IMPROPER RIEMANN INTEGRALS

$$\lim_{\square}\int_{\square}^{\square}\square \overset{?}{=} \int_{\square}^{\square}\lim_{\square}\square$$

Ioannis M. Roussos

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Ioannis M. Roussos
Hamline University
Saint Paul, Minnesota, USA

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Dedication

To:
Ma.-Le. St.-St.

Στὴν:
Μα. − Λέ. Στ. − Στ.

Contents

Acknowledgements

I would like to express my deep thanks to my colleague Professor of Mathematics, Dr. Arthur Guetter. Without him this work would not have been motivated, started and finished. He has also been my LaTeX teacher for the technical typing of this material. Secondly, I would like to thank my student Nathan Davis whose fluent expertise in LaTeX and figure making have been indispensable and very valuable in this work. He has spent a lot of time for this cause and I feel indebted to him for all his work and help. I also wish to thank some of my friends for their moral support and encouragement.

Prologue

This book is written at the masters level to help students of mathematics, statistics, applied sciences and engineering. Its scope is the improper or generalized Riemann integral, its convergence, principal value, evaluation and its application to science and engineering. Questions, problems and applications involving various improper integrals and series of numbers often emerge in these subjects. At the undergraduate level, results concerning useful improper integrals are mostly taken for granted, provided by an authority or obtained through tables and computer programs or packages. Here we try to give students sufficient knowledge and tools to enable them to answer these questions by themselves and acquire a deeper understanding of this matter and/or prepare them to do so with some further study of the matter.

We try to achieve these goals by explaining the concepts involved, presenting sufficient theory and using a number of theorems, some with their proofs and others without. A complete, general and advanced exposition of this vast area of mathematics would contain a much greater number of theorems and proofs and involve advanced mathematical theories of real and complex analysis, integral transforms, special functions, etc., that lie far beyond the undergraduate and/or masters curriculum.

Whenever possible, we present the material in a self-contained manner. We have proved many results but not all. Sometimes our proofs are not established under the most general conditions that the more advanced theories can provide, but under conditions accessible to the undergraduate and sufficient for application. We also state and use a few advanced general theorems, results and tools from real and complex analysis without proofs. Their complete presentations and rigorous proofs would require taking the graduate level courses on these subjects. Here their statements are adjusted to a level that students can understand and are interpreted in a way so that the students can handle, manipulate and use them efficiently as powerful tools in our list of problems. In this way we avoid stating and proving a great number of criteria and partial results and thus avoid forcing the students into too much searching (a lot of times done by trial and error) for finding out the case they deal with each time and what criterion to apply. Thus, we try to render these advanced mathematical results and tools accessible and useful even to the undergraduate students with sufficient background so that they can use them in fairly straightforward manners in many pertinent problems they may come across in the subjects aforementioned. Moreover, our presentation and use of these advanced and general theorems and results give the undergraduate student a taste of the power of the graduate level mathematics and motivate the interested one to take these courses at the graduate level in due time. We also expose a great number of detailed examples in order to illustrate the concepts and practice a lot with the tools that check convergence of improper integrals and evaluate their exact value when this is possible.

We include many exercises and problems in every section. These are carefully chosen to serve both as practice and for further application. They are representative enough so that the student, on the basis of these, can solve many other exercises and problems not included in this book and also use them in many situations of application. We try to keep the number of exercises at a level so that on the one hand the student does not get lost in a vast sea of exercises and on the other hand the opportunity to practice and learn the material well and apply it is not compromised. A few problems that are lengthy, have several questions and may be hard could be assigned as projects to an individual student or a group of students. Also, the input and help from the teacher or pertinent bibliography may be significant.

Many examples are presented several times in different ways in order to see them from various points of view and see how different methods can give correct answers to the same questions. That is, their solutions are achieved in various ways depending on the context. We also repeat a few problems from section to section and we seek their solutions within the new context. In this way we try to show the students the interconnection of the whole matter, how a given question may be viewed in many ways and within various contexts, and that there are many ways to achieve a correct answer. This is something generally lacking in the undergraduate mathematics education.

This book includes many theorems and methods for checking the convergence and the computation of most improper integrals encountered in applications. The content is sufficient to provide answers to the majority of them. We briefly examine the Laplace transform, Mellin transform and Fourier transform. Except for a few results, we do not develop the theory behind these integral transforms, but we mostly concentrate on their evaluation and some applications. We have omitted other integral transforms, such as the Hankel transform, etc. At this level, we did not include many special and hard integrals such as improper integrals in several variables, elliptic and hyper-elliptic integrals, Dirichlet integrals,

integrals involving special functions such as Bessel functions, hypergeometric functions, asymptotic expansions, methods of steepest descents, etc., and some very special cases of contour integration (Cauchy, Legendre, Mellin, etc.). However, a lot of concrete cases out of these special integrals can still be resolved by making appropriate use of the tools provided here. Also, in advanced mathematics we encounter the singular integrals, which is a whole subject in itself, very important in mathematics and application.

We must say that one will encounter several not fully explained points, indicated in the text by expressions like "justify this," etc. All of them, however, can be justified by the versatile, studious and knowledgeable master-level student. An undergraduate could also clarify all of them with the help of the teacher. The proofs of theorems and results omitted in the text can be found in real analysis, complex analysis and applied mathematics literature.

In conclusion, the useful and practical material of this book is accessible to and can be mastered by any student who has finished a calculus sequence and has taken some multi-variable calculus, basic ordinary differential equations, basic mathematical analysis, complex numbers and the basics of complex variables. Knowing this material, a student may not rely on authority, tables or computer packages to give and understand answers to questions related to this important material in theory and application. On account of all these and its whole content, this book can be used as the text for an undergraduate course or a supplementary text to other courses of mathematics, statistics, applied sciences and engineering. It can also become a very helpful manual and reference to students at the master level and even beyond.

At the senior undergraduate level, this material can be used for a capstone course of a program and also serve for a good review of calculus and basic mathematical and complex analysis. At the starting graduate level, we find many illustrations of several strong tools of real and complex analysis with numerous examples and problems, a good many of which are quite involved. We use these tools, results and theorems, not just in computing examples and solving problems, but also in justifying that our methods of various computations are legitimate.

A student who knows advanced calculus and has learnt the material and problems of this book must be able to verify at least all the integrals numbered 582 – 709 that appear in the pages 448 – 455 of the *CRC Standard Mathematical Tables and Formulae*, by Daniel Zwillinger, 31st Edition, Chapman & Hall / CRC, 2003. Have a look at and practice with them after you have finished studying this book.

At the end of this book, we have collected in a list all the major integrals evaluated one way or another in the text and the major finite and infinite sums in a different list. We did not go into computing and collecting infinite products. That would have been another chapter in the book. As expected we have included a sufficient bibliography, but far from all the bibliography that circulates in the world on these subjects. For the convenience of students and readers, an index of terms and names is also included.

Finally we thank all the people who study and use this book and we kindly ask them, if they encounter a typo or error that has escaped our attention, to communicate it to author for correcting it in a prospective next edition. Also, suggesting new interesting and pertinent problems is highly appreciated.

Dr. Ioannis M. Roussos, Professor of Mathematics
Hamline University, Saint Paul, Minnesota, 2013

Chapter 1

Improper Riemann Integrals

1.1 Definitions and Examples

Many theorems in Mathematics and many applications in science and technology depend on the evaluation and on the properties of improper Riemann integrals. Therefore, we are going to state the definitions of improper or generalized integrals and then discuss their properties. Subsequently, we discuss criteria for checking their existence (or non-existence) and then we develop methods and mathematical techniques we can use in order to evaluate them. Certainly the answers to many important improper Riemann integrals have been tabulated in mathematical handbooks and can also be found with the help of various computer programs, which we can use if we can trust in them, of course. However, these means can never exhaust every interesting case. Hence, the good knowledge of the mathematical theory of how to understand, handle and compute improper integrals, at a higher level, will always remain very important for being able to deal with new cases and checking the accuracy of the answers provided in tables or found by computer programs or packages.

In a regular undergraduate Calculus course we study the Fundamental Theorem of Integral Calculus. This states:

Theorem 1.1.1 (Fundamental Theorem of Integral Calculus) *If a real function* $f : [a, b] \to \mathbb{R}$ *($a < b$ are real numbers) is continuous, then it possesses antiderivatives* $F(x)$*, i.e., functions that satisfy* $F'(x) = f(x)$ *for every* $x \in [a, b]$ *[at the end points we consider the appropriate side derivatives,* $F'_+(a) = f(a)$ *and* $F'_-(b) = f(b)$*]. Any such antiderivative* $F(x)$ *of* $f(x)$ *is necessarily continuous in* (a, b)*, right continuous at* a*, [i.e.,* $F(a) = \lim_{x \to a^+} F(x)$ *], left continuous at* b*, [i.e.,* $F(b) = \lim_{x \to b^-} F(x)$ *], and satisfies*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We emphasize the three hypotheses that must hold in order for this Theorem to be valid:

1. $[a,b]$ is a closed and bounded interval in the real line.
2. $f(x)$ is continuous and therefore, by the extreme value theorem, bounded on $[a,b]$.
3. In the computation $F(b)-F(a)$, $F(x)$ can be any fixed continuous antiderivative.

We know that on $[a,b]$, there are infinitely many antiderivatives of a continuous function $f(x)$, but any two of them differ by a real constant C. Since they are differentiable they are continuous, and since their derivative is the continuous function $f(x)$ they are continuously differentiable.

Under these conditions the integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is called the **proper Riemann**[1] **integral** of $f(x)$ over the interval $[a,b]$. This is well defined and equal to the limit of the **Riemann sums** of $f(x)$ over $[a,b]$, as the maximum length of the subintervals into which we subdivide $[a,b]$, in this well-known process, approaches zero. I.e.,

$$\int_a^b f(x)\,dx = \lim_{\max(\Delta x_k)\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = F(b) - F(a).$$

We can go a bit beyond the undergraduate interpretation of the Fundamental Theorem of Integral Calculus and relax the above hypotheses as follows:

We more generally consider $f : [a,b] \to \mathbb{R}$ **piecewise continuous** and **bounded**. Then its Riemann integral exists. In such a case, we can also find $F(x)$ antiderivative of $f(x)$ which is continuous in (a,b), right continuous at a, left continuous at b and differentiable only at the points of continuity of $f(x)$. At the points of discontinuity of $f(x)$, $F(x)$ may have a left or right derivative but not derivative.

Sometimes $f : [a,b] \to \mathbb{R}$ may be continuous in (a,b), right continuous at a, left continuous at b, but in order to obtain, by means of the usual methods and rules of antidifferentiation, an antiderivative $F(x)$ of $f(x)$ which is continuous in (a,b), right continuous at a, left continuous at b, we may have to make necessary adjustments, by adjusting certain

---

[1] Georg Friedrich Bernhard Riemann, German mathematician, 1826-1866.

constants, at certain points of the interval of integration $[a, b]$. Only then we can guarantee the **result**

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

in such cases.

In fact, the Fundamental Theorem of Calculus proves that if $f(x)$ is a bounded Riemann integrable function on the closed and bounded interval $[a, b]$, then

$$F(x) = \int_a^x f(t)dt \text{ is continuous in } [a, b]$$

and satisfies

$$\frac{d}{dx}F(x)|_{x=w} = f(w) \text{ at all points of continuity } w \text{ of } f(x) \text{ on } [a, b].$$

So, if $f(x)$ is continuous on $[a, b]$, the function $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$ and is continuously differentiable.

The anomaly we discuss here is not due to any deficiency of the Fundamental Theorem of Calculus, but it is created by the standard rules and methods of antidifferentiation. At times, the answers obtained by these rules are not defined at certain points and therefore are discontinuous at these points. To obtain the continuity as the Fundamental Theorem of Calculus guarantees and requires, we must adjust these answers appropriately. To understand this extraordinary situation and be aware of its occurrence, let us study the following example (and see also **Problem 1.1.3**):

Example 1.1.1 We consider the function

$$f(x) = \frac{3}{5 - 4\cos(x)}.$$

This function is defined for every $x \in \mathbb{R}$. It is continuous at every $x \in \mathbb{R}$, bounded $\left[\frac{1}{3} \leq f(x) \leq 3\right]$, periodic with period 2π and even.

When we integrate rational functions of sine and cosine, we usually use the half angle substitution $u = \tan\left(\frac{x}{2}\right)$. (See also **Remark 2 of Example 3.7.13**.) Then we find (work it out)

$$\int \frac{3}{5 - 4\cos(x)}\,dx = 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right] + C.$$

We let $C = 0$ (as we usually do in calculus when we evaluate definite integrals). So, we choose

$$F(x) = 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right].$$

This function is defined for all real $x \neq (2k+1)\pi$, with $k \in \mathbb{Z}$, since at $x = (2k+1)\pi$, with $k \in \mathbb{Z}$, $\tan\left(\frac{x}{2}\right)$ is not defined. At these exceptional points, however, we have

$$\lim_{x\to(2k+1)\pi^-} F(x) = 2\cdot\frac{\pi}{2} = \pi \quad\text{and}\quad \lim_{x\to(2k+1)\pi^+} F(x) = 2\cdot\frac{-\pi}{2} = -\pi.$$

So, at each $x = (2k+1)\pi$, with $k \in \mathbb{Z}$, $F(x)$ has a jump discontinuity with jump $|\pi - (-\pi)| = 2\pi$. Notice also that $F(x)$ is bounded, $[-\pi < F(x) < \pi]$.

Therefore, to evaluate the definite integrals

$$\int_a^b \frac{3}{5 - 4\cos(x)}\,dx = F(b) - F(a), \quad \text{for any} \quad -\pi \leq a,\ b \leq \pi,$$

we can use the continuous antiderivative

$$\bar{F}(x) = \begin{cases} -\pi, & \text{if} \quad x = -\pi^+ \\ 2\arctan\left[3\tan\left(\frac{x}{2}\right)\right], & \text{if} \quad -\pi < x < \pi \\ \pi, & \text{if} \quad x = \pi^-. \end{cases}$$

For example,

$$\int_{-\pi}^{\pi} \frac{3}{5 - 4\cos(x)}\,dx = \bar{F}(\pi^-) - \bar{F}(-\pi^+) = \pi - (-\pi) = 2\pi.$$

(See also **Example 3.4.4**.)

But, we cannot use $F(x)$ or $\bar{F}(x)$ to evaluate definite integrals if a or b does not satisfy $-\pi \leq a,\ b \leq \pi$. For instance, if we use it with $a = -2\pi$ and $b = 4\pi$, we find

$$\int_{-2\pi}^{4\pi} \frac{3}{5 - 4\cos(x)}\,dx = F(4\pi) - F(-2\pi) = 0 - 0 = 0,$$

which is incorrect, since the continuous function $f(x) > 0$, and therefore this definite integral should be > 0.

This error has occurred because the chosen antiderivative over the interval $[-2\pi, 4\pi]$ is discontinuous at the exceptional points examined above.

The correct answer is obtained if we use the adjusted antiderivative

$$F_c(x) = \begin{cases} 2\arctan\left[3\tan\left(\dfrac{x}{2}\right)\right] + 2\pi \left[\!\!\left[\dfrac{x+\pi}{2\pi}\right]\!\!\right], & \text{if } x \neq (2k+1)\pi,\ k \in \mathbb{Z} \\ (2k+1)\pi, & \text{if } x = (2k+1)\pi,\ k \in \mathbb{Z}, \end{cases}$$

where $\left[\!\!\left[\dfrac{x+\pi}{2\pi}\right]\!\!\right]$ is the **integer part** or **floor function** of $\dfrac{x+\pi}{2\pi}$.

This new $F_c(x)$ is now continuous, differentiable and $F_c'(x) = f(x)$ at all points of $[-2\pi, 4\pi]$ as the Fundamental Theorem of Calculus claims and requires. (In fact, this is true at every $x \in \mathbb{R}$. See **Problems 1.1.1** and **1.1.2**.)

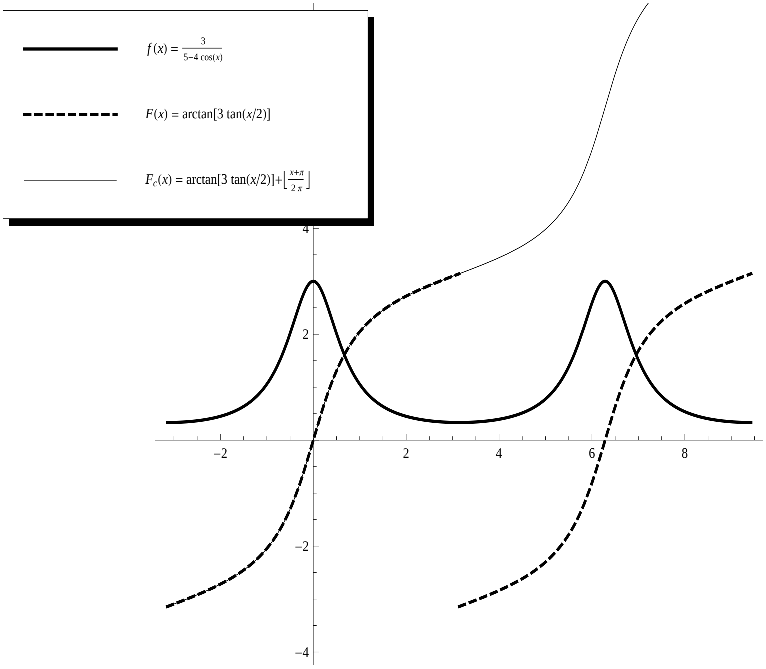

FIGURE 1.1: The three functions in Example 1.1.1

Now, we get:

$$\int_{-2\pi}^{4\pi} \frac{3}{5-4\cos(x)}\,dx = F_c(4\pi) - F_c(-2\pi) = 4\pi - (-2\pi) = 6\pi > 0.$$

This result is the correct one and was also expected since $f(x)$ is $2\pi-$periodic, with integral 2π over $[-\pi, \pi]$ and we have integrated it over an interval of length 6π, i.e., three times its period.

(About periodic functions, in general, see **Problem 1.1.6**.)

▲

Now we continue with the improper or generalized Riemann integrals.

Definition 1.1.1 *An integral of a piecewise continuous real function of a real variable is called an **improper** or **generalized Riemann integral** if at least one of the following three conditions occurs:*

1. *The integrated function is unbounded over the interval of integration.*
2. *The interval of integration is not closed.*
3. *The interval of integration is unbounded.*

In all the pertinent definitions that follow, an improper or generalized Riemann integral of a real piecewise continuous function of a real variable defined over a set $I \subseteq \mathbb{R}$ will be defined to be a certain limit of proper Riemann integrals.

More concretely, we present four cases and definitions in our exposition, each of which may include two or more subcases, that generalize the proper Riemann integrals:

Definition 1.1.2 *Suppose $y = f(x)$ is a real function continuous in $[a, b) \subset \mathbb{R}$, then we define:*

For $b < \infty$

$$\int_a^b f(x)\,dx = \lim_{\rho \to b^-} \int_a^\rho f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx.$$

For $b = \infty$

$$\int_a^\infty f(x)\,dx = \lim_{M \to \infty} \int_a^M f(x)\,dx.$$

Examples

Example 1.1.2

$$\int_{-1}^0 \frac{dx}{\sqrt[3]{x}} = \lim_{\rho \to 0^-} \int_{-1}^\rho x^{\frac{-1}{3}}\,dx = \lim_{\rho \to 0^-} \left[\frac{3}{2} x^{\frac{2}{3}}\right]_{-1}^\rho =$$

$$\lim_{\rho \to 0^-} \left[\frac{3}{2}\rho^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}}\right] = \frac{3}{2} \lim_{\rho \to 0^-} \rho^{\frac{2}{3}} - \frac{3}{2} = \frac{3}{2} \cdot 0 - \frac{3}{2} = -\frac{3}{2}.$$

▲

Example 1.1.3

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{M\to\infty}[\arctan(x)]_1^M =$$

$$\lim_{M\to\infty}[\arctan(M)-\arctan(1)] = \lim_{M\to\infty}\arctan(M)-\frac{\pi}{4} = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}.$$

[π.[2]]

▲

Example 1.1.4

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{M\to\infty}[2\sqrt{x}]_1^M = \lim_{M\to\infty} 2\sqrt{M}-2\sqrt{1} = \infty - 2 = \infty.$$

▲

Example 1.1.5

$$\int_0^\infty \sin(x)\,dx = \lim_{M\to\infty}[-\cos(x)]_0^M =$$

$$\lim_{M\to\infty}[-\cos(M)+\cos(0)] = -\lim_{M\to\infty}\cos(M)+1 = \text{does not exist.}$$

This limit does not exist because $\cos(M)$ oscillates between -1 and 1.

▲

Definition 1.1.3 *Suppose $y = f(x)$ is a real continuous function in $(a,b] \subset \mathbb{R}$. Then we define:*

For $-\infty < a$

$$\int_a^b f(x)\,dx = \lim_{\sigma\to a^+}\int_\sigma^b f(x)\,dx = \lim_{\delta\to 0^+}\int_{a+\delta}^b f(x)\,dx.$$

For $a = -\infty$

$$\int_{-\infty}^b f(x)\,dx = \lim_{N\to-\infty}\int_N^b f(x)\,dx.$$

---

[2] Pi, π, ϖ, Π the sixteenth letter of the Greek alphabet. Symbol of the transcendental number equal to the fixed ratio of the circumference of any circle divided by its diameter. Introduced by Archimedes of Syracuse (in Sicily today in Italy), Greek mathematician, 287-212 B.C.E. It is one of the most important numbers in mathematics, science, technology and applications. $\pi \cong 3.1415926536....$ Archimedes is considered by the great majority of mathematicians to be the greatest mathematician of all times. He is the first who conceived the process of integration by which he found the area of the circle of radius r to be $A = \pi r^2$, the length of its circumference $c = 2\pi r$ and also $\int_{-1}^1 x^2 dx = \frac{2}{3}$. His method was a limiting process of sums of areas or lengths that could be computed elementarily, similar to the Riemann or Riemann-Darboux or Darboux sums of the nineteenth century.

(Jean Gaston Darboux, French mathematician, 1842-1917.)

Examples

Example 1.1.6

$$\int_0^2 \frac{dx}{\sqrt{x}} = \lim_{\sigma \to 0^+} [2\sqrt{x}]_\sigma^2 = 2\sqrt{2} - \lim_{\sigma \to 0^+} 2\sqrt{\sigma} = 2\sqrt{2} - 0 = 2\sqrt{2}.$$

▲

Example 1.1.7

$$\int_{-\infty}^0 e^x \, dx = \lim_{N \to -\infty} [e^x]_N^0 =$$

$$\lim_{N \to -\infty} \left[e^0 - e^N\right] = 1 - \lim_{N \to -\infty} e^N = 1 - 0 = 1.$$

[e.[3]]

▲

Example 1.1.8

$$\int_{-\infty}^1 x^2 \, dx = \lim_{N \to -\infty} \left[\frac{x^3}{3}\right]_N^1 = \lim_{N \to -\infty} \left[\frac{1}{3} - \frac{N^3}{3}\right] =$$

$$\frac{1}{3} - \lim_{N \to -\infty} \frac{N^3}{3} = \frac{1}{3} - (-\infty) = \infty.$$

▲

Definition 1.1.4 *Suppose $y = f(x)$ is a real continuous function in $(a, b) \subseteq \mathbb{R}$. Then we define:*

For $-\infty < a < b < \infty$

$$\int_a^b f(x) \, dx = \lim_{\substack{\rho \to b^- \\ \sigma \to a^+}} \int_\sigma^\rho f(x) \, dx.$$

For $a = -\infty$ and $b = \infty$

$$\int_{-\infty}^{\infty} f(x) \, dx = \lim_{\substack{M \to \infty \\ N \to -\infty}} \int_N^M f(x) \, dx.$$

---

[3] e is the symbol of the transcendental number $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}$. It is one of the most important numbers in mathematics, science, technology and applications. It is the base of the natural logarithms. It was somehow known to the Scottish mathematician John Napier, 1550-1617, but the Swiss mathematicians Jakob (Jacques) Bernoulli, 1654-1705, and Leonhard Euler, 1707-1783, were the ones who recognized its highest significance to mathematics and applications. Its approximate value is: $e \cong 2.718281828459045 \cdots$

For $a = -\infty$ and $b < \infty$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{\substack{\rho\to b^-\\ N\to-\infty}} \int_N^{\rho} f(x)\,dx.$$

For $-\infty < a$ and $b = \infty$

$$\int_a^{\infty} f(x)\,dx = \lim_{\substack{M\to\infty\\ \sigma\to a^+}} \int_{\sigma}^{M} f(x)\,dx.$$

In the above double limits, the two limiting processes are independent of each other in general.

Examples

Example 1.1.9

$$\int_{-1}^{1} \frac{dx}{x^2-1} = \int_{-1}^{1} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx =$$

$$\lim_{\substack{\rho\to 1^-\\ \sigma\to-1^+}} \int_{\sigma}^{\rho} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx =$$

$$\frac{1}{2} \lim_{\substack{\rho\to 1^-\\ \sigma\to-1^+}} \left[\ln|x-1| - \ln|x+1|\right]_{\sigma}^{\rho} =$$

$$\frac{1}{2} \lim_{\substack{\rho\to 1^-\\ \sigma\to-1^+}} \left[\ln|\rho-1| - \ln|\rho+1| - \ln|\sigma-1| + \ln|\sigma+1|\right].$$

We have that

$$\lim_{\rho\to 1^-} \ln|\rho-1| = -\infty, \qquad \lim_{\rho\to 1^-} \ln|\rho+1| = \ln(2),$$

$$\lim_{\sigma\to -1^+} \ln|\sigma-1| = \ln(2), \qquad \lim_{\sigma\to -1^+} \ln|\sigma+1| = -\infty.$$

So, the above improper integral as double limit is

$$\frac{1}{2}[-\infty - \ln(2) - \ln(2) - \infty] = -\infty.$$

▲

Example 1.1.10

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{\substack{M\to\infty \\ N\to-\infty}} [\arctan(x)]_N^M = \lim_{\substack{M\to\infty \\ N\to-\infty}} [\arctan(M) - \arctan(N)] =$$

(since both partial limits exist separately, we get)

$$\lim_{M\to\infty} \arctan(M) - \lim_{N\to-\infty} \arctan(N) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

▲

Example 1.1.11

$$\int_0^{\infty} \frac{dx}{\sqrt{x}} = \lim_{\substack{M\to\infty \\ \sigma\to 0^+}} [2\sqrt{x}]_\sigma^M = \lim_{M\to\infty} 2\sqrt{M} - \lim_{\sigma\to 0^+} 2\sqrt{\sigma} = \infty - 0 = \infty.$$

▲

Example 1.1.12

$$\int_{-\infty}^{0} \frac{dx}{x^2} = \lim_{\substack{\rho\to 0^- \\ N\to-\infty}} \left[\frac{-1}{x}\right]_N^\rho =$$

$$\lim_{\rho\to 0^-} \left(-\frac{1}{\rho}\right) - \lim_{N\to-\infty} \left(\frac{-1}{N}\right) = -(-\infty) - 0 = \infty.$$

▲

Example 1.1.13

$$\int_{-\infty}^{\infty} x\,dx = \lim_{\substack{M\to\infty \\ N\to-\infty}} \left[\frac{x^2}{2}\right]_N^M =$$

$$\lim_{\substack{M\to\infty \\ N\to-\infty}} \left[\frac{M^2}{2} - \frac{N^2}{2}\right] = \infty - \infty = \text{ does not exist.}$$

In fact, if for instance we let $M = \sqrt{N^2+2A}$, where A is any real number such that $N^2 + 2A \geq 0$, then

$$\lim_{\substack{M\to\infty \\ N\to-\infty}} \left[\frac{M^2}{2} - \frac{N^2}{2}\right] = \lim_{N\to-\infty} \frac{2A}{2} = A.$$

So, this double limiting process may produce any real number as limit. Similarly, we can make this double limit equal to $-\infty$ or ∞ or make it oscillate. (Find some limiting processes that produce these results.)

▲

Definition 1.1.5 *Suppose $y = f(x)$ real function continuous defined in the set $[a,c) \cup (c,b] \subset \mathbb{R}$ with a, b finite and at $x = c$, $y = f(x)$ is unbounded, that is, it approaches $\pm\infty$ as x approaches c. Then we define:*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where the two partial integrals have been defined in ***Definitions 1.1.2*** *and* ***1.1.3****.*

Instead of $[a,c) \cup (c,b]$ we could have $(a,c) \cup (c,b) \subset \mathbb{R}$ with a, b finite or infinite. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

in where the two partial integrals have been defined in ***Definitions 1.1.2, 1.1.3*** *and* ***1.1.4****.*

Examples

Example 1.1.14

$$\int_{-2}^{3} \frac{dx}{(x-1)^3} = \int_{-2}^{1} \frac{dx}{(x-1)^3} + \int_{1}^{3} \frac{dx}{(x-1)^3} =$$

$$\lim_{\rho \to 1^-} \int_{-2}^{\rho} \frac{dx}{(x-1)^3} + \lim_{\sigma \to 1^+} \int_{\sigma}^{3} \frac{dx}{(x-1)^3} =$$

$$\lim_{\rho \to 1^-} \left[-\frac{(x-1)^{-2}}{2} \right]_{-2}^{\rho} + \lim_{\sigma \to 1^+} \left[-\frac{(x-1)^{-2}}{2} \right]_{\sigma}^{3} =$$

$$\lim_{\rho \to 1^-} \left[\frac{-1}{2(\rho-1)^2} + \frac{1}{18} \right] + \lim_{\sigma \to 1^+} \left[\frac{-1}{8} + \frac{1}{2(\sigma-1)^2} \right].$$

By manipulating the two limiting processes, this double limit may assume any possible value, finite or infinite. It follows that this improper integral does not exist.

▲

Example 1.1.15 Now we examine the following integral which, we must notice, is improper at $x = 1$:

$$\int_{-2}^{3} \frac{dx}{x-1} = \int_{-2}^{1} \frac{dx}{x-1} + \int_{1}^{3} \frac{dx}{x-1} =$$

$$\lim_{\rho \to 1^-} \int_{-2}^{\rho} \frac{dx}{x-1} + \lim_{\sigma \to 1^+} \int_{\sigma}^{3} \frac{dx}{x-1} =$$

$$\lim_{\rho \to 1^-} \left[\ln|x-1| \right]_{-2}^{\rho} + \lim_{\sigma \to 1^+} \left[\ln|x-1| \right]_{\sigma}^{3} =$$

$$= \lim_{\rho\to 1^-}[\ln|\rho-1| - \ln(3)] + \lim_{\sigma\to 1^+}[\ln(2) - \ln|\sigma-1|] =$$

$$[\ln(0^+) - \ln(3)] + [\ln(2) - \ln(0^+)] =$$

$$[-\infty - \ln(3)] + [\ln(2) - (-\infty)] =$$

$$-\infty + \infty = \text{does not exist!}$$

We could also write

$$\lim_{\rho\to 1^-}[\ln|\rho-1| - \ln(3)] + \lim_{\sigma\to 1^+}[\ln(2) - \ln|\sigma-1|] =$$

$$\ln\left(\frac{2}{3}\right) + \lim_{\substack{\rho\to 1^- \\ \sigma\to 1^+}} \ln\left(\frac{|\rho-1|}{|\sigma-1|}\right).$$

We can easily see that the double limit $\lim\limits_{\substack{\rho\to 1^- \\ \sigma\to 1^+}} \ln\left(\frac{|\rho-1|}{|\sigma-1|}\right)$ can assume any value as $\sigma \to 1^+$ and $\rho \to 1^-$ independently. Therefore, the improper integral $\displaystyle\int_{-2}^{3}\frac{dx}{x-1}$ does not exist.

Notice that the point $x = 1$, at which this integral is improper, is an interior point of the interval of integration $[-2, 3]$. So, if we inadvertently write

$$\int_{-2}^{3}\frac{dx}{x-1} = [\ln|x-1|\,]_{-2}^{3} = \ln(2) - \ln(3) = \ln\left(\frac{2}{3}\right),$$

then we find a wrong answer, and we have made a bad mistake! We must also notice that the antiderivative of $f(x) = \dfrac{1}{x-1}$, $F(x) = \ln|x-1|$, is not defined (or continuous) at $x = 1$, a point inside the interval of integration $[-2, 3]$. (See also **Example 1.1.1**.)

▲

Important Remark for Improper Integrals: As we have seen in the **previous three examples**, whenever the final evaluation of an improper integral takes final formal form $\infty - \infty$, then the improper integral does not exist. By manipulating the limiting processes, we may make it assume any possible value, finite or infinite, and so such an improper integral does not exist. (Obviously $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$.)

This should not be confused with the limits of the indeterminate form $\infty - \infty$. These limits may exist and are resolved by some mathematical manipulation and/or adjusting the well known **L' Hôpital's rule**.[4]

---

[4] Guillaume François Antoine Marquis de L' Hôpital, French mathematician, 1661-1704.

Also, as we will see in numerous cases and examples, when breaking an integral as a sum two integrals one of which is ∞ and the other one is $-\infty$, then this breaking is illegitimate and must be avoided. Otherwise, all sorts of mistakes can occur.

Example 1.1.16 Similarly with the previous example,

$$\text{for } \infty \leq a < 0 < b \leq \infty, \text{ the integral } \int_a^b \frac{dx}{x} \text{ does not exist.}$$

(Verify this by work analogous to the work of the **previous example**.)

▲

Sometimes an integral seems to be improper, whereas it is proper. For instance, let us investigate the following two examples.

Example 1.1.17 The integral

$$\int_{-1}^{1} \frac{\sin(x)}{x}\, dx$$

is proper, even though the function $f(x) = \dfrac{\sin(x)}{x}$ at $x = 0$ takes the indeterminate form $\dfrac{0}{0}$. This is so because, as we know from calculus,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Therefore, $f(x) = \dfrac{\sin(x)}{x}$ is bounded on the interval $[-1, 1]$ and can be continuously defined at $x = 0$, by assigning the value $f(0) = 1$.

This integral can be evaluated, by means of power series, as a series of real numbers. By using the power series expansion of the function $\sin(x)$ we find that the power series of the function $f(x)$ is:

$$f(x) = \frac{\sin(x)}{x} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}, \ \forall\, x \in \mathbb{R}.$$

Since we can integrate a power series, within its interval of convergence, term by term, we get

$$\int_{-1}^{1} \frac{\sin(x)}{x}\, dx =$$

$$\int_{-1}^{1} \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\right] dx = \sum_{n=0}^{\infty} \int_{-1}^{1} \left[(-1)^n \frac{x^{2n}}{(2n+1)!}\right] dx =$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}\right]_{-1}^{1} = \sum_{n=0}^{\infty}(-1)^n \frac{2}{(2n+1)(2n+1)!}.$$

(See also **Problem 3.2.27**.)

▲

Important Remark: In the **previous example**, we have used the fact that we can integrate power series term by term, which means that we can commute the integral $\int_a^b$ with the infinite summation $\sum_{n=0}^{\infty} = \lim\limits_{0\leq k\to\infty}\left(\sum_{n=0}^{k}\right)$. (That is, we can switch the order of integration and the limit process.) Whereas this is always legitimate with integrals of power series when the limits of integration a and b are inside their intervals of convergence, it does not hold in every situation with limits of sequences or series of functions, even if the limits of integration are within the domain of definition of all functions involved. Serious mistakes may occur if such a commutation is performed while it is not valid! (For this see **Section 2.3**, in the **next chapter**.)

Example 1.1.18 As in the **previous example** so the following integral

$$\int_{-3}^{5} \frac{1-\cos(x)}{x^2}\,dx$$

is proper.

Again, at the singular point $x = 0$, the function $g(x) = \dfrac{1-\cos(x)}{x^2}$ is bounded and can be continuously defined by assigning the value $g(0) = \dfrac{1}{2}$. This follows from the fact that under certain hypotheses we can resolve a limit of type $\dfrac{0}{0}$ by using L' Hôpital's rule. Indeed, we have:

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{[1-\cos(x)]'}{(x^2)'} = \lim_{x\to 0}\frac{\sin(x)}{2x} = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

▲

Remark: Under certain necessary conditions, we can resolve limits of the types $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$ by using L' Hôpital's rule. (Remember that not every such limit can be answered by this rule, but only those that satisfy the necessary conditions. Review this rule one more time from a calculus or mathematical analysis book!) These limits may assume any real value, or $\pm\infty$, or may not exist. If this problem arises at a point of a set over which we examine an integral and such a limit is equal to a real number, then the integral is proper with respect to this singular point. Otherwise, it is improper.

1.1.1 Applications

Application 1: In calculus, geometry, differential geometry and other areas we encounter the **logarithmic spirals**. In polar coordinates (r, θ) they are given by the formula

$$r = ae^{b\theta},$$

where $a \neq 0$ and $b \neq 0$ real constants.

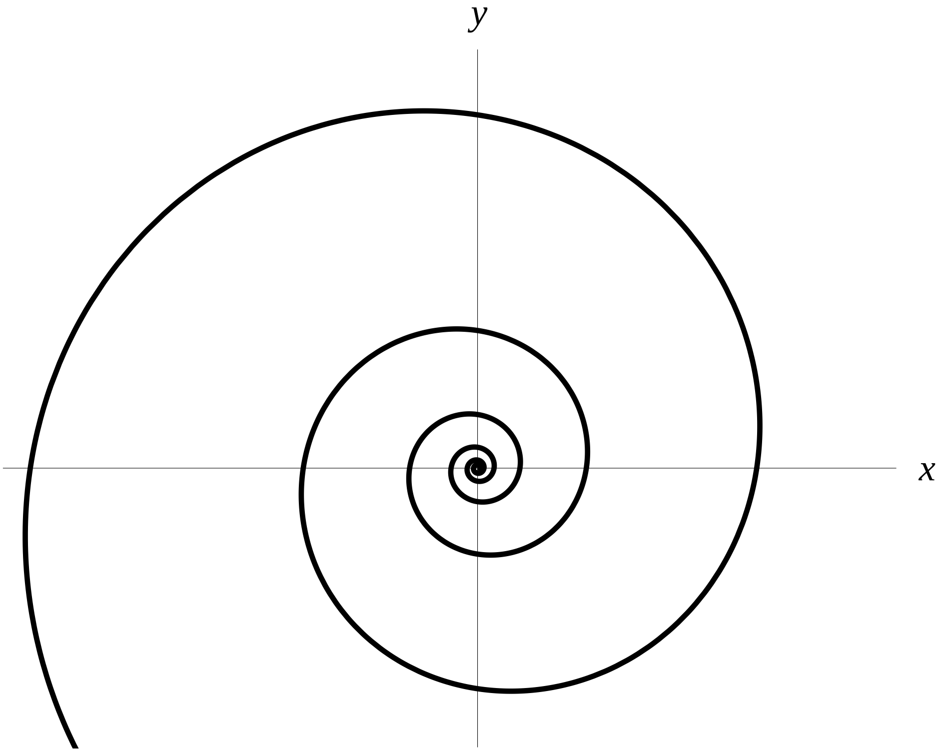

FIGURE 1.2: Logarithmic spiral $\mathbf{r = a\,e^{b\theta}}$

For such a curve $r = f(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, as we learn in calculus, the arc-length is given by

$$L(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Applying this to the logarithmic spiral, we find

$$L(\theta_1,\theta_2)=\int_{\theta_1}^{\theta_2}|a|\sqrt{1+b^2}\,e^{b\theta}d\theta=|a|\frac{\sqrt{1+b^2}}{b}\left(e^{b\theta_2}-e^{b\theta_1}\right).$$

Now for $b>0$ and any $\theta\in\mathbb{R}$ the $L(-\infty,\theta)$ is an improper integral but has finite value. Namely

$$L(-\infty,\theta)=|a|\frac{\sqrt{1+b^2}}{b}\,e^{b\theta}.$$

Similarly for $b<0$ and $\theta\in\mathbb{R}$, we get

$$L(\theta,\infty)=|a|\frac{\sqrt{1+b^2}}{-b}\,e^{b\theta}.$$

Application 2: In physics we learn that the Earth creates around it a conservative gravitational field. If the mass of the Earth is M, then the force W (weight) exerted on a mass m located at distance r from the center of gravity of the Earth, by Newton's law of gravitational attraction, has measure

$$W=-G\,\frac{M\,m}{r^2},$$

where G is the universal gravitational constant. The minus sign has the meaning that the force is directed toward the center of gravity of the Earth.

The gravitational **potential energy** of m at a point P located at distance R from the center of gravity of the Earth O is equal to the work needed to move m from distance R to infinite distance. Then,

$$E=\int_R^\infty W(r)\,dr.$$

Since the gravitational field is conservative (i.e., this integral is independent of the path), we can evaluate E by moving on the straight line OP from R to ∞, where O is considered to be the origin. So,

$$E=\int_R^\infty W(r)\,dr=\int_R^\infty -G\,\frac{M\,m}{r^2}\,dr=-GMm\left[-\frac{1}{r}\right]_R^\infty=\frac{-\,GM}{R}m.$$

The **potential** of the gravitational field of the Earth at any point P at distance R from O is defined to be the above energy E per unit-mass, and so it is

$$U=\frac{E}{m}=-\frac{GM}{R}.$$

Application 3: The decaying law of a radioactive substance is

$$m(t) = m_0 e^{kt},$$

where t is time, $m(t)$ is the radioactive mass remaining after time t, $m_0 = m(0)$ is the initial mass at time $t = 0$ and k is a negative constant representing the percentage rate of decay of the substance.

The mean life of an atom of this substance is

$$\mu = -k \int_0^\infty t e^{kt} dt.$$

We can compute the improper integral and find that the mean life, in fact, is

$$\mu = -k \int_0^\infty t e^{kt} dt = -k \int_0^\infty t\, d\left(\frac{e^{kt}}{k}\right) =$$
$$-k \left[t \frac{e^{kt}}{k}\right]_0^\infty + k \int_0^\infty \frac{e^{kt}}{k} dt = -[0 - 0] + k \left[\frac{e^{kt}}{k^2}\right]_0^\infty = -\frac{1}{k}.$$

[Notice that if $k < 0$, $\lim_{t \to \infty} \left(t\, e^{kt}\right) = 0$ and $\lim_{t \to \infty} \left(e^{kt}\right) = 0$.]

Application 4: The plane curve given implicitly by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

has four cusps at the points $\{(1,0),\ (0,1),\ (-1,0),\ (0,-1)\}$. It is symmetrical about either axis, the lines $y = \pm x$ and about the origin. (See **Figure 1.3**.)

Then, by its symmetries, its arc-length (by the well-known formula from calculus) is going to be

$$L = 4 \int_0^1 \sqrt{1 + (y')^2} dx.$$

By implicit differentiation, we find $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$ and so

$$\sqrt{1 + (y')^2} = \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \sqrt{\frac{1}{x^{\frac{2}{3}}}} = \frac{1}{x^{\frac{1}{3}}} = x^{\frac{-1}{3}}.$$

So, even if the arc-length of this curve is finite, it is given by an improper integral as

$$L = 4 \int_0^1 \sqrt{1 + (y')^2} dx = 4 \int_0^1 x^{\frac{-1}{3}} dx = 4 \left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_0^1 = 4 \cdot \frac{3}{2} = 6.$$

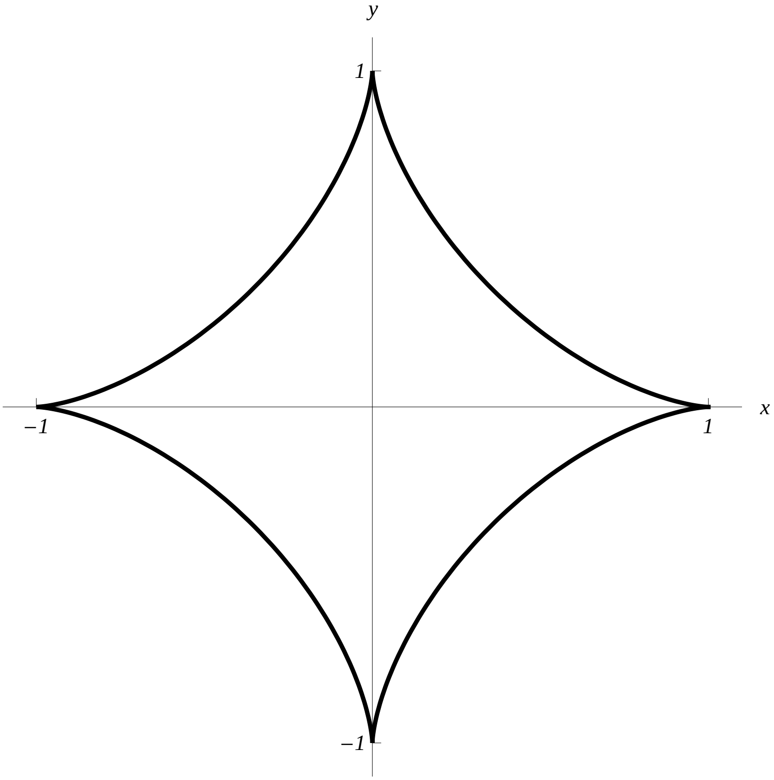

FIGURE 1.3: Astroid $\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1}$

Application 5: If a company expects annual profits $p(t)$, t years from now with interest compounded continuously at an annual interest rate r, then the present value for all future profits, also called **present value of the income stream** $p(t)$, using appropriate Riemann sums, can be shown to be given by the improper integral

$$\text{Present Value} = \int_0^\infty e^{-rt} p(t) dt = \int_0^\infty p(t)\, d\left(\frac{e^{-rt}}{-r}\right) =$$

$$\left[p(t)\frac{e^{-rt}}{-r}\right]_0^\infty + \int_0^\infty \frac{e^{-rt}}{r} dp(t) = \frac{p(0)}{r} + \frac{1}{r}\int_0^\infty e^{-rt}\, dp(t).$$

[We have assumed that $p(\infty)\dfrac{e^{-r\infty}}{-r} = 0$, which is a natural condition.]

If we need to find the present value for a time interval $0 \leq a \leq b$, then we compute the above integral from a to b.

Problems

1.1.1 Study the graphs of the functions $f(x)$, $F(x)$, $\bar{F}(x)$ and $F_c(x)$ of **Example 1.1.1** by using the graphs and the information already provided and the information obtained by studying their first and second derivatives. Compare them with each other and observe the similarities and differences!

1.1.2 Prove that the function $F_c(x)$ in **Example 1.1.1**, is:

(a) Continuous at every $x \in \mathbb{R}$.

(b) Differentiable at every $x \in \mathbb{R}$, by showing:

At the points $x = (2k+1)\pi$ with $k \in \mathbb{Z}$, use the definition of side derivatives and the help of L' Hôpital's rule to resolve the corresponding limits to show $F_c'[(2k+1)\pi] = f[(2k+1)\pi] = \frac{1}{3}$.

Then show that at every $x \in \mathbb{R}$ $F_c'(x) = f(x)$, where $f(x)$ is the given function in this **example**.

1.1.3 Check that for any real constant C

$$\frac{d}{dx}\left[\frac{1}{3}\arctan\left(\frac{3x(1-x^2)}{x^4-4x^2+1}\right)+C\right] = \frac{x^4+1}{x^6+1} > 0.$$

Using this, we find

$$\int_0^1 \frac{x^4+1}{x^6+1}\,dx = \left[\frac{1}{3}\arctan\left(\frac{3x(1-x^2)}{x^4-4x^2+1}\right)\right]_0^1 = 0 - 0 = 0.$$

An integral of a positive function is 0(?)! Explain what has happened. (Similar problem as in **Example 1.1.1**.)

1.1.4 Someone wrote $\displaystyle\int_{-2}^{2} \frac{1}{x}\,dx = [\ln(|x|)]_{-2}^{2} = \ln(2) - \ln(2) = 0$, which is wrong. Find the error and explain why. (See **Example 1.1.16**.)

1.1.5 Let

$$f(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le 1 \\ \\ 2, & \text{if} \quad 1 < x \le 2. \end{cases}$$

Then $\displaystyle\int_0^2 f(x)dx = 3$ and an antiderivative of $f(x)$ is

$$F(x) = \begin{cases} x, & \text{if} \quad 0 \le x \le 1 \\ \\ 2x, & \text{if} \quad 1 < x \le 2. \end{cases}$$

But, $F(2) - F(0) = 4 - 0 = 4 \neq 3$. Why has this happened?

Find an antiderivative $G(x)$ of $f(x)$ such that $G(2) - G(0) = 3$, the correct answer.

1.1.6 Project on periodic real functions.

In this project, without loss of generality, we consider real functions of a real variable $y = f(x) \ : \ \mathbb{R} \longrightarrow \mathbb{R}$.

Such a function is called **periodic** if there is a real number $q \neq 0$ such that $f(x) = f(x+q), \ \forall \ x \in \mathbb{R}$. This number q is called **a period of the real function** $y = f(x)$. Otherwise, $y = f(x)$ is called **non-periodic**.

Obviously, $q = 0$ satisfies this condition for every function. So, $q = 0$ does not tell us anything about any function. We can call $q = 0$ **trivial period** for any function.

Also, if $y = f(x)$ is a constant function, then obviously any real number $q \in \mathbb{R}$ is a period of it.

[In integrals that we study in this text, at times, we use **properties (7.)** and **(8.)** below. With this opportunity, we try to present a more complete exposition of the periodic real functions. See also **Problem 1.2.31.**]

1. If functions $y = f(x)$ and $y = g(x)$ have a common period q and $c \in \mathbb{R}$ is a constant, then prove that the functions $f + g$, $f - g$, $c \cdot f$, $f \cdot g$, $\dfrac{f}{g}$ and $f \circ g$ have q as a period. For the composition $f \circ g$ we can relax one hypothesis. Which one and why?

2. For any q period of $y = f(x)$, prove that $-q$ and in general any kq with $k \in \mathbb{Z}$ (integer) is another period.

   If moreover r is any other period of $y = f(x)$ (including the trivial one), then $q \pm r$ is also a period, but qr and $\dfrac{q}{r}$ may not be periods of $y = f(x)$.

3. For any real numbers $s \neq t$, we define a so-called **Dirichlet**[5] **function** $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

   $$y = f(x) = \begin{cases} s, & \text{if } x = \text{rational} \\ \\ t, & \text{if } x = \text{irrational.} \end{cases}$$

   In the literature, many times, we encounter such a function with $s = 0$ and $t = 1$ which is the **characteristic function of the**

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[5] Johann Peter Gustav Lejeune Dirichlet, German mathematician, 1805-1859.

irrationals in $\mathbb{R}$, $\chi_{\mathbb{R}-\mathbb{Q}}$, or with $s = 1$ and $t = 0$ which is **the characteristic function of the rationals** in $\mathbb{R}$, $\chi_{\mathbb{Q}}$.

For any Dirichlet function $y = f(x)$, prove:
(a) It is nowhere continuous.
(b) It is even [i.e., $f(-x) = f(x)$].
(c) It is periodic and any rational number $r \in \mathbb{Q}$ is a period.
(d) Any irrational number $w \in \mathbb{R} - \mathbb{Q}$ is not a period of $y = f(x)$.

4. If $y = f(x)$ possesses a point of continuity (i.e., there is an $x_0 \in \mathbb{R}$ such that $\lim\limits_{x \to x_0} f(x) = f(x_0)\,[= f(\lim\limits_{x \to x_0} x)]$) and a sequence of non-zero periods ($q_n \neq 0$) with $n \in \mathbb{N}$ such that $\lim\limits_{n \to \infty} q_n = 0$, then $y = f(x)$ is identically constant.

5. If $y = f(x)$ is periodic, non-constant and possesses a point of continuity, then it cannot have a sequence of distinct periods that converges to zero.

   Then prove that such a function has a minimum positive period and any other period is an integer multiple of it.

   I.e., if $p :=$ inf{ positive periods of $y = f(x)$ }, then $p > 0$ and $f(x) = f(x + p)$, $\forall\ x \in \mathbb{R}$. Hence,

$$p := \inf\{\text{ positive periods of } y = f(x)\,\} = \\ \min\{\text{ positive periods of } y = f(x)\,\}.$$

   Moreover, for any other period q of $y = f(x)$, there is $k \in \mathbb{Z}$ such that $q = kp$.

   In such a case, this number p is unique and we call it **the period of the real function** $y = f(x)$. Then the function $y = f(x)$ is called $p-$**periodic**.

6. Give an example of two $p-$periodic functions $y = f(x)$ and $y = g(x)$ such that the period q of their sum $f+g$, or difference $f-g$, or product fg and/or ratio $\dfrac{f}{g}$ is not p. What are the possible answers that the ratio $\dfrac{q}{p}$ may assume?

7. If a periodic function $y = f(x)$ is Riemann integrable, then for any $a \in \mathbb{R}$ and any of its periods $q \in \mathbb{R}$ the integral

$$\int_a^{a+q} f(x)\,dx$$

   is fixed, that is, independent of $a \in \mathbb{R}$. (For $q = 0$ this is trivially true regardless.)

8. Suppose $f : [0,\infty) \longrightarrow \mathbb{R}$ is a periodic function with period $p > 0$ which is Riemann integrable in every interval $[0, M]$, with $M > 0$. Then for every $u \geq 0$ prove

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{p}\int_0^p f(x)\,dx = \frac{1}{p}\int_u^{u+p} f(x)\,dx.$$

[Hint: For any $x > 0$, consider $n := \left[\left[\dfrac{x}{p}\right]\right]$ the integer part of $\dfrac{x}{p}$. Then use:

$$\frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}\left[\sum_{k=0}^{n-1}\int_{kp}^{(k+1)p} f(t)dt + \int_{np}^{x} f(t)dt\right],$$

the previous result, the inequality $\dfrac{1}{p} > \dfrac{n}{x} > \frac{n}{(n+1)p}$ (prove it first), the **Squeeze Lemma**, etc.]

9. If $y = f(x)$ is periodic and differentiable, then its derivative $y' = f'(x)$ is also periodic with the same periods as $y = f(x)$ and it is zero at at least one point in every interval of length greater than or equal to any positive period.

10. Give an example of a periodic function $y = f(x)$ whose integral

$$F(x) = \int_0^x f(t)\,dt$$

is not periodic.

11. Give examples of periodic functions $y = f(x)$ with two irrational periods a and b, such that $\dfrac{a}{b}$ is rational.

12. Give an example of two non-periodic functions whose composition is periodic.

13. Can the composition of a periodic function and a non-periodic function in either order be periodic?

14. If $y = f(x)$ is a continuous and periodic function with an irrational period q, then prove that the set $f(\mathbb{Z}) := \{f(n) \mid n \in \mathbb{Z}\}$ is dense in its range $f(\mathbb{R}) := \{f(x) \mid x \in \mathbb{R}\}$). That is, between any two different numbers in the range $f(\mathbb{R})$ there is a number of the set $f(\mathbb{Z})$. This is equivalent to the fact that any number in the range $f(\mathbb{R})$ is the limit of a sequence in $f(\mathbb{Z})$. (The latter statement is easier to prove.)

15. Using the previous result and the properties of the trigonometric functions $y = \cos(x)$ and $y = \sin(x)$, prove that the sets $\{\cos(n) \mid n \in \mathbb{N}\}$ and $\{\sin(n) \mid n \in \mathbb{N}\}$ are dense in the range $[-1, 1]$.

16. If $y = f(x)$ is periodic and has two periods a and b such that $\dfrac{a}{b}$ is irrational, then it has a sequence of different periods that converges to zero. In such a case, in order for $y = f(x)$ to be non-constant [by (4.) above] it must be discontinuous everywhere.

17. Give (construct) an example of a non-constant function with two periods a and b such that $\dfrac{a}{b}$ is irrational.

[Hint: In **Items (11.), (13.)** and **(14.)** you can use the following fact: "For any irrational number t the set $\mathbb{Z} + t\mathbb{Z}$ is dense in $\mathbb{R}$. I.e., between any two different real numbers there is a number of the form $k + tl$, with k and l integers." You may provide a proof of this fact, but if you cannot, just use it readily.[6]]

1.1.7 Project on the modified Dirichlet function.

Part I:

For any rational number $r \in \mathbb{Q}$ we consider two integers $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ such that: $q > 0$, p and q have no common factors except the trivial 1 [i.e., $\gcd(p, q) = 1$] and $r = \dfrac{p}{q}$. For any integer m (including $m = 0$) we have $m = \dfrac{m}{1}$ and so $p = m$ and $q = 1$. We call such a representation of the rational number r **reduced representation**.

With this in mind we define the so-called **modified Dirichlet or Riemann function** $g : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$y = g(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ rational in reduced representation} \\ \\ 0, & \text{if } x = \text{irrational.} \end{cases}$$

Now prove:

1. $\forall\, x \in \mathbb{R},\ 0 \le g(x) \le 1, \quad g^{-1}(\{0\}) = \mathbb{R} - \mathbb{Q} \quad$ and $g^{-1}(\{1\}) = \mathbb{Z}$.
2. $\forall\, w \in \mathbb{R},\ \lim\limits_{\substack{x \to w \\ x \neq w}} g(x) = 0.$

---

[6] This is a result of Leopold Kronecker, German mathematician, 1823-1891.

3. $y = g(x)$ is continuous at every x irrational and discontinuous at every x rational.[7]

4. $y = g(x)$ is even [i.e., $g(-x) = g(x)$].

5. $y = g(x)$ is periodic and the set of its periods is exactly $\mathbb{Z}$.

6. $y = g(x)$ is nowhere differentiable.

Part II:

We consider the function $y = h(x)$ to be the restriction of $y = g(x)$ on the closed interval $[0, 1]$. (In general we could consider any interval $[a, b]$, where $-\infty < a < b < \infty$, but we use $[0, 1]$ without loss of generality.) I.e., $h : [0, 1] \longrightarrow \mathbb{R}$ is defined by:

$$y = h(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ rational in reduced representation in } [0,1] \\ \\ 0, & \text{if } x = \text{ irrational in } [0,1]. \end{cases}$$

Then:

1. Prove that $y = h(x)$ is Riemann integrable and $\displaystyle\int_0^1 h(x)dx = 0$.

   (Read again the definition of "**Riemann integrable function**" and/or some criteria of "**Riemann integrability**" from appropriate books of Mathematical Analysis and apply them to this function.)

2. Define $u : [0, 1] \longrightarrow \mathbb{R}$ by

$$y = u(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ \\ 0, & \text{if } x = 0. \end{cases}$$

   Prove that $y = u(x)$ is Riemann integrable and $\displaystyle\int_0^1 u(x)dx = 1$.

3. Prove that

$$y = (u \circ h)(x) = \chi_{[0,1]\cap\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x = \text{rational in } [0,1] \\ \\ 0, & \text{if } x = \text{irrational in } [0,1]. \end{cases}$$

   Then prove that the composition of these two Riemann integrable functions, $u \circ h = \chi_{[0,1]\cap\mathbb{Q}}$, is not a Riemann integrable function.

---

[7] This is an example of a real function which is continuous at exactly the irrational numbers. We can prove that there is no real function which is continuous at exactly the rational numbers.

4. However, $u \circ h = \chi_{[0,1]\cap\mathbb{Q}}$ is the point-wise limit of a sequence of Riemann integrable functions, defined as follows:

   Let $[0,1]\cap\mathbb{Q} = \{r_1, r_2, r_3, \ldots,\}$ be an enumeration of the rational numbers in $[0,1]$. Then $\forall\ n \in \mathbb{N}$ define

$$y = v_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ \\ 0, & \text{if } x \in [0,1] - \{r_1, r_2, \ldots, r_n\}. \end{cases}$$

   Now prove that: $\forall\ n \in \mathbb{N}$, $y = v_n(x)$ is a Riemann integrable function with $\int_0^1 v_n(x)dx = 1$ and $\forall\ x \in [0,1]$, $\lim_{n\to\infty} v_n(x) = \chi_{[0,1]\cap\mathbb{Q}}(x)$.

5. Finally, prove that: $\lim_{m\to\infty}\left[\lim_{n\to\infty}\cos^{2n}(m!\pi x)\right] = \chi_{[0,1]\cap\mathbb{Q}}(x)$.

   That is, $\chi_{[0,1]\cap\mathbb{Q}}(x)$ is an iterated limit of a double limit process of bounded continuous functions.

(See also **Theorem 2.3.5** and **Examples 2.3.14** and **2.3.15**.)

1.2 Cauchy Principal Value

In some cases we can define the so-called **Cauchy**[8] **principal value** or simply **principal value** of an improper integral. This is a certain symmetrical limit and it is defined in the following four situations:

Definition 1.2.1 *If the integral is improper simply because the set of integration is $\mathbb{R} = (-\infty, \infty)$, then we define its principal value to be:*

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx \overset{def}{=} \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Definition 1.2.2 *If the set of integration is $[a,c)\cup(c,b]$, where $a < c < b$ finite real numbers, and the integral becomes improper at c, then we define its principal value to be:*

$$P.V. \int_a^b f(x)\,dx \overset{def}{=} \lim_{\epsilon\to 0^+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right].$$

---

[8] Augustin Louis Cauchy, French mathematician, 1789-1857.

Definition 1.2.3 *If both situations of the previous two definitions occur, i.e., we have improper integrals over* $(-\infty, c) \cup (c, \infty)$*, with* $c \in \mathbb{R}$*, then we combine the two definitions and we define the principal value of this improper integral to be:*

$$P.V. \int_{\infty}^{\infty} f(x)\,dx \overset{def}{=} \lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} \left[\int_{-R}^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{R} f(x)\,dx \right].$$

Definition 1.2.4 *If the set of integration is the finite open interval* (a, b) *(*$a < b$ *are finite real numbers), and the integral is improper just because the interval is open at both endpoints, then we define the principal value of this improper integral to be:*

$$P.V. \int_{a}^{b} f(x)\,dx \overset{def}{=} \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b-\epsilon} f(x)\,dx.$$

Again we see that the principal values are obtained by symmetrical limiting processes and therefore are special. However, they turn out to be very useful in mathematics and applications. We will see applications of the principal value in many sections that follow.

Examples

Example 1.2.1

$$\text{P.V.} \int_{-\infty}^{\infty} x\,dx = \lim_{R \to \infty} \int_{-R}^{R} x\,dx = \lim_{R \to \infty} \left[\frac{x^2}{2} \right]_{-R}^{R} =$$

$$\lim_{R \to \infty} \left[\frac{R^2}{2} - \frac{(-R)^2}{2} \right] = \lim_{R \to \infty} 0 = 0.$$

▲

Example 1.2.2

$$\text{P.V.} \int_{-\infty}^{\infty} x^2\,dx = \lim_{R \to \infty} \int_{-R}^{R} x^2\,dx = \lim_{R \to \infty} \left[\frac{x^3}{3} \right]_{-R}^{R} =$$

$$\lim_{R \to \infty} \left[\frac{R^3}{3} - \frac{(-R)^3}{3} \right] = \lim_{R \to \infty} \frac{2R^3}{3} = \infty.$$

▲

Example 1.2.3 (a) As we have seen in **Example 1.1.16** the integral

$\int_{-1}^{1} \frac{dx}{x}$ does not exist, but

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{x} = \lim_{\epsilon \to 0^+} \left[\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} \right] =$$

$$\lim_{\epsilon \to 0^+} \left([\ln|x|]_{-1}^{-\epsilon} + [\ln|x|]_{\epsilon}^{1} \right) = \lim_{\epsilon \to 0^+} (\ln\epsilon - \ln 1 + \ln 1 - \ln\epsilon) = \lim_{\epsilon \to 0^+} 0 = 0.$$

(b) Similarly $\int_{-2}^{3} \frac{dx}{x}$ does not exist, but

$$\text{P.V.} \int_{-2}^{3} \frac{dx}{x} = \lim_{\epsilon \to 0^+} \left[\int_{-2}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{3} \frac{dx}{x} \right] = \lim_{\epsilon \to 0^+} \left([\ln|x|]_{-2}^{-\epsilon} + [\ln|x|]_{\epsilon}^{3} \right) =$$

$$\lim_{\epsilon \to 0^+} (\ln\epsilon - \ln 2 + \ln 3 - \ln\epsilon) = \lim_{\epsilon \to 0^+} \ln\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2}\right).$$

(c) Also $\int_{-\infty}^{\infty} \frac{dx}{x}$ does not exist, but

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x} =$$

$$\lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} \left[\int_{-R}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{R} \frac{dx}{x} \right] = \lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} \left([\ln|x|]_{-R}^{-\epsilon} + [\ln|x|]_{\epsilon}^{R} \right) =$$

$$\lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} (\ln\epsilon - \ln R + \ln R - \ln\epsilon) = \lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} 0 = 0.$$

▲

Example 1.2.4

$$\text{P.V.} \int_{-2}^{3} \frac{dx}{(x-1)^3} = \lim_{\epsilon \to 0^+} \left[\int_{-2}^{1-\epsilon} \frac{dx}{(x-1)^3} + \int_{1+\epsilon}^{3} \frac{dx}{(x-1)^3} \right] =$$

$$\lim_{\epsilon \to 0^+} \left(\left[\frac{-1}{2(x-1)^2} \right]_{-2}^{1-\epsilon} + \left[\frac{-1}{2(x-1)^2} \right]_{1+\epsilon}^{3} \right) =$$

$$\lim_{\epsilon \to 0^+} \left(\frac{-1}{2\epsilon^2} + \frac{1}{18} - \frac{1}{8} + \frac{1}{2\epsilon^2} \right) = \frac{1}{18} - \frac{1}{8} = \frac{-5}{72}.$$

▲

Example 1.2.5

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=$$

$$\lim_{R\to\infty}[\arctan(x)]_{-R}^{R}=\lim_{R\to\infty}[\arctan(R)-\arctan(-R)]=$$

$$\lim_{R\to\infty}\arctan(R)-\lim_{R\to\infty}\arctan(-R)=\frac{\pi}{2}-(-\frac{\pi}{2})=\pi,$$

since both of the two partial limits exist.

▲

By the above **definitions** and **examples** we conclude the following:

(a) If the improper integral exists, then all limiting processes give the same answer which is the value of the improper integral, and so the principal value also exists and it is equal to the improper integral.

(b) If the improper integral does not exist, then its principal value may or may not exist.

(c) If the principal value does not exist, then the improper integral does not exist either, since the principal value is one of the limiting processes.

Thus, the principal value of an improper integral constitutes a proper generalization of the improper integral. When we know á-priori that the improper integral exists, we can evaluate it by just computing its principal value, especially when the computation of this symmetric limit is easier than any other way.

The following **definitions** and immediate **results** are also useful:

(a) If $y=f(x)$ is an **odd function** in $\mathbb{R}$, i.e., by definition $\forall\, x\in\mathbb{R},\ f(-x)=-f(x)$, then

$$\text{P.V.}\int_{-\infty}^{\infty}f(x)\,dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx=0.$$

(b) If $y=f(x)$ is an **even function** in $\mathbb{R}$, i.e., by definition $\forall\, x\in\mathbb{R},\ f(-x)=f(x)$, then

$$\text{P.V.}\int_{-\infty}^{\infty}f(x)\,dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx=$$

$$2\lim_{R\to\infty}\int_{0}^{R}f(x)\,dx=2\lim_{R\to\infty}\int_{-R}^{0}f(x)\,dx.$$

In general, a function $y=f(x)$, where $f\ :\ \mathbb{R}\longrightarrow\mathbb{R}$, is **odd about a point** $c\in\mathbb{R}$, if by definition

$$\forall\, u\in\mathbb{R},\ \ f(c-u)=-f(c+u),\ \text{ or }\ \forall\, x\in\mathbb{R},\ \ f(2c-x)=-f(x).$$

Also, $y = f(x)$ is **even about a point** $c \in \mathbb{R}$, if by definition

$$\forall\ u \in \mathbb{R},\ \ f(c-u) = f(c+u),\ \text{ or } \forall\ x \in \mathbb{R},\ \ f(2c-x) = f(x).$$

Now consider $y = f(x)$ a function defined in $(-\infty, c) \cup (c, \infty)$, with $c \in \mathbb{R}$. We have:

(c) If $y = f(x)$ is an **odd function about** c [we let $f(c) = 0$, in this case], then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)\,dx + \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)\,dx = 0.$$

(d) If $y = f(x)$ is an **even function about** c, then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)\,dx + \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)\,dx =$$

$$2 \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{c+\epsilon}^{R} f(x)\ dx = 2 \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \int_{-R}^{c-\epsilon} f(x)\,dx.$$

(e) **Rule of translate or shift:** Consider a function

$$f\ :\ [a, a+r] \longrightarrow \mathbb{R}, \quad \text{with} \quad r > 0.$$

The translate or shift of $y = f(x)$ at the interval $[b, b+r]$ is given by

$$y = f(x - b + a), \quad \text{with} \quad b \le x \le b + r.$$

Problems

1.2.1 Give all the reasons as to why the following integrals are improper:

$$I_1 = \int_0^{\infty} \ln(x)\,dx, \qquad I_2 = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx,$$

$$I_3 = \int_0^{\infty} \frac{\cos(x)}{x}\,dx, \qquad I_4 = \int_0^{\infty} \frac{1}{2x-1}\,dx,$$

$$I_5 = \int_0^{\infty} \frac{\sin(5x)}{e^{2x}-1}\,dx, \qquad I_6 = \int_0^{\infty} x^n e^{-x}\,dx, \quad n \in \mathbb{Z}.$$

In **problems 1.2.2-1.2.22** compute the given improper integrals. (Prove that they are equal to $-\infty$, or $+\infty$, or the provided value and/or you find their values. In some of these problems you have to distinguish different cases depending on the values of the parameters involved.)

1.2.2

$$\int_0^\infty e^{-\mu x}\,dx, \quad \text{where } \mu \in \mathbb{R} \text{ constant.}$$

1.2.3

$$\int_1^\infty x^\alpha \ln(x)\,dx \qquad \text{and} \qquad \int_0^1 x^\alpha \ln(x)\,dx,$$

where α is a real constant.

1.2.4

$$\int_{-\infty}^\infty \frac{|x|}{x^2+1}\,dx \qquad \text{and} \qquad \text{P.V.}\int_{-\infty}^\infty \frac{|x|}{x^2+1}\,dx.$$

1.2.5

$$\int_0^9 \frac{x}{(x-3)^2}\,dx \qquad \text{and} \qquad \text{P.V.}\int_0^9 \frac{x}{(x-3)^2}\,dx.$$

1.2.6

$$\int_a^\infty \frac{dx}{x^p},$$

where $a > 0$ and p are real constants.

1.2.7

$$\int_2^\infty \frac{dt}{t[\ln(t)]^p},$$

where $p \geq 1$ is a real constant.

1.2.8

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(x)\,dx \qquad \text{and} \qquad \text{P.V.}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(x)\,dx.$$

1.2.9

$$\int_0^\infty \frac{dx}{(x+1)^3} \qquad \text{and} \qquad \int_0^\infty \frac{dx}{(x-1)^3}.$$

1.2.10

$$\int_1^\infty \frac{dx}{(x-2)^3} \qquad \text{and} \qquad \text{P.V.}\int_1^\infty \frac{dx}{(x-2)^3}.$$

1.2.11

$$\int_{-\infty}^{\infty} |x|e^{-x^2}\,dx.$$

1.2.12

$$\int_{-\infty}^{\infty} e^{-x}\cos(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} e^{-x}\sin(x)\,dx.$$

1.2.13

$$\int_{3}^{\infty} \frac{dx}{x^2+x-2} = \frac{1}{3}\ln\left(\frac{5}{2}\right).$$

1.2.14

$$\int_{0}^{\infty} e^{-\alpha x}\sin(\beta x)\,dx = \frac{\beta}{\alpha^2+\beta^2},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ constants.

(See also **Problem 3.7.40**.)

1.2.15

$$\int_{a}^{\infty} e^{-\alpha x}\sin(\beta x)\,dx,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $a \in \mathbb{R}$ constants.

(See also **Problem 3.7.40**.)

1.2.16

$$\int_{0}^{\infty} e^{-\alpha x}\cos(\beta x)\,dx = \frac{\alpha}{\alpha^2+\beta^2},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ constants.

(See also **Problem 3.7.40**.)

1.2.17

$$\int_{a}^{\infty} e^{-\alpha x}\cos(\beta x)\,dx,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $a \in \mathbb{R}$ constants.

(See also **Problem 3.7.40**.)

1.2.18

$$\int_{0}^{1} x^{p-1}\,dx,$$

where p is a real constant.

1.2.19

$$\text{(a)}\quad \int_0^a \ln(x)\,dx \qquad \text{and} \qquad \int_a^\infty \ln(x)\,dx,$$

$$\text{(b)}\quad \int_0^a \ln^2(x)\,dx \qquad \text{and} \qquad \int_a^\infty \ln^2(x)\,dx,$$

where $a > 0$ is a real constant.

1.2.20

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2ax + b^2} = \frac{\pi}{\sqrt{b^2 - a^2}}$$

for $b > |a|$ real constants.

1.2.21 By making two successive appropriate u-substitutions, prove that

$$\int_3^\infty \frac{dx}{x\ln(x)\ \left[\ln\left[\ln(x)\right]\right]^2} = \frac{1}{\ln\left[\ln(3)\right]}.$$

1.2.22 By making two successive appropriate u-substitutions, prove that

$$\int_3^\infty \frac{dx}{x\ln(x)\ \ln\left[\ln(x)\right]} = \infty.$$

1.2.23 Show that the integral

$$\int_{-1}^{1} \frac{e^x - 1}{x}\,dx$$

is proper and find its value as a series of real numbers.

1.2.24 Compute the integrals

$$\int_{-2}^{2} \frac{1-\cos(x)}{x^2}\,dx \qquad \text{and} \qquad \int_{-3}^{5} \frac{1-\cos(x)}{x^2}\,dx$$

as series of real numbers.

[Hint: Use the power series of $\cos(x)$, etc.]

1.2.25 Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Prove:
(a) If $y = f(x)$ is odd, then $f(0) = 0$.
(b) If $y = f(x)$ is both odd and even, then it is identically zero.
(c) If $y = f(x)$ is differentiable and odd, then $f'(x)$ is even.
(d) If $y = f(x)$ is differentiable and even, then $f'(x)$ is odd.

1.2.26 (a) Prove that with compositions of functions we have the rules:

$$\begin{gathered}\text{even} \circ \text{even} = \text{even}\\ \text{even} \circ \text{odd} = \text{even}\\ \text{odd} \circ \text{even} = \text{even}\\ \text{odd} \circ \text{odd} = \text{odd}.\end{gathered}$$

(b) Prove that with multiplications of functions we have the rules:

$$\begin{gathered}\text{even} \cdot \text{even} = \text{even}\\ \text{even} \cdot \text{odd} = \text{odd}\\ \text{odd} \cdot \text{even} = \text{odd}\\ \text{odd} \cdot \text{odd} = \text{even}.\end{gathered}$$

1.2.27 Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then prove that any two of the following statements imply the third one:

(a) $f(x)$ is odd about $x = 0$, i.e., $f(-x) = -f(x)$.

(b) $f(x)$ is odd about $x = c$, i.e., $f(2c - x) = -f(x)$ or $f(c - u) = -f(c + u)$.

(c) $f(x)$ is $2c-$periodic, i.e., $f(x + 2c) = f(x)$.

1.2.28 Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then prove that any two of the following statements imply the third one:

(a) $f(x)$ is even about $x = 0$, i.e., $f(-x) = f(x)$.

(b) $f(x)$ is even about $x = c$, i.e., $f(2c - x) = f(x)$ or $f(c - u) = f(c + u)$.

(c) $f(x)$ is $2c-$periodic, i.e., $f(x + 2c) = f(x)$.

1.2.29 Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then prove:

(a) If $f(x)$ is odd about $x = 0$ and even about $x = c$, then $f(x)$ is $4c-$periodic, i.e., $f(x + 4c) = f(x)$.

(b) If $f(x)$ is even about $x = 0$ and odd about $x = c$, then $f(x)$ is $4c-$periodic, i.e., $f(x + 4c) = f(x)$.

(c) Notice that here we cannot have results similar to the results of the **two previous problems**. Why?

1.2.30 Consider any $c > 0$ and

$$f(x) = x(c - x), \quad 0 \le x \le c.$$

Show that the extension of this function all over $(-\infty, \infty)$, such that the extended function is odd about both $x = 0$ and $x = c$, is given by:

$$\forall\ n \in \mathbb{Z},\ \text{ for } nl \le x \le (n{+}1)c,\ \text{ then }\ f(x) = (-1)^n (x{-}nc)\left[(n+1)c - x\right].$$

[Hint: Extend the function to $[-c, 0]$ as odd and then use **Problem 1.2.27** and the **Rule of shift, (e)**, above, etc.]

1.2.31 For any $a \in \mathrm{R}$ and $p > 0$, consider any real function $y = f(x)$ defined in the interval $[a, a + p)$ or $(a, a + p]$.

(a) Extend this function to the whole R periodically with a period equal to p.

(b) Give an example in which the period of the extended function in (**a**) is less than p.

(c) If we consider $f(x)$ defined on the closed interval $[a, a+p]$, then give an example of a function $f(x)$ which cannot be extended as a periodic function.

(d) Under what condition a real function $f(x)$ defined on the closed interval $[a, a+p]$ can be extended as a periodic function to the whole R and with a period equal to p?

1.3 Some Criteria of Existence

We have defined the improper integrals as certain limits. These limits may or may not exist. When such a limit exists, we say that the **improper integral exists** or it is **convergent**. If the limit does not exist, then we say that the **improper integral does not exist** or it is **divergent**.

In the previous definitions, for more generality, the real value function $y = f(x)$ was considered to be piecewise continuous rather than continuous. (In the most general theory of integration developed in Advanced Real Analysis, we deal with more general integrals of a "very large" class of functions, the class of the **measurable functions**. We study these in an advanced course of real analysis.)

Necessary and sufficient conditions for the existence of improper integrals are developed in advanced calculus, mathematical analysis and real analysis. Most of these are beyond the scope of this book. So, we will content ourselves with the few criteria stated in this section, some of which are reminiscent to criteria for the convergence of infinite series in calculus. These criteria are sufficient and powerful enough to give answers about existence or non-existence (convergence or non-convergence) questions for almost all the interesting improper integrals of mathematics and scientific applications at this primary level.

Definition 1.3.1 ***Non-standard Definition:*** *In this book, we shall call a function to be a "**nice function**" if it is piecewise continuous in its domain of definition with finitely many discontinuities, each of which is of the following three types:*

1. *Jump discontinuity with finite or infinite jump.*
2. *Essential discontinuity because the limit of the function, as x approaches the point of discontinuity, is $\pm\infty$.*

3. *Essential discontinuity, such that the limit of the function, as x approaches the point of discontinuity, does not exist and is not $\pm\infty$. (The function oscillates.) In this case, we shall assume the extra condition that the function is bounded in some interval containing the point of the essential discontinuity.*

The continuous functions are of course a subset of this set of nice functions since they have zero discontinuities in their domain. We use this non-standard term of "nice functions" for short, so that we do not have to repeat these conditions whenever we need them throughout this book. So, from now on we must remember what we mean by this non-standard term of "nice function" whenever we refer to it.

When a jump discontinuity has infinity jump or the limit of the function at the point of the discontinuity is $\pm\infty$, then the function is unbounded. In general, the discontinuity is essential if the limit of the function at the point of the discontinuity does not exist.

Also the **domain** of definition of such a function is going to be denoted by a capital letter like A, where $A \subseteq \mathbb{R}$ is any set that we have already encountered in the definitions of the **previous two sections** and/or any nice set that we have already dealt with in an undergraduate calculus course (e.g., a bounded closed interval, a finite union of bounded closed intervals, etc.).

Theorem 1.3.1 (Comparison Test with Non-negative Functions) *Let f and g be two nice functions defined in a set $A \subseteq \mathbb{R}$ (as we have indicated in the previous paragraph) and satisfying the inequality $0 \leq f(x) \leq g(x)$. Then we have:*

(a) If $\int_A g(x)\,dx$ exists, then $\int_A f(x)\,dx$ exists.

["Exists" for a non-negative function $0 \leq f(x) \leq g(x)$ means that the integral assumes a finite non-negative value.]

In this case we have the inequality

$$0 \leq \int_A f(x)\,dx \leq \int_A g(x)\,dx < \infty.$$

(b) If $\int_A f(x)\,dx$ does not exist, then $\int_A g(x)\,dx$ does not exist.

["Does not exist" for non-negative functions $0 \leq f(x) \leq g(x)$ means that the integrals are infinite, that is, their values are equal ∞. So, in this case we have $\int_A f(x)\,dx = \infty = \int_A g(x)\,dx$.]

Proof The proof of this criterion is rather obvious, since for any closed interval $[p, q] \subseteq A$ and for any nice function satisfying the inequality $0 \leq f(x) \leq g(x)$, by basic calculus we have

$$0 \leq \int_p^q f(x)\,dx \leq \int_p^q g(x)\,dx$$

and the limiting processes preserve the $\leq$ inequalities.

Then, to prove **claim (a)** and **claim (b)** of the Theorem, we respectively use the facts

$$\int_A g(x)\,dx < \infty \qquad \text{and} \qquad \int_A f(x)\,dx = \infty.$$

(**Note**: This criterion is reminiscent of the Comparison Test for convergence of non-negative series.)

Examples

Example 1.3.1 Prove that

$$\int_0^\infty e^{-x^2}\,dx \ \text{ exists.}$$

(In other words, it is convergent or equals a finite value.)

Consider the continuous function $f(x) = e^{-x^2}$ on $[0, \infty)$ and define

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le 1 \\ \\ e^{-x} & \text{for } 1 \le x < \infty. \end{cases}$$

Then $0 < f(x) \le g(x)$, $\forall\, x \in [0, \infty)$ and

$$\int_0^\infty g(x)\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x}\,dx = \int_0^1 e^{-x^2}\,dx + \left[-e^{-x}\right]_1^\infty =$$
$$(\text{finite value}) + [0 - (-e^{-1})] = (\text{finite value}).$$

Therefore,

$$\int_0^\infty f(x)\,dx = \int_0^\infty e^{-x^2}\,dx < \int_0^\infty g(x)\,dx < \infty$$

and so $\int_0^\infty f(x)\,dx$ is finite.

In the above inequality we have used the strictly less because $f(x) < g(x)$, $\forall\ x \in (1, \infty)$.

In the same way we can prove that

$$\int_{-\infty}^\infty e^{-x^2}\,dx \ \text{ exists,}$$

or we can use the fact that $f(x) = e^{-x^2}$ with $x \in \mathbb{R}$ is an even function and so

$$\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx.$$

▲

Note: There is no explicit antiderivative of e^{-x^2}. This is proven by **Liouville's**[9] **theory for finding antiderivatives in terms of elementary functions**. This theory involves a good knowledge of complex analysis.

Example 1.3.2 Prove that

$$\int_2^\infty \frac{1}{\ln(x)}\,dx = \infty,$$

i.e., it diverges.

On $[2, \infty)$ we have that $0 < \dfrac{1}{x} < \dfrac{1}{\ln(x)}$.

Now

$$\int_2^\infty \frac{1}{x}\,dx = [\ln(x)]_2^\infty = \infty.$$

Therefore,

$$\int_2^\infty \frac{1}{\ln(x)}\,dx = \infty.$$

Again, there is no explicit antiderivative for $\displaystyle\int \frac{1}{\ln(x)}\,dx$.

▲

Now we state the Integral Test, already known from calculus for checking the convergence or divergence of certain infinite series. This is stated as follows:

Theorem 1.3.2 (Integral Test) *Let $y = f(x)$ be a nice, positive, decreasing function defined on an interval $[k, \infty)$, where k is an integer. [That is: $\forall\, x \in [k, \infty)$, $f(x) > 0$ and if $k \le x_1 \le x_2 < \infty$ then $f(x_1) \ge f(x_2)$.] We let $a_n = f(n)$ for $n = k, k+1, k+2, \ldots$. Then*

$$\int_k^\infty f(x)\,dx \quad \textit{converges (diverges)}$$

if and only if

$$\sum_{n=k}^\infty a_n \quad \textit{converges (diverges).}$$

(The **proof** of this criterion can be found in any good calculus book.)

Note: For positive functions and positive series, respectively, the $\displaystyle\int_k^\infty f(x)\,dx$ and $\sum_{n=k}^\infty a_n$ diverges means it is equal to ∞.

---

[9] Joseph Liouville, French mathematician, 1804-1882.

Remark: Whereas in calculus the Integral Test is mainly used to check the convergence or divergence of a positive series that satisfies the hypotheses of this criterion, here we use it in the converse way to check the convergence or divergence of an improper Riemann integral under these hypotheses. So we need to prove that the respective positive series converges or diverges. To this end we employ any different criterion that gives an answer for the series, among all those someone can find in books of advanced calculus or real analysis. Review these criteria one more time. For instance, we remark that under the conditions of the Integral Test the following three criteria are often very convenient.

Theorem 1.3.3 (Cauchy Positive Series CondensationTheorem) *Suppose that* $a_1 \geq a_2 \geq a_3 \geq \; ... \; \geq 0$ *is a decreasing sequence of non-negative numbers. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges (diverges)}$$

if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \; ... \; \text{converges (diverges).}$$

Theorem 1.3.4 (Absolute Root Test. (Cauchy)) *Consider a series of real numbers* $\sum_{n=k}^{\infty} a_n$. *Suppose the following limit exists or is* ∞

$$0 \leq \lim_{n \to \infty} \sqrt[n]{|a_n|} = \rho \leq \infty.$$

Then:

(1) If $0 \leq \rho < 1$, *the series* $\sum_{n=k}^{\infty} a_n$ *converges absolutely and therefore it converges.*

(2) If $1 < \rho \leq \infty$, *the series* $\sum_{n=k}^{\infty} a_n$ *diverges.*

(3) If $\rho = 1$, *the test is inconclusive.*

Theorem 1.3.5 (Absolute Ratio Test. (D' Alembert)) [10] *Consider a series of real numbers* $\sum_{n=k}^{\infty} a_n$. *Suppose the following limit exists or is* ∞

$$0 \leq \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \rho \leq \infty.$$

---

[10] Jean Le Rond d' Alembert, French mathematician, 1717-1783.

Then:

(1) If $0 \le \rho < 1$, *the series* $\sum_{n=k}^{\infty} a_n$ *converges absolutely and therefore it converges.*

(2) If $1 < \rho \le \infty$, *the series* $\sum_{n=k}^{\infty} a_n$ *diverges.*

(3) If $\rho = 1$, *the test is inconclusive.*

Remark: The root and ratio tests as presented here are not stated in the most general form; one can find them in a good mathematical analysis book. In such a book these tests are stated in terms of the **liminf** and **limsup** of sequences of real numbers for achieving the most more general results. Study this material from a good mathematical analysis book.

Examples

Example 1.3.3 The results presented in this example are straightforward, but because they are very useful we find them at times under the name "**p-Test**." When combined with other tests it can answer a lot of questions on convergence or divergence of integrals rather easily.

Since $\forall\ p \in \mathbb{R}$ the antiderivative of the function

$$f(x) = x^{-p} = \frac{1}{x^p}, \quad x \in (0, \infty)$$

is

$$F(x) = \begin{cases} \dfrac{x^{-p+1}}{-p+1} + c & \text{if } \ p \neq 1 \\[2ex] \ln(|x|) + c & \text{if } \ p = 1, \end{cases}$$

where c is an arbitrary constant, we obtain the following easy but useful **results**:

Let $0 < k < \infty$ be a constant. Then:

$$(1) \quad \int_k^{\infty} \frac{dx}{x^p} = \begin{cases} \infty & \text{if } p \le 1 \\[2ex] \dfrac{1}{k^{p-1}(p-1)} & \text{if } p > 1. \end{cases}$$

$$(2) \quad \int_0^{k} \frac{dx}{x^p} = \begin{cases} \dfrac{k^{1-p}}{1-p} = \dfrac{1}{k^{p-1}(1-p)} & \text{if } p < 1 \\[2ex] \infty & \text{if } p \ge 1. \end{cases}$$

So, by both **(1)** and **(2)**

$$\forall\ p \in \mathbb{R} \quad \text{we have:} \quad \int_0^\infty \frac{dx}{x^p} = \infty.$$

We now obtain the two byproducts:
(1) For any $a < b$ and $p < 1$ real constants

$$\int_a^b \frac{dx}{(x-a)^p} = \int_0^{b-a} \frac{dt}{t^p} = \int_a^b \frac{dx}{(b-x)^p} = \frac{(b-a)^{1-p}}{1-p}.$$

(2) For any any a, b and p real constants

$$\int_a^\infty \frac{dx}{(x-a)^p} = \int_0^\infty \frac{dt}{t^p} = \int_{-\infty}^b \frac{dx}{(b-x)^p} = \infty.$$

An **example of using the p-Test** is in proving the convergence of the integral

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \int_0^1 \frac{\sin^2(x)}{x^2}\,dx + \int_1^\infty \frac{\sin^2(x)}{x^2}\,dx.$$

(The á-priori splitting of this integral is legitimate since the integrand function is positive.)

For the part

$$\int_0^1 \frac{\sin^2(x)}{x^2}\,dx$$

we observe

$$\lim_{x\to 0} \frac{\sin^2(x)}{x^2} = 1,$$

a fact that makes the integral proper and therefore finite.

For the second part, we observe

$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx < \int_1^\infty \frac{1}{x^2}\,dx = 1$$

and so this integral is convergent, by the **Comparison Test 1.3.1** and the p-Test with $p = 2 > 1$.

On the other hand, the integral

$$\int_0^\infty \frac{\sin^2(x)}{x^3}\,dx = \int_0^1 \frac{\sin^2(x)}{x^3}\,dx + \int_1^\infty \frac{\sin^2(x)}{x^3}\,dx$$

diverges because

$$\int_0^1 \frac{\sin^2(x)}{x^3}\,dx = \infty.$$

This is so because near $x = 0$, $\dfrac{\sin^2(x)}{x^3} = \dfrac{\sin^2(x)}{x^2} \cdot \dfrac{1}{x}$ behaves like $b \cdot \dfrac{1}{x}$ for some constant $b > 0$. That is, we can find a constant $0 < k \leq 1$ such that for $0 < b = \dfrac{1}{2} < 1$ we have $\dfrac{\sin^2(x)}{x^2} \cdot \dfrac{1}{x} > \dfrac{1}{2} \cdot \dfrac{1}{x}$ for all $0 < x < k$. Then

$$\int_0^1 \frac{\sin^2(x)}{x^3}\, dx > \int_0^k \frac{1}{2} \cdot \frac{1}{x}\, dx = \frac{1}{2} \cdot [\ln(k) + \infty] = \infty.$$

▲

Example 1.3.4 Limit Comparison Test.

In this example we present a method analogous to the limit comparison test for positive series. We find the limit of the ratio of two positive functions as we approach a singularity or ±∞ and then we make an appropriate comparison of their integrals. We illustrate this with the following examples.

(1) Prove that $\forall\ q \in \mathbb{R}$, $\displaystyle\int_1^\infty x^q e^{-x}\, dx$ **converges.**

In $[1, \infty)$ we compare the positive function $f(x) := x^q e^{-x}$ with the positive function $g(x) := \dfrac{1}{x^2}$ by taking the limit

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} x^{q+2} e^{-x} = \lim_{x \to \infty} \frac{x^{q+2}}{e^x} = 0 \ \text{ (by L' Hôpital's rule, e.g.).}$$

Therefore, there is a constant $k > 1$ such that $\dfrac{f(x)}{g(x)} < 1$ if $k \leq x < \infty$ or $0 < f(x) < g(x) < 1$ if $k \leq x < \infty$. But then

$$\int_k^\infty x^q e^{-x} < \int_k^\infty \frac{1}{x^2}\, dx = \left[\frac{-1}{x}\right]_k^\infty = \frac{1}{k}.$$

Hence,

$$\int_1^\infty x^q e^{-x} dx = \int_1^k x^q e^{-x}\, dx + \int_k^\infty x^q e^{-x}\, dx < \text{finite} + \frac{1}{k} < \infty$$

and therefore converges.

(2) Prove that $\forall\ q > -1$, $\displaystyle\int_0^1 x^q e^{-x}\, dx$ **converges.** (Notice that when $q \geq 0$ the integral is proper and so finite.)

When $-1 < q < 0$, the positive function $f(x) := x^q e^{-x}$ has singularity at $x = 0$ (it tends to ∞ as $x \longrightarrow 0^+$). So, for $-1 < q < 0$, we compare $f(x)$ with the positive function $g(x) := \dfrac{1}{x^q}$ by taking the limit

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} e^{-x} = 1.$$

Therefore, there is a constant $0 < k < 1$ such that $\dfrac{f(x)}{g(x)} < 2$ if $0 < x \leq k$, i.e., $0 < f(x) < 2g(x)$ if $0 < x \leq k$. But, then $q + 1 > 0$ and

$$\int_0^k x^q e^{-x}\, dx < \int_0^k 2\frac{1}{x^q}\, dx = 2\left[\frac{x^{q+1}}{q+1}\right]_0^k = \frac{2k^{q+1}}{q+1}.$$

Hence, when $-1 < q < 0$,

$$\int_0^1 x^q e^{-x}\, dx = \int_0^k x^q e^{-x}\, dx + \int_k^1 x^q e^{-x}\, dx < \frac{2k^{q+1}}{q+1} + \text{finite} < \infty$$

and therefore converges. So,

$$\int_0^1 x^q e^{-x}\, dx \quad \text{converges for all } q > -1.$$

(3) Prove that $\forall\ q \leq -1$, $\displaystyle\int_0^1 x^q e^{-x}\, dx (= \infty)$ diverges.

In $(0, 1]$ the positive function $f(x) := x^q e^{-x}$ has singularity at $x = 0$ (it tends to ∞ as $x \longrightarrow 0^+$). We compare it with the positive function $g(x) := x^q$ by taking the limit

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} e^{-x} = 1.$$

Therefore, there is a constant $0 < k < 1$ such that $\dfrac{f(x)}{g(x)} > \dfrac{1}{2}$ if $0 < x \leq k$, i.e., $f(x) > \dfrac{1}{2}g(x)$ if $0 < x \leq k$. But then (by the **previous example**)

$$\int_0^k x^q e^{-x}\, dx \geq \int_0^k \frac{1}{2} \cdot x^q\, dx = \infty.$$

Hence,

$$\int_0^k x^q e^{-x}\, dx = \infty$$

and therefore

$$\int_0^1 x^q e^{-x}\,dx = \int_0^k x^q e^{-x}\,dx + \int_k^1 x^q e^{-x}\,dx = \infty + \text{finite} = \infty$$

diverges for all $q \leq -1$.

▲

Example 1.3.5 We would like to prove that the integral

$$\int_2^\infty \frac{dx}{[\ln(x)]^{\ln(x)}}$$

converges.

One way to do this is to use the **Integral Test**, since same we easily observe the function

$$f(x) = \frac{1}{[\ln(x)]^{\ln(x)}}, \quad \text{on} \quad [2, \infty),$$

is positive for $x \geq 2$ and decreasing for $x \geq 3$. (Also, its limit is zero as $x \longrightarrow \infty$.)

So, to prove that this integral converges (is finite) we must prove that the positive series

$$\sum_{n=2}^\infty \frac{1}{[\ln(n)]^{\ln(n)}}$$

converges. This is done as follows:

We use the **Cauchy Condensation Theorem, 1.3.3**, all the hypotheses of which are satisfied. (Check this.) So, we must prove that the series

$$\sum_{k=2}^\infty \frac{2^k}{\left[\ln\left(2^k\right)\right]^{\ln(2^k)}} = \sum_{k=2}^\infty \frac{2^k}{[k\ln(2)]^{k\ln(2)}},$$

converges.

We prove that the latter series converges by using the **Absolute Root Test**. Indeed:

$$\rho := \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{2^k}{[k\ln(2)]^{k\ln(2)}}} = \lim_{k\to\infty} \frac{2}{[k\ln(2)]^{\ln(2)}} = 0 < 1.$$

Hence, the last series converges and so the initial series converges too. Therefore, the given improper integral converges by the **Integral Test**.

Another way to prove that the given integral converges is the following: Using the substitution $t = \ln(x) \iff x = e^t$ in the above integral we get

$$\int_2^\infty f(x)dx = \int_{\ln(2)}^\infty \frac{e^t}{t^t}\,dt.$$

Then we observe that the function $g(t) = \dfrac{e^t}{t^t}$ is positive for $t \geq \ln(2)$ and decreasing for $t \geq 3$. (Also, its limit is zero as $x \longrightarrow \infty$.) Moreover, the series

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n}$$

converges.

For the convergence of this series we can use, e.g., the **Root Test** to get

$$\lim_{n \to \infty} \sqrt[n]{\frac{e^n}{n^n}} = \lim_{n \to \infty} \frac{e}{n} = 0 < 1.$$

So, the **Integral Test** applies and the improper integral converges.

▲

Example 1.3.6 With work similar to the **previous example**, we can prove that

$$\int_3^{\infty} \frac{dx}{[\ln[\ln(x)]]^{\ln(x)}}$$

converges.

But,

$$\int_3^{\infty} \frac{dx}{[\ln(x)]^{\ln[\ln(x)]}}$$

diverges. (Work out the details for both proofs.)

▲

Now we consider a nice function $y = f(x)$ with positive and negative values, defined in a set A. We say that the improper integral of $f(x)$ over $A \subseteq \mathbb{R}$ exists if $\displaystyle\int_A f(x)\,dx$ is a finite (real) value. However, in such a situation, we distinguish the following two cases and definitions:

Definition 1.3.2 *We say that the improper integral of a nice function $f(x)$ over a set A exists or converges* ***absolutely*** *if*

$$\int_A |f(x)|\,dx$$

is equal to a finite non-negative value.

Otherwise, $\int_A |f(x)|\,dx = \infty$ and then we say that the improper integral of $f(x)$ over the set A ***diverges absolutely****.*

Definition 1.3.3 *We say that the improper integral of a nice function $f(x)$ over a set A converges* ***conditionally*** *if*

$$\int_A f(x)\,dx$$

is equal to a finite real value (and so this improper integral exists), but it diverges absolutely.

Now we state the absolute convergence test which claims that absolute convergence implies convergence, but not vice-versa.

Theorem 1.3.6 (Absolute Convergence Test) *We consider a nice function $f(x)$ defined in a set A.*

(a) If

$$\int_A |f(x)|\,dx \quad \textit{exists}$$

then

$$\int_A f(x)\,dx \quad \textit{exists.}$$

(b) In ***Case (a)*** *we also have the inequality*

$$\left|\int_A f(x)\,dx\right| \leq \int_A |f(x)|\,dx.$$

(c) The converse of this test is not true.

Proof (a) The inequality $0 \leq |f(x)| - f(x) \leq 2|f(x)|$ is valid for all $x \in A$. Now we apply the **Non-negative Comparison Test (Theorem 1.3.1)**, and with a straightforward manipulation we obtain the result.

(b) The relation

$$\left|\int_A f(x)\,dx\right| \leq \int_A |f(x)|\,dx$$

is valid, because it is valid for any nice function and any closed interval $[p, q] \subseteq A$, as we have learnt in a calculus course.

(c) In the sequel we shall see several examples that disprove the converse.

Before we present concrete examples using the **Absolute Convergence Test** and/or other tests, we need to clear out some things about what is **legitimate splitting of integrals into smaller parts**. So, begin with the following example.

Example 1.3.7 Consider the improper integral

$$\int_a^\infty f(x)dx = \lim_{\substack{N\to\infty \\ \sigma\to a^+}} \int_\sigma^N f(x)dx,$$

being improper at both endpoints only. That is, $f(x)$ is nice in the open interval (a,∞) and on any closed and bounded (finite) subinterval of (a,∞) its integral is proper.

Then, if we pick any two fixed numbers b and c such that $a < b < c < \infty$, we can always write

$$\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^\infty f(x)dx.$$

In this equality the first and the last summands are improper integrals only at the lower and the upper limit of integration, respectively. The middle summand is a proper integral.

The justification of such a splitting of this improper integral goes as follows: In taking the limits in this improper integral we do not lose anything by keeping $a < \sigma < b$ and $c < N < \infty$. Also, the following equality is always valid

$$\int_\sigma^N f(x)dx = \int_\sigma^b f(x)dx + \int_b^c f(x)dx + \int_c^N f(x)dx,$$

because all the integrals of this equality are proper. Therefore,

$$\int_a^\infty f(x)dx = \lim_{\substack{N\to\infty \\ \sigma\to a^+}} \int_\sigma^N f(x)dx =$$

$$\lim_{\substack{N\to\infty \\ \sigma\to a^+}} \left[\int_\sigma^b f(x)dx + \int_b^c f(x)dx + \int_c^N f(x)dx\right] =$$

$$\lim_{\sigma\to a^+} \int_\sigma^b f(x)dx + \int_b^c f(x)dx + \lim_{N\to\infty} \int_c^N f(x)dx =$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^\infty f(x)dx.$$

For the same reasons, if $a < a_1 < a_2 < ... < a_n < a_{n+1} < \infty$, for any $n \in \mathbb{N}$, we can write

$$\int_a^\infty f(x)dx = \int_a^{a_1} f(x)dx + \sum_{k=1}^{n} \int_{a_k}^{a_{k+1}} f(x)dx + \int_{a_{n+1}}^\infty f(x)dx.$$

▲

Now we continue with a simple, nevertheless useful, lemma that tells us when and how we can break an improper integral into denumerable summations of appropriately chosen smaller pieces. As we know from calculus, it is always possible to break a proper integral into summations of countably (finitely or denumerably) many smaller proper integrals. But, even though we can write an improper integral as a summation of finitely many smaller parts, as this was done in the previous example, not all splittings into denumerable summations of smaller integrals are legitimate. The following lemma describes conditions under which these denumerable summations are valid.

Lemma 1.3.1 ***(a)*** *Let $y = f(x)$ be a function on the interval $[a, c)$, where $-\infty < a < c$ and $c \in \mathbb{R}$ or $c = \infty$. Consider any (strictly) increasing sequence $a = a_0 < a_1 < a_2 < \ldots$ with $\lim\limits_{n\to\infty} a_n = c$. Then:*

If

(a_1) $\quad \displaystyle\int_a^c f(x)\,dx$ *exists and it is equal to a real number L,*

or

(a_2) $\quad f(x) \geq 0, \ \forall \ x \in [a, c),$

or

(a_3) $\quad f(x) \leq 0, \ \forall \ x \in [a, c),$

then

$$\int_a^c f(x)\,dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx\ .$$

(b) *Let $y = f(x)$ be a function on the interval $(c, b]$, where $c < b < \infty$ and $c \in \mathbb{R}$ or $c = -\infty$. Consider any (strictly) decreasing sequence $b = b_0 > b_1 > b_2 > \ldots$ with $\lim\limits_{n\to\infty} b_n = c$. Then:*

If

(b_1) $\quad \displaystyle\int_c^b f(x)\,dx$ *exists and it is equal to a real number L,*

or

(b_2) $\quad f(x) \geq 0, \ \forall \ x \in (c, b],$

or

(b_3) $\quad f(x) \leq 0, \ \forall \ x \in (c, b],$

then

$$\int_c^b f(x)\,dx = \sum_{n=0}^{\infty} \int_{b_{n+1}}^{b_n} f(x)\,dx\ .$$

Proof We shall prove **Case (a)** only, for **Case (b)** is just analogous. Also, **Subcase ($\mathbf{a_3}$)** is analogous to **Subcase ($\mathbf{a_2}$)** and so it suffices to prove only the **Subcases ($\mathbf{a_1}$)** and **($\mathbf{a_2}$)**.

In **Subcase ($\mathbf{a_1}$)**, we assume that $\displaystyle\int_a^c f(x)\,dx = L$ for some real number L. Then

$$\int_a^c f(x)\,dx = \lim_{R\to c^-} \int_a^R f(x)\,dx = L.$$

Since the limit exists, any legitimate limiting process whatsoever gives the number L as value of the limit. Therefore, for any increasing sequence with limit c ($a_n \uparrow c$ as $n \to \infty$) we have

$$\int_a^c f(x)\,dx = \lim_{n\to\infty} \int_{a=a_0}^{a_n} f(x)\,dx = L.$$

But, since we can break any proper integral into a finite sum of successive smaller proper integrals, we get, for $n \geq 1$

$$\int_{a=a_0}^{a_n} f(x)\,dx = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(x)\,dx.$$

Then putting the last two equations together, we obtain

$$\int_a^c f(x)\,dx = \lim_{n\to\infty} \int_{a=a_0}^{a_n} f(x)\,dx =$$

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(x)\,dx \ = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx \ = L.$$

In **Subcase ($\mathbf{a_2}$)**, we assume that $f(x) \geq 0, \ \forall \ x \in [a, c)$. If it happens that $\displaystyle\int_a^c f(x)\,dx$ exis, then we invoke the previous subcase and the proof is over. If the integral does not exist, since $f(x) \geq 0$ this means that

$$\int_a^c f(x)\,dx = \lim_{R\to c^-} \int_a^R f(x)\,dx = +\infty.$$

Now, given any R such that $a < R < c$, since $\lim\limits_{n\to\infty} a_n = c$, we can pick a term a_{n+1} of the sequence such that $R \leq a_{n+1}$. Then by the non-negativity of the function, we have

$$\int_a^R f(x)\,dx \leq \int_a^R f(x)\,dx + \int_R^{a_{n+1}} f(x)\,dx =$$

$$\int_a^{a_{n+1}} f(x)\,dx = \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} f(x)\,dx.$$

Also, for any given term a_{n+1} of the sequence, where $n = in\mathbb{N}_0$, we can pick a real R such that $a < R \leq a_{n+1}$ to obtain again a similar inequality.

Since

$$\lim_{R\to c^-} \int_a^R f(x)\,dx = \infty,$$

$a < R \leq a_{n+1}$ and $f(x) \geq 0$, we get

$$\infty = \lim_{R\to c^-} \int_a^R f(x)\,dx \leq \lim_{n\to\infty} \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} f(x)\,dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx.$$

Therefore,

$$\int_a^c f(x)\,dx = \lim_{R\to c^-} \int_a^R f(x)\,dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx = \infty.$$

Examples

Example 1.3.8 and **Remark**: The **previous lemma** does not apply in either of the two cases of $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ when these improper integrals do not exist.

For instance, in **Example 1.1.5** we have seen that the integral

$$\int_0^\infty \sin(x)\,dx \text{ does not exist.}$$

If we now let $a_n = 2n\pi$ for $n = 0, 1, 2, ...,$ which satisfies all the requirements of the lemma, then we get

$$\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \sin(x)\ dx = \sum_{n=0}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \sin(x)\ dx = \sum_{n=0}^{\infty} 0 = 0,$$

even though the integral itself does not exist.

With $a_n = n\pi$ for $n = 0,\ 1,\ 2,\ \ldots,$ we get

$$\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \sin(x)\ dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \sin(x)\ dx$$
$$= \sum_{n=0}^{\infty} 2(-1)^n = \text{does not exist,}$$

an answer different from the one found before.

(For your own practice, find some other sequences $(a_n)_{n\in\mathbb{N}}$ that satisfy the requirements of the lemma and yield other values for the respective infinite summation in this example.)

▲

Example 1.3.9 The improper integral

$$\int_0^\infty \frac{\sin(x)}{x^2+1}\,dx$$

is absolutely convergent and therefore convergent.

To show this, we notice

$$\int_0^\infty \left|\frac{\sin(x)}{x^2+1}\right| dx < \int_0^\infty \frac{1}{x^2+1}\, dx = \frac{\pi}{2}.$$

So,

$$\frac{-\pi}{2} < \int_0^\infty \frac{\sin(x)}{x^2+1}\, dx < \frac{\pi}{2}.$$

▲

Example 1.3.10 Let

$$f(x) = \frac{\sin(x)}{n+1} \quad \text{for} \quad n\pi \le x \le (n+1)\pi, \;\; n = 0,\ 1,\ 2,\ \dots .$$

With the help of **Lemma 1.3.1**, **Case** ($\mathbf{a_2}$), applied to the non-negative function $|f(x)|$, we obtain

$$\int_0^\infty |f(x)|\, dx =$$

$$\sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{n+1}\, dx = 2\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) = \infty.$$

So, the improper integral $\int_0^\infty f(x)\, dx$ diverges absolutely, and we cannot claim anything about its conditional convergence yet.

Since we do not know that this integral converges (exists), we cannot apply **Lemma 1.3.1** at this point in order to say that

$$\int_0^\infty f(x)\, dx \overset{?}{=}$$

$$\sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{n+1}\, dx = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) = 2\ln(2).$$

To prove that it converges (conditionally) we must use a different argument. If convergence is established first, then we can use **Lemma 1.3.1** in order to claim that this equation is correct and the value of the improper integral is $2\ln(2)$.

So, to prove that this integral converges, we proceed as follows: For any $R > 0$ there is an integer $k \ge 0$ such that $k\pi \le R < (k+1)\pi$. Then, $(R \longrightarrow \infty) \iff (k \longrightarrow \infty)$ (prove this "iff" as an exercise!) and by definition we have

$$\int_0^\infty f(x)\,dx \overset{def}{=}$$

$$\lim_{R\to\infty}\int_0^R f(x)\,dx = \lim_{R\to\infty}\left[\int_0^{(k+1)\pi} f(x)\,dx - \int_R^{(k+1)\pi} f(x)\,dx\right] =$$

$$\lim_{R\to\infty}\left\{2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^k\frac{1}{k+1}\right]-\int_R^{(k+1)\pi} f(x)\,dx\right\}.$$

But, the two partial limits inside the bracket exist. For the first one, we know

$$\lim_{R\to\infty} 2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^k\frac{1}{k+1}\right] = 2\ln(2).$$

For the second one, we have

$$\lim_{R\to\infty}\left[\int_R^{(k+1)\pi} f(x)\,dx\right] = 0,$$

since we easily observe that

$$0 < \left|\int_R^{(k+1)\pi} f(x)\,dx\right| \leq \int_{k\pi}^{(k+1)\pi} |f(x)|\,dx = \frac{2}{k+1} \longrightarrow 0,$$

$$\text{as} \quad k \longrightarrow \infty \iff R \longrightarrow \infty.$$

Since these two partial limits exist, we can take their difference to obtain

$$\int_0^\infty f(x)\,dx \overset{def}{=} \lim_{R\to\infty}\int_0^R f(x)\,dx =$$

$$2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\right) - 0 = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 2\ln(2).$$

Hence, this improper integral converges conditionally to the number $2\ln(2)$.

▲

Example 1.3.11 Let

$$g(x) = \frac{\sin(x)}{(n+1)^2} \quad \text{for} \quad n\pi \leq x \leq (n+1)\pi,\ n = 0,1,2,...$$

For the absolute convergence, we can apply **Case ($\mathbf{a_2}$)** of **Lemma 1.3.1** to the non-negative function $|g(x)|$ to find

$$\int_0^\infty |g(x)|\,dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{(n+1)^2}\,dx =$$

$$2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...\right) = 2\cdot\frac{\pi^2}{6} = \frac{\pi^2}{3}.$$

(See **Corollary 3.7.3 of Example 3.7.22** and **Problem 3.7.36**.) Hence, this improper integral converges absolutely, and so it converges.

Now, by **Case (a$_3$)** of the **Lemma 1.3.1** it is legitimate to say that

$$\int_0^\infty g(x)\,dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{(n+1)^2}\,dx =$$

$$2\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...\right) = 2\cdot\frac{\pi^2}{12} = \frac{\pi^2}{6}$$

(see **Problem 3.7.41**). This is the actual finite value of this integral.

▲

The ideas in **Example 1.3.10** motivate us to state the following useful lemma about convergence (existence) and estimation or actual evaluation of improper integrals. Its proof is omitted as analogous to the series of arguments presented in the solution of this example. (You can write it out for practice.) This lemma can also be used to justify the splitting of improper integrals into an infinite summation of smaller parts, and so it should be viewed together with **Lemma 1.3.1**.

Lemma 1.3.2 *(a) Let $y = f(x)$ be a function on an interval $[a, c)$, where $-\infty < a < c$ and $c \in \mathbb{R}$ or $c = \infty$. Consider any (strictly) increasing sequence $a = a_0 < a_1 < a_2 < ...$ with $\lim_{n\to\infty} a_n = c$.*

We assume:

(a_1) $$\lim_{n\to\infty} \sum_{k=0}^{n} \int_{a_n}^{a_{n+1}} f(x)\,dx = l, \quad \text{with} \quad -\infty \le l \le \infty.$$

(a_2) *For any real number R such that $a < R < c$ and the unique $k \in \mathbb{N}$ such that $a_k \le R < a_{k+1}$ [k depends on R and is unique since the sequence (a_n) is strictly increasing] we have:*

$$\lim_{R\to c^-} \int_R^{a_{k+1}} f(x)\,dx = 0.$$

Then,

$$\int_a^c f(x)\,dx = \sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx = l.$$

(b) Let $y = f(x)$ *be a function on an interval* $(c, b]$*, where* $c < b < \infty$ *and* $c \in \mathbb{R}$ *or* $c = -\infty$*. Consider any (strictly) decreasing sequence* $b = b_0 > b_1 > b_2 > \ldots$ *with* $\lim\limits_{n\to\infty} b_n = c$*.*

We assume:

$$(b_1) \qquad \lim_{n\to\infty} \sum_{k=0}^{n} \int_{b_{n+1}}^{b_n} f(x)\, dx = l, \quad \text{with} \quad -\infty \le l \le \infty.$$

(b_2) *For any real number* R *such that* $c < R < b$ *and the unique* $k \in \mathbb{N}$ *such that* $b_{k+1} < R \le b_k$ *[k depends on R and is unique since the sequence* (b_n) *is strictly decreasing] we have:*

$$\lim_{R\to c^+} \int_{b_{k+1}}^{R} f(x)\, dx = 0.$$

Then,

$$\int_c^b f(x)\, dx = \sum_{n=0}^{\infty} \int_{b_{n+1}}^{b_n} f(x)\, dx = l.$$

(For the **proof**, imitate the solutions of the **two previous examples**. For a kind of counterexample, see **Problem 2.2.19**.)

Remark 1: Given that in **this lemma** the $\int_{a_n}^{a_{n+1}} f(x)\, dx$ is assumed to be proper and therefore finite and R is any number in the interval $[a_n, a_{n+1})$, we can only assume that $\lim\limits_{R\to c^-} \int_R^{a_{n+1}} f(x)\, dx = 0$ and not another finite number, since for R close enough to a_{n+1} we can make this partial integral as close to zero as we wish.

Remark 2: Notice that **assumption** (a_2) of **this lemma** fails in **Example 1.1.5**. As we have seen in that example and in **Example 1.3.8**, the integral $\int_0^\infty \sin(x)\, dx$ does not exist.

We conclude this section with the very important and powerful criterion of Cauchy. We interpret it in the following way:

Theorem 1.3.7 (Cauchy Test) *Let* $y = f(x)$ *be a nice function on* $[a, c)$*, where* $a \in \mathbb{R}$ *and* $c \in \mathbb{R}$ *with* $a < c$*, or* $c = \infty$*. Consider the following three statements:*

(a) $\int_a^c f(x)\, dx$ *converges.*

(b) $\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{R}:\ a \le N < c$ *such that* $\forall\, p \in \mathbb{R}$ *and* $\forall\, q \in \mathbb{R}$ *such that* $N \le p,\ q < c$*, we have*

$$\left| \int_p^q f(x)\, dx \right| < \epsilon.$$

(c) $\forall\ r \in [a, c)$, *the integral* $\int_a^r f(x)\,dx$ *exists.*
Then we have:
(I) (a) implies (b).
(II) If (c) holds, then the converse of (I) is true, i.e., (b) implies (a).

Remark 1: We have analogous results for $\int_c^b f(x)\,dx$, on $(c, b]$, with $c < b$ in $\mathbb{R}$, or $c = -\infty$. (Write down these results explicitly, for practice.)

Remark 2: **Hypothesis (c)** is needed in **Part (II)** as seen in the **the example** that follows the proof. It is valid when $f(x)$ is continuous, or bounded in $[a, c)$, and in some other situations. In most applications, we are interested in using **Part (II)**.

Proof (I) We assume

$$\int_a^c f(x)\,dx = \lim_{M \to c^-} \int_a^M f(x)\,dx = L$$

exists as a real finite value L.

Then we let

$$F(M) = \int_a^M f(x)dx,\ \ \forall\ M \in [a, c).$$

By our assumption, the function $F(M)$ is well defined on $[a, c)$ and it is continuous in the variable M. (From calculus, we already know that the definite integral of a nice function is continuous with respect to its upper limit.) Also,

$$\lim_{M \to c^-} F(M) = L.$$

Now, $\forall\, \epsilon > 0$, we consider $\frac{\epsilon}{2} > 0$ and we use the analytical definition of the existence of a limit to claim that:

$\exists\ N\ :\ \ a \le N < c$ such that $\forall\ M\ :\ \ N \le M < c$ the inequality $|F(M) - L| < \frac{\epsilon}{2}$ is true.

Then, for any p and q: $N \le p$, $q < c$ we get

$$|F(q) - F(p)| = |F(q) - F(p) + L - L| \le$$
$$|F(q) - L| + |F(p) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since

$$|F(q) - F(p)| = \left| \int_a^q f(x)\,dx - \int_a^p f(x)\,dx \right| = \left| \int_p^q f(x)\,dx \right|,$$

we obtain the claim:

$$\forall\ \epsilon > 0,\ \exists\ N,\ a \le N < c \ \text{ such that, for any } p \text{ and } q \text{ in } \mathbb{R}:$$
$$N \le p,\, q < c \implies \left| \int_p^q f(x)\,dx \right| < \epsilon.$$

(II) By **hypothesis (c)** the function $F(r)$, as defined above in the proof of (I), is well defined on $[a, c)$.

Then, the hypotheses of this converse implication are translated as follows:

$$\forall\ \epsilon > 0,\ \exists\ N,\ a \le N < c \text{ such that, for any } p \text{ and } q\text{:}$$
$$N \le p,\, q < c \implies |F(q) - F(p)| < \epsilon.$$

By the **Cauchy General Criterion for convergence in the real line**[11] we readily obtain that $\lim\limits_{M\to\infty} F(M)$ exists as a finite real value.

Therefore, $\displaystyle\int_a^\infty f(x)\,dx$ exists, i.e., it is a finite real value.

---

[11] This **Cauchy criterion for convergence in the real line** that we have invoked here claims:

A sequence of real numbers (x_n) converges to a real number x if and only if it is a **Cauchy sequence**. This equivalence, by definition, is written in terms of positive ϵ's as follows:

$$[\exists\ x \in \mathbb{R} :\ \forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} :\ \forall\ n \ge N \implies |x_n - x| \le \epsilon] \iff$$
$$[\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}\ :\ \forall\ m \in \mathbb{N},\ \forall\ n \in \mathbb{N}\ :\ (m \ge N,\ n \ge N) \implies |x_m - x_n| \le \epsilon].$$

That is,

$$\left[\exists \lim_{n\to\infty} x_n \in \mathbb{R}\right] \iff \left[\lim_{\substack{m\to\infty\\ n\to\infty}} |x_m - x_n| = 0\right].$$

This very important criterion is a very powerful tool and it is equivalent to the completeness of the real numbers. It can be found in any book of mathematical analysis or advanced calculus.

Examples

Example 1.3.12 In the **Cauchy Test**, above, for the converse of *(I)* in **Part (II)**, **hypothesis (c)** is necessary. For instance, we let

$$f(x) = \begin{cases} 5, & \text{if} \quad x = 0 \\ \dfrac{1}{x}, & \text{if} \quad 0 < x < 1 \\ 0, & \text{if} \quad x \geq 1. \end{cases}$$

Obviously, $f(x)$ is a nice piecewise continuous function on $[0, \infty)$, and

$$\int_0^\infty f(x)dx = \ln(1) - \ln(0^+) = 0 - (-\infty) = \infty,$$

i.e., **(a)** is false.

But **(b)** is true since for any $\epsilon > 0$ we can pick $N \geq 1$. This happens because **(c)** fails, and so the function $F(r)$, in the above proof, is not well defined for all $r \in [0, \infty)$.

Another example is

$$g(x) = \begin{cases} 5, & \text{if} \quad x = 0 \\ \dfrac{1}{x}, & \text{if} \quad -1 < x \neq 0 < 1 \\ 0, & \text{if} \quad x \geq 1. \end{cases}$$

$g(x)$ is a nice piecewise continuous function on $[-1, \infty)$, and

$$\int_{-1}^\infty g(x)dx = -\infty + \infty = \text{does not exist},$$

i.e., (a) is false. But again **(b)** is true since for any $\epsilon > 0$ we can pick $N \geq 1$.

▲

Example 1.3.13 In this example (see also **Examples 2.2.8** and **3.7.33**), we will prove the following two important **results**:

(a) $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\, dx$ exists (converges to a finite real value).

(b) This integral does not converge absolutely and so it converges conditionally.

Proof of (a): The continuous function $f(x) = \dfrac{\sin(x)}{x}$ in $(0, \infty)$ can be continuously extended to $x = 0$ by letting $f(0) = 1$, since $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$. So, the **integral (a)** is improper only because it is taken over an unbounded interval.

Using integration by parts, we get that for any $0 < p < q$,

$$\int_p^q \frac{\sin(x)}{x}\, dx = \int_p^q \frac{-1}{x}\, d\cos(x) = \frac{\cos p}{p} - \frac{\cos q}{q} - \int_p^q \frac{\cos(x)}{x^2}\, dx.$$

Thus

$$\left| \int_p^q \frac{\sin(x)}{x}\, dx \right| \le \frac{1}{p} + \frac{1}{q} + \int_p^q \frac{|\cos(x)|}{x^2}\, dx \le \frac{1}{p} + \frac{1}{q} + \int_p^q \frac{1}{x^2}\, dx =$$

$$\frac{1}{p} + \frac{1}{q} + \left[\frac{-1}{x} \right]_p^q = \frac{2}{p} \to 0, \quad \text{as} \quad p \to \infty.$$

Then $\forall\ \epsilon > 0$, if we pick any $q > p > \dfrac{2}{\epsilon}$, we get $\left| \displaystyle\int_p^q \frac{\sin(x)}{x}\, dx \right| < \epsilon$.

Since here **condition (c)** of the Cauchy Test is valid, $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\, dx$ exists as a finite value, by the **Cauchy Test, 1.3.7**.

Proof of (b): This integral does not converge absolutely. Indeed, $|f(x)| \ge 0$, and so by **Lemma 1.3.1** we have

$$\int_0^\infty |f(x)|\, dx = \int_0^\infty \frac{|\sin(x)|}{x}\, dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}\, dx.$$

Now for all $n = 0,\ 1,\ 2,\ 3,\ \dots$, we have

$$\int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}\, dx \ge \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin(x)|\, dx = \frac{2}{(n+1)\pi}.$$

So,

$$\int_0^\infty |f(x)|\, dx = \int_0^\infty \frac{|\sin(x)|}{x}\, dx \ge \sum_{n=0}^\infty \frac{2}{(n+1)\pi} =$$

$$\frac{2}{\pi} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) = \infty.$$

Therefore,

$$\int_0^\infty \frac{|\sin(x)|}{x}\, dx = \infty.$$

Remark 1: By observing that the function $f(x) = \dfrac{\sin(x)}{x}$ is an even and continuous function over all $\mathbb{R}$, we also obtain that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = 2 \int_{0}^{\infty} \frac{\sin(x)}{x}\, dx.$$

Therefore, this improper integral over all $\mathbb{R}$ converges conditionally, too.

Remark 2: The same results of conditional convergence can be obtained for the improper integrals

$$\int_{0}^{\infty} \frac{\sin(\beta x)}{x}\, dx = \operatorname{sign}(\beta) \cdot \int_{0}^{\infty} \frac{\sin(u)}{u}\, du$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(\beta x)}{x}\, dx = \operatorname{sign}(\beta) \cdot \int_{-\infty}^{\infty} \frac{\sin(u)}{u}\, du,$$

for any real constant $\beta \neq 0$. (If $\beta = 0$ the integrals are obviously zero.)

The equality is obtained by letting $u = \beta x$ and the definition

$$\operatorname{sign}(\beta) = \begin{cases} +1, & \text{if} \quad \beta > 0 \\ \\ -1, & \text{if} \quad \beta < 0. \end{cases}$$

Remark 3: **Lemma 1.3.2** can also be used to prove convergence, except the **Cauchy Test** is more efficient.

▲

Example 1.3.14 The **Fresnel**[12] **integral** $\displaystyle\int_{0}^{\infty} \sin\left(x^2\right) dx$ converges conditionally.

We have

$$\int_{0}^{\infty} \sin\left(x^2\right) dx \overset{def}{=} \lim_{0<R\to\infty} \int_{0}^{R} \sin\left(x^2\right) dx.$$

We want to show that this limit exists. That is, every limiting process gives the same finite answer.

We let $x^2 = u$ or $x = \sqrt{u}$ and $dx = \dfrac{du}{2\sqrt{u}}$. We notice that $\displaystyle\lim_{u\to 0} \frac{\sin(u)}{\sqrt{u}} = 0$, and so we do not introduce any singularity at $x = 0$.

---

[12] Augustin Jean Fresnel, French mathematician and physicist, 1788-1827.

Then, for any $q > p > 0$, using integration by parts, we get

$$\int_p^q \sin\left(x^2\right) dx = \int_{p^2}^{q^2} \sin(u)\, \frac{du}{2\sqrt{u}} = \frac{1}{2}\int_{p^2}^{q^2} \frac{1}{\sqrt{u}}\, d\left[-\cos(u)\right] =$$

$$\frac{-1}{2}\left[\frac{\cos(u)}{\sqrt{u}}\right]_{p^2}^{q^2} + \frac{1}{2}\int_{p^2}^{q^2} \cos(u)\cdot d\left(u^{\frac{-1}{2}}\right) =$$

$$\frac{-1}{2}\left[\frac{\cos(q^2)}{q} - \frac{\cos(p^2)}{p}\right] - \frac{1}{4}\int_{p^2}^{q^2} \frac{\cos(u)}{u^{\frac{3}{2}}}\, du.$$

Therefore, by the properties of absolute value combined with inequalities and integrals and the fact that $|\cos(u)| \leq 1,\ \forall\ u \in \mathbb{R}$, we have

$$\left|\int_p^q \sin\left(x^2\right) dx\right| \leq \frac{1}{2}\left[\frac{1}{q} + \frac{1}{p}\right] + \frac{1}{4}\int_{p^2}^{q^2} \frac{1}{u^{\frac{3}{2}}}\, du =$$

$$\frac{1}{2}\left[\frac{1}{q} + \frac{1}{p}\right] - \frac{1}{2}\left[\frac{1}{\sqrt{u}}\right]_{p^2}^{q^2} = \frac{1}{p}.$$

Now, as in the **previous example**, $\forall\ \epsilon > 0$ we choose $p > \dfrac{1}{\epsilon}$ to get that $\forall\ q > p > \dfrac{1}{\epsilon}$ to guarantee the validity of the inequality

$$\left|\int_p^q \sin\left(x^2\right) dx\right| < \epsilon.$$

Since for any $r > 0$ the integral $\displaystyle\int_0^r \sin\left(x^2\right) dx$ exists, by the **Cauchy Test**

$$\int_0^\infty \sin\left(x^2\right) dx$$

converges to a finite real value.

Now we prove that the integral diverges absolutely. Again we let $x^2 = u$ to obtain

$$\int_0^\infty \left|\sin\left(x^2\right)\right|\, dx = \frac{1}{2}\int_0^\infty \left|\frac{\sin(u)}{\sqrt{u}}\right| du >$$

$$> \frac{1}{2}\int_0^1 \frac{|\sin(u)|}{\sqrt{u}}\, du + \frac{1}{2}\int_1^\infty \frac{|\sin(u)|}{u}\, du = \text{finite} + \infty = \infty,$$

by the analysis of the **previous example**! (All the steps here are directly justified by the definitions and the fact that we work with a non-negative function.)

So, the **Fresnel integral** $\int_0^\infty \sin\left(x^2\right) dx$ converges but diverges absolutely. Therefore, it converges conditionally.

With parallel work and analogous adjustments, we also obtain that the other **Fresnel integral**

$$\int_0^\infty \cos\left(x^2\right) dx.$$

converges conditionally. (See and solve **Problem 1.3.3** below.)

▲

Problems

1.3.1 Give examples of functions with discontinuities of the three types that we have stated in the non-standard **Definition 1.3.1**.

1.3.2 (1) Prove:

$$\text{(a)} \quad \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi\sqrt{3}}{9}, \qquad \text{(b)} \quad \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}.$$

[Hint: You need partial fractions and the integral rules with the natural logarithm and arc-tangent.]

(2) Use the results of the previous part and apply integration by parts to prove:

$$\text{(a)} \quad \int_0^\infty \frac{dx}{\left(1+x^3\right)^2} = \frac{4\pi\sqrt{3}}{27}, \qquad \text{(b)} \quad \int_0^\infty \frac{dx}{\left(1+x^4\right)^2} = \frac{3\pi\sqrt{2}}{16}.$$

1.3.3 The **Fresnel cosine integral** is

$$\int_0^\infty \cos\left(x^2\right) dx.$$

Prove that it exists but diverges absolutely.

In **the problems (1.3.4-1.3.11)** below, check in any possible way the existence or non-existence of the given improper integrals.

1.3.4

$$\text{(a)}\quad \int_0^\infty \frac{dx}{\sqrt{1+x^3}}, \qquad \text{(b)}\quad \int_0^\infty \frac{dx}{\sqrt{1+x^4}}.$$

1.3.5

$$\text{(a)}\quad \int_0^\infty \frac{\sin^2(x)}{x}\,dx, \quad \text{(b)}\quad \int_0^\infty \frac{\sin^2(x)}{x^2}\,dx.$$

1.3.6

$$\text{(a)}\quad \int_0^\infty \frac{\sin^3(x)}{x}\,dx, \quad \text{(b)}\quad \int_0^\infty \frac{\sin^3(x)}{x^2}\,dx, \quad \text{(c)}\quad \int_0^\infty \frac{\sin^3(x)}{x^3}\,dx.$$

1.3.7

$$\text{(a)}\quad \int_0^\infty \frac{e^{-x}\sin(x)}{x}\,dx, \quad \text{(b)}\quad \int_0^\infty \frac{x dx}{\sqrt{1+x^3}}.$$

1.3.8

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}-1}\,dx, \quad \text{where } a\in\mathbb{R} \text{ and } b>0 \text{ constants.}$$

1.3.9

$$\text{(a)}\quad \int_2^\infty \frac{dx}{[\ln(x)]^p}, \quad \text{(b)}\quad \int_1^\infty \frac{\sin(x)}{x^p}\,dx, \quad \text{(c)}\quad \int_1^\infty \frac{\cos(x)}{x^p}\,dx,$$

where $p>0$ constant.

1.3.10

$$\int_{-\infty}^\infty e^{\frac{-(x-\mu)^2}{\sigma^2}}\,dx, \quad \text{where } \mu\in\mathbb{R} \text{ and } \sigma\neq 0 \text{ constants.}$$

1.3.11

$$\text{(a)}\quad \int_1^{10} \frac{dx}{\sqrt{x-1}}, \qquad \text{(b)}\quad \int_0^1 \frac{dx}{\sqrt[3]{x-1}}.$$

1.3.12 Prove that the integrals

$$\text{(a)}\quad \int_0^\infty \sin\left(e^x\right)dx, \quad \text{(b)}\quad \int_0^\infty \cos\left(e^x\right)dx$$

and

$$\text{(c)} \quad \int_0^\infty \frac{\sin(x)}{\sqrt{x^2+1}}dx, \quad \text{(d)} \quad \int_0^\infty \frac{\cos(x)}{\sqrt{x^2+1}}dx$$

do not converge absolutely but only conditionally.

[Hint: For the first two, use $u = e^x$ and the **Cauchy Test, 1.3.7**.]

1.3.13 Prove that $\forall \ n \geq 0$ integer the following integrals exist:

$$\text{(a)} \quad \int_{-\infty}^\infty x^n e^{-x^2} dx, \quad \text{(b)} \quad \int_{-\infty}^\infty x^n e^{-|x|} dx.$$

If the integer is odd, then their values are zero.

1.3.14 Prove that for every $n \in \mathbb{N}$ the inequality $\sqrt{\dfrac{n}{2}} < \sqrt[n]{n!} < \dfrac{n+1}{2}$ holds and $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty$.

1.3.15 If (a_n) with $n \in \mathbb{N}$ is a sequence of positive numbers and the limits

$$\lim_{n\to\infty} \sqrt[n]{a_n} \quad \text{and} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exist, then prove that they are equal.

Chapter 2

Real Analysis Techniques

In this chapter, we present some Real Analysis Techniques for the computation of the precise value of some important improper integrals. All possible techniques are too many and too advanced to be included here. Also, at times we do not expose all possible mathematical rigor and generality. We relegate this to a course of Advanced Real Analysis.

2.1 Calculus Techniques

Here we present the very useful **Euler[1]-Poisson[2]-Gauß[3] Integral**.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{\infty} e^{-x^2}\,dx = 2\int_{-\infty}^{0} e^{-x^2}\,dx = \sqrt{\pi}. \tag{2.1}$$

By **Example 1.3.1**, it exists, that is, it equals to a positive finite value. The first two equalities are due to the fact that the function $f(x) = e^{-x^2}$ is positive and even over $(-\infty, \infty) = \mathbb{R}$.

This integral is very useful in various applications in mathematics, physics, engineering, probability and statistics. Many times, it is calculated in a multi-variable calculus course. To find its precise value, we work as follows:

(a) We evaluate the double integral

$$\iint\limits_{\overline{D(0,a)}} e^{-(x^2+y^2)}\,dxdy$$

where $\overline{D(0,a)} = \{\,(x,y)\,|\,x^2+y^2 \le a^2\}$ is the closed (circular) disc of center $(0,0)$ and radius $a > 0$.

---

[1] Leonhard Euler, Swiss mathematician, 1707-1783.

[2] Simon Denis Poisson, French mathematician, 1781-1840.

[3] Johann Carl Friedrich Gauß, German mathematician, 1777-1855.

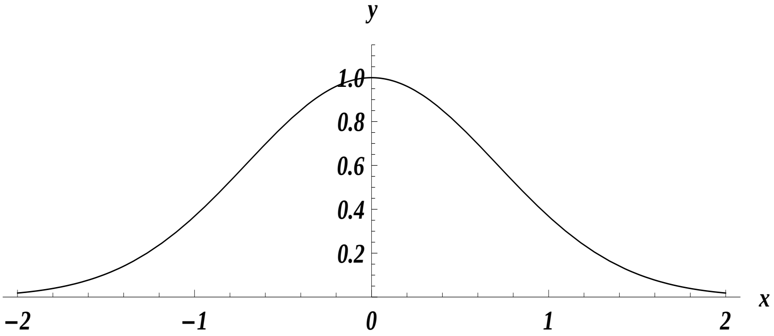

FIGURE 2.1: Function $\mathbf{y = e^{-x^2}}$

It is usually more convenient to switch to polar coordinates when we work with circular discs with center the origin. So,

$$x = r\cos(\theta),\;\; y = r\sin(\theta),\;\; x^2 + y^2 = r^2,\;\; dxdy = rdrd\theta$$

and

$$\overline{D(0,a)} = \{\,(r,\theta)\,|\,0 \le r \le a,\;\, 0 \le \theta \le 2\pi\}.$$

Hence the integral in polar coordinates $r,\ \theta$ is

$$\iint\limits_{\overline{D(0,a)}} e^{-(x^2+y^2)}\,dxdy = \int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta =$$

$$2\pi\left[\frac{-e^{-r^2}}{2}\right]_0^a = \pi\left(1 - e^{-a^2}\right).$$

(b) Take the limit as $a \to \infty$ to find

$$\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dxdy = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi(1-0) = \pi.$$

[Since the function $e^{-(x^2+y^2)}$ is continuous and positive in $\mathbb{R}^2$, advanced integration theory proves that it is integrable and for any non-negative integrable function any legitimate limit process yields the unique non-negative real or $+\infty$ value of its integral. See **Section 2.4 condition I**.]

(c) Now we view the integral

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dxdy,$$

as the limit of integrals over the rectangles $R_a = [-a,a] \times [-a,a]$, as $0 < a \to \infty$, i.e.,

$$\lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)}\,dxdy = \pi,$$

or

$$\lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dxdy = \pi.$$

Then

$$\lim_{a\to\infty} \left(\int_{-a}^{a} e^{-x^2}\,dx \cdot \int_{-a}^{a} e^{-y^2}\,dy\right) = \pi.$$

(d) In **Example 1.3.1** we have proved that

$$\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{exists.}$$

So, the last equation can be rewritten as

$$\left(\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2}\,dx\right) \cdot \left(\lim_{a\to\infty} \int_{-a}^{a} e^{-y^2}\,dy\right) = \pi.$$

These two limits are the same, and so

$$\left(\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi.$$

Finally, since $\int_{-\infty}^{\infty} e^{-x^2}\,dx > 0$ as an integral of a positive function, we can take square roots of both sides of the last equality to get the **result**

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Remark: Since $f(x) = e^{-x^2}$ is an even function over $\mathbb{R}$, we have that

$$\int_{0}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Important Note: If a function of two variables is non-negative or non-positive (i.e., does not change sign), then its double integral over a domain can be manipulated in any way and iterated in any order without affecting the final result. Otherwise, we would need the integral of its absolute value over its domain to exist or some other conditions that we will study later in **Section 2.4**. In the above example, the function $e^{-(x^2+y^2)}$ in $\mathbb{R}^2$ is positive. Therefore, our manipulations and iterations do not alter the existence and the uniqueness of the final answer.

Since the integral $\displaystyle\int_0^x e^{-t^2}\,dt$ cannot be found in closed form, we define, by means of the integral computed here, and use the following two standard functions in theory and application.

(1) The **error function** $\mathrm{erf}(x)$

$$\forall\ x \in \mathbb{R}, \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt. \tag{2.2}$$

Notice

$$\lim_{x \to \infty} \mathrm{erf}(x) = 1.$$

This function is very important to application and has been tabulated.

(2) The **complementary error function** $\mathrm{erfc}(x)$

$$\forall\ x \in \mathbb{R}, \quad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt.$$

Notice

$$\lim_{x \to \infty} \mathrm{erfc}(x) = 0.$$

For another useful integral representation of either function see and adjust **Example 2.2.14**.

2.1.1 Applications

Application 1: In probability and statistics, any **real continuous random variable** X takes values x in a set A, where $A \subseteq \mathbb{R}$ has nonempty interior. To such a random variable, we associate a **probability density function** $y = f(x)$, i.e., a function by which we find the probabilities

$$Pr(a \le x \le b) = \int_a^b f(x)dx,$$

for any real constants a and b with $a \le b$. Notice that

$$Pr(a < x < b) = Pr(a < x \le b) = Pr(a \le x < b) = Pr(a \le x \le b).$$

We set $f(x) = 0$ for every $x \in \mathbb{R} - A$. Then without loss of generality

$$f\ :\ \mathbb{R} \longrightarrow \mathbb{R}$$

satisfies the following three properties:

1. $y = f(x)$ is piecewise continuous.
2. $f(x) \geq 0, \quad \forall\ x \in \mathbb{R}$.
3. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now we define the $r^{\mathbf{th}}$ **moment about the origin of** X to be

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x)\,dx,$$

for $r = 0,\ 1,\ 2,\ 3,\ \ldots$. We are always interested in knowing:

(a) The **expected value** or the **mean value** or simply the **mean** or **average value** of X denoted and defined by:

$$\mu := E(X) := \mu'_1.$$

(b) The **variance** of X denoted and defined by:

$$var(X) := V(X) := \sigma^2 := \mu'_2 - \mu^2.$$

(c) The **moment-generating function** of X denoted and defined by:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Now, for any real constants μ and $\sigma > 0$, we define the function

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty.$$

We can prove that this function qualifies as a probability density function. It obviously satisfies **properties (1.)** and **(2.)** above, and we must prove **property (3.)**.

We use the change of variables $u = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$ and the **Integral (2.1)** to get:

$$\int_{-\infty}^{\infty} n(x;\mu,\sigma)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx =$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\,\sigma\,du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1.$$

If a real random variable $X = x$ has probability density function the function $n(x;\mu,\sigma)$, we say that X has the **normal distribution** with mean value μ and variance σ^2. When $\mu=0$ and $\sigma=1$, then

$$n(x;0,1)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\quad\text{for}\quad -\infty<x<\infty$$

and we say that X has the **standard normal distribution**.

We must check that indeed $E(X)=\mu$ and $var(X)=\sigma^2$ for this definition to be accurate. We use the same change of variables as before, and we have:

$$E(X)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx=$$

$$\frac{\sqrt{2}\,\sigma}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sqrt{2}\,\sigma u+\mu)e^{-u^2}\,du=\frac{1}{\sqrt{\pi}}(0+\mu)\,\sqrt{\pi}=\mu.$$

So,

$$E(X)=\mu.$$

Next,

$$\mu_2'=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^2\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx=$$

$$\frac{\sqrt{2}\,\sigma}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sqrt{2}\,\sigma u+\mu)^2e^{-u^2}\,du=$$

$$\frac{1}{\sqrt{\pi}}\left(\sqrt{\pi}\mu^2+0+2\sigma^2\int_{-\infty}^{\infty}u^2e^{-u^2}du\right).$$

We use integration by parts to find

$$\int_{-\infty}^{\infty}u^2\,e^{-u^2}du=-\int_{-\infty}^{\infty}u\;d\left(\frac{e^{-u^2}}{2}\right)=$$

$$\left[-u\frac{e^{-u^2}}{2}\right]_{-\infty}^{\infty}+\frac{1}{2}\int_{-\infty}^{\infty}e^{-u^2}du=0+\frac{\sqrt{\pi}}{2}.$$

So,

$$\mu_2'=\frac{1}{\sqrt{\pi}}\left(\sqrt{\pi}\mu^2+2\sigma^2\frac{\sqrt{\pi}}{2}\right)=\mu^2+\sigma^2.$$

Therefore,

$$var(X)=\mu_2'-\mu^2=\mu^2+\sigma^2-\mu^2=\sigma^2.$$

We can also find the moment-generating function of X. We have:

$$M_X(t) = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx =$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[-2xt\sigma^2+(x-\mu)^2\right]} dx.$$

We complete the square to write

$$-2xt\sigma^2 + (x-\mu)^2 = \left[x - \left(\mu + t\sigma^2\right)\right]^2 - 2\mu t\sigma^2 - t^2\sigma^4.$$

Then

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2} \left\{ \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{x-(\mu+t\sigma^2)}{\sigma}\right]^2} dx \right\} =$$

$$e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot 1 = e^{\mu t + \frac{1}{2}t^2\sigma^2}.$$

So,

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}.$$

Application 2: In thermodynamics, we learn that the average speed of molecules of an ideal gas is given by the formula

$$v_{av} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{\frac{3}{2}} \int_0^{\infty} v^3 e^{-\frac{Mv^2}{2RT}} dv,$$

where M is the molecular weight of the gas, R is a constant that depends on the gas under consideration, T is the absolute temperature and v is the molecular speed.

We can evaluate the improper integral involved and find the average speed in a closed form. For convenience, we write $a = \frac{M}{2RT}$, constant, and then we apply integration by parts. That is,

$$\int_0^{\infty} v^3 e^{-\frac{Mv^2}{2RT}} dv = \int_0^{\infty} v^3 e^{-av^2} dv = \int_0^{\infty} v^2 d\left(\frac{e^{-av^2}}{-2a}\right) =$$

$$\left[-\frac{v^2 e^{-av^2}}{2a}\right]_0^{\infty} + \int_0^{\infty} \frac{e^{-av^2}}{2a} 2v dv = 0 + \frac{1}{a}\int_0^{\infty} d\left[\frac{e^{-av^2}}{-2a}\right] =$$

$$\frac{1}{a}\left[\frac{e^{-av^2}}{-2a}\right]_0^{\infty} = \frac{1}{a}\left(0 - \frac{1}{-2a}\right) = \frac{1}{2a^2}.$$

Hence the average speed of the molecules is

$$v_{av} = \frac{4}{\sqrt{\pi}}\, a^{\frac{3}{2}} \frac{1}{2a^2} = \frac{2}{\sqrt{a\pi}} = \frac{2\sqrt{2RT}}{\sqrt{\pi M}} = 2\sqrt{\frac{2RT}{\pi M}}.$$

Problems

In **problems 2.1.1-2.1.10** find the precise values of the given improper integrals.

2.1.1

$$\int_{-\infty}^{\infty} e^{-3x^2}\,dx, \qquad \int_{-\infty}^{0} e^{-3x^2}\,dx \qquad \text{and} \qquad \int_{0}^{\infty} e^{-3x^2}\,dx.$$

2.1.2

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}}\,dx, \qquad \int_{\mu}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}}\,dx \quad \text{and} \quad \int_{-\infty}^{\mu} e^{-\frac{(x-\mu)^2}{\sigma^2}}\,dx,$$

where μ and $\sigma \neq 0$ are real constants.

2.1.3

$$\int_{0}^{\infty} xe^{-x^2}\,dx, \qquad \int_{-\infty}^{0} xe^{-x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} xe^{-x^2}\,dx.$$

2.1.4

$$\int_{0}^{\infty} x^2e^{-x^2}\,dx, \qquad \int_{-\infty}^{0} x^2e^{-x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} x^2e^{-x^2}\,dx.$$

2.1.5

$$\int_{-\infty}^{\infty} x^3e^{-x^2}\,dx, \qquad \int_{0}^{\infty} x^3e^{-x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{0} x^3e^{-x^2}\,dx.$$

2.1.6

$$\int_{0}^{\infty} xe^{-x}\,dx, \qquad \int_{-\infty}^{0} xe^{-x}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} xe^{-x}\,dx.$$

2.1.7

$$\int_{0}^{\infty} x^2e^{-x}\,dx, \qquad \int_{-\infty}^{\infty} x^2e^{-x}\,dx \quad \text{and} \quad \int_{-\infty}^{0} x^2e^{-x}\,dx.$$

2.1.8

$$\int_0^\infty x^3e^{-x}\,dx, \qquad \int_{-\infty}^0 x^3e^{-x}\,dx \quad \text{and} \quad \int_{-\infty}^\infty x^3e^{-x}\,dx.$$

2.1.9

$$\int_{-\infty}^\infty xe^{-x^4}\,dx, \qquad \int_{-\infty}^0 xe^{-x^4}\,dx \quad \text{and} \quad \int_0^\infty xe^{-x^4}\,dx.$$

[Hint: Let $u = x^2$, etc.]

2.1.10

$$\int_{-\infty}^\infty \sqrt{5}\,e^{-(x-10)^2}\,dx, \int_{-\infty}^{10} \sqrt{5}\,e^{-(x-10)^2}\,dx \,\text{and} \int_{10}^\infty \sqrt{5}\,e^{-(x-10)^2}\,dx.$$

2.1.11 If $\alpha > 0$ constant, then prove

$$\int_0^\infty e^{-\alpha x^2}\,dx = \int_{-\infty}^0 e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$

Hence, for $\alpha > 0$ constant we have

$$\int_{-\infty}^\infty e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}\;.$$

2.1.12 Write the erf(x) and erfc(x) as power series, with $x \in \mathbb{R}$.

2.1.13 Let

$$f(x) = \int_0^x e^{-t^2}\,dt.$$

Prove

$$\int_0^\infty e^{-x^2+f(x)}\,dx = e^{\frac{\sqrt{\pi}}{2}} - 1.$$

2.1.14 (a) Prove that for any real constant a

$$\int_0^\infty e^{\frac{-a}{x^2}}\,dx = \infty.$$

(b) If a and b are real constants, explain why we cannot split the integral

$$\int_0^\infty \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right)\,dx$$

as the difference

$$\int_0^{\infty} e^{\frac{-a}{x^2}}\,dx - \int_0^{\infty} e^{\frac{-b}{x^2}}\,dx.$$

(c) If $a \geq 0$ and $b \geq 0$ constants, then prove that

$$\int_0^{\infty} \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right) dx = \sqrt{\pi b} - \sqrt{\pi a}.$$

[Hint: Use $u = \dfrac{1}{x}$, integration by parts, L' Hôpital's rule, **Problem 2.1.11**, etc.]

(d) If $a > b \geq 0$ constants, then prove that

$$\int_0^{\infty} \left(e^{\frac{a}{x^2}} - e^{\frac{b}{x^2}}\right) dx = \infty.$$

2.1.15 (a) Show that

$$\int_0^1 \ln(u)\,du = -1, \quad \int_0^1 |\ln(u)|\,du = 1 \quad \text{and} \quad \int_{-2}^2 \ln|u|\,du = 4[\ln(2) - 1].$$

(b) Show that $\forall\ m = 0,\ 1,\ 2,\ 3,\ ...$ integer

$$\int_0^1 \ln^m(u)\ du = (-1)^m m! \qquad \text{and} \qquad \int_0^1 |\ln(u)|^m\ du = m!.$$

(See also **Problems 2.6.4** and **2.6.7**.)

[Hint: Use an integral formula or induction. Remember $0! = 1$.]

2.1.16 (a) Use the known facts $0 < \sin(x) < x$ for $0 < x \leq \dfrac{\pi}{2}$, $\lim_{x \to 0^+} \dfrac{\sin(x)}{x} = 1^-$, and **Problem 2.1.15 (a)** to prove that

$$I = \int_0^{\pi/2} \ln[\sin(x)]\,dx$$

exists and has value < -1.

(b) Use the appropriate trigonometric properties of sine and cosine and u-substitutions to show that the integral I in **(a)** is also equal to

$$I = \int_{\pi/2}^{\pi} \ln[\sin(x)]\,dx = \int_0^{\pi/2} \ln[\cos(x)]\,dx = \int_{\pi/2}^{\pi} \ln[|\cos(x)|]\,dx$$

and

$$I = \int_0^1 \frac{\ln(u)}{\sqrt{1-u^2}}\,du = \frac{1}{2}\int_{-1}^1 \frac{\ln(|u|)}{\sqrt{1-u^2}}\,du.$$

(c) Prove the relation

$$2I = \int_0^{\pi/2} \ln\left[\frac{\sin(2x)}{2}\right] dx.$$

(d) Consider the relation in **(c)** and prove that the common value of the integrals in **(a)** and **(b)** is

$$I = -\frac{\pi}{2}\ln(2).$$

(e) Prove that $\forall\ a > 0$

$$\int_0^{\pi/2} \ln[a\sin(x)]dx = \int_0^{\pi/2} \ln[a\cos(x)]dx = \frac{\pi}{2}\ln\left(\frac{a}{2}\right).$$

(f) Prove

$$\int_0^{\pi} \ln[\sin(x)]\,dx = \int_0^{\pi} \ln[|\cos(x)|]\,dx = -\pi\ln(2).$$

[See also **Problems 3.2.26 (d), 3.7.88** and **Example 3.7.47**.]

2.1.17 Show

$$(a) \quad \int_0^{\pi/4} \ln[\sin(\theta)]\ d\theta = \int_{\pi/4}^{\pi/2} \ln[\cos(\theta)]\ d\theta,$$

$$(b) \quad \int_{\pi/4}^{\pi/2} \ln[\sin(\theta)]\ d\theta = \int_0^{\pi/4} \ln[\cos(\theta)]\ d\theta.$$

[Hint: Use **Problem 2.1.16** for existence and let $u = \frac{\pi}{2} - \theta$.]

2.1.18 (a) Show

$$\int_0^{\pi/2} \ln\left[\tan(x)\right]\ dx$$

exists and is equal to 0.

(b) Then show

$$\int_0^{\pi} \ln\left[|\tan(x)|\right]\ dx = 0.$$

[Hint: Use **Problem 2.1.16** above.]

2.1.19 (a) Show that

$$\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx$$

exists and equals 0.

(See also **Example 3.7.44** and **Problem 3.7.90**.)

[Hint: Use $x = \tan(w)$ and **Problems 2.1.18** and **2.1.15, (a)**, above.]

(b) Prove

$$\frac{\pi}{4} - \frac{\ln(2)}{2} < \int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx < 1.$$

(See also **Example 3.7.45**.)

[Hint: $\forall\ x > 0,\ 1 - \dfrac{1}{x} \le \ln(x) \le x - 1$. Prove this inequality!]

2.1.20 Prove

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx = \frac{\pi\ln(2)}{8}.$$

[Hint: Notice that this integral is proper. Use first $x = \tan(w)$ and then prove and use the relation $\sin(w) + \cos(w) = \sqrt{2}\sin(\frac{\pi}{4} + w) = \sqrt{2}\cos(\frac{\pi}{4} - w)$. You may also need **Problem 2.1.17**.]

2.1.21 Prove

$$\int_{-1}^\infty \frac{\ln(x+1)}{x^2+1}\,dx = \frac{3\pi\ln(2)}{8}.$$

[Hint: Look at the hint of **Problem 2.1.20**.]

2.1.22 Show that

$$\int_0^\infty \frac{\ln(x)}{(x^2+1)^2}\,dx = -\frac{\pi}{4}.$$

(See also **Problem 3.7.92**.)

[Hint: Use **Problem 2.1.19 (a)** and integration by parts.]

2.1.23 For any $r \in \mathbb{R}$ prove that

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^r(x)}\,dx \overset{(x=\frac{\pi}{2}-u)}{=} \int_0^{\frac{\pi}{2}} \frac{\tan^r(u)}{1+\tan^r(u)}\,du \overset{\left[\tan(u)=\frac{1}{\cot(u)}\right]}{=}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cot^r(u)}{1+\cot^r(u)}\,du = \frac{\pi}{4}.$$

2.1.24 If $\alpha > 0$ and $\beta > 0$, prove the two general formulae

$$\text{(a)} \quad \int_0^\infty \frac{1}{\beta^2 + x^2}\, dx = \frac{\pi}{2\beta} \quad \text{and} \quad \text{(b)} \quad \int_0^\infty \frac{\ln(\alpha x)}{\beta^2 + x^2}\, dx = \frac{\pi}{2\beta} \ln(\alpha\beta).$$

(See also **Problem 3.7.90**.)

[Hint: For the first integral, use arc-tangent. For the second integral, use **Problem 2.1.19** and adjust.]

2.1.25 Consider the integral $\int_0^\infty e^{-x} \ln(x)\, dx$.

Prove:

(a) This integral converges absolutely.
(b) This integral is equal to

$$\int_0^1 \frac{e^{-t} + e^{\frac{-1}{t}} - 1}{t}\, dt.$$

(See also **Problem 2.3.6**.)
(c) The integral in **(b)** is proper.
(d) The value of these two equal integrals is negative.[4]

2.2 Integrals Dependent on Parameters

For difficult integrals (proper or improper) that depend on parameters, we may use the **Techniques of Continuity and Differentiability**, as illustrated in this section. Again, let $A \subseteq \mathbb{R}$ be any typical set used in the definitions of the improper integrals, which we have examined in **Chapter 1, Section 1.1**.

---

[4] In fact, with advanced mathematics, we can prove that this common value is $-\gamma < 0$, where γ is the **Euler-Mascheroni constant** or simply **Euler's constant** defined to be

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right] \simeq 0.57721566... > 0.$$

(See also **Problems 2.3.5** and **2.3.6**.)
(Lorenzo Mascheroni, Italian mathematician, 1750-1800.)
We still do not know if γ is rational or irrational, algebraic or transcendental.

We consider continuous or piecewise continuous functions $f(x,t)$ with $x \in A$ and t in some interval $I \subseteq \mathbb{R}$. If we consider the integral (proper or improper)

$$\int_A f(x,t)\,dx, \quad \forall\ t \in I,$$

then we call it an **integral with a parameter**, namely t.

We define the set

$$J = \{t \mid t \in I \quad : \quad \int_A f(x,t)\,dx \ \text{ exists}\} \subseteq I.$$

If this set is non-empty ($J \neq \emptyset$), this integral defines a function $F(t)$ on the set J and t is now viewed as a variable. Namely

$$F(t) = \int_A f(x,t)\,dx, \ \text{ with } t \in J.$$

Depending on the definition of $f(x,t)$ and the set A, we have $\emptyset \subseteq J \subseteq I$. So, if $J = \emptyset$, there is nothing to talk about. Otherwise, in the **Theorem** that follows, we address the interesting case where $J = I \subseteq \mathbb{R}$ is an interval of the form (α, β) or $[\alpha, \beta]$ or $[\alpha, \beta)$ or $(\alpha, \beta]$, where $-\infty \leq \alpha < \beta \leq \infty$.

From calculus, we know that when the integral is proper, hence $I = [a, b]$ is a closed and bounded interval with $-\infty < a < b < \infty$ constant real numbers, $f(x,t)$ is continuous and $\dfrac{\partial f(x,t)}{\partial t}$ is continuous, then $F(t)$ is differentiable and therefore continuous. But in case of an improper integral, even if $f(x,t)$ is continuous or differentiable, it does not follow automatically that $F(t)$ is continuous or differentiable at a given point $t_0 \in I$. To guarantee these outcomes, we need some extra conditions. Here we are going to state quite a general version of a Theorem for the continuity and differentiability of $F(t)$. Variations and generalizations of this Theorem may be found in advanced books, along with various proofs. Here we concentrate on the correct use of this Theorem as a tool for computing integrals, and we omit its proof as being above the level of this book. We like to refer to it as the **Main Theorem** of this section.

Theorem 2.2.1 *Let $f(x,t)$ be a real function "nice enough" in $x \in A \subseteq \mathbb{R}$ and continuous in $t \in I \subseteq \mathbb{R}$ where I is an interval of the form (α, β) or $[\alpha, \beta]$ or $[\alpha, \beta)$ or $(\alpha, \beta]$, with $-\infty \leq \alpha < \beta \leq \infty$.*

(I) ***Continuity:*** *Suppose that there exists a real function $g(x) \geq 0$, nice in A, such that*

$$|f(x,t)| \leq g(x), \ \forall\ x \in A \ \textit{ and } \forall\ t \in I, \ \textit{ and } \int_A g(x)dx < \infty.$$

Then the function

$$F(t) = \int_A f(x,t)\,dx, \;\; \textit{with } t \in I,$$

is a continuous real valued function in I. (At an endpoint of I, the continuity is understood as the suitable left or right side continuity.) So, $F(t)$ satisfies

$$\forall\ t_0 \in I, \qquad \lim_{I \ni\, t \to t_0} F(t) = F(t_0) = F(\lim_{I \ni\, t \to t_0} t)$$

or

$$\lim_{I \ni\, t \to t_0} \int_A f(x,t)\,dx = \int_A f(x,t_0)\,dx = \int_A \lim_{I \ni\, t \to t_0} f(x,t)\,dx.$$

Under these conditions the same result is (obviously) true for the real valued function

$$G(t) := \int_A |f(x,t)|\,dx, \quad t \in I.$$

(II) ***Differentiability****: Suppose that $\dfrac{\partial f(x,t)}{\partial t}$ exists for $t \in I$ and $x \in A$ and suppose that there exists a real function $g(x) \geq 0$ nice in A such that*

$$\left|\frac{\partial f(x,t)}{\partial t}\right| \leq g(x), \;\forall\ x \in A \;\; \textit{and}\; \forall\ t \in I, \;\; \textit{and} \;\; \int_A g(x)dx < \infty.$$

Then

$$F(t) = \int_A f(x,t)\,dx$$

is

$$\frac{dF}{dt}(t) = F'(t) = \int_A \frac{\partial f(x,t)}{\partial t}\,dx.$$

Remarks: On the previous theorem, we can make the following remarks.

1. The condition $\displaystyle\int_A g(x) < \infty$ implies the absolute convergence of $\displaystyle\int_A f(x,t)dx, \ \forall\ \ t \in I.$
2. The power of this Theorem and use of parameter(s) in integrals are illustrated in several examples that follow.

3. To check continuity and differentiability of $F(t)$, we need to check either property point by point for any point t, where $\alpha < t < \beta$. (Continuity and differentiability are local properties or pointwise properties.) So, we keep in mind that in order to do this, many times, we simply take any random $t \in (\alpha, \beta)$ and then a "small" interval $[c, d]$ or $(\alpha, d]$ or $[c, \beta)$ containing t and subset of (α, β). Then it is easier and more convenient to work over this new smaller subinterval of (α, β) for finding an appropriate choice of the function $g(x)$ over this subinterval only.

4. Sometimes we find a $g(x)$ for $f(x,t)\cdot$(a damping factor), as we shall see in some examples that follow.

5. The first part of the Theorem is essentially due to Weierstraß.[5] Both parts of this Theorem have been generalized in various ways by the Lebesgue[6] theory of integration. This result is stronger than those which require the uniform convergence of an integral dependent on a parameter as we encounter in other expositions. (See **Definition 2.3.4**.)

6. The second part of the **Theorem** proves the **Leibniz[7] rule** for differentiation of Riemann integrals over bounded closed intervals. This states:

   If $f(x,t)$ is continuous in (x,t) and continuously differentiable in t, where $x \in [a,b] \subset \mathbb{R}$ and $t \in (\alpha, \beta) \subseteq \mathbb{R}$, then

   $$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx, \quad \forall\ t \in (\alpha, \beta).$$

   Combining this rule with the **chain rule** (for the differentiation of composition of functions), we obtain the **general Leibniz rule**. This states:

   If $u(t)$ and $v(t)$ are differentiable real valued functions and $f(x,t)$

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[5] Karl Theodor Wilhelm Weierstraß, German mathematician, 1815-1897.

[6] Henri Lon Lebesgue, French mathematician, 1875-1941.

[7] Gottfried Wilhelm von Leibniz, German mathematician and philosopher, 1646-1716.

satisfies the above conditions in every interval $[u(t), v(t)]$ *of* x*, then*

$$\frac{d}{dt}\left[\int_{u(t)}^{v(t)} f(x,t)\,dx\right] =$$

$$\int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\left[v(t),t\right]\cdot v'(t) - f\left[u(t),t\right]\cdot u'(t).$$

(A direct proof, not invoking the **Main Theorem, 2.2.1,** can be found in advanced calculus books.)

7. We can use the **Leibniz Rule** to evaluate new definite integrals from known ones that depend on parameters.

Examples using the Leibniz Rule

Example 2.2.1 From the known integral

$$\int_0^b \frac{dx}{1+ax} = \frac{1}{a}\ln(1+ab)$$

with parameter $a > 0$ and upper limit $b > 0$ constant (check that this answer is correct), we obtain the following new integral formula

$$\int_0^b \frac{x dx}{(1+ax)^2} = \frac{1}{a^2}\ln(1+ab) - \frac{b}{a(1+ab)}$$

by differentiating both sides of the above equality with respect to the parameter a. (Compute the derivatives and confirm the correctness of the answer stated here.)

▲

Example 2.2.2 From the known integral

$$\int_0^b \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{b}{a}\right)$$

with parameter $a > 0$ and upper limit $b > 0$ constant (check that this answer is correct), we obtain the new integral formula

$$\int_0^b \frac{dx}{(a^2+x^2)^3} = \frac{b}{8a^4}\left[\frac{5a^2+3b^2}{(a^2+b^2)^2} + \frac{3}{ab}\arctan\left(\frac{b}{a}\right)\right]$$

by differentiating **twice** with respect to the parameter a and making the necessary adjustments. (Perform all missing steps and computations!)

▲

Example 2.2.3 Using calculus (the integration of rational expressions of sines and cosines), we find the integral formula

$$F(a,b)=\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2(x)+b^2\sin^2(x)}=\frac{\pi}{2ab}$$

with parameters $a>0$ and $b>0$. (We may achieve this result by the method of integrating rational functions of $\sin(x)$ and $\cos(x)$. Open a calculus book that contains this section and review this method one more time. See also **Examples 2.6.18**, **3.5.15** and **3.8.3**.)

Now, by differentiating with respect to the parameter a and multiplying by b and then differentiating with respect to the parameter b and multiplying by a, adding the results and dividing by $-2ab$, we obtain the following new integral formula

$$\frac{1}{-2ab}\left[b\frac{\partial F(a,b)}{\partial a}+a\frac{\partial F(a,b)}{\partial b}\right]=$$

$$\int_0^{\pi/2}\frac{dx}{\left[a^2\cos^2(x)+b^2\sin^2(x)\right]^2}=\frac{\pi\left(a^2+b^2\right)}{4(ab)^3}.$$

▲

Example 2.2.4 For your own practice, find known proper integrals with parameters from various sources, like tables, apply differentiation and necessary manipulation to derive new integral formulae, as we did in the previous three examples.

▲

Examples using the Main Theorem of this Section

Example 2.2.5 Technique of Continuity. (Compare the result obtained here with **Examples 3.7.7** and **3.7.8**.)

We illustrate this technique with the following example that requires a preliminary result. Using Complex Analysis methods (see **Examples 3.7.7** and **3.7.8**) we can prove the following result that we are going to use here:

For all m integers such that $0\le m<n$, we have:

$$\int_0^{\infty}\frac{x^{2m}}{1+x^{2n}}\,dx=\frac{\pi}{2n}\cdot\frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

Taking this fact for granted then by means of the **Continuity Part** of the above **Main Theorem, 2.2.1,** we prove the following very important and general result: For all $p\in\mathbb{R}$ such that $0<p<1$, we have:

$$\int_0^{\infty}\frac{t^{p-1}}{1+t}\,dt=\int_0^{\infty}\frac{1}{(1+t)t^{1-p}}\,dt=\frac{\pi}{\sin(p\pi)}.$$

(Also notice the convenient equivalent expression in **Problem 2.2.3**. See also **property (8.)** in **Section 2.6.2** for another method.)

The **proof** proceeds as follows: In the above known integral, we perform the substitution $x = t^{\frac{1}{2n}}$ to obtain

$$\int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{1+t}\,dt = \frac{\pi}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

Next, we shall show that the function

$$F(p) = \int_0^\infty \frac{t^{p-1}}{t+1}\,dt$$

is continuous in $p \in (0,1)$.

To this end, we first observe that the function $f(t,p) = \dfrac{t^{p-1}}{1+t}$ is obviously continuous for $0 < t < \infty$ and $0 < p < 1$.

We consider any $0 < p_1 < p_2 < 1$ and restrict the parameter p to the interval $(p_1, p_2) \subset (0,1)$.

Now we define the function

$$g(t) = \begin{cases} \dfrac{t^{p_1-1}}{1+t} & \text{if } 0 < t \le 1 \\[2ex] \dfrac{t^{p_2-1}}{1+t} & \text{if } 1 \le t < \infty. \end{cases}$$

This is a positive function, such that for all $0 < t < \infty$ and for all $p_1 < p < p_2$ it holds:

$$|f(t,p)| = f(t,p) \le g(t).$$

Also,

$$\int_0^\infty g(t)\,dt = \int_0^1 \frac{t^{p_1-1}}{t+1}\,dt + \int_1^\infty \frac{t^{p_2-1}}{t+1}\,dt.$$

We drop t from the denominator of the first partial integral and 1 from the denominator of the second one to obtain the inequality

$$\int_0^\infty g(t)\,dt < \int_0^1 t^{p_1-1}\,dt + \int_1^\infty t^{p_2-2}\,dt = \left[\frac{t^{p_1}}{p_1}\right]_0^1 + \left[\frac{t^{p_2-1}}{p_2-1}\right]_1^\infty =$$
$$\left(\frac{1}{p_1} - 0\right) + \left(0 - \frac{1}{p_2-1}\right) = \frac{1}{p_1} + \frac{1}{1-p_2} < \infty,$$

(positive finite value).

Then by the **Continuity Part** of the **Main Theorem, 2.2.1,** we conclude that the function

$$F(p) = \int_0^\infty \frac{t^{p-1}}{t+1}\,dt \quad \text{is continuous for all } p \in (p_1, p_2).$$

Since this consideration can apply to any $0 < p < 1$, by picking $0 < p_1 < p < p_2 < 1$, we have that $F(p)$ is continuous at every $p \in (0,1)$, or else it is a continuous function of $p \in (0,1)$.

Now, we **claim** that any $p \in (0,1)$ is the limit of a sequence of numbers of the form $q_{mn} := \dfrac{2m+1}{2n}$ with $n > m \geq 0$ integers.

To justify this claim, it is enough to show that between any two numbers of the interval $(0,1)$ there is a number of the form $\dfrac{2m+1}{2n}$ with $n > m \geq 0$ integers.

First, consider $p < q$ in $\mathbb{R}$. Then $0 < q - p$, and so there are integers $n > 0$ such that $2n(q-p) > 3$. Therefore, there is at least one odd integer $2m+1$ strictly between $2nq$ and $2np$. That is,

$$2nq > 2m+1 > 2np \quad \text{and so} \quad q > \frac{2m+1}{2n} > p.$$

Hence, between any two real numbers $p < q$ there is a fraction of the form $\dfrac{2m+1}{2n}$, with m and $n > 0$ integers.

If now $0 < p < q < 1$ and

$$1 > q > \frac{2m+1}{2n} > p > 0,$$

then without loss of generality we can consider $2m+1 > 0$ and $2n > 0$ and so $m \geq 0$ and $n > 0$. Also, since $1 > \dfrac{2m+1}{2n}$, we get $2n > 2m+1$ and so $n > m$.

So, we can find $0 \leq m < n$ integers, as the conditions in this problem require, such that for any $0 < p < q < 1$ we have the inequality

$$0 < p < \frac{2m+1}{2n} < q < 1,$$

which takes care of the **justification of the claim**.

We re-index q_{mn} as p_k with $k = 1, 2, 3, \dots$, so that $\lim\limits_{k\to\infty} p_k = p$. Then by the continuity of $F(p)$ we have that

$$F(p) = \lim_{k\to\infty} F(p_k).$$

Hence,

$$\forall\ p\ :\ 0<p<1,$$
$$\int_0^\infty \frac{t^{p-1}}{1+t}\,dt = \lim_{k\to\infty}\int_0^\infty \frac{t^{p_k-1}}{1+t}\,dt = \lim_{k\to\infty}\frac{\pi}{\sin\left(p_k\pi\right)} = \frac{\pi}{\sin\left(p\pi\right)},$$

finishing the proof.

▲

Example 2.2.6 Using the **previous example**, we obtain the following important **general results**:

(a) For all a and b **real numbers** such that $b \neq 0$ and $0 < \dfrac{a+1}{b} < 1$, we find:

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{1}{|b|}\cdot\frac{\pi}{\sin\left(\frac{a+1}{b}\pi\right)}.$$

(See also **Examples 3.7.7** and **3.7.8**.)

This is obtained by making the u-substitution $u = x^b$, or $x = u^{1/b}$. So, $du = bx^{b-1}dx$, and we adjust the limits to $u(0) = 0^b = 0$, $u(\infty) = \infty^b = \infty$ if $b > 0$ and to $u(0) = 0^b = \infty$, $u(\infty) = \infty^b = 0$ if $b < 0$. Hence,

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{1}{|b|}\int_0^\infty \frac{u^{\frac{a+1}{b}-1}}{1+u}\,du.$$

Since $0 < \dfrac{a+1}{b} < 1$, we apply the **previous example** to obtain:

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{1}{|b|}\cdot\frac{\pi}{\sin\left(\frac{a+1}{b}\pi\right)}.$$

For **example**, if $a = -\frac{1}{3}$ and $b = 1$, we find

$$\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x}\,dx = \frac{\pi}{\sin\left[\left(-\frac{1}{3}+1\right)\pi\right]} = \frac{\pi}{\sin\left(\frac{2\pi}{3}\right)} = \frac{2\sqrt{3}\,\pi}{3}.$$

Similarly,

$$\int_0^\infty \frac{x^{\sqrt{3}}}{1+x^\pi}\,dx = \frac{1}{\pi}\cdot\frac{\pi}{\sin\left(\frac{\sqrt{3}+1}{\pi}\cdot\pi\right)} = \frac{1}{\sin(1+\sqrt{3})}.$$

(b) In particular, for all $m \geq 0$ and $n \geq 2$ **integers** such that $0 \leq m < n-1$, we have:

$$\int_0^\infty \frac{x^m}{1+x^n}\,dx = \frac{1}{n}\cdot\frac{\pi}{\sin\left(\frac{m+1}{n}\pi\right)}.$$

For **example**, if $m = 0$ and $n = 4$, we find

$$\int_0^\infty \frac{dv}{1+v^4} = \frac{1}{4}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi\sqrt{2}}{4}.$$

(Compare this method with the elementary method in **Problem 1.3.2** for both questions of its **Part (1)**. This integral is also discussed in **Example 2.4.1** using partial fractions and in **Problem 2.6.20** using the Beta and Gamma functions.)

