & I_{87} &= \mathbf{v} \circ \mathbf{B}^2 \mathbf{v} & I_{88} &= \mathbf{v} \circ \mathbf{MBv} \\
 I_{89} &= \mathbf{v} \circ \mathbf{Cv} & I_{90} &= \mathbf{v} \circ \mathbf{C}^2 \mathbf{v} & I_{91} &= \mathbf{v} \circ \mathbf{MCv} \\
 I_{92} &= \mathbf{v} \circ \mathbf{ABv} & I_{93} &= \mathbf{v} \circ \mathbf{ACv} & I_{94} &= \mathbf{v} \circ \mathbf{BCv} \\
 I_{95} &= \mathbf{v} \circ \mathbf{W}^2 \mathbf{v} & I_{96} &= \mathbf{v} \circ \mathbf{MWv} & I_{97} &= (\mathbf{v} \circ \boldsymbol{\tau}) \mathbf{v} \circ \mathbf{W} \boldsymbol{\tau} \\
 I_{98} &= \mathbf{v} \circ \mathbf{AWv} & I_{99} &= \mathbf{v} \circ \mathbf{BWv} & I_{100} &= \mathbf{v} \circ \mathbf{CWv} \\
 I_{101} &= \mathbf{v} \circ \mathbf{X}^2 \mathbf{v} & I_{102} &= \mathbf{v} \circ \mathbf{MXv} & I_{103} &= (\mathbf{v} \circ \boldsymbol{\tau}) \mathbf{v} \circ \mathbf{X} \boldsymbol{\tau} \\
 I_{104} &= \mathbf{v} \circ \mathbf{AXv} & I_{105} &= \mathbf{v} \circ \mathbf{BXv} & I_{106} &= \mathbf{v} \circ \mathbf{CXv} \\
 I_{107} &= \mathbf{v} \circ \mathbf{WXv} & & & &
 \end{aligned} \tag{A.87}$$

A.3.2 Vector invariants

Including one symmetric tensor

Vector invariants of one symmetric tensor **A** and one vector **v**:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \mathbf{Mv} & \mathbf{f}_3 &= (\mathbf{v} \circ \boldsymbol{\tau}) \boldsymbol{\tau} \\
 \mathbf{f}_4 &= \mathbf{Av} & \mathbf{f}_5 &= \mathbf{A}^2 \mathbf{v} & \mathbf{f}_6 &= (\mathbf{MA} - \mathbf{AM}) \mathbf{v}
 \end{aligned} \tag{A.88}$$

Vector invariants of one symmetric tensor **A**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned}
 \mathbf{f}_7 &= \mathbf{Wv} & \mathbf{f}_8 &= \mathbf{W}^2 \mathbf{v} & \mathbf{f}_9 &= (\mathbf{MW} + \mathbf{WM}) \mathbf{v} \\
 \mathbf{f}_{10} &= (\mathbf{AW} + \mathbf{WA}) \mathbf{v} & & & &
 \end{aligned} \tag{A.89}$$

Including two symmetric tensors

Vector invariants of two symmetric tensors **A** and **B**, and one vector **v**:

$$\begin{aligned}
 \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \mathbf{Mv} & \mathbf{f}_3 &= (\mathbf{v} \circ \boldsymbol{\tau}) \boldsymbol{\tau} \\
 \mathbf{f}_4 &= \mathbf{Av} & \mathbf{f}_5 &= \mathbf{A}^2 \mathbf{v} & \mathbf{f}_6 &= (\mathbf{MA} - \mathbf{AM}) \mathbf{v} \\
 \mathbf{f}_7 &= \mathbf{Bv} & \mathbf{f}_8 &= \mathbf{B}^2 \mathbf{v} & \mathbf{f}_9 &= (\mathbf{MB} - \mathbf{BM}) \mathbf{v}
 \end{aligned} \tag{A.90}$$

Vector invariants of two symmetric tensors **A** and **B**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned}
 \mathbf{f}_{10} &= \mathbf{Wv} & \mathbf{f}_{11} &= \mathbf{W}^2 \mathbf{v} & \mathbf{f}_{12} &= (\mathbf{MW} + \mathbf{WM}) \mathbf{v} \\
 \mathbf{f}_{13} &= (\mathbf{AW} + \mathbf{WA}) \mathbf{v} & \mathbf{f}_{14} &= (\mathbf{BW} + \mathbf{WB}) \mathbf{v} & &
 \end{aligned} \tag{A.91}$$

Vector invariants of two symmetric tensors **A** and **B**, two skew symmetric tensors **W** and **X**, and one vector **v**: Add

$$\begin{aligned} \mathbf{f}_{15} &= \mathbf{X}\mathbf{v} & \mathbf{f}_{16} &= \mathbf{X}^2\mathbf{v} & \mathbf{f}_{17} &= (\mathbf{MX} + \mathbf{XM})\mathbf{v} \\ \mathbf{f}_{18} &= (\mathbf{AX} + \mathbf{XA})\mathbf{v} & \mathbf{f}_{19} &= (\mathbf{BX} + \mathbf{XB})\mathbf{v} \\ \mathbf{f}_{20} &= (\mathbf{WX} - \mathbf{XW})\mathbf{v} \end{aligned} \quad (\text{A.92})$$

Including three symmetric tensors

Vector invariants of three symmetric tensors **A**, **B**, and **C**, and one vector **v**:

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{v} & \mathbf{f}_2 &= \mathbf{M}\mathbf{v} & \mathbf{f}_3 &= (\mathbf{v} \circ \boldsymbol{\tau})\boldsymbol{\tau} \\ \mathbf{f}_4 &= \mathbf{A}\mathbf{v} & \mathbf{f}_5 &= \mathbf{B}\mathbf{v} & \mathbf{f}_6 &= \mathbf{C}\mathbf{v} \\ \mathbf{f}_7 &= \mathbf{A}^2\mathbf{v} & \mathbf{f}_8 &= \mathbf{B}^2\mathbf{v} & \mathbf{f}_9 &= \mathbf{C}^2\mathbf{v} \\ \mathbf{f}_{10} &= (\mathbf{MA} - \mathbf{AM})\mathbf{v} & \mathbf{f}_{11} &= (\mathbf{MB} - \mathbf{BM})\mathbf{v} & \mathbf{f}_{12} &= (\mathbf{MC} - \mathbf{CM})\mathbf{v} \end{aligned} \quad (\text{A.93})$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, one skew symmetric tensor **W**, and one vector **v**: Add

$$\begin{aligned} \mathbf{f}_{13} &= \mathbf{W}\mathbf{v} & \mathbf{f}_{14} &= \mathbf{W}^2\mathbf{v} & \mathbf{f}_{15} &= (\mathbf{MW} + \mathbf{WM})\mathbf{v} \\ \mathbf{f}_{16} &= (\mathbf{AW} + \mathbf{WA})\mathbf{v} & \mathbf{f}_{17} &= (\mathbf{BW} + \mathbf{WB})\mathbf{v} & \mathbf{f}_{18} &= (\mathbf{CW} + \mathbf{WC})\mathbf{v} \end{aligned} \quad (\text{A.94})$$

Vector invariants of three symmetric tensors **A**, **B**, and **C**, two skew symmetric tensors **W** and **X**, and one vector **v**: Add

$$\begin{aligned} \mathbf{f}_{19} &= \mathbf{X}\mathbf{v} & \mathbf{f}_{20} &= \mathbf{X}^2\mathbf{v} & \mathbf{f}_{21} &= (\mathbf{MX} + \mathbf{XM})\mathbf{v} \\ \mathbf{f}_{22} &= (\mathbf{AX} + \mathbf{XA})\mathbf{v} & \mathbf{f}_{23} &= (\mathbf{BX} + \mathbf{XB})\mathbf{v} & \mathbf{f}_{24} &= (\mathbf{CX} + \mathbf{XC})\mathbf{v} \\ \mathbf{f}_{25} &= (\mathbf{WX} - \mathbf{XW})\mathbf{v} \end{aligned} \quad (\text{A.95})$$

A.3.3 Symmetric tensor invariants

Including one symmetric tensor

Symmetric tensor invariants of one symmetric tensor **A**:

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I} & \mathbf{H}_2 &= \mathbf{M} & \mathbf{H}_3 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_4 &= \mathbf{A} & \mathbf{H}_5 &= \mathbf{A}^2 \\ \mathbf{H}_6 &= \mathbf{MA} + \mathbf{AM} & \mathbf{H}_7 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \end{aligned} \quad (\text{A.96})$$

Symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{H}_8 &= \mathbf{W}^2 & \mathbf{H}_9 &= \mathbf{MW} - \mathbf{WM} & \mathbf{H}_{10} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} + \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{11} &= \mathbf{AW} - \mathbf{WA} \end{aligned} \quad (\text{A.97})$$

Including two symmetric tensors

Symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I} & \mathbf{H}_2 &= \mathbf{M} & \mathbf{H}_3 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_4 &= \mathbf{A} & \mathbf{H}_5 &= \mathbf{B} \\ \mathbf{H}_6 &= \mathbf{A}^2 & \mathbf{H}_7 &= \mathbf{B}^2 \\ \mathbf{H}_8 &= \mathbf{MA} + \mathbf{AM} & \mathbf{H}_9 &= \mathbf{MB} + \mathbf{BM} \\ \mathbf{H}_{10} &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_{11} &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{12} &= \mathbf{AB} + \mathbf{BA} \end{aligned} \quad (\text{A.98})$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{H}_{13} &= \mathbf{W}^2 & \mathbf{H}_{14} &= \mathbf{M}\mathbf{W} - \mathbf{W}\mathbf{M} & \mathbf{H}_{15} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} + \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{16} &= \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} & \mathbf{H}_{17} &= \mathbf{B}\mathbf{W} - \mathbf{W}\mathbf{B} \end{aligned} \quad (\text{A.99})$$

Symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned} \mathbf{H}_{18} &= \mathbf{X}^2 & \mathbf{H}_{19} &= \mathbf{M}\mathbf{X} - \mathbf{X}\mathbf{M} & \mathbf{H}_{20} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} + \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{21} &= \mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{A} & \mathbf{H}_{22} &= \mathbf{B}\mathbf{X} - \mathbf{X}\mathbf{B} \\ \mathbf{H}_{23} &= \mathbf{X}\mathbf{W} + \mathbf{W}\mathbf{X} \end{aligned} \quad (\text{A.100})$$

Including three symmetric tensors

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{I} & \mathbf{H}_2 &= \mathbf{M} & \mathbf{H}_3 &= \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_4 &= \mathbf{A} & \mathbf{H}_5 &= \mathbf{B} & \mathbf{H}_6 &= \mathbf{C} \\ \mathbf{H}_7 &= \mathbf{A}^2 & \mathbf{H}_8 &= \mathbf{B}^2 & \mathbf{H}_9 &= \mathbf{C}^2 \\ \mathbf{H}_{10} &= \mathbf{M}\mathbf{A} + \mathbf{A}\mathbf{M} & \mathbf{H}_{11} &= \mathbf{M}\mathbf{B} + \mathbf{B}\mathbf{M} & \mathbf{H}_{12} &= \mathbf{M}\mathbf{C} + \mathbf{C}\mathbf{M} \\ \mathbf{H}_{13} &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_{14} &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} & \mathbf{H}_{15} &= \boldsymbol{\tau} \otimes \mathbf{C}\boldsymbol{\tau} + \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{16} &= \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} & \mathbf{H}_{17} &= \mathbf{A}\mathbf{C} + \mathbf{C}\mathbf{A} & \mathbf{H}_{18} &= \mathbf{B}\mathbf{C} + \mathbf{C}\mathbf{B} \end{aligned} \quad (\text{A.101})$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{H}_{19} &= \mathbf{W}^2 & \mathbf{H}_{20} &= \mathbf{M}\mathbf{W} - \mathbf{W}\mathbf{M} & \mathbf{H}_{21} &= \boldsymbol{\tau} \otimes \mathbf{W}\boldsymbol{\tau} + \mathbf{W}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{22} &= \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} & \mathbf{H}_{23} &= \mathbf{B}\mathbf{W} - \mathbf{W}\mathbf{B} & \mathbf{H}_{24} &= \mathbf{C}\mathbf{W} - \mathbf{W}\mathbf{C} \end{aligned} \quad (\text{A.102})$$

Symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned} \mathbf{H}_{25} &= \mathbf{X}^2 & \mathbf{H}_{26} &= \mathbf{M}\mathbf{X} - \mathbf{X}\mathbf{M} & \mathbf{H}_{27} &= \boldsymbol{\tau} \otimes \mathbf{X}\boldsymbol{\tau} + \mathbf{X}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{H}_{28} &= \mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{A} & \mathbf{H}_{29} &= \mathbf{B}\mathbf{X} - \mathbf{X}\mathbf{B} & \mathbf{H}_{30} &= \mathbf{C}\mathbf{X} - \mathbf{X}\mathbf{C} \\ \mathbf{H}_{31} &= \mathbf{X}\mathbf{W} + \mathbf{W}\mathbf{X} \end{aligned} \quad (\text{A.103})$$

A.3.4 Skew symmetric tensor invariants

Including one symmetric tensor

Skew symmetric tensor invariants of one symmetric tensor **A**:

$$\begin{aligned} \mathbf{Z}_1 &= \mathbf{M}\mathbf{A} - \mathbf{A}\mathbf{M} & \mathbf{Z}_2 &= \mathbf{M}\mathbf{A}^2 - \mathbf{A}^2\mathbf{M} & \mathbf{Z}_3 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\ \mathbf{Z}_4 &= \mathbf{M}\mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M}\mathbf{A} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A}\mathbf{M} - \mathbf{A}\mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} \end{aligned} \quad (\text{A.104})$$

Skew symmetric tensor invariants of one symmetric tensor **A** and one skew symmetric tensor **W**: Add

$$\begin{aligned} \mathbf{Z}_5 &= \mathbf{W} & \mathbf{Z}_6 &= \mathbf{M}\mathbf{W} + \mathbf{W}\mathbf{M} & \mathbf{Z}_7 &= \mathbf{M}\mathbf{W}^2 - \mathbf{W}^2\mathbf{M} \\ \mathbf{Z}_8 &= \mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} \end{aligned} \quad (\text{A.105})$$

Including two symmetric tensors

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**:

$$\begin{aligned}
 \mathbf{Z}_1 &= \mathbf{MA} - \mathbf{AM} & \mathbf{Z}_2 &= \mathbf{MA}^2 - \mathbf{A}^2\mathbf{M} & \mathbf{Z}_3 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{Z}_4 &= \mathbf{MB} - \mathbf{BM} & \mathbf{Z}_5 &= \mathbf{MB}^2 - \mathbf{B}^2\mathbf{M} & \mathbf{Z}_6 &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} - \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{Z}_7 &= \mathbf{AB} - \mathbf{BA} \\
 \mathbf{Z}_8 &= \mathbf{MA}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{MA} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{AM} - \mathbf{AM}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} \\
 \mathbf{Z}_9 &= \mathbf{MB}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{MB} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{BM} - \mathbf{BM}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B}
 \end{aligned} \tag{A.106}$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 \mathbf{Z}_{10} &= \mathbf{W} & \mathbf{Z}_{11} &= \mathbf{MW} + \mathbf{WM} & \mathbf{Z}_{12} &= \mathbf{MW}^2 - \mathbf{W}^2\mathbf{M} \\
 \mathbf{Z}_{13} &= \mathbf{AW} + \mathbf{WA} & \mathbf{Z}_{14} &= \mathbf{BW} + \mathbf{WB}
 \end{aligned} \tag{A.107}$$

Skew symmetric tensor invariants of two symmetric tensors **A** and **B**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 \mathbf{Z}_{15} &= \mathbf{X} & \mathbf{Z}_{16} &= \mathbf{MX} + \mathbf{XM} & \mathbf{Z}_{17} &= \mathbf{MX}^2 - \mathbf{X}^2\mathbf{M} \\
 \mathbf{Z}_{18} &= \mathbf{AX} + \mathbf{XA} & \mathbf{Z}_{19} &= \mathbf{BX} + \mathbf{XB} \\
 \mathbf{Z}_{20} &= \mathbf{WX} - \mathbf{XW}
 \end{aligned} \tag{A.108}$$

Including three symmetric tensors

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**:

$$\begin{aligned}
 \mathbf{Z}_1 &= \mathbf{MA} - \mathbf{AM} & \mathbf{Z}_2 &= \mathbf{MA}^2 - \mathbf{A}^2\mathbf{M} & \mathbf{Z}_3 &= \boldsymbol{\tau} \otimes \mathbf{A}\boldsymbol{\tau} - \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{Z}_4 &= \mathbf{MB} - \mathbf{BM} & \mathbf{Z}_5 &= \mathbf{MB}^2 - \mathbf{B}^2\mathbf{M} & \mathbf{Z}_6 &= \boldsymbol{\tau} \otimes \mathbf{B}\boldsymbol{\tau} - \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{Z}_7 &= \mathbf{MC} - \mathbf{CM} & \mathbf{Z}_8 &= \mathbf{MC}^2 - \mathbf{C}^2\mathbf{M} & \mathbf{Z}_9 &= \boldsymbol{\tau} \otimes \mathbf{C}\boldsymbol{\tau} - \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
 \mathbf{Z}_{10} &= \mathbf{AB} - \mathbf{BA} & \mathbf{Z}_{11} &= \mathbf{AC} - \mathbf{CA} & \mathbf{Z}_{12} &= \mathbf{BC} - \mathbf{CB} \\
 \mathbf{Z}_{13} &= \mathbf{MA}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{A}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{MA} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{AM} - \mathbf{AM}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{A} \\
 \mathbf{Z}_{14} &= \mathbf{MB}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{B}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{MB} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{BM} - \mathbf{BM}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{B} \\
 \mathbf{Z}_{15} &= \mathbf{MC}\boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{C}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{M} + \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{MC} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{CM} - \mathbf{CM}\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \mathbf{M}\boldsymbol{\tau} \otimes \boldsymbol{\tau}\mathbf{C}
 \end{aligned} \tag{A.109}$$

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and one skew symmetric tensor **W**: Add

$$\begin{aligned}
 \mathbf{Z}_{16} &= \mathbf{W} & \mathbf{Z}_{17} &= \mathbf{MW} + \mathbf{WM} & \mathbf{Z}_{18} &= \mathbf{MW}^2 - \mathbf{W}^2\mathbf{M} \\
 \mathbf{Z}_{19} &= \mathbf{AW} + \mathbf{WA} & \mathbf{Z}_{20} &= \mathbf{BW} + \mathbf{WB} & \mathbf{Z}_{21} &= \mathbf{CW} + \mathbf{WC}
 \end{aligned} \tag{A.110}$$

Skew symmetric tensor invariants of three symmetric tensors **A**, **B**, and **C**, and two skew symmetric tensors **W** and **X**: Add

$$\begin{aligned}
 \mathbf{Z}_{22} &= \mathbf{X} & \mathbf{Z}_{23} &= \mathbf{MX} + \mathbf{XM} & \mathbf{Z}_{24} &= \mathbf{MX}^2 - \mathbf{X}^2\mathbf{M} \\
 \mathbf{Z}_{25} &= \mathbf{AX} + \mathbf{XA} & \mathbf{Z}_{26} &= \mathbf{BX} + \mathbf{XB} & \mathbf{Z}_{27} &= \mathbf{CX} + \mathbf{XC} \\
 \mathbf{Z}_{28} &= \mathbf{WX} - \mathbf{XW}
 \end{aligned} \tag{A.111}$$

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Appendix B

Representation for fourth-order constants

Here we look at representation of fourth-order tensor moduli $\mathbf{E}$ subject to symmetry conditions. Given a symmetry group $\mathcal{G}$ we are looking for the general form of $\mathbf{E}$ for which

$$\mathbf{E} : \mathbf{A} = \mathbf{M}[\mathbf{E} : (\mathbf{M}^T \mathbf{A} \mathbf{M})] \mathbf{M}^T \quad (\text{B.1})$$

for each transformation $\mathbf{M}$ in $\mathcal{G}$, and for any second-order tensor $\mathbf{A}$. We only consider orthogonal $\mathbf{M}$. Given $\mathbf{E} = E_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, $\mathbf{M} = M_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ in a base $\mathbf{e}_i$ of orthogonal unit vectors, we can write this as

$$E_{ijkl} A_{kl} = M_{im} [E_{mnop} (M_{qo} A_{qr} M_{rp})] M_{jn} = M_{im} M_{jn} M_{ko} M_{lp} E_{mnop} A_{kl}, \quad (\text{B.2})$$

which requires that

$$E_{ijkl} = M_{im} M_{jn} M_{ko} M_{lp} E_{mnop}. \quad (\text{B.3})$$

We assume that $\mathbf{E}$ is such that for any second-order tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{A}^T : \mathbf{E} : \mathbf{B} = \mathbf{A} : \mathbf{E} : \mathbf{B}^T = \mathbf{B} : \mathbf{E} : \mathbf{A}. \quad (\text{B.4})$$

This indicates that given $\mathbf{E} = E_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ written in any base $\mathbf{e}_i$, the components of $\mathbf{E}$ are assumed to have the properties

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}, \quad (\text{B.5})$$

therefore reducing the number of independent components of $\mathbf{E}$ to 21. The requirement that $\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{A}^T : \mathbf{E} : \mathbf{B}$ indicates that the first two bases of $\mathbf{E}$ be those associated with symmetric second-order tensors. There are six base tensors for a symmetric second-order tensor that we can write as

$$\mathbf{E}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1, \quad (\text{B.6})$$

$$\mathbf{E}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2, \quad (\text{B.7})$$

$$\mathbf{E}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (\text{B.8})$$

$$\mathbf{E}_4 = \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2, \quad (\text{B.9})$$

$$\mathbf{E}_5 = \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, \quad (\text{B.10})$$

$$\mathbf{E}_6 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1. \quad (\text{B.11})$$

The requirement that $\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{A} : \mathbf{E} : \mathbf{B}^T$ states that base vectors three and four also should be given by the base vectors $\mathbf{E}_i$ for a symmetric second-order tensor. These two conditions combined, therefore require that $\mathbf{E}$ be expressible by bases of the form $\mathbf{E}_i \otimes \mathbf{E}_j$, which are a total of 36 bases. The final requirement that $\mathbf{A} : \mathbf{E} : \mathbf{B} = \mathbf{B} : \mathbf{E} : \mathbf{A}$ states that $\mathbf{E}$ is symmetric in the interchange of the first two bases with bases three and four. This allows us to write the 21 base tensors for fourth-order tensor $\mathbf{E}$ as $\mathbf{A}_{ij}$ for $j \geq i$ and define them as

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{E}_i \otimes \mathbf{E}_j & \text{for } j = i, \\ \mathbf{E}_i \otimes \mathbf{E}_j + \mathbf{E}_j \otimes \mathbf{E}_i & \text{for } j > i. \end{cases} \quad (\text{B.12})$$

Therefore, the general expression for $\mathbf{E}$ without any material symmetry conditions added can be written as

$$\begin{aligned}\mathbf{E} = & \mu_{11}\mathbf{A}_{11} + \mu_{12}\mathbf{A}_{12} + \mu_{13}\mathbf{A}_{13} + \mu_{14}\mathbf{A}_{14} + \mu_{15}\mathbf{A}_{15} + \mu_{16}\mathbf{A}_{16} \\ & + \mu_{22}\mathbf{A}_{22} + \mu_{23}\mathbf{A}_{23} + \mu_{24}\mathbf{A}_{24} + \mu_{25}\mathbf{A}_{25} + \mu_{26}\mathbf{A}_{26} \\ & + \mu_{33}\mathbf{A}_{33} + \mu_{34}\mathbf{A}_{34} + \mu_{35}\mathbf{A}_{35} + \mu_{36}\mathbf{A}_{36} \\ & + \mu_{44}\mathbf{A}_{44} + \mu_{45}\mathbf{A}_{45} + \mu_{46}\mathbf{A}_{46} \\ & + \mu_{55}\mathbf{A}_{55} + \mu_{56}\mathbf{A}_{56} \\ & + \mu_{66}\mathbf{A}_{66},\end{aligned}\tag{B.13}$$

where μ_{ij} , for $j \geq i$, are 21 scalar constants. In matrix form, these constants can be represented as an upper triangular matrix, so we will write them in the following array form

$$\begin{array}{cccccc}\mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} \\ & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} \\ & & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36} \\ & & & \mu_{44} & \mu_{45} & \mu_{46} \\ & & & & \mu_{55} & \mu_{56} \\ & & & & & \mu_{66}\end{array}\tag{B.14}$$

The following has been extracted from the articles by Spencer,¹ Suhubi,² and Gurtin.³ The following tensor transformations, described in an orthogonal base $\mathbf{e}_i$ of unit vectors, are used to describe the different symmetries:

$$\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i, \tag{B.15}$$

$$\mathbf{C} = -\mathbf{I}, \tag{B.16}$$

$$\mathbf{R}_1 = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.17}$$

$$\mathbf{R}_2 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.18}$$

$$\mathbf{R}_3 = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.19}$$

$$\mathbf{D}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.20}$$

$$\mathbf{D}_2 = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.21}$$

$$\mathbf{D}_3 = -\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{B.22}$$

$$\mathbf{T}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2, \tag{B.23}$$

$$\mathbf{T}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, \tag{B.24}$$

$$\mathbf{T}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3 + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \tag{B.25}$$

$$\mathbf{M}_1 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1, \tag{B.26}$$

$$\mathbf{M}_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_2, \tag{B.27}$$

$$\mathbf{S}_1 = -\frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{\sqrt{3}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{B.28}$$

$$\mathbf{S}_2 = -\frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - \frac{\sqrt{3}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{B.29}$$

where $\mathbf{I}$ is the identity transformation, $\mathbf{C}$ is the central inversion, $\mathbf{R}_i$ is the reflection about the plane normal to $\mathbf{e}_i$, $\mathbf{D}_i$ is a 180° rotation about $\mathbf{e}_i$, $\mathbf{T}_i$ is a reflection about the plane

¹Spencer, A. J. M., *Theory of Invariants*, in *Continuum Physics: Volume I-Mathematics*, Edited by A. C. Eringen, Academic Press, New York, 1971.

²Suhubi, E. S., *Thermoelastic Solids*, in *Continuum Physics: Volume II-Continuum Mechanics of Single-Substance Bodies*, Edited by A. C. Eringen, Academic Press, New York, 1975.

³Gurtin, M. E., The linear theory of elasticity, in *Mechanics of Solids*, Edited by C. Truesdell, Springer-Verlag, Berlin, Heidelberg, 1972.

Table B.1: Crystal classes.

Triclinic system	Class #	Type 1
<i>Pedial</i>	1	$\mathbf{I}$
<i>Pinacoidal</i>	2	$\mathbf{I}, \mathbf{C}$
Monoclinic system	Class #	Type 2
<i>Domatic</i>	3	$\mathbf{I}, \mathbf{R}_1$
<i>Sphenoidal</i>	4	$\mathbf{I}, \mathbf{D}_1$
<i>Prismatic</i>	5	$\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{D}_1$
Rhombic system	Class #	Type 3
<i>Rhombic – pyramidal</i>	6	$\mathbf{I}, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1$
<i>Rhombic – disphenoidal</i>	7	$\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$
<i>Rhombic – dipyramidal</i>	8	$\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$
Tetragonal system	Class #	Type 4
<i>Tetragonal – disphenoidal</i>	9	$\mathbf{I}, \mathbf{D}_3, (\mathbf{D}_1, \mathbf{D}_2) \cdot \mathbf{T}_3$
<i>Tetragonal – pyramidal</i>	10	$\mathbf{I}, \mathbf{D}_3, (\mathbf{R}_1, \mathbf{R}_2) \cdot \mathbf{T}_3$
<i>Tetragonal – dipyramidal</i>	11	$\mathbf{I}, \mathbf{C}, \mathbf{R}_3, \mathbf{D}_3, (\mathbf{R}_1, \mathbf{R}_2, \mathbf{D}_1, \mathbf{D}_2) \cdot \mathbf{T}_3$
Tetragonal system	Class #	Type 5
<i>Tetragonal – scalenohedral</i>	12	$(\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{T}_3)$
<i>Ditetragonal – pyramidal</i>	13	$(\mathbf{I}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{T}_3)$
<i>Tetragonal – trapezohedral</i>	14	$\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, (\mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \cdot \mathbf{T}_3$
<i>Ditetragonal – dipyramidal</i>	15	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{T}_3)$
Cubic system	Class #	Type 6
<i>Tetartoidal</i>	16	$(\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{M}_1, \mathbf{M}_2)$
<i>Diploidal</i>	17	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{M}_1, \mathbf{M}_2)$
Cubic system	Class #	Type 7
<i>Hextetrahedral</i>	18	$(\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{M}_1, \mathbf{M}_2)$
<i>Gyroidal</i>	19	$(\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{M}_1, \mathbf{M}_2),$ $(\mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \cdot (\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$
<i>Hexoctahedral</i>	20	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot$ $(\mathbf{I}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{M}_1, \mathbf{M}_2)$

that contains $\mathbf{e}_i$ and the sum of the other two base vectors (i.e., their bisection), $\mathbf{M}_1$ and $\mathbf{M}_2$ are rotations by, respectively, 120° and 240° around the diagonal $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, and $\mathbf{S}_1$ and $\mathbf{S}_2$ are rotations by, respectively, 120° and 240° around $\mathbf{e}_3$.

There are 32 crystal classes that can be organized into 11 types, each type having a different representation for $\mathbf{E}$. These are defined by the naming system and transformations described in Tables B.1 and B.2.

B.1 Isotropic

For an isotropic material, the symmetry class is the full group of orthogonal transformations. The expression for $\mathbf{E}$ is given by

$$\mathbf{E} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu \mathbf{e}_i \otimes \mathbf{e}_j \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i). \quad (\text{B.30})$$

Table B.2: Crystal classes-Continued.