(c) If **both integers** m and n are **even**, then the integrand is an **even function** over all $\mathbb{R}$, and in such a case we also get:

For all m and n **even** integers such that $0 \le m < n-1$, we have:

$$\int_{-\infty}^\infty \frac{x^m}{1+x^n}\,dx = 2\int_0^\infty \frac{x^m}{1+x^n}\,dx = \frac{2}{n}\cdot\frac{\pi}{\sin\left(\frac{m+1}{n}\pi\right)}.$$

For **example**, if $m = 2$ and $n = 8$, we find

$$\int_{-\infty}^\infty \frac{v^2\,dv}{1+v^8} = \frac{2}{8}\cdot\frac{\pi}{\sin\left(\frac{2+1}{8}\pi\right)} =$$

$$\frac{\pi}{4\sin\left(\frac{3}{8}\pi\right)} = \frac{\pi}{4\cos\left(\frac{\pi}{8}\right)} = \frac{\pi}{4\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}} = \frac{\pi}{2\sqrt{2+\sqrt{2}}}.$$

(d) If m is **odd** and n is **even**, then the integrand is an **odd function** over all $\mathbb{R}$, and in such a case we also get:

For all integers m **odd** and n **even** such that $0 \le m < n-1$, we have:

$$\int_{-\infty}^\infty \frac{x^m}{1+x^n}\,dx = 0.$$

▲

Example 2.2.7 Let us consider

$$\int_0^\infty \frac{7x^2}{2+9x^5}\,dx = \frac{7}{2}\int_0^\infty \frac{x^2}{1+\frac{9}{2}x^5}\,dx = 3.5\int_0^\infty \frac{x^2}{1+\left(\sqrt[5]{4.5}\;x\right)^5}\,dx.$$

Now we set $u = \sqrt[5]{4.5}\;x$ to find

$$\int_0^\infty \frac{7x^2}{2+9x^5}\,dx = 3.5\cdot\frac{1}{\sqrt[5]{(4.5)^3}}\int_0^\infty \frac{u^2}{1+u^5}\,du = \frac{3.5}{\sqrt[5]{(4.5)^3}}\cdot\frac{1}{5}\cdot\frac{\pi}{\sin\left(\frac{2+1}{5}\,\pi\right)}.$$

Therefore,

$$\int_0^\infty \frac{7x^2}{2+9x^5}\,dx = \frac{0.7}{\sqrt[5]{(4.5)^3}}\cdot\frac{\pi}{\sin(0.6\,\pi)} =$$

$$\frac{7\,\pi}{10\sqrt[5]{(4.5)^3}\,\sin(0.6\,\pi)} \simeq 0.937808....$$

(In this kind of situation, it may be more convenient to keep the numeric fractions instead of changing them to decimals.)

▲

Example 2.2.8 Technique combining Continuity and Differentiability. (The result obtained here is also obtained by complex analysis in **Example 3.7.33**.)

In **Example 1.3.13**, we have shown that the integral

$$\int_0^\infty \frac{\sin(x)}{x}\,dx$$

converges conditionally.

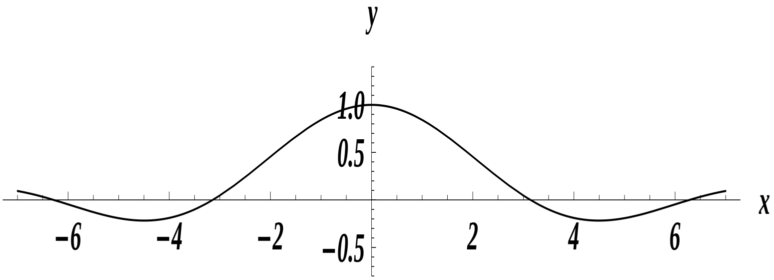

FIGURE 2.2: Function $\mathbf{y = \frac{\sin(x)}{x}}$

In **Remark 2** of the **that example**, we have seen that by letting $u = \beta x$ for any real constant $\beta \neq 0$, we find

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx = \operatorname{sign}(\beta)\cdot\int_0^\infty \frac{\sin(u)}{u}\,du.$$

Therefore, this integral converges conditionally. (Remember that $\operatorname{sign}(\beta) = +1$ if $\beta > 0$ and $\operatorname{sign}(\beta) = -1$ if $\beta < 0$. For $\beta = 0$, the integral is trivially zero.)

Here, we are going to evaluate the integral

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx$$

for any real constant $\beta \neq 0$.

To apply the **Main Theorem, 2.2.1,** we need some absolute convergence and limit process. To this end, for any $\beta \in \mathbb{R}$ fixed, we consider this integral to be a part of the following more general integral

$$F(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx,\ \forall\ \alpha \geq 0.$$

The parameter $\alpha \in I := [0, \infty)$ of this integral is considered to be the variable of the function $F(\alpha)$.

If $\alpha = 0$, we notice that the integral

$$F(0) = \int_0^\infty \frac{\sin(\beta x)}{x}\, dx$$

exists, as we have already proved in **Example 1.3.13** and its **Remark 2**.

If $\alpha > 0$, we shall prove that the integral $F(\alpha)$ exists because it converges absolutely. We can show this as follows:

For any fixed $\beta \in \mathbb{R}$, the function $\dfrac{\sin(\beta x)}{x}$ is continuously defined at $x = 0$ by assigning the value $f(0) = \beta$. This follows by L' Hôpital's rule, since

$$\lim_{x \to 0} \frac{\sin(\beta x)}{x} = \lim_{x \to 0} \frac{\beta \cos(\beta x)}{1} = \frac{\beta}{1} = \beta.$$

Next, at ∞ we get

$$\lim_{x \to \infty} \frac{\sin(\beta x)}{x} = 0.$$

Then, by a simple modification of the Extreme Value Theorem for Continuous Functions (study it one more time and make the modification in this situation), we get that there exists a finite constant M such that:

$$0 < M = \operatorname*{maximum}_{0 \le x < \infty} \left| \frac{\sin(\beta x)}{x} \right| < \infty.$$

Hence $\left| e^{-\alpha x} \dfrac{\sin(\beta x)}{x} \right| \le M \cdot e^{-\alpha x}, \ \forall\ x \in [0, \infty)$. Then

$$\int_0^\infty \left| e^{-\alpha x} \frac{\sin(\beta x)}{x} \right| dx \le \int_0^\infty M e^{-\alpha x}\, dx = \frac{M}{\alpha}.$$

Finally, for $\alpha > 0$, the integral

$$F(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin(\beta x)}{x}\, dx$$

converges absolutely, and therefore it converges.

For $\alpha = 0$, as we explained at the beginning, the integral

$$F(0) = \int_0^\infty \frac{\sin(\beta x)}{x}\, dx$$

converges conditionally.

In the sequel, we will firstly show that for any given $\beta \neq 0$, the real valued function $F(\alpha)$ is continuous for all $0 \leq \alpha < \infty$. Notice here that the parameter interval is $I = [0, \infty)$ including $\alpha = 0$. (When $\beta = 0$ then $F(\alpha)$ is obviously continuous, since in this case $F(\alpha) = 0$, constant for all α.)

Secondly, for any given $\beta \neq 0$ fixed, we will evaluate $F(\alpha)$ for any $\alpha > 0$. Then we will compute the limit of $F(\alpha)$ as $\alpha \to 0^+$, and so by the **Continuity Part** of the **Main Theorem, 2.2.1,** we will find the value of the integral $\displaystyle\int_0^\infty \frac{\sin(\beta x)}{x}\,dx$, which exists (as we already knew).

(**Note**: This kind of method is used quite often with improper integrals. We imbed the given integral into a more general one which we can evaluate, and then we manipulate the value of the more general integral to find the value of the given particular integral.)

To prove the continuity of $F(\alpha)$ in α, with $\alpha \in I = [0, \infty)$, we must apply **Part (I)** of the **Main Theorem, 2.2.1**. We therefore need to find an appropriate function $g(x)$, as in the Theorem. This is not so immediate in this example because we include the value $\alpha = 0$, and so we cannot find a good $g(x)$ that works throughout the whole $I = [0, \infty)$.

Therefore, we are going to work as follows: For any given $\beta \neq 0$ fixed and $0 \leq \alpha < \infty$, we first write

$$F(\alpha) = \int_0^1 e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx + \int_1^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx.$$

That is, we write the integral $F(\alpha)$ as a sum of these two smaller parts. Then we need to prove that each of these parts is itself a continuous function of α. In doing so, we may use the **Continuity Part** of the **Main Theorem, 2.2.1,** applied to some or all of the smaller parts of this integral.

$$(*) \qquad \int_0^1 e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx$$

is a **continuous** function of α, for all $0 \leq \alpha < \infty$.

This is so because we can apply the **Continuity Part** of the **Main Theorem, 2.2.1,** if, for example, we pick

$$g(x) = M, \quad \text{where} \quad 0 < M = \underset{0 \leq x \leq 1}{\text{maximum}}\frac{|\sin(\beta x)|}{x} < \infty,$$

for which

$$\int_0^1 g(x)dx = M < \infty.$$

Notice, of course, that $\forall\ \alpha \geq 0$ constant, $0 < e^{-\alpha x} \leq 1,\ \forall\ x \in [0, \infty)$ and M is a positive finite constant, guaranteed by the Extreme Value

Theorem for Continuous Functions on a closed and bounded interval $[a, b]$.

Next, the second partial integral of $F(\alpha)$

$$\int_1^\infty e^{-\alpha x} \frac{\sin(\beta x)}{x}\, dx$$

must be first transformed in the following way: We perform integration by parts after choosing

$$u = \frac{1}{x} \qquad \text{and} \qquad dv = e^{-\alpha x} \sin(\beta x) dx.$$

The indefinite integration of dv up to a constant, which we take here to be zero, gives:

$$v(x) = \frac{-e^{-\alpha x} [\alpha \sin(\beta x) + \beta \cos(\beta x)]}{\alpha^2 + \beta^2}.$$

We let $\cos(\phi) = \dfrac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$ and $\sin(\phi) = \dfrac{\beta}{\sqrt{\alpha^2 + \beta^2}}$ to implicitly define the function $\phi(\alpha)$ (remember $\beta \neq 0$ is fixed). By adding to ϕ multiples of 2π whenever we go across such multiples, the function $\phi(\alpha)$ can be defined to be continuous in α. So, we can rewrite

$$v(x) = v(x; \alpha) = \frac{-e^{-\alpha x} \sin[\beta x + \phi(\alpha)]}{\sqrt{\alpha^2 + \beta^2}}$$

and $v(x) = v(x; \alpha)$ is continuous in $\alpha \geq 0$, since $\beta \neq 0$.

Now performing integration by parts, we obtain

$$\int_1^\infty e^{-\alpha x} \frac{\sin(\beta x)}{x}\, dx = \frac{e^{-\alpha} \sin(\beta + \phi)}{\sqrt{\alpha^2 + \beta^2}} - \int_1^\infty \frac{e^{-\alpha x} \sin(\beta x + \phi)}{x^2 \sqrt{\alpha^2 + \beta^2}}\, dx.$$

We observe that the function

$$(**) \qquad \frac{e^{-\alpha} \sin[\beta + \phi(\alpha)]}{\sqrt{\alpha^2 + \beta^2}}$$

is **continuous** in α, for all $0 \leq \alpha < \infty$, because $\beta \neq 0$.

Next, the integral

$$(***) \qquad \int_1^\infty \frac{e^{-\alpha x} \sin(\beta x + \phi)}{x^2 \sqrt{\alpha^2 + \beta^2}}\, dx$$

is **continuous** in α for $0 \leq \alpha < \infty$, because we can apply the **Continuity Part (I) of the Main Theorem, 2.2.1,** if we choose $g(x) = \dfrac{1}{|\beta| x^2}$.

Finally, $F(\alpha)$ is **continuous** in α, for $0 \leq \alpha < \infty$, since it is the sum of the three continuous functions stated in **(*)**, **(**)**, **(***)** above.

Now we are going to evaluate the integral $F(\alpha)$ for any $\alpha > 0$. To do this, we look at the integral as a function of the parameter β. That is, we let

$$H(\beta) = \int_0^\infty e^{-\alpha x} \frac{\sin(\beta x)}{x}\, dx \quad \text{with} \quad \beta \in \mathbb{R}.$$

As we proved earlier, $H(\beta)$ exists $\forall\ \beta \in \mathbb{R}$ and $\forall\ \alpha \geq 0$. (We trivially observe $H(0) = 0,\ \forall\ \alpha \geq 0$.)

By the **Differentiability Part (II)** of the **Main Theorem, 2.2.1,** we can compute the derivative $\dfrac{d}{d\beta}[H(\beta)]$ by differentiating under the integral sign because:

$$\frac{d}{d\beta}\left[e^{-\alpha x}\frac{\sin(\beta x)}{x}\right] = e^{-\alpha x}\cos(\beta x),$$

and so for fixed given $\alpha > 0$, regardless of $\beta \in \mathbb{R}$, this derivative is absolutely bounded by the positive function $g(x) = e^{-\alpha x}$, whose integral $\displaystyle\int_0^\infty e^{-\alpha x}\, dx = \frac{1}{\alpha}$ is finite. Therefore, the **Differentiability Part** of the **Main Theorem, 2.2.1,** applies to obtain:

$$\frac{d}{d\beta}[H(\beta)] =$$

$$\int_0^\infty e^{-\alpha x}\frac{d}{d\beta}\left[\frac{\sin(\beta x)}{x}\right] dx = \int_0^\infty e^{-\alpha x}\cos(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2}.$$

Since $H(0) = 0$, we find that $H(\beta)$ satisfies the initial value-problem

$$\frac{d}{d\beta}[H(\beta)] = \frac{\alpha}{\alpha^2+\beta^2} \qquad \text{and} \qquad H(0) = 0.$$

The solution of this initial value-problem is found easily to be

$$H(\beta) = \int_0^\beta \frac{\alpha}{\alpha^2+u^2}\, du = \arctan\left(\frac{\beta}{\alpha}\right).$$

So, we have obtained the important **result** that for all $\alpha > 0$ and $\beta \in \mathbb{R}$ constants, we have

$$\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\, dx = \arctan\left(\frac{\beta}{\alpha}\right).$$

Hence, by continuity with respect to α in $[0,\infty)$, we obtain

$$F(0)=\lim_{\alpha\to 0^+}F(\alpha)=\lim_{\alpha\to 0^+}\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx,$$

or

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx=\lim_{\alpha\to 0^+}\arctan\left(\frac{\beta}{\alpha}\right)=\begin{cases}\dfrac{\pi}{2} & \text{if } \beta>0\\[2ex] 0 & \text{if } \beta=0\\[2ex] -\dfrac{\pi}{2} & \text{if } \beta<0.\end{cases}$$

(For **example**, $\int_0^\infty \frac{\sin(x)}{x}\,dx=\int_0^\infty \frac{\sin(2x)}{x}\,dx=\frac{\pi}{2}$, etc.)

Also, since the function $\frac{\sin(\beta x)}{x}$ is even in $\mathbb{R}$, we have:

$$\int_{-\infty}^\infty \frac{\sin(\beta x)}{x}\,dx=\begin{cases}\pi & \text{if } \beta>0\\[2ex] 0 & \text{if } \beta=0\\[2ex] -\pi & \text{if } \beta<0.\end{cases}$$

▲

Four consequences of the previous result:

Example 2.2.9 In the course of computation in the **previous example**, we have shown the following useful integral **result** that we sometimes find in integral tables:

$$\forall\ \alpha>0 \ \text{ and } \ \beta\in\mathbb{R} \ \text{ constants,} \quad \int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx=\arctan\left(\frac{\beta}{\alpha}\right).$$

▲

Example 2.2.10 In advanced problems and computations, many times we need to write the function of **absolute value in the integral form**

$$|\beta|=\frac{2\beta}{\pi}\int_0^\infty \frac{\sin(\beta x)}{x}\,dx, \quad \text{for} \quad -\infty<\beta<\infty.$$

(E.g., see **Application 1** of **Subsection 3.7.6**.) ▲

Example 2.2.11

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \int_0^\infty \sin^2(x)\,d\left(\frac{-1}{x}\right) =$$

$$\left[\frac{-\sin^2(x)}{x}\right]_0^\infty + \int_0^\infty \frac{2\sin(x)\cos(x)}{x}\,dx = 0 + \int_0^\infty \frac{\sin(2x)}{x}\,dx = \frac{\pi}{2}.$$

▲

Example 2.2.12

$$\int_0^\infty \frac{1-\cos(x)}{x^2}\,dx = \int_0^\infty \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}\,dx \overset{x=2u}{=} \int_0^\infty \frac{\sin^2(u)}{u^2}\,dx = \frac{\pi}{2}.$$

▲

In **Example 2.2.8**, we used the technique of differentiability. We continue with another important example illustrating this technique.

Example 2.2.13 Technique of Differentiability. (Compare this example with **Problem 2.2.38**. The result obtained here is also obtained by complex analysis in **Example 3.7.17** and **Problem 3.7.24**.)

For any $\alpha > 0$ constant and any given $\beta \in \mathbb{R}$ constant, the integral

$$I(\beta) = \int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx$$

converges absolutely, since by **Problem 2.1.11**

$$\int_0^\infty \left|e^{-\alpha x^2}\cos(\beta x)\right|\,dx \le \int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} < \infty.$$

This also proves that $I(\beta)$ is a continuous function for all $\beta \in \mathbb{R}$.

(Here, we are interested in the finiteness of this integral regardless of β and not in its exact value. The exact value proves finiteness, but most of the times we can prove finiteness without knowing the exact value.)

Next, we show that $I(\beta)$ is differentiable. We have:

$$\frac{d}{d\beta}\left[e^{-\alpha x^2}\cos(\beta x)\right] = -xe^{-\alpha x^2}\sin(\beta x).$$

Then for any real constant β

$$\left|-xe^{-\alpha x^2}\sin(\beta x)\right| \le xe^{-\alpha x^2}$$

and for any $\alpha > 0$ if we let $u = \alpha x^2$, we find

$$\int_0^\infty xe^{-\alpha x^2}\,dx = \int_0^\infty e^{-u}\,\frac{du}{2\alpha} = \frac{1}{2\alpha}[-e^{-u}]_0^\infty = \frac{1}{2\alpha}\cdot 1 = \frac{1}{2\alpha} < \infty\,.$$

So, we can apply **Part(II)** of the **Main Theorem, 2.2.1,** to get

$$\frac{d}{d\beta}\left[I(\beta)\right] = \int_0^\infty e^{-\alpha x^2}(-x)\sin(\beta x)\,dx = \int_0^\infty \sin(\beta x)\ d\left(\frac{e^{-\alpha x^2}}{2\alpha}\right) =$$

$$\left[e^{-\alpha x^2}\frac{\sin(\beta x)}{2\alpha}\right]_0^\infty - \frac{\beta}{2\alpha}\int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx = -\frac{\beta}{2\alpha}I(\beta).$$

Therefore, $I(\beta)$ satisfies the **homogeneous first-order ordinary differential equation**

$$\frac{d}{d\beta}\left[I(\beta)\right] = -\frac{\beta}{2\alpha}\,I(\beta).$$

By **Problem 2.1.11**, $I(\beta)$ also satisfies the **initial condition**

$$I(0) = \int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{\sqrt{\alpha}}\int_0^\infty e^{-u^2}\,du = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$

The solution of the initial value-problem

$$\frac{d}{d\beta}\left[I(\beta)\right] = -\frac{\beta}{2\alpha}\,I(\beta)$$
$$I(0) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

is obtained easily by separating the variables $\dfrac{dI}{I} = -\dfrac{\beta}{2\alpha}\,d\beta$ when $I(\beta) \neq 0$ and then integrating. We finally find:

$$I(\beta) = \int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\ e^{\frac{-\beta^2}{4\alpha}},$$

for any $-\infty < \beta < \infty$ and any $\alpha > 0$. (Fill in the details of the integrations.)

Note: Since $e^{-\alpha x^2}\cos(\beta x)$ is an even function in $(-\infty, \infty)$,

$$\int_{-\infty}^\infty e^{-\alpha x^2}\cos(\beta x)\,dx = \sqrt{\frac{\pi}{\alpha}}\ e^{\frac{-\beta^2}{4\alpha}}.$$

(Compare this with **Example 3.7.17** and **Problem 3.7.24**.)

▲

Example 2.2.14 In this example, we prove that for any $a \geq 0$ and $k \geq 0$ constants but not both zero

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-kx}\sin(a\sqrt{x})}{x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ku^2}\sin(au)}{u}\,du = \operatorname{erf}\left(\frac{a}{2\sqrt{k}}\right)$$

and thus obtain **an integral representation of the error function**, defined by **(2.2)**. We easily see that the two integrals exist and are equal by means of the change of variables $x = u^2$.

We observe that for $k = 0$, by letting $x = u^2$

$$\frac{1}{\pi}\int_0^\infty \frac{\sin(a\sqrt{x})}{x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{\sin(au)}{u}\,du = \frac{2}{\pi}\frac{\pi}{2} = 1 = \mathrm{erf}(\infty).$$

(Also for $k = \infty$, we find that both integrals are $0 = \mathrm{erf}(0)$.)

For $0 < k < \infty$ constant, the two integrals converge absolutely and are continuous in a. Indeed, for all $x \in \mathbb{R}$ $\left|\frac{\sin(a\sqrt{x})}{x}\right| \leq \frac{a}{\sqrt{x}}$, and then in any open finite interval (b, c) containing a we have:

$$\int_0^\infty \left|\frac{e^{-kx}\sin(a\sqrt{x})}{x}\right| dx \leq$$
$$\int_0^\infty \frac{a\,e^{-kx}}{\sqrt{x}}\,dx < a\int_0^1 \frac{1}{\sqrt{x}}\,dx + a\int_1^\infty e^{-kx}dx =$$
$$a\left(2 + \frac{e^{-k}}{k}\right) < c\left(2 + \frac{e^{-k}}{k}\right) < \infty.$$

Also, for all $u \in \mathbb{R}$, $\left|\frac{\sin(au)}{u}\right| \leq a$, and then in any open finite interval (b, c) containing a (by **Problem 2.1.11**) we have:

$$\int_0^\infty \left|\frac{e^{-ku^2}\sin(au)}{u}\right| du \leq \int_0^\infty a e^{-ku^2}\,du = \frac{a}{2}\sqrt{\frac{\pi}{k}} < \frac{c}{2}\sqrt{\frac{\pi}{k}} < \infty.$$

So, by **Part (I)** of the **Main Theorem, 2.2.1,** the two integrals are continuous in $a \geq 0$.

Since the two integral are equal, we are going to work with the second one. We let

$$I(a) = \frac{2}{\pi}\int_0^\infty \frac{e^{-ku^2}\sin(au)}{u}\,du.$$

It is legitimate to differentiate with respect to a under the integral sign since we have:

$$(1)\ \ \frac{d}{da}\left[\frac{e^{-ku^2}\sin(au)}{u}\right] = e^{-ku^2}\cos(au), \quad (2)\ \ \left|e^{-ku^2}\cos(au)\right| \leq e^{-ku^2}$$

and

$$(3)\ \ \int_0^\infty e^{-ku^2}du = \frac{1}{2}\sqrt{\frac{\pi}{k}} < \infty$$

(by **Problem 2.1.11**).

Then, by **Part (II)** of the **Main Theorem, 2.2.1,**

$$\frac{dI(a)}{da} = \frac{2}{\pi}\int_0^\infty e^{-ku^2}\cos(au)\,du = \frac{2}{\pi}\frac{\sqrt{\pi}}{2\sqrt{k}}\,e^{-\frac{a^2}{4k}} = \frac{1}{\sqrt{\pi k}}\,e^{-\frac{a^2}{4k}}$$

(by **Example 2.2.13**).

Since $I(0) = 0$, by letting $v = \dfrac{b}{2\sqrt{k}}$, we obtain

$$I(a) = \frac{1}{\sqrt{\pi k}}\int_0^a e^{-\frac{b^2}{4k}}\,db = \frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{2\sqrt{k}}} e^{-v^2}\,dv = \operatorname{erf}\left(\frac{a}{2\sqrt{k}}\right).$$

If now we let $t = \frac{a}{2\sqrt{k}}$ or $a = 2\sqrt{k}t$, we find the formulae

$$\operatorname{erf}(t) = \frac{1}{\pi}\int_0^\infty \frac{e^{-kx}\sin(2t\sqrt{kx})}{x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ku^2}\sin(2\sqrt{k}tu)}{u}\,du,$$

for $t \geq 0$ and $k > 0$, which are two useful **integral representations of the error function**.

▲

Example 2.2.15 Find the following **Frullani type integral** (see also **Example 2.5.2**). :

$$F(t) := \int_0^\infty \frac{\arctan(tx) - \arctan(x)}{x}\,dx,$$

where $t \geq 0$. (With the **Frullani type integrals**, we deal extensively in **Section 2.5**.)

Obviously $F(1) = 0$. Next, by the Mean Value Theorem for derivatives, we have that

$$f(x,t) := \frac{\arctan(tx) - \arctan(x)}{x} = \frac{tx - x}{x}\cdot\frac{1}{1+c^2} = (t-1)\frac{1}{1+c^2},$$

where c is a number between the numbers tx and x. So,

$$|f(x,t)| \leq |t-1|\left[\frac{1}{1+x^2} + \frac{1}{1+(tx)^2}\right],$$

which has a finite integral for all $t \geq 0$. Therefore, $F(t)$ exists for all $t \geq 0$.

Now we consider $a > 0$ and then, for any $t \geq a$, we have:

$$\left|\frac{\partial}{\partial t}f(x,t)\right| = \frac{1}{1+(tx)^2} \leq \frac{1}{1+(ax)^2}.$$

Since

$$\int_0^\infty \frac{1}{1+(ax)^2}\,dx = \frac{\pi}{2a}$$

is finite, by the differentiability part of the **Main Theorem, 2.2.1,** we have that for $t \geq a$

$$F'(t) = \int_0^\infty \frac{1}{1+(tx)^2}\,dx = \frac{\pi}{2t}.$$

Therefore, $F(t) = \dfrac{\pi}{2}\ln(t) + C$. But $F(1) = 0$ and so $C = 0$. Finally, $F(t) = \dfrac{\pi}{2}\ln(t)$.

▲

Problems

2.2.1 Let $\epsilon \in (0, +\infty]$ and $f : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function, i.e., with derivatives of all orders in $(-\epsilon, \epsilon)$, and with the value $f(0) = 0$.

Define $F : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} \dfrac{f(x)}{x} & \text{if } x \neq 0, \\ \\ f'(0) & \text{if } x = 0. \end{cases}$$

Prove:

(a) $F(x)$ is continuous in $(-\epsilon, \epsilon)$ and $\mathcal{C}^\infty$ in $(-\epsilon, \epsilon) - \{0\}$.

(b)

$$F'(0) = \frac{f''(0)}{2}.$$

(c)

$$F''(0) = \frac{f'''(0)}{3}.$$

(d)

$$F(x) = \int_0^1 f'(xt)\,dt.$$

(e) Use **(d)** to justify why $F(x)$ is $\mathcal{C}^\infty$ in $(-\epsilon, \epsilon)$.

(f)

$$F^{(n)}(0) = \frac{f^{(n+1)}(0)}{n+1}, \;\; \forall\; n \in \mathbb{N}.$$

(g) Prove that

$$f(x)=\begin{cases} e^{\frac{-1}{x}} & \text{if } x>0,\\ \\ 0 & \text{if } x\le 0\end{cases}$$

is $\mathcal{C}^\infty$ in $(-\infty,\infty)$ and find $F^{(n)}(0),\ \forall\ n\in\mathbb{N}$.

(h) Prove that

$$f(x)=\begin{cases} e^{\frac{-1}{x^2}} & \text{if } x\neq 0,\\ \\ 0 & \text{if } x=0\end{cases}$$

is $\mathcal{C}^\infty$ in $(-\infty,\infty)$ and find $F^{(n)}(0),\ \forall\ n\in\mathbb{N}$.

2.2.2 (a) If in **Example 2.2.5** we let $p=0$ or $p=1$, then prove, respectively,

$$\int_0^\infty \frac{1}{t(1+t)}\,dt=\infty \ \text{ and } \ \int_0^\infty \frac{1}{1+t}\,dt=\infty.$$

(b) Do these results extend the formula found in that example correctly?

2.2.3 Justify why the result of **Example 2.2.5** can also be written as

$$\forall\ 0<b<1,\ \int_0^\infty \frac{1}{t^b(1+t)}\,dt=\frac{\pi}{\sin(b\pi)}.$$

2.2.4 Find the exact values of the integrals

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx \quad\text{and}\quad \int_0^\infty \frac{x^{-1.45}}{1+x^{3.8}}\,dx.$$

2.2.5 Find the exact values of the integrals

$$\int_0^\infty \frac{1}{\sqrt[3]{x}\,(1+x^2)}\,dx \quad\text{and}\quad \int_0^\infty \frac{x^e}{1+x^{2\pi}}\,dx.$$

2.2.6 Find the exact values of the integrals

$$\int_0^\infty \frac{4x^{\frac{2}{5}}}{2+3x^{\frac{7}{3}}}\,dx \quad\text{and}\quad \int_0^\infty \frac{x^{\frac{7}{3}}}{1000+3000x^{\frac{8}{3}}}\,dx.$$

2.2.7 Find the exact values of the integrals

$$\int_0^\infty \frac{5x^{\frac{2}{3}}}{-2-3x^{\frac{12}{5}}}\,dx \quad \text{and} \quad \int_0^\infty \frac{-10x^{-2}}{2+3x^{-5}}\,dx.$$

2.2.8 For $n = 2,\ 3,\ 4,\ 5,\ 6$, find the exact values of the integral

$$\int_0^\infty \frac{1}{1+x^n}\,dx.$$

2.2.9 For $n = 4$ and $n = 6$ find the exact values of the two integrals:

$$(1) \quad \int_0^\infty \frac{x^2}{1+x^n}\,dx, \qquad (2) \quad \int_{-\infty}^\infty \frac{x^2}{1+x^n}\,dx.$$

2.2.10 Find the exact value of the integral

$$\int_0^\infty \frac{1-2x+3x^2-4x^3+5x^4}{1+x^6}\,dx.$$

2.2.11 (a) For **integers** $m \geq n-1$, prove:

$$\int_0^\infty \frac{x^m}{1+x^n}\,dx = \infty.$$

(b) For integers $m \geq n-1$ with m **odd** and n **even**, prove:

$$\int_{-\infty}^\infty \frac{x^m}{1+x^n}\,dx = \text{does not exist},$$

but its principal value is

$$\text{P.V.} \int_{-\infty}^\infty \frac{x^m}{1+x^n}\,dx = 0.$$

(c) For **any** integer m and any **odd** integer n, prove:

$$\int_{-\infty}^\infty \frac{x^m}{1+x^n}\,dx = \text{does not exist}.$$

(Notice that when n is odd then $x = -1$ is a singular point.)

2.2.12 (a) For real numbers $0 < a < b$, prove:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{bx}}\,dx = \frac{1}{b}\cdot\frac{\pi}{\sin\left(\frac{a}{b}\pi\right)}.$$

(Make the u-substitution $u = e^{bx}$, etc.)

(b) For $a = 0$ and $b \in \mathbb{R}$, or for $0 \le b \le a$ reals, prove:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{bx}}\,dx = \infty.$$

2.2.13 Use the **previous problem** to find the exact values of the integrals:

(a) $$\int_{-\infty}^{\infty} \frac{e^{2x}}{1+e^{5x}}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{-2x}}{1+e^{-5x}}\,dx.$$

(b) For $b < a < 0$ reals, find the general formula for the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{bx}}\,dx.$$

(c) $$\int_{-\infty}^{\infty} \frac{3e^{2x}}{1+10e^{5x}}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{3e^{-2x}}{1+10e^{-5x}}\,dx.$$

(d) $$\int_{-\infty}^{\infty} \frac{3e^{2x}}{2+10e^{5x}}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{3e^{-2x}}{5+10e^{-5x}}\,dx.$$

2.2.14 Knowing $\displaystyle\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$, prove directly that

$$\forall \ \beta \in \mathbb{R}, \qquad \int_0^{\infty} \frac{\sin(\beta x)}{x}\,dx = \begin{cases} \dfrac{\pi}{2} & \text{if } \beta > 0 \\ 0 & \text{if } \beta = 0 \\ -\dfrac{\pi}{2} & \text{if } \beta < 0. \end{cases}$$

2.2.15 Knowing $\displaystyle\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx \left[= \int_0^{\infty} \frac{1-\cos^2(x)}{x^2}\,dx\right] = \frac{\pi}{2}$, prove directly that

$$\forall \ \alpha \in \mathbb{R}, \qquad \int_0^{\infty} \frac{\sin^2(\alpha x)}{x^2}\,dx = \left[\int_0^{\infty} \frac{1-\cos^2(\alpha x)}{x^2}\,dx\right] = |\alpha|\frac{\pi}{2}.$$

2.2.16 Use a half angle trigonometric identity and the **previous problem** to prove that for any real constant a

$$\text{(a)}\ \int_0^\infty \frac{1-\cos(ax)}{x^2}\,dx = |a|\frac{\pi}{2},\quad \text{(b)}\ \int_0^\infty \frac{\cos(ax)[1-\cos(ax)]}{x^2}\,dx = 0.$$

(See also the relative part of **Example 3.7.34** and **Problem 3.7.69**.)

2.2.17 Prove that for any $a \in \mathbb{R}$,

$$\int_0^\infty \frac{\sin(a\sqrt{x})}{x}\,dx = \begin{cases} \pi, & \text{if} \quad a > 0 \\ 0, & \text{if} \quad a = 0 \\ -\pi, & \text{if} \quad a < 0. \end{cases}$$

2.2.18 As we prove the continuity of $\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx$ in $\alpha \geq 0$, for any given $\beta \in \mathbb{R}$ fixed (see **Example 2.2.8**), in the same way prove that

$$\int_0^\infty e^{-cu}\frac{\sin(\beta u)}{\sqrt{u}}\,du$$

is continuous in $c \in [0,\infty)$, for any given $\beta \in \mathbb{R}$ fixed. (We need this in **Example 2.4.1**.)

2.2.19 Let

$$f(x) = \frac{\sin(x)}{x}, \quad \text{for} \quad x \in [0,\infty).$$
$$f^+(x) = \max\{f(x), 0\}, \quad \text{for} \quad x \in [0,\infty).$$
$$f^-(x) = \min\{f(x), 0\}, \quad \text{for} \quad x \in [0,\infty).$$

(a) Show that:

$$f^+(x) + f^-(x) = f(x), \quad \text{for} \quad x \in [0,\infty),$$
$$\int_0^\infty f^+(x)\,dx = \infty \quad \text{and} \quad \int_0^\infty f^-(x)\,dx = -\infty.$$

(b) Do the three results in **(a)** violate the fact that

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}\,?$$

(c) Check $f(x)$ against **Lemma 1.3.1** (what holds and what fails).

Explain your answers fully.

2.2.20 Either find the following integrals or prove that they do not exist:

(a) $\displaystyle\int_0^\infty \frac{\sin(x^2)}{x}\,dx, \qquad \int_{-\infty}^0 \frac{\sin(x^2)}{x}\,dx$ and $\displaystyle\int_{-\infty}^\infty \frac{\sin(x^2)}{x}\,dx.$

(b) $\displaystyle\int_0^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right)dx, \int_{-\infty}^0 \frac{1}{x}\sin\left(\frac{1}{x}\right)dx$ and $\displaystyle\int_{-\infty}^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right)dx.$

(c) $\displaystyle\int_0^\infty \sin\left(\frac{1}{x}\right)dx, \qquad \int_{-\infty}^0 \sin\left(\frac{1}{x}\right)dx$ and $\displaystyle\int_{-\infty}^\infty \sin\left(\frac{1}{x}\right)dx.$

2.2.21 Find the integral

$$\int_0^\infty \frac{\sin(ax)\cos(bx)}{x}\,dx$$

for $a \neq b$ and for $a = b$.

[Hint: $\sin(u)\cos(v) = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$.]

2.2.22 (a) Prove that for any $p \leq 0$ the integral

$$\int_0^\infty \frac{\cos(x)}{x^p}\,dx$$

does not exist.

[Hint: For $p \leq 0$ this integral does not exist because "near" infinity it oscillates "badly."]

(b) Prove that for any $0 < p < 1$, the integral

$$\int_0^\infty \frac{\cos(x)}{x^p}\,dx$$

converges conditionally but not absolutely.

[Hint: For conditional convergence, use the **Cauchy Test, 1.3.7**. For absolute divergence, split the integral into appropriate series.]

(c) Prove that for any $p \geq 1$

$$\int_0^\infty \frac{\cos(x)}{x^p}\,dx = \infty.$$

[Hint: Remember $\lim\limits_{x\to 0^+} \cos(x) = 1^-$. For $p \geq 1$, the integral becomes infinite near $x = 0$.]

(d) Prove that for $0 < p < 1$, the integral function

$$F(p) = \int_0^\infty \frac{\cos(x)}{x^p}\,dx$$

is continuous in p.

[Hint: You may consider

$$\int_0^\infty \frac{\cos(x)}{x^p}\,dx = \int_0^1 \frac{\cos(x)}{x^p}\,dx + \int_1^\infty \frac{d\,\sin(x)}{x^p} = \int_0^1 \frac{\cos(x)}{x^p}\,dx - \sin(1) + p\int_1^\infty \frac{\sin(x)}{x^{p+1}}\,dx.$$

Then for any given $0 < p < 1$, we consider $0 < k < p < l < 1$. Then for $0 < x < 1$ we have $\left|\dfrac{\cos(x)}{x^p}\right| \leq \dfrac{1}{x^l}$, and for $x \geq 1$, we have $\left|\dfrac{\sin(x)}{x^{p+1}}\right| \leq \dfrac{1}{x^{k+1}}$, etc.]

(See also **Example 2.6.7**.)

2.2.23 (a) Prove that for any $p \leq 0$, the integral

$$\int_0^\infty \frac{\sin(x)}{x^p}\,dx$$

does not exist.

[Hint: For $p \leq 0$ this integral does not exist because near infinity it oscillates “badly”.]

(b) Prove that for any $0 < p \leq 1$, the integral

$$\int_0^\infty \frac{\sin(x)}{x^p}\,dx$$

converges conditionally but not absolutely.

[Hint: For conditional convergence, use the Cauchy Test and remember that for $0 < p \leq 1$, the integral is proper at $x = 0$. For absolute divergence, split the integral into appropriate series.]

(c) Prove that for any $1 < p < 2$, the integral

$$\int_0^\infty \frac{\sin(x)}{x^p}\, dx$$

converges absolutely.

[Hint: For the absolute convergence here, split this integral over $[0,1] \cup (1,\infty)$, and for the first part remember: $\left|\dfrac{\sin(x)}{x}\right| \leq 1$.]

(d) Prove that for any $p \geq 2$

$$\int_0^\infty \frac{\sin(x)}{x^p}\, dx = \infty.$$

[Hint: Remember $\lim\limits_{x\to 0^+} \sin(x) = 0^+$ and $\lim\limits_{x\to 0^+} \dfrac{\sin(x)}{x} = 1^-$. For $p \geq 2$, the integral becomes infinite near $x = 0$.]

(e) Prove that for $0 < p < 2$, the integral function

$$G(p) = \int_0^\infty \frac{\sin(x)}{x^p}\, dx$$

is continuous in p.

[Hint: You may consider

$$\int_0^\infty \frac{\sin(x)}{x^p}\, dx = \int_0^1 \frac{\sin(x)}{x}\frac{1}{x^{p-1}}\, dx - \int_1^\infty \frac{d\cos(x)}{x^p} =$$
$$\int_0^1 \frac{\sin(x)}{x}\frac{1}{x^{p-1}}\, dx - \cos(1) - p\int_1^\infty \frac{\cos(x)}{x^{p+1}}\, dx.$$

Remember that $\left|\dfrac{\sin(x)}{x}\right| \leq 1$ for all x. Then for any given $0 < p < 2$, we consider $-1 < p-1 < l < 1$ and $0 < k < p < 2$. Then for $0 < x < 1$ we have $\left|\dfrac{\sin(x)}{x}\dfrac{1}{x^{p-1}}\right| \leq \dfrac{1}{x^l}$, and for $x \geq 1$, we have $\left|\dfrac{\cos(x)}{x^{p+1}}\right| \leq \dfrac{1}{x^{k+1}}$, etc.]

(See also **Example 2.6.7**.)

2.2.24 (a) Give two reasons for which the following integral

$$\int_{-\infty}^\infty \frac{\cos(x)}{x}\, dx$$

is improper. Now use the **combined Definition 1.2.3 of the principal value** and prove that the principal value of this improper integral is zero. Lastly, prove that the integral itself does not exist and therefore does not converge absolutely.

(b) Now consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx$$

and answer the following questions: For how many and what reasons is it improper? What is its principal value? What is its value? Does it converge absolutely?

2.2.25 (a) Prove that for all $a \geq 0$

$$\int_0^{\infty} \frac{\sin(x)}{x+a}\, dx$$

converges conditionally to a positive value.

(b) Prove that for all $a > 0$

$$\int_0^{\infty} \frac{\sin(x)}{x+a}\, dx = \int_0^{\infty} \frac{\sin(ax)}{x+1}\, dx.$$

2.2.26 (a) Prove that for all $a > 0$

$$\int_0^{\infty} \frac{\cos(x)}{x+a}\, dx$$

converges conditionally.

(b) Prove that for all $c > 0$

$$\int_c^{\infty} \frac{\cos(x)}{x}\, dx$$

converges conditionally.

(c) Prove

$$\int_0^{\infty} \frac{\cos(x)}{x}\, dx = +\infty.$$

(d) Prove that for all $a > 0$

$$\int_0^{\infty} \frac{\cos(x)}{x+a}\, dx = \int_0^{\infty} \frac{\cos(ax)}{x+1}\, dx.$$

2.2.27 (a) Let

$$I(\alpha) = \int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2}\, dx.$$

Show that it is legitimate to take the derivative of $I(\alpha)$ and also $I'(\alpha) = 0$. Then show that

$$I(\alpha) = \frac{\sqrt{\pi}}{2}.$$

(b) Use **(a)** to prove

$$\int_0^\infty e^{-\left(x^2+\alpha^2 x^{-2}\right)} dx = \frac{\sqrt{\pi}}{2}\, e^{-2\alpha}.$$

2.2.28 Imitate **Example 2.2.15** to prove the following Frullani integral:

$$\ln(t) = \int_0^\infty \frac{e^{-x} - e^{-tx}}{x}\, dx.$$

(See also **Example 2.5.1**.)

2.2.29 For $\alpha \geq 0$, consider the function

$$F(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{x^2+1}\, dx.$$

(a) Prove that $F(0) = \frac{\pi}{2}$ and $\lim\limits_{\alpha\to\infty} F(\alpha) = 0$.
(b) Show that $0 < F(\alpha) \leq \frac{\pi}{2}$, for all $\alpha \geq 0$.
(c) Show that $F(\alpha)$ is continuous in $0 \leq \alpha < \infty$.
(d) Show that if $\alpha > 0$, then

$$F(\alpha) = \alpha \int_0^\infty e^{-\alpha x} \arctan(x)\, dx.$$

(e) Show that at $\alpha = 0^+$ the

$$F(\alpha) = \alpha \int_0^\infty e^{-\alpha x} \arctan(x)\, dx$$

is of the form $0 \cdot \infty$ and

$$\lim_{\alpha\to 0^+} \alpha \int_0^\infty e^{-\alpha x} \arctan(x)\, dx = \frac{\pi}{2}.$$

(f) Show that for $\alpha > 0$, $F(\alpha)$ is twice continuously differentiable.

[Hint: For any $\alpha > 0$, consider a p such that $0 < p < \alpha$ and use the **Main Theorem, 2.2.1,** on the interval $[p, \infty)$.]

(g) Prove that $\lim_{\alpha \to 0^+} F'(\alpha) = -\infty$ and $\lim_{\alpha \to 0^+} F''(\alpha) = \infty$.

[Hint: Observe that $\int_0^\infty \frac{x}{x^2+1}\, dx = \infty$ and $\int_0^\infty \frac{x^2}{x^2+1}\, dx = \infty$.]

(h) Prove that for $\alpha > 0$, the function $F(\alpha)$ satisfies the ordinary differential equation $F''(\alpha) + F(\alpha) = \frac{1}{\alpha}$.

(i) Prove that $\lim_{\alpha \to 0^+} F''(\alpha) = \infty$ by using **(a)**, **(c)** and **(h)** only.

(j) Use the method of variation of parameters to find that the general solution of the ordinary differential equation in **(h)** is

$$F(\alpha) = c_1 \cos(\alpha) + c_2 \sin(\alpha) - \cos(\alpha) \int_0^\alpha \frac{\sin(t)}{t}\, dt - \sin(\alpha) \int_\alpha^\infty \frac{\cos(t)}{t}\, dt$$

where c_1 and c_2 are arbitrary real constants.

The integrals $\mathrm{Si}(\alpha) := \int_0^\alpha \frac{\sin(t)}{t}\, dt$ and $\mathrm{Ci}(\alpha) := \int_\alpha^\infty \frac{\cos(t)}{t}\, dt$ are very important and cannot be found in closed form. They define two new functions, called **integral sine** and **integral cosine**, respectively.

(k) Use **(a)** to prove that $c_1 = \frac{\pi}{2}$ and $c_2 = 0$. That is, the final answer for the given integral with parameter $F(\alpha)$ is

$$F(\alpha) = \frac{\pi}{2} \cos(\alpha) - \cos(\alpha) \int_0^\alpha \frac{\sin(t)}{t}\, dt - \sin(\alpha) \int_\alpha^\infty \frac{\cos(t)}{t}\, dt.$$

(l) Prove that for $\alpha \geq 0$, the function $F(\alpha)$ found in **(k)** can also be written as

$$F(\alpha) = \cos(\alpha) \int_\alpha^\infty \frac{\sin(t)}{t}\, dt - \sin(\alpha) \int_\alpha^\infty \frac{\cos(t)}{t}\, dt =$$
$$\int_\alpha^\infty \frac{\sin(t-\alpha)}{t}\, dt = \int_0^\infty \frac{\sin(u)}{u+\alpha}\, du = \int_0^\infty \frac{\sin(\alpha u)}{u+1}\, du.$$

(m) For $\alpha > 0$, justify why it is legitimate to take the derivative of

$F(\alpha)$ and show

$$F'(\alpha) = \int_0^\infty \frac{xe^{-\alpha x}}{x^2+1}\,dx = \frac{\alpha}{2}\int_0^\infty e^{-\alpha x}\ln\left(x^2+1\right)\,dx =$$
$$\sin(\alpha)\left[\frac{\pi}{2} - \int_0^\alpha \frac{\sin(t)}{t}\,dt\right] + \cos(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\sin(\alpha)\int_\alpha^\infty \frac{\sin(t)}{t}\,dt + \cos(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\int_\alpha^\infty \frac{\cos(t-a)}{t}\,dt = \int_0^\infty \frac{\cos(u)}{u+a}\,du = \int_0^\infty \frac{\cos(\alpha u)}{u+1}\,du.$$

(See also **Problems 2.2.25**, **2.2.26** and **3.7.64**.)

2.2.30 Let

$$F(t) = \int_0^\infty te^{-tx}\,dx, \quad \text{for} \quad 0 \le t < \infty.$$

Show

(a) $F(0) = 0$ and $F(t) = 1, \ \forall \ t > 0$.

(b) F is continuous at every $t > 0$.

(c) F is discontinuous at $t = 0$. In fact, $\lim\limits_{t \to 0^+} F(t) \neq F(\lim\limits_{t \to 0^+} t) = F(0)$.

(d) Explain why the **Continuity Part** of the **Main Theorem, 2.2.1,** does not apply.

2.2.31 Prove that for all constants $B \ge 0$, $h \ge 0$ and $H > 0$, the integral

$$\int_0^{\frac{\pi}{H}} \frac{\sqrt{1+4H^2h^2+B^2+2B\sin(2Hs)}\ [1+B\sin(2Hs)]}{1+B^2+2B\sin(2Hs)}\,ds$$

has a positive value. Then examine separately its continuity when one of the constants approaches zero while the other two are fixed.

2.2.32 (a) Prove that for integers $n \ge m > 1$, the integral

$$\int_0^\infty \frac{\sin^n(x)}{x^m}\,dx$$

converges absolutely.

(b) Now for any integer $n \ge 1$, explain what happens with respect to the convergence (conditional and/or absolute) of the integral

$$\int_0^\infty \frac{\sin^n(x)}{x}\,dx.$$

[Hint: E.g., we have seen that $\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$, but this integral diverges absolutely.

Also prove $\int_0^\infty \frac{\sin^2(x)}{x}\,dx = \infty$ and $\int_0^\infty \frac{\sin^4(x)}{x}\,dx = \infty$, and so these integrals of positive integrands diverge, etc.]

2.2.33 Compute precisely each of the following integrals:

$$\text{(a)} \quad \int_0^\infty \frac{\sin^3(x)}{x^k}\,dx \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{\sin^3(x)}{x^k}\,dx \qquad \text{with} \;\; k = 1, 3.$$

(For $k = 2$, see **Problem 2.5.14**.)

$$\text{(b)} \quad \int_0^\infty \frac{\sin^4(x)}{x^l}\,dx \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{\sin^4(x)}{x^l}\,dx \qquad \text{with} \;\; l = 1, 2, 4.$$

(For $k = 3$, see **Problem 2.5.14**.)

[Hint: Prove and use trigonometric identities such as:

$$\begin{aligned} &(1) \quad \sin^3(x) = \frac{-1}{4}\sin(3x) + \frac{3}{4}\sin(x), \\ &(2) \quad \sin^4(x) = \sin^2(x)[1 - \cos^2(x)], \\ &(3) \quad \sin(x)\cos(x) = \frac{1}{2}\sin(2x). \end{aligned}$$

E.g.: to derive the first of these trigonometric identities, expand the $\sin(3x) = \sin(x + 2x) = ...$, etc.

Use appropriate integration by parts as many times as necessary, results previously obtained and the **previous problem**. Finally, find

$$\int_0^\infty \frac{\sin^3(x)}{x}\,dx = \frac{\pi}{4}, \qquad \int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}$$

$$\int_0^\infty \frac{\sin^4(x)}{x}\,dx = \infty, \qquad \int_0^\infty \frac{\sin^4(x)}{x^2}\,dx = \frac{\pi}{4}, \qquad \int_0^\infty \frac{\sin^4(x)}{x^4}\,dx = \frac{\pi}{3}.]$$

(c) Use a half angle formula to prove

$$\int_{-\infty}^\infty \frac{[1 - \cos(x)]^2}{x^4} dx = 2\int_0^\infty \frac{[1 - \cos(x)]^2}{x^4} dx = 2\int_{-\infty}^0 \frac{[1 - \cos(x)]^2}{x^4} dx = \frac{\pi}{3}.$$

2.2.34 Use any results established in the text and in the problems and appropriate trigonometric identities to prove

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{\sin(ax)\sin(bx)}{x^2} dx = \pi \min\{a,\ b\},$$

$$\text{(b)} \quad \int_{-\infty}^{\infty} \frac{\sin^2(ax)\sin^2(bx)}{x^4} dx = \frac{\pi}{2} \min\{a,\ b\},$$

where without loss of generality $a \geq 0$ and $b \geq 0$ are real constants.

[For **(a)**, see also **Example 3.7.35**.]

[Hint: In **(a)** you may begin with integration by parts.]

2.2.35 Obviously,

$$0 \leq \frac{\sin^4(x)}{x^4} = \frac{\sin^2(x)[1 - \cos^2(x)]}{x^4} =$$

$$\frac{\sin^2(x) - \sin^2(x)\cos^2(x)}{x^4} = \frac{\sin^2(x)}{x^4} - 4\frac{\sin^2(2x)}{(2x)^4}.$$

So,

$$0 < \int_0^{\infty} \frac{\sin^4(x)}{x^4}\, dx =$$

$$\int_0^{\infty} \left[\frac{\sin^2(x)}{x^4} - 4\frac{\sin^2(2x)}{(2x)^4}\right] dx =$$

$$\int_0^{\infty} \frac{\sin^2(x)}{x^4}\, dx - 2\int_0^{\infty} \frac{\sin^2(2x)}{(2x)^4}\, d(2x) =$$

$$\int_0^{\infty} \frac{\sin^2(x)}{x^4}\, dx - 2\int_0^{\infty} \frac{\sin^2(u)}{u^4}\, du =$$

$$\int_0^{\infty} \frac{\sin^2(x)}{x^4}\, dx - 2\int_0^{\infty} \frac{\sin^2(x)}{x^4}\, dx = -\int_0^{\infty} \frac{\sin^2(x)}{x^4}\, dx < 0,$$

wow and woe! Find where the error has occurred and explain why.

2.2.36 (See also **Problem 2.6.11**.) (a) Prove that for $p > 1$, the integrals

$$\int_0^{\infty} \sin(x^p)\, dx \quad \text{and} \quad \int_0^{\infty} \cos(x^p)\, dx$$

converge conditionally.

(b) Analyze what happens when $p \leq 1$. (For each of the two integrals, examine the following cases: $p = 1$, $0 < p < 1$, $p = 0$ and $p < 0$ separately.)

2.2.37 Justify why

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx = 0,$$

where $a \in \mathbb{R}$ and $b \neq 0$ constants.

2.2.38 (a) Prove that for $\alpha > 0$ and $\beta \in \mathbb{R}$ constants, the integral

$$I(\beta) = \int_0^{\infty} e^{-\alpha x^2} \sin(\beta x)\,dx$$

converges absolutely, and so it exists.

(b) Imitate the work done in **Example 2.2.13** to find the initial value-problem that this integral satisfies.

(c) Solve this initial value-problem to find

$$I(\beta) = \frac{1}{2\alpha}\, e^{-\frac{\beta^2}{4\alpha}} \int_0^{\beta} e^{\frac{\rho^2}{4\alpha}}\,d\rho.$$

(We cannot put this result in closed form. Compare with **Problem 3.7.25**.)

(d) Find the exact numeric value of

$$I(\beta) = \int_{-\infty}^{\infty} e^{-\alpha x^2} \sin(\beta x)\,dx, \quad \text{where } \alpha > 0, \ \beta \in \mathbb{R} \text{ constants},$$

and justify your answer.

2.2.39 Prove that for $\alpha > 0$, $\beta \neq 0$ and $c \geq 2$ constants

$$I(\beta) = \int_0^{\infty} e^{-\alpha x} \frac{\sin(\beta x)}{x^c}\,dx = \operatorname{sign}(\beta)\cdot\infty$$

and

$$I(\beta) = \int_0^{\infty} e^{-\alpha x^2} \frac{\sin(\beta x)}{x^c}\,dx = \operatorname{sign}(\beta)\cdot\infty.$$

If $c < 2$, prove that these integrals are finite.

2.2.40 Prove that for $\alpha > 0$, $\beta \neq 0$ and $c \geq 1$ constants

$$I(\beta) = \int_0^{\infty} e^{-\alpha x} \frac{\cos(\beta x)}{x^c}\,dx = \infty$$

and

$$I(\beta) = \int_0^{\infty} e^{-\alpha x^2} \frac{\cos(\beta x)}{x^c}\,dx = \infty.$$

If $c < 1$, prove that these integrals are finite.

2.3 Commuting Limits with Integrals and Derivatives

This section consists of two parts. These parts are related to the two parts of continuity and differentiability of the **Main Theorem, 2.2.1,** of this chapter.

In the first part, we give the definitions of point-wise and uniform convergence of sequences and series of functions and of improper integrals. Then, we state the most important theorems concerning the most applicable sufficient conditions for commuting limits and integrals.

In the second part, we state the most important theorems concerning the most applicable sufficient conditions for commuting limits and derivatives of sequences and series of functions. Again, we need the definitions of point-wise and uniform convergence of sequences and series of functions.

We do not give the proofs. These can be found in practically all books of mathematical and/or real analysis. The interested reader who has not learnt these theorems yet is recommended to study them in a good book on these subjects.

2.3.1 Commuting Limits and Integrals

In a calculus course, we have learnt that we can integrate power series term by term. This means that we can commute the integral $\int_c^d$ with the infinite summation $\sum_{n=0}^{\infty} = \lim\limits_{0\leq k\to\infty}\left(\sum_{n=0}^{k}\right)$, or as we say, we can switch the order of integration and the limit process of the infinite summation. Hence, if the power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, taken with center $a = 0$ without loss of generality, converges in the open interval $(-r, r) \subseteq \mathbb{R}$, where $r > 0$ or $r = \infty$, and $[c, d] \subset (-r, r)$ then

$$\int_c^d f(x)\,dx = \int_c^d \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \int_c^d a_n x^n\,dx =$$

$$\sum_{n=0}^{\infty} a_n \left[\frac{x^{n+1}}{n+1}\right]_c^d = \sum_{n=0}^{\infty} a_n \left[\frac{d^{n+1} - c^{n+1}}{n+1}\right].$$

In this situation, the integral $\displaystyle\int_{-r}^{r} f(x)\,dx$ could be an improper Riemann integral at either or both end points. When this is the case, the integral is treated by means of the limiting processes we have seen so far.

Notation: In what follows, we use the following notation: $\mathbb{N} = \{1,\ 2,\ 3,\ 4,\ ...\}$, the set of natural numbers, and $\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3,\ 4, ...\}$.

In **the important remark** immediately following **Example 1.1.17**, we have indicated that, whereas this commuting is always legitimate with integrals of power series and limits of integration in the open interval of convergence of the power series, it is not valid in every situation with limits of sequences or series of functions, even if the limits of integration are within the domain of definition of all functions involved. Serious mistakes may occur if such a commutation is performed while not valid! For instance:

Example 2.3.1 The functions

$$f_n(x) = nxe^{-nx^2}, \quad \forall \quad n \in \mathbb{N}$$

are all continuous at every $x \in \mathbb{R}$ and therefore Riemann integrable over any interval $[a, b] \subset \mathbb{R}$.

Now, for $x = 0$, we have that $f_n(0) = 0$ for every $n \in \mathbb{N}$, and for any $x \neq 0$, we find that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{\frac{d}{dn}(nx)}{\frac{d}{dn}(e^{nx^2})} =$$

$$\lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = \lim_{n\to\infty} \frac{1}{xe^{nx^2}} = \frac{1}{\pm\infty} = 0.$$

Therefore, for every $x \in \mathbb{R}$, this sequence of functions converges to the continuous functions $f(x) = 0$, i.e.,

$$\lim_{n\to\infty} f_n(x) = f(x) = 0, \ \forall \ x \in \mathbb{R}.$$

Now we evaluate the proper Riemann integrals of all $f_n(x)$

$$\int_0^1 f_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx = \left[-\frac{e^{-nx^2}}{2}\right]_0^1 = \frac{1-e^{-n}}{2}$$

and of $f(x) = 0$

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0.$$

Hence

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{1-e^{-n}}{2} = \frac{1}{2}$$

and

$$\int_0^1 f(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

So, we see that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

Therefore, this is an example in which we cannot commute (switch) the order of limit and integration. Otherwise, the mistake would be imminent.

We could have used the interval $A = [0, \infty)$ instead of $[0, 1]$ to deal with improper Riemann integrals, getting again

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^\infty \lim_{n\to\infty} f_n(x)\,dx.$$

But, on the interval $B = [1, \infty)$, we get

$$\lim_{n\to\infty}\int_1^\infty f_n(x)\,dx = \lim_{n\to\infty}\left[\frac{-e^{-nx^2}}{2}\right]_1^\infty = \lim_{n\to\infty}\left[\frac{0-(-e^{-n})}{2}\right] =$$

$$\lim_{n\to\infty}\left[\frac{e^{-n}}{2}\right] = 0 = \int_1^\infty 0\,dx = \int_1^\infty f(x)\,dx = \int_1^\infty \lim_{n\to\infty} f_n(x)\,dx.$$

I.e., with $B = [1, \infty)$, we have: $\lim_{n\to\infty}\int_1^\infty f_n(x)\,dx = \int_1^\infty \lim_{n\to\infty} f_n(x)\,dx$ is valid.

▲

This example shows that we cannot commute (switch) the order of limit and integral in general, something that we can do freely with power series as long as we stay inside their intervals of convergence. We also see that the legitimacy of this commutation could depend on the interval of integration. Thus, in this part of this section, we expose conditions under which this commuting is legitimate. But first we need the following two definitions.

Definition 2.3.1 *Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f_n : A \to \mathbb{R}$, $\forall\, n \in \mathbb{N}$ be a sequence of real functions on A. Suppose there is a function $f : A \to \mathbb{R}$ such that $\forall\, x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$. Then we say that* ***the sequence of real functions $(f_n)_{n\in\mathbb{N}}$ converges point-wise to the function f in the set A, or the function f is the point-wise limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ in the set A.***

We write $f_n(x) \xrightarrow{pw} f(x)$ in A, as $n \to \infty$, or $\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x)$ in A, or simply $f_n(x) \longrightarrow f(x)$ in A, as $n \to \infty$, or $\lim_{n\to\infty} f_n = f$ in A.

The condition $\forall\ x \in A,\ \lim\limits_{n\to\infty} f_n(x) = f(x)$ can be equivalently expressed by the $\varepsilon-$condition:

$$\forall\ \epsilon > 0,\ \ \exists\ N := N(\epsilon, x) \in \mathbb{N} :\ \ \forall\ n \in \mathbb{N},$$
$$(n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon)\,.$$

Example 2.3.2 We let

$$f_n : (-1, 1] \to \mathbb{R},\ f_n(x) = x^n,\ \ \forall\ n \in \mathbb{N}$$

and

$$f : (-1, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

We readily see that the sequence of functions $(f_n)_{N\in\mathbb{N}}$ converges point-wise to the function f in the set $A = (-1, 1]$.

We also observe that all the functions $(f_n)_{N\in\mathbb{N}}$ are continuous on $(-1, 1]$, whereas the function f is not, since it has a jump discontinuity at $x = 1$. So, **the point-wise limit of continuous functions may or may not be a continuous function**. But, this f, with only one discontinuity at $x = 1$, is Riemann integrable on $(-1, 1]$.

▲

Remark: There are more advanced examples in which a sequence of continuous functions has point-wise limit a function which is not even Riemann integrable, let alone continuous. We are not going to explore these examples here. [See **Problem 1.1.7 Part II, (5)**.]

Definition 2.3.2 *Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f_n : A \to \mathbb{R},\ \forall\ n \in \mathbb{N}$ be a sequence of real functions. Suppose there is a function $f : A \to \mathbb{R}$ such that*

$$\forall\ \epsilon > 0,\ \ \exists\ N := N(\epsilon) \in \mathbb{N} :\ \ \forall\ n \in \mathbb{N}$$
$$[(n \geq N\ \ and\ \ x \in A) \Longrightarrow |f_n(x) - f(x)| < \epsilon]\,.$$

Then we say that ***the sequence of real functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function f in the set A, or the function f is the uniform limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ in the set A.***

We write $\lim\limits_{n\to\infty} f_n(x) \overset{un}{=} f(x)$ on A, or $\lim\limits_{n\to\infty} f_n = f$ uniformly on A, or $f_n \xrightarrow{un} f$ as $n \longrightarrow \infty$ on A.

Remark: (a) In the point-wise convergence, the convergence $\lim\limits_{n\to\infty} f_n(x) = f(x)$ is checked $\forall\ x \in A$ point by point as a convergence of a sequence of real numbers. The $N \in \mathbb{N}$ depends on both the á-priori chosen $\epsilon > 0$ and the individual $x \in A$. But, in the uniform convergence, the $N \in \mathbb{N}$ in the definition depends only on the á-priori chosen $\epsilon > 0$ and not on the individual $x \in A$, i.e., for any given $\epsilon > 0$ it remains the same for all $x \in A$. Therefore, the uniform convergence depends not only on the functions of the sequence but also on their common domain $\emptyset \neq A \subseteq \mathbb{R}$.

(b) If $B \subset A$ and the convergence is uniform in A, then it is automatically uniform in B. But, it can happen that the convergence is uniform in B, even though it is not uniform in A. We will see such an example in the sequel.

(c) We observe that the uniform convergence implies the point-wise convergence, that is $\forall\ x \in A,\ \lim_{n\to\infty} f_n(x) = f(x)$, but not vice versa, as we show in some examples.

The **Definition 2.3.2** implies the following corollary:

Corollary 2.3.1 *The condition*

$$\forall\ \epsilon > 0,\ \ \exists\ N := N(\epsilon) \in \mathbb{N} :\ \ \forall\ \ n \in \mathbb{N}$$
$$[(n \geq N\ \ \textit{and}\ \ x \in A) \Longrightarrow |f_n(x) - f(x)| < \epsilon]$$

in the ***Definition 2.3.2*** *of uniform convergence is equivalent to*

$$\lim_{n\to\infty} \left[\max_{x\in A} |f_n(x) - f(x)| \right] = 0.$$

In many situations, this equivalent condition is very convenient in proving uniform convergence. We also get the **negative result** that: **If this limit is not zero, then the convergence is not uniform**.

Example 2.3.3 In the **previous example** with

$$f_n : (-1, 1] \to \mathbb{R},\ f_n(x) = x^n,\ \ \forall\ n \in \mathbb{N}$$

and

$$f : [-1, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0, & \text{if } -1 < x < 1 \\ \\ 1, & \text{if } x = 1 \end{cases}$$

the convergence is not uniform but only point-wise. To see this, by means of the definition, we can pick any $0 < \epsilon < 1$. Then for any given $n \in \mathbb{N}$ if x satisfies $\sqrt[n]{\epsilon} < |x| < 1$, $f(x) = 0$ and

$$|f_n(x) - f(x)| = |x^n - 0| = |x^n| > \epsilon,\ \ \forall\ x :\ \ \sqrt[n]{\epsilon} < |x| < 1.$$

However, the convergence is uniform on any interval $[a, b] \subset (-1, 1)$. To see this, we let $k = \max\{|a|, |b|\} \in (-1, 1)$, and we notice that $0 \leq k < 1$ and

$$f(x) = 0 \text{ and } \forall\ n \in \mathbb{N},\ \max_{x\in[a,b]} |f_n(x) - f(x)| = \max_{x\in[a,b]} |f_n(x)| = k^n.$$

Since $0 \leq k < 1$, $\lim\limits_{n\to\infty} k^n = 0$. So, by the **previous corollary**, we get

$$\lim_{n\to\infty} f_n(x) \overset{un}{=} f(x) \equiv 0, \quad \text{on} \quad [a, b] \subset (-1, 1).$$

▲

Example 2.3.4 We let

$$f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{\sin(nx)}{\sqrt{n}},\ \forall\ x \in \mathbb{R}$$

and

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = 0,\ \forall\ x \in \mathbb{R}.$$

Then $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions that converges uniformly to the function f in $\mathbb{R}$. Indeed,

$$\lim_{n\to\infty} \left[\max_{x\in\mathbb{R}} (|f_n(x) - f(x)|)\right] = \lim_{n\to\infty} \left[\max_{x\in\mathbb{R}} \left(\left|\frac{\sin(nx)}{\sqrt{n}} - 0\right|\right)\right] =$$

$$\lim_{n\to\infty} \left[\max_{x\in\mathbb{R}} \left(\left|\frac{\sin(nx)}{\sqrt{n}}\right|\right)\right] = \lim_{n\to\infty} \left(\left|\frac{\pm 1}{\sqrt{n}}\right|\right) = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}\right) = 0.$$

▲

Example 2.3.5 In **Example 2.3.1**, with the sequence of functions $f_n(x) = nxe^{-nx^2}$ for $n \in \mathbb{N}$ and $x \in A = [0, \infty) \subset \mathbb{R}$, we had seen that this sequence converges point-wise to the continuous functions $f(x) = 0$ for every $x \in A = [0, \infty)$. Let us now check if this point-wise convergence is or is not uniform in $A = [0, \infty)$ by evaluating the $\lim\limits_{n\to\infty} \left[\max\limits_{x\in A} (|f_n(x) - f(x)|)\right]$.

To find this maximum, we take the derivatives of these functions

$$f_n'(x) = (n - 2n^2x)e^{-nx^2},\ \forall\ n \in \mathbb{N}.$$

This derivative is zero only at $x = \dfrac{1}{2n} \in A$. Then $\forall\ n \in \mathbb{N}$ the maximum of the function $f_n(x)$ on $A = [0, \infty)$ occurs at this point because $\forall\ n \in \mathbb{N}$, $f_n(x) \geq 0$, $f_n(0) = 0$ and $\lim\limits_{x\to\infty} f_n(x) = 0$. Therefore,

$$\lim_{n\to\infty} \left[\max_{x\in A} (|f_n(x) - f(x)|)\right] = \lim_{n\to\infty} \left[\max_{x\in A} (|f_n(x) - 0|)\right] =$$

$$\lim_{n\to\infty} \left[\left(\left|f_n\left(\frac{1}{2n}\right)\right|\right)\right] = \lim_{n\to\infty} \left(\frac{1}{2e^{\frac{1}{4n}}}\right) = \frac{1}{2e^0} = \frac{1}{2} \neq 0.$$

Since this limit is not zero, the convergence is not uniform in $A = [0, \infty)$.

But, the convergence is uniform in $B = [1, \infty)$. This is so because for each $n \in \mathbb{N}$ $f_n(x)$ is decreasing in $[1, \infty)$ and

$$\lim_{n\to\infty}\left[\max_{x\in B}\left(|f_n(x) - f(x)|\right)\right] = \lim_{n\to\infty}\left[\max_{x\in B}\left(|f_n(x) - 0|\right)\right] =$$

$$\lim_{n\to\infty}\left[\max_{x\in B}\left(|f_n(x)|\right)\right] = \lim_{n\to\infty}\left[(|f_n(1)|)\right] = \lim_{n\to\infty}\left(\frac{n}{e^n}\right) = 0.$$

▲

The definitions of point-wise and uniform convergence extend in analogous ways to series of functions and improper integrals that depend on parameters over a set $\emptyset \neq A \subseteq \mathbb{R}$. On the basis of **Definitions 2.3.1** and **2.3.2**, we state the definitions:

Definition 2.3.3 *For a series of real functions, we define:*

$$\text{(a)} \quad \sum_{n=0}^{\infty} f_n(x) \overset{pw}{=} f(x) \text{ for } x \in A \overset{def}{\Longleftrightarrow}$$

$$\lim_{0\leq k\to\infty} \sum_{n=0}^{k} f_n(x) \overset{pw}{=} f(x) \text{ for } x \in A,$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} f_n(x) \overset{un}{=} f(x) \text{ for } x \in A \overset{def}{\Longleftrightarrow}$$

$$\lim_{0\leq k\to\infty} \sum_{n=0}^{k} f_n(x) \overset{un}{=} f(x) \text{ for } x \in A.$$

Definition 2.3.4 *If $A = [a, \infty) \subset \mathbb{R}$ and we have the improper integral $\int_A f(x,t)dx$ where t is a real parameter in a set $T \subseteq \mathbb{R}$, we define:*

$$\text{(a)} \quad \phi(t) \overset{pw}{=} \int_a^{\infty} f(x,t)dx \text{ for } t \in T \overset{def}{\Longleftrightarrow}$$

$$\forall\, t \in T, \quad \lim_{0\leq M\to\infty} \int_a^{M} f(x,t)dx \overset{pw}{=} \phi(t),$$

$$\text{(b)} \quad \phi(t) \overset{un}{=} \int_a^{\infty} f(x,t)dx \text{ for } t \in T \overset{def}{\Longleftrightarrow}$$

$$\lim_{0\leq M\to\infty} \int_a^{M} f(x,t)dx \overset{un}{=} \phi(t), \text{ in } T.$$

These definitions can be translated by means of $\epsilon > 0$, $\delta > 0$, as before. Also, in the case of improper integrals, we can use other types of domains of integration and not just $A = [a, \infty) \subset \mathbb{R}$. (Practice by translating these four definitions by using the $\epsilon > 0$ conditions and also by writing these definitions in the other cases of improper integrals.)

For uniform convergence of series of functions, the following criterion is very convenient.

Theorem 2.3.1 (Weierstraß M-Test for uniform convergence) *We consider $A \subseteq \mathbb{R}$ and a sequences of functions $f_n : A \longrightarrow \mathbb{R}$, for all $n \in \mathbb{N}_0$, that satisfies the following condition:*

$$\forall\ n \in \mathbb{N}_0,\ \exists\ \text{constant } M_n \geq 0\ :\ \forall\ x \in A,\ |f_n(x)| \leq M_n$$

and

$$\sum_{n=0}^{\infty} M_n < \infty.$$

Then the series of functions $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on A to some function $f : A \longrightarrow \mathbb{R}$ (which sometimes we can compute explicitly).

(The **proof** of **this Theorem** can be found in advanced calculus or mathematical analysis books.)

Example 2.3.6 Pick any number $-1 < p < 1$ and define

$$f_n : [-\pi, \pi] \longrightarrow \mathbb{R}\ :\ f_n(\theta) = p^n \cos(n\theta),\ \forall\ n \in \mathbb{N}_0.$$

Then with $M_n = |p|^n$, we have:

$$\forall\ n \in \mathbb{N}_0 \text{ and } \forall\ \theta \in [-\pi, \pi],\ |f_n(\theta)| \leq M_n$$

and

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |p|^n = \frac{1}{1 - |p|} < \infty.$$

Therefore, by the **Weierstraß M-Test**, the series

$$\sum_{n=0}^{\infty} f_n(\theta) = \sum_{n=0}^{\infty} p^n \cos(n\theta)$$

converges uniformly to some function on $[-\pi, \pi]$.

(In this way, we obtain the same result if $f_n(\theta) = p^n \sin(n\theta)$. See also **Problem 3.5.32**. You may be able to solve it now. Otherwise, solve it when you study **Section 3.5**.)

▲

Theorem 2.3.2 *Let* $\lim_{n\to\infty} f_n \overset{un}{=} f$ *in a bounded set* A, $\emptyset \neq A \subset \mathbb{R}$, *that is,* $\emptyset \neq A \subseteq (u, v)$ *for some real numbers* u *and* v. *Then we have the following three results:*

(a) If all f_n*'s are continuous in* A, *then* f *is continuous in* A *and therefore all these functions are Riemann integrable.*

(b) If all f_n*'s are Riemann integrable in* A, *then* f *is Riemann integrable in* A *(not necessarily continuous).*

(c) In either ***Case (a)*** *or* ***(b)***

$$\lim_{n\to\infty}\left[\int_A f_n(x)\,dx\right] = \int_A \lim_{n\to\infty}\left[f_n(x)\right]\,dx = \int_A f(x)\,dx.$$

Remark 1: The boundedness of the set A is not necessary in **(a)**, since continuity is a local property, but it is necessary in **(b)** and **(c)**.

For example, we consider $\forall\ n \in \mathbb{N},\ \ f_n(x) = \dfrac{1}{n},\ \ \forall\ x \in \mathbb{R}$ (constant functions). This sequence of functions converges uniformly to the function $f(x) = 0$ in $\mathbb{R}$. Now, computing the respective improper integrals, we find:

$$\forall\ n \in \mathbb{N},\ \ \int_{\mathbb{R}} f_n(x) = \infty \quad \text{and} \quad \int_{\mathbb{R}} f(x) = 0,$$

that is, equality fails.

Also, if **(c)** is valid the convergence does not have to be uniform. See **Example 2.3.7** below.

Remark 2: As we have seen in **Part II of Problem 1.1.7, Item (4.)**, the point-wise limit of a sequence of Riemann integrable functions may not be a Riemann integrable function necessarily. But, as we see here, the uniform limit is! By **Part II of Problem 1.1.7, Item (5.)**, even the point-wise limit of a sequence of continuous functions may not be a Riemann integrable function.

Corollary 2.3.2 *Suppose* (f_n), $n \in \mathbb{N}$ *is a sequence of Riemann integrable functions on an interval* $[a, b)$, *where* $\infty < a < b < \infty$, *such that* $f_n \longrightarrow f$ *as* $n \longrightarrow \infty$ *point-wise in* $[a, b)$, *and* $\forall\ r,\ a < r < b,\ f_n \longrightarrow f$ *as* $n \longrightarrow \infty$ *uniformly in* $[a, r]$. *Then* $f(x)$ *is Riemann integrable and*

$$\lim_{n\to\infty}\left[\int_a^b f_n(x)\,dx\right] = \int_a^b \lim_{n\to\infty}\left[f_n(x)\right]\,dx = \int_a^b f(x)\,dx.$$

Corollary 2.3.3 *Let* $\lim_{n\to\infty} f_n \overset{pw}{=} f$ *in a bounded set* A, $\emptyset \neq A \subset \mathbb{R}$, *that is,* $\emptyset \neq A \subseteq (a, b)$ *for some real numbers* a *and* b. *Then we have the following three results:*

(a) If all f_n*'s are continuous in* A *but* f *is not continuous in* A, *then the convergence is not uniform (only point-wise).*

(b) If all f_n's are Riemann integrable in A but f is not Riemann integrable in A, then the convergence is not uniform (only point-wise).

(c) If $\lim\limits_{n\to\infty}\left[\int_A f_n(x)\,dx\right] \neq \int_A \lim\limits_{n\to\infty}[f_n(x)]\,dx = \int_A f(x)\,dx$, *then the convergence is not uniform (only point-wise).*

Example 2.3.7 In **Examples 2.3.2** and **2.3.3**, we had

$$f_n : (-1,1] \to \mathbb{R},\ f_n(x) = x^n,\ \ \forall\ n \in \mathbb{N},$$

$$f : (-1,1] \to \mathbb{R},\ f(x) = \begin{cases} 0, & \text{if } -1 < x < 1 \\ \\ 1, & \text{if } x = 1 \end{cases}$$

and $\lim\limits_{n\to\infty} f_n(x) \overset{pw}{=} f(x)$ in $A = (-1,1] \subset \mathbb{R}$.

Since all the functions $(f_n)_{n\in\mathbb{N}}$ are continuous on $(-1,1]$ whereas the function f is not (it has a jump discontinuity at $x = 1$), we conclude that the point-wise convergence in **Examples 2.3.2** and **2.3.3** is not uniform over $(-1,1]$.

But, for $n \in \mathbb{N}$,

$$\int_{-1}^{1} f_n(x)dx = \frac{1-(-1)^{n+1}}{n+1} \to \int_{-1}^{1} f(x)dx = 0, \quad \text{as} \quad n \to \infty.$$

So, the converse of **Theorem 2.3.2, (c)** fails. ▲

Example 2.3.8, In **Example 2.3.1**, we had the sequence of the continuous functions $f_n(x) = nxe^{-nx^2}$, $n \in \mathbb{N}$, on the interval $[0,1]$. We saw that:

$$\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x) = 0, \quad \forall\ x \in A = [0,1] \subset \mathbb{R}$$

and

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0\,dx.$$

Therefore, the point-wise convergence in **Example 2.3.1** is not uniform. ▲

Next, we state three big theorems of advanced real analysis. We will adjust them to the level of this text; therefore, we do not state them in the greatest possible generality, and we do not include their proofs. But, they can easily and efficiently be used in applications. All that someone needs to do is to check the stated hypotheses, which are fairly straightforward. So, at this level, we can at least learn how to use them as efficient and powerful tools.

In these theorems, we give sufficient conditions under which we can switch the order between limit and integral. We consider sequences of Riemann integrable functions, and we require their limit functions to be Riemann integrable, too. We do so because, as we have seen in **Part II of project Problem 1.1.7**, **Items (3.)** and **(4.)**, the point-wise limit of a sequence of Riemann integrable functions is not always a Riemann integrable function.

In advanced real analysis, we use the class of Lebesgue integrable functions which contains the class of Riemann integrable functions. Now, it is always the case that the point-wise limit of a sequence of Lebesgue integrable functions is a Lebesgue integrable function. Therefore, such a requirement on the point-wise limit function, in the general interpretation of these Theorems within the Lebesgue theory of integration, is not necessary because it follows as a result from the already stated hypotheses.

Theorem 2.3.3 (Lebesgue's Monotone Convergence Theorem) *We consider a sequence of real functions $(f_n)_{n\in\mathbb{N}}$ in a set $\emptyset \neq A \subseteq \mathbb{R}$ that satisfies the following three conditions:*

(a) $f_1(x) \leq f_2(x) \leq f_3(x) \leq f_4(x) \leq \dots \leq \infty, \ \forall\, x \in A$ *(increasing sequence of real functions in A).*

(b) $\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x)$ *in A.*

(c) All f_n's are Riemann integrable in A and also f is Riemann integrable in A (possibly in the generalized Riemann sense).

Then, under these conditions, we have:

$$(1) \quad \forall\ n \in \mathbb{N}, \quad \int_A f_n(x)\,dx \leq \int_A f_{n+1}(x)\,dx \leq \int_A f(x)\,dx$$

and

$$(2) \quad \lim_{n\to\infty} \int_A f_n(x)\,dx = \int_A \lim_{n\to\infty} f_n(x)\,dx = \int_A f(x)\,dx.$$

Theorem 2.3.4 (Lebesgue's Dominated Convergence Theorem) *We consider a sequence of real functions (f_n), $n \in \mathbb{N}$ in a set A, $\emptyset \neq A \subseteq \mathbb{R}$, that satisfies the following three conditions:*

(a) $\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x)$ *in A.*

(b) All f_n's are Riemann integrable in A and also f is Riemann integrable in A (possibly in the generalized Riemann sense).

(c) There exists a Riemann integrable function (possibly in the generalized Riemann sense) $g : A \to [0,\infty]$ such that

$$\int_A g(x)\,dx < \infty \quad \text{and} \quad \forall\ n \in \mathbb{N},\ |f_n(x)| \leq g(x),\ \forall\, x \in A.$$

*(In this context, $g(x)$ is called the **dominating function**.)*

Then, under these conditions, we have:

$$(1) \quad |f(x)| \le g(x), \;\; \forall \, x \in A,$$

$$(2) \quad \lim_{n\to\infty} \int_A |f_n(x) - f(x)| \, dx = 0,$$

and

$$(3) \quad \lim_{n\to\infty} \int_A f_n(x) \, dx = \int_A \lim_{n\to\infty} f_n(x) \, dx = \int_A f(x) \, dx.$$

Remark 1: If in the **previous two Theorems** we have series of real functions $\sum_{n=0}^{\infty} f_n(x) \overset{pw}{=} f(x)$ for all $x \in A$ which as sequences of the initial partial sums satisfy the same conditions, then we obtain the analogous results for series.

Remark 2: The **Continuity Part** of the **Main Theorem, 2.2.1,** is essentially Lebesgue's dominated convergence Theorem, where the parameter t plays the role of the index $n \in \mathbb{N}$.

Example 2.3.9 For all $n \in \mathbb{N}$, find

$$\lim_{n\to\infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx.$$

We have that the functions $f_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-2x}$, with $n \in \mathbb{N}$, satisfy the following conditions:

(a) $f_1(x) \le f_2(x) \le f_3(x) \le f_4(x) \le ... \le \infty, \; \forall \, x \in A = [0, \infty)$, because for any $x \ge 0$ the sequence $a_n := \left(1 + \dfrac{x}{n}\right)^n$ is non-decreasing. (Prove this as an exercise!)

(b) We know that (prove it one more time)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Then we get

$$\lim_{n\to\infty} f_n(x) \overset{pw}{=} \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n e^{-2x} = e^x e^{-2x} = e^{-x}.$$

(c) All the functions f_n's and the limit function $f(x) = e^{-x}$ are continuous in $A = [0, \infty)$ and therefore Riemann integrable (in the generalized sense).

Then by **Lebesgue's monotone convergence Theorem** , we get

$$\lim_{n\to\infty}\int_0^{\infty}\left(1+\frac{x}{n}\right)^n e^{-2x}dx=\int_0^{\infty}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n e^{-2x}dx=$$
$$\int_0^{\infty} e^{-x}dx=[-e^{-x}]_0^{\infty}=0-(-1)=1.$$

(See also **Problem 2.3.2**.) ▲

Example 2.3.10 In the **previous example**, we can use **Lebesgue's dominated convergence Theorem** instead of **Lebesgue's monotone convergence Theorem**, with dominating function $g(x) = e^{-x}$. (Check that all the conditions of the two Theorems are satisfied.)

▲

Example 2.3.11 Working as in the **previous two examples**, we can achieve the following result:

$$\lim_{n\to\infty}\int_0^{n}\left(1+\frac{x}{n}\right)^n e^{-2x}dx=1.$$

Here, we consider the sequence of Riemann integrable functions defined by

$$f_n\ :\ \mathbb{R}\longrightarrow\mathbb{R}\ :\ \ \mathbb{R}\ni x\longrightarrow f_n(x)=\begin{cases}\left(1+\dfrac{x}{n}\right)^n e^{-2x}, & \text{if }\ 0\le x\le n\\[2ex] 0, & \text{if }\ n<x<\infty,\end{cases}$$

for all $n = 1, 2, 3, 4, \ldots$. Then we take

$$\lim_{n\to\infty}\int_0^{\infty} f_n(x)\,dx.$$

Now we can use either **Lebesgue's monotone convergence Theorem** or **Lebesgue's dominated convergence Theorem** to switch limit and integral sign and obtain the result, as we did in the previous two examples. (Fill in the details.)

(See also **Problem 2.3.2**.) ▲

Example 2.3.12 Suppose that for all $n \in \mathbb{N}$, we have Riemann integrable functions satisfying:

$$\forall\ x\in[a,b],\ \ 0\le f_n(x)\le c\ \text{and}\ \lim_{n\to\infty} f_n(x)\overset{pw}{=} f(x)=0.$$

Then we can apply **Lebesgue's dominated convergence Theorem**, with dominating function $g(x) = c \geq 0$ for all $x \in [a, b]$, since

$\int_a^b g(x)\,dx = \int_a^b c\,dx = c(b-a) < \infty$. Hence, in such a situation, we get

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b 0\,dx = 0.$$

▲

We also state Lebesgue's criterion for Riemann integrable functions in $\mathbb{R}$. But first we need:

Definition 2.3.5 *A set A subset of $\mathbb{R}$ ($A \subset \mathbb{R}$) has Lebesgue measure zero if for every $\epsilon > 0$ there are open intervals whose union contains A and the sum of their lengths is less than ϵ.*

Example 2.3.13 Every countable $A \subset \mathbb{R}$ has measure zero.

Suppose $A = \{a_1,\ a_2,\ a_3,\ a_4,\ \dots\}$ and consider any $\epsilon > 0$. Then we consider the sequence of open intervals

$$I_n = \left(a_n - \frac{\epsilon}{2^{n+2}}, a_n + \frac{\epsilon}{2^{n+2}}\right), \quad n \in \mathbb{N}.$$

Obviously,

$$A \subset \bigcup_{n=1}^{\infty} I_n$$

and for the sum of the lengths we have

$$\sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2} < \epsilon.$$

(There are also uncountable subsets of $\mathbb{R}$ with Lebesgue measure zero. A classical example is the Kantor set. Have a look at it in the bibliography.)

▲

Now we state:

Theorem 2.3.5 (Lebesgue's Criterion for Riemann Integrability) *Let $a < b$ be real numbers and suppose $f : [a,b] \longrightarrow \mathbb{R}$ is bounded, on the closed and bounded real interval $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if the set of discontinuity points of f has Lebesgue measure zero.*

Corollary 2.3.4 *Let $f \circ g$ be a composition of two real bounded functions defined on an interval $[a,b]$. Then, if one function is continuous and the other is Riemann integrable, the composition is Riemann integrable.*

Remark: **The corollary** fails if both are Riemann integrable. See **project Problem 1.1.7, Part II**.

Example 2.3.14 The Dirichlet function (see also **Problem 1.1.7**)

$$y = f(x) = \chi_{[0,1]\cap\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x = \text{rational in } [0,1] \\ \\ 0, & \text{if } x = \text{irrational in } [0,1] \end{cases}$$

is bounded and discontinuous at every point of the closed interval $[0,1]$. Since the Lebesgue measure of $[0,1]$ is the length of this interval, i.e., $1 > 0$, this function is not Riemann integrable. This result can also be checked elementarily, for the upper and lower sums of this function have constant difference 1.

▲

Example 2.3.15 The **Riemann Dirichlet function**

$$y = h(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ rational in reduced representation in } [0,1] \\ \\ 0, & \text{if } x = \ \text{irrational in } [0,1] \end{cases}$$

(see also **Problem 1.1.7**) is bounded and discontinuous exactly at the rational points of the closed interval $[0,1]$. Since the rational numbers are countable, the Lebesgue measure of $[0,1]\cap\mathbb{Q}$ is 0, and so this function is Riemann integrable.

▲

We also state the following inequality Theorem, which is a restricted version of Fatou's[8] Theorem. In the Lebesgue theory of integration, this is a key result and is stated in the most general terms such as liminfimum instead of limit here. So, we modify it to our context and level.

Theorem 2.3.6 (Fatou's Theorem) *We consider a sequence of non-negative Riemann integrable real functions* (f_n), $n \in \mathbb{N}$ *in a set* $\emptyset \neq A \subseteq \mathbb{R}$ *such that* $\forall\ x \in A$, $\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x)$ *is a Riemann integrable function and*

$$\lim_{n\to\infty} \int_A f_n(x)\,dx \quad \textit{exists.}$$

Then we have the inequality

$$\int_A \lim_{n\to\infty} f_n(x)\,dx = \int_A f(x)dx \leq \lim_{n\to\infty} \int_A f_n(x)\,dx.$$

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[8] Pierre Fatou, French mathematician, 1878–929.

Example 2.3.16 Strict inequality may occur in **Fatou's Theorem**.

$$\forall \;\; n \in \mathbb{N} \quad \text{we let} \quad f_n(x) = \begin{cases} 1, & \text{if} \quad n-1 \le x \le n \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then we have:

$$\lim_{n\to\infty} f_n(x) \overset{pw}{=} 0,$$

$$\int_{-\infty}^{\infty} f_n(x)dx = 1, \quad \forall \;\; n \in \mathbb{N} \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = 0.$$

Hence

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} f(x)dx = 0 < \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = 1.$$

▲

Example 2.3.17 Another example with a strict inequality in **Fatou's Theorem** is the following:

$$\forall \;\; n \in \mathbb{N} \quad \text{we let} \quad f_n(x) = \begin{cases} n, & \text{if} \quad 0 \le x \le \frac{1}{n} \\ \\ 0, & \text{if} \quad \frac{1}{n} < x \le 2. \end{cases}$$

Then we have:

$$\lim_{n\to\infty} f_n(x) \overset{pw}{=} \begin{cases} 0, & \text{if} \quad 0 < x \le 2 \\ \\ \infty, & \text{if} \quad x = 0, \end{cases}$$

and so

$$\int_0^2 f_n(x)dx = 1, \quad \forall \;\; n \in \mathbb{N} \quad \text{and} \quad \lim_{0<\epsilon\to 0^+} \int_\epsilon^2 f(x)dx = 0.$$

Hence

$$\int_0^2 \lim_{n\to\infty} f_n(x)\, dx = \int_0^2 f(x)dx = 0 < \lim_{n\to\infty} \int_0^2 f_n(x)\, dx = 1.$$

▲

We also have the following well-known and useful theorem for series of functions.

Theorem 2.3.7 ***[Beppo-Levi***[9]*] We consider a sequence of real functions $(f_n)_{n\in\mathbb{N}}$ in a set $\emptyset \neq A \subseteq \mathbb{R}$ that satisfies the following two conditions:*

(a) All f_n's are Riemann integrable in A.

(b) $\displaystyle\sum_{n=0}^{\infty}\int_A |f_n(x)|dx < \infty$.

Then under these two conditions, there is a function $f(x)$ defined in A [at some points of A, $f(x)$ may be $\pm\infty$] such that

$$\sum_{n=0}^{\infty} f_n(x) \overset{pw}{=} f(x) \quad in \quad A.$$

(c) If, moreover, $f(x)$ is Riemann integrable in A (possibly in the generalized Riemann sense), then we get:

$$\int_A |f(x)|\, dx < \infty,$$

and

$$\sum_{n=0}^{\infty}\left[\int_A f_n(x)\, dx\right] = \int_A \left[\sum_{n=0}^{\infty} f_n(x)\right]\, dx = \int_A f(x)\, dx.$$

Example 2.3.18 For all $n = 0,\ 1,\ 2,\ 3,\ ...$ integer, we let

$$f_n(x) = \frac{x^n}{n+1} \quad \text{with} \quad x \in A = [-1, 1].$$

Each of these functions is continuous and therefore Riemann integrable.

On account of the absolute value and the result in **Corollary 3.7.3 of Example 3.7.22** and/or **Problem 3.7.36**, we find:

$$\sum_{n=0}^{\infty}\int_A |f_n(x)|dx = 2\sum_{n=0}^{\infty}\left[\frac{x^{n+1}}{(n+1)^2}\right]_0^1 = 2\sum_{m=1}^{\infty}\frac{1}{m^2} = 2\cdot\frac{\pi^2}{6} = \frac{\pi^2}{3} < \infty.$$

Then, by the **Beppo-Levi Theorem**, we conclude that the series $\displaystyle\sum_{n=0}^{\infty} f_n(x)$ converges to a function $f(x)$, and we can commute integration with summation.

If we use the **Weierstraß M-Test (Theorem 2.3.1)** , we prove that $f(x)$ is finite and continuous in the interval $[-1, 1)$. At $x = 1$,

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[9] Beppo-Levi, Italian mathematician, 1875-1961.

we find that $f(1) = \sum_{m=1}^{\infty} \frac{1}{m} = \infty$. So, the $\int_{-1}^{1} f(x)dx$ is an improper (generalized) Riemann integral.

In fact, by using the power series

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad \forall\ -1 \le x \le 1,$$

we find that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = -\frac{\ln(1-x)}{x}, \quad \forall\ -1 \le x \le 1.$$

This function is Riemann integrable in the generalized sense.

Part (c) of the **Beppo-Levi Theorem, 2.3.7**, guarantees

$$\int_A |f(x)|dx < \infty$$

and also, in view of the result in **Problem 3.7.38**, we obtain

$$\int_A f(x)dx =$$

$$\int_A \left[\sum_{n=0}^{\infty} f_n(x)\right] dx = \sum_{n=0}^{\infty} \left[\int_A f_n(x)dx\right] = \sum_{n=0}^{\infty} \left[\frac{x^{n+1}}{(n+1)^2}\right]_{-1}^{1} =$$

$$\sum_{n=0}^{\infty} \frac{1-(-1)^{n+1}}{(n+1)^2} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2} = 2 \cdot \frac{\pi^2}{8} = \frac{\pi^2}{4} < \infty.$$

Therefore, we have

$$\int_{-1}^{1} \frac{-\ln(1-x)}{x}\, dx = \frac{\pi^2}{4}.$$

Along these lines, we also obtain the following two integrals:

$$\int_{0}^{1} \frac{-\ln(1-x)}{x}\, dx = \frac{\pi^2}{6}$$

and

$$\int_{-1}^{0} \frac{-\ln(1-x)}{x}\, dx = \frac{\pi^2}{12}.$$

(See also and compare with **Problem 3.7.44**.)

▲

2.3.2 Commuting Limits and Derivatives

We conclude this section by saying a few words about **commuting limits and derivatives**. This material is also important like the material about commuting limits and integrals before. It is implicitly related to the **Differentiability Part** of the **Main Theorem, 2.2.1**.

In a calculus course, we have learnt that we can differentiate power series term by term. This means that we can commute the derivative $\frac{d}{dx}$ with the infinite summation $\sum_{n=0}^{\infty} = \lim_{0 \leq k \to \infty} \left(\sum_{n=0}^{k} \right)$, that is, we can switch the order of differentiation and the limit process of the infinite summation. Hence, if the power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, with center $a = 0$ without loss of generality, converges in the open interval $(-r, r) \subseteq \mathbb{R}$, where $r > 0$ or $r = \infty$, and $u \in (-r, r)$ then

$$\frac{d}{dx}|_{x=u} \left[f(x) \right] = \frac{d}{dx}|_{x=u} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \frac{d}{dx}|_{x=u} \left(a_n x^n \right) =$$
$$\sum_{n=0}^{\infty} a_n n x^{n-1}|_{x=u} = \sum_{n=0}^{\infty} a_n n u^{n-1} = \sum_{n=1}^{\infty} n a_n u^{n-1}.$$

At the endpoints $-r$ and r, anything can happen. Sometimes the appropriate side-derivatives exist.

Whereas this commuting is always legitimate with derivatives of power series at any point in the interval of convergence, it is not valid in every situation with limits of sequences or series of functions and derivatives taken at points within the domain of definition of all functions involved. Serious mistakes may occur if such a commutation is performed while not valid! For instance:

Example 2.3.19 In **Example 2.3.4**, we saw that the sequence of functions

$$f_n : \mathbb{R} \to \mathbb{R} \;\; f_n(x) = \frac{\sin(nx)}{\sqrt{n}}, \;\; \forall \; x \in \mathbb{R},$$

converges uniformly to the function

$$f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = 0, \;\; \forall \; x \in \mathbb{R}.$$

Let us now examine the derivatives:

$$\frac{d}{dx} \left[f_n(x) \right] = \frac{d}{dx} \left[\frac{\sin(nx)}{\sqrt{n}} \right] = \sqrt{n} \cos(nx), \;\; \forall \; x \in \mathbb{R}$$

and

$$\frac{d}{dx}f(x) = \frac{d}{dx}(0) = 0, \ \forall \ x \in \mathbb{R}.$$

We see that

$$\lim_{n\to\infty}\frac{d}{dx}|_{x=0}\left[f_n(x)\right] = \lim_{n\to\infty}\frac{d}{dx}|_{x=0}\left[\frac{\sin(nx)}{\sqrt{n}}\right] =$$
$$\lim_{n\to\infty}\sqrt{n}\cos(nx)|_{x=0} = \lim_{n\to\infty}\sqrt{n}\cos(0) = \lim_{n\to\infty}\sqrt{n}\cdot 1 = \infty,$$

but

$$\frac{d}{dx}|_{x=0}\left[f(x)\right] = 0.$$

Therefore, we see that

$$\lim_{n\to\infty}\frac{d}{dx}|_{x=0}\left[f_n(x)\right] = \infty \neq 0 = \frac{d}{dx}|_{x=0}\left[f(x)\right].$$

Hence, in this example, even though the sequence of the differentiable functions $(f_n)_{n\in\mathbb{N}}$ converges to a differentiable function f uniformly in $\mathbb{R}$, we cannot switch the order of limit and differentiation.

▲

By **this example**, we see that we need some stronger conditions to guarantee the legitimacy of commuting (switching) the order of a limit process and taking the derivative (differentiating). The most-seen general theorem in the literature, or slight variations of it, is the following:

Theorem 2.3.8 *Suppose that a sequence of real functions* $(f_n)_{n\in\mathbb{N}}$ *on an interval* $(a,b) \subseteq \mathbb{R}$ *satisfies the following conditions:*

(a) All functions f_n*'s are differentiable.*

(b) There exists a point $x_0 \in (a,b)$ *such that the sequence of real numbers* $f_n(x_0)$ *converges. (We require convergence at at least one point of the domain.)*

(c) The sequence of the derivative functions $f_n'(x)$ *converges uniformly to some function* $g(x)$ *on* (a,b)*, as* $n \to \infty$*. I.e.,* $\exists$ $g(x)$ *real function on* (a,b)*, such that*

$$\lim_{n\to\infty} f_n'(x) \overset{un}{=} g(x) \ \ in \ \ (a,b).$$

Then: The sequence of $f_n(x)$*'s converges uniformly to a differentiable function* $f(x)$ *on* (a,b)*, as* $n \to \infty$*, that is,*

$$\lim_{n\to\infty} f_n(x) \overset{un}{=} f(x) \ \ in \ \ (a,b),$$

and

$$\left[\lim_{n\to\infty} f_n(x)\right]' = \lim_{n\to\infty} f_n'(x),$$

or

$$f'(x) = g(x).$$

Remark: If a power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $R > 0$, then the radius of convergence of the power series $\sum_{k=0}^{\infty} k\, b_k(x-c)^{k-1}$ is also R. (Prove this by noticing $\lim_{k\to\infty} \sqrt[n]{k} = 1$, or $\lim_{k\to\infty} \frac{k+1}{k} = 1$!)

Then the sequences of the initial partial sums $S_n(x) = \sum_{k=0}^{n} b_k(x-c)^k$ and $S'_n(x) = \sum_{k=1}^{n} k\, b_k(x-c)^{k-1}$ of the two power series, respectively, satisfy all of the conditions (and even more) of the above Theorem in their common open interval of convergence. So, we apply the above Theorem in their common open interval of convergence to prove that **the derivative of a power series is equal to the power series of the derivatives of its terms, at any** $x \in (c-R, c+R)$.

Example 2.3.20 Condition (b) is necessary. For instance, if we let $\forall\, n \in \mathbb{N}\ f_n(x) = n,\ \forall\, x \in \mathbb{R}$, then the **conditions (a)** and **(c)** are satisfied as $f'_n(x) = 0,\ \forall\, x \in \mathbb{R}$ and $\forall\, n \in \mathbb{N}$. So, as $n \to \infty$ the sequence of the derivative functions converges uniformly in $\mathbb{R}$ to the differentiable function $g(x) = 0,\ \forall\, x \in \mathbb{R}$. But

$$\lim_{n\to\infty} f_n(x) \ \text{ does not exist at any point } \ x \in \mathbb{R},$$

and therefore we cannot claim any of the conclusions of the Theorem.

▲

Example 2.3.21 Condition (c) is necessary. For instance, as we saw before the sequence of functions

$$f_n : \mathbb{R} \to \mathbb{R},\ \ f_n(x) = \frac{\sin(nx)}{\sqrt{n}},\ \ \forall\, x \in \mathbb{R},$$

converges uniformly to the function

$$f : \mathbb{R} \to \mathbb{R},\ \ f(x) = 0,\ \ \forall\, x \in \mathbb{R},$$

and we could not switch the order of taking limit and derivative.

We observe that the derivative functions

$$\frac{d}{dx}\left[f_n(x)\right] = \frac{d}{dx}\left[\frac{\sin(nx)}{\sqrt{n}}\right] = \sqrt{n}\cos(nx),\ \ \forall\, x \in \mathbb{R}$$

do not converge even point-wise to any differentiable function, let alone uniformly. For example, at the points $k\pi,\ \ k \in \mathbb{Z}$, we have

$$\lim_{n\to\infty} \frac{d}{dx}|_{x=k\pi}\,[f_n(x)] = \lim_{n\to\infty} \frac{d}{dx}|_{x=k\pi} \left[\frac{\sin(nx)}{\sqrt{n}}\right] =$$

$$\lim_{n\to\infty} \sqrt{n}\cos(nx)|_{x=k\pi} = \lim_{n\to\infty} \sqrt{n}\cos(nk\pi) =$$

$$\begin{cases} +\infty, \ \text{ if } k \text{ is even} \\ \\ \text{does not exist, if } k \text{ is odd.} \end{cases}$$

▲

Example 2.3.22 Let us examine the situation with the functions

$$f_n : (-1, 1) \to \mathbb{R},\ f_n(x) = x^n,\ \ \forall\ n \in \mathbb{N}$$

and their point-wise limit

$$f : (-1, 1) \to \mathbb{R},\ f(x) = 0.$$

We have

$$\forall\ n \in \mathbb{N},\ \ \frac{d}{dx} f_n(x) = nx^{n-1},\ \ -1 < x < 1.$$

We can easily prove

$$\lim_{n\to\infty} \frac{d}{dx}\,[f_n(x)] \overset{pw}{:=} g(x) = 0,\ \ -1 < x < 1.$$

The convergence here is only point-wise, not uniform. (Prove this as a calculus exercise! See **Problem 2.3.1**.)

Obviously,

$$\frac{d}{dx}\,[f(x)] = \frac{d}{dx}\,(0) = 0,\ \ -1 < x < 1.$$

So, here we do not have uniform convergence of the sequence of the derivatives of the functions, and so we have lost the uniform converge of the sequence of the functions $(f_n)_{n\in\mathbb{N}}$ on $(-1, 1)$, otherwise claimed by the **Theorem**.

▲

Problems

2.3.1 Prove that for $n \in \mathbb{N}$ and $x \in (-1, 1)$

$$\lim_{n\to\infty} \left(nx^{n-1}\right) \overset{pw}{=} 0$$

and this convergence is not uniform over $(-1,1)$.

Is the convergence uniform on any closed interval $[a,b] \subset (-1,1)$? Prove your answer!

2.3.2 Prove

$$\text{(a)} \lim_{n\to\infty} \int_0^\infty \left(1-\frac{x}{n}\right)^n e^{\frac{x}{2}}\,dx = 2$$

and

$$\text{(b)} \lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n e^{\frac{x}{2}}\,dx = 2.$$

[Hint: Imitate the ideas of **Examples 2.3.9** and **2.3.11**.]

2.3.3 For all $n \in \mathbb{N}$, consider the real functions

$$f_n(x) = \frac{nx^2}{nx^2+1}, \qquad x \in \mathbb{R}.$$

(a) Prove that $\forall\ x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) \overset{pw}{=} f(x) = \begin{cases} 1, & \text{if} \quad x \neq 0 \\ \\ 0, & \text{if} \quad x = 0. \end{cases}$$

(b) Prove that the convergence in **(a)** is not uniform over the whole $\mathbb{R}$.

(c) Prove that in any set $A_\varepsilon = (-\infty, -\varepsilon] \cup [\varepsilon, \infty) \subset \mathbb{R}$, where $\varepsilon > 0$, the convergence in **(a)** is uniform.

(d) Compute the derivatives $f_n'(x)$ for all $n \in \mathbb{N}$. (Simplify.)

(e) Prove that $\forall\ x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n'(x) \overset{pw}{=} 0.$$

(f) Prove that the convergence in **(e)** is not uniform.

(g) Prove

$$\lim_{n\to\infty} f_n'(x) \neq \left[\lim_{n\to\infty} f_n(x)\right]'.$$

2.3.4 Consider a sequence of real numbers (a_n) with $n \in \mathbb{N}_0$, such that $a_n \geq 0$ (non-negative) for all $n \in \mathbb{N}_0$ and $\displaystyle\sum_{n=0}^{\infty} a_n = s \in \mathbb{R}$.

(a) Prove that for every $t \in R$, $f(t) := \displaystyle\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ converges absolutely.

(b) Use **Theorem 2.3.3** and the fact that $\int_0^\infty t^n e^{-t} dt = n!$ to prove that

$$\int_0^\infty f(t)\, e^{-t}\, dt = s.$$

2.3.5 Justify each equality of the following integral representations of the Euler-Mascheroni constant γ.

(See also **Problem 2.1.25** and its **footnote**. At some steps you need to perform an appropriate u-substitution. Some ideas in **Examples 2.3.9** and **2.3.11** may help at some equalities.)

$$-\int_0^\infty e^{-x} \ln(x)\; dx = \int_0^\infty -e^{-x} \ln(x)\; dx =$$

$$\int_0^1 -e^{-x} \ln(x)\; dx + \int_1^\infty -e^{-x} \ln(x)\; dx =$$

$$\int_0^1 \ln(x)\; de^{-x} + \int_1^\infty \ln(x)\; de^{-x} =$$

$$\int_0^1 \ln(x)\; d\left(e^{-x} - 1\right) + \int_1^\infty \ln(x)\; de^{-x} =$$

$$\int_0^1 \frac{1 - e^{-x}}{x}\, dx - \int_1^\infty \frac{e^{-x}}{x}\, dx =$$

$$\lim_{n\to\infty} \left[\int_0^1 \frac{1 - (1 - \frac{x}{n})^n}{x}\, dx - \int_1^n \frac{(1 - \frac{x}{n})^n}{x}\, dx\right] =$$

$$\lim_{n\to\infty} \left[\int_0^{\frac{1}{n}} \frac{1 - (1 - t)^n}{t}\, dt - \int_{\frac{1}{n}}^1 \frac{(1 - t)^n}{t}\, dt\right] =$$

$$\lim_{n\to\infty} \left[\int_0^{\frac{1}{n}} \frac{1 - (1 - t)^n}{t}\, dt + \int_{\frac{1}{n}}^1 \left[\frac{1 - (1 - t)^n}{t} - \frac{1}{t}\right] dt\right] =$$

$$\lim_{n\to\infty} \left[\int_0^1 \frac{1 - (1 - t)^n}{t}\, dt - \int_{\frac{1}{n}}^1 \frac{1}{t}\, dt\right] =$$

$$\lim_{n\to\infty} \left[\int_0^1 \frac{1 - t^n}{1 - t}\, dt + \ln\left(\frac{1}{n}\right)\right] =$$

$$\lim_{n\to\infty} \left[\int_0^1 \left(t^n + t^{n-1} + ... + t + 1\right)\, dt + \ln\left(\frac{1}{n}\right)\right] =$$

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{2} + 1 - \ln(n)\right) := \gamma,$$

by the definition of γ. (E.g., see footnote **to Problem 2.1.25**.)

2.3.6 Show the useful result

$$\int_0^\infty -e^{-x}\ln(x)\;dx = \int_0^1 \frac{1-e^{-x}}{x}\,dx - \int_1^\infty \frac{e^{-x}}{x}\,dx =$$

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_0^1 \frac{e^{\frac{-1}{t}}}{t}\,dt = \int_0^1 \frac{1-e^{-t}-e^{\frac{-1}{t}}}{t}\,dt.$$

(See also **Problem 2.1.25**.)

2.4 Double Integral Technique

As we have seen in **Section 2.1**, the double integrals can be used to evaluate improper integrals. Here we investigate this technique much further. It consists of imbedding a given improper integral into an appropriate double integral. Then we evaluate the double integral, usually by switching the order of integration. (Therefore, we could name this tecnique as: **"Technique of Switching the Order of Integration in Double Integrals."**) From the result we find about the double integral we can now find the value of the originally given improper integral.

In a course of multi-variable calculus, we see that if a function of two variables $z = f(x,y)$ is defined on the closed rectangle $\mathcal{R} = [a,b]\times[c,d]$, where $a < b$ and $c < d$ real numbers, and it is integrable and bounded (e.g., when it is continuous, but not only continuous), then we have

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

But, if $z = f(x,y)$ is not bounded in the closed rectangle $\mathcal{R}$ (and therefore not continuous in $\mathcal{R}$), and its double integrals—one for each order of integration—exist, it is not automatic that they are equal. Two often-encountered examples (counterexamples) are the one in **Problem 2.4.7** and the following:

We consider $\mathbb{R} = [0,1]\times[0,1]$ and

$$f(x,y) = \begin{cases} \dfrac{x-y}{(x+y)^3}, & \text{if} \quad (x,y) \in (0,1]\times(0,1] \\ \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Then

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right] dx = \int_0^1 \left[\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy\right] dx =$$
$$= \int_0^1 \left\{\int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dy\right\} dx = \dots = \frac{1}{2},$$

whereas

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right] dy = \dots = \frac{-1}{2}.$$

Also,

$$\int_0^1 \left[\int_0^x \frac{x-y}{(x+y)^3}\,dy\right] dx = \infty$$

and

$$\int_0^1 \left[\int_0^y \frac{x-y}{(x+y)^3}\,dx\right] dy = -\infty.$$

(Carry out the details to convince yourselves. See **Problem 2.4.6**.)

On this example, we notice the following five things:

1. $\lim\limits_{0<x=y\to 0} f(x,y) = 0.$
2. $\lim\limits_{\substack{0<x\to 0\\ y=0}} f(x,y) = +\infty.$
3. $\lim\limits_{\substack{0<y\to 0\\ x=0}} f(x,y) = -\infty.$
4. $f(x,y)$ is discontinuous at $(0,0)$, no matter what value we assign to $f(0,0)$.
5. Near $(0,0)$ the function $f(x,y)$, besides the value 0, assumes unbounded positive and unbounded negative values.

This example and many more like it show that we need extra conditions to guarantee that the two iterated integrals of a double integral give the same result, which is the value of the double integral. Especially over infinite domains and with unbounded integrable functions that change sign, the situation becomes, at times, quite complicated. So, in this section, we are going to state the most important applicable conditions under which the two iterated integrals of a double integral give the same result.

Suppose now that we have a continuous or piecewise continuous function $z = f(x,y)$ where (x,y) is in a region $R \subseteq \mathbb{R}^2$, which is bounded or unbounded and not necessarily closed and/or open. The function $f(x,y)$

may also be bounded or unbounded in this region R. We have already seen similar situations with improper integrals in previous sections. By letting $f(x,y) = 0,\ \forall\ (x,y) \in \mathbb{R}^2 - R$, we may consider $f(x,y)$ to be defined over all $\mathbb{R}^2$ and piecewise continuous.

After mentioning these adjustments, we now need some convenient conditions that guarantee the equality:

$$\int\!\!\int_{\mathbb{R}^2} f(x,y)\,dxdy =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx. \tag{2.3}$$

If $f(x,y)$ is defined in the closed rectangle $[a,b]\times[c,d]$, then we may use the numbers a,b,c,d as the limits of the double integration, instead of the $\pm\infty$. We could do the same if the rectangle were open or partially open, in which case some open limits are allowed to be finite or $\pm\infty$.

Real analysis proves that some **convenient conditions** which guarantee the validity of equality **(2.3)** (written without loss of generality over the whole $\mathbb{R}^2$) are the following:

1. **Condition I.**

$$f(x,y) \geq 0,\ \ \forall\ \ (x,y) \in \mathbb{R}^2,$$

or

$$f(x,y) \leq 0,\ \ \forall\ \ (x,y) \in \mathbb{R}^2.$$

Notice that in this case, the three parts of equality **(2.3)** may be all ∞ or $-\infty$, respectively. So, if the function is non-negative or non-positive (i.e., does not change sign) we can freely switch the order of integration in any way we would like without altering the answer.

Otherwise, we may use any one of the following three **convenient conditions**, which in real analysis are proven to be **equivalent**. I.e., any one of them implies the other two.

2. **Condition II.**

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |f(x,y)|\,dx\right]dy < \infty.$$

3. **Condition III.**

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |f(x,y)|\,dy\right]dx < \infty.$$

4. **Condition IV.**

$$\iint\limits_{\mathbb{R}^2} |f(x,y)|\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|\,dxdy < \infty.$$

Here, the double integral is the limit of the double **Riemann Sums** of $|f(x,y)|$, as the norms of the double partitions approach 0.

The **equivalent conditions II and III** were proved by Tonelli.[10] The **IV equivalent condition** was proved by Fubini.[11]

Under any of the above four conditions, the results of the double integration in either order are equal, and for piecewise continuous functions their common value is the value of the double integral as the limit of the corresponding double Riemann Sums. If **condition I** is not met, then the most convenient conditions to check and use in applications are **II** and **III**. **Condition IV** is convenient for more theoretical results. So, if any of the above four conditions is satisfied, we have:

$$\iint\limits_{\mathbb{R}^2} f(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [f(x,y)]\,\,dxdy =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx.$$

We must be careful when any one of these conditions is violated. Apart from the fact that switching the order of integration may result in two different answers, even a transformation of coordinates (change of variables) may result in a different answer. For example, solve **Problem 2.4.5**.

Example 2.4.1 We illustrate this technique by evaluating the famous **Fresnel Integrals**, which are useful in optics and road construction:

$$\int_0^{\infty} \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4} \quad \text{and} \quad \int_0^{\infty} \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

We will evaluate the first one. The second one is done by the same type of work, and so we leave it to the reader as a practicing exercise whose solution is analogous to the one presented here. (See **Problem 2.6.10** and compare with **Example 3.7.16**.)

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[10] Leonida Tonelli, Italian mathematician, 1885-1946.

[11] Guido Fubini, Italian mathematician, 1879-1943.

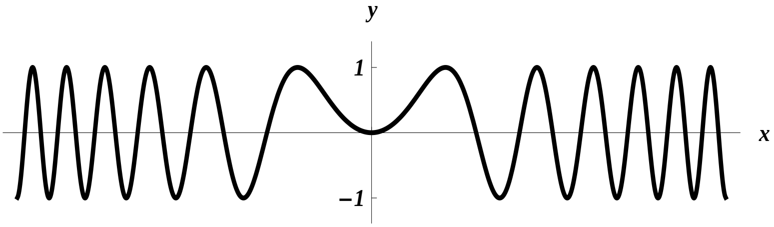

FIGURE 2.3: Function $\mathbf{y = sin(x^2)}$

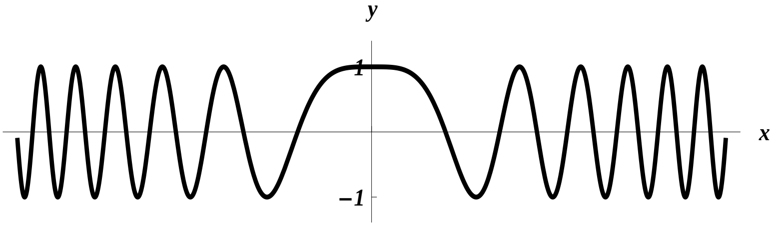

FIGURE 2.4: Function $\mathbf{y = cos(x^2)}$

In **Example 1.3.14**, we proved that the integral converges conditionally, and if we make the change of variables $u = x^2$, we get

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du.$$

Next, in **Section 2.1**, **Problem 2.1.11**, we proved that for any $u > 0$

$$\int_0^\infty e^{-uv^2}\,dv = \frac{1}{2}\sqrt{\frac{\pi}{u}} \qquad \Longleftrightarrow \qquad \frac{1}{\sqrt{u}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-uv^2}\,dv.$$

Hence

$$\begin{gathered}\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du = \int_0^\infty \sin(u)\frac{1}{\sqrt{u}}\,du = \\ \int_0^\infty \sin(u)\left(\int_0^\infty \frac{2}{\sqrt{\pi}}\,e^{-uv^2}\,dv\right)du = \\ \frac{2}{\sqrt{\pi}}\int_0^\infty\left(\int_0^\infty e^{-uv^2}\sin(u)\,du\right)dv.\end{gathered}$$

At this point, if we could justify the switching of the order of integration, the calculation of this double integral would be rather simple, and we would finish the proposed integral at this point without any further pains, since by **Problem 1.2.14** we have:

$$\int_0^{\infty} e^{-v^2u} \sin(u)\, du = \frac{1}{1+v^4}.$$

Then we can use partial fractions

$$\frac{1}{1+v^4} =$$
$$\frac{1}{(v^2+\sqrt{2}v+1)(v^2-\sqrt{2}v+1)} = \frac{Av+B}{v^2+\sqrt{2}v+1} + \frac{Cv+D}{v^2-\sqrt{2}v+1},$$

and after computing A, B, C and D, we eventually find, by means of natural logarithm and arc-tangent, that

$$\int \frac{dv}{1+v^4} = K+$$
$$\frac{1}{4\sqrt{2}}\left[\ln\left(\frac{v^2+v\sqrt{2}+1}{v^2-v\sqrt{2}+1}\right) + 2\arctan(v\sqrt{2}+1) + 2\arctan(v\sqrt{2}-1)\right]$$

with K the constant of integration. So,

$$\int_0^{\infty} \frac{dv}{1+v^4} = \frac{1}{4\sqrt{2}}\left[0 + 2\cdot\frac{\pi}{2} + 2\cdot\frac{\pi}{2} - 0 - 2\cdot\frac{\pi}{4} - 2\cdot(-\frac{\pi}{4})\right] = \frac{\pi\sqrt{2}}{4}.$$

[The method of partial fractions is elementary but many times involves some lengthy computations. To find this integral faster, we could use **Example 2.2.6 (b)**. See also **Problem 2.6.20**.]

Therefore, if all things performed above were legitimate, the **Fresnel Integral** would be

$$\int_0^{\infty} \sin(x^2)\, dx = \frac{1}{2}\frac{2}{\sqrt{\pi}}\frac{\pi\sqrt{2}}{4} = \frac{\sqrt{2\pi}}{4}.$$

But to ascertain this result, we must justify the change of order of integration. This cannot be justified by **condition I**, since the function $e^{-uv^2}\sin(u)$ changes sign.

Using the inequality $|e^{-v^2u}\sin(u)| \leq e^{-uv^2}$ in order to check **Condition II** or **III** does not yield the desired result because,

$$\int_0^{\infty}\left[\int_0^{\infty} e^{-uv^2}\, dv\right] du =$$
$$\int_0^{\infty} \frac{1}{2}\sqrt{\frac{\pi}{u}}\, du = \frac{\sqrt{\pi}}{2}\int_0^{\infty}\frac{1}{\sqrt{u}}\, du = \sqrt{\pi}\ \left[\sqrt{u}\right]_0^{\infty} = \infty$$

and

$$\int_0^\infty \left[\int_0^\infty e^{-uv^2}\, du\right] dv = \int_0^\infty \frac{1}{v^2}\, dv = \left[-\frac{1}{v}\right]_0^\infty = -0 + \infty = \infty.$$

To bypass this difficulty, we must use an indirect way. We succeed in doing so by introducing an extra mitigating multiplicative factor, namely e^{-cu}, with $c \geq 0$ acting as a parameter. That is, we consider the more general integral

$$\int_0^\infty e^{-cu}\frac{\sin(u)}{\sqrt{u}}\, du.$$

This integral is continuous in $c \in [0, \infty)$ (see **Problem 2.2.18**).

So, for any $c \geq 0$, we have that

$$\int_0^\infty e^{-cu}\frac{\sin(u)}{\sqrt{u}}\, du = \frac{2}{\sqrt{\pi}}\int_0^\infty \left(\int_0^\infty e^{-(c+v^2)u}\sin(u)\, dv\right) du. \qquad (2.4)$$

Observe

$$\left|e^{-(c+v^2)u}\sin(u)\right| \leq e^{-(c+v^2)u}.$$

When $c > 0$, since the function $e^{-(c+v^2)u}$ is positive, by **condition I**, we get

$$\int_0^\infty \int_0^\infty e^{-(c+v^2)u}\, dudv = \int_0^\infty \left[\frac{e^{-(c+v^2)u}}{-(c+v^2)}\right]_0^\infty dv =$$

$$\int_0^\infty \frac{1}{c+v^2}\, dv = \left[\frac{1}{\sqrt{c}}\arctan\left(\frac{v}{\sqrt{c}}\right)\right]_0^\infty = \frac{\pi}{2\sqrt{c}} < \infty.$$

So, if $c > 0$, we have

$$\int_0^\infty \left(\int_0^\infty \left|e^{-(c+v^2)u}\sin(u)\right| du\right) dv < \frac{\pi}{2\sqrt{c}} < \infty.$$

Therefore, by the **Tonelli Conditions II or III**, we are allowed to switch the order of integration in the double integral **(2.4)** above (without the absolute value). Thus, using **Problem 1.2.14**, we obtain

$$\int_0^\infty e^{-cu}\frac{\sin(u)}{\sqrt{u}}\, du = \qquad (2.5)$$

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \left(\int_0^\infty e^{-(c+v^2)u}\sin(u)\, du\right) dv = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dv}{1+(c+v^2)^2}.$$

Since

$$\forall \;\; c \geq 0, \quad 0 < \frac{1}{1+(c+v^2)^2} \leq \frac{1}{1+v^4} \quad \text{and} \quad \int_0^\infty \frac{dv}{1+v^4} = \frac{\pi\sqrt{2}}{4},$$

we also conclude that the integral

$$\int_0^\infty \frac{dv}{1+(c+v^2)^2} \quad \text{is continuous as a function of} \quad c \geq 0.$$

Then, in equality **(2.5)**, we let $c \to 0^+$ to get

$$\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dv}{1+v^4} = \frac{2}{\sqrt{\pi}} \cdot \frac{\pi\sqrt{2}}{4} = \frac{\sqrt{2\pi}}{2}.$$

This finally verifies that the **Fresnel integral** is equal to

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin(u)}{\sqrt{u}}\,du = \frac{1}{2} \cdot \frac{\sqrt{2\pi}}{2} = \frac{\sqrt{2\pi}}{4}.$$

Remark 1: Similar work shows

$$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

Remark 2: Note that since $\sin(x^2)$ and $\cos(x^2)$ are even functions in $\mathbb{R}$, then

$$\int_{-\infty}^\infty \sin(x^2)\,dx = 2\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{2}.$$

and

$$\int_{-\infty}^\infty \cos(x^2)dx = 2\int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{2}.$$

Also for any real number $a \neq 0$, by means of u-substitution, we have

$$\int_{-\infty}^\infty \sin(ax^2)dx = 2\int_0^\infty \sin(ax^2)dx = \operatorname{sign}(a)\sqrt{\frac{2\pi}{2|a|}}$$

and

$$\int_{-\infty}^\infty \cos(ax^2)dx = 2\int_0^\infty \cos(ax^2)dx = \sqrt{\frac{2\pi}{2|a|}}.$$

[See also **Problem 2.4.14, (a)**.]

Remark 3: We observe that, using **Problem 1.2.14**, for $\beta > 0$ and $c > 0$ constants, the equality **(2.5)** is written more generally

$$2\int_0^{\infty} e^{-cx^2}\sin(\beta x^2)\,dx = \int_0^{\infty} e^{-cu}\,\frac{\sin(\beta u)}{\sqrt{u}}\,du = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{\beta\,dv}{\beta^2+(c+v^2)^2}.$$

(See **Problem 3.7.13** for the final closed answer.)

Remark 4: The integrals defined by

$$S(x) := \int_0^x \sin\left(t^2\right)dt \quad \text{and} \quad C(x) := \int_0^x \cos\left(t^2\right)dt$$

are called **Fresnel sine integral** and **Fresnel cosine integral**, respectively. Some authors like to use $\frac{\pi}{2}\,t^2$ instead of just t^2 in the argument. So, check what definition the book you study uses.

▲

Application

In the theory of curves in $\mathbb{R}^2$, the natural equation of a curve C is its curvature κ given as a function of the arc-length parameter. I.e., $\kappa = f(s)$, where s is the arc-length parameter of C.

Then, up to the rigid motions (isometries) of the plane, the curve is given by

$$C(s) = \left[\int_{\star}^{s}\cos\left(\int_{\star\star}^{t} f(u)\,du\right)dt\,,\ \int_{\star}^{s}\sin\left(\int_{\star\star}^{t} f(u)\,du\right)dt\right]$$

with the frame $\{T,\,N\}$ of the tangent and normal unit vectors positively oriented.

So, the curve with $\kappa = 2a^2 s$, where $a > 0$ constant, i.e., the curvature is a positive multiple of s, without loss of generality is given by

$$C(s) = \left[\int_0^s \cos\left(a^2t^2\right)dt\,,\ \int_0^s \sin\left(a^2t^2\right)dt\right].$$

Such a curve is called **clothoid** or **spiral of Cornu (see Figure 2.5)** .[12] Cornu used it in optics for diffraction problems and Fresnel for lens designing. It was also known to Leonhard Euler and Jakob Bernoulli.

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[12] Marie Alfred Cornu, French mathematician, 1841-1902.

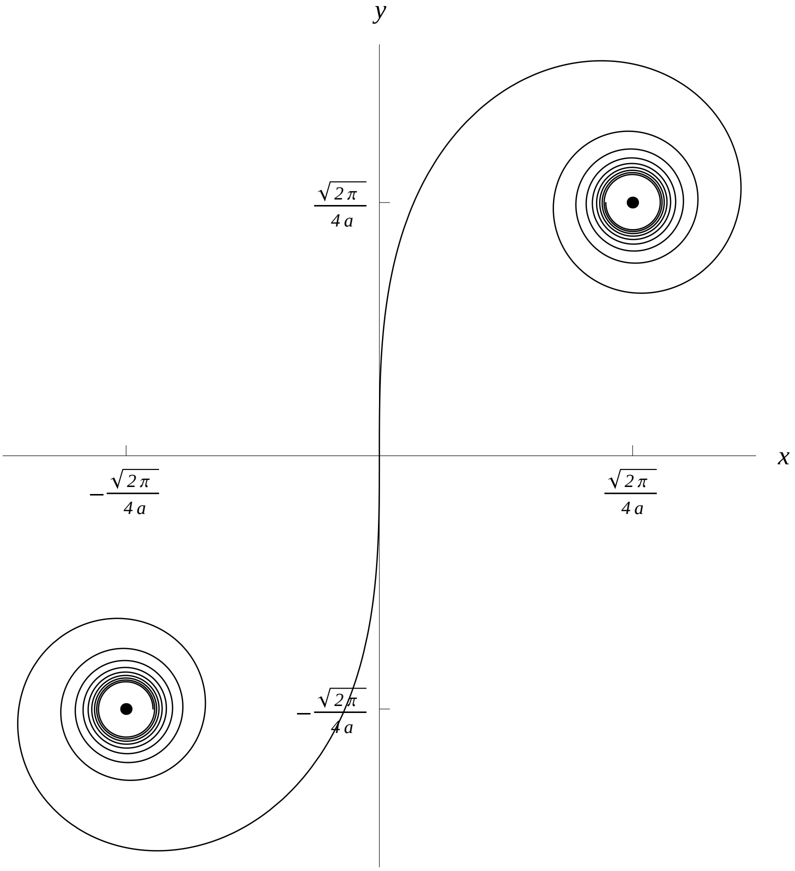

FIGURE 2.5: Clothoid or Spiral of Cornu

This curve is very important in designing and constructing railways and exits and entrances of highways.

We observe: (1) $C(0) = 0$. (2) The curve is symmetrical about the origin $C(-s) = -C(s)$. (3) Using $v = at$ and the values of the Fresnel integrals, found in the **previous example**, we find that

$$\lim_{s\to\infty} C(s) = \left(\frac{\sqrt{2\pi}}{4a}, \frac{\sqrt{2\pi}}{4a}\right) \quad \text{and} \quad \lim_{s\to-\infty} C(s) = \left(-\frac{\sqrt{2\pi}}{4a}, -\frac{\sqrt{2\pi}}{4a}\right).$$

Problems

2.4.1 Prove the relation

$$\frac{1}{x} = \int_0^{\infty} e^{-xt}\,dt, \quad \text{for} \quad x > 0.$$

Use it to prove

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}.$$

[Hint: Use appropriate proper double integrals as in calculus. When the double integrals are proper, the conditions for switching the order of integration are valid. Then take limits.]

2.4.2 Prove completely that

$$\int_0^{\infty} \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

2.4.3 Use the power series of $\sin(x)$ and $\cos(x)$ to express the following two integrals as power series:

$$S(x) := \int_0^{x} \sin\left(u^2\right)\,du \quad \text{and} \quad C(x) := \int_0^{x} \cos\left(u^2\right)\,du.$$

2.4.4 Evaluate the integrals

$$\int_0^{\infty} \frac{\cos(x)}{\sqrt{x}}\,dx \qquad \text{and} \qquad \int_0^{\infty} \frac{\sin(x)}{\sqrt{x}}\,dx.$$

2.4.5 (a) By looking at the double improper integral

$$\int_0^{\infty}\int_0^{\infty} \sin(x^2+y^2)\,dxdy$$

as the limit of double proper integrals over the rectangles $[0,a]\times[0,b]$ as $a\to\infty$ and $b\to\infty$, use the values of the Fresnel integrals to find that its value obtained in this way is $\frac{\pi}{4}$.

(b) What happens if you try to evaluate this integral by using polar coordinates?

(c) Can you explain the discrepancy between the results of **(a)** and **(b)**?

2.4.6 Justify the results of the initial example of this section:

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right]dx = \int_0^1 \left[\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy\right]dx =$$

$$\int_0^1 \left[\int_0^1 \left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right)dy\right]dx = \ldots = \frac{1}{2}$$

whereas

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right]dy = \ldots = \frac{-1}{2}.$$

Next,

$$\int_0^1 \left[\int_0^x \frac{x-y}{(x+y)^3}\,dy\right]dx = \infty$$

and

$$\int_0^1 \left[\int_0^y \frac{x-y}{(x+y)^3}\,dx\right]dy = -\infty.$$

Also show that near $(0,0)$ the function $f(x,y) = \dfrac{x-y}{(x+y)^3}$ assumes values near 0, equal to 0, unbounded positive and unbounded negative.

2.4.7 Let

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0) \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that near $(0,0)$, $f(x,y)$ assumes values near 0, equal to 0, unbounded positive and unbounded negative.

(b) Write $x^2 - y^2 = 2x^2 - (x^2+y^2)$ and then break the fraction to prove that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx\right)dy = -\frac{\pi}{4}.$$

(c) Similarly, prove

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy\right)dx = \frac{\pi}{4}.$$

(d) Also, prove

$$\int_0^1 \left(\int_0^x f(x,y)\,dy\right)dx = \infty \quad \text{and} \quad \int_0^1 \left(\int_0^y f(x,y)\,dx\right)dy = -\infty.$$

2.4.8 Consider $a > 0$ and $b \in \mathbb{R}$ constants. Use a **Tonelli condition** to justify the change of order of integration in

$$\int_0^\infty \left(\int_0^1 e^{-ay} \sin(2bxy)\, dx \right) dy$$

and then prove

$$\int_0^\infty e^{-ay}\, \frac{1-\cos(2by)}{2y}\, dy = \int_0^\infty e^{-ay}\, \frac{\sin^2(by)}{y}\, dy = \frac{1}{4} \ln\left(\frac{a^2+4b^2}{a^2} \right).$$

Prove that this formula is also correct for $\alpha = 0$, as $\infty = \infty$.
(See also and compare with **Problem 2.5.18**.)

2.4.9 Consider $a \geq 0$ and $b \in \mathbb{R}$ constants. Then:
(a) Without computing, explain why for every $0 < N < \infty$

$$\int_0^N \left(\int_0^1 e^{-axy} \sin(by)\, dx \right) dy = \int_0^1 \left(\int_0^N e^{-axy} \sin(by)\, dy \right) dx.$$

(b) Now, perform the inner integration in both double integrals in **(a)** and write the obtained equation.

(c) Take limit as $N \to \infty$ of the equal expressions found in **(b)**, and prove: For $a \geq 0$ and $b \in \mathbb{R}$ constants, we have

$$I(\alpha) := \int_0^\infty e^{-ay} \frac{\sin(by)}{y}\, dy = \frac{\pi}{2} - \arctan\left(\frac{a}{b}\right) = \arctan\left(\frac{b}{a}\right).$$

[Hint: In **(c)**, commuting the limit and the integral in the second side of the equation obtained in **(b)**, you need to use either **Part (I)** of the **Main Theorem, 2.2.1,** or **Definition 2.3.2** and **Theorem 2.3.2**.]

(d) The integral $I(\alpha)$ found in **(c)** can also be found by the following steps:
(1) Use the **Main Theorem, 2.2.1,** and **Problem 1.2.14** to find

$$\frac{d}{d\alpha}\left[I(\alpha)\right].$$

(2) What is $I(0)$ and why?
(3) Now find $I(\alpha)$.

(**Note**: This was also found by a different method inside the computation of **Example 2.2.8** and stated in **Example 2.2.9**. It is also reported in many integral tables. See also and compare with **Problem 2.5.18**.)

2.4.10 (a) For $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants, prove

$$J(\alpha) := \int_0^\infty e^{-\alpha u} \frac{\sin^2(\beta u)}{u^2}\, du = \beta \arctan\left(\frac{2\beta}{\alpha}\right) - \frac{\alpha}{4} \ln\left(\frac{4\beta^2}{\alpha^2} + 1\right).$$

[Hint: Use appropriate integration by parts and the results in **Problems 2.4.8** and **2.4.9**. For $\alpha = 0$, as usual, you may use a continuity argument.]

(b) The integral $J(\alpha)$ found above can also be found by the following steps:
(1) Use the **Main Theorem, 2.2.1,** and **Problem 2.4.8** to find

$$\frac{d}{d\alpha}\left[J(\alpha)\right].$$

(2) Find $J(0)$. (See **Problem 2.2.15**).
(3) Now find $J(\alpha)$.

2.4.11 Change the integral[13]

$$\int_{x=0}^\infty \int_{x=y}^\infty e^{-(x-y)^2} \sin^2(x^2+y^2) \frac{x^2-y^2}{(x^2+y^2)^2}\, dy\, dx$$

to polar coordinates and prove that it is equal to

$$\frac{1}{16}[4\arctan(2) - \ln(5) - 2\pi] \approx -0.2165018.$$

2.4.12 For k and l real constants, prove:

$$\text{(a)If} l \geq k+1, \quad \int_0^\infty e^{-x} \frac{\sin^k(x)}{x^l}\, dx = +\infty.$$

$$\text{(b)If} l \geq 1, \quad \int_0^\infty e^{-x} \frac{\cos^k(x)}{x^l}\, dx = +\infty.$$

2.4.13 For $l = 0,\ 1,\ 2$ and 3, find

$$\int_0^\infty e^{-x} \frac{\sin^3(x)}{x^l}\, dx.$$

---

[13] American Mathematical Monthly, Problem 11650, Vol. 119, Number 6, June-July 2012.

[Hint: $\sin^3(x) = \frac{-1}{4}\sin(3x) + \frac{3}{4}\sin(x)$, etc.]

2.4.14 (a) For any real numbers $a \neq 0$, b and c, use the known identify $ax^2+bx+c = a(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$, the Fresnel integrals, and appropriate trigonometric formulae, to prove

$$\int_{-\infty}^{\infty} \sin(ax^2+bx+c)dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\operatorname{sign}(a)\cos\left(\frac{4ac-b^2}{4a}\right) + \sin\left(\frac{4ac-b^2}{4a}\right)\right]$$

and

$$\int_{-\infty}^{\infty} \cos(ax^2+bx+c)dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\cos\left(\frac{4ac-b^2}{4a}\right) - \operatorname{sign}(a)\sin\left(\frac{4ac-b^2}{4a}\right)\right].$$

(b) Use **(a)** and the appropriate trionometric formulae, to compute the integrals

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \sin(-2x^2)\cos(4x+6)dx, \\ I_2 &= \int_{-\infty}^{\infty} \sin(3x^2)\sin(-2x+1)dx, \\ I_3 &= \int_{-\infty}^{\infty} \cos(x^2)\cos(-4x+1)dx. \end{aligned}$$

2.5 Frullani Integrals

A special category of improper integrals are the Frullani[14] or Cauchy-Frullani integrals. They have the **general type**

$$I(a,b) = \int_0^{\infty} \frac{f(bx)-f(ax)}{x}\,dx \quad \text{where} \quad 0 < a,\ b < \infty,$$

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[14] Giuliano Frullani, Italian mathematician, 1795-1834.

where $f : (0, \infty) \longrightarrow \mathbb{R}$ is a nice function not necessarily continuous but satisfies the following **condition** which we assume **throughout this section**:

$$(\maltese) \quad \int_\lambda^\mu f(x)\,dx \quad \text{exists}, \quad \forall \ 0 < \lambda < \mu < \infty.$$

(For functions continuous in $(0, \infty)$, this condition is obviously fulfilled.)

We observe that $I(a, a) = 0$ and $I(a, b) = -I(b, a)$ for any $0 < a,\ b < \infty$. We must also observe that for $I(a, b)$ to exist when $0 < a \neq b < \infty$, the limits $\lim\limits_{x \to 0^+} [f(bx) - f(ax)]$ and $\lim\limits_{x \to \infty} [f(bx) - f(ax)]$ must either be zero or oscillate about zero. These conditions are necessary (but not sufficient, as we will see in examples and problems that follow). Otherwise, i.e., if either condition or both is/are not valid, then $I(a, b)$ is either $\pm\infty$ or does not exist (oscillates depending on the two limiting processes). (Prove this assertion, using the **Limit Comparison Test, Example 1.3.4,** and the fact that for $0 < r,\ c < \infty$, $\displaystyle\int_0^r \frac{1}{x}\,dx = \infty$ and $\displaystyle\int_c^\infty \frac{1}{x}\,dx = \infty$.)

In order to state the important general results and the hypotheses under which are valid, we begin with the following[15]:

Preliminary Computation

Consider any $0 < a,\ b < \infty$. Using the standing assumption **($\maltese$)** for $f(x)$, we can write

$$I(a, b) =$$

$$\lim_{\substack{h \to \infty \\ \epsilon \to 0^+}} \int_\epsilon^h \frac{f(bx) - f(ax)}{x}\,dx = \lim_{\substack{h \to \infty \\ \epsilon \to 0^+}} \left[\int_\epsilon^h \frac{f(bx)}{x}\,dx - \int_\epsilon^h \frac{f(ax)}{x}\,dx \right].$$

Performing the substitutions $t = bx$ and $t = ax$ in the first and second integral, respectively, and using the standing assumption **($\maltese$)** for

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[15] In the exposition of this material, we follow bibliography: Ostrowski 1949 and Agnew 1951.

$f(x)$, we find

$$I(a,b) = \lim_{\substack{h\to\infty \\ \epsilon\to 0^+}} \left[\int_{b\epsilon}^{bh} \frac{f(t)}{t} dt - \int_{a\epsilon}^{ah} \frac{f(t)}{t}\, dt \right] =$$

$$\lim_{\substack{h\to\infty \\ \epsilon\to 0^+}} \left[\int_{b\epsilon}^{ah} \frac{f(t)}{t} dt + \int_{ah}^{bh} \frac{f(t)}{t} dt - \int_{a\epsilon}^{b\epsilon} \frac{f(t)}{t}\, dt - \int_{b\epsilon}^{ah} \frac{f(t)}{t} dt \right] =$$

$$\lim_{\substack{h\to\infty \\ \epsilon\to 0^+}} \left[\int_{ah}^{bh} \frac{f(t)}{t} dt - \int_{a\epsilon}^{b\epsilon} \frac{f(t)}{t}\, dt \right].$$

[For this splitting of the integral to be legitimate, we need the **condition (✠)** and then the two extreme integrals cancel each other.]

If the individual limits exist as real numbers, or are ±∞, we can write

$$I(a,b) = \lim_{h\to\infty} \int_{ah}^{bh} \frac{f(t)}{t} dt - \lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(t)}{t}\, dt. \tag{2.6}$$

The case $\infty - \infty$ means that the $I(a,b)$ does not exist.

For giving more practical results, we will deal with these two limits under certain conditions. We assume that $f(x)$ is continuous in $(0,\delta)$ for some $\delta > 0$ and in (μ, ∞) for some $\mu > 0$.

In the first limit, we use the substitution $t = e^u$ and consider h large enough so that $ah > \mu$ and $bh > \mu$ to invoke the Mean Value Theorem for integrals. Then we find

$$\int_{ah}^{bh} \frac{f(t)}{t} dt = \int_{\ln(ah)}^{\ln(bh)} f(e^u) du = [\ln(bh) - \ln(ah)]\, f(e^c) = \ln\left(\frac{b}{a}\right) f(\zeta),$$

where c is between $\ln(ah)$ and $\ln(bh)$ or $\zeta = e^c$ is between ah and bh.

When $h \to \infty$ then $\zeta \to \infty$ and

$$\lim_{h\to\infty} \int_{ah}^{bh} \frac{f(t)}{t} dt = \ln\left(\frac{b}{a}\right) \cdot \lim_{\zeta\to\infty} f(\zeta) = \ln\left(\frac{b}{a}\right) f(\infty)$$

when $f(\infty) := \lim\limits_{x\to\infty} f(x)$ is a real number or ±∞.

Now using the substitution $t = e^{-u}$ in the second limit, with analogous work, we find

$$\lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(t)}{t} dt = \ln\left(\frac{b}{a}\right) f(0),$$

when $f(0) := \lim\limits_{x\to 0^+} f(x)$ is a real number or ±∞.

This preliminary computation along with the stipulated hypotheses proves the following general theorem:

Theorem 2.5.1 *Let* $f : (0,\infty) \longrightarrow \mathbb{R}$ *be a nice function that satisfies (✠) and* $0 < a,\ b < \infty$. *Assume that* $f(x)$ *is continuous in* $(0,\delta)$ *for some* $\delta > 0$ *and in* (μ,∞) *for some* $\mu > 0$, $\lim\limits_{x\to 0^+} f(x) = f(0) \in [-\infty,\infty]$ *and* $\lim\limits_{x\to\infty} f(x) = f(\infty) \in [-\infty,\infty]$. *Then*

$$I(a,b) = \int_0^{\infty} \frac{f(bx)-f(ax)}{x}\,dx = [f(\infty)-f(0)]\ln\left(\frac{b}{a}\right)$$

which, depending on the value of $f(\infty)-f(0)$, *may be a real number, or* $\pm\infty$, *or does not exist in case of* $\infty-\infty$.

Remark: In general, an improper integral that takes final answer $\infty-\infty$ does not exist. If under the hypotheses of the **above Theorem** one of the limits $f(0)$ and $f(\infty)$ is finite and the other oscillates, or both oscillate, then $I(a,b)$ oscillates and so does not exist.

Letting $b = t$ and $a = 1$, we have the following:

Corollary 2.5.1 *Under the conditions of the theorem and if* $f(\infty) - f(0) \neq 0$, *or* $\pm\infty$, *or* $\infty-\infty$, *or does not oscillate, we obtain the following integral formula for* $\ln(t)$*:*

$$\ln(t) = \frac{1}{f(\infty)-f(0)}\int_0^{\infty}\frac{f(tx)-f(x)}{x}\,dx = \frac{1}{f(0)-f(\infty)}\int_0^{\infty}\frac{f(x)-f(tx)}{x}\,dx.$$

Remark: Integral expressions of $\ln(t)$ are useful in applications to special integrals and functions. Obviously, besides $b = t$ and $a = 1$, other combinations can give the result of the **corollary**.

Examples

In the examples that follow here and throughout this section and also in the problems that follow, make sure that the functions involved satisfy the prerequisite **condition (✠)**, when necessary. Otherwise, indicate the opposite or the counterexample.

Example 2.5.1 For $0 < a, b < \infty$,

$$\int_0^{\infty}\frac{e^{-bx}-e^{-ax}}{x}\,dx = -\ln\left(\frac{b}{a}\right) = \ln\left(\frac{a}{b}\right)$$

since $f(x) = e^{-x}$ satisfies the conditions of **Theorem 2.5.1**, $f(\infty) = 0$ and $f(0) = 1$.

(The á-priori existence of this integral can be worked out as in **Example 2.2.15**.)

From this example, we have the integral expression of $\ln(t)$

$$\ln(t) = \int_0^\infty \frac{e^{-x} - e^{-tx}}{x}\, dx.$$

(See also **Problem 2.2.28**.)

Also, if we put $t = e^{-x}$, we find

$$\int_0^1 \frac{t^{a-1} - t^{b-1}}{\ln(t)}\, dt = \ln\left(\frac{a}{b}\right).$$

▲

Example 2.5.2 For $0 < a, b < \infty$,

$$\int_0^\infty \frac{\arctan(bx) - \arctan(ax)}{x}\, dx = \frac{\pi}{2}\ln\left(\frac{b}{a}\right)$$

since $f(x) = \arctan(x)$ satisfies the conditions of **Theorem 2.5.1**, $f(\infty) = \frac{\pi}{2}$ and $f(0) = 0$.

(The á-priori existence of this integral is worked out in **Example 2.2.15**.)

From this example, we have the integral expression of $\ln(t)$

$$\ln(t) = \frac{2}{\pi}\int_0^\infty \frac{\arctan(tx) - \arctan(x)}{x}\, dx.$$

(See also **Example 2.2.15**.)

Remark: Any example similar to this (and the previous one) may also be carried out in the following elementary way, independent of the knowledge of the theory of the Frullani integrals. We have:

$$\int_0^\infty \frac{\arctan(bx) - \arctan(ax)}{x}\, dx = \int_0^\infty \frac{1}{x}\left[\arctan(ux)|_{u=a}^{u=b}\right] dx =$$
$$\int_0^\infty \left[\int_a^b \frac{1}{1+(xu)^2}\, du\right] dx = \int_a^b \left[\int_0^\infty \frac{1}{1+(xu)^2}\, dx\right] du =$$
$$\int_a^b \frac{1}{u}\left[\arctan(ux)|_{x=0}^{x=\infty}\right] du = \int_a^b \frac{1}{u}\frac{\pi}{2}\, du = \frac{\pi}{2}\ln\left(\frac{b}{a}\right).$$

(The switching of the order of integration in the double integral is justified by **positivity condition I** in **Section 2.4**, for instance.)

(Solve the **previous example** by applying this method. Also, notice that with **Examples 2.5.5** and **2.5.6**, this method does not work.)

Hence as a **rule**, we observe that in evaluating such integrals by performing these steps, we need to be able to: **(1)** apply the **fundamental theorem of calculus, 1.1.1**, to the derivative of $f(x)$ [e.g., when $f'(x)$ is continuous] and **(2)** switch the order of integration in the double integral that follows.

▲

Example 2.5.3 For $0 < a, b < \infty$

$$\int_0^\infty \frac{e^{bx} - e^{ax}}{x}\,dx = (\infty - 1)\ln\left(\frac{b}{a}\right) = \begin{cases} +\infty & \text{if } b > a \\ 0 & \text{if } b = a \\ -\infty & \text{if } b < a, \end{cases}$$

a result that can also be proven elementarily.

▲

Example 2.5.4 For $0 < a \neq b < \infty$ and $f(x) = \dfrac{e^x}{x}$ on $(0, \infty)$, we get

$$\int_0^\infty \frac{\frac{e^{bx}}{bx} - \frac{e^{ax}}{ax}}{x}\,dx = (\infty - \infty)\ln\left(\frac{b}{a}\right) = \text{does not exist},$$

a result that can also be proven elementarily.

▲

If we investigate the preliminary computation further, we will see that under certain additional conditions we can evaluate $I(a, b)$ even if one of the limits $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ does not exist due to oscillation and the other limit exists or is $\pm\infty$.

In fact, let us assume:

1. $f(x)$ is continuous in $(0, \delta)$ for some $\delta > 0$.
2. $\lim\limits_{x\to 0^+} f(x) \in [-\infty, \infty]$ or oscillates.
3. $\displaystyle\int_c^\infty \frac{f(x)}{x}\,dx$ is finite for some $c > 0$.

[We assume nothing about $\lim\limits_{x\to\infty} f(x)$.]

As in the preliminary computation, **hypothesis (1)** implies

$$\lim_{\epsilon\to 0^+}\int_{a\epsilon}^{b\epsilon}\frac{f(t)}{t}dt=\ln\left(\frac{b}{a}\right)f(0).$$

Using the **Cauchy Test 1.3.7**, **hypothesis (2)** implies

$$\lim_{h\to\infty}\int_{ah}^{bh}\frac{f(t)}{t}dt=0.$$

Therefore,

$$I(a,b)=\int_0^\infty\frac{f(bx)-f(ax)}{x}\,dx=-f(0)\ln\left(\frac{b}{a}\right).$$

[If the limit $f(0)$ oscillates, the integral does not exist.]

Similarly, if we assume:

1. $f(x)$ is continuous in (μ,∞) for some $\mu>0$.
2. $\lim\limits_{x\to\infty}f(x)\in[-\infty,\infty]$ or oscillates.
3. $\displaystyle\int_0^r\frac{f(x)}{x}\,dx$ is finite for some $r>0$.

[We assume nothing about $\lim\limits_{x\to 0^+}f(x)$.]

Then again

$$\lim_{\epsilon\to 0^+}\int_{a\epsilon}^{b\epsilon}\frac{f(t)}{t}dt=0$$

[by the **Cauchy Test 1.3.7** and its **Remark 1**], and we find:

$$\int_0^\infty\frac{f(bx)-f(ax)}{x}\,dx=f(\infty)\ln\left(\frac{b}{a}\right).$$

[If the limit $f(\infty)$ oscillates, the integral does not exist.]

So, we have proven the following:

Theorem 2.5.2 *Let* $f:(0,\infty)\longrightarrow\mathbb{R}$ *be a nice function that satisfies (✠) and* $0<a,\ b<\infty$.

(I) If $f(x)$ *is continuous in* $(0,\delta)$ *for some* $\delta>0$, $\lim\limits_{x\to 0^+}f(x)\in[-\infty,\infty]$ *or oscillates, and* $\displaystyle\int_c^\infty\frac{f(x)}{x}\,dx$ *is finite for some* $c>0$, *then*

$$\int_0^\infty\frac{f(bx)-f(ax)}{x}\,dx=-f(0)\,\ln\left(\frac{b}{a}\right).$$

[If the limit $f(0) = \lim_{x\to 0^+} f(x)$ *oscillates, the integral does not exist.]*

(II) If $f(x)$ *is continuous in* (μ, ∞) *for some* $\mu > 0$, $\lim_{x\to\infty} f(x) \in [-\infty, \infty]$ *or oscillates, and* $\int_0^r \frac{f(x)}{x}\,dx$ *is finite for some* $r > 0$, *then*

$$\int_0^\infty \frac{f(bx) - f(ax)}{x}\,dx = f(\infty)\ln\left(\frac{b}{a}\right).$$

[If the limit $f(\infty) = \lim_{x\to\infty} f(x)$ *oscillates, the integral does not exist.]*

Letting $b = t$ and $a = 1$, we have the following:

Corollary 2.5.2 *Under the conditions of the two cases of the Theorem and if the limits* $f(\infty)$ *and* $f(0)$ *are not* 0, *or* $\pm\infty$, *and do not oscillate, we obtain the following integral formulae for* $\ln(t)$, *respectively:*

$$(I)\quad \ln(t) = \frac{1}{f(\infty)}\int_0^\infty \frac{f(tx) - f(x)}{x}\,dx = \frac{1}{-f(\infty)}\int_0^\infty \frac{f(x) - f(tx)}{x}\,dx.$$

$$(II)\quad \ln(t) = \frac{1}{-f(0)}\int_0^\infty \frac{f(tx) - f(x)}{x}\,dx = \frac{1}{f(0)}\int_0^\infty \frac{f(x) - f(tx)}{x}\,dx.$$

Examples

Example 2.5.5[16] For $0 < a,\ b < \infty$,

$$\int_0^\infty \frac{\sin(bx) - \sin(ax)}{x}\,dx = 0,$$

since $f(0) = \sin(0) = 0$ and by **Example 2.2.8**, $f(x) = \sin(x)$ satisfies the conditions of **Theorem 2.5.2 (I)**.

This result was already evident in view of the general result in **Example 2.2.8**, since in this case, we can split the integral and we get

$$\int_0^\infty \frac{\sin(bx) - \sin(ax)}{x}\,dx = \int_0^\infty \frac{\sin(bx)}{x} - \int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

Again, by **Theorem 2.5.2 (II)**, we also find

$$\int_0^\infty \frac{\sin\left(\frac{1}{bx}\right) - \sin\left(\frac{1}{ax}\right)}{x}\,dx = 0.$$

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[16] For more examples on the Frullani integrals, see bibliography: Albano, Amdeberhan, Beyerstedt and Moll 2010 and Gradshteyn and Ryzhik 2007.

Here $\lim_{x\to\infty}\sin\left(\frac{1}{x}\right)=\sin(0)=0$, but $\lim_{x\to 0^+}\sin\left(\frac{1}{x}\right)$ does not exist (see **Problem 2.5.5**). Instead, by **Example 2.2.8**, we have the condition

$$\int_0^\infty \frac{\sin\left(\frac{1}{x}\right)}{x}\,dx \overset{u=\frac{1}{x}}{=} \int_0^\infty \frac{\sin(u)}{u}\,du = \frac{\pi}{2}$$

and so the Theorem applies.

▲

Example 2.5.6 For $0 < a,\ b < \infty$,

$$\int_0^\infty \frac{\cos(bx)-\cos(ax)}{x}\,dx = \ln\left(\frac{a}{b}\right)$$

since $f(0)=\cos(0)=1$ and by **Problem 2.2.26, (b)**, $f(x)=\cos(x)$ satisfies the conditions of **Theorem 2.5.2 (I)**.

In this example, the splitting of integral about the minus "−" is not legitimate because then it becomes $\infty-\infty$.

Remark: The integrals in this and the previous example cannot be computed by the method described in the **remark** of **Example 2.5.2.** (Check this to see where the method fails.)

From this example, we have the integral expression of $\ln(t)$

$$\ln(t)=\int_0^\infty \frac{\cos(x)-\cos(tx)}{x}\,dx.$$

(See also **Problem 2.5.3**.)

Again, by **Theorem 2.5.2 (II)** we also find

$$\int_0^\infty \frac{\cos\left(\frac{1}{bx}\right)-\cos\left(\frac{1}{ax}\right)}{x}\,dx = \ln\left(\frac{b}{a}\right).$$

Here $\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)=\cos(0)=1$, but $\lim_{x\to 0^+}\cos\left(\frac{1}{x}\right)$ does not exist (see **Problem 2.5.5**). Instead, by **Problem 2.2.26, (b)**, we have the condition

$$\forall\ r>0,\quad \int_0^r \frac{\cos\left(\frac{1}{x}\right)}{x}\,dx \overset{u=\frac{1}{x}}{=} \int_{\frac{1}{r}}^\infty \frac{\cos(u)}{u}\,du \quad \text{converges conditionally,}$$

and so the Theorem applies.

▲

We continue with an interesting and useful theorem and two corollaries in which we do not impose any concrete assumptions on the limits

$\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to\infty} f(x)$, but we require a particular mean value to exist. Before we state and prove the Theorem, we must define this mean value and make an important remark.

We consider $f:(0,\infty)\longrightarrow\mathbb{R}$ a nice function that, as always in this section, satisfies **condition (✠)**. For any $\tau>0$, we define the mean value

$$M_\tau(f) := \lim_{\tau<x\to\infty}\frac{1}{x}\int_\tau^x f(t)\,dt.$$

We assume that $M_\rho(f)$ exists (as finite value) for some $\rho>0$. Then for any $\sigma>0$ (either $0<\rho\le\sigma$ or $0<\sigma\le\rho$) the mean value $M_\sigma(f)$ exists and $M_\sigma(f)=M_\rho(f)$. Therefore, if the mean value exists for some $\rho>0$, we can drop the index and simply write $M(f)$ for a function f that satisfies **(✠)**.

This observation follows immediately from the fact that by **hypothesis (✠)** the integral $\displaystyle\int_\rho^\sigma f(t)\,dt$ is finite, and so

$$\lim_{\{\rho,\sigma\}<x\to\infty}\frac{1}{x}\int_\rho^\sigma f(t)\,dt=0.$$

Now we state and prove the following theorem without ambiguity.

Theorem 2.5.3 *Suppose $f:(0,\infty)\longrightarrow\mathbb{R}$ is a nice function that satisfies* (✠) *and such that $\displaystyle\int_0^\delta\frac{f(x)}{x}\,dx$ exists for some $\delta>0$.*

We fix a $\rho>0$ and assume that the mean value

$$M(f) := \lim_{\rho<x\to\infty}\frac{1}{x}\int_\rho^x f(t)\,dt$$

exists (as finite value).

Then for any $a,\ b>0$ we have

$$\int_0^\infty\frac{f(bx)-f(ax)}{x}\,dx = M(f)\,\ln\left(\frac{b}{a}\right).$$

Proof Since $f(x)$ is a nice function such that $\displaystyle\int_0^\delta\frac{f(x)}{x}\,dx$ exists for some $\delta>0$, then [by the **Cauchy Test, 1.3.7**, and its **Remark 1**] for $a,\ b>0$

$$\lim_{\epsilon\to 0^+}\int_{a\epsilon}^{b\epsilon}\frac{f(t)}{t}dt=0.$$

Since $\rho > 0$, by **(✠)** we have $\int_\rho^x f(t)\,dt$ exists $\forall\ x \geq \rho > 0$, and so we can define

$$F(x) := \frac{1}{x}\int_\rho^x f(t)\,dt, \quad \forall\ x \geq \rho > 0.$$

Then, under the given hypotheses, we conclude that $F(x)$ is continuous for all $x \geq \rho$, $[xF(x)]' = f(x)$ at the points of continuity of $f(x)$ in $[\rho, \infty)$, and $\lim\limits_{\rho < x \to \infty} F(x) = M(f)$.

We consider any $h > 0$ such that $ah > \rho$ and $bh > \rho$. Applying integration by parts, we get

$$\int_{ah}^{bh} \frac{f(t)}{t}\,dt = \int_{ah}^{bh} \frac{1}{t} d[tF(t)] = F(bh) - F(ah) + \int_{ah}^{bh} \frac{F(t)}{t}\,dt.$$

Using the substitution $t = e^u$ and the Mean Value Theorem for integrals, we find

$$\int_{ah}^{bh} \frac{f(t)}{t}\,dt =$$

$$F(bh) - F(ah) + \int_{\ln(ah)}^{\ln(bh)} F(e^u)\,du = F(bh) - F(ah) + F(\zeta)\ln\left(\frac{b}{a}\right),$$

where $\zeta = e^c$ is between ah and bh.

If $h \to \infty$, then $ah \to \infty$, $bh \to \infty$, and $\zeta \to \infty$. Since $M(f)$ exists, we get

$$\lim_{h\to\infty} \int_{ah}^{bh} \frac{f(t)}{t}\,dt = M(f)\ln\left(\frac{b}{a}\right).$$

Finally, by equation **(2.6)** in the preliminary computation, we get

$$I(a,b) := \int_0^\infty \frac{f(bx) - f(ax)}{x}\,dx =$$

$$\lim_{h\to\infty} \int_{ah}^{bh} \frac{f(t)}{t} dt - \lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(t)}{t}\,dt = M(f)\ln\left(\frac{b}{a}\right).$$

Corollary 2.5.3 *Under the hypotheses of* ***Theorem 2.5.3*** *we get:*

(1) If $\lim\limits_{x\to\infty} f(x) = f(\infty)$ *is a real number, then* $M(f) = f(\infty)$ *(see* ***Problem 2.5.6****) and so*

$$\int_0^\infty \frac{f(bx) - f(ax)}{x}\,dx = f(\infty)\ln\left(\frac{b}{a}\right).$$

[So, ***Theorem 2.5.3*** *generalizes* ***Theorem 2.5.2 (II)****.]*

(2) If $f(x)$ *is periodic with period* $p > 0$ *and* $\int_0^p f(x)\,dx$ *exists, then*

$$M(f) = \frac{1}{p}\int_0^p f(x)\,dx = \frac{1}{p}\int_u^{u+p} f(x)\,dx, \quad \forall\ u \geq 0$$

[see ***Problem 1.1.6, Item (8.)****, and make an easy adjustment of its proof] and so*

$$\int_0^\infty \frac{f(bx) - f(ax)}{x}\,dx =$$

$$\frac{\ln\left(\frac{b}{a}\right)}{p}\int_0^p f(x)\,dx = \frac{\ln\left(\frac{b}{a}\right)}{p}\int_u^{u+p} f(x)\,dx,\ (u \geq 0).$$

(3) For $b = t$ *and* $a = 1$*, we get the corresponding integral representations of* $\ln(t)$ *(as before).*

If in **Theorem 2.5.3** we replace the hypothesis $\int_0^\delta \frac{f(x)}{x}\,dx$ exists for some $\delta > 0$ by $f(x)$ is continuous in $(0, \delta]$ for some $\delta > 0$ and $\lim_{x \to 0^+} f(x) = f(0) \in [-\infty, \infty]$, then by adjusting the proof of the Theorem and using the preliminary computation, we obtain:

Corollary 2.5.4 *Suppose* $f\ :\ (0, \infty) \longrightarrow \mathbb{R}$ *is a nice function that satisfies* **(✠)***, continuous in* $(0, \delta]$ *for some* $\delta > 0$ *and* $\lim_{x \to 0^+} f(x) = f(0) \in [-\infty, \infty]$ *or oscillates.*

We fix a $\rho > 0$ *and assume that the mean value*

$$M(f) := \lim_{x \to \infty} \frac{1}{x}\int_\rho^x f(t)\,dt$$

exists (as finite value).

Then for any $a,\ b > 0$ *we have*

$$\int_0^\infty \frac{f(bx) - f(ax)}{x}\,dx = [M(f) - f(0)]\ \ln\left(\frac{b}{a}\right).$$

[If the limit $f(0)$ *oscillates, the integral does not exist.]*

[For $b = t$ *and* $a = 1$*, we get the corresponding integral representations of* $\ln(t)$ *(as before).]*

Example 2.5.7 In **Problem 2.1.16 (f)**, we have proved

$$\int_0^\pi \ln|\cos(x)|dx = -\pi\ \ln(2).$$

The function $f(x) = \ln|\cos(x)|$ is continuous on $\left[0, \dfrac{\pi}{4}\right]$ [notice $f(0) = 0$] and $\displaystyle\lim_{x\to 0^+} \frac{f(x)}{x} = 0$ (use L' Hôpital's rule). So, $\displaystyle\int_0^{\pi/4} \frac{f(x)}{x}\, dx$ exists.

Since $f(x)$ is periodic with period π [by **Theorem 2.5.3** and **Part (2)** of **Corollary 2.5.3**, or **Corollary 2.5.4**], we get

$$M(f) = \frac{1}{\pi}\left[-\pi \ln(2)\right] = -\ln(2).$$

Then, ∀ $a > 0$ and $b > 0$ we obtain

$$\int_0^\infty \ln\left|\frac{\cos(bx)}{\cos(ax)}\right| \frac{1}{x}\, dx = -\ln(2)\, \ln\left(\frac{b}{a}\right) = \ln(2)\, \ln\left(\frac{a}{b}\right).$$

[Notice: $f(x)$ is not continuous in $(0, \infty)$ and $\displaystyle\lim_{x\to\infty} f(x)$ does not exist due to oscillation, but both the mean value $M(f)$ and the integral exist.]

▲

Problems

2.5.1 For real numbers $0 < a,\ b < \infty$, prove:

$$\int_0^\infty \frac{\arctan\left(\dfrac{x}{b}\right) - \arctan\left(\dfrac{x}{a}\right)}{x}\, dx = \frac{\pi}{2} \ln\left(\frac{a}{b}\right).$$

From this, write the corresponding integral expression of $\ln(t)$.

2.5.2 For real numbers $0 < a,\ b < \infty$, prove:

$$\int_0^\infty \frac{\dfrac{1}{1+(bx)^2} - \dfrac{1}{1+(ax)^2}}{x}\, dx = \ln\left(\frac{a}{b}\right).$$

From this, write the corresponding integral expression of $\ln(t)$.

2.5.3 Establish an á-priori existence of

$$\int_0^\infty \frac{\cos(x) - \cos(tx)}{x}\, dx [= \ln(t)],$$

without using the result in **Example 2.5.6**.

Does this integral converge absolutely?

2.5.4 (a) For real numbers $0 < a,\ b < \infty$, prove:

$$\int_0^\infty \left[b \sin\left(\frac{1}{bx}\right) - a \sin\left(\frac{1}{ax}\right)\right] dx = \ln\left(\frac{b}{a}\right).$$

(b) What is the corresponding integral expression of $\ln(t)$?
(c) Also prove:

$$\int_0^\infty \left[b \cos\left(\frac{1}{bx}\right) - a \cos\left(\frac{1}{ax}\right)\right] dx = \begin{cases} +\infty, & \text{if } \ b > a > 0 \\ 0, & \text{if } \ b = a > 0 \\ -\infty, & \text{if } \ 0 < b < a. \end{cases}$$

2.5.5 (a) If a function $f(x)$ is continuous in an interval $(0, \delta)$ for some $\delta > 0$, $\displaystyle\int_0^\delta \frac{f(x)}{x}\, dx$ exists and $\displaystyle\lim_{x\to 0^+} f(x)$ exists, then prove $\displaystyle\lim_{x\to 0^+} f(x) = 0$.

(b) If a function $f(x)$ is continuous in an interval (r, ∞) for some $r > 0$, $\displaystyle\int_r^\infty \frac{f(x)}{x}\, dx$ exists and $\displaystyle\lim_{x\to\infty} f(x)$ exists, then prove $\displaystyle\lim_{x\to\infty} f(x) = 0$.

2.5.6 Consider a nice function $f : (0, \infty) \longrightarrow \mathbb{R}$ such that for some $\rho \geq 0$ $\displaystyle\int_\rho^A f(x)dx$ exists $\forall\ A > \rho$ and $\displaystyle\lim_{x\to\infty} f(x) = f(\infty) \in [-\infty, \infty]$. Prove

$$M(f) := \frac{1}{x}\int_\rho^x f(t)\, dt = f(\infty).$$

2.5.7 For $0 < a,\ \ b < \infty$, write and evaluate the Frullani integrals $I(a, b)$ of the functions: $f(x) = \dfrac{e^x}{e^x + 1}$, $g(x) = x^p\, e^{-x}$, $u(x) = \sin(x^p)$, $v(x) = \cos(x^p)$, where $p \geq 1$. Justify your answers.

2.5.8 (a) In **Problem 2.1.16 (f)**, we have proved

$$\int_0^\pi \ln|\sin(x)|dx = -\pi\ \ln(2).$$

However, show that for $0 < a,\ \ b < \infty$

$$\int_0^\infty \ln\left|\frac{\sin(bx)}{\sin(ax)}\right| \frac{1}{x}\, dx = \begin{cases} +\infty, & \text{if } \ b > a > 0 \\ 0, & \text{if } \ a = b > 0 \\ -\infty, & \text{if } \ 0 < b < a. \end{cases}$$

(b) Use **Problem 2.1.18** to prove the same result for $\ln|\tan(x)|$ in the place of $\ln|\sin(x)|$.

2.5.9 For $0 < a,\ b < \infty$, show

$$\int_0^\infty \frac{|\sin(bx)| - |\sin(ax)|}{x}\,dx = \frac{2}{\pi}\ln\left(\frac{b}{a}\right)$$

and

$$\int_0^\infty \frac{|\cos(bx)| - |\cos(ax)|}{x}\,dx = \left(\frac{2}{\pi} - 1\right)\ln\left(\frac{b}{a}\right).$$

(See also **Problem 2.5.11**.)

2.5.10 Use the result in **Problem 2.6.18** to prove that for any $0 < \alpha < 1$ and $a,\ b > 0$

$$J(\alpha) := \int_0^\infty \frac{|\tan(bx)|^\alpha - |\tan(ax)|^\alpha}{x}\,dx = \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)}\ln\left(\frac{b}{a}\right).$$

Notice that $J(0) = 0$ and explain why $J(\alpha)$ has discontinuity at $\alpha = 0$ when $a \neq b$.

Investigate the integral obtained by replacing tangent with cotangent cot for the same and/or new range of α.

2.5.11 Use the result in **Problem 2.6.19** to prove that for any $\alpha < 1$ and $a,\ b > 0$

$$J(\alpha) := \int_0^\infty \frac{|\sec(bx)|^\alpha - |\sec(ax)|^\alpha}{x}\,dx = \left[\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)} - 1\right]\ln\left(\frac{b}{a}\right).$$

Replace secant with cosecant and investigate the new integral.

2.5.12 Consider the function $f(x) = \dfrac{\sin(x)}{x}$ on the interval $(0, \infty)$.

(a) Prove that this function satisfies all of the conditions of both **Theorems 2.5.1** and **2.5.2** for $x \in (0, \infty)$.

(b) For any real numbers $0 < a,\ b < \infty$, write and compute the Frullani integral of this function.

2.5.13 For real numbers $0 < a,\ b < \infty$, prove:

$$\int_0^\infty \frac{\frac{\sin^2(bx)}{bx} - \frac{\sin^2(ax)}{ax}}{x}\,dx = 0.$$

Without using this result show directly that this integral is proper over any interval $[0, \varepsilon]$, with $\varepsilon > 0$.

2.5.14 Prove that

$$\int_0^\infty \frac{\sin^3(x)}{x^2}\,dx = \frac{3}{4}\ln(3) \quad \text{and} \quad \int_0^\infty \frac{\sin^4(x)}{x^3}\,dx = \ln(2).$$

Then evaluate the following four integrals:

$$\int_{-\infty}^0 \frac{\sin^3(x)}{x^2}\,dx, \qquad \int_{-\infty}^\infty \frac{\sin^3(x)}{x^2}\,dx,$$

$$\int_{-\infty}^0 \frac{\sin^4(x)}{x^3}\,dx, \qquad \int_{-\infty}^\infty \frac{\sin^4(x)}{x^3}\,dx.$$

(Compare with **Problem 2.2.33**.)

[Hint: Use integration by parts, trigonometric identities and **Example 2.5.6**.]

2.5.15 For $i = 1,\ 2,\ 3,\ \ldots,\ k$, we consider any positive numbers $a_i > 0$ and any real numbers A_i that satisfy the condition $\sum\limits_{i=1}^{k} A_i = 0$.

(a) Let $y = f(x)$ be a function that satisfies the conditions of the Frullani integrals of **Theorem 2.5.1** and $f(0) \neq f(\infty)$. Then, use the **Corollary 2.5.1** to prove

$$\int_0^\infty \frac{\sum_{i=1}^{k} A_i\, f(a_i x)}{x}\,dx = \frac{1}{f(\infty) - f(0)} \sum_{i=1}^{k} A_i \ln(a_i).$$

(b) Use **(a)** to show

$$\int_0^\infty \frac{A_1 \arctan(a_1 x) + A_2 \arctan(a_2 x) + ... + A_k \arctan(a_k x)}{x}\,dx =$$

$$\frac{2}{\pi}\left[A_1 \ln(a_1) + A_2 \ln(a_2) + \ldots + A_k \ln(a_k)\right].$$

(c) Let $y = f(x)$ be a function that satisfies the conditions of the Frullani integrals of **Theorem 2.5.2** and $f(0) \neq 0$. Then, the **Corollary 2.5.2** to prove

$$\int_0^\infty \frac{\sum_{i=1}^{k} A_i\, f(a_i x)}{x}\,dx = -\frac{1}{f(0)} \sum_{i=1}^{k} A_i \ln(a_i).$$

(d) Use **(c)** to show

$$\int_0^\infty \frac{A_1 \cos(a_1 x) + A_2 \cos(a_2 x) + A_3 \cos(a_3 x) + ... + A_k \cos(a_k x)}{x}\,dx =$$

$$-A_1 \ln(a_1) - A_2 \ln(a_2) - A_3 \ln(a_3) - \ldots - A_k \ln(a_k).$$

(J. Wolstenholme[17])

2.5.16 State the necessary conditions and give the proofs so that for any $\epsilon > 0$ and for all $0 < a,\ b < \infty$

$$(1) \qquad \int_{\epsilon}^{\infty} \frac{f(bx) - f(ax)}{x}\, dx = [f(\infty) - f(\epsilon)] \ln\left(\frac{b}{a}\right),$$

or

$$(2) \qquad \int_{\epsilon}^{\infty} \frac{f(bx) - f(ax)}{x}\, dx = -f(\epsilon) \ln\left(\frac{b}{a}\right).$$

2.5.17 (a) Use the **previous problem** to prove that for any $\epsilon > 0$ and any $0 < a,\ b < \infty$, we have

$$\int_{\epsilon}^{\infty} \frac{\dfrac{\cos^2(bx)}{bx} - \dfrac{\cos^2(ax)}{ax}}{x}\, dx = \frac{\cos^2(\epsilon)}{\epsilon} \ln\left(\frac{a}{b}\right).$$

(b) What happens when $\epsilon \to 0^+$?

(c) Without using this result, show directly that this integral is improper over any interval $[0, \varepsilon]$, with $\varepsilon > 0$ and its value is $+\infty$ when $a > b$.

2.5.18 For $\beta \geq 0$ and $a,\ b$ real constants, show:

$$(\text{a}) \qquad I(\beta) := \int_{0}^{\infty} e^{-\beta x} \frac{\cos(ax) - \cos(bx)}{x}\, dx = \frac{1}{2} \ln\left(\frac{\beta^2 + b^2}{\beta^2 + a^2}\right).$$

If we replace $\cos(ax) - \cos(bx)$ with $\cos(ax) + \cos(bx)$, then prove that the new integral is equal to ∞.

$$(\text{b})\ J(\beta) := \int_{0}^{\infty} e^{-\beta x} \frac{\sin(ax) \pm \sin(bx)}{x}\, dx = \arctan\left(\frac{a}{\beta}\right) \pm \arctan\left(\frac{b}{\beta}\right).$$

[Hint: These integrals are not Frullani type integrals (why?). Use **Problems 2.4.8** and **2.4.9** and/or the result in **Example 2.2.9**, after you justify the splitting of the integral $J(\beta)$.]

(c) The integral $I(\beta)$ found in **(a)** can also be found by the following steps: (1) Use the **Main Theorem, 2.2.1,** and **Problem 1.2.16** to find $\dfrac{d}{d\beta}[I(\beta)]$. (2) What is $I(0)$ and why? (3) Now find $I(\beta)$.

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[17] Joseph Wolstenholme, English mathematician, 1829-1891. He computed the above integral using contour integration in complex analysis.

(d) The integral $J(\beta)$ found in **(b)** can also be found by the following steps: (1) Use the **Main Theorem, 2.2.1,** and **Problem 1.2.14** to find $\dfrac{d}{d\beta}\,[J(\beta)]$. (2) What is $J(0)$ and why? (3) Now find $J(\beta)$.

(e) Knowing $I(\beta)$ and $J(\beta)$ á-priori, explain how you can use them to find the integrals in **Examples 2.5.5** and **2.5.6**, i.e., to prove: For all $0 < a,\ b < \infty$, we have

$$\int_0^\infty \frac{\cos(bx) - \cos(ax)}{x}\,dx = \ln\left(\frac{a}{b}\right) \text{ and } \int_0^\infty \frac{\sin(bx) - \sin(ax)}{x}\,dx = 0.$$

[Hint: You must justify why you can let $\beta = 0$ by using a continuity argument, i.e., switching limit and integral.]

2.5.19 For $\beta \geq 0$ and $a,\ b$ real constants, find:

$$\text{(a)} \quad \int_0^\infty e^{-\beta x}\,\frac{\cos(ax)\cos(bx)}{x}\,dx, \quad \text{(b)} \quad \int_0^\infty e^{-\beta x}\,\frac{\sin(ax)\sin(bx)}{x}\,dx,$$

$$\text{(c)} \quad \int_0^\infty e^{-\beta x}\,\frac{\cos(ax)\sin(bx)}{x}\,dx, \quad \text{(d)} \quad \int_0^\infty e^{-\beta x}\,\frac{\sin(ax)\cos(bx)}{x}\,dx.$$

[Hint: Use appropriate trigonometric identities to split the trigonometric products and **Problem 2.5.18 (a)** and **(b)**.]

2.5.20 For $\beta \geq 0$ and $a,\ b$ real constants, use **Problem 2.5.18, (a)** and **(b)** to find the values of the following four integrals:

$$\text{(a)} \int_0^\infty e^{-\beta x}\,\frac{\cos^2(ax) - \cos^2(bx)}{x}\,dx, \quad \int_0^\infty e^{-\beta x}\,\frac{\sin^2(ax) - \sin^2(bx)}{x}\,dx,$$

$$\text{(b)} \int_0^\infty e^{-\beta x}\,\frac{\cos^3(ax) - \cos^3(bx)}{x}\,dx, \quad \int_0^\infty e^{-\beta x}\,\frac{\sin^3(ax) - \sin^3(bx)}{x}\,dx.$$

[Hint: The two integrals in **(a)** can be found easier by using the double angle trigonometric formulae or using **Problems 2.4.8** and **2.4.9**. For the two integrals in **(b)**, use the trigonometric formulae that reduce the cubic powers to first powers, etc.]

2.5.21 For $a,\ b$ real constants, prove

$$\int_{-\infty}^\infty \frac{\cos(bx) - \cos(ax)}{x^2}\,dx = 2\int_0^\infty \frac{\cos(bx) - \cos(ax)}{x^2}\,dx = (|a| - |b|)\pi$$

and

$$\int_0^{\infty} \frac{\sin(bx) - \sin(ax)}{x^2}\,dx =$$

$$-\int_{-\infty}^{0} \frac{\sin(bx) - \sin(ax)}{x^2}\,dx = \begin{cases} +\infty, & \text{if } b > a \\ 0, & \text{if } b = a \\ -\infty, & \text{if } b < a. \end{cases}$$

2.6 The Real Gamma and Beta Functions

In this section, we study the real Gamma[18] and Beta[19] functions and their fundamental properties. The complex Gamma and Beta functions and the great number of their properties and applications are studied in advanced complex analysis. This chapter of complex analysis is outside the scope of this text. These special functions are very important in Mathematics, Statistics, Engineering and Science. They were first defined and used by Euler.

2.6.1 The Gamma Function

The **Gamma function** is defined as the improper integral of a real parameter p

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x}\,dx.$$

This is also called **Euler's integral of the second kind**. We notice that the integrand is always non-negative, and this integral is obviously improper since the interval of integration, $[0, \infty)$, is unbounded. When $0 < p < 1$, the integral is improper for one more reason: The integrand becomes $+\infty$ at $x = 0^+$. (In complex analysis, the real p is replaced with the complex variable $z = x + iy$, and so we must rename the dummy variable with a letter other than x.)

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[18] Gamma, γ, Γ, the third letter of the Greek alphabet.

[19] Beta, β, B, the second letter of the Greek alphabet.

In **Example 1.3.4** we have established the convergence of this integral for all $p > 0$ and its divergence ($= \infty$) for all $p \leq 0$. So, the study of the Gamma function begins with examining the properties of this integral for $p \in (0, \infty)$.

We are also going to provide some preliminary estimates which, besides reproving the existence of $\Gamma(p)$ for all $0 < p < \infty$, allow us to see what happens when p approaches 0^+ or ∞.

Preliminary Estimates

In proving various facts about the Gamma function, it is convenient to write

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx = \int_0^1 x^p \frac{1}{xe^x}\, dx + \int_1^\infty x^p \frac{1}{xe^x} dx. \tag{2.7}$$

With the help of this relation, we will derive some preliminary estimates for the Gamma function. These estimates are useful in proving some results and solving some problems.

(I) We estimate the first part of the integral in **(2.7)**, namely:

$$\int_0^1 x^{p-1} e^{-x} dx = \int_0^1 x^p \frac{1}{xe^x} dx \quad \text{for all} \quad p > 0.$$

This integral is improper when $0 < p < 1$ and is proper when $p \geq 1$.

We observe the following inequality:

$$\forall\ 0 < p < \infty \text{ and } \forall\ 0 \leq x \leq 1, \;\; x^{p-1} e^{-1} \leq x^{p-1} e^{-x} \leq x^{p-1}.$$

Therefore,

$$\frac{1}{e} \int_0^1 x^{p-1}\, dx < \int_0^1 x^{p-1} e^{-x}\, dx < \int_0^1 x^{p-1}\, dx.$$

So,

$$\forall\ 0 < p < \infty, \quad \frac{1}{ep} < \int_0^1 x^{p-1} e^{-x}\, dx < \frac{1}{p}.$$

(II) We now continue with the estimation of the second part of the integral in **(2.7)**, namely:

$$\int_1^\infty x^{p-1} e^{-x} dx = \int_1^\infty x^p \frac{1}{xe^x} dx.$$

This integral is improper because of the infinite interval of integration. We distinguish two cases:

Case 1: $0 < p < 1$**.**

By the following inequality (which deserves mentioning on its own merit; therefore, prove it as an exercise)

$$\forall\ x > 0, \quad 1 - \frac{1}{x} \le \ln(x) \le x - 1,$$

we see that for any $x \ge 1$ we have $0 \le \ln(x) \le x - 1$, and so we obtain: $\forall\ x \ge 1$ and $\forall\ 0 < p < 1$,

$$e^{-2x+1} \le e^{-x-\ln(x)} = x^{-1}e^{-x} \le x^{p-1}e^{-x} \le e^{-x}.$$

Therefore, $\forall\ 0 < p < 1$,

$$\int_1^\infty e^{-2x+1}\,dx < \int_1^\infty x^{p-1}e^{-x}\,dx < \int_1^\infty e^{-x}\,dx$$

and so

$$\forall\ 0 < p < 1, \quad \frac{1}{2e} < \int_1^\infty x^{p-1}e^{-x}\,dx < \frac{1}{e}.$$

Case 2: $1 \le p < \infty$.

We let $n = [\![p-1]\!]$, the integer part of $p - 1$. Since $p \ge 1$, we have that $n \ge 0$ integer and $n \le p - 1 < n + 1$. So, we have the inequality

$$\int_1^\infty x^n e^{-x}\,dx \le \int_1^\infty x^{p-1}e^{-x}\,dx < \int_1^\infty x^{n+1}e^{-x}\,dx.$$

Applying n integrations by parts in the first integral and $n+1$ integrations by parts in the third, we obtain

$$\frac{1 + n + n(n-1) + n(n-1)(n-2) + ... + n! + n!}{e} \le$$

$$\int_1^\infty x^{p-1}e^{-x}\,dx <$$

$$\frac{1 + (n+1) + (n+1)n + (n+1)n(n-1) + ... + (n+1)! + (n+1)!}{e}.$$

Finally, we have achieved the following **preliminary estimates of the Gamma function**:

$$\textbf{(1)} \quad \forall\ 0 < p < 1, \quad \frac{p+2}{2ep} = \frac{1}{2e} + \frac{1}{ep} < \Gamma(p) < \frac{1}{p} + \frac{1}{e} = \frac{e+p}{ep},$$

obtained by **(I)** and **(II, Case 1)**.

From these estimates, we also obtain the fact:

$$\lim_{p \to 0^+} \Gamma(p) = \infty.$$

(2) $\forall\ 1 \le p < \infty$,

$$\frac{1}{ep} + \frac{1 + n + n(n-1) + n(n-1)(n-2) + ... + n! + n!}{e} < \Gamma(p) <$$

$$\frac{1}{p} + \frac{1 + (n+1) + (n+1)n + (n+1)n(n-1) + ... + (n+1)! + (n+1)!}{e},$$

where $n = [\![p-1]\!]$, obtained by **(I)** and **(II, Case 2)**.

If $p \to \infty$, then $[\![p-1]\!] = n \to \infty$, and from these estimates we obtain the fact:

$$\lim_{p\to\infty} \Gamma(p) = \infty.$$

Some Basic Properties and Values of the Gamma Function

(Γ, 1). For $0 < p < \infty$, $\Gamma(p)$ is a continuous function of p.

Proof We must use the **Continuity Part** of **Theorem 2.2.1**. Since continuity is a local property, for any given $0 < p < \infty$ we fix any p_1, p_2 such that $0 < p_1 < p < p_2 < \infty$. Then we use the Theorem by choosing the non-negative function

$$g(x) = \begin{cases} x^{p_1-1}\, e^{-x}, & \text{if } 0 < x \leq 1 \\ \\ x^{p_2-1}\, e^{-x}, & \text{if } 1 \leq x < \infty. \end{cases}$$

Then

$$\left|x^{p-1}e^{-x}\right| = x^{p-1}e^{-x} < g(x)$$

and

$$\int_0^\infty g(x)dx < \Gamma(p_1) + \Gamma(p_2) < \infty.$$

Now the result follows from the **Continuity Part** of **Theorem 2.2.1**. (Work out any missing details.)

(Γ, 2). For $0 < p < \infty$, $\Gamma(p)$ is infinitely differentiable with $\mathbf{n^{th}}$ **order derivative**

$$\Gamma^{(n)}(p) = \int_0^\infty x^{p-1}e^{-x}[\ln(x)]^n\, dx, \;\; \forall\; n = 0,\; 1,\; 2,\; 3, \ldots.$$

Proof (See also **project Problem 2.6.42**.) For the first derivative and similarly with any derivative thereon, we use the **Differentiability Part** of **Theorem 2.2.1**. Since differentiability is a local property, for any given $0 < p < \infty$ we fix any p_1, p_2 such that $0 < p_1 < p < p_2 < \infty$. We deal with the function

$$\left|\frac{\partial}{\partial p}\left(x^{p-1}e^{-x}\right)\right| = x^{p-1}e^{-x}\left|\ln(x)\right|.$$

Then we use the Theorem by choosing the non-negative function

$$g(x) = \begin{cases} x^{p_2}e^{-x}, & \text{if } 1 \leq x < \infty \\ \\ x^{p_1-1}[-\ln(x)], & \text{if } 0 < x < 1. \end{cases}$$

We see that

$$\left|\frac{\partial}{\partial p}\left(x^{p-1}e^{-x}\right)\right| = x^{p-1}e^{-x}\,|\ln(x)| < g(x)$$

and we also observe

$$\int_1^\infty g(x)dx = \int_1^\infty x^{p_2}e^{-x}\,dx < \Gamma(p_2+1) < \infty.$$

For the other part of the integral of $g(x)$, we use u-substitution and integration by parts to obtain

$$\int_0^1 x^{p_1-1}[-\ln(x)]\,dx = \int_0^1 x^{p_1-1}\ln\left(\frac{1}{x}\right)\,dx \overset{u=\frac{1}{x}}{=}$$
$$\int_\infty^1 u^{1-p_1}\ln(u)\,\frac{du}{-u^2} = \int_1^\infty u^{-1-p_1}\ln(u)\,du = \frac{1}{p_1^2} < \infty.$$

Hence

$$\int_0^\infty g(x)dx < \infty$$

and the result follows. (Supply the missing details.)

Note: According to **Problem 2.1.25** and its **footnote** and **Problem 2.3.5**

$$\Gamma'(1) = \int_0^\infty e^{-x}\ln(x)\,dx = -\gamma < 0,$$

where γ is the Euler constant defined to be

$$\gamma = \lim_{n\to\infty}\left[\left(\sum_{k=1}^n \frac{1}{k}\right) - \ln(n)\right] \simeq 0.57721566... > 0.$$

Make a note of this result, for it is needed in the proofs of many important and difficult results on special integrals and special functions.

(Γ, 3). The Gamma function is analytic. That is, it can be expressed as a power series locally.

[Hint: Use the Taylor[20] Power Series Theorem as we know it from calculus and then show that for any fixed point $0 < p_0 < \infty$ the Taylor remainder

$$R_n = \frac{\Gamma^{(n+1)}(p^*)}{(n+1)!}\,(p-p_0)^{n+1},$$

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[20] Brook Taylor, English mathematician, 1685-1731.

with some p^* between p and p_0, approaches zero as $n \to \infty$. Then show that the radius of convergence of the obtained power series is $R = p_0$.]

($\boldsymbol{\Gamma}$, 4). For $0 < p < \infty$, $\Gamma(p)$ is strictly convex. This follows by **($\boldsymbol{\Gamma}$, 2)** since

$$\Gamma''(p) = \int_0^{\infty} x^{p-1} e^{-x} [\ln(x)]^2 \, dx > 0.$$

Hence the second derivative of the Gamma function is strictly positive, and therefore the Gamma function is strictly convex (or concave up, as we say in calculus).

($\boldsymbol{\Gamma}$, 5). $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} \, dx \overset{x=u^2}{=}$$
$$2\int_0^{\infty} e^{-u^2} \, du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi},$$

by the integral **(2.1)** in section **2.1**.

($\boldsymbol{\Gamma}$, 6). $\Gamma(1) = 1$ and $\Gamma(2) = 1$.

Proof

$$\Gamma(1) = \int_0^{\infty} e^{-x} \, dx = \left[-e^{-x}\right]_0^{\infty} = 0 - (-1) = 1,$$

$$\Gamma(2) = \int_0^{\infty} x e^{-x} \, dx = \left[-xe^{-x}\right]_0^{\infty} + \int_0^{\infty} e^{-x} \, dx = 0 + 1 = 1.$$

($\boldsymbol{\Gamma}$, 7). By **($\boldsymbol{\Gamma}$, 4)** and **($\boldsymbol{\Gamma}$, 6)**, we conclude that the Gamma function has a unique local minimum, which is also global minimum in $(0, \infty)$, at a number r between 1 and 2. An estimate of this number is

$$r = 1.461632144968362341262659542325721328468196...$$

and the global minimum of the Gamma function in $(0, \infty)$ is approximately

$$\Gamma(r) = 0.8856031944108887002788159005825887332079 5....$$

We then conclude that $\Gamma'(r) = 0$, $\Gamma(p)$ is strictly decreasing [and so $\Gamma'(p) < 0$] for $0 < p < r$ and strictly increasing [and so $\Gamma'(p) > 0$] for $r < p < \infty$.

(Γ, 8). $\mathbf{\Gamma(p+1) = p\,\Gamma(p)}$ for $\mathbf{p > 0}$.
Proof

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = \;\text{(use integration by parts)} =$$

$$\left[-x^p e^{-x}\right]_0^\infty + \int_0^\infty p\,x^{p-1}e^{-x}\,dx = 0 + p\int_0^\infty x^{p-1}e^{-x}\,dx = p\,\Gamma(p).$$

We can use this recursive relation as

$$\Gamma(p) = \frac{1}{p}\,\Gamma(p+1)$$

to extend the Gamma function to the negative non-integer real numbers (recursively).

For instance, if $p = -\frac{2}{3}$, we find

$$\Gamma\left(-\frac{2}{3}\right) = -\frac{3}{2}\Gamma\left(\frac{1}{3}\right).$$

In the same way, keeping in mind that $\Gamma(0^+) = \infty$ and $\Gamma(1) = 1$, we get that the real Gamma function approaches $\pm\infty$ as p approaches a negative integer. More precisely

$$\text{for } n \leq 0 \quad \text{integer,} \quad \Gamma(n^\pm) = (-1)^n(\pm\infty).$$

For instance: $\Gamma(0^-) = -\infty$, $\Gamma(-1^+) = -\infty$, $\Gamma(-1^-) = +\infty$, etc.

(See **Figure 2.6**.)

(Γ, 9). For $\mathbf{p = 0, 1, 2, 3, ...}$, $\mathbf{\Gamma(p+1) = p\,!}$.
Proof Do p iterations of **(Γ, 8)** and use **(Γ, 6)**.

We now notice that the Gamma function is an analytic function in $(0, \infty)$ and contains all the factorials $n!$ of the integers $n = 0,\ 1,\ 2,\ 3, ...$ in the range of its values. Therefore, the Gamma function is used to define the **factorials of all real numbers greater than** -1. In fact, **we define**:

$$\forall\, p > -1, \quad p\,! = \Gamma(p+1).$$

For **example**:

$$\Gamma(20) = 19!,$$

and

$$0! = \Gamma(1) = 1, \quad 1! = \Gamma(2) = 1, \quad \text{and so on.}$$

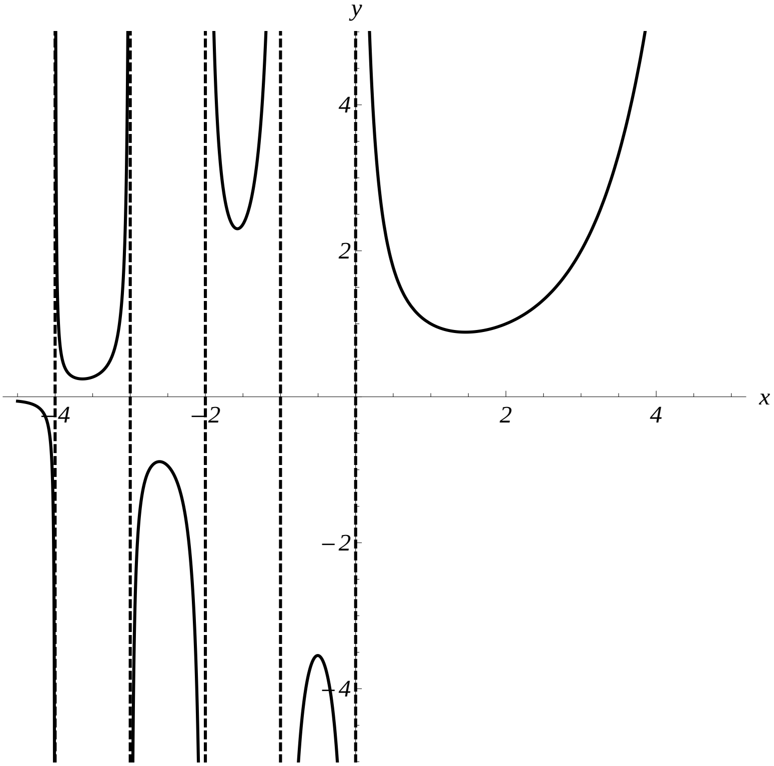

FIGURE 2.6: Graph of the Gamma function and its vertical asymptotes at $\mathbf{x = n}$, where $\mathbf{n = 0, -1, -2, -3, \ldots}$.

Since by **(Γ, 8)** and **(Γ, 5)**, we obtain

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi},$$

we write $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$.

It is clear now that if $\mathbf{p} > 0$ **is not an integer**, then we can evaluate $\Gamma(p)$ by means of the value of $\Gamma(p - [\![p]\!])$, where $[\![p]\!]$ is the integer part of p. That is, all values $\Gamma(p)$ may be found in terms of the factorials of integers and the values of $\Gamma(p)$ for $0 < p < 1$. (Years ago, there were in use extensive tables of values of $\Gamma(p)$ for many $0 < p < 1$.)

(Γ, 10).

$$\Gamma\left(p+\frac{1}{2}\right)=\Gamma\left(p-\frac{1}{2}+1\right)=\left(p-\frac{1}{2}\right)\Gamma\left(p-\frac{1}{2}\right)=$$
$$\left(\frac{2p-1}{2}\right)\Gamma\left(p-\frac{1}{2}\right)= \ \dots \ \text{etc.}$$

(See **Problem 2.6.14** for the final answer.)

(Γ, 11). For $x > 0$ and $p > 0$, by making the substitution $v = xu$, we obtain

$$\frac{1}{x^p}=\frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}e^{-xu}du, \quad \text{or} \quad \int_0^\infty u^{p-1}e^{-xu}du=\frac{\Gamma(p)}{x^p}.$$

If $p = 1,\ 2,\ 3, \dots$ integer, then

$$\int_0^\infty u^{p-1}e^{-xu}du=\frac{(p-1)!}{x^p}.$$

Note: We state without proof the following **result**:

With the help of the Gamma function, we can prove the renowned **Stirling's**[21] **formula** which is useful for the computation of the factorials of large natural numbers. This says that if n is a **large** natural number, then we have the following approximate equality:

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(See **project Problem 2.6.43** at the end of this **section**.)

Examples

Example 2.6.1

$$\int_0^\infty x^7\, e^{-x}dx=\Gamma(8)=7!=5040.$$

▲

Example 2.6.2

$$\int_0^\infty x^4e^{-2x}dx \overset{u=2x}{=} \int_0^\infty \frac{u^4}{16}\cdot e^{-u}\frac{du}{2}=\frac{1}{32}\int_0^\infty u^4e^{-u}du=$$
$$\frac{1}{32}\cdot\Gamma(5)=\frac{1}{32}\cdot 4!=\frac{24}{32}=\frac{3}{4}.$$

---

[21] James Stirling, Scottish mathematician, 1692-1770.

Or, by **(Γ, 11)**, we directly find

$$\int_0^\infty x^4 e^{-2x} dx = \int_0^\infty x^{5-1} e^{-2x} dx = \frac{\Gamma(5)}{2^5} = \frac{4!}{32} = \frac{24}{32} = \frac{3}{4}.$$

▲

Example 2.6.3 By **(Γ, 11)**, e.g., we find:

$$\text{(a)} \quad \int_0^\infty x^{4.57} e^{-3.5x} dx = \int_0^\infty x^{5.57-1} e^{-3.5x} dx = \frac{\Gamma(5.57)}{(3.5)^{5.57}},$$

$$\text{(b)} \quad \int_0^\infty x^{0.5} e^{-0.2x} dx = \int_0^\infty x^{1.5-1} e^{-0.2x} dx =$$
$$\frac{\Gamma(1.5)}{(0.2)^{1.5}} = \frac{0.5\sqrt{\pi}}{0.2\sqrt{0.2}} = 2.5\sqrt{5\pi} = 9.908318244\ldots.$$

▲

Example 2.6.4

$$\Gamma(4.01) = \Gamma(3.01+1) = 3.01 \cdot \Gamma(3.01) =$$
$$3.01 \cdot 2.01 \cdot 1.01 \cdot 0.01 \cdot \Gamma(0.01) = 0.06110601 \cdot \Gamma(0.01).$$

▲

Example 2.6.5

$$\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy \overset{x=y^2}{=} \int_0^\infty x^{\frac{1}{4}}\, e^{-x} \frac{dx}{2\sqrt{x}} = \frac{1}{2} \int_0^\infty x^{\frac{-1}{4}}\, e^{-x} dx =$$
$$\frac{1}{2} \int_0^\infty x^{\frac{3}{4}-1}\, e^{-x} dx = \frac{1}{2} \cdot \Gamma\left(\frac{3}{4}\right).$$

▲

Example 2.6.6

$$\int_0^\infty \sqrt[3]{y}\, e^{-y^2} dy \overset{x=y^2}{=} \int_0^\infty x^{\frac{1}{6}}\, e^{-x} \frac{dx}{2\sqrt{x}} =$$
$$\frac{1}{2} \int_0^\infty x^{\frac{-1}{3}}\, e^{-x} dx = \frac{1}{2} \int_0^\infty x^{\frac{2}{3}-1}\, e^{-x} dx = \frac{1}{2} \cdot \Gamma\left(\frac{2}{3}\right).$$

▲

Example 2.6.7 We can use **(Γ, 11)** and double integration to show the following two useful **results**:

$$\text{(1)} \quad \text{For} \quad 0 < p < 2, \qquad \int_0^\infty \frac{\sin(x)}{x^p}\, dx = \frac{\pi}{2\Gamma(p) \sin\left(\frac{p\pi}{2}\right)}, \tag{2.8}$$

$$(2)\quad \text{For}\quad 0 < p < 1, \quad \int_0^\infty \frac{\cos(x)}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos\left(\frac{p\pi}{2}\right)}. \tag{2.9}$$

(See also **Problems 2.6.38** and **3.7.14**.)

In **Problems 2.2.22** and **2.2.23**, we have proved that both of these generalized Riemann integrals exist and are continuous in p within the respective intervals of p. We have stated that for $0 < p < 1$, the integrals are conditionally convergent. (They do not exist as Lebesgue integrals, but this is outside the scope of this text!) We have also seen that the first integral is conditionally convergent for $p = 1$ and absolutely convergent for $1 < p < 2$.

For all the other positive values of p, they are equal to infinity because of the singularity near $x = 0$. For $p \leq 0$, they do not exist because they oscillate "badly" "near" infinity.

Here, we prove the first one, and in analogous way we prove the second one. In **($\boldsymbol{\Gamma}$, 11)**, we have

$$\forall\ x > 0, \quad \forall\ p > 0, \quad \frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-xu} du$$

and so

$$\int_0^\infty \frac{\sin(x)}{x^p}\,dx = \lim_{M\to\infty} \int_0^M \frac{1}{\Gamma(p)} \int_0^\infty \left(u^{p-1} e^{-xu}\,du\right)\,\sin(x)\,dx =$$
$$\frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^M \left(\int_0^\infty u^{p-1} e^{-xu}\,du\right) x\,\frac{\sin(x)}{x}\,dx.$$

Since

$$\forall\ x \in \mathbb{R}, \quad \left|\frac{\sin(x)}{x}\right| \leq 1,$$

for the positive function $h(x,u) = u^{p-1}e^{-xu}\,x$ in $(0, M] \times (0, \infty)$, we get

$$\int_0^M \frac{1}{\Gamma(p)} \left(\int_0^\infty u^{p-1} e^{-xu}\,du\right) x\,dx =$$
$$\int_0^M \frac{1}{x^p}\,x\,dx = \left[\frac{x^{2-p}}{2-p}\right]_0^M = \frac{M^{2-p}}{2-p} < \infty, \quad \forall\ 0 < p < 2.$$

Hence, by the **Tonelli conditions, Section 2.4**, we can switch the order of integration and obtain

$$\int_0^\infty \frac{\sin(x)}{x^p}\,dx = \frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1} \left(\int_0^M e^{-xu} \sin(x)\,dx \right) du =$$
$$\frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1} \left[\frac{-e^{-xu}[u\sin(x)+\cos(x)]}{u^2+1} \right]_0^M du = \tag{2.10}$$
$$\frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty u^{p-1}\, \frac{-e^{-Mu}[u\sin(M)+\cos(M)]+1}{u^2+1}\,du =$$
$$\frac{1}{\Gamma(p)} \lim_{M\to\infty} \int_0^\infty \left[\frac{u^{p-1}}{u^2+1} - \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \right] du.$$

Now, independently of the limit, by the result in **Example 2.2.6**, we have

$$\forall\ 0<p<2, \quad \int_0^\infty \frac{u^{p-1}}{u^2+1}\,du = \frac{1}{2}\,\frac{\pi}{\sin(\frac{p\pi}{2})}.$$

Also, $\forall\ u>0$ we have, $e^{-Mu} \to 0$ as $M \to \infty$. Thus,

$$\forall\ u>0, \quad \forall\ 0<p<2, \quad \lim_{M\to\infty} \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \overset{pw}{=} 0$$

(see **Definition 2.3.1**).

Since $u>0$ and $M\to\infty$, for $M\geq 1$, we have

$$\left| \frac{u^{p-1}e^{-Mu}[u\sin(M)+\cos(M)]}{u^2+1} \right| <$$
$$\frac{u^{p-1}e^{-u}(u+1)}{u^2+1} < u^p e^{-u} + \frac{u^{p-1}}{u^2+1}.$$

Therefore, for any p such that $0<p<2$, we consider the positive function $g(u) = u^p e^{-u} + \dfrac{u^{p-1}}{u^2+1} > 0$ on $(0,\infty)$. This function is independent of M and satisfies

$$\int_0^\infty g(u)du = \int_0^\infty \left(u^p e^{-u} + \frac{u^{p-1}}{u^2+1} \right) du =$$
$$\int_0^\infty u^p e^{-u}\,du + \int_0^\infty \frac{u^{p-1}}{u^2+1}\,du = \Gamma(p+1) + \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{p\pi}{2}\right)} < \infty,$$

by the definition of the $\Gamma(p+1)$ and the result in **Example 2.2.6** (as we have also used it above).

So, for any given p, such that $0<p<2$, by **Part (I)** of the **Main**

Theorem, 2.2.1, we obtain

$$\lim_{M\to\infty}\int_0^\infty \frac{u^{p-1}e^{-Mu}\ [u\sin(M)+\cos(M)]}{u^2+1}\,du =$$
$$\int_0^\infty \lim_{M\to\infty}\frac{u^{p-1}e^{-Mu}\ [u\sin(M)+\cos(M)]}{u^2+1}\,du = \int_0^\infty 0\,du = 0.$$

This limit, along with the result in **Example 2.2.6** and relation **(2.10)**, prove equality **(2.8)**, that is, result **(1)** of this example.

Notice that both formulae **(2.8)** and **(2.9)** are discontinuous at $p = 0$. I.e., at $p = 0$ the two sides of the formulae do not agree. (Check this! See also **Problems 2.6.38** and **3.7.14**.)

▲

Example 2.6.8 The following two more general results are very useful:

(1) for α and $\beta \neq 0$ real constants such that $-1 < \dfrac{\alpha+1}{\beta} < 1$, we have

$$\int_0^\infty x^\alpha \sin(x^\beta)\,dx = \frac{1}{|\beta|}\cdot\frac{\pi}{2\cdot\Gamma\left(1-\dfrac{\alpha+1}{\beta}\right)\cdot\sin\left[\left(1-\dfrac{\alpha+1}{\beta}\right)\dfrac{\pi}{2}\right]},$$

and

(2) for α and $\beta \neq 0$ real constants such that $0 < \dfrac{\alpha+1}{\beta} < 1$, we have

$$\int_0^\infty x^\alpha \cos(x^\beta)\,dx = \frac{1}{|\beta|}\cdot\frac{\pi}{2\cdot\Gamma\left(1-\dfrac{\alpha+1}{\beta}\right)\cdot\cos\left[\left(1-\dfrac{\alpha+1}{\beta}\right)\dfrac{\pi}{2}\right]}.$$

These are obtained from the two **Results (2.8)** and **(2.9)** of the **previous example** by making the u-substitution $u = x^\beta$. (Work this out!)

▲

2.6.2 The Beta Function

The Beta function is defined as the following integral

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

with two real parameters p and q. This is also called **Euler's integral of the first kind.** We will shortly see that the Beta function is closely related to the Gamma function. (In complex analysis, p and q are considered complex variables.)

Basic Properties of the Beta Function

(B, 1). This integral converges to a positive finite value and it is continuous for $p > 0$ and $q > 0$. It diverges otherwise, i.e., it becomes infinite.

This result is obtained by proving the following four cases:

(a) For $p \geq 1$ and $q \geq 1$, the integral is a proper integral of a continuous function on $[0, 1]$. So, its value is positive finite, in this case.

(b) For $p \leq 0$ or $q \leq 0$, we have $B(p, q) = \infty$.

(c) If $0 < p < 1$ and $0 < q < 1$, then we split the integral about $x = \frac{1}{2}$ to easily obtain convergence to a positive finite value.

(d) For $0 < p < 1$ and $q \geq 1$, or $p \geq 1$ and $0 < q < 1$, we also obtain convergence to a positive finite value easily.

(B, 2). $\forall\, p > 0,\ \forall\, q > 0, \quad B(p, q) = B(q, p).$

This is obtained by the change of variables $u = 1 - x$.

(B, 3). The Beta function satisfies

$$\forall\, p > 0,\ \forall\, q > 1, \quad B(p, q) = \frac{q-1}{p} B(p, q-1) - \frac{q-1}{p} B(p, q).$$

[Hint: This relation is obtained by first using integration with parts, and then we replace x^p with $x^{p-1} - x^{p-1}(1 - x)$.]

Similarly,

$$\forall\, p > 1,\ \forall\, q > 0, \quad B(p, q) = \frac{p-1}{q} B(p-1, q) - \frac{p-1}{q} B(p, q).$$

Therefore, solving these two relations for $B(p, q)$, we obtained two **recursive formulae of the Beta function**

$$\forall\, p > 0,\ \forall\, q > 1, \quad B(p, q) = \frac{q-1}{p+q-1} B(p, q-1),$$

and

$$\forall\, p > 1,\ \forall\, q > 0, \quad B(p, q) = \frac{p-1}{p+q-1} B(p-1, q).$$

(See also **Problem 2.6.21**.)

(B, 4). By using the **previous property**, or directly, we can prove two more **recursive formulae of the Beta function**

$$\forall\ p\ > 0,\ \forall\ q > 1, \quad B(p,q) = \frac{q-1}{p} B(p+1, q-1),$$

and

$$\forall\ p\ > 1,\ \forall\ q > 0, \quad B(p,q) = \frac{p-1}{q} B(p-1, q+1).$$

(See also **Problem 2.6.21**.)

(B, 5). We have the following five integral representations of the Beta function: $\forall\ p\ > 0$ and $\forall\ q > 0$

$$(I) \qquad B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\, du.$$

This is obtained by making the change of variables $x = \dfrac{u}{1+u}$.

By the symmetry of the Beta function, or by analogous substitution, we also have

$$(II) \qquad B(p,q) = \int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}}\, du.$$

If we add equations (I) and (II) and divide by 2, we get

$$(III) \qquad B(p,q) = \frac{1}{2} \int_0^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}}\, du.$$

We observe that by making the substitution $u = \dfrac{1}{v}$ in the first integral below, we obtain

$$\int_0^1 \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}}\, du = \int_1^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}}\, du.$$

Therefore, by (III), we get

$$(IV) \qquad B(p,q) = \int_0^1 \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}}\, du$$

and

$$(V) \qquad B(p,q) = \int_1^\infty \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}}\, du.$$

(B, 6). $\forall\ p\ > 0,\ \forall\ q > 0, \quad B(p,q) = 2\displaystyle\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta.$

This is obtained by letting $0 \le x = \sin^2(\theta) \le 1$.

(B, 7). Relation of the Beta function to the Gamma function:

$$\forall\ p > 0,\ \forall\ q > 0,\ B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

This follows by letting $x = u^2$ in $\Gamma(p)$ and $x = v^2$ in $\Gamma(q)$ to get

$$\Gamma(p)\Gamma(q) = 4\left(\int_0^\infty u^{2p-1}e^{-u^2}\,du\right)\left(\int_0^\infty v^{2q-1}e^{-v^2}\,dv\right) =$$
$$4\int_0^\infty\int_0^\infty u^{2p-1}v^{2q-1}e^{-(u^2+v^2)}\,dudv.$$

Now we use polar coordinates $u = r\cos\theta,\ v = r\sin\theta$ and **(B, 6)** to find $\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q)$, and the result follows.

From this result and **(Γ, 9)**, we get the convenient byproduct: For $m \ge 0$ and $n \ge 0$ integers, we have

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx =$$
$$B(m+1,n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} =$$
$$\frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\binom{m+n}{m}} = \frac{1}{(m+n+1)\binom{m+n}{n}}.$$

(See also **Problem 2.6.16**.)

(B, 8). The Beta and Gamma functions satisfy:

$$\forall\ p:\quad 0 < p < 1,\quad B(p,1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

To prove this, we let $q = 1-p$ ($\Longleftrightarrow p+q = 1$) in **(B, 5, I)** and use the result of **Example 2.2.5** to find first

$$\forall\ 0 < p < 1,\ B(p,1-p) = \int_0^\infty \frac{u^{p-1}}{1+u}\,du = \frac{\pi}{\sin(p\pi)}.$$

Since $\Gamma(p+q) = \Gamma(1) = 1$, by **(B, 7)**, we get that

$$\forall\ p:\ \ 0 < p < 1, \quad B(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

We can rewrite the last relation in various ways. For example, replacing p with $\frac{1}{2} + p$, we obtain:

$$\forall\ p:\ \ -\frac{1}{2} < p < \frac{1}{2},$$
$$B\left(\frac{1}{2} + p, \frac{1}{2} - p\right) = \Gamma\left(\frac{1}{2} + p\right)\Gamma\left(\frac{1}{2} - p\right) = \frac{\pi}{\cos(p\pi)},$$

and so on.

(B, 9). $\forall\ p > 0,\ \ \forall\ q > 0,\ \ \forall\ r > 0$ constants, we have

$$\int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}}\, du = \frac{1}{(r+1)^p \cdot r^q}\, B(p, q).$$

To obtain this result, use the change of variables $u = \dfrac{rx}{r+1-x}$. (Work this out!)

So, we have obtained the following expression of the Beta function:

$$\forall\ p > 0, \quad \forall\ q > 0, \quad \forall\ r > 0,$$
$$B(p, q) = (r+1)^p \cdot r^q \int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}}\, du.$$

(B, 10). We can use the Beta function to obtain the so-called **Gamma function duplication formula**.

We have the relation

$$2^{2p} \int_0^{\pi/2} \sin^{2p}(x)\cos^{2p}(x)\, dx = \int_0^{\pi/2} \sin^{2p}(2x)\, dx \overset{u=2x}{=}$$
$$\frac{1}{2}\int_0^{\pi} \sin^{2p}(u)\, du = \int_0^{\pi/2} \sin^{2p}(u)\, du.$$

Then by **(B, 6)**, after ignoring the $1/2$ in both sides, we find

$$2^{2p} \cdot B\left(p + \frac{1}{2}, p + \frac{1}{2}\right) = B\left(p + \frac{1}{2}, \frac{1}{2}\right).$$

So, by **(B, 7)**

$$2^{2p}\cdot\frac{\Gamma^{2}\left(p+\frac{1}{2}\right)}{\Gamma(2p+1)}=\frac{\Gamma\left(p+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(p+1)}.$$

By **(Γ, 8)**, $\Gamma(x+1)=x\Gamma(x)$ for any $x>0$, and **(Γ, 5)**, $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$, we find

$$2^{2p}\cdot\frac{\Gamma^{2}\left(p+\frac{1}{2}\right)}{2p\,\Gamma(2p)}=\frac{\Gamma\left(p+\frac{1}{2}\right)\cdot\sqrt{\pi}}{p\,\Gamma(p)}.$$

Simplifying this and solving for $\Gamma(2p)$ we get:

$$\Gamma(2p)=\frac{2^{2p-1}}{\sqrt{\pi}}\cdot\Gamma(p)\cdot\Gamma\left(p+\frac{1}{2}\right)=\frac{2^{2p-\frac{1}{2}}}{\sqrt{2\pi}}\cdot\Gamma(p)\cdot\Gamma\left(p+\frac{1}{2}\right).$$

This formula, for obvious reasons, is called the **Gamma function duplication formula** (found by A. - M. Legendre[22]). [See and compare with **Problem 2.6.33** and its **footnote** and **Problem 2.6.42, Item (11.)**.]

From this formula, we obtain the useful formula

$$\Gamma\left(p+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2p)}{2^{2p-1}\Gamma(p)}.$$

By this and **(Γ, 5)**, we also find that

$$\lim_{p\to 0^{+}}\frac{\Gamma(2p)}{\Gamma(p)}=\frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{\pi}}=\frac{\sqrt{\pi}}{2\sqrt{\pi}}=\frac{1}{2}.$$

[See and compare with **Problem 2.6.42, Item (12.)**.]

(B, 11). For any $p>0$ fixed, we apply **Theorem 2.3.4** with dominating function $g(x)=(1-x)^{p-1}\geq 0$, for $0\leq x\leq 1$, to find

$$\lim_{0<q\to\infty}B(p,q)=0^{+}.$$

Therefore, by **(B, 7)**,

$$\lim_{0<q\to\infty}\frac{\Gamma(p+q)}{\Gamma(q)}=\lim_{0<q\to\infty}\frac{\Gamma(p)}{B(p,q)}=\frac{\Gamma(p)}{0^{+}}=+\infty.$$

[See also and compare with **Problem 2.6.42, Item (6.)**.]

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[22] Adrien-Marie Legendre, French mathematician, 1752 -1833.

(B, 12). For any $p > 0$ fixed, we have $\lim\limits_{0<q\to\infty} q^p \cdot B(p,q) = \Gamma(p)$.

To prove this, we notice that for any $p > 0$ and any $q > 0$, we have

$$q^p \cdot B(p,q) = \int_0^1 (qx)^{p-1}(1-x)^{q-1}\, d(qx) \overset{u=qx}{=}$$
$$\int_0^q u^{p-1}\left(1-\frac{u}{q}\right)^{q-1} du.$$

But $\left(1-\dfrac{u}{q}\right)^{q-1} \uparrow e^{-u}$, as $q \longrightarrow \infty$. So, as we did in **Example 2.3.11** or in **Problem 2.3.2**, we obtain

$$\lim_{0<q\to\infty} q^p \cdot B(p,q) = \int_0^\infty u^{p-1}e^{-u}du = \Gamma(p).$$

This result implies the following important result for the Gamma function: For any $p \in \mathbb{R}$ fixed, we have

$$\lim_{0<\,q\,\to\,\infty} \frac{\Gamma(p+q)}{\Gamma(q)\cdot q^p} = 1.$$

For $p > 0$, we use **(B, 7)** to get

$$\lim_{0<q\to\infty} \frac{\Gamma(p)}{q^p \cdot B(p,q)} = \lim_{0<\,q\,\to\,\infty} \frac{\Gamma(p+q)}{\Gamma(q)\cdot q^p} = 1.$$

For $p = 0$, we observe this result is trivially true.

For $p = -r < 0$ (and so $r > 0$), we manipulate the case with p positive to get that for given $r > 0$, we have

$$\lim_{0<\,q\,\to\,\infty} \frac{\Gamma(p-r)}{\Gamma(q)\cdot q^{-r}} = \lim_{0<\,q\,\to\,\infty} \frac{\left[\dfrac{(q-r)+r}{q-r}\right]^r}{\dfrac{\Gamma[(q-r)+r]}{(q-r)^r\cdot\Gamma(q-r)}} = \frac{1}{1} = 1.$$

[See also **Problems 2.6.42, Item (6.)**, and **2.6.43, (a)**.]

Examples

A very large number of improper and/or complicated integrals are reduced to values of the Gamma and Beta functions. Many books on this subject contain a great number of them. Here we present the following:

Example 2.6.9

$$\int_0^1 x^{17}\,dx =$$

$$\int_0^1 (1-x)^{17}\,dx = B(18,1) = B(1,18) = \frac{\Gamma(18)\Gamma(1)}{\Gamma(19)} = \frac{17!\cdot 0!}{18!} = \frac{1}{18}.$$

▲

Example 2.6.10

$$\int_0^1 x^{17}(1-x)^{33}\,dx = \int_0^1 x^{33}(1-x)^{17}\,dx = B(18,34) =$$

$$B(34,18) = \frac{\Gamma(18)\Gamma(34)}{\Gamma(52)} = \frac{17!\cdot 33!}{51!} = \frac{1}{51\cdot\binom{50}{17}} = \frac{1}{51\cdot\binom{50}{33}}.$$

▲

Example 2.6.11

$$\int_0^1 x^{\frac{1}{3}}(1-x)^{\frac{7}{5}}\,dx = B\left(\frac{4}{3},\frac{12}{5}\right) = \frac{\Gamma(\frac{4}{3})\Gamma(\frac{12}{5})}{\Gamma(\frac{4}{3}+\frac{12}{5})} = \frac{\Gamma(\frac{4}{3})\Gamma(\frac{12}{5})}{\Gamma(\frac{56}{15})}.$$

Now this can be evaluated approximately by using tables of the Gamma function or computer means.

▲

Example 2.6.12 By **(B, 8)**, we readily get

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin(\frac{1}{4}\pi)} = \pi\sqrt{2}.$$

So,

$$\Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}, \qquad \text{or} \qquad \Gamma\left(\frac{1}{4}\right) = \frac{\pi\sqrt{2}}{\Gamma(\frac{3}{4})}.$$

▲

Example 2.6.13 We use **(B, 5)**, **(B, 7)** and **(B, 8)** to find

$$\int_0^\infty \frac{\sqrt[4]{x}}{(1+x)^2}\,dx = B\left(\frac{3}{4},\frac{5}{4}\right) = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{5}{4}+\frac{3}{4})} = \frac{\Gamma(1+\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(2)} =$$

$$\frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{1} = \frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{4}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{1}{4}\cdot\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{\pi\sqrt{2}}{4}.$$

▲

Example 2.6.14 We use **(B, 6)**, **(B, 7)** and **(B, 8)** to find

$$\int_0^{\pi/2} \sin^{\frac{5}{2}}(x)\cos^{\frac{3}{2}}(x)\,dx =$$

$$\frac{1}{2}\cdot B\left(\frac{7}{4},\frac{5}{4}\right) = \frac{1}{2}\cdot\frac{\Gamma(\frac{7}{4})\Gamma(\frac{5}{4})}{\Gamma(3)} = \frac{1}{2}\cdot\frac{\Gamma(1+\frac{3}{4})\cdot\Gamma(1+\frac{1}{4})}{2!} =$$

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{3}{64}\cdot\frac{\pi}{\sin(\frac{\pi}{4})} = \frac{3\pi\sqrt{2}}{64}.$$

▲

Example 2.6.15 By letting $q = \dfrac{1}{2}$ in **(B, 6)**, then by **(B, 7)** we get that for $p > 0$ constant

$$\int_0^{\pi/2} \sin^{2p-1}(\theta)\,d\theta = \frac{1}{2}B\left(p,\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}.$$

From this, by the symmetry and the positivity of sine in $[0, \pi]$, we also obtain

$$\int_0^{\pi} \sin^{2p-1}(\theta)\,d\theta = B\left(p,\frac{1}{2}\right) = \frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}.$$

Similarly for $q > 0$ constant

$$\int_0^{\pi/2} \cos^{2q-1}(\theta)\,d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)}.$$

In this case, we cannot extend the integral over $[0, \pi]$ when the exponent is not an integer, since the cosine is negative in $\left(\dfrac{\pi}{2}, \pi\right]$. But, if the exponent is an even integer, then the integral over $[0, \pi]$ is twice the previous integral, and if the exponent is odd, the integral over $[0, \pi]$ is zero.

(See also and compare with **Problem 2.6.25**.)

▲

Example 2.6.16 By **(B, 6)**, **(B, 7)** and **Problem 2.6.14** for $m \geq 0$ and $n \geq 0$ integers, we find

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{1}{2}\cdot B\left(m+\frac{1}{2}, n+\frac{1}{2}\right) =$$

$$\frac{1}{2}\cdot\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(m+n+1)} = \frac{\pi}{2^{2m+2n+1}}\cdot\frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

The integral over $[0, \pi]$ is twice this integral.

(See also **Problem 2.6.25**.)

▲

Example 2.6.17 In general, for $p > 0$ and $q > 0$ by **(B, 6)** and **(B, 7)**, we have that:

$$\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta = \frac{1}{2}B(p,q) = \frac{1}{2}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(See also **Problem 2.6.25**.)

▲

Example 2.6.18 $\forall\ C > 0,\ \ \forall\ D > 0,\ \ \forall\ p > 0,\ \ \forall\ q > 0$ constants, we have

$$\int_0^{\pi/2} \frac{\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)}{\left[C\,\sin^2(\theta) + D\,\cos^2(\theta)\right]^{p+q}}\,d\theta = \frac{B(p,q)}{2\,C^p\cdot D^q}.$$

This result is obtained by making the change of variables $u = \sin^2(\theta)$ and adjusting the integral obtained to **(B, 9)**. (Work this out!)

(See also and compare with **Examples 2.2.3**, **3.5.15** and **3.8.3**.)

▲

2.6.3 Applications

Application 1: In physics, with the help of differential equations, we find that the actual period T of a simple pendulum of length l is given by

$$T = 4\sqrt{\frac{l}{g}}\cdot\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2\sin^2(\psi)}}$$

where g is the acceleration of gravity at the place where the pendulum swings and $k = \sin\left(\frac{\theta_0}{2}\right)$, with $\theta_0 > 0$ the maximum swing angle that the pendulum makes with the vertical axis passing through the fixed point on which the one end of the pendulum is firmly attached. On the other end, a mass m is attached which is much greater than the mass of the rod or the wire of suspension of the pendulum, which is considered negligible. Also, we assume that apart from gravity no resistance or any other force of any sort is exerted on the pendulum as it swings. I.e., it performs free swings under the force of gravity.

When $\theta_0 > 0$ is very close to zero (small swing), then we can approximately consider $k \simeq 0$, and the period is given by the known much

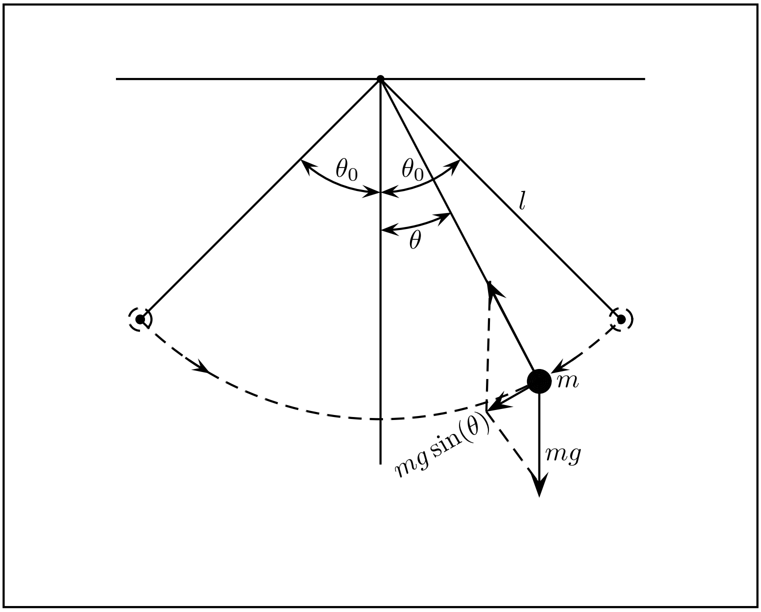

FIGURE 2.7: Pendulum in motion

simpler but approximate law

$$T \simeq 2\pi\sqrt{\frac{l}{g}}.$$

The special integral

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2\sin^2(\psi)}},$$

where $0 < k < 1$, that appears in this physical law, is a **complete elliptic integral of the first kind in its Legendre form**.

With the help of the Gamma function, we will evaluate this integral when $\theta_0 = \dfrac{\pi}{2}$ and so $0 < k = \sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}} < 1$. We have

$$K\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2(\phi)}}.$$

By letting $\sin(\phi) = \sqrt{2}\sin\left(\dfrac{\theta}{2}\right)$, this integral changes to (work out this substitution carefully)

$$\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos(\theta)}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\cos^{\frac{-1}{2}}(\theta)\,d\theta.$$

By the **properties (B, 6)** and **(B, 7)** of the Beta and Gamma functions and the result of **Example 2.6.12**, this is equal to

$$\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot B\left(\frac{1}{2},\frac{1}{4}\right) = \frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2}+\frac{1}{4})} = \frac{1}{2\sqrt{2}}\frac{\sqrt{\pi}\,\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{\Gamma^{\,2}(\frac{1}{4})}{4\sqrt{\pi}}.$$

Therefore,

$$K\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2(\phi)}} = \frac{\Gamma^{\,2}\left(\frac{1}{4}\right)}{4\sqrt{\pi}}.$$

So, the precise period of a pendulum with swing angle $\dfrac{\pi}{2}$ is

$$T = \sqrt{\frac{l}{g\pi}}\cdot\Gamma^{\,2}\left(\frac{1}{4}\right).$$

The derivation of the above formula for the actual period is as follows: We look at the provided **Figure, 2.7,** and we use Newton's law "mass × acceleration = force." Since $\theta = \theta(t)$ in this circular motion, we get

$$ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta), \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta).$$

To integrate this second-order ordinary differential equation, we first multiply both sides by $\dfrac{d\theta}{dt}$, i.e.,

$$\frac{d^2\theta}{dt^2}\cdot\frac{d\theta}{dt} = -\frac{g}{l}\sin(\theta)\cdot\frac{d\theta}{dt}, \quad \text{or} \quad \frac{1}{2}\frac{d}{dt}\left[\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{g}{l}\frac{d[\cos(\theta)]}{dt}.$$

We integrate once and obtain

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{l}\cos(\theta) + c.$$

We assume that the pendulum is released at time $t = 0$ from the position $\theta = \theta_0$. When $t = 0$, $\dfrac{d\theta}{dt}|_{t=0} = 0$ (zero velocity at the start), and so

$$c = -\frac{g}{l}\cos(\theta_0).$$

Therefore,

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}} \cdot \sqrt{\cos(\theta) - \cos(\theta_0)}.$$

When the pendulum goes from $\theta = \theta_0$ to $\theta = 0$, $\dfrac{d\theta}{dt}$ is negative because θ is decreasing and the time needed is one quarter of the period. So,

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}} \cdot \int_{\theta_0}^{0} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}}.$$

Using the half angle trigonometric formulae, we find

$$T = 2\sqrt{\frac{l}{g}} \cdot \int_{0}^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}.$$

Finally, we use the change of variables

$$\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta_0}{2}\right)\sin(\psi)$$

to find

$$T = 4\sqrt{\frac{l}{g}} \cdot \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2\sin^2(\psi)}},$$

where $k = \sin\left(\dfrac{\theta_0}{2}\right)$.

Application 2: In advanced geometry, analysis, partial differential equations (especially elliptic), potential theory, etc., we need to know the volume and the surface area of balls in the Euclidean spaces.

In $\mathbb{R}^n$, $n = 1,\ 2,\ 3,\ \dots$, the n**-dimensional closed ball** with center a point $\overrightarrow{c} = (c_1,\ c_2,\ \dots,\ c_n) \in \mathbb{R}^n$ and radius $R > 0$ is the set given by

$$B_n\left(\overrightarrow{c}, R\right) = \left\{ \overrightarrow{x} = (x_1,\ x_2,\ \dots,\ x_n) \in \mathbb{R}^n \quad \middle| \quad \sum_{i=1}^{n}(x_i - c_i)^2 \le R^2 \right\}.$$

The surface or this ball, denoted here by S_n or $S_n\left(\overrightarrow{c}, R\right)$, has dimension $n-1$ and is called the $(n-1)$**-dimensional sphere** with center $\overrightarrow{c}$ and radius R. This is the subset of $\mathbb{R}^n$ analytically given by

$$S_n\left(\overrightarrow{c}, R\right) = \left\{ \overrightarrow{x} = (x_1,\ x_2,\ \dots,\ x_n) \in \mathbb{R}^n \quad \middle| \quad \sum_{i=1}^{n}(x_i - c_i)^2 = R^2 \right\}.$$

With the help of the Gamma function, we will prove

$$\textbf{Volume}\left[B_n\left(\overrightarrow{c}, R\right)\right] = \frac{2R^n\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$$

and

$$\textbf{Area}\left[S_n\left(\overrightarrow{c}, R\right)\right] = \frac{2R^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

For $n = 1,\ 2,\ 3$, we can check that these formulae give the correct answers. When $n = 1$, the center $\overrightarrow{c} = (c_1)$ is just a real number c on the real line. So, $B_1(c, R)$ is the closed interval $[c - R, c + R] \subset \mathbb{R}$. Its length, $2R$, is viewed as its volume and its two point set boundary, $\{c - R,\ c + R\}$, as its surface. The formulae for $n = 1$ give

$$\text{Volume}\left[B_1(c, R)\right] = \frac{2R\pi^{\frac{1}{2}}}{1\Gamma\left(\frac{1}{2}\right)} = \frac{2R\sqrt{\pi}}{\sqrt{\pi}} = 2R$$

and

$$\text{Area}\left[S_1(c, R)\right] = \frac{2R^{\,0}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{2\sqrt{\pi}}{\sqrt{\pi}} = 2.$$

When $n = 2$, we deal with the usual circle with center $\overrightarrow{c} = (c_1,\ c_2)$ and radius R. Its area, πR^2, is viewed as its volume and the length of its circumference, $2\pi R$, as its surface. The formulae for $n = 2$ give

$$\text{Volume}\left[B_2\left(\overrightarrow{c}, R\right)\right] = \frac{2R^2\pi^{\frac{2}{2}}}{2\Gamma\left(\frac{2}{2}\right)} = \frac{R^2\pi}{1} = \pi R^2$$

and

$$\text{Area}\left[S_2\left(\overrightarrow{c}, R\right)\right] = \frac{2R^{\,1}\pi^{\frac{2}{2}}}{\Gamma\left(\frac{2}{2}\right)} = \frac{2R\pi}{1} = 2\pi R.$$

When $n = 3$, we get the usual ball in $\mathbb{R}^3$ with center $\overrightarrow{c} = (c_1,\ c_2,\ c_3)$ and radius R. Its volume is $\dfrac{3\pi}{4}R^3$, and the area of its boundary surface, which is the usual sphere with center $\overrightarrow{c}$ and radius R in $\mathbb{R}^3$, is $4\pi R^2$. The formulae for $n = 3$ give

$$\text{Volume}\left[B_3\left(\overrightarrow{c}, R\right)\right] = \frac{2R^3\pi^{\frac{3}{2}}}{3\Gamma\left(\frac{3}{2}\right)} = \frac{2R^3\pi^{\frac{3}{2}}}{3\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4R^3\pi^{\frac{3}{2}}}{3\sqrt{\pi}} = \frac{4\pi}{3}R^3$$

and

$$\text{Area}\left[S_3\left(\overrightarrow{c}, R\right)\right] = \frac{2R^{2\ 1}\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} = \frac{2R^2\pi^{\frac{3}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4R^2\pi^{\frac{3}{2}}}{\sqrt{\pi}} = 4\pi R^2.$$

So, we need to prove the formulae for $n \geq 4$. We will prove the volume formula first. Without loss of generality, we assume $\overrightarrow{c} = \overrightarrow{0}$ the origin of $\mathbb{R}^n$. Then, as we know,

$$\text{Volume}\left[B_n\left(\overrightarrow{0}, R\right)\right] = \int_{B_n\left(\overrightarrow{0}, R\right)} 1 \; dx_1 dx_2 \, \dots \, dx_n.$$

We use the parametrization of $B_n\left(\overrightarrow{0}, R\right)$ in $\mathbb{R}^n$, which is the extension of spherical coordinates as we know them in $\mathbb{R}^3$. These coordinates are

$$0 \leq \rho \leq R, \quad 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi_j \leq \pi \text{ for } j = 1,\ 2,\ \dots,\ n-2,$$

and satisfy

$$x_1 = \rho\cos(\phi_1), \quad x_2 = \rho\sin(\phi_1)\cos(\phi_2), \quad x_3 = \rho\sin(\phi_1)\sin(\phi_2)\cos(\phi_3), \\ \dots, \quad x_{n-1} = \rho\sin(\phi_1)\dots\sin(\phi_{n-2})\cos(\theta) \\ \text{and} \quad x_n = \rho\sin(\phi_1)\dots\sin(\phi_{n-2})\sin(\theta).$$

The **Jacobian**[23] **determinant** of this change of variables is

$$\frac{\partial(x_1,\, x_2,\, x_3,\, \dots, x_{n-1}, x_n)}{\partial(\rho,\, \phi_1,\, \phi_2,\, \dots, \phi_{n-2},\, \theta)} = \\ \rho^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\dots\sin^2(\phi_{n-3})\sin(\phi_{n-2}),$$

as it is verified either inductively or by writing it explicitly and pulling the common factors out of the determinant. (Here the Jacobian matrix is in **Hessenberg**[24] **form**[25] and so its determinant is computed fairly easily.)

Since this Jacobian is positive, the volume is

$$\text{Volume}\left[B_n\left(\overrightarrow{c}, R\right)\right] = \text{Volume}\left[B_n\left(\overrightarrow{0}, R\right)\right] = \int_{B_n\left(\overrightarrow{0}, R\right)} 1 \; dx_1 dx_2 \, \dots \, dx_n = \\ \int_0^R \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \rho^{n-1}\sin^{n-2}(\phi_1)\dots\sin(\phi_{n-2})d\theta d\phi_1\dots d\phi_{n-2}d\rho = \\ \frac{2\pi R^n}{n} \cdot \int_0^\pi \sin^{n-2}(\phi)d\phi \cdot \int_0^\pi \sin^{n-3}(\phi)d\phi \dots \int_0^\pi \sin(\phi)d\phi = \\ \frac{2\pi R^n}{n} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \dots \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

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[23] Carl Gustav Jacob Jacobi or Carolus Gustavus Iacobus Iacobi, German mathematician, 1804-1851.

[24] Karl Adolf Hessenberg, German mathematician and engineer, 1904-1959.

[25] A square matrix is in **Hessenberg form** if all entries (i, j) with $j \geq i + 2$, or (i, j) with $1 \leq j \leq i - 2$, are zero.

(by using the second integral of **Example 2.6.15**. See also **Problem 2.6.34** for another way of computing this.).

Since $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, after simplifying we get

$$\text{Volume}\left[B_n\left(\overrightarrow{c},R\right)\right]=\frac{2\pi R^n}{n}\cdot\frac{\Gamma^{n-2}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}=\frac{2R^n\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}.$$

We observe that

$$\text{Volume}\left[B_n\left(\overrightarrow{c},R\right)\right]=\text{Volume}[B_n\left(\overrightarrow{0},R\right)]=\int_{B_n\left(\overrightarrow{0},R\right)}1\;dx_1dx_2\,\ldots\,dx_n=$$

$$\int_0^R\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\rho^{n-1}\sin^{n-2}(\phi_1)\ldots\sin(\phi_{n-2})d\theta d\phi_1\ldots d\phi_{n-2}d\rho=$$

$$\int_0^R\int_{S_n\left(\overrightarrow{0},R\right)}dS\,d\rho,$$

where dS is the surface element of $S_n\left(\overrightarrow{0},R\right)$.

Therefore,

$$\frac{d}{dR}\left\{\text{Volume}\left[B_n\left(\overrightarrow{c},R\right)\right]\right\}=$$

$$\frac{d}{dR}\left[\int_0^R\int_{S_n\left(\overrightarrow{0},R\right)}dS\,d\rho\right]=\int_{S_n\left(\overrightarrow{0},R\right)}dS=\text{Area}[S_n\left(\overrightarrow{c},R\right)],$$

and so

$$\text{Area}\left[S_n\left(\overrightarrow{c},R\right)\right]=\frac{2R^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

[The area in $\mathbb{R}^n$ may be viewed as $(n-1)$-dimensional volume in $\mathbb{R}^n$.]

We now observe that since $\left(\left[\left[\frac{n}{2}\right]\right]-1\right)!\leq\Gamma\left(\frac{n}{2}\right)$, we get the rather counterintuitive results

$$\text{(a)}\quad\lim_{n\to\infty}\left\{\text{Volume}\left[B_n\left(\overrightarrow{c},R\right)\right]\right\}=\lim_{n\to\infty}\left[\frac{2R^n\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}\right]=0,\ \text{ and}$$

$$\text{(b)}\quad\lim_{n\to\infty}\left\{\text{Area}[S_n\left(\overrightarrow{c},R\right)]\right\}=\lim_{n\to\infty}\left[\frac{2R^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\right]=0.$$

(See also **Problem 2.6.35**.)

A **corollary** of this application is the computation of the **volume** of an n**-dimensional ellipsoid** with semi-axes $a_i > 0$ for $i = 1,\ 2,\ \dots,\ n$

$$E_n = \left\{ \overrightarrow{x} = (x_1,\, x_2,\, \dots, x_i) \quad \Big| \quad \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \le 1 \right\}.$$

We make the change of variable

$$u_i = \frac{x_i}{a_i} \iff x_i = a_i u_i$$

for $i = 1,\ 2,\ \dots,\ n$, and we use the volume of the unit ball found above. Then we find

$$\text{Volume}\,[E_n] = \frac{2\, a_1 a_2 \dots a_n\, \pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}.$$

(The surface area of such an ellipsoid is a much harder matter.)

Application 3: Referring to definitions in **Application 1 of Subsection 2.1.1**, the function

$$g(x : \alpha, \beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1} e^{-\frac{x}{\beta}} & \text{for} \quad x > 0 \\ \\ 0 & \text{for} \quad x \le 0, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ constants, is a probability density function.

Indeed, it is piecewise continuous and greater than or equal to zero everywhere. We only need to show that its integral over $\mathbb{R}$ is equal to 1. We let $u = \dfrac{x}{\beta}$, and we have:

$$\int_{-\infty}^{\infty} g(x : \alpha, \beta) dx =$$

$$\int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \beta^{\alpha-1} u^{\alpha-1} e^{-u} \beta\, du =$$

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = \frac{1}{\Gamma(\alpha)}\, \Gamma(\alpha) = 1.$$

If a random variable $X = x > 0$ has density function $g(x : \alpha, \beta)$, we say that it has the **gamma distribution**. We can compute its expected value, variance and moment-generating function.

For all $r = 0,\ 1,\ 2,\ 3,\ \dots,$ we have that the moment of order r

about the origin is

$$\mu'_r = \int_0^\infty x^r \frac{1}{\beta^\alpha \Gamma(\alpha)} \; x^{\alpha-1} e^{-\frac{x}{\beta}} dx =$$
$$\frac{\beta^r}{\Gamma(\alpha)} \int_0^\infty u^{\alpha+r-1} e^{-u} du = \frac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}.$$

So, the expected value is

$$E(X) = \mu = \mu'_1 = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta.$$

Similarly, we find

$$\mu'_2 = \frac{\beta^2\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\beta^2(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2.$$

Therefore, the variance is

$$var(X) = \sigma^2 = \mu'_2 - (\mu'_1)^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2.$$

The moment-generating function is

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha\Gamma(\alpha)} \; x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\frac{x(1-\beta t)}{\beta}} dx.$$

We use the substitution $v = \dfrac{x(1-\beta t)}{\beta}$ to obtain

$$M_X(t) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty \beta^{\alpha-1} v^{\alpha-1} (1-\beta t)^{-\alpha+1} e^{-v} \beta (1-\beta t)^{-1} dv =$$
$$\frac{(1-\beta t)^{-\alpha}\beta^\alpha}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty v^{\alpha-1} e^{-v} dv = \frac{(1-\beta t)^{-\alpha}\beta^\alpha\Gamma(\alpha)}{\beta^\alpha\Gamma(\alpha)} = (1-\beta t)^{-\alpha}.$$

Application 4: In physics, physical and/or quantum chemistry, study of gasses, thermodynamic, etc., improper integrals appear in various laws and computations. For example, finding the average kinetic and potential energy of the **H** (hydrogen) atom at an excited state uses repeatedly the integral

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}},$$

where $n = 0, 1, 2, 3, 4, \ldots$, integer and $a > 0$ constant. This is a standard integral rule of the Gamma function. See **property ($\boldsymbol{\Gamma}$, 11)** and its extension in **Problem 2.6.5** for more generality.

In this application, we skip the derivations of formulae and the meanings of various physics constants and terms, and we concentrate on the mathematics of the integrals involved. We consider a particular example of an excited state of the **H** (hydrogen) atom for which the average value of the kinetic energy is given by the following improper integral:

$$\text{Average Kinetic Energy} = -\frac{\hbar^2}{2m_e}\cdot\frac{1}{32\pi a_0^3}\times\int_0^{2\pi} d\theta\int_0^{\pi}\sin(\phi)d\phi$$

$$\int_0^\infty\left(2-\frac{\rho}{a_0}\right)e^{-\frac{\rho}{2a_0}}\frac{1}{\rho^2}\frac{d}{d\rho}\left\{\rho^2\frac{d}{d\rho}\left[\left(2-\frac{\rho}{a_0}\right)e^{-\frac{\rho}{2a_0}}\right]\right\}\rho^2 d\rho =$$

$$\frac{\hbar^2}{2m_e}\cdot\frac{1}{32\pi a_0^3}\cdot 2\pi\cdot 2\times\int_0^\infty\left(2-\frac{\rho}{a_0}\right)\frac{e^{-\frac{\rho}{a_0}}}{4\,a_0^3\,\rho}(16a_0-10a_0\rho+\rho^2)\rho^2 d\rho.$$

We break the integral into four smaller integrals. This is legitimate because each of these four integrals exists. Then we simplify and get

$$\text{Average Kinetic Energy} = -\frac{\hbar^2}{2m_e}\cdot\frac{1}{8a_0^3}\times\left(\frac{9}{a_0^2}\int_0^\infty\rho^2 e^{-\frac{\rho}{a_0}}d\rho\right.$$

$$\left.-\frac{8}{a_0}\int_0^\infty\rho e^{-\frac{\rho}{a_0}}d\rho-\frac{3}{a_0^3}\int_0^\infty\rho^3 e^{-\frac{\rho}{a_0}}d\rho+\frac{1}{4a_0^4}\int_0^\infty\rho^4 e^{-\frac{\rho}{a_0}}d\rho\right).$$

We use the standard Gamma integral we wrote above, and after direct computation we find

$$\text{Average Kinetic Energy} = \frac{\hbar^2}{8m_e a_0^2}.$$

The average value of the potential energy of this atom is given by the following integral:

$$\text{Average Potential Energy} = -\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{32\pi a_0^3}\times\int_0^{2\pi}d\theta\int_0^{\pi}\sin(\phi)d\phi\,\cdot$$

$$\int_0^\infty\left(2-\frac{\rho}{a_0}\right)e^{-\frac{\rho}{2a_0}}\frac{1}{\rho}\left(2-\frac{\rho}{a_0}\right)e^{-\frac{\rho}{2a_0}}\rho^2 d\rho = -\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{32\pi a_0^3}\cdot 2\pi\cdot 2\times$$

$$\left(4\int_0^\infty\rho e^{-\frac{\rho}{a_0}}d\rho-\frac{4}{a_0}\int_0^\infty\rho^2 e^{-\frac{\rho}{a_0}}d\rho+\frac{1}{a_0^2}\int_0^\infty\rho^3 e^{-\frac{\rho}{a_0}}d\rho\right)=$$

$$-\frac{e^2}{4\pi\epsilon_0}\frac{1}{8a_0^3}\left(2a_0^2\right) = -\frac{e^2}{16\pi\epsilon_0 a_0}.$$

Problems

2.6.1 In our lists of the properties of the Gamma and the Beta functions, the proofs of some claims were left as exercises or incomplete with or without hints. Identify all these claims and provide their full proofs.

2.6.2 Prove that:

(a) For $\alpha \in \mathbb{R}$
$$\lim_{0<p\to 0^+} p^{\alpha}\,\Gamma(p) = \begin{cases} 1, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha < 1. \end{cases}$$

(b) For $\alpha \in \mathbb{R}$ and $\forall\ 0 < b < \infty$,

$$\int_0^b p^{\alpha}\Gamma(p)\,dp = \begin{cases} \infty, & \text{if } \alpha \le 0, \\ \text{finite} > 0, & \text{if } \alpha > 0. \end{cases}$$

(c) $\forall\ p \ge 8,\ \Gamma(p) > e^p$.

(d) $\forall\ \alpha \in \mathbb{R},\ \lim\limits_{0<p\to\infty} \dfrac{\Gamma(p)}{p^{\alpha}} = \infty$.

(e) $\forall\ \alpha \in \mathbb{R}$ and $\forall\ 0 < a < \infty,\ \displaystyle\int_a^{\infty} \frac{\Gamma(p)}{p^{\alpha}}\,dp = \infty$.

(f) $\forall\ \alpha \in \mathbb{R},\ \lim\limits_{p\to 0^+} \Gamma(p)\,\sin(\alpha p) = \alpha$.

(g) $\forall\ n \in \mathbb{N},\ \lim\limits_{p\to\infty} \dfrac{\Gamma(p)}{\Gamma(p+n)} = 0$. (Prove this directly when $n \in \mathbb{N}$.)

[Otherwise, for any $n \in \mathbb{R}$, appeal to **property (B, 12)** or **project Problems 2.6.42, Item (6.)**, and **2.6.43, (a)**.]

(h) For $r \in \mathbb{R}$ and $\forall\ 0 < a < \infty$,

$$\int_a^{\infty} \frac{\Gamma(x)}{\Gamma(x+r)}\,dx = \begin{cases} \infty, & \text{if } 0 \le r \le 1, \\ \text{finite} > 0, & \text{if } r > 1. \end{cases}$$

Examine what happens with this integral when $r < 0$.

[Hint: See **properties (B, 11)** and **(B, 12)** or **Problems 2.6.42, Item (6.)** and **2.6.43, (a)**.]

(i) For $n = 0,\ 1,\ 2,\ 3,\ \ldots,\ \lim\limits_{0<p\to 0^+} p^{n+1}\Gamma^{(n)}(p) = (-1)^n n!$.

[$\Gamma^{(n)}(p)$ is the n^{th} derivative of the Gamma function.]

(j) For any $p > 0$ fixed, $\lim\limits_{0<q\to 0^+} B(p,q) = \Gamma(0^+) = +\infty$.

(k) $\lim_{(p,\,q)\to(0^+,\,0^+)} B(p,q) = +\infty.$

2.6.3 Prove that:

(a) If $p \geq 1$ and $q > 1$ or if $p > 1$ and $q \geq 1$, then $\Gamma(p)\Gamma(q) < \Gamma(p+q)$.

(b) If $p = q = 1$, then $\Gamma(p)\Gamma(q) = \Gamma(p+q)$.

(c) If $0 < p \leq 1$ and $0 < q < 1$ or if $0 < p < 1$ and $0 < q \leq 1$, then $\Gamma(p)\Gamma(q) > \Gamma(p+q)$.

(d) If p and q satisfy other combinations of inequalities, then $\Gamma(p)\Gamma(q)$ and $\Gamma(p+q)$ may satisfy different inequalities from those above.

In fact, using **properties (B, 11)** and **(B, 12)** or **Problems 2.6.42, Item (6.)** and **2.6.43, (a)**, prove that for any $0 < p < 1$ there is $q > 1$ such that any of the following three relations can be achieved:

(1) $\Gamma(p)\Gamma(q) > \Gamma(p+q)$,

(2) $\Gamma(p)\Gamma(q) < \Gamma(p+q)$,

(3) $\Gamma(p)\Gamma(q) = \Gamma(p+q)$. (Obviously, we have analogous conclusions for a given $0 < q < 1$ or $q \geq 1$, etc.)

(e) Besides $p = q = 1$, there are infinitely many $p > 0$ and $q > 0$ such that $B(p,q) = 1$.

[Hint: Use the relation between the Beta and Gamma functions and the definition of the Beta function.]

2.6.4 (a) Use the change of variables $x = -\ln(u)$ to prove that for $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = \int_0^1 [-\ln(u)]^{p-1}du = \int_0^1 |\ln(u)|^{p-1}du.$$

(b) Now prove

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{-\frac{1}{2}} dx = \int_0^1 \frac{1}{\sqrt{-\ln(x)}}\,dx = \int_0^1 \frac{1}{\sqrt{|\ln(x)|}}\,dx = \sqrt{\pi}$$

and

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\frac{1}{2}} dx = \int_0^1 \sqrt{-\ln(x)}\,dx = \int_0^1 \sqrt{|\ln(x)|}\,dx = \frac{\sqrt{\pi}}{2}.$$

[See also **Problems 2.1.15**, **2.6.7** and **3.7.47**, and **properties (Γ, 5)** and **(Γ, 9)**.]

(c) If we let $x = e^u$ in the definition of the Gamma function to prove that

$$\forall \ p > 0, \quad \Gamma(p) = \int_{-\infty}^{\infty} e^{pu - e^u} du.$$

2.6.5 [Extension of **property (Γ, 11)**.] (a) For $\alpha > -1$, $\beta > 0$ and $c > 0$ constants, use $u = \beta x^c$ to prove that

$$\int_0^\infty x^\alpha e^{-\beta x^c}\, dx = \frac{1}{c\beta^{\frac{\alpha+1}{c}}} \Gamma\left(\frac{\alpha+1}{c}\right).$$

E.g., check that $\displaystyle\int_0^\infty x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4}$.

(b) Use **(a)** to prove

$$\forall \ p > 0 \quad \Gamma(p) = 2^{1-p} \int_0^\infty u^{2p-1} e^{-\frac{1}{2}u^2} du$$

and (for $c = 1$)

$$\int_0^\infty x^\alpha e^{-\beta x}\, dx = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}.$$

2.6.6 Use **(Γ, 11)** or the **previous problem** to prove the following integral representations of the related infinite sums $\forall \ p \geq 0$ (for $0 \leq p \leq 1$ we get equalities of the type $\infty = \infty$):

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^\infty \frac{x^{p-1}}{e^x - 1}\, dx =$$

$$\frac{1}{\Gamma(p)} \int_1^\infty \frac{\ln^{p-1}(u)}{(u-1)u}\, du = \frac{1}{\Gamma(p)} \int_0^1 \frac{[-\ln(v)]^{p-1}}{1-v}\, dv.$$

2.6.7 Prove that for $n = 0,\ 1,\ 2,\ 3,\ ...$ integers and $\alpha > -1$ constant

$$\int_0^1 x^\alpha [\ln(x)]^n\, dx = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}.$$

(See also **Problems 2.1.15** and **2.6.4**.)

[Hint: Let $x = e^{-u}$, etc.]

2.6.8 Imitate **Example 2.6.7** and provide the proof of the **Result 2.9**, i.e.,

$$\int_0^{\infty} \frac{\cos(x)}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(\frac{p\pi}{2})}\,, \quad \text{for } 0 < p < 1.$$

2.6.9 Use the results of **Example 2.6.8** and compute the integrals

$$I_1 = \int_0^{\infty} x^2 \sin(x^4)\,dx \qquad I_2 = \int_0^{\infty} \sqrt{x}\cos(x^3)\,dx$$
$$I_3 = \int_0^{\infty} x^{-1/2} \sin(x^{-5/2})\,dx \qquad I_4 = \int_0^{\infty} x^{-2}\cos(x^{-3})\,dx.$$

2.6.10 With the help of the Gamma function, compute the values of both **Fresnel integrals**

$$\int_0^{\infty} \sin(x^2)\,dx \quad \text{and} \quad \int_0^{\infty} \cos(x^2)\,dx.$$

2.6.11 Investigate the values of the two integrals

$$\int_0^{\infty} \sin(x^p)\,dx \quad \text{and} \quad \int_0^{\infty} \cos(x^p)\,dx,$$

for all real values of p.

(See also **Problem 2.2.36**.)

2.6.12 Find the values of

$$\int_0^{\infty} x^{10} e^{-x}\,dx \quad \text{and} \quad \int_0^{\infty} x^{\frac{5}{2}} e^{-\frac{15}{22}x}\,dx.$$

2.6.13 Prove that

$$\int_0^{2} x\sqrt[3]{8 - x^3}\,dx = \frac{16\pi}{9\sqrt{3}}.$$

[Hint: Let $x^3 = 8u$, etc.]

2.6.14 Prove directly, or by induction, or by using the byproduct of the **duplication formula** in **(B, 10)**, that:

(a) For $n = 0,\ 1,\ 2,\ 3,\ ...,$ we have

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\,n!} = \frac{(2n)!\,\sqrt{\pi}}{4^n n!}.$$

This expression for $n = 1,\ 2,\ 3, \ldots$ simplifies to

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}.$$

(b) Use the definition of $\Gamma(p)$ for $p < 0$ (have a look at it if you have forgotten it) to prove that for $m = 1,\ 2,\ 3,\ 4, \ldots$

$$\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m\, 2^m}{1\cdot 3\cdot 5\cdots(2m-1)}\sqrt{\pi} = \frac{(-1)^m\, 2^{2m}\, m!}{(2m)!}\sqrt{\pi}.$$

2.6.15 Compute

$$\int_0^1 t^4(1-t)^5\,dt \quad \text{and} \quad \int_0^1 x^4(1-x)^{97}\,dx.$$

2.6.16 For any $m \geq 0$ and $n \geq 0$ integers, use the Beta and Gamma functions and the **Binomial Theorem** to prove

$$\sum_{k=0}^{n}\frac{(-1)^k}{m+k+1}\,[nCk] = \sum_{l=0}^{m}\frac{(-1)^l}{n+l+1}\,[mCl] =$$

$$\frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\,[(m+n)Cm]} = \frac{1}{(m+n+1)\,[(m+n)Cn]},$$

where $nCk = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are the combination numbers.

2.6.17 Compute

$$\int_0^1 \sqrt[4]{t(1-t)}\,dt \quad \text{and} \quad \int_0^1 \sqrt[5]{x^3(1-x)^8}\,dx.$$

2.6.18 For all $-1 < \alpha < 1$, prove the two equalities

$$\text{(a)} \quad \int_0^{\pi} |\tan(x)|^{\alpha}\,dx = 2\int_0^{\pi/2}\tan^{\alpha}(x)\,dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)},$$

$$\text{(b)} \quad \int_0^{\pi} |\cot(x)|^{\alpha}\,dx = 2\int_0^{\pi/2}\cot^{\alpha}(x)\,dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}.$$

Show that the formulae give the right answer (∞) even when $\alpha = \pm 1$ and evaluate the results for $\alpha = \dfrac{1}{2}$ and $\alpha = \dfrac{1}{3}$.

2.6.19 For all $\alpha < 1$, prove the two equalities

$$\text{(a)} \quad \int_0^{\pi} |\sec(x)|^{\alpha}\,dx = 2\int_0^{\pi/2} \sec^{\alpha}(x)\,dx = \sqrt{\pi}\cdot\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)},$$

$$\text{(b)} \quad \int_0^{\pi} |\csc(x)|^{\alpha}\,dx = 2\int_0^{\pi/2} \csc^{\alpha}(x)\,dx = \sqrt{\pi}\cdot\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}.$$

Show that the formulae give the right answer (∞) even when $\alpha = 1$ and evaluate the results for $\alpha = \dfrac{1}{2}$ and $\alpha = \dfrac{1}{3}$.

2.6.20 Using the properties of the Beta and Gamma functions and $v^2 = \tan(x)$, prove [see also **Examples 2.2.6 (b)** and **2.4.1**]

$$\int_0^{\infty} \frac{dv}{1+v^4} = \frac{\pi\sqrt{2}}{4}.$$

2.6.21 Use **property (Γ, 8) of the Gamma function** and **property (B, 7) of the Beta function** to prove the **recursive properties, (B, 3) and (B, 4), of the Beta function**.

2.6.22 Compute

$$\int_0^{\infty} \frac{e^{-t}}{t^{\frac{5}{4}}}\,dt \quad \text{and} \quad \int_0^{\infty} \frac{e^{-8t}}{(5t)^{\frac{1}{4}}}\,dt.$$

2.6.23 Use $x = b\tan(\theta)$ to prove that if $b > 0$, then

$$\int_0^{b} \frac{dx}{\sqrt{b^4 - x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{2\pi}}.$$

2.6.24 Use $x^2 = b^2\tan(\theta)$ to prove that if $b > 0$, then

$$\int_0^{\infty} \frac{dx}{\sqrt{b^4 + x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{\pi}}.$$

2.6.25 (a) Using the Beta and Gamma functions, prove the following result which we frequently see in many calculus books and integral tables.

For the integers $p = 0,\ 1,\ 2,\ 3,\ \ldots$, we have:

$$\int_0^{\pi/2} \sin^p(\phi)\, d\phi = \int_0^{\pi/2} \cos^p(\phi)\, d\phi =$$

$$\begin{cases} \dfrac{\pi}{2}, & \text{if } p = 0 \\[2ex] 1, & \text{if } p = 1 \\ \dfrac{1 \cdot 3 \cdot 5 \ldots (p-1)}{2 \cdot 4 \cdot 6 \ldots p} \cdot \dfrac{\pi}{2} = \dfrac{p\,!}{2^p \left[\left(\frac{p}{2}\right)!\right]^2} \cdot \dfrac{\pi}{2} = \dfrac{\binom{p}{\frac{p}{2}}}{2^p} \cdot \dfrac{\pi}{2}, & \text{if } p = \text{even int.} \geq 2 \\[2ex] \dfrac{2 \cdot 4 \cdot 6 \ldots (p-1)}{1 \cdot 3 \cdot 5 \ldots p} = \frac{2^{p-1}\left[\left(\frac{p-1}{2}\right)!\right]^2}{p\,!} = \frac{2^{p-1}}{p\binom{p-1}{\frac{p-1}{2}}}, & \text{if } p = \text{ odd integer} \geq 3. \end{cases}$$

(b) Prove this result by means of calculus and induction.

2.6.26 Prove that for $p > 0$, $q > 0$ and $t \in \mathbb{R}$

$$\int_0^1 e^{xt} x^{p-1} (1-x)^{q-1} dx = \sum_{n=0}^{\infty} \frac{\Gamma(n+p)\,\Gamma(q)}{n!\,\Gamma(n+p+q)}\, t^n.$$

2.6.27 (a) Prove that for $a > 0$, $p > 0$ and $q > 0$

$$\int_0^{\frac{1}{a}} x^{p-1} \left(1 - ax\right)^{q-1} dx = \frac{1}{a^p}\, B(p,q).$$

(b) What happens when $0 < a \longrightarrow 0^+$ or $0 < a \longrightarrow +\infty$?

2.6.28 By letting $x = e^{-t}$, prove that for all $p > 0$ and $q > 0$

$$B(p,q) = \int_0^{\infty} e^{-p\,t} \left(1 - e^{-t}\right)^{q-1} dt$$

and show $\displaystyle\int_0^{\infty} e^{-3t} \left(1 - e^{-t}\right)^5 dt = \frac{1}{168}$.

2.6.29 By letting $x = u^2$, prove that for all $p > 0$ and $q > 0$

$$B(p,q) = 2 \int_0^1 u^{2p-1} \left(1 - u^2\right)^{q-1} du$$

and show $\displaystyle\int_0^1 u^9 \left(1 - u^2\right)^8 du = \frac{1}{12870}$.

2.6.30 If $b > a$, $p > -1$ and $q > -1$, use $u = \dfrac{1}{b-a}(x-a)$ to prove

$$\int_a^b (x-a)^p (b-x)^q dx = (b-a)^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.$$

2.6.31 (a) Prove that for all a and b real, such that $a > b > 0$,

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(x)]^{2a}}\, dx = 4^{a-1}\, B(a+b,\, a-b) = 4^{a-1}\, \frac{\Gamma(a+b)\,\Gamma(a-b)}{\Gamma(2a)}.$$

[Hint: Change to exponentials, simplify, use $u = e^{-2t}$ and **(B, 5, IV)**.]

(b) So, for any $a > 0$

$$\int_0^\infty \frac{1}{[\cosh(x)]^{2a}}\, dx = 4^{a-1}\, \frac{[\Gamma(a)]^2}{\Gamma(2a)}.$$

(c) For $n = 1,\ 2,\ 3,\ 4,\ 5,\ 6$, use **(b)** to find the values of

$$\int_0^\infty \frac{1}{[\cosh(x)]^n}\, dx.$$

(d) Make the substitution $y = rx$ and use **(a)** to obtain the general integral for $r > 0$, $b > 0$ and $a > \dfrac{b}{r}$

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(rx)]^{2a}}\, dx =$$

$$\frac{4^{a-1}}{r}\, B\left(a + \frac{b}{r},\, a - \frac{b}{r}\right) = \frac{4^{a-1}}{r}\, \frac{\Gamma\left(a + \frac{b}{r}\right)\, \Gamma\left(a - \frac{b}{r}\right)}{\Gamma(2a)}.$$

2.6.32 Prove that for all a and b real, such that $a > b > -1$,

$$\int_0^\infty \frac{\sinh^b(x)}{\cosh^a(x)}\, dx = \frac{1}{2} B\left(\frac{b+1}{2},\, \frac{a-b}{2}\right) = \frac{1}{2}\, \frac{\Gamma\left(\frac{b+1}{2}\right)\, \Gamma\left(\frac{a-b}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}.$$

2.6.33 (a) Verify **the triplication formula of the Gamma function,**[26]

$$\forall\ p > 0,\ \Gamma(3p) = \frac{3^{3p-1/2}}{2\pi}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{3}\right)\,\Gamma\left(p+\frac{2}{3}\right).$$

[See and compare with **duplication property (B, 10) of the Gamma function** and **Problem 2.6.42, Item (11.)**.]

(b) Use this formula and **(B, 8)** to prove

$$\lim_{p\to 0^+}\frac{\Gamma(3p)}{\Gamma(p)} = \frac{1}{3}.$$

[See and compare with **Problem 2.6.42, Item (12.)**.]

2.6.34 For any $n = 1,\ 2,\ 3,\ \dots$, develop the integral in the equality

$$\pi^{\frac{n}{2}} = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^n$$

as a multiple integral over $\mathbb{R}^n$ in the spherical coordinates that we have used in **Application 2** [in a way analogous to the way of obtaining the **Integral (2.1)** in **Section 2.1**] to obtain that

$$\pi^{\frac{n}{2}} = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\int_0^{\pi}\dots\int_0^{\pi}\int_0^{2\pi}\sin^{n-2}(\phi_1)\dots\sin(\phi_{n-2})d\theta d\phi_1\dots d\phi_{n-2},$$

and so the **area of the** $(n-1)$**-dimensional unit sphere** is

$$\text{Area}\,[S_n(1)] = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

2.6.35 (a) See **Application 2** and prove that both series

$$\sum_{n=0}^{\infty}\text{Volume}\,[B_n(\overrightarrow{c},R)] \quad \text{and} \quad \sum_{n=0}^{\infty}\text{Area}\,[S_n(\overrightarrow{c},R)] \quad \text{converge.}$$

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[26] The duplication and triplication formulae are subcases of the **Gauß multiplication formula of the Gamma function.** This states: $\forall\ p > 0$ and $\forall\ n \in \mathbb{N}$,

$$\Gamma(np) = \frac{n^{np-1/2}}{(2\pi)^{\frac{n-1}{2}}}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{n}\right)\Gamma\left(p+\frac{2}{n}\right)\dots\Gamma\left(p+\frac{n-1}{n}\right).$$

For a proof, see **Problem 2.6.42, Item (11.)**.

[Hint: Use the **Ratio Test, 1.3.5,** and **Problem 2.6.2, (g)** or **property (B, 12)** or **Problem 2.6.42, Item (6.)** or **Problem 2.6.43, (a)**.]

(b) Prove that the sum of the areas of the $(2k-1)$-dimensional spheres of radius R in $\mathbb{R}^{2k}$ is

$$\sum_{k=1}^{\infty} \text{Area}\left[S_{2k}\left(\overrightarrow{c}, R\right)\right] = 2\pi R e^{\pi R^2}$$

and the sum of the volumes of the even (≥ 2) dimensional balls of radius R is

$$\sum_{k=1}^{\infty} \text{Volume}\left[B_{2k}\left(\overrightarrow{c}, R\right)\right] = e^{\pi R^2} - 1.$$

(c) Prove that the sum of the areas of the $(2k)$-dimensional spheres of radius R in $\mathbb{R}^{2k+1}$ is

$$\sum_{k=0}^{\infty} \text{Area}\left[S_{2k+1}\left(\overrightarrow{c}, R\right)\right] = 2\sum_{k=0}^{\infty} k!\, \frac{\left(4\pi R^2\right)^k}{(2k)!}$$

and the sum of the volumes of the odd (≥ 1) dimensional balls of radius R is

$$\sum_{k=0}^{\infty} \text{Volume}\left[B_{2k+1}\left(\overrightarrow{c}, R\right)\right] = 2R\sum_{k=0}^{\infty} \frac{k!}{2k+1}\, \frac{\left(4\pi R^2\right)^k}{(2k)!}.$$

2.6.36 For $n \geq 2$ integer the n**-dimensional right circular cone** in $\mathbb{R}^n$, with base radius $R > 0$, height $h > 0$ and axis of symmetry the x_1-non-negative axis, is the set

$$C_n = \left\{ \overrightarrow{x} = (x_1,\, x_2,\, \ldots, x_n) \quad | \quad \frac{h}{R}\sqrt{x_2^2 + x_3^2 + \ldots + x_n^2} \leq x_1 \leq 1 \right\}.$$

Prove that

$$\text{Volume}\left[C_n\right] = \frac{2hR^{n-1}\pi^{\frac{n-1}{2}}}{n(n-1)\Gamma\left(\frac{n-1}{2}\right)}.$$

(When $n = 2$ the respective, cone is an isosceles triangle with base $2R$ and height h, and this formula gives its area.)

2.6.37 (a) Prove that for any $n \geq 2$ integer

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \prod_{k=1}^{n} \Gamma\left(\frac{k}{n}\right) =$$

$$\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{n-1}{n}\right) \Gamma\left(\frac{n}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

(See also **Problem 2.6.42, Item (11.)**.)

[Hint: Use the result in **Problem 3.2.26 (c)** and **(B, 8)**.]

(b) Prove

$$\int_0^1 \ln[\Gamma(x)]dx = \frac{1}{2}\ln(2\pi) = \ln\left(\sqrt{2\pi}\right).$$

[Hint: Take the logarithm of both sides in **(a)**, form a Riemann sum and take its limit.]

(c) Use **property (Γ, 8)** to prove that for $p \geq 1$

$$\int_p^{p+1} \ln[\Gamma(x)]dx = p\ln(p) - (p-1)\ln(p-1) - 1 + \int_{p-1}^{p} \ln[\Gamma(x)]dx.$$

(d) For $p \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots,\ \}$, prove

$$\int_p^{p+1} \ln[\Gamma(x)]dx = p\ln(p) - p + \ln\left(\sqrt{2\pi}\right),$$

which is also true for $p = 0^+$, by **(b)**.

(e) Use **(d)** to prove

$$\int_1^2 \ln[\Gamma(x)]\,dx = -1 + \ln\left(\sqrt{2\pi}\right) = -0.0810615\ldots$$

and also to compute $\displaystyle\int_2^3 \ln[\Gamma(x)]\,dx$.

(f) For $n = 1,\ 2,\ 3, \ldots$, find a formula for

$$\int_{0^+}^{n} \ln[\Gamma(x)]\,dx.$$

(See also **Problem 2.6.42, Item (13.)**.)

2.6.38 (See also **Problem 3.7.14**.) Use the results of **Example 2.6.7** and **property (B, 8)** to prove that for any $0 < m < 1$

$$\text{(a)} \quad \int_0^\infty \frac{\cos(x)}{x^{1-m}}dx = \Gamma(m)\cos\left(\frac{m\pi}{2}\right),$$

$$\text{(b)} \quad \int_0^\infty \frac{\sin(x)}{x^{1-m}}dx = \Gamma(m)\sin\left(\frac{m\pi}{2}\right).$$

2.6.39 The **t-Student**[27] **statistical sampling distribution with ν degrees of freedom** where ν is a positive integer has probability density given by

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{\frac{-(\nu+1)}{2}}, \quad \text{for } -\infty < t < \infty.$$

Prove the following:

(a) $f(t)$ is a positive even function and

$$\int_{-\infty}^{\infty} f(t)\,dt = 1.$$

(b) For $1 \le n \le \nu - 1$ odd integer

$$\int_{-\infty}^{\infty} t^n f(t)\,dt = 0.$$

(c) For $n \ge \nu$ odd integer

$$\text{P.V.}\int_{-\infty}^{\infty} t^n f(t)\,dt = 0.$$

(d) For $n \ge \nu$ odd integer

$$\int_{-\infty}^{\infty} t^n f(t)\,dt = \text{does not exist.}$$

(e) For $n \ge \nu$ even integer

$$\int_{-\infty}^{\infty} t^n f(t)\,dt = \infty.$$

(f) For $2 \le n \le \nu - 1$ even integer, the integral

$$\int_{-\infty}^{\infty} t^n f(t)\,dt$$

exists. Then, evaluate its value.

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[27] Student is the pseudonym of William Sealy Gosset, English statistician who discovered the very important t-Student distribution for small statistical samples, 1876-1937.

[Hint: Making appropriate u-substitutions may be necessary in some of the above questions.]

2.6.40 The F-statistical sampling distribution[28] **with ν_1 and ν_2 degrees of freedom (or parameters)** where ν_1 and ν_2 are positive integers has probability density given by

$$g(f) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\,\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} f^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1}{\nu_2} f\right)^{\frac{-(\nu_1+\nu_2)}{2}}, \quad \text{for } 0 < f < \infty,$$

and

$$g(f) = 0 \quad \text{for } -\infty < f \leq 0.$$

(Notice here that the letter f is used as the variable of the function g.)

(a) Prove $g(f)$ is non-negative and

$$\int_{-\infty}^{\infty} g(f)\,df = 1.$$

(b) For $\nu_2 > 2$ integer, prove

$$\int_{-\infty}^{\infty} f g(f)\,df = \frac{\nu_2}{\nu_2 - 2}.$$

(c) Find the positive integers ν_2 for which the integral

$$\int_{-\infty}^{\infty} f^2 g(f)\,df$$

exists. Then, evaluate it.

[Hint: The transformation $u = \left(1 + \frac{\nu_1}{\nu_2} f\right)^{-1}$ may be useful.]

2.6.41 In a second-semester calculus course, we see **Newton's**[29] **binomial series**, which is very useful to applications. That is: For any real number p as the exponent and for all $-1 < x < 1$, we have the power series expansion

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p\,(p-1)\ldots(p-n+1)}{n!}\, x^n.$$

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[28] The F-distribution is very important for small statistical samples. It was discovered independently by Sir Ronald Aylmer Fisher, famous English geneticist and statistician, 1890-1962, and George Waddel Snedecor, American statistician, 1881-1974.

[29] Sir Isaac Newton, English mathematician and physicist, one of the greatest mathematicians and scientists of all time, 1643-1727.

(Review this very important power series one more time from a good calculus book.)

(a) Prove that this expansion agrees with the **Binomial Theorem** when $p \geq 0$ integer.

(b) Prove that if $p < 0$ real and $-1 < x < 1$, then we can also write

$$(1-x)^p = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n-p)}{n!\,\Gamma(-p)}\, x^n.$$

Why in this formula have we considered negative real exponents only and not every real exponent?

2.6.42 Project on Gamma function and some inequalities for which you may need to consult the bibliography.

Let $f : A \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}$ be piecewise continuous integrable functions over an appropriate set $A \subseteq \mathbb{R}$.

When we study the integrals of such functions, in a more complete theory, for every number $1 \leq s < \infty$, we define:

$$\|f\|_s := \left[\int_A |f(x)|^s dx\right]^{\frac{1}{s}}.$$

For the special case $s = \infty$, we define

$$\|f\|_\infty := \underset{x \in A}{\text{Maximum}} |f(x)|.$$

(Similar definitions are for g. These definitions can be generalized in the larger class of measurable functions, in a more general setting.)

1. Prove that for any two real numbers and/or infinity s and t such that $1 \leq s,\ t \leq \infty$ and $\dfrac{1}{s} + \dfrac{1}{t} = 1$, we have the inequality:

$$\int_A |f(x) \cdot g(x)|\, dx = \int_A |f(x)| \cdot |g(x)|\, dx \leq \left[\int_A |f(x)|^s dx\right]^{\frac{1}{s}} \cdot \left[\int_A |g(x)|^t dx\right]^{\frac{1}{t}}$$

or using the above notation

$$\|f \cdot g\|_1 \leq \|f\|_s \cdot \|g\|_t. \tag{2.11}$$

This holds as **equality** if and only if there are real numbers α and β such that $|\alpha| + |\beta| > 0$ (i.e., not both α and β are 0) and $\alpha |f|^s = \beta |g|^{\,t}$ (and so $\alpha \cdot \beta \geq 0$, that is, α and β have the same sign).

This inequality is usually called **Hölder's[30] inequality** in the

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[30] Otto Ludwig Hölder, German mathematician, 1859-1937.

mathematical literature. However, it was also discovered independently by Viktor Yakovlevich Bunyakovsky[31] a few years earlier. Therefore, many times it is called **Bunyakovsky-Hölder's inequality**.

The exponents s and t as above are called **conjugate exponents**. Notice that $s = 1$ is conjugate to $t = \infty$ and vice-versa. In this case, the proof of the inequality is easier than in the other cases. Also, the case of equal conjugate exponents $s = t = 2$ is a very important one in theory and applications. In this case, the inequality is also called **Cauchy-Schwarz[32] inequality** and generalizes the homonymous inequality in Linear Algebra with inner products.

We can use the **inequality 2.11** for establishing absolute convergence of integrals. For, if we prove that the right side is finite ($\|f\|_s \cdot \|g\|_t < \infty$), then we obtain the absolute convergence (and therefore the convergence) of the integral in the left side, $\int_A |f(x) \cdot g(x)|\, dx = \int_A |f(x)| \cdot |g(x)|\, dx < \infty$.

2. Use Cauchy-Schwarz inequality, **Problem 2.1.15** and the facts that if $x \geq 1$ then $0 \leq \ln(x) \leq x - 1$ and if $0 \leq x \leq 1$ then $\frac{1}{e} \leq e^{-x} \leq 1$ to prove that for any $p > \frac{1}{2}$ and any $n = 1, 2, 3, ...$ integer, we have

$$\left|\Gamma^{(n)}(p)\right| < \int_0^\infty \left|x^{p-1} e^{-x} \left[\ln(x)\right]^n\right| dx =$$
$$\int_0^\infty \left|x^{p-1} e^{\frac{-x}{2}}\right| \cdot \left|e^{\frac{-x}{2}} \left[\ln(x)\right]^n\right| dx < \ldots <$$
$$\left[\Gamma(2p-1)\right]^{1/2} \left[(2n)! + \frac{\Gamma(2n+1)}{e}\right]^{1/2}.$$

So, here we prove that: For $p > \frac{1}{2}$ the integral that represents the n^{th} derivative of the Gamma function, for all integers $n \geq 1$, converges absolutely, and so it exists.

3. We have already said the integral that represents the n^{th} derivative of the Gamma function, for all integers $n \geq 1$, exists for all $p > 0$ and not just for $p > \frac{1}{2}$ that we examined in the previous item.

Prove the absolute convergence of this integral for any $0 < p < 1$

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[31] Viktor Yakovlevich Bunyakovsky, Russian mathematician, 1804-1889.

[32] Hermann Amandus Schwarz, German mathematician, 1843-1921.

(or even any $0 < p$) by picking an integer $k \geq 2$ such that $\dfrac{1}{k} < p$, and use Hölder's inequality with conjugate exponents $s = \dfrac{k}{k-1}$ and $t = k$ to derive an inequality similar to the previous one by completing

$$\left|\Gamma^{(n)}(p)\right| < \int_0^\infty \left|x^{p-1}e^{-x}\left[\ln(x)\right]^n\right| dx =$$

$$\int_0^\infty \left|x^{p-1}e^{\frac{-x}{2}}\right| \cdot \left|e^{\frac{-x}{2}}\left[\ln(x)\right]^n\right| dx < \ldots <$$

$$\left[\int_0^\infty \left|x^{\frac{kp-1}{k-1}-1}e^{\frac{-kx}{2(k-1)}}\right| dx\right]^{(k-1)/k} \cdot \left[\int_0^\infty \left|e^{\frac{-kx}{2}}\left[\ln(x)\right]^{nk}\right| dx\right]^{1/k} \ldots$$

and then use **Problems 2.6.5** and **2.1.15**, etc. (The choice of k to be an integer is not necessary. We can find analogous upper bounds for any $k > 0$ such that $kp > 1$.)

4. Use the above Cauchy-Schwarz inequality appropriately to prove that for any $n = 1,\ 2,\ 3, \ldots$ integer, the derivatives of the Gamma function satisfy the strict inequality

$$\left[\Gamma^{\,(n)}(p)\right]^2 < \Gamma(p) \cdot \Gamma^{\,(2n)}(p).$$

5. Use the previous result to prove that the $\ln\left[\Gamma(p)\right]$ is strictly convex by showing that its second derivative is positive. I.e., prove:

$$\left[\ln\left[\Gamma(p)\right]\right]'' = \frac{\Gamma''(p) \cdot \Gamma(p) - \left[\Gamma'(p)\right]^2}{\Gamma(p)^2} > 0.$$

   [So, here we prove that not only the Gamma function itself is convex, as we have seen in **($\boldsymbol{\Gamma}$, 4)**, but also the logarithm of the Gamma function is also convex.]

6. Prove that for any $p \in \mathbb{R}$ fixed

$$\lim_{0 < q \to \infty} \frac{\Gamma(p+q)}{\Gamma(q) \cdot q^p} = 1.$$

   [See also **property (B, 12)** and compare with **Problem 2.6.43, (a)**.]

   [Hint: Either appeal to **property (B, 12)** or provide a different proof as follows:

First, prove this directly when $p \geq 0$ integer, using **property (Γ, 8)**. Then, prove that the given fraction is eventually (i.e., for large enough q's) increasing in p, when $p > 0$. (You may do this directly or prove that the derivative of the fraction with respect to p is eventually positive.) Then, for any non-integer $0 < p$, if $[\![p]\!]$ is the integer part of p, we have that $[\![p]\!] < p < [\![p]\!] + 1$ and use the **Squeeze Lemma** from calculus.

Finally, reduce the case $p < 0$ to the case $p > 0$.]

Then prove that for any $p > 0$ fixed

$$\lim_{0 < q \to \infty} \frac{\Gamma(q)}{\Gamma(q+p)} = 0^+ \quad \text{or} \quad \lim_{0 < q \to \infty} \frac{\Gamma(q+p)}{\Gamma(q)} = +\infty$$

and

$$\lim_{0 < q \to \infty} q^p \cdot B(p,q) = \Gamma(p).$$

[See also and compare with **properties (B, 11)** and **(B, 12)**.]

7. Prove that for $p > 0$ and $n \in \mathbb{N}$

$$\Gamma(p) = \lim_{n \to \infty} \frac{n!\, n^p}{p\,(p+1)(p+2)\dots(p+n)}.$$

[Hint: You may use the previous result!]

8. **A characterization of the Gamma function.**

Let $f : (0,\infty) \longrightarrow (0,\infty)$ be a function with the following three properties:

(a) $f(1) = 1$, (b) $f(p+1) = p\,f(p)$ and (c) $\ln[f(p)]$ is convex.

Prove that $f(p) = \Gamma(p)$. (I.e., these properties characterize the Gamma function.)

[Hint: Prove that such a function $f(p)$ is equal to

$$\lim_{n \to \infty} \frac{n!\, n^p}{p\,(p+1)(p+2)\dots(p+n)}$$

and therefore is unique. Then it is equal to the Gamma function, since the Gamma function satisfies these properties!]

9. Show that there is no function $f : [0,\infty) \longrightarrow \mathbb{R}$ (i.e., f is defined at 0), satisfying: (a) $f(1) = 1$ and (b) $f(p+1) = p\,f(p)$.

10. Given $n \in \mathbb{N}$, for all $q > 0$, define the function

$$g(q) = \frac{n^{q-1/2}}{(2\pi)^{\frac{n-1}{2}}}\,\Gamma\left(\frac{q}{n}\right)\Gamma\left(\frac{q+1}{n}\right)\Gamma\left(\frac{q+2}{n}\right)\ldots\Gamma\left(\frac{q+n-1}{n}\right).$$

Prove $g(q) = \Gamma(q)$.

[Hint: Show that $g(q)$ satisfies the three characteristic **conditions (a), (b)** and **(c)** of the Gamma function, stated in **(8)** before. You also need the result of **Problem 2.6.37, (a)**.]

11. Use the result in **(10)** to give a proof to the **Gauß multiplication formula of the Gamma function.**

$\forall\ n \in \mathbb{N}$ and $\forall\ p > 0$, the Gamma function satisfies

$$\Gamma(np) = \frac{n^{np-1/2}}{(2\pi)^{\frac{n-1}{2}}}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{n}\right)\Gamma\left(p+\frac{2}{n}\right)\ldots\Gamma\left(p+\frac{n-1}{n}\right).$$

[Notice: With $p = \frac{1}{n}$, we get back the result in **Problem 2.6.37 (a)**. See also and compare with **(B, 10)** and **Problem 2.6.33**.]

12. Prove:

$$\text{(a)}\quad \lim_{n\to\infty}\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1+\frac{2}{n}\right)\ldots\Gamma\left(1+\frac{n-1}{n}\right) = 0.$$

[Hint: Use the result in **(10)** before.]

$$\text{(b)}\quad \forall\ n \in \mathbb{N},\quad \lim_{p\to 0^+}\frac{\Gamma(np)}{\Gamma(p)} = \frac{1}{n}.$$

[Hint: Use the result in **(10)** before and **Problem 2.6.37 (a)**.]

$$\text{(c)}\quad \forall\ \alpha > 0,\quad \lim_{p\to 0^+}\frac{\Gamma(\alpha p)}{\Gamma(p)} = \frac{1}{\alpha}.$$

[Hint: Using **(b)**, prove this for $\alpha = \frac{k}{l} > 0$ rational, with $k \in \mathbb{N}$ and $l \in \mathbb{N}$, and then use the density of the rationals in the reals and the continuity of the Gamma function.]

13. Prove that $\forall\ p > 0$

$$\lim_{n\to\infty}\left\{\frac{1}{n}\ln\left[\frac{\Gamma(np)}{n^{np-\frac{1}{2}}}\right]\right\} = -\ln\left(\sqrt{2\pi}\right) + \int_p^{p+1}\ln[\Gamma(x)]\,dx.$$

(In relation to this, see **Problem 2.6.37**.)

14. In a common real analysis course, the material concerning **Hölder's inequality** is usually combined with the following:

    For any $1 \leq s \leq \infty$. prove that the real number $\|f\|_s$, defined above, satisfies the following four properties of what we call to be the properties that define a **norm function**:

    (a) $\|f\|_s \geq 0$ (**positivity**)

    (b) $\|f\|_s = 0 \Longleftrightarrow f \equiv 0$ (**non-degeneracy**)

    (c) $\forall\ c \in \mathbb{R},\ \ \|c \cdot f\|_s = |c| \cdot \|f\|_s$ (**semi-linearity**)

    (d) $\|f + g\|_s \leq \|f\|_s + \|g\|_s$ (**triangle inequality**).

    The real number $\|f\|_s$ is called the **$\mathcal{L}_s$-norm** of the function f.

    In this context, **the triangle inequality** [in (d)] is also called **Minkowski's**[33] **inequality**. This holds as **equality** if and only if there are real numbers α and β such that $\alpha \cdot \beta \geq 0$ (i.e., the two numbers have the same sign), $|\alpha| + |\beta| > 0$ (i.e., not both α and β are 0) and $\alpha f = \beta g$.

    When $s = 1$ or $s = \infty$, the proof of Minkowski's inequality is easy. (Write the inequalities explicitly and argue about their proofs.)

    To prove Minkowski's inequality when $1 < s < \infty$, we need Hölder's inequality.

    We can use Minkowski's inequality for establishing absolute convergence of integrals. For, if we prove that the right side is finite, then we obtain the convergence (finiteness) of the integral in the left side. For example, prove:

$$\int_0^1 \left(\frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[5]{x}} \right)^{3/2} dx < \left[2^{2/3} + \left(\frac{10}{7} \right)^{2/3} \right]^{3/2}.$$

2.6.43 Project on Stirling's Formula. Find and consult appropriate bibliography and provide a detailed proof of the general Stirling's formula

$$\lim_{0 < x \to \infty} \frac{\Gamma(x+1)}{\left(\dfrac{x}{e}\right)^x \sqrt{2\,\pi\,x}} \;=\; 1,$$

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[33] Hermann Minkowski, German mathematician, 1864-1909.

or, for $x > 0$

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\,\left(\frac{x}{e}\right)^{x}[1 + R(x)],$$

where $R(x)$ is a remainder that satisfies the asymptotic relation

$$|R(x)| \leq C \cdot \frac{1}{x}, \quad \text{as} \quad x \longrightarrow \infty,$$

for some constant $C > 0$. Use this to prove:

(a) If p is any real constant, then

$$\lim_{0< q\to\infty} \frac{\Gamma(p+q)}{\Gamma(q)\cdot q^p} = 1.$$

[See also **property (B, 12)** and **Problem 2.6.42, Item (6.)**.]

(b) For any $n \in \mathbb{N}$

$$\lim_{0<n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^{n}\sqrt{2\,\pi\,n}} = 1.$$

(c) For $n \in \mathbb{N}$ **large**, we have the following approximate equality:

$$n! \cong \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^{n}.$$

Also, for all $n \geq 1$, we have the inequality

$$1 < \frac{n!}{\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^{n}} \leq \frac{e}{2\pi} = 1.08444\cdots.$$

2.6.44 Project on Derivatives of the Gamma function. Find and consult appropriate bibliography to study the derivative and/or the logarithmic derivative of the Gamma and prove:

$$\text{(a)} \quad \Gamma'(1) = -\gamma,$$

$$\text{(b)} \quad \Gamma'(2) = 1 - \gamma$$

$$\text{(c)} \quad \Gamma''(1) = \frac{\pi^2}{6} + \gamma^2,$$

where $\gamma = \lim\limits_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566... > 0$ is the **Euler-Mascheroni constant**.

Important Note

In the **theory** and the **problems** of the **sections** covered thus far, we have seen that when computing improper integrals, we must justify:

1. Switching limit and integral. See **Theorem 2.2.1 Part I** and **Section 2.3 Part I**, etc.
2. Switching derivative and integral. See **Theorem 2.2.1 Part II** and **Section 2.3 Part II**, etc.
3. Switching the order of iterated integrals in a double integral. See **Section 2.4**, etc.
4. Changing coordinates in a double integral. See **Problem 2.4.5**, etc.

We must keep these points in mind. Otherwise, errors may ensue.

2.7 A Brief Overview of Laplace Transform

In this brief overview, we are not going to develop the theory of Laplace transform and its applications. There are a lot of books developing fully or nearly fully this important subject of mathematics and application. Here, we would like to give students the basic ingredients of this very important subject, so that with the examples, applications and problems provided they could acquire a taste and motivation on this subject, and to enable them to study a specialized book on their own if they want or need to do so.

The Laplace transform is a very powerful tool for solving applied problems, initial value and boundary value problems of ordinary and partial differential equations, etc. Therefore, plenty of good books on this subject and extensive tables containing hundreds of evaluated Laplace transforms of very important and frequently encountered functions have been published. Among all these functions, we see many special ones. These include the **Bessel[34] functions**, **Heaviside[35] step functions**, **shift functions**, **Dirac[36] impulse functions**, various other **special functions**, **convolutions of functions** (a very interesting, elegant and important topic), etc.

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[34] Friedrich Wilhelm Bessel, German astronomer and mathematician, 1784-1846.

[35] Oliver Heaviside, English engineer and mathematician, 1850-1925.

[36] Paul Adrien Maurice Dirac, English physicist and mathematician, 1902-1984.

2.7.1 Laplace Transform

Definition 2.7.1 *Given a nice real function* $y = f(x)$ *defined on* $[0, \infty)$ *or* $(0, \infty)$*, we define its* ***Laplace***[37] ***Transform*** *to be the following improper integral with one parameter* s*:*

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

Let us call D the set of all $s \in \mathbb{R}$ for which this improper integral converges. If $D \neq \emptyset$, then this improper integral defines a real valued function of s on the set D. The parameter s is the independent variable of this newly obtained function, which we write by $\mathcal{L}\{f(x)\}(s)$, and we call it the **Laplace Transform of** $f(x)$. If $D = \emptyset$, then the **Laplace Transform of** $f(x)$ **does not exist**.

If $y = f(x)$ is a nice real function defined on $[a, \infty)$ or (a, ∞), with $0 < a < \infty$, then to find its Laplace transform we use the same definition as above after setting $f(x) = 0$ on $[0, a)$ or $[0, a]$, respectively.

Since the Laplace transforms of functions are improper Riemann integrals with parameters, we can use and apply the theory and the results about improper integrals with parameters along with all the different related techniques when we study and compute Laplace transforms. We now continue with a few definitions, results and problems:

If $A = [a, \infty)$ or $A = (a, \infty)$ where $a \geq 0$ and

$$\int_A |f(x)| dx < \infty,$$

i.e., $f(x)$ is **absolutely integrable** over A, then the Laplace transform of $f(x)$ exists (converges absolutely) $\forall\ s \geq 0$.

This follows immediately from the **Absolute Convergence Test, 1.3.6**, since $\forall\ s \geq 0$ and $\forall\ x \in A$ it holds: $|e^{-sx} f(x)| \leq |f(x)|$. (The **Cauchy Test, 1.3.7**, can also be used to prove this result.) (In this context, look at **Problem 2.7.5** and provide its solution.)

Next, the functions of **exponential order in** $[0, \infty)$ **or** $(0, \infty)$ form a very large class of functions, sufficient for most needs of applications. They are defined as follows:

Definition 2.7.2 *A real function* $y = f(x)$ *on* $[0, \infty)$ *or* $(0, \infty)$ *is defined to be* ***of exponential order u*** *if there are constants* $u \geq 0$*,* $M \geq 0$ *and* $A \geq 0$*, such that*

$$|f(x)| \leq M e^{ux}, \text{ for all } x \in [A, \infty).$$

---

[37] Pierre-Simon Laplace, French mathematician, 1749-1827.

For a function $f(x)$ of exponential order u in $[0, \infty)$ with constants $M \geq 0$ and $A \geq 0$ as in the above definition, we have that

$$\int_0^\infty e^{-sx} f(x)\, dx = \int_0^A e^{-sx} f(x)\, dx + \int_A^\infty e^{-sx} f(x)\, dx.$$

But for all $s > u$, we get that the second integral

$$\int_A^\infty e^{-sx} f(x)\, dx$$

converges absolutely because:

$$\int_A^\infty e^{-sx} |f(x)|\, dx \leq$$

$$\int_A^\infty e^{-sx} M e^{ux}\, dx = M \left[\frac{e^{(u-s)x}}{u-s} \right]_A^\infty = M \frac{e^{(u-s)A}}{s-u} < \infty.$$

Then, we conclude that

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx, \quad \text{exists for all} \quad s > \mu,$$

$$\overset{iff}{\Longleftrightarrow}$$

$$\int_0^A e^{-sx} f(x)\, dx, \quad \text{exists for all} \quad s > v, \quad \text{for some} \quad v > 0,$$

where $\mu > \max\{u, v\}$ constant. (Compare with **Example 2.7.2**.)

Since the interval $[0, A]$ is closed and bounded, we observe that these integrals exist when, for instance, $f(x)$ is continuous, or bounded, or absolutely integrable, etc., on $[0, A]$. (See **Problem 2.7.5**.)

Examples

Example 2.7.1 All bounded functions on $[0, \infty)$ are of exponential order. M is the bound of the $|f(x)|$ on $[0, \infty)$ (so, $A = 0$) and $u = 0$ (the smallest u).

All powers x^n with $n \in \mathbb{N}$, and therefore all polynomials in $[0, \infty)$ are of exponential order. This is so because $x^n = e^{n \ln(x)} \leq e^{-n} e^{nx}$ for all $x \in [0, \infty)$ [since $\ln(x) \leq x - 1$ for all $x > 0$]. Therefore, $M = e^{-n}$ and $u = n$.

Also, all functions $C\, e^{kx}$ on $[0, \infty)$ with C and k any real constants are of exponential order. Take $M = |C|$ and $u = |k|$.

The functions $\exp(x^\alpha)$ for all real $\alpha > 1$ are not of exponential order as it can easily be verified. The same is true for the function $\exp(e^x)$, etc.

▲

Example 2.7.2 The function $f(x) = \dfrac{1}{x}$ with $x \in (0, \infty)$ is of exponential order, since $|f(x)| \leq x$ for all $x \in [1, \infty)$. (So, here: $M = 1$, $u = 1$ and $A = 1$.)

But,

$$\mathcal{L}\left\{\frac{1}{x}\right\}(s) = \int_0^\infty e^{-sx}\frac{1}{x}\,dx$$

does not exist, since we easily see that

$$\forall \;\; s \in \mathbb{R}, \qquad \int_0^1 e^{-sx}\frac{1}{x}\,dx = \infty.$$

If we now choose a constant $B > 0$ (e.g., $B = 1$) and define

$$g(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < B \\ \\ \dfrac{1}{x}, & \text{if} \quad B \leq x < \infty, \end{cases}$$

then

$$\mathcal{L}\{g(x)\}(s) = \int_0^\infty e^{-sx} g(x)\,dx = \int_B^\infty e^{-sx}\frac{1}{x}\,dx$$

exists (in fact, converges absolutely) for all $s > 0$.

▲

The Laplace transform is a **linear operator** in the set of functions for which it exists. That is, for any nice real functions $f(x)$ and $h(x)$ on $[0, \infty)$ whose Laplace transforms exist and for any constant $c \in \mathbb{R}$, the Laplace transform of the functions $(f + h)(x) = f(x) + h(x)$ and $(cf)(x) = cf(x)$ exists and satisfies the two **linearity properties**:

(a) $\mathcal{L}\{(f + h)(x)\}(s) = \mathcal{L}\{f(x) + h(x)\}(s) = \mathcal{L}\{f(x)\}(s) + \mathcal{L}\{h(x)\}(s)$,

(b) $\mathcal{L}\{(cf)(x)\}(s) = \mathcal{L}\{cf(x)\}(s) = c\,\mathcal{L}\{f(x)\}(s)$.

It is rather obvious that if $f : [0, \infty) \longrightarrow \mathbb{R}$ is a nice function and its Laplace transform $\mathcal{L}\{f(x)\}(s) = \displaystyle\int_0^\infty e^{-sx} f(x)\,dx$ exists, then it is unique.

With higher mathematics, we can prove that the Laplace transform is a **one-to-one (injective) operator** in the set of continuous functions for which it exists. That is, if $f(x)$ and $g(x)$ are two real continuous functions on $[0, \infty)$, their Laplace transforms exist and $f \neq g$, then $\mathcal{L}\{f(x)\} \neq \mathcal{L}\{g(x)\}$. Or else, if $f(x)$ and $g(x)$ are two real continuous functions on $[0, \infty)$, their Laplace transforms exist and $\mathcal{L}\{f(x)\} = \mathcal{L}\{g(x)\}$, then $f = g$.

Examples

In the problems that follow, we study some additional properties of the Laplace transform and evaluate a good number of basic and advanced Laplace transforms. (Have a look at the problems and read them at least.) In the examples that follow, we see the switching of order of a double integration, the use of power series, the convolutions and the Dirac delta functions.

Example 2.7.3 Prove

$$\mathcal{L}\{\operatorname{erf}(x)\}(s) = \frac{e^{\frac{s^2}{4}}}{s} \operatorname{erfc}\left(\frac{s}{2}\right).$$

By the **definition of the error function**, relation **(2.2)**, we have

$$\mathcal{L}\{\operatorname{erf}(x)\}(s) = \int_0^\infty e^{-sx} \left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du\right) dx.$$

Since all functions involved are positive, we can switch the order of integration (**condition I** in **Section 2.4** and refer to the provided **Figure, 2.8,**) to get

$$\mathcal{L}\{\operatorname{erf}(x)\}(s) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left(\int_u^\infty e^{-sx} dx\right) du =$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \frac{e^{-su}}{s} du = \frac{2}{\sqrt{\pi} s} \int_0^\infty e^{-u^2+su} du =$$

$$\frac{2}{\sqrt{\pi} s} \int_0^\infty e^{-(u+\frac{s}{2})^2+\frac{s^2}{4}} du = \frac{2e^{\frac{s^2}{4}}}{\sqrt{\pi} s} \int_0^\infty e^{-(u+\frac{s}{2})^2} du =$$

$$\frac{2e^{\frac{s^2}{4}}}{\sqrt{\pi} s} \int_{\frac{s}{2}}^\infty e^{-v^2} dv = \frac{e^{\frac{s^2}{4}}}{s} \operatorname{erfc}\left(\frac{s}{2}\right).$$

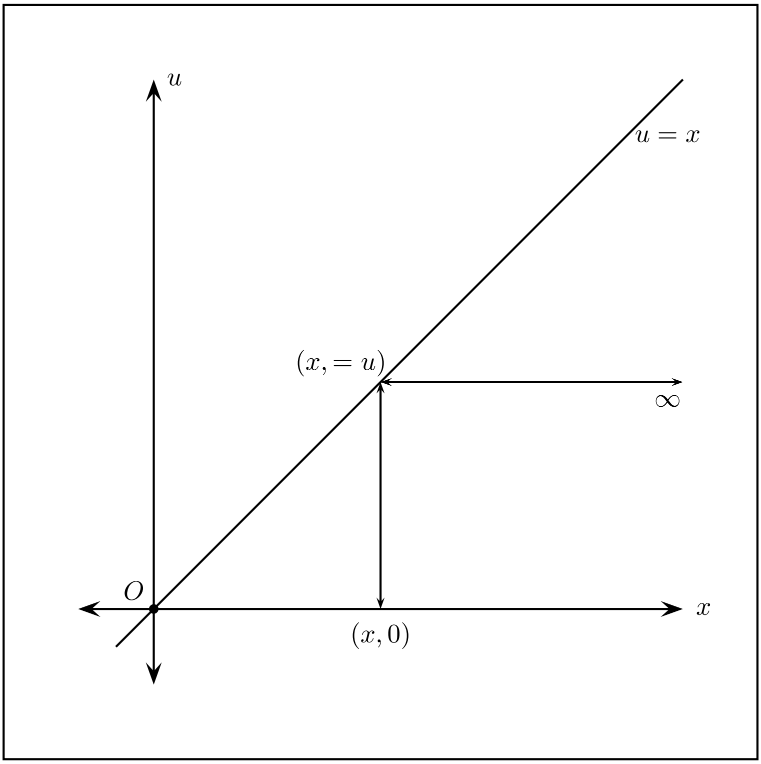

FIGURE 2.8: For switching integration in Example 2.7.3

In general for $a > 0$ constant, by the **Rule (10.)** of the **table in Problem 2.7.18**, we have

$$\mathcal{L}\{\mathrm{erf}(ax)\}(s) = \frac{e^{\frac{s^2}{4a^2}}}{as}\,\mathrm{erfc}\left(\frac{s}{2a}\right).$$

For example,

$$\mathcal{L}\{\mathrm{erf}(2x)\}(s) = \frac{e^{\frac{s^2}{16}}}{2s}\,\mathrm{erfc}\left(\frac{s}{4}\right).$$

Since the **complementary error function** is $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$, using the **linearity property** of the Laplace transform and **Rule (1.)** of the **table in Problem 2.7.12**, we find that its Laplace transform is

$$\mathcal{L}\{\mathrm{erfc}(ax)\}(s) = \frac{1}{s} - \frac{e^{\frac{s^2}{4a^2}}}{as}\,\mathrm{erfc}\left(\frac{s}{2a}\right),$$

where $a > 0$ constant and $s > 0$ is the variable.

▲

Example 2.7.4 Assume that

$$F(s) := \mathcal{L}\{f(x)\}(s) \quad \text{and} \quad G(s) := \mathcal{L}\{g(x)\}(s)$$

are the Laplace transforms of two real functions $f(x)$ and $g(x)$ on $[0, \infty)$, for $s > (\geq) a \geq 0$, where a is constant.

Then, we multiply to get

$$F(s)G(s) = \left\{ \int_0^\infty e^{-su} f(u) du \right\} \cdot \left\{ \int_0^\infty e^{-sv} g(v) dv \right\} =$$

$$= \int_0^\infty g(v) dv \int_0^\infty e^{-s(u+v)} f(u) du.$$

Now we let $u + v = t$ to get

$$F(s)G(s) = \int_0^\infty g(v) dv \int_v^\infty e^{-st} f(t - v) dt.$$

We switch the order of integration (use **Figure 2.8** with analogous labels) and obtain

$$F(s)G(s) = \int_0^\infty e^{-st} \left[\int_0^t f(t - v) g(v) dv \right] dt.$$

We denote the particular integral that has appeared here, by

$$(f * g)(t) := \int_0^t f(t - v) g(v) dv$$

and call it **convolution** of the functions f and g (in the Laplace transform context).

Hence, we have proved the rule that

$$\mathcal{L}\{(f * g)(x)\}(s) = \mathcal{L}\{f(x)\}(s) \cdot \mathcal{L}\{g(x)\}(s),$$

i.e., **the Laplace transform of the convolution of two functions (as defined above) is the product of their Laplace transforms**.

Application: If $f \equiv 1$, we get

$$\mathcal{L}\left\{ \int_0^t g(x) dx \right\}(s) = \mathcal{L}\left\{ \int_0^t 1 g(x) dx \right\}(s) = \mathcal{L}\left\{ \int_0^t 1 * g(x) dx \right\}(s) =$$

$$\mathcal{L}\{1\}(s) \cdot \mathcal{L}\{g(x)(s)\} = \frac{1}{s} \cdot \mathcal{L}\{g(x)\}(s),$$

which is **Rule (6.)** of the **table in Problem 2.7.18**.

▲

Example 2.7.5 Some properties of the convolution are the following:

(a) It is **commutative**, i.e., $f * g = g * f$. This is immediately obtained by the change of variables $t - v = u$ to get

$$\int_0^t f(t-v)g(v)dv = \int_0^t f(u)g(t-u)du.$$

(b) It is straightforward that $a(f*g) = (af)*g = f*(ag)$ for any real constant a. In particular, $0 * g = 0 = f * 0$.

(c) It is **linear** with respect to each function position. I.e., for any real constants a and b, we have:

$$(af_1 + bf_2) * g = a(f_1 * g) + b(f_2 * g) \quad \text{and}$$
$$f * (ag_1 + bg_2) = a(f * g_1) + b(f * g_2).$$

The proof is immediate from the definition.

(d) The convolution is **associative**, i.e., $(f*g)*h = f*(g*h)$. The proof goes as follows:

$$[(f*g)*h](t) = \int_0^t \left[\int_0^{t-r} f(t-r-s)g(s)ds\right] h(r)dr.$$

We let $s + r = w$ and we have

$$[(f*g)*h](t) = \int_0^t \left[\int_r^t f(t-w)g(w-r)dw\right] h(r)dr.$$

We switch the order of integration, and we obtain associativity

$$[(f*g)*h](t) = \int_0^t f(t-w)\left[\int_0^w g(w-r)h(r)dr\right] dw = [f*(g*h)](t).$$

(e) Notice that in general

$$(f*1)(x) = \int_0^x f(x-w)dw \neq f(x).$$

E.g.,

$$(\sin *1)(x) = \int_0^x \sin(x-w)\cdot 1dw = [\cos(x-w)]_0^x =$$
$$\cos(0) - \cos(x) = 1 - \cos(x) \neq \sin(x).$$

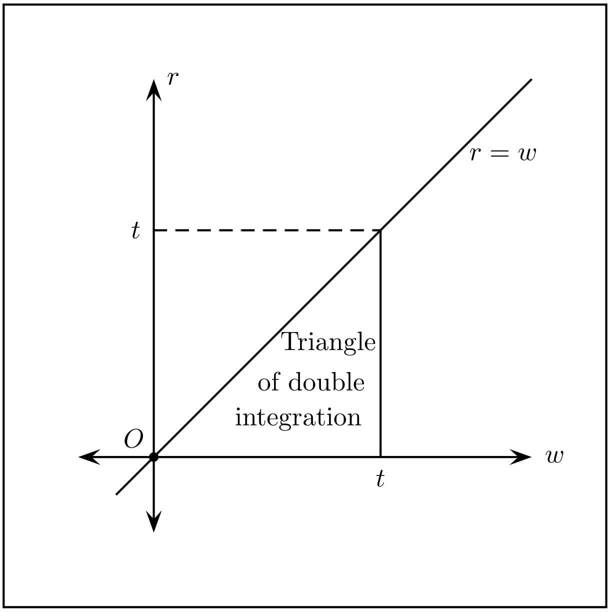

FIGURE 2.9: For switching integration in Example 2.7.5 (d)

(f) If $f(x)$ and $g(x)$ are absolutely integrable and one of them continuous, then $(f * g)(x)$ is continuous. The proof follows from the definition of continuity and **Theorem 2.2.1 Part (I)** or **Theorem 2.3.4** and its **remark**.

(g) Suppose $f(x)$ and $g(x)$ are continuous real functions on $[0, \infty)$. Then $(f * g)(x) = 0$ for all $x \in [0, \infty)$ if and only if $f(x) = 0$ or $g(x) = 0$ for all $x \in [0, \infty)$, i.e.,

$$f * g \equiv 0 \quad \Longleftrightarrow \quad f \equiv 0 \quad \text{or} \quad g \equiv 0.$$

This result is a famous Theorem proved by E. C. Titchmarsh[38] in 1926. The proof is lengthy and hard.

▲

Example 2.7.6 We consider the set of functions

$$\mathbf{C} = \{f \ : \ [0, \infty) \longrightarrow \mathbb{R} \quad \text{continuous}\}.$$

In the previous example, we have seen that the operation convolution is a closed operation and behaves like a nice algebraic multiplication

---

[38] Edward Charles "Ted" Titchmarsh, English mathematician, 1899-1963.

in this set. The only property left is the unit or neutral element for this operation. As we have seen above this is not the constant function $f(x) \equiv 1$.

Here we explain that the unit or neutral element of convolution is an object that we call the **Dirac delta function**. Let us see how we arrive at its definition in the context of the Laplace transform and the convolution we study here.

For every $\epsilon > 0$, we consider the step function

$$D_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & \text{if} \quad 0 \le x \le \epsilon \\ \\ 0, & \text{if} \quad \epsilon < x \end{cases}$$

and any function $f \ : \ [0, \infty) \longrightarrow \mathbb{R}$ which is continuous in $[0, \alpha)$, for some $\alpha > 0$. Then we have:

(a)

$$\lim_{\epsilon \to 0} D_\epsilon(x) \overset{pw}{=} \begin{cases} \infty, & \text{if} \quad x = 0 \\ \\ 0, & \text{if} \quad x \ne 0. \end{cases}$$

(b) For every $\epsilon > 0$

$$\int_0^\infty D_\epsilon(x)dx = 1 \quad \text{and so} \quad \lim_{\epsilon \to 0} \int_0^\infty D_\epsilon(x)dx = 1.$$

(c) For $0 < \epsilon < \alpha$, we use the Mean Value Theorem for integrals to get

$$\int_0^\infty D_\epsilon(x) f(x)dx = \int_0^\epsilon \frac{1}{\epsilon} f(x)dx = \epsilon \frac{1}{\epsilon} f\left(x_\epsilon\right) = f\left(x_\epsilon\right),$$

for some $0 \le x_\epsilon \le \epsilon$. Then, by the continuity of $f(x)$ in $[0, \alpha)$, we obtain

$$\lim_{\epsilon \to 0^+} \int_0^\infty D_\epsilon(x) f(x)dx = f\left(\lim_{\epsilon \to 0^+} x_\epsilon\right) = f(0).$$

So, **(a)**, **(b)** and **(c)** suggest to define the symbol $\delta(x)$, $\forall \ x \in \mathbb{R}$, by

$$\delta(x) = \begin{cases} \ne 0 \ (= \infty?) \ ?, & \text{if} \quad x = 0 \\ \\ 0, & \text{if} \quad x \ne 0 \end{cases}$$

and by stipulating that it satisfies the following **two properties**:

$$(\mathbf{1}) \qquad \int_0^\infty \delta(x)dx = 1.$$

(2) For every $f : [0,\infty) \longrightarrow \mathbb{R}$ continuous in $[0,\alpha)$, for some $\alpha > 0$,

$$\int_0^\infty f(x)\delta(x)dx = f(0).$$

We call $\delta(x)$ **Dirac delta function** or **unit impulse function**, even though, as we saw above, it is not a function in the classical meaning of the word but rather an operator derived from function processes.

For any $a > 0$, we also consider the shift $\delta(x-a)$ of $\delta(x)$, and we obtain

$$\forall\ a > 0,\ \int_0^\infty f(x)\delta(x-a)dx = \int_{-a}^\infty f(u+a)\delta(u)du =$$
$$\int_0^\infty f(u+a)\delta(u)du = f(0+a) = f(a).$$

Now, for all $x \in [0,\infty)$ and any $f(x) \in \mathbf{C}$, we get

$$(f*\delta)(x) = \int_0^x f(x-u)\delta(u)du = \int_0^\infty f(x-u)\delta(u)du = f(x-0) = f(x).$$

Therefore, $f * \delta = f = \delta * f$, that is, **the unit element for the commutative operation of convolution in the set C is the Dirac delta function**.

From these properties, we can easily find the Laplace transforms of $\delta(x)$ and its shifts $\delta(x-a)$ for $a > 0$. Namely:

$$\mathcal{L}\{\delta(x)\}(s) = \int_0^\infty e^{-xs}\delta(x)dx = e^{-0s} = 1$$

and

$$\mathcal{L}\{\delta(x-a)\}(s) = \int_0^\infty e^{-xs}\delta(x-a)dx = e^{-as}.$$

▲

Example 2.7.7 Prove that $\mathcal{L}\{\sin(\sqrt{x})\}(s) = \dfrac{\sqrt{\pi}e^{\frac{-1}{4s}}}{2s^{\frac{3}{2}}}$.

We use the power series of $\sin(x)$ to obtain

$$\sin(\sqrt{x}) = \sum_{n=0}^\infty \frac{(-1)^n(\sqrt{x})^{2n+1}}{(2n+1)!} = \sum_{n=0}^\infty \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!}.$$

We use **Rule (6.)** of the **table in Problem 2.7.12** and the result of **Problem 2.6.14 (a)** to find

$$\mathcal{L}\{\sin(\sqrt{x})\}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\Gamma\left(n+1+\frac{1}{2}\right)}{s^{n+1+\frac{1}{2}}} =$$

$$\frac{1}{s^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{[2(n+1)]!\sqrt{\pi}}{4^{n+1}(n+1)!} = \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+1)}{n! 4(n+1)} \left(\frac{1}{4s}\right)^n =$$

$$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1}{4s}\right)^n = \frac{\sqrt{\pi}\, e^{\frac{-1}{4s}}}{2s^{\frac{3}{2}}}.$$

We can do analogous work with $\cos(\sqrt{x})$. We use the power series of $\cos(x)$ to obtain

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}.$$

Then by **Rule (5.)** of the **table in Problem 2.7.12**, we obtain

$$\mathcal{L}\{\cos(\sqrt{x})\}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{n!}{s^{n+1}} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)\dots(2n)} \frac{1}{s^n}.$$

▲

Application: We can use known Laplace transforms to compute improper integrals.

E.g.: We consider $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and any $k > 0$ constant. Then

$$\int_0^{\infty} e^{-kx} \sinh(x) \sin(x)\, dx =$$

$$\frac{1}{2} \int_0^{\infty} e^{-kx} e^{x} \sin(x)\, dx - \frac{1}{2} \int_0^{\infty} e^{-kx} e^{-x} \sin(x)\, dx =$$

$$\frac{1}{2} \mathcal{L}\{e^x \sin(x)\}(k) - \frac{1}{2} \mathcal{L}\{e^{-x} \sin(x)\}(k) =$$

[use **(9.)** of the **table in Problem 2.7.12**]

$$\frac{1}{2}\left[\frac{1}{(k-1)^2 + 1^2} - \frac{1}{(k+1)^2 + 1^2}\right] = \frac{2k}{k^4 + 4}.$$

For example,

$$\int_0^\infty e^{-3x}\ \sinh(x)\sin(x)\,dx = \frac{6}{85}.$$

Similarly,

$$\int_0^\infty e^{-kx}\frac{\sinh(x)\sin(x)}{x}\,dx =$$

$$\frac{1}{2}\int_0^\infty e^{-kx}\frac{e^x\sin(x)}{x}\,dx - \frac{1}{2}\int_0^\infty e^{-kx}\frac{e^{-x}\sin(x)}{x}\,dx =$$

[use **(9.)** of the **table in Problem 2.7.18**]

$$\frac{1}{2}\int_k^\infty \mathcal{L}\{e^x\sin(x)\}(u)du - \frac{1}{2}\int_k^\infty \mathcal{L}\{e^{-x}\sin(x)\}(u)du =$$

$$\frac{1}{2}\int_k^\infty \frac{1}{(u-1)^2+1^2}du - \frac{1}{2}\int_k^\infty \frac{1}{(u+1)^2+1^2}du =$$

$$\frac{1}{2}\left[\arctan(v)\right]_{k-1}^{k+1} = \frac{1}{2}[\arctan(k+1) - \arctan(k-1)] =$$

$$\frac{1}{2}\arctan\left(\frac{2}{k^2}\right).$$

For example,

$$\int_0^\infty e^{-\sqrt{2}x}\ \frac{\sinh(x)\sin(x)}{x}\,dx = \frac{1}{2}\arctan(1) = \frac{\pi}{8}.$$

To find the Laplace transform of a given function, we use:
(1) The definition, for primitive cases.
(2) The **linearity Properties (a)** and **(b)** above.
(3) Already known Laplace transforms of other functions.
(4) The rules in the **table of Problem 2.7.12**.
(5) The rules in the **table of Problem 2.7.18**.
(6) Other rules not listed in the two tables above.
(7) Limit processes with known results, especially when parameters are involved.
(8) Double integration and switching order.
(9) Power series.
(10) Tables of Laplace transforms with or without adjustments, if we can trust them, of course. (Sometimes there are human errors and/or typos in tables, and so we may need to check the readily available answers, especially when we use them, for crucial applications.)

(11) Special computer packages, if we can trust them of course. (Keep in mind that human errors are always possible, even with computers and/or computer packages. Therefore, we must check the answers provided carefully.)

Problems

2.7.1 Provide the missing details in **Examples 2.7.1**.

2.7.2 Prove that $f(x) = \ln(x)$, where $x > 0$ is of exponential order.

2.7.3 Provided that all integrals involved exist, prove the two **linearity properties (a)** and **(b)** of the Laplace transform.

2.7.4 (a) Construct an example of a continuous function $y = f(x)$ defined on $[0, \infty)$, such that it is absolutely integrable, i.e.,

$$\int_0^\infty |f(x)| dx < \infty,$$

but not of exponential order.

(b) Give an example of a continuous function $y = f(x)$ defined on $[0, \infty)$ of exponential order and such that

$$\int_0^\infty |f(x)| dx = \infty.$$

(c) Give an example of a continuous function $y = f(x)$ defined on $[0, \infty)$ of exponential order and such that

$$\int_0^\infty |f(x)| dx < \infty.$$

2.7.5 (a) Prove that if $y = f(x)$ is a nice function defined on $A = [a, \infty)$, where $a \geq 0$, and it is **absolutely integrable**, or **square integrable**, i.e.,

$$\int_A |f(x)| dx < \infty, \quad \text{or} \quad \int_A |f(x)|^2 dx < \infty,$$

respectively, then the Laplace transform of $f(x)$ converges absolutely (and therefore it exists) $\forall \ s \geq 0$.

(b) Then also prove:

$$\lim_{s \to \infty} \mathcal{L}\{f(x)\}(s) = 0$$

and the convergence is uniform, i.e., $\forall\ \epsilon > 0,\ \exists\ K > 0\ :\ \forall\ s > K$ we have $|\mathcal{L}\{f(x)\}(s)| < \epsilon$.

(c) Prove the same results as in **(a)** and **(b)** for $s \geq u$, if for some $u \geq 0$ we replace the absolute and square integrability of $f(x)$ with the conditions

$$\int_A e^{-ux}|f(x)|dx < \infty, \quad \text{or} \quad \int_A e^{-ux}|f(x)|^2 dx < \infty.$$

[Hint: You may use **Lebesgue's Dominated Convergence Theorem 2.3.4**, **Hölder inequality**, **project Problem 2.6.42** and the **Cauchy criterion for convergence** as it was done in the **Cauchy Test, Theorem 1.3.7**.]

2.7.6 If $f(x)$ is a "nice" function of exponential order in $[0,\infty)$ (you may want to express some "nice" conditions that $f(x)$ should satisfy on an interval $[0,B]$, with $B > 0$ constant), then prove that the Laplace transform of $f(x)$ exists (in fact, converges absolutely) on the interval $[c,\infty)$ or (c,∞), where c is the infimum of all possible u's> 0 that may be used in **Definition 2.7.2** and so $c \geq 0$.

Then also prove:

$$\lim_{s\to\infty} \mathcal{L}\{f(x)\}(s) = 0$$

and the convergence is uniform, i.e., $\forall\ \epsilon > 0,\ \exists\ K > 0\ :\ \forall\ s > K$ we have $|\mathcal{L}\{f(x)\}(s)| < \epsilon$.

2.7.7 (a) Show that $f(x) = e^{x^2}$ on $[0,\infty)$ is not of exponential order and does not have Laplace transform.

(b) Show that $\forall\ n \in \mathbb{N}$, the function e^{-x^n} is of exponential order and its Laplace transform exists on $[0,\infty)$.

2.7.8 Look at the definition of the **Gamma function** $\Gamma(p)$. Notice that it may be considered as the value of the Laplace transform of a certain function for a special choice of s. Identify this function and the choice of s.

2.7.9 Give Laplace transform interpretations to the following three improper integrals (found in **Problems 1.2.14**, **1.2.16** and **2.4.9**) with parameters $\alpha > 0$ and $\beta \in \mathbb{R}$:

$$\int_0^\infty e^{-\alpha x}\sin(\beta x)\,dx = \frac{\beta}{\alpha^2+\beta^2},$$

$$\int_0^\infty e^{-\alpha x}\cos(\beta x)\,dx = \frac{\alpha}{\alpha^2+\beta^2},$$

$$\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx = \arctan\left(\frac{\beta}{\alpha}\right).$$

In other words, find the functions and their corresponding Laplace transforms hidden in these integrals. Justify your answers.

2.7.10 The following results can be used to find the Laplace transforms of some functions. In each one, find the function and its corresponding Laplace transform. Explain your answers.

$$\text{(a)}\quad I(\beta) = \int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{\frac{-\beta^2}{4\alpha}}$$

for any $-\infty < \beta < \infty$ and any $\alpha > 0$ (found in **Example 2.2.13**).

$$\text{(b)}\quad \int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}},$$

where $\alpha > 0$ (found in **Problem 2.1.11**).

[Hint: Let $x^2 = t$ and then observe and interpret what you get.]

2.7.11 The following results, found in **Problem 2.5.18**, can be used to find the Laplace transforms of some functions. In each one, find the function and its corresponding Laplace transform. Explain your answers.

For $\beta \geq 0$ and $a,\ b$ real constants:

$$\text{(a)}\quad \int_0^\infty e^{-\beta x}\frac{\cos(ax)-\cos(bx)}{x}\,dx = \frac{1}{2}\ln\left(\frac{\beta^2+b^2}{\beta^2+a^2}\right).$$

$$\text{(b)}^{\pm}\quad \int_0^\infty e^{-\beta x}\frac{\sin(ax)\pm\sin(bx)}{x}\,dx = \arctan\left(\frac{a}{\beta}\right) \pm \arctan\left(\frac{b}{\beta}\right).$$

2.7.12 Verify the results of the **following table**:

	Function $h(x),\ x \in [0, \infty)$	Laplace transform $\mathcal{L}\{h(x)\}(s)$
1.	a (=constant)	$\dfrac{a}{s}, \quad s > 0$
2.	e^{ax}	$\dfrac{1}{s-a}, \quad s > a$
3.	$\sin(ax)$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
4.	$\cos(ax)$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
5.	$x^n, \quad n = 0, 1, 2, 3, ...$	$\dfrac{n!}{s^{n+1}}, \quad s > 0, \quad [\,0! = \Gamma(1) = 1\,]$
6.	$x^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0, \quad [\,p\,! = \Gamma(p+1)\,]$
7.	$\sinh(ax)$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh(ax)$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9.	$e^{ax}\sin(bx)$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{ax}\cos(bx)$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$x^n e^{ax}, \quad n = 0, 1, 2, 3, ...$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$

2.7.13 Use the **table in Problem 2.7.12** to compute the Laplace transforms of the following functions:

$$\text{(a)} \qquad \frac{-2}{3} + e^{-5x} + 3\sin(2x) - 7\cos(\sqrt{5}x)$$

$$\text{(b)} \qquad 2x^3 - 5x^{\frac{2}{3}} + e^{3x}\cos(5x) - e^{-2x}\sin(12x) + 5$$

$$\text{(c)} \qquad 5\cosh(-3x) + 7\sinh(2x)$$

$$\text{(d)} \qquad \sqrt{5x} + (10x)^{\frac{-1}{3}}$$

$$\text{(e)} \qquad x^3e^{-3x} - 5e^{2x}\sin(-5x) - 6.$$

2.7.14 Modify the results of **Problem 2.2.29** to prove that for $\alpha > 0$ constant, we have:

$$(1) \quad \mathcal{L}\left\{\frac{1}{x^2+\alpha^2}\right\}(s) = \frac{1}{\alpha}\left[\cos(\alpha s)\left(\frac{\pi}{2} - \mathrm{Si}(\alpha s)\right) - \sin(\alpha s)\,\mathrm{Ci}(\alpha s)\right].$$

$$(2) \qquad \mathcal{L}\left\{\frac{x}{x^2+\alpha^2}\right\}(s) = \sin(\alpha s)\left[\frac{\pi}{2} - \mathrm{Si}(\alpha s)\right] + \cos(\alpha s)\,\mathrm{Ci}(\alpha s).$$

$$(3)\, \mathcal{L}\left\{\arctan(\alpha x)\right\}(s) =$$

$$\frac{1}{s}\left[\cos\left(\frac{s}{\alpha}\right)\left(\frac{\pi}{2} - \mathrm{Si}\left(\frac{s}{\alpha}\right)\right) - \sin\left(\frac{s}{\alpha}\right)\,\mathrm{Ci}\left(\frac{s}{\alpha}\right)\right].$$

$$(4) \quad \mathcal{L}\left\{\ln\left(x^2+\alpha^2\right)\right\}(s) =$$

$$\frac{2\ln(\alpha)}{s} + \frac{2\alpha}{s}\sin(\alpha s)\left[\frac{\pi}{2} - \mathrm{Si}(\alpha s)\right] + \cos(\alpha s)\,\mathrm{Ci}(\alpha s).$$

2.7.15 (a) If $y = f(x)$ defined on $[0, \infty)$ is **periodic with period** p and for some $u \geq 0$ the integral

$$\int_0^p e^{-sx} f(x)\,dx \quad \text{exists}, \quad \forall\ s \geq u,$$

then, for all $s \geq u$, prove

$$\mathcal{L}\{f(x)\}(s) = \frac{1}{1-e^{-sp}}\int_0^p e^{-sx}f(x)\,dx = \frac{e^{sp}}{e^{sp}-1}\int_0^p e^{-sx}f(x)\,dx.$$

(b) Use this rule to find the Laplace transforms of $\sin(ax)$ and $\cos(ax)$, where $a > 0$ constant [as they appear in the **table in Problem, 2.7.12, Rules (3.) and (4.)**].

(c) Explain why this rule does not work with $\tan(ax)$ and $\cot(ax)$, where $a > 0$ constant?

2.7.16 (a) For any $b \in \mathbb{R}$ constant, prove directly that the **Heaviside unit step function** defined by

$$H_b(x) := H(x-b) := \begin{cases} 0 & \text{if} \quad x < b \\ \\ 1 & \text{if} \quad x \geq b \end{cases}$$

[so, $H_0(x) = H(x-0) = H(x)$] has Laplace transform

$$\mathcal{L}\{H_b(x)\}(s) = \frac{e^{-bs}}{s}, \quad \text{for all} \quad s > 0.$$

(b) For $b \geq 0$, this rule also follows by **Rule (1.)** of the **table in Problem 2.7.12** and **Rule (5.)** of the **table insssss Problem 2.7.18**. Check this!

(c) Find the Laplace transforms of

$$3H(x-5) \quad \text{and} \quad -7H(x+5).$$

(d) Let

$$p(x) = \begin{cases} 1 & \text{if} \ \ 0 \leq a < (\leq)\ x < (\leq)\ b \\ \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\mathcal{L}\{p(x)\}(s) = \frac{-e^{-bs} + e^{-as}}{s}.$$

(e) Find the Laplace transforms of

$$q(x) = \begin{cases} \sqrt{5.78} & \text{if} \ \ 2.45 < x \leq 12.34 \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$r(x) = \begin{cases} \sqrt[3]{5.78} & \text{if } \ 0 \le x < 123.45 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(f) Prove that for any $x \in \mathbb{R}$, its integer part or floor function satisfies

$$[\![x]\!] = \sum_{n=1}^{\infty} H_n(x) = \sum_{n=1}^{\infty} H(x-n).$$

(g) Generalize **(f)** to

$$\left[\!\!\left[\frac{x}{p}\right]\!\!\right] = \sum_{n=1}^{\infty} H_{pn}(x) = \sum_{n=1}^{\infty} H(x-pn),$$

where $x \in \mathbb{R}$ and $p \neq 0$.

2.7.17 (a) Use the known result

$$\Gamma\,'(1) = \int_0^{\infty} e^{-x} \ln(x)\, dx = -\gamma,$$

(where $\gamma = \lim\limits_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right] \simeq 0.57721566... > 0$ is the **Euler-Mascheroni constant**, see **Problem 2.1.25** and its **footnote** and **Problem 2.3.5**) to prove that for any $s > 0$ the Laplace transform of $\ln(x)$ is given by

$$\mathcal{L}\{\ln(x)\}(s) = \int_0^{\infty} e^{-xs} \ln(x) dx = \frac{-[\gamma + \ln(s)]}{s}.$$

(b) Find the Laplace transform of $g(x) = 6\ln\left(10\,x^2\right)$.

(c) Given that

$$\Gamma\,''(1) = \int_0^{\infty} e^{-x} \ln^2(x)\, dx = \frac{\pi^2}{6} + \gamma^2,$$

prove that for any $s > 0$ the Laplace transform of $\ln^2(x)$ is given by

$$\mathcal{L}\left\{\ln^2(x)\right\}(s) = \int_0^{\infty} e^{-xs} \ln^2(x) dx = \frac{\pi^2}{6s} + \frac{[\gamma + \ln(s)]^2}{s}.$$

(d) Find the Laplace transform of $g(x) = 6\ln^2\left(10\,x^2\right)$.

2.7.18 Verify the following general properties-rules of the Laplace transform in the **following table**:

	Function $h(x)$ $x \in [0, \infty)$ or $x \in (0, \infty)$	Laplace transform $\mathcal{L}\left\{h(x)\right\}(s)$, for $s > k \geq 0$, k constant
1.	$f'(x)$	$s\mathcal{L}\{f(x)\}(s) - f(0)$
2.	$f''(x)$	$s^2\mathcal{L}\{f(x)\}(s) - sf(0) - f'(0)$
3.	$f^{(n)}(x)$	$s^n\mathcal{L}\{f(x)\}(s) -$ $s^{n-1}f(0) - ... - f^{(n-1)}(0)$
4.	$e^{ax}f(x)$ with a constant	$\mathcal{L}\{f(x)\}(s-a)$ for $s > k + a$ (shift by a in the Laplace transform)
5.	$H_b(x) \cdot f(x-b)$ shift of $f(x)$ by constant $b \geq 0$ $[H_b(x) =$ Heaviside function$]$	$e^{-bs}\mathcal{L}\{f(x)\}(s)$ for $s > k$
6.	$\displaystyle\int_0^x f(t)\,dt$	$\displaystyle\frac{1}{s}\,\mathcal{L}\{f(x)\}(s)$
7.	$\displaystyle\int_x^\infty f(t)\,dt$	$\displaystyle\frac{1}{s}\left[\int_0^\infty f(t)\,dt - \mathcal{L}\{f(x)\}(s)\right]$
8.	$x^n f(x), \quad n = 0, 1, 2, 3, ...$	$\displaystyle(-1)^n\,\frac{d^n}{ds^n}\mathcal{L}\{f(x)\}(s)$
9.	$\displaystyle\frac{f(x)}{x}$	$\displaystyle\int_s^\infty \mathcal{L}\{f(x)\}(u)\,du$
10.	$f(ax)$ $a > 0$ constant	$\displaystyle\frac{1}{a}\,\mathcal{L}\{f(x)\}\left(\frac{s}{a}\right)$
11.	$(-x)^n f(x), \quad n = 0, 1, 2, 3, ...$	$\displaystyle\frac{d^n}{ds^n}\mathcal{L}\{f(x)\}(s)$

(Notice that the results of some previous and/or following problems and of some examples provide important rules that can be added to the two **tables in Problems 2.7.12** and **2.7.18**. Check these problems and examples and attach their additional rules to the tables, thus creating a more complete collection of Laplace transforms rules.)

2.7.19 Combine the **Rules (5.)** and **(10.)** of the **table in Problem 2.7.18** to prove that for $a > 0$ and $b \geq 0$ constants,

$$\mathcal{L}\{f(ax-b)\}(s) = \frac{1}{a}\, e^{\frac{-bs}{a}}\; \mathcal{L}\{f(x)\}\left(\frac{s}{a}\right).$$

2.7.20 Use **Rule (9.)** of the **table in Problem 2.7.18** to prove that the Laplace transform of

$$f(x) = \frac{\sin(\beta x)}{x}$$

is

$$\mathcal{L}\{f(x)\}(s) = \frac{\pi}{2} - \arctan\left(\frac{s}{\beta}\right) = \arctan\left(\frac{\beta}{s}\right),$$

as it was already referred in **Problem 2.7.9**.

2.7.21 Use the two **tables in Problems 2.7.12 and 2.7.18** to compute the Laplace transforms of the following functions for which we assume that $x > 0$ or $x \geq 0$. If some function does not have Laplace transform, then explain why.

(a) $x^2 \cos(5x)$, (b) $(-x)^3 \sinh(2x)$,

(c) $\dfrac{e^{-3x}\sin(-5x)}{x}$, (d) $\dfrac{\sin(x)}{x}$,

(e) $\displaystyle\int_0^x \frac{\sin(t)}{t}\, dt$, (f) $\dfrac{\cos(x)}{x}$,

(g) $\displaystyle\int_x^\infty \frac{\cos(t)}{t}\, dt$, (h) $3e^{-2x}H(x-5)$,

(i) $-7e^{3x}H(x+5)$, (j) $H(x+5)\cos(x+10)$.

2.7.22 (a) Take $a > 0$ constant and define the functions

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < a \\[2ex] \dfrac{\cos(x)}{x}, & \text{if } \ x \geq a \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } 0 \le x < a \\ \displaystyle\int_x^{\infty} \frac{\cos(t)}{t}\,dt, & \text{if } \ x \ge a. \end{cases}$$

Find the Laplace transforms of both $f(x)$ and $g(x)$. (If you cannot find them in closed form, at least justify why they exist, then leave them in integral form and/or look at an advanced Laplace transform table.)

(b) Prove that

$$\mathcal{L}\left\{\frac{1-\cos(x)}{x}\right\}(s) = \int_s^{\infty}\left(\frac{1}{t} - \frac{t}{1+t^2}\right)dt = \frac{1}{2}\ln\left(1+\frac{1}{s^2}\right).$$

(**Careful**: We cannot use the **additive property** of Laplace transform and of the integral here! Why?)

2.7.23 Let $a > 0$. Define

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } \ x \ge a \\ 0 & \text{if } \ 0 \le x < a. \end{cases}$$

1. Prove that $\mathcal{L}\{f(x)\}(s)$ exists for all $s > 0$, but it cannot be found in closed form from its definition.
2. Find the derivative $\dfrac{d}{ds}\mathcal{L}\{f(x)\}(s)$.
3. Use what you have found in the previous item to prove

$$\forall \ s > 0, \quad \mathcal{L}\{f(x)\}(s) = -\ln(s) - \sum_{n=1}^{\infty}\frac{(-1)^n a^n}{n!}\frac{s^n}{n} + c,$$

   where c is a constant.

4. Prove that an expression of the constant c is

$$c = \int_a^{\infty}\frac{e^{-x}}{x}\,dx + \sum_{n=1}^{\infty}\frac{(-1)^n a^n}{n\cdot n!}.$$

2.7.24 (**Extension of the previous problem**.) Let $a > 0$ and any $k = 2,\ 3,\ 4,\ \cdots$. Define

$$f(x) = \begin{cases} \dfrac{1}{x^k} & \text{if } \ x \ge a \\ 0 & \text{if } \ 0 \le x < a. \end{cases}$$

1. Prove that $\mathcal{L}\left\{f(x)\right\}(s)$ exists for all $s \geq 0$.
2. Find the k^{th} derivative of $\mathcal{L}\left\{f(x)\right\}(s)$.
3. For $s > 0$, find an expression of $\mathcal{L}\left\{f(x)\right\}(s)$ depending on k constants.
4. Explain how you can determine the k constants.

2.7.25 For given real constants a and b, prove that

$$\mathcal{L}\left\{\frac{e^{ax}-e^{bx}}{x}\right\}(s) = \int_s^\infty \left(\frac{1}{t-a} - \frac{1}{t-b}\right) dt = \ln\left|\frac{s-b}{s-a}\right|.$$

2.7.26 Project: Search the bibliography on Laplace transform and study the Laplace transforms of the:
(a) **Bessel functions**.
(b) **Legendre polynomials**.
(c) **Shifted functions**.
(d) **Dirac impulse functions**.
(e) **Convolutions of functions**.

2.7.2 Inverse Laplace Transform

As we have already said, on continuous functions the Laplace transform is a one-to-one operator. That is, the Laplace transforms of two different continuous functions are different. The same is essentially true on nice discontinuous functions in which we may allow two functions to be different at "a few" exceptional points (points of discontinuity). So, we can give the following definition:

Definition 2.7.3 *Given $f(x)$ with Laplace transform $g(s) = \mathcal{L}\{f(x)\}(s)$, we call $f(x)$ the **Inverse Laplace Transform** of $g(s)$, and we write*

$$\mathcal{L}^{-1}\{g(s)\}(x) = f(x).$$

Since $\mathcal{L}^{-1}$ is the inverse of the linear operator $\mathcal{L}$, it is a linear operator itself. (See **Problem 2.7.27**.) Generically speaking, if we know the Laplace transform of a function, then by applying to it the inverse Laplace transform we recover the function. For this purpose, many extensive tables and computer libraries have been created, so that from the Laplace transform we can find the function readily.

In theory and application, we essentially have and use the following scheme:

$$f(x) \xrightarrow{\mathcal{L}} \mathcal{L}\{f(x)\}(s) = g(s) \xrightarrow{\mathcal{L}^{-1}} \mathcal{L}^{-1}\{g(s)\}(x) = f(x).$$

Examples

Example 2.7.8 By **Rule (2.)** of the **table in Problem 2.7.12**, we have

$$\mathcal{L}\left\{e^{-5x}\right\}(s) = \frac{1}{s+5}.$$

Therefore,

$$f(x) = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}(x) = e^{-5x}.$$

Or, according to the above scheme,

$$e^{-5x} \xrightarrow{\mathcal{L}} \mathcal{L}\left\{e^{-5x}\right\}(s) = \frac{1}{s+5} \xrightarrow{\mathcal{L}^{-1}} \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}(x) = e^{-5x}.$$

▲

Example 2.7.9 Let

$$\mathcal{L}\{f(x)\}(s) = \frac{3}{s^2+4}.$$

Then by adjusting **Rule (3.)** of the **table in Problem 2.7.12**, we find that

$$f(x) = \frac{3}{2}\sin(2x).$$

So, in this example the above scheme is

$$\frac{3}{2}\sin(2x) \xrightarrow{\mathcal{L}} \mathcal{L}\left\{\frac{3}{2}\sin(2x)\right\}(s) = \frac{3}{s^2+4} \xrightarrow{\mathcal{L}^{-1}}$$
$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\}(x) = \frac{3}{2}\sin(2x).$$

▲

Example 2.7.10 Using the partial fraction decomposition, we find

$$\frac{4s^2+12}{s(s^2+4)} = \frac{3}{s} + \frac{s}{s^2+4}.$$

Then by using **linearity** and **Rules (1.)** and **(4.)** of the **table in Problem 2.7.12**, we find that

$$f(x) = 3 + \cos(2x).$$

Now, by **Rule (4.)** of the **table in Problem 2.7.18**, we find that the Laplace transform of

$$g(x) = e^{-5x}[3+\cos(2x)] = e^{-5x}f(x)$$

is

$$\mathcal{L}\left\{e^{-5x}[3+\cos(2x)]\right\}(s)=\frac{4(s+5)^2+12}{(s+5)[(s+5)^2+4]}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{4(s+5)^2+12}{(s+5)[(s+5)^2+4]}\right\}(x)=e^{-5x}[3+\cos(2x)].$$

▲

Example 2.7.11 Prove

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}(x)=\frac{a}{2\sqrt{\pi}}\,x^{\frac{-3}{2}}e^{\frac{-a^2}{4x}},$$

where $a>0$ constant. (See also **Problem 3.7.109**.)

If $\mathcal{L}\{f(x)\}(s)=g(s)$, then we have $\mathcal{L}\{f'(x)\}(s)=sg(s)-f(0)$ [by **(1)** of the **table in Problem 2.7.18**]. So,

$$\text{if}\quad f(0)=0,\quad \mathcal{L}^{-1}\{sg(s)\}(x)=f'(x).$$

Then, with

$$f(x):=\operatorname{erfc}\left(\frac{a}{2}\sqrt{x}\right)=1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{2\sqrt{x}}}e^{-u^2}du$$

we have $f(0)=1-1=0$ and, by **Problem 3.7.110**,

$$g(s):=\mathcal{L}\{f(x)\}(s)=\frac{e^{-a\sqrt{s}}}{s}.$$

So, we get

$$\mathcal{L}^{-1}\{s\cdot g(s)\}(x)=\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}=f'(x)=$$

$$\frac{d}{dx}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{a}{2\sqrt{x}}}e^{-u^2}du\right)=\frac{a}{2\sqrt{\pi}}\,x^{\frac{-3}{2}}e^{\frac{-a^2}{4x}}.$$

▲

2.7.3 Applications

Application 1: Here we remark that the Laplace transform may be used to give complete solutions of initial value problems with ordinary differential equations. E.g., see the last seven of the problems that follow. However, there are cases in which it does not find the complete solution or even the solution at all. We study these situations in the following two examples.

(1) The power series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

converges absolutely for all real x's (prove this!) and defines the so-called **Bessel function of the first kind of order zero**.

The function $J_0(x)$ was originally derived as a power series solution of the **Bessel's differential equation of order** 0[39]

$$xy'' + y' + xy = 0,$$

which is a homogenous equation with $x = 0$ a **regular singular point**.

Bessel derived this differential equation while studying problems of planetary motion. Ever since, the Bessel differential equations of all orders appear in mathematics, application and engineering, and their solutions are very rich of properties and are so important that they have been tabulated.

We find the Laplace transform of $J_0(x)$ by integrating term by term and summing up

$$\mathcal{L}\{J_0(x)\}(s) =$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}\right\}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{2^{2n}} \mathcal{L}\left\{x^{2n}\right\}(s) =$$

(use the table in **Problem 2.7.12**)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{2^{2n}} \frac{(2n)!}{s^{2n+1}} = \frac{1}{s} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{s^{2n}} =$$

(use the convergent binomial series for $s > 1$, **Problem 2.6.41**)

$$\frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{s}\left(\frac{s^2}{1+s^2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}},$$

which is defined for all $s \geq 0$.

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[39] **Bessel's differential equation of order** $\nu \geq 0$ is $\dfrac{d^2u}{dz^2} + \dfrac{1}{z}\dfrac{du}{dz} + \left(1 - \dfrac{\nu^2}{z^2}\right) = 0$. It has a bounded solution which is called **the Bessel function of the first kind of order** ν and an unbounded solution, an expression of which is called **the Bessel function of the second kind of order** ν. So, with $\nu = 0$, we obtain the equation we examine above.