Hexagonal system	Class #	Type 8
<i>Trigonal – pyramidal</i>	21	$\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2$
<i>Rhombohedral</i>	22	$(\mathbf{I}, \mathbf{C}) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
Hexagonal system	Class #	Type 9
<i>Ditrigonal – pyramidal</i>	23	$(\mathbf{I}, \mathbf{R}_1) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Trigonal – trapezohedral</i>	24	$(\mathbf{I}, \mathbf{D}_1) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Hexagonal – scalenohedral</i>	25	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{D}_1) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
Hexagonal system	Class #	Type 10
<i>Trigonal – dipyramidal</i>	26	$(\mathbf{I}, \mathbf{R}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Hexagonal – pyramidal</i>	27	$(\mathbf{I}, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Hexagonal – dipyramidal</i>	28	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_3, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
Hexagonal system	Class #	Type 11
<i>Ditrigonal – dipyramidal</i>	29	$(\mathbf{I}, \mathbf{R}_1, \mathbf{R}_3, \mathbf{D}_2) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Dihexagonal – pyramidal</i>	30	$(\mathbf{I}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Hexagonal – trapezohedral</i>	31	$(\mathbf{I}, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$
<i>Dihexagonal – dipyramidal</i>	32	$(\mathbf{I}, \mathbf{C}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3) \cdot (\mathbf{I}, \mathbf{S}_1, \mathbf{S}_2)$

B.2 Transversely isotropic

For a transversely isotropic material, the symmetry class was given in the previous appendix. Assuming the axis of transverse isotropy is given by $\mathbf{e}_3$, the expression for $\mathbf{E}$ is given by five constants in the form

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & 0 \\
 & \mu_{11} & \mu_{13} & 0 & 0 & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & 0 \\
 & & & & & \frac{1}{2}(\mu_{11} - \mu_{12})
 \end{array} \quad (\text{B.31})$$

B.3 Crystal classes

There are 32 crystal classes that can be organized into 11 types, each type having a different representation for $\mathbf{E}$. These can be defined by the following naming system and transformations

B.3.1 Type 1: Triclinic

The expression for $\mathbf{E}$ is given by all 21 constants represented by

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} \\
 & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} \\
 & & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36} \\
 & & & \mu_{44} & \mu_{45} & \mu_{46} \\
 & & & & \mu_{55} & \mu_{56} \\
 & & & & & \mu_{66}
 \end{array} \quad (\text{B.32})$$

B.3.2 Type 2: Monoclinic

The expression for **E** is given by 13 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & \mu_{16} \\
 & \mu_{22} & \mu_{23} & 0 & 0 & \mu_{26} \\
 & & \mu_{33} & 0 & 0 & \mu_{36} \\
 & & & \mu_{44} & \mu_{45} & 0 \\
 & & & & \mu_{55} & 0 \\
 & & & & & \mu_{66}
 \end{array} \tag{B.33}$$

B.3.3 Type 3: Rhombic

The expression for **E** is given by 9 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & 0 \\
 & \mu_{22} & \mu_{23} & 0 & 0 & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{55} & 0 \\
 & & & & & \mu_{66}
 \end{array} \tag{B.34}$$

B.3.4 Type 4: Tetragonal

The expression for **E** is given by 7 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & \mu_{16} \\
 & \mu_{11} & \mu_{13} & 0 & 0 & -\mu_{16} \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & 0 \\
 & & & & & \mu_{66}
 \end{array} \tag{B.35}$$

B.3.5 Type 5: Tetragonal

The expression for **E** is given by 6 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & 0 \\
 & \mu_{11} & \mu_{13} & 0 & 0 & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & 0 \\
 & & & & & \mu_{66}
 \end{array} \tag{B.36}$$

B.3.6 Types 6 and 7: Cubic

The expression for **E** is given by 3 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{12} & 0 & 0 & 0 \\
 & \mu_{11} & \mu_{12} & 0 & 0 & 0 \\
 & & \mu_{11} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & 0 \\
 & & & & & \mu_{44}
 \end{array} \tag{B.37}$$

B.3.7 Type 8: Hexagonal

The expression for **E** is given by 7 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & 0 \\
 & \mu_{11} & \mu_{13} & -\mu_{14} & -\mu_{15} & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & -\mu_{15} \\
 & & & & \mu_{44} & \mu_{14} \\
 & & & & & \frac{1}{2}(\mu_{11} - \mu_{12})
 \end{array} \quad (\text{B.38})$$

B.3.8 Type 9: Hexagonal

The expression for **E** is given by 6 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & 0 & 0 \\
 & \mu_{11} & \mu_{13} & -\mu_{14} & 0 & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & \mu_{14} \\
 & & & & & \frac{1}{2}(\mu_{11} - \mu_{12})
 \end{array} \quad (\text{B.39})$$

B.3.9 Types 10 and 11: Hexagonal

The expression for **E** is given by 5 constants represented as

$$\begin{array}{cccccc}
 \mu_{11} & \mu_{12} & \mu_{13} & 0 & 0 & 0 \\
 & \mu_{11} & \mu_{13} & 0 & 0 & 0 \\
 & & \mu_{33} & 0 & 0 & 0 \\
 & & & \mu_{44} & 0 & 0 \\
 & & & & \mu_{44} & 0 \\
 & & & & & \frac{1}{2}(\mu_{11} - \mu_{12})
 \end{array} \quad (\text{B.40})$$

Appendix C

Basic equations

In this appendix, we provide the equations for the balance laws, their description in general curvilinear coordinates, and their description in rectangular, cylindrical, and spherical coordinate systems. In each case, we look at the descriptions in the differential forms, both in the current and reference configurations. The usually used cylindrical and spherical coordinates are not curvilinear systems, but they are related to companion curvilinear systems, which are described in each case. The notation used for rectangular, cylindrical, and spherical coordinate systems is shown in Figure C.1.

C.1 Basic equations

C.1.1 Kinematics

The position of a particle in the reference configuration is denoted by $\mathbf{X}$ and in the current configuration is denoted by $\mathbf{x}$, with the displacement denoted by $\mathbf{u}$ and given by

$$\mathbf{x} = \mathbf{X} + \mathbf{u}. \quad (\text{C.1})$$

The vector $\mathbf{x}$ describes the motion of a body if it is given by a function of position in the reference configuration $\mathbf{X}$ and of time t , so that $\mathbf{x}(\mathbf{X}, t)$. Such a description that provides a function as a function of $(\mathbf{X}, t)$ is known as a material description, also known as a Lagrangian description. In this case, the differential change in $\mathbf{x}$ can result from a change in $\mathbf{X}$ and from a change in t so that we can write the differential change in $\mathbf{x}$ as

$$d\mathbf{x} = \mathbf{Grad}(\mathbf{x})d\mathbf{X} + \frac{\partial \mathbf{x}}{\partial t}dt, \quad (\text{C.2})$$

$$d\mathbf{x} = d\mathbf{X}\nabla_{\mathbf{X}}(\mathbf{x}) + \frac{\partial \mathbf{x}}{\partial t}dt, \quad (\text{C.3})$$

where $\mathbf{Grad}(\mathbf{x})$ is the gradient of the motion relative to $\mathbf{X}$, known as the deformation gradient, $\nabla_{\mathbf{X}}(\mathbf{x}) = [\mathbf{Grad}(\mathbf{x})]^T$, and $\frac{\partial \mathbf{x}}{\partial t}$ is the velocity. We denote these by

$$\mathbf{F} = \mathbf{Grad}(\mathbf{x}) = [\nabla_{\mathbf{X}}(\mathbf{x})]^T, \quad (\text{C.4})$$

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}, \quad (\text{C.5})$$

and write

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} + \mathbf{v}dt. \quad (\text{C.6})$$

If the velocity is given in terms of a *material description*, we can write it as $\mathbf{v}(\mathbf{X}, t)$ and a differential change in velocity is given by a change in the reference position and in time

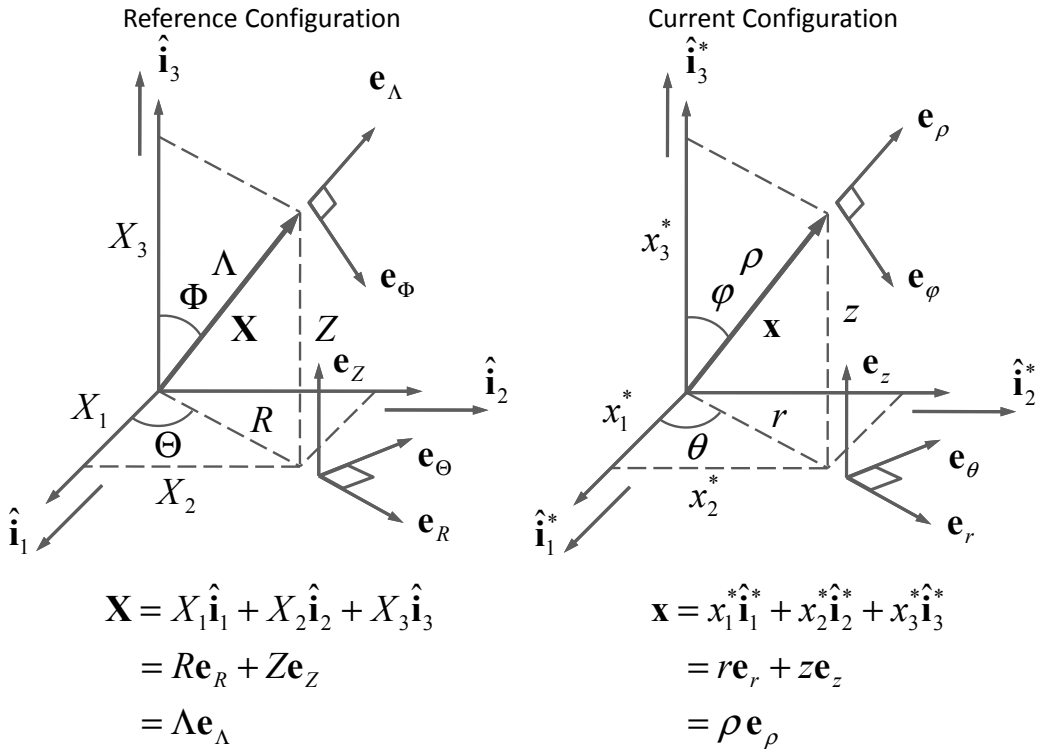


Figure C.1: Notation used for describing rectangular, cylindrical, and spherical coordinate systems in the reference and current configurations.

so that we can write

$$d\mathbf{v} = \mathbf{Grad}(\mathbf{v})d\mathbf{X} + \frac{\partial \mathbf{v}}{\partial t}dt, \quad (\text{C.7})$$

$$d\mathbf{v} = d\mathbf{X} \nabla_{\mathbf{X}}(\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}dt, \quad (\text{C.8})$$

where $\mathbf{Grad}(\mathbf{v})$ is the gradient with respect to $\mathbf{X}$ of $\mathbf{v}$ and $\frac{\partial \mathbf{v}}{\partial t}$ is the acceleration $\mathbf{a}$. We can write these as

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}[\mathbf{Grad}(\mathbf{x})] = \mathbf{Grad}(\mathbf{v}), \quad (\text{C.9})$$

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \dot{\mathbf{v}}, \quad (\text{C.10})$$

so that we have

$$d\mathbf{v} = \dot{\mathbf{F}}d\mathbf{X} + \mathbf{a}dt. \quad (\text{C.11})$$

If the velocity is given in terms of a *spatial description*, we can write it as $\mathbf{v}(\mathbf{x}, t)$ and a differential change in velocity is given by a change in the current position and in time so

that we can write

$$d\mathbf{v} = \mathbf{grad}(\mathbf{v})d\mathbf{x} + \frac{\partial \mathbf{v}}{\partial t}dt, \quad (\text{C.12})$$

$$d\mathbf{v} = d\mathbf{x}\nabla_{\mathbf{x}}(\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}dt, \quad (\text{C.13})$$

where $\mathbf{grad}(\mathbf{v})$ is the velocity gradient with respect to changes in the current position denoted by $\mathbf{L}$ and $\frac{\partial \mathbf{v}}{\partial t}$ is the change in velocity with respect to time at a given current point so that we can write

$$d\mathbf{v} = \mathbf{L}d\mathbf{x} + \frac{\partial \mathbf{v}}{\partial t}dt = d\mathbf{x}\mathbf{L}^T + \frac{\partial \mathbf{v}}{\partial t}dt, \quad (\text{C.14})$$

since $\nabla_{\mathbf{x}}(\mathbf{v}) = [\mathbf{grad}(\mathbf{v})]^T = \mathbf{L}^T$. When the velocity is given in a spatial description, we can use the material mapping to obtain the acceleration as

$$\mathbf{a} = \mathbf{L}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = \mathbf{v}\nabla_{\mathbf{x}}(\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}. \quad (\text{C.15})$$

There is a relation between the velocity gradient, the deformation gradient, and the rate of the deformation gradient given by

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \quad (\text{C.16})$$

The velocity gradient is separated into its symmetric and skew-symmetric parts, which are, respectively, called the deformation rate tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$, so that $\mathbf{L} = \mathbf{D} + \mathbf{W}$ where

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \quad (\text{C.17})$$

The primary measure of deformation is $\mathbf{F}$ and of deformation rate is $\dot{\mathbf{F}}$. We sometimes use the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{I}$ and its rate $\dot{\mathbf{H}} = \dot{\mathbf{F}}$ as the associate measures. These are defined in terms of the displacement as

$$\mathbf{H} = \mathbf{Grad}(\mathbf{u}), \quad \dot{\mathbf{H}} = \mathbf{Grad}(\dot{\mathbf{u}}) = \mathbf{Grad}(\mathbf{v}). \quad (\text{C.18})$$

A number of measures for stretch are used that include the right Cauchy stretch (“strain”) tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and the left Cauchy stretch (“strain”) tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. A number of strain tensors also are used, including the Green or Lagrangian strain $\mathbf{E}$ and the Almansi strain $\mathbf{A}$ defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \quad \mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}). \quad (\text{C.19})$$

The relation between the deformation rate $\mathbf{D}$, the Green strain rate $\dot{\mathbf{E}}$, and the rate of right Cauchy stretch $\dot{\mathbf{C}}$ is given by

$$\dot{\mathbf{C}} = 2\dot{\mathbf{E}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}. \quad (\text{C.20})$$

The volume ratio is denoted by J and is equal to the determinant of $\mathbf{F}$ so that

$$J = \det(\mathbf{F}). \quad (\text{C.21})$$

There is a relation between the rate of change of the volume and the velocity gradient given as

$$\text{tr}(\mathbf{L}) = \frac{\dot{J}}{J}. \quad (\text{C.22})$$

The temperature θ can be provided in a *material description* by a function $\theta(\mathbf{X}, t)$. In this case, the differential change in θ is given by

$$\begin{aligned} d\theta &= \mathbf{Grad}(\theta) \circ d\mathbf{X} + \frac{\partial \theta}{\partial t}dt \\ &= \mathbf{G} \circ d\mathbf{X} + \dot{\theta}dt, \end{aligned} \quad (\text{C.23})$$

where $\mathbf{G} = \mathbf{Grad}(\theta) = \nabla_{\mathbf{X}}(\theta)$ is the temperature gradient of θ with respect to motions in the reference configuration, and $\dot{\theta}$ is the material time derivative of temperature. If the temperature is given in terms of the *spatial description* as $\theta(\mathbf{x}, t)$, the change of temperature can result from changes in the current position and in time. This gives

$$\begin{aligned} d\theta &= \mathbf{grad}(\theta) \circ d\mathbf{x} + \frac{\partial \theta}{\partial t} dt \\ &= \mathbf{g} \circ d\mathbf{X} + \frac{\partial \theta}{\partial t} dt, \end{aligned} \quad (\text{C.24})$$

where $\mathbf{g} = \mathbf{grad}(\theta) = \nabla_{\mathbf{x}}(\theta)$ is the gradient of θ with respect to motions in the current configuration and $\frac{\partial \theta}{\partial t}$ is the spatial time derivative of temperature. The two gradients are related through the relation

$$\mathbf{G} = \mathbf{gF}. \quad (\text{C.25})$$

C.1.2 Balance laws in current configuration

The conservation of mass, balance of linear momentum, balance of energy, and the entropy production inequality in differential form for every point are given by

$$\rho J = \rho_o, \quad (\text{C.26})$$

$$\text{div}(\mathbf{T}^T) + \rho \mathbf{b} = \rho \mathbf{a}, \quad (\text{C.27})$$

$$-\text{div}(\mathbf{q}) + \rho r + \text{tr}(\mathbf{TL}) = \rho \dot{e}, \quad (\text{C.28})$$

$$\rho \dot{\psi} - \text{tr}(\mathbf{TL}) + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (\text{C.29})$$

where ρ is the current density, J is volume ratio, ρ_o is the density in the reference configuration, $\mathbf{x}$ is the current position, $\mathbf{T}$ is the Cauchy stress, $\mathbf{b}$ is specific body force, $\mathbf{a}$ is the acceleration, $\mathbf{q}$ is the heat flux vector, r is the specific radiation/heat generation, $\mathbf{L}$ is the velocity gradient, e is the specific internal energy, ψ is the specific free energy, η is the specific entropy, θ is the temperature, and $\mathbf{g}$ is the temperature gradient with respect to motions in the current configuration. In this equation $\text{div} = \text{div}_{\mathbf{x}}$ denotes the divergence with respect to current position. The balance of angular momentum can be satisfied by taking the Cauchy stress to be symmetric. For these expressions, we have the relations

$$\text{tr}(\mathbf{TL}) = \text{tr}(\mathbf{TD}), \quad (\text{C.30})$$

$$e = \psi + \theta \eta. \quad (\text{C.31})$$

C.1.3 Balance laws in reference configuration

The balance laws also can be written a form that must be satisfied for points in the reference configuration. The balance of mass, linear momentum, work and energy, and the entropy production inequality can be written in this case as

$$\rho_o J = \rho_o, \quad (\text{C.32})$$

$$\text{Div}(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o \mathbf{a}, \quad (\text{C.33})$$

$$-\text{Div}(\mathbf{q}_o) + \rho_o r + \text{tr}(\mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \dot{e}, \quad (\text{C.34})$$

$$\rho_o \dot{\psi} - \text{tr}(\mathbf{T}_o \dot{\mathbf{F}}) + \rho_o \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q}_o \circ \mathbf{G} \leq 0, \quad (\text{C.35})$$

where $\mathbf{T}_o$ is the nominal stress, $\mathbf{q}_o$ is the nominal heat flux, and $\mathbf{G}$ is the temperature gradient with respect to motions in the reference configuration. In this equation, $\text{Div} = \text{div}_{\mathbf{X}}$ denotes the divergence with respect to the reference position.

The following relations exist between the different variables

$$\mathbf{T}_o = J\mathbf{F}^{-1}\mathbf{T}, \quad (\text{C.36})$$

$$\mathbf{q}_o = J\mathbf{F}^{-1}\mathbf{q}. \quad (\text{C.37})$$

The symmetry of $\mathbf{T}$ requires that the nominal stress have the property

$$\mathbf{F}\mathbf{T}_o = (\mathbf{F}\mathbf{T}_o)^T. \quad (\text{C.38})$$

C.2 Curvilinear coordinates

A general curvilinear coordinate system is given by the coordinates α_i and position vector $\mathbf{X}$ described in terms of these coordinates (i.e., $\mathbf{X}(\alpha_1, \alpha_2, \alpha_3)$). The base vectors for this system are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{X}}{\partial \alpha_i}. \quad (\text{C.39})$$

The metric tensor $g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$ is an array (a matrix) of numbers describing the relation between the base vectors of the coordinate system. The components of the inverse matrix to this are denoted as g_{ij}^{-1} such that $\delta_{im} = g_{ij}g_{jm}^{-1}$.

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \circ \mathbf{b} = (a_i \mathbf{e}_i) \circ (b_j \mathbf{e}_j) = a_i b_j g_{ij}. \quad (\text{C.40})$$

The trace of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\text{tr}(\mathbf{AB}) = \text{tr}[(A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j)(B_{mn} \mathbf{e}_m \otimes \mathbf{e}_n)] = \text{tr}(A_{ij} B_{mn} g_{jm} \mathbf{e}_i \otimes \mathbf{e}_n) = A_{ij} B_{mn} g_{jm} g_{in}. \quad (\text{C.41})$$

A general vector $\mathbf{a}$ in such a system is written as $\mathbf{a} = a_i \mathbf{e}_i$, and any second-order vector $\mathbf{B}$ is written in terms of this base as $\mathbf{B} = B_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$. In such a system, the gradient of a scalar ψ , a vector $\mathbf{a}$ and a second-order tensor $\mathbf{B}$ is given by

$$\text{Grad}(\psi) = \nabla_{\mathbf{X}}(\psi) = \frac{\partial \psi}{\partial \alpha_i} \bar{\mathbf{e}}_i = \frac{\partial \psi}{\partial \alpha_i} g_{ij}^{-1} \mathbf{e}_j, \quad (\text{C.42})$$

$$\begin{aligned} \text{Grad}(\mathbf{a}) &= [\nabla_{\mathbf{X}}(\mathbf{a})]^T = \frac{\partial \mathbf{a}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i = \frac{\partial \mathbf{a}}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial a_m \mathbf{e}_m}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial a_m}{\partial \alpha_i} \mathbf{e}_m \otimes g_{ij}^{-1} \mathbf{e}_j + a_m \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j, \end{aligned} \quad (\text{C.43})$$

$$\begin{aligned} \text{Grad}(\mathbf{B}) &= [\nabla_{\mathbf{X}}(\mathbf{B}^T)]^T = \frac{\partial \mathbf{B}}{\partial \alpha_i} \otimes \bar{\mathbf{e}}_i = \frac{\partial \mathbf{B}}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial B_{mn} \mathbf{e}_m \otimes \mathbf{e}_n}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial B_{mn}}{\partial \alpha_i} \mathbf{e}_m \otimes \mathbf{e}_n \otimes g_{ij}^{-1} \mathbf{e}_j + B_{mn} \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \otimes \mathbf{e}_n \otimes g_{ij}^{-1} \mathbf{e}_j + B_{mn} \mathbf{e}_m \otimes \frac{\partial \mathbf{e}_n}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j. \end{aligned} \quad (\text{C.44})$$

The divergence of a vector $\mathbf{a}$ and a second-order tensor $\mathbf{B}$ is given as

$$\begin{aligned} Div(\mathbf{a}) &= \frac{\partial \mathbf{a}}{\partial \alpha_i} \circ \bar{\mathbf{e}}_i = \frac{\partial \mathbf{a}}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) = \frac{\partial a_m \mathbf{e}_m}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \\ &= \frac{\partial a_m}{\partial \alpha_i} \mathbf{e}_m \circ (g_{ij}^{-1} \mathbf{e}_j) + a_m \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \\ &= \frac{\partial a_m}{\partial \alpha_i} + a_m \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j), \end{aligned} \quad (\text{C.45})$$

$$\begin{aligned} Div(\mathbf{B}) &= \frac{\partial \mathbf{B}}{\partial \alpha_i} \circ \bar{\mathbf{e}}_i = \frac{\partial \mathbf{B}}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) = \frac{\partial B_{mn} \mathbf{e}_m \otimes \mathbf{e}_n}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \\ &= \frac{\partial B_{mn}}{\partial \alpha_i} \mathbf{e}_m [\mathbf{e}_n \circ (g_{ij}^{-1} \mathbf{e}_j)] + B_{mn} \frac{\partial \mathbf{e}_m}{\partial \alpha_i} [\mathbf{e}_n \circ (g_{ij}^{-1} \mathbf{e}_j)] + B_{mn} \mathbf{e}_m \left[\frac{\partial \mathbf{e}_n}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \right] \\ &= \frac{\partial B_{mn}}{\partial \alpha_n} \mathbf{e}_m + B_{mn} \frac{\partial \mathbf{e}_m}{\partial \alpha_n} + B_{mn} \mathbf{e}_m \left[\frac{\partial \mathbf{e}_n}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \right]. \end{aligned} \quad (\text{C.46})$$

The expression for the curl of a vector $\mathbf{a}$ and a second-order tensor $\mathbf{A}$ is given by

$$Curl(\mathbf{a}) = -[\nabla_{\mathbf{x}} \times (\mathbf{a})]^T = -\frac{\partial \mathbf{a}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i = -\frac{\partial \mathbf{a}}{\partial \alpha_i} \times (g_{ij}^{-1} \mathbf{e}_j), \quad (\text{C.47})$$

$$Curl(\mathbf{A}) = -[\nabla_{\mathbf{x}} \times (\mathbf{A}^T)]^T = -\frac{\partial \mathbf{A}}{\partial \alpha_i} \times \bar{\mathbf{e}}_i = -\frac{\partial \mathbf{A}}{\partial \alpha_i} \times (g_{ij}^{-1} \mathbf{e}_j). \quad (\text{C.48})$$

C.2.1 Position, velocity, and acceleration

The current position $\mathbf{x}$ of material points is given by the equation $\mathbf{x} = \mathbf{X} + \mathbf{u}$, where $\mathbf{u}$ is the displacement. If we use the coordinate system α_i to describe the position of a particle in the reference configuration and use β_i to describe the position in the current configuration, we can write the two curvilinear systems as

$$d\mathbf{X} = d\alpha_i \mathbf{e}_i, \quad (\text{C.49})$$

$$d\mathbf{x} = d\beta_i \mathbf{e}_i^*, \quad (\text{C.50})$$

where the base vectors are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{X}}{\partial \alpha_i}, \quad (\text{C.51})$$

$$\mathbf{e}_i^* = \frac{\partial \mathbf{x}}{\partial \beta_i}. \quad (\text{C.52})$$

In this system

$$\mathbf{X} = X_i \mathbf{e}_i = X_i^* \mathbf{e}_i^*, \quad (\text{C.53})$$

$$\mathbf{x} = x_i \mathbf{e}_i = x_i^* \mathbf{e}_i^*, \quad (\text{C.54})$$

$$\mathbf{u} = u_i \mathbf{e}_i = u_i^* \mathbf{e}_i^*, \quad (\text{C.55})$$

where the terms with “*” represent the current configuration so that, for example, X_i^* are the components of $\mathbf{X}$ in the base vectors $\mathbf{e}_i^*$ used for the current configuration. The expressions for particle velocity and acceleration are given by

$$\mathbf{v} = v_i \mathbf{e}_i = v_i^* \mathbf{e}_i^* = \dot{u}_i \mathbf{e}_i, \quad (\text{C.56})$$

$$\mathbf{a} = a_i \mathbf{e}_i = a_i^* \mathbf{e}_i^* = \dot{v}_i \mathbf{e}_i = \ddot{u}_i \mathbf{e}_i. \quad (\text{C.57})$$

If $\beta_i(\alpha_1, \alpha_2, \alpha_3, t)$ is the material mapping describing the location in the current configuration of a point which is at coordinates α_i in the reference configuration, then it can be shown that

$$\mathbf{v} = v_i^* \mathbf{e}_i^* = \frac{\partial \beta_i}{\partial t} \mathbf{e}_i^* = \dot{\beta}_i \mathbf{e}_i^*, \quad (\text{C.58})$$

where $\dot{\beta}_i$ is the material time derivative. The expression for acceleration can be evaluated when using spatial coordinates $(\beta_1, \beta_2, \beta_3, t)$ to describe the velocity $\mathbf{v}$. This expression is

$$\begin{aligned} \mathbf{a} &= \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \beta_i} \frac{\partial \beta_i}{\partial t} \\ &= \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \beta_i} v_i^* \\ &= \frac{\partial v_i^*}{\partial t} \mathbf{e}_i^* + v_i^* \left[\frac{\partial v_j^*}{\partial \beta_i} \mathbf{e}_j^* + v_j^* \frac{\partial \mathbf{e}_j^*}{\partial \beta_i} \right] \\ &= \left[\frac{\partial v_i^*}{\partial t} + v_j^* \frac{\partial v_i^*}{\partial \beta_j} \right] \mathbf{e}_i^* + v_i^* v_j^* \frac{\partial \mathbf{e}_i^*}{\partial \beta_j}. \end{aligned} \quad (\text{C.59})$$

C.2.2 Deformation, velocity, and temperature gradients

The expressions for the deformation gradient, deformation gradient rate, and velocity gradient are given in curvilinear coordinates by

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j = \frac{\partial x_m}{\partial \alpha_i} \mathbf{e}_m \otimes g_{ij}^{-1} \mathbf{e}_j + x_m \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \mathbf{e}_i^* \otimes \mathbf{e}_k, \end{aligned} \quad (\text{C.60})$$

$$\begin{aligned} \mathbf{F}^{-1} &= \frac{\partial \mathbf{X}}{\partial \beta_i} \otimes g_{ij}^{*-1} \mathbf{e}_j^* = \frac{\partial X_m^*}{\partial \beta_i} \mathbf{e}_m^* \otimes g_{ij}^{*-1} \mathbf{e}_j^* + X_m^* \frac{\partial \mathbf{e}_m^*}{\partial \beta_i} \otimes g_{ij}^{*-1} \mathbf{e}_j^* \\ &= \frac{\partial \alpha_i}{\partial \beta_j} g_{jk}^{*-1} \mathbf{e}_i \otimes \mathbf{e}_k^*, \end{aligned} \quad (\text{C.61})$$

$$\begin{aligned} \dot{\mathbf{F}} &= \frac{\partial \mathbf{v}}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j = \frac{\partial v_m}{\partial \alpha_i} \mathbf{e}_m \otimes g_{ij}^{-1} \mathbf{e}_j + v_m \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j \\ &= \frac{\partial^2 u_m}{\partial t \partial \alpha_i} \mathbf{e}_m \otimes g_{ij}^{-1} \mathbf{e}_j + \frac{\partial u_m}{\partial t} \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \otimes g_{ij}^{-1} \mathbf{e}_j, \end{aligned} \quad (\text{C.62})$$

$$\begin{aligned} \mathbf{L} &= \frac{\partial \mathbf{v}}{\partial \beta_i} \otimes g_{ij}^{*-1} \mathbf{e}_j^* = \frac{\partial v_m^*}{\partial \beta_i} \mathbf{e}_m^* \otimes g_{ij}^{*-1} \mathbf{e}_j^* + v_m \frac{\partial \mathbf{e}_m^*}{\partial \beta_i} \otimes g_{ij}^{*-1} \mathbf{e}_j^* \\ &= \frac{\partial \dot{\beta}_m}{\partial \beta_i} \mathbf{e}_m^* \otimes g_{ij}^{*-1} \mathbf{e}_j^* + \dot{\beta}_m \frac{\partial \mathbf{e}_m^*}{\partial \beta_i} \otimes g_{ij}^{*-1} \mathbf{e}_j^*. \end{aligned} \quad (\text{C.63})$$

Note that $\dot{\beta}_m$ is the material time derivative of β_m .