Then, by the uniqueness of the Laplace transform (or its real analyticity in this particular example), we can say

$$\mathcal{L}\{J_0(x)\}(s) = \frac{1}{\sqrt{1+s^2}}, \quad \text{for all} \quad s \geq 0.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{1+s^2}}\right\}(x) = J_0(x).$$

We could also take the Laplace transform of the equation itself, and, using the appropriate rules in the **table in Problem 2.7.18**, we find

$$\mathcal{L}\left\{xy'' + y' + xy\right\}(s) = 0, \quad \text{or}$$
$$\left[-\left(\mathcal{L}\left\{y''\right\}\right)' + \mathcal{L}\{y'\} - \left(\mathcal{L}\{y\}\right)'\right](s) = 0.$$

For convenience, we put $Y(s) = \mathcal{L}\{y\}(s)$, use the **table in Problem 2.7.18**, and we find

$$-\left[s^2Y(s) - sy(0) - y'(0)\right]' + sY(s) - y(0) - Y'(s) = 0, \quad \text{or}$$
$$(s^2+1)Y'(s) + sY(s) = 0.$$

The last equation is a homogeneous separable ordinary differential equation of first order for $Y(s)$, which can be solved directly by separating the variables and integrating both sides, etc., to find

$$Y(s) = \frac{c}{\sqrt{1+s^2}},$$

where c is an arbitrary constant.

Therefore, a solution $y(x)$ of $xy'' + y' + xy = 0$ is

$$y(x) = \mathcal{L}^{-1}\left\{Y(s)\right\}(x) = \mathcal{L}^{-1}\left\{\frac{c}{\sqrt{1+s^2}}\right\}(x) = cJ_0(x),$$

which we can directly verify that it is a solution by plugging it into the given differential equation and differentiating the power series term by term.

So, with $c = 1$, both methods gave the same answer

$$\mathcal{L}\{J_0(x)\}(s) = \frac{1}{\sqrt{1+s^2}}, \quad \text{for all} \quad s \geq 0$$

and one correct solution, namely $J_0(x)$, to this Bessel differential equation.

Since the above Bessel equation is a linear homogeneous differential equation of second order, it must have another solution linearly independent of $J_0(x)$, which was not found by the Laplace transform method. This happened because, for the other solution, the Laplace transform of at least one of xy'' or y' or xy does not exist. In fact, if we find the other solution by means of power series, we will see that it is unbounded and the Laplace transform of at least one of these expressions does not exist.

In a course of differential equations, we learn other methods for finding the evasive second solution in which we use the solution found here. So, the Laplace transform method may not always give the complete solution to a problem with a differential equation but a partial one, which in turn helps in finding the complete solution.

We have just seen that the method of the Laplace transform did not find both solutions of an ordinary differential equation of second order. Whenever this happens, we may not be able to solve a particular initial value problem by using the Laplace transform method.

(2) The method may exhibit even more serious difficulty. Sometimes it is not able to find any solution at all. For instance, the initial value problem

$$\begin{cases} y'(t) - y(t) = (2t-1)e^{t^2}, \quad \text{for} \quad t > 0 \\ \\ y(0) = 2 \end{cases}$$

has unique (complete) solution $y(t) = e^t + e^{t^2}$ (found easily, as we can easily check it).

However, the method of the Laplace transform cannot find this solution because

$$\mathcal{L}\left\{(2t-1)e^{t^2}\right\}(s) = \int_0^\infty e^{-st}(2t-1)e^{t^2}\,dt = \int_0^\infty (2t-1)e^{t(t-s)}dt$$

does not exist. (This integral does not converge for any $s \geq 0$.)

[**Note**: This solution, however, can be found by the Mikusiński[40] general operator theory (1950-1951). In this theory, the results in **Examples 2.7.5** and **2.7.6** play a fundamental role, especially Titchmarsh Theorem. This separate chapter of mathematics lies outside the scope of this book.]

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[40] Jan Mikusiński, Polish mathematician, 1913-1987.

Application 2: We consider the following initial boundary value problem with a partial differential equation:

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \ \ (k>0 \ \text{constant}), \quad \text{for } \ x>0 \ \text{ and } \ t>0\\
&u(x,0)=0, \quad \text{for } \ x\geq 0\\
&u(0,t)=a \ \text{ constant}, \quad \text{for } \ t>0\\
&u(\infty,t)=\lim_{x\to\infty} u(x,t)=0, \quad \text{for } \ t>0.
\end{aligned}$$

This is a mathematical model of the problem of heat convection along an insulated and uniform rod of infinite length placed on the positive x-axis and with initial point at the origin. $u(x,t)$ is the temperature at position x at time t. k is a positive constant that depends on the uniform material of the rod which is a good heat conductor. (In reality, we have a rod considered very long.)

At the beginning, when $t=0$, the temperature at every point of the rod is zero. Afterwards, a source emits heat at the beginning of the rod in such a way that:
(1) The temperature at the beginning of the rod ($x=0$) stays constant equal a for every $t>0$.
(2) The temperature at the endpoint of the rod ($x=\infty$) is always the initial temperature, which is zero.

We are going to solve this problem by means of the Laplace transform. We let

$$U(x,s)=\mathcal{L}\left\{u(x,t)\right\}(s)=\int_0^{\infty} e^{-st}u(x,t)dt$$

be the Laplace transform of $u(x,t)$ with respect to t.

Then, by the first initial boundary condition, **Rule (1.)** in the **table in Problem 2.7.18** and differentiating under the integral sign with respect to x twice, from the partial differential equation we find

$$sU(x,s)=k\,\frac{\partial^2 U(x,s)}{\partial x^2}.$$

As we see, this has general solution

$$U(x,s)=c_1(s)\,e^{x\sqrt{\frac{s}{k}}}+c_2(s)\,e^{-x\sqrt{\frac{s}{k}}}.$$

The last two conditions change to

$$\begin{aligned}
&U(0,s)=\frac{a}{s},\\
&U(\infty,t)=\lim_{x\to\infty} U(x,s)=0.
\end{aligned}$$

From these, we conclude that we must have $c_1(s) = 0$ and $c_2(s) = \dfrac{a}{s}$. Therefore, the Laplace transform of the solution of this problem is

$$U(x,s) = \frac{a}{s}\, e^{-x\sqrt{\frac{s}{k}}}.$$

Then, by **Problem 3.7.110**, we obtain

$$u(x,t) = \mathcal{L}^{-1}\left\{U(x,s)\right\}(t) = a \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right) = a\left[1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right],$$

which is easily verified to be a solution satisfying the partial differential equation and the imposed conditions.

We remark that the function

$$u(x,t) + v(x,t)$$

with $u(x,t)$ the solution just found and $v(x,t) = \dfrac{cx}{\sqrt{t^3}}\, e^{\frac{-x^2}{4kt}}$, where $c \neq 0$ constant, is another solution of this problem, as we can directly check. This solution was not found by the Laplace transform method. We also notice that it is not a bounded solution. E.g., if $t = x^2 \to 0$, then $v(x,t) \to \infty$.

If we require that the solutions of the problem are bounded, a thing natural to assume, then we obtain the uniqueness of the solution $u(x,t)$ found above.

Application 3: We want to solve the initial value problem

$$\begin{aligned}
&\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 6 = C \cdot \delta(t - t_0),\\
&y(0) = 0,\\
&\frac{dy(0)}{dt} = 0.
\end{aligned}$$

Problems like this arise in studying electrical circuits or mechanical oscillators. At an “instant” $t_0 \geq 0$ a voltage or force pulse of size C, constant, is applied to the circuit or the oscillator, and we would like to find the electric current or the size of the oscillation, respectively, at time $t \geq t_0$.

We use the respected rules of the Laplace transform from the **table in Problem 2.7.18** and the Laplace transform of the Dirac delta function from **Example 2.7.6**. If the Laplace transform of the solution of the problem is $Y(s)$, then the differential equation together with the initial conditions give

$$Y(s) = \frac{Ce^{-t_0 s}}{s^2 + 4s + 6} = \frac{C\, e^{-t_0 s}}{(s+2)^2 + (\sqrt{2})^2}.$$

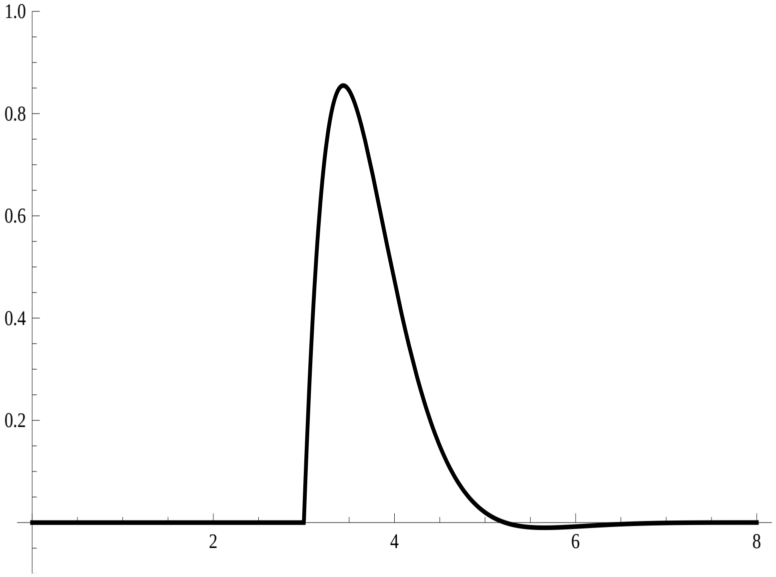

FIGURE 2.10: Pulse in application 3, with $t_0 = 3$ and $C = 5$

By the inverse process and using the **table in Problem 2.7.12**, we first find

$$\mathcal{L}^{-1}\left\{\frac{C}{\sqrt{2}}\cdot\frac{\sqrt{2}}{[s-(-2)]^2+(\sqrt{2}\,)^2}\right\}(t)=\frac{C}{\sqrt{2}}\cdot e^{-2t}\sin\left(\sqrt{2}\,t\right).$$

Then, by **Rule (4.)** in the **table in Problem 2.7.18**, the solution of the problem is

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}(t)=H_{t_0}(t)\frac{C}{\sqrt{2}}\cdot e^{-2(t-t_0)}\sin\left[\sqrt{2}\,(t-t_0)\right]=$$
$$\begin{cases} 0, & \text{if} \quad t<t_0 \\ \dfrac{C}{\sqrt{2}}\cdot e^{-2(t-t_0)}\sin\left[\sqrt{2}\,(t-t_0)\right], & \text{if} \quad t\ge t_0, \end{cases}$$

a graph of which, with $t_0 = 3$ and $C = 5$, is seen in **Figure 2.10**.

We conclude this section with the following:

To find the inverse Laplace transform of a given Laplace transform, we use:

(1) The definition (especially for primitive cases).

(2) For direct computations of the inverse Laplace transforms, see **Subsection 3.7.10** and its problems.

(3) The linearity properties.

(4) Already known inverse Laplace transforms.

(5) The inverse rules in the **table in Problem 2.7.12**.

(6) The inverse rules in the **table in Problem 2.7.18**.

(7) Other rules not listed in the two tables above.

(8) Limit processes with known results, especially when parameters are involved.

(9) Tables of Laplace and inverse Laplace transforms with or without adjustments, if we can trust them, of course. (Sometimes there are human errors and/or typos in tables, and so you may need to check the readily available answers.)

(10) Special computer packages, if we can trust them, of course. (Keep in mind that human errors are always possible with computers and computer packages.)

Problems

2.7.27 Prove that the Inverse Laplace Transform is a linear operator. Do this by proving the following general result, encountered in Linear Algebra and elsewhere:

Let V and W be two vector spaces over a field F and $T : V \longrightarrow W$ a one-to-one and onto linear operator. Then prove that the set-theoretic inverse $T^{-1} : W \longrightarrow V$ is also a one-to-one and onto linear operator.

2.7.28 Compute the inverse Laplace transforms of the following four functions and write the corresponding four schemes as in **Examples 2.7.9** and **2.7.10**:

$$\text{(a)}\quad \frac{-2}{s}+\frac{2}{3s^2+5}-\frac{2s}{s^2+8}, \qquad \text{(b)}\quad \frac{4}{(s+2)^5}-\frac{2s}{s^2-6}+\frac{3}{s^2-9},$$

$$\text{(c)}\quad \frac{3}{2s-5}+\frac{10}{s^6}, \qquad \text{(d)}\quad \frac{2s}{s^2+5s-2}-\frac{s-5}{s^2-3s+9}.$$

2.7.29 Use partial fraction decomposition to find the inverse Laplace transform of

$$\text{(a)}\quad \frac{8s^2-4s+12}{s(s^2+4)} \quad \text{and} \quad \text{(b)}\quad \frac{s^2+2s+3}{(s^2+4s+5)(s^2+2s+10)}.$$

2.7.30 Find the inverse Laplace transform of

$$\text{(a)}\quad \frac{(s-5)e^{-2s}}{s^2-5s+6} \quad\text{and}\quad \text{(b)}\quad \frac{2e^{-3s}-5e^{-4s}}{s}.$$

2.7.31 Use **Problem 2.7.17** to prove that

$$\text{(a)}\quad \mathcal{L}^{-1}\left\{\frac{\ln(s)}{s}\right\}(x) = -[\ln(x)+\gamma]$$

and

$$\text{(b)}\quad \mathcal{L}^{-1}\left\{\frac{\ln^2(s)}{s}\right\}(x) = [\ln(x)+\gamma]^2 - \frac{\pi^2}{6}.$$

2.7.32 Verify that the initial value-problem

$$\begin{cases} y'(x) + p\,y(x) = r(x), & p \text{ is a constant} \\ y(0) = y_0 & (y_0 \text{ is constant}) \end{cases}$$

has solution

$$y(x) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{r(x)\}(s) + y_0}{s+p}\right\}(x).$$

2.7.33 Find the solution of the initial value-problem

$$\begin{cases} y'(x) - 5\,y(x) = e^x\cos(x) \\ \\ y(0) = -2. \end{cases}$$

2.7.34 Verify that the initial value-problem

$$\begin{cases} y''(x) + p\,y'(x) + q\,y(x) = r(x), & p,\ q \text{ are constants} \\ y(0) = y_0 & (y_0 \text{ is constant}) \\ y'(0) = y_0' & (y_0' \text{ is constant}) \end{cases}$$

has solution

$$y(x) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{r(x)\}(s) + (s+p)y_0 + y_0'}{s^2+ps+q}\right\}(x).$$

2.7.35 Find the solution of the initial value-problem

$$\begin{cases} y''(x) - 4\,y'(x) + 3\,y(x) = 2\sin(x) + 5\cos(2x) \\ \\ y(0) = 1 \\ \\ y'(0) = \dfrac{1}{2}. \end{cases}$$

2.7.36 Find the solution of the initial value-problem

$$\begin{cases} y''(x) + 2\,y'(x) + y(x) = 4e^{-x} \\ \\ y(0) = 2 \\ \\ y'(0) = -1. \end{cases}$$

2.7.37 Find the solution of the initial value-problem

$$\begin{cases} y''(x) + 4y(x) = \begin{cases} 3, & 0 \le x < \pi \\ \\ 0, & \pi \le x < \infty \end{cases} \\ \\ y(0) = 2 \\ \\ y'(0) = -1. \end{cases}$$

2.7.38 Find the general formula for the solution of the following third order initial value-problem:

$$\begin{cases} y'''(x) + p\,y''(x) + q\,y'(x) + c\,y(x) = r(x), & p,\ q,\ c \text{ are constants} \\ y(0) = y_0 & (y_0 \text{ is constant}) \\ y'(0) = y_0' & (y_0' \text{ is constant}) \\ y''(0) = y_0'' & (y_0'' \text{ is constant}). \end{cases}$$

Next, solve the two initial value problems

$$\begin{cases} y'''(x) - 2y''(x) + 2y' + 3y(x) = 4e^x \\ \\ y(0) = y'(0) = y''(0) = 1 \end{cases}$$

and

$$\begin{cases} y'''(x) + y'(x) = e^{4x} \\ \\ y(0) = y'(0) = y''(0) = 0. \end{cases}$$

2.7.39 Project: Put together and prove the results in **Examples 2.7.5** and **2.7.6**, especially Titchmarsh Theorem of convolution, to demonstrate that the set

$$\mathbf{C} = \{f \ : \ [0, \infty) \longrightarrow \mathbb{R} \quad \text{continuous}\}$$

equipped with the usual operation of addition $(+)$ and the operation of convolution $(*)$ is an algebraic integral domain and so it can be extended to an algebraic field. This is a field of operators whose unit element is the Dirac delta function.

Chapter 3

Complex Analysis Techniques

Complex Analysis provides several powerful methods for proving existence (convergence) and evaluating proper and improper integrals. In this chapter, we are going to investigate the most important ones. But, in order to state and use these powerful tools, we need to know some necessary complex analysis preliminaries.

3.1 Basics of Complex Variables

3.1.1 Basic Definitions and Operations

A **complex number** z is an expression of the form

$$z = a + bi,$$

where i is a new symbol with the property $i^2 = -1$, and both a and b are real numbers.

Since $i^2 < 0$, the new symbol i cannot represent a real number. In fact, any expression of the form bi with b real number represents what we call an **imaginary number**. So, $i = 1 \cdot i$ is an imaginary number. Also, for any real number b, we require the legitimacy of the following computation:

$$(bi)^2 = (bi)(bi) = (bb)(ii) = b^2 \cdot i^2 = b^2 \cdot (-1) = -\,b^2 (\leq 0).$$

Then $(\pm i)^2 = (\pm 1)^2 i^2 = 1(-1) = -1$. Hence, we say that the real number -1 has two imaginary square roots, the $\pm i$. In the context of the complex numbers, we write

$$\sqrt{-1} = \pm i, \quad \text{which means } +\sqrt{-1} = +i \text{ and } -\sqrt{-1} = -i.$$

The real number a is called the **real part** of the complex number $z = a + bi$, and the real number b is called the **imaginary part**. We write

$$a = \mathrm{Re}(z) \quad \text{and} \quad b = \mathrm{Im}(z).$$

Equality of two complex numbers z and w is defined by

$$z = w \quad \overset{def}{\Longleftrightarrow} \quad \mathrm{Re}(z) = \mathrm{Re}(w) \quad \text{and} \quad \mathrm{Im}(z) = \mathrm{Im}(w).$$

Notice:

$$\begin{gathered} 0 = 0 + 0i = 0i, \\ 1 = 1 + 0i, \\ z = a \in \mathbb{R} \quad \Longleftrightarrow \quad \mathrm{Im}(z) = 0 \quad \Longleftrightarrow \quad z = a + 0i, \\ z = bi \quad \text{pure imaginary} \quad \Longleftrightarrow \quad \mathrm{Re}(z) = 0 \quad \Longleftrightarrow \quad z = 0 + bi. \end{gathered}$$

The operations of **addition**, **subtraction** and **multiplication** of complex numbers are defined in the obvious way, as shown in the following examples.

Example 3.1.1 If $z = 2 + 3i$ and $w = -1 + 7i$, we have:

$$\begin{aligned} z + w &= (2 + 3i) + (-1 + 7i) = (2 - 1) + (3 + 7)i = 1 + 10i \\ z - w &= (2 + 3i) - (-1 + 7i) = (2 + 1) + (3 - 7)i = 3 - 4i \\ z \cdot w &= (2 + 3i) \cdot (-1 + 7i) = (-2 + 21i^2) + (-3 + 14)i = -23 + 11i. \end{aligned}$$

▲

That is, if in general $z = a + bi$ and $w = c + di$, then

$$\begin{aligned} z + w &= (a + bi) + (c + di) = (a + c) + (b + d)i \\ z - w &= (a + bi) - (c + di) = (a - c) + (b - d)i \\ z \cdot w &= (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i. \end{aligned}$$

In the notation of the multiplication, we may drop the dot "·".

We observe: $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$, $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$ and $\mathrm{Re}(z-w) = \mathrm{Re}(z) - \mathrm{Re}(w)$, $\mathrm{Im}(z-w) = \mathrm{Im}(z) - \mathrm{Im}(w)$.

For the **division**, we need first to define the conjugate of a complex number. The **conjugate** of a complex number $z = a + bi$ is given by the complex number

$$\bar{z} = a - bi.$$

Note that

$$\begin{gathered} \mathrm{Re}(\bar{z}) = +\mathrm{Re}(z) \\ \mathrm{Im}(\bar{z}) = -\mathrm{Im}(z) \\ z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0. \end{gathered}$$

The function $z \in \mathbb{C} \longrightarrow \bar{z} \in \mathbb{C}$ is one-to-one and onto. We called it **conjugation of complex numbers**. Notice that if we compose it with itself, we get the identity function, i.e., $\bar{\bar{z}} = z$. Review the **basic properties of conjugation** listed in **Problem 3.1.6**. They are easy to verify. (See also **Problem 3.2.16**.)

We also define the **absolute value** of a complex number $z = a + bi$. This is defined by:

$$|z| = |a + ib| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

It is also called **modulus** or **magnitude** or **length** of the complex number z. For **example**, $|i| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$ and $|2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$.

It follows immediately that

$$\begin{gathered} |\pm \bar{z}| = |\pm z| = |\pm (a \pm ib)| = \sqrt{a^2 + b^2}, \\ \forall \;\; z \in \mathbb{C}, \;\; |z| \geq 0, \\ |z| = 0 \iff z = 0 \iff \bar{z} = 0, \\ |z|^2 = |\bar{z}|^2 = z\bar{z} = a^2 + b^2. \end{gathered}$$

We now define the **division** of two complex numbers. We begin with an example:

Example 3.1.2 If $z = 2 + 3i$ and $w = -1 + 7i$, we have division in the following way:

$$\frac{z}{w} = \frac{2+3i}{-1+7i} = \frac{2+3i}{-1+7i} \cdot \frac{-1-7i}{-1-7i} = \frac{19-17i}{(-1)^2+7^2} = \frac{19-17i}{50} = \frac{19}{50} - \frac{17}{50}i.$$

▲

In general, when $z = a + bi$ and $w = c + di \neq 0 + 0i = 0$, the **division** $\frac{z}{w}$ $(w \neq 0)$ is defined by

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2} = \frac{ac + bd + (-ad + bc)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{(-ad + bc)}{c^2 + d^2} i.$$

So, the **reciprocal** of any non-zero complex number $z = a + ib \neq 0$ is

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{1}{|z|^2} \cdot \bar{z} = \frac{a - bi}{a^2 + b^2}.$$

All the operations thus defined in the complex numbers satisfy all the properties of the corresponding operations in the real numbers, as we can directly check. So, associativity, commutativity, distribution, reciprocals, and neutral elements are legitimate to use when applicable.

The following most important **properties of the absolute value** in the set of complex numbers are easily verified. (See **Problems 3.1.2, 3.1.3** and **3.1.4**.) We assume that z and w are any complex numbers and any denominators involved are not zero. Then

$$
\begin{gathered}
|z \cdot w| = |z| \cdot |w| \\
\left| \frac{z}{w} \right| = \frac{|z|}{|w|} \\
|z^n| = |z|^n, \ \forall \ n \in \mathbb{Z} \\
|z + w| \leq |z| + |w| \\
\pm(|z| - |w|) \leq ||z| - |w|| \leq |z \pm w| \leq |z| + |w| \\
\max\{|\text{Re } z|, |\text{Im } z|\} \leq |z| \leq |\text{Re } z| + |\text{Im } z|.
\end{gathered}
$$

The inequality $|z+w| \leq |z|+|w|$ is called **triangle inequality**. This holds as an equality if and only if $z = 0$, or $w = 0$, or z is a positive real multiple of w (or w is a positive real multiple of z). (See **Problem 3.1.3**.)

Remark 1: From these properties, we may produce others as corollaries. (See **Problems 3.1.8** and **3.1.17**.)

Remark 2: For $r > 0$, the equation $|z| = |a + bi| = \sqrt{a^2 + b^2} = r$ is equivalent to $a^2 + b^2 = r^2$, which in the (a, b)-plane represents a circle with center the origin and radius r.

3.1.2 Representations and Roots of Complex Numbers

(1) Geometric and Vector Representations of Complex Numbers

We can give a **geometric interpretation of complex numbers** by associating the complex number $z = a + bi$ with the point (a, b) in the 2-dimensional Euclidean plane $\mathbb{R}^2$. We then write that $\mathbb{C} \approx \mathbb{R}^2$. The horizontal axis is called the **real axis**, and the vertical axis is called the **imaginary axis**. The **origin** is $O = (0, 0) = 0 + 0i$. See **Figure 3.1**.

Given a number $z = a + bi$, corresponding to the point (a, b), note that the **conjugate** $\bar{z} = a - bi$ corresponds to the point $(a, -b)$. Thus, geometrically, we get $\bar{z}$ from z by a reflection in the real axis.

The $|z| = |a + bi| = \sqrt{a^2 + b^2}$ is the distance of the point (a, b) from the origin.

Also, notice that the **distance of two points** $P = (a, b)$ and $Q = (c, d)$ in the complex plane is given by

$$d(P, Q) = \sqrt{(c - a)^2 + (d - b)^2} = |z_2 - z_1| = |z_1 - z_2|,$$

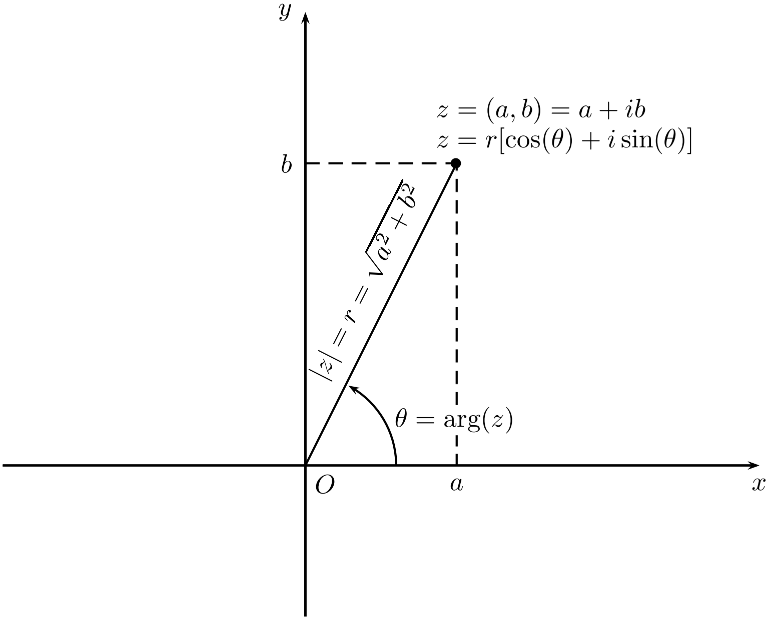

FIGURE 3.1: Complex Numbers and Trigonometric Form

where $z_1 = a + ib$ and $z_2 = c + id$.

In this way, using the complex numbers and their properties, we can describe and solve a lot of geometrical problems in the plane. There are books in the literature that have undertaken such a task. E.g., see bibliography: Dodge 1972, Schwerdtfeger 1979 and Zwikker 2005.

For instance, a circle with center the point $C = (c, d)$ or the complex number $w = c + id$ and radius $r \geq 0$ is the set of complex numbers $z = x + iy$, such that

$$\{z \in \mathbb{C} \ \mid \ |z-w| = r\} = \left\{z = x + iy \in \mathbb{C} \ \mid \ \sqrt{(x-c)^2 + (y-d)^2} = r\right\},$$

etc.

A complex number $z = a + ib$ may also be viewed as the vector $\overrightarrow{OP}$, where $P = (a, b)$. If $w = c + id$ is another complex number viewed as the vector $\overrightarrow{OQ}$, where $Q = (c, d)$, then we have:

1. $z + w = (a + c) + i(b + d) = [(a + c), (b + d)] = \overrightarrow{OP} + \overrightarrow{OQ}$.
2. $z - w = (a - c) + i(b - d) = [(a - c), (b - d)] = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP}$.
3. The **length of the vector** is $\|\overrightarrow{OP}\| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$.

4. The **inner product** is $\overrightarrow{OP} \cdot \overrightarrow{OQ} = ac + bd = \mathrm{Re}(z\bar{w})$.

   So, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are perpendicular iff $\mathrm{Re}(z\bar{w}) = 0$.

5. If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are non-zero vectors (i.e., $z \neq 0$ and $w \neq 0$) and $0 \leq \alpha \leq \pi \pmod{2\pi}$ is the angle between them, then

$$\cos(\alpha) = \frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{\|\overrightarrow{OP}\| \cdot \|\overrightarrow{OQ}\|} = \frac{ac + bd}{\sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}} = \frac{\mathrm{Re}(z\bar{w})}{|z||w|}.$$

6.

$$|z - w|^2 = (a - c)^2 + (b - d)^2 = a^2 + c^2 + b^2 + d^2 - 2(ac + bd) =$$
$$|z|^2 + |w|^2 - 2\mathrm{Re}(z\overline{w}) = |z|^2 + |w|^2 - 2|z||w|\cos(\alpha),$$

   or

$$\|\overrightarrow{PQ}\|^2 = \|\overrightarrow{QP}\|^2 = \|\overrightarrow{OP} - \overrightarrow{OQ}\|^2 =$$
$$\|\overrightarrow{OP}\|^2 + \|\overrightarrow{OQ}\|^2 - 2\|\overrightarrow{OP}\| \cdot \|\overrightarrow{OQ}\|\cos(\alpha),$$

   which is the trigonometric **law of cosines** in the triangle OPQ.

In this way, we can derive the various rules that pertain to vector calculations by means of complex numbers.

(2) Trigonometric Form of Complex Numbers

Since each point (a, b) in the plane $\mathbb{R}^2$ is identified by the complex number $z = a + ib \in \mathbb{C}$, we can rewrite z, as we say, in the **trigonometric form of the complex number**. This is done as follows:

The point (a, b) has polar coordinate representation (r, θ), where $r = \sqrt{a^2 + b^2}$, and if $(a, b) \neq (0, 0)$, then θ is determined $\pmod{2\pi}$ by the relations:

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

If $z = (a, b) = (0, 0)$, then $r = 0$ and θ is indeterminate.

Note that θ can have infinitely many values for any specific point (a, b), because it is determined $\pmod{2\pi}$. So, if a θ is used in the representation of a complex number z in trigonometric form, then any $\theta + 2k\pi$ with $k \in \mathbb{Z}$ can be used as well, without altering the number z.

From these relations we have

$$a = r\cos(\theta) \quad \text{and} \quad b = r\sin(\theta),$$

so that we can write the complex number z as

$$z = (a, b) = a + bi = r[\cos(\theta) + i\sin(\theta)].$$

This representation of z is called the **trigonometric form of** z.

We observe that the **radius** $r = |z| = \sqrt{a^2 + b^2}$ is the **absolute value** or **modulus** or **magnitude** or **length** of z.

We call the angle θ **argument** of z, and we write $\arg(z) = \theta$. See **Figure 3.1**.

Notice that $|\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$ for all θ.

(3) Some Important Remarks on the Argument

As we have seen, the $\arg(z) = \theta$ for any $z \neq 0$ is determined $(\bmod 2\pi)$. Then it is convenient to consider a fixed initial interval of values of the argument of complex numbers, such that all values in it are sufficient to represent any complex number $z \in \mathbb{C} - \{0\}$ in trigonometric form **uniquely**.

If $z = 0$, then $r = |z| = |0| = 0$, and any angle can serve as argument of $z = 0$. Thus, $z = 0$ cannot have unique trigonometric representation no matter how we restrict θ (unless we restrict it at just one value).

Many authors prefer this initial interval of values of the argument to be $0 \leq \theta < 2\pi$. Others prefer the $-\pi \leq \theta < \pi$ or $-\pi < \theta \leq \pi$. These assignments are convenient but arbitrary.

In general, we could use any half-open and half-closed interval of length 2π to begin with. Such an interval provides sufficient values for the angle θ to represent any complex number $z \neq 0$ in trigonometric form **uniquely**.

The argument that varies in such a fixed preassigned interval is called **principal argument**. Here, to write the principal argument, we use the notation $\mathbf{Arg}(z)$ or $\mathbf{Arg}_0(z)$.

Then, to find all possible values of $\arg(z)$ for any $0 \neq z \in \mathbb{C}$, we add $2k\pi$ to $\mathrm{Arg}(z)$, where $k \in \mathbb{Z}$. These values are precisely what we can call all possible arguments of $0 \neq z \in \mathbb{C}$. The notation $\mathbf{Arg}_0(z)$ for the principal argument is justified because if we use $k = 0$ we find $\mathrm{Arg}(z)$. Similarly, any other argument is obtained by using any integer $k \in \mathbb{Z}$, which we denote as $\mathbf{Arg}_k(z)$.

Hence, we define the **argument** (or we can also call it **total argument**) **of a complex number** as the set of values:

$$\arg(0) = \{\theta \mid \theta \in \mathbb{R}\}$$

and if $z \neq 0$, then

$$\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}(z) + 2k\pi \mid k \in \mathbb{Z}\}.$$

Any value singled out of these possible values is simply called **an argument of** z. (We will see more important remarks on the continuity and discontinuity of the argument in **Subsection 3.5.3**.)

Note: Pay attention to the fact that it is not always true that $\arg(z) = \theta = \arctan\left(\dfrac{b}{a}\right)$ even though $\tan(\theta) = \dfrac{b}{a}$. Also, notice if $a = 0$ and $b > 0$, then $\theta = \dfrac{\pi}{2} \pmod{2\pi}$. If $a = 0$ and $b < 0$, then $\theta = -\dfrac{\pi}{2} \pmod{2\pi}$.

We illustrate the definition of the $\operatorname{Arg}(z)$ and the remarks on it by the following examples:

Example 3.1.3 Let $(a,b) = (-1,\sqrt{3}) = -1 + i\sqrt{3}$. Then $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$, $\sin(\theta) = \dfrac{\sqrt{3}}{2} > 0$ and $\cos(\theta) = \dfrac{-1}{2} < 0$. Therefore, θ is a second quadrant angle. In fact, $\theta = \dfrac{2\pi}{3} \pmod{2\pi}$.

We observe that if the initial interval for the principal argument is $[0,2\pi)$, or $(0,2\pi]$, or $[-\pi,\pi)$, or $(-\pi,\pi]$ and $z = -1 + i\sqrt{3}$, then $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \dfrac{2\pi}{3}$, in all these cases.

We also have: $\tan(\theta) = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$ but $\arctan(-\sqrt{3}) = -\dfrac{\pi}{3} \neq \dfrac{2\pi}{3} \pmod{2\pi}$.

Therefore, if we use arc-tangent or another inverse trigonometric function to determine the principal argument, **we may need to add a corrective constant**. In this example, we find that an argument of the complex number $z = -1 + i\sqrt{3}$ (or its principal argument, depending on what we declare as initial interval for the principal argument) is

$$\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \theta = \frac{2\pi}{3} = -\frac{\pi}{3} + \pi = \arctan(-\sqrt{3}) + \pi.$$

That is, the corrective constant is π.

▲

Example 3.1.4 If an argument is found to be $\theta = \dfrac{4\pi}{3}$, then in $[0,2\pi)$ we have $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \dfrac{4\pi}{3}$. But in $[-\pi,\pi)$ or in $(-\pi,\pi]$, we have $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = -\dfrac{2\pi}{3}$.

Similarly, if $\theta = \pi$, then: In $[0,2\pi)$ we have $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \pi$. In $[-\pi,\pi)$ we have $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \pi - 2\pi = -\pi$. In $(-\pi,\pi]$ we have $\operatorname{Arg}(z) = \operatorname{Arg}_0(z) = \pi$.

▲

(4) Some Important Usages of the Trigonometric Form

The trigonometric form of z gives useful interpretations of equality, multiplication, division, powers and roots of complex numbers and renders the pertinent computations easy. So, suppose

$$z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)] \quad \text{and} \quad z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)],$$

where $r_1 \geq 0$ and $r_2 \geq 0$ are two given complex numbers in trigonometric form.

In this form, we obtain the following basic results:

(1) For **equality**, we have:

$$\{\, z_1 = z_2, \text{ or } r_1[\cos(\theta_1) + i\sin(\theta_1)] = r_2[\cos(\theta_2) + i\sin(\theta_2)] \,\} \overset{\text{iff}}{\Longleftrightarrow}$$
$$\text{either} \quad \{r_1 = r_2 = 0 \quad \text{and} \quad \theta_1,\ \theta_2 \quad \text{are any angles}\},$$
$$\text{or} \quad \{r_1 = r_2 > 0 \quad \text{and} \quad \cos(\theta_1) + i\sin(\theta_1) = \cos(\theta_2) + i\sin(\theta_2)\,\}.$$

From this, we get that the following rule of **equality of complex numbers in their trigonometric forms**:

$$z_1 = z_2 \quad \overset{\text{iff}}{\Longleftrightarrow} \quad \text{either} \quad r_1 = r_2 = 0 \quad \text{and} \quad \theta_1,\ \theta_2 \quad \text{are any angles,}$$
$$\text{or } r_1 = r_2 > 0 \text{ and } \theta_1 = \theta_2 \pmod{2\pi}, \text{ i.e., } \theta_1 = \theta_2 + 2k\pi \text{ with } k \in \mathbb{Z}.$$

(Prove these rules as an exercise!)

(2) For **conjugation**, we have:

$$\bar{z} = \overline{r[\cos(\theta) + i\sin(\theta)]} = \bar{r}\ \overline{[\cos(\theta) + i\sin(\theta)]} =$$
$$r[\cos(\theta) - i\sin(\theta)] = r[\cos(-\theta) + i\sin(-\theta)].$$

That is, to find the conjugate of $z = r[\cos(\theta) + i\sin(\theta)]$, we change the θ to $-\theta$ and simplify.

(3) For the **multiplication**, we have:

$$z_1 z_2 = r_1 r_2\,[\cos(\theta_1) + i\sin(\theta_1)] \cdot [\cos(\theta_2) + i\sin(\theta_2)] =$$
$$r_1 r_2\,[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\,(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))]$$
$$= r_1 r_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]\,.$$

Hence, the product $z_1 z_2$ in trigonometric form has absolute value the product of the absolute values of z_1 and z_2 and argument the sum of their arguments (angles) $\pmod{2\pi}$.

(4) For the **division**, when $z_2 \neq 0$, we have:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{\cos(\theta_1) + i\sin(\theta_1)}{\cos(\theta_2) + i\sin(\theta_2)} =$$

$$\frac{r_1}{r_2}\left\{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i[\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)]\right\}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

Hence, the division $\dfrac{z_1}{z_2}$ in trigonometric form has absolute value the quotient of the absolute values of z_1 and $z_2 \neq 0$ and argument the difference of the argument of the numerator minus the argument of the denominator $\pmod{2\pi}$.

(5) Powers with Integer Exponents and De Moivre Formula

To find the **powers** of a complex number $z = a + bi$ with integer exponents, we may use the **Binomial Theorem:** , For all exponents $n = 0,\ 1,\ 2,\ 3,\ \dots,$

$$(a + bi)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}(bi)^k = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k i^k,$$

where $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$. We remember that $0! = 1$ and so $\dbinom{n}{0} = 1$. For $k \in \mathbb{N}$, we simplify

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots3\cdot2\cdot1}.$$

The powers of i with integer exponents k are simplified by the rule

$$i^k = \begin{cases} +1, & \text{if } k = 4m \\ +i, & \text{if } k = 4m+1 \\ -1, & \text{if } k = 4m+2 \\ -i, & \text{if } k = 4m+3, \end{cases}$$

where m is the integer quotient of k divided by 4. (See **Problem 3.1.5**.)

Example 3.1.5 We have

$$z^2 = (a + bi)^2 = a^2 - b^2 + 2abi = (a-b)(a+b) + 2abi,$$

$$z^3 = (a+bi)^3 =$$
$$(a^3 - 3ab^2) + (3a^2b - b^3)i = a(a^2 - 3b^2) + b(3a^2 - b^2)i =$$
$$a(a - \sqrt{3}\ b)(a + \sqrt{3}\ b) + b(\sqrt{3}\ a - b)(\sqrt{3}\ a + b)i,$$

$$z^4 = (a+bi)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k i^k = \ \dots\ ,$$

and so on.

▲

The binomial method for computing powers of complex numbers is not efficient for "large" exponents. An efficient method uses the trigonometric form of the complex number z as follows:

A simple inductive argument using n multiplications of

$$z = r[\cos(\theta) + i\sin(\theta)]$$

by itself, for any positive integer n, results in the so-called **De Moivre**[1] **formula**

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

If $z \neq 0$, this formula is easily extended to $n = 0$, in which case both sides are equal to 1, and to n a negative integer, since $z^{-k} = \dfrac{1}{z^k}$ for k positive integer.

So, the **De Moivre Formula** is valid $\forall\ n \in \mathbb{N} = \{1,\ 2,\ 3,\ 4,\ \dots\}$ regardless of z, and if $z \neq 0$, then it is valid for all integers $n \in \mathbb{Z} = \{0,\ \pm1, \pm2,\ \pm3,\ \pm4,\ \dots\}$.

(6) Roots of Complex Numbers

Now we study the **roots** of complex numbers. First $\forall n \in \mathbb{N} = \{1,\ 2,\ 3,\ 4, ...\}$ we consider the equation

$$z^n = 1 = 1[\cos(0) + i\sin(0)].$$

Letting $z = \rho[\cos(\phi) + i\sin(\phi)]$, in trigonometric form and so $\rho \geq 0$, by the De Moivre formula we find

$$z^n = \rho^n[\cos(n\phi) + i\sin(n\phi)] = 1 = 1[\cos(0) + i\sin(0)].$$

---

[1] Abraham De Moivre, French mathematician, 1667-1754.

So, $\rho^n = 1$, $\cos(n\phi) = \cos(0)$ and $\sin(n\phi) = \sin(0)$. Therefore, $\rho = \sqrt[n]{1} = 1 > 0$ and $\phi = \dfrac{0 + 2k\pi}{n} = \dfrac{2k\pi}{n}$, with $k \in \mathbb{Z}$. So,

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right), \quad k \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$, by the division identity, we have $k = nq + l$ with $q \in \mathbb{Z}$ and $l = 0,\ 1,\ 2, \ldots, n-1$. So, for $q \in \mathbb{Z}$ and $l = 0,\ 1,\ 2,\ \ldots,\ n-1$

$$z_{ql} = \cos\left(\frac{2l\pi}{n} + 2q\pi\right) + i \sin\left(\frac{2l\pi}{n} + 2q\pi\right).$$

Since the functions cos and sin are 2π-periodic, we find

$$z_l = \cos\left(\frac{2l\pi}{n}\right) + i \sin\left(\frac{2l\pi}{n}\right), \quad l = 0,\ 1,\ 2, \ldots, n-1.$$

These n numbers are pairwise different. (Check this and make sure you understand it, as an exercise!)

Finally, **the equation**

$$z^n = 1$$

has n different distinct solutions given by the formulae

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right), \quad k = 0,\ 1,\ 2, \ldots, n-1.$$

These n different distinct solutions are called the n **n$^{\text{th}}$ roots of unity**. Notice that $z_0 = 1$, for all $n \in \mathbb{N}$.

Example 3.1.6 The **first root** of unity is trivially itself, a fact true for any complex number.

The **second or square roots of unity** are easily found to be 1 and -1.

The **third or cubic roots of unity** are easily computed to be 1, $-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ and $-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.

The **fourth or quadric roots of unity** are easily computed to be 1, i, -1 and $-i$.

(Check these results and compute the five **fifth roots of unity**.)

▲

Next, we consider any complex number

$$w = a + ib = r[\cos(\theta) + i\sin(\theta)] \neq 0 + i0 = 0,$$

where $r = |w| > 0$ and θ are the magnitude and the argument of w, respectively, as defined above. θ could be any argument, but if it is convenient, we may use the unique one in the interval $[0, 2\pi)$, in radians.

Working as we did with the roots of unity and using the **De Moivre formula**, we find that the equation

$$z^n = \rho^n[\cos(n\phi) + i\sin(n\phi)] = w = a + ib = r[\cos(\theta) + i\sin(\theta)]$$

has again n different distinct solutions which are given by the formulae

$$z_k = \sqrt[n]{w} = \sqrt[n]{r}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right],$$
$$k = 0, 1, 2, \ldots, n-1,$$

where $\sqrt[n]{r} > 0$ is the positive $\mathrm{n^{th}}$ root of the positive number $r > 0$. (Work this out one more time as before, for practice!)

These n different distinct solutions are called **the n $\mathbf{n^{th}}$ roots of $w \in \mathbb{C}$.**

Note: If $n \in \mathbb{N}$, then all the $\mathrm{n^{th}}$ roots of $w = 0 + i\,0 = 0$ are equal to $0 + i\,0 = 0$, since $|0 + i\,0| = 0$.

Also, notice that in calculus, $\forall\ n \in \mathbb{N}$ and $\forall\ a \geq 0$ the symbol $\sqrt[n]{a} = a^{\frac{1}{n}} = p$ represents the unique real value $p \geq 0$ such that $p^n = p \cdot p \cdot \ldots \cdot p = a$ (n times). But in the general complex analysis, the symbol $w^{\frac{1}{n}} = \sqrt[n]{w}$, for any complex number $w \neq 0$, represents n different distinct complex values, namely the n $\mathrm{n^{th}}$ toots of w. (Study and solve completely **Problem 3.1.18**!)

Example 3.1.7 Find the three cubic roots of $w = 2 - 3i$.

We have that $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. We may take θ to be the unique angle of the fourth quadrant in $[0, 2\pi)$ with

$$\cos(\theta) = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \text{and} \qquad \sin(\theta) = \frac{-3}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}.$$

Then we find that $\theta = 2\pi - \arccos\left(\frac{2}{\sqrt{13}}\right) \simeq 5.3003916$ radians, approximately.

So, we have that the three cubic roots of $w = 2 - 3i$ are:

$$z_0 = \sqrt[3]{\sqrt{13}}\left[\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right] = \sqrt[6]{13}\left[\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right],$$

and similarly

$$z_1 = \sqrt[6]{13}\left[\cos\left(\frac{2\pi+\theta}{3}\right) + i\sin\left(\frac{2\pi+\theta}{3}\right)\right],$$

$$z_2 = \sqrt[6]{13}\left[\cos\left(\frac{4\pi+\theta}{3}\right) + i\sin\left(\frac{4\pi+\theta}{3}\right)\right].$$

From these, we can evaluate the approximate values of z_0, z_1 and z_2, in seven decimal digits, to be:

$$z_0 \simeq -0.2986283 + i\ 1.5040465,$$

$$z_1 \simeq -1.1532283 - i\ 1.0106429,$$

$$z_2 \simeq +1.4518566 - i\ 0.4934035.$$

Notice that we can avoid the approximate answers by doing more work in the following way: We need to express the $\cos\left(\frac{\theta}{3}\right)$ and $\sin\left(\frac{\theta}{3}\right)$ in terms of the known numbers $\cos(\theta) = \frac{2\sqrt{13}}{13}$ and $\sin(\theta) = \frac{-3\sqrt{13}}{13}$. To do this, we use the trigonometric formulae $\cos(\theta) = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$ and $\sin(\theta) = 3\sin\left(\frac{\theta}{3}\right) - 4\sin^3\left(\frac{\theta}{3}\right)$. Then, we use the formulae for solving cubic polynomial equations by radicals (e.g., see bibliography: Dickson 1949, 46-51). After this, we use the expansion formulae of $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$ in the three third roots of $w = 2 - 3i$ obtained and calculate them accurately in terms of radicals. (See **Problem 3.1.13**!)

This work is quite lengthy and cumbersome even when $n = 3$ or $n = 4$ and thus impractical. For $n \geq 5$, there are not even formulae for solving any general polynomial of degree $n \geq 5$.

Only for $n = 2$, is such a work quick and practical, as we shall see in the **next Subsection, 3.1.3**.

▲

(7) Argument Revisited

To accurately understand the properties of the argument, we must first define the following: Suppose $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$ non-empty sets and $z \in \mathbb{C}$. Then we define the following subsets of $\mathbb{C}$:

1. The **sum subset**

$$S + T = \{w = s + t \mid s \in S \text{ and } t \in T\}.$$

2. The **difference subset**

$$S - T = \{w = s - t \mid s \in S \text{ and } t \in T\}.$$

3. The **product subset**

$$S \cdot T = \{w = s \cdot t \mid s \in S \text{ and } t \in T\}.$$

4. The **multiple subset**

$$z \cdot S = \{z\} \cdot S = \{w = z \cdot s \mid s \in S\}.$$

(**Note**: If one of the sets S and T is empty, then these new sets are also empty.)

If $S = \{w\}$ (one element set), then we may simply write $w+T,\ w-T$ for $\{w\}+T$ and $\{w\}-T$, respectively (as we see in the multiple subset).

We notice that for the set of integers $\mathbb{Z}$, any $k \in \mathbb{Z}$ and any $\emptyset \neq M \subseteq \mathbb{Z}$, we have the easy relations:

$$\begin{gathered}
1 \cdot \mathbb{Z} = \mathbb{Z} \\
(-1) \cdot \mathbb{Z} = -\mathbb{Z} = \mathbb{Z} \\
\mathbb{Z} \cdot \mathbb{Z} = \mathbb{Z} \\
k \cdot \mathbb{Z} \subseteq \mathbb{Z} \quad \text{and equality holds iff} \quad k = \pm 1 \\
M \cdot \mathbb{Z} \subseteq \mathbb{Z} \\
k + \mathbb{Z} = \mathbb{Z} \\
k - \mathbb{Z} = \mathbb{Z} \\
\pm M \pm \mathbb{Z} = \mathbb{Z} \\
\mathbb{Z} + \mathbb{Z} = \mathbb{Z} \\
\mathbb{Z} - \mathbb{Z} = \mathbb{Z} \\
\pm \mathbb{Z} \pm \mathbb{Z} \pm \ldots \pm \mathbb{Z} = \mathbb{Z}.
\end{gathered}$$

Using these definitions and notations, we can write the sets of $\arg(z)$ for any $z \in \mathbb{C}$ as

$$\arg(0) = \{\theta \mid \theta \in \mathbb{R}\} = \mathbb{R} \subset \mathbb{C},$$

and if $z \neq 0$, then

$$\begin{gathered}
\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}(z) + 2k\pi \mid k \in \mathbb{Z}\} = \\
\mathrm{Arg}_0(z) + 2\pi\mathbb{Z} = \mathrm{Arg}(z) + 2\pi\mathbb{Z} \subset \mathbb{C}.
\end{gathered}$$

In view of what we have seen, for any $z,\ z_1,\ z_2 \in \mathbb{C}$, any $k \in \mathbb{Z}$ and any $n \in \mathbb{N}$, we can easily prove the following set relations, which we can easily check:

$$\arg(\bar{z}) = -\arg(z)$$
$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$
$$\text{If } \ z_2 \neq 0, \ \text{ then } \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$
$$\arg\left(z^k\right) = k \cdot \mathrm{Arg}_0(z) + 2\pi\mathbb{Z} \supseteq k \cdot \arg(z)$$

and equality holds if and only if $k = \pm 1$.

If the n n$^{\text{th}}$ roots of $z \neq 0$ are given by

$$w_k = \sqrt[n]{|z|}\left[\cos\left(\frac{\mathrm{Arg}(z) + 2k\pi}{n}\right) + i\,\sin\left(\frac{\mathrm{Arg}(z) + 2k\pi}{n}\right)\right],$$

with $k = 0,\ 1,\ 2, \ldots n-1$, then

$$\arg(w_k) = \frac{\mathrm{Arg}(z) + 2k\pi}{n} + 2\pi\mathbb{Z}, \quad \text{for} \quad k = 0,\ 1,\ 2, \ldots n-1.$$

The union of these sets is

$$\arg\left(z^{\frac{1}{n}}\right) = \left\{\frac{\mathrm{Arg}(z) + 2k\pi}{n} \mid k = 0,\ 1,\ 2,\ \ldots\ n-1\right\} + 2\pi\mathbb{Z}.$$

Notice that if $\mathrm{Arg}(z) \in [0, 2\pi)$, then $\dfrac{\mathrm{Arg}(z) + 2k\pi}{n} \in [0, 2\pi)$ for all $k = 0,\ 1,\ 2,\ 3 \ldots n-1$.

Problems

3.1.1 For $z = 3 - 2i$ and $w = 3 + i$, compute $z + w, z - w, zw$ and z/w. Also compute $|z|, |w|, |zw|$ and $|z/w|$.

3.1.2 For any $z \in \mathbb{C}$, prove

$$\max\{|\mathrm{Re}\ z|,\ |\mathrm{Im}\ z|\} \leq |z| \leq |\mathrm{Re}\ z| + |\mathrm{Im}\ z|.$$

In equivalent form with real numbers, this inequality says that for any two real numbers a and b, we have

$$\max\{|a|,\ |b|\} \leq \sqrt{a^2 + b^2} \leq |a| + |b|.$$

Give a geometric interpretation of this inequality.

3.1.3 Prove the **triangle inequality** $|z + w| \leq |z| + |w|$. Then prove that it holds as an equality if and only if $z = 0$, or $w = 0$, or z is a positive real multiple of w, or w is a positive real multiple of z.

3.1.4 For all complex numbers z and w prove

$$\pm\left(|z| - |w|\right) \leq \left||z| - |w|\right| \leq |z \pm w| \leq |z| + |w|.$$

3.1.5 Show that for $n \in \mathbb{Z}$

$$(-1)^n = \begin{cases} +1 & \text{if } n = 2k\text{, that is, } n \text{ is even} \\ -1 & \text{if } n = 2k+1\text{, that is, } n \text{ is odd} \end{cases}$$

$$i^n = \begin{cases} +1 & \text{if } n = 4k \\ +i & \text{if } n = 4k+1 \\ -1 & \text{if } n = 4k+2 \\ -i & \text{if } n = 4k+3 \end{cases}$$

$$\cos(n\pi) = \left.\begin{cases} +1 & \text{if } n = 2k \\ -1 & \text{if } n = 2k+1 \end{cases}\right\} = (-1)^n$$

$$\sin(n\pi + \frac{\pi}{2}) = \left.\begin{cases} +1 & \text{if } n = 2k \\ -1 & \text{if } n = 2k+1 \end{cases}\right\} = (-1)^n$$

3.1.6 For any complex numbers $z, w \in \mathbb{C}$ prove the following **basic properties of conjugation**.

1. $\bar{\bar{z}} = z$
2. $\overline{z \pm w} = \bar{z} \pm \bar{w}$
3. $\overline{zw} = \bar{z}\bar{w}$
4. $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$
5. $\forall\ n \in \mathbb{Z}, \quad \overline{z^n} = (\bar{z})^n$
6. $|\bar{z}| = |z|$
7. $\arg(\bar{z}) = -\arg(z), \quad (\text{mod } 2\pi)$
8. $\text{Re}(\bar{z}) = \frac{1}{2}(z + \bar{z}) = \text{Re}(z)$
9. $\text{Im}(\bar{z}) = \frac{-1}{2i}(z - \bar{z}) = \frac{i}{2}(z - \bar{z}) = -\text{Im}(z)$

10. $z \in \mathbb{R} \Longleftrightarrow z = \bar{z}$.

11. z is pure imaginary $\Longleftrightarrow z = -\bar{z}$.

12. $\bar{z} = 0 \Longleftrightarrow z = 0 \Longleftrightarrow |z| = 0 \Longleftrightarrow |\bar{z}| = 0$.

13. $\bar{z} = z^{-1} \Longleftrightarrow |z| = 1$.

3.1.7 Give the geometric interpretation of finding the conjugate $\bar{z}$ of a complex number z.

3.1.8 Given any $z,\ w \in \mathbb{C}$, prove the two identities:

$$\begin{aligned} &\text{(a)} \qquad |1 - z\bar{w}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2). \\ &\text{(b)} \qquad |z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2. \end{aligned}$$

Give the geometric interpretation of the equality in (b).

3.1.9 Give the geometric interpretation of multiplying the complex number z by -1.

3.1.10 Give the geometric interpretation of multiplying the complex number z by i and then by $-i$.

[Hint: Note that for i, $r = 1$ and $\theta = \dfrac{\pi}{2}$.]

3.1.11 Give the geometric interpretation of forming the reciprocal $1/z$ of the complex number z. Distinguish between the cases $|z| < 1$, $|z| = 1$ and $|z| > 1$. How about $z = 0$?

3.1.12 Find all solutions to the equations $z^3 - 8 = 0$ and $z^4 = 1 - i$.

3.1.13 Find the exact answers in **Example 3.1.7** by following the steps we have described at the end of it.

3.1.14 Find all solutions to the equations $z^n = -1$, $z^n = i$ and $z^n = -i$.

3.1.15 For any solution $z \neq 1$ of the equation $z^n = 1$, show that $1 + z + z^2 + \ldots + z^{n-1} = 0$.

3.1.16 Find all solutions to the equations:

$$\begin{aligned} &\text{(a)} \qquad z^4 - 30z^2 + 289 = 0, \qquad &&\text{(b)} \qquad (z + 1)^n + (z - 2)^n = 0, \\ &\text{(c)} \qquad z^8 + z^4 + 1 = 0, \qquad &&\text{(d)} \qquad z^9 + z^6 + z^3 + 1 = 0. \end{aligned}$$

3.1.17 Prove that for all $z \in \mathbb{C}$ and any $n \in \mathbb{N}$, we have $\left| z^{\frac{1}{n}} \right| = |z|^{\frac{1}{n}}$.

3.1.18 (a) If $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$, then prove that $z^{\frac{p}{q}} = \sqrt[|q|]{z^{\text{sign}(q)\cdot p}}$ and this has at most $|q|$ values.

(b) Show that this has exactly $|q|$ values iff p and q are relatively prime.

(c) Give an example in which the number of values is less than $|q|$.

3.1.19 Let $z \in \mathbb{C}$, $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots\ \omega_{k-1}$ be the k k$^{\text{th}}$ roots of 1. Prove that the k k$^{\text{th}}$ roots of z^k are given by $z\omega_0,\ z\omega_1,\ \ldots\ z\omega_{k-1}$.

3.1.20 Let $k \in \mathbb{N}$, $\omega_0,\ \omega_1,\ \ldots\ \omega_{k-1}$ be the k k$^{\text{th}}$ roots of 1 and let ψ be a k$^{\text{th}}$ root of -1. Prove that the k k$^{\text{th}}$ roots of -1 are given by $\psi\omega_0,\ \psi\omega_1,\ \ldots\ \psi\omega_{k-1}$.

Generalize this problem to the roots of any $z \in \mathbb{C}$.

3.1.21 For any $z \in \mathbb{C} - \{0\}$, any $n \in \mathbb{Z} - \{0\}$ and $k \in \mathbb{N} - \{1\}$, we let A be the set of the n^{th} roots of z and B the set of the $(kn)^{\text{th}}$ roots of z^k. Prove that $A \subsetneq B$.

Use this to explain why the reasoning $-1 = \sqrt[3]{-1} = \sqrt[6]{(-1)^2} = \sqrt[6]{1} = 1$ is false. Where is the mistake?

3.1.22 Show that all solutions to the equation $z^n = 1$ are vertices of a canonical (regular) n-gon inscribed in the unit circle, with one vertex the point $(1, 0)$.

3.1.23 For a fixed point z_1, what are the sets of points z satisfying the relations $|z-z_1| = r$, $|z-z_1| < r$, $|z-z_1| \le r$, $|z-z_1| > r$ and $|z-z_1| \ge r$, where $r > 0$?

3.1.24 Let $z = \sqrt{3} - i$ and $w = 2 + 2i$. Find the absolute values, the principal arguments in $[0, 2\pi)$ and the total arguments of: z, w, $z + w$, $z - w$, zw, $\dfrac{z}{w}$, z^{10}, $(zw)^{10}$, $(zw)^{-10}$ and $z^{\frac{1}{10}}$.

3.1.25 For two different fixed points z_1 and z_2, what is the set of points z satisfying the equation $|z - z_1| + |z - z_2| = r$, where $r > 0$?

3.1.26 For two different fixed points z_1 and z_2, what is the set of points z satisfying the equation $|z - z_1| - |z - z_2| = r$, where $r > 0$?

3.1.27 (I) Use the **De Moivre formula** along with the **Binomial Theorem**

$$[\cos(x)+i\sin(x)]^n = \cos(nx)+i\sin(nx) = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(x)\cdot i^k\cdot\sin^k(x)$$

to find the $\cos(nx)$ and $\sin(nx)$ for $n=2$, $n=3$ and $n=4$ in terms of $\cos(x)$ and $\sin(x)$.

(II) Now prove the following important general trigonometric formulae for any $n \geq 0$ integer, where by $\left[\left[\frac{n}{2}\right]\right]$ we denote the integer part of $\frac{n}{2}$. Also, remember that if $l > n$ integers, then $\binom{n}{l} = 0$:

(a)

$$\cos(nx) = \sum_{k=0}^{[\![\frac{n}{2}]\!]}(-1)^k\binom{n}{2k}\cos^{n-2k}(x)\sin^{2k}(x).$$

Use this to prove that for any $n \geq 0$ integer, $\cos(nx)$ can be expressed as a polynomial of $\cos(x)$. Find this polynomial.

(b)

$$\sin(nx) = \sum_{k=0}^{[\![\frac{n}{2}]\!]}(-1)^k\binom{n}{2k+1}\cos^{n-2k-1}(x)\sin^{2k+1}(x).$$

Use this to prove that for any $n \geq 1$ **odd** integer, $\sin(nx)$ can be expressed as a polynomial of $\sin(x)$. Find this polynomial.

Then prove that for any $n \geq 2$ **even** integer, $\sin(nx)$ is a polynomial of both $\sin(x)$ and $\cos(x)$ (i.e., it cannot be a polynomial of just $\sin(x)$). Find this polynomial.

(c)

$$\tan(nx) = \frac{\sum_{k=0}^{[\![\frac{n}{2}]\!]}(-1)^k\binom{n}{2k+1}\tan^{2k+1}(x)}{\sum_{k=0}^{[\![\frac{n}{2}]\!]}(-1)^k\binom{n}{2k}\tan^{2k}(x)}.$$

(d) If $\arctan(x) = \sqrt{2}$, find $\tan(4x)$ and $\tan(5x)$.

(III) In the last formula, replace the "tan" with $\frac{1}{\cot}$ to find the corresponding formula for the $\cot(nx)$.

3.1.3 Square Roots without De Moivre

Here we will study the function of square root in $\mathbb{R}$ and $\mathbb{C}$ without using the De moivre Formula and the roots of complex numbers as they were expressed in trigonometric forms before. It is important to study this function in Cartesian coordinates and see its connection with some aspects discussed in **Subsections 3.5.3** and **3.5.4** later. So, we begin first with:

1: Square Roots in $\mathbb{R}$

Definition 3.1.1 *A square root of a non-negative real number $x \geq 0$ is defined to be any real number r such that $r^2 = x$.*

Since $\forall\, r \in \mathbb{R}$, we have that $r^2 \geq 0$ the condition $x \geq 0$, in this definition, is necessary.

If $x = 0$, then $r^2 = r \cdot r = 0$, and so $r = 0$ necessarily.

If $x > 0$, then by looking at the continuous polynomial function $f(r) = r^2 - x$, whose domain is all $\mathbb{R}$, we see that

$$f(0) = -x < 0 \quad \text{and} \quad f\left(x + \frac{1}{2}\right) = \left(x + \frac{1}{2}\right)^2 - x = x^2 + \frac{1}{4} > 0.$$

Therefore, when $x > 0$, by the **Intermediate Value Theorem** (study this theorem one more time), there is a real solution r of the equation $r^2 - x = 0$, which is equivalent to $r^2 = x(> 0)$, such that $0 < r < x + \frac{1}{2}$.

Since $(-1)^2 = +1$, if r is a square root of x, then $-r = -1 \cdot r$ is also a square root of x. These two square roots are equal if and only if

$$r = -r \Longleftrightarrow r = 0 \Longleftrightarrow x = 0.$$

So, if $r > 0$, then $-r < 0$, and so any real number $x > 0$ has at least two different square roots, one positive and the other its opposite negative.

By looking at $r^2 = x$ as $r^2 - x = 0$, i.e., as a polynomial equation of degree two in the variable (indeterminate) r over the real numbers, we know from algebra that this equation cannot have more than two solutions. In conclusion:

Any positive real number has exactly two different square roots, one positive and the other its opposite negative. As we have already said, **the square roots of 0 are both equal to** 0, which is also seen by the fact that the solutions of $r^2 = 0$, or $(r - 0)(r - 0) = 0$, are $r_1 = r_2 = 0$.

Since $r^2 \geq 0,\ \forall\, r \in \mathbb{R}$, a negative real number cannot have a square root which is a real number. The problem of the square root(s) of a negative real number is taken care of by introducing the symbol i such that

$$i^2 = -1\ (< 0).$$

Therefore, $i \neq 0$ and i cannot be a real number, since for any real number $a \neq 0$ we always have: $a^2 > 0$. We say that i is an **imaginary number**.

Then for $x < 0$, we have that $-x = |x| > 0$. Hence by the discussion above, we have two opposite real numbers $\pm\, r \neq 0$ which are the two square roots of $-x = |x| > 0$, i.e., $(\pm\, r)^2 = |x| = -x > 0$.

Definition 3.1.2 *We define the square roots of a negative number $x < 0$ to be the imaginary numbers $i \cdot (\pm\, r) = \pm\, i \cdot r$, where r is the positive square root of $|x| = -x > 0$ (defined above).*

Now, when we use the **square root symbol** $\sqrt{}$ with any real number $x \in \mathbb{R}$ and we write

$$\sqrt{x},$$

then we mean the number $+r \geq 0$ **if** $x \geq 0$ and the imaginary number $+i \cdot r$, with $r > 0$, **if** $x < 0$, where r was the number found (or defined) above. So, this notation means

$$\sqrt{x} = +\sqrt{x}.$$

That is why when x is a real variable or an unknown real number, we have

$$\sqrt{x^2} = |x| \geq 0,$$

and it is not equal to simply x or $\pm x$.

To indicate the negative square root $-r$ when $x > 0$ or the imaginary root $-i \cdot r$ when $x < 0$, we explicitly write:

$$-\sqrt{x}.$$

To indicate both square roots together, we write:

$$\pm\sqrt{x}.$$

In conclusion, we have obtained the **result**: For any real number $x \in \mathbb{R}$, we have:

$$\text{Square roots of } x = \begin{cases} \pm\sqrt{0} = 0, & \text{if} \quad x = 0 \\ \pm\sqrt{x} = \pm r, & \text{if} \quad x > 0 \\ \pm i\sqrt{-x} = \pm ir, & \text{if} \quad x < 0 \end{cases}$$

where $r \in \mathbb{R}$ was defined above. (For example: $\sqrt{-1} = i$, $-\sqrt{-1} = -i$ and $\pm\sqrt{-1} = \pm\, i$, respectively.)

(Remember, finally, that this is the way we must understand and use the **square root symbol** $\sqrt{}$ with real numbers.) Now, we continue with:

2: Square Roots in $\mathbb{C}$

We consider that $\mathbb{R} \subset \mathbb{C}$ under the identification: If $x \in \mathbb{R}$, then $x = x + i0 \in \mathbb{C}$. Then we have

Definition 3.1.3 *For any $z = x + iy \in \mathbb{C}$, where $x,\ y \in \mathbb{R}$, we define a square root of z to be any complex number $w = u + iv$ such that*

$$w^2 = z \quad \text{or} \quad (u + iv)^2 = x + iy.$$

By developing the last equation, we find

$$u^2 - v^2 + 2iuv = x + iy,$$

and so we get the following quadratic system

$$\left\{ \begin{array}{c} u^2 - v^2 = x \\ 2uv = y \end{array} \right\} \tag{3.1}$$

of two equations in two unknowns, u and v.

If $z = 0 = 0 + i0$, i.e., $x = y = 0$, then the system

$$\left\{ \begin{array}{c} u^2 - v^2 = 0 \\ 2uv = 0 \end{array} \right\}$$

is easily solved, and we find the unique solution $u = v = 0$. So, $w = 0 = 0 + i0$ is the only square root of $z = 0 = 0 + i0$.

Otherwise, when $z = x + iy \neq 0 + i0$, i.e., $x \neq 0$ or $y \neq 0$, we work as follows: We observe that

$$(u^2 + v^2)^2 = (u^2 - v^2)^2 + 4u^2v^2 = x^2 + y^2$$

and since $u^2 + v^2 \geq 0$, we get that

$$u^2 + v^2 = \sqrt{x^2 + y^2},$$

that is, we have only one equation with the + sign in front of the square root. (The − sign is excluded from this situation.)

Now, we solve the system

$$\left\{ \begin{array}{c} u^2 - v^2 = x \\ \\ u^2 + v^2 = \sqrt{x^2 + y^2} \end{array} \right\},$$

and we find

$$\begin{pmatrix} u^2 = \dfrac{1}{2}\left(x + \sqrt{x^2+y^2}\right) \geq 0 \\ \\ v^2 = \dfrac{1}{2}\left(-x + \sqrt{x^2+y^2}\right) \geq 0 \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} u = \pm\sqrt{\dfrac{1}{2}\left(x + \sqrt{x^2+y^2}\right)} \\ \\ v = \pm\sqrt{\dfrac{1}{2}\left(-x + \sqrt{x^2+y^2}\right)} \end{pmatrix}.$$

Since $2uv = y$, the following conditions must be satisfied:

1. If $y > 0$, then both u and v are not zero and they have the same sign.
2. If $y < 0$, then both u and v are not zero and they have opposite sign.
3. If $y = 0$, then $uv = 0$, and so either $u = 0$ or $v = 0$ (inclusive or).

To incorporate the first two cases in one formula, we introduce the function

$$y \in \mathbb{R} - \{0\} \longrightarrow \frac{y}{|y|} = \begin{cases} +1, & \text{if} \quad y > 0 \\ \\ -1, & \text{if} \quad y < 0. \end{cases}$$

We observe that this function is not defined at $y = 0$. At $y = 0$, it has a jump equal to $|+1-(-1)| = 2$, and therefore it cannot be extended (or defined) continuously at $y = 0$.

Now, taking into account what we have found for possible values of u and v together with the above three conditions, we find that the initial system **(3.1)** above has only **two solutions** (and not four, as someone prematurely may have thought) **which are**:

(a) **If $y \neq 0$, then:**

$$\begin{pmatrix} u = \pm\sqrt{\dfrac{1}{2}\left(x + \sqrt{x^2+y^2}\right)} \\ \\ v = \pm\dfrac{y}{|y|}\sqrt{\dfrac{1}{2}\left(-x + \sqrt{x^2+y^2}\right)} \end{pmatrix}$$

where the + is coupled with the + and the − with the −.

(b) **If** $y = 0$, **then**:

1. $u = v = 0$, when $x = y = 0$
2. $u = 0$, when $x < 0$ and $v = \pm\sqrt{-x}$,
3. $v = 0$, when $x > 0$ and $u = \pm\sqrt{x}$.

In conclusion, we have found the following result: The two square roots of a complex number: $z = x + iy$ are

$$\sqrt{z} = \sqrt{x+iy} = u + iv = u(x,y) + iv(x,y) =$$

$$\begin{cases} \pm\left[\sqrt{\frac{1}{2}\left(x+\sqrt{x^2+y^2}\right)} + i\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)}\right], & \text{if } y \neq 0 \\ \pm\sqrt{x}, & \text{if} \quad y = 0,\ x \geq 0 \\ \pm i \cdot \sqrt{-x}, & \text{if} \quad y = 0,\ x \leq 0. \end{cases}$$

So, a complex number $z = x + iy$ has two square roots. For $z = 0$, the two square roots coincide and are equal to 0.

Note: In complex analysis, we write the function

$$f(z) = \sqrt{z}, \quad \text{with} \quad z \in \mathbb{C}.$$

Unless otherwise declared, this usually means the two-valued-complex function with answers the two values found above. So, we must keep in mind the context (real or complex) in which we use the **square root symbol** $\sqrt{\quad}$.

Important remark: Since the function

$$H : \mathbb{R} \longrightarrow \mathbb{R} \quad : \quad y \longrightarrow H(y) = \frac{y}{|y|} = \begin{cases} +1 & \text{if} \quad y > 0 \\ \text{undefined} & \text{if} \quad y = 0 \\ -1 & \text{if} \quad y < 0 \end{cases}$$

cannot be defined continuously at $y = 0$ (i.e., no matter how we define it at $y = 0$, it will always be discontinuous at $y = 0$), we observe that the two square roots of $z = x + iy$ found above are discontinuous at

$$y = 0 \quad \text{when} \quad v(x,y) = \sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)} \neq 0.$$

This happens when

$$v(x,0)=\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2}\right)}=\sqrt{\frac{1}{2}\left(-x+|x|\right)}\neq 0 \quad\Longleftrightarrow\quad x=-|x|<0,$$

that is, when and only when x is negative.

So, the two square roots of $z=x+iy$ as functions of x and y are:

$$\textbf{(I)}\qquad f_+(z)=f_+(x+iy)=+\sqrt{z}=+\sqrt{x+iy}=$$

$$u_+ + iv_+ = u_+(x,y)+iv_+(x,y)=$$

$$\begin{cases} +\left[\sqrt{\frac{1}{2}\left(x+\sqrt{x^2+y^2}\right)}+i\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)}\right], & \text{if } y\neq 0\\ +\sqrt{x}, & \text{if } \ y=0,\ x\geq 0\\ +i\cdot\sqrt{-x}, & \text{if } \ y=0,\ x\leq 0 \end{cases}$$

and

$$\textbf{(II)}\qquad f_-(z)=f_-(x+iy)=-\sqrt{z}=-\sqrt{x+iy}=$$

$$u_- + iv_- = u_-(x,y)+iv_-(x,y)=$$

$$\begin{cases} -\left[\sqrt{\frac{1}{2}\left(x+\sqrt{x^2+y^2}\right)}+i\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)}\right], & \text{if } y\neq 0\\ -\sqrt{x}, & \text{if } \ y=0,\ x\geq 0\\ -i\cdot\sqrt{-x}, & \text{if } \ y=0,\ x\leq 0. \end{cases}$$

Both square root functions **(I)** and **(II)** are discontinuous along the negative x-axis (semi-axis). For example, with the square root (I) when $x<0$ and $0<y\longrightarrow 0^+$, we get

$$f_+(z)\longrightarrow i\sqrt{-x}=\sqrt{-x}\,e^{i\frac{\pi}{2}},$$

whereas when $x<0$ and $0>y\longrightarrow 0^-$, we get

$$f_+(z)\longrightarrow -i\sqrt{-x}=\sqrt{-x}\,e^{i\frac{3\pi}{2}}.$$

So, with $x < 0$ ($x \neq 0$) these two limits are not equal, since $i\sqrt{-x} \neq -i\sqrt{-x}$. We also observe that these limits have the same magnitudes, equal to $\sqrt{-x}$, but different arguments $\frac{\pi}{2} \neq \frac{3\pi}{2}$. I.e., if we rewrite this square root function in trigonometric form, we observe a jump discontinuity in the argument along the negative x-axis. We observe that the absolute value of this jump is $\left|\frac{3\pi}{2} - \frac{\pi}{2}\right| = \pi$, which is half of 2π.

We have analogous results for the other square root $f_-(z)$. (Check them out in this case!)

Example 3.1.8: Find the square roots of the following complex numbers z : 10, -10, 100, -100, i, $-i$, $-3+4i$, $-3-4i$, $2-5i$ and $2+3i$.

Answers:

1. $z = 10 \Longrightarrow \sqrt{z} = \pm\sqrt{10}$.
2. $z = -10 \Longrightarrow \sqrt{z} = \pm i\sqrt{10}$.
3. $z = 100 \Longrightarrow \sqrt{z} = \pm\sqrt{100} = \pm 10$.
4. $z = -100 \Longrightarrow \sqrt{z} = \pm i\sqrt{100} = \pm i \cdot 10$.
5. Let $z = i$. Then $x = 0, y = 1, \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$ and $\frac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{i} = \pm\left(\sqrt{\frac{1}{2}(0+1)} + i\sqrt{\frac{1}{2}(-0+1)}\right) =$$
$$\pm\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

6. Let $z = -i$. Then $x = 0$, $y = -1$, $\sqrt{x^2 + y^2} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$ and $\frac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{-i} = \pm\left(\sqrt{\frac{1}{2}(0+1)} - i\sqrt{\frac{1}{2}(-0+1)}\right) =$$
$$\pm\left(\sqrt{\frac{1}{2}} - i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right).$$

7. Let $z = -3 + 4i$. Then $x = -3$, $y = 4$, $\sqrt{x^2+y^2} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$ and $\dfrac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{-3+4i} = \pm\left(\sqrt{\frac{1}{2}(-3+5)} + i\sqrt{\frac{1}{2}[-(-3)+5]}\right) =$$
$$\pm\left(\sqrt{1} + i\sqrt{4}\right) = \pm(1+2i).$$

8. Let $z = -3 - 4i$. Then $x = -3$, $y = -4$, $\sqrt{x^2+y^2} = \sqrt{(-3)^2+(-4)^2} = \sqrt{25} = 5$ and $\dfrac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{-3-4i} = \pm\left(\sqrt{\frac{1}{2}(-3+5)} - i\sqrt{\frac{1}{2}[-(-3)+5]}\right) =$$
$$\pm\left(\sqrt{1} - i\sqrt{4}\right) = \pm(1-2i).$$

9. Let $z = 2 - 5i$. Then $x = 2$, $y = -5$, $\sqrt{x^2+y^2} = \sqrt{2^2+(-5)^2} = \sqrt{29}$ and $\dfrac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{2-5i} = \pm\left(\sqrt{\frac{1}{2}\left(2+\sqrt{29}\right)} - i\sqrt{\frac{1}{2}\left(-2+\sqrt{29}\right)}\right).$$

10. Let $z = -2 + 3i$. Then $x = -2$, $y = 3$, $\sqrt{x^2+y^2} = \sqrt{(-2)^2+3^2} = \sqrt{13}$ and $\dfrac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{-2+3i} = \pm\left(\sqrt{\frac{-2+\sqrt{13}}{2}} + i\sqrt{\frac{2+\sqrt{13}}{2}}\right).$$

▲

Problems

3.1.28 Apply the square root formula twice to compute the four fourth roots of the complex numbers: $z = 3 - 2i$, $z = -3 - 2i$, $z = -3 + 2i$ and $z = 3 + 2i$.

3.1.29 In a precalculus course, we learn that $\sqrt{x^2} = |x|$, $\forall\ x \in \mathbb{R}$. What is valid with $\sqrt{z^2}$, $\forall\ z \in \mathbb{C}$? Explain.

3.1.30 Find the formulae for the four fourth roots of a complex number $z = x + iy$ in terms of the cartesian coordinates x and y. (I.e., do not use the trigonometric form of z.)

3.1.31 Solve and factor completely the polynomials:

$$\begin{array}{llll} \text{(a)} & 3z^4 - 8z^2 - 5, & \text{(b)} & 3z^4 - 9z^2 - 5, \\ \text{(c)} & 3z^4 - 2z^2 - 5, & \text{(d)} & 3z^4 - 2z^2 + 5. \end{array}$$

3.1.32 Consider the **De Moivre formula** for the two square roots

$$\sqrt{z} = \sqrt{r}\left[\cos\left(\frac{\theta + k2\pi}{2}\right) + i\sin\left(\frac{\theta + k2\pi}{2}\right)\right]$$

of a complex number $z = x + iy \neq 0$, where $r = \sqrt{x^2 + y^2}$, $\cos(\theta) = \dfrac{x}{r}$, $\sin(\theta) = \dfrac{y}{r}$ and $k = 0, 1$.

Show that these formulae can be transformed to the formulae **(I)** and **(II)** for the $\sqrt{z} = \sqrt{x + iy}$, found above, in terms of x and y.

[Hint: Use the appropriate half angle trigonometric formulae.]

3.1.33 (a) Locate and explain the error made in

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1.$$

(b) Conclude when the rule $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ (the high school algebra rule about square roots) is valid with complex numbers a and b.

(See also **Problem 3.5.11**.)

3.2 Power Series—a Quick Review

A **real power series** in the real variable x and **based at** a, or **with center** a, is an infinite sum or a limit of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = \lim_{K \to \infty} \sum_{n=0}^{K} c_n (x - a)^n.$$

The c_n's are real numbers called the **coefficients of the power series**.

Notice that any power series at $x = a$ attains the value c_0. In the following Theorem, we summarize the results on power series that we need in this text. These results can be found in calculus or real analysis.

Theorem 3.2.1 *If the power series converges for an* $x_0 \neq a$*, then there is a real number* $R \in \mathbb{R}$*, called the* **radius of convergence of the power series***, such that* $0 < |x_0 - a| \leq R \leq \infty$ *and the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges absolutely (and therefore it converges) for all* x *such that* $|x - a| < R$ *and diverges for all* x *such that* $|x - a| > R$*. We call the open interval* $(a - R, a + R)$ *the* ***open interval of convergence*** *of the power series.*

If the power series converges only for $x = a$*, attaining the value* c_0*, then* $R = 0$*.*

If the power series converges for all real numbers $x \in \mathbb{R}$*, then* $R = \infty$*.*

If $0 < R < \infty$*, the power series may or may not converge at either or both endpoints* $x = a \pm R$ *of its open interval of convergence. (We must check each endpoint individually for convergence.)*

If $R > 0$*, then on any closed subinterval* $[c, d] \subset (a - R, a + R)$ *the power series converges uniformly.*

If $R > 0$*, we define the function*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad \text{for } a - R < x < a + R.$$

This function is differentiable and integrable in the open interval $(a - R, a + R)$ *with*

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \quad \text{for } a - R < x < a + R,$$

and

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}, \quad \text{for } a - R < x < a + R.$$

Hence a power series is differentiable and integrable term by term in its ***open*** *interval of convergence, and both of the new series obtained have the same radius of converge. (That is, we can switch the order of summation with differentiation or integration.)*

Definition 3.2.1 *If a function* f *can be written as a real power series based at* a *and with radius of convergence* $R > 0$*, we say that* f *is* **real analytic at** a*.*

After some thinking, we can prove that if a function f is analytic at a, with radius of convergence $R > 0$, then it is analytic at every point of the open interval of convergence $(a - R, a + R)$ or, as we say, is analytic in $(a - R, a + R)$. I.e., we can write f as a power series with center any point $c \in (a - R, a + R)$ and radius of convergence $R_c > 0$.

The **Absolute Ratio Test, 1.3.5**, can be used to find the radius of convergence of a power series as follows: Given the power series $\sum_{n=0}^{\infty} c_n(x - a)^n$, we find the limit

$$0 \le \rho = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} \le \infty,$$

provided this limit exists. Then the **radius of convergence** is equal to

$$R = \frac{1}{\rho} = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}. \tag{3.2}$$

Also, by the **Absolute Root Test, 1.3.4**, we can prove that

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}, \tag{3.3}$$

provided again that the limit exists. This root expression of the radius of convergence is sometimes more convenient than the ratio one. (See **Problem 3.2.4**.)

Example 3.2.1 The **geometric series**

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

has radius of convergence 1 and so converges on the interval $(-1, 1)$. (It diverges at both 1 and -1.) We can manipulate this power series to get power series of other functions. For instance, for $-1 < x < 1$, we derive the following five results:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x},$$

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2},$$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\ln(1-x),$$

$$f(-x^2) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1+x^2},$$

$$\int_0^x f(-t^2)\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \tan^{-1}(x).$$

In each of these examples, the convergence or divergence at each of the endpoints $x = -1$ and $x = 1$ is checked separately. (Do this checking!)

▲

Power series can be computed by the Taylor Power Series Theorem (as studied in a calculus or real analysis course) for many standard elementary functions. To do this for a real function of a real variable $y = f(x)$ at some point a of its domain as center, we compute the coefficients by

$$c_n = \frac{f^{(n)}(a)}{n!}, \ \forall \ n = 0, 1, 2, 3, ...$$

and then we calculate the radius of convergence by **(3.2)** or **(3.3)**. If in particular the center $a = 0$, then the power series we obtain is called the **Maclaurin**[2] **series** of $f(x)$.

For instance, we have seen that for $a = 0$ and $-\infty < x < \infty$, we have:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.$$

(Compute these Maclaurin power series once more!)

Product of Power Series: It can be shown that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R_1 and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with radius of convergence R_2, then the product of the two functions also has a power series, given by

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n,$$

and this power series has radius of convergence for $R = \min\{R_1, R_2\}$.

---

[2] Colin Maclaurin, Scottish mathematician, 1698-1746.

Everything that we have stated so far about real power series holds with no change if the real number a and the real variable x are replaced with a complex number c and the complex variable z and the coefficients c_n's are allowed to be complex numbers, in general, and thus producing complex power series.

The radius of convergence R of a complex power series, obtained in this way, is again computed by **(3.2)** or **(3.3)**. The open interval of convergence is now replaced with $D(c, R) := \{z \mid |z - c| < R\}$ the open disc of center c and radius R in the complex plane. The complex series converges absolutely (and therefore it converges) for any z in this open disc. Also, the convergence is uniform on any closed sub-disc $\overline{D(c, r)} := \{z \mid |z - c| \leq r\}$, where $0 < r < R$.

When $0 < R < \infty$, the boundary points of the disc, which make up $C(c, R) := \{z \mid |z - c| = R\}$ the circle of center c and radius R in the complex plane, must be checked separately for convergence, just like when we check separately the endpoints of the interval of converge in a calculus course.

[The theory of convergence of complex sequences and series of numbers or functions in the complex plane $\mathbb{C}$ or the extended complex plane $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a whole fundamental chapter in a complete course of complex analysis. It involves several definitions, topological considerations and many criteria. (Here ∞ is the complex infinity. We talk about it a little in the next section.) In this book, we are going to use just what we have stated in order to practice with problem solving, instead of focusing on the underlying theory.]

In this way, the previous examples of the real power series of the real functions $e^x, \sin(x)$ and $\cos(x)$ give the definitions of their extensions into the complex plane. I.e., for any complex number z, we define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

We observe that these three power series **converge absolutely** for any complex number z. That is, if in the power series we replace z with $|z|$ the new series still converge, for any $z \in \mathbb{C}$, as this follows from the absolute convergence of the obtained real series. (In the **problems that follow**, we will see similar definitions of other functions and some results that we can prove easily.)

The **Series Rearrangement Theorem of B. G. F. Riemann** says that: *"The convergence of a series is absolute if and only if any rearrangement of the terms of the series, or any way of summing it up, does not change the final limit sum."* (For the proof of this, we need some advanced concepts and results on convergence from mathematical analysis. See, e.g., Rudin 1976, 75-78.)

Definition 3.2.2 *The complex function*

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

*defined for all $z \in \mathbb{C}$ is called **complex exponential function** or simply **exponential function**.*

We can easily prove:

$$\begin{aligned} e^{z+w} &= e^z e^w, \ \forall\ z,\ w \in \mathbb{C} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i}, \ \forall\ z \in \mathbb{C} \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2}, \ \forall\ z \in \mathbb{C}. \end{aligned}$$

The functions $\sin(z)$ and $\cos(z)$ are called the **complex sine** and **complex cosine**.

In the problems (**Problem 3.2.5**, **Problem 3.2.6**, **Problem 3.2.8**, **Problem 3.2.11**, etc.) we state and suggest to prove some important and rather immediate properties of the exponential function. Here we state the following with some auxiliary hints:

Using the rule for the powers of i (see **Problem 3.1.5**) and the power series of cosine and sine, we derive **Euler's formula**:

$$\forall\ y \in \mathbb{R} \quad \text{we have:} \quad e^{iy} = \sum_{m=0}^{\infty} \frac{i^m y^m}{m!} =$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} = \cos(y) + i\sin(y).$$

Then

$$\forall\ z = x + iy \in \mathbb{C}, \quad e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i\sin(y)].$$

This is the trigonometric form of e^z.

So,

$$\begin{gathered}\forall\ y \in \mathbb{R}, \quad \left|e^{iy}\right| = \sqrt{\cos^2(y) + \sin^2(y)}\ = 1,\\ \forall\ z = x + iy \in \mathbb{C}, \quad |e^z| = \left|e^{x+iy}\right| = e^x > 0,\\ \forall\ z = x + iy \in \mathbb{C}, \quad e^z \neq 0,\\ \forall\ z = x + iy \in \mathbb{C}, \quad \arg\left(e^z\right) = y + 2\pi\mathbb{Z}.\end{gathered}$$

Now, we can easily prove that

$$e^z = 1 \quad \Longleftrightarrow \quad z = 2k\pi\, i, \text{ with } k \in \mathbb{Z}.$$

Next, we observe that if $0 \neq w \in \mathbb{C}$ written in trigonometric form

$$w = r[\cos(\theta) + i\sin(\theta)],$$

for some argument θ, since $0 < r = e^{\ln(r)}$, using Euler's formula we get

$$0 \neq w = e^{\ln(r)}e^{i\theta} = e^{\ln(r)+i\theta} = e^{\ln(r)+i\theta+i2k\pi}, \quad \forall\ k \in \mathbb{Z}.$$

This expression of $w \in C - \{0\}$ is called the **exponential form of a complex number** $w \neq 0$. We see that $0 = 0 + i0$ does not have exponential form.

We have already seen that the **domain** of the exponential function is $\mathbb{C}$, and we now observe that its **range** is $\mathbb{C} - \{0\}$. In fact, the equation $e^z = w$, for any given $w \neq 0$, has infinitely many solutions in z, whereas $e^z = 0$ has no solution.

The **trigonometric functions** in the complex plane are defined by:

$$\begin{gathered}\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = -i\,\frac{e^{iz} - e^{-iz}}{2},\ \forall\ z \in \mathbb{C},\\ \cos(z) = \frac{e^{iz} + e^{-iz}}{2},\ \forall\ z \in \mathbb{C},\\ \tan(z) = \frac{\sin(z)}{\cos(z)} = -i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}},\ \forall\ z \in \mathbb{C},\\ \cot(z) = \frac{\cos(z)}{\sin(z)} = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}},\ \forall\ z \in \mathbb{C},\\ \sec(z) = \frac{1}{\cos(z)} = \frac{2}{e^{iz} + e^{-iz}},\ \forall\ z \in \mathbb{C},\\ \csc(z) = \frac{1}{\sin(z)} = \frac{2\,i}{e^{iz} - e^{-iz}},\ \forall\ z \in \mathbb{C}.\end{gathered}$$

(When a denominator is zero, the answer is the complex infinity.)

The **trigonometric functions** as defined here for all complex numbers are the extensions of the real trigonometric functions to the complex plane. Therefore, they satisfy all the **trigonometric identities** that we learn in a trigonometry or a precalculus class. We can check the validity of the extension of each of these identities as an exercise. With more complex analysis, this becomes automatic.

Also, we can easily prove that the **range of the complex** $\sin(z)$ **and** $\cos(z)$, (etc.), is not just the interval $[-1,1]$, which is the range of the real $\sin(x)$ and $\cos(x)$, (etc), but the whole complex plane $\mathbb{C}$, (etc). For instance, if $z = iy$ with $y \in \mathbb{R}$, then $\sin(z) = \sin(iy) = \dfrac{e^{-y} - e^{y}}{2i}$. But $\left| \dfrac{e^{-y} - e^{y}}{2i} \right| = \dfrac{|e^{-y} - e^{y}|}{2} \to \infty$, as $y \to \pm\infty$. (Provide similar work for $\cos(z)$, etc.)

The **hyperbolic functions** in the complex plane are defined by:

$$\begin{aligned}
\sinh(z) &= \frac{e^{z} - e^{-z}}{2}, \ \forall\, z \in \mathbb{C}, \\
\cosh(z) &= \frac{e^{z} + e^{-z}}{2}, \ \forall\, z \in \mathbb{C}, \\
\tanh(z) &= \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \ \forall\, z \in \mathbb{C}, \\
\coth(z) &= \frac{\cosh(z)}{\sinh(z)} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}, \ \forall\, z \in \mathbb{C}, \\
\operatorname{sech}(z) &= \frac{1}{\cosh(z)} = \frac{2}{e^{z} + e^{-z}}, \ \forall\, z \in \mathbb{C}, \\
\operatorname{csch}(z) &= \frac{1}{\sinh(z)} = \frac{2}{e^{z} - e^{-z}}, \ \forall\, z \in \mathbb{C}.
\end{aligned}$$

(When a denominator is zero, the answer is the complex infinity.)

All the identities that we learn in trigonometry, precalculus and calculus carry through to complex variables. Examine the problems that follow to find some additional identities and some **identities and relations** satisfied by the **trigonometric and hyperbolic functions**.

Definition 3.2.3 *A function that can be written as convergent complex power series at all points of its domain is called a* ***complex analytic function****.*

We observe that the exponential, trigonometric and hyperbolic functions are complex analytic.

Problems

3.2.1 Use the **Ratio Test** to show that the interval of convergence for the geometric series is $(-1, 1)$. Show that the series diverges at either $x = 1$ or $x = -1$.

3.2.2 Use the **Ratio Test** to show that the open interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

is $(-1, 1)$. Does this series converge at either $x = 1$ or $x = -1$?

3.2.3 Use the **Ratio Test** to show that the interval of convergence for the series used to define $e^x, \sin(x)$, and $\cos(x)$ is $(-\infty, \infty)$.

3.2.4 Let (c_n) with $n \in \mathbb{N}_0$ be a sequence of complex numbers. Prove that when both limits $\lim\limits_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$ and $\lim\limits_{n\to\infty} \sqrt[n]{|c_n|}$ exist, then they are equal.

3.2.5 Use the power series expansion of e^z, the power series product and the **Binomial Theorem** to show that

$$e^{z+w} = e^z e^w$$

and therefore

$$e^z e^w = e^{z+w} = e^{w+z} = e^w e^z.$$

3.2.6 (a) Use the appropriate power series and the **Riemann Absolute Convergence Rearrangement Theorem** to show that for y real

$$e^{iy} = \cos(y) + i \sin(y) \quad (\textbf{Euler's formula}).$$

(b) Now use the **previous problem** to show that if $z = x + iy$, then

$$e^z = e^{x+iy} = e^x \left[\cos(y) + i \sin(y)\right].$$

(This relation can alternatively be used to extend the exponential function to the whole complex plane.)

(c) Prove $\overline{e^z} = e^{\bar{z}}$.

(d) Prove that the set

$$\{e^z \mid -\infty < x < \infty \text{ and } y \in \mathbb{R} \text{ fixed}\}$$

is the infinite open half line starting at the origin and forming angle y, (mod 2π), with the positive x-semi-axis. Each point of this half line is attained once.

(e) Prove that for any $a < b$ real such that $b - a = 2\pi$, either of the two following sets

$$\{e^z \mid x \in \mathbb{R} \text{ fixed and } a < y \leq b\}$$

or

$$\{e^z \mid x \in \mathbb{R} \text{ fixed and } a \leq y < b\}$$

is a complete circle, each point of which is attained once. What is the center and the radius of this circle?

(f) For any $a < b$ real such that $b - a = 2\pi$, the horizontal strips

$$\{z \mid z = x + iy \text{ and } -\infty < x < \infty\text{and } a < y \leq b\}$$

or

$$\{z \mid z = x + iy \text{ and } -\infty < x < \infty\text{and } a \leq y < b\}$$

are mapped by the exponential function function $f(z) = e^z$ in one-to-one manner onto the set $\mathbb{C} - \{0\}$.

So, prove that $f(\mathbb{C})$ is $\mathbb{C} - \{0\}$ obtained infinitely many times, once for every $k \in \mathbb{Z}$.

3.2.7 Using the power series definitions given above for $e^x, \sin(x)$ and $\cos(x)$, show that

$$\frac{d}{dx}\, e^x = e^x$$
$$\frac{d}{dx}\, \sin(x) = \cos(x)$$
$$\frac{d}{dx}\, \cos(x) = -\sin(x).$$

(We can make a list of the derivatives of all the trigonometric and hyperbolic functions after we study the next section.)

3.2.8 (a) Prove that for z, complex number,

$$e^z = 1 \iff z = 2k\pi i, \ \ k \in \mathbb{Z},$$

where $\mathbb{Z} = \{\,0, \ \pm 1, \ \pm 2, \ \pm 3, \ \dots\,\}$ is the set of integer numbers.
(b) Now set each of the functions $\cos(z), \sin(z), \cosh(z), \sinh(z)$ equal to 0 and find all the infinitely many solutions of the four equations.
(c) Prove that the equation $e^z = 0$ has no solution in the complex numbers.

3.2.9 Use **Euler's Formula** to find the n $\mathrm{n^{th}}$-roots of e^z for any complex number $z = x + iy$ and any $n \in \mathbb{N}$.

3.2.10 Write the $\mathrm{n^{th}}$ power and the n $\mathrm{n^{th}}$-roots of any complex number $z = x + iy \neq 0$ in exponential forms for all $n \in \mathbb{N}$.

3.2.11 (a) Use **Euler's Formula** to show that

$$\begin{aligned} \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2}. \end{aligned}$$

(b) These equations are used to define the trigonometric functions $\sin(z)$ and $\cos(z)$ in the whole complex plane, given by the same formulae by means of exponentials.

Write these two formulae for any complex number $z = a + ib$ with replacing x by $z = a + ib$ to obtain the **definitions of complex** $\sin(z)$ **and** $\cos(z)$ and show that they coincide with their power series representations stated earlier.

(c) Write the **definitions** of the remaining four **complex trigonometric functions** for any complex number $z = a + ib$ in terms of $\sin(z)$ and $\cos(z)$ and in terms of exponentials.

(d) Find: $|e^{iy}|, \ \forall\ y \in \mathbb{R}$ and $|e^z| = |e^{x+iy}|, \ \forall\ z = x + iy \in \mathbb{C}$.

(e) Prove: $|e^{iu}| = 1 \iff u \in \mathbb{R}$.

(f) Prove that for any integer $n \geq 0$ and any x real

$$\sum_{k=0}^{n} \sin(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

If x is an integer multiple of 2π, the answer has the form $\frac{0}{0}$, but the formula is still valid if we resolve it by L' Hôpital's rule.

(g) Prove that for any integer $n \geq 0$ and any x real

$$\sum_{k=0}^{n} \cos(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \cos\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

If x is an integer multiple of 2π, the answer has the form $\frac{0}{0}$, but the formula is still valid if we resolve it by L' Hôpital's rule.

3.2.12 Consider the two **hyperbolic functions**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

(a) Show that

$$\begin{aligned}\sinh(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\\ \cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.\end{aligned}$$

(b) Make the natural extensions of the above definitions and the power series to any complex number $z = a + ib$ in $\mathbb{C}$, by replacing x with z. Prove that the exponential expressions and the power series that you obtain coincide.

(c) Write the **definitions** of the remaining four **complex hyperbolic functions** for any complex number $z = a + ib$ in terms of $\sinh(z)$ and $\cosh(z)$ and in terms of exponentials.

3.2.13 Use the properties and the Maclaurin series of the functions e^z, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$ to find their Taylor series with center any point $a \in \mathbb{C}$.

3.2.14 Show the useful identities for any complex number $z = x + iy$:

$$\begin{aligned}\cos^2(z) + \sin^2(z) &= 1\\ \cosh^2(z) - \sinh^2(z) &= 1\\ \sec^2(z) - \tan^2(z) &= 1\\ \operatorname{sech}^2(z) + \tanh^2(z) &= 1\\ \csc^2(z) - \cot^2(z) &= 1\\ -\operatorname{csch}^2(z) + \coth^2(z) &= 1\end{aligned}$$

$$\begin{aligned}\sin(z) &= -i\,\sinh(iz)\\ \cos(z) &= \cosh(iz)\\ \sinh(z) &= -i\,\sin(iz)\\ \cosh(z) &= \cos(iz)\\ \sin(iz) &= i\,\sinh(z)\\ \cos(iz) &= \cosh(z)\\ \sinh(iz) &= i\,\sin(z)\\ \cosh(iz) &= \cos(z)\\ \sin(z) &= \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)\\ \cos(z) &= \cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)\\ \sinh(z) &= \sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)\\ \cosh(z) &= \cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)\end{aligned}$$

$$
\begin{aligned}
|\sin(z)|^2 &= \sin^2(x) + \sinh^2(y) \\
|\cos(z)|^2 &= \cos^2(x) + \sinh^2(y) \\
|\sinh(z)|^2 &= \sin^2(y) + \sinh^2(x) \\
|\cosh(z)|^2 &= \cos^2(y) + \sinh^2(x).
\end{aligned}
$$

3.2.15 The hyperbolic functions exhibit many identities analogous to trigonometric identities that we find in a book of trigonometry (up to + or − signs). (We have written down a few of them in the **previous problem**.) Search the bibliography and find a collection of such identities. Take a look at them and verify a few of them for practice!

3.2.16 Prove the following properties of conjugation and then write and prove the analogous properties for the remaining trigonometric and hyperbolic functions:

$$
\begin{aligned}
e^{\bar{z}} &= \overline{e^{z}} \\
\cos(\bar{z}) &= \overline{\cos(z)} \\
\sin(\bar{z}) &= \overline{\sin(z)} \\
\tan(\bar{z}) &= \overline{\tan(z)} \\
\cot(\bar{z}) &= \overline{\cot(z)} \\
\sec(\bar{z}) &= \overline{\sec(z)} \\
\csc(\bar{z}) &= \overline{\csc(z)} \\
\cosh(\bar{z}) &= \overline{\cosh(z)} \\
\sinh(\bar{z}) &= \overline{\sinh(z)} \\
\tanh(\bar{z}) &= \overline{\tanh(z)} \\
\coth(\bar{z}) &= \overline{\coth(z)} \\
\operatorname{sech}(\bar{z}) &= \overline{\operatorname{sech}(z)} \\
\operatorname{csch}(\bar{z}) &= \overline{\operatorname{csch}(z)}.
\end{aligned}
$$

3.2.17 Show the useful identities for any complex number $z = x + iy$:

$$
\begin{aligned}
\tan(iz) &= i\ \tanh(z) \\
\cot(iz) &= -i\ \coth(z) \\
\tanh(iz) &= i\ \tan(z) \\
\coth(iz) &= -i\ \cot(z)
\end{aligned}
$$

$$\begin{aligned}
\sec(iz) &= \operatorname{sech}(z)\\
\csc(iz) &= -i\operatorname{csch}(z)\\
\operatorname{sech}(iz) &= \sec(z)\\
\operatorname{csch}(iz) &= -i\ \csc(z).
\end{aligned}$$

3.2.18 From **Example 3.2.1**, find the power series of $\ln(1-x)$ and then use it to find the power series for $\ln(1+x)$ and $\ln\left(\dfrac{1+x}{1-x}\right)$. Write all these results in the complex variable z.

3.2.19 (a) If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$, prove the formulae for the following two finite sums that can also be used to estimate infinite series:

$$\sum_{k=1}^{n} \sin[a+(k-1)d] = \frac{\sin\left[a+\frac{(n-1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)},$$

$$\sum_{k=1}^{n} \cos[a+(k-1)d] = \frac{\cos\left[a+\frac{(n-1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}.$$

(b) Apply both formulae for $d = a$ and $d = 2a$ and prove:

$$\sum_{k=1}^{n} \sin(ka) = \frac{\sin\left[\frac{(n+1)a}{2}\right]\sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)},$$

$$\sum_{k=1}^{n} \cos(ka) = \frac{\cos\left[\frac{(n+1)a}{2}\right]\sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)},$$

$$\sum_{k=1}^{n} \sin[(2k-1)a] = \frac{\sin^2(na)}{\sin(a)},$$

$$\sum_{k=1}^{n} \cos[(2k-1)a] = \frac{\sin(2na)}{2\sin(a)}.$$

(c) Replace a with $2a$ in the first two equations of **(b)** to prove

$$\sum_{k=1}^{n} \sin(2ka) = \frac{\sin[(n+1)a]\sin(na)}{\sin(a)},$$

$$\sum_{k=1}^{n} \cos(2ka) = \frac{\cos[(n+1)a]\sin(na)}{\sin(a)}.$$

(d) Prove that

$$\cos\left(\frac{\pi}{17}\right) + \cos\left(\frac{3\pi}{17}\right) + ... + \cos\left(\frac{15\pi}{17}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{23}\right) + \cos\left(\frac{3\pi}{23}\right) + ... + \cos\left(\frac{21\pi}{23}\right) = \frac{1}{2}.$$

[Hint: Use the complex expressions of sine and cosine as in the text or in **Problem 3.2.11** and the formulae for geometric sums.]

3.2.20 (a) If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$, prove

$$\sum_{k=1}^{n} \sin^2[a + (k-1)d] = \frac{n}{2} - \frac{\cos[2a + (n-1)d]\sin(nd)}{2\sin(d)}.$$

(b) Apply this formula for $d = a$ and $d = 2a$ to prove

$$\sum_{k=1}^{n} \sin^2(ka) = \frac{n}{2} - \frac{\cos[(n+1)a]\sin(na)}{2\sin(a)},$$

$$\sum_{k=1}^{n} \sin^2[(2k-1)a] = \frac{n}{2} - \frac{\cos(2na)\sin(2na)}{2\sin(2a)}.$$

(c) Apply this formula with $2d$ in the places of both a and d [or simply put $2d$ for a in the first formula of (b)] to prove

$$\sum_{k=1}^{n} \sin^2(2kd) = \frac{n}{2} - \frac{\cos[2(n+1)d]\sin(2nd)}{2\sin(2d)}.$$

(d) Find the analogous formulae for the sums

$$\sum_{k=1}^{n} \cos^2[a + (k-1)d],$$

$$\sum_{k=1}^{n} \cos^2(ka),$$

$$\sum_{k=1}^{n} \cos^2[(2k-1)a],$$

$$\sum_{k=1}^{n} \cos^2(2ka).$$

[Hint: Use the complex expressions of sine and cosine as in the text or in **Problem 3.2.11** and the formulae for geometric sums.]

3.2.21 Use the geometric series to show that if $-\frac{\pi}{6} < x < \frac{\pi}{6}$ (real), then

$$\sum_{n=1}^{\infty} 2^{n-1} \sin^n(x) = \frac{\sin(x)}{1 - 2\sin(x)}.$$

3.2.22 (a) Use the appropriate **telescopic series partial sum**

$$\sum_{k=1}^{n} [f(k) - f(k+1)] = f(1) - f(n+1)$$

to show

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^{n+2}}\right) \cos\left(\frac{3\pi}{2^{n+2}}\right) = \frac{1}{2}.$$

(b) Now for $z \in \mathbb{C}$, prove that $\tan(z) = \cot(z) - 2\cot(2z)$ and then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan\left(\frac{z}{2^n}\right) = \frac{1}{z} - \cot(z).$$

[Hint: In **(b)** you may use an appropriate **telescopic series partial sum** of the form $\sum_{k=1}^{n} [f(k+1) - f(k)] = f(n+1) - f(1)$.]

3.2.23 For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that

$$\arctan\left[\frac{x}{1 + n(n+1)x^2}\right] = \arctan[(n+1)x] - \arctan(nx)$$

and then

$$\sum_{n=1}^{\infty} \arctan\left[\frac{x}{1 + n(n+1)x^2}\right] =$$

$$\begin{cases} \frac{\pi}{2} - \arctan(x) = \arctan\left(\frac{1}{x}\right), & \text{if} \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -\frac{\pi}{2} - \arctan(x) = \arctan\left(\frac{1}{x}\right), & \text{if} \quad x < 0. \end{cases}$$

Justify the discontinuity at $x = 0$.

3.2.24 Let $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots,\ \omega_{k-1}$ be the k k^th roots of unity $(= 1)$. Prove

$$\omega_0^j + \omega_1^j + \ldots + \omega_{k-1}^j = \begin{cases} 0, & \text{if} \quad j \in \{1,\ 2,\ \ldots\ k-1\} + k\mathbb{Z} \\ \\ k, & \text{if} \quad j \in k\mathbb{Z}. \end{cases}$$

Generalize this problem to the roots of any $z \in \mathbb{C}$.

[Hint: Use the exponential forms of the roots of unity and the finite geometric sum.]

3.2.25 Let $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots,\ \omega_{k-1}$ be the k k^th roots of unity $(= 1)$. Prove

$$\begin{gathered} \omega_0 + \omega_1 + \ldots + \omega_{k-1} = 0 \\ \omega_0\omega_1 + \omega_0\omega_2 + \ldots + \omega_{k-2}\omega_{k-1} = 0 \\ \omega_0\omega_1\omega_2 + \ldots + \omega_{k-3}\omega_{k-2}\omega_{k-1} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_0\omega_1 \ldots \omega_{k-2} + \ldots + \omega_1\omega_2 \ldots \omega_{k-1} = 0 \\ \omega_0\omega_1 \ldots \omega_{k-1} = (-1)^{k+1}. \end{gathered}$$

Generalize this problem to the roots of any $z \in \mathbb{C}$.

[Hint: Use Vieta's[3] formulae for polynomials.]

3.2.26 (a) Consider the n^th root of unity $(= 1)$

$$\zeta = e^{\frac{2i\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right).$$

Prove that all the n n^th roots of unity $(= 1)$ are given by $\zeta_k = \zeta^k$ for $k = 0,\ 1,\ 2,\ 3,\ \ldots,\ n-1$.

(b) Prove

$$\prod_{k=1}^{n-1}(1-\zeta^k) = \prod_{k=1}^{n-1}(1-\zeta_k) = n.$$

[Hint: Notice that

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1) = (x-1)\prod_{k=1}^{n-1}(x-\zeta^k),$$

etc.]

---

[3] François Viéte (Latin: Franciscus Vieta), French mathematician, 1540-1603.

(c) Let $m = \left[\left[\frac{n}{2}\right]\right]$ be the integer part of $\frac{n}{2}$. Prove

$$\prod_{k=1}^{m} \sin\left(\frac{k\pi}{n}\right) = \sqrt{\frac{n}{2^{n-1}}} \quad \text{and} \quad \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

(See also **Problem 2.6.37**.)

[Hint: First prove

$$|1 - \zeta^k| = 2\left|\sin\left(\frac{k\pi}{n}\right)\right|.$$

Then use **(b)**, the symmetries of the values of sine, $\sin\left(\frac{k\pi}{n}\right) > 0$ for $k = 1,\ 2, \ldots, n-1$, etc.]

(d) Use **(c)** to prove

$$\int_0^{\pi} \ln[\sin(x)]\, dx = 2\int_0^{\frac{\pi}{2}} \ln[\sin(x)]\, dx = -\pi \ln(2).$$

(See also **Problems 2.1.16, 3.7.88** and **Example 3.7.47**.)

3.2.27 The **integral sine** is given by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$

Find a power series for the integrand and then integrate term by term to get a power series for $\mathrm{Si}(x)$.

[Compare with **Example 1.1.17**. See also **Problem 2.2.29 (j)**.]

3.2.28 The **real Riemann zeta function** is defined by the series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Prove:
(a) This series diverges ($= \infty$) for any $x \le 1$.
(b) This series converges point-wise for all $x > 1$.
(c) This series converges uniformly on any interval $[a, \infty)$ with $a > 1$.
(d) The convergence is not uniform on the open interval $(1, \infty)$.

[**Note** that this is not a power series. Replacing x with the complex z, we obtain the complex Riemann zeta function for all z with $\mathrm{Re}(z) > 1$, which is very important in complex analysis and number theory.]

3.3 Limits, Continuity and Derivatives

In this exposition, we do not plan to present all the topological aspects of the complex plane and the implied results on the complex functions, sequences and series. These important results, if not known by now, can be studied from a book of complex analysis. (See our bibliography.) As far as the limits are concerned, we mention what we need for the goals of this text. So, we have:

$$z = x+iy \to z_0 = x_0+iy_0 \;\Leftrightarrow\; (x,y) \to (x_0,y_0) \;\Leftrightarrow\; x \to x_0 \text{ and } y \to y_0.$$

Since $|z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, this follows from the inequality

$$\max\{|x-x_0|, |y-y_0|\} \le \sqrt{(x-x_0)^2 + (y-y_0)^2} \le |x-x_0| + |y-y_0|,$$

for all complex numbers $z = x + iy$ and $z_0 = x_0 + iy_0$.

Similarly, by the same inequality, we have

$$g(z) = u(x,y) + iv(x,y) \longrightarrow a + ib \in \mathbb{C}, \quad \text{as}$$
$$z = x + iy \longrightarrow z_0 = x_0 + iy_0 \in \mathbb{C},$$

if and only if

$$u(x,y) \longrightarrow a, \quad \text{as} \quad (x,y) \longrightarrow (x_0,y_0), \quad \text{and}$$
$$v(x,y) \longrightarrow b, \quad \text{as} \quad (x,y) \longrightarrow (x_0,y_0).$$

We understand that if

$$\lim_{z \to z_0} g(z) = \lim_{x+iy \to x_0+iy_0} [u(x,y) + iv(x,y)] = a + ib \in \mathbb{C},$$

then the answer $a + ib$, to this limit, is always the same no matter in what way $z = x + iy$ approaches the constant $z_0 = x_0 + iy_0$ in the complex plane $\mathbb{C}$. There are, of course, infinitely many ways and paths that $z = x + iy$ can follow to reach $z_0 = x_0 + iy_0$. We can express this by the following general and mathematically rigorous definition:

$$\lim_{z \to z_0} f(z) = L \in \mathbb{C} \overset{def}{\Longleftrightarrow} \forall\, \epsilon > 0,\ \exists\, \delta > 0 \ :\ |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon.$$

So: **If in such a limit two different ways of approaching the number (constant, point) $z_0 = x_0 + iy_0$ yield two different answers, then this limit does not exist.**

In the real line $\mathbb{R}$, we distinguish two infinities, the $+\infty$ and $-\infty$. They are answers to certain real limits. (Suppose $a > 0$ real constant, then $\frac{a}{0^{\pm}} = \pm\infty$ and $\frac{-a}{0^{\pm}} = \mp\infty$. The limit form $\frac{0}{0}$ is indeterminate, that is, any answer, real number, $\pm\infty$, or does not exist, is possible. The same is true for the indeterminate limit $\frac{\pm\infty}{\pm\infty}$.)

In the complex plane $\mathbb{C}$, some limits have answers $a \pm i\infty$, where $a \in \mathbb{R}$, or $\pm\infty + ib$, where $b \in \mathbb{R}$, and/or $\pm\infty \pm i\infty$. In the complex plane, all of these unbounded answers are considered to be one infinity, the complex infinity (∞). So, in the complex plane, we do not distinguish between various infinities, unless we need to declare its specific form out of these **eight** stated choices. For any $c \in \mathbb{C}$, we have $\frac{c}{\infty} = 0$, but $\frac{\infty}{\infty}$ is indeterminate.

A complex function has limit a complex number at the complex infinity if by definition

$$\lim_{z\to\infty} f(z) = L \in \mathbb{C} \overset{def}{\iff} \forall\ \epsilon > 0,\ \exists\ R > 0\ :\ |z| > R \Longrightarrow |f(z) - L| < \epsilon.$$

Therefore, if $\lim_{z\to\infty} f(z) = L \in \mathbb{C}$ and for any $R > 0$ we define

$$f_R(z) = \begin{cases} f(z), & \text{if} \quad |z| > R \\ \\ L, & \text{if} \quad |z| \leq R \end{cases}$$

the convergence $\lim_{R\to\infty} f_R(z) = L$ is uniform in z. We use this fact very often, but a lot of times tacitly, when we perform complex integration, especially when we switch limits and integrals over finite ranges. So, in the sequel, we must keep in mind this fact, especially when we emphasize that we have a limit as z approaches the complex infinity.

The definition and the properties of continuous functions from $\mathbb{C}$ to $\mathbb{C}$ transfer the same as with the real functions of a real variable. So:

Definition 3.3.1 *A complex function $f : D \longrightarrow \mathbb{C}$ of one complex variable z, where the domain $D \subseteq \mathbb{C}$, is **continuous at** $z_0 \in D$ if*

$$\lim_{z\to z_0} f(z) = f(z_0)\ [= f(\lim_{z\to z_0} z)].$$

*It is called **continuous in** D if it is continuous at every point of D.*

The sum, difference, multiplication, division and composition of continuous functions are continuous. For the division, we exclude the points at which the divisor function (denominator) is zero.

Example 3.3.1 Consider the complex function $f(z) = \dfrac{z^2}{z+1}$.

If $z_0 \neq -1$, by continuity we get

$$\lim_{z \to z_0} \frac{z^2}{z+1} = \frac{z_0^2}{z_0+1},$$

otherwise,

$$\lim_{z \to -1} \frac{z^2}{z+1} = \infty.$$

▲

Example 3.3.2 Consider $z = x + iy$ and the complex function

$$f(z) = \frac{\text{Re}(z^2)}{z\bar{z}} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Then:

$$\lim_{z=x+i0 \to 0+i0} f(z) = \lim_{z=x+i0 \to 0+i0} \frac{x^2}{x^2} = 1,$$

and

$$\lim_{z=0+iy \to 0+i0} f(z) = \lim_{z=0+iy \to 0+i0} \frac{-y^2}{y^2} = -1.$$

Therefore, $\lim\limits_{z=x+iy \to 0+i0} f(z)$ does not exist.

(See also **Examples 3.6.5** and **3.6.7**.)

▲

Example 3.3.3 One way to prove that $f(z) = z^2 = (x+iy)^2$ is continuous at every point $z_0 = x_0 + iy_0 \in \mathbb{C}$ is the following:

$$0 \leq |z^2 - z_0^2| = |z + z_0| \cdot |z - z_0| \leq$$
$$(|x + x_0| + |y + y_0|) \cdot (|x - x_0| + |y - y_0|) \to (2x_0 + 2y_0) \cdot (0 + 0) = 0,$$

as $z = x + iy \to z_0 = x_0 + iy_0$, or as ($x \to x_0$ and $y \to y_0$), or as $(x, y) \to (x_0, y_0)$.

Therefore, $z^2 \to z_0^2$ as $z \to z_0$, i.e., the function $f(z) = z^2$ is continuous at every point $z_0 \in \mathbb{C}$. (Prove this in a different way, too.)

▲

We continue with the following:

Definition 3.3.2 *A complex-valued function f of one complex variable z is said to be* **analytic** *or* **complex analytic** *at a point $c \in \mathbb{C}$ if we can write*

$$f(z) = \sum_{n=0}^{\infty} b_n (z - c)^n,$$

and this complex power series has (maximum) radius of convergence some $0 < R \leq \infty$, i.e., the power series converges for all z such that $|z - z_0| < R$ and diverges for any z such that $|z - z_0| > R$. The complex number c is called the ***center of the power series.*** .

We denote the **open disc** with center c and radius R by $D(c, R) = \{z \in \mathbb{C} \mid |z - c| < R\}$ and the **closed disc** with center c and radius R by $\overline{D(c, R)} = \{z \in \mathbb{C} \mid |z - c| \leq R\}$. Both of them have boundary the **circle** with center c and radius R, $C(c, R) = \{z \in \mathbb{C} \mid |z - c| = R\}$.

So, in the above **definition**, the complex power series of a complex function $f(z)$ analytic at c, i.e., $f(z) = \sum_{n=0}^{\infty} b_n (z - c)^n$, converges for any $z \in D(c, R)$ and diverges for any $z \in \mathbb{C} - \overline{D(c, R)}$ (the complement of the closed disc). At a point of their boundary $C(c, R)$ anything can happen and must be checked individually. By the Absolute Root or Ratio Tests **the convergence is absolute in** $D(c, R)$. If at a point of $C(c, R)$ the power series converges, the convergence may or may not be absolute.

Such a function is differentiable at any $z \in D(c, R)$ for $R > 0$. As in calculus we can prove so here that its derivative is

$$f'(z) = \sum_{n=0}^{\infty} n b_n (z - c)^{n-1} = \sum_{n=1}^{\infty} n b_n (z - c)^{n-1}.$$

Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ or $\lim_{n\to\infty} \frac{n+1}{n} = 1$, the radius of convergence of this new complex power series is also R, that is, the same as the radius of convergence of the complex power series we started with. [See **Problem 3.3.1 (a)**.] Then to prove this result, we can use the complex analog of **Theorem 2.3.8**, applied point-wise to $S_n(z) = \sum_{k=0}^{n} b_k (z - c)^k$ and $S'_n(z) = \sum_{k=1}^{n} k\, b_k (z - c)^{k-1}$ for any z in the common disc of convergence $D(c, R)$, with $R > 0$.

Inductively, the k^{th} order derivative of $f(z)$, where $k \in \mathbb{N}$, is

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1) b_n (z - c)^{n-k} =$$

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) b_n (z - c)^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} b_n (z - c)^{n-k}$$

and the radius of convergence of this series is again $R > 0$. So, such a function is infinitely differentiable in $D(c, R)$.

In fact, such a function $f(z)$ is analytic at any $z_0 \in D(c, R)$. To show this, we consider the power series

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

with center z_0 and radius of convergence $r = R - |z_0 - c| > 0$, so that $D(z_0, r) \subseteq D(c, R)$ and coefficients a_n's (given by Taylor's formula)

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{k!} \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) b_n (z_0 - c)^{n-k},$$
$$\forall \quad k = 0,\ 1,\ 2,\ 3,\ \dots\ .$$

By the convergence of the initial power series in $D(c, R)$, all of these coefficients are finite numbers. The argument that this power series $g(z)$ converges and it is equal to $f(z)$ in $D(z_0, r)$, and so radius of convergence ρ is at least r (i.e., $\rho \geq r$), is accessible at this level but not that trivial. So, we skip it here for the sake of brevity. The interested reader may consult Cartan 1973, 37-39, in the bibliography. In the proof, he uses the absolute convergence of power series and the Binomial Theorem, and he also gives an example in which $\rho > r$.

In general, for any complex function $w = f(z)$, we define the complex derivative of $f(z)$ by:

Definition 3.3.3 *The* ***complex derivative*** *or simply the* ***derivative*** *or the* ***first derivative of a complex function*** *$f(z)$ at a point z is defined (in the same way as we have seen for real functions) to be*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

In this definition, we recognize that h is a complex number that tends to $0 = 0 + i0$. Equivalently, $h = s + it \to 0$, or $|h| \to 0$, or $\sqrt{s^2 + t^2} \to 0$, or $(s, t) \to (0, 0)$, that is, h is in a disc of radius r around 0, and $r \to 0$. If this limit does not exist at some point, then we say that $f(z)$ does not have a (complex) derivative at that point.

Definition 3.3.4 *A complex function $f : D \longrightarrow \mathbb{C}$, where $D \subseteq \mathbb{C}$ is open, that possesses complex derivative at every point point $z \in D$ is called a* ***holomorphic function*** *in its domain D.*

It turns out that the concepts "**complex analytic**," i.e., complex power series, and "**holomorphic**" in a domain D are equivalent. We will discuss this fact more and use it in what follows.

We now write $z = x + iy$ and $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where the functions u and v are real functions of two real variables. Since the limit, in the definition of the (complex) derivative, exists if and only if it yields the same value no matter how $h = s + it$ approaches $0 = 0 + i0$, i.e., $(s, t) \longrightarrow (0, 0)$, we have the following:

(I) First, we choose $h = s + i0$ in the limit of the derivative of $f(z)$ to obtain

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{s\to 0} \frac{f(x+s+iy) - f(x+iy)}{s} =$$
$$\lim_{s\to 0} \frac{u(x+s,y) + iv(x+s,y) - u(x,y) - iv(x,y)}{s} =$$
$$\lim_{s\to 0} \frac{u(x+s,y) - u(x,y)}{s} + \lim_{s\to 0} i\frac{v(x+s,y) - v(x,y)}{s} =$$
$$\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

(II) Next, we choose $h = 0 + it$ in the limit of the derivative of $f(z)$ to obtain

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{t\to 0} \frac{f[x+i(y+t)] - f(x+iy)}{t} =$$
$$\lim_{t\to 0} \frac{u(x,y+t) + iv(x,y+t) - u(x,y) - iv(x,y)}{it} =$$
$$\lim_{t\to 0} \frac{u(x,y+t) - u(x,y)}{it} + \lim_{t\to 0} i\frac{v(x,y+t) - v(x,y)}{it} =$$
$$\frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

From the computations **(I)** and **(II)**, we see that:
(1) If the complex derivative of $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ exists, then the partial derivatives of $u(x,y)$ and $v(x,y)$ do exist.
(2) If the derivative exists, then the two limits in **(I)** and **(II)** must be equal, and thus we obtain the following **two formulae for the complex derivative**

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y). \tag{3.4}$$

(3) The equality of the two limits in **(I)** and **(II)** forces the validity of the following **two necessary conditions**

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

These two first-order partial differential equations are called the **Cauchy-Riemann equations or conditions in Cartesian coordinates x and y** (even though they were known to and used by d' Alembert and Euler earlier). Both together form a system of two first-order partial differential equations.

The existence of the partial derivatives of $u(x,y)$ and $v(x,y)$ and the two equations that must satisfy are only **necessary conditions** for the complex derivative to exist. On their account, we also obtained two additional formulae for the derivative of $f(z)$, i.e.,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) - i\frac{\partial u}{\partial y}(x, y) = \frac{\partial v}{\partial y}(x, y) + i\frac{\partial v}{\partial x}(x, y). \tag{3.5}$$

The **general necessary and sufficient conditions for the existence of the complex derivative of a complex function** are the following:

"*A complex function* $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, *with* $z = x + iy \in D \subseteq \mathbb{C}$ *open, has complex derivative at a point* $z_0 = x_0 + iy_0 \in D$ *if and only if (1) the two real functions* $u(x, y)$ *and* $v(x, y)$ *are differentiable*[4] *at* $z_0 = (x_0, y_0)$ *and (2) they satisfy the Cauchy-Riemann conditions at* $z_0 = (x_0, y_0)$*.*"

(This result is usually proven in a course of complex analysis. It says that differentiability and the Cauchy-Riemann conditions imply the existence of complex derivative and vice-versa. Even though its proof is not hard, we omit it here for the sake of brevity but make a note of this result. Also, see and compare with **Problem 3.3.13**.)

For keeping the exposition simpler, we assume for sufficiency that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann conditions and are of class $\mathcal{C}^1$ in D. The $\mathcal{C}^1$ condition is a bit stronger than the simple differentiability, but we do not lose much under this assumption.

Even though the complex derivative is defined in a way analogous to the derivative of a real function of a real variable, the Cauchy-Riemann necessary conditions impose a strong restriction for the complex derivative to exist.[5] The complex derivative exists iff u and v are differentiable and satisfy the Cauchy-Riemann conditions.

The **Cauchy-Riemann equations or conditions expressed in polar coordinates**

$$(r, \theta) \longrightarrow z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$$

---

[4] A function $g : \mathcal{R} \longrightarrow \mathbb{R}^m$, where $\mathcal{R} \subseteq \mathbb{R}^n$ open, $m \in \mathbb{N}$ and $n \in \mathbb{N}$ is **differentiable at** $\overrightarrow{x}_0 \in \mathcal{R}$ if by definition there exists a matrix of real numbers $A_{m\times n} = (a_{ij})$ such that $\lim\limits_{\overrightarrow{x} \to \overrightarrow{x}_0} \frac{g(\overrightarrow{x}) - g(\overrightarrow{x}_0) - A(\overrightarrow{x} - \overrightarrow{x}_0)}{\|\overrightarrow{x} - \overrightarrow{x}_0\|} = 0$. In such a case, the i^{th} row of $A_{m\times n}$ consists of the partial derivatives of the i^{th} component of g. However, it can happen that the partial derivatives of g at a point $\overrightarrow{x}_0$ exist but g is not differentiable at $\overrightarrow{x}_0$. But, if all the partial derivatives, except possibly one, are continuous, then g is differentiable. (Review this material again!)

[5] This is because in the complex field there are two linearly independent directions, the real and the imaginary, whereas in the real field there is only the real one. Then all linear combinations of 1 and i with real coefficients form the set of complex numbers, which turns out to be an algebraic field.

are very useful in many situations.

We have: If a complex function, with real and imaginary part expressed in polar coordinates,

$$f(z) = f\left(re^{i\theta}\right) = f\left\{r[\cos(\theta) + i\sin(\theta)]\right\} = u(r,\theta) + iv(r,\theta)$$

has complex derivative $f'(z)$, then

$$r\,u_r(r,\theta) = v_\theta(r,\theta) \quad \text{and} \quad r\,v_r(r,\theta) = -u_\theta(r,\theta).$$

Also, the derivative in polar coordinates at a point

$$z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)$$

is computed to be

$$\begin{gathered} f'(z) = f'\left(re^{i\theta}\right) = e^{-i\theta}[u_r(r,\theta) + iv_r(r,\theta)] = \\ [\cos(\theta) - i\sin(\theta)] \cdot [u_r(r,\theta) + iv_r(r,\theta)] = \\ u_r(r,\theta)\cos(\theta) + v_r(r,\theta)\sin(\theta) + i[v_r(r,\theta)\cos(\theta) - u_r(r,\theta)\sin(\theta)], \ \text{etc.} \end{gathered}$$

[See and solve **Problem 3.3.8**. The polar form of the Cauchy-Riemann equations and the complex derivative is naturally useful when we work in **polar coordinates** (r,θ).]

Most rules of complex derivatives are analogous to those we learn in calculus. For instance, using either the definition of the derivative $f'(z)$ as a limit, or expression **(3.4)** or **(3.5)**, or the expression in polar coordinates found above, or differentiating power series, we can directly prove the following rules:

1. If a function $w = f(z)$ has derivative at z_0, then it is continuous at z_0.
2. For any constant $c \in \mathbb{C}$, $\dfrac{d(c)}{dz} = 0$.
3. For any integer $n \in \mathbb{Z}$, $\dfrac{d(z^n)}{dz} = nz^{n-1}$.
4. For any complex $z \in \mathbb{C}$, $\dfrac{d(e^z)}{dz} = e^z$.
5. **Sum and/or difference rule**: Provided the derivatives involved exist we have

$$\frac{d}{dz}[f(z) \pm g(z)] = \frac{d}{dz}[f(z)] \pm \frac{d}{dz}[g(z)] = f'(z) \pm g'(z).$$

6. **Factoring a multiplicative constant rule**: Provided the derivatives involved exist we have for any complex constant c

$$\frac{d}{dz}[c \cdot f(z)] = c \cdot \frac{d}{dz}[f(z)] = c \cdot f'(z).$$

7. **Product rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}[f(z) \cdot g(z)] =$$
$$\frac{d}{dz}[f(z)] \cdot g(z) + f(z) \cdot \frac{d}{dz}[g(z)] = f'(z) \cdot g(z) + f(z) \cdot g'(z).$$

8. **Reciprocal rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}\left[\frac{1}{g(z)}\right] = \frac{-\dfrac{d}{dz}\,[g(z)]}{g^2(z)} = \frac{-g'(z)}{g^2(z)}.$$

9. **Quotient rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] =$$
$$\frac{\dfrac{d}{dz}\,[f(z)] \cdot g(z) - f(z) \cdot \dfrac{d}{dz}\,[g(z)]}{g^2(z)} = \frac{f'(z) \cdot g(z) - f(z) \cdot g'(z)}{g^2(z)}.$$

10. **Chain rule**: Provided the derivatives involved exist, and if we set $w = g(z)$, we have

$$\frac{d}{dz}[f \circ g](z) = \frac{d}{dw}[f(w)] \cdot \frac{d}{dz}[g(z)] = [f' \circ g](z) \cdot g'(z) = f'[g(z)] \cdot g'(z).$$

11. **Inverse function rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}[f^{-1}(z)] = \frac{1}{\dfrac{df}{dz}\,[f^{-1}(z)]} = \frac{1}{[f' \circ f^{-1}](z)} = \frac{1}{f'[f^{-1}(z)]}.$$

12.
$$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z).$$

13.
$$\frac{d}{dz}\sinh(z) = \cosh(z) \quad \text{and} \quad \frac{d}{dz}\cosh(z) = \sinh(z).$$

14. We can write the derivatives of all six **complex trigonometric** and all six **complex hyperbolic functions** and the derivatives of their **inverses** by replacing the real variable x with the complex variable z in the rules we learn in calculus. Write and prove these rules by using rules stated above!

Remember that if $f'(c)$ exists, then $f(z)$ is continuous at $c \in \mathbb{C}$. The proof is analogous to the one in calculus. (Have a look at it and write one for this case.) The converse is not true.

Example 3.3.4 Consider the function $f(z) = \overline{z}$. This is continuous at every point of the complex plane $\mathbb{C}$. (Prove this as an exercise.) But,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h} = \text{does not exist.}$$

(E.g., when $h = s + 0i$ this limit is 1, and when $h = 0 + ti$ it is -1.)

Therefore, the simple complex continuous function $f(z) = \overline{z}$ does not have complex derivative.

We easily observe that the Cauchy-Riemann necessary conditions are not satisfied. (Check this!)

So, by the chain rule stated above, all complex functions that involve $\overline{z}$ in a non-trivial way do not have complex derivatives.

▲

Problems

3.3.1 (a) If a complex power series with center $c \in \mathbb{C}$

$$\sum_{n=0}^{\infty} b_n (z-c)^n$$

has radius of convergence $R > 0$, then prove that its derivative

$$\sum_{n=0}^{\infty} n b_n (z-c)^{n-1} = \sum_{n=1}^{\infty} n b_n (z-c)^{n-1}$$

and its indefinite integral

$$C + \sum_{n=0}^{\infty} b_n \frac{(z-c)^{n+1}}{n+1}$$

also have radius of convergence R.

(b) What is the radius of convergence of the second derivative

$$\sum_{n=0}^{\infty} n(n-1)b_n(z-c)^{n-2} = \sum_{n=2}^{\infty} n(n-1)b_n(z-c)^{n-2}, \text{ etc.}$$

of the complex power series in **(a)**?

3.3.2 Consider $z = x + iy$ and the complex function

$$f(z) = \frac{\mathrm{Im}(z^2)}{z\bar{z}}.$$

Prove that $\lim\limits_{z=x+iy \to 0+i0} f(z)$ does not exist.

3.3.3 Prove that $\lim\limits_{z=x+iy \to 0+i0} e^{\frac{1}{z}}$ and $\lim\limits_{z\to\infty} e^z$ do not exist.

3.3.4 Consider the exponential function of the variable $z = x + iy$

$$f(z) = e^z = e^x \cos(y) + ie^x \sin(y).$$

(a) What function is defined by f when $y = 0$, that is, $z = x + i0 = x$ is real?

(b) Prove $f'(z) = e^z$.

[Hint: Use the formula of $f'(z)$ found in equation **(3.4)** or **(3.5)**.]

3.3.5 Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Prove that $\mathrm{Re}[f(z)]$ or $\mathrm{Im}[f(z)]$ is constant if and only if $f(z)$ is constant.

3.3.6 If for a complex function $f(z)$ the $f'(c)$ exists at $c \in \mathbb{C}$, prove that $f(z)$ is continuous at $c \in \mathbb{C}$.

The converse is not true. Give a counterexample!

3.3.7 (a) Prove that $f'(z) = 0$ in an open connected or path connected (see **Definition 3.4.2** in the next section) region[6] $R \subseteq \mathbb{C}$ if and only if $f(z) \equiv c$ in R, for some constant $c \in \mathbb{C}$.

(b) Give an example of a complex function $f(z)$ such that $f'(z) = 0$ in an open region which is not connected (e.g., it consists of at least two

---

[6] In order to avoid too much topology and too many topological definitions in topological and/or metric spaces, you may think of a **connected set** intuitively as a set in the plane which consists of one piece. More accurately and for our purposes in this text, we can consider it to be a set in $\mathbb{C}$ in which we can join any two points in it by a continuous path lying entirely in the set. We also call such a set a **path connected set**. See **Definition 3.4.2** in the next section.

open disjoint subsets) and $f(z)$ is not identically constant.

[Hint: In **(a)** use the formula of $f'(z)$ found in equation **(3.4)** or **(3.5)** and multi-variable calculus to show that the functions $u(x,y)$ and $v(x,y)$ are constant in R. Or, use the chain rule with paths. See the beginning of the next section.]

3.3.8 (a) Prove that **in polar coordinates**

$$(r,\theta) \longrightarrow z = r\,[\cos(\theta) + i\sin(\theta)] = re^{i\theta}$$

the Cauchy-Riemann conditions or equations for a complex function written as

$$f(z) = f\left(re^{i\theta}\right) = f\,\{r\,[\cos(\theta) + i\sin(\theta)]\} = u(r,\theta) + i\,v(r,\theta)$$

for which $f'(z)$ exists become

$$ru_r = v_\theta \quad \text{and} \quad r\,v_r = -u_\theta.$$

(b) Prove that at a point $z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$, the complex derivative of a complex function $f(z)$ in polar coordinates is given by

$$\begin{gathered} f'(z) = f'\left(re^{i\theta}\right) = e^{-i\theta}[u_r(r,\theta) + iv_r(r,\theta)] = \\ [\cos(\theta) - i\sin(\theta)] \cdot [u_r(r,\theta) + iv_r(r,\theta)] = \\ u_r(r,\theta)\cos(\theta) + v_r(r,\theta)\sin(\theta) + i[v_r(r,\theta)\cos(\theta) - u_r(r,\theta)\sin(\theta)], \ \text{etc.} \end{gathered}$$

[Hint: Use the relations

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right) + (\text{appropriate constant})$$

(valid locally). Then prove and use the differential operators

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{x}{r} \cdot \frac{\partial}{\partial r} - \frac{y}{r^2} \cdot \frac{\partial}{\partial \theta} = \cos(\theta) \cdot \frac{\partial}{\partial r} - \frac{\sin(\theta)}{r} \cdot \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{y}{r} \cdot \frac{\partial}{\partial r} + \frac{x}{r^2} \cdot \frac{\partial}{\partial \theta} = \sin(\theta) \cdot \frac{\partial}{\partial r} + \frac{\cos(\theta)}{r} \cdot \frac{\partial}{\partial \theta}. \end{aligned}$$

Simplify each step of your computations.]

(c) Verify the Cauchy-Riemann conditions in polar coordinates for the exponential function

$$e^z = e^{re^{i\theta}} = e^{r[\cos(\theta) + i\sin(\theta)]} = e^{r\cos(\theta)}\{\cos[r\sin(\theta)] + i\sin[r\sin(\theta)]\}.$$

(d) Apply the above rule of derivatives in polar coordinates to show $\left(z^2\right)' = 2z$ and $\left(\dfrac{1}{z}\right)' = \dfrac{-1}{z^2}$.

3.3.9 Consider the function $z^2 = x^2 - y^2 + i2xy$. Show that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ satisfy the Cauchy-Riemann equations.

Check the same thing for z^3.

3.3.10 The function $f(z) = \mathrm{Re}(z) = x$ is a real analytic and therefore continuous function. Show that it is not a holomorphic function.

Show the same thing for the following functions:

$g(z) = z\bar{z}, \quad h(z) = \bar{z}, \quad p(z) = |z| \quad \text{and} \quad q(z) = \mathrm{Im}(z) = y.$

[Hint: Check the Cauchy-Riemann conditions.]

3.3.11 Use the Cauchy-Riemann equations to show that if $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$ is holomorphic in an open set $D \subseteq \mathbb{C}$, then the two functions $u(x,y)$ and $v(x,y)$ are **harmonic** in D, or else each of them satisfies the **Laplace equation**

$$\Delta w := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

in D, provided that the functions u and v are of class $\mathcal{C}^2$, i.e., they are twice continuously differentiable.

Next, under the same conditions, prove that the function $g(x,y) := |f(z)|^2 = u^2(x,y) + v^2(x,y)$ satisfies the relation

$$\Delta g = 2\left|\vec{\nabla}(u)\right|^2 + 2\left|\vec{\nabla}(v)\right|^2 = 4\left|\vec{\nabla}(u)\right|^2 = 4\left|\vec{\nabla}(v)\right|^2 \geq 0,$$

where $\vec{\nabla}(u)$ is the gradient of u, etc.

Then prove

$$\Delta g \equiv 0 \quad \text{in} \quad D \quad \Longleftrightarrow \quad f \quad \text{is constant in} \quad D.$$

3.3.12 Prove that the Laplacian of a real sufficiently differentiable function $w = w(x,y)$ of two variables x and y, written in polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, has the form

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

[Hint: Use the relations and the operators in the hint in **Problem 3.3.8**, along with the chain rule on the function $w(x, y)$. Simplify completely at each step.]

3.3.13 Consider the function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & \text{if} \quad z \neq 0 \\ \\ 0, & \text{if} \quad z = 0. \end{cases}$$

Then:

(a) With $z = x + iy$, write $f(z)$ explicitly in the form $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.

(b) Prove directly that for every $z \in \mathbb{C}$, $f(z)$ is continuous but does not have complex derivative at z.

(c) Prove that both $u(x, y)$ and $v(x, y)$ have partial derivatives at every $(x, y) \in \mathbb{R}^2$.

(d) Prove that at the point $(x_0, y_0) = (0, 0)$ the partial derivatives of $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann conditions.

(e) Prove that both partial derivatives of both $u(x, y)$ and $v(x, y)$ are discontinuous at $(x_0, y_0) = (0, 0)$.

(f) Prove directly that both functions $u(x, y)$ and $v(x, y)$ are not differentiable at $(x_0, y_0) = (0, 0)$.

(g) Prove directly that both functions $u(x, y)$ and $v(x, y)$ do not satisfy the Cauchy-Riemann conditions at every point $(x, y) \neq (0, 0)$.

(h) Prove directly that both functions $u(x, y)$ and $v(x, y)$ have continuous partial derivatives at every point $(x, y) \neq (0, 0)$, and therefore they are differentiable at such a point.

3.4 Line Integrals in the Complex Plane

Definition 3.4.1 *A **complex path** (or continuous complex path) in the complex plane $\mathbb{C}$ is a function $C : [a, b] \to R \subseteq \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a bounded closed interval of the real numbers, defined by*

$$C(t) = z(t) = x(t) + iy(t),$$

such that $z(t)$ is continuous, i.e., both $x(t)$ and $y(t)$ are continuous, at every $t \in [a, b]$. (At a and b, we consider the appropriate side continuities.)

If the derivative

$$C'(t) = z'(t) = x'(t) + iy'(t)$$

*exists for all $t \in [a, b]$, we say that $C(t)$ is a **differentiable complex path** on $[a, b]$. (At a and b we consider the appropriate side derivatives.)*

*If $C'(t)$ is continuous, i.e., $z'(t)$ is continuous, i.e., both $x'(t)$ and $y'(t)$ are continuous, then we say that $C(t)$ is a **continuously differentiable complex path.***

Definition 3.4.2 *A non-empty set $\emptyset \neq U \subseteq \mathbb{C}$ is called **path connected** if for any two points p and q in U there is a continuous path $C : [a, b] \to U \subseteq \mathbb{C}$, such that $C(a) = p$, $C(b) = q$ and $C([a, b]) \subseteq U$, that is, the whole path lies in the set U.*

Important remark: In this definition, if the set U is open and path connected, then we can prove that the path $C(t)$ can be chosen to be a continuously differentiable complex path. This involves some standard arguments of general topology and analysis, which we skip here. Otherwise, we would have to write a whole chapter on the topological facts of the complex plane. In fact, any open and connected subset of the complex plane is path connected, and the paths joining any two points of it can be chosen to be not just continuous but also continuously differentiable. We can use this important fact freely in the sequel.

Now we consider a complex function

$$f : R \longrightarrow \mathbb{C} \ \ : \ \ z = x + iy \in R \longrightarrow w = f(z) = u(x, y) + iv(x, y)$$

of the complex variable $z = x + iy$ from a region $R \subseteq \mathbb{C}$ into $\mathbb{C}$ and a complex path in R

$$C : [a, b] \longrightarrow R \subseteq \mathbb{C} \ \ : \ \ t \in [a, b] \longrightarrow C(t) = x(t) + iy(t) = z(t).$$

Therefore, the composition $(f \circ C)(t) = f[C(t)]$ is possible for all $t \in [a, b]$.

Next, we assume that $C'(t) = x'(t) + iy'(t) = z'(t)$ is continuous, i.e., C is a **continuously differentiable path**. Then, along this path we define the **complex line integral** as follows:

$$\int\limits_C f(z)\, dz = \int_a^b f[C(t)] d[C(t)] =$$

$$\int_a^b f[x(t) + iy(t)] \cdot d[x(t) + iy(t)] = \int_a^b f[x(t) + iy(t)] \cdot [x'(t) + iy'(t)]\, dt.$$

So, if $f(z) = u(x,y) + iv(x,y)$ is continuous over C and $C(t)$ is continuously differentiable, then by multiplying out we find

$$\int_C f(z)\,dz = \int_a^b \{u[x(t),y(t)] + iv[x(t),y(t)]\}[x'(t) + iy'(t)]\,dt =$$

$$\int_a^b \{u[x(t),y(t)]x'(t) - v[x(t),y(t)]y'(t)\}\,dt +$$

$$i\int_a^b \{u[x(t),y(t)]y'(t) + v[x(t),y(t)]x'(t)\}\,dt.$$

(By the continuity of all functions involved here, both final integrals exist.)

If $C(t)$ is continuous but does not possess derivatives at finitely many values $t_i \in [a,b]$ for $i = 1, 2, 3, ..., n$, (usually due to corners), but $C'(t)$ is continuous at all other points t of $[a,b]$, we say that $C(t)$ is a **continuous and piecewise continuously differentiable path**. Suppose, without loss of generality, that $a = t_0 \le t_1 \le t_2 \le t_3 \le ... \le t_n \le b = t_{n+1}$. Then the complex line integral is naturally defined to be the sum of the integrals over the individual smooth pieces. That is,

$$\int_C f(z)\,dz = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} f[x(t) + iy(t)] \cdot [x'(t) + iy'(t)]\,dt.$$

If a path C is a **union of continuously differentiable paths** C_i for $i = 1, 2, 3, \ldots, n$ (successive or disjoint or both), i.e., $C = \bigcup_{i=1}^{n} C_i$, then we simply call it a **piecewise continuously differentiable path**. In such a case, for any complex function $f(z)$ we have again

$$\int_C f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz.$$

We evaluate this integral by parametrizing each C_i, etc.

This definition suggests to use the $+$ instead of the $\cup$, in this context. That is, we prefer to write $C = \sum_{i=1}^n C_i = C_1 + C_2 + \ldots + C_n$ instead of $C = \bigcup_{i=1}^{n} C_i$. We are going to use this notation from now on.

Important Properties of the Complex Line Integral:

Here we state five properties of the complex line integrals that we are going to use in the material that follows. The proofs of the first four are like the proofs of the corresponding properties of line integrals, found in any book of multi-variable calculus, with some straightforward adjustments to our context. The fifth one is harder and we prove it here.

(1) The rule of u-substitution with definite integrals

$$\int_a^b f[u(x)]u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

verifies that the value of $\int\limits_C f(z)\,dz$ **is independent of the parametrization** of the path C.

(2) The line integral switches sign if we switch direction of traversing the path. If two paths have exactly the same range in $\mathbb{C}$ but are traversed in opposite directions, then we call them **opposite paths**. If we denote one of them by C, then we denote its opposite by $-C$. Then

$$\int\limits_{-C} f(z)\,dz = -\int\limits_C f(z)\,dz.$$

So, the direction of the motion along the path is significant for the sign of the value of the line integral but not for its absolute value.

(3) For any complex path C and any complex functions $f(z)$ and $g(z)$, we have the usual **linearity property of the integral**

$$\int\limits_C (f+g)(z)dz = \int\limits_C f(z)dz + \int\limits_C g(z)dz.$$

One way to prove this is to perform the computations in both sides separately and see that they yield the same result.

(4) For any complex path C, any complex function $f(z)$ and any complex number $c = a + ib$, we have the usual **linearity property of the integral**

$$\int\limits_C (cf)(z)dz = c\int\limits_C f(z)dz.$$

Again, one way to prove this is to perform the computations in both sides separately and see that they yield the same result.

(5) For any complex path C and any complex function $f(z)$, we have the **important inequality**

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|\,|dz|.$$

The **properties (1)-(4)** are straightforward and follow from the definitions and carrying out the computations. Also, **property (5)** is fairly easy in the real case, as it has been shown in the calculus courses. But, proving **property (5**) in the complex case becomes tricky. So, we present:

Proof of (5) It is sufficient to prove this for C a continuously differentiable path. That is,

$$C : [a,b] \longrightarrow R \subseteq \mathbb{C} \quad : \quad t \in [a,b] \longrightarrow C(t) = x(t) + iy(t) = z(t),$$

such that

$$C'(t) = x'(t) + iy'(t) = z'(t)$$

exists for all $t \in [a,b]$ and it is continuous. Then

$$\int\limits_C f(z)\,dz = \int_a^b f[z(t)]\cdot\, d[z(t)] = \int_a^b f[z(t)]\cdot z'(t)\,dt.$$

We let

$$I = \int\limits_C f(z)\,dz = \int_a^b f[z(t)]\cdot z'(t)\,dt$$

and

$$J = \int\limits_C |f(z)|\;|dz| = \int_a^b |f[z(t)]\cdot z'(t)|\;dt, \quad \text{where}$$

$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ and $d\,[z(t)] = z'(t)\,dt = [x'(t) + iy'(t)]\,dt$.

Obviously, $0 \leq |I| < \infty$, $0 \leq J < \infty$ and we must prove $|I| \leq J$. If $I = 0$, this is trivially valid.

If $I \neq 0$, we write $I = |I|e^{i\theta}$. [I.e., $\theta = \text{Arg}(I)$.] We let $c = \dfrac{|I|}{I} = e^{-i\theta}$, a complex constant with $|c| = 1$. Then:

$$0 < \left|\int_a^b f[z(t)]\cdot z'(t)\,dt\right| = |I| =$$

$$cI = c\int_a^b f[z(t)]\cdot z'(t)\,dt = \int_a^b c\cdot f[z(t)]\cdot z'(t)\,dt.$$

We write

$$c\cdot f[z(t)]\cdot z'(t) = p(t) + iq(t)\;(= \text{real part} + i\cdot \text{imaginary part})\,.$$

Then, $0 < |I| = cI$ implies

$$0 < \left| \int_a^b f[z(t)] \cdot z'(t)\, dt \right| = \int_a^b cf[z(t)] \cdot z'(t)\, dt = \int_a^b p(t)\, dt + i \int_a^b q(t)\, dt.$$

Since this is positive real, we conclude

$$\int_a^b q(t)\, dt = 0.$$

Also, by the properties of the absolute value, we get

$$p(t) \leq |c \cdot f[z(t)] \cdot z'(t)| = |c|\, |f[z(t)]|\, |z'(t)| = \\ 1 \cdot |f[z(t)]|\, |z'(t)| = |f[z(t)]|\, |z'(t)|.$$

By the previous two observations and the inequality properties of real integrals, we eventually find

$$0 < |I| = \left| \int_a^b f[z(t)] \cdot z'(t)\, dt \right| = \int_a^b p(t)\, dt \leq \\ \int_a^b |f[z(t)]|\, |z'(t)|\, dt = \int\limits_C |f(z)|\, |dz| = J.$$

If $C = C_1 + C_2 + \ldots + C_n$ is a path with each piece C_i continuously differentiable, this inequality is valid on each C_i. Then we get

$$\left| \int_C f(z) dz \right| = \left| \sum_{i=1}^n \int_{C_i} f(z) dz \right| \leq \sum_{i=1}^n \left| \int_{C_i} f(z) dz \right| \leq \\ \sum_{i=1}^n \int_{C_i} |f(z)|\, |dz| = \int_C |f(z)|\, |dz|.$$

Examples

Example 3.4.1 Let $f(z) = z^2 = x^2 - y^2 + i2xy$ and C be the unit circle $\{z \in \mathbb{C} : \quad |z| = 1\}$ traversed in the positive direction (counterclockwise). We indicate this by writing C^+. Then, a parametrization of C^+ is

$$z(t) = \cos(t) + i \sin(t) \quad \text{with} \quad 0 \leq t \leq 2\pi.$$

Hence

$$dz(t) = [-\sin(t) + i \cos(t)]\, dt$$

and

$$f[z(t)] = \cos^2(t) - \sin^2(t) + i2\cos(t)\sin(t).$$

So,

$$\int\limits_{C^+} f(z)\,dz =$$

$$\int_0^{2\pi} \{[\cos^2(t) - \sin^2(t)][-\sin(t)] - 2\cos(t)\sin(t)\cos(t)\}\,dt +$$
$$i\int_0^{2\pi} \{[\cos^2(t) - \sin^2(t)]\cos(t) + 2\cos(t)\sin(t)[-\sin(t)]\}\,dt =$$
$$\int_0^{2\pi} [-\cos(2t)\sin(t) - \sin(2t)\cos(t)]\,dt +$$
$$i\int_0^{2\pi} [\cos(2t)\cos(t) - \sin(2t)\sin(t)]\,dt =$$
$$\int_0^{2\pi} [-\sin(2t+t)]\,dt + i\int_0^{2\pi} \cos(2t+t)\,dt =$$
$$\int_0^{2\pi} [-\sin(3t)]\,dt + i\int_0^{2\pi} \cos(3t)\,dt = 0 + i0 = 0.$$

Note: In general, we may have to reside in this lengthy way in terms of $x(t)$ and $y(t)$ in order to compute complex line integrals. This is especially the case when $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is not given as an explicit function of z and the path C is not given as a complex expression of the parameter t. Instead, either one is given as some function of x and y.

In this example, however, we observe that $f(z) = z^2$ and $C(t) = e^{it}$, and so both have *explicit complex expressions in terms of the complex variable z and the parameter t. Then the line integral can be computed faster*, as follows:

$$\int\limits_{C^+} f(z)\,dz = \int_0^{2\pi} \left(e^{it}\right)^2\, de^{it} = \int_0^{2\pi} e^{2it} i e^{it}\,dt = i\int_0^{2\pi} e^{3it}\,dt =$$
$$\frac{i}{3i}\left[e^{3it}\right]_0^{2\pi} = \frac{1}{3}\left(e^{6\pi i} - e^0\right) = \frac{1}{3}(1-1) = 0.$$

From now on, any time we can do so, we will employ this shorter complex computation method to evaluate the complex line integral. The initial lengthier way will be used in all other cases.

▲

Example 3.4.2 Let C^+ be as before the positive unit circle with center the origin, and

$$\text{(a)}\quad f(z) = \frac{1}{z}, \qquad \text{(b)}\quad g(z) = \frac{1}{z^2}.$$

Then:

$$\text{(a)}\quad \int\limits_{C^+} f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\,de^{it} = \int_0^{2\pi} e^{-it} i e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

If we travel the path of the unit circle in the negative direction, indicated by the opposite path of C^+, i.e., $C^- = -C^+$, then we get

$$\int\limits_{C^-} f(z)\,dz = \int\limits_{C^-} \frac{1}{z}\,dz = -2\pi i.$$

$$\text{(b)}\qquad \int\limits_{C^+} g(z)\,dz = \int_0^{2\pi} e^{-2it} i e^{it}\,dt = i\int_0^{2\pi} e^{-it}\,dt =$$

$$\frac{i}{-i}\left[e^{-it}\right]_0^{2\pi} = -1\left(e^{-2\pi i} - e^0\right) = -(1-1) = 0.$$

Note: *Every complex line integral can be separated into real and imaginary parts to yield two real line integrals.* This strategy is very effective when computing real integrals. For instance, from these two examples, we get:

$$\text{(a)}\qquad \int\limits_{C^+} \frac{1}{z}\,dz = \int\limits_{C^+} \frac{xdx + ydy}{x^2+y^2} + i\int\limits_{C^+} \frac{-ydx + xdy}{x^2+y^2} = 0 + 2\pi i.$$

So, we find the following two real line integrals

$$\int\limits_{C^+} \frac{xdx + ydy}{x^2+y^2} = 0 \qquad \text{and} \qquad \int\limits_{C^+} \frac{-ydx + xdy}{x^2+y^2} = 2\pi.$$

Also,

$$\int\limits_{C^-} \frac{xdx + ydy}{x^2+y^2} = -0 = 0 \qquad \text{and} \qquad \int\limits_{C^-} \frac{-ydx + xdy}{x^2+y^2} = -2\pi.$$

Similarly,

$$\text{(b)}\ \int\limits_{C^+} \frac{1}{z^2}\,dz = \int\limits_{C^+} \frac{(x^2-y^2)dx + 2xydy}{(x^2+y^2)^2} + i\int\limits_{C^+} \frac{-2xydx + (x^2-y^2)dy}{(x^2+y^2)^2} = 0.$$

Then both real and imaginary parts are equal to zero, i.e.,

$$\int_{C\pm} \frac{(x^2-y^2)dx+2xydy}{(x^2+y^2)^2}=0 \qquad \text{and} \qquad \int_{C\pm} \frac{-2xydx+(x^2-y^2)dy}{(x^2+y^2)^2}=0.$$

We can, of course, verify these results directly with the methods of calculus. However, this efficient technique to evaluate complicated real line integrals gives answers easily and quickly.

▲

Example 3.4.3 We consider the points $P=(-1,1)$ and $Q=(1,2)$ and the complex function $f(z)=f(x+iy)=x+y-ix^2$. We would like to compute

$$\int_F f(z)dz = \int_F (x+y-ix^2)\,d(x+iy) = \int_F (x+y-ix^2)\,(dx+idy)$$

where F is the straight segment PQ, moving from P to Q.

A straight segment like this, in complex analysis, is denoted by $[z_1, z_2]$, where $z_1=-1+i$ is the complex number representing the initial point P and $z_2=1+2i$ is the complex number representing the terminal point Q.

Here, we have no choice but to integrate using the long method. The usual parametrization of a segment like F is

$$\begin{gathered} F=[z_1,z_2]=\{z=(1-t)z_1+tz_2 \mid 0\le t\le 1\}= \\ \{z=(1-t)(-1+i)+t(1+2i) \mid 0\le t\le 1\}= \\ \{z=(-1+2t)+i(1+t) \mid 0\le t\le 1\}. \end{gathered}$$

Then, along F, $x=-1+2t$, $y=1+t$, $x+y=3t$, $x^2=1-4t+4t^2$, $dz=(2+i)dt$, and we find

$$\int_F f(z)\,dz = \int_F (x+y-ix^2)\,d(x+iy) =$$

$$\int_0^1 [3t-i(1-4t+4t^2)](2+i)dt = (2+i)\left[\frac{3t^2}{2}-i\left(t-2t^2+\frac{4t^3}{3}\right)\right]_0^1 =$$

$$(2+i)\left[\frac{3}{2}-i\left(1-2+\frac{4}{3}\right)-(0+i0)\right] = (2+i)\left(\frac{3}{2}-\frac{i}{3}\right) = \frac{10}{3}+i\frac{5}{6}.$$

Now we take a different path L that joins P and Q and compute the path integral of the same function. We let $O=(0,0)$ the origin, and we pick L to be the union of the two straight segments PO and OQ. We write

$$L=[z_1, 0+i0]\cup[0+i0z_2]=[-1+i, 0+i0]\cup[0+i0, 1+2i].$$

As we have already said, in this context we prefer to use the $+$ instead of the $\cup$. So, from now on, we are going to use the notation with the $+$. That is, we prefer $L = [-1+i, 0+i0] + [0+i0, 1+2i]$ instead of $L = [-1+i, 0+i0] \cup [0+i0, 1+2i]$.

A parametrization of this new path L is

$$L = \{z = (t-1)(-1+i) \mid 0 \le t \le 1\} + \{z = (1+i2)t \mid 0 \le t \le 1\} =$$
$$\{z = -t+1+i(t-1) \mid 0 \le t \le 1\} + \{z = t+i2t \mid 0 \le t \le 1\}.$$

Along this new path, we compute the integral

$$\int\limits_L (x+y-ix^2)\,(dx+idy) =$$
$$\int_0^1 [0-i(-t+1)^2]\,(-1+i)dt + \int_0^1 (t+2t)-it^2)\,(1+i2)dt =$$
$$(-1+i)(-i)\int_0^1 (t^2-2t+1)dt + (1+i2)\int_0^1 (3t-it^2)\,dt =$$
$$= (1+i)\left[\frac{t^3}{3}-t^2+t\right]_0^1 + (1+i2)\left[\frac{3t^2}{2}-i\frac{t^3}{3}\right]_0^1 =$$
$$(1+i)\left[\frac{1}{3}-1+1-0\right] + (1+i2)\left[\frac{3}{2}-\frac{i}{3}-0\right] = \frac{5}{2}+i3.$$

We see that the two path integrals of the given function $f(z)$ are different, even though both paths start at P and terminate at Q.

Along the path opposite to F, written as $-F$ from Q to P, the line integral has the opposite value, i.e.,

$$\int\limits_{-F} (x+y-ix^2)\,(dx+idy) = -\left(\frac{10}{3}+i\frac{5}{6}\right),$$

and similarly for $-L$.

▲

Example 3.4.4 Compute $\displaystyle\int\limits_{C^+} \frac{1}{z}\,dz$, where C^+ is the circle of center -2 and radius 1, travelled in the positive direction.

A parametrization of the path C^+ is

$$z(t) = -2+e^{it} \qquad \text{with} \qquad 0 \le t \le 2\pi.$$

So,

$$\int\limits_{C^+} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{-2+e^{it}}\,d(-2+e^{it}) = i\int_0^{2\pi} \frac{e^{it}dt}{-2+e^{it}}.$$

To make the denominator of $\dfrac{e^{it}}{e^{it}-2}$ real, we multiply both numerator and denominator by the conjugate of the denominator, which is

$$\overline{e^{it}-2} = \overline{e^{it}} - \overline{2} = \overline{e^{it}} - 2 = e^{-it} - 2.$$

Then

$$\frac{e^{it}}{e^{it}-2} = \frac{1-2e^{it}}{1+4-2(e^{it}+e^{-it})} = \frac{1-2\cos(t)-i2\sin(t)}{5-4\cos(t)}.$$

So,

$$\int\limits_{C+} \frac{1}{z}\,dz = 2\int_0^{2\pi} \frac{\sin(t)}{5-4\cos(t)}dt + i\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}dt.$$

We now compute both integrals by elementary methods. In the first integral, we use the substitution $u = \cos(t)$ to find

$$\int_0^{2\pi} \frac{\sin(t)}{5-4\cos(t)}dt = -\int_1^1 \frac{du}{5-4u} = 0.$$

The second integral can be written as

$$\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}\,dt = \frac{1}{2}\int_0^{2\pi} \frac{5-4\cos(t)-3}{5-4\cos(t)}\,dt =$$
$$\frac{1}{2}\int_0^{2\pi} \left(1-\frac{3}{5-4\cos(t)}\right)\,dt = \pi - \frac{3}{2}\int_0^{2\pi} \frac{dt}{5-4\cos(t)}.$$

Now, in order to use the substitution $w = \tan(u/2)$, tangent of half angle, in this rational expression of $\cos(t)$ (and $\sin(t)$), we need the dummy variable to vary in $[-\pi, \pi]$, because tangent is undefined (discontinuous) at $\pm\pi/2$. (See **Example 1.1.1**.) So, we let first $u = t - \pi$, and then using the substitution the last integral becomes

$$-\frac{3}{2}\int_{-\pi}^{\pi} \frac{du}{5+4\cos(u)} = -\frac{3}{2}\cdot\frac{2}{3}\left[\arctan\left(\frac{1}{3}\tan\left(\frac{u}{2}\right)\right)\right]_{-\pi}^{\pi} =$$
$$-\arctan(\infty) + \arctan(-\infty) = \frac{-\pi}{2} + \frac{-\pi}{2} = -\pi.$$

Therefore,

$$\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}dt = \pi - \pi = 0$$

and finally

$$\int\limits_{C+} \frac{1}{z}\,dz = 0.$$

▲

3.5 Cauchy-Goursat Theorem and Consequences

3.5.1 Complex Preliminaries and Notation

The result 0 that we found in **Examples 3.4.1** and **3.4.4** were not accidents. We must notice and remember the following three conditions.

(1) The functions $w = z^2$ and $w = \dfrac{1}{z}$ were both defined at every point of the respective paths and at every point inside the regions enclosed by these paths.

(2) The paths were **closed**, that is, they had the same initial and terminal points and they were **simple**, that is, they had no self-intersections. So, each of them **enclosed a region** of the complex plane without ambiguity.

We call the region of the complex plane enclosed by a simple closed path **the set of its interior points** or **the inside of the simple closed path**. We call the unbounded region of the plane not enclosed by such a path **the set of its exterior points** or **the outside of the simple closed path**.

(3) We must observe that the functions $w = z^2$ and $w = \dfrac{1}{z}$ have complex derivatives $\dfrac{dw}{dz} = 2z$ and $\dfrac{dw}{dz} = \dfrac{-1}{z^2}$, respectively, which exist **at every point** of the corresponding paths and the enclosed regions. (As we see, for $w = \dfrac{1}{z}$ the troublesome point $z = 0$ is not in that region.) So, $w = z^2$ and $w = \dfrac{1}{z}$ are holomorphic. So, according to the most important fact of Complex Analysis that we are going to state right after, they are complex analytic, i.e., they are locally represented as convergent complex power series. Namely, for the **two examples**, we have:

(a) In $\{z \in \mathbb{C} \mid |z| \leq 1\}$, $f(z) = z^2 = 0 + 0z + 1z^2 + 0z^3 + ...$ is already in power series form.

(b) In $\{z \in \mathbb{C} \mid |z + 2| \leq 1\}$, with the help of the geometric series, we have that

$$g(z) = \frac{1}{z} = \frac{1}{(z+2)-2} = \frac{-1}{2}\frac{1}{1-\left(\dfrac{z+2}{2}\right)} =$$

$$\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z+2}{2}\right)^n = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}(z+2)^n = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}[z-(-2)]^n.$$

In fact, this power series is convergent for all z's such that $\left|\dfrac{z+2}{2}\right| < 1$, that is, all z's in the open disc $D(-2,2) := \{z \ : \ |z+2| < 2\}$.

Also, $D(-2,2)$ purely contains the region of the closed disc $\overline{D(-2,1)} = \{z \ : \ |z+2| \leq 1\}$ enclosed by the given path.

Notice that $z = 0$ does not belong to the open disc $D(-2,2)$.

Now we state:

Theorem 3.5.1 ("Most Important Fact" of Complex Analysis) *Every complex function $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$ which is defined and holomorphic in an open region $R \subseteq \mathbb{C}$, that is, the complex derivative $\dfrac{dw}{dz} = f'(z)$ exists at every point $z \in R$, is locally expressed as a power series in the region R. I.e., for every point $z_0 \in R$ and every $r > 0$ such that the open disc $D(z_0, r) \subseteq R$, there are complex coefficients a_0, a_1, a_2, a_3, ... such that*

$$w = f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall\, z \in D(z_0, r) \subseteq R\,.$$

Therefore, this power series with center z_0 and radius $r > 0$ converges, and it is equal to $f(z)$ at every point $z \in D(z_0, r) \subseteq R$. (In particular, $f(z_0) = a_0$.)

Proof We can just use this well-known fact of Complex Analysis without proving it. Here, we simply provide a proof-sketch under the extra assumption that $u(x,y)$ and $v(x,y)$ are of class $\mathcal{C}^2$, that is, twice continuously differentiable. (For details, see **Appendix 3.5.8** at the end of this section.)

We have seen (**Problem 3.3.11**) that if $w = f(z) = u(x,y) + iv(x,y)$ is holomorphic in R, i.e., $f'(z)$ exists at every point of the region R, then by the Cauchy-Riemann conditions, and under the assumption that $u(x,y)$ and $v(x,y)$ are twice continuously differentiable, both $u(x,y)$ and $v(x,y)$ are harmonic in R. I.e.,

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

The solutions of these Laplace equations in discs can be obtained by the **Poisson integral formula (3.14)** (obtained in **Appendix 3.5.8** at the end of this section).

This formula proves that both $u(x,y)$ and $v(x,y)$ are analytic functions, that is, power series in x and y, in discs (proven in **Theorem 3.5.8**).

But the harmonic functions u and v are closely related by means of the Cauchy-Riemann conditions in such a way that the two power series that represent them in any given disc, $D(z_0, r)$ with $r > 0$, can be added together according to the rule $u(x, y) + iv(x, y)$ and then rewritten so that their sum is a power series in the complex variable z. This power series is equal to the complex holomorphic function

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

(proven in **Theorem 3.5.10**).

Hence, $w = f(z)$ is a complex power series in any disc $D(z_0, r) \subseteq R$ and therefore complex analytic in R.

Remark 1: We have just seen that this fundamental result of Complex Analysis can be established through the **solutions of the Laplace equation in discs** by the **Poisson integral formula**. Proving first the so-called **Cauchy integral-formula**, which we will see soon in **Subsection 3.5.7**, from the existence of $f'(z)$ and then obtaining the analyticity of $f(z)$ as a result, is the standard way followed in most advanced books of complex analysis. In this way, we avoid the extra condition that requires $u(x, y)$ and $v(x, y)$ to be twice continuously differentiable. In such a complex analysis book, this hypothesis is immediately obtained from the analyticity of $f(z)$.

This approach needs several preliminaries that we cannot develop at this level and thus establish the Cauchy integral formula from the existence of the $f'(z)$ before the analyticity of $f(z)$. At this level, we are content with this extra condition for it is not very restrictive to applications and simplifies the approach to the subject matter and the proof involved a lot.

In this course for beginners, and our brief and oriented toward applications exposition, we follow Weierstraßś approach on this subject. Hence, from the **power series representation** we derive the **Cauchy integral formula** in **Subsection 3.5.7**. In more advanced courses of complex analysis, the classic approach is to establish the Cauchy integral formula first and then the power series representation. But, both approaches are essentially equivalent.

Our goal here is not to develop a rigorous and advanced complex analysis course. This would need many more mathematical prerequisites. As we have said, we want to expose to the novices the most important tools that complex analysis can provide in computing various proper and improper Riemann integrals.

Remark 2: For a power series, we observe

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = 0, \quad \forall\, z \in D(z_0, r) \iff a_n = 0, \ \forall\, n \in \mathbb{N}_0.$$

Because the power series, claimed by **Theorem 3.5.1**, can be found by the Taylor series formula in the complex variable z, in exactly the same way as we have used it in calculus, we also called them **Taylor series**. That is,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \forall\ z \in D(z_0, r),\ (r > 0).$$

Hence, the previous coefficients a_0, a_1, a_2, a_3, ... satisfy the relation

$$a_n = \frac{f^{(n)}(z_0)}{n!},\ \ \forall\ \ n = 0, 1, 2, 3, \ldots.$$

The radius of convergence is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}},$$

provided that these limits exist.

[When these two limits exist, prove as an exercise that they are equal. (See **Problem 3.2.4**.) Also, do not forget that in the real line we have the limit rules $\frac{1}{0^+} = +\infty$ and $\frac{1}{+\infty} = 0^+$.]

The power series converges for every z inside the disc $D(z_0, R)$. In fact, it converges absolutely. But, for z on the boundary of the disc, $\partial D(z_0, R) := \{z \mid z \in \mathbb{C}\ :\ |z - z_0| = R\}$, the power series may converge or may diverge. In general, a boundary point must be checked individually.

Notice that for complex analytic or holomorphic functions, we do not need to check if Taylor's Remainder converges to zero as $n \to \infty$. We can straightly compute the power series, and then we must find its radius of convergence.

Many times there are various shortcuts for finding these complex power series. These depend on the case and already known results. For instance, to find the power series of the functions $\cos(z)$, $\sin(z)$, $\cosh(z)$, $\sinh(z)$, we can more easily use their definitions by means of the exponential function $\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Another way is to use the power series of real functions and then replace the real variable with the complex variable, as we have seen in **Section 3.2**.

In the following list, we cite a few basic examples of power series along with their domain of convergence, representing some frequently used complex analytic functions. These power series have center $z_0 = 0$. When the center of the power series is $z_0 = 0$, we call the power series a **Maclaurin series**. This is a special case of the Taylor series. (Convergence at boundary points is checked point by point, in general!)