The expression for temperature gradient $\mathbf{g}$ with respect to motions in the current configuration written in curvilinear coordinates is given by

$$\mathbf{g} = \mathbf{grad}(\theta) = \nabla_{\mathbf{x}}(\theta) = \frac{\partial \theta}{\partial \beta_i} g_{ij}^{*-1} \mathbf{e}_j^*. \quad (\text{C.64})$$

The expression for the (nominal) temperature gradient $\mathbf{G}$ with respect to motions in the reference configuration written in curvilinear coordinates is given by

$$\mathbf{G} = \mathbf{Grad}(\theta) = \nabla_{\mathbf{X}}(\theta) = \frac{\partial \theta}{\partial \alpha_i} g_{ij}^{-1} \mathbf{e}_j. \quad (\text{C.65})$$

C.2.3 Laplacian

The expression for the Laplacian of a scalar field is given by $\nabla^2(\psi) = \text{div}[\nabla(\psi)]$. If ψ is a function of the curvilinear coordinates α_i used in the reference configuration, then

$$\begin{aligned}\nabla_{\mathbf{X}}^2(\psi) &= \text{div}_{\mathbf{X}}[\nabla_{\mathbf{X}}(\psi)] \\ &= \text{div}_{\mathbf{X}} \left[\frac{\partial \psi}{\partial \alpha_i} g_{ij}^{-1} \mathbf{e}_j \right] \\ &= \frac{\partial}{\partial \alpha_m} \left[\frac{\partial \psi}{\partial \alpha_i} g_{ij}^{-1} \mathbf{e}_j \right] \circ g_{mn}^{-1} \mathbf{e}_n \\ &= \frac{\partial^2 \psi}{\partial \alpha_m \partial \alpha_i} g_{im}^{-1} - \frac{\partial \psi}{\partial \alpha_i} g_{ij}^{-1} \left(\mathbf{e}_j \circ \frac{\partial \mathbf{e}_n}{\partial \alpha_m} \right) g_{mn}^{-1}.\end{aligned}\quad (\text{C.66})$$

C.2.4 Balance laws in current configuration

The balance laws written in differential form using the base vectors $\mathbf{e}_i^*$ for points in the current configuration are written as

$$\rho J = \rho_o, \quad (\text{C.67})$$

$$\frac{\partial T_{ji}^*}{\partial \beta_j} \mathbf{e}_i^* + T_{ji}^* \frac{\partial \mathbf{e}_j^*}{\partial \beta_i} + T_{ji}^* \mathbf{e}_j^* \left[\frac{\partial \mathbf{e}_i^*}{\partial \beta_m} \circ (g_{mn}^{*-1} \mathbf{e}_n^*) \right] + \rho b_i^* \mathbf{e}_i^* = \rho a_i^* \mathbf{e}_i^*, \quad (\text{C.68})$$

$$-\frac{\partial q_m^*}{\partial \beta_m} - q_m^* \frac{\partial \mathbf{e}_m^*}{\partial \beta_i} \circ (g_{ij}^{*-1} \mathbf{e}_j^*) + \rho r + T_{ij}^* L_{mn}^* g_{jm}^* g_{in}^* = \rho \dot{e}, \quad (\text{C.69})$$

$$\rho \dot{\psi} - T_{ij}^* L_{mn}^* g_{jm}^* g_{in}^* + \rho \eta \dot{\theta} + \frac{1}{\theta} q_i g_j g_{ij} \leq 0. \quad (\text{C.70})$$

C.2.5 Balance laws in reference configuration

The differential expression for the balance laws written in the base $\mathbf{e}_i$ used for describing points in the reference configuration are given by

$$\rho J = \rho_o, \quad (\text{C.71})$$

$$\frac{\partial T_{o,nm}}{\partial \alpha_n} \mathbf{e}_m + T_{o,nm} \frac{\partial \mathbf{e}_m}{\partial \alpha_n} + T_{o,nm} \mathbf{e}_m \left[\frac{\partial \mathbf{e}_n}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) \right] + \rho_o b_i \mathbf{e}_i = \rho_o a_i \mathbf{e}_i, \quad (\text{C.72})$$

$$-\frac{\partial q_{o,m}}{\partial \alpha_m} - q_{o,m} \frac{\partial \mathbf{e}_m}{\partial \alpha_i} \circ (g_{ij}^{-1} \mathbf{e}_j) + \rho_o r + T_{o,ij} \dot{F}_{mn} g_{jm} g_{in} = \rho_o \dot{e}, \quad (\text{C.73})$$

$$\rho_o \dot{\psi} - T_{o,ij} \dot{F}_{mn} g_{jm} g_{in} + \rho_o \eta \dot{\theta} + \frac{1}{\theta} q_{o,i} G_j g_{ij} \leq 0. \quad (\text{C.74})$$

C.3 Rectangular coordinates

The rectangular coordinate system is defined by three fixed orthonormal base vectors $\hat{\mathbf{i}}_i$ used to describe positions in the reference configuration, and three fixed orthonormal base vectors $\hat{\mathbf{i}}_i^*$ used to describe position in the current configuration. As such, we are considering using different rectangular coordinates to describe the two positions in the two configurations. To make things consistent with our development, we can take $\alpha_i = X_i$ to be the components of $\mathbf{X}$ in the reference configuration and $\beta_i = x_i^*$ to be the components of $\mathbf{x}$ in the current configuration. In general we can write

$$\mathbf{X} = X_i \hat{\mathbf{i}}_i = X_i^* \hat{\mathbf{i}}_i^*, \quad \mathbf{x} = x_i \hat{\mathbf{i}}_i = x_i^* \hat{\mathbf{i}}_i^*. \quad (\text{C.75})$$

The description of the coordinate system is given in Figure C.1.

The position vector in the reference configuration is given by $\mathbf{X} = X_i \hat{\mathbf{i}}_i$, with $\alpha_i = X_i$ being the curvilinear coordinates and the vectors $\mathbf{e}_i = \hat{\mathbf{i}}_i$ being the associated base vectors. The assumption here is that the same constant measure is used for all three directions and that $\mathbf{e}_i$ constructs a constant orthonormal base. The metric tensor for this base is $g_{ij} = \delta_{ij}$ and its inverse is $g_{ij}^{-1} = \delta_{ij}$. This results in the dual base becoming equal to the base and so we have

$$\bar{\mathbf{e}}_i = \hat{\mathbf{i}}_i. \quad (\text{C.76})$$

For the current configuration, we take $\beta_i = x_i^*$ as the curvilinear coordinates so $\mathbf{e}_i^* = \hat{\mathbf{i}}_i^*$ become the associated base vectors with $g_{ij}^* = g_{ij}^{*-1} = \delta_{ij}$, and we have the dual base given by

$$\bar{\mathbf{e}}_i^* = \hat{\mathbf{i}}_i^*. \quad (\text{C.77})$$

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \circ \mathbf{b} = (a_i \hat{\mathbf{i}}_i) \circ (b_j \hat{\mathbf{i}}_j) = a_i b_i. \quad (\text{C.78})$$

The trace of the product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\text{tr}(\mathbf{AB}) = A_{ij} B_{ji}. \quad (\text{C.79})$$

A general vector $\mathbf{a}$ in such a system is written as $\mathbf{a} = a_i \hat{\mathbf{i}}_i$, and any second-order vector $\mathbf{B}$ is written in terms of this base as $\mathbf{B} = B_{ij} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j$. Since $\hat{\mathbf{i}}_i$ are constant vectors, in such a system the gradient of a scalar field ψ , a vector field $\mathbf{a}$ and a second-order tensor field $\mathbf{B}$ is given by

$$\mathbf{Grad}(\psi) = \nabla_{\mathbf{X}}(\psi) = \frac{\partial \psi}{\partial X_i} \hat{\mathbf{i}}_i, \quad (\text{C.80})$$

$$\begin{aligned} \mathbf{Grad}(\mathbf{a}) &= [\nabla_{\mathbf{X}}(\mathbf{a})]^T = \frac{\partial \mathbf{a}}{\partial X_i} \otimes \hat{\mathbf{i}}_i = \frac{\partial a_m \hat{\mathbf{i}}_m}{\partial X_i} \otimes \hat{\mathbf{i}}_i \\ &= \frac{\partial a_m}{\partial X_i} \hat{\mathbf{i}}_m \otimes \hat{\mathbf{i}}_i, \end{aligned} \quad (\text{C.81})$$

$$\begin{aligned} \mathbf{Grad}(\mathbf{B}) &= [\nabla_{\mathbf{X}}(\mathbf{B}^T)]^T = \frac{\partial \mathbf{B}}{\partial X_i} \otimes \hat{\mathbf{i}}_i = \frac{\partial B_{mn} \hat{\mathbf{i}}_m \otimes \hat{\mathbf{i}}_n}{\partial X_i} \otimes \hat{\mathbf{i}}_i \\ &= \frac{\partial B_{mn}}{\partial X_i} \hat{\mathbf{i}}_m \otimes \hat{\mathbf{i}}_n \otimes \hat{\mathbf{i}}_i. \end{aligned} \quad (\text{C.82})$$

The divergence of a vector $\mathbf{a}$ and a second-order tensor $\mathbf{B}$ is given as

$$\begin{aligned} \text{Div}(\mathbf{a}) &= \text{div}_{\mathbf{X}}(\mathbf{a}) = \frac{\partial \mathbf{a}}{\partial X_i} \circ \hat{\mathbf{i}}_i = \frac{\partial a_m \hat{\mathbf{i}}_m}{\partial X_i} \circ \hat{\mathbf{i}}_i = \frac{\partial a_m}{\partial X_i} (\hat{\mathbf{i}}_m \circ \hat{\mathbf{i}}_i) \\ &= \frac{\partial a_m}{\partial X_m}, \end{aligned} \quad (\text{C.83})$$

$$\begin{aligned} \text{Div}(\mathbf{B}) &= \text{div}_{\mathbf{X}}(\mathbf{B}) = \frac{\partial \mathbf{B}}{\partial X_i} \circ \hat{\mathbf{i}}_i = \frac{\partial B_{mn} \hat{\mathbf{i}}_m \otimes \hat{\mathbf{i}}_n}{\partial X_i} \circ \hat{\mathbf{i}}_i = \frac{\partial B_{mn}}{\partial X_i} \hat{\mathbf{i}}_m (\hat{\mathbf{i}}_n \circ \hat{\mathbf{i}}_i) \\ &= \frac{\partial B_{mn}}{\partial X_n} \hat{\mathbf{i}}_m. \end{aligned} \quad (\text{C.84})$$

The expression for the curl of a vector $\mathbf{a}$ is given by

$$\begin{aligned} \text{Curl}(\mathbf{a}) &= -\frac{\partial \mathbf{a}}{\partial X_j} \times \hat{\mathbf{i}}_j = -\frac{\partial a_i}{\partial X_j} \hat{\mathbf{i}}_i \times \hat{\mathbf{i}}_j = -\frac{\partial a_i}{\partial X_j} \epsilon_{ijk} \hat{\mathbf{i}}_k \\ &= \left(\frac{\partial a_3}{\partial X_2} - \frac{\partial a_2}{\partial X_3} \right) \hat{\mathbf{i}}_1 + \left(\frac{\partial a_1}{\partial X_3} - \frac{\partial a_3}{\partial X_1} \right) \hat{\mathbf{i}}_2 + \left(\frac{\partial a_2}{\partial X_1} - \frac{\partial a_1}{\partial X_2} \right) \hat{\mathbf{i}}_3. \end{aligned} \quad (\text{C.85})$$

C.3.1 Position, velocity, and acceleration

The current position of material points is given by the equation $\mathbf{x} = \mathbf{X} + \mathbf{u}$. If we use the coordinate system α_i to describe the position of a particle in the reference configuration and use β_i to describe the position in the current configuration, we can write the two curvilinear systems as

$$d\mathbf{X} = d\alpha_i \mathbf{e}_i = dX_i \hat{\mathbf{i}}_i, \quad (\text{C.86})$$

$$d\mathbf{x} = d\beta_i \mathbf{e}_i^* = dx_i^* \hat{\mathbf{i}}_i^*, \quad (\text{C.87})$$

where the base vectors are given by

$$\mathbf{e}_i = \frac{\partial \mathbf{X}}{\partial \alpha_i} = \frac{\partial \mathbf{X}}{\partial X_i} = \hat{\mathbf{i}}_i, \quad (\text{C.88})$$

$$\mathbf{e}_i^* = \frac{\partial \mathbf{x}}{\partial \beta_i} = \frac{\partial \mathbf{x}}{\partial x_i^*} = \hat{\mathbf{i}}_i^*. \quad (\text{C.89})$$

In this system

$$\mathbf{X} = X_i \hat{\mathbf{i}}_i = X_i^* \hat{\mathbf{i}}_i^*, \quad (\text{C.90})$$

$$\mathbf{x} = x_i \hat{\mathbf{i}}_i = x_i^* \hat{\mathbf{i}}_i^*, \quad (\text{C.91})$$

$$\mathbf{u} = u_i \hat{\mathbf{i}}_i = u_i^* \hat{\mathbf{i}}_i^*, \quad (\text{C.92})$$

where the terms with “*” represent the current configuration so that, for example, X_i^* are the components of $\mathbf{X}$ in the base vectors $\mathbf{e}_i^* = \hat{\mathbf{i}}_i^*$ used for the current configuration. The expressions for particle velocity and acceleration are given by

$$\mathbf{v} = v_i \hat{\mathbf{i}}_i = v_i^* \hat{\mathbf{i}}_i^* = \dot{u}_i \hat{\mathbf{i}}_i, \quad (\text{C.93})$$

$$\mathbf{a} = a_i \hat{\mathbf{i}}_i = a_i^* \hat{\mathbf{i}}_i^* = \dot{v}_i \hat{\mathbf{i}}_i = \ddot{u}_i \hat{\mathbf{i}}_i. \quad (\text{C.94})$$

If $x_i^*(X_1, X_2, X_3, t)$ are the material mappings describing the location in the current configuration of a point that is at coordinates X_i in the reference configuration, then it can be shown that

$$\mathbf{v} = v_i^* \hat{\mathbf{i}}_i^* = \frac{\partial x_i^*}{\partial t} \hat{\mathbf{i}}_i^* = \dot{x}_i^* \hat{\mathbf{i}}_i^*. \quad (\text{C.95})$$

When using a spatial description of the velocity in terms (x_1^*, x_2^*, x_3^*, t) , the expressions for acceleration can be calculated from

$$\mathbf{a} = \left(\frac{\partial v_i^*}{\partial t} + v_j^* \frac{\partial v_i^*}{\partial x_j^*} \right) \hat{\mathbf{i}}_i^*. \quad (\text{C.96})$$

If we use the same coordinate system for the two configurations, then we can set $(\alpha_1, \alpha_2, \alpha_3)$ equal to (X_1, X_2, X_3) for the coordinates of $\mathbf{X}$ in $\hat{\mathbf{i}}_i$ and we can set $(\beta_1, \beta_2, \beta_3)$ equal to (x_1, x_2, x_3) for the coordinates of $\mathbf{x}$ in $\hat{\mathbf{i}}_i^* = \hat{\mathbf{i}}_i$.

C.3.2 Deformation, velocity, and temperature gradients

The expression for the deformation gradient, deformation gradient rate, and velocity

gradient is given in curvilinear coordinates by

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j = \frac{\partial x_i^*}{\partial X_j} \hat{\mathbf{i}}_i^* \otimes \hat{\mathbf{i}}_j, \quad (\text{C.97})$$

$$\mathbf{F}^{-1} = \frac{\partial X_i}{\partial x_j^*} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j^* = \frac{\partial X_i^*}{\partial x_j^*} \hat{\mathbf{i}}_i^* \otimes \hat{\mathbf{i}}_j^* \quad (\text{C.98})$$

$$\begin{aligned} \dot{\mathbf{F}} &= \frac{\partial v_i}{\partial X_j} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j = \frac{\partial v_i^*}{\partial X_j} \hat{\mathbf{i}}_i^* \otimes \hat{\mathbf{i}}_j \\ &= \frac{\partial^2 u_i}{\partial t \partial X_j} \hat{\mathbf{i}}_i \otimes \hat{\mathbf{i}}_j, \end{aligned} \quad (\text{C.99})$$

$$\mathbf{L} = \frac{\partial v_i^*}{\partial x_j^*} \hat{\mathbf{i}}_i^* \otimes \hat{\mathbf{i}}_j^* = \frac{\partial \dot{x}_i^*}{\partial x_j^*} \hat{\mathbf{i}}_i^* \otimes \hat{\mathbf{i}}_j^*. \quad (\text{C.100})$$

If one uses the same rectangular coordinate system to describe points in both the reference and current configurations, the distinction between components and directions for the two coordinates disappears and we can set $X_i^* = X_i$, $x_i^* = x_i$ and $\hat{\mathbf{i}}_i^* = \hat{\mathbf{i}}_i$.

The expression for temperature gradient $\mathbf{g}$ with respect to motions in the current configuration written in curvilinear coordinates is given by

$$\mathbf{g} = \mathbf{grad}(\theta) = \nabla_{\mathbf{x}}(\theta) = \frac{\partial \theta}{\partial x_i^*} \hat{\mathbf{i}}_i^*. \quad (\text{C.101})$$

The expression for the (nominal) temperature gradient $\mathbf{G}$ with respect to motions in the reference configuration written in curvilinear coordinates is given by

$$\mathbf{G} = \mathbf{Grad}(\theta) = \nabla_{\mathbf{X}}(\theta) = \frac{\partial \theta}{\partial X_i} \hat{\mathbf{i}}_i. \quad (\text{C.102})$$

C.3.3 Laplacian

The expression for the Laplacian of a scalar field is given by $\nabla^2(\psi) = \text{div}[\nabla(\psi)]$. If ψ is a function of the curvilinear coordinates X_i used in the reference configuration, then

$$\nabla_{\mathbf{X}}^2(\psi) = \frac{\partial^2 \psi}{\partial X_i \partial X_i}. \quad (\text{C.103})$$

C.3.4 Balance laws in current configuration

The balance laws in differential form written in terms of components in the $\hat{\mathbf{i}}_i^*$ base (e.g., $\mathbf{T} = T_{mn}^* \hat{\mathbf{i}}_m^* \otimes \hat{\mathbf{i}}_n^*$) are given by

$$\rho J = \rho_o, \quad (\text{C.104})$$

$$\frac{\partial T_{ji}^*}{\partial x_j^*} \hat{\mathbf{i}}_i^* + \rho b_i^* \hat{\mathbf{i}}_i^* = \rho a_i^* \hat{\mathbf{i}}_i^*, \quad (\text{C.105})$$

$$-\frac{\partial q_i^*}{\partial x_i^*} + \rho r + T_{ij}^* L_{ji}^* = \rho \dot{e}, \quad (\text{C.106})$$

$$\rho \dot{\psi} - T_{ij}^* L_{ji}^* + \rho \eta \dot{\theta} + \frac{1}{\theta} q_i^* g_i^* \leq 0. \quad (\text{C.107})$$

C.3.5 Balance laws in reference configuration

We use the curvilinear coordinates X_i and the components of the tensors in the base $\hat{\mathbf{i}}_i$ (e.g., $\mathbf{T}_o = T_{o-mn}\hat{\mathbf{i}}_m \otimes \hat{\mathbf{i}}_n$) to get the differential form of the balance laws written as

$$\rho J = \rho_o, \quad (C.108)$$

$$\frac{\partial T_{o-ji}}{\partial X_j} \hat{\mathbf{i}}_i + \rho_o b_i \hat{\mathbf{i}}_i = \rho_o a_i \hat{\mathbf{i}}_i, \quad (C.109)$$

$$-\frac{\partial q_{o-i}}{\partial X_i} + \rho_o r + T_{o-ij} \dot{F}_{ji} = \rho_o \dot{e}, \quad (C.110)$$

$$\rho_o \dot{\psi} - T_{o-ij} \dot{F}_{ji} + \rho_o \eta \dot{\theta} + \frac{1}{\theta} q_{o-i} G_i \leq 0. \quad (C.111)$$

C.4 Cylindrical coordinates

The cylindrical coordinate system is defined by the coordinates (r, θ, z) with the position vector $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$. The commonly used cylindrical base vectors do not describe a curvilinear system, but there is a direct relation between them and a curvilinear system that can be used to obtain the following relations. The resulting curvilinear base vectors for this system are given by

$$\mathbf{e}_1 = \mathbf{e}_r, \quad \mathbf{e}_2 = r\mathbf{e}_\theta, \quad \mathbf{e}_3 = \mathbf{e}_z. \quad (C.112)$$

The metric tensor and its inverse for this base are given by

$$[g] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [g]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (C.113)$$

We can calculate the dual base $\bar{\mathbf{e}}_i = g_{ij}^{-1} \mathbf{e}_j$ to get

$$\bar{\mathbf{e}}_1 = \mathbf{e}_r, \quad \bar{\mathbf{e}}_2 = \frac{1}{r} \mathbf{e}_\theta, \quad \bar{\mathbf{e}}_3 = \mathbf{e}_z. \quad (C.114)$$

The relations relating the components for a curvilinear system and the related cylindrical system can be found for a vector $\mathbf{a}$ and a second-order tensor $\mathbf{B}$ by looking at the expressions

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} a_r & a_\theta & a_z \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}, \quad (C.115)$$

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} B_{rr} & B_{r\theta} & B_{rz} \\ B_{\theta r} & B_{\theta\theta} & B_{\theta z} \\ B_{zr} & B_{z\theta} & B_{zz} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \end{aligned} \quad (C.116)$$

This provides the relations

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_r & \frac{1}{r} a_\theta & a_z \end{bmatrix}, \quad (C.117)$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} B_{rr} & \frac{1}{r} B_{r\theta} & B_{rz} \\ \frac{1}{r} B_{\theta r} & \frac{1}{r^2} B_{\theta\theta} & \frac{1}{r} B_{\theta z} \\ B_{zr} & \frac{1}{r} B_{z\theta} & B_{zz} \end{bmatrix}. \quad (C.118)$$

Using the results from curvilinear systems, and by constructing the equivalent curvilinear system to the cylindrical system, we can find the expressions for the gradient and divergence in a cylindrical system. The gradient of the scalar ψ and the vector $\mathbf{a}$ are given by

$$\mathbf{grad}(\psi) = \frac{\partial \psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \mathbf{e}_\theta + \frac{\partial \psi}{\partial z} \mathbf{e}_z, \quad (\text{C.119})$$

$$\mathbf{grad}(\mathbf{a}) = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial a_r}{\partial r} & \frac{1}{r} \frac{\partial a_r}{\partial \theta} - \frac{1}{r} a_\theta & \frac{\partial a_r}{\partial z} \\ \frac{\partial a_\theta}{\partial r} & \frac{1}{r} \frac{\partial a_\theta}{\partial \theta} + \frac{1}{r} a_r & \frac{\partial a_\theta}{\partial z} \\ \frac{\partial a_z}{\partial r} & \frac{1}{r} \frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \quad (\text{C.120})$$

The divergence of vector field $\mathbf{a}$ and second-order tensor field $\mathbf{B}$ is given as

$$\text{div}(\mathbf{a}) = \frac{\partial a_r}{\partial r} + \frac{1}{r} \frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z} + \frac{1}{r} a_r, \quad (\text{C.121})$$

$$\text{div}(\mathbf{B}) = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial B_{rr}}{\partial r} + \frac{1}{r} \frac{\partial B_{r\theta}}{\partial \theta} + \frac{\partial B_{rz}}{\partial z} + \frac{1}{r} (B_{rr} - B_{\theta\theta}) \\ \frac{\partial B_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial B_{\theta\theta}}{\partial \theta} + \frac{\partial B_{\theta z}}{\partial z} + \frac{1}{r} (B_{r\theta} + B_{\theta r}) \\ \frac{\partial B_{zr}}{\partial r} + \frac{1}{r} \frac{\partial B_{z\theta}}{\partial \theta} + \frac{\partial B_{zz}}{\partial z} + \frac{1}{r} B_{zr} \end{bmatrix}. \quad (\text{C.122})$$

The expression for the curl of a vector $\mathbf{a}$ is given by

$$\text{curl}(\mathbf{a}) = \left(\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left(\frac{\partial r a_\theta}{\partial r} - \frac{\partial a_r}{\partial \theta} \right) \mathbf{e}_z. \quad (\text{C.123})$$

C.4.1 Position, velocity, and acceleration

The current position of material points is given by the equation $\mathbf{x} = \mathbf{X} + \mathbf{u}$, where

$$\mathbf{X} = R \mathbf{e}_R + Z \mathbf{e}_Z, \quad (\text{C.124})$$

$$\mathbf{x} = r \mathbf{e}_r + z \mathbf{e}_z, \quad (\text{C.125})$$

$$\mathbf{u} = u_R \mathbf{e}_R + u_\theta \mathbf{e}_\theta + u_Z \mathbf{e}_Z = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z, \quad (\text{C.126})$$

where (r, θ, z) are the current coordinates of a point that is at the coordinates (R, Θ, Z) in the reference configuration. In this case, we are considering only particles represented in cylindrical coordinates both in the reference and current configurations, even though the two cylindrical coordinates are different. The expressions for particle velocity and acceleration are given by

$$\mathbf{v} = \dot{u}_R \mathbf{e}_R + \dot{u}_\theta \mathbf{e}_\theta + \dot{u}_Z \mathbf{e}_Z, \quad (\text{C.127})$$

$$\mathbf{a} = \ddot{u}_R \mathbf{e}_R + \ddot{u}_\theta \mathbf{e}_\theta + \ddot{u}_Z \mathbf{e}_Z. \quad (\text{C.128})$$

$$(\text{C.129})$$

The expressions for velocity and acceleration for a particle given in a cylindrical coordinate system are given by expressions

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z, \quad (\text{C.130})$$

and

$$\mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \mathbf{e}_r + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \mathbf{e}_\theta + \ddot{z} \mathbf{e}_z, \quad (\text{C.131})$$

where in this expression (r, θ, z) represent position of a material particle as a function of time. If we use a spatial description of velocity, where the velocity is given in terms of

the spatial coordinates and time as $\mathbf{v}(r, \theta, z, t)$, the same expression for acceleration can be written as

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}, \quad (\text{C.132})$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}, \quad (\text{C.133})$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}. \quad (\text{C.134})$$

$$(\text{C.135})$$

C.4.2 Deformation, velocity, and temperature gradients

The expression for the deformation gradient and its inverse are given by

$$\begin{aligned} \mathbf{F} &= \mathbf{e}_i^* \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \otimes \mathbf{e}_k \\ &= \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ R\mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \frac{1}{R}\mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & r \frac{\partial \theta}{\partial \Theta} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}, \end{aligned} \quad (\text{C.136})$$

$$\begin{aligned} \mathbf{F}^{-1} &= \mathbf{e}_i \frac{\partial \alpha_i}{\partial \beta_j} g_{jk}^{*-1} \otimes \mathbf{e}_k^* \\ &= \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_Z \end{bmatrix} \begin{bmatrix} \frac{\partial R}{\partial r} & \frac{1}{r} \frac{\partial R}{\partial \theta} & \frac{\partial R}{\partial z} \\ R \frac{\partial \theta}{\partial r} & R \frac{\partial \theta}{\partial \theta} & R \frac{\partial \theta}{\partial z} \\ \frac{\partial Z}{\partial r} & \frac{1}{r} \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \end{aligned} \quad (\text{C.137})$$

The expressions for deformation gradient rate and velocity gradient are given by

$$\dot{\mathbf{F}} = \text{Grad}(\mathbf{v}) = \begin{bmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_Z \end{bmatrix} \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R} \frac{\partial v_R}{\partial \Theta} - \frac{1}{R} v_\Theta & \frac{\partial v_R}{\partial Z} \\ \frac{\partial v_\Theta}{\partial R} & \frac{1}{R} \frac{\partial v_\Theta}{\partial \Theta} + \frac{1}{R} v_R & \frac{\partial v_\Theta}{\partial Z} \\ \frac{\partial v_Z}{\partial R} & \frac{1}{R} \frac{\partial v_Z}{\partial \Theta} & \frac{\partial v_Z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_Z \end{bmatrix}, \quad (\text{C.138})$$

$$\mathbf{L} = \text{grad}(\mathbf{v}) = \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{1}{r} v_\theta & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r} v_r & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix}. \quad (\text{C.139})$$

Since there is a conflict of notation between the θ used for temperature and the θ used for the second coordinate axis, *we will use θ^* in this section to denote temperature*. The expression for temperature gradient in cylindrical coordinates is given by

$$\mathbf{g} = \text{grad}(\theta^*) = \nabla_{\mathbf{x}}(\theta^*) = \frac{\partial \theta^*}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \theta^*}{\partial \theta} \mathbf{e}_\theta + \frac{\partial \theta^*}{\partial z} \mathbf{e}_z, \quad (\text{C.140})$$

so that the components of $\mathbf{g}$ are given by

$$g_r = \frac{\partial \theta^*}{\partial r}, \quad g_\theta = \frac{1}{r} \frac{\partial \theta^*}{\partial \theta}, \quad g_z = \frac{\partial \theta^*}{\partial z}. \quad (\text{C.141})$$

The nominal temperature gradient is given by

$$\mathbf{G} = \mathbf{Grad}(\theta^*) = \nabla_{\mathbf{x}}(\theta^*) = \frac{\partial \theta^*}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial \theta^*}{\partial \Theta} \mathbf{e}_\Theta + \frac{\partial \theta^*}{\partial Z} \mathbf{e}_Z, \quad (\text{C.142})$$

so that the components of $\mathbf{G}$ are given by

$$G_R = \frac{\partial \theta^*}{\partial R}, \quad G_\Theta = \frac{1}{R} \frac{\partial \theta^*}{\partial \Theta}, \quad G_Z = \frac{\partial \theta^*}{\partial Z}. \quad (\text{C.143})$$

C.4.3 Laplacian

The expression for the Laplacian of a scalar field ψ described in the reference configuration coordinates (R, Θ, Z) is given by

$$\nabla_{\mathbf{x}}^2(\psi) = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \Theta^2} + \frac{\partial^2 \psi}{\partial Z^2}. \quad (\text{C.144})$$

The Laplacian of the same function written in terms of the current configuration coordinates (r, θ, z) is given by

$$\nabla_{\mathbf{x}}^2(\psi) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}. \quad (\text{C.145})$$

C.4.4 Balance laws in current configuration

The balance of linear momentum in cylindrical coordinates written in the current configuration is given as the three component equations

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) + \rho b_r = \rho a_r, \quad (\text{C.146})$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2}{r} T_{r\theta} + \rho b_\theta = \rho a_\theta, \quad (\text{C.147})$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r} T_{rz} + \rho b_z = \rho a_z. \quad (\text{C.148})$$

In the expression for balance of work and energy and the entropy production inequality we have

$$\begin{aligned} \text{tr}(\mathbf{TL}) &= T_{rr}L_{rr} + T_{r\theta}L_{\theta r} + T_{rz}L_{zr} \\ &\quad + T_{\theta r}L_{r\theta} + T_{\theta\theta}L_{\theta\theta} + T_{\theta z}L_{z\theta} \\ &\quad + T_{zr}L_{rz} + T_{z\theta}L_{\theta z} + T_{zz}L_{zz}. \end{aligned} \quad (\text{C.149})$$

In the expression of balance of work and energy, we have

$$\text{div}(\mathbf{q}) = \frac{\partial q_r}{\partial r} + \frac{1}{r} \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} + \frac{1}{r} q_r. \quad (\text{C.150})$$

In the expression for the entropy production inequality, we have

$$\mathbf{q} \circ \mathbf{g} = q_r \frac{\partial \theta^*}{\partial r} + \frac{1}{r} q_\theta \frac{\partial \theta^*}{\partial \theta} + q_z \frac{\partial \theta^*}{\partial z}, \quad (\text{C.151})$$

where θ^* is used for denoting temperature.