1. $\dfrac{1}{1-z}=\displaystyle\sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3+...+z^n+..., \quad \forall\ z:\ |z|<1.$
2. $e^z=\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+..., \qquad \forall\ z\in\mathbb{C}.$
3. $\cos(z)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-...+(-1)^n\frac{z^{2n}}{(2n)!}+...,$

   $\forall\ z\in\mathbb{C}.$
4. $\sin(z)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}=z-\frac{z^3}{3!}+\frac{z^5}{5!}-...+$

   $(-1)^n\dfrac{z^{2n+1}}{(2n+1)!}+..., \qquad \forall\ z\in\mathbb{C}.$
5. $\cosh(z)=\displaystyle\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}=1+\frac{z^2}{2!}+\frac{z^4}{4!}+...+\frac{z^{2n}}{(2n)!}+..., \quad \forall\ z\in\mathbb{C}.$
6. $\sinh(z)=\displaystyle\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+...+\frac{z^{2n+1}}{(2n+1)!}+...,$

   $\forall\ z\in\mathbb{C}.$
7. $\arccos(z)=\dfrac{\pi}{2}-\displaystyle\sum_{n=0}^{\infty}\binom{\frac{1}{2}+n-1}{n}\frac{z^{2n+1}}{2n+1}$

   $=\dfrac{\pi}{2}-\dfrac{z}{1}-\dfrac{1}{2}\cdot\dfrac{z^3}{3}-\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{z^5}{5}-\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{z^7}{7}-..., \quad \forall\ z:\ |z|<1.$
8. $\arcsin(z)=\displaystyle\sum_{n=0}^{\infty}\binom{\frac{1}{2}+n-1}{n}\frac{z^{2n+1}}{2n+1}$

   $=\dfrac{z}{1}+\dfrac{1}{2}\cdot\dfrac{z^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{z^5}{5}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{z^7}{7}-..., \quad \forall\ z:\ |z|<1.$
9. $\arctan(z)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+...,$

   $\forall\ z:\ |z|<1.$

(10. For the very useful power series of **complex logarithms** and **complex binomial power series**, see, respectively, **Sections 3.5.3** and **3.5.4**, where we define the complex logarithms and the complex powers.)

Now, we return to the previous discussion. In **Example 3.4.2 (a)**, the function $f(z) = \dfrac{1}{z}$ was integrated over the unit circle with center the origin, $C(0,1) = \{z \ : \ |z| = 1\}$, in the positive direction. The result of this line integral was found to be

$$\int\limits_{C^+(0,1)} \frac{1}{z}\, dz = 2\pi i.$$

This path is simple, and closed, and encloses the open unit disc: $D(0,1) = \{z \ : \ |z| < 1\}$, which is inside the region $R = D(0,2)$. But, at the origin $z = 0$, which is a point of this region R, the function $f(z) = \dfrac{1}{z}$ is not defined, let alone having derivative. There is no power series representation of this function with center $z_0 = 0$.

Similarly, given any $r > 0$ and any $z_0 \in \mathbb{C}$, we let $C(z_0, r) = \{z \ : \ |z - z_0| = r\}$ be the circle of center z_0 and radius r. This path encloses the open disc $D(z_0, r) = \{z \ : \ |z - z_0| < r\}$, with center z_0 and radius r, which is inside the region $R = D(0, 2r)$. Then, moving along this path in the positive direction, we obtain

$$\int\limits_{C^+(z_0,r)} \frac{1}{z - z_0}\, dz = 2\pi i.$$

(Check this in an analogous way as before.)

But, in **Example 3.4.2 (b)**, we found that

$$\int\limits_{C(0,1)} \frac{1}{z^2}\, dz = 0,$$

even though the function $g(z) = \dfrac{1}{z^2}$ is also not defined at the origin $z = 0$. Similarly, for any $r > 0$, we have

$$\int\limits_{C^\pm(z_0,r)} \frac{1}{(z - z_0)^2}\, dz = 0.$$

We must examine carefully what is going on here. But before we do that, we need to introduce the following **notation**, which we are going to use from now on. Given any $r > 0$ and any $z_0, \ z_1, z_2, \ \in \mathbb{C}$, we denote:

1. **Closed straight segment** in the direction from $z_1 \in \mathbb{C}$ to $z_2 \in \mathbb{C}$:
   $[z_1, z_2] = \{z \ : \ z = (1-t)z_1 + tz_2, \ 0 \le t \le 1\}$
2. **Circle** with center z_0 and radius r:
   $C(z_0, r) = \{z \ : \ |z - z_0| = r\}$

3. **Open disc** with center z_0 and radius r:
   $D(z_0, r) = \{z \ : \ |z - z_0| < r\}$

4. **Closed disc** with center z_0 and radius r:
   $\overline{D(z_0, r)} = \{z \ : \ |z - z_0| \leq r\} = D(z_0, r) \cup C(z_0, r)$

5. **Open center-punctured disc** with center z_0 and radius r:
   $D^o(z_0, r) = D(z_0, r) - \{z_0\} = \{z \ : \ 0 < |z - z_0| < r\}$

6. **Closed center-punctured disc** with center z_0 and radius r:
   $\overline{D(z_0, r)}^{\,o} = \overline{D(z_0, r)} - \{z_0\} = \{z \ : \ 0 < |z - z_0| \leq r\} = D^o(z_0, r) \cup C(z_0, r)$

7. **Circle** with center z_0 and radius r travelled in the **positive direction**:
   $C^+(z_0, r) = \left\{z = z_0 + re^{it}, \ 0 \leq t \leq 2\pi\right\}$

8. **Circle** with center z_0 and radius r travelled in the **negative direction**:
   $C^-(z_0, r) = \left\{z = z_0 + re^{-it}, \ 0 \leq t \leq 2\pi\right\}$

9. **Complex line integral** of a complex function $w = f(z)$ along a **closed path** C in the positive or negative direction: $\displaystyle\oint_{C^\pm} f(z)\, dz$.

Notice that for a **closed path** C, i.e., a path with initial and terminal points the same, the opposite of C^+ is simply $-C^+ = C^-$.

3.5.2 Cauchy-Goursat Theorem

In the **previous part of this section**, we discussed the **Taylor and Maclaurin series**. We also saw **Theorem 3.5.1**, which we called *the "Most Important Fact" of Complex Analysis*. According to that, if $f(z)$ is defined and holomorphic in an open region $R \subseteq \mathbb{C}$ and $D(z_0, r) \subseteq R$, for some $r > 0$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ \forall \ z \in D(z_0, r),$$

i.e., $f(z)$ is a Taylor series in the open disc $D(z_0, r)$.

Hence, an **antiderivative** of $f(z)$ is

$$F(z) = c + \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n + 1}, \ \ \forall \ z \in D(z_0, r),$$

where $c \in \mathbb{C}$ is a constant, that is, $F'(z) = f(z)$, since we can differentiate power series term by term in the disc of convergence.

Similarly, if in the same region we consider a function given by the series

$$g(z) = \sum_{n=2}^{\infty} b_n (z - z_0)^{-n}, \ \forall \ z \in D^o(z_0, r),$$

(we begin summation with $n = 2$ and not 1), then $g(z)$ has antiderivative

$$G(z) = c + \sum_{n=2}^{\infty} b_n \frac{(z - z_0)^{-n+1}}{-n+1}, \ \forall \ z \in D^o(z_0, r),$$

where $c \in \mathbb{C}$ is constant.

The function $F(z)$ is defined in every disc $D(z_0, r) \subseteq R$ in which $f(z)$ is defined, and it is continuous in this disc. Similarly, $G(z)$ is defined in $D^o(z_0, r)$ and is continuous. So, along any circle $C(z_0, \rho)$ with $0 < \rho < r$ or in any open set A inside the $D^o(z_0, r)$ containing $C(z_0, \rho)$ [i.e., $C(z_0, \rho) \subset A$ (open) $\subset D^o(z_0, r)$], the functions $F(z)$ and $G(z)$ are defined and continuous.

The existence of a continuous antiderivative is the **crucial fact** that applies in **Example 3.4.2 (b)**, in order to get

$$\int_{C(0,1)} \frac{1}{z^2}\, dz = 0.$$

That is, the function $g(z) = \frac{1}{z^2}$ has antiderivative the function $G(z) = \frac{-1}{z}$ defined and continuous in $D^o(0, 1)$.

In **Example 3.4.2 (a)**, however, we have that

$$\int_{C(0,1)} \frac{1}{z}\, dz = 2\pi i.$$

The non-zero answer happened because, as we shall shortly see in **Subsection 3.5.3** about complex logarithms, the function $f(z) = \frac{1}{z}$ does not have antiderivative which is both defined and continuous everywhere in $D^o(0, 1)$. In general, we have the following **important theorem**:

Theorem 3.5.2 *Suppose* $w = f(z)$ *has a continuous antiderivative* $F(z)$ *in an open set* $R \subseteq \mathbb{C}$, *and* $C : t \longrightarrow z(t)$, *with* $a \leq t \leq b$, *is a path (simple or not simple) contained in* R *and joining the complex* *numbers (points)* $z_a = z(a) \in R$ *and* $z_b = z(b) \in R$. *Then*

$$\int\limits_C f(z)\,dz = F(z_b) - F(z_a),$$

and so the path integral is independent of the path.

If C *is closed, then* $z_a = z_b$, *and this integral is zero.*

Proof Suppose $z(t),\ \ a \le t \le b$ is a parametrization of the closed path C, such that $z(a) = z(b)$. Let $F(z)$ be a continuous antiderivative of $f(z)$ in the open set that contains the closed path C.

Then, $F[z(a)] = \lim\limits_{t\to a^+} F[z(t)],\ \lim\limits_{t\to b^-} F[z(t)] = F[z(b)]$ and $F[z(a)] = f[z(b)]$. Thus, by the chain rule

$$d\,\{F[z(t)]\} = f[z(t)]\,dz(t),$$

and so

$$\int\limits_C f(z)\,dz = \int_a^b f[z(t)]\,dz(t) = \int_a^b \frac{d}{dt}F[z(t)]\,dt =$$

$$F[z(b)] - F[z(a)] = F(z_b) - F(z_a).$$

If C is closed, then $z_a = z_b$, and this integral is $F(z_b) - F(z_a) = 0$.

A special case of this general Theorem is the following:

Corollary 3.5.1 *For any complex function of the form*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=2}^{\infty} b_n (z - z_0)^{-n},\ \ \forall\ z \in D^o(z_0, r)$$

(notice that the power with exponent $n = -1$ *is missing) and any continuous and piecewise continuously differentiable path* C *inside* $D^o(z_0, r)$, *we have that*

$$\int\limits_C f(z)\,dz = 0.$$

Proof In the discussion before the above Theorem, we explained why such a function has a continuous antiderivative in the center-punctured disc $D^o(z_0, r)$, and so the result follows from the Theorem.

Remark: In **this corollary**, if $b_n = 0$ for all $n = 2,\ 3,\ 4,\ \dots$, then the function $f(z)$ is complex analytic (or holomorphic) in the whole disc, and we can replace the center-punctured disc $D^o(z_0, r)$ with the whole disc $D(z_0, r)$.

Example 3.5.1 The function $f(z) = z^2$ has as continuous antiderivatives the functions $F(z) = \frac{z^3}{3} + c$, where c is any constant, in the whole $\mathbb{C}$. So, for any path C joining the numbers $-3i$ and 9, we have

$$\int\limits_C z^2\,dz = \int_{-3i}^{9} z^2\,dz = \left[\frac{z^3}{3}\right]_{-3i}^{9} = \frac{9^3}{3} - \frac{(-3i)^3}{3} = 243 - 9i.$$

Similarly, along any path that does not contain 0, i.e., in the open set $\mathbb{C} - \{0 + i0\}$, and joins $-3i$ and 9, we find

$$\int_{-3i}^{9} \frac{1}{z^2}\,dz = \left[-\frac{1}{z}\right]_{-3i}^{9} = \frac{-1}{9} - \frac{-1}{-3i} = \frac{-1+3i}{9}.$$

Also, along any path joining $-1+3i$ and $9-2i$, we find

$$\int_{-1+3i}^{9-2i} \sin(z)\,dz = [\cos(z)]_{-1+3i}^{9-2i} = \cos(9-2i) - \cos(-1+3i) =$$
$$\frac{e^{i(9-2i)} + e^{-i(9-2i)} - e^{i(-1+3i)} - e^{-i(-1+3i)}}{2} =$$
$$\frac{e^{2+9i} + e^{-2-9i} - e^{-3-i} - e^{3+i}}{2}.$$

(See also **Example 3.5.4**.)

▲

Now we present a version of the fundamental Cauchy-Goursat Theorem that we will use in this text:

Theorem 3.5.3 (Cauchy-Goursat Theorem) [7] *Let $R \subseteq \mathbb{C}$ be an open region and let $f : R \longrightarrow \mathbb{C}$ be a complex* ***holomorphic*** *function [i.e., $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at every point $z \in R$]. Then for any simple closed path C in R for which all of its interior points are in R, we have*

$$\int\limits_C f(z)\,dz = 0.$$

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[7] There are more general versions of this Theorem, sometimes called the Global Cauchy Theorem, in which the paths are not simple but homotopic to a point, or the domains have holes and the paths may contain holes in their interior, etc., but we do not need these generalizations in our goals here. The interested reader can consult the bibliography in complex analysis.

Proof We observe that if $R = D(z_0, r)$ is an open disc for some center $z_0 \in \mathbb{C}$ and radius $r > 0$, then we have the power series representation of the holomorphic function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for some appropriate complex numbers $a_n, n = 0, 1, 2, ...$ This has a continuous antiderivative in $R = D(z_0, r)$ the power series

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1},$$

and the result follows from the **previous general theorem** or its **corollary**.

So, we assume that $R \neq D(z_0, r)$ (refer to **Figure 3.2**). In this general case and at this level, we can prove the Theorem under the assumption that $u(x, y)$ and $v(x, y)$ are of class $\mathcal{C}^1$ (i.e., their first order partial derivatives exist and are continuous). This extra assumption is not restrictive for the usual applications.

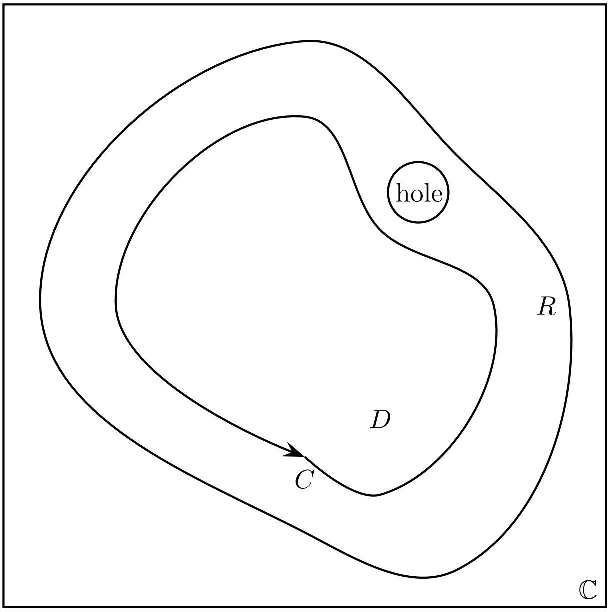

FIGURE 3.2: Region $R \subseteq \mathbb{C}$, Contour C in R with interior D

We let D be the interior of C in R. Then

$$\oint\limits_{C^{\pm}} f(z)dz = \oint\limits_{C^{\pm}} [u(x,y)+iv(x,y)](dx+idy) =$$

$$\oint\limits_{C^{\pm}} (udx - vdy) + i \oint\limits_{C^{\pm}} (vdx + udy).$$

Under all assumptions imposed here, we can apply the analytic form of Green's[8] Theorem in the region D, as we see it in calculus.[9] So, we have

$$\oint\limits_{C^{\pm}} f(z)dz = \pm\left[\int\!\!\int\limits_{D} (-v_x - u_y)dxdy + i\int\!\!\int\limits_{D} (u_x - v_y)dxdy\right].$$

But, the **Cauchy-Riemann conditions**,

$$u_x = v_y \text{ and } u_y = -v_x,$$

hold in $D \cup C$ since $f'(z)$ exists at every $z \in D \cup C$. But then,

$$\oint\limits_{C^{\pm}} f(z)dz = \pm\left[\int\!\!\int\limits_{D} 0\,dxdy + i\int\!\!\int\limits_{D} 0\,dxdy\right] = 0 + i0 = 0,$$

and the proof is finished!

Remark 1: This result was known first to Gaußbut was published first by Cauchy. A bit later, Goursat[10] relaxed some hypotheses and improved the proof that Cauchy had provided. His proof is somewhat involved but not difficult, and it is more general because it is based on the existence of $f'(z)$ only. Nothing is mentioned about $u(x,y)$ and $v(x,y)$ let alone to be of class C^1.

Remark 2: The **Cauchy-Goursat Theorem** claims that, under

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[8] George Green, English mathematician, 1793-1841.

[9] **Green's Theorem:** Let $P(x,y)$ and $Q(x,y)$ be two real functions of two variables which are continuously differentiable at every point of a bounded region G and every point of its boundary ∂G, in the plane $\mathbb{R}^2$. We also assume that the boundary ∂G is piecewise differentiable. Then:

$$\oint\limits_{\partial G^{\pm}} P\,dx + Q\,dy = \pm\int\!\!\int\limits_{G} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

[10] Edouard Jean-Baptiste Goursat, French mathematician, 1858-1936.

its hypotheses, the non-existence of holes in the interior of the simple closed path "contour" C is a sufficient condition to draw its conclusion.

However, this condition is not necessary because we have seen the example

$$\int\limits_{C(0,1)} \frac{1}{z^2}\, dz = 0,$$

and in this example the origin forms a hole inside the unit circle $C(0,1)$, since the function $f(z) = \dfrac{1}{z^2}$ is not defined at $z = 0$.

In this example, we apply **Theorem 3.5.2** because $F(z) = \dfrac{-1}{z}$ is a continuous antiderivative of $f(z)$ in $\overline{D}^{\,o}(0,1)$.

But, as we will see in the following subsection, we cannot claim and do the same things for the function $g(z) = \dfrac{1}{z}$ in $\overline{D}^{\,o}(0,1)$ for which we have seen that

$$\int\limits_{C^+(0,1)} \frac{1}{z}\, dz = 2\pi i.$$

On the one hand, the **Cauchy-Goursat Theorem** does not apply here, and on the other hand, the function $g(z) = \dfrac{1}{z}$ does not have a continuous antiderivative in $\overline{D}^{\,o}(0,1)$, as we explain in the subsection that follows.

Notice that one condition of the **Cauchy-Goursat Theorem** was that all interior points of the simple closed path C belonged to the open region R. This motivates the following definition:

Definition 3.5.1 *A region $R \subseteq \mathbb{C}$ is called* ***simply connected*** *if for every simple closed path C lying in R the inside of C is a subset of R.*

In view of this definition, we state an important consequence of the Cauchy-Goursat Theorem. We will not prove it, since under our approach in this text we will not use it. Its proof is not hard, and the interested reader can easily locate it in the bibliography.

Theorem 3.5.4 *Let $f : R \longrightarrow \mathbb{C}$ be a holomorphic function in an open, connected and simply connected region $R \subseteq \mathbb{C}$. We pick any $z_0 \in R$, and for any $z \in R$ we define*

$$F(z) = \int_C f(z)dz, \tag{3.6}$$

where C is any piecewise differentiable path lying in R and joining z_0 and z. Then $F(z)$ is a well-defined function, i.e., independent of the chosen path, and it is a continuous antiderivative of $f(z)$, i.e., $F'(z) = f(z)$.

Under these hypotheses, if C is any closed path inside the region R, then

$$\int_C f(z)\,dz = 0.$$

The converse of the Cauchy-Goursat Theorem is also true. It is called Morera's[11] Theorem, and we state it without its proof, which is not hard. It can be found with its proof in any good book of complex analysis. Many books choose for C's only triangles for a bit more generality.

Theorem 3.5.5 (Morera's Theorem) *Let $R \subseteq \mathbb{C}$ be an open region and let $f : R \longrightarrow \mathbb{C}$ be a complex continuous function [$f(z) = f(x+iy) = u(x, y) + iv(x, y)$].*

If for any simple closed piecewise differentiable path C in R for which all of its interior points are in R, we have

$$\int\limits_C f(z)\,dz = 0,$$

then the function $f(z)$ is ***holomorphic*** *in R. I.e., $f'(z)$ exists at every point $z \in R$.*

[In fact, $f(z) = F'(z)$ and so $f'(z) = F''(z)$, where $F(z)$ is given by ***(3.6)*** *above. Since $F(z)$ is holomorphic, then any derivative of $F(z)$ is also holomorphic.]*

3.5.3 Complex Logarithm

Someone motivated by the fact $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ in the realm of the real functions may be tempted to claim that the function $f(z) = \dfrac{1}{z}$ has as a continuous antiderivative in any $D^o(0, r)$ with $r > 0$, the function $F(z) = \log(z)$ that we will define in what follows, whereas we have seen in **Example 3.4.2 (a)**

$$\int\limits_{C(0,1)} \frac{1}{z}\,dz = 2\pi i \neq 0.$$

Is this a contradiction to **Theorems 3.5.2** and **3.5.3**? The answer is not In the sequel, we define and understand what we mean by $F(z) = \log(z)$ in the complex plane and along any continuous closed path that encloses the origin in its interior. We shall see that this function is defined continuously only locally. That is, we cannot define the complex $\log(z)$

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[11] Giacinto Morera, Italian mathematician and engineer, 1856-1909.

in any continuous manner everywhere in any $D^o(0, r)$ with $r > 0$ or in $\mathbb{C} - \{0\}$ for the same matter.

So, we must understand the definition of the **complex logarithmic function**. We remember that the trigonometric representation of complex numbers is $z = re^{i\theta}$, where $r = |z|$ and the angle θ is an argument of z. As we have seen in **Subsection 3.1.2, Parts (2)** and **(3)**, when $z \neq 0$, this angle is determined (mod 2π). To define and evaluate $\log(z)$ for any $z \neq 0$, we use the exponential form $z = e^{\ln r + i\theta}$, where θ is determined (mod 2π).

We begin as follows: Given a $z \in \mathbb{C} - \{0\}$, we have $r = |z| > 0$. Then, using an argument θ and **Problem 3.2.8**, we find

$$z = re^{i\theta} = e^{\ln(r)} e^{i\theta} \cdot 1 = e^{\ln(r)} e^{i\theta} e^{i2k\pi} =$$
$$e^{\ln(r) + i\theta + i2k\pi} = e^{\ln(r) + i(\theta + 2k\pi)},$$

where $k \in \mathbb{Z}$. So, there are infinitely many values

$$\{w_k = \ln(r) + i(\theta + 2k\pi) \mid k \in \mathbb{Z}\},$$

such that $e^{w_k} = z$. In fact, these are precisely all the possible complex numbers $w = w_k$ for which $e^{w_k} = z$ (prove this claim!). So, here we give the definition:

Definition 3.5.2 *For any $z \in \mathbb{C} - \{0\}$* ***the complex logarithm*** $\log(z)$ *of z is defined to be* ***the set*** *of all complex values w such that $e^w = z$.* ***A complex logarithm*** *or* ***a branch of logarithm*** *of z is any particular value w_k taken out of the set of all possible values.*

Remarks on this definition:

1. For any $z \in \mathbb{C} - \{0\}$, we pick a principal argument $\theta = \operatorname{Arg}(z)$ in a fixed initial interval and we have

   $$w = \log(z) = \{w_k = \ln(r) + i(\theta + 2k\pi) \mid k = 0,\ \pm 1,\ \pm 2,\ \pm 3, ...\},$$

   that is, the $\log(z)$ is the set of the complex numbers

   $$\log(z) =$$
   $$\ln(r) + i(\theta + 2\pi\mathbb{Z}) = \ln(r) + i\,\arg(z) = \ln(r) + i\,\operatorname{Arg}(z) + i\,2\pi\mathbb{Z}.$$

   In many concrete applications, we must designate and fix an initial interval for the principal arguments $\theta = \operatorname{Arg}(z)$.

   I.e., the complex $\log(z)$ has infinitely many values obtained by replacing $\theta[= \operatorname{Arg}(z)]$ with $\theta + 2k\pi$ with k an integer. This happens because e^z is not one-to-one in $\mathbb{C}$. We have seen that $e^{2k\pi i} = 1$ if and only if $k \in \mathbb{Z}$, and so for each $k \in \mathbb{Z}$ we get a different value in the set $\arg(z)$, i.e., $\arg(z)$ contains countably infinite values.

We say that $w = \log(z)$ is a "**multi-value function**" with values the infinitely many complex numbers found above.

In local coordinates, if $z = x + iy$, we can write

$$\log(z) = \ln\left(\sqrt{x^2 + y^2}\right) + i\left[\arctan\left(\frac{y}{x}\right) + c + 2k\pi\right], \qquad (3.7)$$

where $k \in \mathbb{Z}$ and c is a corrective constant to obtain the correct principal argument (as in **Example 3.1.3**).

2. Obviously, $\log(z)$ cannot be defined at the origin $z = 0$ because $e^z \neq 0$, $\forall\, z \in \mathbb{C}$. [Also, there is no $\ln(0)$.]

3. For any $z \in \mathbb{C} - \{0\}$, we have that

$$e^{\log(z)} = e^w = \{e^{w_k} \mid k \in \mathbb{Z}\} = \{z\}.$$

Hence, we simply write $e^{\log(z)} = z$.

4. For any $w \in \mathbb{C}$, we have that

$$\log(e^w) = w + i2\pi\mathbb{Z} \supset \{w\}.$$

5. If we designate an initial interval $[a, a + 2\pi)$ or $(a, a + 2\pi]$, where $a \in \mathbb{R}$, for the principal argument $\mathrm{Arg}_0(z)$ [see **Subsection 3.1.2, Parts (2)** and **(3)**], then $\forall\ k \in \mathbb{Z}$, we write

$$\log_{(k)}(z) = w_k = \ln(r) + i[\mathrm{Arg}_0(z) + 2k\pi]$$

[(k) is an index, in parentheses, so that it is not confused with a logarithmic base].

As in equation **(3.7)** above, in local coordinates $z = x + iy$, we write

$$\log_{(k)}(z) = \ln\left(\sqrt{x^2 + y^2}\right) + i\left[\arctan\left(\frac{y}{x}\right) + c + 2k\pi\right], \ \ \forall\ k \in \mathbb{Z},$$

where c is an appropriate corrective constant.

So,

$$e^{\log_{(k)}(z)} = z, \ \ \forall\ k \in \mathbb{Z}.$$

We call all the different values $w_k = \log_{(k)}(z)$ of the complex $\log(z)$ the **branches of the complex logarithm**. Here, k is simply an index written as a subscript and should not be confused with some logarithmic base, which for the complex logarithm is always e.

Hence, when we work with the complex logarithm, we must keep in mind whether we use all of its values as a set of numbers or one value of a particular branch.

Example 3.5.2 In calculus and real analysis, we learn $\ln(e) = 1$. That is, when we write $\ln(e)$, by definition, we mean the unique real answer, such that $e^{\ln(e)} = e$.

However, in the general complex analysis $\log(e)$ is the following set of complex numbers:

$$\log(e) = \{\, \ln(e) + i(0 + 2k\pi) = 1 + i2k\pi \ \mid\ k \in \mathbb{Z} \,\} = 1 + i\,2\pi\mathbb{Z},$$

where we have designated as an initial interval of principal arguments the $[0, 2\pi)$, let us say.

For each $k \in \mathbb{Z}$, we have a branch of logarithm. For instance, if $k = 0$, we have $\log_{(0)}(e) = 1$. This is the real value that we learn in a calculus course. If $k = -10$, then $\log_{(-10)}(e) = 1 - i\,20\pi$, which is a complex number. Similarly, $\log_{(10)}(e) = 1 + i\,20\pi$.

▲

About continuity and discontinuity of the argument and the complex logarithm

The angles $\theta = \arg(z)$ are measured from the positive x-axis. Let us here designate the interval $(-\pi, \pi]$ to be the interval in which the principal argument $\text{Arg}_0(z)$ varies. With these values of $\theta = \text{Arg}_0(z)$, we can write any complex number $z \neq 0$ in trigonometric or exponential form uniquely. We observe that for $\theta = \pi$, we achieve all non-positive numbers in the complex plane $\mathbb{C}$.

Now, any negative number may be approached in many ways, but here we consider two: first by a sequence of complex numbers of the upper half plane, and second by a sequence of complex numbers of the lower half plane.

So, we consider any $a < 0$ and $z_n = r_n e^{i\theta_n}$ $n = 1,\ 2,\ 3,\ ...$ a sequence of complex numbers in the upper half plane such that $0 < \theta_n < \pi,\ \ n = 1,\ 2,\ 3,\ ...$ and $z_n \longrightarrow a$, as $n \longrightarrow \infty$. Then $r_n \longrightarrow |a|$ and $\theta_n \longrightarrow \pi$, as $n \longrightarrow \infty$. So, with an initial interval of $\text{Arg}_0(z)$ the interval $(-\pi, \pi]$, we have

$$\log(z_n) = \ln(r_n) + i\theta_n \longrightarrow \ln(|a|) + i\pi, \ \text{ as } \ n \longrightarrow \infty.$$

We now consider $\zeta_n = \rho_n e^{i\phi_n}$ $n = 1, 2, 3, ...$ a sequence of complex numbers in the lower half plane such that $-\pi < \phi_n < 0,\ \ n = 1, 2, 3, ...,$

and $\zeta_n \longrightarrow a$, as $n \longrightarrow \infty$. Then $\rho_n \longrightarrow |a|$ and $\phi_n \longrightarrow -\pi$, as $n \longrightarrow \infty$. So, with an initial interval of $\mathrm{Arg}_0(z)$ the interval $(-\pi, \pi]$, we have

$$\log(\zeta_n) = \ln(\rho_n) + i\phi_n \longrightarrow \ln(|a|) - i\pi, \text{ as } n \longrightarrow \infty.$$

Thus, even though

$$\lim_{n\to\infty} z_n = a \text{ and } \lim_{n\to\infty} \zeta_n = a,$$

i.e., z_n and ζ_n have the same limit as $n \longrightarrow \infty$, we have that

$$\lim_{n\to\infty} \log(z_n) = \ln(|a|) + i\pi \neq \ln(|a|) - i\pi = \lim_{n\to\infty} \log(\zeta_n).$$

Since the two limits are different, the $\lim_{z\to a} \log(z)$ does not exist. Therefore, the $\log(z)$ we have considered here cannot be extended continuously to any number $a \leq 0$. In other words, $\log(z)$ is not a continuous function in $\mathbb{C}$ and, in fact, is not continuous in any punctured disc $D^o(0, r)$, $r > 0$.

We see that apart from the origin at which $\log(z)$ cannot be defined in any way, $w = \log(z)$ has a jump, equal to $2\pi i$, at every point of the negative x-axis. This discontinuity is due to the discontinuities of the $\arg(z)$ along the non-positive x-axis, if we take the interval $(-\pi, \pi]$ to be the interval of the principal argument $\mathrm{Arg}_0(z)$.

So, $\arg(z)$ and hence $\log(z)$ are continuously defined in the region $\mathbb{C} - \{a \mid a \leq 0\} = \mathbb{C} - \{\text{non-positive } x\text{-axis}\}$ and not in the whole $\mathbb{C}$! This means that in any situation in which we need to use continuous $\arg(z)$ and $\log(z)$, we must exclude the non-positive x-axis from $\mathbb{C}$, and then the principal argument must vary in the interval $(-\pi, \pi)$ and not in $(-\pi, \pi]$. That is, a maximal domain in $\mathbb{C}$ in which the principal argument and the logarithm are continuous is not the whole $\mathbb{C}$ but the $\mathbb{C} - \{\text{non-positive } x\text{-axis}\}$.

Instead of the non-positive x-axis, we could have excluded from $\mathbb{C}$ any closed half line starting at the origin $\{z = re^{i\phi} \mid r \geq 0 \text{ and } \phi \in \mathbb{R} \text{ constant}\}$. For instance, we can take away the non-negative x-axis by choosing as the initial interval of continuous $\mathrm{Arg}_0(z)$ to be the interval $(0, 2\pi)$. (If we take the initial interval to be $[0, 2\pi)$, then we take into account the whole $\mathbb{C}$, and discontinuities occur along the non-negative x-axis.) Advanced theorems of complex analysis prove that we can also exclude other curves that start at the origin as well. But the half lines starting at the origin are good enough for the scope of this book and most applications.

With $\{a \mid a \leq 0\} = \{\text{non-positive } x\text{-axis}\}$ excluded from $\mathbb{C}$, we can define the function $w = \log(z)$ continuously in $\mathbb{C} - \{a \mid a \leq 0\} = \mathbb{C} - \{\text{non-positive } x\text{-axis}\}$, if $-\pi < \mathrm{Arg}_0(z) < \pi$. In this way, we achieve a continuous piece of the multi-value function $w = \log(z)$ out of the infinitely many pieces of the complex log. We call such a piece a **continuous branch of** $w = \log(z)$. As we have already said, we achieve continuity at the expense of losing the whole $\mathbb{C}$. For this continuous branch of log, the non-positive x-axis is called the **branch cut**. The origin is the common point of all these infinitely many branch cuts, and it is called **branch point** or **branching point** of the complex log.

Now, **the derivative of any continuous branch** (continuity is a necessary condition for the existence of derivative) of the complex function $w = \log(z)$ can be found by the chain rule and using the relations $e^{\log_{(k)}(z)} = z$, for any $k \in \mathbb{Z}$, and $\dfrac{d}{dz}(e^z) = e^z$. [Or, we can directly use the derivative rule for inverse functions. Or, we can locally use equations **(3.7)** and **(3.4)** or **(3.5)**.] With $\log(z)$ representing all the continuous branches $\{\log_{(k)}(z) \mid \ k \in \mathbb{Z}\}$, we find

$$\frac{d}{dz}[\log(z)] = \frac{1}{z}.$$

Next, by computing the derivatives of all orders of $f(z) = \log(1+z)$ at $z = 0$, we derive the following Maclaurin power series for the continuous branch of $f(z) = \log(1+z)$ for which $\log(1) = 0$:

$$\log(1+z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ..., \ \forall\ z :\ |z| < 1.$$

This is the power series of the branch $w_0 = \log_{(0)}(1+z)$ (i.e., $k = 0$) if the $\mathrm{Arg}_0(z)$ varies in $(-\pi, \pi)$.

To obtain the power series of any branch $w_k = \log_{(k)}(1+z)$, $k \in \mathbb{Z}$, we must add the constant $2k\pi i$ to this power series.

The basic **properties of the complex logarithms** have been listed together with the related properties of the complex power functions in the **Section 3.5.4**, in which we also explain the **adjustment of the argument** for both. Next, we present the following:

Example 3.5.3 Find all complex values of the complex logarithms $\log(1)$, $\log(5)$, $\log(-5)$ and $\log(5-2i)$, with initial arguments in $(-\pi, \pi]$.

Since [by **Part (a)** of **Problem 3.2.8**]

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) = 1 \iff \theta = 2k\pi \text{ with } k \text{ integer},$$

then we have that

$$\log(1) = \{2k\pi i \ \mid \ k \in \mathbb{Z}\}.$$

So, for $k = 0$, we find $\log_{(0)}(1) = \ln(1) = 0$ (as we know it from calculus).

Next, we have $5 = 5e^{i0}$, $-5 = 5e^{i\pi}$ and $5 - 2i = \sqrt{5^2 + (-2)^2}\, e^{i\phi} = \sqrt{29}\, e^{i\phi}$, with $\phi \in (-\pi, \pi]$ the unique IV quadrant angle determined by $\cos(\phi) = \dfrac{5}{\sqrt{29}}$ and $\sin(\phi) = \dfrac{-2}{\sqrt{29}}$. Therefore, $\phi = \arcsin\left(\dfrac{-2}{\sqrt{29}}\right) \in (-\pi, \pi]$.

So, we find

$$\log(5) = \{\ln(5) + i(2k\pi + 0) = \ln(5) + i\, 2k\pi \ \mid\ k \in \mathbb{Z}\},$$

$$\log(-5) = \{\ln(5) + i(2k\pi + \pi) = \ln(5) + i(2k + 1)\pi \ \mid\ k \in \mathbb{Z}\},$$

$$\log(5 - 2i) = \{\ln(\sqrt{29}) + i(2k\pi + \phi) = \frac{\ln(29)}{2} + i(2k\pi + \phi) \ \mid\ k \in \mathbb{Z}\}.$$

If all initial arguments were measured in $[0, 2\pi)$, then the answers would be:

$$\log(1) = \{2k\pi i \ \mid\ k \in \mathbb{Z}\},$$

$$\log(5) = \{\ln(5) + i(2k\pi + 0) = \ln(5) + i\, 2k\pi \ \mid\ k \in \mathbb{Z}\},$$

$$\log(-5) = \{\ln(5) + i(2k\pi + \pi) = \ln(5) + i(2k + 1)\pi \ \mid\ k \in \mathbb{Z}\},$$

$$\log(5 - 2i) = \{\ln(\sqrt{29}) + i(2k\pi + \psi) = \frac{\ln(29)}{2} + i(2k\pi + \psi) \ \mid\ k \in \mathbb{Z}\},$$

with $\psi \in [0, 2\pi)$ the unique IV quadrant angle determined by $\cos(\psi) = \dfrac{5}{\sqrt{29}}$ and $\sin(\psi) = \dfrac{-2}{\sqrt{29}}$. Therefore, $\psi = 2\pi + \arcsin\left(\dfrac{-2}{\sqrt{29}}\right) \in [0, 2\pi)$ in this case. (That is, the first three answers of both cases coincide, but the fourth one is different.)

▲

Example 3.5.4 In the **open right half plane**, i.e., $\mathrm{Re}(z) > 0$, the function

$$f(z) = \frac{z^3 + 5z^2 - 3z + 6}{z^2}$$

has as a continuous antiderivative the function

$$F(z) = \frac{z^2}{2} + 5z - 3\log_{(0)}(z) - \frac{6}{z},$$

where we have picked the principal branch of the complex logarithm $\log_{(0)}(z)$ with principal argument in $[-\pi, \pi)$.

So, for any path C lying in the open right half plane $[\mathrm{Re}(z) > 0]$ and joining the numbers $1+i$ and $2-5i$, by **Theorem 3.5.2**, we find

$$\int_C f(z)\,dz = \int_{1+i}^{2-5i} \frac{z^3+5z^2-3z+6}{z^2} =$$

$$\left[\frac{z^2}{2} + 5z - 3\log_{(0)}(z) - \frac{6}{z}\right]_{1+i}^{2-5i} =$$

$$\left[\frac{(2-5i)^2}{2} + 5(2-5i) - 3\log_{(0)}(2-5i) - \frac{6}{2-5i}\right] -$$

$$\left[\frac{(1+i)^2}{2} + 5(1+i) - 3\log_{(0)}(1+i) - \frac{6}{1+i}\right] =$$

$$\left[\frac{-21-20i}{2} + 10 - 25i - 3\ln\left(\sqrt{29}\right) - 3\arcsin\left(\frac{-5}{\sqrt{29}}\right) i - \frac{12+30i}{29}\right] -$$

$$\left[i + 5 + 5i - 3\ln\left(\sqrt{2}\right) - 3\frac{\pi}{4}\, i - 3 + 3i\right] =$$

$$-\frac{169}{58} - \frac{3}{2}\ln\left(\frac{29}{2}\right) + i\left[\frac{3\pi}{4} + 3\arcsin\left(\frac{5\sqrt{29}}{29}\right) - \frac{1306}{29}\right].$$

But on the any circle $C = C^+(0,r)$, $r > 0$, positively oriented, we get

$$\int_{C^+(0,\,r)} f(z)\,dz = (-3)\cdot 2\pi i = -6\pi i.$$

▲

3.5.4 Complex Power Functions

We need to say a few words about the **multi-value complex power function**. Besides its own interest, we are going to use it in several cases in the sequel. This multi-value complex function is:

$$f(z) = z^\alpha,$$

where, in general, $\alpha \in \mathbb{C}$ is constant and $z \in \mathbb{C}$ is the complex variable. This function is defined by means of the complex $\log(z)$ and the exponential function e^z in the following way:

If $z \neq 0$, we define

$$z^\alpha = e^{\alpha\log(z)} \overset{def}{=} e^{\alpha\ln(|z|) + i\alpha\arg(z)},$$

where, as we have already seen,

$$\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}_0(z)\} + 2\pi\mathbb{Z}$$

is a countable set of numbers. So, this function is **a multi-value complex function**, i.e., its output is **a set of numbers**, in general. We can choose particular values from this set as the continuous branches of the power function. In fact, we pick continuous branches of the complex logarithm, and then we find the corresponding continuous branches of the power.

In general, we may consider any domain $\mathbb{C} - \{\text{a ray from the origin}\}$, and so the principal argument $\mathrm{Arg}_0(z)$ is in an open interval $(\phi, 2\pi + \phi)$, where ϕ is some angle.

Example 3.5.5 In calculus and real analysis, we learn that $e^{\sqrt{2}}$ is a unique real value. However, in the general complex analysis, $e^{\sqrt{2}}$, viewed as a complex power, is the following set of numbers:

$$e^{\sqrt{2}} = \{\ e^{[\log(e)]\sqrt{2}}\ \} =$$
$$\{\, e^{[1+i(0+2k\pi)]\sqrt{2}}\ \mid\ k \in \mathbb{Z}\,\} = \{\, e^{(1+i2k\pi)\sqrt{2}}\ \mid\ k \in \mathbb{Z}\,\}.$$

For each $k \in \mathbb{Z}$, we have a branch of this complex power. For instance, if $k = 0$, we find the unique real answer $e^{\sqrt{2}}$, as we know it in calculus or real analysis. But, if, for instance, $k = -10$, then we find the complex number $e^{\sqrt{2}(1-i20\pi)} = e^{\sqrt{2}}[\,\cos(20\pi\sqrt{2}) - i\sin(20\pi\sqrt{2})\,]$ and so on.

Note: It would be appropriate to introduce notations indicating if the expression $e^{\sqrt{2}}$, and any expression a^b with a and b real numbers, is considered as a real power with a unique real number as answer, if such a real number exists, or as a complex power with infinitely many answers, out of which we can pick particular branches which we can indicate with indices $k \in \mathbb{Z}$, But in order to avoid too crowded notations, we are not going to introduce and use such notations. We must simply keep in mind when expanding complex powers, what we mean with the complex $\log(z)$ or $\log_{(k)}(z)$ with $k \in \mathbb{Z}$.

▲

It does not always hold that each branch of the complex logarithm will yield a different branch of the complex power. This depends on the exponent. For instance: As we have seen in **Section 3.1**, if $\alpha = n \in \mathbb{Z}$ (i.e., a **is an integer**), then $f(z) = z^n \left(= z^{\frac{n}{1}}\right)$ is a unique-value function. Indeed:

If $z = 0$ and $n \in \mathbb{N}$, then we have $0^n = 0 \cdot 0 \cdots 0 = 0$, by the original definition of a power with a natural exponent. (For $n = 0,\ -1,\ -2, \ldots$, the power 0^n is not defined.)

If $z \neq 0$ and $n \in \mathbb{Z}$, then we obtain

$$f(z) = z^n = e^{n\log(z)} = e^{n[\ln(|z|)+i\arg(z)]} = e^{n\left[\ln(|z|)+i\mathrm{Arg}_0(z)+2k\pi i\right]} =$$
$$e^{n\ln(|z|)+i\,n\mathrm{Arg}_0(z)+2kn\pi i} = e^{n\ln(|z|)} \cdot e^{in\mathrm{Arg}_0(z)} \cdot e^{2kn\pi i} =$$

$$= |z|^n \cdot \{\cos[n\mathrm{Arg}_0(z)] + i\sin[n\mathrm{Arg}_0(z)]\} \cdot 1 =$$
$$\{|z| \cdot \{\cos[\mathrm{Arg}_0(z)] + i\sin[\mathrm{Arg}_0(z)]\}\}^n =$$
$$\begin{cases} z^n = z \cdot z \cdot z \cdots z, & \text{if} \quad n > 0 \\ 1, & \text{if} \quad n = 0 \\ z^n = \frac{1}{z} \cdot \frac{1}{z} \cdot \frac{1}{z} \cdots \frac{1}{z}, & \text{if} \quad n < 0, \end{cases}$$

where the last two steps follow from the **De Moivre formula** [**Subsection 3.1.2**, **Part (5)**]. This answer coincides with the original definition of the power function with integer exponent $n \in \mathbb{Z}$ and base $z \neq 0$.

In general, if the exponent α **is not an integer**, then the function $f(z) = z^\alpha$ is a multi-value function. Indeed:

If $\alpha = \dfrac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$, i.e., α is **rational**, then $f(z) = z^\alpha$ has at most $|q|$ values. The $|q|$ $|q|^{\text{th}}$ roots of $z^{\text{sign}(q)\cdot p}$ repeat as k runs $\mathbb{Z}$ in the expression $2k\pi i$. If p and q are relatively prime, then there are exactly $|q|$ different values. (See **Problem 3.1.18**.)

Example 3.5.6 (a) For any integer $n = \dfrac{n}{1}$, we have just seen that $f(z) = z^n$ has exactly one value.

(b) The n$^{\text{th}}$ order roots of complex numbers, as we have studied them in **Subsection 3.1.2, Part (6)**, can be viewed in the context of the complex power function. For every $n \in \mathbb{N}$ and any $z \neq 0$ complex number, the power $z^{\frac{1}{n}}$ represents n different complex numbers, namely the n n$^{\text{th}}$ roots of z.

(c) When $\alpha = \dfrac{1}{2}$, we have seen in **Subsection 3.1.3** that the complex square root function has two values. Here we find the two continuous branches of $\sqrt{z} = z^{\frac{1}{2}}$.

We have that for $z \neq 0$

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\ln|z| + \frac{i}{2}\arg(z)} = \sqrt{|z|}e^{\frac{i}{2}\arg(z)}.$$

We consider as continuous $\mathrm{Arg}_0(z)$ the one in the initial interval $(0, 2\pi)$. That is, we work in the open domain

$$\mathbb{C} - \{\text{non-negative } x\text{-axis}\} \subset \mathbb{C}.$$

Thus, we obtain two distinct branches: one for $0 < \arg(z) < 2\pi$ and another for $2\pi < \arg(z) < 4\pi$. For all the other values of the $\arg(z)$, the branches of $\sqrt{z} = z^{\frac{1}{2}}$ repeat, and any one of them is equal to one of these two. Indeed:

If we call $\mathrm{Arg}_0(z)$ the argument for the first branch, then we have the following two distinct branches of $\sqrt{z} = z^{\frac{1}{2}}$:

$$(1)\ \sqrt{z} = z^{\frac{1}{2}} = \sqrt{|z|}\, e^{i\dfrac{\mathrm{Arg}_0(z)}{2}}, \quad \text{when} \quad 0 < \arg(z) := \mathrm{Arg}_0(z) < 2\pi.$$

$$(2)\ \sqrt{z} = z^{\frac{1}{2}} = \sqrt{|z|}\, e^{i\dfrac{\mathrm{Arg}_0(z)+2\pi}{2}}, \quad \text{when}$$
$$2\pi < \arg(z) = \arg_1(z) = \mathrm{Arg}_0(z) + 2\pi < 4\pi.$$

E.g., let $z = -2 = 2e^{i\arg(-2)}$. Then $\mathrm{Arg}_0(-2) = \pi$, and the **first** branch gives $\sqrt{-2} = \sqrt{2}e^{i\frac{\pi}{2}} = +\sqrt{2}\,i$, whereas the **second** branch gives $\sqrt{-2} = \sqrt{2}e^{i\frac{\pi+2\pi}{2}} = \sqrt{2}e^{i\frac{3\pi}{2}} = -\sqrt{2}\,i$. Both values are the known values of $\sqrt{-2} = \pm\sqrt{2}\,i$.

Remark: In this example, the exponent $\alpha = \frac{1}{2}$ is positive. It makes sense to define $\sqrt{0} = 0$. This extension holds for positive exponents only. That is, if $\alpha > 0$, then $0^\alpha = 0$. We observe that all the continuous branches of powers with positive exponents meet at $z = 0$. If the exponents are negative, we do not make any such extension unless we set all answers to be ∞.

▲

As we have seen in **Example 3.5.5**, the power $e^{\sqrt{2}}$ represents infinitely many different values in the complex domain. In fact, this is the case with any complex power $f(z) = z^\alpha$ with exponent α an **irrational real**. This follows from the fact that whenever α is irrational the expression $e^{i\alpha\, 2k\pi}$ yields a different value for each different $k \in \mathbb{Z}$. Indeed: If $e^{i\alpha\, 2k\pi} = e^{i\alpha\, 2l\pi}$ with k and l integers, then $e^{i\alpha\, 2(k-l)\pi} = 1$. Therefore, $\alpha\, 2(k-l)\pi = 2m\pi$ for some integer m [see **Part (a)** of **Problem 3.2.8**]. Then, $\alpha(k-l) = m$. Since α is irrational, this is possible only if $k = l$ and $m = 0$. (Otherwise, $k \neq l$ and then $\alpha = \dfrac{m}{k-l}$ would be a rational number, which is a contradiction.)

Example 3.5.7 Find the continuous branches of

$$z^{\sqrt{2}} = e^{\sqrt{2}[\ln|z| + i\arg(z)]} = |z|^{\sqrt{2}} e^{i\sqrt{2}\arg(z)} = |z|^{\sqrt{2}} e^{i\sqrt{2}\,[\mathrm{Arg}_0(z)+2k\pi]}.$$

Since $\sqrt{2}$ is irrational, there are countably infinitely many continuous branches of this function obtained by the countably infinitely many branches of the $\arg(z)$. E.g., we consider $\mathrm{Arg}_0(z)$ in $(0, 2\pi)$, and then for every integer $k = 0,\ \pm1,\ \pm2,\ \pm3, \ldots$ we get a continuous complex argument such that $2k\pi < \arg(z) < 2(k+1)\pi$.

▲

If α is a **complex number with Im**$(\alpha) \neq$ **0**, then, again, $f(z) = z^\alpha$ has infinitely many answers. The continuous branches of this function are obtained by using the continuous branches of the complex $\log(z)$. We illustrate this in the following:

Example 3.5.8 Find the value of

$$(1-i)^{(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln(\sqrt{2})+i\arg(1-i)\right]}$$

corresponding to $\mathrm{Arg}_0(1-i) = \dfrac{7\pi}{4} \in [0, 2\pi)$.

We have

$$\begin{gathered}(1-i)^{(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln(\sqrt{2})+i\frac{7\pi}{4}\right]} = \\ e^{\left[2\ln(\sqrt{2})-3\cdot\frac{7\pi}{4}\right]+i\left[3\ln(\sqrt{2})+2\cdot\frac{7\pi}{4}\right]} = e^{\left[\ln(2)-\frac{21\pi}{4}\right]+i\left[\frac{3}{2}\ln(2)+\frac{14\pi}{4}\right]} = \\ e^{\left[\ln(2)-\frac{21\pi}{4}\right]}\left\{\cos\left[\frac{3}{2}\ln(2)+\frac{14\pi}{4}\right]+i\sin\left[\frac{3}{2}\ln(2)+\frac{14\pi}{4}\right]\right\} = \\ e^{\left[\ln(2)-\frac{21\pi}{4}\right]}\left\{\cos\left[\frac{3}{2}\ln(2)+\frac{3\pi}{2}\right]+i\sin\left[\frac{3}{2}\ln(2)+\frac{3\pi}{2}\right]\right\} = \\ e^{\left[\ln(2)-\frac{21\pi}{4}\right]}\left\{\sin\left[\frac{3}{2}\ln(2)\right]-i\cos\left[\frac{3}{2}\ln(2)\right]\right\} = \\ \frac{2}{e^{\frac{21\pi}{4}}}\left\{\sin\left[\frac{3}{2}\ln(2)\right]-i\cos\left[\frac{3}{2}\ln(2)\right]\right\}.\end{gathered}$$

All the infinitely many values of this complex power are

$$\begin{gathered}(1-i)^{(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln(\sqrt{2})+i\left(\frac{7\pi}{4}+2k\pi\right)\right]} = \dots = \\ \frac{2}{e^{3\left(\frac{7}{4}+2k\right)\pi}}\left\{\sin\left[\frac{3}{2}\ln(2)\right]-i\cos\left[\frac{3}{2}\ln(2)\right]\right\}\end{gathered}$$

for $k = 0,\ \pm1,\ \pm2,\ \pm3, \dots$.

▲

Using the chain rule, the derivative rules for exponentials and the complex logarithm, we obtain the following two **derivative rules**:

(1) For any constant $\alpha \in \mathbb{C}$, the derivative of a continuous branch of $f(z) = z^\alpha$ is

$$\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}, \qquad (z \neq 0).$$

(2) If we take the derivative with respect to α (considering now α to be the variable), we have that the derivative of a continuous branch of $f(z) = z^\alpha$ is

$$\frac{d}{d\alpha}(z^\alpha) = z^\alpha \log(z), \quad (z \neq 0).$$

In the usual way, we also obtain the **complex binomial series** with complex exponents

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \alpha \in \mathbb{C}, \quad \forall\, z : \ |z| < 1,$$

where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!}$, $\forall\, n \in \mathbb{N}$. Whenever $(1+z)^\alpha$ is a multi-value function, this power series represents the continuous branch which is equal to 1 at $z = 0$.

3.5.5 Properties of Complex Logarithms and Powers

Since a complex logarithm and a complex power are a set of complex numbers in general, we would like to accurately state and understand the **basic properties of complex logarithms and complex powers** as sets of numbers.

Let z, z_1 and z_2 be in $\mathbb{C} - \{0\}$. With $\log(z)$, $\log(z_1)$ and $\log(z_2)$, we mean the sets of all values of their **complex logarithms**. Similarly, the **complex powers** z^a and z^b with a and b in $\mathbb{C}$ mean the sets of their values.

Using the properties of the argument, the operations among subsets of $\mathbb{C}$ that we have seen in **Subsection 3.1.2, Parts (2), (3)** and **(7)**, and the definitions and remarks of **Subsection 3.5.3**, we can directly prove that **the corresponding sets** of these complex numbers satisfy the following properties:

1.
$$\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2).$$

2.
$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2).$$

3.
$$\log\left(z^a\right) = a \cdot \log(z) + i\, 2\pi\mathbb{Z} \supseteq a \cdot \log(z).$$

4.
$$z^a \cdot z^b = z^{a+b}.$$

5.
$$z^a \cdot z^{-b} = z^{a-b}.$$

6.
$$(z_1 \cdot z_2)^a = z_1^a \cdot z_2^a.$$

7.
$$\left(\frac{z_1}{z_2}\right)^a = z_1^a \cdot z_2^{-a}.$$

8.
$$\left(z^a\right)^b = z^{ab} \cdot e^{ib2\pi\mathbb{Z}} \supseteq z^{ab}.$$

Notice that, in general,

$$\left(z^a\right)^b = z^{ab} \cdot e^{ib2\pi\mathbb{Z}} \neq \left(z^b\right)^a = z^{ab} \cdot e^{ia2\pi\mathbb{Z}},$$

unless a and b are especially chosen. Besides the choices $a = b \in \mathbb{C}$, or both a and b are integers, not necessarily equal, find some other choices of $a \neq b$ for which we obtain equality. (See **Problem 3.5.12**.)

Also, we must notice that when we designate a common initial interval $(r, r + 2\pi)$, $r \in \mathbb{R}$, [e.g., $(-\pi, \pi]$, $[-\pi, \pi)$, $[0, 2\pi]$, etc.] for the continuous principal arguments of z_1 and z_2, then $\mathrm{Arg}_0(z_1) \pm \mathrm{Arg}_0(z_2)$, etc., may not be in this interval. We may need to add or subtract an appropriate constant to these results in order to return into $(r, r + 2\pi)$. (See the next example.)

Examples

Example 3.5.9 Let $(a, a + 2\pi) = (-\pi, \pi)$, i.e., $a = -\pi$.

Suppose $\mathrm{Arg}_0(z_1) = \frac{3\pi}{4}$ and $\mathrm{Arg}_0(z_2) = \frac{2\pi}{3}$. Then $\mathrm{Arg}_0(z_1) + \mathrm{Arg}_0(z_2) = \frac{3\pi}{4} + \frac{2\pi}{3} = \frac{17\pi}{12}$. This number is outside $(-\pi, \pi)$. But, if we subtract 2π from it, we find $\frac{17\pi}{12} - 2\pi = -\frac{7\pi}{12}$, which is in $(-\pi, \pi)$.

Similarly, if $\mathrm{Arg}_0(z_1) = -\frac{3\pi}{4}$ and $\mathrm{Arg}_0(z_2) = \frac{5\pi}{6}$, then $\mathrm{Arg}_0(z_1) - \mathrm{Arg}_0(z_2) = -\frac{3\pi}{4} - \frac{5\pi}{6} = -\frac{19\pi}{12}$. This number is outside $(-\pi, \pi)$. But, if we add 2π to it, we find $-\frac{19\pi}{12} + 2\pi = \frac{7\pi}{12}$, which is in $(-\pi, \pi)$.

In the same way, we make similar adjustments in situations similar to this!

▲

Example 3.5.10 Find the continuous branches of $\frac{1}{\sqrt[3]{z}} = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}}$.

We have that for $z \neq 0$

$$\frac{1}{\sqrt[3]{z}} = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = e^{\frac{-1}{3}\ln|z| - \frac{i}{3}\arg(z)} = \frac{1}{\sqrt[3]{|z|}}\, e^{-\frac{i}{3}\arg(z)}.$$

We consider as continuous $\operatorname{Arg}_0(z)$ the one in the initial interval $(0, 2\pi)$. That is, we work in the open domain

$$\mathbb{C} - \{\text{non-negative } x\text{-axis}\} \subset \mathbb{C}.$$

Thus, we obtain three distinct branches: one for $0 < \arg(z) < 2\pi$, another for $2\pi < \arg(z) < 4\pi$ and another for $4\pi < \arg(z) < 6\pi$. For all other values of the $\arg(z)$, the branches of $\frac{1}{\sqrt[3]{z}} = \frac{1}{z^{\frac{-1}{3}}}$ repeat, and any one of them is equal to one of these three. Indeed:

For the first branch, we use the $\operatorname{Arg}_0(z)$ and find the following three branches of $z^{\frac{-1}{3}} = \frac{1}{z^3}$:

$$w_1 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\operatorname{Arg}_0(z)}{3}},$$

$$\text{when} \quad 0 < \arg(z) := \operatorname{Arg}_0(z) < 2\pi,$$

$$w_2 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\operatorname{Arg}_0(z)+2\pi}{3}}, \quad \text{when} \quad 2\pi < \arg_1(z) < 4\pi,$$

$$w_3 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\operatorname{Arg}_0(z)+4\pi}{3}}, \quad \text{when} \quad 4\pi < \arg_2(z) < 6\pi.$$

For instance, let $z = -8 = 8e^{i\arg(-8)}$ with $\operatorname{Arg}_0(-8) = \pi$. So, we want to evaluate the three values of $(-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}}$:

The **first** branch gives

$$w_1 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{2}e^{-i\frac{\pi}{3}} = \frac{1}{2}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] =$$

$$\frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right] = \frac{1}{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{4}(1 - i\sqrt{3}).$$

The **second** branch gives

$$w_2 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{\sqrt[3]{|-8|}}e^{-i\frac{\pi+2\pi}{3}} = \frac{1}{2}[\cos(\pi) + i\sin(\pi)] = -\frac{1}{2}.$$

The **third** branch gives

$$w_3 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{2}e^{-i\frac{\pi+4\pi}{3}} = \frac{1}{2}\left[\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right] =$$

$$\frac{1}{2}\left[\cos\left(\frac{5\pi}{3}\right) - i\sin\left(\frac{5\pi}{3}\right)\right] = \frac{1}{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{4}(1 + i\sqrt{3}).$$

▲

Example 3.5.11 Find all the values of

$$i^{\,i} = e^{i\log(i)} = e^{i[\ln|1| + i\arg(i)]} = e^{-\arg(i)}.$$

Since all possible values of $\arg(i) = \dfrac{\pi}{2} + 2k\pi$, we find that all the values of the complex power $i^{\,i}$ are the real numbers $e^{-\left(\frac{\pi}{2}+2k\pi\right)}$, with $k = 0,\ \pm 1,\ \pm 2,\ \pm 3, \dots$.

▲

Example 3.5.12 Find the infinitely many values of the complex power

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)\log(-2+i)}.$$

We let $\mathrm{Arg}_0(-2+i) = \theta$ be the unique II-quadrant angle $\theta \in [0, 2\pi)$ determined by $\cos(\theta) = \dfrac{-2}{\sqrt{5}}$ and $\sin(\theta) = \dfrac{1}{\sqrt{5}}$ and measured in radians. (We can approximate this θ to several decimal digits.) Then

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)[\ln(\sqrt{5}) + i(\theta + 2k\pi)]} =$$
$$e^{\frac{3\ln(5)}{2} + 2\theta + 4k\pi}\left\{\cos[-\ln(5) + 3\theta + 6k\pi] + i\sin[-\ln(5) + 3\theta + 6k\pi]\right\} =$$
$$e^{\frac{3\ln(5)}{2} + 2\theta + 4k\pi}\left\{\cos[-\ln(5)]\cos(3\theta) - \sin[-\ln(5)]\sin(3\theta) +\right.$$
$$\left. i\left\{\sin\left[-\ln(5)\right]\cos(3\theta) + \cos\left[-\ln(5)\right]\sin(3\theta)\right\}\right\}.$$

But, by triple angle formulae, we have

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) = 4\cdot\left(\frac{-2}{\sqrt{5}}\right)^3 - 3\cdot\left(\frac{-2}{\sqrt{5}}\right) =$$
$$4\cdot\left(\frac{-8}{5\sqrt{5}}\right) - 3\cdot\left(\frac{-2}{\sqrt{5}}\right) = \frac{-2\sqrt{5}}{25},$$

$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta) = 3\cdot\frac{1}{\sqrt{5}} - 4\cdot\left(\frac{1}{\sqrt{5}}\right)^3 =$$
$$\frac{3}{\sqrt{5}} - \frac{4}{5\sqrt{5}} = \frac{11\sqrt{5}}{25}.$$

Substituting these values with $k \in \mathbb{Z}$ and simplifying, we find

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)\log(-2+i)} =$$
$$e^{2\theta + 4k\pi}\cdot\left\{-2\cos[\ln(5)] + 11\sin[\ln(5)] + i\left\{2\sin[\ln(5)] + 11\cos[\ln(5)]\right\}\right\}.$$

▲

Example 3.5.13 Find all solutions of the equation $\sin(z) = -2$.

By the definition of sine, we have

$$\frac{e^{iz} - e^{-iz}}{2i} = -2.$$

Manipulating this algebraically, we find

$$\left(e^{iz}\right)^2 + 4ie^{iz} - 1 = 0,$$

and by the quadratic formula, we get

$$e^{iz} = \left(-2 \pm \sqrt{3}\right) i.$$

Therefore,

$$iz = \log\left[\left(-2 \pm \sqrt{3}\right) i\right].$$

Since both $-2 \pm \sqrt{3}$ are negative, we can choose $\mathrm{Arg}_0[(-2 \pm \sqrt{3})i] = \frac{3\pi}{2} \in [0, 2\pi)$. (If these numbers have different arguments, we must consider them apart in two different formulae.)

Then we get

$$z = \frac{1}{i}\log\left[\left(-2 \pm \sqrt{3}\right) i\right] = -i\log\left[\left(-2 \pm \sqrt{3}\ \right) i\right] =$$
$$-i\left[\ln\left|-2 \pm \sqrt{3}\right| + \left(2k + \frac{3}{2}\right)\pi i\right] = \left(2k + \frac{3}{2}\right)\pi - i\ln\left(2 \mp \sqrt{3}\right),$$

with $k \in \mathbb{Z}$.

If we observe that $2 + \sqrt{3} = \frac{1}{2 - \sqrt{3}}$, then we can write

$$z := z_k^{\pm} = \left(2k + \frac{3}{2}\right)\pi \pm i\ln\left(2 + \sqrt{3}\right), \quad \forall\ k \in \mathbb{Z}.$$

So, the above equation has infinitely many complex solutions, one for each integer $k = 0,\ \pm 1,\ \pm 2, \ldots$. The solutions were expected to be complex since we have set the sine function equal to $-2 \notin [-1, 1]$ (the range of the real sine). If we plug all of these numbers back into $\sin(z)$, we will easily verify that we get -2.

In this example, we could have found the multi-value function inverse of sine, in general, and then plug in the number -2.

That is, if $w = \sin^{-1}(z)$, then $z = \sin(w)$. Thus,

$$\frac{e^{iw} - e^{-iw}}{2i} = z \quad \text{or} \quad \left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0.$$

So, by the quadratic formula, we find

$$e^{iw} = iz \pm (1 - z^2)^{\frac{1}{2}}$$

and then

$$w = \sin^{-1}(z) = -i \log \left[iz \pm (1 - z^2)^{\frac{1}{2}} \right].$$

This has infinitely many answers produced by the infinitely many values of the complex logarithm.

If by $(1 - z^2)^{\frac{1}{2}}$ we mean the two-value function of the power with exponent $\frac{1}{2}$, then we can replace the ± with simply + in the above formula.

For $z = -2$, we find

$$\sin^{-1}(-2) = -i \log \left[-i2 \pm (-3)^{\frac{1}{2}} \right] = -i \log \left[i \left(-2 \pm \sqrt{3} \right) \right]$$

which are the values we have found above, etc.

If we plug $z = 1$, a value within the real range of sine, we find

$$\sin^{-1}(1) = -i \log(i) = -i \left[\ln(1) + i \left(\frac{\pi}{2} + 2k\pi \right) \right] =$$
$$-ii \left(\frac{\pi}{2} + 2k\pi \right) = \frac{\pi}{2} + 2k\pi,$$

with $k \in \mathbb{Z}$. In this case, all values are real and are those that we know from elementary trigonometry.

[Repeat the same thing with $z = -\frac{1}{2}$ to find all the infinitely many real values for

$$\sin^{-1}\left(-\frac{1}{2}\right) = \begin{cases} \dfrac{5\pi}{6} + 2k\pi \\ \\ \dfrac{7\pi}{6} + 2k\pi, \end{cases}$$

with $k \in \mathbb{Z}$. These are the values that we know from elementary trigonometry.]

Remark: In the same way, as in this example, we can find the general formulae of the **inverse functions of the trigonometric and hyperbolic functions (multi-value functions)**. Then, given any complex number, we can plug it into these formulae to compute the corresponding infinitely many answers. For example:

$$\cos^{-1}(z) = -i \log \left[z \pm (z^2 - 1)^{\frac{1}{2}} \right],$$
$$\tan^{-1}(z) = \frac{i}{2} \log \left(\frac{1 - iz}{1 + iz} \right), \quad z \neq \pm i,$$

etc.

(See also **Problem 3.5.14**.)

▲

3.5.6 Consequence

Going back to $f(z) = \dfrac{1}{z}$ in the $D^o(0,r)$ with $r > 1$ and $C = C(0,1)$, we now see that this function does not have a continuous antiderivative along C. In fact, at $a = -1 \in \mathbb{C}$, the $\log(z)$ has a jump equal to $+2\pi i$ when we cross $a = -1$ as we travel along C. This jump is precisely equal to

$$\oint_{C^+(0,1)} \frac{1}{z}\, dz = +2\pi i.$$

Similarly, we have that for any $r > 0$ and any $z_0 \in \mathbb{C}$

$$\oint_{C^\pm(z_0,r)} \frac{1}{z - z_0}\, dz = \pm 2\pi i.$$

Therefore, for any $r > 0$ and any complex number c fixed, we have

$$\oint_{C^\pm(0,r)} \frac{c}{z}\, dz = \pm 2c\pi i,$$

and similarly, if $z_0 \in \mathbb{C}$ fixed

$$\oint_{C^\pm(z_0,r)} \frac{c}{z - z_0}\, dz = \pm 2c\pi i.$$

So, now we observe that when we integrate a function written as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \;\; \forall\, z \in D^o(z_0, r),$$

along a closed path C inside a punctured-disc $D^o(z_0, r)$, with $r > 0$, only the term $\dfrac{b_1}{z - z_0}$ may contribute a non-zero result to the integral, but all the other additive terms of the series contribute zero.

In fact, for the scope of this book, we use the following (partial) **Theorem**:

Theorem 3.5.6 *Let C be a simple, closed, continuous, piecewise continuously differentiable path in $\mathbb{C}$ and $f(z)$ be a function given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \;\; \forall\, z \in \mathbb{C} - \{z_0\},$$

where $z_0 \in \mathbb{C}$ fixed ***but not on the path*** *C. Then:*

(a) If z_0 lies inside the region that C encloses, then

$$\oint_{C^{\pm}} f(z)\,dz = \pm 2b_1\pi i.$$

(b) If z_0 lies outside the region that C encloses, then

$$\oint_{C^{\pm}} f(z)\,dz = 0.$$

Proof (Refer to **Figure 3.3**, below.) By the previous discussion, it is enough to prove this theorem for the function

$$f(z) = \frac{b_1}{z - z_0}.$$

We let D be the interior of the simple closed path C, that is, the open region that C encloses.

(a) Assume z_0 lies inside the region D. Then this function $f(z)$ does not have a continuous antiderivative in $D - \{z_0\}$.

We take $r > 0$ small enough so that C and $C(z_0, r)$ do not intersect or touch. Call $R = D - \overline{D(z_0, r)}$. This is the part of D lying between C and the circle $C(z_0, r)$.

As we have already seen, for the circle $C(z_0, r)$, we get

$$\oint_{C^{\pm}(z_0,r)} \frac{b_1}{z - z_0}\,dz = \pm \int_0^{2\pi} b_1 \frac{1}{z_0 + re^{i\theta} - z_0}\,d(re^{i\theta}) =$$

$$\pm b_1 \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}\,d\theta = \pm b_1 i \int_0^{2\pi} d\theta = \pm 2b_1\pi i,$$

and so

$$\oint_{C^{\mp}(z_0,r)} \frac{b_1}{z - z_0}\,dz = \mp 2b_1\pi i.$$

Next, in order to apply Green's Theorem in R, we must be careful with the relative orientation of the two pieces of the boundary of R, and we need to write our function as

$$\frac{1}{z - z_0} = u(x, y) + iv(x, y) =$$

$$\left(\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} + i\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\right)$$

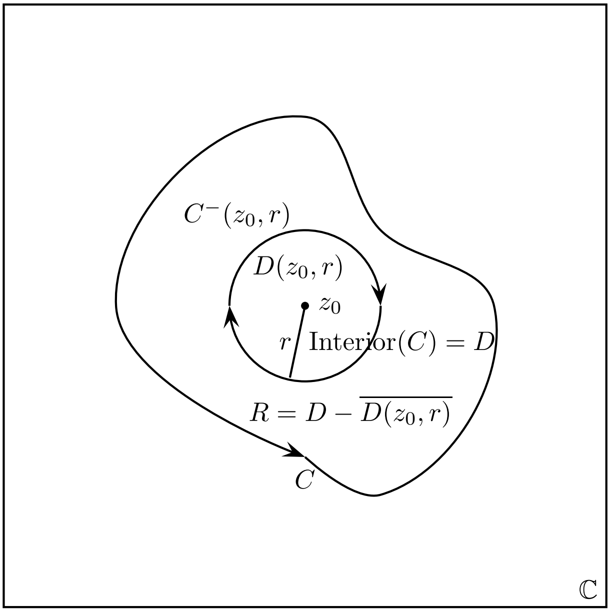

FIGURE 3.3: Contour C in $\mathbb{C}$ with point z_0 in its interior D

where $z = x + iy$ and $z_0 = x_0 + iy_0$. Then we have:

$$\oint_{\partial R} \frac{b_1}{z - z_0}\, dz = \oint_{C^{\pm}} \frac{b_1}{z - z_0}\, dz + \oint_{C^{\mp}(z_0,r)} \frac{b_1}{z - z_0}\, dz =$$

$$= b_1 \oint_{C^{\pm}} [u(x,y) + iv(x,y)](dx + idy) +$$

$$b_1 \oint_{C^{\mp}(z_0,r)} [u(x,y) + iv(x,y)](dx + idy) =$$

$$b_1 \left[\oint_{C^{\pm}} (udx - vdy) + i \oint_{C^{\pm}} (vdx + udy) \right] +$$

$$b_1 \left[\oint_{C^{\mp}(z_0,r)} (udx - vdy) + i \oint_{C^{\mp}(z_0,r)} (vdx + udy) \right].$$

Therefore,

$$\oint\limits_{\partial R} \frac{b_1}{z-z_0}\,dz =$$

$$b_1\left[\oint\limits_{C^{\pm}} + \oint\limits_{C^{\mp}(z_0,r)}\right](udx - vdy) + b_1 i\left[\oint\limits_{C^{\pm}} + \oint\limits_{C^{\mp}(z_0,r)}\right](vdx + udy).$$

Then by Green's Theorem in the region R (as we have seen it in calculus), we get

$$\oint\limits_{\partial R} \frac{b_1}{z-z_0}\,dz = \pm b_1\left[\iint\limits_{R}(-v_x - u_y)dxdy + i\iint\limits_{R}(u_x - v_y)dxdy\right].$$

But, in R, the function $\frac{1}{z-z_0}$ has complex derivative at every point, equal to $\frac{-1}{(z-z_0)^2}$. (Notice that $z_0 \notin R$.) Therefore, it is holomorphic in R, and so the **Cauchy-Riemann conditions** hold for this function in R. That is,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Therefore, both double integrals above are equal to zero, and so

$$\oint\limits_{\partial R} \frac{b_1}{z-z_0} = \oint\limits_{C^{\pm}} \frac{b_1}{z-z_0}\,dz + \oint\limits_{C^{\mp}(z_0,r)} \frac{b_1}{z-z_0}\,dz = 0.$$

Since

$$\oint\limits_{C^{\pm}(z_0,r)} \frac{b_1}{z-z_0}\,dz = \mp 2b_1\pi i,$$

we finally find

$$\oint\limits_{C^{\pm}} \frac{b_1}{z-z_0}\,dz = -(\mp 2b_1\pi i) = \pm 2b_1\pi i.$$

(b) Assume $z_0 \notin D \cup C$, that is, it lies outside the closed region $D \cup C$ that C encloses. Since $f'(z) = \frac{-1}{(z-z_0)^2}$ is defined at every point of $D \cup C$, then by the **Cauchy-Goursat Theorem, 3.5.3,** we immediately get

$$\oint\limits_{C^{\pm}} f(z)\,dz = 0.$$

[In this case, we could also apply **Theorem 3.5.2** because in the bounded region $D \cup C$ the function $f(z)$ has a continuous antiderivative which is any continuous branch of the function $\log(z - z_0)$.]

Remark: If in the **previous Theorem** $z_0 \in C$, then the $\oint\limits_{C^\pm} f(z)\,dz$ is dealt individually. Its existence, non-existence and evaluation depend on the given $f(z)$ and C.

Examples

Example 3.5.14 Let C be the boundary of the square $\{z = x + iy \ : \ |x| \leq 2,\ |y| \leq 2\}$. Then, due to **Theorem 3.5.6, Part (a)**, all the following integrals give the same value:

$$\oint\limits_{C^\pm} \frac{1}{z}\,dz = \oint\limits_{C^\pm} \frac{1}{z-1}\,dz = \oint\limits_{C^\pm} \frac{1}{z-i}\,dz = \oint\limits_{C^\pm} \frac{1}{z+1}\,dz = \oint\limits_{C^\pm} \frac{1}{z+i}\,dz = \oint\limits_{C^\pm} \frac{1}{z-(1-i)}\,dz = \pm 2\pi i.$$

Similarly, by the **same Theorem, Part (b)**, we have:

$$\oint\limits_{C^\pm} \frac{1}{z-3}\,dz = \oint\limits_{C^\pm} \frac{1}{z-3i}\,dz = \oint\limits_{C^\pm} \frac{1}{z+3}\,dz = \oint\limits_{C^\pm} \frac{1}{z+3i}\,dz = \oint\limits_{C^\pm} \frac{1}{z-(4-5i)}\,dz = 0.$$

▲

Example 3.5.15 For any $a > 0$ and $b > 0$ constants, let C be the ellipse $\{z = x + iy \ : \ x = a\cos\theta,\ y = b\sin\theta,\ 0 \leq \theta \leq 2\pi\}$. Then, due to **Theorem 3.5.6**, we have that

$$\oint\limits_{C^+} \frac{1}{z}\,dz = 2\pi i.$$

From this, we get

$$\int_0^{2\pi} \frac{-a\sin(\theta) + ib\cos(\theta)}{a\cos(\theta) + ib\sin(\theta)}\,d\theta = \int_0^{2\pi} \frac{(b^2 - a^2)\sin(\theta)\cos(\theta) + iab}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\,d\theta = 2\pi i = 0 + 2\pi i.$$

Separating real and imaginary parts, we get the following two real integrals:

$$\int_0^{2\pi} \frac{(b^2 - a^2)\sin(\theta)\cos(\theta)}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 0$$

and

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 2\pi.$$

Therefore, we also obtain the following two real integrals:

$$\int_0^{2\pi} \frac{\sin(\theta)\cos(\theta)}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 0$$

and

$$\int_0^{2\pi} \frac{1}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = \frac{2\pi}{ab}.$$

So,

$$\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = \frac{\pi}{2ab}.$$

(Compare this with **Examples 2.2.3 2.6.18**, and **3.8.3**. Also, for comparison, try to evaluate the second integral by pure calculus methods!)

▲

From the two **previous examples**, we get a sense of the power of the **Theorem 3.5.6**.

3.5.7 Cauchy Integral Formula

Let $w = f(z)$ be a holomorphic function in an open region $R \subseteq \mathbb{C}$ and C be a simple, closed, continuous and piecewise continuously differentiable path in R. Let z_0 be located inside the region $D \subset R \subseteq \mathbb{C}$ that C encloses and let $0 < r < p$, where $p > 0$ and such that $D(z_0, p) \subseteq D$. Then from **Theorem 3.5.1**, we know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ \forall\ z \in D(z_0, p),$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} \in \mathbb{C}, \ \forall\ n = 0,\ 1,\ 2,\ ....$

So, $f(z_0) = a_0$ and

$$\frac{f(z)}{z - z_0} = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}, \ \ \forall\ z \in D(z_0, p).$$

Hence

$$\oint\limits_{C^+(z_0,r)} \frac{f(z)}{z-z_0}\,dz = a_0 2\pi i = 2\pi i f(z_0).$$

As in the proof of **Theorem 3.5.6**, we can use Green's Theorem to get that for any simple, closed, continuous and piecewise continuously differentiable path C around the point z_0, we have that

$$\oint\limits_{C^+} \frac{f(z)}{z-z_0}\,dz = \oint\limits_{C^+(z_0,r)} \frac{f(z)}{z-z_0}\,dz.$$

Therefore,

$$\oint\limits_{C^+} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

This is the powerful and useful result:

Theorem 3.5.7 (Cauchy integral formula) *Let $w = f(z)$ be a holomorphic function in an open region $G \subseteq \mathbb{C}$ and C be any simple, closed, continuous and piecewise continuously differentiable path in G. We let $D :=$ the inside of C, and we assume $D \subseteq G$. In particular, this is true when G is simply connected. Then for any $z_0 \in D$, we have that*

$$\oint\limits_{C^+} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) \iff f(z_0) = \frac{1}{2\pi i}\oint\limits_{C^+} \frac{f(z)}{z-z_0}\,dz. \tag{3.8}$$

This Theorem says that all the values of a holomorphic function obtained at the interior points of the path C, as in the Theorem, can be completely determined by the values of the function at just the points of C, via the stated formula.

Under the conditions of this Theorem, if $z_0 \in G-(D\cup C)$, then, by the **Cauchy-Goursat Theorem, 3.5.3**, the above integral is equal to zero, because with $z_0 \in G-(D\cup C)$, $\dfrac{f(z)}{z-z_0}$ is holomorphic in $D\cup C$.

If C^+ is a positively oriented circle with center z_0 and radius $R>0$ such that $\overline{D(z_0,R)} \subseteq G$, then we have that along C^+ $z = z_0 + Re^{it}$ with $0 \le t \le 2\pi$ and $dz = Rie^{it}dt$. In this case, the **Cauchy integral formula** gives

$$\oint\limits_{C^+} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} f\left(z_0 + Re^{it}\right)dt = 2\pi i f(z_0) \iff$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + Re^{it}\right)dt. \tag{3.9}$$

This last equation, for obvious reasons, is called **the Mean Value Property of Holomorphic Functions**. The value of a holomorphic function at the center of a disc is the average or mean value of its values along the boundary of the disc.

Note: In our exposition, we use the fact that a holomorphic function is complex analytic (i.e., complex power series) and then we derive the Cauchy integral formula. Most books of complex analysis prove the Cauchy integral formula for holomorphic functions first by basic principles and then working in the direction opposite to our approach (in which they make use of the geometric power series) establish the complex analyticity. There are several proofs of this very fundamental result. For another proof, see **Problem 3.5.18 (a)**.

Examples

Example 3.5.16 If $C = C(0,1)$, then by **(3.8)** we find

$$\oint_{C^+} \frac{\cos(z)}{z}\, dz = 2\pi i \cos(0) = 2\pi i.$$

Also, by **(3.9)** we find

$$\frac{1}{2\pi} \int_0^{2\pi} \cos\left(e^{it}\right) dt = \cos(0) = 1.$$

Similarly,

$$\oint_{C^+} \frac{\sin(z)}{z}\, dz = 2\pi i \sin(0) = 0,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \sin\left(e^{it}\right) dt = \sin(0) = 0.$$

Also,

$$\oint_{C^+} \frac{z^2}{z - \frac{1}{2}}\, dz = 2\pi i \left(\frac{1}{2}\right)^2 = \frac{\pi i}{2},$$

since $\frac{1}{2} \in D(0,1)$, but we do not apply **(3.9)** to this example because $\frac{1}{2}$ is not the center of the disc $D(0,1)$.

If now $C = C(2, \sqrt{5})$, then by **(3.8)** and **(3.9)** we respectively obtain

$$\oint_{C^+} \frac{\sin\left(e^z\right)}{z - 2}\, dz = 2\pi i \sin\left(e^2\right)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(e^{2+\sqrt{5}e^{it}}\right)\, dt = \sin\left(e^2\right).$$

▲

Example 3.5.17 If $C = C(0,2)$, then

$$\oint\limits_{C^+} \frac{dz}{z^2-1} = \oint\limits_{C^+} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) dz = \frac{1}{2}(2\pi i - 2\pi i) = 0$$

$$\oint\limits_{C^+} \frac{dz}{z^2-8} = 0,$$

since $\pm\sqrt{8}$ lie outside the $D(0,2)$ and so $\frac{1}{z^2-8}$ is holomorphic in $D(0,2)$.

Along the same path, we also have

$$\oint\limits_{C^+} \frac{dz}{z^2+2z-3} = \oint\limits_{C^+} \frac{dz}{(z+3)(z-1)} = 2\pi i \cdot \frac{1}{1+3} = \frac{\pi i}{2}.$$

▲

Under the same conditions on $f(z)$, by considering z_0 as parameter in the **Cauchy integral formula**, we can differentiate k times with respect to z_0 and under the integral sign to obtain the more **general formula**:

$$\oint\limits_{C^+} \frac{f(z)}{(z-z_0)^{k+1}}\, dz = \frac{2\pi i f^{(k)}(z_0)}{k!}, \quad \forall\ k = 0,\ 1,\ 2,\ 3,\ \ldots, \tag{3.10}$$

where C and z_0 are as in **Theorem 3.5.7**, above. That is, we start with the Cauchy integral formula ($k = 0$) and then we take derivative k times with respect to z_0, where $k = 1,\ 2,\ 3, \ldots$, and we find:

$$\oint\limits_{C^+} \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0),$$

$$\oint\limits_{C^+} \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i f'(z_0),$$

$$\oint\limits_{C^+} \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i f''(z_0)}{2} = \pi i f''(z_0),$$

$$\oint\limits_{C^+} \frac{f(z)}{(z-z_0)^4}\, dz = \frac{2\pi i f'''(z_0)}{6} = \frac{\pi i f'''(z_0)}{3},$$

and so on for all order derivatives, as the above general formula states.

If $C = C^+(z_0, R)$ under the same conditions as in **equation (3.9)**, then **equation (3.10)** becomes

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + Re^{it}\right) e^{-ikt}\,dt = \frac{R^k f^{(k)}(z_0)}{k!},\ \forall\ k = 0,\ 1,\ 2,\ \ldots. \quad (3.11)$$

Examples

Example 3.5.18 If $C = C(0,1)$, then by **(3.10)** we find

$$\oint\limits_{C^+} \frac{\sin(z)}{z^2}\,dz = 2\pi i \sin'(z)|_{z=0} = 2\pi i\cos(0) = 2\pi i,$$

$$\oint\limits_{C^+} \frac{e^z}{z^2}\,dz = 2\pi i(e^z)'|_{z=0} = 2\pi i e^0 = 2\pi i \cdot 1 = 2\pi i,$$

and by **(3.11)** we find

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(e^{it}\right) e^{-it}\,dt = \frac{\cos(0)}{1!} = 1,$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{e^{it}-it}\,dt = \frac{e^0}{1!} = 1.$$

Similarly, with $C = C(2,3)$ we find

$$\oint\limits_{C^+} \frac{\sin(z)}{(z-2)^4}\,dz = \frac{2\pi i \sin^{(3)}(2)}{3!} = \frac{-2\pi i\sin(2)}{6} = \frac{-\pi i\sin(2)}{3},$$

$$\oint\limits_{C^+} \frac{1}{(z-2)^3}\,dz = \frac{2\pi i(1'')|_{z=2}}{2!} = \pi i \cdot 0 = 0,$$

$$\oint\limits_{C^+} \frac{\cos(z)}{z^2}\,dz = 2\pi i\cos'(z)|_{z=2} = -2\pi i\sin(2),$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(2+3e^{it}\right) e^{-3it}\,dt = \frac{3^3\sin^{(3)}(2)}{3!} = \frac{-9\pi i\sin(2)}{2},$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-2it}\,dt = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\left(2+3e^{it}\right) e^{-it}\,dt = \frac{3\cos'(2)}{2!} = \frac{3}{2}\sin(2).$$

▲

Example 3.5.19 If $C = C(0,2)$, then

$$\oint_{C^+} \frac{e^z}{(z-1)^3}\,dz = \frac{2\pi i\,(e^z)''\,|_{z=1}}{2!} = \frac{2\pi i e}{2} = \pi e i,$$

$$\oint_{C^+} \frac{z^3}{(z-i)^2}\,dz = 2\pi i(z^3)'|_{z=i} = 2\pi i 3i^2 = -6\pi i,$$

$$\oint_{C^+} \frac{z^3}{(z+i)^5}\,dz = \frac{2\pi i(z^3)^{(4)}|_{z=-i}}{4!} = \frac{0}{24} = 0,$$

$$\oint_{C^+} \frac{(z+1)^7}{[z-(1+i)]^{10}}\,dz = \frac{2\pi i[(z+1)^7]^{(9)}|_{z=1+i}}{9!} = 0.$$

▲

Problems

3.5.1 Compute $\displaystyle\int_C z\overline{z}\,dz$ along the parabola $y = x^2$, for$-2 \leq x \leq 4$.

3.5.2 Compute the four integrals:

$$I_1 = \int_{[-i,i]} z\,dz, \qquad I_2 = \int_{[-i,i]} |z|\,dz,$$

$$I_3 = \int_{[1+i,-2+3i]} \frac{1}{z}\,dz, \qquad I_4 = \int_{[1+i,-2+3i]} \frac{1}{\overline{z}}\,dz.$$

3.5.3 Consider as the initial interval of $\operatorname{Arg}_0(z)$ the $[0, 2\pi)$ and prove the discontinuity of $\arg(z)$ and $\log(z)$ along the non-negative x-axis.

[Hint: Imitate what we did in the text with initial interval $[-\pi, \pi)$.]

3.5.4 Find all complex values of $\log(5i)$, $\log(-5i)$ and $\log(-3+5i)$.

3.5.5 Pick any principal argument $a < \operatorname{Arg}(z) \leq b$ or $a \leq \operatorname{Arg}(z) < b$ such that $b - a = 2\pi$ (a, b are real constants). Prove that for each $k \in \mathbb{Z}$

the corresponding branch of the complex logarithm maps the set $\mathbb{C}-\{0\}$ in one-to-one manner onto the horizontal strip

$$\{z \mid z = x + i(y + 2k\pi) \text{ and } -\infty < x < \infty\text{and } a < y \leq b\} \quad \text{or}$$
$$\{z \mid z = x + i(y + 2k\pi) \text{ and } -\infty < x < \infty\text{and } a \leq y < b\},$$

respectively.

3.5.6 Verify the Cauchy-Riemann conditions for the complex logarithm $f(z) = \log(z)$, $z \neq 0$, in both cartesian and polar coordinates. Then, using the expressions of the complex derivative of $f(z)$ in either coordinate system, derive $\frac{d}{dz}[\log(z)] = \frac{1}{z}$.

3.5.7 Find all complex values of $(-3)^{\frac{1}{6}}$, $(-5i)^{\pi}$ and $(-3+5i)^{2-i}$.

3.5.8 Find all complex continuous branches of the functions $z^{\frac{1}{4}}$, $z^{-\frac{1}{4}}$, z^{π} and z^{i}.

3.5.9 (a) For any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and any complex number z, prove the set-relations (with non-positive exponents assume $z \neq 0$)

$$z^{\frac{p}{q}} = \left(z^{\frac{1}{q}}\right)^{p} \subseteq (z^{p})^{\frac{1}{q}}.$$

(b) Compute all complex answers to $(-5)^{\frac{3}{5}}$, $(2-6i)^{-\frac{4}{5}}$, $(1+i)^{\frac{4}{-5}}$ and $i^{\frac{-4}{5}}$.

3.5.10 We start with the true relation $e^{2\pi i} = 1$ and continue as follows:

$$\left(e^{2\pi i}\right)^{\frac{1}{6}} = 1^{\frac{1}{6}} \quad \Longrightarrow \quad e^{\frac{\pi}{3}i} = 1 \quad \Longrightarrow \quad \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = 1 \quad \Longrightarrow$$
$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\,?? \quad \text{Find the mistake!}$$

3.5.11 (a) Locate and explain the error made in

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = -1.$$

(See also **Problem 3.1.33**.)

(b) Conclude when the elementary algebra rule about square roots $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ is valid if a and b are complex numbers.

3.5.12 Prove that for a, b in $\mathbb{C}$, $e^{ib2\pi\mathbb{Z}} = e^{ia2\pi\mathbb{Z}}$ if and only if the following two conditions hold:

(1) $\forall\ m \in \mathbb{Z},\ \exists\ n \in \mathbb{Z}\ :\ am - bn \in \mathbb{Z}.$
(2) $\forall\ k \in \mathbb{Z},\ \exists\ l \in \mathbb{Z}\ :\ bk - al \in \mathbb{Z}.$

Besides the choices $a = b \in \mathbb{C}$, or both a and b integers (in $\mathbb{Z}$) not necessarily equal, find some other choices of $a \neq b$ for which we obtain equality.

3.5.13 Set each of the functions $\cos(z), \sin(z), \cosh(z), \sinh(z)$ equal to 1 and find all solutions of each equation. Then, do the same thing by setting them equal to -1 and 0.

[Hint: Imitate **Example 3.5.13**. For 0, the work is easier.]

3.5.14 Imitate **Example 3.5.13** to find all of the solutions of the following equations:

$$\begin{aligned} \cos(z) &= 2 & \sin(z) &= -3 \\ \tan(z) &= 2 & \cot(z) &= -3 \\ \sec(z) &= 2 & \csc(z) &= -3 \\ \cosh(z) &= -4 & \sinh(z) &= 5 \\ \tanh(z) &= -4 & \coth(z) &= 5 \\ \operatorname{sech}(z) &= -4 & \operatorname{csch}(z) &= 5. \end{aligned}$$

You may want to find the **inverse functions of the trigonometric and hyperbolic functions** (multi-value functions) in general, then plug in the given numbers and simplify your computations.

3.5.15 Prove that the **range** of the **complex** $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$ is the whole complex plane $\mathbb{C}$.

What is the **range** of e^z?

Prove that the **range** of the **complex** $\tan(z)$ and $\cot(z)$ is the set $\mathbb{C} - \{\pm i\}$.

3.5.16 Find the Taylor series of $\dfrac{1}{a - z}$ with center $c \neq a$ ($a,\ c \in \mathbb{C}$) and its radius of convergence. Do the same for e^z.

[Hint: $\dfrac{1}{a - z} = \dfrac{1}{a - c} \cdot \dfrac{1}{1 - \dfrac{z - c}{a - c}}$ and $e^z = e^c e^{z-c}$.]

3.5.17 (a) Find the Maclaurin series of $\log(9 + z)$ and its radius of convergence.

(b) Find the Taylor series of $\log(z)$ and $\dfrac{1}{\sqrt{z}}$ with center $c = 9$ and their radii of convergence.

[Hint: (a) $9+z=9\left(1+\frac{z}{9}\right)$.

(b) $z=9\left(1+\frac{z-9}{9}\right)$ and $\frac{1}{\sqrt{z}}=\frac{1}{3}\left(1+\frac{z-9}{9}\right)^{\frac{-1}{2}}$.]

3.5.18 (a) If $f(z)$ is holomorphic in an open region $\mathcal{R}\subseteq\mathbb{C}$ and for some $r>0$ the circle $C(z_0,r)$ is inside $\mathcal{R}$, use just the continuity of $f(z)$ to prove that

$$\lim_{r\to 0}\int\limits_{C^-(z_0,r)}\frac{f(z)}{z-z_0}\,dz=-2\pi i f(z_0).$$

(Notice the negative orientation.)

Then, for any $\mathcal{C}^+$ simple closed contour in $\mathcal{R}$ with z_0 in its interior, use Green's Theorem on the path $\mathcal{C}^+ + C^-(z_0,r)$ and its interior, for $r>0$ small enough, and the Cauchy-Riemann conditions for the holomorphic function $g(z)=\frac{f(z)}{z-z_0}:=U(x,y)+iV(x,y)$, for $z\neq z_0$, to prove the **Cauchy integral formula** stated in **Theorem 3.5.7**.

(b) Use the Cauchy integral formula and its derivatives with respect to z_0 to derive the **Cauchy estimates of the derivatives of $f(z)$ at z_0**

$$\left|f^{(n)}(z_0)\right|\leq\frac{n!}{r^n}\max_{|z-z_0|=r}|f(z)|,\quad\forall\ \ n=0,\ 1,\ 2,\ \ldots$$

and for every $r>0$ such that the circle $C(z_0,r)$ is inside the open region $\mathcal{R}$.

Then, if $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in a disc $D(z_0,R)\subseteq\mathcal{R}$, with $R>0$, prove that for every r such that $0<r<R$, we have:

$$|a_n|\leq\frac{\max\limits_{|z-z_0|=r}|f(z)|}{r^n},\quad\forall\ \ n=0,\ 1,\ 2,\ \ldots.$$

(c) For $n=0$, we obtain

$$|\,f(z_0)|\leq\max_{|z-z_0|=r}|f(z)|.$$

Use this to justify the claim: If $f(z)$ is holomorphic in an open region $R\subseteq\mathbb{C}$ and $\overline{D(z_0,r)}\subset R$, then the maximum of $|f(z)|$ over $\overline{D(z_0,r)}$ is obtained on the boundary $C(z_0,r)$ of the disc, and so $|f(z)|$ cannot have a strict interior maximum in R.

(This property of holomorphic functions is called **Maximum Modulus Principle**. This principle can be refined to the fact that $|f(z)|$ cannot have any interior maximum, strict or non-strict, in R, unless $|f(z)|$ is identically constant in R. The proof of this claim is more complicated and is relegated to a more advanced course of analysis.)

(d) A holomorphic function over the entire $\mathbb{C}$ is called an **entire function**. [E.g., $f(z) = e^z$.] Use the Cauchy estimates to prove that if $f(z)$ is an entire **bounded** function, i.e., there exists a $B > 0$ such that $\forall\ z \in \mathbb{C}\ |f(z)| \leq B$, then $f(z)$ is identically constant.

(e) Obviously, every polynomial $P(z)$ with complex coefficients is an entire function. Prove that if degree$[P(z)] \geq 1$, then $P(z)$ has a root in $\mathbb{C}$, i.e., there is $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$, thus proving the **Fundamental Theorem of Algebra**.
[Hint: If not, then $f(z) = \dfrac{1}{P(z)}$ would be an entire bounded function in $\mathbb{C}$, i.e., there is a constant $B \geq 0$ such that $|f(z)| \leq B$ for every $z \in \mathbb{C}$. Then $P(z)$ would be identically constant, which is a contradiction!]

(f) If $f(z)$ is an entire function and for an $n \in \mathbb{N}$ it satisfies

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 0,$$

then $f^{(n)}(z) = 0$ at every $z \in \mathbb{C}$, and so $f(z)$ is a polynomial of degree at most $n - 1$.

3.5.19 Evaluate the integral

$$\oint_C \frac{z^2 - 5z + 7}{(z-1)^3}\, dz$$

along any simple, closed, continuous and piecewise continuously differentiable path C, that has $z_0 = 1$ in the inside, by using:
(a) The Cauchy integral formula.
(b) Partial fraction decomposition.

3.5.20 With $C = C(0, 2)$ evaluate the following six integrals:

$$I_1 = \oint_{C^+} \frac{dz}{z^2 + z + 1} \qquad I_2 = \oint_{C^+} \frac{|z|e^z}{z^2}\, dz$$

$$I_3 = \oint_{C^+} \frac{dz}{z^2(z^2 + 16)} \qquad I_4 = \oint_{C^+} \frac{z^2 - 1}{z^2 + 1}\, dz$$

$$I_5 = \oint_{C^+} z\overline{z}\, dz \qquad I_6 = \oint_{C^+} \frac{1}{\overline{z}}\, dz.$$

3.5.21 Evaluate the following ten integrals by using **equation (3.8)** or **equation (3.10):**

$$I_1 = \oint_{C^+(0,1)} \frac{e^z}{z^n}\,dz, \ \ n \in \mathbb{Z} \qquad I_2 = \oint_{C^+(2i,1)} \frac{e^z}{z^n}\,dz, \ \ n \in \mathbb{Z}$$

$$I_3 = \oint_{C^+(0,1)} \frac{z}{z-2}\,dz \qquad I_4 = \oint_{C^+(2,1)} \frac{z}{z-2}\,dz$$

$$I_5 = \oint_{C^+(0,3)} \frac{4e^z}{z-2}\,dz \qquad I_6 = \oint_{C^+(0,3)} \frac{5z}{z+2}\,dz$$

$$I_7 = \oint_{C^+(0,4)} \frac{\sin(z)}{(z+3)^3}\,dz \qquad I_8 = \oint_{C^+(0,4)} \frac{\tan(z)}{(z+1)^3}\,dz$$

$$I_9 = \oint_{C^+(0,1)} \frac{\sin(z)}{z^n}\,dz, \ \ n \in \mathbb{Z} \qquad I_{10} = \oint_{C^+(0,1)} \frac{\cos(z)}{z^n}\,dz, \ \ n \in \mathbb{Z}.$$

Now apply **equation (3.9)** or **equation (3.11)** to those integrals, from the above, on which either of these equations can apply and find the corresponding integrals.

3.5.22 Notice that the integrals

$$\oint_{C^+(0,2)} \frac{z}{z-2}\,dz \qquad \text{and} \qquad \oint_{C^+(1,1)} \frac{z}{z-2}\,dz$$

are improper and cannot be found by the Cauchy integral formula, since $z = 2$ is on the paths of integration. Use convenient parametrizations of the paths to evaluate these integrals or to show that they do not exist. Do the same for their principal values.

3.5.23 If $C = [\infty - \pi i, 0 - \pi i] + [-\pi i, \pi i] + [0 + \pi i, \pi i + \infty]$, then prove

$$\int_C e^{e^z}\,dz = 2\pi i.$$

3.5.24 Let C be the boundary of the square

$$\{z = x + iy \ : \ |x| \le 2, \ |y| \le 2\}.$$

Now evaluate the following ten integrals:

$$I_1 = \oint\limits_{C^+} \frac{\cos(z)}{z}\,dz \qquad I_2 = \oint\limits_{C^+} \frac{\cos(z)}{z^4}\,dz$$

$$I_3 = \oint\limits_{C^+} \frac{z^2}{2z+3}\,dz \qquad I_4 = \oint\limits_{C^+} \frac{\cosh^2(z)}{z^2}\,dz$$

$$I_5 = \oint\limits_{C^+} \frac{dz}{z^2+2} \qquad I_6 = \oint\limits_{C^+} \frac{dz}{(z^2+2)^2}$$

$$I_7 = \oint\limits_{C^+} \frac{\cos(z)}{z-\frac{\pi}{6}}\,dz \qquad I_8 = \oint\limits_{C^+} \frac{e^z}{z(z+1)}\,dz$$

$$I_9 = \oint\limits_{C^+} \frac{z}{z(z-i)}\,dz \qquad I_{10} = \oint\limits_{C^+} \frac{z}{z(z+i)}\,dz.$$

[Hint: You may use the Cauchy integral formulae, but in some of them you may need partial fractions first.]

3.5.25 Let $a > 0$ and C any path joining the numbers of the x-axis $-a$ and a, all other points of which are in the upper half plane. Prove

$$\int_C z^i\,dz = \frac{a\,(1+e^{-\pi})\,\{\cos[\ln(a)] + i\sin[\ln(a)]\}}{2}\cdot(1-i).$$

[Hint: Take as branch cut the non-positive y-axis and principal argument in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Then use as an antiderivative of $f(z) = z^i$ the $F(z) = \frac{z^{i+1}}{i+1} = \frac{z^{i+1}}{2}(1-i)$.]

3.5.26 The **Rodrigues formula**[12] for the **Legendre polynomials** is usually written as

$$P_n(z) = \frac{1}{2^n\,n!}\,\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right], \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \ldots .$$

(a) If C is a smooth simple closed path around a point $z \in \mathbb{C}$, use the **Cauchy integral formula** to prove

$$P_n(z) = \frac{1}{2\pi i}\,\frac{1}{2^n}\int_{C^+} \frac{(w^2-1)^n}{(w-z)^{n+1}}\,dw, \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \ldots .$$

---

[12] Benjamin Olinde Rodrigues, French mathematicians, 1795-1851.

(b) Let C^+ be the circle of center z and radius $\sqrt{|z^2-1|}$. Show that

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\left[z+\sqrt{z^2-1}\,\cos(\theta)\right]^n d\theta, \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \ldots .$$

[Hint: Use **Part (a)**, the convenient parametrization of the circle, the $z^2-1 = |z^2-1|e^{i\phi}$ for some fixed $0 \leq \phi < 2\pi$, the periodicity of the integrand as a function of θ and **Item (7.)** of **Problem 1.1.6**.

Another way is to work backwards and use **Example 3.6.14**, **Problem 3.6.17** and **Lemma 3.8.1**. So, redo this problem in this way when you have studied these topics.]

3.5.27 If $C = C(0,2)$, then by **Theorem 3.5.6**, we have that

$$\oint_{C^+}\frac{dz}{z-1} = 2\pi i.$$

Show that by making the substitution $u(t) = 2e^{it}$ where $0 \leq t \leq 2\pi$, we obtain

$$\oint_{C^+}\frac{dz}{z-1} = \int_0^{2\pi}\frac{2ie^{it}dt}{2e^{it}-1}.$$

Now let $v = e^{it}$, so that $v(0) = 1,\ v(2\pi) = 1$ and $dv = ie^{it}dt$. Then

$$\oint_{C^+}\frac{dz}{z-1} = \int_0^{2\pi}\frac{2ie^{it}dt}{2e^{it}-1} = \int_1^1\frac{2dv}{2v-1} = 0 \neq 2\pi i.$$

Find and explain the mistake!

3.5.8 Appendix

The Potential or Laplace Equation in a Disc and the Poisson Integral Formula

We have seen in **Problem 3.3.12** that the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

of a function $u = u(x,y)$ of two variables x and y, written in polar coordinates r, θ, such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, has the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

This form of Δu is convenient when solving the Laplace equation in the disc $x^2 + y^2 < a^2$, with prescribed boundary data. Here, $a > 0$ is the radius of the disc whose center is at the origin. If the center is not the origin, then we apply a shift transformation of the coordinates so that the center is the new origin. That is, without loss of generality, we consider the problem

$$\begin{cases} \Delta u(x,y) = 0, & x^2 + y^2 < a^2, \\ \\ u(x,y) = h(x,y), & x^2 + y^2 = a^2, \end{cases} \tag{3.12}$$

which in polar coordinates takes the form

$$\begin{cases} \Delta v(r,\theta) = 0, & 0 \le r < a, \quad -\pi < \theta \le \pi \\ \\ v(a,\theta) = f(\theta), & -\pi < \theta \le \pi, \end{cases} \tag{3.13}$$

where $v(r,\theta) = u[r\cos(\theta), r\sin(\theta)]$ and $f(\theta) = h[a\cos(\theta), a\sin(\theta)]$.

Here, we consider a harmonic $\mathcal{C}^2$ function[13] $u(x,y) = v(r,\theta)$ defined everywhere inside the disc $0 \le r < a$. Therefore, as continuous, it is bounded on any smaller closed disc $0 \le r \le b < a$ around the origin.

Then we have

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \qquad 0 \le r < a, \quad -\pi < \theta \le \pi$$
$$v(a,\theta) = f(\theta), \quad -\pi < \theta \le \pi.$$

To solve this boundary value problem we use the method of separation of variables. That is, we write $v(r,\theta) = R(r)\Theta(\theta)$ and plug it into the equation. Then for $v(r,\theta) \ne 0$, we find

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

Hence,

$$-\left(r^2\frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = \mu,$$

where μ is a separation constant. So, we must consider three cases: μ positive, zero and negative.

Case $\mu = \lambda^2 > 0$:

In this case, we have $\Theta'' - \lambda^2\Theta = 0$, which has the general solution

$$\Theta(\theta) = Ae^{\lambda\theta} + Be^{-\lambda\theta}.$$

---

[13] A function is of class $\mathcal{C}^2$ if its second partial derivatives exist and are continuous.

Since $\Theta(\theta)$ must be 2π periodic, in order to have continuous solutions defined at every point in the disc $r < a$, we immediately find that we must have $A = 0 = B$, or $\Theta \equiv 0$. This results in $v \equiv 0$, and so this case does not produce any non-trivial solutions.

Case $\mu = 0$:

In this case, $\Theta'' = 0$, which has the general solution

$$\Theta(\theta) = A\theta + B.$$

This solution is 2π periodic only if $A = 0$, so that $\Theta(\theta) = B = B \cdot 1$. Therefore, we consider the constant function

$$\Theta_0(\theta) = 1.$$

In this case, the differential equation for $R(r)$ has the form

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = 0 \quad \text{or} \quad rR'' + R' = 0.$$

Two linearly independent solutions of this equation are

$$R(r) = 1 \quad \text{and} \quad R(r) = \ln(r).$$

Since $\lim\limits_{r \to 0+} \ln(r) = \infty$, this solution does not qualify to be a part of the solution of the problem posed here. We accept only the solution $R(r) = 1$. Therefore, we obtain only one basic solution for the problem, namely:

$$v_0(r, \theta) = 1 \cdot 1 = 1.$$

Case $\mu = -\lambda^2 < 0$:

In this case, $\Theta'' + \lambda^2\Theta = 0$, which has two independent solutions

$$\Theta(\theta) = \cos(\lambda\theta) \quad \text{and} \quad \Theta(\theta) = \sin(\lambda\theta).$$

Since we work in a disc, these solutions along with their first and second derivatives must be 2π-periodic. This happens if and only if

$$\cos(\lambda 2\pi) = \cos(\lambda\, 0) = \cos(0) = 1 \iff$$
$$2\lambda\pi = 2n\pi, \quad n \in \mathbb{Z} \iff \lambda = n \in \mathbb{Z}.$$

Also, $\lambda = n \in \mathbb{Z}$ satisfies the 2π-periodicity of $\sin(\lambda\theta)$, and we do not obtain any other λ's from it either. So, all λ's are precisely the integer numbers.

We had already found the basic solution $v_0(r, \theta) = 1$, which is also obtained by $\cos(\lambda\theta)$ when $\lambda = n = 0$ and $R(r) = 1$, and since sine is odd and cosine is even, we do not lose anything if we consider

$$0 < \lambda = n = 1,\ 2,\ 3,\ \ldots$$

only. So, we consider only:

$$\Theta_n(\theta) = \cos(n\theta) \text{ and } \Theta_n(\theta) = \sin(n\theta) \text{ with } \lambda = n = 1,\ 2,\ 3, \ldots .$$

Then, $R(r)$ satisfies the second-order ordinary differential equation

$$r^2R'' + rR' - n^2R = 0.$$

[For $n = 0$, this reduces to $rR'' + R' = 0$, which was studied above and gave the solution $R(r) = 1$.]

This linear homogeneous ordinary differential equation is of **Euler type**. To solve it, we try solutions of the form $R(r) = r^\alpha$, where α is some constant exponent to be determined. Then we obtain:

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = 0.$$

Hence,

$$\left[\alpha(\alpha-1) + \alpha - n^2\right] r^\alpha = 0,$$

from which we find

$$\alpha^2 - n^2 = 0, \quad \text{and so} \quad \alpha = \pm\, n.$$

So, for $n = 1,\ 2,\ 3, \ldots$ we obtain the two independent solutions for $R(r)$

$$R_n(r) = r^n \quad \text{and} \quad R_n(r) = r^{-n}.$$

But, for $n = 1,\ 2,\ 3, \ldots$, we have $\lim\limits_{r\to 0+} r^{-n} = \infty$. Therefore, these solutions do not qualify for being a part of the final solution.

Hence, this case has produced the following basic solutions for **Problem (3.13)**:

$$v_n(r,\theta) = r^n\cos(n\theta) \text{ and } v_n(r,\theta) = r^n\sin(n\theta), \text{ for } n = 1,\ 2,\ 3, \ldots .$$

Putting the last two cases together, we observe that all the basic solutions of this problem are

$$r^n\cos(n\theta), \text{ for } n = 0,\ 1,\ 2, \ldots \text{ and } r^n\sin(n\theta), \text{ for } n = 1,\ 2,\ 3, \ldots .$$

Then, by the linearity of the Laplacian Δ, the general solution of $\Delta v(r,\theta) = 0$ in the disc $0 \le r < a$ is a linear combination of these basic solutions. That is,

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left[a_nr^n\cos(n\theta) + b_nr^n\sin(n\theta)\right],$$

with a_n and b_n real constants.

To satisfy the boundary condition $v(r,\theta) = f(\theta),\ -\pi < \theta \leq \pi$, we must have

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n a^n \cos(n\theta) + b_n a^n \sin(n\theta)\right].$$

This means that $a_0, a_n a^n, b_n a^n,\ n = 1,\ 2,\ 3, \ldots$ are the Fourier[14] coefficients of the 2π periodic function $f(\theta)$. So, for $n = 1,\ 2,\ 3, \ldots$ we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi \\ a_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\phi) \cos(n\phi)\, d\phi \\ b_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\phi) \sin(n\phi)\, d\phi. \end{aligned}$$

The Fourier series theory proves that if $f(\theta)$ is continuous and 2π-periodic, then this Fourier series solution $u(r,\theta)$ is indeed a solution of the boundary value problem **(3.12)** or **(3.13)**. Also, by the theory of harmonic functions, we prove that the solution is unique. (These two statements need proofs, which we must skip as going far beyond the material in this text.) By these two facts, we are certain that we have completely and uniquely found the solution of the boundary value **Problem (3.13)** and therefore the solution of **Problem (3.12)**.

As far as finding the solution of **Problem (3.12)** or **(3.13)**, we could stop at this point. However, we continue because the Fourier series solution for $v(r,\theta)$ found above can be put in a very interesting compact integral formula, the so called **Poisson integral formula**. To achieve this, we first replace the integral representations of the Fourier coefficients into the series for $v(r,\theta)$, and we find

$$\begin{aligned} v(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi + \\ \sum_{n=1}^{\infty} \frac{1}{\pi a^n} \left[\int_{-\pi}^{\pi} f(\phi) \cos(n\phi)\, d\phi\right] r^n \cos(n\theta) + \\ \sum_{n=1}^{\infty} \frac{1}{\pi a^n} \left[\int_{-\pi}^{\pi} f(\phi) \sin(n\phi)\, d\phi\right] r^n \sin(n\theta). \end{aligned}$$

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[14] Jean Baptiste Joseph Fourier, French mathematician, 1768-1830.

We factor out the fraction $\dfrac{1}{2\pi}$ and write

$$v(r,\theta) =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[1+\sum_{n=1}^{\infty}\frac{2}{a^n}r^n\left[\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)\right]\right]\,d\phi =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[-1+2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)]\right]\,d\phi.$$

Now, we use **Euler's formula** for the cosine and the summation of the geometric series to obtain

$$2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)] = 2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\frac{e^{in(\theta-\phi)}+e^{-in(\theta-\phi)}}{2} =$$

$$\frac{1}{1-\dfrac{r}{a}\,e^{i(\theta-\phi)}}+\frac{1}{1-\dfrac{r}{a}\,e^{-i(\theta-\phi)}} = \cdots = \frac{2\left[a^2-ar\cos(\theta-\phi)\right]}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

Therefore,

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[-1+\frac{2\left[a^2-ar\cos(\theta-\phi)\right]}{a^2+r^2-2ar\cos(\theta-\phi)}\right]\,d\phi,$$

which eventually becomes

$$v(r,\theta) = \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)}\,d\phi. \tag{3.14}$$

This formula for the solution $v(r,\theta)$ is called the **Poisson integral formula** for the solution of the Laplace equation in the disc. (See also **Problem 3.5.35**.)

In the Poisson integral formula, the boundary value function $f(\phi)$ may in general be just absolutely integrable, that is, $\displaystyle\int_{-\pi}^{\pi}|f(\phi)|\,d\phi < \infty$, and not necessarily 2π-periodic and/or continuous as we required earlier.

Definition 3.5.3 *The expression*

$$P_{(a,\phi)}(r,\theta) = \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} =$$

$$-1+2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)] = 1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)],$$

where $0 \le r < a$ and θ, ϕ are real constants, derived above as a function of (r,θ) for any given (a,ϕ), is called the ***Poisson kernel*** *in the disc $D(0,a)$.*

Remark: We observe that as a real function of two variables $0 \le r < a$ and $0 \le \theta \le 2\pi$ the $P_{(a,\phi)}(r,\theta)$ is a real analytic function since it is a composition of finitely many real analytic functions.

We also observe that

$$\lim_{0<r\to a^-} P_{(a,\phi)}(r,\phi) = +\infty.$$

(Check this for yourselves!)

Theorem 3.5.8 *The harmonic function which is the solution of the boundary value problem*

$$\begin{aligned} \Delta u(x,y) &= 0, && x^2+y^2 < a^2, \\ u(x,y) &= h(x,y), && x^2+y^2 = a^2, \end{aligned}$$

or in polar coordinates

$$\begin{aligned} \Delta v(r,\theta) &= 0, && 0 \le r < a,\ -\pi < \theta \le \pi \\ v(a,\theta) &= f(\theta), && -\pi < \theta \le \pi, \end{aligned}$$

where $v(r,\theta) = u[r\cos(\theta), r\sin(\theta)]$ *with* $f(\theta) = h[a\cos(\theta), a\sin(\theta)]$ *and* $f(\theta)$ *absolutely integrable, is real analytic inside the disc* $r < a$*. Therefore, all harmonic functions* $u(x,y)$ *are analytic inside their domain of definition.*

Proof It follows from the Poisson integral formula, the analyticity of the Poisson kernel and the fact that we can switch the integral $\int_{-\pi}^{\pi}$ with the infinite summation when we write the Poisson kernel as an infinite series in two variables. For instance, all hypotheses of **Theorem 2.3.4** are fulfilled. (See also **Example 2.3.6**.)

In this context, we also state (without proof) the following Theorem[15] because it is very important and relevant.

Theorem 3.5.9 *At all points* $0 \le \phi \le 2\pi$ *at which* $f(\theta)$ *is continuous, we have*

$$\lim_{(r,\theta)\to(a,\phi)} v(r,\theta) = f(\phi).$$

So, if $f(\theta)$ *is continuous for all* $0 \le \theta \le 2\pi$ *(and so it must be* 2π*-periodic), then the solution* $v(r,\theta)$ *of* ***Problem (3.12)*** *or* ***(3.13)*** *obtained as Fourier series or by the Poisson integral formula extends continuously to all points of the boundary of the disc* $D(0,a)$ *with values equal to the boundary data, i.e.,* $v(a,\theta) = f(\theta)$ *for all* $0 \le \theta \le 2\pi$*.*

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[15] For the **proof**, see the bibliography, e.g., Weinberger 1965, sections 24 and 25.

Remark: The unbounded solutions

$$\ln(r),\ r^{-n},\ n = 1, 2, 3, \dots$$

found for $R(r)$ must be used when we solve the Laplace equation in an annulus, since we stay away from $r = 0$. The same is valid for any domain that does not contain a neighborhood of the origin.

The Mean Value Property

By the Poisson integral formula, we find that

$$u(0,0) = v(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi.$$

Similarly, for any $0 < \rho \leq a$ fixed we get that

$$u(0,0) = v(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\phi)\, d\phi.$$

That is, the value of u (or v) at the center of the disc is equal to the mean (average) value of u (or v) along the circumference of any circle with center the center of the disc and radius $0 < \rho \leq a$.

In general, we consider a harmonic function $u : U \longrightarrow \mathbb{R}$, where $U \subseteq \mathbb{R}^2$ open. Then, for any point $P = (x_0, y_0) \in U$, any $r > 0$ such that $D(P, r) \subset U$ and any $0 \leq \rho \leq r$, we let

$$v(\rho,\phi) := u[x_0 + \rho\cos(\phi), y_0 + \rho\sin(\phi)]$$

and we analogously have

$$\forall\ \ 0 < \rho \leq r, \quad u(P) = u(x_0, y_0) = v(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\phi)\, d\phi.$$

This property of harmonic functions is called the **mean value property of harmonic functions**.

Example 3.5.20 We can easily check that $u(x,y) = \ln(x^2 + y^2)$ is a harmonic function in $\mathbb{R}^2 - \{(0,0)\}$. So, by the mean value property at the point $(x_0, y_0) = (2,3)$ and with $r \geq \rho = 1$, we get

$$\int_{-\pi}^{\pi} \ln[14 + 4\cos(\phi) + 6\sin(\phi)]\, d\phi = 2\pi\ln(13) = 16.11605212\dots .$$

▲

From the above mean value property, we also obtain

$$\int_0^r v(0,\theta)\rho\, d\rho = \frac{1}{2\pi}\int_0^r \int_{-\pi}^{\pi} v(\rho,\phi)\, d\phi\, \rho\, d\rho,$$

or

$$u(P) = v(0,\theta) = \frac{1}{\pi\, r^2}\int\int_{D(P,\, r)} u(x,y)\, dA,$$

where $dA = dxdy = \rho d\rho$ is the area element in $\mathbb{R}^2$. That is, we have obtained the following **equivalent mean value property** of harmonic functions:

Let $u : U \longrightarrow \mathbb{R}$, where $U \subseteq \mathbb{R}^2$ open is harmonic. Then, for any point $P = (x_0, y_0)$ and $r > 0$ such that $D(P,\, r) \subset U$, we have

$$u(x_0,y_0) = \frac{1}{\text{Area}[D(P,\, r)]}\int\int_{D(P,\, r)} u(x,y)\, dA.$$

(By differentiation with respect to r, we obtain the converse.)

Application: Suppose $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a harmonic and square integrable function. Then, for any $P = (x_0, y_0) \in \mathbb{R}^2$ and $r > 0$, by the mean value property and **Hölder's inequality in** $\mathbb{R}^2$ (see also **project Problem 2.6.42, Item (1.)**), we have

$$|u(x_0,y_0)| = \frac{1}{\text{Area}[D(P,\, r)]}\int\int_{D(P,\, r)} |u(x,y)|\cdot 1\, dA \leq$$

$$\frac{1}{\text{Area}[D(P,\, r)]}\left[\int\int_{D(P,\, r)} u^2(x,y)\, dA\right]^{\frac{1}{2}}\left[\int\int_{D(P,\, r)} 1^2\, dA\right]^{\frac{1}{2}}.$$

Now, let $C = \int_{\mathbb{R}^2} u^2(x,y)\, dA \geq 0$, a finite constant by hypothesis. Then, as $r \longrightarrow \infty$, we get

$$|u(x_0,y_0)| \leq \frac{1}{\text{Area}[D(P,\, r)]}\sqrt{C}\,\{\text{Area}[D(P,\, r)]\}^{\frac{1}{2}} = \sqrt{\frac{C}{\pi r}} \longrightarrow 0.$$

Therefore, $u(x_0, y_0) = 0$ at any $P = (x_0, y_0) \in \mathbb{R}^2$. So, the only harmonic function defined at every point of R^2 which is square integrable is $u \equiv 0$.

Problems

3.5.28 Check that the functions $R(r) = 1$ and $R(r) = \ln(r)$ are linearly independent solutions to the Laplace equation in polar coordinates.

3.5.29 Find all solutions $v(r)$ to the Laplace equation $\Delta v = 0$ in polar coordinates that are independent of θ, and all solutions $v(\theta)$ that are independent of r.

3.5.30 Prove that the Poisson kernel, **Definition 3.5.3**, is harmonic in the disc $r < a$.

3.5.31 Find the function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ to which the series of functions in **Example 2.3.6** converges uniformly on $[-\pi, \pi]$.

3.5.32 Prove that the integral of the Poisson kernel, **Definition 3.5.3**, over $[-\pi, \pi]$ is equal to 2π. (You may use the **Weierstraß M-Test, Theorem 2.3.1**, to justify the term-by-term integration and use the theory of **Section 2.3**. Also, compare with **Example 3.8.2**.)

3.5.33 Prove

$$\int_{-\pi}^{\pi} \arctan\left[\frac{1+\sqrt{2}\cos(\phi)}{\sqrt{3}+\sqrt{2}\sin(\phi)}\right] d\phi = \frac{\pi^2}{3}.$$

3.5.34 Solve the Laplace equation $\Delta u = 0$ in the annulus
$0 < R_1 < r < R_2 < \infty$ with boundary conditions:
(a) $u(R_1, \theta) = 0$ and $u(R_2, \theta) = f(\theta)$.
(b) $u(R_1, \theta) = g(\theta)$ and $u(R_2, \theta) = 0$.
(c) $u(R_1, \theta) = g(\theta)$ and $u(R_2, \theta) = f(\theta)$.

3.5.35 Let $w = Re^{i\phi}$ and $z = re^{i\theta}$ with $0 \le r < R$.

(a) Prove

$$|w-z|^2 = (w-z)(\overline{w-z}) = (w-z)(\overline{w}-\overline{z}) = R^2 - 2Rr\cos(\theta-\phi) + r^2.$$

(b) Prove

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{1}{2}\left[\frac{w+z}{w-z} + \overline{\left(\frac{w+z}{w-z}\right)}\right] =$$
$$\frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}}\right) = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2},$$

the Poisson kernel.

(c) Find the $\operatorname{Im}\left(\dfrac{w+z}{w-z}\right)$.

(d) Prove

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{w}{w-z} - \frac{w}{w - \dfrac{R^2}{\overline{z}}}.$$

(e) Prove that $dw = iw\,d\phi$ and that the number $\dfrac{R^2}{\overline{z}}$ lies outside the closed disc $\overline{D(0,R)}$.

(f) Use **(d)**, **(e)**, the **Cauchy integral formula** (**Theorem 3.5.7**) and **(b)** above, to prove that if $f(z)$ is a holomorphic function in an open region R that contains the closed disc $\overline{D(0,R)}$ and $z = re^{i\theta}$, where $0 \le r \le R$, then

$$f(z) = f\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f\left(Re^{i\phi}\right)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi.$$

(g) Under the same assumptions as in **(f)** and for a closed disc such that $\overline{D(a,R)} \subset R$, the formula in **(f)** becomes

$$f(a+z) = f\left(a + re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f\left(a + Re^{i\phi}\right)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi.$$

(h) Use **equation (3.14)**, **(b)**, **(f)** and **(g)** to prove that any real harmonic function $u(x,y)$ in an open region R containing the closed disc $D(0,R)$, and if $z = x + iy = re^{i\theta}$, satisfies

$$u(x,y) = u(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\right) u\left(Re^{i\phi}\right)\, d\phi =$$
$$\mathrm{Re}\left[\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\right) u\left(Re^{i\phi}\right)\, d\phi\right]$$

and analogously for any $D(a,R) \subseteq R$. Therefore, prove that any real harmonic function $u(x,y)$ in an open region R is the real part of a holomorphic function.

Application to Holomorphic Functions

Suppose we have the solutions $u(r,\theta)$ and $v(r,\theta)$ of two boundary value problems of the type **(3.12)** or **(3.13)** that we solved before, given in polar variables (r,θ) by the series

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right]$$

and

$$v(r,\theta) = c_0 + \sum_{n=1}^{\infty} \left[c_n r^n \cos(n\theta) + d_n r^n \sin(n\theta)\right],$$

where the coefficients a_n's, b_n's, c_n's, d_n's are real constants. So, these two functions, $u(r,\theta)$ and $v(r,\theta)$, are harmonic.

We moreover assume that $u(r,\theta)$ and $v(r,\theta)$ satisfy the Cauchy-Riemann conditions (written here in polar coordinate form)

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad u_\theta = -r v_r.$$

Then, developing the Cauchy-Riemann conditions for these functions (given in the above series forms), we find that:

$$\forall\ n = 1,\ 2,\ 3,\ \dots \quad a_n = d_n \quad \text{and} \quad b_n = -c_n$$

(check this easily). So,

$$u(r,\theta) + iv(r,\theta) =$$

$$a_0 + ic_0 + \sum_{n=1}^{\infty} \{a_n r^n [\cos(n\theta) + i\sin(n\theta)] + b_n r^n [\sin(n\theta) - i\cos(n\theta)]\} =$$

$$= a_0 + ic_0 +$$

$$\sum_{n=1}^{\infty} \{a_n r^n [\cos(n\theta) + i\sin(n\theta)] i b_n r^n [\cos(n\theta) + i\sin(n\theta)]\} =$$

$$a_0 + ic_0 + \sum_{n=1}^{\infty} \{(a_n - ib_n) r^n [\cos(n\theta) + i\sin(n\theta)]\} = C_0 + \sum_{n=1}^{\infty} C_n z^n,$$

where $C_0 = a_0 + ic_0$, $C_n = a_n - ib_n$, and by the **De Moivre formula** we have:

$$r^n [\cos(n\theta) + i\sin(n\theta)] = z^n = (x+iy)^n.$$

Therefore, the function

$$f(z) = f\left(r[\cos(\theta) + i\sin(\theta)]\right) = u(r,\theta) + iv(r,\theta) = C_0 + \sum_{n=1}^{\infty} C_n z^n,$$

or

$$f(z) = f(x+iy) =$$

$$C_0 + \sum_{n=1}^{\infty} C_n z^n = C_0 + \sum_{n=1}^{\infty} C_n (x+iy)^n = U(x,y) + iV(x,y),$$

is a complex power series of the complex variable z, and therefore it is holomorphic. So, we have:

Theorem 3.5.10 *If the functions $u(x,y)$ and $v(x,y)$ are harmonic in a disc $D(0,a)$, ($a > 0$ constant) and they satisfy the Cauchy-Riemann conditions, then the function $u(x,y) + iv(x,y)$ is a holomorphic function $f(z)$ in the complex variable $z = x + iy$.*

3.6 Roots, Singularities, Residues

3.6.1 Definitions, Laurent Expansion and Examples

We consider an open region $R \subseteq \mathbb{C}$ and a holomorphic function $f : R \longrightarrow \mathbb{C}$, that is, $f'(z)$ exists $\forall\ z \in\ R$. Since R is open, for any $z \in R$ there is a number $r = r(z) > 0$ such that $D(z, r) \subseteq R$. That is, all points of R are interior points. Also, by **Theorem 3.5.1**, $f(z)$ is a power series, $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$, in any disc $D(z_0, r) \subseteq R$, with $r > 0$.

Definition 3.6.1 *We call a $z_0 \in R$ **root of order or multiplicity** $m = 1,\ 2,\ 3,\ ...$ of $f(z)$, if there is $r > 0$ such that in the open disc $D(z_0, r) \subseteq R$ we can write*

$$f(z) = (z - z_0)^m g(z), \quad \forall\ z \in D(z_0, r),$$

where $g(z)$ is a holomorphic function in $D(z_0, r)$ and $g(z_0) \neq 0$.

*A root of order (or multiplicity) 1 is also called **a simple root**.*

This definition implies immediately that $z_0 \in R$ **is a root of order or multiplicity** $m \geq 1$ of the holomorphic function $f(z)$ if and only if $f(z_0) = f'(z_0) = f''(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$. This is a convenient condition to detect the order m of a root. It is easily seen by differentiating the $f(z) = (z - z_0)^m g(z)$ m times and using $g(z_0) \neq 0$.

(Many times, we need to find the order of a root. Usually this condition is convenient to use. Give a complete proof of this "if and only if" claim, using the analyticity of the functions $f(z)$ and $g(z)$. See also **Example 3.6.12** for a special case.)

Example 3.6.1 $z_0 = 0$ is a root of order $m = 3$ of the holomorphic function $f(z) = z^2 \sin z$, which is defined in all $\mathbb{C}$.

Indeed: $f(0) = 0^2 \sin(0) = 0 \cdot 0 = 0$,
$f'(0) = 2z \sin(z) + z^2 \cos(z)|_{z=0} = 0 + 0 = 0$,
$f''(0) = 2 \sin(z) + 4z \cos(z) - z^2 \sin(z)|_{z=0} = 0 + 0 - 0 = 0$ and
$f^{(3)}(0) = 6 \cos(z) - 6z \sin(z) - z^2 \cos(z)|_{z=0} = 6 - 0 - 0 = 6 \neq 0$.

All the other roots of $f(z)$ are the real numbers $z_k = k\pi$, with $k \neq 0$ integer, which are the other roots of $\sin(z)$. All of them have order $m = 1$.

Indeed: $f(z_k) = z_k^2 \sin(z_k) = k^2\pi^2 \sin(k\pi) = k^2\pi^2 \cdot 0 = 0$ and
$f'(z_k) = 2z \sin(z) + z^2 \cos(z)|_{z_k} = 0 + k^2\pi^2(-1)^k = k^2\pi^2(-1)^k \neq 0$.

▲

Next we consider an open region $R \subseteq \mathbb{C}$ and z_0 (an interior) point of R. We define:

Definition 3.6.2 *Suppose that* $z_0 \in R$ *and* $f : R - \{z_0\} \longrightarrow \mathbb{C}$ *is a holomorphic function [that is,* $f(z)$ *is defined at every point of* R *except* z_0*]. Then we call the point* z_0 *an* ***isolated singularity*** *of* $f(z)$*.*

The region R in this definition could for convenience be just an open disc $D(z_0, r)$, for some $r > 0$. I.e., $f(z)$ is defined in the punctured disc $R = D^\circ(z_0, r)$.

Definition 3.6.3 *An isolated singularity is called* ***a removable singularity*** *or* ***non-essential singularity*** *if we can extend the function* $f(z)$ *to the point* z_0 *(i.e., we can define* $f(z)$ *at* z_0*) so that the extended function is holomorphic in the entire* R *(including* z_0*).*

If we cannot do this, then this singularity is called a ***non-removable*** *or* ***essential singularity****.*

So, if z_0 is a removable singularity of $f(z)$, then at $z = z_0$ we can assign an appropriate value $f(z_0)$ of $f(z)$ so that the complex derivative $f'(z_0)$, exists at $z = z_0$ too. If z_0 is a non-removable singularity, it is impossible to achieve such a thing.

Definition 3.6.4 *A singularity* z_0 *of* $f(z)$ *which is not isolated is called* ***a non-isolated singularity.***

So, if z_0 is a non-isolated singularity of $f(z)$, then $\forall\ r > 0$ such that $D(z_0, r) \subseteq R$ there exists $z_r \in D(z_0, r)$ such that $z_r \neq z_0$ and z_r is a singularity of $f(z)$. Therefore, there exists a sequence (z_n), $n \in \mathbb{N}$, of singularities of $f(z)$, converging to z_0, as $n \to \infty$.

Now we state a partial version of the Laurent Series Expansion Theorem around an isolated singularity, which is sufficient for the scope of this text.

Theorem 3.6.1 (Laurent Series Expansion, a partial version) *We consider an open region* $R \subseteq \mathbb{C}$ *and* $z_0 \in R$ *an isolated singularity of a holomorphic function* $f : R - \{z_0\} \longrightarrow \mathbb{C}$*. Then we can find* $0 < r \leq \infty$*, such that* $D(z_0, r) = \{ z \in \mathbb{C} \ : \ |z - z_0| < r\} \subseteq R$*, and the following expansion of* $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \qquad (3.15)$$
$$\forall\ z \in D^o(z_0, r) = D(z_0, r) - \{z_0\}$$

holds for appropriate complex coefficients a_n*'s and* b_n*'s, and it is unique.*

This series expansion is called **the Laurent**[16] **series expansion of** $f(z)$ **with center** z_0**.** (This expansion was also known to Weierstraß.)

---

[16] Pierre Alphonse Laurent, French mathematician, 1813-1854.

For the sake of brevity, we omit the proof even though we have developed enough tools to provide it. (Consult any pertinent bibliography, in which you can find the most general statement of this Theorem and its proof.)

From the form of this expansion, we easily observe (for instance, using **Theorem 3.5.6**, or the **Cauchy integral formula 3.5.7** etc.) that the **coefficients of the Laurent expansion** can be found by

$$a_n = \frac{1}{2\pi i} \oint\limits_{C^+(z_0,\,\rho)} f(z)(z-z_0)^{-(n+1)}\,dz, \quad \forall\ n = 0,\ 1,\ 2,\ 3, \ldots$$

and

$$b_n = \frac{1}{2\pi i} \oint\limits_{C^+(z_0,\,\rho)} f(z)(z-z_0)^{(n-1)}\,dz, \quad \forall\ n = 1,\ 2,\ 3,\ 4,\ \ldots,$$

where $0 < \rho < r$. These coefficients, evaluated by these formulae, do not depend on the choice of ρ, $0 < \rho < r$, and so the Laurent series expansion of a function about a given center z_0 is unique.

The Laurent series expansion **(3.15)** reduces to the Taylor series expansion of $f(z)$ with center $z = z_0$ when $f(z)$ has no singularity in $D(z_0, r)$, i.e., $f(z)$ is holomorphic at z_0 and/or z_0 is a removable singularity of $f(z)$. In such a case, $b_n = 0$, $\forall\ n \in \mathbb{N}$ and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, $\forall\ n \in \mathbb{N}_0$. $f(z)$ is a power series with center z_0 in the whole disc $D(z_0, r)$ and $f(z_0) = a_0$.

In general, some or all of the a_n's and/or some or all b_n's may be zero. If all the b_n's are zero, then $f(z)$ has a removable singularity at z_0. In such a case, $f(z_0) = a_0$ and $f(z)$ is holomorphic at $z = z_0$, also.

This version of this theorem is not the most general one, but it is sufficient for the needs we face in this text. The interested reader can find the theorem in various intermediate-or advanced-level books of complex analysis. In the general version, $f(z)$ is holomorphic in the domain

$$\mathbb{C} - \{\text{a set of isolated singularities}\}, \quad \text{i.e.,}$$
$$R = \mathbb{C} - \{\text{isolated singularities of } f(z)\},$$

and the center z_0 may be any point, regular or singular for $f(z)$. Then, the Laurent expansion may be found in any annulus $\{\, z \in \mathbb{C}\ :\ \rho < |z| < r\} \subseteq \mathbb{C}$ that does not contain any singularity, and $0 \leq \rho < r \leq \infty$ are appropriate constants.

The maximum $r > 0$ for which expansion **(3.15)** holds is the distance of z_0 to the nearest singularity of $f(z)$ other than z_0. It is strictly positive because the singularity z_0 is isolated. I.e., in this case, the radius of convergence is the distance of the center z_0 of the Laurent series to the nearest isolated essential singularity. If there is no singularity in $\mathbb{C}$ other than z_0, then $r = \infty$. Also, if $f(z)$ has no singularity in the whole $\mathbb{C}$, then its Taylor series with any center has radius of convergence ∞.

If z_0 is a non-isolated singularity, then we cannot obviously have Laurent series expansion of $f(z)$ with center z_0. That is, we do not have Laurent series expansions with centers non-isolated essential singularities.

Definition 3.6.5 *In the Laurent expansion of $f(z)$ with center the isolated singularity z_0, the coefficient b_1 is special and is called **the residue of** $f(z)$ **at** z_0.*

Notation: We write

$$b_1 = \operatorname*{Res}_{z=z_0} f(z), \quad \text{or} \quad b_1 = \operatorname*{Res}_{z=z_0} [f(z)].$$

Important Remark: We have seen in **Subsection 3.5.6**, and we shall see it again in the sequel, that the residues are very important in complex integration. We must also keep in mind that we do not have residues for non-isolated essential singularities. We cannot define them in such a case, because there is no Laurent series expansion around a non-isolated singularity. Also, at an isolated removable singularity, the Laurent series becomes a Taylor series, and so the residue at an isolated removable singularity is equal to zero.

Question: Why have we singled out b_1 and given it a special name?

Answer: This is because we have already seen that b_1 plays the most important role in the line integral along any simple, closed, continuous and piecewise continuously differentiable path C that encloses z_0. Namely, we have seen several times in previous sections that

$$\oint_{C^{\pm}} f(z)\, dz = \pm 2\pi i b_1 = \pm 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

So, finding the isolated singularities and their corresponding residues is the most important piece of information in computing complex line integrals and in turn two real line integrals.

A theorem characterizing removable singularities is the following:

Theorem 3.6.2 *(Riemann) Let $R \subseteq \mathbb{C}$ be an open region, z_0 be an interior point of R and $f : R - \{z_0\} \longrightarrow \mathbb{C}$ be a holomorphic function. Then the following three statements are equivalent.*

(a) The isolated singularity z_0 of $f(z)$ is removable.

(b) The $\lim_{z\to z_0} f(z)$ *exists as a complex number* L.

(c) There is some $r > 0$ *such that* $\overline{D(z_0, r)} \subset R$ *and* $f(z)$ *is bounded on the closed punctured disc* $\overline{D(z_0, r)}^{\,o}$.

Proof (a) $\Longrightarrow$ (b) Since we assume that z_0 is a removable singularity of $f(z)$, by definition we can define $f(z)$ at z_0 such that $f(z_0) = L$ and $f'(z_0)$ exists. So, $f(z)$ is continuous in all R, and therefore $\lim_{z\to z_0} f(z) = L$ exists.

(b) $\Longrightarrow$ (c) Suppose $\lim_{z\to z_0} f(z) = L \in \mathbb{C}$ exists as a complex number L. Then at $z = z_0$, we define $f(z)$ by $f(z_0) = L$. This assignment makes $f(z)$ continuous in all R.

Since R is an open set, there is $\rho > 0$ such that $D(z_0, \rho) \subseteq R$. Then for any $0 < r < \rho$, we have $\overline{D(z_0, r)} \subseteq R$. Then $f(z)$ is continuous on the closed and bounded set $\overline{D(z_0, r)}$, and so, by the Extreme Value Theorem, is bounded on $\overline{D(z_0, r)}$.

Another way to show this boundedness is by the definition of limit. We have that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - L| < \epsilon$. So, for any $0 < r < \delta$ we get that if $|z - z_0| < r$, then $|f(z)| < |L| + \epsilon$. Therefore, $f(z)$ is bounded on $\overline{D(z_0, r)}$ by $|L| + \epsilon$.

(c) $\Longrightarrow$ (a) We suppose that there is some $r > 0$ such that $\overline{D(z_0, r)} \subset R$ and $f(z)$ is bounded on closed punctured disc $\overline{D(z_0, r)}^{\,o}$. Then in R, we define

$$g(z) = \begin{cases} (z - z_0)^2 f(z), & \text{if} \quad z \neq z_0, \\ \\ 0, & \text{if} \quad z = z_0. \end{cases}$$

Obviously, $g'(z)$ exists for all $z \neq z_0$. At $z = z_0$, we observe that if $z \in \overline{D(z_0, r)}^{\,o}$, then we have

$$\lim_{z_0 \neq z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} (z - z_0) f(z) = 0,$$

since $\lim_{z\to z_0} (z - z_0) = 0$ and $f(z)$ is bounded on $\overline{D(z_0, r)}^{\,o}$. Hence, $g'(z_0) = 0$ also exists. That is, $g(z)$ is holomorphic in all R.

Then $g(z)$ is a power series in $D(z_0, r)$. I.e.,

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots, \quad \forall\ z \in D(z_0, r).$$

Since $g(z_0) = 0$ and $g'(z_0) = 0$, we have $c_0 = 0$ and $c_1 = 0$. So,

$$g(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \dots = (z - z_0)^2 f(z), \quad \forall\ z \in D(z_0, r).$$

Hence,

$$f(z) = c_2 + c_3(z - z_0) + c_4(z - z_0)^2 + \dots, \quad \forall\ z \in D(z_0, r).$$

I.e., $f(z)$ is a power series and so holomorphic in $D(z_0, r)$, with $f(z_0) = c_2$. That is, $f(z)$ is extended to z_0 holomorphically, and so z_0 is a removable singularity. [Finally, $f(z)$ is holomorphic in the whole R.]

Remark: The proof of the **Part** [$\underline{\textbf{(c)} \Longrightarrow}$ **(a)**] carries through if we replace the boundedness of $f(z)$ on $\overline{D(z_0, r)}^{\,o}$ with the more general condition:

$\exists\ M > 0$ *and* $\alpha < 1$ *real constants, such that*

$$|f(z)| \leq \frac{M}{|z - z_0|^{\alpha}}, \ \ \forall\ \ z \in \overline{D(z_0, r)}^{\,o}.$$

From this Theorem, we immediately have the following:

Corollary 3.6.1 *Let $R \subseteq \mathbb{C}$ be open, $A \subset R$ be a finite subset of R. We assume that the function $f : R \longrightarrow \mathbb{C}$ is continuous in the whole R and holomorphic in $R - A$. Then $f(z)$ is holomorphic in the whole R (i.e., it has complex derivative even at the exceptional points of A).*

Important Remark: Another way, besides invoking the **above theorem**, to **prove this corollary** is achieved by invoking **Morera's Theorem, 3.5.5,** and check that it applies. Moreover, Morera's Theorem proves this corollary even when the exceptional set A contains not only finitely many isolated points of R but even segments of paths in R on which the function $f(z)$ is continuous, but we know nothing about the existence of complex derivatives on them. Eventually, $f(z)$ has complex derivative even on those exceptional segments.

Definition 3.6.6 *An isolated singularity z_0 of an otherwise holomorphic function $f(z)$ in a disc $D(z_0, r)$ $(r > 0)$ is called a **pole of order** $m \geq 1$ of $f(z)$, if in the **Laurent expansion (3.15)***

$$b_n = 0, \ \forall\ n \geq m + 1, \quad \textit{but} \quad b_m \neq 0.$$

The expression

$$\sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n} = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \ldots + \frac{b_1}{z - z_0}$$

*is called the **principal part of the pole** z_0.*

*A pole of order one is also called a **simple pole**.*

We observe that a pole is an essential (non-removable) singularity. If in the Laurent series infinitely many b_n's are non-zero, then the singularity z_0 is essential but not a pole.

In analogy with the definition of a root of order m, we easily observe that ***this definition of a pole is equivalent to the condition***

$$(z-z_0)^m f(z) = g(z) \Longleftrightarrow f(z) = (z-z_0)^{-m} g(z),$$

where $g(z)$ *is a holomorphic function in a disc* $D(z_0, r)$, *for some* $r > 0$, *and* $(b_m =) g(z_0) \neq 0$.

(Many times, we need to find the order of a pole. Usually this condition is convenient to use. Give a complete proof of this claim. See also **Example 3.6.12** for a special case.)

A theorem characterizing a pole is the following:

Theorem 3.6.3 *Let* $D \subseteq \mathbb{C}$ *be open,* $z_0 \in D$ *and* $f : D - \{z_0\} \longrightarrow \mathbb{C}$ *be a holomorphic function. Then* z_0 *is a pole of* $f(z)$ *if and only if* $\lim\limits_{z \to z_0} f(z) = \infty$.

Proof ($\Longrightarrow$) This is immediately true since for any function the principal part of a pole approaches ∞, as $z \longrightarrow z_0$, and also the power series part of the **Laurent expansion (3.15)** is equal to the constant a_0 at z_0.

($\Longleftarrow$) If $\lim\limits_{z \to z_0} f(z) = \infty$, then $f(z)$ cannot have any roots in a disc $D(z_0, \rho) \subseteq D$ for some $\rho > 0$ and also $\lim\limits_{z \to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\infty} = 0$.

Therefore, for any $0 < r < \rho$, the function $g(z) := \dfrac{1}{f(z)}$ is bounded on $\overline{D(z_0, r)}$, and it is holomorphic in $\overline{D(z_0, r)} - \{z_0\}$.

So, $g(z_0) = 0$, and by the **previous theorem** $g(z)$ is holomorphic in $\overline{D(z_0, r)}$. Then we let $k \geq 1$ be the order of this root of $g(z)$, and by definition there is a holomorphic function $h(z)$ in $D(z_0, r)$ and

$$g(z) = (z-z_0)^k h(z), \ \ \forall \ z \in D(z_0, r) \ \text{ and with } \ h(z_0) \neq 0.$$

Then for sufficiently small $r > 0$, we have $h(z) \neq 0$, $\forall \ z \in D(z_0, r)$, and

$$f(z) = (z-z_0)^{-k} \frac{1}{h(z)}.$$

Since $\dfrac{1}{h(z)}$ is holomorphic in $D(z_0, r)$, it is a power series

$$\frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad \forall \ z \in D(z_0, r),$$

with $c_0 \neq 0$.

Hence, we have: $c_0 \neq 0$ and $\forall \; z \in D(z_0, r)$

$$f(z) = (z - z_0)^{-k} \sum_{n=0}^{\infty} c_n (z - z_0)^n =$$

$$\frac{c_0}{(z - z_0)^k} + \ldots + \frac{c_{k-1}}{z - z_0} + c_k + c_{k+1}(z - z_0) + c_{k+2}(z - z_0)^2 + \ldots ,$$

thus proving that z_0 is a pole of $f(z)$ with order k.

Because of this result we say that any polynomial of degree ≥ 1 has a pole at the complex infinity. For any such polynomial $p(z)$, we obviously have $\lim\limits_{z \to \infty} p(z) = \infty$. (See also **Corollary 3.7.1**.)

In view of the **previous two theorems**, we conclude that z_0 is an essential singularity of $f(z)$ which is not a pole if and only if the $\lim\limits_{z \to z_0} f(z)$ oscillates among the complex numbers. Hence, we have shown that if $D \subseteq \mathbb{C}$ is an open set and $z_0 \in D$ is an isolated singularity of a holomorphic function $f(z)$ in $D - \{z_0\}$, then:

1. z_0 is a removable singularity of $f(z) \Longleftrightarrow \lim\limits_{z \to z_0} f(z) := L \in \mathbb{C}$.

2. z_0 is a pole of $f(z) \Longleftrightarrow \lim\limits_{z \to z_0} f(z) = \infty$ (complex infinity).

3. z_0 is an essential singularity but not a pole of $f(z) \Longleftrightarrow \lim\limits_{z \to z_0} f(z)$ oscillates in $\mathbb{C}$.

Also, in view of the **Laurent expansion (3.15)**, the **previous two theorems** and comments, we conclude that:

1. z_0 is a removable singularity of $f(z) \Longleftrightarrow$ in **(3.15)** $b_n = 0, \forall \; n \in \mathbb{N}$.

2. z_0 is a pole of $f(z) \Longleftrightarrow$ in **(3.15)** $b_n \neq 0$, for finitely many n's. The order of the pole is the maximum n for which $b_n \neq 0$.

3. z_0 is an essential singularity but not a pole of $f(z) \Longleftrightarrow$ in **(3.15)** $b_n \neq 0$, for infinitely many n's.

Examples

Example 3.6.2 The function $f(z) = e^z$ is holomorphic in the entire $\mathbb{C}$. (We know $f'(z) = e^z, \; \forall \; z \in \mathbb{C}$). Therefore, $f(z) = e^z$ has no singularity, and at any point $z_0 \in \mathbb{C}$ the residue of $f(z) = e^z$ is zero ($b_1 = 0$).

▲

Example 3.6.3 The function $f(z) = \dfrac{z^4 - 1}{z - 1}$ has an isolated singularity at $z = 1$. This isolated singularity is removable because

$$\lim_{1 \neq z \to 1} f(z) = \lim_{z \to 1} \left(z^3 + z^2 + z + 1\right) = 4.$$

So, the extended function

$$g(z) = \begin{cases} f(z), & \text{if} \quad z \neq 1 \\ \\ 4, & \text{if} \quad z = 1 \end{cases}$$

is not only continuous but also holomorphic in the whole $\mathbb{C}$. In fact,

$$\frac{dg(z)}{dz}|_{z=1} = (3z^2 + 2z + 1)|_{z=1} = 6.$$

The residue of $f(z)$ at $z = 1$ is zero (i.e., $b_1 = 0$).

▲

Example 3.6.4 The function $f(z) = \dfrac{e^z - z - 1}{z^2}$ has an isolated singularity at $z = 0$. This isolated singularity is removable because

$$\lim_{\substack{z \to 0 \\ z \neq 0}} f(z) = \lim_{z \to 0} \left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots\right) = \frac{1}{2}.$$

So, the extended function

$$g(z) = \begin{cases} f(z), & \text{if} \quad z \neq 0 \\ \\ \dfrac{1}{2}, & \text{if} \quad z = 0 \end{cases}$$

is not only continuous but also holomorphic in $\mathbb{C}$. In fact,

$$\frac{dg(z)}{dz}|_{z=0} = \frac{1}{3!} = \frac{1}{6}.$$

The residue of $f(z)$ at $z = 0$ is zero ($b_1 = 0$).

▲

Example 3.6.5 Since

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...\right),$$

we have that $z_0 = 0$ is a root of $\sin(z)$ of order one. The $\sin(z)$ is holomorphic in the entire $\mathbb{C}$. Therefore, at any $z_0 \in \mathbb{C}$, the residue of $\sin(z)$ is $b_1 = 0$.

Also, the function

$$g(z) = \frac{\sin(z)}{z}, \quad \forall \ z \neq 0,$$

has an isolated singularity at $z = 0$. This singularity is removable. Indeed, we observe that

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ..., \quad \forall \ z \neq 0,$$

and so we can extend $g(z)$ at $z = 0$ by defining $g(0) = 1$. The extended $g(z)$ is a power series which is defined and converges for all $z \in \mathbb{C}$. Therefore, it is holomorphic in the entire $\mathbb{C}$. The residue of $g(z)$ at $z_0 = 0$ is $b_1 = 0$.

But,

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - ..., \quad \forall \ z \neq 0.$$

Therefore, $\sin\left(\frac{1}{z}\right)$ has an isolated singularity at $z_0 = 0$, which is not removable and it is not a pole. The corresponding residue is $b_1 = 1$.

Also,

$$\lim_{z \to 0} \sin\left(\frac{1}{z}\right) = \text{does not exist. (This limit oscillates!)}$$

▲

Example 3.6.6 The function

$$f(z) = \frac{(z-1)^{10}}{z-3}$$

has a root of order 10 at $z_0 = 1$ and a singularity at $z_0 = 3$. To find the corresponding residue, we must observe that

$$f(z) = \frac{(z-1)^{10}}{z-3} = \frac{[(z-3)+2]^{10}}{z-3} =$$

$$\frac{\sum_{k=0}^{10} \binom{10}{k} (z-3)^{10-k} 2^k}{z-3} = \left[\sum_{k=0}^{9} 2^k \binom{10}{k} (z-3)^{10-k-1}\right] + \frac{2^{10}}{z-3} =$$

$$\left[\sum_{k=0}^{9} 2^k \binom{10}{k} (z-3)^{9-k}\right] + \frac{2^{10}}{z-3}.$$

So, $z_0 = 3$ is a pole of order one with corresponding residue $b_1 = 2^{10} = 1024$. Also,

$$\lim_{z\to 3} f(z) = \infty.$$

▲

Example 3.6.7 We know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + ...$$

is defined everywhere in $\mathbb{C}$, and it has no root and no singularity. But,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} - ...$$

is not defined at $z_0 = 0$ and cannot be extended to it as a holomorphic function. Therefore, $z_0 = 0$ is an essential singularity, which is not a pole. The corresponding residue is $b_1 = 1$.

Also,

$$\lim_{z\to 0} e^{\frac{1}{z}} = \text{does not exist.}$$

This limit oscillates! For instance: If $x \in \mathbb{R}$, then

$$\lim_{x\to 0^+} e^{\frac{1}{x}} = e^{\infty} = \infty$$

and

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = e^{-\infty} = 0.$$

Similarly, if $y \in \mathbb{R}$, then

$$\lim_{y\to 0^\pm} e^{\frac{1}{iy}} = \lim_{y\to 0^\pm} e^{\frac{-i}{y}} = \lim_{y\to 0^\pm} \left[\cos\left(\frac{1}{y}\right) - i\sin\left(\frac{1}{y}\right)\right] = \text{does not exist.}$$

▲

Example 3.6.8 We have seen that the continuous branch of $\log(z)$

$$f(z) = \log(z), \quad z \in \mathbb{C} - \{a \mid a \leq 0\}$$

is holomorphic, since

$$\frac{d}{dz}\log(z) = \frac{1}{z}, \quad z \in \mathbb{C} - \{a \mid a \leq 0\}.$$

Here, we consider $-\pi < \operatorname{Arg}(z) < \pi$.

So, for this branch of the complex logarithm, all the points of the non-positive x-axis, or the whole branch cut, are singularities. This function is discontinuous at each of these points, and therefore it is impossible to be extended holomorphically at any one of them. (Continuity is a necessary condition for holomorphicity.) These singularities form a continuous closed half line starting at the origin, and therefore they are not isolated. If z_0 is one of these singularities, then for any $r > 0$ the disc $D(z_0, r)$ obviously contains the segment $(z_0 - r, z_0 + r)$ which consists of singularities. Keep in mind: **There are no residues for non-isolated singularities**.

▲

Example 3.6.9 The function

$$f(z) = \frac{1}{1 - e^{\frac{1}{z}}}$$

has singularities at the points $z_0 = 0$ and $\frac{1}{2n\pi i}$, $n = \pm 1, \pm 2, \pm 3, \ldots$.

$z_0 = 0$ is a non-isolated singularity because $\frac{1}{2n\pi i} \to 0$ as $n \to \pm\infty$. So, we cannot find the residue of $f(z)$ at $z_0 = 0$. All the other singularities are isolated, and we can compute the corresponding residues.

▲

Example 3.6.10 Let $f(z)$ be a holomorphic function in some region R which contains z_0 as interior point and $f(z_0) \neq 0$. Then the function

$$g(z) = \frac{f(z)}{(z - z_0)^m}, \quad z \in R - \{z_0\}$$

has a pole of order m at z_0. This follows from the fact that there is $r > 0$ such that $D(z_0, r) \subseteq R$ and

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad z \in D(z_0, r),$$

where $c_n = \frac{f^{(n)}(z_0)}{n!}$ are the **Taylor coefficients** of $f(z)$ at z_0 and $n = 0, 1, 2, 3, \ldots$. So,

$$g(z) = \frac{c_0}{(z - z_0)^m} + \frac{c_1}{(z - z_0)^{m-1}} + \ldots + \frac{c_{m-2}}{(z - z_0)^2} + \frac{c_{m-1}}{z - z_0} + \\ c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \ldots.$$

Since $c_0 = f(z_0) \neq 0$, by definition, z_0 is a pole of order m for the function $g(z)$. Its residue at $z = z_0$ is

$$\operatorname*{Res}_{z=z_0} g(z) = c_{m-1} = \frac{f^{(m-1)}(z_0)}{(m - 1)!}.$$

Also,

$$\lim_{z \to z_0} g(z) = \infty.$$

▲

Example 3.6.11 The polynomial

$$p(z) = z^2 + z + 1$$

has two solutions

$$z = \frac{-1 \pm i\sqrt{3}}{2}.$$

So,

$$f(z) = \frac{1}{z^2 + z + 1} = \frac{1}{\left(z - \frac{-1+i\sqrt{3}}{2}\right)\left(z - \frac{-1-i\sqrt{3}}{2}\right)}$$

has two singularities at the points $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Using the geometric series, we find that the residue at $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$ is

$$\frac{1}{\frac{-1+i\sqrt{3}}{2} - \frac{-1-i\sqrt{3}}{2}} = \frac{1}{i\sqrt{3}} = \frac{-i\sqrt{3}}{3}$$

and the residue at $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$ is

$$\frac{1}{\frac{-1-i\sqrt{3}}{2} - \frac{-1+i\sqrt{3}}{2}} = \frac{1}{-i\sqrt{3}} = \frac{i\sqrt{3}}{3}.$$

▲

Example 3.6.12 Let $f(z)$ have a root of order k and $g(z)$ have a root of order l at z_0, where $f(z)$ and $g(z)$ are both holomorphic functions in some region which contains z_0 as an interior point. Then, there is some $r > 0$ such that

$$f(z) = (z - z_0)^k \, p(z), \quad \forall \ z \in D(z_0, r),$$

with $p(z)$ holomorphic in $D(z_0, r)$ and $p(z_0) \neq 0$. Similarly, there is some $\rho > 0$ such that

$$g(z) = (z - z_0)^l \, q(z), \quad \forall \ z \in D(z_0, \rho),$$

with $q(z)$ holomorphic in $D(z_0, \rho)$ and $q(z_0) \neq 0$.

Then we have the **rule**: At $z = z_0$, the function

$$\frac{f(z)}{g(z)} \quad \text{has} \quad \begin{cases} \text{Regular point (not a root),} & \text{if } k = l. \\ \\ \text{Root of order } k - l, & \text{if } k > l. \\ \\ \text{Isolated singularity, pole of order } m = l - k, & \text{if } k < l. \end{cases}$$

In the latter case, for $\sigma = \min\{r, \rho\}$ we have

$$\frac{f(z)}{g(z)} = \frac{p(z)}{(z - z_0)^{l-k} q(z)}, \quad \forall \ z \in D^o(z_0, \sigma).$$

If $l - k = 1$, that is, z_0 is a simple pole, then

$$b_1 = \operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{p(z_0)}{q(z_0)}.$$

If $l - k > 1$, then by **Example 3.6.10**, the residue is

$$b_1 = \operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{\left[\frac{p(z)}{q(z)}\right]^{(l-k-1)} \Big|_{z=z_0}}{(l - k - 1)!}.$$

In the next section, we develop several ways to compute the residues of functions and of various fractions $\dfrac{f(z)}{g(z)}$ in particular.

▲

3.6.2 Five Ways to Evaluate Residues

Suppose that $w = f(z)$ is a holomorphic function in $D^o(z_0, r)$ for some $r > 0$. Then z_0 is an isolated singularity of $f(z)$. We want to have some convenient ways to compute its residue at z_0, denoted by $\operatorname*{Res}_{z=z_0} f(z)$.

Way 1. We have already seen that if C is any simple, closed, continuous and piecewise continuously differentiable path in $D^o(z_0, r)$ that encloses z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_{C^+} f(z)\, dz.$$

In particular, we can choose C to be any convenient circumference of a circle $C(z_0, \rho)$ with $\rho < r$.

Way 2. If in any way we manage to find the Laurent Series of $f(z)$ with center z_0

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n},\ \ \forall\ z\in D^o(z_0,r),$$

then we readily have

$$\operatorname*{Res}_{z=z_0}f(z)=b_1.$$

Way 3. (**Example 3.6.10** revisited.) Suppose that

$$f(z)=\frac{h(z)}{(z-z_0)^m},\ \ \forall\ z\in D^o(z_0,r),$$

with $h(z)$ holomorphic in all $D(z_0,r)$, $h(z_0)\neq 0$ and m some positive integer (fixed). In this situation, z_0 is a pole of order m. Then we can use the formula:

$$\operatorname*{Res}_{z=z_0}f(z)=\frac{1}{(m-1)!}\cdot\frac{d^{m-1}}{dz^{m-1}}\left[h(z)\right]\Big|_{z=z_0}=$$
$$\frac{1}{(m-1)!}\cdot\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]|_{z=z_0}.$$

This follows from the fact that $h(z)$ can be written as a power series

$$h(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n,\ \ \forall\ z\in D^o(z_0,r),$$

with

$$c_n=\frac{h^{(n)}(z_0)}{n!}.$$

So, we have that

$$\operatorname*{Res}_{z=z_0}f(z)=c_{m-1}=\frac{h^{(m-1)}(z_0)}{(m-1)!}=\frac{1}{(m-1)!}\cdot\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]|_{z=z_0}.$$

Remark: There is no such formula for an isolated essential singularity that it is not a pole! (This is another substantial difference between a pole and an isolated essential singularity which is not a pole!)

Way 4. Sometimes the function $h(z)$ in **Way 3** is not expressed in a convenient closed form, and so the computation of its derivatives is not efficient. Then, to find the $\operatorname*{Res}_{z=z_0}f(z)$ we use a manipulation similar to the one explained in the **following example**. In fact, with this method we compute the whole principal part of a pole $z = z_0$. Many times, in complex analysis, we need the whole principal part and not just the residue of a pole!

Example 3.6.13 We consider the function

$$f(z) = \frac{\cot(z)}{z^2} = \frac{\cos(z)}{z^2 \sin(z)} =$$
$$\frac{\cos(z)}{z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}} = \frac{\cos(z)}{z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}}.$$

Since $\cos(0) = 1$, $f(z)$ has singularity at $z_0 = 0$.

If we set

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n},$$

this power series is convergent $\forall\ z \in \mathbb{C}$ (use, for instance, the **Ratio Test** to prove this), and $g(0) = 1$. So, there is some $\epsilon_1 > 0$ such that $g(z) \neq 0, \forall\ z \in D(0, \epsilon_1)$. Now, to compare with **Way 3**, we let

$$h(z) = \frac{\cos(z)}{g(z)}, \ \ \forall\ z \in D(0, \epsilon_1).$$

We have that $h(0) = \frac{1}{1} = 1 \neq 0$. So, there is $0 < \epsilon \leq \epsilon_1$, such that $h(z) \neq 0$, and it is holomorphic $\forall\ z \in D(0, \epsilon)$. Then in $D(0, \epsilon)$, we can write $h(z)$ as power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \neq 0, \ \ \forall\ z \in D(0, \epsilon),$$

(with $c_0 = h(0) = 1$). Since $f(z) = \frac{h(z)}{z^3}$, we get

$$f(z) = \frac{c_0}{z^3} + \frac{c_1}{z^2} + \frac{c_2}{z} + c_3 + c_4 z + c_5 z^2 + c_6 z^3 + \dots .$$

We rewrite this as

$$f(z) = \frac{b_3}{z^3} + \frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots .$$

We now see that $z_0 = 0$ is a pole of $f(z)$ of order three and we must compute the residue b_1. For this, we work as follows. First, we see that

$$b_3 = z^3 f(z)|_{z=0} = \frac{z\cos(z)}{\sin(z)}|_{z=0} = \frac{0}{0} =?.$$

At this point, we use the **complex L' Hôpital's rule** that is applied in exactly the same way as in the real case of an **indeterminate limit of the form** $\frac{0}{0}$, that we know from calculus. **(Remember to simplify the expressions to the fullest every time you apply L' Hôpital's rule).** So,

$$b_3 = \frac{[z\cos(z)]'}{[\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z)}{\cos(z)}\Big|_{z=0} = \frac{1-0}{1} = 1.$$

Then,

$$b_2 = z^2\left[f(z) - \frac{1}{z^3}\right]\Big|_{z=0} =$$
$$\left(\frac{\cos(z)}{\sin(z)} - \frac{1}{z}\right)\Big|_{z=0} = \frac{z\cos(z) - \sin(z)}{z\sin(z)}\Big|_{z=0} =$$
$$\frac{[z\cos(z) - \sin(z)]'}{[z\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z) - \cos(z)}{\sin(z) + z\cos(z)}\Big|_{z=0} =$$
$$\frac{[-z\sin(z)]'}{[\sin(z) + z\cos(z)]'}\Big|_{z=0} = \frac{-\sin(z) - z\cos(z)}{\cos(z) + \cos(z) - z\sin(z)}\Big|_{z=0} = \frac{0}{2} = 0.$$

Finally,

$$b_1 = z\left[f(z) - \frac{1}{z^3} - \frac{0}{z^2}\right]\Big|_{z=0} =$$
$$\left(\frac{\cos(z)}{z\sin(z)} - \frac{1}{z^2}\right)\Big|_{z=0} = \frac{z\cos(z) - \sin(z)}{z^2\sin(z)}\Big|_{z=0} =$$
$$\frac{[z\cos(z) - \sin(z)]'}{[z^2\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z) - \cos(z)}{2z\sin(z) + z^2\cos(z)}\Big|_{z=0} =$$
$$\frac{[-\sin(z)]'}{[2\sin(z) + z\cos(z)]'}\Big|_{z=0} = \frac{-\cos(z)}{2\cos(z) + \cos(z) - z\sin(z)}\Big|_{z=0} = \frac{-1}{3}.$$

So,

$$\operatorname*{Res}_{z=0} f(z) = \frac{-1}{3}.$$

By analyzing the method of this rather complicated example, we could write a general algorithm for finding the residues in such cases.

▲

Way 5. Here we redo the **previous example** in a way that combines the initial part of **Way 4** and the formula of **Way 3**. This way is also convenient in complicated cases.

So, we reconsider the **previous example**. There, we came up with

$$f(z) = \frac{\cot(z)}{z^2} = \frac{1}{z^3}\frac{\psi(z)}{g(z)},$$

where $\psi(z) = \cos(z)$ and, as before,

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \dots .$$

Then $\psi(0) = \cos(0) = 1$, $\psi'(0) = -\sin(0) = 0$, $\psi''(0) = -\cos(0) = -1$, etc., and $g(0) = 1$, $g'(0) = 0$, $g''(0) = \frac{-2}{6} = \frac{-1}{3}$, etc.

So, we conclude that $z = 0$ is a pole of order three and then, according to the formula in **Way 3**, we get:

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(3-1)!} \left[z^3 f(z)\right]'' \big|_{z=0} = \frac{1}{2!} \left[\frac{\psi(z)}{g(z)}\right]'' \big|_{z=0} =$$

$$\frac{1}{2} \left[\psi(z) g^{-1}(z)\right]'' \big|_{z=0} = \frac{1}{2} \left[\psi''(z) g^{-1}(z) - 2\psi'(z) g^{-2}(z) g'(z) + \right.$$

$$\left. 2\psi(z) g^{-3}(z) (g')^2(z) - \psi(z) g^{-2}(z) g''(z)\right] \big|_{z=0} = \frac{1}{2} \left[\psi''(0) g^{-1}(0) - \right.$$

$$\left. 2\psi'(0) g^{-2}(0) g'(0) + 2\psi(0) g^{-3}(0) (g')^2(0) - \psi(0) g^{-2}(0) g''(0)\right] =$$

$$\frac{1}{2} \left[-1 \cdot 1 - 2 \cdot 0 \cdot 1 \cdot 0 + 2 \cdot 1 \cdot 1 \cdot 0 - 1 \cdot 1 \cdot \left(\frac{-1}{3}\right)\right] =$$

$$\frac{1}{2} \left(-1 + \frac{1}{3}\right) = -\frac{1}{3}.$$

Long Division of Power Series and Residues

The **example** of computing the residue of

$$f(z) = \frac{\cot(z)}{z^2} = \frac{\cos(z)}{z^2 \sin(z)} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{\displaystyle z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+3}}$$

at the pole $z = 0$, treated in **Ways 4** and **5** before, can be dealt with faster with the long division of power series.

We perform a few steps of the long division of the two Maclaurin series written in increasing order of the powers of z

$$f(z) = \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots}{z^3 - \frac{z^5}{6} + \frac{z^7}{120} - \dots},$$

using the usual scheme that we use in a calculus or analysis course and in applications. For this particular example, we need three quick steps, and we find that the initial five terms of the quotient are

$$z^{-3} - \frac{1}{3} z^{-1} - \frac{1}{45} z \dots .$$

(The terms that do not appear have coefficients equal to zero. Check this for yourselves.)

Faster than in **Ways 4** and **5** above, we find again that the whole **principal part of the pole** is

$$z^{-3} + 0z^{-2} - \frac{1}{3}z^{-1}$$

and

$$\operatorname*{Res}_{z=0} f(z) = \frac{-1}{3}.$$

Leibniz Rule for Higher Derivatives of Products

To compute higher derivatives of products, something that appears often while computing residues, we find it convenient to use the **Leibniz rule for higher derivatives of products**. This is stated as follows:

For any two differentiable functions $f(z)$ and $g(z)$ with high order derivatives, the n$^{\text{th}}$ derivative of their product is

$$\frac{d^n}{dz^n}[f(z) \cdot g(z)] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}[f(z)] \cdot \frac{d^k}{dz^k}[g(z)], \tag{3.16}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We remember that $0! = 1$ and so $\binom{n}{0} = 1$. For $k \in \mathbb{N}$, we simplify

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 3 \cdot 2 \cdot 1}.$$

This rule is easily proven by mathematical induction.

Examples

Example 3.6.14 For $a,\ b,\ c,\ d \in \mathbb{C}$ constants and $n,\ l,\ m, \in \mathbb{N}$, by the Leibniz rule for derivatives of products, we have

$$\frac{d^n}{dz^n}[(az+b)^l \cdot (cz+d)^m] =$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}[(az+b)^l] \cdot \frac{d^k}{dz^k}[(cz+d)^m] =$$

$$\sum_{k=0}^{n} \binom{n}{k} l\,(l-1)\dots(l-n+k+1)a^{n-k}(az+b)^{l-n+k} \ \times$$

$$m(m-1)\dots(m-k+1)c^k(cz+d)^{m-k}.$$

Notice that if p and q are in $\mathbb{N}$, then the product $p\,(p-1)\ldots(p-q+1)$ will be zero for all $q \geq p+1$.

▲

Example 3.6.15 For $f(z) = \dfrac{z+1}{z-2}$, find the $\operatorname*{Res}_{z=2} f(z)$.

(a) Let $C = C^{+}(2, r)$ be the positively oriented circle of center $z_0 = 2$ and radius some $r > 0$. Then

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{z+1}{z-2}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{2+re^{i\theta}+1}{2+re^{i\theta}-2} rie^{i\theta}\,d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (3 + re^{i\theta})\,d\theta = \frac{1}{2\pi}(3\cdot 2\pi + 0) = 3.$$

(b) We have $f(z) = \dfrac{z+1}{z-2} = \dfrac{z-2+3}{z-2} = 1 + \dfrac{3}{z-2}$. So,

$$\operatorname*{Res}_{z=2} f(z) = b_1 = 3.$$

(c) By **Example 3.6.12**, $z = 2$ is a pole of order one, and so we can apply the formula in **Way 3** to find

$$b_1 = (z-2)f(z)|_{z=2} = (z+1)|_{z=2} = 2+1 = 3.$$

▲

Example 3.6.16 Find the

$$\operatorname*{Res}_{z=2}\left[\frac{z^2-2z+5}{(z-2)^3}\right].$$

We see that $z = 2$ is an isolated singularity for the given rational function (fraction of two polynomials), since at $z = 2$ the numerator is $2^2 - 2\cdot 2 + 5 = 5 \neq 0$. We can use any of the ways of the previous example to find that

$$\operatorname*{Res}_{z=2}\left[\frac{z^2-2z+5}{(z-2)^3}\right] = b_1 = 1.$$

For instance,

$$f(z) = \frac{z^2-2z+5}{(z-2)^3} = \frac{z^2-4z+4+2z-4+5}{(z-2)^3} =$$

$$\frac{(z-2)^2+2(z-2)+5}{(z-2)^3} = \frac{1}{z-2} + \frac{2}{(z-2)^2} + \frac{5}{(z-2)^3}.$$

So,

$$b_1 = 1.$$

Or, by **Example 3.6.12**, $z = 2$ is a pole of order three, and so we can apply the formula in **Way 3** to find

$$b_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-2)^3 f(z)\right]\Big|_{z=2} = \frac{1}{2}\frac{d^2}{dz^2}(z^2-2z+5)|_{z=2} = \frac{1}{2}\cdot(2) = 1.$$

▲

Example 3.6.17 Find the

$$\operatorname*{Res}_{z=1}\left[\frac{e^z}{(z-1)^2}\right].$$

Since $e^1 = e \neq 0$, by **Example 3.6.12**, $z = 1$ is a pole of order two, and so we can apply the formula in **Way 3** to find the residue

$$\operatorname*{Res}_{z=1}\left[\frac{e^z}{(z-1)^2}\right] = \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{e^z}{(z-1)^2}\right]\Big|_{z=1} = 1\cdot(e^z)'\Big|_{z=1} = e.$$

Also, using the power series expansion of e^z about $z = 1$, we get

$$e^z = e\cdot e^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}.$$

Therefore,

$$f(z) = e\sum_{n=0}^{\infty}\frac{(z-1)^{n-2}}{n!} = e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \sum_{n=2}^{\infty}\frac{(z-1)^{n-2}}{n!}\right].$$

So,

$$\operatorname*{Res}_{z=1}\left[\frac{e^z}{(z-1)^2}\right] = e\cdot 1 = e.$$

▲

Example 3.6.18 Find the residue of

$$f(z) = \exp\left(z + \frac{1}{z}\right)$$

at $z = 0$.