C.4.5 Balance laws in reference configuration

The balance of linear momentum in cylindrical coordinates written in the reference configuration is given as the three component equations

$$\frac{\partial T_{o\_RR}}{\partial R} + \frac{1}{R} \frac{\partial T_{o\_R\Theta}}{\partial \Theta} + \frac{\partial T_{o\_ZR}}{\partial Z} + \frac{1}{R} (T_{o\_RR} - T_{o\_R\Theta}) + \rho_o b_R = \rho_o a_R, \quad (C.152)$$

$$\frac{\partial T_{o\_R\Theta}}{\partial R} + \frac{1}{R} \frac{\partial T_{o\_R\Theta}}{\partial \Theta} + \frac{\partial T_{o\_Z\Theta}}{\partial Z} + \frac{1}{R} (T_{o\_R\Theta} + T_{o\_R\Theta}) + \rho_o b_\Theta = \rho_o a_\Theta, \quad (C.153)$$

$$\frac{\partial T_{o\_RZ}}{\partial R} + \frac{1}{R} \frac{\partial T_{o\_RZ}}{\partial \Theta} + \frac{\partial T_{o\_ZZ}}{\partial Z} + \frac{1}{R} T_{o\_RZ} + \rho_o b_Z = \rho_o a_Z. \quad (C.154)$$

In the expression for balance of work and energy and the entropy production inequality, we have

$$\begin{aligned} tr(\mathbf{T}_o \dot{\mathbf{F}}) &= T_{o\_RR} \dot{F}_{RR} + T_{o\_R\Theta} \dot{F}_{\Theta R} + T_{o\_RZ} \dot{F}_{ZR} \\ &\quad + T_{o\_R\Theta} \dot{F}_{R\Theta} + T_{o\_R\Theta} \dot{F}_{\Theta\Theta} + T_{o\_R\Theta} \dot{F}_{Z\Theta} \\ &\quad + T_{o\_RZ} \dot{F}_{RZ} + T_{o\_RZ} \dot{F}_{\Theta Z} + T_{o\_RZ} \dot{F}_{ZZ}. \end{aligned} \quad (C.155)$$

In the expression of balance of work and energy, we have

$$Div(\mathbf{q}_o) = \frac{\partial q_{o\_R}}{\partial R} + \frac{1}{R} \frac{\partial q_{o\_R}}{\partial \Theta} + \frac{\partial q_{o\_Z}}{\partial Z} + \frac{1}{R} q_{o\_R}. \quad (C.156)$$

In the expression for the entropy production inequality, we have

$$\mathbf{q}_o \circ \mathbf{G} = q_{o\_R} \frac{\partial \theta^*}{\partial R} + \frac{1}{R} q_{o\_R} \frac{\partial \theta^*}{\partial \Theta} + q_{o\_Z} \frac{\partial \theta^*}{\partial Z}, \quad (C.157)$$

where θ^* is used for denoting temperature.

C.5 Spherical coordinates

The spherical coordinate system is defined by the coordinates (ρ, ϕ, θ) , with the position vector given by $\mathbf{x} = \rho \mathbf{e}_\rho$. The commonly used spherical base vectors do not describe a curvilinear system, but there is a direct relation between this base and a curvilinear base that can be used to obtain the following expressions. The resulting base vectors for a curvilinear system using these variables is given by

$$\mathbf{e}_1 = \mathbf{e}_\rho, \quad \mathbf{e}_2 = \rho \mathbf{e}_\phi, \quad \mathbf{e}_3 = \rho \sin(\phi) \mathbf{e}_\theta. \quad (C.158)$$

The metric tensor associated with this curvilinear base is given by

$$[g] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2(\phi) \end{bmatrix}, \quad [g]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2 \sin^2(\phi)} \end{bmatrix}. \quad (C.159)$$

One can write the dual base by using $\bar{\mathbf{e}}_i = g_{ij}^{-1} \mathbf{e}_j$ to get

$$\bar{\mathbf{e}}_1 = \mathbf{e}_\rho, \quad \bar{\mathbf{e}}_2 = \frac{1}{\rho} \mathbf{e}_\phi, \quad \bar{\mathbf{e}}_3 = \frac{1}{\rho \sin(\phi)} \mathbf{e}_\theta. \quad (C.160)$$

The relation between the components of an equivalent curvilinear system and that of a spherical system for a vector $\mathbf{a}$ and for a second-order tensor $\mathbf{B}$ are obtained from the relations

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} a_\rho & a_\phi & a_\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}, \quad (\text{C.161})$$

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \end{bmatrix} \begin{bmatrix} B_{\rho\rho} & B_{\rho\phi} & B_{\rho\theta} \\ B_{\phi\rho} & B_{\phi\phi} & B_{\phi\theta} \\ B_{\theta\rho} & B_{\theta\phi} & B_{\theta\theta} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}. \end{aligned} \quad (\text{C.162})$$

This provides the needed connections

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_\rho & \frac{1}{\rho}a_\phi & \frac{1}{\rho \sin(\phi)}a_\theta \end{bmatrix}, \quad (\text{C.163})$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} B_{\rho\rho} & \frac{1}{\rho}B_{\rho\phi} & \frac{1}{\rho \sin(\phi)}B_{\rho\theta} \\ \frac{1}{\rho}B_{\phi\rho} & \frac{1}{\rho^2}B_{\phi\phi} & \frac{1}{\rho^2 \sin(\phi)}B_{\phi\theta} \\ \frac{1}{\rho \sin(\phi)}B_{\theta\rho} & \frac{1}{\rho^2 \sin(\phi)}B_{\theta\phi} & \frac{1}{\rho^2 \sin^2(\phi)}B_{\theta\theta} \end{bmatrix}. \quad (\text{C.164})$$

Using the results obtained for curvilinear systems and by constructing the equivalent curvilinear system to the spherical system, we can find the expressions for the gradient and divergence in a spherical system. The gradient of the scalar field ψ and the vector field $\mathbf{a}$ is given by

$$\text{grad}(\psi) = \frac{\partial\psi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi + \frac{1}{\rho \sin(\phi)}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta, \quad (\text{C.165})$$

$$\begin{aligned} \text{grad}(\mathbf{a}) &= \begin{bmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \end{bmatrix} \\ &\begin{bmatrix} \frac{\partial a_\rho}{\partial\rho} & \frac{1}{\rho}\frac{\partial a_\rho}{\partial\phi} - \frac{1}{\rho}a_\phi & \frac{1}{\rho \sin(\phi)}\frac{\partial a_\rho}{\partial\theta} - \frac{1}{\rho}a_\theta \\ \frac{\partial a_\phi}{\partial\rho} & \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{1}{\rho}a_\rho & \frac{1}{\rho \sin(\phi)}\frac{\partial a_\phi}{\partial\theta} - \frac{\cos(\phi)}{\rho \sin(\phi)}a_\theta \\ \frac{\partial a_\theta}{\partial\rho} & \frac{1}{\rho}\frac{\partial a_\theta}{\partial\phi} & \frac{1}{\rho \sin(\phi)}\frac{\partial a_\theta}{\partial\theta} + \frac{1}{\rho}(a_\rho + \frac{\cos(\phi)}{\sin(\phi)}a_\phi) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}. \end{aligned} \quad (\text{C.166})$$

The divergence of vector $\mathbf{a}$ and second-order tensor $\mathbf{B}$ is given as

$$\text{div}(\mathbf{a}) = \frac{\partial a_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{1}{\rho \sin(\phi)}\frac{\partial a_\theta}{\partial\theta} + \frac{1}{\rho}\left[2a_\rho + \frac{\cos(\phi)}{\sin(\phi)}a_\phi\right], \quad (\text{C.167})$$

$$\begin{aligned} \text{div}(\mathbf{B}) &= \begin{bmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \end{bmatrix} \\ &\begin{bmatrix} \frac{\partial B_{\rho\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial B_{\rho\phi}}{\partial\phi} + \frac{1}{\rho \sin(\phi)}\frac{\partial B_{\rho\theta}}{\partial\theta} + \frac{1}{\rho}(2B_{\rho\rho} - B_{\phi\phi} - B_{\theta\theta} + \frac{\cos(\phi)}{\sin(\phi)}B_{\rho\phi}) \\ \frac{\partial B_{\phi\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial B_{\phi\phi}}{\partial\phi} + \frac{1}{\rho \sin(\phi)}\frac{\partial B_{\phi\theta}}{\partial\theta} + \frac{1}{\rho}[B_{\rho\phi} + 2B_{\phi\rho} + \frac{\cos(\phi)}{\sin(\phi)}(B_{\phi\phi} - B_{\theta\theta})] \\ \frac{\partial B_{\theta\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial B_{\theta\phi}}{\partial\phi} + \frac{1}{\rho \sin(\phi)}\frac{\partial B_{\theta\theta}}{\partial\theta} + \frac{1}{\rho}[B_{\rho\theta} + 2B_{\theta\rho} + \frac{\cos(\phi)}{\sin(\phi)}(B_{\phi\theta} + B_{\theta\phi})] \end{bmatrix}. \end{aligned} \quad (\text{C.168})$$

The expression for the curl of a vector $\mathbf{a}$ is given by

$$\begin{aligned} \text{curl}(\mathbf{a}) &= \frac{1}{\rho \sin(\phi)}\left[\frac{\partial a_\theta \sin(\phi)}{\partial\phi} - \frac{\partial a_\phi}{\partial\theta}\right]\mathbf{e}_\rho \\ &+ \frac{1}{\rho}\left[\frac{1}{\sin(\phi)}\frac{\partial a_\rho}{\partial\theta} - \frac{\partial \rho a_\theta}{\partial\rho}\right]\mathbf{e}_\phi + \frac{1}{\rho}\left[\frac{\partial \rho a_\phi}{\partial\rho} - \frac{\partial a_\rho}{\partial\phi}\right]\mathbf{e}_\theta. \end{aligned} \quad (\text{C.169})$$

C.5.1 Position, velocity, and acceleration

The current position of material points is given by the equation $\mathbf{x} = \mathbf{X} + \mathbf{u}$ for

$$\mathbf{X} = \Lambda \mathbf{e}_\Lambda, \quad (\text{C.170})$$

$$\mathbf{x} = \rho \mathbf{e}_\rho, \quad (\text{C.171})$$

$$\mathbf{u} = u_\Lambda \mathbf{e}_\Lambda + u_\Phi \mathbf{e}_\Phi + u_\Theta \mathbf{e}_\Theta = u_\rho \mathbf{e}_\rho + u_\phi \mathbf{e}_\phi + u_\theta \mathbf{e}_\theta, \quad (\text{C.172})$$

where (ρ, ϕ, θ) are the current coordinates of a point that is at the coordinates (Λ, Φ, Θ) in the reference configuration. Here and in what follows we use two spherical coordinate systems, one for the reference configuration and one for the current configuration. The expressions for particle velocity and acceleration are given by

$$\mathbf{v} = \dot{u}_\Lambda \mathbf{e}_\Lambda + \dot{u}_\Phi \mathbf{e}_\Phi + \dot{u}_\Theta \mathbf{e}_\Theta, \quad (\text{C.173})$$

$$\mathbf{a} = \ddot{u}_\Lambda \mathbf{e}_\Lambda + \ddot{u}_\Phi \mathbf{e}_\Phi + \ddot{u}_\Theta \mathbf{e}_\Theta. \quad (\text{C.174})$$

$$(\text{C.175})$$

The expression for velocity in spherical coordinates also is given by

$$\mathbf{v} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\phi} \mathbf{e}_\phi + \rho \dot{\theta} \sin(\phi) \mathbf{e}_\theta, \quad (\text{C.176})$$

and the expression for acceleration is given by

$$\begin{aligned} \mathbf{a} = & [\ddot{\rho} - \rho \dot{\phi}^2 - \rho \dot{\theta}^2 \sin^2(\phi)] \mathbf{e}_\rho \\ & + [r \dot{\rho} \dot{\phi} + \rho \ddot{\phi} - \rho \dot{\theta}^2 \sin(\phi) \cos(\phi)] \mathbf{e}_\phi \\ & + [2 \dot{\rho} \dot{\theta} \sin(\phi) + 2 \rho \dot{\phi} \dot{\theta} \cos(\phi) + \rho \ddot{\theta} \sin(\phi)] \mathbf{e}_\theta, \end{aligned} \quad (\text{C.177})$$

where in this expression (ρ, ϕ, θ) represent the position of a material particle as a function of time. If we use a spatial description of velocity, where the velocity is given in terms of the spatial coordinates and time as $\mathbf{v}(\rho, \phi, \theta, t)$, the same expression for acceleration can be written as

$$a_\rho = \frac{\partial v_\rho}{\partial t} + v_\rho \frac{\partial v_\rho}{\partial \rho} + \frac{v_\phi}{\rho} \frac{\partial v_\rho}{\partial \phi} + \frac{v_\theta}{\rho \sin(\phi)} \frac{\partial v_\rho}{\partial \theta} - \frac{v_\phi^2 + v_\theta^2}{\rho}, \quad (\text{C.178})$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_\rho \frac{\partial v_\phi}{\partial \rho} + \frac{v_\phi}{\rho} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{\rho \sin(\phi)} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\rho v_\phi}{\rho} - \frac{\cos(\phi)}{\sin(\phi)} \frac{v_\theta^2}{\rho}, \quad (\text{C.179})$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_\rho \frac{\partial v_\theta}{\partial \rho} + \frac{v_\phi}{\rho} \frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta}{\rho \sin(\phi)} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\rho v_\theta}{\rho} + \frac{\cos(\phi)}{\sin(\phi)} \frac{v_\phi v_\theta}{\rho}. \quad (\text{C.180})$$

C.5.2 Deformation, velocity, and temperature gradients

The expression for the deformation gradient and its inverse are given by

$$\begin{aligned}
 \mathbf{F} &= \mathbf{e}_i^* \frac{\partial \beta_i}{\partial \alpha_j} g_{jk}^{-1} \otimes \mathbf{e}_k \\
 &= \begin{bmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \rho \sin(\phi) \mathbf{e}_\theta \end{bmatrix} \begin{bmatrix} \frac{\partial \rho}{\partial \Lambda} & \frac{\partial \rho}{\partial \Phi} & \frac{\partial \rho}{\partial \Theta} \\ \frac{\partial \phi}{\partial \Lambda} & \frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial \Theta} \\ \frac{\partial \theta}{\partial \Lambda} & \frac{\partial \theta}{\partial \Phi} & \frac{\partial \theta}{\partial \Theta} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\Lambda^2} & 0 \\ 0 & 0 & \frac{1}{\Lambda^2 \sin^2(\Phi)} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\Lambda \\ \Lambda \mathbf{e}_\Phi \\ \Lambda \sin(\Phi) \mathbf{e}_\Theta \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \rho \sin(\phi) \mathbf{e}_\theta \end{bmatrix} \begin{bmatrix} \frac{\partial \rho}{\partial \Lambda} & \frac{\partial \rho}{\partial \Phi} & \frac{\partial \rho}{\partial \Theta} \\ \frac{\partial \phi}{\partial \Lambda} & \frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial \Theta} \\ \frac{\partial \theta}{\partial \Lambda} & \frac{\partial \theta}{\partial \Phi} & \frac{\partial \theta}{\partial \Theta} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\Lambda \\ \frac{1}{\Lambda} \mathbf{e}_\Phi \\ \frac{1}{\Lambda \sin(\Phi)} \mathbf{e}_\Theta \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \end{bmatrix} \begin{bmatrix} \frac{\partial \rho}{\partial \Lambda} & \frac{1}{\Lambda} \frac{\partial \rho}{\partial \Phi} & \frac{1}{\Lambda \sin(\Phi)} \frac{\partial \rho}{\partial \Theta} \\ \rho \frac{\partial \phi}{\partial \Lambda} & \frac{\rho}{\Lambda} \frac{\partial \phi}{\partial \Phi} & \frac{\rho}{\Lambda \sin(\Phi)} \frac{\partial \phi}{\partial \Theta} \\ \rho \sin(\phi) \frac{\partial \theta}{\partial \Lambda} & \frac{\rho \sin(\phi)}{\Lambda} \frac{\partial \theta}{\partial \Phi} & \frac{\rho \sin(\phi)}{\Lambda \sin(\Phi)} \frac{\partial \theta}{\partial \Theta} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\Lambda \\ \mathbf{e}_\Phi \\ \mathbf{e}_\Theta \end{bmatrix}, \quad (\text{C.181})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{F}^{-1} &= \mathbf{e}_i \frac{\partial \alpha_i}{\partial \beta_j} g_{jk}^{*-1} \otimes \mathbf{e}_k^* \\
 &= \begin{bmatrix} \mathbf{e}_\Lambda & \mathbf{e}_\Phi & \mathbf{e}_\Theta \end{bmatrix} \begin{bmatrix} \frac{\partial \Lambda}{\partial \rho} & \frac{1}{\rho} \frac{\partial \Lambda}{\partial \phi} & \frac{1}{\rho \sin(\phi)} \frac{\partial \Lambda}{\partial \theta} \\ \Lambda \frac{\partial \Phi}{\partial \rho} & \frac{\Lambda}{\rho} \frac{\partial \Phi}{\partial \phi} & \frac{\Lambda}{\rho \sin(\phi)} \frac{\partial \Phi}{\partial \theta} \\ \Lambda \sin(\Phi) \frac{\partial \Theta}{\partial \rho} & \frac{\Lambda \sin(\Phi)}{\rho} \frac{\partial \Theta}{\partial \phi} & \frac{\Lambda \sin(\Phi)}{\rho \sin(\phi)} \frac{\partial \Theta}{\partial \theta} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}. \quad (\text{C.182})
 \end{aligned}$$

The expressions for deformation gradient rate and velocity gradient are given by

$$\begin{aligned}
 \dot{\mathbf{F}} &= \text{Grad}(\mathbf{v}) = \begin{bmatrix} \mathbf{e}_\Lambda & \mathbf{e}_\Phi & \mathbf{e}_\Theta \end{bmatrix} \\
 &\quad \begin{bmatrix} \frac{\partial v_\Lambda}{\partial \Lambda} & \frac{1}{\Lambda} \frac{\partial v_\Lambda}{\partial \Phi} - \frac{1}{\Lambda} v_\Phi & \frac{1}{\Lambda \sin(\Phi)} \frac{\partial v_\Lambda}{\partial \Theta} - \frac{1}{\Lambda} v_\Theta \\ \frac{\partial v_\Phi}{\partial \Lambda} & \frac{1}{\Lambda} \frac{\partial v_\Phi}{\partial \Phi} + \frac{1}{\Lambda} v_\Lambda & \frac{1}{\Lambda \sin(\Phi)} \frac{\partial v_\Phi}{\partial \Theta} - \frac{\cos(\Phi)}{\Lambda \sin(\Phi)} v_\Theta \\ \frac{\partial v_\Theta}{\partial \Lambda} & \frac{1}{\Lambda} \frac{\partial v_\Theta}{\partial \Phi} & \frac{1}{\Lambda \sin(\Phi)} \frac{\partial v_\Theta}{\partial \Theta} + \frac{1}{\Lambda} (v_\Lambda + \frac{\cos(\Phi)}{\sin(\Phi)} v_\Phi) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\Lambda \\ \mathbf{e}_\Phi \\ \mathbf{e}_\Theta \end{bmatrix}, \quad (\text{C.183})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{L} &= \text{grad}(\mathbf{v}) = \begin{bmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \end{bmatrix} \\
 &\quad \begin{bmatrix} \frac{\partial v_\rho}{\partial \rho} & \frac{1}{\rho} \frac{\partial v_\rho}{\partial \phi} - \frac{1}{\rho} v_\phi & \frac{1}{\rho \sin(\phi)} \frac{\partial v_\rho}{\partial \theta} - \frac{1}{\rho} v_\theta \\ \frac{\partial v_\phi}{\partial \rho} & \frac{1}{\rho} \frac{\partial v_\phi}{\partial \phi} + \frac{1}{\rho} v_\rho & \frac{1}{\rho \sin(\phi)} \frac{\partial v_\phi}{\partial \theta} - \frac{\cos(\phi)}{\rho \sin(\phi)} v_\theta \\ \frac{\partial v_\theta}{\partial \rho} & \frac{1}{\rho} \frac{\partial v_\theta}{\partial \phi} & \frac{1}{\rho \sin(\phi)} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{\rho} (v_\rho + \frac{\cos(\phi)}{\sin(\phi)} v_\phi) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}. \quad (\text{C.184})
 \end{aligned}$$

Since there is a conflict of notation between the θ used for temperature and the θ used for the second coordinate axis, we will use θ^* in this section to denote temperature. The expression for temperature gradient for cylindrical coordinates is given by

$$\mathbf{g} = \text{grad}(\theta^*) = \nabla_{\mathbf{x}}(\theta^*) = \frac{\partial \theta^*}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial \theta^*}{\partial \phi} \mathbf{e}_\phi + \frac{1}{\rho \sin(\phi)} \frac{\partial \theta^*}{\partial \theta} \mathbf{e}_\theta, \quad (\text{C.185})$$

so that the components of $\mathbf{g}$ are given by

$$g_\rho = \frac{\partial \theta^*}{\partial \rho}, \quad g_\phi = \frac{1}{\rho} \frac{\partial \theta^*}{\partial \phi}, \quad g_\theta = \frac{1}{\rho \sin(\phi)} \frac{\partial \theta^*}{\partial \theta}. \quad (\text{C.186})$$

The nominal temperature gradient is given by

$$\mathbf{G} = \text{Grad}(\theta^*) = \nabla_{\mathbf{X}}(\theta^*) = \frac{\partial \theta^*}{\partial \Lambda} \mathbf{e}_\Lambda + \frac{1}{\Lambda} \frac{\partial \theta^*}{\partial \Phi} \mathbf{e}_\Phi + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial \theta^*}{\partial \Theta} \mathbf{e}_\Theta, \quad (\text{C.187})$$

so that the components of $\mathbf{G}$ are given by

$$G_{\Lambda} = \frac{\partial \Theta^*}{\partial \Lambda}, \quad G_{\Phi} = \frac{1}{\Lambda} \frac{\partial \Theta^*}{\partial \Phi}, \quad G_{\Theta} = \frac{1}{\Lambda \sin(\Phi)} \frac{\partial \Theta^*}{\partial \Theta}. \quad (\text{C.188})$$

C.5.3 Laplacian

The expression for the Laplacian of a scalar field ψ described in the reference configuration coordinates (Λ, Φ, Θ) is given by

$$\nabla_{\mathbf{x}}^2(\psi) = \frac{1}{\Lambda^2} \frac{\partial}{\partial \Lambda} \left(\Lambda^2 \frac{\partial \psi}{\partial \Lambda} \right) + \frac{1}{\Lambda^2 \sin(\Phi)} \frac{\partial}{\partial \Phi} \left[\sin(\Phi) \frac{\partial \psi}{\partial \Phi} \right] + \frac{1}{\Lambda^2 \sin^2(\Phi)} \frac{\partial^2 \psi}{\partial \Theta^2}. \quad (\text{C.189})$$

The Laplacian of the same function written in terms of the current configuration coordinates (r, θ, z) are given by

$$\nabla_{\mathbf{x}}^2(\psi) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left[\sin(\phi) \frac{\partial \psi}{\partial \phi} \right] + \frac{1}{\rho^2 \sin^2(\phi)} \frac{\partial^2 \psi}{\partial \theta^2}. \quad (\text{C.190})$$

C.5.4 Balance laws in current configuration

The balance of linear momentum in spherical coordinates in the current configuration is given as the three component equations

$$\begin{aligned} & \frac{\partial T_{\rho\rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\phi\rho}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial T_{\theta\rho}}{\partial \theta} \\ & + \frac{1}{\rho} \left[2T_{\rho\rho} - T_{\phi\phi} - T_{\theta\theta} + T_{\phi\rho} \frac{\cos(\phi)}{\sin(\phi)} \right] + \rho^* b_{\rho} = \rho^* a_{\rho}, \end{aligned} \quad (\text{C.191})$$

$$\begin{aligned} & \frac{\partial T_{\rho\phi}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial T_{\theta\phi}}{\partial \theta} \\ & + \frac{1}{\rho} \left[3T_{\rho\phi} + (T_{\phi\phi} - T_{\theta\theta}) \frac{\cos(\phi)}{\sin(\phi)} \right] + \rho^* b_{\phi} = \rho^* a_{\phi}, \end{aligned} \quad (\text{C.192})$$

$$\begin{aligned} & \frac{\partial T_{\rho\theta}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial T_{\theta\theta}}{\partial \theta} \\ & + \frac{1}{\rho} \left[3T_{\rho\theta} + 2T_{\theta\phi} \frac{\cos(\phi)}{\sin(\phi)} \right] + \rho^* b_{\theta} = \rho^* a_{\theta}, \end{aligned} \quad (\text{C.193})$$

where because of a conflict in notation we use ρ^* to denote the current density. In the expression for balance of work and energy and the entropy production inequality, we have

$$\begin{aligned} \text{tr}(\mathbf{T}\mathbf{L}) &= T_{\rho\rho}L_{\rho\rho} + T_{\rho\phi}L_{\phi\rho} + T_{\rho\theta}L_{\theta\rho} \\ &+ T_{\phi\rho}L_{\rho\phi} + T_{\phi\phi}L_{\phi\phi} + T_{\phi\theta}L_{\theta\phi} \\ &+ T_{\theta\rho}L_{\rho\theta} + T_{\theta\phi}L_{\phi\theta} + T_{\theta\theta}L_{\theta\theta}. \end{aligned} \quad (\text{C.194})$$

In the expression of balance of work and energy, we have

$$\text{div}(\mathbf{q}) = \frac{1}{\rho^2} \frac{\partial \rho^2 q_{\rho}}{\partial \rho} + \frac{1}{\rho \sin(\phi)} \frac{\partial q_{\phi} \sin(\phi)}{\partial \phi} + \frac{1}{\rho \sin(\phi)} \frac{\partial q_{\theta}}{\partial \theta}. \quad (\text{C.195})$$

In the expression for the entropy production inequality, we have

$$\mathbf{q} \circ \mathbf{g} = q_{\rho} \frac{\partial \theta^*}{\partial \rho} + \frac{1}{\rho} q_{\phi} \frac{\partial \theta^*}{\partial \phi} + \frac{1}{\rho \sin(\phi)} q_{\theta} \frac{\partial \theta^*}{\partial \theta}, \quad (\text{C.196})$$

where θ^* is used for denoting temperature.

C.5.5 Balance laws in reference configuration

The balance of linear momentum in spherical coordinates in the reference configuration is given as the three component equations

$$\begin{aligned} & \frac{\partial T_{o-\Lambda\Lambda}}{\partial \Lambda} + \frac{1}{\Lambda} \frac{\partial T_{o-\Phi\Lambda}}{\partial \Phi} + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial T_{o-\Theta\Lambda}}{\partial \Theta} \\ & + \frac{1}{\Lambda} \left[2T_{o-\Lambda\Lambda} - T_{o-\Phi\Phi} - T_{o-\Theta\Theta} + T_{o-\Phi\Lambda} \frac{\cos(\Phi)}{\sin(\Phi)} \right] + \rho_o b_\Lambda = \rho_o a_\Lambda, \end{aligned} \quad (C.197)$$

$$\begin{aligned} & \frac{\partial T_{o-\Lambda\Phi}}{\partial \Lambda} + \frac{1}{\Lambda} \frac{\partial T_{o-\Phi\Phi}}{\partial \Phi} + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial T_{o-\Theta\Phi}}{\partial \Theta} \\ & + \frac{1}{\Lambda} \left[2T_{o-\Lambda\Phi} + T_{o-\Phi\Lambda} + (T_{o-\Phi\Phi} - T_{o-\Theta\Theta}) \frac{\cos(\Phi)}{\sin(\Phi)} \right] + \rho_o b_\Phi = \rho_o a_\Phi, \end{aligned} \quad (C.198)$$

$$\begin{aligned} & \frac{\partial T_{o-\Lambda\Theta}}{\partial \Lambda} + \frac{1}{\Lambda} \frac{\partial T_{o-\Phi\Theta}}{\partial \Phi} + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial T_{o-\Theta\Theta}}{\partial \Theta} \\ & + \frac{1}{\Lambda} \left[2T_{o-\Lambda\Theta} + T_{o-\Theta\Lambda} + (T_{o-\Theta\Phi} + T_{o-\Phi\Theta}) \frac{\cos(\Phi)}{\sin(\Phi)} \right] + \rho_o b_\Theta = \rho_o a_\Theta. \end{aligned} \quad (C.199)$$

In the expression for balance of work and energy and the entropy production inequality, we have

$$\begin{aligned} tr(\mathbf{T}_o \dot{\mathbf{F}}) &= T_{o-\Lambda\Lambda} \dot{F}_{\Lambda\Lambda} + T_{o-\Lambda\Phi} \dot{F}_{\Phi\Lambda} + T_{o-\Lambda\Theta} \dot{F}_{\Theta\Lambda} \\ &+ T_{o-\Phi\Lambda} \dot{F}_{\Lambda\Phi} + T_{o-\Phi\Phi} \dot{F}_{\Phi\Phi} + T_{o-\Phi\Theta} \dot{F}_{\Theta\Phi} \\ &+ T_{o-\Theta\Lambda} \dot{F}_{\Lambda\Theta} + T_{o-\Theta\Phi} \dot{F}_{\Phi\Theta} + T_{o-\Theta\Theta} \dot{F}_{\Theta\Theta}. \end{aligned} \quad (C.200)$$

In the expression of balance of work and energy, we have

$$Div(\mathbf{q}_o) = \frac{1}{\Lambda^2} \frac{\partial \Lambda^2 q_\Lambda}{\partial \Lambda} + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial q_\Phi \sin(\Phi)}{\partial \Phi} + \frac{1}{\Lambda \sin(\Phi)} \frac{\partial q_\Theta}{\partial \Theta}. \quad (C.201)$$

In the expression for the entropy production inequality, we have

$$\mathbf{q}_o \circ \mathbf{G} = q_\Lambda \frac{\partial \Theta^*}{\partial \Lambda} + \frac{1}{\Lambda} q_\Phi \frac{\partial \Theta^*}{\partial \Phi} + \frac{1}{\Lambda \sin(\Phi)} q_\Theta \frac{\partial \Theta^*}{\partial \Theta}, \quad (C.202)$$

where θ^* is used for denoting temperature.

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The Mechanical and Thermodynamical Theory of Plasticity

Born out of fifteen years of courses and lectures on continuum mechanics, nonlinear mechanics, continuum thermodynamics, viscoelasticity, plasticity, crystal plasticity and thermodynamic plasticity, ***The Mechanical and Thermodynamical Theory of Plasticity*** represents one of the most extensive and in-depth treatises on the mechanical and thermodynamical aspects of plastic and viscoplastic flow. Suitable for student readers and experts alike, it offers a clear and comprehensive presentation of multi-dimensional continuum thermodynamics to both aid in initial understanding and introduce and explore advanced topics.

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