We have that

$$\exp\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(z+\frac{1}{z}\right)^n =$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,z^k\frac{1}{z^{n-k}}\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\,z^{2k-n}\right].$$

We observe that $z = 0$ is an essential singularity which is not a pole, since there are infinitely many negative exponents. In this example, the sought residue is the total coefficient of $z^{-1} = \frac{1}{z}$.

A non-zero coefficient of $\frac{1}{z}$ is achieved whenever $2k - n = -1$, or $n = 2k + 1$, i.e., whenever n is odd. If $n = 2k + 1$, for $k = 0,\ 1,\ 2, \ldots,$ the corresponding summand of the coefficient of $\frac{1}{z}$ is written as

$$\frac{1}{k!(n-k)!} = \frac{1}{k!(2k+1-k)!} = \frac{1}{k!(k+1)!} = \frac{1}{(k!)^2(k+1)}.$$

By adding all of these summands for $k = 0,\ 1,\ 2, \ldots$, we find

$$\operatorname*{Res}_{z=0}\ \exp\left(z + \frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = \sum_{k=0}^{\infty} \frac{1}{(k!)^2(k+1)}.$$

▲

Example 3.6.19 For any $a \in R$ constant, find the residues of

$$f(z) = \sin\left(\frac{1}{z}\right) \cdot e^{az}$$

at $z = 0$.

We have that for $z \neq 0$

$$f(z) = \left(\frac{1}{z} - \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{5!} \cdot \frac{1}{z^5} - \ldots\right) \times$$
$$\left[1 + az + \frac{1}{2!}(az)^2 + \frac{1}{3!}(az)^3 + \frac{1}{4!}(az)^4 + \ldots\right].$$

So, we observe that we can get $\frac{1}{z}$ in infinitely many ways which give as its coefficient

$$\operatorname*{Res}_{z=0} f(z) = 1 - \frac{a^2}{2! \cdot 3!} + \frac{a^4}{4! \cdot 5!} - \frac{a^6}{6! \cdot 7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n\, a^{2n}}{(2n)! \cdot (2n+1)!}.$$

▲

Example 3.6.20 Find the residues of

$$f(z) = \frac{z-2}{z^2(z-1)}$$

at $z = 0$ and $z = 1$.

By **Example 3.6.12**, $z = 0$ is a pole of order two and $z = 1$ is a pole of order one. So, we can apply the formula in **Way 3** to find

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\frac{d}{dz}\left[z^2 f(z)\right]\big|_{z=0} = \frac{d}{dz}\left(\frac{z-2}{z-1}\right)\big|_{z=0} = \frac{(z-1)-(z-2)}{(z-1)^2}\Big|_{z=0} = \frac{1}{1} = 1,$$

and similarly

$$\operatorname*{Res}_{z=1} f(z) = \left[(z-1)f(z)\right]\big|_{z=1} = \frac{z-2}{z^2}\Big|_{z=1} = \frac{-1}{1} = -1.$$

▲

Example 3.6.21 Find the residues of

$$f(z) = \frac{1}{(1+z^2)^{n+1}}$$

at $z = -i$ and $z = +i$, for $n = 0, 1, 2, 3, \ldots$.

Since

$$f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z-i)^{n+1}(z+i)^{n+1}},$$

we have that $z = -i$ and $z = +i$ are both poles of order $n+1$. Then,

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left[(z-i)^{n+1}f(z)\right]\big|_{z=i} = \frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{1}{(z+i)^{n+1}}\right]\big|_{z=i} = \left[\frac{(-1)^n(n+1)(n+2)\ldots(2n)}{n!}\cdot\frac{1}{(z+i)^{2n+1}}\right]\big|_{z=i} = \frac{(-1)^n(n+1)(n+2)\ldots(2n)}{n!}\cdot\frac{1}{(2i)^{2n+1}} = \frac{(-1)^n 1\cdot 2\ldots n(n+1)(n+2)\ldots(2n)}{(n!)^2\,2^{2n+1}\,i^{2n}\,i} = \frac{(2n)!}{(n!)^2\,2^{2n+1}\,i} = \frac{-i(2n)!}{(n!)^2\,2^{2n+1}}.$$

Similarly,

$$\operatorname*{Res}_{z=-i} f(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left[(z+i)^{n+1}f(z)\right]\big|_{z=-i} = \frac{(-1)^n(2n)!}{(n!)^2\,2^{2n+1}\,(-i)^{2n}(-i)} = \frac{-(2n)!}{(n!)^2\,2^{2n+1}\,i} = \frac{i(2n)!}{(n!)^2\,2^{2n+1}}.$$

▲

Example 3.6.22 Find the residues of

$$f(z) = \frac{z^{2m}}{1+z^{2n}}$$

at the roots of the denominator. Here $m,\ n = 1, 2, 3, \ldots$.

The roots of the denominator are the complex numbers that satisfy the equation

$$z^{2n} = -1 = e^{\pi i}.$$

So, these roots are the $2n$ $(2\mathrm{n})^{\mathrm{th}}$ roots of -1

$$z_k = e^{i\frac{(2k+1)\pi}{2n}}, \quad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ \ldots,\ 2n-1.$$

These roots are **simple**, that is, of order one, and so they are simple poles for the given function $f(z)$. Thus, for $k = 0,\ 1,\ 2,\ 3, \ldots,\ 2n-1$ we have

$$\operatorname*{Res}_{z=z_k} f(z) = [(z-z_k)f(z)]\,\big|_{z=z_k} = \left[\frac{(z-z_k)z^{2m}}{1+z^{2n}}\right]\big|_{z=z_k} =$$

$$\text{(use L' Hôpital's rule)} \quad \frac{z^{2m} + (z-z_k)2mz^{2m-1}}{2nz^{2n-1}}\Big|_{z=z_k} =$$

$$\frac{z_k^{2m}}{2nz_k^{2n-1}} = \left(\text{remember } z_k^{2n} = -1\right) \frac{-1}{2n} z_k^{2m+1} = \frac{-1}{2n} e^{i\frac{(2k+1)(2m+1)}{2n}\pi}.$$

(Notice how we use the **complex L' Hôpital's rule** in this example and imitate it in similar situations.)

▲

Problems

3.6.1 (a) Find the roots of $\sin(z)$ and their orders.

(b) Explain why $\sin(z)$ cannot be equal to $f^2(z)$ for some holomorphic function $f\ :\ \mathbb{C} \longrightarrow \mathbb{C}$.

(c) Do **(a)** and **(b)** for $\cos(z)$.

3.6.2 What kind of point is $z = 0$ for each function?

$$f(z) = \frac{\sin^5(z)}{z^2},$$

$$g(z) = \frac{\sin^2(z)}{z^5},$$

$$h(z) = \frac{\sin^5(z)}{z^5}.$$

[Hint: Find $\lim\limits_{z\to 0} \dfrac{\sin^5(z)}{z^5}$.]

In **problems 3-15** that follow, **compute the residues** of the given functions at the given **singular points**. First, justify why the given points are **isolated singularities** of the given functions. Otherwise, justify why they are **non-isolated singularities**.

3.6.3

$$f_1(z) = \frac{z+1}{(z-1)^3(z-4)}, \quad \text{at} \quad z_0 = 1 \quad \text{and} \quad z_0 = 4.$$

3.6.4

$$f_2(z) = \frac{1}{z^2+2z+2},$$

at the two roots of the denominator. (Find these roots first.)

3.6.5

$$f_4(z) = \frac{1}{e^z - 1},$$

at the infinitely many roots of the denominator. Find these roots and their orders first, then justify why they are isolated singularities and compute the corresponding residues.

3.6.6

$$f_3(z) = \frac{1}{\sin\left(\frac{1}{z^2}\right)}, \quad \text{at} \quad z_0 = 0.$$

3.6.7

$$f_5(z) = \frac{z^2}{1+z^4}, \quad \text{at all roots of the denominator.}$$

(Find these roots first. In exponential form they are more convenient.)

3.6.8

$$f_6(z) = \frac{1}{1+z^5}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

3.6.9

$$f_7(z) = \frac{1}{1+z^{10}}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

3.6.10

$$f_8(z) = \frac{1}{z^2\sin(z)}, \quad \text{at} \quad z_0 = 0.$$

3.6.11

$$f_9(z) = \frac{e^z - 1}{\sin^3(z)}, \quad \text{at} \quad z_0 = 0.$$

3.6.12

$$f_{10}(z) = \frac{1}{z^5 - 1}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

3.6.13

$$f_{11}(z) = \frac{z^2}{z^4 - 1}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

3.6.14

$$f_{12}(z) = \frac{\cot(z)}{z} = \frac{\cos(z)}{z\sin(z)}, \quad \text{at all roots of the denominator.}$$

3.6.15

$$f_{13}(z) = \frac{\cot(z)}{z^3} = \frac{\cos(z)}{z^3\sin(z)}, \quad \text{at all roots of the denominator.}$$

3.6.16 Prove:

(a) $\lim\limits_{z\to\infty} e^{-z} = \lim\limits_{z\to\infty} \dfrac{1}{e^z}$ does not exist.

(b) $\forall\ a \in \mathbb{C},\ \lim\limits_{z\to\infty} \dfrac{z}{z^2 \pm a} = 0.$

3.6.17 (a) Prove the equalities

$$\left[z+\sqrt{z^2-1}\,\frac{w+\frac{1}{w}}{2}\right]^n\frac{1}{iw}=\frac{\left(\sqrt{z^2-1}\,w^2+2zw+\sqrt{z^2-1}\,\right)^n}{i2^nw^{n+1}}=$$

$$\begin{cases}\dfrac{\left[\sqrt{z^2-1}\left(w+\dfrac{z+1}{\sqrt{z^2-1}}\right)\left(w+\dfrac{z-1}{\sqrt{z^2-1}}\right)\right]^n}{i\,2^n\,w^{n+1}}, & \text{if} \quad z\neq\pm1\\[2ex] \dfrac{1}{iw}, & \text{if} \quad z=1\\[2ex] \dfrac{(-1)^n}{iw}, & \text{if} \quad z=-1.\end{cases}$$

(b) Consider z as a constant and w as a variable in the expression in **(a)**. Find the order of the pole $w=0$ and compute the corresponding residue in each of the three cases.

3.6.18 Use mathematical induction to prove the **Leibniz rule for higher derivatives of products (3.16)**.

3.6.19 Consider the function

$$f(z)=\frac{1}{1-e^{\frac{1}{z}}}$$

in $\mathbb{C}$.

(a) Show that $f(z)$ has singularities at the points $z_0=0$ and $z_n=\dfrac{1}{2n\pi i}$ with $n=\pm1,\ \pm2,\ \pm3,\ \dots$.

(b) Show that $z_0=0$ is a non-isolated singularity and all the other singularities are isolated.

(c) Compute the residues at the isolated singularities.

3.6.20 Study the example treated in "**Way 4**" of evaluating residues and write a general algorithm that always computes the residues when the singularity is a pole of order m, in the way described there.

3.6.21 Let Log(z) (with capital L) mean the holomorphic branch of the complex $\log(z)$ defined in the domain

$$D:=\mathbb{C}-\{x\mid x\leq 0\}=\{\,(r,\,\theta)\mid r\geq 0,\ \text{and}\ -\pi<\theta<\pi\,\}.$$

(a) Prove that the composite complex logarithmic function

$$h(z):=\mathrm{Log}(1-z)=\mathrm{Log}(1-x-iy)$$

is defined in the domain

$$D := \mathbb{C} - \{x \mid 1 - x \leq 0\} = \mathbb{C} - \{x \mid x \geq 1\}$$

and is holomorphic.

(b) Show that in this domain D

$$\operatorname{Re} h(z) = \ln |1 - z| \quad \text{and} \quad -\pi < \operatorname{Im} h(z) < \pi.$$

(c) Now consider the function $g(z) := \dfrac{h(z)}{z}$ in this domain D and prove that it has a removable singularity at $z = 0$ and non-isolated singularities at all real $x \geq 1$.

(d) What can you say about the residues of $g(z)$ at these singularities?

3.6.22 On the basis of what you know, prove the following result:

Let $r > 0$, $c \in \mathbb{C}$ and $f : D(c, r) \longrightarrow \mathbb{C}$ be a holomorphic function. There is an infinite sequence of complex numbers (z_n), $n \in \mathbb{N}$, in $D(c, r)$ such that $f(z_n) = 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} z_n = c$ if and only if $f(z) = 0$ for all $z \in D(c, r)$.

3.7 Contour Integration and Integrals

In this section, we will integrate holomorphic (complex analytic) functions along simple, closed, continuous and piecewise continuously differentiable paths. From now on, we shall call such paths **contours**.

3.7.1 Residue Theorem and Examples

The main tool of integration in the sequel is the following version of a generalization of **Theorem 3.5.6**. (In more advanced literature, we can find more general settings of this Theorem. For our level and purposes, the one presented here is enough.)

Theorem 3.7.1 (Residue Theorem) *Let $R \subseteq \mathbb{C}$ be an open region and $w = f(z)$ be a complex holomorphic function in $R - \{z_1, z_2, z_3, ..., z_n, \ldots, z_l\}$, where $1 \leq n \leq l$ and each point of the finite exceptional set $\{z_1, z_2, z_3, ..., z_l\}$ is an isolated singularity of $f(z)$.*

Then, for any C simple, closed and piecewise continuously differentiable contour such that no singularity of $f(z)$ is on C and C encloses the isolated singularities z_1, z_2, z_3, ..., z_n but no other singularity, then the following integral equality holds:

$$\oint_{C^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

(If $l > n$, the singularities $\{z_{l+1}, \ldots, z_l\}$ lie outside C and play no role in this result!)

Remark: If $n = 0$, then the empty summation is equal to zero, and we are in the case of the **Cauchy-Goursat Theorem, 3.5.3.**

In advanced literature, we find more general settings of this important Theorem, in which the contour may not be simple, and apart from the singularities the domain may have other holes within the contour, etc. We refer the interested reader to advanced bibliography on complex analysis. For our level and purposes, the version of this Theorem presented here is enough.

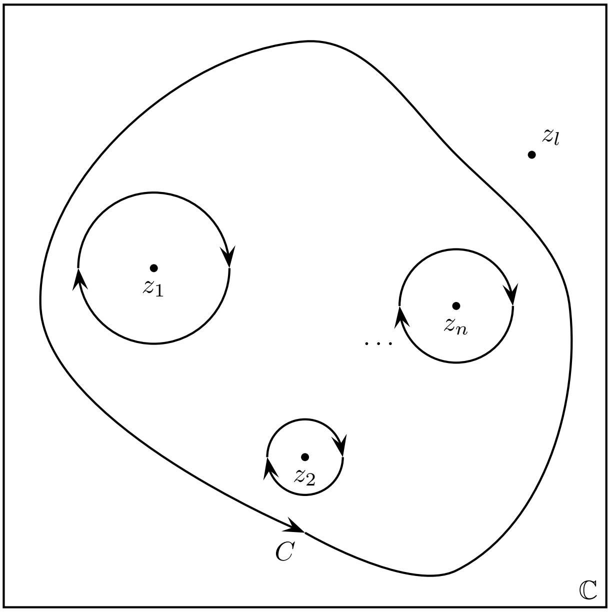

FIGURE 3.4: Contour and discs

Proof (Refer to **Figure 3.4**) We consider discs $D(z_k, r_k)$, where $r_k > 0$, for $k = 1,\ 2,\ 3,\ ...,\ n$ small enough to lie in the inside of C and to be pairwise disjoint. Then, we apply Green's Theorem to the function $f(z)$ in the domain

$$\left\{ (\text{inside of } C) - \bigcup_{k=1}^{n} D(z_k, r_k) \right\} \subset R \subseteq \mathbb{C}$$

in the same way as we did in **Theorem 3.5.6**. That Theorem dealt with only one singularity in the whole $\mathbb{C}$. Now, there may be any finite number of singularities in the region $R \subseteq \mathbb{C}$. These generalizations do not change the course of the proof, and the result follows in the same way as it did then. (Study that proof one more time.)

We continue with the following definition:

Definition 3.7.1 *Let $w = f(z)$ be a complex function of $z \in \mathbb{C}$ holomorphic in $\mathbb{C} - \overline{D(0,R)} = \{z \in C \quad | \quad |z| > R\}$ [the complement of the closed disc $\overline{D(0,R)}$], for some $R > 0$. We call* ***residue at infinity*** *of $w = f(z)$ the quantity $\underset{z=\infty}{Res}\, f(z) := -\underset{z=0}{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$.*

The following theorem is interesting but of little use to applications:

Theorem 3.7.2 *Let $f(z)$ be a holomorphic function in $\mathbb{C} - \{a_1,\ a_2,\ \ldots,\ a_n\}$, where $n \geq 0$ integer, with singularities (isolated) at the points $a_1,\ a_2,\ \ldots,\ a_n$. Then,*

$$\underset{z=a_1}{Res}\, f(z) + \underset{z=a_2}{Res}\, f(z) + \ldots + \underset{z=a_n}{Res}\, f(z) + \underset{z=\infty}{Res}\, f(z) = 0.$$

Proof We consider any $R > 0$ for which $|a_i| < R$, for all $i = 1,\ 2,\ \ldots,\ n$ and $|a_i| > \frac{1}{R}$, for all $a_i \neq 0$. Then, by **Definition 3.6.5** or **Subsection 3.6.2, Way 1**, we have

$$\underset{z=\infty}{Res}\, f(z) := -\underset{z=0}{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{1}{2\pi i}\int_{C^+\left(0,\frac{1}{R}\right)} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz.$$

We use the change of variables $w = \frac{1}{z}$, and we find

$$\underset{z=\infty}{Res}\, f(z) = \frac{1}{2\pi i}\int_{C^-(0,R)} f(w)dw = -\frac{1}{2\pi i}\int_{C^+(0,R)} f(w)\, dw =$$
$$-[\underset{z=a_1}{Res}\, f(z) + \underset{z=a_2}{Res}\, f(z) + \ldots + \underset{z=a_n}{Res}\, f(z)],$$

and the result follows.

Corollary 3.7.1 *The residue at infinity of an entire function, in particular of a polynomial, is zero.*

Examples

Example 3.7.1 Find the integral

$$\oint\limits_{C^+} \frac{5z-3}{z(z-2)}\,dz$$

if: (a) $C := C(0,1)$, (b) $C := C(0,3)$ and (c) $C := C(4,1)$.

First, we compute the residues of the function $f(z) = \dfrac{5z-3}{z(z-2)}$ at the isolated singularities $z_0 = 0$ and $z_0 = 2$. At all other points of $\mathbb{C}$, this function is defined and holomorphic.

$$\operatorname*{Res}_{z=0} f(z) = [zf(z)]|_{z=0} = \frac{5z-3}{z-2}\Big|_{z=0} = \frac{-3}{-2} = \frac{3}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = [(z-2)f(z)]|_{z=2} = \frac{5z-3}{z}\Big|_{z=2} = \frac{7}{2}.$$

So, by the **Residue Theorem, 3.7.1**, we have:
(a) Since $z = 0$ is the only singularity inside $C := C(0,1)$, then

$$\oint\limits_{C^+(0,1)} \frac{5z-3}{z(z-2)}\,dz = 2\pi i \cdot \frac{3}{2} = 3\pi i.$$

(b) Now, both singularities are inside $C := C(0,3)$, and so

$$\oint\limits_{C^+(0,3)} \frac{5z-3}{z(z-2)}\,dz = 2\pi i\left(\frac{3}{2} + \frac{7}{2}\right) = 10\pi i.$$

(c) Finally, inside $C := C(4,1)$ there are no singularities of $f(z)$. Therefore, the line integral is zero, i.e.,

$$\oint\limits_{C^+(4,1)} \frac{5z-3}{z(z-2)}\,dz = 0.$$

Remark: If we parametrize each of these C's and then separate the real and imaginary parts of each complex line integral, we obtain two real integrals in each case.

▲

Example 3.7.2 Evaluate the integral

$$\oint\limits_{C^+} \frac{dz}{e^z - 1}$$

if $C = C(0, 3\pi)$. Then, compute the two real integrals resulting from it by using the parametrization of $C = C(0, 3\pi)$ given by $z = 3\pi e^{i\theta}$ with $0 \leq \theta \leq 2\pi$.

Inside $C = C(0, 3\pi)$, the function $f(z) = \dfrac{dz}{e^z - 1}$ has three singularities, namely $-2\pi i$, 0 and $2\pi i$, which are simple roots of the denominator. Using one of the methods, that we have seen, we find that each of the three residues is equal to 1. E.g.,

$$\operatorname*{Res}_{z=-2\pi i} f(z) = \{[z - (-2\pi i)]f(z)\}\big|_{z=-2\pi i} = \frac{z + 2\pi i}{e^z - 1}\Big|_{z=-2\pi i} =$$

$$\text{(use L' Hôpital's Rule)} \quad \frac{1}{e^z}\Big|_{z=-2\pi i} = \frac{1}{e^{-2\pi i}} = \frac{1}{1} = 1.$$

So,

$$\oint\limits_{C+} \frac{dz}{e^z - 1} = 2\pi i(1 + 1 + 1) = 6\pi i.$$

From this integral, we can now evaluate two real ones, if we use the parametrization $z = 3\pi e^{i\theta}$, with $0 \leq \theta \leq 2\pi$, of $C = C(0, 3\pi)$ and then separate the real and imaginary parts. So,

$$\int_0^{2\pi} \frac{3\pi i e^{i\theta} d\theta}{e^{3\pi e^{i\theta}} - 1} = 6\pi i, \ \text{ or } \ \int_0^{2\pi} \frac{e^{i\theta} d\theta}{e^{3\pi e^{i\theta}} - 1} = 2.$$

Now we replace $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and separate real and imaginary parts to get

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\cos\left[\theta - 3\pi\sin(\theta)\right] - \cos(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right] + e^{6\pi\cos(\theta)}}\, d\theta = 2$$

and

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\sin\left[\theta - 3\pi\sin(\theta)\right] - \sin(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right] + e^{6\pi\cos(\theta)}}\, d\theta = 0.$$

(Who could have imagined these real integrals?)

▲

Example 3.7.3 Any polynomial $P(z) = a_n z^n + a_{n-1} z^1 + \ldots + a_1 z + a_0$, with $n \geq 0$ integer, has $\operatorname*{Res}_{z=\infty} P(z) = 0$.

This follows from **Theorem 3.7.2**, since any polynomial has no singularities.

But, we can also see it directly from **Definitions 3.6.5** and **3.7.1**,

since

$$\operatorname*{Res}_{z=\infty} P(z) := -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\,P\left(\frac{1}{z}\right)\right] =$$
$$-\operatorname*{Res}_{z=0}\frac{1}{z^2}\left(a_0+\frac{a_1}{z}+\ldots+\frac{a_{n-1}}{z^{n-1}}+\frac{a_n}{z^n}\right)=$$
$$\operatorname*{Res}_{z=0}\left(\frac{0}{z}-\frac{a_0}{z^2}-\frac{a_1}{z^3}-\ldots-\frac{a_{n-1}}{z^{n+1}}-\frac{a_n}{z^{n+2}}\right)=0.$$

▲

Example 3.7.4 Show that for $m \geq 1$ integer and $a \in \mathbb{C}$ constant

$$\operatorname*{Res}_{z=\infty}\left[\frac{1}{(z-a)^m}\right] = \begin{cases} -1, & \text{if} \quad m=1 \\ \\ 0, & \text{if} \quad m \geq 2. \end{cases}$$

This follows by direct computation of the residue of

$$\frac{-1}{z^2}\,\frac{1}{\left(\frac{1}{z}-a\right)^m} = \frac{-z^{m-2}}{(1-az)^m}$$

at $z=0$. This is -1 when $m=1$ and 0 when $m \geq 2$. (Check this. This result follows also from **Theorem 3.7.2**.)

▲

Problems

3.7.1 Evaluate

$$\oint_{C^+} \frac{dz}{z^2+2z+2}$$

if: (a) $C = C(0,1)$, (b) $C = C(0,5)$ and (c) $C = C(100,1)$.

3.7.2 Evaluate

$$\oint_{C^+} \frac{dz}{(z-1)^3(z-4)} \quad \text{and} \quad \oint_{C^+} \frac{z-2}{z^2(z-1)}\,dz$$

if the simple closed contour C is: (a) $C = C(0,2)$, (b) $C = C(0,5)$ and (c) $C = C\left(3, \dfrac{3}{2}\right)$.

3.7.3 Evaluate

$$\oint_{C^+} \frac{e^z}{z}\,dz \quad \text{and} \quad \oint_{C^+} \frac{e^z}{z-1}\,dz$$

when $C = C(0,2)$. Then, obtain two real integrals from each of these complex line integrals.

3.7.4 If $C = C(0,3)$, evaluate

$$\oint_{C^+} \frac{e^z}{(z-2)(z+4)}\,dz \quad \text{and} \quad \oint_{C^+} \frac{\sin(z)}{z^2}\,dz.$$

3.7.5 Evaluate

$$\oint_{C^+} \frac{e^{z^2}}{z-2}\,dz \quad \text{and} \quad \oint_{C^+} \frac{e^z}{z-2}\,dz$$

when C is the boundary of the rectangle $\{z = x+iy \mid 0 \leq x \leq 4,\ -2 \leq y \leq 2\}$. Then, find the two corresponding real integrals.

3.7.6 If $C = C(0,4)$, evaluate

$$\oint_{C^+} \frac{dz}{\sin(z)} \quad \text{and} \quad \oint_{C^+} \frac{dz}{\sinh(z)}.$$

3.7.2 Contour Integration and Improper Real Integrals

We are going to use integrations of complex functions along appropriately chosen contours to evaluate improper real integrals and integrals of Fourier type. This method is very powerful, for it computes very difficult integrals and at the same time proves their existence. Choosing the correct contour(s) and then applying the **Residue Theorem, 3.7.1**, is an art that takes some experience. Then we take the appropriate limits to obtain the real improper integral. We are going to analyze the most important cases of such integral techniques which are sufficient for the needs of an undergraduate student. We also present in detail several examples in order to practice and to get a feeling of what is going on. Not all possible cases, encountered in the bibliography, can be presented in this exposition. Some of them are above the level of this text and they require a higher level course of complex analysis. Several useful lemmata, depending on the particular cases, will be proven and used in order to speed up the various methods of the evaluations of the respective integrals.

Example 3.7.5 The integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan(\infty) - \arctan(-\infty) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$$

has been computed elementarily. For easy practice, we will use contour integration to establish this result.

To this end, we consider the complex function $f(z) = \dfrac{1}{1+z^2}$ in $\mathbb{C}$. The denominator has two simple roots, the $+i$ and the $-i$, which are isolated singularities, poles of order one. So,

$$\operatorname*{Res}_{z=i} f(z) = [(z-i)f(z)]\big|_{z=i} = (z-i)\frac{1}{(z-i)(z+i)}\Big|_{z=i} = \frac{1}{2i}.$$

We consider any $R > 1$ and the contour $C = [-R, R] + S_R^+$ consisting of two parts: (1) the straight segment of the x-axis $[-R, R]$ from $-R$ to R, and (2) the positively oriented upper half of $C(0, R)$, denoted by S_R^+. See **Figure 3.5**. These parts are respectively parametrized by:

(1) $\{z = x + 0i \mid -R \leq x \leq R\}$ and (2) $\{z = Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$. (From now on, we will use these notations in similar situations.)

We have chosen the contour C in this way, so that at least one of the singularities, namely the $z = +i$, is enclosed in it. Then, we apply **the Residue Theorem, 3.7.1**, to find

$$\oint_{C^+} \frac{dz}{z^2+1} = 2\pi i \frac{1}{2i} = \pi.$$

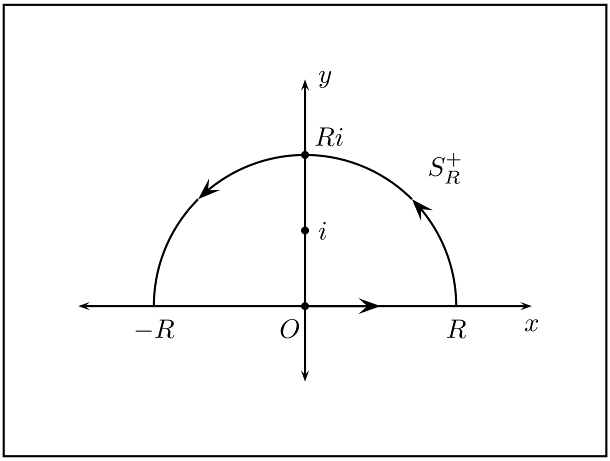

FIGURE 3.5: Contour 1 for Example 3.7.5

But, then

$$\pi = \oint\limits_{C^+} \frac{dz}{z^2+1} = \oint\limits_{[-R,R]} \frac{dz}{z^2+1} + \oint\limits_{S_R^+} \frac{dz}{z^2+1} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}.$$

This equality is valid for all $R > 1$.

Then, we let $R \to \infty$. In this limit process, the constant π does not change, and the two partial integrals become:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

and

$$\lim_{R\to\infty} \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} = 0.$$

The latter limit needs a proof of course. Such a proof is the following. The inequality

$$0 \leq \left| \int_C f(z)dz \right| \leq \int_C |f(z)|\,|dz|$$

for the complex variable $z \in \mathbb{C}$ and C any path in $\mathbb{C}$ is always valid. (See **property (5)** of the complex line integral in **Section 3.4**.) Using this property and the properties of the absolute value of complex numbers (see **Subsection 3.1.1**, **Problems 3.1.2, 3.1.3**, **3.1.4**, etc.), we get:

$$0 \leq \lim_{R\to\infty} \left| \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} \right| \leq \lim_{R\to\infty} \int_0^{\pi} \left| \frac{Rie^{i\theta}}{1+R^2e^{i2\theta}} \right| d\theta =$$

$$= \lim_{R\to\infty} \int_0^{\pi} \frac{\left|Rie^{i\theta}\right|}{\left|1+R^2e^{i2\theta}\right|} d\theta = \lim_{R\to\infty} \int_0^{\pi} \frac{R}{\left|1+R^2e^{i2\theta}\right|} d\theta \leq$$

$$\lim_{R\to\infty} \int_0^{\pi} \frac{R}{\left||1|-|R^2e^{i2\theta}|\right|} d\theta = \lim_{R\to\infty} \int_0^{\pi} \frac{R}{|1-R^2|} d\theta =$$

$$\lim_{1<R\to\infty} \int_0^{\pi} \frac{R}{R^2-1} d\theta = \lim_{1<R\to\infty} \frac{R}{R^2-1}\pi = 0.$$

So, by the Squeeze Theorem for limits, we obtain

$$\lim_{R\to\infty} \left| \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} \right| = 0.$$

Then, by the property of limits

$$\lim_{t\to a} |A_t| = 0 \iff \lim_{t\to a} A_t = 0,$$

we have

$$\lim_{R\to\infty}\int_0^{\pi}\frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}=0.$$

Thus, we have finally proven

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\pi.$$

We observe that by having taken the **symmetric limit** as $R\to\infty$, we have obtained the evaluation of the **principal value** of this real improper integral. Since we can easily prove (using real methods) that this integral exists, its principal value is equal to its value.

Remark: Since $f(x)=\dfrac{1}{1+x^2}$ is an even function in $\mathbb{R}$, we also get:

$$\int_{-\infty}^{0}\frac{dx}{1+x^2}=\int_0^{\infty}\frac{dx}{1+x^2}=\frac{\pi}{2},$$

or

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=2\int_{-\infty}^{0}\frac{dx}{1+x^2}=2\int_0^{\infty}\frac{dx}{1+x^2}.$$

▲

Example 3.7.6 Similarly, we find

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx=2\int_{-\infty}^{0}\frac{x^2}{1+x^4}\,dx=2\int_0^{\infty}\frac{x^2}{1+x^4}\,dx=\frac{\pi}{\sqrt{2}}=\frac{\pi\sqrt{2}}{2},$$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^6}=2\int_{-\infty}^{0}\frac{dx}{1+x^6}=2\int_0^{\infty}\frac{dx}{1+x^6}=\frac{2\pi}{3},$$

$$\int_0^{\infty}\frac{dx}{1+x^n}=\frac{\pi}{n\sin\left(\frac{\pi}{n}\right)},\quad n=2,\,3,\,4,\ldots.$$

All these integrals are special cases of the general integral

$$\int_0^{\infty}\frac{x^j}{1+x^l}\,dx=\frac{\pi}{l\sin\left(\frac{(j+1)\pi}{l}\right)},$$

where $j=0,\ 1,\ 2,\ ...$ and $l>j+1$ integer.

We are going to prove this general integral formula in the sequel. But before we do that, we need to present two lemmata, which we are going to use in the proof in order to facilitate the involved computations. These lemmata are very useful in many similar situations.

▲

In most of the results and examples that follow, we end up with limits of integrals on closed subintervals of $[0, 2\pi]$. These limits are uniform, and thus we apply **Theorem 2.3.2**. The uniform convergence in $\theta \in [a, b] \subseteq [0, 2\pi]$ is a byproduct of the existence of a certain limit as z approaches the complex infinity.

Lemma 3.7.1 *Let $w = f(z)$ be a complex function defined and continuous in an open subset of $\mathbb{C}$ of the type*

$$\mathbb{C} - \{\textit{finitely many points of } \ \mathbb{C}\}$$

and satisfying the condition

$$\lim_{z\to\infty} zf(z) = 0$$

where ∞ is the complex infinity.

For any $0 \le \theta_1 \le \theta_2 \le 2\pi$ and $R > 0$, we consider the closed or open arc of the $C(0, R)$ from θ_1 to θ_2, denoted by A_R. This arc, as a path, can be parametrized by

$$A_R = \left\{z = Re^{i\theta}, \quad \theta_1 \le \theta \le \theta_2\right\}$$

when it is closed, and by

$$A_R = \left\{z = Re^{i\theta}, \quad \theta_1 < \theta < \theta_2\right\}$$

when it is open. Then, in either case we have

$$\lim_{R\to\infty} \int\limits_{A_R} f(z)\,dz = \lim_{R\to\infty} \int_{\theta_1}^{\theta_2} f(Re^{i\theta})R\,i\,e^{i\theta}d\theta = 0,$$

and the convergence in this limit is uniform.

Proof We take $R > 0$ large enough so that all finitely many points of $\mathbb{C}$ at which $f(z)$ is not defined are located inside the circle $C(0, R)$.

We let

$$M(R) = \operatorname*{Maximum}_{z\in A_R} |f(z)|.$$

Then, by the condition $\lim\limits_{z\to\infty} z\,f(z) = 0$, we get $\lim\limits_{R\to\infty} R\,M(R) = 0$. Consequently,

$$\left|\int\limits_{A_R} f(z)\,dz\right| = \left|\int_{\theta_1}^{\theta_2} f(Re^{i\theta})R\,i\,e^{i\theta}d\theta\right| \le \int_{\theta_1}^{\theta_2} \left|f(Re^{i\theta})\right| R\cdot 1\cdot\, d\theta \le$$

$$R\,M(R)(\theta_2 - \theta_1) \to 0, \quad \text{as} \quad R \to \infty.$$

Hence,

$$\lim_{R\to\infty}\int\limits_{A_R} f(z)\,dz = \lim_{R\to\infty}\int_{\theta_1}^{\theta_2} f(Re^{i\theta})R\,i\,e^{i\theta}d\theta = 0.$$

The convergence is uniform by the **Weierstraß M-Test, Theorem 2.3.1**, for instance.

Lemma 3.7.2 *Let $w = f(z)$ be a complex function defined and holomorphic in the open subset of $\mathbb{C}$*

$$\mathbb{C} - \{z_1,\, z_2,\, z_3,\, ...,\, z_k\}$$

and satisfying the condition

$$\lim_{z\to\infty} zf(z) = 0,$$

*where ∞ is the complex infinity. We assume that **each** of the k exceptional points z_1, z_2, z_3, ..., z_k is an isolated singularity of $f(z)$ and **not on the real axis**.*

*(a) If **all** the exceptional points located in the upper half plane are the z_1, z_2, z_3, ..., z_l, where $1 \le l \le k$, then*

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\sum_{j=1}^{l}\underset{z=z_j}{Res} f(z).$$

*(b) If $0 \le l < k$, that is, the z_{l+1}, z_{l+2}, ..., z_k are **precisely** the exceptional points located in the lower half plane, then we also have*

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = -2\pi i\sum_{j=l+1}^{k}\underset{z=z_j}{Res} f(z).$$

Proof (a) As in **Example 3.7.5** of this **section**, we let $C = [-R, R] + S_R^+$, where S_R^+ is the upper half of the circle $C(0, R)$ positively oriented and $R > 0$ is large enough so that C encloses all z_1, z_2, z_3, ..., z_l. Then, by the **Residue Theorem, 3.7.1**, we get

$$\oint\limits_{C^+} f(z)\,dz = 2\pi i\sum_{j=1}^{l}\operatorname*{Res}_{z=z_j} f(z),$$

or

$$\int_{-R}^{R} f(x)\,dx + \int\limits_{S_R^+} f(z)\,dz = 2\pi i\sum_{j=1}^{l}\operatorname*{Res}_{z=z_j} f(z).$$

Now apply **Lemma 3.7.1** with $A_R = S_R^+$ and take there (symmetric) limit as $R \to \infty$. The result

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j} f(z)$$

follows immediately.

(b) The second result is obtained in a similar way by considering the contour $C = [-R, R] + T_R^-$, with $R > 0$ large enough and T_R^- the lower half of the circle $C(0, R)$ negatively oriented. The minus $(-)$ sign in this second equality is necessary because, in this case, we traverse C in the negative (clockwise) direction.

In the **next two examples**, we examine and evaluate three cases of integrals, even though the first two can be drawn as byproducts of the third one. We do this in order to practice with and emphasize the choice of the contour from case to case and to also observe the similarities and the differences of these three cases.

Example 3.7.7 (Compare with **Examples 2.2.5** and **2.2.6**.) We use the **previous Lemma, 3.7.2,** to prove:
Result: For all integers l and j such that: $l > j + 1$ and $j = 0, 1, 2, 3, 4, ...$, we have:

$$\int_0^{\infty} \frac{x^j}{1 + x^l}\, dx = \frac{\pi}{l \sin\left[\frac{(j+1)\pi}{l}\right]}.$$

The function $\dfrac{z^j}{1 + z^l}$ has finitely many (isolated) singularities, namely the roots of the denominator $1 + z^l$, which are all simple.

If l is odd, then the root $z = -1$ is in the real axis, but this does not matter because we integrate over $[0, \infty)$.

Instead of taking into account all of these singularities, in this case, we consider only one, namely

$$z_0 = e^{\frac{\pi i}{l}}.$$

We enclose it by the three-piece positively oriented contour

$$C^+ = [0, R] + A_R^+ + [Re^{\frac{2\pi i}{l}}, 0],$$

where $[0, R]$ and $[Re^{\frac{2\pi i}{l}}, 0]$ are straight segments and

$$A_R^+(\theta) = \left\{ z = Re^{i\theta} \quad | \quad 0 \le \theta \le \frac{2\pi}{l} \right\}$$

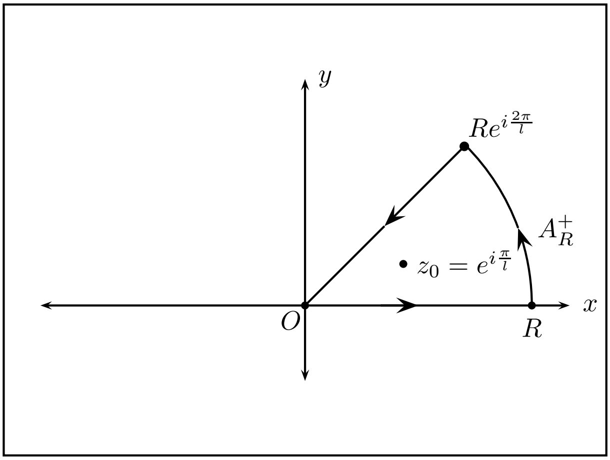

FIGURE 3.6: Contour 2 for Example 3.7.7

is an arc. We have that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{(z-z_0)z^j}{1+z^l}\Big|_{z=z_0} \left(= \frac{0}{0}\right) = \frac{z^j + j(z-z_0)z^{j-1}}{lz^{l-1}}\Big|_{z=z_0} =$$

$$\frac{z_0^j}{lz_0^{l-1}} = \frac{z_0^{j+1}}{l(-1)} = \frac{-e^{i\frac{(j+1)\pi}{l}}}{l}.$$

So,

$$\oint\limits_{C^+} \frac{z^j dz}{1+z^l} = \int\limits_{[0,R]} \frac{z^j dz}{1+z^l} + \int\limits_{A_R^+} \frac{z^j dz}{1+z^l} + \int\limits_{[Re^{\frac{2\pi i}{l}},0]} \frac{z^j dz}{1+z^l} = -2\pi i \frac{e^{i\frac{(j+1)\pi}{l}}}{l}.$$

First, since $\lim\limits_{z\to\infty} \frac{z \cdot z^j}{1+z^l} = 0$, by **Lemma 3.7.1**, we have that

$$\oint\limits_{A_R^+} \frac{z^j dz}{1+z^l} \to 0, \text{ as } R \to \infty.$$

Second,

$$\lim_{R\to\infty} \int\limits_{[0,R]} f(z)\,dz = \int_0^\infty \frac{x^j dx}{1+x^l}.$$

Third,

$$\int_{[0,Re^{\frac{2\pi i}{l}}]} f(z)\,dz =$$

$$\int_0^R \frac{x^j e^{i\frac{2\pi j}{l}} e^{\frac{2\pi i}{l}}\,dx}{1+x^l e^{2\pi i}} = \int_0^R \frac{x^j e^{i\frac{2\pi (j+1)}{l}}\,dx}{1+x^l} = e^{i\frac{2\pi (j+1)}{l}} \int_0^R \frac{x^j\,dx}{1+x^l}.$$

Taking limit as $R \to \infty$, we get

$$\lim_{R\to\infty} \int_{[Re^{\frac{2\pi i}{l}},0]} f(z)\,dz = -e^{i\frac{2\pi (j+1)}{l}} \int_0^\infty \frac{x^j\,dx}{1+x^l}.$$

Therefore, by using the **Residue Theorem, 3.7.1**, and the three-part-limit (as $R \to \infty$) for $\oint_{C+} \frac{z^j\,dz}{1+z^l}$, we get

$$\int_0^\infty \frac{x^j\,dx}{1+x^l} - e^{i\frac{2\pi (j+1)}{l}} \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$

$$\left(1 - e^{i\frac{2\pi (j+1)}{l}}\right) \int_0^\infty \frac{x^j\,dx}{1+x^l} dx = -2\pi i \frac{e^{i\frac{(j+1)\pi}{l}}}{l}.$$

So, by Euler's formula (see **Section 3.2**)

$$\left[1 - \cos\left(\frac{2(j+1)\pi}{l}\right) - i\sin\left(\frac{2(j+1)\pi}{l}\right)\right] \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$

$$\frac{-2\pi i}{l}\left[\cos\left(\frac{(j+1)\pi}{l}\right) + i\sin\left(\frac{(j+1)\pi}{l}\right)\right].$$

Using the double angle formulae in trigonometry, we find

$$\left[2\sin^2\left(\frac{(j+1)\pi}{l}\right) - 2i\sin\left(\frac{(j+1)\pi}{l}\right)\cos\left(\frac{(j+1)\pi}{l}\right)\right] \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$

$$= 2\sin\left(\frac{(j+1)\pi}{l}\right) \cdot \left[\sin\left(\frac{(j+1)\pi}{l}\right) - i\,\cos\left(\frac{(j+1)\pi}{l}\right)\right] \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$

$$\frac{2\pi}{l}\left[\sin\left(\frac{(j+1)\pi}{l}\right) - i\cos\left(\frac{(j+1)\pi}{l}\right)\right].$$

Separating the real and imaginary parts, or simply dividing by the bracket in front of the integral, we find the important result:

Result: For all integers l and j such that $l > j + 1$ and $j = 0, 1, 2, 3, 4, ...$ we have:

$$\int_0^\infty \frac{x^j}{1+x^l}\,dx = \frac{\pi}{l\,\sin\left[\frac{(j+1)\pi}{l}\right]}.$$

Remark 1: Here, the contour integration method proves the existence of the integral and evaluates it at the same time.

Remark 2: If $l = j + 1$ and $j = 0, 1, 2, 3, 4, ...$, then the result extends as

$$\int_0^\infty \frac{x^j}{1+x^{j+1}}\,dx = \left[\frac{1}{j+1}\ln\left|1+x^{j+1}\right|\right]_0^\infty = \infty = \frac{\pi}{(j+1)\sin(\pi^+)} = \frac{\pi}{0^+}.$$

Remark 3: If $j = 2m \geq 0$ and $l = 2n$ are **both even integers**, then the integrand is an even function, and by this result we also get

$$\int_{-\infty}^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{2\pi}{2n\,\sin\left[\frac{(2m+1)\pi}{2n}\right]} = \frac{\pi}{n\,\sin\left[\frac{(2m+1)\pi}{2n}\right]},$$

where n is integer such that $2n > 2m + 1$ and $m = 0,\ 1,\ 2,\ 3,\ 4, \ldots$.

Remark 4: If the integer $j = 2k + 1$ is odd and the integer $l = 2n$ is even, then the integrand is an odd function, and so

$$\text{P.V.}\int_{-\infty}^\infty \frac{x^{2k+1}}{1+x^{2n}}\,dx = 0.$$

Remark 5: If the integer $l = 2n+1$ is odd, then the integral and/or its principal value

$$\int_{-\infty}^\infty \frac{x^j}{1+x^l}\,dx$$

needs investigation, since it also becomes improper at $x = -1$. (See **Theorem 3.7.3**.) Depending on the value of j and l, various answers may be obtained.

▲

Example 3.7.8 (Compare with Example 2.2.5.) The result in the **previous example** generalizes to the following important **result**:

For $l = 1,\ 2,\ 3,\ 4, ...$ integer and $\alpha \in \mathbb{R} - \mathbb{Z}$ such that $0 < \alpha + 1 < l$

(or $-1 < \alpha < l - 1$), we have: $$\int_0^\infty \frac{x^\alpha}{1+x^l}\,dx = \frac{\pi}{l\,\sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

For example, if $\alpha = 1/2$ and $l = 2$, we find:

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = \frac{\pi}{2\sin\left[\frac{(\frac{1}{2}+1)\pi}{2}\right]} = \frac{\pi}{2\cdot\frac{\sqrt{2}}{2}} = \frac{\pi\sqrt{2}}{2}.$$

The case $\alpha \in \mathbb{Z}$ was studied in the **previous example**. This general integral formula is also useful in the study of the Gamma function.

The **proof** of this result for α **not an integer** is more complicated than before. The reason is that in such a case the power function is defined by:

$$z^\alpha = e^{\alpha \log(z)}.$$

To obtain continuous branches of $\log(z)$, we must introduce appropriate branch cuts. So, the function

$$f(z) = \frac{z^\alpha}{1+z^l}$$

has as isolated singularities all the roots of the denominator and as non-isolated singularities a whole branch cut, which can be taken to be a closed half line starting at the origin.

Under the conditions $0 < \alpha+1 < l$ imposed on l and α, this function satisfies the condition of **Lemma 3.7.2**:

$$\lim_{z\to\infty} zf(z) = 0.$$

When $\alpha \in \mathbb{R} - \mathbb{Z}$, this integral is dealt with in the following two **cases**:

(a) $l = 2, 3, 4, ...$, and **(b)** $l = 1$.

Case (a): $l = 2, 3, 4, \ldots$. In this case, besides the roots of the denominator, the contour must also avoid the non-isolated singularities of the branch cut and the origin. We achieve this by considering the branch cut to be the closed lower y-semi-axis $\{z = 0 + iy \mid y \le 0\}$.

Then, we choose the following contour C, this time consisting of four parts (in order to avoid the origin and the branch cut). We consider any $0 < r < 1 < R < \infty$, and we define

$$C = [r, R] + A_R^+ + [Re^{\frac{2\pi i}{l}}, re^{\frac{2\pi i}{l}}] + A_r^-,$$

where we have the straight segments $[r, R]$, $[Re^{\frac{2\pi i}{l}}, re^{\frac{2\pi i}{l}}]$ and the arcs

$$A_R^+(\theta) = \left\{ Re^{i\theta} \quad \middle| \quad 0 \le \theta \le \frac{2\pi}{l} \right\}$$

and

$$A_r^-(\theta) = \left\{ re^{i\theta} \quad \middle| \quad \frac{2\pi}{l} \ge \theta \ge 0 \right\},$$

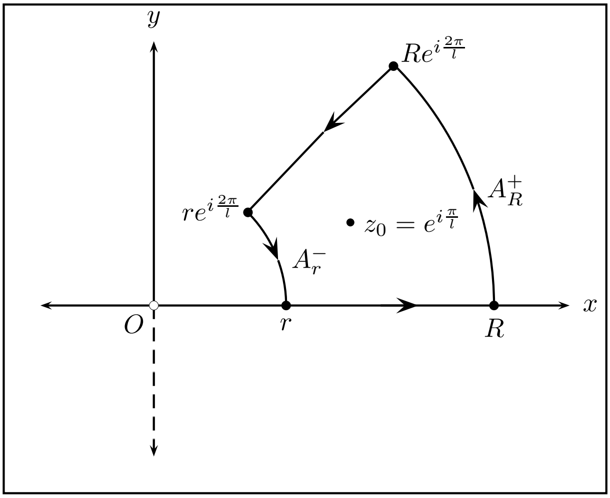

FIGURE 3.7: Contour 3 for Example 3.7.8, Case (a)

as in **Figure 3.7**.

We now follow the same method as in the previous example, but we take two limits: one as $R \to \infty$ and the other as $r \to 0^+$.

Again, by **Lemma 3.7.1**, we have that

$$\lim_{R\to\infty} \int\limits_{A_R^+} \frac{z^\alpha}{1+z^l}\,dz = 0.$$

Under the conditions $0 < r < 1$ and $\alpha > -1$, we have that

$$\lim_{r\to 0^+} \int\limits_{A_r^-} \frac{z^\alpha}{1+z^l}\,dz = 0.$$

This follows easily from

$$\left|\int\limits_{A_r^-} \frac{z^\alpha}{1+z^l}\,dz\right| = \left|\int_{2\pi/l}^{0} \frac{\left(re^{i\theta}\right)^\alpha}{1+\left(re^{i\theta}\right)^l} ire^{i\theta}\,d\theta\right| \le$$

$$\int_0^{2\pi/l} \left|\frac{r^\alpha e^{i\alpha\theta}}{1+r^l e^{il\theta}}\,i\,r\,e^{i\theta}\right|\,d\theta \le \int_0^{2\pi/l} \frac{r^{\alpha+1}}{1-r^l}\,d\theta = \frac{2\pi}{l}\,\frac{r^{\alpha+1}}{1-r^l} \longrightarrow \frac{0}{l} = 0,$$

$$\text{as} \quad 0 < r \longrightarrow 0^+.$$

Having done this, the remaining computation is exactly the same as in the **previous example**. (Finish it!) So, we have obtained the following:

Result:

$$\forall\ l = 2,\ 3,\ 4, \ldots, \quad \text{and} \quad \forall\ \alpha \in \mathbb{R}\ :\ -1 < \alpha < l-1$$
$$\int_0^\infty \frac{x^\alpha}{1+x^l}\,dx = \frac{\pi}{l\,\sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

Case (b) $l = 1$. [This case is substantially different from **Case (a)**.] Here, we must prove that for all $-1 < \alpha < 0$ real, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \frac{\pi}{\sin[(\alpha+1)\pi]} = \frac{-\pi}{\sin(\alpha\pi)} = \frac{\pi}{\sin(-\alpha\pi)}.$$

Notice that the only root of the denominator is $z = -1$. We take as the branch cut the closed half line of the non-negative real semi-axis $\{z = x + 0i \mid x \geq 0\}$. In **Figure 3.8**, this is indicated by the **dashed line**.[17] That is, we have chosen the positive continuous argument $0 < \arg(z) < 2\pi$. So, we write $z = -1 = e^{i\pi}$, and then

$$\operatorname*{Res}_{z=-1}\left(\frac{z^\alpha}{1+z}\right) = (-1)^\alpha = \left(e^{i\pi}\right)^\alpha = e^{i\alpha\pi}.$$

We now pick numbers r and R such that $0 < r < 1 < R < \infty$ and as appropriate contour

$$C = [r,R] + A_R^+ + [R,r] + A_r^-,$$

where

$$A_R^+(\theta) = Re^{i\theta}, \quad 0 < \theta < 2\pi \quad \text{and} \quad A_r^-(\theta) = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

Notice the strict inequalities in the interval of θ, and pay attention to the direction of each part of the contour.

By the **Residue Theorem, 3.7.1**, we obtain

$$\oint\limits_{C^+} \frac{z^\alpha}{1+z}\,dz =$$

$$\int\limits_{[r,R]} \frac{z^\alpha dz}{1+z} + \int\limits_{A_R^+} \frac{z^\alpha dz}{1+z} + \int\limits_{[R,r]} \frac{z^\alpha dz}{1+z} + \int\limits_{A_r^-} \frac{z^\alpha dz}{1+z} = 2\pi i e^{i\alpha\pi}, \qquad (3.17)$$

---

[17] **Note**: In this text, branch cuts in contours are indicated by dashed lines. The same is true for arcs that are depicted but not used.

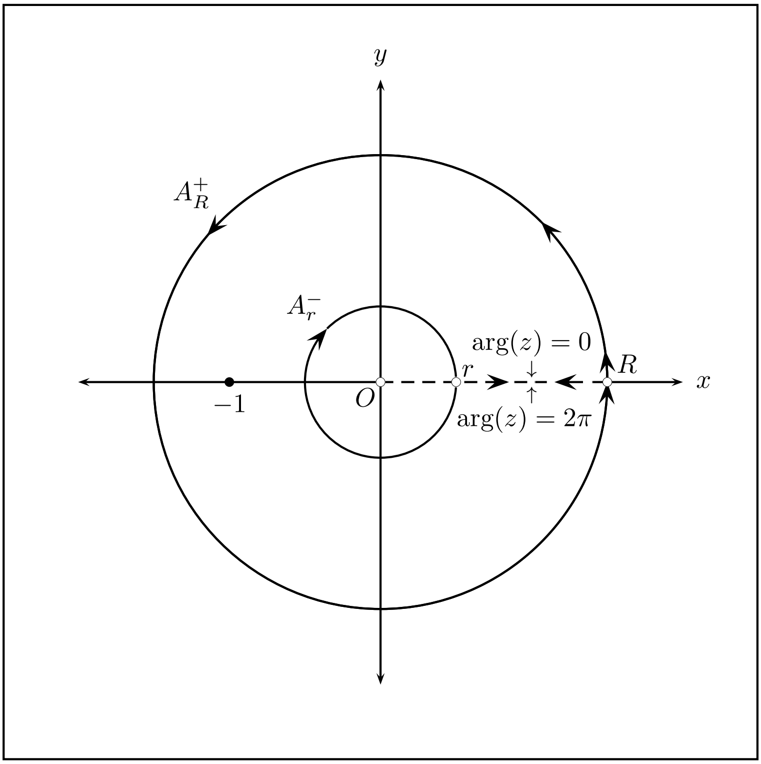

FIGURE 3.8: Contour 4 for Example 3.7.8, Case (b)

and we must take limits as $R \to \infty$ and as $r \to 0^+$.

By **Lemma 3.7.1**, we get

$$(1) \qquad \lim_{R\to\infty} \int\limits_{A_R^+} f(z)\,dz = 0.$$

Again, under the condition $-1 < \alpha < 0$ and using the parametrization of A_r^- as in **Case (a)**, we prove that

$$(2) \qquad \lim_{r\to 0^+} \int\limits_{A_r^-} \frac{z^\alpha}{1+z}\,dz = 0,$$

in exactly the same way.

Next, we must compute the two partial integrals along the branch cut, that is, over the intervals $[r, R]$ and $[R, r]$. Someone may think that the two integrals cancel because we integrate over opposite intervals. However, this is not so, because the function is discontinuous along the branch cut. Its limits are different when we approach the branch cut from above (in the upper half plane) vs approaching it from below (in the lower half plane). Notice also the indication of the two different arguments in **Figure 3.8**.

In the case of the (positive) segment $[r, R]$, as we travel along the contour the arc A_r^- indicates[18] that we approach the branch cut from above, and so the limit of the $\arg(z)$ is 0. Hence, along $[r, R]$ we compute the real integral $\int_r^R \frac{\left(xe^{0i}\right)^\alpha}{1+x}\,dx \int_r^R \frac{x^\alpha}{1+x}\,dx$. Taking limits, we get

$$(3) \qquad \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{x^\alpha}{1+x}\,dx = \int_0^\infty \frac{x^\alpha}{1+x}\,dx.$$

But, in the case of the (negative) segment $[R, r]$, we approach the branch cut from below, as the arc A_R^+ indicates, and so the limit of the $\arg(z)$ is 2π. Hence, along $[R, r]$ we must compute the integral $\int_R^r \frac{(xe^{2\pi i})^\alpha}{1+x}\,dx$. The integrand now is different from the one along $[r, R]$ above, and this happens because $-1 < \alpha < 0$ is not an integer and so $e^{2\pi\alpha i} \neq 1$. Taking limits we get

$$(4) \qquad \lim_{\substack{r\to 0\\ R\to\infty}} \int_R^r \frac{(xe^{2\pi i})^\alpha}{1+x}\,dx = -e^{2\pi i\alpha} \int_0^\infty \frac{x^\alpha}{1+x}\,dx.$$

Now, by taking the indicated limits in **(3.17)** by the four computed pieces **(1)**, **(2)**, **(3)** and **(4)**, we find

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx - e^{2\pi i\alpha} \int_0^\infty \frac{x^\alpha}{1+x}\,dx =$$

$$\left(1 - e^{2\pi i\alpha}\right) \int_0^\infty \frac{x^\alpha}{1+x}\,dx = 2\pi i e^{i\alpha\pi}.$$

---

[18] **Note**: We emphasize all these details because **in this Figure and all the figures with branch cuts that follow**, in order to simplify the drawing of contours and writing of too many limits with a lot of ϵ's and δ's, we give the figure of the limiting contour of the contours that other books draw and use. Make a note of this, and also have a look at other books to be careful see this difference. Also in this book be aware as to how the contours are made and how you approach the branch cuts as you travel along the contours in the direction indicated.

So, for any $-1 < \alpha < 0$ real, we have:

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \frac{2\pi i e^{i\alpha\pi}}{1-e^{2\pi i\alpha}} = \frac{\pi}{\frac{e^{-i\alpha\pi}-e^{i\alpha\pi}}{2i}} = \frac{\pi}{-\frac{e^{i\alpha\pi}-e^{-i\alpha\pi}}{2i}} = \frac{\pi}{-\sin(\alpha\pi)} = \frac{\pi}{\sin(-\alpha\pi)} = \frac{\pi}{\sin[(\alpha+1)\pi]}.$$

▲

Remark 1: For $\alpha \geq 0$ or $\alpha \leq -1$

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \infty.$$

Indeed, if $\alpha \geq 0$, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx \geq \int_0^1 \frac{x^\alpha}{1+x}\,dx + \int_1^\infty \frac{1}{1+x}\,dx = \int_0^1 \frac{x^\alpha}{1+x}\,dx + \infty = \infty,$$

and if $\alpha \leq -1$, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx \geq \int_0^1 \frac{x^\alpha}{2}\,dx + \int_1^\infty \frac{x^\alpha}{1+x}\,dx = \infty + \int_1^\infty \frac{x^\alpha}{1+x}\,dx = \infty.$$

Remark 2: For the real integral

$$\int_{-\infty}^0 \frac{x^\alpha}{1+x}\,dx$$

or its principal value, we need x^α to be defined as real. So, we need α to be a reduced rational with an odd denominator.

We notice that this integral is improper at $-\infty$ and $x = -1$. Essentially, each case needs a separate investigation. For example, if $\alpha = 0$, then

$$\text{P.V.} \int_{-\infty}^0 \frac{1}{1+x}\,dx \quad \text{does not exist,}$$

and therefore the integral itself does not exist.

Remark 3: The integrals of **Example 3.7.7** and **Case (a) of Example 3.7.8** can be drawn from this integral by making the u-substitution: $u = x^l \iff x = u^{\frac{1}{l}}$ and then $dx = \frac{1}{l}u^{\frac{1}{l}-1}du$. (Work it out.)

Remark 4: If $-1 < \alpha < 0$ and $b > 0$, then using the change of variables $x = bu$ and so $dx = b\,du$, we find

$$\int_0^\infty \frac{x^\alpha}{b+x}\,dx = b^\alpha \int_0^\infty \frac{u^\alpha}{1+u}\,du = \frac{b^\alpha \pi}{\sin(-\alpha\pi)}.$$

Example 3.7.9

$$\int_0^\infty \frac{x^{\frac{-1}{3}}}{10+x}\,dx = \frac{10^{\frac{-1}{3}}\pi}{\sin\frac{\pi}{3}} = \frac{2\pi\sqrt[3]{100}\sqrt{3}}{30}.$$

▲

Example 3.7.10 To find

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,dx,$$

we perform the u-substitution

$$u = x^{\sqrt{2}} \Longleftrightarrow x = u^{\frac{1}{\sqrt{2}}} \quad \text{and} \quad dx = \frac{1}{\sqrt{2}}u^{\frac{1}{\sqrt{2}}-1}du.$$

So,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{u^{\frac{3}{2\sqrt{2}}-1}}{1+u}\,du = \infty,$$

since $\alpha = \frac{3}{2\sqrt{2}} - 1 > 0$.

▲

Example 3.7.11 To find

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{3}}}\,dx,$$

we perform the u-substitution

$$u = x^{\sqrt{3}} \Longleftrightarrow x = u^{\frac{1}{\sqrt{3}}} \quad \text{and} \quad dx = \frac{1}{\sqrt{3}}u^{\frac{1}{\sqrt{3}}-1}du.$$

So,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{3}}}\,dx = \frac{1}{\sqrt{3}}\int_0^\infty \frac{u^{\frac{3}{2\sqrt{3}}-1}}{1+u}\,du = \frac{1}{\sqrt{3}}\frac{\pi}{\sin\left(\frac{\sqrt{3}}{2}\pi\right)},$$

since $-1 < \alpha = \frac{3}{2\sqrt{3}} - 1 < 0$.

▲

Example 3.7.12 To find

$$\int_0^\infty \frac{x^8}{1+x^{4\pi}}\,dx,$$

we perform the u-substitution

$$u = x^{4\pi} \Longleftrightarrow x = u^{\frac{1}{4\pi}} \quad \text{and} \quad dx = \frac{1}{4\pi}u^{\frac{1}{4\pi}-1}du.$$

So,

$$\int_0^\infty \frac{x^8}{1+x^{4\pi}}\,dx = \frac{1}{4\pi}\int_0^\infty \frac{u^{\frac{9}{4\pi}-1}}{1+u}\,du = \frac{1}{4\pi}\frac{\pi}{\sin\left[\left(1-\frac{9}{4\pi}\right)\pi\right]} = \frac{1}{4\sin\left(\frac{9}{4}\right)},$$

since $-1 < \alpha = \dfrac{9}{4\pi} - 1 < 0$.

▲

Now, we present an interesting mathematical application of **Examples 3.7.7 and 3.7.8.**

Application: For $n \in \mathbb{N}$

$$f_n(x) = \begin{cases} \dfrac{x^n - 2x + 1}{x^{2n} - 1}, & \text{if} \quad x \neq 1 \\[2ex] \dfrac{n-2}{2n}, & \text{if} \quad x = 1 \end{cases}$$

and

$$I_n = \int_0^\infty f_n(x)dx.$$

(a) For $n = 1$, we simplify and easily find that $I_1 = -\infty$.

(b) For $n \geq 2$, without simplifying, we use the result in **Example 3.7.7** and we find that $I_n = 0$. (See also, *Mathematics Magazine*, Problem 1912, Vol. 86, Number 1, February 2013.)

To prove this, we notice that $z = 1$ is a removable singularity of $f_n(z)$, where $z = x + iy$ the complex variable. Then, for $n \geq 2$, we choose $R > 1$ and the contour

$$C^+ = [0, R] + A_R^+(\theta) + \left[Re^{\frac{3\pi i}{2n}}, 0\right],$$

where $A_R^+(\theta) = \left\{ z = Re^{i\theta} | 0 \leq \theta \leq \dfrac{3\pi}{2n} \right\}$.

We observe that $f_n(z)$ has a unique pole of order one in the interior of this contour, which is the $2n^{\text{th}}$ root of $z_* = e^{\frac{3\pi i}{n}}$. The corresponding residue is $\underset{\mathrm{Z}=\mathrm{Z}}{\mathrm{Res}}\ f_n(z) = -\frac{e^{\frac{2\pi i}{n}}}{n}$.

Then, we apply the **Residue Theorem, 3.7.1,** and take the limit as $R \to \infty$ to obtain the result. (Fill in the details.)

(c) Also, we easily find that

$$f(x) \overset{pw}{:=} \lim_{r\to 0^+} f_n(x) = \begin{cases} 2x-1, & \text{if} \quad 0 \le x < 1 \\ \frac{1}{2}, & \text{if} \quad x = 1 \\ 0, & \text{if} \quad x > 1. \end{cases}$$

Therefore, the convergence is not uniform and $\int_0^\infty f(x)dx = 0$.

(d) Using the result **Example 3.7.8** and making the necessary adjustments, we find that $I_\alpha = 0$ for all real numbers $\alpha > 1$.

Now, in the **following example**, we examine and evaluate another kind of integral, in which we use branch cuts and the contours have some segments parallel to the branch cuts.

Example 3.7.13 We want to evaluate the integral

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{a+\sin(\theta)}, \quad \forall\ a > 1 \quad \text{real constant.}$$

We work with the complex function

$$f(z) = \frac{1}{(z+a)\sqrt{z^2-1}}.$$

This has an isolated singularity at $z_0 = -a$, which is a simple pole, and non-isolated singularities along the branch cut(s) that we are going to choose for making $\sqrt{z^2-1}$ a continuous function in the open domain $\mathbb{C} - \{\text{the union of the branch cut(s)}\}$.

We have

$$\sqrt{z^2-1} = (z^2-1)^{\frac{1}{2}} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}.$$

In view of these two factors, we choose convenient branch cuts that shift $z-1$ and $z+1$ to the origin. We choose

$$\mathbb{C} - [+1, +\infty) \quad \text{for} \quad \sqrt{z-1} \qquad \text{and} \qquad \mathbb{C} - [-1, +\infty) \quad \text{for} \quad \sqrt{z+1}.$$

The intersection of these branch cuts is $[+1, +\infty)$ and their union is $[-1, +\infty)$, both intervals of the real axis.

The corresponding restrictions on the arguments are

$$0 < \arg(z-1) < 2\pi \quad \text{and} \quad 0 < \arg(z+1) < 2\pi.$$

Then, by the definition of the non-integer powers through complex logarithms, we have

$$\sqrt{z^2-1} = |z^2-1|^{\frac{1}{2}} e^{i\frac{\arg(z^2-1)}{2}} = \sqrt{|z^2-1|}\, e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}.$$

Now, for any $0 < r < \min\{1,\ a-1\}$ and any $(1 <)a < R$, we consider the contour

$$\begin{gathered} C^+ = [R, 1+r]^- + T_r^- + [1-r, -1+r]^- + A_r^- \\ +[-1+r, 1-r]^+ + S_r^- + [1+r, R]^+ + A_R^+, \end{gathered}$$

consisting of two positive segments, two negative segments, three negative arcs and one positive arc, as in **Figure 3.9**. The arcs are given analytically by:

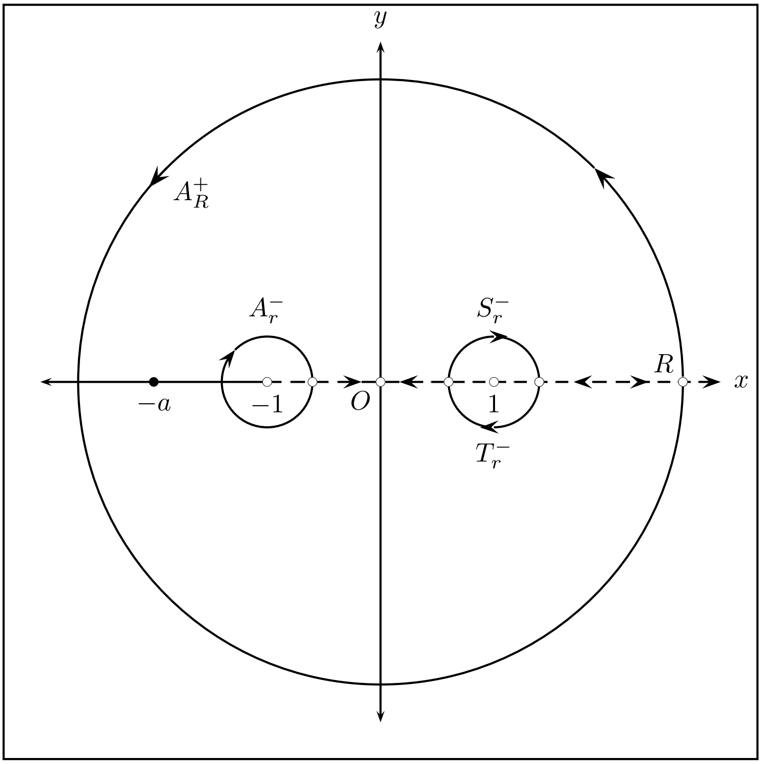

FIGURE 3.9: Contour 5 for Example 3.7.13

$$\begin{aligned} T_r^- &= \left\{z = 1 + re^{i\theta} \mid 2\pi > \theta > \pi\right\} \\ A_r^- &= \left\{z = -1 + re^{i\theta} \mid 2\pi > \theta > 0\right\} \\ S_r^- &= \left\{z = 1 + re^{i\theta} \mid \pi > \theta > 0\right\} \\ A_R^+ &= \left\{z = Re^{i\theta} \mid 0 < \theta < 2\pi\right\}. \end{aligned}$$

We must understand that **the positive segments are approached by staying in the upper half plane, whereas the negative segments are approached by staying in the lower half plane**. Then, the arguments of $\arg(z-1)$ and $\arg(z+1)$ along these segments are:

$$\begin{aligned}
&\text{Along } [R, 1+r]^- : && \arg(z-1) = \arg(z+1) = 2\pi\\
&\text{Along } [1-r, -1+r]^- : && \arg(z-1) = \pi, \quad \arg(z+1) = 2\pi\\
&\text{Along } [-1+r, 1-r]^+ : && \arg(z-1) = \pi, \quad \arg(z+1) = 0\\
&\text{Along } [1+r, R]^+ : && \arg(z-1) = \arg(z+1) = 0.
\end{aligned}$$

Next, the residue at $z_0 = -a$ is

$$\operatorname*{Res}_{z_0=-a} f(z) = f(z)(z+a)|_{z_0=-a} = \frac{1}{\sqrt{z^2-1}}\Big|_{z_0=-a} =$$

$$\frac{1}{\sqrt{|z^2-1|}\,e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}}\Big|_{z_0=-a} = \frac{1}{\sqrt{|(-a)^2-1|}\,e^{i\frac{\arg(-a-1)+\arg(-a+1)}{2}}} =$$

$$\frac{1}{\sqrt{|a^2-1|}\,e^{i\frac{\pi+\pi}{2}}} = \frac{1}{\sqrt{|a^2-1|}\,e^{i\pi}} = \frac{1}{-\sqrt{|a^2-1|}}.$$

Now, we integrate $f(z)$ along C^+, and then we take limit as $R \to \infty$ and $r \to 0^+$. Thus, by **the Residue Theorem, 3.7.1**, the final result is equal to

$$2\pi i \cdot \frac{1}{-\sqrt{|a^2-1|}} = \frac{-2\pi i}{\sqrt{|a^2-1|}} = \frac{-2\pi i}{\sqrt{a^2-1}}, \quad \text{when} \quad a > 1 \quad \text{constant.}$$

Since $\lim_{z\to\infty} zf(z) = 0$, by **Lemma 3.7.1**, we have $\lim_{R\to\infty} \int_{A_R^+} f(z)\,dz = 0$.

Also, we easily prove that

$$\lim_{r\to 0^+} \int_{T_r^-} f(z)\,dz = \lim_{r\to 0^+} \int_{A_r^-} f(z)\,dz = \lim_{r\to 0^+} \int_{S_r^-} f(z)\,dz = 0.$$

Now, on $[R, 1+r]^-$ we have

$$f(z) = f(x) = \frac{1}{(x+a)\sqrt{|x^2-1|}\,e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \tag{3.18}$$

$$\frac{1}{(x+a)\sqrt{x^2-1}\,e^{i\frac{2\pi+2\pi}{2}}} = \frac{1}{(x+a)\sqrt{x^2-1}\,e^{i\,2\pi}} = \frac{1}{(x+a)\sqrt{x^2-1}}.$$

Similarly, on $[1+r, R]^+$ we have

$$f(z) = f(x) = \frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \tag{3.19}$$
$$\frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\frac{0+0}{2}}} = \frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\,0}} = \frac{1}{(x+a)\sqrt{x^2-1}}.$$

Hence, on these two opposite segments the function $f(z)$ remains the same, and so the two integrals over these opposite segments cancel each other.

[**Note**: We expect this phenomenon to happen along the common part of two branch cuts, each of which corresponds to a factor of a function under consideration. Along the common part of the branch cuts, the function extends continuously, since crossing the common part changes both factors in a way that their product remains unchanged.

We add that in such a case the function remains also holomorphic along the common part of the two branch cuts. This is a consequence of **Morera's Theorem, 3.5.5**, since, by a simple limit argument, the holomorphicity of $f(z)$ in $\mathbb{C} - \{\text{union of the branch cuts}\}$ and its continuity in $\{\mathbb{C} - \{\text{union of the branch cuts}\}\} \cup \{\text{intersection of the branch cuts}\}$ imply

$$\int_C f(z)dz = 0$$

for any simple closed contour C such that C and its interior are inside the open region
$\{\mathbb{C} - \{\text{union of the branch cuts}\}\} \cup \{\text{intersection of the branch cuts}\}$.]

Next, we need to work out the two remaining integrals over the opposite segments $[1-r, -1+r]^-$ and $[-1+r, 1-r]^+$ only. On $[1-r, -1+r]^-$ we have:

$$f(z) = f(x) =$$
$$\frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi+2\pi}{2}}} =$$
$$\frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{3\pi}{2}}} = \frac{1}{-i(x+a)\sqrt{1-x^2}}.$$

Similarly, on $[-1+r, 1-r]^+$ we have:

$$f(z) = f(x) =$$
$$\frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi+0}{2}}} =$$

$$= \frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi}{2}}} = \frac{1}{i(x+a)\sqrt{1-x^2}}.$$

Finally, we find

$$\int_{1}^{-1} \frac{1}{-i(x+a)\sqrt{1-x^2}}\, dx + \int_{-1}^{1} \frac{1}{i(x+a)\sqrt{1-x^2}}\, dx = \frac{-2\pi i}{\sqrt{|a^2-1|}},$$

or

$$\int_{-1}^{1} \frac{1}{(x+a)\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{1}{(x+a)\sqrt{1-x^2}} dx = i\frac{-2\pi i}{\sqrt{|a^2-1|}} = \frac{2\pi}{\sqrt{|a^2-1|}}.$$

So, we have eventually found: For all $a > 1$ real constant

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{a+\sin(\theta)} = \frac{\pi}{\sqrt{a^2-1}}.$$

Remark 1: When $|a| \leq 1$, this integral is improper at $x = a$ and does not exist. E.g., for $a = 0$, we get

$$\int_{-1}^{0} \frac{dx}{x\sqrt{1-x^2}} \leq \int_{-1}^{0} \frac{dx}{x} = -\infty \quad \text{and} \quad \int_{0}^{1} \frac{dx}{x\sqrt{1-x^2}} \geq \int_{0}^{1} \frac{dx}{x} = \infty,$$

and so

$$\int_{-1}^{1} \frac{dx}{x\sqrt{1-x^2}} = \text{does not exist.}$$

But,

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{x\sqrt{1-x^2}} = 0,$$

since we integrate an odd function over an interval symmetrical about the origin.

For $a = -\frac{1}{2}$, we have

$$\int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} = \int_{-\frac{3}{2}}^{\frac{1}{2}} \frac{du}{u\sqrt{1-\left(u+\frac{1}{2}\right)^2}} = \text{does not exist,}$$

and similarly

$$\int_{-1}^{1} \frac{dx}{(1+x)\sqrt{1-x^2}} = +\infty \quad \text{and} \quad \int_{-1}^{1} \frac{dx}{(-1+x)\sqrt{1-x^2}} = -\infty,$$

etc. We examine the principal value in the **next example**.

Remark 2: The integral evaluated in this **example** may be considered as an integral of a rational function of $\sin(\theta)$ and $\cos(\theta)$, which in a calculus course is treated with the change of variables

$$u = \tan\left(\frac{\theta}{2}\right) \quad \Longleftrightarrow \quad \theta = 2\arctan(u),$$

called **tangent of half- angle substitution**. From this, we find:

$$d\theta = \frac{2}{1+u^2}du,\ \sin[2\arctan(u)] = \frac{2u}{1+u^2},\ \cos[2\arctan(u)] = \frac{1-u^2}{1+u^2}.$$

These results change an (indefinite) integral of a rational function of $\sin(\theta)$ and $\cos(\theta)$ into an (indefinite) integral of a rational function of u, which is usually computed by partial fraction decomposition, etc. Then, from this indefinite integral, we compute the definite.

For example, we find that

$$\int \frac{d\theta}{-1+\sin(\theta)} = \frac{2}{\tan\left(\frac{\theta}{2}\right)-1} + C.$$

So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{-1+\sin(\theta)} =$$

$$\frac{2}{\tan\left(\frac{\pi}{4}\right)-1} - \frac{2}{\tan\left(\frac{-\pi}{4}\right)-1} = \frac{2}{0^-} - \frac{2}{-2} = -\infty + 1 = -\infty.$$

Also,

$$\int \frac{d\theta}{1+\sin(\theta)} = \frac{-2}{\tan\left(\frac{\theta}{2}\right)+1} + C,$$

and so

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{1+\sin(\theta)} =$$

$$\frac{-2}{\tan\left(\frac{\pi}{4}\right)+1} - \frac{-2}{\tan\left(\frac{-\pi}{4}\right)+1} = \frac{-2}{2} - \frac{-2}{0^+} = -1 - (-\infty) = \infty.$$

Using this method, we can compute proper and improper integrals and principal values of the rational functions of $\sin(\theta)$ and $\cos(\theta)$.

Remark 3: See **Section 3.8.1** for a different treatment of some integrals of this kind along with an easier uniform treatment of integrals of rational functions of sines and cosines on the interval $[0, 2\pi]$ or more generally over an interval of length 2π.

▲

We would like to examine the principal value of the integral in the **previous example** when $|a| < 1$. But first, in case of isolated simple poles, we need the following convenient lemma, which we will use in the sequel. As we see, in this example a is a simple isolated pole.

Lemma 3.7.3 *Let the complex function $f(z)$ have a **simple pole** at $z_0 \in \mathbb{C}$ and let $A_\theta(z_0, \epsilon) = \{z = z_0 + \epsilon e^{i\phi} \mid \theta_0 \leq \phi \leq \theta_0 + \theta\}$ be a circular arc centered at z_0, of radius $\epsilon > 0$ and angle θ, where θ_0 is a fixed angle. (See **Figure 3.10**.) Then,*

$$\lim_{\epsilon \to 0} \int_{A_\theta(z_0,\epsilon)} f(z)dz = \theta i \operatorname*{\mathit{Res}}_{z=z_0} f(z).$$

Proof Let $\operatorname*{Res}_{z=z_0} f(z) = b_1$. Since $z = z_0$ is a simple pole for $f(z)$, there is $r > 0$ such that in the closed disc $\overline{D(z_0, r)}$ we have

$$f(z) = \frac{b_1}{z - z_0} + g(z)$$

with $g(z)$ a holomorphic function and therefore bounded. So, there is $M > 0$ such that

$$|g(z)| \leq M, \ \forall \ z \in \overline{D(z_0, r)}.$$

Thus,

$$\int_{A_\theta(z_0,\epsilon)} f(z)dz = \int_{A_\theta(z_0,\epsilon)} \frac{b_1}{z - z_0}dz + \int_{A_\theta(z_0,\epsilon)} g(z)dz.$$

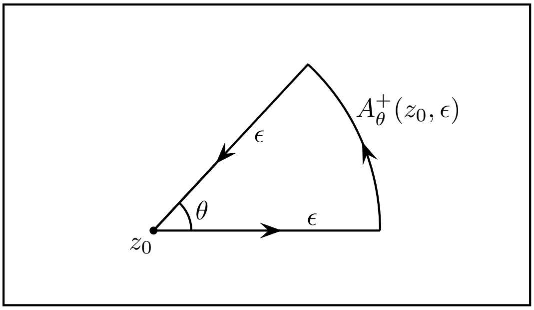

FIGURE 3.10: Contour 6 for Lemma 3.7.3

But,

$$\int_{A_\theta(z_0,\epsilon)} \frac{b_1}{z-z_0}dz = b_1 \int_{\theta_0}^{\theta_0+\theta} \frac{\epsilon i e^{i\phi}}{\epsilon e^{i\phi}} d\phi = b_1 \theta\, i,$$

and for $0 < \epsilon < r$

$$\left| \int_{A_\theta(z_0,\epsilon)} g(z)dz \right| \leq \int_{A_\theta(z_0,\epsilon)} |g(z)||dz| \leq M\epsilon\, \theta \longrightarrow 0, \text{ as } \epsilon \longrightarrow 0.$$

Finally,

$$\lim_{\epsilon \to 0} \int_{A_\theta(z_0,\epsilon)} f(z)dz = b_1\theta\, i + 0 = \theta i \operatorname*{Res}_{z=z_0} f(z),$$

and the proof is finished!

Example 3.7.14 We will show that if $|a| < 1$

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{(a+x)\sqrt{1-x^2}} = \text{P.V.} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)} = 0.$$

Let us work with $a = -\dfrac{1}{2}$, and the same work proves the result for every a such that $|a| < 1$. We observe

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} =$$

$$\lim_{\delta \to 0^+} \left[\int_{-1}^{\frac{1}{2}-\delta} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} + \int_{\frac{1}{2}+\delta}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} \right] =$$

$$\lim_{\delta \to 0^+} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}-\delta} \frac{d\theta}{-\frac{1}{2}+\sin(\theta)} + \int_{\frac{\pi}{6}+\delta}^{\frac{\pi}{2}} \frac{d\theta}{-\frac{1}{2}+\sin(\theta)} \right].$$

We consider again the **previous example**, and we observe that $z = \dfrac{1}{2}$ is on the branch cut and behaves like a "simple pole" for either functions, similar to **(3.18)** and **(3.19)**,

$$f^{\pm}(z) = \frac{1}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}},$$

where the + and − indicate that $f(z)$ is in the upper half plane and in the lower half plane, respectively, as we cannot cross the branch cut on the segment $-1 < x < 1$.

To find the principal value, we use a contour that is the one we use in the **previous example** with two additional semicircles around $x = \dfrac{1}{2}$ and the consequent modification of the segments in between. Now we choose $0 < r < \dfrac{1}{4}$ and the contour

$$C_*^+ = [R, 1+r]^- + T_r^- + [1-r, \frac{1}{2}+r]^- + T_{*r}^- + [\frac{1}{2}-r, -1+r]^- + A_r^-$$
$$+[-1+r, \frac{1}{2}-r]^+ + S_{*r}^- + [\frac{1}{2}+r, 1-r]^+ + S_r^- + [1+r, R]^+ + A_R^+.$$

The two new arcs around $x = \dfrac{1}{2}$ are

$$T_{*r}^- = \left\{ z = \frac{1}{2} + re^{i\theta} \mid 2\pi > \theta > \pi \right\}$$
$$S_{*r}^- = \left\{ z = \frac{1}{2} + re^{i\theta} \mid \pi > \theta > 0 \right\}.$$

The other arcs are the same as before.

We also observe that there are no singularities inside this contour and so no residues to compute. Therefore,

$$\int_{C_*^+} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = 0.$$

The question is what the newly introduced arcs contribute to the integral. We examine this now. **Lemma 3.7.3** can apply here, and we find that in the lower half plane

$$\int_{T_{*r}^-} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = (\pi - 2\pi)i\frac{1}{\sqrt{\left|\left(\frac{1}{2}\right)^2 - 1\right|}}e^{i\frac{\arg\left(\frac{1}{2}-1\right)+\arg\left(\frac{1}{2}+1\right)}{2}} =$$
$$-\pi i\frac{1}{\sqrt{\frac{3}{4}}}e^{i\frac{\pi+2\pi}{2}} = -\pi i\frac{1}{\sqrt{\frac{3}{4}}}(-i) = -\frac{2\pi}{\sqrt{3}} = -\frac{2\pi\sqrt{3}}{3}.$$

In the upper half plane, we get

$$\int_{S_{*r}^-} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = (0 - \pi)i\frac{1}{\sqrt{\left|\left(\frac{1}{2}\right)^2 - 1\right|}}e^{i\frac{\arg\left(\frac{1}{2}-1\right)+\arg\left(\frac{1}{2}+1\right)}{2}} =$$
$$-\pi i\frac{1}{\sqrt{\frac{3}{4}}}e^{i\frac{\pi+0}{2}} = -\pi i\frac{1}{\sqrt{\frac{3}{4}}}i = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

Finally, the net contribution of the new arcs is $-\dfrac{2\pi}{\sqrt{3}} + \dfrac{2\pi}{\sqrt{3}} = 0$. Therefore,

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2} + x\right)\sqrt{1 - x^2}} = 0,$$

and the same thing is true if we replace $-\dfrac{1}{2}$ with any a such that $|a| < 1$.

Remark: This result can also be obtained by elementary integration if we use the antiderivative (up to a constant)

$$\int \frac{d\theta}{a + \sin(\theta)} = \frac{1}{\sqrt{1 - a^2}} \ln \left| \frac{a \tan\left(\frac{\theta}{2}\right) + 1 - \sqrt{1 - a^2}}{a \tan\left(\frac{\theta}{2}\right) + 1 + \sqrt{1 - a^2}} \right|,$$

when a is a real constant with $|a| < 1$.

▲

We continue with a technical and convenient theorem for integrals of the special type $\displaystyle\int_0^{\infty} x^r f(x)dx$, where r is a real constant but not an integer. Some integrals of this type can also be evaluated by other methods, but sometimes this theorem is convenient and efficient.

Theorem 3.7.3 *We consider a complex function $f(z)$ and a real number r that satisfy the following hypotheses:*

1. *$f(z)$ is analytic in $\mathbb{C} - \{z_1, z_2, \ldots, z_n\}$, $n \geq 0$ integer, where $z_1, z_2, \ldots, z_n$ are (isolated) singularities of $f(z)$.*
2. *All $z_1, z_2, \ldots, z_n$ are not on the (strictly) positive x-axis.*
3. *$r > -1$ is not an integer.*
4. *There are constants $M_1 > 0$, $R_1 > 0$, $b > r + 1$, such that if $|z| \geq R_1$, then $|f(z)| \leq \dfrac{M_1}{|z|^b}$.*
5. *There are constants $M_2 > 0$, $R_2 > 0$, $d < r + 1$, such that if $0 < |z| \leq R_2$, then $|f(z)| \leq \dfrac{M_2}{|z|^d}$.*

Then,

$$\int_0^{\infty} x^r f(x)dx = \frac{-\pi e^{-r\pi i}}{\sin(r\pi)} \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i}[z^r f(z)],$$

and the integral converges absolutely.

Here, $z^r = e^{r\log(z)}$ using the branch $0 < Arg(z) < 2\pi$.

These hypotheses were properly imposed so that the proof works out. We omit the proof here, but the interested reader can find it in the bibliography, e.g: Marsden and Hoffman 1987, 304-307.

We can directly check that the hypotheses of this Theorem are satisfied when $f(z) = \dfrac{P(z)}{Q(z)}$ is a rational function, where $P(z)$ and $Q(z)$ are polynomials of degrees $p \geq 0$ and $q \geq 0$, respectively, and satisfy the following hypotheses:

1. We assume that $P(z)$ and $Q(z)$ have no common factors, or else $f(z)$ is completely simplified.
2. $Q(z)$ has no zero on the positive x-axis.
3. The number r is not an integer and satisfies $-1 < r < q - p - 1$.
4. If $Q(0) = 0$ and, as root of $Q(z)$, $z = 0$ has order $m \geq 1$, then we also need $m < r + 1$.

We illustrate this result with the following example:

Example 3.7.15 We would like to check if the integral

$$\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2 + x + 1} dx$$

satisfies the conditions of **Theorem 3.7.3** and evaluate it.

(1) The function $f(z) = \dfrac{1}{z^2 + z + 1}$ is analytic in $\mathbb{C} - \{e^{\frac{2\pi}{3}},\ e^{\frac{4\pi}{3}}\}$, and the numbers $z_1 = \frac{2\pi}{3}$, $z_2 = \frac{4\pi}{3}$ are isolated singularities (poles of order 1).

(2) The numbers $z_1 = e^{\frac{2\pi}{3}}$, $z_2 = e^{\frac{4\pi}{3}}$ are not on the positive x-axis.

(3) $r = \dfrac{1}{3} > -1$ is not an integer.

(4) Since

$$\left| \frac{f(z)}{\frac{1}{z^2}} \right| = \left| \frac{z^2}{z^2 + z + 1} \right| \longrightarrow 1, \quad \text{as} \quad z \longrightarrow \infty,$$

we conclude that there is $R_1 > 0$ such that if $|z| \geq R_1$, then $\left| \dfrac{f(z)}{\frac{1}{z^2}} \right| \leq 2$ or $|f(z)| \leq \dfrac{2}{|z|^2}$. So, we can pick $M_1 = 2$ and $b = 2$.

(5) Since

$$\left| \frac{f(z)}{\frac{1}{z}} \right| = \left| \frac{z}{z^2 + z + 1} \right| \longrightarrow 0, \quad \text{as} \quad z \longrightarrow 0,$$

we conclude that there is $R_2 > 0$ such that if $0 < |z| \leq R_2$, then $\left|\frac{f(z)}{\frac{1}{z}}\right| \leq 1$ or $|f(z)| \leq \frac{1}{|z|}$. So, we can pick $d = 1 < \frac{1}{3} + 1$ and $M_2 = 1$. So, all the hypotheses of **Theorem 3.7.3** are met.

Hence,

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx = \frac{-\pi e^{-\frac{\pi}{3}i}}{\sin\left(\frac{\pi}{3}\right)}\left\{\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] + \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right]\right\}.$$

Now, since

$$f(z) = \frac{1}{(z-z_1)(z-z_2)},$$

we get that

$$\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] = \frac{e^{\frac{2\pi}{9}i}}{z_1 - z_2} \quad \text{and} \quad \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right]\frac{e^{\frac{4\pi}{9}i}}{z_2 - z_1}$$

within the branch $0 < \operatorname{Arg}(z) < 2\pi$.

Then,

$$\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] + \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right] = \frac{e^{\frac{2\pi}{9}i} - e^{\frac{4\pi}{9}i}}{e^{\frac{2\pi}{3}i} - e^{\frac{4\pi}{3}i}}.$$

Putting these together, we find

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx =$$

$$\frac{-\pi e^{-\frac{\pi}{3}i}}{\sin\left(\frac{\pi}{3}\right)} \cdot \frac{e^{\frac{2\pi}{9}i} - e^{\frac{4\pi}{9}i}}{e^{\frac{2\pi}{3}i} - e^{\frac{4\pi}{3}i}} = \frac{\pi\left(-e^{\frac{-\pi}{9}i} + e^{\frac{\pi}{9}i}\right)}{\sin\left(\frac{\pi}{3}\right)2i\frac{\sqrt{3}}{2}} = \frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right).$$

(See also **Problem 3.7.98** for another method.)

▲

The integrals in **Theorem 3.7.3** are immediately related to the Mellin[19] transform of a function.

Definition 3.7.2 *The* ***Mellin transform*** *of a real function* $y = f(x)$ *with* $0 < x < \infty$ *or* $0 \leq x < \infty$ *[we consider* $f(x) = 0$ *for* $x < 0$*] is defined by*

$$\mathcal{M}\{f(x)\}(s) := \phi(s) = \int_0^\infty x^{s-1}f(x)\,dx$$

for all those $s's$ *for which this integral exists.*

---

[19] Robert Hjalmar Mellin, Finnish mathematician, 1854-1933.

This transform has a lot of applications in mathematics, engineering and computer science. The inverse transform is

$$\mathcal{M}^{-1}\{\phi(s)\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \phi(s)\, ds$$

for an appropriate constant c. We state that the Mellin transform exists if $x^k f(x)$ is absolutely integrable on $(0, \infty)$ for some $k > 0$. Then, the inverse transform also exists for $c > k$.

Several integrals that we have evaluated so far and involve the product of a function with some power x^α may be viewed as Mellin transforms. For example, the $\Gamma(p)$ is the Mellin transform of $f(x) = e^{-x}$, with $x \in (0, \infty)$. Also, in **Example 3.7.8**, we have proved

For $l = 1,\ 2,\ 3,\ 4, \ldots$ integer and $\alpha \in \mathbb{R}$ such that $l > \alpha + 1 > 0$ (or $l - 1 > \alpha > -1$) we have: $\displaystyle\int_0^\infty \frac{x^\alpha}{1 + x^l}\, dx = \frac{\pi}{l \sin\left[\frac{(\alpha+1)\pi}{l}\right]}$.

Then, under these conditions, this integral can be viewed as

$$\mathcal{M}\left\{\frac{1}{1 + x^l}\right\}(\alpha + 1) = \frac{\pi}{l \sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

Replacing $\alpha + 1$ with s, we obtain the Mellin transform

$$\mathcal{M}\left\{\frac{1}{1 + x^l}\right\}(s) = \frac{\pi}{l \sin\left[\frac{(s)\pi}{l}\right]}.$$

Many other such integrals in the examples and the problems of this text may be translated as Mellin transforms in an analogous way.

Now, we are going to evaluate the **Fresnel integrals** of **Examples 1.3.14, 2.4.1** and **Problem 2.6.10** by using contour integration. To do this, we first need to prove the so-called **Jordan's**[20] **Lemma**. This lemma is also very useful in the computation of Fourier transforms, as we shall see soon.

Lemma 3.7.4 (Jordan's Lemma) *If* $0 \leq \theta_1 < \theta_2 \leq \pi$ *and* $\mu > 0$, *then*

$$\int_{\theta_1}^{\theta_2} e^{-\mu \sin(t)} dt < \frac{\pi}{\mu}.$$

---

[20] Marie Ennemond Camille Jordan, French mathematician, 1838-1922.

Proof Since $e^{-\mu\sin(t)} > 0$, we obtain

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \le \int_0^{\pi} e^{-\mu\sin(t)}dt = \int_0^{\pi/2} e^{-\mu\sin(t)}dt + \int_{\pi/2}^{\pi} e^{-\mu\sin(t)}dt.$$

Using $u = \pi - t$, we find

$$\int_{\pi/2}^{\pi} e^{-\mu\sin(t)}dt = \int_0^{\pi/2} e^{-\mu\sin(t)}dt.$$

So,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \le 2\int_0^{\pi/2} e^{-\mu\sin(t)}dt.$$

But, for $0 \le t \le \dfrac{\pi}{2}$, we have that $\dfrac{2}{\pi}t \le \sin(t)$. This inequality is seen graphically, since $\sin(t)$ is a concave function in the interval $\left[0, \dfrac{\pi}{2}\right]$, because $\sin''(t) = -\sin(t) \le 0$ for $0 \le t \le \dfrac{\pi}{2}$. Therefore, $y = \sin(t)$ is greater than or equal to the straight segment function $y = \dfrac{2}{\pi}t$ in $\left[0, \dfrac{\pi}{2}\right]$.

So, $e^{-\mu\sin(t)} \le e^{-\frac{2\mu t}{\pi}}$ for all $t \in \left[0, \dfrac{\pi}{2}\right]$. Hence,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}\,dt \le 2\int_0^{\pi/2} e^{-\frac{2\mu t}{\pi}}\,dt = 2\left(\frac{-\pi}{2\mu}\right)\left[e^{-\frac{2\mu t}{\pi}}\right]_0^{\frac{\pi}{2}} =$$

$$\frac{-\pi}{\mu}\left(e^{-\mu} - 1\right) = \frac{\pi}{\mu}\left(1 - e^{-\mu}\right) < \frac{\pi}{\mu}.$$

Remark: Jordan's Lemma also implies the following inequalities:
(1)If $\pi \le \theta_1 < \theta_2 \le 2\pi$ and $\lambda > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{\lambda\sin(t)}dt < \frac{\pi}{\lambda}.$$

(2) If $-\dfrac{\pi}{2} \le \theta_1 < \theta_2 \le \dfrac{\pi}{2}$ and $\sigma > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{-\sigma\cos(t)}dt < \frac{\pi}{\sigma}.$$

(3) If $\dfrac{\pi}{2} \le \theta_1 < \theta_2 \le \dfrac{3\pi}{2}$ and $\tau > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{\tau\cos(t)}dt < \frac{\pi}{\tau}.$$

Example 3.7.16 Now we can compute the **Fresnel integrals** by using contour integration techniques. (Compare this with **Example 2.4.1**.)

$$\int_0^\infty \cos\left(x^2\right) dx = \int_0^\infty \sin\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

We consider $f(z) = e^{-z^2}$, which is holomorphic in $\mathbb{C}$. This function has no singularities in $\mathbb{C}$ and so, by **the Cauchy-Goursat Theorem, 3.5.3**, its integral along any closed, continuous and piecewise continuously differentiable path is zero.

Then, for any $R > 0$, we consider the contour

$$C = [0, R] + A_R^+ + [Re^{\frac{i\pi}{4}}, 0],$$

where A_R^+ is the positively oriented arc parametrized by $z = Re^{i\theta}$ with $0 \le \theta \le \dfrac{\pi}{4}$. Then,

$$\oint\limits_{C^+} e^{-z^2} dz = \int\limits_{[0,\,R]} e^{-z^2} dz + \int\limits_{A_R^+} e^{-z^2} dz + \int\limits_{[Re^{\frac{i\pi}{4}},\,0]} e^{-z^2} dz = 0.$$

Now,

$$\int\limits_{[0,R]} e^{-z^2} dz = \int_0^R e^{-x^2} dx,$$

and, as we have found in **Section 2.1**, **(2.1)**,

$$\lim_{R\to\infty} \int_0^R e^{-x^2} dx = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Next, since $e^{\frac{i\pi}{2}} = i$ and along $[Re^{\frac{i\pi}{4}}, 0]$ we have $z = xe^{\frac{i\pi}{4}}$ with $R > x > 0$, we get

$$\int\limits_{[Re^{\frac{i\pi}{4}},\,0]} e^{-z^2} dz = \int_R^0 e^{-x^2 e^{\frac{i\pi}{2}}} e^{\frac{i\pi}{4}} dx = -e^{\frac{i\pi}{4}} \int_0^R e^{-x^2 i} dx.$$

Lastly, on A_R^+ we get

$$\int\limits_{A_R^+} e^{-z^2} dz = iR \int_0^{\frac{\pi}{4}} e^{-R^2 e^{2ix}} e^{ix} dx.$$

We observe that

$$e^{-R^2 e^{2ix}} = e^{-R^2[\cos(2x) + i\sin(2x)]} = e^{-R^2\cos(2x)} e^{-iR^2\sin(2x)}.$$

Therefore,

$$\left|e^{-R^2 e^{2ix}} e^{ix}\right| = e^{-R^2 \cos(2x)}.$$

So, we have

$$\left|\int\limits_{A_R^+} e^{-z^2} dz\right| \leq \int\limits_{A_R^+} \left|e^{-z^2}\right| |\, dz \,| \leq \int_0^{\pi/4} e^{-R^2 \cos(2\phi)} R \, d\phi \overset{\left(u=-\phi+\frac{\pi}{4}\right)}{=}$$

$$R \int_0^{\pi/4} e^{-R^2 \sin(2u)} du \overset{(v=2u)}{=} \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin(v)} dv <$$

$$\text{(by **Jordan's Lemma 3.7.4**)} \quad \frac{R}{2} \cdot \frac{\pi}{R^2} = \frac{\pi}{2R}.$$

Since $\lim\limits_{R\to\infty} \dfrac{\pi}{2R} = 0$, we get

$$\lim_{R\to\infty} \int\limits_{A_R^+} e^{-z^2} dz = 0.$$

So, as $R \to \infty$, we finally get

$$\frac{\sqrt{\pi}}{2} - e^{\frac{i\pi}{4}} \int_0^\infty e^{-ix^2} dx = 0.$$

Then,

$$\int_0^\infty \left[\cos\left(x^2\right) - i \sin\left(x^2\right)\right] dx =$$

$$\frac{\sqrt{\pi}}{2} e^{\frac{-i\pi}{4}} = \frac{\sqrt{\pi}}{2} \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2\pi}}{4} (1-i).$$

Now, we separate the real and imaginary parts of this equality and obtain the final result

$$\int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4} = \int_0^\infty \sin\left(x^2\right) dx.$$

Remark 1: This computation was carried out even though we knew nothing about the convergence of these integrals. So, the contour integration evaluates and proves the existence of the integral at the same time.

Remark 2: In our approach here, to compute the Fresnel integrals, we use the integral $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ that was computed earlier with the help of double integration in **Section 2.1**.

There is also the reverse approach. By choosing an appropriate contour, different from the one in this example, we can compute the Fresnel integrals first, and then from them we can compute $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.[21] ▲

Example 3.7.17 In this example, we show that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(4x)dx = e^{-4}\sqrt{\pi}.$$

(This is generalized in **Problem 3.7.24**, which is essentially the same with **Example 2.2.13**, but the methods used are different.)

For any $R > 0$, we consider the rectangle contour in **Figure 3.11**

$$C^+ = [-R, R] + [R, R+2i] + [R+2i, -R+2i] + [-R+2i, -R].$$

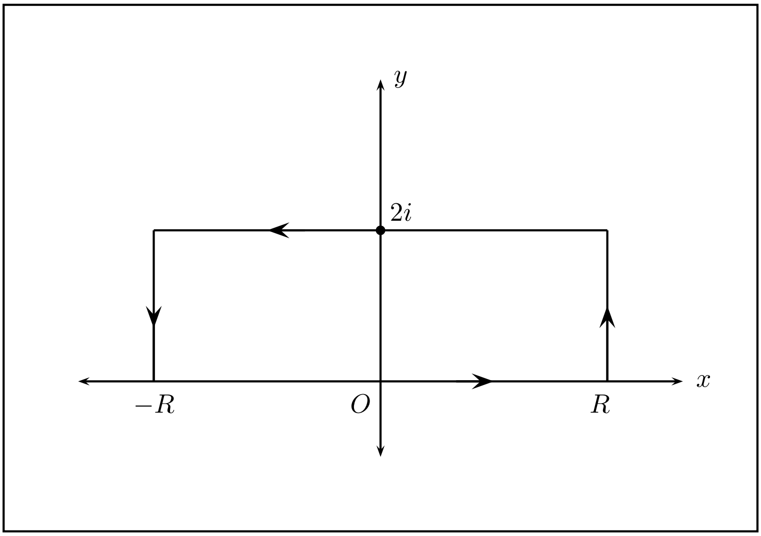

FIGURE 3.11: Contour 7 for Example 3.7.17

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[21] E.g., see the bibliography: Marsden and Hoffman 1987 and 1993.
In these two books and others, such as Copson 1948, Markushevich 1977, Brown and Churchill 2008, etc., one can find many interesting and complicated examples and/or exercises of contour integration.

Again, as in the previous example, we have

$$\oint\limits_{C+} e^{-z^2} dz = 0,$$

and so

$$\int_{-R}^{R} e^{-x^2} dx + \int_{0}^{2} e^{-(R+iy)^2} i\, dy +$$
$$\int_{R}^{-R} e^{-(x+2i)^2} dx + \int_{2}^{0} e^{-(-R+iy)^2} i\, dy = 0.$$

Then, we have

$$\left| \int_{0}^{2} e^{-(R+iy)^2} i\, dy \right| \leq \int_{0}^{2} \left| e^{-R^2} e^{-2Riy} e^{y^2} \right| dy =$$
$$\int_{0}^{2} e^{-R^2} e^{y^2} dy \leq 2e^{-R^2} e^{4} \to 0, \quad \text{as} \quad R \to \infty.$$

Similarly,

$$\left| \int_{2}^{0} e^{-(-R+iy)^2} i dy \right| \leq 2e^{-R^2} e^{4} \to 0, \quad \text{as} \quad R \to \infty.$$

Hence, taking limit as $R \to \infty$, we get

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(x+2i)^2} dx.$$

Thus, by the equality of the real parts and the **Integral (2.1)** in **Section 2.1**, we have

$$\operatorname{Re}\left[\int_{-\infty}^{\infty} e^{-(x+2i)^2} dx \right] = \operatorname{Re}\left[\int_{-\infty}^{\infty} e^{-x^2} dx \right] = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Developing this, we get

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(4x) dx = e^{-4} \sqrt{\pi}.$$

Remark 1: Since $f(x) = e^{-x^2} \cos(4x)$ is even, we get

$$\int_{-\infty}^{0} e^{-x^2} \cos(4x) dx = \int_{0}^{\infty} e^{-x^2} \cos(4x) dx =$$
$$\frac{e^{-4}\sqrt{\pi}}{2} \simeq 0.01623181234006586\ldots.$$

Remark 2: The equality of the imaginary parts gives

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(4x)dx = 0,$$

a fact already known, since the function $f(x) = e^{-x^2}\sin(4x)$ is odd and the integral exists.

Remark 3: For any $R > 0$, if we integrate e^{-z^2} over the contour

$$C = [0, R] + [R, R + 2i] + [R + 2i, 2i] + [2i, 0](= \text{rectangle})$$

and then take the limit as $R \to \infty$, using the **Integral (2.1)** and the **Error Function (2.2)** (**Section 2.1**) and the first remark above, we find

$$\int_0^{\infty} e^{-x^2}\sin(4x)dx = e^{-4}\int_0^2 e^{y^2}dy =$$
$$\frac{\sqrt{\pi}}{2e^4}\,\mathrm{erf}(2) \simeq 0.3013403889237924\ldots.$$

(Check this. Keep in mind that the indefinite integral $\int e^{y^2}dy$ cannot be found in closed form by means of elementary functions. So, we compute $\int_0^2 e^{y^2}dy$ numerically.)

▲

Problems

3.7.7 Consider $f(z) = \dfrac{e^{-az}}{z^2+1}$ with $a \in \mathbb{R}$ constant and $C(0, R)$ the circumference of the circle with center the origin and radius $R > 0$.

(a) If $a \neq 0$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)}|zf(z)|\right] = +\infty.$$

(b) If $a = 0$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)}|zf(z)|\right] = 0.$$

(c) If $a > 0$ real constant and $A_R = \{z = Re^{i\theta} \mid 0 \leq \theta \leq \pi/2\}$, the arc of the circle $C(0, R)$ between the angles 0 and $\pi/2$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in A_R}|zf(z)|\right] = 0.$$

3.7.8 Conclude the integrals of **Example 3.7.7** and **Case (a) of Example 3.7.8** from the integral of **Case (b) of Example 3.7.8**.

3.7.9 Compute the following ten integrals by whichever methods you prefer. You may use direct appropriate contour integrations or any other correct methods and integral formulae already used or proved in previous examples, etc.

$$I_1 = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx \qquad I_2 = \int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx$$

$$I_3 = \int_0^{\infty} \frac{dx}{1+x^n}, \quad \forall\, n = 2, 3, ... \qquad I_4 = \int_0^{\infty} \frac{\sqrt[3]{x^2}}{1+x^2}\,dx$$

$$I_5 = \int_0^{\infty} \frac{\sqrt[3]{x^5}}{1+x^3}\,dx \qquad I_6 = \int_0^{\infty} \frac{dx}{\sqrt[3]{x}(x+1)}$$

$$I_7 = \int_0^{\infty} \frac{x^{\frac{-1}{2}}}{100+x}\,dx \qquad I_8 = \int_0^{\infty} \frac{x^{\frac{5}{2}}}{100+x^4}\,dx$$

$$I_9 = \int_{-1}^{1} \frac{dx}{(x+10)\sqrt{1-x^2}} \qquad I_{10} = \int_{-\pi/2}^{\frac{\pi}{2}} \frac{d\theta}{20+\sin(\theta)}\,.$$

3.7.10 Using contour integration, show that for all $n = 0,\ 1,\ 2,\ ...,$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{n+1}} = \frac{\pi(2n)!}{(n!)^2 2^{2n}}.$$

[Hint: Use **Example 3.6.21**.]

3.7.11 (a) Using contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x-i)(x-2i)}\,dx.$$

(b) Separate the real and imaginary parts to find the two corresponding real integrals.

3.7.12 Prove that the integral

$$\int_0^{\infty} \frac{3x^2-2}{x^4+5x^2+6}\,dx$$

exists, and then use contour integration to evaluate it.

3.7.13 Use contour integration to prove:

(a) If $\beta \in \mathbb{R}$ and $c > 0$ constants

$$\int_{-\infty}^{\infty} e^{-cx^2} \sin(\beta x^2)\, dx =$$

$$2\int_{0}^{\infty} e^{-cx^2} \sin(\beta x^2)\, dx = \int_{0}^{\infty} e^{-cu}\, \frac{\sin(\beta u)}{\sqrt{u}}\, du =$$

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\beta\, dv}{\beta^2 + \left(c + v^2\right)^2} = \operatorname{sign}(\beta) \sqrt{\frac{\pi}{2}} \sqrt{\frac{-c + \sqrt{\beta^2 + c^2}}{\beta^2 + c^2}}.$$

(See also **Example 2.4.1**, **Remark 3**.)

(b) If $\beta \in \mathbb{R}$ and $c > 0$ constants

$$\int_{-\infty}^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx =$$

$$2\int_{0}^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx = \int_{0}^{\infty} e^{-cu}\, \frac{\cos(\beta u)}{\sqrt{u}}\, du =$$

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\left(c + v^2\right)\, dv}{\beta^2 + \left(c + v^2\right)^2} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{c + \sqrt{\beta^2 + c^2}}{\beta^2 + c^2}}.$$

3.7.14 Solve **Problem 2.6.38** by integrating the function $f(z) = \dfrac{e^{iz}}{z^{1-m}}$, $(0 < m < 1)$, along the contour

$$C^+ = [r, R] + A^+ + [Ri, ri] + A^-,$$

where $0 < r < R$ real constants, $A^+(\theta) = \{z = Re^{i\theta} \mid 0 \le \theta \le \frac{\pi}{2}\}$ and $A^-(\theta) = \{z = re^{i\theta} \mid \frac{\pi}{2} \ge \theta \ge 0\}$, and then take limits as $r \longrightarrow 0$ and $R \longrightarrow \infty$. (See also **Example 2.6.7**.)

3.7.15 Prove that the two integrals

$$\int_{0}^{\infty} \frac{\sqrt{x}}{x^2 \pm x + 1}\, dx$$

exist, and then use contour integration to evaluate them.

[Hint: Notice that $x^3 \pm 1 = (x \pm 1)(x^2 \mp x + 1)$. So, for easier computations, you may use exponential representations of the isolated singularities which are the cubic roots of ± 1. Use the **contour of Figure 3.8** of **Example 3.7.8, Case (b)**, and work analogously. Or, use **Theorem 3.7.3**.]

3.7.16

$$\text{(a) Prove:} \quad \int_0^\infty \frac{\sqrt{x}}{x^3+x^2+x+1}\,dx = \frac{\pi}{2}\,(\sqrt{2}-1).$$

$$\text{(b) Evaluate:} \quad \int_0^\infty \frac{\sqrt[3]{x}}{x^3+x^2+x+1}\,dx.$$

[Hint: Use $f(z) = \dfrac{1}{z^3+z^2+z+1} = \dfrac{1}{(z-i)(z+i)(z+1)}$ in **Theorem 3.7.3**.]

3.7.17 Prove

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+4x+8}\,dx = \frac{\pi\sqrt{2}}{2} \cdot \frac{\sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\pi\sqrt{6}}{6}\sqrt{2-\sqrt{3}}.$$

3.7.18 (a) If $-1 < a \neq 0 < 1$ and $b > 0$, prove

$$\int_0^\infty \frac{x^a}{x^2+b^2}\,dx = \frac{\pi\, b^{a-1}}{2\cos\left(\frac{\pi a}{2}\right)}.$$

(b) If $a \neq 0$ and $a \neq 1$ and $-1 < a < 2$, prove

$$\int_0^\infty \frac{x^a}{x^3+1}\,dx = \frac{-\pi\left\{1+2\cos\left[\frac{2\pi(a+1)}{3}\right]\right\}}{3\sin(\pi a)}.$$

3.7.19 Prove that if $-1 < a < 1$ and $0 < \theta < \pi$,

$$\int_0^\infty \frac{x^a}{x^2+2x\cos(\theta)+1}\,dx = \frac{\pi\sin(a\theta)}{\sin(a\pi)\sin\theta}.$$

Determine the answer when $a = 0$, by using L' Hôpital's rule. Can the answer be correctly extended to $\theta = 0$ and $\theta = \pi$, in this case?

[Hint: Use the **contour of Figure 3.8** of **Example 3.7.8, Case (b)**.]

3.7.20 If $0 < a \leq b$ and $0 < c < 2$, prove

$$\int_a^b \left(\frac{b-x}{x-a}\right)^{c-1} \frac{dx}{x} = \frac{\pi}{\sin(c\pi)}\left[1-\left(\frac{b}{a}\right)^{c-1}\right].$$

3.7.21 For any $a > 1$ constant, find the general integral formula for

$$\int_{-1}^{1} \frac{dx}{(a-x)\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{a-\sin(\theta)}.$$

3.7.22 (Practice problem!) Prove that

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2}$$

in the following two ways:

(1) By letting $x = \sec(u)$ (elementary way).

(2) By considering the complex function

$$f(z) = \frac{1}{z\sqrt{z^2-1}}$$

and do work analogous to **Example 3.7.13**.

Notice that

$$\frac{1}{\sqrt{z^2-1}} = \frac{1}{\sqrt{z-1}} \cdot \frac{1}{\sqrt{z+1}}.$$

Take branch cuts $[1, \infty)$ and $(-\infty, -1]$ for the first and the second factor, respectively. So,

$$\frac{1}{\sqrt{z-1}} = \frac{1}{\sqrt{|z-1|}} e^{i\frac{\arg(z-1)}{2}} \quad \text{with} \quad 0 < \arg(z-1) < 2\pi$$

and

$$\frac{1}{\sqrt{z+1}} = \frac{1}{\sqrt{|z+1|}} e^{i\frac{\arg(z+1)}{2}} \quad \text{with} \quad -\pi < \arg(z+1) < \pi.$$

Notice that $z = 0$ is a simple pole of $f(z)$ with residue $-i$ (compute this).

As a **contour**, use the one provided in **Figure 3.12** for appropriate $r > 0$ and $R > 0$. Find the correct arguments of the complex numbers involved along the segments of the branch cuts, as you approach them staying in the upper half plane and/or staying in the lower half plane.

Then, use the **Residue Theorem, 3.7.1**, and take the appropriate limits. The limits along the three arcs are zero. The limits along the four straight segments parallel to the branch cuts are the same. Finally, we get

$$4\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = 2\pi,$$

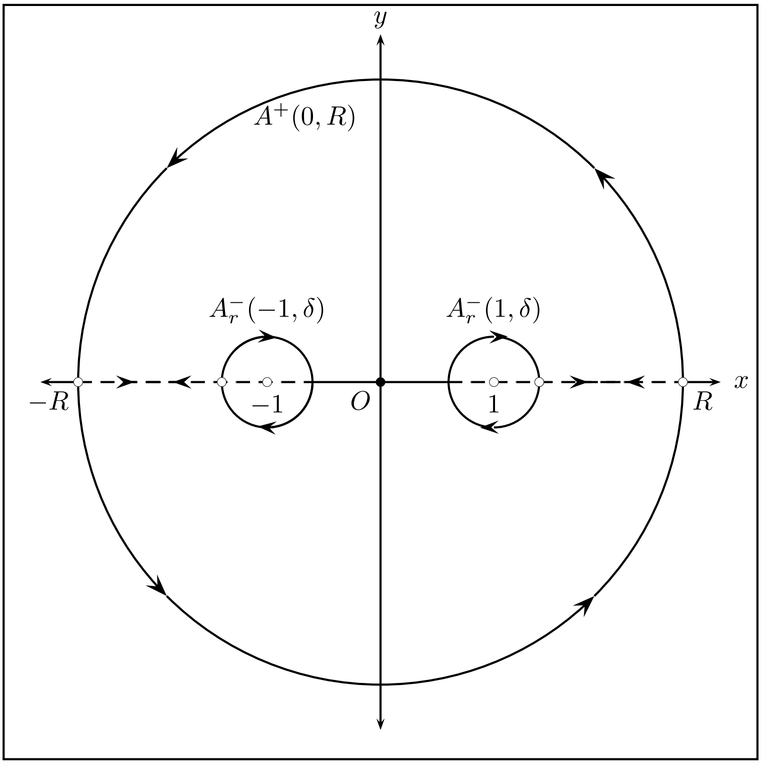

FIGURE 3.12: Contour 8, for practice Problem 3.7.21

and so the result follows.

[This problem, even though easy by means of the substitution in **Way 1**, is suggested in order to practice with contour integration, branch cuts and correct arguments!]

3.7.23 Prove that

$$\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)\sqrt{1-x^2}}$$

does not exist, but

$$\text{P.V.}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)\sqrt{1-x^2}} = \frac{2\ln(2+\sqrt{3})}{\sqrt{3}}.$$

3.7.24 For $a > 0$ and $b > 0$ constants, prove

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}}{a}e^{\frac{-b^2}{4a^2}},$$

and so

$$\int_{-\infty}^{0} e^{-a^2x^2}\cos(bx)\,dx = \int_{0}^{\infty} e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}}{2a}e^{\frac{-b^2}{4a^2}}.$$

(Compare this with **Examples 2.2.13** and **3.7.17**.)

[Hint: First, notice

$$e^{-a^2x^2}\cos(bx) = e^{-\frac{b^2}{4a^2}}\,\mathrm{Re}\left[e^{-\left(ax+\frac{b}{2a}i\right)^2}\right],$$

and then for any $R > 0$ choose as an appropriate rectangular contour

$$C = [-R, R] + [R, R + \frac{b}{2a}i] + [R + \frac{b}{2a}i, -R + \frac{b}{2a}i] + [-R + \frac{b}{2a}i, -R]$$

and consider the integral $\displaystyle\oint_{C^+} e^{-z^2}\,dz$.]

3.7.25 Work as in the **previous problem** or **Example 3.7.17** and its **Remark 3** to prove that for $a \neq 0$ and $b \in \mathbb{R}$ constants

$$\int_{0}^{\infty} e^{-a^2x^2}\sin(bx)dx = \frac{1}{2a^2}e^{-\frac{b^2}{4a^2}}\int_{0}^{b} e^{\frac{t^2}{4a^2}}\,dt.$$

(Compare with **Problem 2.2.38**.)

3.7.26 Use contour integration to prove

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sz}}{\sqrt{z+1}}dz = 2i\sqrt{\pi}\,\frac{e^{-s}}{\sqrt{s}},$$

where $s > 0$ and $\gamma > 0$ constants. [$(\gamma - i\infty, \gamma + i\infty)$ is the infinite vertical straight line $x = \gamma$.]

[Hint: Choose as the branch cut the interval $(-\infty, -1]$ and an appropriate contour.]

3.7.3 Infinite Isolated Singularities and Integrals

Suppose we need to evaluate

$$\int_{-\infty}^{\infty} f(x)\, dx,$$

by means of residues, where $f(z)$ is holomorphic in the upper closed half plane $\mathrm{Im}(z) \geq 0$, or in the lower closed half plane $\mathrm{Im}(z) \leq 0$, except at a set of (countably) infinite isolated singularities $A = \{z_n \mid n \in \mathbb{N}\}$.

Let us assume that we work in the **upper closed half plane** $[\mathrm{Im}(z) \geq 0]$ under the following hypotheses:

1. We assume that there are no singularities of $f(z)$ on the x-axis. That is, $\mathrm{Im}(z_n) \neq 0,\ \forall\ n \in \mathbb{N}$.
2. Since the singularities are isolated, there is no convergent subsequence of $(z_n)_{n\in\mathbb{N}}$, and they can be ordered so that $|z_1| \leq |z_2| \leq |z_3| \leq \ldots$. Then we have: $|z_n| \to \infty$, as $n \to \infty$.
3. For any positive integer l, we can find a number $R_l > 0$ and a simple closed contour C_l with interior D_l in the upper closed half plane such that:
   - (a) $R_l \uparrow \infty$, as $l \to \infty$ (strictly increasing to ∞).
   - (b) $\forall\ l \in \mathbb{N},\ C_l \cap A = \emptyset$, that is, no singularity lies on any of the contours.
   - (c) Since the singularities are isolated, any interior D_l contains finitely many of them. Assume $|A \cap D_l| = k_l > 0$ integer and let $A \cap D_l = \{z_1, z_2, \ldots, z_{k_l}\}$.
   - (d) $C_l = [-R_l, R_l] \cup P_l$ and $P_l \cap \{x-\text{axis}\} = \{-R_l, R_l\}$ (set of two elements). That is, the contour C_l consists of two parts: the closed segment $[-R_l, R_l]$ of the x-axis (and C_l has no other points of the x-axis) and a curve P_l in the upper half plane.
   - (e) For any $k \neq l$, we have $P_k \cap P_l = \emptyset$, that is, the parts of the contours not on the x-axis are pairwise disjoint.
   - (f) If $k < l$, then $D_k \subsetneq D_l$, that is, the interiors of the contours are in strict increasing order.
   - (g) $A \subset \bigcup\limits_{l\in\mathbb{N}} D_l$, that is, all singularities are in the infinite union of the interiors of the contours.
4. Finally, we assume

$$\lim_{l\to\infty} \int\limits_{P_l} f(z)\, dz = 0.$$

Then, under the above hypotheses (conditions), by the **Residue Theorem, 3.7.1**, we get

$$\oint_{C_l^+} f(z)\,dz = 2\pi\,i \sum_{k=1}^{k_l} \operatorname*{Res}_{z=z_k} f(z),$$

and so

$$\int_{-R_l}^{R_l} f(x)\,dx = 2\pi\,i \sum_{k=1}^{k_l} \operatorname*{Res}_{z=z_k} f(z) - \int_{P_l} f(z)\,dz.$$

Taking limits as $l \longrightarrow \infty$, by the last hypothesis, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi\,i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z),$$

provided that the integral exists and the series converges.

More generally, we have computed the principal value of the integral, if this principal value exists, since the limit is taken over the symmetrical intervals $[-R_l, R_l] \longrightarrow (-\infty, \infty)$, as $l \longrightarrow \infty$. I.e.,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi\,i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z),$$

provided that the principal value of the integral exists and the series converges.

This infinite series of residues may or may not converge even if the integral exists. There may be cases in which the integral exists but the series oscillates. For example, see **Problem 3.7.27**. **But, if the integral exists or its principal value exists and the series converges, the above equalities are valid, and thus we have evaluated the integral or its principal value by this series of numbers.**

With analogous hypotheses and work in the **lower closed half plane** $\operatorname{Im}(z) \leq 0$, we find

$$\int_{-\infty}^{\infty} f(x)\,dx = -2\pi\,i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z), \text{ where } \operatorname{Re}(z_n) < 0,\ \forall\ n \in \mathbb{N}$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = -2\pi\,i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z), \text{ where } \operatorname{Re}(z_n) < 0,\ \forall\ n \in \mathbb{N},$$

provided that the integral or its principal value exists and the

series converges.

Example 3.7.18 We easily observe that the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{sech}(x)}{1+x^2}\,dx$$

exists. We will try to evaluate it as a convergent infinite series.

The corresponding complex function $f(z) = \dfrac{1}{(1+z^2)\cosh(z)}$ has singularities at the solutions of $(1+z^2)\cosh(z) = 0$. These are

$$z = \pm i \ \text{ and } \ z = \frac{2k+1}{2}\pi\, i \ \text{ with } \ k \in \mathbb{Z}\,.$$

No singularity lies on the x-axis, and there are infinitely many **isolated** singularities in either of the half planes.

We choose to work in the upper closed half plane in which the singularities are:

$$i,\ \frac{1}{2}\pi\, i,\ \frac{3}{2}\pi\, i,\ \frac{5}{2}\pi\, i,\ \dots\,.$$

These are simple poles and their **absolute values tend to** ∞.

For $l = 1,\ 2,\ 3, \dots$, we pick numbers $0 < R_l = l\pi \uparrow \infty$ and simple closed contours $C_l^+ = [-R_l, R_l] \cup S_{R_l}^+$, where $P_l := S_{R_l}^+ = \{z = R_l e^{i\theta} \mid 0 \le \theta \le \pi\}$ is the upper positively oriented semicircle with center the origin and radius $R_l = l\pi$.

The numbers $l\pi$ and contours C_l^+ satisfy the conditions stated at the beginning of this subsection. In fact, the first three are obvious, but we need to verify the fourth one. So, in some way we need to verify

$$\lim_{l\to\infty} \int\limits_{P_l = S_{R_l}^+} f(z)\,dz = 0.$$

Here, we cannot apply **Lemma 3.7.1** because the condition $\lim\limits_{z\to\infty} z f(z) = 0$ does not hold due to the existence of infinitely many poles that tend to ∞. So, we must verify **hypothesis (4)** for our particular choice of the contours!

For these particular paths $P_l = S_{R_l}^+$, we can prove that

$$\lim_{l\to\infty} \left[R_l \max_{z\in S_{R_l}^+} |f(z)| \right] = 0.$$

Then, as we prove **Lemma 3.7.1**, in the same way we obtain **hypothesis (4)**. So, we have:

$$R_l \max_{z\in S^+_{R_l}} |f(z)| = R_l \max_{z\in S^+_{R_l}} \frac{1}{|1+z^2||\cosh(z)|} \leq$$

$$l\pi \max_{z\in S^+_{R_l}} \left[\frac{1}{|z|^2-1}\cdot\frac{1}{|\cosh(z)|}\right] = \frac{l\pi}{l^2\pi^2-1}\max_{z\in S^+_{R_l}}\frac{1}{|\cosh(z)|} =$$

$$\frac{l\pi}{l^2\pi^2-1}\frac{1}{\min\limits_{z\in S^+_{R_l}}|\cosh(z)|}.$$

Since for any $x = x+yi$ we have $|\cosh(z)|^2 = \cos^2(y)+\sinh^2(x)$ (see **Problem 3.2.14**), and x, y satisfy $x^2+y^2 = l^2\pi^2$, with $0 \leq y \leq l\pi$, we might think of using the Lagrange[22] multipliers method to find the minimum of $|\cosh(z)|^2$. Unfortunately, the derived system of three equations in three unknowns cannot be solved easily. So, we must find a way to estimate a lower bound of $|\cosh(z)|^2$ (from which we get a lower bound of $|\cosh(z)|$), which, hopefully, is greater than (not equal to) zero. This is done as follows:

If $l\pi - \frac{\pi}{4} \leq y \leq l\pi$, then $\frac{1}{2} \leq \cos^2(y) \leq 1$.

If $0 \leq y < l\pi - \frac{\pi}{4}$, then for $l \geq 1$ we have

$$|x| \geq \sqrt{l^2\pi^2 - \left(l-\frac{1}{4}\right)\pi^2} = \pi\sqrt{\frac{l}{2}-\frac{1}{16}} = \pi\sqrt{\frac{8l-1}{16}} =$$

$$\frac{\pi}{4}\sqrt{8l-1} \geq \frac{\pi}{4}\sqrt{7}.$$

Since for $x \geq 0$, $\sinh(x) \geq 0$ and increasing, we have

$$\sinh^2(x) = \frac{e^{2x}+e^{-2x}-2}{4} \geq \frac{e^{\frac{\pi}{2}\sqrt{7}}+e^{-\frac{\pi}{2}\sqrt{7}}-2}{4} > 15.45\ldots\ .$$

Therefore, in either case, we have that

$$|\cosh(z)|^2 = \cos^2(y)+\sinh^2(x) > \frac{1}{2}, \quad \text{and so} \quad \frac{1}{|\cosh(z)|} < \sqrt{2}.$$

We then conclude

$$0 < R_l \max_{z\in S^+_{R_l}} |f(z)| < \frac{l\pi\sqrt{2}}{l^2\pi^2-1} \longrightarrow 0, \text{ as } l \longrightarrow \infty.$$

---

[22] Joseph-Louis Lagrange, considered to be a French mathematician, Italian born and named Giuseppe Luigi Lagrancia or Lagrangia, 1736-1813.

Hence, for the chosen contours, we obtain

$$\lim_{l\to\infty} \int\limits_{P_l=S^+_{R_l}} f(z)\,dz = 0\cdot\pi = 0.$$

We have observed that the singularities are simple poles. Now, we must calculate their residues.

$$\operatorname*{Res}_{z=i} f(z) =$$

$$\frac{z-i}{(z^2+1)\cosh(z)}|_{z=i} = \frac{1}{(z+i)\cosh(z)}|_{z=i} = \frac{1}{2i\,\cosh(i)} = \frac{1}{2i\,\cos(1)}.$$

For any $n \geq 1$ integer, we have:

$$\operatorname*{Res}_{z=\frac{2n-1}{2}\pi\,i} f(z) = \frac{z-\frac{2n-1}{2}\pi\,i}{(z^2+1)\cosh(z)}|_{z=\frac{2n-1}{2}\pi\,i} = \frac{1}{(z^2+1)\sinh(z)}|_{z=\frac{2n-1}{2}\pi\,i} =$$

$$\frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]\sinh\left(\frac{2n-1}{2}\pi\,i\right)} = \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]\,i\,\sin\left(\frac{2n-1}{2}\pi\right)} =$$

$$\frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right](-1)^{n-1}\,i} = \frac{(-1)^n}{\left[\left(\frac{2n-1}{2}\right)^2\pi^2-1\right]\,i}.$$

So, finally,

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)\cosh(x)} = 2\pi\,i\left\{\frac{1}{2i\cos(1)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{i\left[\left(n-\frac{1}{2}\right)^2\pi^2-1\right]\,i}\right\} =$$

$$\frac{\pi}{\cos(1)}+2\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2-1},$$

and we see that this series converges absolutely.

Since $f(x) = \dfrac{1}{(1+x^2)\cosh(x)}$ is an even function, we also get

$$\int_{-\infty}^{0}\frac{dx}{(1+x^2)\cosh(x)} = \int_{0}^{\infty}\frac{dx}{(1+x^2)\cosh(x)} =$$

$$\frac{\pi}{2\cos(1)}+\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2-1}.$$

▲

Example 3.7.19 Notice

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)\sinh(x)} = 0,$$

since the function $f(x) = \dfrac{1}{(1+x^2)\sinh(x)}$ is odd. But,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\,dx$$

does not exist. [Prove this using $\sinh(0^+) = 0^+$ and $\sinh(0^-) = 0^-$, which imply $f(0^+) = +\infty$ and $f(0^-) = -\infty$. Next, $\lim\limits_{x\to 0} \dfrac{\sinh(x)}{x} = 1$, as $x \to 0$, and $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x}$ does not exist. Now, use the **p-Test, Example 1.3.3**.]

Here, the corresponding complex function has a singularity on the x-axis at $z = 0$.

(Continue in **Example 3.7.21**.)

▲

Isolated Singularities on Coordinate Axis and Cauchy Principal Value

We begin by proving again the **result**

$$\int_0^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2 \pi^2 - 1},$$

using **another method**, which is especially convenient with problems in which $f(z)$ has finitely or infinitely many **isolated simple poles**. In such a case, **Lemma 3.7.3** is a useful and convenient tool. In this new proof, we use the contour C_k in **Figure 3.13**.

(With **isolated poles of order greater than one** and other types of **isolated singularities**, we work case by case.)

We have already seen that in the closed upper half plane the function

$$f(z) = \frac{1}{(1+z^2)\cosh(z)}$$

has simple poles at

$$i,\ \frac{1}{2}\pi\, i,\ \frac{3}{2}\pi\, i,\ \frac{5}{2}\pi\, i,\ \dots,$$

with corresponding residues:

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2i\,\cos(1)}$$

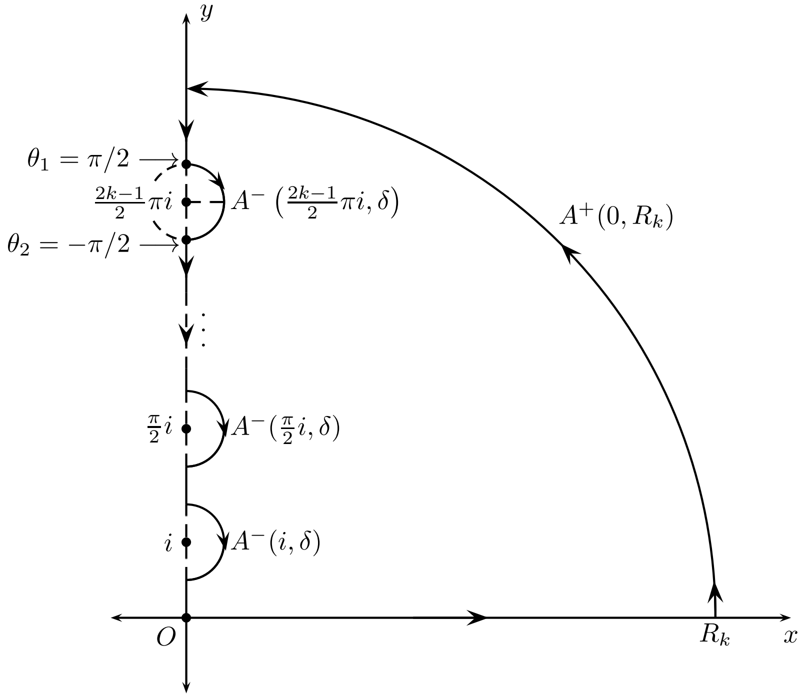

FIGURE 3.13: Contour 9 for the result examined here

and for any $k \geq 1$ integer

$$\operatorname*{Res}_{z=\frac{2k-1}{2}\pi\, i} f(z) = \frac{(-1)^k}{\left[\left(\frac{2k-1}{2}\right)^2 \pi^2 - 1\right] i}.$$

We have also seen if $R_k = k\pi$ and $A_k^+(0, R_k) = \{z = R_k e^{i\alpha} \mid 0 \leq \alpha \leq \frac{\pi}{2}\}$, then

$$\lim_{k\to\infty} R_k \max_{z\in A_k^+(0,R_k)} |f(z)| = 0.$$

We take $0 < \delta < \frac{1}{2}\left(\frac{\pi}{2} - 1\right) \simeq 0.2853982...$ small enough so that all the small semicircles of C_k^+, **Figure 3.13**, with centers the poles of $f(z)$ and radius δ, are pairwise disjoint. Given that there are no singularities of $f(z)$ in the interior of this contour, by the **Residue Theorem, 3.7.1**, we have:

$$\int_{C_k^+} f(z)dz = 0.$$

We must observe:

1. Along any negatively oriented semicircle, as above, the angle difference is $\theta_2 - \theta_1 = (-\frac{\pi}{2}) - (\frac{\pi}{2}) = -\pi$.
2. For any interval $[ui, vi]$ of the imaginary axis that does not contain any pole, the integral

   $$\int_u^v f(z)dz = \int_u^v \frac{d(iy)}{[1+(iy)^2]\cosh(iy)} = i\int_u^v \frac{dy}{(1-y^2)\cos(y)}$$

   is pure imaginary and so it does not affect the real integral

   $$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)}.$$

Then, by **Lemma 3.7.3** and taking the limit as $k \to \infty$, we find

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} + (-\pi i)\left[\frac{1}{2\cos(1)i} + \sum_{n=1}^\infty \frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2\pi^2 - 1\right]i}\right] = 0,$$

and so we obtain the **result**

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi\sum_{n=1}^\infty \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2 - 1}.$$

Important Remarks

1. Here, we first prove that

   $$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi\sum_{n=1}^\infty \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2 - 1},$$

   and then we use the evenness of the integrand function to conclude

   $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{\cos(1)} + 2\pi\sum_{n=1}^\infty \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2 - 1}.$$

2. If there were isolated singularities w's of $f(z)$ in the interior of the contours C_k^+'s, then we should add the $2\pi i \sum\limits_w \operatorname*{Res}\limits_{z=w} f(z)$ in the second side of the equality.

3. Since by construction of the contours C_k^+'s we avoid a symmetrical interval $(p - \delta i, p + \delta i)$ around each pole p of $f(z)$ located in the upper closed half plane, by letting $\delta \to 0$ we obtain:

$$\text{P.V.} \int_0^\infty f(iy)\, d(iy) = \text{P.V.} \int_0^\infty \frac{d(iy)}{[1 + (iy)^2] \cosh(iy)} =$$
$$\text{P.V. } i \int_0^\infty \frac{dy}{(1 - y^2) \cos(y)} = 0,$$

   and so
$$\text{P.V.} \int_0^\infty \frac{dy}{(1 - y^2) \cos(y)} = 0.$$

   I.e., the **Cauchy principal value** of this integral is equal to 0.

4. The integrand function has infinitely many isolated simple poles on the negative imaginary half axis. Then, in the same way, we get

$$\text{P.V.} \int_{-\infty}^0 \frac{dy}{(1 - y^2) \cos(y)} = 0.$$

   This with the evenness of $f(y) = \dfrac{1}{(1 - y^2) \cos(y)}$ implies

$$\text{P.V.} \int_{-\infty}^\infty \frac{dy}{(1 - y^2) \cos(y)} = 0.$$

5. If the integral exists, then its equal to its **Cauchy principal value**. In this example, the integral of $f(z)$ along the upper closed half imaginary axis, $[0i, \infty i)$, does not exist. (Prove this!)

6. Let

$$A = [0i, (1 - \delta)i] \cup \left[(1 + \delta)i, \left(\frac{\pi}{2} - \delta\right) i\right] \cup$$
$$\left(\bigcup_{l=1}^{\infty} \left[\left(\frac{2l - 1}{2}\pi + \delta\right) i, \left(\frac{2l + 1}{2}\pi - \delta\right) i\right]\right).$$

Then, we have:

$$(a) \qquad \int_A f(z)dz = 0,$$

$$(b) \qquad \lim_{0\leq R\to\infty} \int_{A\cap[0i,Ri]} f(z)dz = 0,$$

$$(c) \qquad \text{P.V.} \int_0^\infty f(z)dz = 0.$$

7. Analogous remarks and computations can be made when there are **finitely or infinitely many isolated singularities** on the x-axis or the y-axis.

We end this subsection by adding a result about the Cauchy principal value. In view of **Lemmata 3.7.1, 3.7.2 and 3.7.3**, we write and can directly prove the theorem that follows on **evaluating the Cauchy principal value of a certain type of integrals**.

If a function $f(x)$ has finitely many isolated singularities (x_i), with $i = 1,\ 2,\ \dots,\ n \in \mathbb{N}$, on the x-axis, we pick small $\epsilon > 0$ and big $R > 0$ and then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx =$$

$$\lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \left[\int_{-R}^{x_1-\epsilon} f(x)dx + \sum_{i=1}^{n} \int_{x_i-\epsilon}^{x_i+\epsilon} f(x)dx + \int_{x_n+\epsilon}^{R} f(x)dx \right].$$

Now, we state:

Theorem 3.7.4 *Suppose a complex function $f(z)$ satisfies the conditions of **Lemma 3.7.1** or **Lemma 3.7.2** with the exception that it is allowed to have finitely many **simple** (isolated) **poles** on the x axis. Then we have:*

$$P.V. \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum [\textit{residues of } f(z) \textit{ in the upper half plane}] +$$
$$\pi i \sum [\textit{residues of } f(z) \textit{on the real axis}]$$

or

$$P.V. \int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum [\textit{residues of } f(z) \textit{ in the lower half plane}] -$$
$$\pi i \sum [\textit{residues of } f(z) \textit{ on the real axis}].$$

(The **proof** follows from **Lemmata 3.7.1, 3.7.2 and 3.7.3**.)

Example 3.7.20 Find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 - 1}\, dx.$$

The complex function $f(z) = \dfrac{1}{z^4 - 1} = \dfrac{1}{(z-1)(z+1)(z-i)(z+i)}$ has two simple poles on the x-axis, the numbers $z = 1$ and $z = -1$, one simple pole, $z = i$, in the upper half plane and satisfies the conditions of **Theorem 3.7.4**. The corresponding residues are:

$$\begin{aligned}
\operatorname*{Res}_{z=1} f(z) &= \frac{1}{2(1-i)(1+i)} = \frac{1}{4}, \\
\operatorname*{Res}_{z=-1} f(z) &= \frac{1}{-2(-1-i)(-1+i)} = \frac{-1}{4}, \\
\operatorname*{Res}_{z=i} f(z) &= \frac{1}{(i-1)(i+1)2i} = \frac{-1}{4i}.
\end{aligned}$$

Therefore, by **Theorem 3.7.4**, we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 - 1}\, dx = 2\pi i \left(\frac{-1}{4i}\right) + \pi i \left(\frac{1}{4} - \frac{1}{4}\right) = \frac{-\pi}{2}.$$

▲

Example 3.7.21 We continue with **Example 3.7.19**. There we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\, dx = 0, \quad \text{but}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\, dx \quad \text{does not exist.}$$

The singularities of $f(z) = \dfrac{1}{(1+z^2)\sinh(z)}$ are the roots of the denominator $g(z) := (1+z^2)\sinh(z)$, namely: $z = \pm i$ and $z = k\pi i$ with $k \in \mathbb{Z}$. These roots of $g(z)$ are simple, and so they are simple poles of $f(z)$.

For $k = 0$, we get the pole $z = 0$, which is on the x-axis, and

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

We work in the upper half plane, and so

$$\operatorname*{Res}_{z=i} f(z) = \frac{-1}{2\sin(1)} = \frac{-1}{2}\csc(1).$$

At $z = k\pi i$, with $k \in \mathbb{N}$, we find

$$\operatorname*{Res}_{z=k\pi i} f(z) = \frac{1}{(1-k^2\pi^2)(-1)^k} = \frac{(-1)^{k-1}}{\pi^2\left[k^2 - \left(\frac{1}{\pi}\right)^2\right]}.$$

Therefore, by **Theorem 3.7.4**, and since the above principal value is zero, it must be

$$\pi i \cdot 1 + 2\pi i \cdot \left(\frac{-1}{2}\right)\csc(1) + 2\pi i \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi^2\left[k^2 - \left(\frac{1}{\pi}\right)^2\right]} =$$

$$\pi i \left\{ 1 - \csc(1) + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\left[k^2 - \left(\frac{1}{\pi}\right)^2\right]} \right\} = 0.$$

In fact, this sum is zero by the result in **Problem 3.7.41, (b)**.

Thinking in the reverse way, we can use **Theorem 3.7.4** and the obvious fact that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)\sinh(x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{a^2+x^2}\,dx = 0$$

and derive the result of **Problem 3.7.41 (b)** for any $a \notin \mathbb{Z}$.

▲

Application: Here, we use the results of **Problem 3.7.35** below in a statistical application. If we assume that some measurements $X = x$ have normal distribution (see **Subsection 2.1.1, Application 1**)

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty,$$

and we want to find the average or mean value of $Y = \dfrac{1}{x}$, then by **Problem 3.7.35 (a)** this mean value does not exist. So, if we experiment with many such measurements in an irregular or a random way, we can get any awkward results, in general.

If, however, we make many measurements in a way that they are evenly or almost evenly distributed about the mean μ, or we have strong natural reasons to suppose so, then using principal values, by **Problem**

3.7.35 (d) with $a = \sqrt{2}\,\sigma$, we find

$$E(Y) = \frac{1}{\sigma\sqrt{2\pi}}\ P.V. \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{x}\, dx\ =$$

$$\frac{e}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{x^2+2\sigma^2} \left\{ x\ \cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right] - a\ \sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\} dx.$$

For example, if $\mu = 2$ and $\sigma = \dfrac{1}{\sqrt{2}} \simeq 0.70711\ldots$ and we make a large number of approximately evenly distributed measurements, as above, about the mean $\mu = 2$, 3000 of them let us say, then, by **Problem 3.7.35 (e)**, we find

$$E(Y) \simeq \frac{1}{\sigma\sqrt{2\pi}} \cdot 1.06822\ldots =$$

$$\frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} \cdot 1.06822\ldots = \frac{1}{\sqrt{\pi}} \cdot 1.06822\ldots \simeq 0.60268\ldots\ .$$

If $\mu = 0$, the principal value of the expected value of $Y = \dfrac{1}{x}$ of many measurements symmetrical about zero is zero, as expected.

Problems

3.7.27 (a) Prove directly that

$$\int_{-\infty}^{\infty} \frac{1}{\cosh(x)}\, dx = \int_{-\infty}^{\infty} \operatorname{sech}(x)\, dx = \pi =$$

$$2\int_{0}^{\infty} \frac{1}{\cosh(x)}\, dx = 2\int_{0}^{\infty} \operatorname{sech}(x)\, dx =$$

$$2\int_{-\infty}^{0} \frac{1}{\cosh(x)}\, dx = 2\int_{-\infty}^{0} \operatorname{sech}(x)\, dx,$$

but this integral cannot be computed by the infinite series of the respected residues.

(b) Prove that the integral

$$\int_{-\infty}^{\infty} \frac{x}{\cosh(x)}\, dx$$

exists absolutely and so is equal to zero as an integral of an odd function. But, explain why this integral cannot be computed by the infinite series of the respected residues, as we have seen for **Example 3.7.18**.

3.7.28 Since $\frac{z}{\sinh(z)}|_{z=0} = 1$, the singularity of $f(z) = \frac{z}{(1+z^4)\sinh(z)}$ at $z = 0$ is removable. Then, prove

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^4)\sinh(x)}\,dx = \int_{-\infty}^{\infty} \frac{x}{1+x^4}\ \operatorname{csch}(x)\,dx =$$
$$\frac{\pi\ \sinh\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right)+\sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4\pi^4+1}.$$

3.7.29 For $f(x) = \frac{x}{\sinh(x)}$, prove that its integral over $\mathbb{R}$ exists and

$$0 < \int_{-\infty}^{\infty} \frac{x}{\sinh(x)}\,dx = \int_{-\infty}^{\infty} x\operatorname{csch}(x)\,dx < 3+\sqrt{e},$$

but it cannot be evaluated by the infinite series of residues.

3.7.30 For $-\pi < a < \pi$, prove that

$$\int_{-\infty}^{\infty} \frac{\sinh(ax)}{\sinh(\pi x)}\,dx = \tan\left(\frac{a}{2}\right).$$

3.7.31 Evaluate the integrals

$$\text{(a)} \qquad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)[\cos(x)+4]}$$

and

$$\text{(b)} \qquad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)[\sin(x)+2]}$$

by means of infinite series.

3.7.32 (A result of **Legendre A.- M.**. See also **Example 3.7.24** and **Problem 3.7.48**.)

(a) Integrate the complex functions

$$f(z) = \frac{e^{aiz}}{e^{2\pi z}-1} \quad \text{and} \quad g(z) = \frac{e^{-aiz}}{e^{2\pi z}-1},$$

where $a \in \mathbb{C}$ such that $|\mathrm{Im}(a)| < 2\pi$, over the positively oriented **contour** provided in **Figure 3.14**, use **Lemma 3.7.3**, let $\epsilon \to 0^+$ and $R \to \infty$ and then subtract the two results to prove:

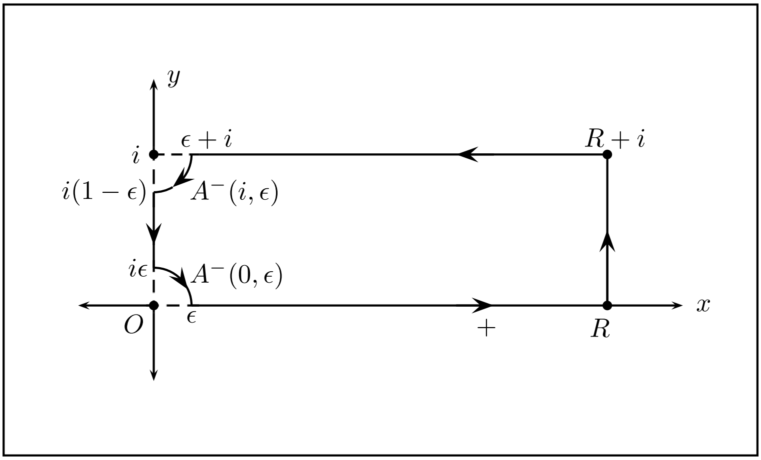

FIGURE 3.14: Contour 10 for Problem 3.7.32 (a)

$$\int_0^\infty \frac{\sin(ax)}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}.$$

(The condition $|\mathrm{Im}(a)| < 2\pi$ is used in showing that the limit of the integral of either function over the interval of constant length $[R+0i, R+1i]$ is zero, as $R \to \infty$.)

(b) Now, manipulate the constants α and β in $\mathbb{C}$, such that $\mathrm{Re}(\beta) > 0$ and $|\mathrm{Im}(\alpha)| < \mathrm{Re}(\beta)$, to prove

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x}-1}\,dx = \frac{\pi}{2\beta}\coth\left(\frac{\alpha\pi}{\beta}\right) - \frac{1}{2a}.$$

(c) From the processes in **(a)** and **(b)**, show the byproduct

$$\lim_{\epsilon\to 0^+}\left\{\lim_{r\to\infty}\left[2\,\sinh\left(\frac{2\pi\alpha}{\beta}\right)\int_\epsilon^r \frac{\cos(\alpha u)}{e^{\beta u}-1}\,du\right] + \int_\epsilon^{\frac{2\pi}{\beta}-\epsilon} \sinh(\alpha u)\cot\left(\frac{\beta u}{2}\right)\,du\right\} = 0.$$

(d) Prove that under the same conditions on the constants α and β, we have

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x}+1}\,dx = -\frac{\pi}{2}\operatorname{csch}\left(\frac{\alpha\pi}{\beta}\right)+\frac{1}{2\alpha}.$$

3.7.33 (See also **Example 3.7.24** and **Problem 3.7.48**.) Imitate or use the **previous problem** and some identities in **Problems 3.2.14** and **3.2.17** to prove that for constants a and a in $\mathbb{C}$, such that $\operatorname{Re}(b) > 0$ and $|\operatorname{Re}(a)| < \operatorname{Re}(b)$, we have

$$\text{(a)}\quad \int_0^\infty \frac{\sinh(ax)}{e^{bx}-1}\,dx = -\frac{\pi}{2b}\cot\left(\frac{a\pi}{b}\right)+\frac{1}{2a}$$

and

$$\text{(b)}\quad \int_0^\infty \frac{\sinh(ax)}{e^{bx}+1}\,dx = \frac{\pi}{2b}\csc\left(\frac{a\pi}{b}\right)-\frac{1}{2a}.$$

3.7.34 Find the principal values:

$$\text{(a)}\quad \text{P.V.}\int_{-\infty}^\infty \frac{dx}{x^2-1}(=0),\quad \text{(b)}\quad \text{P.V.}\int_{-\infty}^\infty \frac{dx}{x^3-1}(=\frac{\pi}{\sqrt{3}}),$$
$$\text{(c) P.V.}\int_{-\infty}^\infty \frac{dx}{x^3+1},\ \text{(d) P.V.}\int_{-\infty}^\infty \frac{xdx}{x^3-1},\ \text{(e) P.V.}\int_{-\infty}^\infty \frac{xdx}{x^3+1}.$$

3.7.35 The following results, besides being interesting for their sake, can be useful to problems in probability, statistics and expected values with the normal distribution, as we have seen in the application above.

(a) For any $\mu \in \mathbb{R}$ and any $a > 0$, prove

$$\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\,dx$$

does not exist.

(b) Consider any $b \in \mathbb{R}-\{0\}$, any $a > 0$ and $0 < \epsilon < R$ constants. Integrate the function $f(z) = \dfrac{e^{-\frac{(z-\mu)^2}{a^2}}}{z}$ along the contour

$$C = [-R,-\epsilon]+S_\epsilon^- +[\epsilon,R]+[R,R+bi]+[R+bi,-R+bi]+[-R+bi,-R],$$

take limits as $R \to \infty$ and $\epsilon \to 0^+$ and use **Lemma 3.7.3** [$z = 0$ is a simple pole for $f(z)$] to prove

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx =$$

$$e^{\frac{b^2}{a^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2} \left\{ x \cos\left[\frac{2b(x-\mu)}{a^2}\right] - b \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx,$$

(the same answer for all b's) and

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2} \left\{ b \cos\left[\frac{2b(x-\mu)}{a^2}\right] + x \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx =$$

$$\operatorname{sign}(b)\, \pi e^{\frac{-(\mu^2+b^2)}{a^2}}.$$

(c) What do you observe in

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2} \left\{ b \cos\left[\frac{2b(x-\mu)}{a^2}\right] + x \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx =$$

$$\operatorname{sign}(b)\, \pi e^{\frac{-(\mu^2+b^2)}{a^2}}$$

when $b \to 0$?

(d) Replace b with ab or a (a and b as before) to get the following two simpler forms of

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx =$$

$$e^{b^2} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+(ab)^2} \left\{ x \cos\left[\frac{2b(x-\mu)}{a}\right] - ab \sin\left[\frac{2b(x-\mu)}{a}\right] \right\} dx,$$

(the same answer for all b's) and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx =$$

$$e \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+a^2} \left\{ x \cos\left[\frac{2(x-\mu)}{a}\right] - a \sin\left[\frac{2(x-\mu)}{a}\right] \right\} dx.$$

(e) Let $a = 1$ and $\mu = 2$. Estimate that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-(x-2)^2}}{x}\, dx = 1.06822\ldots.$$

From this, guess the estimate of

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-(x+2)^2}}{x}\, dx.$$

(f) With $a > 0$, $b \neq 0$ and μ as before, integrate

$$e^{\frac{b^2}{a^2}} \cdot \frac{e^{-\frac{(z-\mu)^2}{a^2}}}{z^2 + b^2} \left\{ z \cos\left[\frac{2b(z-\mu)}{a^2}\right] - b \sin\left[\frac{2b(z-\mu)}{a^2}\right] \right\}$$

along the rectangular contour

$$C = [-R, R] + [R, R + ci] + [R + ci, -R + ci] + [-R + ci, -R],$$

where $c > |b| > 0$ constant, and observe that the imaginary part of this integral is $\pi e^{-\frac{\mu^2}{a^2}}$ (and so independent of b).

[Hint: You do not need to carry out all the computations. Use the **Residue Theorem, 3.7.1**.]

(g) With $a > 0$, $b \neq 0$ and μ as before, integrate

$$\frac{e^{-\frac{(z-\mu)^2}{a^2}}}{z^2 + b^2} \left\{ b \cos\left[\frac{2b(z-\mu)}{a^2}\right] + z \sin\left[\frac{2b(z-\mu)}{a^2}\right] \right\}$$

along the rectangular contour

$$C = [-R, R] + [R, R + ci] + [R + ci, -R + ci] + [-R + ci, -R],$$

where $c > |b| > 0$ constant, and observe that the real part of this integral is $\operatorname{sign}(b)\, \pi e^{\frac{-(\mu^2+b^2)}{a^2}}$.

[Hint: You do not need to carry out all the computations. Use the **Residue Theorem, 3.7.1**.]

(h) For any $\mu \in \mathbb{R}$ and any $a > 0$, prove

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2}\, dx = \infty = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2}\, dx.$$

3.7.4 Infinite Isolated Singularities and Series

We consider a holomorphic function $f(z)$ in $\mathbb{C} - A$, where $A \subset \mathbb{C}$ is a countable set of isolated singularities of $f(z)$.

Suppose we can find simple closed contours (circles, squares, parallelograms, etc.) C_l with interior D_l, such that:

(a) $\mathbb{C} = \bigcup_{l=1}^{\infty} D_l$, and

(b) $\lim_{l \to \infty} \oint_{C_l} f(z)\, dz = 0$.

Then,

$$\sum_{w \in A} \operatorname*{Res}_{z=w} f(z) = 0.$$

This allows us to evaluate certain infinite series, as we do in the following:

Example 3.7.22 For any $a \in \mathbb{C}$, such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer), prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a),$$

from which we immediately obtain the **result**:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a} \left[\pi \coth(\pi a) - \frac{1}{a} \right].$$

We consider the function

$$f(z) = \frac{1}{z^2 + a^2} \cot(\pi z),$$

which has simple poles at $z = \pm a\,i$ and $z = n$ for all $n \in \mathbb{Z}$.

Convenient contours, in this case, are the (positively oriented) squares with vertices $\pm(n + \frac{1}{2}) \pm (n + \frac{1}{2})i$, for $n = 1, 2, 3, \ldots$. (See **Figure 3.15**.)

The residues are:

$$\operatorname*{Res}_{z=a\,i} f(z) = \frac{1}{2a\,i} \cot(\pi a\,i) = \frac{-1}{2a} \coth(\pi a),$$

$$\operatorname*{Res}_{z=-a\,i} f(z) = \frac{1}{-2a\,i} \cot(-\pi a\,i) = \frac{-1}{2a} \coth(\pi a),$$

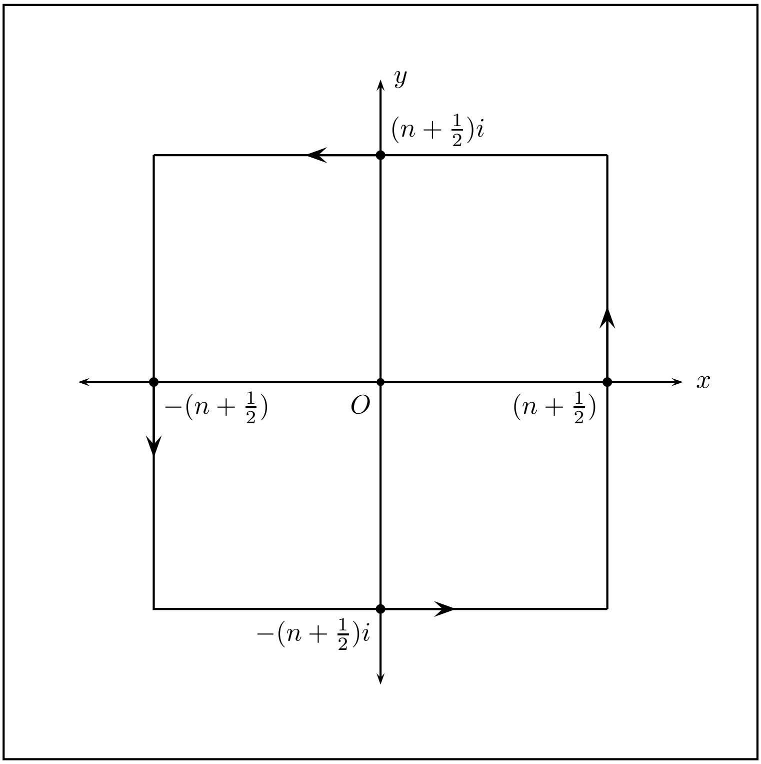

FIGURE 3.15: Contour 11 for Example 3.7.22

and ∀ $n \in \mathbb{Z}$, we have

$$\operatorname*{Res}_{z=n} f(z) = \frac{(z-n)\cos(\pi z)}{(z^2+a^2)\sin(\pi z)}|_{z=n} =$$

$$\frac{(-1)^n}{n^2+a^2}\frac{1}{\pi\ \cos(\pi z)}|_{z=n} = \frac{1}{\pi\,(n^2+a^2)}.$$

Now, we must check if for any $n \in \mathbb{N}$

$$\lim_{n\to\infty} \oint\limits_{C_n^+} f(z)dz = 0.$$

Obviously,

$$\oint\limits_{C_n^+} f(z)dz = I_1(n) + I_2(n) + I_3(n) + I_4(n),$$

with

$$I_1(n) = \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left[x - \left(n+\frac{1}{2}\right) i\right]^2 + a^2} \cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\} dx,$$

$$I_2(n) = \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left[\left(n+\frac{1}{2}\right) + y\,i\right]^2 + a^2} \cot\left\{\pi\left[\left(n+\frac{1}{2}\right) + y\,i\right]\right\} dy,$$

$$I_3(n) = \int_{\left(n+\frac{1}{2}\right)}^{-\left(n+\frac{1}{2}\right)} \frac{1}{\left[x + \left(n+\frac{1}{2}\right) i\right]^2 + a^2} \cot\left\{\pi\left[x + \left(n+\frac{1}{2}\right) i\right]\right\} dx,$$

$$I_4(n) =$$

$$\int_{\left(n+\frac{1}{2}\right)}^{-\left(n+\frac{1}{2}\right)} \frac{1}{\left[-\left(n+\frac{1}{2}\right) + y\,i\right]^2 + a^2} \cot\left\{\pi\left[-\left(n+\frac{1}{2}\right) + y\,i\right]\right\} dy.$$

We will show that for $j = 1,\ 2,\ 3,\ 4$

$$\lim_{n\to\infty} I_j(n) = 0.$$

We will do this for $j = 1$ and 2, and the work is similar with $j = 3$ and 4. For $n \geq a$, we have

$$|I_1(n)| \leq$$

$$\int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left|x - \left(n+\frac{1}{2}\right) i\right|^2 - a^2} \left|\cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right| dx =$$

$$\int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{x^2 + \left(n+\frac{1}{2}\right)^2 - a^2} \left|\cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right| dx.$$

But (see **Problem 3.2.14**), we have

$$\left|\cos\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right|^2 = \cos^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]$$

and

$$\left|\sin\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right|^2 = \sin^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right].$$

Therefore,

$$\left|\cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right| = \sqrt{\frac{\cos^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]}{\sin^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]}} \longrightarrow$$

$$\sqrt{1} = 1, \quad \text{as} \quad n \longrightarrow \infty.$$

So, for n large $\left|\cot\left[\pi\left(x-\left(n+\frac{1}{2}\right)i\right)\right]\right|<2$.

Hence, for n large

$$|I_1(n)| \le \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{2}{x^2+\left(n+\frac{1}{2}\right)^2-a^2}\,dx =$$

$$\left[\frac{2}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\arctan\left[\frac{x}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]\right]_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} =$$

$$\frac{2}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\cdot\left\{\arctan\left[\frac{n+\frac{1}{2}}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]-\right.$$

$$\left.\arctan\left[-\frac{n+\frac{1}{2}}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]\right\} \longrightarrow \frac{2}{\infty}\left[\arctan(1)-\arctan(-1)\right] =$$

$$\frac{2}{\infty}\left(\frac{\pi}{4}-\frac{-\pi}{4}\right)=\frac{\pi}{\infty}=0, \quad \text{as} \quad n\longrightarrow\infty.$$

Similarly, for n large, we have

$$|I_2(n)| \le \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{y^2+\left(n+\frac{1}{2}\right)^2-a^2}\,dy \longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty$$

and

$$I_3(n)\longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty \qquad \text{and} \qquad I_4(n)\longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty.$$

Therefore,

$$\lim_{n\to\infty}\oint_{C_n} f(z)dz = 0.$$

Then, $\sum_{w\in A}\operatorname*{Res}_{z=w} f(z)=0$, and so

$$2\cdot\frac{-1}{2a}\cdot\coth(\pi a)+\sum_{n=-\infty}^{\infty}\frac{1}{\pi(n^2+a^2)}=0.$$

Hence, we obtain the following:

Result:

$$\forall\, a \in \mathbb{C}, \qquad \sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{a}\coth(\pi a),$$

and so

$$\forall\, a \,\epsilon\, \mathbb{C}, \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a), -\frac{1}{2a^2} = \frac{1}{2a}\left[\pi\coth(\pi a) - \frac{1}{a}\right].$$

*(*When $a = ni$, with $n\epsilon\mathbb{Z}$, the two equations take the form $\infty = \infty$.)

Corollary 3.7.2 *Notice that this final sum can be rewritten as*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi a\,\cosh(\pi a) - \sinh(\pi a)}{2a^2\sinh(\pi a)}.$$

Corollary 3.7.3 *By letting a be real in the* ***above corollary*** *and applying L' Hôpital's rule as $a \to 0^+$, we find* ***Euler's sum***[23]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(For a direct proof, see **Problem 3.7.36**.)

Corollary 3.7.4 *Replacing a with $ai \in \mathbb{C}$, where $a \notin \mathbb{Z}$, we find the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a}\left[\frac{1}{a} - \pi\cot(\pi a)\right].$$

(See also **Problems 3.7.37** and **3.7.61**.)

▲

Example 3.7.23 We suppose that $a \notin \mathbb{Z}$ and rewrite the result of the **previous corollary** as

$$\pi\cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2-n^2} = \sum_{\substack{n=-\infty \\ n\neq 0}} \frac{a}{n(a-n)}.$$

(The third expression yields the second one by associating the terms of the opposite indices.)

---

[23] For even exponents, we can prove $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$, $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$, etc.

The $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is not known yet. It still remains a famous open problem. In 1979, the Greek-French mathematician Roger Apéry, 1916-1994, proved that this number is irrational. It still remains open whether it is algebraic or transcendental. These sums are related to the **Riemann zeta function** and the **Bernoulli numbers**, after Jakob Bernoulli.

We are going to manipulate this triple equation to derive some interesting results. (See also **Problem 3.7.61.**)

We notice that

$$\frac{a}{n(a-n)} = \frac{1}{a-n} + \frac{1}{n}.$$

So, by considering the symmetric summation about $n = 0$ and associating the terms with apposite indices, the fractions $\frac{1}{n}$'s cancel, and we find

$$\pi \cot(\pi a) - \frac{1}{a} = \lim_{m \to \infty} \sum_{\substack{n=-m \\ n \neq 0}}^{m} \frac{1}{a-n},$$

or

$$\pi \cot(\pi a) = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{a-n}. \tag{3.20}$$

Now, we work with

$$\lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{a-n}.$$

We associate the even and odd indices separately, and we obtain

$$\sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{a-n} = \sum_{n=-k}^{k} \frac{1}{a-2n} - \sum_{n=-(k+1)}^{k+1} \frac{1}{a-1-2n} + \frac{1}{a-2k-3} =$$
$$= \frac{1}{2} \sum_{n=-k}^{k} \frac{1}{\frac{a}{2}-n} - \frac{1}{2} \sum_{n=-(k+1)}^{k+1} \frac{1}{\frac{a-1}{2}-n} + \frac{1}{a-2k-3}.$$

We take the limit as $k \longrightarrow \infty$, use **(equation) 3.20** above and find

$$\lim_{k \to \infty} \sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{a-n} = \frac{\pi}{2} \cot\left(\frac{\pi a}{2}\right) - \frac{\pi}{2} \cot\left[\frac{\pi(a-1)}{2}\right] = \dots$$
$$= \frac{\pi}{\sin(\pi a)} = \pi \csc(\pi a).$$

Hence, we have obtained the important formula

$$\frac{\pi}{\sin(\pi a)} = \pi \csc(\pi a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-n}$$

(since this alternating series converges).

Now, if we let $a = \dfrac{1}{2}$, we find the **Leibniz formula for** π [which can also be derived by using the power series expansion of $\arctan(z)$, let $z = 1$ and use **Abel's**[24] **Lemma**[25]]

$$\pi = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{1}{2} - n} = 2\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} + \sum_{n=-1}^{-\infty} \frac{(-1)^{n+1}}{2n-1}\right] =$$
$$4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

and also with different bracketing

$$\pi = 2 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{4} - n^2} = 2 + 2\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots\right).$$

▲

Example 3.7.24 By means of series summation, we will evaluate the integral

$$\int_0^{\infty} \frac{\sin(ax)}{e^{bx} + c}\, dx,$$

where a, b and c are appropriate complex constants. In the process, we will discover the conditions that a, b and c must satisfy.

First, we rewrite the $\sin(ax)$ by its exponential form, and then we expand the integrand $\dfrac{\sin(ax)}{e^{bx} + c}$ as an infinite series by means of the geometric series, as follows:

$$\frac{\sin(ax)}{e^{bx} + c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right) e^{-bx}\, \frac{1}{1 + ce^{-bx}}.$$

To apply the geometric series expansion to the last fraction, we need minimum condition $\left|ce^{-bx}\right| < 1$ for all $x \in (0, \infty)$. Therefore, we must stipulate the conditions:

(1) $\operatorname{Re}(b) > 0$

---

[24] Niels Henrik Abel, Norwegian mathematician, 1802-1829.

[25] **Abel's Lemma**: Suppose the function defined by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{converges for} \quad -1 < x < 1$$

(radius of convergence is 1).

If the series $\sum_{n=0}^{\infty} c_n$ converges, then $\lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} c_n = f(1)$.
So, $f(x)$ converges at $x = 1$, and it is also (left) continuous at $x = 1$.

(2) $|c| \leq 1$
Then, we have:

$$\frac{\sin(ax)}{e^{bx}+c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)e^{-bx}\sum_{n=0}^{\infty}\left(-ce^{-bx}\right)^{n} =$$
$$\frac{1}{2i}\sum_{n=0}^{\infty}(-c)^{n}\left(e^{[ia-b(n+1)]x} - e^{-[ia+b(n+1)]x}\right). \tag{3.21}$$

To integrate the final sum on $[0, \infty)$ term by term, we need the condition

(3) $|\mathrm{Im}(a)| < (0+1)\mathrm{Re}(b) = \mathrm{Re}(b).$

Under this condition, we can apply **Theorem 2.3.4**. We observe that the function $f(x) = \dfrac{\sin(ax)}{e^{bx}-1}$ is continuous for $x > 0$ and approaches the value $\dfrac{a}{b}$, as $x \longrightarrow 0$. (Use L' Hôpital's Rule. If we put $c \neq 1$ in the place of -1, then the value of the new function is directly 0.) Therefore, $B := \max\limits_{0 \leq x \leq 1}|f(x)|$ is finite. So, we can use as dominating function the following function

$$g(x) = \begin{cases} B, & \text{if} \quad 0 \leq x \leq 1 \\ \\ \dfrac{e^{|Im(a)|x}}{e^{Re(b)x}-1}, & \text{if} \quad 1 < x < \infty. \end{cases}$$

Now, we integrate the above series, in **(3.21)**, term by term and use the fact that for any real numbers u, v, α and β the formula

$$\int_{u}^{v} e^{(\alpha+i\beta)x}dx = \frac{e^{v(\alpha+i\beta)} - e^{u(\alpha+i\beta)}}{\alpha+i\beta}$$

is valid (**Problem 3.7.40**).

Then, after a few straightforward computations, we arrive at the useful result:

Result: *If a, b and c are complex constants, such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| \leq 1$, then*

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx =$$

$$a\sum_{n=0}^{\infty}(-c)^n\frac{1}{a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2}. \tag{3.22}$$

Corollary 3.7.5 *This formula implies the following nine* ***corollaries****:*

(A) *For $c = -1$, $Re(b) > 0$ and $|Im(a)| < Re(b)$, by using the result of* ***Example 3.7.22****, we find*

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}-1}\,dx = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{1}{n^2+\left(\frac{a}{b}\right)^2} =$$

$$\frac{a}{b^2}\left[\frac{\pi}{2\frac{a}{b}}\coth\left(\frac{\pi a}{b}\right) - \frac{1}{2\left(\frac{a}{b}\right)^2}\right] = \frac{\pi}{2b}\coth\left(\frac{\pi a}{b}\right) - \frac{1}{2a},$$

which agrees with the result in ***Problem 3.7.32 (b)****.*

(B) *Similarly, for $c = 1$, by using* ***Problem 3.7.41 (a)****, we obtain the result in* ***Problem 3.7.32 (d)****, i.e:*

$$\text{If } |Im(a)| < Re(b), \text{ then: } \int_0^\infty \frac{\sin(ax)}{e^{bx}+1}\,dx = -\frac{\pi}{2b}\,csch\left(\frac{\pi a}{b}\right) + \frac{1}{2a}.$$

(C) *For $c = 0$, only the first term of the infinite sum survives, under the rule $(-c)^0 = 1$, and we find:*

$$\text{If } |Im(a)| < Re(b), \text{ then: } \int_0^\infty \sin(ax)\,e^{-bx}\,dx = \frac{a}{a^2+b^2},$$

which agrees with the result in ***Problem 1.2.14****.*

(D) *With a, b and c real constants, such that $b > 0$, and $|c| < 1$, we obtain the values of other real integrals by the corresponding infinite sums.*

(E) *We assume $a \neq 0$. Then, we divide both sides of the formula by a and take the limit as $a \longrightarrow 0$. (Notice that switching limit and integral and limit and sum is legitimate here.) Then, since $\lim\limits_{\alpha\to 0}\dfrac{\sin(\alpha x)}{\alpha} = x$, we obtain the* ***result****:*

If b and c are complex constants, such that Re$(b) > 0$ and $|c| \leq 1$, then

$$\int_0^\infty \frac{x}{e^{bx}+c}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2}. \tag{3.23}$$

For Re$(b) > 0$ and $c = -1$, by the result of ***Corollary 3.7.3 of Example 3.7.22****, we obtain*

$$\int_0^\infty \frac{x}{e^{bx}-1}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{b^2}\frac{\pi^2}{6}. \tag{3.24}$$

(See and compare with ***Problem 2.6.6****.)*

For Re$(b) > 0$ and $c = 1$ and the result of ***Problem 3.7.41 (c)****, we obtain the* ***result***

$$\int_0^\infty \frac{x}{e^{bx}+1}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} = \frac{1}{b^2}\frac{\pi^2}{12}. \tag{3.25}$$

[See also and compare with ***Problem 3.7.62 (5)****.]*

(F) *If in the equations* ***(3.23)****,* ***(3.24), and (3.25)*** *above we let $u = e^x$ or $x = \ln(u)$, then we find, respectively:*

If b and c are complex constants, such that Re$(b) > 0$, and $|c| \leq 1$, then

$$\int_1^\infty \frac{\ln(u)}{(u^b+c)u}\,du = \frac{1}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2},$$

$$\int_1^\infty \frac{u}{(u^b-1)u}\,du = \frac{1}{b^2}\frac{\pi^2}{6} \quad \text{and} \quad \int_1^\infty \frac{\ln(u)}{(u^b+1)u}\,du = \frac{1}{b^2}\frac{\pi^2}{12}.$$

(G) *If in* ***(F)*** *we let $u = \dfrac{1}{x}$, then we find, respectively:*

If b and c are complex constants, such that Re$(b) > 0$, and $|c| \leq 1$, then

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{c+x^b}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1-x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12} \quad \text{and} \quad \int_0^1 \frac{-x^{b-1}\ln(x)}{1+x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12}.$$

(H) *If a, b c, and d are complex constants, such that Re$(b) > 0$, $|$Im$(a)| <$ Re(b), $d \neq 0$ and $\left|\dfrac{c}{d}\right| \leq 1$, then the initial integral is dealt by*

$$\int_0^\infty \frac{\sin(ax)}{de^{bx}+c}\,dx = \frac{1}{d}\int_0^\infty \frac{\sin(ax)}{e^{bx}+\frac{c}{d}}\,dx,$$

and we use ***equation (3.22)*** *with $\dfrac{c}{d}$ in the place of c.*

We work analogously for a similar situation with ***equation (3.23)*** *in* ***(E)*** *above.*

(I) *If in* ***equation (3.22)*** *we replace a with ia and use the identity* $\sin(iz) = i\sinh(z)$, *we find:*

If a, b and c are complex constants, such that $Re(b) > 0$, $|Re(a)| < Re(b)$ and $|c| \leq 1$, then

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx}+c}\,dx = a\sum_{n=0}^\infty (-c)^n \frac{1}{-a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2-\left(\frac{a}{b}\right)^2}.$$

From this, we can obtain again ***equations (3.23), (3.24)*** *and* ***(3.25)****. (See also* ***Problem 3.7.59****.)*

▲

Example 3.7.25 Working as in the **previous Example, 3.7.24**, with cosine in the place of sine, we find:

If a, b and c are complex constants, such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| < 1$, then

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}, \tag{3.26}$$

and we have analogous remarks as in ***(H)*** of the **previous corollary**.

Notice that here we have $|c| < 1$, in general. If $|c| = 1$, we need to check the formula for the individual c. For instance, with $c = 1$ we get

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2},$$

which converges by the **Alternating Series Test**. For example, we have

$$\int_0^\infty \frac{\cos(x)}{e^x+1}\,dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2+1}, \quad \text{etc.}$$

Also, for $a = 0$ and b such that $\text{Re}(b) > 0$, we find

$$\int_0^\infty \frac{1}{e^{bx}+1}\,dx \overset{x=\ln(u)}{=} \int_1^\infty \frac{1}{u(u^b+1)}\,du = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} = \frac{\ln(2)}{b},$$

which can also be verified by the substitution $v = u^b$.

But, if $c = -1$,

$$\sum_{n=1}^\infty \frac{n}{n^2+\left(\frac{a}{b}\right)^2}$$

diverges. In this case, besides the infinite interval of integration, the integral becomes also improper at $x = 0$.

If we take the derivative of **equation (3.26)** with respect to a, we find: *If a, b and c are complex constants, such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| < 1$, then*

$$\int_0^\infty \frac{x\sin(ax)}{e^{bx}+c}\,dx = \frac{2a}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{\left[n^2+\left(\frac{a}{b}\right)^2\right]^2}.$$

If now we divide this equation by a and take the limit as $a \longrightarrow 0$, we find: *Under the conditions $Re(b) > 0$ and $|c| < 1$, we have*

$$\int_0^\infty \frac{x^2}{e^{bx}+c}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^3},$$

and so

$$\int_0^\infty \frac{x^2}{e^{bx}+1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3}$$

and

$$\int_0^\infty \frac{x^2}{e^{bx}-1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

Putting $x = \ln(u)$ in the last two equalities, with $b > 0$ real, we get

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b+1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3}$$

and

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b-1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

If now we let $u = \dfrac{1}{x}$ in the above two integrals, with $b > 0$ real, we obtain

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1+x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3}$$

and

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1-x^b}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

If in **equation (3.26)** we replace a with ia and use the identity $\cos(iz) = \cosh(z)$, we find: *If a, b and c are complex constants, such that $Re(b) > 0$, $|Re(a)| < Re(b)$ and $|c| < 1$, then*

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2-\left(\frac{a}{b}\right)^2}.$$

▲

Example 3.7.26 We consider the integral

$$I := \int_0^{\infty} \frac{\sin(ax)}{e^{bx} + c}\, dx,$$

where $a \in \mathbb{R}$, $b > 0$ and $c \geq -1$. Under these conditions, the integral converges absolutely. (The proof is easy!)

In **Example 3.7.24**, we have found an infinite sum formula for it when $-1 \leq c \leq 1$. Here, we will find a formula for $c > 1$. To this end, we work as follows:

So, we consider $c > 1$ and we write

$$I := \int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx} + c}\, dx + \int_{\frac{\ln(c)}{b}}^{\infty} \frac{\sin(ax)}{e^{bx} + c}\, dx.$$

Then we work with each part separately. So, we have:

$$\int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx} + c}\, dx = \frac{1}{c} \int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{1 + \frac{e^{bx}}{c}}\, dx.$$

Since $\dfrac{e^{bx}}{c} < 1$ when $0 \leq x < \dfrac{\ln(c)}{b}$, we have the geometric series

$$\frac{1}{1 + \frac{e^{bx}}{c}} = \sum_{n=0}^{\infty} \left(\frac{-e^{bx}}{c} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n}\, e^{nbx}.$$

The convergence is uniform on every interval $[0, r]$, with $0 < r < \dfrac{\ln(c)}{b}$, by the **Weierstraß M-Test, Theorem 2.3.1**, since with $c > 1$ the positive series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{e^{br}}{c} \right)^n$ has a finite sum.

So, by **Corollary 2.3.2 of Theorem 2.3.2**, we can integrate term by term below, and with the help of the well-known formulae in **Problem 3.7.40 (d)**, we find:

$$\int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx} + c}\, dx = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n} \int_0^{\frac{\ln(c)}{b}} \sin(ax) e^{nbx}\, dx =$$

$$\frac{a}{c} \sum_{n=0}^{\infty} \frac{(-1)^n}{(a^2 + b^2 n^2)\, c^n} + \frac{b}{c} \sin\left[\frac{a \ln(c)}{b} \right] \sum_{n=0}^{\infty} \frac{(-1)^n n}{a^2 + b^2 n^2} -$$

$$\frac{a}{c} \cos\left[\frac{a \ln(c)}{b} \right] \sum_{n=0}^{\infty} \frac{(-1)^n}{a^2 + b^2 n^2}.$$

For the second part of the integral I, we use the substitution $x = u + \dfrac{\ln(c)}{b}$, and we find

$$\int_{\frac{\ln(c)}{b}}^{\infty} \frac{\sin(ax)}{e^{bx}+c}\,dx =$$

$$\frac{1}{c}\cos\left[\frac{a\ln(c)}{b}\right]\int_0^{\infty}\frac{\sin(au)}{e^{bu}+1}\,du + \frac{1}{c}\sin\left[\frac{a\ln(c)}{b}\right]\int_0^{\infty}\frac{\cos(au)}{e^{bu}+1}\,du.$$

Then, using **formulae (3.22)** and **(3.26)** with $c = 1$, we find

$$\int_{\frac{\ln(c)}{b}}^{\infty} \frac{\sin(ax)}{e^{bx}+c}\,dx =$$

$$\frac{a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{a^2+b^2n^2} + \frac{b}{c}\sin\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^{n-1}n}{a^2+b^2n^2}.$$

Adding the two partial results found, we finally obtain the following:

Result: *For $a \in \mathbb{R}$, $b > 0$ and $c > 1$, we have*

$$I := \int_0^{\infty}\frac{\sin(ax)}{e^{bx}+c}\,dx = \frac{1-\cos\left[\frac{a\ln(c)}{b}\right]}{ac} +$$

$$\frac{a}{c}\sum_{n=1}^{\infty}\frac{(-1)^n}{(a^2+b^2n^2)c^n} + \frac{2a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{a^2+b^2n^2}$$

*which is also valid for $c = 1$, as in **Corollary 3.7.5 (B) of Example 3.7.24**.*

If now we divide by a in the above result and take the limit as $a \longrightarrow 0$, we find the following:

Result: *For $b > 0$ and $c > 1$, we have*

$$\int_0^{\infty}\frac{x}{e^{bx}+c}\,dx = \frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} =$$

$$\frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\frac{\pi^2}{12}$$

*which is also valid for $c = 1$, as in **equality (3.25)**.*

▲

Remark 1: As we have already seen in **Examples 3.7.18** (treated twice), **3.7.24, 3.7.25, 3.7.26** and in several **problems**, we can compute improper integrals or their principal values as infinite sums and/or infinite sums as improper integrals or their principal values. To obtain these kinds of formulae, we use partial fractions, the geometric series, **Problem 3.7.40, (b)**, **Theorem 2.3.4**, the **Residue Theorem, 3.7.1**, etc. (See also **Problem 3.7.55** below.)

Remark 2: All of the examples studied so far and those that will be studied in the sequel show the power of contour integration in obtaining formulae for difficult improper integrals and infinite sums very strongly. They also show that the choice of contour is a very important piece of art that needs a great deal of experience.

Problems

3.7.36 Find **Euler's sum** $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ directly by considering the complex function $f(z) = \frac{\cot(z)}{z^2}$, contours analogous to the ones used in **Example 3.7.22**, computing $\operatorname*{Res}_{z=0}[f(z)] = \frac{-1}{3}$ (see **Example 3.6.13**) and $\operatorname*{Res}_{z=n\pi}[f(z)] = \frac{1}{n^2\pi^2}, \ \forall \ n \in \mathbb{Z}$.

(See also **Corollary 3.7.3 of Example 3.7.22**.)

3.7.37 Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a}\left[\frac{1}{a} - \pi\cot(\pi a)\right]$$

by considering the complex function $f(z) = \frac{\coth(\pi z)}{z^2 + a^2}$ and working in a way similar to the one in **Example 3.7.22**.

(See also **Corollary 3.7.4 of Example 3.7.22**.)

3.7.38 Use **Euler's sum** $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to prove the three sums

(a) $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$, (b) $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, (c) $\sum_{k=1}^{\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2} = \frac{\pi^2}{2}$.

3.7.39 Prove

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2-\left(\frac{1}{2}\right)^2}=\frac{\pi}{4}.$$

[Hint: Use some already proven results or the power series of $\arctan(x)$.]

3.7.40 (a) Prove that for any complex number $\alpha+i\beta\neq 0+i0$ (α and β real) fixed, we have

$$\int e^{(\alpha+i\beta)x}dx=\frac{e^{(\alpha+i\beta)x}}{\alpha+i\beta}+w,$$

where $w=c+id$ is a complex constant.

(b) Use **(a)** to prove that for any $u,v\in\mathbb{R}$ and for any complex number $\alpha+i\beta\neq 0+i0$ (α and β are real) fixed, we have

$$\int_u^v e^{(\alpha+i\beta)x}dx=\frac{e^{v(\alpha+i\beta)}-e^{u(\alpha+i\beta)}}{\alpha+i\beta}.$$

(c) Prove that when $\alpha+i\beta=0+i0$, the answers in **(a)** and **(b)** are, respectively, $x+w$ and $v-u$.

(d) Use **(a)** to prove that when $\alpha+i\beta\neq 0+i0$,

$$\int e^{\alpha x}\cos(\beta x)\,dx=\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\cos(\beta x)+\beta\sin(\beta x)\right]+C_1$$

and

$$\int e^{\alpha x}\sin(\beta x)\,dx=\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta x)-\beta\cos(\beta x)\right]+C_2$$

where α and β are fixed real numbers and C_1, C_2 are the real constants of integration.

(In this way, you avoid integration by parts in these types of indefinite integrals! See **Problems 1.2.14** and **1.2.16**.)

3.7.41 (a) For any $a\in\mathbb{C}$, such that $a\neq ni$, with $n\in\mathbb{Z}$, use the function $f(z)=\dfrac{\csc(\pi z)}{z^2+a^2}$ to find

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a}\operatorname{csch}(\pi a),$$

from which we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a}\operatorname{csch}(\pi a) - \frac{1}{2a^2}.$$

(b) Now, replace a with $ai \in \mathbb{C}$, where $a \notin \mathbb{Z}$, to prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2-a^2} = \frac{-\pi}{a}\csc(\pi a),$$

from which we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\csc(\pi a).$$

(See also **Example 3.7.21**.)

(c) Now, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

3.7.42 Find the integrals: (a) $\displaystyle\int_0^\infty \frac{\sin(2x)}{7e^{3x} \pm 5}\,dx$, (b) $\displaystyle\int_0^\infty \frac{x}{7e^{3x} \pm 5}\,dx$.

3.7.43 Find the conditions on the complex constants a, b and c so that you can compute the integral

$$\int_0^\infty \frac{\sin^2(ax)}{e^{bx}+c}\,dx$$

as a series. Find this series and the byproducts of the obtained result.

[Hint: Follow the steps of **Example 3.7.24** and some parts in **Corollary 3.7.5**.]

3.7.44 Use the results of **Corollary 3.7.5 (E)** or **(F)** of **Example 3.7.24** to verify the following five integrals:

$$I_1 = \int_1^\infty \frac{\ln(u)}{(u+1)u}\,du = \frac{\pi^2}{12}, \qquad I_2 = \int_1^\infty \frac{\ln(u)}{(u-1)u}\,du = \frac{\pi^2}{6},$$

$$I_3 = \int_1^\infty \frac{\ln(u)}{(u^2+1)u}\,du = \frac{\pi^2}{48}, \qquad I_4 = \int_1^\infty \frac{\ln(u)}{(u^2-1)u}\,du = \frac{\pi^2}{24},$$

$$I_5 = \int_1^\infty \frac{\ln(u)}{u^2-1}\,du = \frac{\pi^2}{8}.$$

[See also and compare with **Examples 2.3.18, 3.7.44** and **Problem 3.7.62 (5)**.]

3.7.45 Use the **previous problem** to verify the following integrals:

$$I_1 = \int_0^1 \frac{\ln(v)}{1+v}\,dv = \frac{-\pi^2}{12}, \qquad I_2 = \int_0^1 \frac{\ln(v)}{1-v}\,dv = \frac{-\pi^2}{6},$$
$$I_3 = \int_0^1 \frac{v\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{24}, \qquad I_4 = \int_0^1 \frac{\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{8}.$$

3.7.46 Use the **previous problem** (and some elementary integration rules and techniques if necessary) to verify the following six integrals:

$$I_1 = \int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}, \qquad I_2 = \int_0^1 \frac{\ln(1-x)}{x}\,dx = \frac{-\pi^2}{6},$$
$$I_3 = \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{\pi^2}{4}, \qquad I_4 = \int_0^1 \frac{x\ln(x)}{1-x}\,dx = 1-\frac{\pi^2}{6},$$
$$I_5 = \int_0^1 \ln(x)\ln(1-x)\,dx = 2-\frac{\pi^2}{6},$$
$$I_6 = \int_0^1 \ln(x)\ln(1+x)\,dx = 2-2\ln(2)-\frac{\pi^2}{12}.$$

3.7.47 Prove

$$I_1 = \int_0^1 \ln\left[\ln\left(\frac{1}{x}\right)\right]dx = -\,\gamma, \qquad I_2 = \int_0^1 \frac{dx}{\sqrt{-\ln(x)}} = \sqrt{\pi}.$$

[Hint: You may need to use some properties of the Gamma function and its first derivative. See also **footnote** of **Problem 2.1.25** and **Problems 2.3.5, 2.6.4** and **2.6.43**.]

3.7.48 Use the four summation results of (1) **Example 3.7.22**, (2) its **Corollary 3.7.4**, (3) **Problem 3.7.41** and (4) the integral summation result of **Example 3.7.24** to derive the four general integrals claimed in **Problems 3.7.32** and **3.7.33**.

3.7.49 Use $f(z) = \dfrac{1}{z^3\cos(\pi z)}$ to find

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{1}{8}\sum_{n=0}^\infty \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{32}, \quad \text{and so} \quad \sum_{n=0}^\infty \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{4}.$$

3.7.50 Explain why the method of **Example 3.7.22** (or any other method presented in this **section**) cannot be used to figure out the following sums:

$$\text{(a)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^3 + a^3}, \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{n^3 + a^3}, \quad \text{where} \quad a \in \mathbb{C} - \mathbb{Z}.$$

$$\text{(b)} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^3 + a^3}, \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 + a^3}, \quad \text{where} \quad a \in \mathbb{C} - \mathbb{Z}.$$

$$\text{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{n^3}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.$$

3.7.51 (a) If $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} = \alpha$, then prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{8}{7}\,\alpha$.

(b) If $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k)^3} = \beta$, then prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = 8\beta$.

(So, $\alpha = 7\beta$.)

3.7.52 (a) For any $a \in \mathbb{C} - \mathbb{Z}$, use $f(z) = \dfrac{\cot(\pi z)}{(z+a)^2}$ to find

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \pi^2 \csc^2(\pi a) = \frac{\pi^2}{\sin^2(\pi a)}.$$

(b) Now, for any $a \in \mathbb{C} - \mathbb{Z}$, use $f(z) = \dfrac{\csc(\pi z)}{(z+a)^2}$ to find

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \csc(\pi a)\cot(\pi a) = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

3.7.53 (a) For $a \in \mathbb{C} - \mathbb{Z}$ and $b \in \mathbb{C} - \mathbb{Z}$, use $f(z) = \dfrac{\cot(\pi z)}{(z+a)(z+b)}$ to prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\cot(\pi a) - \cot(\pi b)\right].$$

Use this to solve **Problems 3.7.39** and **3.7.52 (a)**.

(b) Then, prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2+a^2} = \frac{\pi}{2ai}\left\{\cot[\pi(z-ai)] - \cot[\pi(z+ai)]\right\}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2-a^2} = \frac{\pi}{2a}\left\{\cot[\pi(z-a)] - \cot[\pi(z+a)]\right\}.$$

(c) Find the summations

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 \pm n+1} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2 \pm n-1}.$$

[(d) We can use **Results (a) and (b)** to derive several known summations. Spot and justify some of these summations in this way.]

3.7.54 (a) For $a \in \mathbb{C} - \mathbb{Z}$ and $b \in \mathbb{C} - \mathbb{Z}$, use $f(z) = \dfrac{\csc(\pi z)}{(z+a)(z+b)}$ to prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\csc(\pi a) - \csc(\pi b)\right].$$

Use this result to solve **Problem 3.7.52 (b)**.

(b) Then, prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2+a^2} = \frac{\pi}{2ai}\left\{\csc[\pi(z-ai)] - \csc[\pi(z+ai)]\right\}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2-a^2} = \frac{\pi}{2a}\left\{\csc[\pi(z-a)] - \csc[\pi(z+a)]\right\}.$$

(c) Also, find the summations

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 \pm n+1} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 \pm n-1}.$$

(d) How much is

$$\sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right)?$$

[(e) We can use **Results (a) and (b)** to derive several known summations. Spot and justify some of these summations in this way.]

3.7.55 Use partial fractions, **Problem 3.7.40, (b)**, the geometric series and **Theorem 2.3.4** to prove

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(3n+1)} = \frac{1}{2}\int_0^{\infty} \frac{e^{-x}\left(e^{-x} - e^{\frac{-x}{3}}\right)}{1-e^{-x}}\,dx.$$

Then, use the substitution $u = e^{-x}$ to find

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(3n+1)} = -\frac{1}{2}\left(1 + \int_0^1 \frac{u^{\frac{1}{3}}-1}{1-u}\,du\right).$$

3.7.56 Prove

$$\sum_{k=0}^{\infty} \frac{2k+1}{k^2(k+1)} = \frac{\pi^2}{6} + 1.$$

3.7.57 (a) Find

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^4}.$$

Then, prove

$$\sum_{n=1}^{\infty} \frac{1}{1+n^4} = \frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right) + \sinh\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

How much is

$$\sum_{n=0}^{\infty} \frac{1}{1+n^4}\ ?$$

[Hint: Use $f(z) = \dfrac{\cot(\pi z)}{1+z^4}$.]

(b) Find

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^4}.$$

Then, prove

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^4} = \frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right) + \sinh\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

How much is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^4} \ ?$$

[Hint: Use $f(z) = \dfrac{\csc(\pi z)}{1+z^4}$.]

3.7.58 (a) Prove

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n^2}{n^3+1} = \frac{1}{3}\left[1 - \ln(2) + \pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right)\right].$$

(b) Check the relation

$$\frac{z^2}{z^3+1} = \frac{1}{3}\left(\frac{1}{z+1} + \frac{2z-1}{z^2-z+1}\right),$$

and, knowing that

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

use the result in **(a)** to prove

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = \pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right).$$

3.7.59 Imitate the process of **Example 3.7.24** to find the conditions that the complex constants a, b and c must satisfy in order to evaluate the integral

$$\int_0^{\infty} \frac{\sinh(ax)}{e^{bx}+c}\, dx$$

as an infinite summation. In the process, find this infinite summation.

Obtain the same result by using the results of **Example 3.7.24** and the identity $\sinh(z) = -i\sin(iz)$.

Then, check your results against the results of **Problem 3.7.33**.

Replace $\sin(x)$ in **Example 3.7.24** or $\sinh(x)$ here with $\cos(x)$ or $\cosh(x)$ to obtain analogous results in any possible way. (See also **Example 3.7.25**.)

3.7.60 For $a \in \mathbb{R}$, $b > 0$ and $c \geq 1$, imitate the process of **Example 3.7.26** to find the integral

$$\int_0^{\infty} \frac{\cos(ax)}{e^{bx}+c}\, dx$$

as an infinite summation.

3.7.61 (a) For any $a \in \mathbb{C} - \mathbb{Z}$, compute all the residues of the function $f(z) := \dfrac{\pi \cot(\pi z)}{z(z-a)}$, integrate it along the square contours C_n^+, $n \in \mathbb{N}$, of **Example 3.7.22** and take the limit as $n \longrightarrow \infty$ to re-derive the result of **Corollary 3.7.4 of Example 3.7.22** and **Example 3.7.23**

$$\pi \cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}.$$

(b) Let $g(a) := \pi \cot(\pi a) - \dfrac{1}{a}$, with $a \in \mathbb{C}$. Prove that

$$\lim_{a \to 0} g(a) = 0.$$

So, by **Riemann's Theorem, 3.6.2,** and/or its **Corollary 3.6.1**, $g(a)$ is also analytic at $a = 0$.

Then, integrate $g(a)$ (with respect to a) from 0 to any $z \in \mathbb{C}$ by means of a continuous antiderivative and using $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$, with $\mathrm{Arg}(z) \in [-\pi, \pi)$, to obtain

$$\mathrm{Log}\left[\frac{\sin(\pi z)}{z}\right] = \log(\pi) + \sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right),$$

where $\log(\pi) = \mathrm{Log}(\pi) + 2k\pi i$ is a complex logarithm of π, for some $k \in \mathbb{Z}$, not necessarily the $\mathrm{Log}(\pi)$, i.e., k may not be zero.

{Hint: To establish this relation, we need a combination of justifications. We will sketchily provide them here, and the reader supplies the missing details.

First, $\pi \cot(\pi a)$ and $\dfrac{1}{a}$ have continuous antiderivatives in $\mathbb{C} - \{x$-non-positive axis$\}$ the $\mathrm{Log}\,[\sin(\pi a)]$ and $\mathrm{Log}(a)$, respectively. Then,

$$\mathrm{Log}\,[\sin(\pi a)] - \mathrm{Log}(a) = \mathrm{Log}\left[\frac{\sin(\pi a)}{a}\right] + 2k\pi i,$$

where $k = 0$, or 1, or -1, since the principal argument $\mathrm{Arg}(z)$ is in $[-\pi, \pi)$.

Then, the suggested integration from 0 to z, knowing that at $\displaystyle\lim_{a \to 0} \frac{\sin(\pi a)}{a} = \pi > 0$ and $\mathrm{Log}(\pi) = \ln(\pi) + 0i$, gives

$$\mathrm{Log}\left[\frac{\sin(\pi z)}{z}\right] - \mathrm{Log}(\pi).$$

Next, the sum

$$\sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$$

converges absolutely and uniformly on any closed disc $\overline{D(0,R)}$, for any $R > 0$. This is so because, given an $R > 0$, we pick the integer $m = [\![R]\!] + 1$, and then for any $a \in \overline{D(0,R)}$ we have

$$\sum_{n=m}^{\infty} \left| \frac{2a}{a^2 - n^2} \right| \leq \sum_{n=m}^{\infty} \frac{2|a|}{n^2 - |a|^2} \leq \sum_{n=m}^{\infty} \frac{2|R|}{n^2 - |R|^2} < \infty.$$

So, the series converges absolutely. The uniform convergence follows from the **Weierstraß M-Test, Theorem 2.3.1**.

So, switching integration and summation below is legitimate by the uniform convergence, and we get

$$\int_0^z \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} da = \sum_{n=1}^{\infty} \int_0^z \frac{2a}{a^2 - n^2} da =$$

$$\lim_{m \to \infty} \sum_{n=1}^{m} \int_0^z \frac{2a}{a^2 - n^2} da = \lim_{m \to \infty} \sum_{n=1}^{m} \int_0^z \operatorname{Log}\left(1 - \frac{a^2}{n^2}\right) da =$$

$$\lim_{m \to \infty} \left\{ \sum_{n=1}^{m} \left[\operatorname{Log}\left(1 - \frac{z^2}{n^2}\right) - \operatorname{Log}(1) \right] + h_m 2\pi i \right\} =$$

$$\lim_{m \to \infty} \left\{ \sum_{n=1}^{m} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right) + h_m 2\pi i \right\}$$

for some integer h_m. But, this limit exists, and therefore after m large enough we must have $h_m = h$ constant integer. [By the way, notice that for all $z \in \mathbb{C}$, $\lim_{n \to \infty} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right) = \operatorname{Log}(1) = 0 + 0i$.]

So, the second side takes the form

$$\sum_{n=1}^{\infty} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right) + h 2\pi i.$$

Then, both sides yield the equality

$$\operatorname{Log}\left[\frac{\sin(\pi z)}{z}\right] - \operatorname{Log}(\pi) = \sum_{n=1}^{\infty} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right) + h 2\pi i,$$

or, if we set $\log(\pi) = \text{Log}(\pi) + h2\pi$, we eventually find

$$\text{Log}\left[\frac{\sin(\pi z)}{z}\right] = \text{Log}(\pi) + h2\pi i + \sum_{n=1}^{\infty} \text{Log}\left(1 - \frac{z^2}{n^2}\right) =$$

$$\log(\pi) + \sum_{n=1}^{\infty} \text{Log}\left(1 - \frac{z^2}{n^2}\right).\}$$

(c) Conclude that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Let $z = \frac{1}{2}$ to find $\frac{1}{\pi}$ as an infinite product!
Change z to iz to obtain:

$$\sinh(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right).$$

Now, put $\frac{z}{\pi}$ in the place of z in both formulae above to find the two corresponding formulae.

3.7.62 Project: Examine the following work and provide the missing details and the omitted parts of the proofs provided.

(1) For $a \in \mathbb{C}$ and $k \in \mathbb{N}$, we set

$$z^k - a^k = (z - \omega_0 a)(z - \omega_1 a)\ldots(z - \omega_{k-1} a),$$

where $\omega_\rho = \cos\left(\frac{2\rho\pi}{k}\right) + i\sin\left(\frac{2\rho\pi}{k}\right) = e^{\frac{i2\rho\pi}{k}}$, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ are the k k$^{\text{th}}$ roots of unity$(= 1)$.

(2) If $a \neq 0$, using partial fractions we have

$$\frac{1}{z^k - a^k} = \sum_{\rho=0}^{k-1} \frac{A_\rho}{z - \omega_\rho a}, \quad \text{and so} \quad 1 = \sum_{\rho-0}^{k-1} A_\rho \frac{z^k - a^k}{z - \omega_\rho a}.$$

Then, for $z = \omega_\sigma a$ ($\sigma = 0,\ 1,\ 2,\ \ldots,\ k-1$), we obtain

$$1 = A_\sigma \frac{kz^{k-1}}{1}|_{z=\omega_\sigma a} = A_\sigma k \omega_\sigma^{k-1} a^{k-1} = A_\sigma \frac{ka^{k-1}}{\omega_\sigma}, \quad (\omega_\sigma^k = 1).$$

So,

$$A_\sigma = \frac{\omega_\sigma}{ka^{k-1}} \quad \text{for} \quad \sigma = 0,\ 1,\ 2,\ \ldots,\ k-1.$$

Hence,

$$\frac{1}{z^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \frac{\omega_\rho}{z - \omega_\rho a}.$$

(3) Then, if we pick z and a such that $\operatorname{Re}(z-a) > 0$, we have

$$\frac{1}{z^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty e^{-(z-\omega_\rho a)x} dx.$$

(4) We choose $z = n \in \mathbb{N}$ and $0 < |a| < 1$. Then, $\operatorname{Re}(n-a) > 0$, $\forall\ n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty \sum_{n=1}^{\infty} e^{-(n-\omega_\rho a)x} dx =$$

$$\frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty \frac{e^{-(1-\omega_\rho a)x}}{1 - e^{-x}} dx = \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{ka^{k-1}(e^x - 1)} dx.$$

Since $\sum_{\rho=0}^{k-1} \omega_\rho^m = 0$ for $m = 1,\ 2,\ \ldots,\ k-1$ and $\sum_{\rho=0}^{k-1} \omega_\rho^k = k$, by using L' Hôpital's Rule $(k-1)$ times in a row, we find

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \lim_{a \to 0} \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{ka^{k-1}(e^x - 1)} dx = \int_0^\infty \frac{x^{k-1}}{(k-1)!\,(e^x - 1)} dx.$$

(Switching of integral and sum is legitimate. For the **zeta function**, see **Problem 3.2.28**. For $k = 1$, we obtain $\infty = \infty$.)

Then, by switching the integral and the summation (which is legitimate here), we obtain

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \int_0^\infty \frac{x^{k-1}}{(k-1)!} \sum_{n=1}^{\infty} e^{-nx} dx = \frac{1}{\Gamma(k)} \sum_{n=1}^{\infty} \int_0^\infty x^{k-1} e^{-nx} dx.$$

(5) The **previous equation** is a result that follows from the following more general result.

Use the **property of the Gamma function**, $\mathbf{(\Gamma, 11)}$,

$$\forall\ x > 0,\ \forall\ p > 0,\ \ \frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-xu} du$$

and observe that switching of integral and sum is legitimate to obtain:

$\forall\ p > 1$, we have:

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \sum_{n=1}^{\infty} \int_0^\infty x^{p-1} e^{-nx} dx = \frac{1}{\Gamma(p)} \int_0^\infty \frac{x^{p-1}}{e^x - 1} dx.$$

By the way, the change of variables $x = \ln(u)$ transforms this formula to

$$\forall\ p > 1,\ \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_1^{\infty} \frac{[\ln(u)]^{p-1}}{u(u-1)}\, du,$$

and then the change of variables $u = \dfrac{1}{v}$ transforms this formula to

$$\forall\ p > 1,\ \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{1} \frac{[-\ln(v)]^{p-1}}{1-v}\, dv.$$

(6) Now, for any $m \in \mathbb{N}$ and any $a \in \mathbb{C}$ such that $0 < |m - a| < 1$, we similarly obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \sum_{n=1}^{m} \frac{1}{n^k - a^k} + \int_0^{\infty} \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\omega_\rho a)x}}{k a^{k-1}(e^x - 1)}\, dx.$$

(7) Next, if we pick any kth root of -1, ψ let us say, and we replace a with $-a$, we find

$$\forall\ 0 < |a| < 1,\ \sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{\psi \omega_\rho a x}}{k a^{k-1}(e^x - 1)}\, dx$$

and

$$\forall\ m \in \mathbb{N},\ \forall\ a \in \mathbb{C}\ :\ 0 < |m - a| < 1,$$
$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \sum_{n=1}^{m} \frac{1}{n^k + a^k} + \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\psi\omega_\rho a)x}}{k a^{k-1}(e^x - 1)}\, dx.$$

We observe that, in this case, the last two formulae are also valid for $a = m \in \mathbb{N}$.

3.7.5 Fourier Type Integrals

In this section, we deal with the **Fourier type integrals** and computation of Fourier transforms. The complex Fourier type integrals considered here are mainly computed by using the following two lemmata.

Lemma 3.7.5 *Let $w = f(z)$ be a complex function defined and continuous in an open subset of $\mathbb{C}$ of the type*

$$\mathbb{C} - \{\textit{finitely many points of } \ \mathbb{C}\}$$

and satisfying the condition

$$\lim_{z\to\infty} f(z) = 0$$

where ∞ *is the complex infinity. Then:*

(a) For every, $\mu > 0$, $R > 0$ *and* S_R *the upper half of* $C(0,R)$, *we have*

$$\lim_{R\to\infty} \int\limits_{S_R} f(z)e^{i\mu z}dz = 0.$$

(b) For every, $\lambda < 0$, $R > 0$ *and* T_R *the lower half of* $C(0,R)$, *we have*

$$\lim_{R\to\infty} \int\limits_{T_R} f(z)e^{i\lambda z}dz = 0.$$

Proof We prove **(a)**. [**(b)** is done similarly with the necessary adjustments.]

We take $R > 0$ greater than the lengths of all the finitely many exceptional points at which $f(z)$ is not defined. We let

$$M(R) = \underset{z\in S_R}{\text{Maximum}} \left| f(z) \right|.$$

Then, by the hypothesis $\lim\limits_{z\to\infty} f(z) = 0$, we get

$$\lim_{R\to\infty} M(R) = 0.$$

Hence,

$$\left| \int\limits_{S_R} f(z)e^{i\mu z}dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})e^{i\mu R[\cos(\theta)+i\sin(\theta)]}Rie^{i\theta}d\theta \right| \leq$$

$$\int_0^{\pi} \left| f(Re^{i\theta}) \right| R\, e^{-\mu R\sin(\theta)}d\theta < \quad \text{(by **Jordan's Lemma 3.7.4**)}$$

$$M(R)R\frac{\pi}{\mu R} = M(R)\frac{\pi}{\mu} \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty,$$

and this implies the result. (The convergence is uniform.)

Lemma 3.7.6 *Let* $w = f(z)$ *be a complex function defined and holomorphic in the open subset of* $\mathbb{C}$

$$\mathbb{C} - \{z_1, z_2, z_3, ..., z_k\} \quad \textit{with} \quad k \geq 1$$

and satisfying the condition

$$\lim_{z\to\infty} f(z) = 0.$$

where ∞ is the complex infinity. We assume that ***each*** *of the k exceptional points z_1, z_2, z_3, ..., z_k is an isolated singularity of $f(z)$ and is not on the real axis.*

(a) If ***all*** *the exceptional points located in the upper half plane are the z_1, z_2, z_3, ..., z_l with $1 \leq l \leq k$, then for any $\mu > 0$ we have*

$$P.V. \int_{-\infty}^{\infty} f(x)e^{i\mu x}\, dx = 2\pi i \sum_{j=1}^{l} \underset{z=z_j}{Res} \left[f(z)e^{i\mu z}\right].$$

(b) If the z_{l+1}, z_{l+2}, ..., z_k are ***all*** *the exceptional points located in the lower half plane with $0 \leq l < k$, then for any $\lambda < 0$ we have*

$$P.V. \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\, dx = -2\pi i \sum_{j=l+1}^{k} \underset{z=z_j}{Res} \left[f(z)e^{i\lambda z}\right].$$

Proof We observe that $f(z)$ and $f(z)e^{i\mu z}$ [or $f(z)e^{i\lambda z}$] have exactly the same singularities. Then, the results follow from **the Residue Theorem, 3.7.1**, and **Lemma 3.7.5** applied to $f(z)e^{i\mu z}$ [or $f(z)e^{i\lambda z}$].

Example 3.7.27 If $a > 0$ and $b \neq 0$ constants, compute

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2}\, dx = \int_{-\infty}^{\infty} \frac{\cos(ax) + i\sin(ax)}{x^2 + b^2}\, dx.$$

The parameter b appears in the square, and so, without loss of generality, we assume that $b > 0$. Otherwise, we use $|b| > 0$.

First of all for, $b > 0$, this integral exists because

$$\int_{-\infty}^{\infty} \left|\frac{e^{iax}}{x^2 + b^2}\right| dx < \int_{-\infty}^{\infty} \frac{1}{x^2 + b^2}\, dx = \frac{1}{b}\pi,$$

i.e., it converges absolutely. So, its value and its principal value coincide.

If $b = 0$, this integral does not exist because $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2}\, dx = \infty$, since near $x = 0$ the integrand behaves like $\dfrac{1}{x^2}$. Also, we have $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2}\, dx = \infty - \infty$ does not exist, since near $x = 0$ the integrand behaves like $\dfrac{\pm 1}{x}$.

The function $f(z) = \dfrac{1}{z^2 + b^2}$ satisfies the condition $\lim_{z \to \infty} f(z) = 0$ and has two singularities $z_1 = bi$ and $z_2 = -bi$ not on the real axis. So, this function is holomorphic in $\mathbb{C} - \{z_1,\ z_2\}$.

We consider $z_1 = bi$ the singularity located in the upper half plane $(b > 0)$. Then,

$$\operatorname*{Res}_{z=bi}\left[f(z)e^{iaz}\right] = \frac{(z-bi)e^{iaz}}{(z-bi)(z+bi)}\Big|_{z=bi} = \frac{e^{iabi}}{2bi} = \frac{e^{-ab}}{2ib}.$$

So, by **Lemma 3.7.6, (a)**, we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx = 2\pi i\frac{e^{-ab}}{2ib} = \frac{\pi}{b}\,e^{-ab} = \frac{\pi}{be^{ab}}.$$

Remark: From this integral, we get the following two real integrals: For $a > 0$ and $b > 0$

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{be^{ab}} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx = 0.$$

In the first integral, the integrand function is even, and so

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{0}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}.$$

This formula also gives the correct answer for $a = 0$. I.e., this integral is continuous in a at $a = 0$. Moreover, by the evenness of the integrand, the integral does not change if we replace a with $-a$, and therefore we must use $|a|$ in the second side. Finally, we have that for any $b > 0$ and $a \in \mathbb{R}$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{0}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-|a|b}.$$

In the second integral, the integrand function is odd and the integral exists. So, we could á-priori say that its value is immediately zero. But, the integral

$$\int_{0}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx,$$

where $a > 0$ and $b > 0$ constants, cannot be found in a nice closed form or by basic means, even though it converges absolutely.

(See also **Example 3.7.31** in which the initial integral here is viewed as a Fourier transform.)

▲

Example 3.7.28 We consider the integral

$$F(a) := \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx,$$

where $a \geq 0$ and $b > 0$.

We observe that the function $f(x) := \dfrac{\sin(ax)}{x(x^2+b^2)}$ is even in $\mathbb{R}$, [i.e., $f(-x) = f(x), \ \forall \ x \in \mathbb{R}$] and absolutely bounded by the function $g(x) := \dfrac{a}{x^2+b^2} \geq 0$ [i.e., $|f(x)| \leq g(x), \ \forall \ x \in \mathbb{R}$]. Since $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = \frac{a\pi}{b}$ is finite, $F(a)$ converges absolutely for all $a \geq 0$. For $a = 0$, we have $F(0) = 0$.

Now, $\forall \ a \in \mathbb{R}$

$$\left|\frac{\partial}{\partial a}\left[\frac{\sin(ax)}{x(x^2+b^2)}\right]\right| = \left|\frac{\cos(ax)}{x^2+b^2}\right| \leq \frac{1}{x^2+b^2}$$

with

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+b^2)}\,dx = \frac{\pi}{b},$$

finite and independent of a.

Then, by the differentiation **Part II of Theorem 2.2.1** and the **previous example**, we get

$$\frac{d\,F(a)}{da} = F'(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}e^{-ab}.$$

Therefore,

$$F(a) = \frac{\pi}{b} \cdot \frac{e^{-ab}}{-b} + C.$$

Since $F(0) = 0$, we have that $C = \dfrac{\pi}{b^2}$, and so if $a >$ and $b > 0$, we have

$$F(a) := \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{b^2} \cdot \left(1 - e^{-ab}\right).$$

Notice that $F(-a) = -F(a)$. So,

$$\forall \ a \in \mathbb{R}, \ \ F(a) := \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \operatorname{sign}(a)\frac{\pi}{b^2} \cdot \left(1 - e^{-|a|b}\right).$$

As we said above, $f(x)$ is even in $\mathbb{R}$. So,

$$\int_{0}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \int_{-\infty}^{0} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{2b^2} \cdot \left(1 - e^{-ab}\right).$$

Remark: We have computed the above integral $F(a)$ by combining real and complex analyses. We can also compute it quickly by pure complex analysis if we use the theorem that follows. [Try the next theorem on this example, with $f(z) = \dfrac{e^{iaz}}{z(z^2+b^2)}$!]

Pure real analysis can compute this integral but with extensive arguments. Using **Theorem 2.2.1**, as we have used it above, and **Theorem 2.3.8**, in order to prove that $F''(a) = \int_{-\infty}^{\infty} \frac{-x\sin(ax)}{x^2+b^2}\, dx$ as a uniform limit of derivatives, we can show that for $a \geq 0$, $F(a)$ satisfies the initial value problem:

$$\begin{cases} \frac{1}{b^2}\ F''(a) - F(a) = \frac{-\pi}{b^2} \\ \\ F(0) = 0 \\ \\ F'(0) = \frac{\pi}{b}, \end{cases}$$

whose solution is $F(a) = \frac{\pi}{b^2} \cdot \left(1 - e^{-ab}\right)$, etc.

By this example, we see how powerful the methods of complex analysis are in comparison to real analysis ones.

▲

Example 3.7.29 If $a \geq 0$ and $b > 0$, the differentiation **Part II of Theorem 2.2.1** applies immediately if we differentiate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx = \frac{\pi}{b} e^{-ab}$$

with respect to b. Then, for $a \geq 0$ and $b > 0$, we find:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{\left(x^2+b^2\right)^2}\, dx = \frac{\pi}{2b^3} e^{-ab}(1+ab).$$

▲

In view of **Lemmata 3.7.3 and 3.7.6**, we write and can directly prove the theorem that follows, analogous to **Theorem 3.7.4**, on **evaluating the Cauchy principal value of certain types of integrals**.

Theorem 3.7.5 *Suppose a complex function $f(z)$ satisfies the conditions of **Lemma 3.7.6** with the exception that it is allowed to have finitely many (isolated) **simple poles** on the x-axis.*

Then, we have: If $a > 0$ constant

$$P.V. \int_{-\infty}^{\infty} e^{iaz} f(x) dx =$$

$$2\pi i \sum [\text{residues of } e^{iaz} f(z) \text{ in the upper half plane}] +$$

$$\pi i \sum [\text{residues of } e^{iaz} f(z) \text{on the real axis}].$$

If $a < 0$ *constant*

$$P.V. \int_{-\infty}^{\infty} e^{iaz} f(x)dx =$$
$$-2\pi i \sum [\textit{residues of } e^{iaz} f(z) \textit{ in the lower half plane}] -$$
$$\pi i \sum [\textit{residues of } e^{iaz} f(z) \textit{ on the real axis}].$$

(The proof follows from **Lemmata 3.7.3 and 3.7.6**.)

Example 3.7.30 Find the

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^3 - 1}\, dx.$$

The complex function

$$f(z) = \frac{1}{z^3 - 1} = \frac{1}{(z-1)\left(z - e^{\frac{2\pi i}{3}}\right)\left(z - e^{\frac{4\pi i}{3}}\right)}$$

has one simple pole on the x-axis, the number $z = 1$, one simple pole at $z = e^{\frac{2\pi i}{3}}$, in the upper half plane, and satisfies the conditions of **Theorem 3.7.5**. The corresponding residues are

$$\operatorname*{Res}_{z=1} f(z) = \frac{e^i}{\left(1 + \frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(1 + \frac{1}{2} + \frac{\sqrt{3}}{2} i\right)} = \frac{e^i}{3},$$

$$\operatorname*{Res}_{z=e^{\frac{2\pi i}{3}}} f(z) = \frac{e^{ie^{i\frac{2\pi}{3}}}}{\left(e^{\frac{2\pi i}{3}} - 1\right)\left(e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}}\right)} = \ldots =$$
$$\frac{-e^{\frac{-\sqrt{3}}{2}}}{6}\left\{\cos\left(\frac{1}{2}\right) - \sqrt{3}\sin\left(\frac{1}{2}\right) - i\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right)\right]\right\}.$$

Therefore, by **Theorem 3.7.5**, we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^3 - 1}\, dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x) + i\sin(x)}{x^3 - 1}\, dx =$$
$$2\pi i \frac{-e^{\frac{-\sqrt{3}}{2}}}{6}\left\{\cos\left(\frac{1}{2}\right) - \sqrt{3}\sin\left(\frac{1}{2}\right) - i\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right)\right]\right\} +$$
$$\pi i\left(\frac{e^i}{3}\right) = -\frac{\pi}{3} e^{\frac{-\sqrt{3}}{2}}\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) - \sin(1)\right] -$$
$$i\frac{\pi}{3}\left[\cos\left(\frac{1}{2}\right)\sqrt{3}\sin\left(\frac{1}{2}\right) + \cos(1)\right].$$

Separating the real and imaginary parts, we obtain the two real principal values:

$$(1)\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)}{x^3-1}\,dx = -\frac{\pi}{3}\,e^{\frac{-\sqrt{3}}{2}}\left[\sqrt{3}\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)-\sin(1)\right],$$

$$(2)\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{\sin(x)}{x^3-1}\,dx = -\frac{\pi}{3}\left[\cos\left(\frac{1}{2}\right)\sqrt{3}\sin\left(\frac{1}{2}\right)+\cos(1)\right].$$

▲

Now, we want to state the definition of the **Fourier transform** for some classes of functions.

We consider an **absolutely integrable** function $f:\mathbb{R}\longrightarrow\mathbb{R}$, that is,

$$\int_{-\infty}^{\infty}|f(x)|dx<\infty.$$

Then, for any $t\in\mathbb{R}$, the improper integral

$$\int_{-\infty}^{\infty}f(x)e^{itx}dx$$

exists. This follows immediately by the identity $\left|e^{itx}\right|=1$, and so the integral converges absolutely by the absolute integrability of $f(x)$.

So, for the class of the absolutely integrable functions, we define:

Definition 3.7.3 *For any real absolutely integrable function $y=f(x)$, $f:\mathbb{R}\longrightarrow\mathbb{R}$, we define the **Fourier transform** of $y=f(x)$, written as $\hat{f}:=\mathcal{F}[f(x)]:\mathbb{R}\longrightarrow\mathbb{C}$, to be the improper integral*

$$\hat{f}(t)=\mathcal{F}[f(x)](t)=\int_{-\infty}^{\infty}f(x)e^{itx}dx,\quad\forall\ t\in\mathbb{R}\tag{3.27}$$

(which exists for all $t\in\mathbb{R}$).

For a **square integrable** function $f:\mathbb{R}\longrightarrow\mathbb{R}$, i.e.,

$$\int_{-\infty}^{\infty}|f(x)|^2dx<\infty,$$

the improper integral

$$\int_{-\infty}^{\infty}f(x)e^{itx}dx$$

may exist for some t's and may not exist for others or not at all. In this case, we define the Fourier transform of $y=f(x)$ by:

Definition 3.7.4 *For any real square integrable function $y = f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define the* ***Fourier transform*** *of $y = f(x)$, written as $\hat{f} := \mathcal{F}[f(x)] : \mathbb{R} \longrightarrow \mathbb{C}$, to be the principal value*

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = P.V. \int_{-\infty}^{\infty} f(x)e^{itx}dx =$$
$$\lim_{0<M\to\infty} \int_{-M}^{M} f(x)e^{itx}dx, \quad \forall\ t \in \mathbb{R}. \tag{3.28}$$

If the improper integral in **(3.28)** exists, then it is equal to its principal value. So, **Definition 3.7.4** implies, and therefore extends or includes, **Definition 3.7.3**.

In **Lemma 3.7.6**, we have seen a class of functions for which the principal value in **(3.28)** exists. So, for the functions that satisfy the conditions of this lemma, we define their Fourier transform as in **Definition 3.7.4**.

In this book, we mostly concentrate on the evaluation of this integral and some of its basic properties. We will need some results that we will readily take from analysis and we will present a few applications. The large number of properties and applications of this integral belong to a big chapter of analysis, titled **"Fourier Transform."** This is a very important chapter to mathematics, engineering, science, and application.

In many books, **Definitions 3.7.3** and **3.7.4** include the multiplicative constant $\dfrac{1}{\sqrt{2\pi}}$ in front of the integral. That is, by many authors, the **Fourier Transform** is defined by:

$$\tilde{f}(t) = \mathfrak{F}[f(x)](t) =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}}\hat{f}(t), \quad \forall\ t \in \mathbb{R}, \tag{3.29}$$

or

$$\tilde{f}(t) = \mathfrak{F}[f(x)](t) =$$
$$\frac{1}{\sqrt{2\pi}}\ \text{P.V.} \int_{-\infty}^{\infty} f(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}}\hat{f}(t), \quad \forall\ t \in \mathbb{R}. \tag{3.30}$$

Here. we use the symbols $\tilde{f}$ and $\mathfrak{F}$ instead of $\hat{f}$ and $\mathcal{F}$ in the **Definitions 3.7.3** and **3.7.4**. There are some reasons for this constant, but we do not obviously need it for the computation of the integral.

We also notice that

$$\hat{f}(t) = \mathcal{F}[f(x)](t) =$$
$$\int_{-\infty}^{\infty} f(x)e^{itx}dx = \int_{-\infty}^{\infty} f(x)\left[\cos(tx) + i\sin(tx)\right]dx =$$

$$= \int_{-\infty}^{\infty} f(x)\cos(tx)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)\,dx.$$

So, to compute $\hat{f}(t)$, we may compute the last two integrals separately in any possible way, or using already computed integrals, or using the evenness and/or the oddity of certain functions, etc. We call the first integral the **cosine transform of** $f(x)$ and the second the **sine transform of** $f(x)$. Sometimes one may exist but not the other or its principal value may or may not exist, etc.

We observe that on the set of functions that the **Fourier, cosine** and **sine transforms** are well defined (exist) these transforms are **linear**. I.e., assuming that the Fourier transforms involved exist and $a \in \mathbb{R}$, or $\mathbb{C}$, we have:

$$(1) \qquad \mathcal{F}[f(x)+g(x)](t) = \mathcal{F}[f(x)](t) + \mathcal{F}[g(x)](t),$$

$$(2) \qquad \mathcal{F}[af(x)](t) = a\mathcal{F}[f(x)](t),$$

and similarly for the cosine and the sine transforms.

It is rather obvious that if $y = f(x)$ is a nice real function on $\mathbb{R}$, and its Fourier, or sine, or cosine, transform exists, then it is unique.

Also, in our exposition, we will need the following important **results** from advanced analysis.

Result 1: We need to know the behavior of the Fourier transform at ±∞. On this question, we state the following **two key results**:

(a) The first result on this question is the following fundamental Lemma:

Lemma 3.7.7 (Riemann-Lebesgue Lemma) *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an* ***absolutely integrable*** *function, then its Fourier transform $\hat{f}$, given by* ***(3.27)*** *or* ***(3.29)****, is uniformly continuous on $\mathbb{R}$ and*

$$\lim_{t\to\pm\infty} \hat{f}(t) = 0.$$

(A proof of this key result in Fourier analysis can be found in Rudin 1987, Theorem 9.6, 182-183; Weinberger 1965, 316, etc.)

(b) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is **square integrable**, then $\hat{f}$, given by **(3.28)** or **(3.30)**, exists and is square integrable but may not be continuous. Moreover, we have:

$$\int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt = 2\pi \int_{-\infty}^{\infty} |f(x)|^2\,dx (< \infty), \tag{3.31}$$

or

$$\int_{-\infty}^{\infty} |\tilde{f}(t)|^2 \, dt = \int_{-\infty}^{\infty} |f(x)|^2 \, dx (< \infty). \tag{3.32}$$

These two equations are two forms of the so-called **Parseval**[26] **equation**.

Remark: **Equation (3.32**) shows that the Fourier transform $\tilde{f}(t) = \mathfrak{F}[f(x)](t)$ is an isometry in the square integrable functions under the $\|\cdot\|_2$ norm. [See **Problem 2.6.42, Item (14.)**]. This is the main reason why we define $\tilde{f}(t) = \dfrac{1}{\sqrt{2\pi}} \hat{f}(t)$.

So, by **(3.31)** and/or **(3.32)**, if the limits $\lim\limits_{t \to \pm\infty} \hat{f}(t) = \lim\limits_{t \to \pm\infty} \sqrt{2\pi} \tilde{f}(t)$ exist, they must be zero.

[The above results in **(b)** are consequences of the **Plancherel**[27] **Theorem**, a key result in Fourier analysis. See, e.g., Rudin 1987, 185-187.]

For any complex function $h(z)$, we have $|h(z)|^2 = h(z)\overline{h(z)}$. If, now, f and g are two square integrable real or complex functions, then $\pm f \pm g$ and $\pm if \pm ig$ are also square integrable. We apply **(3.31)** and/or **(3.32)** to $f \pm g$ and $f \pm ig$ along with the linearity of the Fourier transform to obtain the four respective Parseval equations. Combining these four equations, we respectively obtain (work this out) the so-called **general Parseval equation**

$$\int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)} \, dt = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx, \tag{3.33}$$

or

$$\int_{-\infty}^{\infty} \tilde{f}(t)\overline{\tilde{f}(t)} \, dt = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx. \tag{3.34}$$

Here, we need to use the complex conjugation, even if f and g are real functions, since the Fourier transform is a complex integral, and so $|\hat{f}(t)|^2 = \hat{f}(t)\overline{\hat{f}(t)}$, etc. Also, the functions f and g may be complex, in general.

The **Parseval equation** and the **general Parseval equation** can be used to compute the improper integral of one side by the other side, when the other side is easier. (See **Examples 3.7.35, 3.7.38, 3.7.39, Problem 3.7.75**, etc.)

Result 2: On sets of functions for which the Fourier transform exists, **the Fourier transform is a one-to-one mapping or operator**.

---

[26] Marc-Antoine Parseval, French mathematician, 1755-1836.

[27] Michel Plancherel, Swiss mathematician, 1885-1967.

So, under some mild conditions, we can retrieve a function from its Fourier transform. Results of this type are called inversion theorems for the Fourier transform. Depending on the conditions, there are several inversion theorems. Here, we state an inversion theorem which we use in some examples and applications in the sequel. This is not the most general inversion theorem but one adjusted to the level of this book. It is quite useful and applicable.

Theorem 3.7.6 (Inversion Theorem) *Suppose that a function $f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, is absolutely integrable or square integrable and piecewise differentiable with $f'(x)$ absolutely integrable. Then,*

$$f(x) = \frac{1}{2\pi}\ P.V. \int_{-\infty}^{\infty} \hat{f}(t) e^{-itx} dt = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(t) e^{-itx} dt$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}}\ P.V. \int_{-\infty}^{\infty} \tilde{f}(t) e^{-itx} dt = \frac{1}{\sqrt{2\pi}} \lim_{M\to\infty} \int_{-M}^{M} \tilde{f}(t) e^{-itx} dt.$$

If, moreover, $\hat{f}$ is absolutely integrable (then $\tilde{f}$ is so), we can drop the principal value, i.e.,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{-itx} dt$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(t) e^{-itx} dt.$$

If such an $f(x)$ has a jump discontinuity at a point $x = \alpha$, then we get

$$\frac{f(\alpha^+) + f(\alpha^-)}{2} = \frac{1}{2\pi}\ P.V. \int_{-\infty}^{\infty} \hat{f}(t) e^{-it\alpha} dt = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(t) e^{-it\alpha} dt$$

or

$$\frac{f(\alpha^+) + f(\alpha^-)}{2} = \frac{1}{\sqrt{2\pi}}\ P.V. \int_{-\infty}^{\infty} \tilde{f}(t) e^{-it\alpha} dt = \frac{1}{\sqrt{2\pi}} \lim_{M\to\infty} \int_{-M}^{M} \tilde{f}(t) e^{-it\alpha} dt.$$

Hence, by this **inversion Theorem**, the **inverse Fourier transform** to either Fourier transform defined by **(3.27)** or **(3.29)** is defined to be

$$\mathcal{F}^{-1}[\hat{f}(t)](x) = 2\pi \int_{-\infty}^{\infty} \hat{f}(t) e^{-itx} dt$$

and

$$\mathfrak{F}^{-1}[\tilde{f}(t)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(t) e^{-itx} dt,$$

respectively.

If we apply these inverse Fourier transforms to the respective Fourier transforms $\hat{f}(t)$ and $\tilde{f}(t)$ of a function $f(x)$, under the conditions stated in the **inversion Theorem**, we retrieve the function $f(x)$.

We observe that the constant $\frac{1}{\sqrt{2\pi}}$ produces a symmetry in the expressions of the Fourier transform and its inverse. Besides some other reasons, this is one of the reasons as to why many authors prefer to put this constant in the definitions.

Examples

Example 3.7.31 Find the **Fourier transform** of

$$f(x) = \frac{1}{x^2 + b^2}, \quad \text{where} \quad b > 0 \quad \text{constant.}$$

In **Example 3.7.27** we have found $\hat{f}(a) = \frac{\pi}{b} e^{-ba}$ for $a > 0$. Now, for $a < 0$, we find

$$\operatorname*{Res}_{z=-bi} \left[f(z) e^{iaz} \right] = \frac{(z+bi) e^{iaz}}{(z-bi)(z+bi)} \Big|_{z=-bi} = \frac{e^{ia(-bi)}}{-2bi} = \frac{e^{ab}}{-2bi}.$$

So, for $a < 0$, by **Lemma 3.7.6, (b)**, we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2} \, dx = -2\pi i \frac{e^{ab}}{-2ib} = \frac{\pi}{b} e^{ab}.$$

For $a = 0$, we straightly compute

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + b^2} \, dx = \frac{\pi}{b}.$$

Putting the three cases $a > 0$, $a = 0$ and $a < 0$ together in one formula, we find that the Fourier transform of

$$f(x) = \frac{1}{x^2 + b^2}, \quad \text{where} \quad b > 0 \quad \text{constant}$$

is

$$\hat{f}(a) = \frac{\pi}{b}\, e^{-b|a|}.$$

Since here $f(x) = \dfrac{1}{x^2+b^2}$ is absolutely integrable, we see that $\hat{f}(a) = \dfrac{\pi}{b}\, e^{-b|a|}$ is uniformly continuous in $\mathbb{R}$ and $\lim\limits_{a \to \pm\infty} f(a) = 0$, as claimed by the **Riemann-Lebesgue Lemma, 3.7.7**.

Notice that $\hat{f}(a)$, $f(x)$ and $f'(x) = \dfrac{-2x}{(x^2+b^2)^2}$ are absolutely integrable in $(-\infty, \infty)$. Then, if we apply the **inversion Theorem, 3.7.6,** we will get the given function $f(x)$ back. Indeed,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi}{b}\, e^{-b|t|}e^{-itx}dt =$$
$$\frac{1}{2\pi}\int_{-\infty}^{0} \frac{\pi}{b}\, e^{bt}e^{-itx}dt + \frac{1}{2\pi}\int_{0}^{\infty} \frac{\pi}{b}\, e^{-bt}e^{-itx}dt =$$
$$\frac{1}{2\pi}\int_{-\infty}^{0} \frac{\pi}{b}\, e^{(b-ix)t}dt + \frac{1}{2\pi}\int_{0}^{\infty} \frac{\pi}{b}\, e^{-(b+ix)t}dt.$$

We use **Problem 3.7.40**, and (for $b > 0$) we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt =$$
$$\frac{1}{2\pi}\frac{\pi}{b}\left(\frac{1}{b-ix} + \frac{1}{b+ix}\right) = \frac{1}{2b}\frac{2b}{x^2+b^2} = \frac{1}{x^2+b^2} = f(x).$$

▲

Example 3.7.32 Compute

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+b^2}dx, \quad \text{where} \quad a > 0 \quad \text{and} \quad b > 0 \quad \text{constants.}$$

The function $f(z) = \dfrac{z}{z^2+b^2}$ satisfies the condition $\lim\limits_{z \to \infty} f(z) = 0$ and has two singularities $z_1 = bi$ and $z_2 = -bi$ not on the real axis. This function is holomorphic in $\mathbb{C} - \{z_1,\ z_2\}$.

We consider $z_1 = bi$ the singularity in the upper half plane. Then,

$$\operatorname*{Res}_{z=bi}\left[f(z)e^{iaz}\right] = \frac{z(z-bi)e^{iaz}}{(z-bi)(z+bi)}\Big|_{z=bi} = \frac{bie^{iabi}}{2bi} = \frac{1}{2e^{ab}}.$$

So, by the **Lemma 3.7.6, (a)**, we get that for $a > 0$

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+b^2}dx = 2\pi i\frac{1}{2e^{ab}} = \frac{\pi i}{e^{ab}} = i\,\pi\, e^{-ab}.$$

Remark 1: From this integral, we get the following two real integrals:

$$\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+b^2}\,dx = 0 \ \text{(odd function, whose improper integral exists)}$$

and

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2}\,dx = \frac{\pi}{e^{ab}} \ \text{(even function, whose improper integral exists).}$$

Remark 2: Working as in the **previous example**, we find that the **Fourier transform** of $f(x) = \dfrac{x}{x^2+b^2}$, where $b > 0$ constant, is

$$\hat{f}(a) = \begin{cases} 0, & \text{if} \quad a = 0, \\ \operatorname{sign}(a)\,\pi i\, e^{-b|a|}, & \text{if} \quad a \neq 0. \end{cases}$$

We observe that this Fourier transform is discontinuous at $a = 0$. In fact, for $a = 0$, we must consider the principal value of the corresponding improper integral, because the improper integral does not exist in the general limiting sense. (Check this out.)

We can easily check that the function $f(x) = \dfrac{x}{x^2+b^2}$ is not absolutely integrable but is square integrable. So, its Fourier transform $\hat{f}(a)$ is defined by means of principal value. It is discontinuous at $a = 0$ and its limits as $x \to \pm\infty$ exist and are equal to zero.

We can also check that the derivative of $f(x) = \dfrac{x}{x^2+b^2}$ and its Fourier transform are absolutely integrable. So, if we apply the **inversion Theorem, 3.7.6**, we will get $f(x)$ back.

▲

Example 3.7.33 (Compare with **Examples 1.3.13 and 2.2.8**.) Verify that

$$\forall\ \alpha > 0 \quad \text{constant}, \qquad \int_0^{\infty} \frac{\sin(\alpha x)}{x}\,dx = \frac{\pi}{2}.$$

Since $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$, we consider the complex function

$$g(z) = \frac{e^{i\alpha z}}{z}.$$

This function has only one singularity at $z = 0$, and so it is holomorphic in $\mathbb{C} - \{0\}$. Since the singularity is located on the real axis, we cannot apply **Lemma 3.7.6**.

This singularity is easily seen to be a simple pole with residue,

$$\operatorname*{Res}_{z=0} g(z) = zg(z)|_{z=0} = e^{i\alpha 0} = 1.$$

Since $\alpha > 0$, we must work in the upper half plane, according to **Lemma 3.7.5**. This is the point at which the positivity of α is used in this problem.

So, we must choose a contour in the upper half closed plane that avoids $z = 0$. Therefore, we consider any $0 < r < R < \infty$ and then an appropriate choice of contour that itself and its interior does not contain $z = 0$ is

$$C = [r, R] + S_R^+ + [-R, -r] + S_r^-,$$

where S_R^+ is the upper half of $C(0, R)$, positively oriented, and S_r^- is the upper half of $C(0, r)$, negatively oriented. (See **Figure 3.16**.)

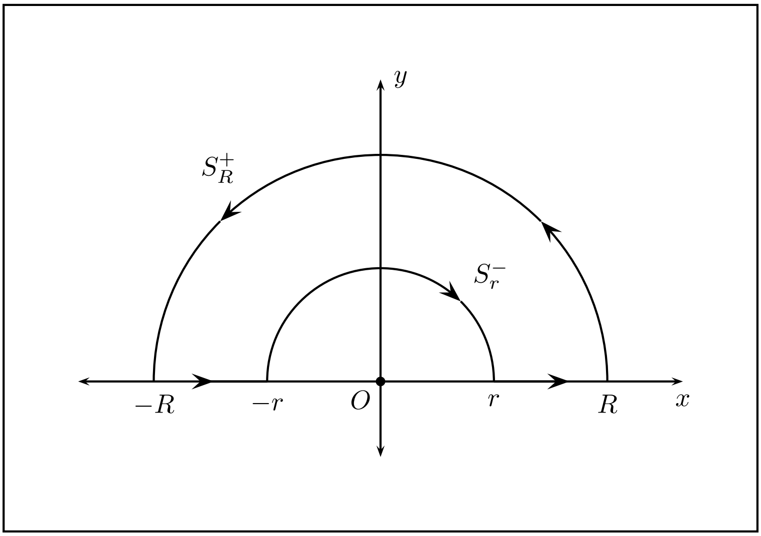

FIGURE 3.16: Contour 12 for Example 3.7.33

Then, by the **Cauchy-Goursat Theorem, 3.5.3**, we have

$$\oint_{C^+} g(z)dz = \int_{[r,R]} g(z)dz + \int_{S_R^+} g(z)dz + \int_{[-R,-r]} g(z)dz + \int_{S_r^-} g(z)dz = 0.$$

We see that

$$\lim_{\substack{r\to 0^+\\ R\to\infty}} \left[\int_{[r,R]} g(z)dz + \int_{[-R,-r]} g(z)dz \right] = \lim_{\substack{r\to 0^+\\ R\to\infty}} \int_{-R}^{-r} g(x)dx +$$

$$+\lim_{\substack{r\to 0^+\\ R\to\infty}}\int_r^R g(x)dx = \int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty}\frac{\cos(\alpha x)+i\sin(\alpha x)}{x}\,dx.$$

Since for $f(z) = \dfrac{1}{z}$ the condition $\lim\limits_{z\to\infty} f(z) = 0$ is satisfied, by **Lemma 3.7.5** we have that

$$\lim_{R\to\infty}\int_{S_R^+} g(z)dz = 0.$$

Next, for all $r > 0$, by **Lemma 3.7.3** we have

$$\int_{S_r^-} g(z)dz = -\pi i \operatorname*{Res}_{z=0} g(z) = -\pi i,$$

and so

$$\lim_{r\to 0^+}\int_{S_r^-} g(z)\,dz = -i\pi.$$

Putting these pieces together, we finally find

$$\int_{-\infty}^{\infty}\frac{\cos(\alpha x)+i\sin(\alpha x)}{x}\,dx - i\pi = 0$$

or

$$\int_{-\infty}^{\infty}\frac{\cos(\alpha x)+i\sin(\alpha x)}{x}\,dx = i\pi.$$

Separating real and imaginary parts, we find

$$\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{x}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty}\frac{\sin(\alpha x)}{x}\,dx = \pi.$$

Since $\dfrac{\sin(\alpha x)}{x}$ is an even function over $\mathbb{R}$, we obtain the **result**:

$$\forall\ \alpha > 0,\ \int_0^{\infty}\frac{\sin(\alpha x)}{x}\,dx = \frac{\pi}{2}.$$

Remark 1: If $\alpha < 0$, then we get

$$\int_0^{\infty}\frac{\sin(\alpha x)}{x}\,dx = -\frac{\pi}{2}.$$

For $\alpha = 0$, we trivially get

$$\int_0^{\infty}\frac{\sin(\alpha x)}{x}\,dx = \int_0^{\infty}\frac{0}{x}\,dx = 0.$$

Remark 2: We present some discussion on these results and the principal value. In **this example**, we, in fact, computed the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{x}\, dx = \pi, \ \ \forall \ \ \alpha > 0,$$

with the help of the particularly chosen contour C.

However, because this integral exists, as proven in **Example 1.3.13**, its principal value is equal to its value. Therefore, its value is equal to π.

But, the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x}\, dx$$

does not exist if all limiting processes are allowed. Whereas the limiting processes as $R \to \pm\infty$ do not matter, because $\lim\limits_{R\to\pm\infty} \dfrac{\cos(\alpha R)}{R} = 0$, problems arise when $r \to 0^{\pm}$ (from right and left). Notice for instance:

$$\int_{0^+}^{1} \frac{\cos(\alpha x)}{x}\, dx = \infty, \qquad \text{whereas} \qquad \int_{-1}^{0^-} \frac{\cos(\alpha x)}{x}\, dx = -\infty.$$

Finally, what we have computed by this integral is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x}\, dx = 0,$$

which is naturally expected, since $\dfrac{\cos(\alpha x)}{x}$ is an odd function over $\mathbb{R}$.

Therefore, what we have accurately found is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = i\pi, \ \ \forall \ \ \alpha > 0.$$

This computation extended to $\alpha = 0$ and $\alpha < 0$ is the Fourier transform of the function $f(x) = \dfrac{1}{x}$, $(x \neq 0)$, i.e.,

$$\hat{f}(\alpha) = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = \begin{cases} i\pi, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha = 0 \\ -i\pi, & \text{if } \alpha < 0. \end{cases}$$

Here, the function $f(x) = \dfrac{1}{x}$, $(x \neq 0)$, is neither absolutely nor square integrable in $\mathbb{R}$ but satisfies the conditions of **Lemma 3.7.6**. So, its Fourier transform is evaluated by the **Principal Value (3.28)** of **Definition 3.28**. We observe that this Fourier transform is not continuous at $\alpha = 0$ and its limit at $\pm\infty$ is not zero.

Note: The same discussion holds true for some previous examples. Essentially, in some of these examples, we compute principal values of integrals. However, if the integrals we consider exist, then their values are equal to their principal values.

▲

Example 3.7.34 (Compare with **Example 2.2.11**. See also **Problem 3.7.66**.) Verify that

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}.$$

For this integral, we consider the function

$$g(z) = \frac{1 + 2iz - e^{2iz}}{z^2} = -\frac{(2i)^2}{2!} - \frac{(2i)^3}{3!}z - ...$$

and we observe that $g(z)$ is holomorphic in $\mathbb{C}$. The origin $z = 0$ is not an essential singularity because we can extend $g(z)$ holomorphically at $z = 0$ by setting

$$g(0) = -\frac{(2i)^2}{2!} = 2.$$

Therefore, for any $R > 0$, we choose the positive contour $C^+ = [-R, R] + S_R^+$, with S_R^+ the upper half of $C(0, R)$ positively oriented. Then, by **the Cauchy-Goursat Theorem, 3.5.3**, we find

$$\oint\limits_{C^+} g(z)\,dz = \int\limits_{[-R,R]} g(z)\,dz + \int\limits_{S_R^+} g(z)\,dz = 0.$$

But, by **Lemma 3.7.1** we get

$$\lim_{R\to\infty} \int\limits_{S_R^+} \frac{1}{z^2}\,dz = 0.$$

Then, by **Lemma 3.7.3**

$$\lim_{R\to\infty} \int\limits_{S_R^+} g(z)\,dz = \lim_{R\to\infty} \int\limits_{S_R^+} \frac{2iz}{z^2}\,dz = 2i \lim_{R\to\infty} \int\limits_{S_R^+} \frac{1}{z}\,dz = 2i \cdot 1 \cdot i\pi = -2\pi,$$

and so, by **Jordan's Lemma, 3.7.4** (fill in the details), we obtain

$$\lim_{R\to\infty} \int\limits_{S_R^+} \frac{e^{2iz}}{z^2}\,dz = 0.$$

Therefore,

$$\lim_{R\to\infty}\int\limits_{S_R^+} g(z)\,dz = \lim_{R\to\infty}\int\limits_{S_R^+}\frac{2iz}{z^2}\,dz = 2i\lim_{R\to\infty}\int\limits_{S_R^+}\frac{1}{z}\,dz = 2i\,i\pi = -2\pi,$$

and so

$$\lim_{R\to\infty}\int\limits_{[-R,R]} g(z)\,dz - 2\pi = 0,$$

or

$$\int_{-\infty}^{\infty}\frac{1+2ix-[\cos(2x)+i\sin(2x)]}{x^2}\,dx = 2\pi.$$

Hence, by separating real and imaginary parts, we get

$$\int_{-\infty}^{\infty}\frac{1-\cos(2x)}{x^2}\,dx = 2\pi, \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{2x-\sin(2x)}{x^2}\,dx = 0.$$

(See also **Problems 2.2.16** and **3.7.69**.)

In the first integral, we use the basic double angle trigonometric identity $\cos(2x) = 1 - 2\sin^2(x)$ to find

$$\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}\,dx = \pi.$$

Since $\dfrac{\sin^2(x)}{x^2}$ is an even function over $\mathbb{R}$, we get

$$\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}.$$

Remark: The second integral is true only as a principal value because it does not exist in any general limiting process. That is,

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{2x-\sin(2x)}{x^2}\,dx = 0.$$

▲

Example 3.7.35 For any $a > 0$ constant, we let

$$f(x) = \begin{cases} 1, & \text{if} \quad |x| < a \\ \\ 0, & \text{if} \quad |x| \geq a. \end{cases}$$

Then, the Fourier transform of $f(x)$ is

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx = \int_{-a}^{a} e^{ix\xi}dx =$$
$$\int_{-a}^{a}[\cos(\xi x) + i\sin(\xi x)]dx = \int_{-a}^{a}\cos(\xi x)dx + i\int_{-a}^{a}\sin(\xi x)]dx =$$
$$2\frac{\sin(a\xi)}{\xi} + i0 = 2\frac{\sin(a\xi)}{\xi}.$$

Also,

$$\tilde{f}(t) = \mathfrak{F}[f(x)](t) = \sqrt{\frac{1}{2\pi}}\ \mathcal{F}[f(x)](t) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}.$$

We can use this result with the **Parseval equation (3.31)**. For $a > 0$, we get

$$\int_{-\infty}^{\infty}\left[2\frac{\sin(a\xi)}{\xi}\right]^2 d\xi = 2\pi\int_{-\infty}^{\infty} f^2(x)\,dx = 2\pi\int_{-a}^{a} 1^2\,dx = 2\pi \cdot 2a.$$

So, we find the already known **result**:
If $a > 0$, then

$$\int_{-\infty}^{\infty}\frac{\sin^2(a\xi)}{\xi^2}\,d\xi = \pi a.$$

[Similarly if we had used **(3.32)**.]

If now for any $b > 0$ constant, we let

$$g(x) = \begin{cases} 1, & \text{if} \quad |x| < b \\ \\ 0, & \text{if} \quad |x| \geq b, \end{cases}$$

then, as before,

$$\hat{g}(t) = 2\frac{\sin(b\xi)}{\xi}.$$

Then, applying the **general Parseval equation, (3.33**), we find

$$\int_{-\infty}^{\infty}\left[2\frac{\sin(a\xi)}{\xi}\right]\left[2\frac{\sin(b\xi)}{\xi}\right] d\xi = 2\pi\int_{-\infty}^{\infty} f(x)g(x)\,dx =$$
$$2\pi\int_{-\min\{a,\,b\}}^{\min\{a,\,b\}} 1 \cdot 1\,dx = 2\pi \cdot 2\min\{a,\,b\}.$$

So, we get

$$\int_{-\infty}^{\infty}\frac{\sin(a\xi)\sin(b\xi)}{\xi^2}\,d\xi = \pi \cdot \min\{a,\,b\}.$$

[Similar work if we had used **(3.34)**. See also **Problem 2.2.34, (a)**.]

▲

3.7.6 Rules and Properties of the Fourier Transform

We have discussed the **linearity properties** of the Fourier transform, its **uniqueness** when it exists, that the Fourier transform is a **one-to-one operator** and an **inversion theorem**. Here, we examine a few additional properties of the Fourier transform that follow from its definition and can be readily used as operational rules in applications.

(1) Suppose that the function $f(x)$ is continuous and piecewise continuously differentiable, $\mathcal{F}[f(x)](t)$ exists for all t and $\lim_{x\to\pm\infty} f(x) = 0$.

Then, for any $R_1 < 0 < R_2$, integrating by parts, we find

$$\int_{R_1}^{R_2} f'(x)e^{itx}dx = \left[f(x)e^{itx}\right]_{R_1}^{R_2} - it\int_{R_1}^{R_2} f(x)e^{itx}dx.$$

Letting $R_1 \to -\infty$ and $R_2 \to \infty$, by the hypotheses, we get

$$\mathcal{F}[f'(x)](t) = \int_{-\infty}^{\infty} f'(x)e^{itx}dx = 0 - it\int_{-\infty}^{\infty} f(x)e^{itx}dx = -it\mathcal{F}[f(x)](t).$$

So, under the stated conditions, we have the **rule for the Fourier transform of the derivative of a function**

$$(1) \quad \mathcal{F}[f'] = -it\mathcal{F}[f].$$

(2) If $f(x)$ is **real**, then $\hat{f}(-t) = \overline{\hat{f}(t)}$.

We have

$$\mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx =$$

$$\int_{-\infty}^{\infty} f(x)\cos(tx)dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)du,$$

from which we get

$$\mathcal{F}[f(x)](-t) = \int_{-\infty}^{\infty} f(x)e^{-itx}dx =$$

$$\int_{-\infty}^{\infty} f(x)\cos(tx)dx - i\int_{-\infty}^{\infty} f(x)\sin(tx)du$$

and

$$\overline{\mathcal{F}[f(x)](t)} = \overline{\int_{-\infty}^{\infty} f(x)e^{itx}dx} =$$

$$\overline{\int_{-\infty}^{\infty} f(x)\cos(tx)dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)du} =$$

$$\int_{-\infty}^{\infty} f(x)\cos(tx)dx - i\int_{-\infty}^{\infty} f(x)\sin(tx)du,$$

that is, the last two results are equal.

We also observe:

$$\text{(a)}\quad \operatorname{Re}\{\mathcal{F}[f(x)](-t)\} = \operatorname{Re}\{\mathcal{F}[f(x)](t)\}$$

and

$$\text{(b)}\quad \operatorname{Im}\{\mathcal{F}[f(x)](-t)\} = -\operatorname{Im}\{\mathcal{F}[f(x)](t)\}.$$

Hence, when $f(x)$ is **real**, the **real part** of $\mathcal{F}[f(x)](t)$ is an **even** function of t and the **imaginary part** of it is an **odd** function of t.

(3) We can compute the derivative of the Fourier transform by differentiating under the integral sign, as long as the differentiability conditions of **Theorem 2.2.1** are met. So,

$$\frac{d}{dt}\{\mathcal{F}[f(x)]\}(t) = \int_{-\infty}^{\infty} f(x)\frac{d}{dt}\left(e^{itx}\right)dx =$$
$$\int_{-\infty}^{\infty} ixf(x)e^{itx}dx = \mathcal{F}[ixf(x)](t),$$

provided that $xf(x)$ is absolutely integrable.

So, provided that $xf(x)$ is absolutely integrable, we have the **derivative of the Fourier transform rule**

$$\frac{d}{dt}\mathcal{F}[f(x)](t) = \mathcal{F}[ixf(x)](t) = i\mathcal{F}[xf(x)](t).$$

(4) The **scaling and shifting rule** is the following:
For any $a \neq 0$ and b real constants, we let $u = ax - b$ and get

$$\mathcal{F}[f(ax-b)](t) = \int_{-\infty}^{\infty} f(ax-b)e^{itx}dx = \frac{1}{a}\int_{-\text{sign}(a)\infty}^{\text{sign}(a)\infty} f(u)e^{\frac{it(u+b)}{a}}du =$$
$$\frac{1}{|a|}\,e^{\frac{ibt}{a}}\int_{-\infty}^{\infty} f(u)e^{\frac{itu}{a}}du = \frac{1}{|a|}\,e^{\frac{ibt}{a}}\mathcal{F}[f(x)]\left(\frac{t}{a}\right).$$

So, for any $a \neq 0$ and b real constants, we have the scaling and shifting rule

$$(4)\quad \mathcal{F}[f(ax-b)](t) = \frac{1}{|a|}\,e^{\frac{ibt}{a}}\mathcal{F}[f(x)]\left(\frac{t}{a}\right).$$

If $|a| = 1$, then we have only shifting, and if $b = 0$, we have only scaling.

(5) For any real constant c, we have

$$\mathcal{F}\left[e^{icx}f(x)\right](t) =$$
$$\int_{-\infty}^{\infty} e^{icx}f(x)e^{itx}dx = \int_{-\infty}^{\infty} f(x)e^{i(t+c)x}dx = \mathcal{F}[f(x)](t+c).$$

So, for any real constant c, we have the **rule of the multiplication by** e^{icx}

$$(5) \quad \mathcal{F}\left[e^{icx}f(x)\right](t) = \mathcal{F}[f(x)](t+c).$$

(6) Rule (5) and the **linearity** of the Fourier transform imply the following two **rules of the multiplication by** $\cos(cx)$ **or** $\sin(cx)$**, where** c **is any real constant**:

$$\mathcal{F}\left[\cos(cx)f(x)\right](t) = \frac{1}{2}\mathcal{F}\left[e^{icx}f(x)\right](t) + \frac{1}{2}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$
$$\frac{1}{2}\left[\mathcal{F}[f(x)](t+c) + \mathcal{F}[f(x)](t-c)\right]$$

and

$$\mathcal{F}\left[\sin(cx)f(x)\right](t) = \frac{1}{2i}\mathcal{F}\left[e^{icx}f(x)\right](t) - \frac{1}{2i}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$
$$\frac{-i}{2}\left[\mathcal{F}[f(x)](t+c) - \mathcal{F}[f(x)](t-c)\right].$$

(7) In the context of the Fourier transform, we define as the **convolution of two real absolutely or square integrable functions** $f(x)$ **and** $g(x)$ **in** $\mathbb{R}$ the integral denoted by

$$(f*g)(x) := \int_{-\infty}^{\infty} f(x-u)g(u)du$$

(or, as in other books, this integral multiplied by the factor $\frac{1}{\sqrt{2\pi}}$).

(a) If $f(x)$ and $g(x)$ are real absolutely integrable functions in $\mathbb{R}$, then $(f*g)(x)$ is absolutely integrable.

This is proven as follows. We notice that for any absolutely integrable function $h(x)$ in $\mathbb{R}$

$$\int_{-\infty}^{\infty} h(x-u)du \overset{x-u=v-x}{=} \int_{-\infty}^{\infty} h(v-x)dv \overset{v-x=y}{=} \int_{-\infty}^{\infty} h(y)dy.$$

Now, for the positive function $|f(x-u)g(u)|$ in $\mathbb{R}$, we have that the iterated integral

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|f(x-u)g(u)|\,dx\right]du = \int_{-\infty}^{\infty}|g(u)|du\int_{-\infty}^{\infty}|f(x-u)|dx =$$
$$\int_{-\infty}^{\infty}|g(u)|du\int_{-\infty}^{\infty}|f(x)|dx$$

is, by hypothesis, a product of two non-negative finite numbers and so non-negative finite.

Then, by the **Tonelli conditions** and **Fubini conditions** (see **Section 2.4**), $f(x-u)g(u)$ is absolutely integrable in $\mathbb{R}^2$, and then we obtain the inequality

$$\int_{-\infty}^{\infty} |(f*g)(x)|\,dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-u)g(u)\,du \right| dx \leq$$
$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |f(x-u)g(u)|du \right] dx = \int_{-\infty}^{\infty} |f(x-u)|dx \int_{-\infty}^{\infty} |g(u)|du =$$
$$\int_{-\infty}^{\infty} |f(x)|dx \int_{-\infty}^{\infty} |g(u)|du.$$

Or, using the notation introduced in **project Problem 2.6.42**, we have:

If $\|f\|_1 < \infty$ and $\|g\|_1 < \infty$, then $\|(f*g)\|_1 \leq \|f\|_1 \cdot \|g\|_1 < \infty$.

(b) If $f(x)$ and $g(x)$ are real square integrable functions in $\mathbb{R}$, then $(f*g)(x)$ is bounded.

Indeed, by **Hölder's Inequality** (see **project Problem 2.6.42**, we get

$$|(f*g)(x)| \leq \int_{-\infty}^{\infty} |f(x-u)g(u)|du \leq$$
$$\left[\int_{-\infty}^{\infty} |f(x-u)|^2 du \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |g(u)|^2 du \right]^{\frac{1}{2}} =$$
$$\left[\int_{-\infty}^{\infty} |f(u)|^2 du \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |g(u)|^2 du \right]^{\frac{1}{2}},$$

which, by hypothesis, is a product of two finite numbers and so finite.

Example 3.7.36 If two functions are absolutely integrable, their convolution may not be bounded. Let

$$f(x) = g(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}}\,, & \text{if } x \in [-1,0) \cup (0,1] \\ \\ 0\,, & \text{otherwise.} \end{cases}$$

Since these functions are non-negative and even, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = 2\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\left[2x^{\frac{1}{2}}\right]_0^1 = 4$$

and so the functions are absolutely integrable. Therefore,

$$\int_{-\infty}^{\infty} |f * g|(x)\,dx \le \int_{-\infty}^{\infty} |f(x)|\,dx \cdot \int_{-\infty}^{\infty} |g(x)|\,dx = 4 \times 4 = 16.$$

Since

$$\int_{-\infty}^{\infty} f^2(x)\,dx = \int_{-\infty}^{\infty} g^2(x)\,dx = 2\int_0^1 \frac{1}{x}\,dx = 2\left[\ln(x)\right]_0^1 = \infty,$$

the functions are not square integrable.

Now,

$$(f * g)(0) = \int_{-\infty}^{\infty} f(0-u)g(u)\,du = \int_{-1}^{1} \frac{1}{|u|}\,du = 2\int_0^1 \frac{1}{u}\,du = \infty,$$

and so $(f * g)(x)$ is not bounded.

▲

(c) The operation of **convolution is commutative**, i.e., $f*g = g*f$. This is easily seen by making the change of variables $x-u=v$ to obtain

$$\int_{-\infty}^{\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(v)g(x-v)dv.$$

(d) It is straightforward that for any real constant a

$$a(f * g) = (af) * g = f * (ag).$$

(e) The convolution is **linear** with respect to each function position. I.e., for any real constants a and b, we have

$$\begin{gathered}(af_1 + bf_2) * g = a(f_1 * g) + b(f_2 * g) \quad \text{and} \\ f * (ag_1 + bg_2) = a(f * g_1) + b(f * g_2).\end{gathered}$$

(f) The convolution is **associative**, i.e., $(f * g) * h = f * (g * h)$. The proof is similar to the proof in **Example 2.7.5**.

(g) Using the two cases of the definitions of the Fourier transform **(3.27)** or **(3.28)**, we can prove that in either case **the Fourier transform of the convolution of two functions is equal to the product of their Fourier transforms**. I.e.,

$$\mathcal{F}[(f * g)(x)](t) = \mathcal{F}[f(x)](t) \cdot \mathcal{F}[g(x)](t).$$

The proof is analogous to the proof with the Laplace transform, **Example 2.7.4**. It is even easier because switching of the order of integration, permissible here by the absolutely or square integrability, is more direct. That is,

$$\mathcal{F}[f(x)](t) \cdot \mathcal{F}[g(x)](t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx \cdot \int_{-\infty}^{\infty} e^{ity} g(y)dy =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(x+y)} f(x)g(y)dxdy,$$

either as an improper integral that converges or as a principal value. We let $x + y = w$ and we find

$$\mathcal{F}[f(x)](t) \cdot \mathcal{F}[g(x)](t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itw} f(w-y)g(y)dwdy =$$
$$\int_{-\infty}^{\infty} e^{itw}\left[\int_{-\infty}^{\infty} f(w-y)g(y)dy\right]dw =$$
$$\int_{-\infty}^{\infty} e^{itw}(f*g)(w)dw = \mathcal{F}[(f*g)(x)](t).$$

We can use this formula and its inverse in applications (in either direction: $A = B$ or $B = A$). I.e.,

$$\mathcal{F}[(f*g)(x)](t) = \mathcal{F}[f(x)](t) \cdot \mathcal{F}[g(x)](t),$$

or

$$(f*g)(x) = \mathcal{F}^{-1}\left\{\mathcal{F}[f(x)](t) \cdot \mathcal{F}[g(x)](t)\right\}(x).$$

Examples

Example 3.7.37 By **Problem 3.7.24**, we have

$$\mathcal{F}\left[e^{-x^2}\right](t) = \int_{-\infty}^{\infty} e^{-x^2} e^{itx}dx = \int_{-\infty}^{\infty} e^{-x^2}\cos(tx)dx = \sqrt{\pi}\, e^{-\frac{t^2}{4}}.$$

Then, by the scaling and shifting **Rule (4)**, we obtain that for any real constants $a > 0$ and b, we have

$$\mathcal{F}\left[e^{-a(x-b)^2}\right](t) = \mathcal{F}\left[e^{-[\sqrt{a}(x-b)]^2}\right](t) =$$
$$\frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{itb\sqrt{a}}{\sqrt{a}}}\, e^{-\frac{t^2}{4a}} = \sqrt{\frac{\pi}{a}}\, e^{itb} e^{-\frac{t^2}{4a}}.$$

Also, by **Rule (6)**, we find that for any real constant c

$$\mathcal{F}\left[e^{-x^2}\sin(cx)\right](t) = \frac{-i}{2}\left[\hat{f}(t+c) - \hat{f}(t-c)\right] =$$

$$= \frac{-i}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} - \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right] = i\sqrt{\pi}\, e^{-\frac{(t^2+c^2)}{4}} \sinh\left(\frac{ct}{2}\right),$$

and similarly

$$\mathcal{F}\left[e^{-x^2}\cos(cx)\right](t) = \frac{1}{2}\left[\hat{f}(t+c) + \hat{f}(t-c)\right] =$$
$$\frac{1}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} + \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right] = \sqrt{\pi}\, e^{-\frac{(t^2+c^2)}{4}} \cosh\left(\frac{ct}{2}\right).$$

▲

Example 3.7.38 We compute

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \int_{-\infty}^{\infty} \frac{e^{-|x|}}{2}\, e^{itx} dx =$$
$$\frac{1}{2}\int_{-\infty}^{0} e^{x(1+it)} dx + \frac{1}{2}\int_{0}^{\infty} e^{x(-1+it)} dx =$$
$$\frac{1}{2}\left[\frac{1}{1+it} - \frac{1}{-1+it}\right] = \frac{1}{1+t^2}.$$

Then, with $f(x) = \dfrac{e^{-|x|}}{2}$, by the convolution rule we have

$$\left(\frac{1}{1+t^2}\right)^2 = \mathcal{F}[(f*f)(x)](t),$$

and so

$$g(x) := \mathcal{F}^{-1}\left[\left(\frac{1}{1+t^2}\right)^2\right](x) = [f*f](x).$$

Therefore, to find the inverse transform $g(x)$, we need to compute the $[f*f](x)$. For $x \leq 0$, we have

$$[f*f](x) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-u|-|u|} du =$$
$$\frac{1}{4}\left(\int_{-\infty}^{x} e^{2u-x} du + \int_{x}^{0} e^{x} du + \int_{-\infty}^{x} e^{-2u+x} du\right) = \frac{1}{4}(1-x)\, e^{x}.$$

Similarly, for $x \geq 0$, we find

$$[f*f](x) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-u|-|u|} du =$$
$$\frac{1}{4}\left(\int_{-\infty}^{0} e^{2u-x} du + \int_{0}^{x} e^{-x} du + \int_{x}^{\infty} e^{-2u+x} du\right) = \frac{1}{4}(1+x)\, e^{-x}.$$

The two parts can be put together as one formula:

$$\forall \;\; x \in \mathbb{R}, \quad g(x) := \mathcal{F}^{-1}\left[\left(\frac{1}{1+t^2}\right)^2\right](x) = \frac{1}{4}(1+|x|)\,e^{-|x|}.$$

Now, we combine the result of the Fourier transforms here and of **Example 3.7.35**, the **inversion Theorem, 3.7.6**, and the **general Parseval equation (3.33)** [or **(3.34)**] to derive the known result in **Examples 2.2.8, 2.2.9** and **Problems 2.4.9, 2.5.18**. We have

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \frac{1}{1+t^2}.$$

If for any $a > 0$ constant, we let

$$f(x) = \begin{cases} 1, & \text{if} \quad |x| < a \\ \\ 0, & \text{if} \quad |x| \geq a, \end{cases}$$

then

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = 2\frac{\sin(at)}{t}.$$

So, by the **inversion theorem**

$$\mathcal{F}[\hat{f}(x)](t) = \mathcal{F}\left[2\frac{\sin(ax)}{x}\right](t) = 2\pi f(t).$$

Then, by the **general Parseval equation (3.33)**, we obtain

$$2\pi \int_{-\infty}^{\infty} \frac{e^{-|x|}}{2} 2\frac{\sin(ax)}{x}\,dx = \int_{-\infty}^{\infty} \frac{1}{1+t^2} 2\pi f(t)\,dt =$$
$$2\pi \int_{-a}^{a} \frac{1}{1+t^2}\,dt = 2\pi \cdot 2\arctan(a).$$

Therefore, for any $a > 0$ we have

$$\int_{-\infty}^{\infty} e^{-|x|}\frac{\sin(ax)}{x}\,dx = 2\arctan(a)$$

or

$$\int_{0}^{\infty} e^{-x}\frac{\sin(ax)}{x}\,dx = \arctan(a).$$

▲

Example 3.7.39 We want to find the Fourier transform of the continuous triangular function

$$g(x) = \begin{cases} 1+x, & \text{if } -1 \le x \le 0 \\ \\ 1-x, & \text{if } 0 \le x \le 1 \\ \\ 0, & \text{if } |x| \le 1. \end{cases}$$

To avoid doing it directly (by definition), we observe that

$$g'(x) = \begin{cases} 1, & \text{if } -1 < x < 0 \\ \\ -1, & \text{if } 0 < x < 1 \\ \\ 0, & \text{if } |x| > 1. \end{cases}$$

Then,

$$\mathcal{F}[g(x)](t) = \int_{-\infty}^{\infty} g(x)e^{itx}\,dx = \int_{-1}^{1} g(x)e^{itx}\,dx = \int_0^1 e^{itx}\,dx + \int_{-1}^0 e^{itx}\,dx = \frac{2i[1-\cos(t)]}{t}.$$

Then, by **Rule (1)** we have

$$\frac{2i[1-\cos(t)]}{t} = -it\mathcal{F}[f(x)](t)$$

and finally

$$\mathcal{F}[f(x)](t) = \frac{2[\cos(t)-1]}{t^2}.$$

Then, by **Parseval equation (3.31)** we get

$$\int_{-\infty}^{\infty} \left\{\frac{2[\cos(t)-1]}{t^2}\right\}^2 dt = 2\pi \int_{-\infty}^{\infty} f^2(x)dx = 2\pi\left[\int_{-1}^0 (1+x)^2\,dx + \int_0^1 (1-x)^2\,dx\right] = 2\pi\left(\frac{1}{3}+\frac{1}{3}\right),$$

and so

$$\int_{-\infty}^{\infty} \frac{[\cos(t)-1]^2}{t^4}\,dt = 2\int_{-\infty}^{0} \frac{[\cos(t)-1]^2}{t^4}\,dt = 2\int_0^{\infty} \frac{[\cos(t)-1]^2}{t^4}\,dt = \frac{\pi}{3}.$$

Then, using $\cos(t) = 1 - 2\sin^2\left(\frac{t}{2}\right)$ and $x = \frac{t}{2}$, we find

$$\int_0^{\infty} \frac{\sin^4(x)}{x^4}\, dx = \frac{\pi}{3}.$$

(See also **Problem 2.2.33 hint**.)

▲

Example 3.7.40 By **Example 3.7.38**, we have

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \frac{1}{1+t^2}.$$

So, for the function

$$f(x) = \frac{1}{x^2 + 6x + 10} = \frac{1}{(x+3)^2 + 1},$$

we apply the **inversion Theorem, 3.7.6**, and **Rule (4)**, with $a = 1$ and $b = -3$, to find

$$\mathcal{F}[f(x)](t) = 2\pi e^{-3it} \frac{e^{-|t|}}{2} = \pi e^{-|t|-3it}.$$

With and $a \neq 0$, for

$$f(x) = \frac{1}{(ax+3)^2 + 1}$$

we find

$$\mathcal{F}[f(x)](t) = \frac{\pi}{|a|}\, e^{-\left|\frac{x}{a}\right| - \frac{3it}{a}}.$$

▲

Example 3.7.41 We let $f(x) = \int_{-\infty}^{x} e^{-(x-u)} g(u)du$, and we would like to find the relation of the Fourier transform of $f(x)$ with the Fourier transform of $g(x)$. We compute

$$\mathcal{F}\left[f(x)\right](t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} e^{-(x-u)} g(u)du\right) e^{itx} dx =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{-(x-u)} g(u) e^{itx} du dx = \int_{-\infty}^{\infty} g(u) e^{u} \left(\int_{u}^{\infty} e^{(-1+it)} dx\right) du =$$

$$\int_{-\infty}^{\infty} g(u) e^{u} \left[\frac{e^{(-1+it)}}{-1+it}\right]_u^{\infty} du = \frac{1}{1-it} \int_{-\infty}^{\infty} g(u) e^{itu} du.$$

That is, we have the relation $\mathcal{F}[f(x)](t) = \frac{1}{1-it}\mathcal{F}[g(x)](t)$.

▲

3.7.7 Applications

Application 1: Here, we give an application of the Fourier transform to an initial boundary value problem with the partial differential equation of the homogeneous wave equation. Namely, we want to solve the two-dimensional problem

$$\begin{aligned}
&u_{yy}(x,y) - u_{xx}(x,y) = 0, \quad \text{for} \quad y > 0,\\
&u(x,0) = 0, \quad \text{for} \quad -\infty < x < \infty,\\
&u_y(x,0) = \begin{cases} 1, & \text{if} \quad |x| < a, \\ \\ 0, & \text{if} \quad |x| \geq a. \end{cases} \qquad \text{where } a > 0 \text{ constant,}
\end{aligned}$$

We consider the Fourier transform of $u(x,y)$ with respect to x. That is,

$$\tilde{u}(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{ix\xi} dx.$$

Then, by taking the Fourier transform with respect to x of the given problem, we obtain (see also **Example 3.7.35**)

$$\begin{aligned}
&\tilde{u}_{yy}(x,y) + \xi^2 \tilde{u}(x,y) = 0, \quad \text{for} \quad y > 0,\\
&\tilde{u}(x,0) = 0, \quad \text{for} \quad -\infty < x < \infty,\\
&\tilde{u}_y(x,0) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}.
\end{aligned}$$

The general solution of the differential equation

$$\tilde{u}_{yy}(x,y) + \xi^2 \tilde{u}(x,y) = 0$$

is $\tilde{u}(\xi,y) = A\cos(\xi y) + B\sin(\xi y)$, where A and B are arbitrary constants. These constants are determined by the two initial conditions. From $\tilde{u}(\xi,0) = 0$, we find $A = 0$, and from $\tilde{u}_y(x,0) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}$, we find $B = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}$. So, the solution of the differential equation that also satisfies the initial conditions is

$$\tilde{u}(\xi,y) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)\sin(\xi y)}{\xi^2}.$$

The $\tilde{u}(\xi,y)$ is absolutely integrable in $(-\infty,\infty)$, and by the **inversion Theorem, 3.7.6**, we find

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin(a\xi)\sin(\xi y)}{\xi^2} e^{-i\xi x}\, d\xi =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\xi)\sin(\xi y)}{\xi^2} e^{-i\xi x}\, d\xi =$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\xi)\sin(\xi y)}{\xi^2} [\cos(\xi x) - i\sin(\xi x)]\, d\xi.$$

This integral is real. Its imaginary part is zero as the integral of an absolutely integrable odd function in $(-\infty, \infty)$. Its real part is the integral of an absolutely integrable even function in $(-\infty, \infty)$. Then we use some elementary trigonometric formulae and find

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\xi)\sin(\xi y)}{\xi^2} \cos(\xi x) d\xi =$$

$$\frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(a\xi)\sin(\xi y)}{\xi^2} \cos(\xi x)\, d\xi =$$

$$\frac{1}{\pi}\int_{0}^{\infty} \left[\frac{\cos(x\xi)\cos[(a-y)\xi]}{\xi^2} - \frac{\cos(x\xi)\cos[(a+y)\xi]}{\xi^2}\right] d\xi =$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \{\cos[(a+x-y)\xi] + \cos[(a-x-y)\xi] -$$

$$\cos[(a+x+y)\xi] - \cos[(a-x+y)\xi]\} \frac{d\xi}{\xi^2}.$$

We use integration by parts and the lower limit process to find

$$u(x,y) = \frac{1}{2\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} \{\cos[(a+x-y)\xi] + \cos[(a-x-y)\xi] -$$

$$\cos[(a+x+y)\xi] - \cos[(a-x+y)\xi]\}\, d\left(\frac{-1}{\xi}\right) =$$

$$\frac{1}{2\pi}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\{\cos[(a+x-y)\epsilon] + \cos[(a-x-y)\epsilon] -$$

$$\cos[(a+x+y)\epsilon] - \cos[(a-x+y)\epsilon]\} +$$

$$\frac{1}{2\pi}\lim_{\epsilon\to 0^+}\left\{-(a+x-y)\int_{\epsilon}^{\infty} \frac{\sin[(a+x-y)\xi]}{\xi} d\xi - \right.$$

$$(a-x-y)\int_{\epsilon}^{\infty} \frac{\sin[(a-x-y)\xi]}{\xi} d\xi$$

$$+(a+x+y)\int_{\epsilon}^{\infty} \frac{\sin[(a+x+y)\xi]}{\xi} d\xi +$$

$$\left.(a-x+y)\int_{\epsilon}^{\infty} \frac{\sin[(a-x+y)\xi]}{\xi} d\xi\right\}.$$

The first limit is zero (by L' Hôpital's rule), and so we find

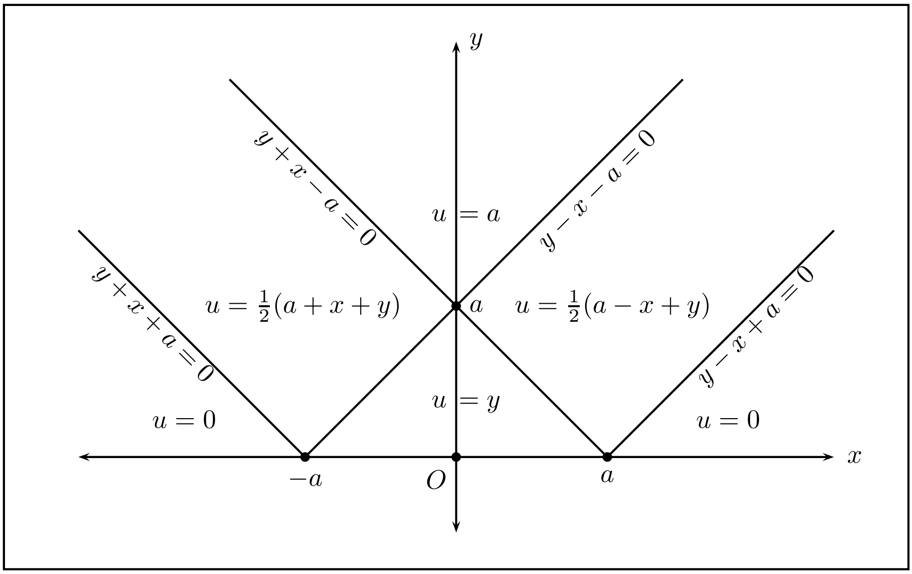

FIGURE 3.17: Sections of the solution in Application 1

$$
\begin{aligned}
u(x,y) = \frac{1}{2\pi}\Big\{ &-(a+x-y)\int_0^{\infty}\frac{\sin[(a+x-y)\xi]}{\xi}d\xi - \\
&(a-x-y)\int_0^{\infty}\frac{\sin[(a-x-y)\xi]}{\xi}d\xi + \\
&(a+x+y)\int_0^{\infty}\frac{\sin[(a+x+y)\xi]}{\xi}d\xi + \\
&(a-x+y)\int_0^{\infty}\frac{\sin[(a-x+y)\xi]}{\xi}d\xi\Big\}.
\end{aligned}
$$

But, from **Example 2.2.10**, we have the **result**

$$\forall\ \beta \in \mathbb{R}, \quad \frac{2}{\pi}\beta\int_0^{\infty}\frac{\sin(\beta\xi)}{\xi}\,dx = |\beta|.$$

Therefore, the final solution is written as

$$u(x,y) = \frac{1}{4}\left[\,|a+x+y| + |a-x+y| - |a+x-y| - |a-x-y|\,\right].$$

See **Figure 3.17** for the behavior of this solution in the upper half plane.

Application 2: We consider the ordinary differential equation of the unknown function $u = u(x)$

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu = 0, \tag{3.35}$$

for which $x = 0$ is a regular singular point. This is **Bessel's differential equation with imaginary argument of order** 0.[28]

By taking the Fourier transform of the equation and using its linearity properties, we find

$$\mathcal{F}\left[x\frac{d^2u}{dx^2}\right](t) + \mathcal{F}\left[\frac{du}{dx}\right](t) - \mathcal{F}[x\,u(x)](t) = 0.$$

We assume that $u(x)$ is absolutely integrable, and for convenience we write $\mathcal{F}[u(x)](t) = \hat{u}(t)$. Then, we use the applicable properties from **Subsection 3.7.6** to obtain

$$\frac{1}{i}\frac{d}{dt}[-t^2\hat{u}(t)] - it\hat{u}(t) - \frac{1}{i}\frac{d}{dt}[\hat{u}(t)] = 0,$$

from which we get the first-order, linear, homogeneous, ordinary, differential equation for $\hat{u}(t)$

$$(1 + t^2)\hat{u}'(t) + t\hat{u}(t) = 0.$$

We solve this differential equation, and we find

$$\hat{u}(t) = c\frac{1}{\sqrt{t^2+1}},$$

where c is an arbitrary constant.

The $\hat{u}(t)$ found is square integrable, and by the **inversion Theorem, 3.7.6**, we get

$$u(x) = \frac{c}{2\pi}\ \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{-itx}}{\sqrt{t^2+1}}\,dt =$$
$$a\cdot\left[\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(tx)}{\sqrt{t^2+1}}\,dt - i\ \text{P.V.}\int_{-\infty}^{\infty}\frac{\sin(tx)}{\sqrt{t^2+1}}\,dt\right],$$

---

[28] **Bessel's differential equation with imaginary argument of order** $\nu \geq 0$ is $\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)u = 0$. Its solutions are called **the Bessel functions with imaginary argument of order** ν. They are referred to by other names, too. With $\nu = 0$, we obtain the equation we have considered above.

where a is an arbitrary constant.

The second principal value is zero since the integrand function is odd. Making the substitution $t = \sinh(\xi)$, we find

$$u(x) = a \cdot \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(tx)}{\sqrt{t^2+1}}\, dt =$$
$$a \cdot \text{P.V.} \int_{-\infty}^{\infty} \cos[x \sinh(\xi)] d\xi.$$

This solution is in integral form. As $x = 0$ is a singular point for the differential equation, we notice that for the solution found $u(0) = \infty$. Apart from $x = 0$, the principal value found converges conditionally. (See **Problem 1.3.12**.) The convergence is slow.

We can check that this solution satisfies the given differential equation. Indeed,

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu(x) =$$
$$a \cdot \ \text{P.V.} \left\{ \int_{-\infty}^{\infty} -x\cos[x\sinh(\xi)]\sinh^2(\xi) d\xi + \right.$$
$$\left. \int_{-\infty}^{\infty} -\sin[x\sinh(\xi)]\sinh(\xi) d\xi - \int_{-\infty}^{\infty} x\cos[x\sinh(\xi)] d\xi \right\} =$$
$$= a \cdot \lim_{R\to\infty} \int_{-R}^{R} \frac{d}{d\xi} \left\{ -\sin[x\sinh(\xi)]\cosh(\xi) \right\} d\xi =$$
$$a \lim_{R\to\infty} \left[-\sin[x\sinh(\xi)]\cosh(\xi) \right]_{-R}^{R} = a \lim_{R\to\infty} 0 = 0,$$

since $\sin[x \sinh(\xi)] \cosh(\xi)$ is an odd function in ξ.

So, regardless of how many hypotheses were satisfied and how many were violated while taking and manipulating the Fourier transforms and applying an inversion theorem, we reach an answer that in the end is a solution to the given differential equation. This tactic is frequently used when we solve applied problems of differential equations. We use the Fourier or Laplace transforms and the inversion theorems without checking the hypotheses under which they apply, and in the end we check if the final answer solves the problem. If it does, we are finished.

Since $\cos[x \sinh(\xi)]$ is even, we can also write the solution as

$$u(x) = b \int_{0}^{\infty} \frac{\cos(tx)}{\sqrt{t^2+1}}\, dt = b \int_{0}^{\infty} \cos[x\sinh(\xi)] d\xi,$$

with b as an arbitrary constant. We also see that the solution $u(x)$ is an even function.

3.7.8 The Fourier Transform with Complex Argument

If in **Definition 3.7.3** [equation **(3.27)** or **(3.29)**] of the Fourier transform of a real function $y = f(x)$ on $\mathbb{R}$ we replace the e^{itx} with $e^{-sx}e^{itx} = e^{(it-s)x} = e^{i(t+is)x} = e^{iwx}$, where $w = t + is$, then we get what we call the **Fourier transform of** $y = f(x)$ **with complex argument the** $w = t + is$.

So, equations **(3.27)** and **(3.29)** with $w = t + is$ become

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \int_{-\infty}^{\infty} f(x)e^{iwx}dx =$$
$$\int_{-\infty}^{\infty} f(x)e^{(-s+it)x}dx = \mathcal{F}[e^{-sx}f(x)](t), \quad \forall\ t \in \mathbb{R}$$

and

$$\tilde{f}(w) = \mathfrak{F}[f(x)](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iwx}dx =$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{(-s+it)x}dx = \mathfrak{F}[e^{-sx}f(x)](t), \quad \forall\ t \in \mathbb{R},$$

respectively.

We observe that $\mathcal{F}[f(x)](w) = \mathcal{F}[e^{-sx}f(x)](t)$, and so for the Fourier transform with complex argument $w = t + is$ to exist, we must require $e^{-sx}f(x)$ to be absolutely integrable. Depending on the value of $s = \mathrm{Im}(w)$, this condition may change, and so $\mathcal{F}[f(x)](w)$ may exist in some areas of the complex plane $\mathbb{C}$ but may not exist in others.

In the areas of the complex plane $\mathbb{C}$ in which the Fourier transforms with complex argument of the functions under consideration exist, all the properties and rules as they are referred to or analyzed in **Subsection 3.7.6** are valid in this new situation. All that we must do is simply replace the real variable t with the complex variable w, and any real constant is now considered to be a complex constant. Thus, we do not need to restate these rules. But, the reader may want to do so and prove them again. All proofs carry through analogously.

Before we study any example and application, we must study some new properties of the Fourier transform with complex argument.

(1) (a) Let us assume that $e^{-sx}f(x)$ is absolutely integrable for $s = a$ and $s = b$, with $a < b$. Then, $e^{-sx}f(x)$ is also absolutely integrable for any $s \in [a, b]$.

This claim follows immediately from the hypotheses and the obvious inequality

$$\int_{-\infty}^{\infty} \left| e^{-sx} f(x) \right| dx = \int_{-\infty}^{0} e^{-sx}|f(x)|dx + \int_{0}^{\infty} e^{-sx}|f(x)|dx \leq$$
$$\int_{-\infty}^{0} e^{-bx}|f(x)|dx + \int_{0}^{\infty} e^{-ax}|f(x)|dx = \text{finite} + \text{finite} = \text{finite } (< \infty).$$

(b) If $e^{-sx}f(x)$ is square integrable for $s = a$ and $s = b$, with $a < b$, then $e^{-sx}f(x)$ is absolutely integrable for any $s \in (a, b)$.

This claim follows from the hypotheses, the **Cauchy-Schwarz inequality** [see **project Problem 2.6.42, Item (1.)**] and the inequalities

$$\int_{-\infty}^{\infty} \left| e^{-sx} f(x) \right| dx = \int_{-\infty}^{0} e^{-sx}|f(x)|dx + \int_{0}^{\infty} e^{-sx}|f(x)|dx =$$
$$\int_{-\infty}^{0} e^{[-b+(b-s)]x}|f(x)|dx + \int_{0}^{\infty} e^{[-a-(s-a)]x}|f(x)|dx =$$
$$\int_{-\infty}^{0} \left[e^{(b-s)x}\right] \cdot \left[e^{-bx}|f(x)|\right] dx + \int_{0}^{\infty} \left[e^{-(s-a)x}\right] \cdot \left[e^{-ax}|f(x)|\right] dx \leq$$
$$\left[\int_{-\infty}^{0} e^{2(b-s)x}dx \cdot \int_{-\infty}^{0} e^{-2bx}|f(x)|^2dx\right]^{\frac{1}{2}} +$$
$$\left[\int_{0}^{\infty} e^{-2(s-a)x}dx \cdot \int_{0}^{\infty} e^{-2ax}|f(x)|^2dx\right]^{\frac{1}{2}} =$$
$$\left[\frac{1}{2(b-s)} \int_{-\infty}^{0} e^{-2bx}|f(x)|^2dx\right]^{\frac{1}{2}} + \left[\frac{1}{2(s-a)} \int_{0}^{\infty} e^{-2ax}|f(x)|^2dx\right]^{\frac{1}{2}} =$$
$$\text{finite } + \text{ finite } = \text{ finite } (< \infty).$$

(2) Under the **condition (a)** or **(b)**, in **(1)**, the $\hat{f}(w) = \mathcal{F}[f(x)](w)$ exists for any $s \in [a, b]$ or any $s \in (a, b)$, respectively. Hence, it exists in the infinite horizontal strip of the complex plane $\mathbb{C}$ respectively defined by

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } a \leq s \leq b\}$$

or

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } a < s < b\}.$$

(3) In such a strip, the convergence is uniform in either s or t since $\left|e^{itx}\right| = 1$ and $e^{-sx}f(x)$ is absolutely integrable. (See **Theorem 2.3.4**, etc.) So, in such a strip, the $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is a continuous function in s and t.

(4) We then consider a maximal open horizontal strip

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } -\infty \leq p < s < q \leq \infty\} \subseteq \mathbb{C}$$

in which $\hat{f}(w) = \mathcal{F}[f(x)](w)$ exists. In such a strip, the Fourier transform $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is a holomorphic function.

This result follows by the **property (3)** above and **Morera's Theorem, 3.5.5**. We consider any simple closed contour $C \subset \mathbb{S}$, and we compute

$$\oint_C \hat{f}(w)dw = \oint_C \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx\,dw =$$
$$\int_{-\infty}^{\infty} f(x) \oint_C e^{iwx}\,dw\,dx = \int_{-\infty}^{\infty} f(x) \cdot 0\,dx = 0.$$

(The switching of the order of the double integration follows from the **Tonelli conditions, Section 2.4**, since $\left|e^{itx}\right| = 1$ and $e^{-sx}f(x)$ is absolutely integrable.)

So, with $e^{-sx}f(x)$ absolutely integrable in $\mathbb{R}$, by the **Riemann-Lebesgue Lemma, 3.7.7**, in a maximal open horizontal strip, in which $-\infty \leq p < s < q \leq \infty$, the limit of the Fourier transform $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is zero as $w \to \infty$, where ∞ here is the complex infinity within the strips and so the convergence is uniform.

Example 3.7.42 We consider any $a > 0$, and we define

$$f(x) = \frac{1}{2a}\, e^{-a|x|} = \begin{cases} \dfrac{1}{2a}\, e^{-ax}, & \text{if } x > 0 \\[2ex] \dfrac{1}{2a}, & \text{if } x = 0 \\[2ex] \dfrac{1}{2a}\, e^{ax}, & \text{if } x < 0. \end{cases}$$

Notice that if $s \leq -a$ or $s \geq a$, then the positive function $e^{-sx}f(x)$ is not absolutely integrable. (Check!) But, for $-a < s < a$, it is absolutely integrable. (Check!)

Then, in the open horizontal strip

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } -a < s < a\},$$

we have

$$\hat{f}(w) = \int_{-\infty}^{\infty} \frac{1}{2a}\, e^{iw} e^{-a|x|} dx =$$
$$\frac{1}{2a} \int_{-\infty}^{0} e^{(it-s+a)x} dx + \frac{1}{2a} \int_{0}^{\infty} e^{(it-s-a)x} dx =$$

$$= \frac{1}{2a}\left[\frac{1}{a+it-s} - \frac{1}{-a+it-s}\right] = \frac{1}{2a}\cdot\frac{-2a}{-a^2-(t+is)^2} = \frac{1}{a^2+w^2}.$$

Hence, in the above maximal open horizontal strip $\mathbb{S}$, we have

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \frac{1}{a^2+w^2}.$$

We see that this Fourier transform with complex argument w is, as expected, holomorphic with respect to w in $\mathbb{S}$. Its limit at infinity is zero.

▲

(5) Since $e^{-sx}f(x)$ is absolutely integrable, the **inversion Theorem, 3.7.6**, can apply to $\hat{f}(w) = \mathcal{F}[f(x)](w) = \mathcal{F}[e^{-sx}f(x)](t)$. Then, we get

$$e^{-sx}f(x) = \frac{1}{2\pi}\ \text{P.V.}\int_{-\infty}^{\infty}\hat{f}(w)e^{-itx}dt = \frac{1}{2\pi}\lim_{M\to\infty}\int_{-M}^{M}\hat{f}(w)e^{-itx}dt.$$

So, with $w = t+is$ we have

$$f(x) = \frac{1}{2\pi}\ \text{P.V.}\int_{-\infty}^{\infty}\hat{f}(t)e^{sx}e^{-itx}dt =$$
$$\frac{1}{2\pi}\ \text{P.V.}\int_{-\infty}^{\infty}\hat{f}(w)e^{-i(t+is)x}dt = \frac{1}{2\pi}\ \text{P.V.}\int_{-\infty}^{\infty}\hat{f}(w)e^{-iwx}dt =$$
$$\frac{1}{2\pi}\lim_{M\to\infty}\int_{-M}^{M}\hat{f}(w)e^{-iwx}dt. \tag{3.36}$$

Since the integration is performed only with respect to t, for any given $s = \text{Im}(w)$, the integrals take place on the horizontal straight line $(-\infty+is, \infty+is)$. Then, if s varies in certain different subintervals of $(-\infty,\infty)$, we may find different $f(x)$'s as the inverse Fourier transforms of $\hat{f}(w)$.

Example 3.7.43 We will apply the **inversion Theorem**, that is, the **inversion formula (3.36)**, to the result of the **previous example**.

For any $a > 0$ and

$$f(x) = \frac{1}{2a}e^{-a|x|},$$

we found (in the **previous example**) that if $-a < s < a$, then

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \frac{1}{a^2+w^2},$$

where $w = t+is$.

Since for any $-a < s < a$, $f(x)$ [or $e^{-sx}f(x)$] satisfies all conditions under which we can apply the **inversion formula (3.36)**, we expect to retrieve the function $f(x)$ if we apply the inversion formula to $\hat{f}(w) = \dfrac{1}{a^2+w^2}$. But, we must expect this only for $-a < s < a$. If we apply the inversion formula for $a < s < \infty$ or $-\infty < s < -a$, then we must expect the possibility of finding different inverse transforms. So, in this example, we plan to explore this inversion.

Case $-a < s < a$**:** We let

$$f(x) = \frac{1}{2\pi}\int_{-\infty+si}^{\infty+si} \frac{e^{-iwx}}{a^2+w^2}\,dt,$$

where $w = t + is$.

Subcase $x > 0$: To find this principal value, we use the negatively oriented contour

$$C^- = [-M+si, M+si] + A_R^-,$$

where $M > 0$ such that $M^2 + s^2 > a^2$, $R = \sqrt{M^2+s^2}$ and A_R^- is the negative arc which is the part of the circle $C(0,R)$, with center the origin and radius R, and below the line $(-\infty + is, \infty + is)$. [So, along this arc $\theta = \mathrm{Arg}(w)$ satisfies $\arcsin\left(\dfrac{s}{M}\right) \geq \theta \geq -\pi - \arcsin\left(\dfrac{s}{M}\right)$.] This contour encloses the number $-ai$, which is a simple pole for the function $e^{-iwx}\hat{f}(w) = \dfrac{e^{-iwx}}{a^2+w^2}$. (Make a figure of the contour.)

We have that

$$\operatorname*{Res}_{z=-ai}\left[e^{-iwx}\hat{f}(w)\right] = \frac{-e^{-ax}}{2ai},$$

and so by the **Residue Theorem, 3.7.1**, we find

$$\frac{1}{2\pi}\int_{C^-}\frac{e^{-iwx}}{a^2+w^2}\,dw = \frac{1}{2\pi}\left\{\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw + \int_{A_R^-}\frac{e^{-iwx}}{a^2+w^2}\,dw\right\} =$$
$$\frac{1}{2\pi}(-2\pi i)\frac{-e^{-ax}}{2ai} = \frac{e^{-ax}}{2a}. \tag{3.37}$$

Now, if $w = u + iv$ along A_R^-, then $v \leq s$, and since $x > 0$, we have $|e^{-iwx}| = |e^{-iux}e^{vx}| \leq e^{sx}$. Thus,

$$\left|\int_{A_R^-}\frac{e^{-iwx}}{a^2+w^2}\,dw\right| \leq \int_{A_R^-}\frac{e^{sx}}{R^2-a^2}\,|dw| <$$
$$\frac{e^{sx}}{R^2-a^2}2\pi R \longrightarrow 0, \quad \text{as } R \longrightarrow \infty.$$

But $R=\sqrt{M^2+s^2}$, and so $R\longrightarrow\infty \iff M\longrightarrow\infty$. Therefore, by taking the limit of the relation **(3.37)** above, as $M\longrightarrow\infty$, we find that

$$\text{for all } x>0,\ \ f(x)=\lim_{M\to\infty}\frac{1}{2\pi}\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw=\frac{e^{-ax}}{2a}.$$

Subcase $x<0$: This case is analogous to the previous subcase except we must use a different contour to obtain zero limit along the arc. The previous contour does not achieve this, as we can easily check, for the integrand approaches infinity in the limit. The contour this time is the positively oriented contour

$$C^{+}=[-M+si,M+si]+A_R^{+},$$

where M and R are the same as before but A_R^{+} is the positive arc of the circle $C(0,R)$, with center the origin and radius R and above the line $(-\infty+is,\infty+is)$. This contour encloses the number ai, which is a simple pole for $e^{-iwx}\hat{f}(w)=\dfrac{e^{-iwx}}{a^2+w^2}$. (Make a figure of the contour.)

We have that

$$\operatorname*{Res}_{z=ai}\left[e^{-iwx}\hat{f}(w)\right]=\frac{e^{ax}}{2ai}.$$

Then, by analogous work as before, we find that

$$\text{for all } x<0,\ \ f(x)=\lim_{M\to\infty}\frac{1}{2\pi}\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw=\frac{e^{ax}}{2a}.$$

Subcase $x=0$: If we plug $x=0$ in both subcases above, we find the same result

$$f(0)=\frac{1}{2a},$$

which, with $-a<s<a$ and $w=t+is$, we can also compute directly by the integral

$$\frac{1}{2\pi}\int_{-\infty+si}^{\infty+si}\frac{1}{a^2+w^2}\,dw.$$

We see that all three subcases can be put in one formula as

$$f(x)=\frac{e^{-a|x|}}{2a},\quad \text{for } x\in\mathbb{R},$$

which is, indeed, the function $f(x)$ we started with in the **previous example**.

Case $s<-a$.

Subcase $x<0$**:** As before, we consider the contour

$$C^{+}=[-M+si,M+si]+A_R^{+}.$$

Notice that in this case the contour encloses both poles $-ai$ and ai. Then, working as in the previous case, we find

$$\text{for all } x<0,\ \ f(x)=\frac{1}{2\pi}2\pi i\left(\frac{e^{ax}}{2ia}-\frac{e^{-ax}}{2ia}\right)=\frac{\sinh(ax)}{a}.$$

Subcase $x>0$: Again, we consider the contour

$$C^{-}=[-M+si,M+si]+A_R^{-}.$$

Notice that in this case the contour does not enclose any singularity (pole). Then, for $x>0$, we find $f(x)=0$.

Both subcases agree at $x=0$, and so we can eventually write

$$f(x)=\begin{cases}\dfrac{\sinh(ax)}{a}, & \text{if } x<0\\[2ex] 0, & \text{if } x\geq 0.\end{cases}$$

Case $s>a$.

This case is analogous to the **previous case**. We use the same contours as above for the subcases $x<0$ and $x>0$ but in reverse order. Eventually, we find

$$f(x)=\begin{cases}\dfrac{-\sinh(ax)}{a}, & \text{if } x>0\\[2ex] 0, & \text{if } x\leq 0.\end{cases}$$

We see that the answers to the inverse Fourier transforms in the three possible different cases of the horizontal strips of s are different.

▲

(6) We are going to prove a result that we tacitly used in the **previous example**.

With $e^{-sx}f(x)$ absolutely integrable in $\mathbb{R}$, in the maximal open horizontal strip

$$\mathbb{S}=\{w=t+is\quad|\quad-\infty<t<\infty\ \text{ and }\ -\infty\leq p<s<q\leq\infty\}\subseteq\mathbb{C}$$

in which $\hat{f}(w)=\mathcal{F}[f(x)](w)$ exists and is holomorphic, the principal value in the **inversion formula (3.36)** is independent of s.

In the proof, we use the fact that the Fourier transform $\hat{f}(w)=\mathcal{F}[f(x)](w)$ has limit zero as $w\to\infty$ (where ∞ is the complex infinity within the strip), as we have already justified in **(4)** above.

We pick any $p < s_1 < s_2 < q$ and consider the contour

$$\begin{gathered} C^+ = [-M + is_1, M + is_1] + [M + is_1, M + is_2] + \\ [M + is_2, -M + is_2] + [-M + is_2, -M + is_1], \end{gathered}$$

which is a positively oriented parallelogram.

Since on C^+ and in its interior the function $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is complex analytic with no singularities, the **Cauchy-Goursat Theorem, 3.5.3**, gives

$$\oint_{C^+} \hat{f}(w)\, dw = 0.$$

Then, we let $M \longrightarrow \infty$ and eventually get

$$\text{P.V.} \int_{-\infty+is_1}^{+\infty+is_1} \hat{f}(w)\, dw = \text{P.V.} \int_{-\infty+is_2}^{+\infty+is_2} \hat{f}(w)\, dw,$$

since by the mentioned condition

$$\lim_{w \to \infty} \hat{f}(w) = 0,$$

the integrals along the constant length($= s_2 - s_1$), finite vertical segments $[M + is_1, M + is_2]$ and $[-M + is_2, -M + is_1]$ become zero, as $M \longrightarrow \infty$ (the complex infinity, and so the convergence is uniform). (Work out the details, as an easy exercise!)

Application:

In **Application 1** of **Subsection 2.7.3**, we saw the Bessel function of order zero

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \tag{3.38}$$

as a solution of Bessel's differential equation of order zero

$$xy'' + y' + xy = 0. \tag{3.39}$$

We see that $J_0(x)$ is an even function in $\mathbb{R}$. We are going to use the method of Fourier transform with complex argument to find an integral representation of it up to a multiplicative constant.

Applying the Fourier transform to this differential equation, as done in **Application 2** of **Subsection 3.7.7**, with the help of the rules

$$\mathcal{F}[f'(x)](w) = -iw\mathcal{F}[f(x)](w)$$

and

$$\frac{d}{dw}\{\mathcal{F}[f(x)]\}(w) = \mathcal{F}[ixf(x)](w),$$

and assuming that all conditions are met to apply them, we find the following first-order linear homogeneous differential equation for the Fourier transform with complex argument of $y(x)$, which here we write by $\hat{y}(w)$:

$$(w^2-1)\frac{d[\hat{y}(w)]}{dw}+w\,\hat{y}(w)=0.$$

The solution of this differential equation is easily found to be

$$\hat{y}(w)=c\left(w^2-1\right)^{\frac{-1}{2}},$$

where c is an arbitrary complex constant.

Now, to find $y(x)$ in integral form, we are going to use the **inversion formula (3.36)**, in **Item (4.)** above, and work in an analogous ways as in **Example 3.7.43**. In the process, we may consider any multiplicative constant to be 1 for convenience.

First of all, we must notice that the function to be used in **inversion formula (3.36)** is

$$z=g(w)=e^{-ixw}\left(w^2-1\right)^{\frac{-1}{2}}=e^{-ixw}(w-1)^{\frac{-1}{2}}\cdot(w+1)^{\frac{-1}{2}}.$$

This has singular points at $w=1$ and $w=-1$ and the fractional power $\frac{1}{2}$ dictates to use two branch cuts, one for each factor. (See also **Example 3.7.13**.)

As convenient branch cuts, that shift $w-1$ and $w+1$ to the origin, we choose the following:

$$\mathbb{C}-[+1,+\infty)\ \text{ for }\ \sqrt{w-1}\quad\text{ and }\quad\mathbb{C}-[-1,+\infty)\ \text{ for }\ \sqrt{w+1}.$$

The intersection of these branch cuts is $[+1,+\infty)$ and their union $[-1,+\infty)$, both intervals of the real axis.

The corresponding restrictions on the arguments are

$$0<\arg(w-1)<2\pi\quad\text{ and }\quad 0<\arg(w+1)<2\pi.$$

Then, by the definition of the non-integer powers through complex logarithms, we finally have

$$z=g(w)=e^{-ixw}\frac{1}{\sqrt{|w^2-1|}}\cdot e^{\frac{-i[\arg(w-1)+\arg(w+1)]}{2}}.$$

Since the singular points and the branch cuts are on the x-axis, in **inversion formula (3.36)**, we must consider the two infinite horizontal strips:

$$\mathbb{S}_1=\{w=t+si\mid\infty<t<\infty\quad\text{and}\quad s>0\}$$

and

$$\mathbb{S}_2 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad s < 0\}.$$

(I) First, we work in the strip $\mathbb{S}_1$. We pick any $s > 0$, and for any $M > 1$ we let $R = \sqrt{M^2 + s^2}$. We have to consider two cases $x < 0$ and $x > 0$. The work is analogous with **Example 3.7.43**.

Case (1): $x < 0$. We consider the contour

$$C^+ = [-M + si, M + si] + A_R^+,$$

where $[-M + si, M + si]$ is a positive horizontal segment and

$$A_R^+ = \left\{w = u + vi = Re^{i\theta} \mid \arcsin\left(\frac{s}{M}\right) \leq \theta \leq \pi - \arcsin\left(\frac{s}{M}\right)\right\}$$

is a positive arc. This is depicted as the upper part in **Figure 3.18**.

Since $g(w)$ has no singularities on C^+ and inside of it, by the **Cauchy-Goursat Theorem, 3.5.3**, we have

$$\oint_{C^+} g(w)\, dw = 0.$$

As in **Example 3.7.43**, with $x < 0$, we find

$$\lim_{M\to\infty} \int_{A_R^+} g(w)\, dw = 0,$$

and so we get

$$\text{P.V.} \int_{-\infty+is}^{\infty+is} g(w)\, dw = \lim_{M\to\infty} \int_{-M+is}^{M+is} g(w)\, dw = \int_{-\infty}^{\infty} g(w)\, dw = 0.$$

So, in $\mathbb{S}_1$ we get $y(x) = 0$ for all $x < 0$, which is a trivial solution for Bessel's equation. So, in this case, we do not obtain anything interesting.

Case (2): $x > 0$. We pick any $0 < r < 1$, M and R as before, and we consider the contour

$$C^- = [-M + si, M + si] + PA_R^- + [R, 1 + r]^- + S_r^+$$
$$+ [1 - r, -1 + r]^- + A_r^+ + [-1 + r, 1 - r]^+ + T_r^+ + [1 + r, R]^+ + A_R^+,$$

depicted as the lower part in **Figure 3.18**.

So, $[-M + si, M + si]$ is a positive horizontal segment as before. The arcs are:

$$\begin{aligned}
PA_R^- &= \left\{w = u + vi = Re^{i\theta} \mid \arcsin\left(\frac{s}{M}\right) \geq \theta > 0\right\}, \\
S_r^+ &= \left\{z = 1 + re^{i\theta} \mid 0 < \theta < \pi\right\}, \\
A_r^+ &= \left\{z = -1 + re^{i\theta} \mid 0 < \theta < 2\pi\right\}, \\
T_r^+ &= \left\{z = 1 + re^{i\theta} \mid \pi < \theta < 2\pi\right\}, \\
A_R^- &= \left\{z = Re^{i\theta} \mid 0 > \theta > -\pi - \arcsin\left(\frac{s}{M}\right)\right\}.
\end{aligned}$$

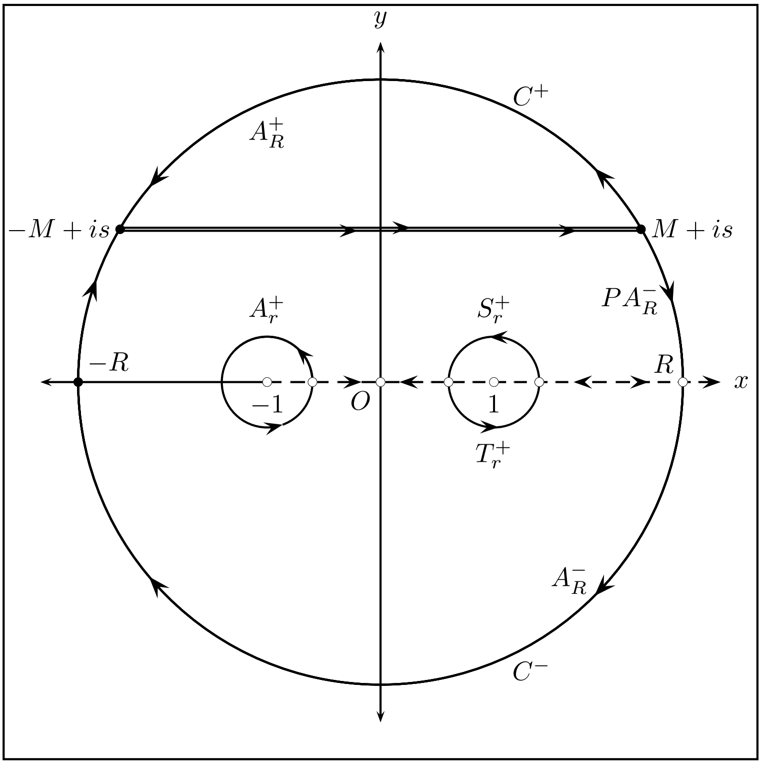

FIGURE 3.18: Contour for Application

The negative segments $[R, 1+r]^-$ and $[1-r, -1+r]^-$ are approached by staying in the upper half plane, whereas the positive segments $[-1+r, 1-r]^+$ and $[1+r, R]^+$ are approached by staying in the lower half plane.

Then, working as in **Examples 3.7.13** and **3.7.43**, with $x > 0$, we find the following:

$$(1) \quad \int_{PA_R^-} g(w)\, dw \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty.$$

$$(2) \quad \int_{A_R^-} g(w)\, dw \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty.$$

Keeping track with the correct arguments, we find

$$(3) \quad \int_{[R,1+r]} g(w)\, dw + \int_{[1+r,R]} g(w)\, dw = 0$$

for any $0 < r < 1$ and $R > 1$. This is so because in the intersection of both branch cuts, $(1, \infty)$, the function $g(w)$ is continuous and therefore holomorphic. Since $[R, 1+r]$ and $[1+r, R]$ are opposite segments the two partial line integrals cancel each other.

We now prove:

$$(4) \qquad \int_{A_r^+} g(w)\, dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

We have $0 < r < 1$ and

$$\int_{A_r^+} g(w)\, dw = \int_{A_r^+} \frac{e^{-ixw}}{\sqrt{w^2-1}} dw$$

with $w = re^{i\theta} + 1 \in A_r^+$ and $0 < \theta < 2\pi$. So,

$$dw = re^{i\theta} d\theta \implies |dw| = r\, d\theta.$$

$$e^{-ixw} = e^{-ix[1+r\cos(\theta)+i\sin(\theta)]} \implies \left|e^{-ixw}\right| = e^{xr\sin(\theta)}.$$

$$w^2 - 1 = r^2 e^{2i\theta} + 2re^{i\theta} + 1 - 1 = r\left(re^{2i\theta} + 2e^{i\theta}\right).$$

So,

$$\left|w^2-1\right| = r\left|re^{2i\theta} + 2e^{i\theta}\right| > r\left|\left|2e^{i\theta}\right| - \left|re^{2i\theta}\right|\right| = r(2-r).$$

Putting these pieces together, we get

$$\left|\int_{A_r^+} g(w)\, dw\right| \le \int_{A_r^+} |g(w)|\, dw < \int_0^{2\pi} \frac{e^{xr\sin(\theta)} r\, d\theta}{\sqrt{r(2-r)}} =$$
$$\int_0^{2\pi} \frac{e^{xr\sin(\theta)} \sqrt{r}\, d\theta}{\sqrt{2-r}} \longrightarrow \int_0^{2\pi} 0\, d\theta = 0, \text{ as } r \longrightarrow 0$$

(by the uniform or the bounded convergence), which implies what we wanted to prove.

In the same way, we prove the next two limits:

$$(5) \qquad \int_{S_r^+} g(w)\, dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

$$(6) \qquad \int_{T_r^+} g(w)\, dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

Now, we compute the integrals over the segments $[1-r, -1+r]$ and

$[-1+r, 1+r]$. We must consider the appropriate arguments in either one. So, $w = t + 0i$, and we have

$$(7) \quad \int_{[1-r,-1+r]} g(w)\,dw =$$

$$\int_{1-r}^{-1+r} e^{-ixt} \left|t^2-1\right|^{\frac{-1}{2}} e^{\frac{-i\left[\mathrm{Arg}(w-1)+\mathrm{Arg}(w+1)\right]}{2}}\,dt =$$

$$\int_{1-r}^{-1+r} [\cos(xt) - i\sin(xt)] \frac{1}{\sqrt{|t^2-1|}} e^{\frac{-i(\pi+\pi)}{2}}\,dt =$$

$$\int_{1-r}^{-1+r} \frac{\cos(xt)}{\sqrt{1-t^2}}(-1)\,dt = \int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt \longrightarrow$$

$$\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \ \text{ as } \ r \longrightarrow 0$$

(the sine integral is zero by the oddity of sine).

Similarly, on the interval $[-1+r, 1+r]$, we get

$$(8) \quad \int_{[-1+r,1-r]} g(w)\,dw =$$

$$\int_{-1+r}^{1-r} e^{-ixt} \left|t^2-1\right|^{\frac{-1}{2}} e^{\frac{-i\left[\mathrm{Arg}(w-1)+\mathrm{Arg}(w+1)\right]}{2}}\,dt =$$

$$\int_{-1+r}^{1-r} [\cos(xt) - i\sin(xt)] \frac{1}{\sqrt{|t^2-1|}} e^{\frac{-i(2\pi+2\pi)}{2}}\,dt =$$

$$= \int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}(+1)\,dt = \int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt \longrightarrow$$

$$\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \ \text{ as } \ r \longrightarrow 0$$

(the sine integral is zero by the oddity of sine).

Next, since

$$\oint_{C^-} g(w)\,dw = 0,$$

we find that for all $M > 1$,

$$\int_{-M+is}^{M+is} g(w)\,dw + 2\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt = 0.$$

So,

$$\text{P.V.}\int g(w)\,dw = \lim_{M\to\infty}\int_{-M+is}^{M+is} g(w)\,dw =$$
$$\int_{-\infty+is}^{\infty+is} g(w)\,dw = -2\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt.$$

Therefore, up to a multiplicative constant, the solution of the above Bessel's differential equation in integral form is

$$y(x) = \int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \quad \text{for all } x > 0.$$

Notice that $\lim\limits_{x\to 0} y(x) = [\arcsin(t)]_{-1}^{1} = \pi$, since limit and integral commute by, e.g., **Part I of Theorem 2.2.1** or **Theorem 2.3.4**.

(II) The work in the strip $\mathbb{S}_2$ is analogous. By choosing the correct contours and cases, we find

$$y(x) = 0, \quad \text{for all } x > 0$$

and

$$y(x) = \int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \quad \text{for all } x < 0.$$

Notice again that $\lim\limits_{x\to 0} y(x) = \pi$. So, we can continuously set $y(0) = \pi$.

Regardless of how many conditions on taking the Fourier transform and applying the inversion formula were kept or violated, the solution found satisfies the differential equation **(3.39)**. We can verify this by plugging the integral found into the equation and then differentiating under the integral sign is legitimate, by **Part II of Theorem 2.2.1**. Indeed,

$$x\frac{d^2}{dx^2}\left[\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt\right] + \frac{d}{dx}\left[\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt\right] + x\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt =$$
$$\int_{-1}^{1}\left[-xt^2\frac{\cos(xt)}{\sqrt{1-t^2}} - t\frac{\cos(xt)}{\sqrt{1-t^2}} + x\frac{\cos(xt)}{\sqrt{1-t^2}}\right]dt =$$
$$\ldots = \int_{-1}^{1}\frac{d}{dt}\left[\sqrt{1-t^2}\,\sin(xt)\right]dt = \left[\sqrt{1-t^2}\,\sin(xt)\right]_{-1}^{1} = 0.$$

Since $J_0(0) = 1$ and $y(0) = \pi$, by **(I)** and **(II)**, we finally have that the integral representation of $J_0(x)$ is

$$J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \quad \text{for all } x \in \mathbb{R}.$$

By the change of variables $t = \sin(\phi)$, we obtain another integral representation

$$J_0(x) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos[x\sin(\phi)]\, d\phi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos[x\sin(\phi)]\, d\phi =$$
$$\frac{1}{\pi}\int_0^{\pi} \cos[x\sin(\phi)]\, d\phi, \quad \text{for all } x \in \mathbb{R}.$$

(See also **Problem 3.7.85**.)

Problems

3.7.63 Using all results and examples of this section, find the values of the following four Fourier type integrals:

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{10x\cos(25x)}{x^2+100}\, dx, \qquad \text{(b)} \quad \int_{-\infty}^{\infty} \frac{10x\sin(25x)}{x^2+100}\, dx,$$

$$\text{(c)} \quad \int_{-\infty}^{\infty} \frac{10\cos(25x)}{x^2+100}\, dx, \qquad \text{(d)} \quad \int_{-\infty}^{\infty} \frac{10\sin(25x)}{x^2+100}\, dx.$$

3.7.64 (a) For $R > 0$, consider the contour $C^+ = [0, R] + A_R^+ + [Ri, 0]$ where A_R^+ is the arc $A_R = \{z = Re^{i\theta} \mid 0 \le \theta \le \pi/2\}$ traveled in the positive direction. For any $\alpha \in \mathbb{R}$, explain why

$$\oint_{C^+} \frac{e^{\alpha i z}}{z+1} dz = 0.$$

(b) Expand the equality in **(a)**, and then for $\alpha > 0$ take the limit as $R \to \infty$ to prove that

$$\text{(a)} \qquad \int_0^{\infty} \frac{\sin(\alpha x)}{x+1}\, dx = \int_0^{\infty} \frac{\sin(x)}{x+\alpha}\, dx =$$
$$\int_0^{\infty} \frac{e^{-\alpha x}}{x^2+1}\, dx = \alpha \int_0^{\infty} e^{-\alpha x} \arctan(x)\, dx.$$

$$\text{(b)} \qquad \int_0^{\infty} \frac{\cos(\alpha x)}{x+1}\, dx = \int_0^{\infty} \frac{\cos(x)}{x+\alpha}\, dx =$$
$$\int_0^{\infty} \frac{xe^{-\alpha x}}{x^2+1}\, dx = \frac{\alpha}{2} \int_0^{\infty} e^{-\alpha x} \log(x^2+1)\, dx.$$

(See also **Problem 2.2.29**.)

3.7.65 Prove that for any $a \in \mathbb{R}$, the limits of $\dfrac{\sin(z)}{z^a}$, $\dfrac{\cos(z)}{z^a}$, $\dfrac{\sinh(z)}{z^a}$, $\dfrac{\cosh(z)}{z^a}$, as $z \to \infty$, do not exist.

3.7.66 Use $f(z) = \dfrac{1 - e^{2iz}}{z^2}$, the **contour in Figure 3.16** and **Lemma 3.7.3** to find

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,dx = \pi \quad \text{and} \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2}\,dx = 0.$$

3.7.67 For $a > 0$, $a = 0$ and $a < 0$ constant, find the two integrals

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x \pm i}\,dx.$$

Then, separate real and imaginary parts to find the two corresponding real integrals in all cases.

If some integrals do not exist, then find their principal values.

3.7.68 For $a > 0$, $a = 0$ and $a < 0$ constant, find the integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)(x-2i)}\,dx.$$

Then, separate real and imaginary parts to find the two corresponding real integrals in all cases.

If some integrals do not exist, then find their principal values.

3.7.69 Prove that $\forall\ a \geq 0$ and $\forall\ b \geq 0$, we have

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = -\pi(a - b).$$

(See also **Problem 2.2.16** and the relative part of **Example 3.7.34**.)

3.7.70 Prove that the integral

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^4 + 2x^2 + 2}\,dx$$

exists and then evaluate it by computing its principal value.

3.7.71 Find the principal values

$$\text{(a)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{2ix}dx}{x-1},\qquad \text{(b)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{3ix}dx}{x+1},$$
$$\text{(c)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{xe^{3ix}dx}{x^2-1},\qquad \text{(d)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2ix}dx}{x^3+1},$$
$$\text{(e)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{xe^{3ix}dx}{x^3+1},\qquad \text{(f)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{x^2e^{-5ix}dx}{x^4-1}.$$

3.7.72 Find the formula for the Fourier type integral

$$\int_{-\infty}^{\infty}\frac{x^m e^{iax}}{x^{2n}+b^2}\,dx$$

with $a \in \mathbb{R}$ and $b > 0$ constants and $m,\ n$ integers satisfying $2n \geq m+1$.

Then, separate real and imaginary parts to find two real integrals.

Check the hypotheses of the **inversion Theorem, 3.7.6**, and apply it appropriately.

3.7.73 If $a > 0$ and μ constants, compute the Fourier type integral

$$\int_{-\infty}^{\infty}\frac{\sin(ax)}{x}e^{i\mu x}\,dx.$$

3.7.74 Use the techniques of this section, like in **Example 3.7.34** along with appropriate adjustments, and trigonometric identities such as:

$$4\sin^3(x) = 3\sin(x) - \sin(3x),\quad 8\sin^4(x) = 3 - 4\cos(2x) + \cos(4x),\quad \text{etc.},$$

to compute the following ten integrals:

$$\text{(1)}\qquad \int_{-\infty}^{\infty}\frac{\sin^3(x)}{x^k}\,dx \qquad \text{with} \qquad k = 1,\ 2,$$

$$\text{(2)}\qquad \int_{-\infty}^{\infty}\frac{\sin^4(x)}{x^l}\,dx \qquad \text{with} \qquad l = 1,\ 2,\ 3,$$

$$\text{(3)}\qquad \int_{0}^{\infty}\frac{\sin^3(x)}{x^k}\,dx \qquad \text{with} \qquad k = 1,\ 2,$$

$$\text{(4)}\qquad \int_{0}^{\infty}\frac{\sin^4(x)}{x^l}\,dx \qquad \text{with} \qquad l = 1,\ 2,\ 3.$$

(Compare with **Problems 2.2.33** and **2.5.14** and the answers provided there.)

3.7.75 Apply the **Parseval equation** separately and the **general Parseval equation** jointly to the Fourier transforms found in **Examples 3.7.31** and **3.7.32** to find three new integrals.

Find also these integrals by differentiating with respect to b the known integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+b^2}\,dx = \frac{\pi}{b}$$

and manipulating the result. Use the differentiation part of **Theorem 2.2.1** to justify that this differentiation is legitimate.

3.7.76 Prove

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,dx = 2\pi.$$

3.7.77 Use **Problem 3.7.24** to prove that for any $a \neq 0$ real constant

$$\mathcal{F}\left[e^{-a^2x^2}\right](t) = \frac{\sqrt{\pi}}{|a|}\,e^{-\frac{t^2}{4a^2}}.$$

3.7.78 Find the Fourier transforms of the following functions in $\mathbb{R}$, or explain if they do not exist:

$$f_1(x) = \frac{1}{x^2+1} \qquad f_2(x) = \frac{x}{x^2+1}$$

$$f_3(x) = \frac{x^3}{(x^2+1)^2} \qquad f_4(x) = e^{-4x^2}$$

$$f_5(x) = e^{-x} \qquad f_6(x) = xe^{-x}.$$

Check and apply the **inversion Theorem, 3.7.6**, to $f_3(x)$ and $f_4(x)$.

[Hint: For the last three functions, you cannot use **Lemma 3.7.6**, since the condition $\lim\limits_{z\to\infty} f(z) = 0$ is not satisfied. Explain why!

In the last three questions, you must directly use

$$\int_{-\infty}^{\infty} f(x)e^{itx}dx = \int_{-\infty}^{\infty} f(x)\left[\cos(tx) + i\sin(tx)\right]dx =$$
$$\int_{-\infty}^{\infty} f(x)\cos(tx)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)\,dx$$

and compute the two integrals in any possible way, or use already computed integrals, or the evenness and/or the oddity of certain functions. See **Problem 3.7.24** for the cosine integral of the even function $f_4(x)$, whereas its sine integral is automatically zero (by the oddity). Justify why the Fourier transforms of $f_5(x)$ and $f_6(x)$ do not exist.]

3.7.79 Let $f(x) = \chi_{[a,b]}(x)$, the characteristic function of an interval $[a, b]$ ($a \leq b$, reals), and similarly $g(x) = \chi_{[c,d]}(x)$. Prove

$$(f * g)(x) = \max\{\, 0,\ \min\{d,\ x - a\} - \max\{c,\ x - d\}\,\}\,.$$

3.7.80 If

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}}\,, & \text{if } x \in [-1, 0) \cup (0, 1] \\ \\ 0\,, & \text{otherwise,} \end{cases}$$

find explicitly $(f * f)(x)$ for $x \in \mathbb{R}$.

[Hint: Consider separate cases: $-\infty < x \leq -2$, $-2 < x \leq -1$, $-1 < x < 0$, $x = 0$, $0 < x \leq 1$, $1 < x \leq 2$, $2 < x < \infty$. In the two extreme cases, the answer is 0. For $x = 0$, $(f * f)(0) = \infty$, etc.]

3.7.81 Let f and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be two absolutely integrable functions. Prove:

(a) If g is continuous, then $f * g$ is continuous.

(b) If g is differentiable and g' is absolutely integrable, then $f * g$ is differentiable.

(Of course, the conditions on f and g can be switched without changing the results on $f * g$.)

3.7.82 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely integrable function, and let function $g : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded. Prove that $f * g$ is bounded. In fact, using the norm notation of **Problem 2.6.42**, prove

$$\|f * g\|_\infty \leq \|f\|_1 \cdot \|g\|_\infty.$$

Can you find the conditions under which this inequality holds as equality?

[Hint: For the last question, g is constant and f does not change sign.]

3.7.83 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely integrable function, and let function $g : \mathbb{R} \longrightarrow \mathbb{R}$ be square integrable. Prove that $f * g$ is square integrable. In fact, using the norm notation of **Problem 2.6.42**, prove

$$\|f * g\|_2 \leq \|f\|_1 \cdot \|g\|_2.$$

Prove that this inequality holds as equality iff either $f \equiv 0$ of $g \equiv 0$.

[Hint: Go through the following computation, justify each step and draw the conclusion:

$$\int_{\mathbb{R}_x} (f * g)^2(x)\, dx =$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} f(x-u) g(u)\, du \right]^2 dx \leq$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|\, du \right]^2 dx =$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)|^{\frac{1}{2}} \cdot |f(x-u)|^{\frac{1}{2}} \cdot |g(u)|\, du \right]^2 dx \leq$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)|\, du \int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|^2\, du \right] dx =$$

$$\|f\|_1 \int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|^2\, du \right] dx =$$

$$\|f\|_1 \int_{\mathbb{R}_u} \left[\int_{\mathbb{R}_x} |f(x-u)| \cdot |g(u)|^2\, dx \right] du =$$

$$\|f\|_1 \int_{\mathbb{R}_u} \left[|g(u)|^2 \int_{\mathbb{R}_x} |f(x-u)|\, dx \right] du =$$

$$\|f\|_1 \cdot \|f\|_1 \int_{\mathbb{R}_u} |g(u)|^2 du = \|f\|_1^2 \cdot \|g\|_2^2 .]$$

3.7.84 Consider the function $\hat{f}(w) = \dfrac{1}{w^3 - 8}$, with $w = t + is$ to be the Fourier transform of some functions. Notice that its singularities are the simple poles $w_0 = 2$, $w_1 = -1 + \sqrt{3}i$ and $w_2 = -1 - \sqrt{3}i$.

Consider the four horizontal strips: (1) $s < -\sqrt{3}$, (2) $-\sqrt{3} < s < 0$, (3) $0 < s < \sqrt{3}$ and (4) $\sqrt{3} < s$, in $\mathbb{C}$, and on each one find the corresponding inverse Fourier transform by applying the **inversion formula (3.36)**.

[Hint: Imitate **Example 3.7.43**. Use appropriate contours, as here we have three poles, one of which is on the real axis.]

3.7.85 Consider Bessel's equation **(3.39)** as in the **application** above. To find an integral representation of its solution, this time choose the branch cuts

$$\mathbb{C} - [+1, +\infty) \quad \text{for} \quad \sqrt{w-1} \quad \text{and} \quad \mathbb{C} - [-1, -\infty) \quad \text{for} \quad \sqrt{w+1}$$

and a contour similar, but not exactly the same, to **Contour 8** in **Figure 3.12** of the practice **Problem 3.7.21**.

Show that these choices do not produce a representation. Can you give reasons why?

3.7.86 Consider Bessel's differential equation with imaginary argument of order 0, **(3.35)**,

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu = 0.$$

Prove that the Fourier transform with complex argument $w = t + is$ of its solution is

$$\hat{u}(w) = \frac{1}{\sqrt{w^2+1}},$$

up to a multiplicative constant.

Notice that $w = -i$ and $w = i$ are singular points. Then, use appropriate branch cuts and contours in the infinite horizontal strips

$$\mathbb{S}_1 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad -\infty < s < -1\},$$

$$\mathbb{S}_2 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad -1 < s < 1\}$$

and

$$\mathbb{S}_3 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad 1 < s < \infty\}$$

to find integral representations of the solution of this differential equation (the Bessel function with imaginary argument of order 0), up to a (complex) multiplicative constant.

3.7.9 Improper Integrals and Logarithms

We begin with the following:

Example 3.7.44 [Compare with **Problems 2.1.19 (a)** and **3.7.90**.] Prove:

$$\int_0^\infty \frac{\ln(x)}{x^2+1}\, dx = 0.$$

We consider the complex function

$$f(z) = \frac{\log(z)}{z^2+1}.$$

This function has non-isolated singularities along a chosen branch cut of the complex $\log(z)$ and isolated singularities at the two roots $z = \pm i$ of the denominator.

We choose as the branch cut the ray of the non-positive imaginary semi-axis $\left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \{-yi \mid y \geq 0\}$ and define the open domain

$$D = \mathbb{C} - \left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \mathbb{C} - \{-yi \mid y \geq 0\} = \mathbb{C} - \{yi \mid y \leq 0\}.$$

Then, $f(z)$ is holomorphic in $D - \{i\}$, and for any $z = re^{i\theta} \in D$ we have

$$\log(z) = \log\left(re^{i\theta}\right) = \ln(r) + i\theta, \text{ with } -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

Next, we must choose a convenient simple closed piecewise differentiable contour in $\mathbb{C}$ that avoids $\left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \{-yi \mid y \geq 0\}$ and $z = i$. Then, we pick numbers r and R such that $0 < r < 1 < R < \infty$, and we let

$$C^+ = [r, R] + S_R^+ + [-R, -r] + S_r^-,$$

where S_R^+ is the upper half of $C(0, R)$, positively oriented, and S_r^- is the upper half of $C(0, r)$, negatively oriented. This contour lies in D and encloses the isolated singularity $z = i$ in its interior. See **Figure 3.19**.

Then,

$$\operatorname*{Res}_{z=i} f(z) = \frac{\log(i)}{2i} = \frac{\ln(1) + \frac{\pi}{2}i}{2i} = \frac{\pi}{4},$$

and by **the Residue Theorem, 3.7.1**, we get

$$\oint_{C^+} f(z)dz =$$

$$\int_{[r,R]} f(z)dz + \int_{S_R^+} f(z)dz + \int_{[-R,-r]} f(z)dz + \int_{S_r^-} f(z)dz = 2\pi i\frac{\pi}{4} = \frac{\pi^2}{2}i.$$

Since $\lim\limits_{z\to\infty} [zf(z)] = \lim\limits_{z\to\infty} \dfrac{z\log(z)}{z^2+1} = 0$ (for the proof of this easy fact, see **Problem 3.7.87**), by **Lemma 3.7.1** we have that

$$\lim_{R\to\infty} \int_{S_R^+} f(z)\, dz = 0.$$

Also,

$$\lim_{r\to 0} \int_{S_r^-} f(z)\, dz = 0,$$

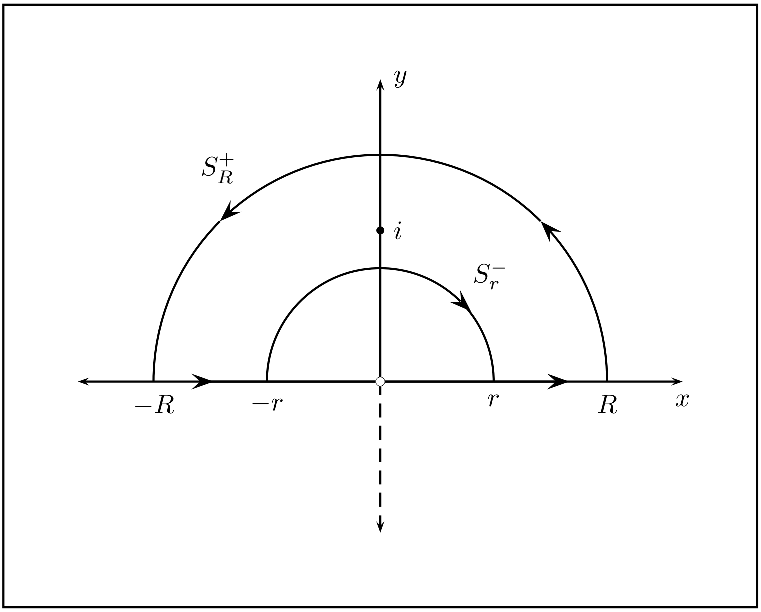

FIGURE 3.19: Contour 13 for Example 3.7.44

as it is seen by

$$\left|\int\limits_{S_r^-} f(z)\,dz\right| = \left|\int_\pi^0 \frac{\log\left(re^{i\theta}\right)}{r^2e^{2i\theta}+1} i\,r\,e^{i\theta}\,d\theta\right| \le \int_0^\pi \left|\frac{\ln(r)+i\theta}{r^2e^{2i\theta}+1} i\,r\,e^{i\theta}\right|\,d\theta =$$

$$\int_0^\pi \frac{\sqrt{\ln^2(r)+\theta^2}}{|r^2e^{2i\theta}+1|}\,r\,d\theta \le \int_0^\pi \frac{\sqrt{\ln^2(r)+\pi^2}}{1-r^2}\,r\,d\theta =$$

$$= \frac{\sqrt{\ln^2(r)+\pi^2}}{1-r^2}\,r\,\pi \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0^+.$$

Therefore,

$$\lim_{\substack{r\to 0^+\\ R\to\infty}} \int_{-R}^{-r} f(z)\,dz + \lim_{\substack{r\to 0^+\\ R\to\infty}} \int_r^R f(z)\,dz = \frac{\pi^2}{2}\,i.$$

Since the complex argument along the negative real semi-axis is π

and along the positive real semi-axis is 0, we find

$$\int_{-\infty}^{0} \frac{\ln(|x|) + i\pi}{x^2+1}\,dx + \int_{0}^{\infty} \frac{\ln(x)}{x^2+1}\,dx = \frac{\pi^2}{2}\,i.$$

Then,

$$2\int_{0}^{\infty} \frac{\ln(x)}{x^2+1}\,dx + \int_{-\infty}^{0} \frac{\pi\, i}{x^2+1}\,dx = \frac{\pi^2}{2}\,i.$$

This answer is purely imaginary. Therefore, its real part is zero. So, we obtain the **result**

$$\int_{0}^{\infty} \frac{\ln(x)}{x^2+1}\,dx = 0.$$

As for the equality of the imaginary parts, this yields

$$\int_{-\infty}^{0} \frac{\pi\, i}{x^2+1}\,dx = \frac{\pi^2}{2}\,i, \quad \text{or} \quad \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \frac{\pi}{2},$$

which is also obtained by a direct elementary computation and use of $\arctan(x)$.

▲

Remark 1: The given integral exists to begin with [see **Problem 2.1.19, (a)**]. Therefore, here we have evaluated its value by means of its principal value.

Remark 2: Using the above integral, for any constants $a > 0$ and $b > 0$, we find that

$$\int_{0}^{\infty} \frac{\ln(ax)}{x^2+b^2}\,dx = \frac{\pi \ln(ab)}{2b}.$$

[We first set $x = bu$ and so $dx = b\,du$, then we use $\ln(abu) = \ln(ab) + \ln(u)$, the result of **this example** and $\arctan(u)$.]

(See also **Problem 3.7.90**.)

Example 3.7.45 As we have seen in **Problem 2.1.19 (b)** [see also **Problem 3.7.89 (a)** below],

$$\frac{\pi}{4} - \frac{\ln(2)}{2} < \int_{1}^{\infty} \frac{\ln(x)}{x^2+1}\,dx = \int_{0}^{1} \frac{-\ln(x)}{x^2+1}\,dx < 1.$$

Here, we will evaluate this integral as a series of real numbers. We use integration by parts and get

$$\int_{0}^{1} \frac{-\ln(x)}{x^2+1}\,dx = \int_{0}^{1} -\ln(x)\,d\arctan(x) = [-\ln(x)\arctan(x)]_0^1 +$$

$$\int_{0}^{1} \frac{\arctan(x)}{x}\,dx = (-0+0) + \int_{0}^{1} \frac{\arctan(x)}{x}\,dx = \int_{0}^{1} \frac{\arctan(x)}{x}\,dx.$$

Since $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, for all $-1 < x < 1$, after integrating term by term, we have

$$\int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^2}.$$

[Since the last series converges (absolutely), by **Abel's Lemma** (see **footnote of Example 3.7.23**) this equality is correct.]

Finally,

$$\int_1^{\infty} \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^2} =$$

$$\text{(by the absolute convergence of this series)}$$

$$\left(\frac{1}{1^2} - \frac{1}{3^2}\right) + \left(\frac{1}{5^2} - \frac{1}{7^2}\right) + \left(\frac{1}{9^2} - \frac{1}{11^2}\right) + \ldots =$$

$$8\sum_{n=0}^{\infty} \frac{2n+1}{(4n+1)^2(4n+3)^2} := G = 0.915965594\ldots,$$

where with the symbol G we denote the so-called **Catalan**[29] **constant**.[30] (See also **Problem 3.7.93**.)

▲

Example 3.7.46 As we work in **Examples 3.7.8 and 3.7.44**, especially **Case (b) of Example 3.7.8**, with $-1 < \alpha < 0$, using the contour of the latter and making the necessary adjustments of the arguments in the presence of the complex log, we find the important **result**:

$$\forall\ \alpha\ :\ -1 < \alpha < 0, \qquad \int_0^{\infty} \frac{x^{\alpha}\ln(x)}{x+1}\,dx = \pi^2\cot(\alpha\pi)\csc(\alpha\pi).$$

[Look at **both examples** once more and make the necessary modifications to derive this result. Also, this result can be obtained by taking the derivative with respect to α of the integral computed in **Example 3.7.8, Case (b)**. Look at this fundamental integral again, justify that this differentiation is legitimate and obtain the result here in this way!]

Notice that this integral is $-\infty$ when $\alpha \leq -1$ and $+\infty$ when $\alpha \geq 0$. (Prove this! See **Problem 3.7.94**.)

---

[29] Eugène Charles Catalan, French-Belgian mathematician, 1814-1894.

[30] The **Catalan constant** appears in many applications from combinatorics, to number theory, to integrals, etc. It can be found in several integral representations besides the ones developed in **the above example**. See **Problem 3.7.93**.

From this result, we obtain the following more general **result**: *For all real numbers* q *and* $r \neq 0$ *satisfying the inequality* $0 < \dfrac{q+1}{r} < 1$, *we have*

$$\int_0^{\infty} \frac{x^q \ln(x)}{x^r + 1}\, dx = -\operatorname{sign}(r)\, \frac{1}{r^2}\, \pi^2 \cot\left(\frac{q+1}{r}\pi\right) \csc\left(\frac{q+1}{r}\pi\right).$$

This follows by letting $u = x^r$ and working it out. (See also **Problems 3.7.94** and **3.7.96** for more variations.)

▲

Example 3.7.47 (Compare with **Problem 2.1.16**.) Prove

$$\int_0^{2\pi} \ln\left|1 - e^{i\theta}\right|\, d\theta = 0.$$

Then, conclude

$$\text{(a)} \qquad \int_0^{2\pi} \ln\left[\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = -2\pi \ln(2)$$

and

$$\text{(b)} \qquad \int_0^{\pi} \ln\left[\sin(u)\right] du = -\pi \ln(2).$$

We consider the holomorphic branch of the complex $\log(z)$

$$\operatorname{Log} \;:\; \mathbb{C} - \{x + 0i \mid x \leq 0\} \longrightarrow \mathbb{R} + i(-\pi, \pi).$$

So, if

$$z = re^{i\theta}, \quad \text{with} \quad \theta \neq \pi \pmod{2\pi},$$

we define

$$\operatorname{Log}(z) = \ln(r) + i\theta, \quad \text{with} \quad -\pi < \theta < \pi.$$

Therefore, the composite function

$$h(z) = \operatorname{Log}(1 - z) = \operatorname{Log}(1 - x - iy)$$

is holomorphic in the domain

$$\mathbb{C} - \{x \mid 1 - x \leq 0\} = \mathbb{C} - \{x \mid x \geq 1\}.$$

That is, for $h(z)$ the branch cut has been translated from $x = 0$ to $x = 1$ along the positive x-axis.

In this domain, we have

$$\operatorname{Re} h(z) = \ln|1 - z| \quad \text{and} \quad -\pi < \operatorname{Im} h(z) < \pi.$$

Also,

$$h(0) = \ln(1) + 0\,i = 0 + 0\,i = 0.$$

So, for every $0 < \epsilon < 1$, we can write $h(z)$ as a power series $h(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ in $D(0, \epsilon)$. Thus, $\dfrac{h(z)}{z} = a_1 + a_2 z + a_3 z^2 + \ldots$. Therefore, the function $\dfrac{h(z)}{z}$ is holomorphic in $D(0, \epsilon)$. Hence, $z = 0$ is not a singularity for $\dfrac{h(z)}{z}$. In fact, we can set $\dfrac{h(z)}{z}\Big|_{z=0} = a_1$.

We now consider $C(0,1)$ and $C(1,r)$ for any $0 < r < \dfrac{1}{2}$. These two circles intersect at two points $e^{i\sigma}$ and $e^{-i\sigma}$ for some $0 < \sigma < \dfrac{\pi}{2}$.

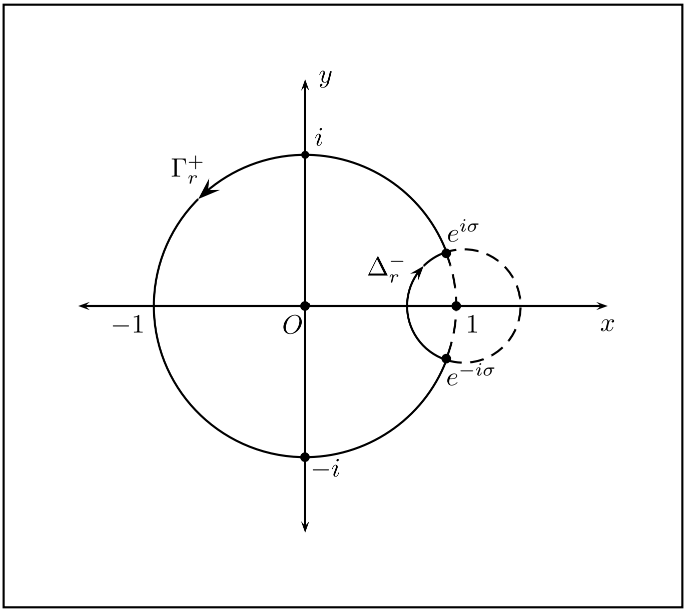

FIGURE 3.20: Contour 14 for Example 3.7.47

We call Γ_r the part of $C(0,1)$ outside $C(1,r)$ and Δ_r the part of $C(1,r)$ inside $C(0,1)$. Then, we let our contour be $C^+ = \Gamma_r^+ + \Delta_r^-$ (see **Figure 3.20**), and we get

$$\oint_{C^+} \frac{h(z)}{z}\, dz = 0.$$

From this, we obtain

$$\int_{\Gamma^+} \frac{h(z)}{z}\,dz = -\int_{\Delta^-} \frac{h(z)}{z}\,dz = \int_{\Delta^+} \frac{h(z)}{z}\,dz.$$

But,

$$\left|\int_{\Delta^+} \frac{h(z)}{z}\,dz\right| = \left|\int_{-\sigma}^{\sigma} \frac{\operatorname{Log}\left[1-\left(1-re^{i\theta}\right)\right]}{1-re^{i\theta}}\,r\,i\,e^{i\theta}\,d\theta\right| =$$

$$\left|\int_{-\sigma}^{\sigma} \frac{\operatorname{Log}\left(re^{i\theta}\right)}{1-re^{i\theta}}\,r\,i\,e^{i\theta}\,d\theta\right| = \left|\int_{-\sigma}^{\sigma} \frac{\ln(r)+i\theta}{1-re^{i\theta}}\,r\,i\,e^{i\theta}\,d\theta\right| \leq$$

$$\leq \int_{-\sigma}^{\sigma} \left|\frac{\ln(r)+i\theta}{1-re^{i\theta}}\,r\,i\,e^{i\theta}\right| d\theta \leq \int_{-\sigma}^{\sigma} \frac{\sqrt{\ln^2(r)+\theta^2}}{|1-re^{i\theta}|}\,r\,d\theta \leq$$

$$\int_{-\sigma}^{\sigma} \frac{\sqrt{\ln^2(r)+\frac{\pi^2}{4}}}{1-r}\,r\,d\theta < \frac{\sqrt{\ln^2(r)+\frac{\pi^2}{4}}}{1-r}\,r\,\pi \to 0, \quad \text{as} \quad r \to 0^+.$$

Hence,

$$\lim_{r\to 0^+} \int_{\Gamma^+} \frac{h(z)}{z}\,dz = 0.$$

Finally,

$$\int_0^{2\pi} \frac{\operatorname{Log}\left(1-e^{i\theta}\right)}{e^{i\theta}}\,i\,e^{i\theta}\,d\theta = 0,$$

or

$$\int_0^{2\pi} \operatorname{Log}\left(1-e^{i\theta}\right)\,d\theta = 0.$$

So,

$$\int_0^{2\pi} \ln\left|1-e^{i\theta}\right|\,d\theta + i\int_0^{2\pi} \arg\left(1-e^{i\theta}\right)\,d\theta = 0,$$

and therefore

$$\int_0^{2\pi} \ln\left|1-e^{i\theta}\right|\,d\theta = 0, \quad \text{and} \quad \int_0^{2\pi} \arg\left(1-e^{i\theta}\right)\,d\theta = 0.$$

But,

$$\left|1-e^{i\theta}\right| = |1-\cos(\theta)-i\sin(\theta)| = \left\{[1-\cos(\theta)]^2+\sin^2(\theta)\right\}^{\frac{1}{2}} =$$

$$\left|2\sin\left(\frac{\theta}{2}\right)\right| = 2\sin\left(\frac{\theta}{2}\right), \quad \forall \quad 0 \leq \theta \leq 2\pi,$$

and so we find

$$\int_0^{2\pi} \ln\left[2\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = 0.$$

Then,

$$\int_0^{2\pi} \ln\left[\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = -\int_0^{2\pi} \ln(2)\, d\theta = -2\pi\ln(2),$$

and by substituting $u = \dfrac{\theta}{2}$ we obtain the final **result**

$$\int_0^{\pi} \ln\left[\sin(u)\right] du = -\pi\ln(2).$$

Remark: In this example, we have also proved that

$$\int_0^{2\pi} \arg\left(1 - e^{i\theta}\right)\, d\theta = 0.$$

(Prove this result directly as a **problem**!)

▲

Problems

3.7.87 Consider $g(z) = \dfrac{\log^k(z)}{z^2+1}$, where $k > 0$, real constant. For every $R > 0$ we consider $C(0, R)$, the circumference of the circle with center the origin and radius R. Prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)} |zg(z)|\right] = 0 \quad \text{and} \quad \lim_{z\to\infty}\left[zg(z)\right] = 0.$$

3.7.88 Use the appropriate result obtained in this **section** to prove that

$$\int_0^{\pi/2} \ln[\sin(u)]\, du = \int_0^{\pi/2} \ln[\cos(u)]\, du = -\frac{\pi}{2}\ln(2).$$

(See also **Problem 2.1.16**.)

3.7.89 (a) Prove that for any $n = 0,\ 1,\ 2,\ \ldots$ non-negative integer

$$\int_0^1 \frac{[\ln(x)]^n}{x^2+1}\, dx = \int_1^\infty \frac{(-1)^n[\ln(x)]^n}{x^2+1}\, dx.$$

(b) Prove that for any $n = 1,\ 3,\ 5,\ \dots$ odd positive integer

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 0.$$

(c) Prove that for any $n = 0,\ 2,\ 4,\ \dots$ even non-negative integer

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 2\int_0^1 \frac{[\ln(x)]^n}{x^2+1}\,dx = 2\int_1^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx.$$

3.7.90 Manipulate the result proven in **Example 3.7.44** to prove

$$\forall \quad a>0,\ b>0 \text{ and } c>0 \quad \int_0^\infty \frac{\ln(cx)}{b^2+(ax)^2}\,dx = \frac{\pi}{2ab}\ln\left(\frac{bc}{a}\right).$$

(See also **Problems 3.1.19 and 2.1.24**.)

3.7.91 Prove

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

Then, use the substitution $x = e^u$ to find

$$\int_{-\infty}^\infty \frac{u^2}{\cosh(u)}\,du = \int_{-\infty}^\infty u^2\operatorname{sech}(u)\,du = \frac{\pi^3}{4}.$$

Also, use $x = \dfrac{1}{u}$ to prove

$$\int_0^1 \frac{\ln^2(x)}{1+x^2}\,dx = \int_1^\infty \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{16}.$$

3.7.92 Prove

$$\int_0^\infty \frac{\ln(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

(See also **Problem 2.1.22**.)

3.7.93 In **Example 3.7.45**, we have introduced the following three integral representations of the **Catalan constant**:

$$G := \int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{\arctan(x)}{x}\,dx.$$

Using appropriate changes of variables and manipulations, verify the following additional five integral representations of G:

$$g_1 = \int_0^1 \int_0^1 \frac{1}{1+x^2y^2}\,dxdy, \qquad g_2 = \int_0^{\frac{\pi}{4}} \frac{t}{\sin(t)\cos(t)}\,dt,$$

$$g_3 = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t}{\sin(t)}\,dt, \qquad g_4 = \int_0^{\frac{\pi}{4}} \ln[\cot(t)]\,dt,$$

$$\text{and} \quad g_5 = \int_0^\infty \arctan\left(e^{-t}\right)\,dt.$$

3.7.94 (a) Prove that the integral

$$\int_0^\infty \frac{x^\alpha \ln(x)}{x+1}\,dx,$$

in **Example 3.7.46**, is $-\infty$ when $\alpha \leq -1$ and $+\infty$ when $\alpha \geq 0$.

(b) Prove that the result of **Example 3.7.46** can be rewritten as

$$\int_0^\infty \frac{x^{\beta-1}\ln(x)}{x+1}\,dx = -\pi^2 \cot(\beta\pi)\csc(\beta\pi)$$

with $0 < \beta < 1$.

(c) Prove that this result can also be written as

$$\int_0^\infty \frac{\ln(x)}{x^\gamma+1}\,dx = -\left(\frac{\pi}{\gamma}\right)^2 \cot\left(\frac{\pi}{\gamma}\right)\csc\left(\frac{\pi}{\gamma}\right),$$

with $\gamma > 1$.

3.7.95 Compute the integrals:

$$I_1 = \int_0^\infty \frac{\ln(x)}{1+x^4}\,dx \qquad I_2 = \int_0^\infty \frac{\sqrt{x}\ln(x)}{1+x^4}\,dx$$

$$I_3 = \int_0^\infty \frac{x^{5/2}\ln(x)}{1+x^4}\,dx \qquad I_4 = \int_0^\infty \frac{x^{-2}\ln(x)}{1+x^{-2}}\,dx$$

$$I_5 = \int_0^\infty \frac{x^{-2}\ln(x)}{1+x^6}\,dx \qquad I_6 = \int_0^\infty \frac{x^{-2/3}\ln(x)}{1+x}\,dx$$

$$I_7 = \int_0^\infty \frac{\ln(x)}{1+x^5}\,dx \qquad I_8 = \int_0^\infty \frac{x^3\ln(x)}{1+x^5}\,dx.$$

3.7.96 Compute the integrals:

$$I_1 = \int_0^\infty \frac{(5x)^{-\frac{1}{4}}\ln(6x)}{7x+8}\,dx \qquad I_2 = \int_0^\infty \frac{(5x)^4\ln(6x)}{(7x)^8+9}\,dx$$

$$I_3 = \int_0^\infty \frac{\ln(10x)}{100+(5x)^2}\,dx \qquad I_4 = \int_0^\infty \frac{x^3\ln(10x)}{64+(2x)^6}\,dx.$$

(See also **Example 3.7.46**.)

3.7.97 Consider the following six integrals:

$$I_1 = \int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}\left(=\frac{2\pi\sqrt{3}}{3}\right),$$

$$I_2 = \int_{-\infty}^{0} \frac{dx}{x^2+x+1}\left(=\frac{4\pi\sqrt{3}}{9}\right), \quad I_3 = \int_{0}^{\infty} \frac{dx}{x^2+x+1}\left(=\frac{2\pi\sqrt{3}}{9}\right),$$

$$I_4 = \int_{-\infty}^{\infty} \frac{dx}{x^2-x+1}, \quad I_5 = \int_{-\infty}^{0} \frac{dx}{x^2-x+1}, \quad I_6 = \int_{0}^{\infty} \frac{dx}{x^2-x+1}.$$

(a) Compute all of them by using calculus.

[Hint: Observe that

$$x^2 \pm x + 1 = \left(x \pm \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2,$$

and then for any $b > 0$ we have

$$\int \frac{1}{(x+a)^2+b^2}\,dx = \frac{1}{b}\arctan\left(\frac{x+a}{b}\right) + C.]$$

(b) Now, compute all of them by using contour integration.

[Hint: For the first integral, use the complex function $f(z) = \dfrac{1}{z^2+z+1}$. For the second integral, use $g(z) = \dfrac{\log(z)}{z^2+z+1}$ or $h(z) = \dfrac{\log(z)}{z^2-z+1}$, whichever is more convenient, with branch cut $-\dfrac{\pi}{2} < \arg(z) < \dfrac{3\pi}{2}$ or another one that you may find more convenient. You may choose contour: $C^+ = [-R,-\delta] + S_\delta^- + [\delta, R] + S_R^+$ for $0 < \delta < R$ chosen appropriately and then let $\delta \to 0$ and $R \to \infty$, etc. do similar work for the remaining integrals.]

3.7.98 Compute the six integrals

$$\int_{-\infty}^{\infty} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx, \quad \int_{-\infty}^{0} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx, \quad \int_{0}^{\infty} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx,$$

by using contour integration. (Just calculus does not seem to work here!)

[Hint: Consider the complex function

$$f(z) = \frac{e^{\frac{1}{3}\log(z)}}{z^2 + z + 1} \quad \text{or} \quad g(z) = \frac{e^{\frac{1}{3}\log(z)}}{z^2 - z + 1}$$

with branch cut $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ or another one that you may find more convenient. You may choose contour:

$$C^+ = [-R, -\delta] + S_\delta^- + [\delta, R] + S_R^+$$

for $0 < \delta < R$ chosen appropriately and then let $\delta \to 0^+$ and $R \to \infty$. For example,

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2 + x + 1} dx = \frac{4\pi}{3} \sin\left(\frac{\pi}{9}\right).$$

(See also **Example 3.7.15** for another method.)]

3.7.99 Observe that $x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$. Then:

(a) Prove $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^3 + x^2 + x + 1}$ does not exist.

[Hint: Examine the singularity at $x = -1$, located on the x-axis, and show that the integral has the form $\infty - \infty$ around it.]

(b) Prove $\displaystyle\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^3 + x^2 + x + 1} = \frac{\pi}{2}.$

[Hint: Use **Theorem 3.7.4**.]

(c) Compute $\displaystyle\int_0^\infty \frac{dx}{x^3 + x^2 + x + 1} \left(= \frac{\pi}{4}\right).$

[Hint: As in **Problem 3.7.97**, consider the complex function

$$f(z) := \frac{\log(z)}{z^3 + z^2 + z + 1} = \frac{\log(z)}{(z+i)(z-i)(z+1)},$$

or use partial fractions for

$$g(x) := \frac{1}{(x+1)(x^2+1)}$$

and calculus.]

3.7.100 Notice $(x^5 - 1) = (x-1)(x^4 + x^3 + x^2 + x + 1)$ and compute the integrals:

$$\text{(a)} \qquad \int_{-\infty}^{\infty} \frac{dx}{x^4 + x^3 + x^2 + x + 1} \left[= \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right)\right],$$

$$\text{(b)} \qquad \int_{0}^{\infty} \frac{dx}{x^4 + x^3 + x^2 + x + 1},$$

$$\text{(c)} \quad \int_{-\infty}^{0} \frac{dx}{x^4 + x^3 + x^2 + x + 1}.$$

[Hint: For **(b)** and **(c)**, work as in the **previous problem** or **Problem 3.7.97**. Here, it is easier to work with the appropriate complex functions and the complex fifth roots of unity.]

3.7.101 For any a real constant, find the principal values or explain if they do not exist:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{iax} dx}{\ln |x|} \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{x e^{iax} dx}{(\ln |x|)^2}.$$

3.7.10 Application to Inverse Laplace Transform

As we have seen in **Section 2.7**, on the classes of the absolutely[31] or square[32] integrable or of exponential order functions $y = f(x)$, with

---

[31] $f(x)$ is **absolutely integrable** if $\int_{-\infty}^{\infty} |f(x)| dx < \infty$.

[32] $f(x)$ is **square integrable** if $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$.

$x \in [0, \infty)$ or $x \in (0, \infty)$, we define the Laplace transform of $f(x)$ by the improper integral

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x) dx,$$

for all s for which it exists.

Under these conditions, there is a real constant c such that

$$F(s) = \mathcal{L}\{f(x)\}(s), \quad \text{exists for all } \ s > c,$$

and, as proven in **Problems 2.7.5** and **2.7.6**,

$$\lim_{s \to \infty} \mathcal{L}\{f(x)\}(s) = 0$$

and the convergence is uniform.

In the context here, we replace the real variable s with the complex variable $\zeta = s + i\tau$, and thus we consider the complex function $F(\zeta)$ for which we require the necessary condition

$$\lim_{\zeta \to \infty} \mathcal{L}\{f(x)\}(\zeta) = 0,$$

where ∞, in this context, is the complex infinity.

If we assume that $e^{-cx} f(x)$ is absolutely or square integrable for some real constant c, then as we have seen in **Subsection 3.7.8, properties (1)** and **(4)**, $e^{-sx} f(x)$ is absolutely integrable for all s such that $c < s < \infty$ and $F(\zeta)$ is holomorphic in the infinite horizontal strip $c < s < \infty$. Now, we state:

Theorem 3.7.7 (Mellin-Bromwich Inversion Theorem) *We consider a continuous function $y = f(x)$ where $x \in [0, \infty)$ or $x \in (0, \infty)$ such that $e^{-cx} f(x)$ is absolutely or square integrable for some real constant c, and so its Laplace transform*

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x) dx,$$

exists for all $s > c$ (or $c \geq 0$).

We consider the complex function $w = F(\zeta)$, where $\zeta = s + i\tau$, and we require that $F(\zeta)$ satisfies the necessary condition

$$\lim_{\zeta \to \infty} \mathcal{L}\{f(x)\}(\zeta) = 0,$$

where ∞ is the complex infinity.

Then, for any $\gamma > c$

$$P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta \tag{3.40}$$

is independent of the constant $\gamma > c$, and if it exists for one choice of γ, then for any $\gamma > c$ we have

$$f(x) = \mathcal{L}^{-1}\left\{F(s)\right\}(x) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta} F(\zeta) d\zeta \ = \tag{3.41}$$

$$\frac{1}{2\pi i} \ P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta = \frac{1}{2\pi} \ P.V. \int_{-\infty}^{\infty} F(\gamma + iy) e^{(\gamma + iy)x} \, dy.$$

In applications $w = F(\zeta)$ is given á-priori, i.e., before $y = f(x)$, in order to retrieve $y = f(x)$ from $w = F(\zeta)$. Under the above conditions, the limit in **(3.41)** exists and is independent of γ. This complex integral formula is called the **Mellin-Bromwich**[33] **inversion integral formula** for the **Laplace transform**. If the function $y = f(x)$ is not continuous at all points, the formula still works at all points at which $y = f(x)$ is continuous. For the existence of the principal value in equation **(3.40)**, see, e.g., **condition (3.)** of the **next Theorem, 3.7.8**, and the **remark** that follows it.

Proof As we have seen, if $e^{-sx} f(x)$ is absolutely integrable for all s such that $c < s < \infty$, then $F(\zeta)$ is holomorphic in the infinite horizontal strip $c < s < \infty$. [See **Subsection 3.7.8, property (4)**.]

If for a choice of $\gamma_1 > c$ the

$$\text{P.V.} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} e^{x\zeta} F(\zeta) d\zeta$$

exists, then it exists for all $\gamma > c$ and is independent of γ. Indeed, for any $\gamma_2 > \gamma_1 > c$ (or $\gamma_1 > \gamma_2 > c$) and any $R > 0$, we have

$$\oint_{C^+} e^{x\zeta} F(\zeta) \, d\zeta = 0,$$

where the path C^+ is the positively oriented parallelogram

$$C^+ = [\gamma_2 - iR, \gamma_2 + iR] + [\gamma_2 + iR, \gamma_1 + iR] +$$
$$[\gamma_1 + iR, \gamma_1 - iR] + [\gamma_1 - iR, \gamma_2 - iR].$$

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[33] Thomas John l'Anson Bromwich, English mathematician, 1875-1929.

[Since on C^+ and in the interior of C^+ the function $e^{x\zeta}F(\zeta)$ is holomorphic, the **Cauchy-Goursat Theorem, 3.5.3**, applies.]

Then, we let $R \longrightarrow \infty$ and eventually get

$$\text{P.V.} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} e^{x\zeta} F(\zeta) d\zeta = \text{P.V.} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} e^{x\zeta} F(\zeta) d\zeta,$$

since by the condition

$$\lim_{\zeta \to \infty} \mathcal{L}\{f(x)\}(\zeta) = \lim_{\zeta \to \infty} F(\zeta) = 0$$

the limits of the integrals along the two finite horizontal segments $[\gamma_2 + iR, \gamma_1 + iR]$ and $[\gamma_1 - iR, \gamma_2 - iR]$ (of constant length $|\gamma_1 - \gamma_2|$) are zero, as $R \longrightarrow \infty$. (Work out the details, as an easy exercise!)

We have defined

$$F(\zeta) = \int_0^\infty e^{-\zeta x} f(x) dx = \int_0^\infty e^{-(s+i\tau)x} f(x) dx, \text{ where } \zeta = s + i\tau.$$

[So, $\tau = 0$ implies $F(s) = \mathcal{L}\{f(x)\}(s)$.] Then, with $f(x) = 0$ for $x < 0$, or $x \leq 0$, the **Fourier transform** of $y = f(x)$, defined by equation **(3.27)**, is

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} f(x) dx = \int_0^\infty e^{-(-i\zeta)x} f(x) dx = F(-i\zeta).$$

Therefore, for $x > 0$ or $x \geq 0$ and for any $s = \gamma > c$, by the **inversion formula (3.36)** for the **Fourier transform with complex argument**, we obtain

$$f(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R+i\gamma}^{R+i\gamma} e^{-i\zeta x} F(-i\zeta) d\zeta,$$

where the integral takes place on the horizontal line $(-\infty + i\gamma, +\infty + i\gamma)$.

We make the change of variables $z = -i\zeta$ and so $d\zeta = \frac{1}{-i} dz = i dz$. Also, since multiplication by $-i$ amounts to a clockwise rotation of ζ by angle $\frac{\pi}{2}$, the new integral will take place on a vertical line. In fact, we get

$$f(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{\gamma+iR}^{\gamma-iR} e^{zx} F(z) i \, dz =$$

$$\frac{-1}{2\pi i} \lim_{R \to \infty} \int_{\gamma+iR}^{\gamma-iR} e^{zx} F(z) \, dz = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma-iR}^{\gamma+iR} e^{zx} F(z) \, dz,$$

which finishes the proof of the Theorem.

Example 3.7.48 We illustrate the use of the **inversion formula (3.41)** by verifying a known example. We can use this formula when the stated conditions are met, the principal value involved exists and, of course, if the formula is convenient enough to produce a result. Otherwise, we use the theorems and the methods of the examples that follow in this subsection.

In the **table of Problem 2.7.12**, **Rule (1.)**, we have that the Laplace transform of $f(x) = a$ (real constant), for $x > 0$ or $x \geq 0$, is

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} a dx = \frac{a}{s}, \quad \text{with} \quad s > 0.$$

By the **inversion formula (3.41)**, we should have

$$a = f(x) = \mathcal{L}^{-1}\left\{F(s)\right\}(x) = \frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy} e^{(\gamma+iy)x}\, dy$$

for any constant $\gamma > 0$, since when $a \neq 0$ the only singularity of $F(\zeta) = \dfrac{a}{\zeta}$ is $z_0 = 0$, and so the constant c in the **previous Theorem** is zero.

Indeed,

$$\frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy} e^{(\gamma+iy)x}\, dy = \frac{ae^{\gamma x}}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{\gamma - iy}{\gamma^2 + y^2} e^{ixy} dy.$$

Using the two Fourier type integrals computed in **Examples 3.7.31** and **3.7.32**, with $\gamma > 0$ and $x > 0$, we find

$$\frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy} e^{(\gamma+iy)x}\, dy =$$

$$\frac{a\gamma e^{\gamma x}}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{\gamma^2 + y^2} e^{ixy} dy - i\frac{ae^{\gamma x}}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{y}{\gamma^2 + y^2} e^{ixy} dy =$$

$$\frac{a\gamma e^{\gamma x}}{2\pi} \frac{\pi}{\gamma} e^{-\gamma x} - i\frac{ae^{\gamma x}}{2\pi}\ i\pi e^{-x\gamma} = \frac{a}{2} + \frac{a}{2} = a,$$

which is correct (and independent of γ).

(As we see, this verification depends on the two Fourier transforms computed in **Examples 3.7.31** and **3.7.32**.)

▲

Next, we state and prove an inversion theorem for the Laplace transform which is not as general as the previous one, but its proof is elementary because it does not depend on the Fourier transform with complex argument. Namely:

Theorem 3.7.8 *Given a complex function $w = F(\zeta)$, we define*

$$c = \inf\{r \mid r \in \mathbb{R} \textit{ such that } F(\zeta) \textit{ is analytic for all } \zeta \textit{ with } Re(\zeta) \geq r\}.$$

Now, we assume

1. $c < +\infty$.
2. $\lim\limits_{\zeta \to \infty} F(\zeta) = 0$, *where ∞ is the complex infinity.*
3. *There is a constant $\gamma \in \mathbb{R}$ constant such that $\gamma > c$ and*

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |F(\zeta)| \cdot |d\zeta| = \int_{-\infty}^{\infty} |F(\gamma + iy)| \, dy < \infty.$$

Then, the integral

$$P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta$$

is independent of the constant $\gamma > c$, and

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta} F(\zeta) d\zeta = \tag{3.42}$$

$$\frac{1}{2\pi i} \; P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta = \frac{1}{2\pi} \; P.V. \int_{-\infty}^{\infty} F(\gamma + iy) e^{(\gamma + iy)x} \, dy,$$

i.e., the limit exists, is independent of γ and gives back the function $f(x)$, whose Laplace transform is $F(s)$.

Remark: The three conditions of this **Theorem** are satisfied in most cases of application. But, they are somewhat restrictive, and so this Theorem is not as general as **Theorem 3.7.7**.

For instance, **condition (3.) of this Theorem** is not satisfied in the **previous example**. Indeed, for $F(\zeta) = \frac{a}{\zeta}$, $c = 0$ and for any $\gamma > 0$, we have

$$\int_{-\infty}^{\infty} \left| \frac{1}{\gamma + iy} \right| dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\gamma^2 + y^2}} \, dy \overset{y = \gamma u}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1 + u^2}} \, du =$$

$$2 \int_0^{\infty} \frac{1}{\sqrt{1 + u^2}} \, du \overset{u = \tan(v)}{=} 2 \int_0^{\frac{\pi}{2}} \sec(v) \, dv =$$

$$\left[2 \ln |\sec(v) + \tan(v)| \right]_0^{\frac{\pi}{2}} = 2(\infty - 0) = \infty.$$

The same is true for $F(\zeta) = \frac{\zeta}{\zeta^2 + a^2}$, where $a > 0$ constant, even though for any $s > 0$, as we have seen in **Problem 2.7.12**, **Rule (4.)**, $f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \cos(ax)$ with $x \geq 0$.

Now, we continue with:

Proof For the choice of $\gamma > c$ in **condition (3.)**, the integral

$$\text{P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta$$

exists because it converges absolutely. Indeed,

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \left| e^{x\zeta} F(\zeta) \right| |d\zeta| =$$

$$\int_{-\infty}^{\infty} \left| e^{x(\gamma + iy)} F(\gamma + iy) \right| dy = e^{x\gamma} \int_{-\infty}^{\infty} |F(\gamma + iy)| \, dy < \infty.$$

Then, this integral exists for any $\gamma > c$ and is independent of the constant $\gamma > c$ by **conditions (1.)** and **(2.)** and the same argument we saw in the proof of **Theorem 3.7.7**.

Now, we consider any $z \in \mathbb{C}$ such that $\text{Re}(z) > c$. Then, take any γ such that $\text{Re}(z) > \gamma > c$, any real $R > |z - \gamma| > 0$ and consider the contour

$$C^+ = L^+_{\gamma R} + H^+_{\gamma R},$$

where

$$L^+_{\gamma R} = \{z = \gamma + iy \quad | \quad -R \leq y \leq R\} = [\gamma + iR, \gamma - iR]$$

is a vertical straight segment with mid-point $\gamma + i\,0$, and

$$H^+_{\gamma R} = \left\{z = \gamma + Re^{i\theta} \quad | \quad -\pi/2 \leq \theta \leq \pi/2\right\}$$

is the semicircle with diameter the segment $L^+_{\gamma R}$ and located to the right of $L^+_{\gamma R}$. (See **Figure 3.21**.)

By the **Cauchy integral formula** (**Theorem 3.5.7**), we have

$$F(z) = \frac{1}{2\pi i} \int_{C^+} \frac{F(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma + iR}^{\gamma - iR} \frac{F(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{H^+_{\gamma R}} \frac{F(\zeta)}{\zeta - z} d\zeta.$$

But,

$$\left| \int_{H^+_{\gamma R}} \frac{F(\zeta)}{\zeta - z} d\zeta \right| \leq \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\left|F\left(\gamma + Re^{i\theta}\right)\right|}{|\gamma + Re^{i\theta} - z|} R \, d\theta \leq$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\left|F\left(\gamma + Re^{i\theta}\right)\right|}{R - |\gamma - z|} R \, d\theta \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty,$$

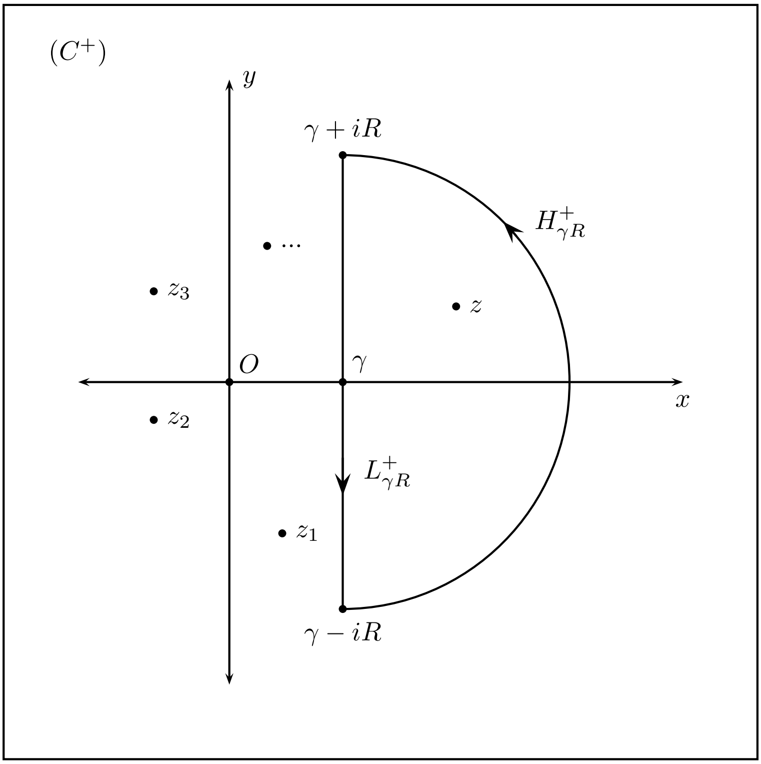

FIGURE 3.21: Contour 15 for Theorem 3.7.8

since $\lim_{R\to\infty} \dfrac{R}{R-|\gamma - z|} = 1$ and $\lim_{\zeta\to\infty} F(\zeta) = 0$ by **hypothesis (2.)**. (The argument is analogous to the proof of **Lemma 3.7.1**.)

Therefore, by letting $R \longrightarrow \infty$, we obtain

$$F(z) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma+iR}^{\gamma-iR} \frac{F(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\ \text{P.V.} \int_{\gamma+i\infty}^{\gamma-i\infty} \frac{F(\zeta)}{\zeta - z}\, d\zeta.$$

Since $\mathrm{Re}(z-\zeta) = \mathrm{Re}(z) - \mathrm{Re}(\zeta) = \mathrm{Re}(z) - \gamma > 0$, by **Problem 3.7.40**, we have

$$-\frac{1}{\zeta - z} = \frac{1}{z-\zeta} = \int_0^\infty e^{-(z-\zeta)x}\, dx.$$

We switch the limits of integration to eliminate the minus $(-)$ sign and obtain

$$F(z) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) \left[\int_0^\infty e^{-(z-\zeta)x}\, dx \right] d\zeta.$$

By **hypothesis (3.)**, we get

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |F(\zeta)| \left[\int_0^\infty \left| e^{-(z-\zeta)x} \right| dx \right] |d\zeta| =$$
$$\int_{-\infty}^{\infty} |F(\gamma + iy)| \frac{1}{\text{Re}(z) - \gamma}\, dy < \infty.$$

Thus, by the **Tonelli conditions** (see **Section 2.4**), we are allowed to switch the order of integration to find

$$F(z) = \int_0^\infty e^{-zx} \left[\frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta \right] dx.$$

Letting

$$f(x) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta,$$

we get

$$F(z) = \int_0^\infty e^{-zx} f(x)\, dx \qquad \text{or} \qquad \mathcal{L}\{f(x)\}(z) = F(z).$$

Therefore,

$$f(x) = \mathcal{L}^{-1}\{F(z)\}(x) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta =$$
$$\frac{1}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} F(\gamma + iy) e^{(\gamma + iy)x}\, dy,$$

and the proof is finished.

Remark: This Theorem can be proven with a slightly relaxed **condition (2.)**. We can assume that there is a $\gamma > c$ and there are denumerably many semicircles

$$C^+_{\gamma R_k} = \left\{ \zeta \mid \zeta = \gamma + R_k e^{i\theta},\ 0 \le \theta \le 2\pi \right\}, \quad k = 1,\ 2,\ 3, \ldots$$

such that $\lim\limits_{k \to \infty} R_k = \infty$ and

$$\lim_{k \to \infty} F(\gamma + R_k e^{i\theta}) = 0, \quad \text{uniformly on } 0 \le \theta \le 2\pi.$$

So, under the conditions of this **Theorem** or, more generally, if the principal value of the above integral exists and is independent of γ, we obtain $f(x)$, the inverse Laplace transform of $F(s)$: I.e.,

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x).$$

We could use the **inversion formula (3.41)** or **(3.42)** to directly recover the function $y = f(x)$, if we are given its Laplace transform $F(s)$. But, the direct computation of the **complex integral** involved is not very convenient, in general. (Even in the simple case of **Example 3.7.48** the direct computation was rather involved.)

However, with the help of this integral formula, we obtain the following convenient computation of the inverse Laplace transform, which many times can apply to cases more general than **Theorem 3.7.8**. For instance, we can apply it to $F(\zeta) = \dfrac{a}{\zeta}$ and $F(\zeta) = \dfrac{\zeta}{\zeta^2 + a^2}$ (referred to in the **remark** before the **proof** of **Theorem 3.7.8**). So, we have the following applicable theorem:

Theorem 3.7.9 *Suppose:*

1. *$F(\zeta)$ is analytic in $\mathbb{C} - \{z_1,\ z_2,\ z_3, ...\}$, where $\{z_1,\ z_2,\ z_3,\ ...\}$ are countably many isolated poles with real parts $Re\,(z_n) < \gamma$, $\forall \quad n = 1,\ 2,\ 3,\ \ldots$, for some constant $\gamma \in \mathbb{R}$.*

2. *There are denumerably many semicircles*

$$H^{+}_{\gamma R_k} = \left\{ \zeta \,|\, \zeta = \gamma + R_k e^{i\theta},\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}, \quad k = 1,\ 2,\ 3, ...$$

   such that $\lim\limits_{k\to\infty} R_k = \infty$, and they contain no poles of $F(\zeta)$.

3. $\lim\limits_{k\to\infty} F(\gamma + R_k e^{i\theta}) = 0$ *uniformly in* $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.

4. *The Mellin-Bromwich inversion integral formula can apply. I.e., the principal value in it exists (and, as we have seen, is independent of γ).*

Then,

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \sum_{n=1}^{\infty} \underset{\zeta = z_n}{Res} \left[F(\zeta) e^{x\zeta} \right],$$

if this sum exists.

Remark: We note that if $\sum\limits_{n=1}^{\infty} \operatorname*{Res}\limits_{\zeta=z_n} \left[F(\zeta)e^{x\zeta}\right]$ exists, then it is independent of the constant γ, as long as all the poles of $F(\zeta)$ lie in the open half plane to the left of the vertical straight line $x = \gamma$. (Why?)

Proof For any $k \in \mathbb{N}$, we consider the simple closed contour

$$C_k^+ = L^+_{\gamma R_k} + H^+_{\gamma R_k}$$

consisting of two pieces: the vertical straight segment

$$L^+_{\gamma R_k}\{z = \gamma + iy \quad | \quad - R_k \leq y \leq R_k\} = [\gamma - iR_k, \gamma + iR_k],$$

with mid-point $\gamma + i0$, and the semicircle

$$H^+_{\gamma R_k} = \left\{\zeta \,|\, \zeta = \gamma + R_k e^{i\theta},\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\},$$

with diameter the straight segment $[\gamma - iR_k, \gamma + iR_k]$ and located to the left of it. (See **Figure 3.22**.)

Then, for any given $x > 0$ fixed, we evaluate the line integral of $F(\zeta)e^{x\zeta}$ along C_k^+ and take its limit as $k \to \infty$. Notice that the set of poles of $F(\zeta)$ is equal to the set of poles of $F(\zeta)e^{x\zeta}$. So, if we let A_k be the set of all poles of $F(\zeta)$ in the interior of the contour C_k^+, then by **the Residue Theorem, 3.7.1**, we have

$$\frac{1}{2\pi i}\int\limits_{C_k^+} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi i}\int\limits_{L^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta + \frac{1}{2\pi i}\int\limits_{H^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = \sum_{z \in A_k} \operatorname*{Res}_{\zeta=z} \left[F(\zeta)e^{x\zeta}\right].$$

In the limit, as $k \to \infty$, we get

$$\lim_{k\to\infty} \sum_{z \in A_k} \operatorname*{Res}_{\zeta=z} \left[F(\zeta)e^{x\zeta}\right] = \sum_{n=1}^{\infty} \operatorname*{Res}_{\zeta=z_n} \left[F(\zeta)e^{x\zeta}\right],$$

and by **condition (4.)** and equation **(3.41)** or **(3.42)**, we obtain

$$\lim_{k\to\infty} \frac{1}{2\pi i}\int\limits_{L^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = \frac{1}{2\pi i}\lim_{k\to\infty}\int_{\gamma-iR_k}^{\gamma+R_k} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi i}\lim_{R_k\to\infty}\int_{\gamma-iR_k}^{\gamma+R_k} F(\zeta)e^{x\zeta}d\zeta = \frac{1}{2\pi i}\ P.V.\int_{\gamma-i\infty}^{\gamma+i\infty} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi}\ P.V.\int_{-\infty}^{\infty} F(\gamma+iy)e^{x(\gamma+iy)}dy = \mathcal{L}^{-1}\{F(s)\}(x) = f(x).$$

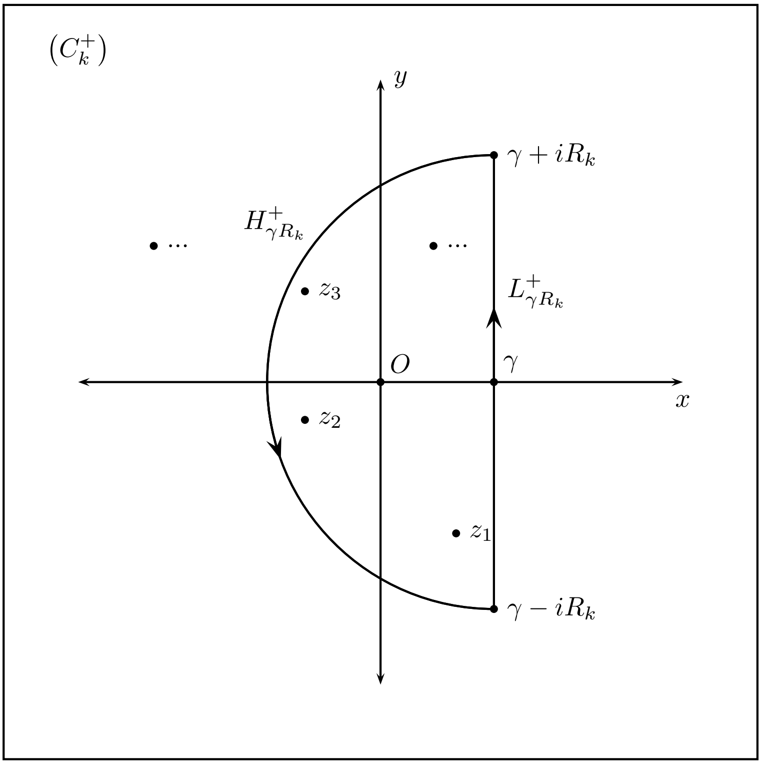

FIGURE 3.22: Contour 16 for Theorem 3.7.9

To finish the proof of the Theorem, we must show

$$\lim_{k\to\infty} \int\limits_{H^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = 0.$$

We have

$$\int\limits_{H^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = \int_{\pi/2}^{3\pi/2} e^{x(\gamma+R_k e^{i\theta})}F\left(\gamma + R_k e^{i\theta}\right) R_k\, i\, e^{i\theta}d\theta.$$

We let

$$M_{R_k} = \underset{\zeta\in H^+_{\gamma R_k}}{\text{Maximum}}\, |F(\zeta)|.$$

By **hypothesis (3.)**,

$$\lim_{k\to\infty} M_{R_k} = 0.$$

For $\zeta = \gamma + R_k e^{i\theta} \in H^+_{\gamma R_k}$, we have $|d\zeta| = |0 + R_k i e^{i\theta} d\theta| = R_k d\theta$ and $\left|e^{x\zeta}\right| = \left|e^{x(\gamma + R_k e^{i\theta})}\right| = e^{x\gamma} e^{x R_k \cos(\theta)}$. Hence, we obtain

$$\left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| \leq$$

$$\int\limits_{H^+_{\gamma R_k}} |F(\zeta)| \left|e^{x\zeta}\right| |d\zeta| \leq e^{x\gamma} M_{R_k} R_k \int_{\pi/2}^{3\pi/2} e^{x R_k \cos(\theta)} d\theta.$$

By using the substitution $\theta = \phi + \dfrac{\pi}{2}$ and **Jordan's Lemma, 3.7.4,** we find that for any $x > 0$

$$\int_{\pi/2}^{3\pi/2} e^{x R_k \cos(\theta)} d\theta = \int_0^{\pi} e^{-x R_k \sin(\phi)} d\phi < \frac{\pi}{R_k x}.$$

So, by the two previous relations, we obtain the inequality

$$\left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| < \frac{e^{x\gamma} M_{R_k} \pi}{x}.$$

Since $\lim\limits_{k\to\infty} M_{R_k} = 0$, we get that for any $x > 0$

$$\lim_{k\to\infty} \left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| = 0 \quad \text{equivalent to} \quad \lim_{k\to\infty} \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta = 0.$$

This proves what we wanted, and the Theorem follows.

Examples

Example 3.7.49 Let us apply the result of this Theorem to **Rule (5.)** in the **table of Problem 2.7.12**, which generalizes **Example 3.7.48** in which $f(x)$ was constant.

We have the function $f(x) = x^n$, for $x > 0$ and $n = 0,\ 1,\ 2,\ \dots$ (if $n = 0$, then $x^0 = 1$). Then,

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^{\infty} x^n e^{-sx} dx = \frac{n!}{s^{n+1}}, \quad \text{with} \quad s > 0,$$

and so

$$F(\zeta) = \frac{n!}{\zeta^{n+1}}, \quad \text{with} \quad \zeta \in \mathbb{C}.$$

This has only one pole of order $n+1$ at $\zeta = 0$.

Since

$$F(\zeta)e^{x\zeta} = \frac{n!}{\zeta^{n+1}} \sum_{k=0}^{\infty} \frac{x^k \zeta^k}{k!} = \sum_{k=0}^{\infty} \frac{n! x^k}{k!} \frac{1}{\zeta^{n+1-k}},$$

for $k = n$, we find the coefficient of $\frac{1}{\zeta}$, or

$$\operatorname*{Res}_{\zeta=0} \left[F(\zeta)e^{x\zeta}\right] = \frac{n! x^n}{n!} = x^n.$$

So, by the **previous Theorem**, for $x > 0$ and $n = 0,\ 1,\ 2,\ \ldots$, we have

$$\mathcal{L}^{-1}\left\{F(s)\right\}(x) = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}(x) = f(x) = x^n.$$

We observe that this method is much faster than directly computing the **integral (3.41)** or **(3.42)**, even when $n = 0$, as we did in **Example 3.7.48**.

▲

Example 3.7.50 Find the function $y = f(x)$ if we are given that

$$F(s) = \mathcal{L}\{f(x)\}(s) = \frac{18}{s^3 + 27}.$$

$F(s)$ obviously satisfies the three conditions of **Theorem 3.7.8**, so we can apply the **inversion integral formula (3.42)**, or more efficiently the result of **Theorem 3.7.9**.

The singularities of the function

$$e^{x\zeta} F(\zeta) = \frac{18 e^{x\zeta}}{\zeta^3 + 27}$$

are the third roots of -27, which are simple isolated poles. These are

$$z_k = 3e^{i\frac{(1+2k)\pi}{3}}, \quad \text{for} \quad k = 0,\ 1,\ 2$$

or

$$z_0 = \frac{3}{2}(1 + \sqrt{3}i), \qquad z_1 = -3, \qquad z_2 = \frac{3}{2}(1 - \sqrt{3}i).$$

Using

$$e^{x\zeta} F(\zeta) = \frac{18 e^{x\zeta}}{\zeta^3 + 27} = \frac{18 e^{x\zeta}}{(\zeta - z_0)(\zeta - z_1)(\zeta - z_2)},$$

we find that the residues of $e^{x\zeta}F(\zeta)$ at these poles are, respectively,

$$\frac{-(1+i\sqrt{3}\,)e^{\frac{3x}{2}(1+i\sqrt{3}\,)}}{3}, \qquad \frac{2e^{-3x}}{3}, \qquad \frac{-(1-i\sqrt{3}\,)e^{\frac{3x}{2}(1-i\sqrt{3}\,)}}{3}.$$

Since

$$\lim_{|\zeta|\to\infty} F(\zeta) = 0$$

and the constant γ can be taken $\gamma > \dfrac{3}{2}$, we can apply **Theorem 3.7.9**.

By adding the three residues and using **Euler's formulae** of exponential in terms of sines and cosines, we find

$$f(x) = \frac{2e^{-3x}}{3} + \frac{2e^{\frac{3x}{2}}}{3}\left[-\cos\left(\frac{3\sqrt{3}}{2}x\right) + \sqrt{3}\sin\left(\frac{3\sqrt{3}}{2}x\right)\right].$$

▲

The next example is very important. We study a more complicated situation in which $F(\zeta)$ has isolated and non-isolated singularities. This constitutes a variation or even generalization of **Theorem 3.7.9**.

As before, we assume:

1. $F(\zeta)$ is analytic for all $\zeta \in \mathbb{C}$ such that $\mathrm{Re}(\zeta) \geq \gamma$ for some constant γ, that is, all the singularities of $F(\zeta)$ have real parts less than γ.
2. $F(\zeta)$ **has isolated and non-isolated singularities**.
3. When applying the **Mellin-Bromwich inversion integral formula, (3.41)** [or **(3.42)**], by integrating along the vertical line segments $L^{+}_{\gamma R} = [\gamma - iR, \gamma + iR]$, for any $R > 0$, etc., we need to use appropriate contours that avoid the non-isolated singularities, and so we must introduce appropriate branch cuts.
4. $\lim\limits_{R\to\infty} F(\gamma + Re^{i\theta}) = 0$ uniformly in θ.

In such a situation, we work as in the **following example**:

Example 3.7.51 For any $a > 0$, we would like to find

$$\mathcal{L}^{-1}\{F(s)\}(x) = f(x) \quad \text{if} \quad F(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}.$$

Thus, we must work with the function $F(\zeta) = \dfrac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}$, where $\zeta \in \mathbb{C}$.

Since the $\sqrt{\zeta}$ has non-isolated singularities, we choose the branch cut to be the closed non-positive real semi-axis $\{\zeta = x + 0i \mid x \leq 0\}$. In this domain $-\pi < \mathrm{Arg}(\zeta) < \pi$ and in its sub-domain $\mathrm{Re}(\zeta) > 0$, $F(\zeta)$ is complex analytic (holomorphic).

So, we can pick any $\gamma > 0$ constant, any $0 < r < \gamma$, any $R > 2\gamma$ and we consider the contour

$$C^+ = [\gamma - iR, \gamma + iR] + S^+_{\gamma R} + [\gamma - R, -r] + A^-_r + [-r, \gamma - R] + T^+_{\gamma R}$$

as in **Figure 3.23** where, apart from the obvious straight segments, we have set

$$S^+_{\gamma R} = \left\{\zeta = \gamma + Re^{i\theta} \;\mid\; \frac{\pi}{2} \leq \theta < \pi\right\},$$
$$A^-_r = \left\{\zeta = Re^{i\theta} \;\mid\; \pi > \theta > -\pi\right\},$$
$$T^-_{\gamma R} = \left\{\zeta = \gamma + Re^{i\theta} \;\mid\; \pi < \theta \leq \frac{3\pi}{2}\right\}.$$

For any $x > 0$ fixed, we integrate the function $e^{x\zeta}F(\zeta) = e^{x\zeta}\dfrac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}$ along this simple closed contour. This line integral is equal to zero because this function has no singularity on C^+ and in its interior, and so the **Cauchy-Goursat Theorem, 3.5.3**, applies. Hence,

$$\int\limits_{C^+} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta +$$
$$\int\limits_{S^+_{\gamma R}} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int_{\gamma - R}^{-r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int\limits_{A^-_r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta +$$
$$\int_{-r}^{\gamma - R} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int\limits_{T^-_{\gamma R}} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = 0. \tag{3.43}$$

Next, we observe

$$\left|\int\limits_{A^-_r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta\right| \leq \int\limits_{A^-_r} \left|e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta\right| \leq$$
$$2\pi r e^{(rx - a\sqrt{r})} r^{\frac{-1}{2}} = 2\pi\sqrt{r}e^{(rx - a\sqrt{r})} \longrightarrow 0, \text{ as } r \to 0.$$

On $S^+_{\gamma R}$ and $T^-_{\gamma R}$, we have that $z = \gamma + Re^{i\theta}$. Then, by **Jordan's**

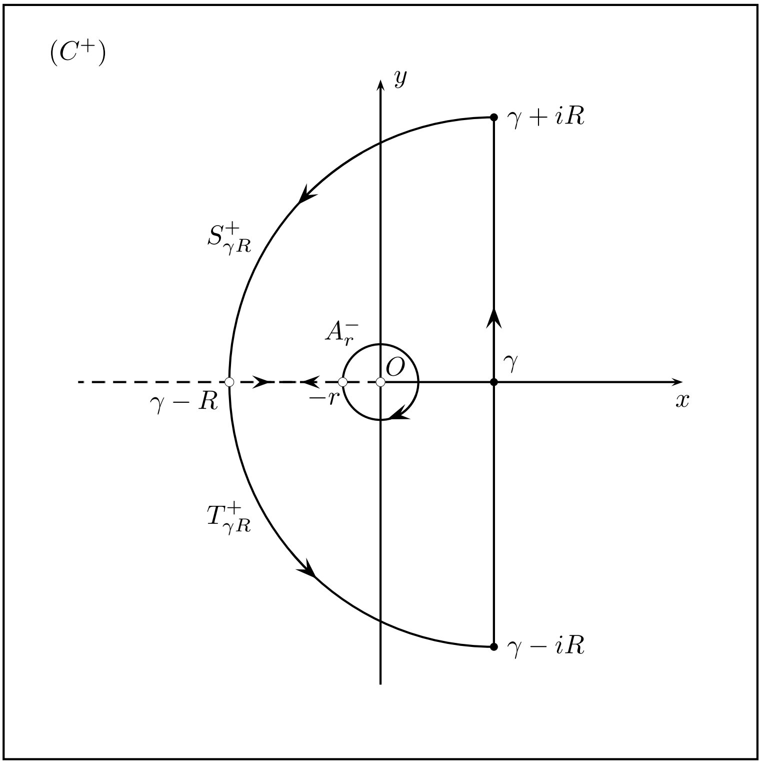

FIGURE 3.23: Contour 17 for Example 3.7.51

Lemma, 3.7.4, we get

$$\left| \int_{S^+_{\gamma R}} e^{x\zeta} \frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta \right| \longrightarrow 0, \quad \text{as} \quad R \to \infty,$$

and

$$\left| \int_{T^-_{\gamma R}} e^{x\zeta} \frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta \right| \longrightarrow 0, \quad \text{as} \quad R \to \infty.$$

(Work out the details.)

Along the segment $[\gamma - R, -r]$, we have that $\zeta = u + 0i = u < 0$ and

$\mathrm{Arg}(\zeta) = \pi$. So,

$$\sqrt{\zeta} = e^{\frac{1}{2}\log(\zeta)} = e^{\frac{1}{2}[\ln(|\zeta|)+i\pi]} = e^{\frac{i\pi}{2}} e^{(\frac{1}{2}\ln(|\zeta|)|} = i\sqrt{|\zeta|} = i\sqrt{|u|}.$$

Therefore, along the segment $[\gamma - R, -r]$, we have

$$e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} = e^{xu}\frac{e^{-ia\sqrt{|u|}}}{i\sqrt{|u|}} = \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}.$$

Then,

$$\int_{\gamma-R}^{-r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{\gamma-R}^{-r} \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}\,du.$$

Similarly, along the segment $[-r, \gamma - R]$, we have that $\zeta = u + 0i = u < 0$ and $\mathrm{Arg}(\zeta) = -\pi$. So,

$$\sqrt{\zeta} = e^{\frac{1}{2}\log(\zeta)} = e^{\frac{1}{2}[\ln(|\zeta|)-i\pi]} = e^{-\frac{i\pi}{2}} e^{(\frac{1}{2}\ln(|\zeta|)|} = -i\sqrt{|\zeta|} = -i\sqrt{|u|}$$

and

$$\int_{-r}^{\gamma-R} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{-r}^{\gamma-R} \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) + i\sin\left(a\sqrt{|u|}\right)\right]}{-i\sqrt{|u|}}\,du.$$

So, by the relation **(3.43)** above, after letting $r \to 0$ and $R \to \infty$, we get

$$\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\,d\zeta =$$
$$\int_{-\infty}^{0} \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}\,du +$$
$$\int_{0}^{-\infty} \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) + i\sin\left(a\sqrt{|u|}\right)\right]}{-i\sqrt{|u|}}\,du =$$
$$-2i\int_{-\infty}^{0} \frac{e^{xu}\cos\left(a\sqrt{|u|}\right)}{\sqrt{|u|}}\,du.$$

We let $u = -v^2$ with $v > 0$, and so $du = -2vdv$, using the result of **Example 2.2.13**, we find

$$\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\,d\zeta = 4i\int_{-\infty}^{0} e^{-xv^2}\cos(av)\,dv =$$
$$4i\int_{0}^{\infty} e^{-xv^2}\cos(av)\,dv = 4i\,\frac{1}{2}\sqrt{\frac{\pi}{x}e^{\frac{-a^2}{4x}}} = 2i\sqrt{\frac{\pi}{x}e^{\frac{-a^2}{4x}}}.$$

Hence, by the **Mellin inversion formula, (3.41)**, we obtain

$$f(x) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta} \frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\, d\zeta = \frac{e^{\frac{-a^2}{4x}}}{\sqrt{\pi x}}, \quad \text{for } x > 0.$$

Finally, we have obtained the rule

$$\forall\ a > 0, \quad f(x) = \mathcal{L}^{-1}\left\{ \frac{e^{-a\sqrt{s}}}{\sqrt{s}} \right\}(x) = \frac{e^{\frac{-a^2}{4x}}}{\sqrt{\pi x}}, \quad \text{for } x > 0.$$

▲

Remark: Another **variation** of this method occurs when the function $F(\zeta)$ has both **isolated** and **non-isolated** singularities. Then, we must combine the methods of **this example** with the method of the **previous example** in order to find $f(x) = \mathcal{L}^{-1}\{F(s)\}(x)$. For the isolated singularities inside the contour C^+, we must use **the Residue Theorem, 3.7.1**.

For instance, $F(\zeta) = \dfrac{\sqrt{\zeta}}{\zeta^2 + 4}$ has non-isolated singularities because of the $\sqrt{\zeta}$, and so we need a branch cut. It also has two isolated singularities at $\zeta = \pm 2i$.

Also, if we deal, e.g., with $F(\zeta) = \dfrac{\sqrt{\zeta}}{\zeta^2 - 4}$, this has two isolated singularities, namely $\zeta = \pm 2$, and one of them, namely $\zeta = -2$, is on the negative x-semi-axis. With the branch cut the non-positive x-semi-axis, we must put two equal circular arcs around $z = -2$, one in the upper half plane and the other in the lower half plane. Then we take limits as the radius approaches zero, etc. zero when we take limits, etc.

Problems

3.7.102 Check which of the three conditions of **Theorem 3.7.8** hold for the functions

$$f_1(\zeta) = e^{-\zeta^2}, \quad f_2(\zeta) = \sin\left(\frac{1}{\zeta}\right), \quad f_3(\zeta) = \frac{5\zeta}{\zeta^2 + 4}, \quad f_4(\zeta) = \frac{5}{\zeta^2 + 4}.$$

3.7.103 Use what we have learnt in **this section** and **Example 3.6.19** to prove that

$$\mathcal{L}^{-1}\left\{ \sin\left(\frac{1}{s}\right) \right\}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)! \cdot (2n+1)!}.$$

In the following **Problems (3.7.104-3.7.108)**, justify why we can use the **Mellin inversion formula (3.41)** or **(3.42)** and use only the methods of this section to find the inverse Laplace transform $f(x)$ if the Laplace transform $F(s)$ is given by:

3.7.104

$$\text{(a)}\quad \frac{5s}{s^2+4}, \qquad\qquad \text{(b)}\quad \frac{5}{s^2+4}.$$

3.7.105

$$\text{(a)}\quad \frac{5s}{s^2-4}, \qquad\qquad \text{(b)}\quad \frac{5}{s^2-4}.$$

3.7.106

$$\text{(a)}\quad \frac{5\sqrt{s}}{s^2+4}, \qquad\qquad \text{(b)}\quad \frac{5\sqrt{s}}{s^2-4}.$$

3.7.107

$$\text{(a)}\quad \frac{5s^3}{s^4-4}, \qquad\qquad \text{(b)}\quad \frac{5(s-2)}{(s+3)(s^2+4s+5)}.$$

3.7.108

$$\text{(a)}\quad \frac{s^2-4}{(s^2+4)^2}, \qquad\qquad \text{(b)}\quad \frac{s^2+4}{(s^2-4)^2}.$$

3.7.109 Imitate the work done in **Example 3.7.51** to prove the following inverse Laplace transform rule:

For any $a > 0$ constant,

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}(x) = \frac{a}{2\sqrt{\pi x^3}}\, e^{\frac{-a^2}{4x}}.$$

So,

$$\mathcal{L}\left\{\frac{a}{2\sqrt{\pi x^3}}\, e^{\frac{-a^2}{4x}}\right\}(s) = e^{-a\sqrt{s}}.$$

3.7.110 Imitate the work done in **Example 3.7.51** to prove the following inverse Laplace transform rule:

For any $a > 0$ constant,

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}(x) = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right).$$

So,

$$\mathcal{L}\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \mathcal{L}\left\{1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \frac{e^{-a\sqrt{s}}}{s}.$$

[Hint: An important difference with **Example 3.7.51** is in

$$\lim_{r\to 0}\int\limits_{A_r^-} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\zeta}d\zeta = \lim_{r\to 0}\int_{\pi}^{-\pi}\frac{e^{rxe^{i\theta}-a\sqrt{r}\,e^{i\frac{\theta}{2}}}}{re^{i\theta}}\,ire^{i\theta}d\theta =$$

$$i\cdot\lim_{r\to 0}\int_{\pi}^{-\pi} e^{rxe^{i\theta}-a\sqrt{r}\,e^{i\frac{\theta}{2}}}\,d\theta = \int_{\pi}^{-\pi} 1d\theta = -2\pi i.$$

In the end, you need to use the integral representation of the error function as done in **Example 2.2.14**.]

3.7.111 Prove that for any real number a, the Laplace transform of

$$f(x) = \frac{1}{\sqrt{x+a}}, \quad \text{where} \quad x > -a,$$

is

$$F(s) = \frac{\sqrt{\pi}\,e^{as}\operatorname{erfc}\left(\sqrt{as}\,\right)}{s}.$$

3.7.112 Find the inverse Laplace transforms of

$$F_1(s) = \frac{\sqrt{s}}{s^2+4}$$

and

$$F_2(s) = \frac{\sqrt{s}}{s^2-4}.$$

[Hint: Refer to the last **remark** of this section above.]

3.8 Definite Integrals with Sines and Cosines

In this section, we study some definite integrals of sines and cosines. There is the case of rational functions of sines and cosines on the whole interval $[0, 2\pi]$ and the case of rational or other functions of sines and cosines on some subintervals of $[0, 2\pi]$. In the first case, we have a more or less clear-cut answer by means of complex analysis and the Residue Theorem. In the other cases besides complex analysis, we can use and combine other techniques.

3.8.1 Rational Functions of Sines and Cosines

Calculus provides a general method for finding indefinite integrals of rational functions of $\sin(\theta)$ and $\cos(\theta)$. According to this method, we perform the so-called *tangent of half angle substitution*, then we use partial fractions decomposition and execute the ensuing computations. In the end, we perform an inverse substitution. (Review this method once more from a good calculus book and then provide the missing details in **Example 1.1.1** and **Remark 2 of Example 3.7.13**.)

Complex functions provide some methods of computing certain definite integrals of rational functions of $\sin(\theta)$ and $\cos(\theta)$, under some mild hypotheses. One method that we have already encountered is the **Cauchy integral Formula** as demonstrated in some examples and problems of **Section 3.5.7**. More general cases of such definite integrals are figured out by the following lemma, which is very convenient and efficient when applicable. So, we have:

Lemma 3.8.1 *Let*

$$g(\theta) = R\left[\sin(\theta), \cos(\theta)\right], \quad \text{with} \quad 0 \le \theta \le 2\pi,$$

be a rational function of $\sin(\theta)$ *and* $\cos(\theta)$. *We set*

$$f(z) = R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \cdot \frac{1}{iz}$$

and suppose that $f(z)$ *has no singularities on the unit circle* $C(0,1)$ *and its singularities inside the open disc* $D(0,1)$ *are isolated and therefore finitely many.*

(a) If all the singularities of $f(z)$ *inside the open disc* $D(0,1)$ *are the complex numbers* $z_1,\ z_2,\ z_3, ...,\ z_n$, $(n \ge 1)$, *then*

$$\int_0^{2\pi} g(\theta)\, d\theta = \int_0^{2\pi} R\left[\sin(\theta), \cos(\theta)\right] d\theta = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

(b) If $f(z)$ *has no singularities inside* $C(0,1)$, *then*

$$\int_0^{2\pi} g(\theta)\, d\theta = \int_0^{2\pi} R\left[\sin(\theta), \cos(\theta)\right] d\theta = 0.$$

[Singularities outside $C(0,1)$ play no role in these two results.]

Proof (a) If $C^+ = C^+(0,1)$, by the **Residue Theorem, 3.7.1**, we have

$$\oint_{C^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

But,

$$\oint_{C^+} f(z)\,dz = \int_0^{2\pi} f\left(e^{i\theta}\right)\,d\left(e^{i\theta}\right) =$$

$$\int_0^{2\pi} R\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}, \frac{e^{i\theta}+e^{-i\theta}}{2}\right)\frac{1}{ie^{i\theta}} ie^{i\theta}\,d\theta = \int_0^{2\pi} R[\sin(\theta),\cos(\theta)]\,d\theta.$$

Thus,

$$\int_0^{2\pi} R[\sin(\theta),\cos(\theta)]\,d\theta = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

(b) This follows immediately from the **Cauchy-Goursat Theorem, 3.5.3**.

Examples

Example 3.8.1 We want to evaluate

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)}$$

with $a > 1$ real constant (so the denominator is never zero).

By **Lemma 3.8.1**, we consider the complex function

$$f(z) = \frac{1}{a + \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)} \cdot \frac{1}{iz} = \frac{2}{z^2 + 2aiz - 1}.$$

The roots of the denominator $z^2 + 2aiz - 1$, evaluated by the quadratic formula, are $z_1 = i(-a+\sqrt{a^2-1})$ and $z_2 = -i(a+\sqrt{a^2-1})$.

The first root is inside the $D(0,1)$ and the second root, outside. Therefore, we need to compute the residue of $f(z)$ at $z = z_1$. So, we have

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{2}{i(-a+\sqrt{a^2-1}) + i(a+\sqrt{a^2-1})} = \frac{1}{i\sqrt{a^2-1}}.$$

Hence,

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = 2\pi i \frac{1}{i\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Remark 1: This result is equal to twice the result of the **Example 3.7.13**, since for $a > 1$

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)}$$

(prove this equality!).

We see that this is an easier and more efficient treatment of **Example 3.7.13**.

Remark 2: This example can also be carried out with $a < -1$, real. In this case, we compute the residue at z_2, and in same way we find

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = \frac{-2\pi}{\sqrt{a^2-1}}.$$

[Another way is

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = -\int_0^{2\pi} \frac{d\theta}{-a+\sin(-\theta)}.$$

We let $u = -\theta$, and with $-a > 1$ we use the first integral computed above.]

So, with $|a| > 1$ real, we get

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = \frac{\text{sign}(a)\cdot 2\pi}{\sqrt{a^2-1}}.$$

▲

Example 3.8.2 Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos(\theta)+a^2}$$

where $a \in \mathbb{C}$ constant with $|a| < 1$.

(This integral is closely related to **Poisson Kernel, Definition 3.5.3**, in the unit disc, if in that definition we put $a = 1$, $r = a$ and $\phi = 0$. Compare these two expressions carefully! See also **Problem 3.5.32**.)

We consider $0 < |a| < 1$ (so, $a \neq 0$) and again use **Lemma 3.8.1** with

$$f(z) = \frac{1}{1 - 2a\frac{1}{2}\left(z + \frac{1}{z}\right) + a^2} \cdot \frac{1}{iz} = \frac{-i}{-az^2 + (a^2+1)\,z - a} =$$

$$\frac{i}{az^2 - (a^2+1)\,z + a} = \frac{i}{(az-1)(z-a)} = \frac{i}{a} \cdot \frac{1}{\left(z - \frac{1}{a}\right)(z-a)}.$$

So, $f(z)$ has isolated singularities $z_1 = a$ inside the $D(0,1)$ and $z_2 = \frac{1}{a}$ outside it.

We compute

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{i}{a} \cdot \frac{1}{z - \frac{1}{a}}\Big|_{z=a} = \frac{i}{a^2-1}.$$

Hence, with $a \in \mathbb{C}$ such that $0 < |a| < 1$, we find

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos(\theta) + a^2} = 2\pi i \frac{i}{a^2-1} = \frac{2\pi}{1-a^2} \quad \text{when} \quad 0 < |a| < 1.$$

We now observe that this formula is trivially true when $a = 0$, and so we can state that the formula is true for any $a \in \mathbb{C}$ such that $0 \le |a| < 1$.

Remark: (a) When $|a| > 1$, we apply this result with $\frac{1}{a}$ in the place of a and we find that (check this)

$$\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos(\theta) + 1} = \frac{2\pi}{a^2-1} \quad \text{when} \quad |a| > 1.$$

(b) Examine this integral when $a = 1$ as an **exercise!** Explain why the integral becomes improper at $\theta = 0$ and $\theta = 2\pi$. What value or principal value did you find eventually?

(c) In harmonic analysis and partial differential equations, for $R > r > 0$, we use the integral

$$\int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos(\theta) + r^2} = \frac{2\pi}{R^2 - r^2}.$$

(Verify this immediately from the result obtained in this example by making the appropriate transformation!)

▲

Note: We must observe that both examples studied before can be figured out by means of the **Cauchy integral Formula** (**Section 3.5.7**).

In both, we use $C = C^+(0,1)$. In the first example, we apply the formula to the function

$$f(z) = \frac{2}{z^2+2aiz-1} = \frac{\frac{2}{z-z_2}}{z-z_1} = \frac{\frac{2}{z+i(a+\sqrt{a^2-1})}}{z-i(-a+\sqrt{a^2-1})},$$

where $a > 1$ real. In the second example, we apply the formula to the function

$$f(z) = \frac{i}{a} \cdot \frac{1}{\left(z-\frac{1}{a}\right)(z-a)} = \frac{i}{a} \cdot \frac{\frac{1}{z-\frac{1}{a}}}{z-a},$$

where $0 < |a| < 1$ complex. (Work out the application of the **Cauchy integral Formula** in both examples.)

This way, using the **Cauchy integral Formula**, is also very convenient here, because there is just one singularity inside the contour $C = C^+(0,1)$. When there are two or more singularities inside $C = C^+(0,1)$, then **Lemma 3.8.1** is more convenient.

In the following example, **Lemma 3.8.1** is more convenient than the **Cauchy integral Formula**.

Example 3.8.3 We prove the result in **Examples 2.2.3**, **2.6.18** and **3.5.15** by means of **Lemma 3.8.1**. That is,

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x)+b^2\sin^2(x)} = \frac{2\pi}{ab}\,,$$

where $a > 0$ and $b > 0$ constants.

For $a = b > 0$, the result is immediate. So, we assume $a \neq b$ and consider

$$f(z) = \frac{1}{\left[a^2\left(\frac{z+\frac{1}{z}}{2}\right)^2 + b^2\left(\frac{z-\frac{1}{z}}{2i}\right)^2\right]iz} = \frac{1}{\left[a^2\left(\frac{z+\frac{1}{z}}{2}\right)^2 - b^2\left(\frac{z-\frac{1}{z}}{2}\right)^2\right]iz} =$$

$$\ldots = \frac{-i4z}{[(a-b)z^2+(a+b)]\cdot[(a+b)z^2+(a-b)]} =$$

$$\frac{-i4z}{[(a-b)z^2+(a+b)]\cdot(a+b)\cdot\left(z-\sqrt{\frac{b-a}{a+b}}\right)\cdot\left(z+\sqrt{\frac{b-a}{a+b}}\right)}.$$

The four roots of the denominator are simple. With $a > 0$ and $b > 0$, two of these roots are outside the disc $\overline{D(0,1)}$, and the roots $z_1 = \sqrt{\dfrac{b-a}{a+b}}$

and $z_2 = -\sqrt{\dfrac{b-a}{a+b}}$ are inside. So, we compute

$$\operatorname*{Res}_{z=z_1} f(z) = ... = \frac{-i}{2ab} \qquad \text{and} \qquad \operatorname*{Res}_{z=z_2} f(z) = ... = \frac{-i}{2ab}.$$

Then, by **Lemma 3.8.1**, we have

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x) + b^2\sin^2(x)} = 2\pi i \left(\frac{-i}{2ab} + \frac{-i}{2ab}\right) = \frac{2\pi}{ab}.$$

Remark 1: This result could be found by using the **Cauchy integral formula** but after we break the function $f(z)$ above in complex partial fractions. So, **Lemma 3.8.1** is more convenient.

Remark 2: From this result, we obtain the following two byproducts: If $A > 0$ and $B > 0$, then

$$\text{(a)} \qquad \int_0^{2\pi} \frac{dx}{A + B\sin^2(x)} =$$

$$\int_0^{2\pi} \frac{dx}{A\cos^2(x) + (A+B)\sin^2(x)} = \frac{2\pi}{\sqrt{A(A+B)}},$$

and

$$\text{(b)} \qquad \int_0^{2\pi} \frac{dx}{A\cos^2(x) + B} =$$

$$\int_0^{2\pi} \frac{dx}{(A+B)\cos^2(x) + B\sin^2(x)} = \frac{2\pi}{\sqrt{(A+B)B}}.$$

▲

3.8.2 Other Techniques with Sines and Cosines

Next, we investigate some techniques of computing various integrals involving $\sin(x)$ and $\cos(x)$, based on various identities. For instance, we use the following **Euler's Formulae**:

$$\begin{aligned} &(1)\ \ e^{ix} = \cos(x) + i\sin(x), \\ &(2)\ \ e^{-ix} = \cos(x) - i\sin(x), \\ &(3)\ \ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \\ &(4)\ \ \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \end{aligned}$$

We can use the fact stated in **Problem 3.7.40** that for any $-\infty < a,\ b < \infty$ real numbers and any non-zero complex number $\alpha + i\beta \neq 0$, the following integral formula

$$\int_a^b e^{(\alpha+i\beta)x} dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha + i\beta}$$

is valid. If $\alpha + i\beta = 0$, then the integral is equal to $b - a$.

From this fact, we immediately get: **For all integers $\boldsymbol{n}$ and $\boldsymbol{m}$**, we have

$$\int_0^{2\pi} e^{-inx} e^{imx} dx = \begin{cases} 0, & \text{if} \quad m \neq n, \\ \\ 2\pi, & \text{if} \quad m = n. \end{cases}$$

Combining this fact with Euler's formulae, we easily obtain the **result**:

For every integer $\boldsymbol{n \neq 0}$, we have

$$\int_0^{2\pi} \cos^2(nx) dx = \int_0^{2\pi} \sin^2(nx) dx = \pi.$$

Notice that for $\boldsymbol{n = 0}$ we trivially obtain that the first integral is equal to 2π whereas the second integral is equal to 0.

We also obtain the following **results**: **For all integers $\boldsymbol{m \neq n}$**, we have:

$$\begin{aligned} &(1) \quad \int_0^{2\pi} \cos(nx)\cos(mx)\, dx = 0, \\ &(2) \quad \int_0^{2\pi} \sin(nx)\cos(mx)\, dx = 0, \\ &(3) \quad \int_0^{2\pi} \sin(nx)\sin(mx)\, dx = 0. \end{aligned}$$

In the computations of integrals with sines and cosines, we may use **algebraic identities**, such as:

1. **Binomial Theorem:** $\forall\ n = 0,\ 1,\ 2,\ ...,$ and $\forall\ z,\ w \in \mathbb{C}$, we have

$$(z + w)^n = \sum_{k=0}^{n} z^{n-k} w^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We remember that $0! = 1$ and so $\binom{n}{0} = 1$. For $k \in \mathbb{N}$, we simplify

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 3 \cdot 2 \cdot 1}.$$

2. $\forall\ n = 0,\ 1,\ 2,\ ...,$ and $\forall\ z \neq w \in \mathbb{C}$, we have

$$\frac{z^n - w^n}{z - w} = z^{n-1} + z^{n-2}w + z^{n-3}w^2 + ... + zw^{n-2} + w^{n-1}.$$

   If $z = w$, then the limit of the first side, as $z \to w$, is equal to the second side if we replace z with w, that is, $nw^{n-1} = (z^n)' \,|_{z=w}$.

3. (Look at various sources of algebraic identities.)

We may also use **identities of combinatorial numbers** and factorials, such as:

1. $\forall\ n \geq k \geq 1$ integers, we have the **recursive identity of the combinatorial numbers**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

2. $\forall\ n \geq k \geq 0$ integers, we have the **symmetry of the combinatorial numbers**

$$\binom{n}{n-k} = \binom{n}{k}.$$

3. $\forall\ n,\ m,\ k,$ positive integers such that $n + m \geq k$, we have

$$\binom{n+m}{k} = \sum_{l=0}^{k} \binom{n}{l}\binom{m}{k-l}.$$

4. (Look at various sources of combinatorial identities.)

Examples

The methods and examples exposed here are rather straightforward. Sometimes they involve some lengthy, but not difficult, computations.

Example 3.8.4 In this example, we consider integrals of powers of $\sin(x)$. (Similar work is done for powers of $\cos(x)$.)

For $n = 0,\ 1,\ 2,\ ...,$ we have

$$\sin^n(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^n = \frac{1}{2^n i^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{(n-2k)ix}.$$

We have already seen that,

$$\int_a^b e^{(n-2k)ix}dx = \begin{cases} \dfrac{e^{(n-2k)ib} - e^{(n-2k)ia}}{(n-2k)i}, & \text{if} \quad n \neq 2k \\ \\ b-a, & \text{if} \quad n = 2k. \end{cases}$$

So, for $0 \leq k \leq n$, we have the partial integrals

$$I_{n,k} := \int_0^{\pi/2} e^{(n-2k)ix}dx = \begin{cases} \dfrac{e^{(n-2k)i\frac{\pi}{2}} - 1}{(n-2k)i}, & \text{if} \quad n \neq 2k \\ \\ \dfrac{\pi}{2}, & \text{if} \quad n = 2k. \end{cases}$$

(a) We consider the **case $\boldsymbol{n = 2m}$ is even**. Then,

$$I_{2m,k} = \begin{cases} \dfrac{[-(-1)^{m-k} + 1]i}{2(m-k)}, & \text{if} \quad m \neq k \\ \\ \dfrac{\pi}{2}, & \text{if} \quad m = k. \end{cases}$$

So, this new method gives

$$\int_0^{\pi/2} \sin^{2m}(x)\,dx = \frac{1}{2^{2m}(-1)^m} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} I_{2m,k}.$$

Since this is a real number, by equating the real parts we finally have

$$\int_0^{\pi/2} \sin^{2m}(x)\,dx = \frac{1}{2^{2m}(-1)^m} \cdot (-1)^m \binom{2m}{m} I_{2m,m} =$$
$$\frac{\pi}{2^{2m+1}} \cdot \frac{(2m)!}{(m!)^2} = \frac{1 \cdot 3 \ldots (2m-1)}{2 \cdot 4 \ldots (2m)} \cdot \frac{\pi}{2}.$$

Remark 1: The imaginary parts must be equal to zero. Thus, we obtain the identity

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} (-1)^k \binom{2m}{k} \frac{1 - (-1)^{m-k}}{m-k} = 0$$

or

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{(-1)^k}{m-k} \binom{2m}{k} = (-1)^m \sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{1}{m-k} \binom{2m}{k}.$$

This identity is not very interesting because we can directly show, by the symmetry of the combinatorial numbers, that each of these two sums is equal to zero. I.e., for any integer $m \geq 1$, we have

$$\sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{(-1)^k}{m-k}\binom{2m}{k} = 0 \qquad \text{and} \qquad \sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{1}{m-k}\binom{2m}{k} = 0.$$

Remark 2: This integral was also found in **Section 2.6, Example 2.6.15**, where we dealt with the **Beta and Gamma functions**. Again, the integral, with the help of **Problem 2.6.14 (a)**, is

$$\int_0^{\pi/2} \sin^{2m}(x)\,dx = \frac{1}{2}B\left(m+\frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} =$$
$$\frac{\pi}{2^{2m+1}}\cdot\frac{(2m)!}{(m!)^2} = \frac{1\cdot 3\ldots(2m-1)}{2\cdot 4\ldots(2m)}\cdot\frac{\pi}{2}.$$

(b) Now, we consider the **case $\boldsymbol{n = 2m+1}$ is odd**. Then, we find

$$I_{2m+1,k} = \frac{(-1)^{m-k}+i}{2m+1-2k}.$$

Therefore,

$$\int_0^{\pi/2} \sin^n(x)\,dx = \int_0^{\pi/2} \sin^{2m+1}(x)\,dx =$$
$$\frac{1}{2^{2m+1}(-1)^m i}\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\frac{(-1)^{m-k}+i}{2m+1-2k} =$$
$$\frac{(-1)^m}{2^{2m+1}}\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\frac{-i(-1)^{m-k}+1}{2m+1-2k}.$$

Separating real and imaginary parts, we find

$$\int_0^{\pi/2} \sin^{2m+1}(x)\,dx = \frac{(-1)^m}{2^{2m+1}}\sum_{k=0}^{2m+1}\frac{(-1)^k}{2m+1-2k}\binom{2m+1}{k}(>0)$$

and the identity

$$\sum_{k=0}^{2m+1}\frac{1}{2m+1-2k}\binom{2m+1}{k} = 0.$$

Remark 3: Again, this integral, in terms of the **Beta and Gamma functions, Section 2.6, Example 2.6.15** and the help of **Problem 2.6.14 (a)**, is

$$\int_0^{\pi/2} \sin^{2m+1}(x)\,dx = \frac{1}{2}B\left(m+1, \frac{1}{2}\right) = \sum_{l=0}^{m} \frac{(-1)^l}{2l+1}\binom{m}{l} =$$

$$\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} = \frac{2^{2m}(m!)^2}{(2m+1)!} = \frac{2^{2m}}{2m+1}\cdot\frac{1}{\binom{2m}{m}} =$$

$$\frac{2\cdot 4\ldots(2m)}{1\cdot 3\ldots(2m+1)} = \frac{(-1)^m}{2^{2m+1}}\sum_{k=0}^{2m+1}\frac{(-1)^k}{2m+1-2k}\binom{2m+1}{k}.$$

Remark 4: We can do similar work with $\cos(x)$ in the place of $\sin(x)$ and with limits of integration $a = 0$ and $b = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$.

▲

Example 3.8.5 We can use **Euler's formulae** to prove

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!}.$$

We have

$$\sin^{2m}(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^{2m} = \frac{1}{2^{2m}(-1)^m}\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}e^{2(m-k)ix}$$

and

$$\cos^{2n}(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n} = \frac{1}{2^{2n}}\sum_{l=0}^{2n}\binom{2n}{l}e^{2(n-l)ix}.$$

So,

$$\sin^{2m}(x)\cos^{2n}(x) = \frac{(-1)^m}{2^{2m+2n}}\sum_{k=0}^{2m}\sum_{l=0}^{2n}(-1)^k\binom{2m}{k}\binom{2n}{l}e^{2(m-k+n-l)ix}.$$

Therefore, the integral is

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{(-1)^m}{2^{2m+2n}}\sum_{k=0}^{2m}\sum_{l=0}^{2n}(-1)^k\binom{2m}{k}\binom{2n}{l}I_{mkln}\,,$$

where

$$I_{mkln} = \begin{cases} \dfrac{(-1)^{m-k+n-l}-1}{2(m-k+n-l)\,i} = \\ \dfrac{\left[1-(-1)^{m-k+n-l}\right]i}{2(m-k+n-l)}, & \text{if} \quad m-k+n-l \neq 0 \\ & \iff l \neq m+n-k \\ \dfrac{\pi}{2}, & \text{if} \quad m-k+n-l = 0 \\ & \iff l = m+n-k. \end{cases}$$

Since the integral is real, the imaginary part of this equation is zero. So, the summation over the indices such that $m-k+n-l \neq 0$ must be zero. That is, we obtain the identity

$$\sum_{\substack{k=0 \\ m+n\neq k+l}}^{2m} \sum_{l=0}^{2n} (-1)^k \binom{2m}{k} \binom{2n}{l} \frac{1-(-1)^{m-k+n-l}}{m-k+n-l} =$$
$$\sum_{\substack{k=0 \\ m+n\neq k+l}}^{2m} \sum_{l=0}^{2n} \binom{2m}{k} \binom{2n}{l} \frac{(-1)^k-(-1)^{m+n-l}}{m-k+n-l} = 0.$$

We observe that only when $m-k+n-l=0$ is there a contribution to the integral. Thus,

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{(-1)^m\pi}{2^{2m+2n+1}} \sum_{\substack{k=0 \\ k=m+n-l}}^{2m} \sum_{l=0}^{2n} (-1)^k \binom{2m}{k}\binom{2n}{l} =$$
$$\frac{(-1)^m\pi}{2^{2m+2n+1}} \sum_{l=0}^{2n} (-1)^{m+n-l} \binom{2m}{m+n-l}\binom{2n}{l} =$$
$$\frac{\pi}{2^{2m+2n+1}} \sum_{l=0}^{2n} (-1)^{n-l} \binom{2m}{m+n-l}\binom{2n}{l}.$$

We could accept this as a final answer. However, someone experienced with **combinatorial identities** can prove that:

If $m \geq 0$ and $n \geq 0$ integers,

$$\sum_{l=0}^{2n} (-1)^{n-l} \binom{2m}{m+n-l}\binom{2n}{l} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

[For the proof, see **Appendix 3.8.3, identity (3.44)**, after the end of **this subsection**.]

Finally, we obtain the answer

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1} m!\,n!\,(m+n)!}.$$

Remark: We have seen that in terms of the **Beta and Gamma functions, Section 2.6, Example 2.6.16,** this integral is

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{1}{2}B\left(m+\frac{1}{2}, n+\frac{1}{2}\right) =$$

$$\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(m+n+1)} = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1} m!\,n!\,(m+n)!}.$$

▲

Example 3.8.6 For any integer n, compute the integral

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx.$$

We notice

$$\frac{\sin(nx)}{\sin(x)} = \frac{\dfrac{e^{inx}-e^{-inx}}{2i}}{\dfrac{e^{ix}-e^{-ix}}{2i}} = \frac{e^{inx}-e^{-inx}}{e^{ix}-e^{-ix}} =$$

$$e^{ix(n-1)} + e^{ix(n-2)}e^{-ix} + e^{ix(n-3)}e^{-2ix} + \dots + e^{-ix(n-1)} =$$

$$e^{ix(n-1)} + e^{ix(n-3)} + e^{ix(n-5)} + \dots + e^{ix(1-n)}.$$

If *n* is even, then every exponent of e^{ix}, in the above expression, is odd, and so the integral is

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \frac{2i}{n-1} + \frac{2i}{n-3} + \frac{2i}{n-5} + \dots + \frac{2i}{3-n} + \frac{2i}{1-n} = 0.$$

If *n* is odd, then exactly one exponent, the middle one, becomes zero. Since the sum of the imaginary numbers involved is zero, in this case we find

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \int_0^{\pi} 1\,dx = \pi, \quad \text{for } n \text{ odd}.$$

Finally, we have rather easily obtained the useful **result**:

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \begin{cases} 0, & \text{if } n \text{ is even} \\ \\ \pi, & \text{if } n \text{ is odd.} \end{cases}$$

Since $\frac{\sin(nx)}{\sin(x)}$ is an even function, we also get

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \begin{cases} 0, & \text{if } n \text{ is even} \\ \\ 2\pi, & \text{if } n \text{ is odd.} \end{cases}$$

(Try to establish this result by other techniques!)

▲

Problems

3.8.1 For $R > r > 0$ and any θ constant, prove that

$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi = 2\pi.$$

3.8.2 For $a \geq 1$, prove that

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\,dx = \pi \ln\left(1 + \frac{1}{a}\right).$$

Then, prove that for $0 < a \leq 1$

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\,dx = \pi \ln(1 + a).$$

[Hint: Integrate the complex function $f(z) = \frac{z}{1 - ae^{-iz}}$ along the rectangle with vertices $[\pm\pi, \pm(\pi + iR)]$, etc.]

3.8.3 Use **Lemma 3.8.1** to prove the result of **Example 2.6.16**

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi}{2^{2m+2n+1}} \cdot \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!},$$

for $m \geq 0$ and $n \geq 0$ integers.

State the two byproducts that we get when $m = 0$ and when $n = 0$.

Examine what happens if we use **Lemma 3.8.1** and one or both exponents of sine and/or cosine is/are odd.

3.8.4 Compute the following integrals by whichever method is more efficient. (Either by using **Lemma 3.8.1** or by using any correct methods and integral formulae already used and proved previously anywhere in the text.)

$$
\begin{aligned}
I_1 &= \int_0^{2\pi} \frac{d\theta}{1 + a\sin(\theta)}, \quad -1 < a < 1, \quad \text{constant},\\
I_2 &= \int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}, \qquad a > 1, \quad \text{constant},\\
I_3 &= \int_0^{2\pi} \frac{d\theta}{1 + a\cos(\theta)}, \quad -1 < a < 1, \quad \text{constant},\\
I_4 &= \int_0^{2\pi} \frac{\cos(2\theta)}{8 + 3\cos(\theta)}\, d\theta,\\
I_5 &= \int_0^{\pi} \frac{\cos(\theta)}{5 + 2\cos(\theta)}\, d\theta,\\
I_6 &= \int_0^{2\pi} \frac{d\theta}{\left[9\cos^2(\theta) + 4\sin^2(\theta)\right]^2}, \qquad \text{(compare with } \textbf{Example 2.2.3}\text{)}\\
I_7 &= \int_0^{2\pi} \frac{dx}{[2 + \cos(x)]\cdot[3 + \cos(x)]} \ \left[= 2\pi\left(\frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{2}}\right)\right],\\
I_8 &= \int_0^{2\pi} \frac{dx}{96\cos^2(x) + 25},\\
I_9 &= \int_0^{2\pi} \frac{dx}{100 + 69\sin^2(x)},\\
I_{10} &= \int_0^{2\pi} \frac{dx}{25\cos^2(x) + 100\sin^2(x)}.
\end{aligned}
$$

3.8.5 For $n \geq 0$ integer and $b = \frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, evaluate the integrals

$$\int_0^b \cos^n(x)\, dx \quad \text{and} \quad \int_0^b \sin^n(x)\, dx.$$

3.8.6 For $n \geq 0$ and $m \geq 0$ integers and $b = \frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, evaluate the integrals

$$\int_0^b \cos^n(x)\cos(mx)\, dx \quad \text{and} \quad \int_0^b \sin^n(x)\sin(mx)\, dx.$$

3.8.7 Evaluate the four integrals

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \sin(x)\sin(2x)\sin(3x)\,dx \quad \left(=\frac{1}{6}\right), \\ I_2 &= \int_0^{\pi} [x\sin(x)]^2\,dx \quad \left(=\frac{\pi^3}{6}-\frac{\pi}{4}\right), \\ I_3 &= \int_0^{\pi} \cos^n(x)\cos(nx)\,dx \quad \left(=\frac{\pi}{2^n}\right), \\ I_4 &= \int_0^{\pi} \sin^n(x)\sin(nx)\,dx \quad \left[=\frac{\pi}{2^n}\sin\left(\frac{n\pi}{2}\right)\right], \end{aligned}$$

where in I_3 and I_4, n is a non-negative integer.

3.8.8 We have proven that

$$\int_0^{\pi/2} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!} =$$
$$\frac{1}{2}B\left(m+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(m+n+1)}.$$

Find the integral formulae when the $2m$ and $2n$ are combined and/or replaced with $2m+1$ and $2n+1$. (There are three more cases. You may give the answers in summation forms if you cannot find them in terms of nice fractions with factorials, etc.) For practice and finding out the correct answer first, write the results in terms of the Beta and Gamma functions and then simplify.

3.8.9 For any integer $m \geq 1$, prove directly

$$\text{(a)} \quad \sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{(-1)^k}{m-k}\binom{2m}{k} = 0, \qquad \text{(b)} \quad \sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{1}{m-k}\binom{2m}{k} = 0.$$

3.8.3 Appendix

In **Example 3.8.5**, we used the identity

$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}, \tag{3.44}$$

where $m \geq n \geq 1$ integers. We would like to give an **elementary proof** of it. Observe that this identity is symmetrical about m and n.[34]

[For simplifying expressions, we use $\binom{k}{r} = \frac{k!}{r!(k-r)!}$ for $0 \leq r \leq k$ integers. For any other integer r, the combination number $\binom{k}{r}$ is 0.]

We let

$$L := \sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l}$$

$$\left[= \sum_{l=0}^{2n}(-1)^{n-l}\frac{(2m)!(2n)!}{(m+n-l)!(m-n+l)!l!(2n-l)!}\right],$$

and so we want to prove

$$L = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}\left[= \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}\right].$$

So,

$$\frac{\binom{m+n}{n}^2}{\binom{2m}{m}\binom{2n}{n}}L = \frac{[(m+n)!]^2}{(2m)!(2n)!}L =$$

$$\sum_{l=0}^{2n}(-1)^{n-l}\frac{[(m+n)!]^2}{l!(2n-l)!(m+n-l)!(m-n+l)!} =$$

$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{m+n}{l}\binom{m+n}{2n-l}. \tag{3.45}$$

By the identity

$$(1-t^2)^{m+n} = (1+t)^{m+n}(1-t)^{m+n}$$

---

[34] These kinds of results have to do with the theory of **Hypergeometric Forms** and **Kummer's Summation Formulae**. (Ernst Eduard Kummer, German mathematician, 1810-1893.) These types of sums are very important in mathematics and applications, and so they have been standardized and tabulated. Also, some computer programs, such as ***Mathematica***, can evaluate them.

and the **Binomial Theorem**, we find

$$\sum_{k=0}^{m+n}\binom{m+n}{k}(-1)^k t^{2k} = \sum_{k_1=0}^{m+n}\binom{m+n}{k_1}t^{k_1}\sum_{k_2=0}^{m+n}(-1)^{k_2}\binom{m+n}{k_2}t^{k_2} =$$
$$\sum_{k_1=0}^{m+n}\sum_{k_2=0}^{m+n}\binom{m+n}{k_1}(-1)^{k_2}\binom{m+n}{k_2}t^{k_1+k_2} = \quad (\text{let } k_1+k_2=k)$$
$$\sum_{k=0}^{2(m+n)}\left[\sum_{k_1=0}^{k}(-1)^{k-k_1}\binom{m+n}{k_1}\binom{m+n}{k-k_1}\right]t^k.$$

By equating the coefficients of t^{2n}, we obtain

$$\binom{m+n}{n}(-1)^n = \sum_{k_1=0}^{2n}(-1)^{2n-k_1}\binom{m+n}{k_1}\binom{m+n}{2n-k_1}. \qquad (3.46)$$

So, by simplifying the $(-1)^n$ and replacing k_1 with l in **(3.46)** and using equation **(3.45)** above, we find $\frac{\binom{m+n}{n}^2}{\binom{2m}{m}\binom{2n}{n}}L = \binom{m+n}{n}$. Therefore, $L = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}$, which finishes the proof of **identity (3.44)**.

Problems

3.8.10 Can you prove the **identity (3.44)** proven in **Appendix 3.8.3** combinatorially?

3.8.11 Prove the following series of equalities

$$\frac{1}{2}B\left(m+1,\frac{1}{2}\right) = \sum_{l=0}^{m}\frac{(-1)^l}{2l+1}\binom{m}{l} =$$
$$\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} = \frac{2^{2m}(m!)^2}{(2m+1)!} = \frac{2^{2m}}{2m+1}\cdot\frac{1}{\binom{2m}{m}} =$$
$$\frac{2\cdot 4\ldots(2m)}{1\cdot 3\ldots(2m+1)} = \frac{(-1)^m}{2^{2m+1}}\sum_{k=0}^{2m+1}\frac{(-1)^k}{2m+1-2k}\binom{2m+1}{k},$$

which appears in **Example 3.8.4 (b)** and **Remark 3**.

Chapter 4

List of Non-elementary Integrals and Sums in Text

4.1 List of Non-elementary Integrals

Here, we cite a list of all major, non-elementary, general and important real or complex integrals that have been evaluated in the text or referred to in problems and footnotes. An addition to this list can be considered all the Laplace, Fourier and inverse transforms that have been evaluated in all relevant sections to which we direct your attention.

1. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

   $$\int_0^\infty e^{-\alpha x} \sin(\beta x)\, dx = \frac{\beta}{\alpha^2 + \beta^2}.$$

   (**Problems 1.2.14, 3.7.40**. **Example 3.7.24 Corollary 3.7.5 (C)**.)

2. If $\alpha > 0$ and $\beta \in \mathbb{R}$ constants,

   $$\int_0^\infty e^{-\alpha x} \cos(\beta x)\, dx = \frac{\alpha}{\alpha^2 + \beta^2}.$$

   (**Problems 1.2.16, 3.7.40**.)

3. If $b > |a|$ real constants,

   $$\int_{-\infty}^\infty \frac{dx}{x^2 + 2ax + b^2} = \frac{\pi}{\sqrt{b^2 - a^2}}.$$

   (**Problem 1.2.20**.)

4. $$\int_0^\infty \frac{dx}{1 + x^3} = \frac{2\pi\sqrt{3}}{9}.$$

   (**Problem 1.3.2**.)

5.
$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}.$$

(**Problems 1.3.2**, **2.6.20**. **Example 2.2.6**.)

6.
$$\int_0^\infty \frac{dx}{(1+x^3)^2} = \frac{4\pi\sqrt{3}}{27}.$$

(**Problem 1.3.2**.)

7.
$$\int_0^\infty \frac{dx}{(1+x^4)^2} = \frac{3\pi\sqrt{2}}{16}.$$

(**Problem 1.3.2**.)

8.
$$\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx = 2\int_{-\infty}^0 e^{-x^2}\,dx = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right).$$

(**Section 2.1**. **Subsection 2.6.1 (Γ, 5)**.)

9. If $\alpha > 0$ constant,
$$\int_{-\infty}^\infty e^{-\alpha x^2}\,dx = 2\int_0^\infty e^{-\alpha x^2}\,dx = 2\int_{-\infty}^0 e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

(**Problem 2.1.11**.)

10. If $a \geq 0$ and $b \geq 0$ constants,
$$\int_0^\infty \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right)dx = \sqrt{\pi b} - \sqrt{\pi a}.$$

(**Problem 2.1.14 (c)**.)

11. If $m = 0, 1, 2, 3, ...$ integer,
$$\int_0^1 \ln^m(u)\,du = (-1)^m m! \qquad \text{and} \qquad \int_0^1 |\ln(u)|^m\,du = m!.$$

(**Problem 2.1.15 (b)**.)

12.

$$\int_0^{\pi} \ln[\sin(x)]\,dx = 2\int_0^{\pi/2} \ln[\sin(x)]\,dx =$$

$$\int_0^{\pi} \ln[|\cos(x)|]\,dx = 2\int_0^{\pi/2} \ln[\cos(x)]\,dx =$$

$$\int_{-1}^{1} \frac{\ln(|u|)}{\sqrt{1-u^2}}\,du = 2\int_0^{1} \frac{\ln(u)}{\sqrt{1-u^2}}\,du = -\pi\ln(2).$$

If $a > 0$ constant,

$$\int_0^{\pi/2} \ln[a\sin(x)]dx = \frac{1}{2}\int_0^{\pi} \ln[a\sin(x)]dx = \frac{\pi}{2}\ln\left(\frac{a}{2}\right).$$

(**Problems 2.1.16, 3.2.26 (d), 3.7.88**. **Example 3.7.47**.)

13.

$$\int_0^{\pi/2} \ln[\tan(x)]\,dx = 0 \quad \text{and} \quad \int_0^{\pi} \ln[|\tan(x)|]\,dx = 0.$$

(**Problem 2.1.18**.)

14.

$$\int_0^{\infty} \frac{\ln(x)}{x^2+1}\,dx = 0.$$

(**Problems 2.1.19**, **3.7.90**. **Example 3.7.44**.)

15.

$$\int_0^{1} \frac{\ln(x+1)}{x^2+1}\,dx = \frac{\pi\ln(2)}{8}.$$

(**Problem 2.1.20**.)

16.

$$\int_{-1}^{\infty} \frac{\ln(x+1)}{x^2+1}\,dx = \frac{3\pi\ln(2)}{8}.$$

(**Problem 2.1.21**.)

17.

$$\int_0^{\infty} \frac{\ln(x)}{(x^2+1)^2}\,dx = -\frac{\pi}{4}.$$

(**Problems 2.1.22**, **3.7.92**.)

18. If $r \in \mathbb{R}$ constant,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^r(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\tan^r(u)}{1+\tan^r(u)}\,du = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cot^r(u)}{1+\cot^r(u)}\,du = \frac{\pi}{4}.$$

(**Problem 2.1.23**.)

19. If $\beta > 0$ constant,

$$\int_0^{\infty} \frac{1}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}.$$

(**Problem 2.1.24**. **Example 3.7.5**.)

20. If $\alpha > 0$ and $\beta > 0$ constants,

$$\int_0^{\infty} \frac{\ln(\alpha x)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\ln(\alpha\beta).$$

(**Problems 2.1.24**, **3.7.90**. **Example 3.7.44 Remark 2**.)

21. If $a > 0$ and b constants,

$$\int_0^{b} \frac{dx}{(a^2+x^2)^3} = \frac{b}{8a^4}\left[\frac{5a^2+3b^2}{(a^2+b^2)^2} + \frac{3}{ab}\arctan\left(\frac{b}{a}\right)\right].$$

(**Example 2.2.2**.)

22. If $a > 0$ and $b > 0$ constants,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2(x)+b^2\sin^2(x)} = \frac{\pi}{2ab},$$

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x)+b^2\sin^2(x)} = \frac{2\pi}{ab}.$$

(**Examples 2.2.3**, **2.6.18**, **3.5.15**, **3.8.3**.)

23. If $a > 0$ and $b > 0$ constants,

$$\int_0^{\pi/2} \frac{dx}{\left[a^2\cos^2(x)+b^2\sin^2(x)\right]^2} = \frac{\pi\left(a^2+b^2\right)}{4(ab)^3}.$$

(**Example 2.2.3**.)

24. If $0 < p < 1$ constant,

$$\int_0^\infty \frac{t^{p-1}}{1+t}\,dt = \int_0^\infty \frac{1}{(1+t)t^{1-p}}\,dt = \frac{\pi}{\sin(p\pi)}.$$

(**Examples 2.2.5**, **3.7.7**, **3.7.8**.)

25. If a, b real constants, $b \neq 0$ and $0 < \dfrac{a+1}{b} < 1$,

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{1}{|b|}\cdot\frac{\pi}{\sin\left(\frac{a+1}{b}\pi\right)}.$$

(**Examples 2.2.6**, **3.7.6**, **3.7.7**, **3.7.8**.)

26.

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx = \begin{cases} \dfrac{\pi}{2} & \text{if } \beta > 0 \\ 0 & \text{if } \beta = 0 \\ -\dfrac{\pi}{2} & \text{if } \beta < 0. \end{cases}$$

(**Examples 1.3.13**, **2.2.8**, **3.7.33**. **Problem 2.2.14**.)

27. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}\,dx = \arctan\left(\frac{\beta}{\alpha}\right).$$

(**Examples 2.2.8, 2.2.9, 3.7.38**. **Problems 2.4.9, 2.5.18**.)

28. If $-\infty < \beta < \infty$ constant,

$$\frac{2\beta}{\pi}\int_0^\infty \frac{\sin(\beta x)}{x}\,dx = |\beta|.$$

(**Example 2.2.10**.)

29.

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx \left[= \int_0^\infty \frac{1-\cos^2(x)}{x^2}\,dx\right] = \int_0^\infty \frac{1-\cos(x)}{x^2}\,dx = \frac{\pi}{2}.$$

(**Examples 2.2.11**, **2.2.12**, **3.7.34**.)

30. If $-\infty < \beta < \infty$ and $\alpha > 0$ constants,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} \cos(\beta x)\, dx = 2\int_0^{\infty} e^{-\alpha x^2} \cos(\beta x)\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-\beta^2}{4\alpha}}.$$

(**Examples 2.2.13**, **3.7.17**. **Problem 3.7.24**.)

31. If $a \geq 0$ and $k \geq 0$ constants not both zero,

$$\frac{1}{\pi}\int_0^{\infty} \frac{e^{-kx}\sin(a\sqrt{x})}{x}\, dx = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-ku^2}\sin(au)}{u}\, du = \operatorname{erf}\left(\frac{a}{2\sqrt{k}}\right).$$

(**Example 2.2.14)**)

32. If $0 < a < b$ constants,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{bx}}\, dx = \frac{1}{b} \cdot \frac{\pi}{\sin\left(\frac{a}{b}\pi\right)}.$$

(**Problem 2.2.12 (a)**.)

33. If $\alpha \in \mathbb{R}$ constant,

$$\int_0^{\infty} \frac{\sin^2(\alpha x)}{x^2}\, dx \left[= \int_0^{\infty} \frac{1-\cos^2(\alpha x)}{x^2}\, dx\right] = \int_0^{\infty} \frac{1-\cos(ax)}{x^2}\, dx = |\alpha|\frac{\pi}{2},$$

$$\int_0^{\infty} \frac{\cos(ax)[1-\cos(ax)]}{x^2}\, dx = 0.$$

(**Problems 2.2.15**, **2.2.16**.)

34. If $a \in \mathbb{R}$ constant,

$$\int_0^{\infty} \frac{\sin(a\sqrt{x})}{x}\, dx = \begin{cases} \pi, & \text{if} \quad a > 0 \\ 0, & \text{if} \quad a = 0 \\ -\pi, & \text{if} \quad a < 0. \end{cases}$$

(**Problem 2.2.17**.)

35. If α real constant,

$$\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

(**Problem 2.2.27 (a)**.)

36. If α real constant,

$$\int_0^\infty e^{-\left(x^2+\alpha^2x^{-2}\right)}\,dx = \frac{\sqrt{\pi}}{2}\,e^{-2\alpha}.$$

(**Problem 2.2.27 (b)**.)

37. If $\alpha \geq 0$ constant,

$$\int_0^\infty \frac{e^{-\alpha x}}{x^2+1}\,dx = \alpha\int_0^\infty e^{-\alpha x}\arctan(x)\,dx =$$
$$\frac{\pi}{2}\cos(\alpha) - \cos(\alpha)\int_0^\alpha \frac{\sin(t)}{t}\,dt - \sin(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\cos(\alpha)\int_\alpha^\infty \frac{\sin(t)}{t}\,dt - \sin(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\int_\alpha^\infty \frac{\sin(t-\alpha)}{t}\,dt = \int_0^\infty \frac{\sin(t)}{t+\alpha}\,dt = \int_0^\infty \frac{\sin(\alpha t)}{t+1}\,dt.$$

(**Problem 2.2.29 (d), (e), (k), (l)**.)

38. If $\alpha > 0$ constant,

$$\int_0^\infty \frac{xe^{-\alpha x}}{x^2+1}\,dx = \frac{\alpha}{2}\int_0^\infty e^{-\alpha x}\ln\left(x^2+1\right)\,dx =$$
$$\sin(\alpha)\left[\frac{\pi}{2} - \int_0^\alpha \frac{\sin(t)}{t}\,dt\right] + \cos(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\sin(\alpha)\int_\alpha^\infty \frac{\sin(t)}{t}\,dt + \cos(\alpha)\int_\alpha^\infty \frac{\cos(t)}{t}\,dt =$$
$$\int_\alpha^\infty \frac{\cos(t-a)}{t}\,dt = \int_0^\infty \frac{\cos(t)}{t+a}\,dt = \int_0^\infty \frac{\cos(\alpha t)}{t+1}\,dt.$$

(**Problem 2.2.29 (m)**.)

39.

$$\int_0^\infty \frac{\sin^3(x)}{x}\,dx = \frac{\pi}{4}.$$

(**Problems 2.2.33 hint**, **3.7.74**.)

40.
$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}.$$

(**Problem 2.2.33 hint**.)

41.
$$\int_0^\infty \frac{\sin^4(x)}{x}\,dx = \infty.$$

(**Problems 2.2.33 hint**, **3.7.74**.)

42.
$$\int_0^\infty \frac{\sin^4(x)}{x^2}\,dx = \frac{\pi}{4}.$$

(**Problems 2.2.33 hint**, **3.7.74**.)

43.
$$\int_0^\infty \frac{\sin^4(x)}{x^4}\,dx = \frac{\pi}{3}.$$

(**Problem 2.2.33 hint**. **Example 3.7.39**.)

44.
$$\int_{-\infty}^\infty \frac{[1-\cos(x)]^2}{x^4}dx = 2\int_0^\infty \frac{[1-\cos(x)]^2}{x^4}dx = 2\int_{-\infty}^0 \frac{[1-\cos(x)]^2}{x^4}dx = \frac{\pi}{3}.$$

(**Problem 2.2.33 (c)**.)

45. If $a \geq 0$ and $b \geq 0$ real constants,
$$\int_{-\infty}^\infty \frac{\sin(ax)\sin(bx)}{x^2}dx = \pi\min\{a,\ b\},$$
$$\int_{-\infty}^\infty \frac{\sin^2(ax)\sin^2(bx)}{x^4}dx = \frac{\pi}{2}\min\{a,\ b\}.$$

(**Problem 2.2.34 (a)**, **(b)**. **Example 3.7.35**.)

46. If $\alpha > 0$ and $\beta \in \mathbb{R}$ constants,
$$\int_0^\infty e^{-\alpha x^2}\sin(\beta x)\,dx = \frac{1}{2\alpha}\,e^{-\frac{\beta^2}{4\alpha}}\int_0^\beta e^{\frac{\rho^2}{4\alpha}}\,d\rho.$$

(**Problem 2.2.38**.)

47.

$$\int_{-1}^{1} \frac{-\ln(1-x)}{x}\,dx = \frac{\pi^2}{4}.$$

$$\int_{0}^{1} \frac{-\ln(1-x)}{x}\,dx = \frac{\pi^2}{6}.$$

$$\int_{-1}^{0} \frac{-\ln(1-x)}{x}\,dx = \frac{\pi^2}{12}.$$

(**Example 2.3.18**.)

48.

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(x^2)\,dx &= 2\int_{0}^{\infty} \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{2},\\ \int_{-\infty}^{\infty} \cos(x^2)\,dx &= 2\int_{0}^{\infty} \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{2}.\end{aligned}$$

If $a \neq 0$ real constant,

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(ax^2)\,dx &= 2\int_{0}^{\infty} \sin(ax^2)\,dx = \operatorname{sign}(a)\sqrt{\frac{\pi}{2|a|}},\\ \int_{-\infty}^{\infty} \cos(ax^2)\,dx &= 2\int_{0}^{\infty} \cos(ax^2)\,dx = \sqrt{\frac{2\pi}{2|a|}}.\end{aligned}$$

(**Fresnel Integrals**. **Examples 2.4.1, 3.7.16**. **Problems 2.4.14 (a), 2.6.10**.)

49. If $a \geq 0$ and $b \in \mathbb{R}$ constants,

$$\int_{0}^{\infty} e^{-ay}\,\frac{1-\cos(2by)}{2y}\,dy = \int_{0}^{\infty} e^{-ay}\,\frac{\sin^2(by)}{y}\,dy = \frac{1}{4}\ln\left(\frac{a^2+4b^2}{a^2}\right).$$

(**Problems 2.4.8**, **2.5.18**.)

50. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_{0}^{\infty} e^{-\alpha u}\,\frac{\sin^2(\beta u)}{u^2}\,du = \beta\arctan\left(\frac{2\beta}{\alpha}\right) - \frac{\alpha}{4}\ln\left(\frac{4\beta^2}{\alpha^2}+1\right).$$

(**Problem 2.4.10**.)

51. If $a \neq 0$, b and c real constants,

$$\int_{-\infty}^{\infty} \sin(ax^2+bx+c)dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\operatorname{sign}(a)\cos\left(\frac{4ac-b^2}{4a}\right)+\sin\left(\frac{4ac-b^2}{4a}\right)\right],$$
$$\int_{-\infty}^{\infty} \cos(ax^2+bx+c)dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\cos\left(\frac{4ac-b^2}{4a}\right)-\operatorname{sign}(a)\sin\left(\frac{4ac-b^2}{4a}\right)\right].$$

(Problem 2.4.14 (a).)

52. If $0 < a, b < \infty$ constants,

$$\int_0^{\infty} \frac{e^{-bx}-e^{-ax}}{x}\,dx = -\ln\left(\frac{b}{a}\right) = \ln\left(\frac{a}{b}\right).$$

If $0 < t$ constant,

$$\int_0^{\infty} \frac{e^{-x}-e^{-tx}}{x}\,dx = \ln(t),$$

$$\int_0^{1} \frac{t^{a-1}-t^{b-1}}{\ln(t)}\,dx = \ln\left(\frac{a}{b}\right).$$

(**Example 2.5.1**. **Problem 2.2.28**.)

53. If $0 < a, b < \infty$ constants,

$$\int_0^{\infty} \frac{\arctan(bx)-\arctan(ax)}{x}\,dx = \frac{\pi}{2}\ln\left(\frac{b}{a}\right).$$

If $0 < t$ constant,

$$\int_0^{\infty} \frac{\arctan(tx)-\arctan(x)}{x}\,dx = \frac{\pi}{2}\ln(t).$$

(**2.2.15**, **Examples 2.5.2**.)

54. If $0 < a,\ b < \infty$ constants,

$$\int_0^{\infty} \frac{\sin(bx)-\sin(ax)}{x}\,dx = 0.$$

(**Example 2.5.5**. **Problem 2.5.18**.)

55. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\cos(bx) - \cos(ax)}{x}\, dx = \ln\left(\frac{a}{b}\right).$$

If $0 < t$ constant,

$$\int_0^\infty \frac{\cos(x) - \cos(tx)}{x}\, dx = \ln(t).$$

(**Example 2.5.6**. **Problem 2.5.18**.)

56. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \ln\left|\frac{\cos(bx)}{\cos(ax)}\right| \frac{1}{x}\, dx = \ln(2)\ln\left(\frac{a}{b}\right).$$

(**Example 2.5.7**.)

57. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\dfrac{1}{1+(bx)^2} - \dfrac{1}{1+(ax)^2}}{x}\, dx = \ln\left(\frac{a}{b}\right).$$

(**Problem 2.5.2**.)

58. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \left[b \sin\left(\frac{1}{bx}\right) - a \sin\left(\frac{1}{ax}\right)\right] dx = \ln\left(\frac{b}{a}\right).$$

(**Problem 2.5.4 (a)**.)

59. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{|\sin(bx)| - |\sin(ax)|}{x}\, dx = \frac{2}{\pi}\ln\left(\frac{b}{a}\right).$$

(**Problem 2.5.9**.)

60. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{|\cos(bx)| - |\cos(ax)|}{x}\, dx = \left(\frac{2}{\pi} - 1\right)\ln\left(\frac{b}{a}\right).$$

(**Problem 2.5.9**.)

61. If $0 < \alpha < 1$ and $a,\ b > 0$ constants,

$$\int_0^\infty \frac{|\tan(bx)|^\alpha - |\tan(ax)|^\alpha}{x}\,dx = \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)}\ln\left(\frac{b}{a}\right).$$

(**Problem 2.5.10**.)

62. If $\alpha < 1$ and $a,\ b > 0$ constants,

$$\int_0^\infty \frac{|\sec(bx)|^\alpha - |\sec(ax)|^\alpha}{x}\,dx = \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)} - 1\right]\ln\left(\frac{b}{a}\right).$$

(**Problem 2.5.11**.)

63. If $0 \le \alpha < 1$ and $a,\ b > 0$ constants,

$$\int_0^\infty \frac{|\sec(bx)|^\alpha - |\sec(ax)|^\alpha}{x}\,dx.$$

(**Problem 2.5.11**.)

64. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\frac{\sin^2(bx)}{bx} - \frac{\sin^2(ax)}{ax}}{x}\,dx = 0.$$

(**Problem 2.5.13**.)

65.

$$\int_0^\infty \frac{\sin^3(x)}{x^2}\,dx = \frac{3}{4}\ln(3).$$

(**Problems 2.5.14**, **3.7.74**.)

66.

$$\int_0^\infty \frac{\sin^4(x)}{x^3}\,dx = \ln(2).$$

(**Problems 2.5.14**, **3.7.74**.)

67. If $\sum_{i=1}^{k} A_i = 0$ for real constants $A_i,\ \ i = 1,\ 2,\ 3,\ \ldots,\ k$ and $a_i > 0,\ \ i = 1,\ 2,\ 3,\ \ldots,\ k$,

$$\int_0^\infty \frac{A_1\cos(a_1x) + A_2\cos(a_2x) + A_3\cos(a_3x) + ... + A_k\cos(a_kx)}{x}\,dx =$$

$$-A_1\ln(a_1) - A_2\ln(a_2) - A_3\ln(a_3) - \ldots - A_k\ln(a_k).$$

(**Problem 2.5.15**.)

68. If $\beta \geq 0$ and $a,\ b$ real constants,

$$\int_0^\infty e^{-\beta x}\,\frac{\cos(ax)-\cos(bx)}{x}\,dx = \frac{1}{2}\ln\left(\frac{\beta^2+b^2}{\beta^2+a^2}\right).$$

(**Problems 2.4.8, 2.4.9, 2.5.18 (a)**. **Example 2.2.9**.)

69. If $\beta \geq 0$ and $a,\ b$ real constants,

$$\int_0^\infty e^{-\beta x}\,\frac{\sin(ax)\pm\sin(bx)}{x}\,dx = \arctan\left(\frac{a}{\beta}\right) \pm \arctan\left(\frac{b}{\beta}\right).$$

(**Problems 2.4.8, 2.4.9, 2.5.18 (b)**. **Example 2.2.9**.)

70. If $a,\ b$ real constants,

$$\int_{-\infty}^\infty \frac{\cos(bx)-\cos(ax)}{x^2}\,dx = 2\int_0^\infty \frac{\cos(bx)-\cos(ax)}{x^2}\,dx = (|a|-|b|)\pi.$$

(**Problem 2.5.21 (a)**.)

71. If $p>0$ constant,

$$\int_0^\infty x^{p-1}e^{-x}\,dx = \Gamma(p).$$

(**Section 2.6**. **Subsection 2.6.1**.)

72.

$$\int_0^\infty \ln(x)\,e^{-x}\,dx = \Gamma'(1) = -\gamma < 0,$$

$$\int_0^\infty x\ln(x)\,e^{-x}\,dx = \Gamma\,'(2) = 1-\gamma > 0,$$

$$\int_0^\infty \ln^2(x)\,e^{-x}\,dx = \Gamma\,''(1) = \frac{\pi^2}{6}+\gamma^2,$$

where $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^n \frac{1}{k} - \ln(n)\right] \simeq 0.57721566... > 0$ is the **Euler-Mascheroni constant**.
(**Subsection 2.6.1 (Γ, 2)**. **Problems 2.1.25, 2.3.5**, **2.6.44, 2.7.17 (a)**, **(c)**, **3.7.47**.)

73. If $p>0$ constant,

$$\int_0^\infty x^p e^{-x}\,dx = \Gamma(p+1) = p\,\Gamma(p).$$

If $n \geq 0$ integer constant,

$$\int_0^{\infty} x^n e^{-x}\, dx = \Gamma(n+1) = n!\,.$$

(**Subsection 2.6.1 ($\boldsymbol{\Gamma}$, 8)** and **($\boldsymbol{\Gamma}$, 9)**.)

74. If $p > 0$ constant,

$$\int_0^{\infty} u^{p-1} e^{-xu} du \ = \frac{\Gamma(p)}{x^p}.$$

(**Subsection 2.6.1 ($\boldsymbol{\Gamma}$, 11)**.)

75. If $0 < p < 2$ constant,

$$\int_0^{\infty} \frac{\sin(x)}{x^p}\, dx = \frac{\pi}{2\Gamma(p)\sin(\frac{p\pi}{2})}.$$

(**Example 2.6.7**. **Problems 2.2.22**, **2.2.23, 2.6.38, 3.7.14**.)

76. If $0 < p < 1$ constant,

$$\int_0^{\infty} \frac{\cos(x)}{x^p}\, dx = \frac{\pi}{2\Gamma(p)\cos(\frac{p\pi}{2})}.$$

(**Example 2.6.7**. **Problems 2.2.22**, **2.2.23**, **2.6.8, 2.6.38, 3.7.14**.)

77. If $\beta \neq 0$ and $-1 < \dfrac{\alpha+1}{\beta} < 1$ constants,

$$\int_0^{\infty} x^{\alpha} \sin(x^{\beta})\, dx = \frac{1}{|\beta|} \cdot \frac{\pi}{2 \cdot \Gamma\left(1 - \dfrac{\alpha+1}{\beta}\right) \cdot \sin\left[\left(1 - \dfrac{\alpha+1}{\beta}\right)\dfrac{\pi}{2}\right]}.$$

(**Example 2.6.8**.)

78. If $\beta \neq 0$ and $0 < \dfrac{\alpha+1}{\beta} < 1$ constants,

$$\int_0^{\infty} x^{\alpha} \cos(x^{\beta})\, dx = \frac{1}{|\beta|} \cdot \frac{\pi}{2 \cdot \Gamma\left(1 - \dfrac{\alpha+1}{\beta}\right) \cdot \cos\left[\left(1 - \dfrac{\alpha+1}{\beta}\right)\dfrac{\pi}{2}\right]}.$$

(**Example 2.6.8**.)

79. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = B(p,q).$$

(**Subsection 2.6.2**.)

80. If $p > 0$ and $q > 0$ constants,

$$\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du == \int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}}\,du = \frac{1}{2}\int_0^\infty \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = B(p,q).$$

(**Subsection 2.6.2 (B, 5)**.)

81. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = \int_1^\infty \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = B(p,q).$$

(**Subsection 2.6.2 (B, 5)**.)

82. If $p > 0$ and $q > 0$ constants,

$$\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta = \frac{B(p,q)}{2}.$$

(**Subsection 2.6.2 (B, 6)**.)

83. If $p > 0$ and $q > 0$ constants,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(**Subsection 2.6.2 (B, 7)**.)

84. If $m \geq 0$ and $n \geq 0$ integer constants,

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx = \frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\binom{m+n}{m}} = \frac{1}{(m+n+1)\binom{m+n}{n}}.$$

(**Subsection 2.6.2 (B, 7)**.)

85. If $p > 0$ constant,

$$\forall\ p:\quad 0 < p < 1,\quad B(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

(**Subsection 2.6.2 (B, 8)**.)

86. If $p > 0$, $q > 0$, $r > 0$ constants,

$$\int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}}\, du = \frac{1}{(r+1)^p \cdot r^q}\, B(p, q).$$

(**Subsection 2.6.2 (B, 9)**.)

87. If $p > 0$ constant,

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right) = \frac{2^{2p-\frac{1}{2}}}{\sqrt{2\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right).$$

(**Subsection 2.6.2 (B, 10)**.)

88. If $p > 0$ constant,

$$\int_0^{\pi/2} \sin^{2p-1}(\theta)\, d\theta = \frac{1}{2} B\left(p, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)}.$$

(**Example 2.6.15**.)

89. If $q > 0$ constant,

$$\int_0^{\pi/2} \cos^{2q-1}(\theta)\, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)}.$$

(**Example 2.6.15**.)

90. If $m \geq 0$ and $n \geq 0$ integers,

$$\int_0^{\pi/2} \sin^{2m}(x) \cos^{2n}(x)\, dx = \frac{1}{2} \cdot B\left(m + \frac{1}{2}, n + \frac{1}{2}\right) =$$

$$\frac{1}{2} \cdot \frac{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{\Gamma(m + n + 1)} = \frac{\pi}{2^{2m+2n+1}} \cdot \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

(**Examples 2.6.16**, **3.8.4**, **3.8.5**. **Problem 3.8.3**.)

91. If $p > 0$ and $q > 0$ real constants,

$$\int_0^{\pi/2} \sin^{2p-1}(\theta) \cos^{2q-1}(\theta)\, d\theta = \frac{1}{2} B(p, q) = \frac{1}{2} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(**Example 2.6.17**.)

92. If $a > 0$, $b > 0$, $p > 0$, $q > 0$ constants,

$$\int_0^{\pi/2} \frac{\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)}{\left[a\,\sin^2(\theta) + b\,\cos^2(\theta)\right]^{p+q}}\,d\theta = \frac{B(p,q)}{2\,a^p \cdot b^q}.$$

(**Example 2.6.18**.)

93.

$$K\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2(\phi)}} = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}.$$

(**Subsection 2.6.3, Application 1**.)

94. If $p > 0$ constant,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = \int_0^1 [-\ln(u)]^{p-1}du = \int_0^1 |\ln(u)|^{p-1}du.$$

(**Problems 2.1.15, 2.6.4**, **2.6.7**, **3.7.47**.)

95. If $\alpha > -1$, $\beta > 0$ and $c > 0$ constants,

$$\int_0^\infty x^\alpha e^{-\beta x^c}\,dx = \frac{1}{c\beta^{\frac{\alpha+1}{c}}}\Gamma\left(\frac{\alpha+1}{c}\right).$$

(**Problem 2.6.5**.)

96. If $p \geq 0$ constant,

$$\int_0^\infty \frac{x^{p-1}}{e^x - 1}\,dx = \int_1^\infty \frac{\ln^{p-1}(u)}{(u-1)u}\,du =$$
$$\int_0^1 \frac{[-\ln(v)]^{p-1}}{1-v}\,dv = \Gamma(p)\sum_{n=1}^\infty \frac{1}{n^p}.$$

(**Problem 2.6.6**.)

97. If $n = 0, 1, 2, 3, ...$ integers and $\alpha > -1$ constant,

$$\int_0^1 x^\alpha[\ln(x)]^n\,dx = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}.$$

(**Problems 2.1.15, 2.6.4, 2.6.7**.)

98. If $n = 0,\ 1,\ 2,\ 3,\ ...$ integer,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{4^n n!} = \frac{1\cdot 3\cdot 5\cdots(2n-1)\,\sqrt{\pi}}{2^n}.$$

(**Problem 2.6.14 (a)**.)

99. If $m = 1,\ 2,\ 3,\ 4,\ ...$ integer,

$$\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m\, 2^m\, \sqrt{\pi}}{1\cdot 3\cdot 5\cdots (2m-1)} = \frac{(-1)^m\, 4^m\, m!\,\sqrt{\pi}}{(2m)!}.$$

(**Problem 2.6.14 (b)**.)

100. If $-1 \le \alpha \le 1$ constant,

$$\int_0^{\pi} |\tan(x)|^{\alpha}\, dx = 2\int_0^{\pi/2} \tan^{\alpha}(x)\, dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}$$

$$\int_0^{\pi} |\cot(x)|^{\alpha}\, dx = 2\int_0^{\pi/2} \cot^{\alpha}(x)\, dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}.$$

(**Problem 2.6.18**.)

101. For all $\alpha \le 1$ constant,

$$\int_0^{\pi} |\sec(x)|^{\alpha}\, dx = 2\int_0^{\pi/2} \sec^{\alpha}(x)\, dx = \sqrt{\pi}\cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}$$

$$\int_0^{\pi} |\csc(x)|^{\alpha}\, dx = 2\int_0^{\pi/2} \csc^{\alpha}(x)\, dx = \sqrt{\pi}\cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}.$$

(**Problem 2.6.19**.)

102. If $b > 0$ constant,

$$\int_0^{b} \frac{dx}{\sqrt{b^4 - x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{2\pi}}.$$

(**Problem 2.6.23**.)

103. If $b > 0$ constant,

$$\int_0^{\infty} \frac{dx}{\sqrt{b^4 + x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{\pi}}.$$

(**Problem 2.6.24**.)

104.

$$\int_0^{\pi/2} \sin^p(\phi)\,d\phi = \int_0^{\pi/2} \cos^p(\phi)\,d\phi =$$

$$\begin{cases} \dfrac{\pi}{2}, & \text{if } p = 0 \\[2ex] 1, & \text{if } p = 1 \\[2ex] \dfrac{1 \cdot 3 \cdot 5 \ldots (p-1)}{2 \cdot 4 \cdot 6 \ldots p} \cdot \dfrac{\pi}{2} = \dfrac{p\,!}{2^p \left[\left(\frac{p}{2}\right)!\right]^2} \cdot \dfrac{\pi}{2} = \dfrac{\binom{p}{\frac{p}{2}}}{2^p} \cdot \dfrac{\pi}{2}, & \\ \qquad\qquad\qquad\qquad \text{if } p = \text{even integer} \geq 2 & \\[2ex] \dfrac{2 \cdot 4 \cdot 6 \ldots (p-1)}{1 \cdot 3 \cdot 5 \ldots p} = \dfrac{2^{p-1}\left[\left(\frac{p-1}{2}\right)!\right]^2}{p\,!} = \dfrac{2^{p-1}}{p\binom{p-1}{\frac{p-1}{2}}}, & \\ \qquad\qquad\qquad\qquad \text{if } p = \text{odd integer} \geq 3. & \end{cases}$$

(**Problem 2.6.25**.)

105. If $a > 0$, $p > 0$ and $q > 0$ constants,

$$\int_0^{\frac{1}{a}} x^{p-1}\,(1 - ax)^{q-1}\,dx = \frac{1}{a^p}\,B(p,q).$$

(**Problem 2.6.27**.)

106. If $p > 0$ and $q > 0$ constants,

$$\int_0^{\infty} e^{-p\,t}\left(1 - e^{-t}\right)^{q-1} dt = B(p,q).$$

(**Problem 2.6.28**.)

107. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 u^{2p-1}\left(1 - u^2\right)^{q-1} du = \frac{B(p,q)}{2}.$$

(**Problem 2.6.29**.)

108. If $b > a$, $p > -1$ and $q > -1$ constants,

$$\int_a^b (x-a)^p (b-x)^q dx = (b-a)^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.$$

(**Problem 2.6.30**.)

109. If $a > b > 0$ constants,

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(x)]^{2a}}\,dx =$$

$$4^{a-1}\,B(a+b,\,a-b) = 4^{a-1}\,\frac{\Gamma(a+b)\,\Gamma(a-b)}{\Gamma(2a)}.$$

(**Problem 2.6.31 (a)**.)

110. If $a > 0$ constant,

$$\int_0^\infty \frac{1}{[\cosh(x)]^{2a}}\,dx = 4^{a-1}\,\frac{[\Gamma(a)]^2}{\Gamma(2a)}.$$

(**Problem 2.6.31 (b)**.)

111. If $r > 0$, $b > 0$ and $a > \frac{b}{r}$ constants,

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(rx)]^{2a}}\,dx =$$

$$\frac{4^{a-1}}{r}\,B\left(a+\frac{b}{r},\,a-\frac{b}{r}\right) = \frac{4^{a-1}}{r}\,\frac{\Gamma\left(a+\frac{b}{r}\right)\,\Gamma\left(a-\frac{b}{r}\right)}{\Gamma(2a)}.$$

(**Problem 2.6.31 (d)**.)

112. If $a > b > -1$ constants,

$$\int_0^\infty \frac{\sinh^b(x)}{\cosh^a(x)}\,dx = \frac{1}{2}B\left(\frac{b+1}{2},\,\frac{a-b}{2}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{b+1}{2}\right)\,\Gamma\left(\frac{a-b}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}.$$

(**Problem 2.6.32**.)

113. If $p > 0$ constant,

$$\Gamma(3p) = \frac{3^{3p-1/2}}{2\pi}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{3}\right)\,\Gamma\left(p+\frac{2}{3}\right).$$

(**Problem 2.6.33**.)

114. If $p > 0$ constant and $n \geq 1$ integer,

$$\Gamma(np) = \frac{n^{np-1/2}}{(2\pi)^{\frac{n-1}{2}}}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{n}\right)\Gamma\left(p+\frac{2}{n}\right)\ldots\Gamma\left(p+\frac{n-1}{n}\right).$$

(**Problems 2.6.33 footnote**, **2.6.42 Item (11.)**.)

115.

$$\int_0^1 \ln[\Gamma(x)]dx = \frac{1}{2}\ln(2\pi) = \ln\left(\sqrt{2\pi}\right).$$

$$\int_1^2 \ln[\Gamma(x)]\,dx = -1 + \ln\left(\sqrt{2\pi}\right) = -0.0810615\ldots.$$

(**Problem 2.6.37 (b)**, **(d)**.)

116. If $0 < m < 1$ constant,

$$\text{(a)} \quad \int_0^\infty \frac{\cos(x)}{x^{1-m}}dx = \Gamma(m)\cos\left(\frac{m\pi}{2}\right),$$

$$\text{(b)} \quad \int_0^\infty \frac{\sin(x)}{x^{1-m}}dx = \Gamma(m)\sin\left(\frac{m\pi}{2}\right).$$

(**Problems 2.6.38**, **3.7.14**.)

117. If $s > 0$ constant,

$$\int_0^\infty e^{-xs}\ln(x)dx = \frac{-[\gamma + \ln(s)]}{s},$$

and

$$\int_0^\infty e^{-xs}\ln^2(x)dx = \frac{\pi^2}{6s} + \frac{[\gamma + \ln(s)]^2}{s},$$

where $\gamma = \lim\limits_{n\to\infty}\left[\sum_{k=1}^n \frac{1}{k} - \ln(n)\right] \simeq 0.57721566... > 0$ is the **Euler-Mascheroni constant**.
(**Problem 2.7.17 (a)**, **(c)**.)

118.

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\cos\left[\theta - 3\pi\sin(\theta)\right] - \cos(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right] + e^{6\pi\cos(\theta)}}\,d\theta = 2.$$

(**Example 3.7.2**.)

119.

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\sin\left[\theta - 3\pi\sin(\theta)\right] - \sin(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right] + e^{6\pi\cos(\theta)}}\,d\theta = 0.$$

(**Example 3.7.2**.)

120. If $-1 < \alpha < 0$ and $b > 0$ constants,

$$\int_0^\infty \frac{x^\alpha}{b+x}\,dx = \frac{b^\alpha\pi}{\sin(-\alpha\pi)}.$$

(**Example 3.7.8 Remark (4)**.)

121. If $|a| > 1$ real constant,

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{a+\sin(\theta)} = \frac{\text{sign}(a)\cdot\pi}{\sqrt{a^2-1}}.$$

(**Examples 3.7.13**, **3.8.1 Remark 2**.)

122.

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx = \frac{4}{3}\sin\left(\frac{\pi}{9}\right).$$

(**Example 3.7.15**. **Problem 3.7.98**.)

123.

$$2\int_0^\infty e^{-x^2}\cos(4x)dx = 2\int_{-\infty}^0 e^{-x^2}\cos(4x)dx =$$
$$\int_{-\infty}^\infty e^{-x^2}\cos(4x)dx = e^{-4}\sqrt{\pi}.$$

(**Examples 2.2.13, 3.7.17**. **Problem 3.7.24**.)

124.

$$\int_0^\infty e^{-x^2}\sin(4x)dx = e^{-4}\int_0^2 e^{y^2}dy =$$
$$\frac{\sqrt{\pi}}{2e^4}\,\text{erf}(2) \simeq 0.3013403889237924\ldots.$$

(**Example 3.7.17 Remark 3**.)

125. If $n = 0,\ 1,\ 2,\ \ldots$, integer,

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^{n+1}} = \frac{\pi(2n)!}{(n!)^2 2^{2n}}.$$

(**Problem 3.7.10**.)

126. If $\beta \in \mathbb{R}$ and $c > 0$ constants,

$$\int_{-\infty}^\infty e^{-cx^2}\sin(\beta x^2)\,dx =$$
$$2\int_0^\infty e^{-cx^2}\sin(\beta x^2)\,dx = \int_0^\infty e^{-cu}\frac{\sin(\beta u)}{\sqrt{u}}\,du =$$
$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\beta\,dv}{\beta^2+(c+v^2)^2} = \text{sign}(\beta)\sqrt{\frac{\pi}{2}}\sqrt{\frac{-c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

(**Problem 3.7.13 (a)**. **Example 2.4.1 Remark 3**.)

127. If $\beta \in \mathbb{R}$ and $c > 0$ constants,

$$\int_{-\infty}^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx =$$

$$2\int_{0}^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx = \int_0^{\infty} e^{-cu}\, \frac{\cos(\beta u)}{\sqrt{u}}\, du =$$

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\left(c+v^2\right)\, dv}{\beta^2 + \left(c+v^2\right)^2} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

(**Problem 3.7.13 (b)**.)

128.

$$\int_0^{\infty} \frac{\sqrt{x}}{x^3+x^2+x+1}\, dx = \frac{\pi}{2}\,(\sqrt{2}-1).$$

(**Problem 3.7.16**.)

129.

$$\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+4x+8}\, dx = \frac{\pi\sqrt{2}}{2} \cdot \frac{\sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\pi\sqrt{6}}{6}\sqrt{2-\sqrt{3}}.$$

(**Problem 3.7.17**.)

130. If $-1 < a < 1$, $a \neq 0$ and $b > 0$ constants,

$$\int_0^{\infty} \frac{x^a}{x^2+b^2}\, dx = \frac{\pi\, b^{a-1}}{2\cos\left(\dfrac{\pi a}{2}\right)}.$$

(**Problem 3.7.18 (a)**.)

131. If $-1 < a < 2$ and $a \neq 0$ or 1 constant

$$\int_0^{\infty} \frac{x^a}{x^3+1}\, dx = \frac{-\pi\left\{1+2\cos\left[\dfrac{2\pi(a+1)}{3}\right]\right\}}{3\sin(\pi a)}.$$

(**Problem 3.7.18 (b)**.)

132. If $-1 < a < 1$ and $0 < \theta < \pi$ constants,

$$\int_0^{\infty} \frac{x^a}{x^2+2x\cos(\theta)+1}\, dx = \frac{\pi\sin(a\theta)}{\sin(a\pi)\sin\theta}.$$

(**Problem 3.7.19**.)

133. If $0 < a \leq b$ and $0 < c < 2$ constants,

$$\int_a^b \left(\frac{b-x}{x-a}\right)^{c-1} \frac{dx}{x} = \frac{\pi}{\sin(c\pi)} \left[1 - \left(\frac{b}{a}\right)^{c-1}\right].$$

(**Problem 3.7.20**.)

134. If $a \neq 0$ and b real constants,

$$\int_{-\infty}^{0} e^{-a^2x^2} \cos(bx)\, dx = \int_0^\infty e^{-a^2x^2} \cos(bx)\, dx =$$
$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-a^2x^2} \cos(bx)\, dx = \frac{\sqrt{\pi}}{2|a|} e^{\frac{-b^2}{4a^2}}.$$

(**Problem 3.7.24**. **Examples 2.2.13**, **3.7.17**.)

135. If $a \neq 0$ and b real constants,

$$\int_0^\infty e^{-a^2x^2} \sin(bx) dx = \frac{1}{2a^2} e^{-\frac{b^2}{4a^2}} \int_0^b e^{\frac{t^2}{4a^2}}\, dt.$$

(**Problem 3.7.25**.)

136.

$$\int_{-\infty}^{0} \frac{dx}{(1+x^2)\cosh(x)} = \int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} =$$
$$\frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n - \frac{1}{2}\right)^2 \pi^2 - 1}.$$

(**Example 3.7.18**. **Lemma 3.7.3**.)

137.

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^4)\sinh(x)}\, dx = \int_{-\infty}^{\infty} \frac{x}{1+x^4}\, \operatorname{csch}(x)\, dx =$$
$$\frac{\pi \sinh\left(\frac{1}{\sqrt{2}}\right) \cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right) + \sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^4\pi^4 + 1}.$$

(**Problem 3.7.28**.)

138. If $-\pi < a < \pi$ constant,

$$\int_{-\infty}^{\infty} \frac{\sinh(ax)}{\sinh(\pi x)}\, dx = \tan\left(\frac{a}{2}\right).$$

(**Problem 3.7.30**.)

139. If $a \in \mathbb{C}$ such that $|\text{Im}(a)| < 2\pi$ constant,

$$\int_0^\infty \frac{\sin(ax)}{e^{2\pi x} - 1}\, dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}.$$

(**Problem 3.7.32 (a)**.)

140. If α and β in $\mathbb{C}$, such that $\text{Re}(\beta) > 0$ and $|\text{Im}(\alpha)| < \text{Re}(\beta)$ constants,

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x} - 1}\, dx = \frac{\pi}{2\beta}\coth\left(\frac{\alpha\pi}{\beta}\right) - \frac{1}{2a}.$$

(**Problem 3.7.32 (b)**. **Example 3.7.24 Corollary 3.7.5 (A)**.)

141. If α and β in $\mathbb{C}$, such that $\text{Re}(\beta) > 0$ and $|\text{Im}(\alpha)| < \text{Re}(\beta)$ constants,

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x} + 1}\, dx = -\frac{\pi}{2}\,\text{csch}\left(\frac{\alpha\pi}{\beta}\right) + \frac{1}{2\alpha}.$$

(**Problem 3.7.32 (d)**. **Example 3.7.24 Corollary 3.7.5 (B)**.)

142. If a and b in $\mathbb{C}$, such that $\text{Re}(b) > 0$ and $|\text{Re}(a)| < \text{Re}(b)$ constants,

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx} - 1}\, dx = -\frac{\pi}{2b}\cot\left(\frac{a\pi}{b}\right) + \frac{1}{2a}.$$

(**Problem 3.7.33 (a)**.)

143. If a and b in $\mathbb{C}$, such that $\text{Re}(b) > 0$ and $|\text{Re}(a)| < \text{Re}(b)$ constants,

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx} + 1}\, dx = \frac{\pi}{2b}\csc\left(\frac{a\pi}{b}\right) - \frac{1}{2a}.$$

(**Problem 3.7.33 (b)**.)

144. If $a > 0$, $\mu \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ constants,

$$P.V. \int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx =$$

$$e^{\frac{b^2}{a^2}} \int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2 + b^2} \left\{x \cos\left[\frac{2b(x-\mu)}{a^2}\right] - b \sin\left[\frac{2b(x-\mu)}{a^2}\right]\right\} dx =$$

$$e^{b^2} \int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2 + (ab)^2} \left\{x \cos\left[\frac{2b(x-\mu)}{a}\right] - ab \sin\left[\frac{2b(x-\mu)}{a}\right]\right\} dx =$$

$$e \int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2 + a^2} \left\{x \cos\left[\frac{2(x-\mu)}{a}\right] - a \sin\left[\frac{2(x-\mu)}{a}\right]\right\} dx.$$

(**Problem 3.7.35 (b), (d)**.)

145. If $a > 0$, $\mu \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ constants,

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2} \left\{ b \cos\left[\frac{2b(x-\mu)}{a^2}\right] + x \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx = \operatorname{sign}(b)\, \pi e^{-\frac{\mu^2+b^2}{a^2}}.$$

(**Problem 3.7.35 (b)**.)

146. If a, b and c such that $\operatorname{Re}(b) > 0$, $|\operatorname{Im}(a)| < \operatorname{Re}(b)$ and $|c| \leq 1$ complex constants,

$$\int_0^{\infty} \frac{\sin(ax)}{e^{bx}+c}\, dx = \sum_{n=0}^{\infty} \frac{a(-c)^n}{a^2+b^2(n+1)^2} = \frac{a}{b^2} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.24**.)

147. If b and c such that $\operatorname{Re}(b) > 0$ and $|c| \leq 1$ complex constants,

$$\int_0^{\infty} \frac{x}{e^{bx}+c}\, dx = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^2},$$

$$\int_0^{\infty} \frac{x}{e^{bx}-1}\, dx = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{b^2} \frac{\pi^2}{6},$$

$$\int_0^{\infty} \frac{x}{e^{bx}+1}\, dx = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{b^2} . \frac{\pi^2}{12}.$$

(**Example 3.7.24 Corollary 3.7.5 (E)**.)

148. If b and c such that $\operatorname{Re}(b) > 0$ and $|c| \leq 1$ complex constants,

$$\int_1^{\infty} \frac{\ln(u)}{(u^b+c)u}\, du = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^2},$$

$$\int_1^{\infty} \frac{u}{(u^b-1)u}\, du = \frac{1}{b^2} \frac{\pi^2}{6},$$

$$\int_1^{\infty} \frac{\ln(u)}{(u^b+1)u}\, du = \frac{1}{b^2} \frac{\pi^2}{12}.$$

(**Example 3.7.24 Corollary 3.7.5 (F)**.)

149. If b and c such that $\mathrm{Re}(b) > 0$ and $|c| \leq 1$ complex constants,

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{c+x^b}\,dx = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1-x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1+x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12}.$$

(**Example 3.7.24 Corollary 3.7.5 (G)**.)

150. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| \leq 1$ complex constants,

$$\int_0^{\infty} \frac{\sinh(ax)}{e^{bx}+c}\,dx = \sum_{n=0}^{\infty}\frac{a(-c)^n}{-a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.24 Corollary 3.7.5 (I)**. **Problem 3.7.59**.)

151. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^{\infty} \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.25**.)

152.

$$\int_0^{\infty} \frac{\cos(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2},$$

$$\int_0^{\infty} \frac{\cos(x)}{e^{x}+1}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+1}.$$

(**Example 3.7.25**.)

153. If b such that $\mathrm{Re}(b) > 0$ complex constant,

$$\int_0^{\infty} \frac{1}{e^{bx}+1}\,dx = \int_1^{\infty}\frac{1}{u(u^b+1)}\,du = \frac{\ln(2)}{b}.$$

(**Example 3.7.25**.)

154. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{x\sin(ax)}{e^{bx}+c}\,dx = \frac{2a}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{\left[n^2+\left(\frac{a}{b}\right)^2\right]^2}.$$

(**Example 3.7.25**.)

155. If b and c such that $\mathrm{Re}(b) > 0$, and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{x^2}{e^{bx}+c}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^3},$$

$$\int_0^\infty \frac{x^2}{e^{bx}+1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_0^\infty \frac{x^2}{e^{bx}-1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 3.7.25**.)

156. If $b > 0$ constant,

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b+1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b-1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 3.7.25**.)

157. If $b > 0$ constant,

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1+x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1-x^b}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 3.7.25**.)

158. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.25**.)

159. If $a \in \mathbb{R}$, $b > 0$ and $c \geq 1$ real constants,

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx = \frac{1-\cos\left[\frac{a\ln(c)}{b}\right]}{ac} + \frac{a}{c}\sum_{n=1}^\infty \frac{(-1)^n}{(a^2+b^2n^2)c^n} + \frac{2a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=1}^\infty \frac{(-1)^{n-1}}{a^2+b^2n^2}.$$

(**Example 3.7.26**.)

160. If $b > 0$ and $c \geq 1$ real constants,

$$\int_0^\infty \frac{x}{e^{bx}+c}\,dx = \frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^\infty \frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\frac{\pi^2}{12}.$$

(**Example 3.7.26**.)

161.

$$\int_1^\infty \frac{\ln(u)}{(u+1)u}\,du = \frac{\pi^2}{12} \qquad \int_1^\infty \frac{\ln(u)}{(u-1)u}\,du = \frac{\pi^2}{6}$$
$$\int_1^\infty \frac{\ln(u)}{(u^2+1)u}\,du = \frac{\pi^2}{48} \qquad \int_1^\infty \frac{\ln(u)}{(u^2-1)u}\,du = \frac{\pi^2}{24}$$
$$\int_1^\infty \frac{\ln(u)}{u^2-1}\,du = \frac{\pi^2}{8}.$$

(**Problem 3.7.44**.)

162.

$$\int_0^1 \frac{\ln(v)}{1+v}\,dv = \frac{-\pi^2}{12} \qquad \int_0^1 \frac{\ln(v)}{1-v}\,dv = \frac{-\pi^2}{6}$$
$$\int_0^1 \frac{v\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{24} \qquad \int_0^1 \frac{\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{8}.$$

(**Problem 3.7.45**.)

163.

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12} \qquad \int_0^1 \frac{\ln(1-x)}{x}\,dx = \frac{-\pi^2}{6}$$
$$\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{\pi^2}{4} \qquad \int_0^1 \frac{x\ln(x)}{1-x}\,dx = 1-\frac{\pi^2}{6}$$

$$\int_0^1 \ln(x)\ln(1-x)\,dx = 2 - \frac{\pi^2}{6}$$

$$\int_0^1 \ln(x)\ln(1+x)\,dx = 2 - 2\ln(2) - \frac{\pi^2}{12}.$$

(**Problem 3.7.46**.)

164. If $a > 0$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}.$$

If $b > 0$ and $a \in \mathbb{R}$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-|a|b},$$

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx = 0.$$

(**Example 3.7.27 Remark**.)

165. If $a \in \mathbb{R}$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = 2\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx =$$

$$2\int_{-\infty}^{0} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \operatorname{sign}(a)\frac{\pi}{b^2}\cdot\left(1 - e^{-|a|b}\right).$$

(**Example 3.7.28**.)

166. If $a \geq 0$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{\left(x^2+b^2\right)^2}\,dx = \frac{\pi}{2b^3}e^{-ab}(1+ab).$$

(**Example 3.7.29**.)

167. If $a > 0$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{x\sin(ax)}{x^2+b^2}\,dx =$$

$$2\int_{-\infty}^{0} \frac{x\sin(ax)}{x^2+b^2}\,dx = \frac{\pi}{e^{ab}}.$$

(**Example 3.7.32 Remark 1**.)

168.

$$\int_{-\infty}^{\infty}\left\{\frac{2[\cos(t)-1]}{t^2}\right\}^2 dt = 2\pi\int_{-\infty}^{\infty} f^2(x)dx =$$
$$2\pi\left[\int_{-1}^{0}(1+x)^2\,dx+\int_{0}^{1}(1-x)^2\,dx\right]=2\pi\left(\frac{1}{3}+\frac{1}{3}\right).$$

(**Example 3.7.39**.)

169. If $a \geq 0$ and $b \geq 0$ constants,

$$\int_{-\infty}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,dx = -\pi(a-b).$$

(**Problems 2.2.16, 3.7.69**. **Example 3.7.34**.)

170.

$$\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{x(1-x^2)}\,dx = 2\pi.$$

(**Problem 3.7.76**.)

171.

$$\int_{1}^{\infty}\frac{\ln(x)}{x^2+1}\,dx = \int_{0}^{1}\frac{-\ln(x)}{x^2+1}\,dx = \int_{0}^{1}\frac{\arctan(x)}{x}\,dx =$$
$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^2} = 8\sum_{n=0}^{\infty}\frac{2n+1}{(4n+1)^2(4n+3)^2} :=$$
$$\textbf{Catalan constant}\ \ G = 0.915965594\ldots.$$

(**Example 3.7.45**. **Problem 3.7.93**.)

172. If $-1 < \alpha < 0$,

$$\int_{0}^{\infty}\frac{x^{\alpha}\ln(x)}{x+1}\,dx = \pi^2\cot(\alpha\pi)\csc(\alpha\pi).$$

(**Examples 3.7.8, 3.7.44, 3.7.46**. **Problem 3.7.94**.)

173. If q and $r \neq 0$ such that $0 < \dfrac{q+1}{r} < 1$ real constants,

$$\int_{0}^{\infty}\frac{x^{q}\ln(x)}{x^{r}+1}\,dx = -\mathrm{sign}(r)\,\frac{1}{r^2}\,\pi^2\cot\left(\frac{q+1}{r}\pi\right)\csc\left(\frac{q+1}{r}\pi\right).$$

(**Examples 3.7.8, 3.7.44, 3.7.46**. **Problems 3.7.94, 3.7.96**.)

174. If $n = 1,\ 3,\ 5,\ \ldots$ odd positive integer,

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 0.$$

(**Problem 3.7.89 (b)**.)

175. If $a > 0$, $b > 0$ and $c > 0$ constants,

$$\int_0^\infty \frac{\ln(cx)}{b^2+(ax)^2}\,dx = \frac{\pi}{2ab}\ln\left(\frac{bc}{a}\right).$$

(**Problems 2.1.19, 2.1.24, 3.7.90**.)

176.

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx = 2\int_0^1 \frac{\ln^2(x)}{1+x^2}\,dx = 2\int_1^\infty \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

(**Problem 3.7.91**.)

177.

$$\int_{-\infty}^\infty \frac{u^2}{\cosh(u)}\,du = \int_{-\infty}^\infty u^2\operatorname{sech}(u)\,du = \frac{\pi^3}{4}.$$

(**Problem 3.7.91**.)

178.

$$\text{P.V.}\int_{-\infty}^\infty \frac{dx}{x^3+x^2+x+1} = \frac{\pi}{2},$$

$$\int_0^\infty \frac{dx}{x^3+x^2+x+1} = \frac{\pi}{4}.$$

(**Problem 3.7.99 (b)**, **(c)**.)

179.

$$\int_{-\infty}^\infty \frac{dx}{x^4+x^3+x^2+x+1} = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right).$$

(**Problem 3.7.100 (a)**.)

180. If $|a| > 1$ real constant,

$$\int_0^{2\pi} \frac{d\theta}{a+\sin(\theta)} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)} = \frac{\operatorname{sign}(a)\cdot 2\pi}{\sqrt{a^2-1}}.$$

(**Examples 3.7.13, 3.8.1 Remark 2**.)

181.

$$\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos(\theta) + 1} = \begin{cases} \dfrac{2\pi}{1-a^2} & \text{when} \quad |a| < 1 \\[2ex] \dfrac{2\pi}{a^2-1} & \text{when} \quad |a| > 1. \end{cases}$$

(**Example 3.8.2**.)

182. If $R > r > 0$ constants,

$$\int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos(\theta) + r^2} = \frac{2\pi}{R^2 - r^2}.$$

(**Example 3.8.2**. **Problem 3.8.1**.)

183. If $A > 0$ and $B > 0$ constants,

$$\text{(a)} \qquad \int_0^{2\pi} \frac{dx}{A + B\sin^2(x)} = \frac{2\pi}{\sqrt{A(A+B)}},$$

$$\text{(b)} \qquad \int_0^{2\pi} \frac{dx}{A\cos^2(x) + B} = \frac{2\pi}{\sqrt{(A+B)B}}.$$

(**Example 3.8.3 Remark 2**.)

184.

$$\int_0^{2\pi} e^{-inx}e^{imx}dx = \begin{cases} 0, & \text{if} \quad m \neq n, \\[2ex] 2\pi, & \text{if} \quad m = n. \end{cases}$$

(**Subsection 3.8.2**.)

185. If n integer,

$$\int_0^{2\pi} \cos^2(nx)dx = \begin{cases} 2\pi, & \text{if} \quad n = 0, \\[2ex] \pi, & \text{if} \quad n \neq 0. \end{cases}$$

(**Subsection 3.8.2**.)

186. If n integer,

$$\int_0^{2\pi} \sin^2(nx)dx = \begin{cases} 0, & \text{if} \quad n = 0, \\[2ex] \pi, & \text{if} \quad n \neq 0. \end{cases}$$

(**Subsection 3.8.2**.)

187. If $m \neq n$ integers,

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = 0.$$

(**Subsection 3.8.2**.)

188. If $m \neq n$ integers,

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0.$$

(**Subsection 3.8.2**.)

189. If $m \neq n$ integers,

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0.$$

(**Subsection 3.8.2**.)

190. If n integer,

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = 2\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \begin{cases} 0, & \text{if } n \text{ is even} \\ \\ 2\pi, & \text{if } n \text{ is odd.} \end{cases}$$

(**Example 3.8.6**.)

191. If $a \geq 1$ constant,

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\,dx = \pi \ln\left(1 + \frac{1}{a}\right).$$

(**Problem 3.8.2**.)

192. If $0 < a < 1$ constant,

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\,dx = \pi \ln(1 + a).$$

(**Problem 3.8.2**.)

193. If n non-negative integer,

$$\int_0^{\pi} \cos^n(x)\cos(nx)\,dx \;=\; \frac{\pi}{2^n}.$$

(**Problem 3.8.7** I_3.)

194. If n non-negative integer,

$$\int_0^{\pi} \sin^n(x)\sin(nx)\,dx \;=\; \frac{\pi}{2^n}\,\sin\left(\frac{n\pi}{2}\right).$$

(**Problem 3.8.7** I_4.)

4.2 List of Non-elementary Sums

Here, we cite a list of all major, non-elementary, general and important real or complex finite or infinite sums that have been evaluated in the text or referred to in problems and footnotes.

1. **Finite geometric sum** for any $z \in \mathbb{C}$,

$$\sum_{n=0}^{k} z^n = 1 + z + z^2 + z^3 + \ldots + z^k =$$

$$\begin{cases} \dfrac{z^{k+1}-1}{z-1} = \dfrac{1-z^{k+1}}{1-z}, & \text{if} \quad z \neq 1 \\ \\ k+1, & \text{if} \quad z = 1. \end{cases}$$

2.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$$

(**Example 1.3.10**. **Problem 3.7.58 (b)**.)

3. If $p \geq 0$ constant,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{x^{p-1}}{e^x - 1}\, dx =$$

$$\frac{1}{\Gamma(p)} \int_1^{\infty} \frac{\ln^{p-1}(u)}{(u-1)u}\, du = \frac{1}{\Gamma(p)} \int_0^{1} \frac{[-\ln(v)]^{p-1}}{1-v}\, dv.$$

(**Problems 2.6.6**, **3.7.62**.)

4.

$$\sum_{k=0}^{n} \frac{(-1)^k}{m+k+1}\, nCk = \sum_{l=0}^{m} \frac{(-1)^l}{n+l+1}\, mCl =$$

$$\frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\cdot(m+n)Cm} =$$

$$\frac{1}{(m+n+1)\cdot(m+n)Cn}.$$

(**Problem 2.6.16**.)

5. If $x \in \mathbb{R}$ and $p \neq 0$ constants, let $\left[\left[\frac{x}{p}\right]\right]$ be the **integer part** or **floor function** of $\frac{x}{p}$ and H_b or H the **Heaviside functions** defined in **Problem 2.7.16 (a)**,
$$\left[\left[\frac{x}{p}\right]\right] = \sum_{n=1}^{\infty} H_{pn}(x) = \sum_{n=1}^{\infty} H(x - pn).$$
(**Problem 2.7.16 (g)**.)

6. If $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \dots\ \omega_{k-1}$ are the k k$^{\text{th}}$ roots of 1,
$$\omega_0^j + \omega_1^j + \dots + \omega_{k-1}^j = \begin{cases} 0, & \text{if} \quad j \in \{1,\ 2,\ \dots\ k-1\} + k\mathbb{Z} \\ k, & \text{if} \quad j \in k\mathbb{Z}. \end{cases}$$
(**Problem 3.2.24**.)

7. If $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \dots\ \omega_{k-1}$ are the k k$^{\text{th}}$ roots of 1,
$$\begin{gathered} \omega_0 + \omega_1 + \dots + \omega_{k-1} = 0 \\ \omega_0\omega_1 + \omega_0\omega_2 + \dots + \omega_{k-2}\omega_{k-1} = 0 \\ \omega_0\omega_1\omega_2 + \dots + \omega_{k-3}\omega_{k-2}\omega_{k-1} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \omega_0\omega_1\dots\omega_{k-2} + \dots + \omega_1\omega_2\dots\omega_{k-1} = 0 \\ \omega_0\omega_1\dots\omega_{k-1} = (-1)^{k+1}. \end{gathered}$$
(**Problem 3.2.25**.)

8. If $n \geq 0$ integer and x real,
$$\cos(nx) = \sum_{k=0}^{\left[\left[\frac{n}{2}\right]\right]} (-1)^k \binom{n}{2k} \cos^{n-2k}(x) \sin^{2k}(x).$$
(**Problem 3.1.27 (II)**.)

9. If $n \geq 0$ integer and x real,
$$\sin(nx) = \sum_{k=0}^{\left[\left[\frac{n}{2}\right]\right]} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(x) \sin^{2k+1}(x).$$
(**Problem 3.1.27 (II)**.)

10. If $n \geq 0$ integer and x real,

$$\tan(nx) = \frac{\sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k+1} \tan^{2k+1}(x)}{\sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k} \tan^{2k}(x)}.$$

(**Problem 3.1.27 (II)**.)

11. If $n \geq 0$ integer and x real,

$$\sum_{k=0}^{n} \sin(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

(**Problem 3.2.11 (d)**.)

12. If $n \geq 0$ integer and x real,

$$\sum_{k=0}^{n} \cos(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \cos\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

(**Problem 3.2.11 (e)**.)

13. **Infinite geometric sum**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + ... + z^n + ...,$$

$$\forall\ z:\ |z| < 1.$$

(**Example 3.2.1**. **Subsection 3.5.1**.)

14.

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad \forall\ z:\ |z| < 1.$$

(**Example 3.2.1**.)

15.

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad \forall\ z:\ |z| < 1.$$

(**Example 3.2.1**.)

16.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} + ...,$$

$$\forall\ z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

17.

$$\cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ... + (-1)^n \frac{z^{2n}}{(2n)!} + ...,$$

$$\forall\ z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

18.

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...+$$

$$(-1)^n \frac{z^{2n+1}}{(2n+1)!} + ..., \quad \forall\ z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

19.

$$\arctan(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + ...,$$

$$\forall\ z: \ |z| < 1.$$

(**Section 3.2**. **Subsection 3.5.1**.)

20.

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ... + \frac{z^{2n}}{(2n)!} + ...,$$

$$\forall\ z \in \mathbb{C}.$$

(**Problem 3.2.12**. **Subsection 3.5.1**.)

21.

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ..., + \frac{z^{2n+1}}{(2n+1)!} + ...,$$

$$\forall\ z \in \mathbb{C}.$$

(**Problem 3.2.12**. **Subsection 3.5.1**.)

22. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin[a + (k-1)d] = \frac{\sin\left[a + \frac{(n-1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}.$$

(**Problem 3.2.19 (a)**.)

23. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \cos[a + (k-1)d] = \frac{\cos\left[a + \frac{(n-1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}.$$

(**Problem 3.2.19 (a)**.)

24. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin(ka) = \frac{\sin\left[\frac{(n+1)a}{2}\right] \sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)}.$$

(**Problems 3.2.11 (d), 3.2.19 (b)**.)

25. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \cos(ka) = \frac{\cos\left[\frac{(n+1)a}{2}\right] \sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)}.$$

(**Problems 3.2.11 (e), 3.2.19 (b)**.)

26. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin[(2k-1)a] = \frac{\sin^2(na)}{\sin(a)}.$$

(**Problem 3.2.19 (b)**.)

27. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \cos[(2k-1)a] = \frac{\sin(2na)}{2\sin(a)}.$$

(**Problem 3.2.19 (b)**.)

28. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin(2ka) = \frac{\sin[(n+1)a]\sin(na)}{\sin(a)}.$$

(**Problem 3.2.19 (c)**.)

29. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \cos(2ka) = \frac{\cos[(n+1)a]\sin(na)}{\sin(a)}.$$

(**Problem 3.2.19 (c)**.)

30. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2[a+(k-1)d] = \frac{n}{2} - \frac{\cos\left[2a+(n-1)d\right]\sin(nd)}{2\sin(d)}.$$

(**Problem 3.2.20 (a)**.)

31. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2(ka) = \frac{n}{2} - \frac{\cos[(n+1)a]\sin(na)}{2\sin(a)}.$$

(**Problem 3.2.20 (b)**.)

32. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2[(2k-1)a] = \frac{n}{2} - \frac{\cos(2na)\sin(2na)}{2\sin(2a)}.$$

(**Problem 3.2.20 (b)**.)

33. If $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2(2kd) = \frac{n}{2} - \frac{\cos[2(n+1)d]\sin(2nd)}{2\sin(2d)}.$$

(**Problem 3.2.20 (c)**.)

34. If $-\dfrac{\pi}{6} < x < \dfrac{\pi}{6}$,

$$\sum_{n=1}^{\infty} 2^{n-1}\sin^n(x) = \frac{\sin(x)}{1-2\sin(x)}.$$

(**Problem 3.2.21**.)

35.

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^{n+2}}\right)\cos\left(\frac{3\pi}{2^{n+2}}\right) = \frac{1}{2}.$$

(**Problem 3.2.22 (a)**.)

36. If $z \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan\left(\frac{z}{2^n}\right) = \frac{1}{z} - \cot(z).$$

(**Problem 3.2.22 (b)**.)

37. If $x \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \arctan\left[\frac{x}{1+n(n+1)x^2}\right] =$$

$$\begin{cases} \dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -\dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right) & \text{if} \quad x < 0. \end{cases}$$

(**Problem 3.2.23**.)

38.

$$\arccos(z) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \frac{z^{2n+1}}{2n+1} =$$

$$\frac{\pi}{2} - \frac{z}{1} - \frac{1}{2} \cdot \frac{z^3}{3} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{z^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{z^7}{7} - ..., \ \forall\, z: \ |z| < 1.$$

(**Subsection 3.5.1**.)

39.

$$\arcsin(z) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \frac{z^{2n+1}}{2n+1} =$$

$$\frac{z}{1} + \frac{1}{2} \cdot \frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{z^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{z^7}{7} - ..., \quad \forall\, z: \ |z| < 1.$$

(**Subsection 3.5.1**.)

40. **Zero branch of complex logarithm**

$$\log_{(0)}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...,$$

$$\forall\, z: \ |z| < 1.$$

(**Subsection 3.5.3**. See also **Section 3.2**.)

41. **Zero branch of complex binomial series**

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \alpha \in \mathbb{C}, \quad \forall\ z:\ |z| < 1,$$

where $\forall \quad n \in \mathbb{N}$,

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!}.$$

(**Subsection 3.5.5**. See also **Problem 2.6.41**.)

42. If $0 \le r < a$ and $\theta,\ \phi$ are real constants,

$$1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)] = -1 + 2\sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)] =$$

$$P_{(a,\phi)}(r,\theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}.$$

(**Definition 3.5.3 of Poisson Kernel and Appendix 3.5.8**.)

43.

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^4)\sinh(x)}\, dx = \int_{-\infty}^{\infty} \frac{x}{1+x^4}\ \operatorname{csch}(x)\, dx =$$

$$\frac{\pi \sinh\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right) + \sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^4\pi^4 + 1}.$$

(**Problem 3.7.28**.)

44. For any $a \in \mathbb{C}$, such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer) constants,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a}\coth(\pi a).$$

(**Example 3.7.22**.)

45. For any $a \in \mathbb{C}$, such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer) constants,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a}\left[\pi\coth(\pi a) - \frac{1}{a}\right].$$

(**Example 3.7.22**.)

46.
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(**Euler Sum**. **Example 3.7.22 Corollary 3.7.3**. **Problem 3.7.36**.)

47.
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(**Example 3.7.22 footnote**.)

48.
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(**Example 3.7.22 footnote**.)

49. If $a \notin \mathbb{Z}$ constant,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a}\left[\frac{1}{a} - \pi\cot(\pi a)\right].$$

(**Examples 3.7.22 Corollary 3.7.4**, **3.7.23**. **Problems 3.7.37**, **3.7.61 (a)**.)

50. If $a \notin \mathbb{Z}$ constant,

$$\pi\cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a}{n(a-n)}.$$

$$\pi\cot(\pi a) - \frac{1}{a} = \lim_{m\to\infty} \sum_{\substack{n=-m \\ n\neq 0}}^{m} \frac{1}{a-n}.$$

$$\pi\cot(\pi a) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{a-n}.$$

$$\frac{\pi}{\sin(\pi a)} = \pi\csc(\pi a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-n}.$$

$$\pi = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{1}{2}-n} = 2\left[1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}+\sum_{n=-1}^{-\infty}\frac{(-1)^{n+1}}{2n-1}\right] = 4\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right).$$

$$\pi = 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{\frac{1}{4}-n^2} = 2+2\left(\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\dots\right).$$

(**Example 3.7.23**. **Problem 3.7.61 (a)**.)

51. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| \le 1$ complex constants,

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx = a\sum_{n=0}^{\infty}(-c)^n\frac{1}{a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.24**.)

52. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| \le 1$ complex constants,

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx}+c}\,dx = \sum_{n=0}^{\infty}\frac{a(-c)^n}{-a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.24 Corollary 3.7.5 (I)**. **Problem 3.7.59**.)

53. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.25**.)

54. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 3.7.25**.)

55.

$$\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{\pi^2}{24}.$$

(**Problem 3.7.38**.)

56.
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

(**Problem 3.7.38**.)

57.
$$\sum_{k=1}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \frac{\pi^2}{2}.$$

(**Problem 3.7.38**.)

58.
$$\sum_{k=0}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - \left(\frac{1}{2}\right)^2} = \frac{\pi}{4}.$$

(**Problems 3.7.39**, **3.7.53 (d)**.)

59. If $a \neq ni$ with $n \in \mathbb{Z}$ complex constant,
$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a}\operatorname{csch}(\pi a).$$

(**Problem 3.7.41 (a)**.)

60. If $a \neq ni$ with $n \in \mathbb{Z}$ complex constant,
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a}\operatorname{csch}(\pi a) - \frac{1}{2a^2}.$$

(**Problem 3.7.41 (a)**.)

61. If $a \notin \mathbb{Z}$ complex constant,
$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2-a^2} = \frac{-\pi}{a}\csc(\pi a).$$

(**Problem 3.7.41 (b)**.)

62. If $a \notin \mathbb{Z}$ complex constant,
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\csc(\pi a).$$

(**Problem 3.7.41 (b)**.)

63.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(**Problem 3.7.41 (c)**.)

64.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{32}.$$

(**Problem 3.7.49**.)

65.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{4}.$$

(**Problem 3.7.49**.)

66. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \pi^2 \csc^2(\pi a) = \frac{\pi^2}{\sin^2(\pi a)}.$$

(**Problem 3.7.52 (a)**.)

67. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \csc(\pi a) \cot(\pi a) = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

(**Problem 3.7.52 (b)**.)

68. If $a \notin \mathbb{Z}$ and $b \notin \mathbb{Z}$ complex constants,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\pi}{b-a} \left[\cot(\pi a) - \cot(\pi b)\right].$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2 + a^2} = \frac{\pi}{2ai} \left\{\cot[\pi(z-ai)] - \cot[\pi(z+ai)]\right\}.$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2 - a^2} = \frac{\pi}{2a} \left\{\cot[\pi(z-a)] - \cot[\pi(z+a)]\right\}.$$

(**Problem 3.7.53 (a), (b)**.)

69. If $a \notin \mathbb{Z}$ and $b \notin \mathbb{Z}$ complex constants,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\csc(\pi a) - \csc(\pi b)\right].$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2+a^2} = \frac{\pi}{2ai}\left\{\csc[\pi(z-ai)] - \csc[\pi(z+ai)]\right\}.$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2-a^2} = \frac{\pi}{2a}\left\{\csc[\pi(z-a)] - \csc[\pi(z+a)]\right\}.$$

(**Problem 3.7.54 (a), (b)**.)

70.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(3n+1)} = \frac{1}{2}\int_0^{\infty} \frac{e^{-x}\left(e^{-x} - e^{\frac{-x}{3}}\right)}{1-e^{-x}}\,dx =$$
$$-\frac{1}{2}\left(1 + \int_0^1 \frac{u^{\frac{1}{3}}-1}{1-u}\,du\right).$$

(**Problem 3.7.55**.)

71.

$$\sum_{k=0}^{\infty} \frac{2k+1}{k^2(k+1)} = \frac{\pi^2}{6} + 1.$$

(**Problem 3.7.56**.)

72.

$$\sum_{n=1}^{\infty} \frac{1}{1+n^4} =$$
$$\frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right) + \sinh\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

(**Problem 3.7.57 (a)**.)

73.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^4} =$$
$$\frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right) + \sinh\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

(**Problem 3.7.57 (b)**.)

74.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n^2}{n^3+1} = \frac{1}{3}\left[1 - \ln(2) + \pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right)\right].$$

(**Problem 3.7.58 (a)**.)

75.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = \pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right).$$

(**Problem 3.7.58 (b)**.)

76.

$$\operatorname{Log}\left[\frac{\sin(\pi z)}{z}\right] = \log(\pi) + \sum_{n=1}^{\infty} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right),$$

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right),$$

$$\sinh(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right).$$

(**Problem 3.7.61**.)

77. If $k \in \mathbb{N}$, $0 < |a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k$^{\text{th}}$ roots of unity$(=1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \int_0^{\infty} \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{k a^{k-1}\,(e^x - 1)}\, dx.$$

(**Problem 3.7.62**.)

78. If $k \in \mathbb{N}$, $m \in \mathbb{N}$, $a \in \mathbb{C}$ such that $0 < |m - a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k^{th} roots of unity$(=1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \sum_{n=1}^{m} \frac{1}{n^k - a^k} + \int_0^{\infty} \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\omega_\rho a)x}}{k a^{k-1}(e^x - 1)}\, dx.$$

(**Problem 3.7.62**.)

79. If $k \in \mathbb{N}$, ψ a k$^{\text{th}}$ root of -1, $0 < |a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k^{th} roots of unity$(=1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{\psi \omega_\rho a x}}{k a^{k-1}(e^x - 1)}\, dx.$$

(**Problem 3.7.62**.)

80. If $k \in \mathbb{N}$, ψ a kth root of -1, $m \in \mathbb{N}$, $a \in \mathbb{C}$ such that $0 < |m-a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k^{th} roots of unity($= 1$),

$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \sum_{n=1}^{m} \frac{1}{n^k + a^k} + \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\psi\omega_\rho a)x}}{ka^{k-1}(e^x - 1)}\, dx.$$

(**Problem 3.7.62**.)

81. If $m \geq 0$ integer,

$$\sum_{k=0}^{2m+1} \frac{1}{2m+1-2k} \binom{2m+1}{k} = 0.$$

(**Example 3.8.4**.)

82. If $m \geq 0$ and $n \geq 0$ integers,

$$\sum_{\substack{k=0 \\ m+n \neq k+l}}^{2m} \sum_{l=0}^{2n} (-1)^k \binom{2m}{k} \binom{2n}{l} \frac{1-(-1)^{m-k+n-l}}{m-k+n-l} =$$

$$\sum_{\substack{k=0 \\ m+n \neq k+l}}^{2m} \sum_{l=0}^{2n} \binom{2m}{k} \binom{2n}{l} \frac{(-1)^k - (-1)^{m+n-l}}{m-k+n-l} = 0.$$

(**Example 3.8.5**.)

83. If $m \geq 0$ and $n \geq 0$ integers,

$$\sum_{l=0}^{2n} (-1)^{n-l} \binom{2m}{m+n-l} \binom{2n}{l} = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

(**Example 3.8.5**. **Appendix 3.8.3**.)

84. If $m \geq 1$ integer,

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{(-1)^k}{m-k} \binom{2m}{k} = 0.$$

(**Example 3.8.5**. **Problem 3.8.9 (a)**.)

85. If $m \geq 1$ integer,

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{1}{m-k} \binom{2m}{k} = 0.$$

(**Example 3.8.5**. **Problem 3.8.9 (b)**.)

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Mathematics

Improper Riemann Integrals gives readers the prerequisites and tools to understand the convergence, principal value, and evaluation of the improper/generalized Riemann integral. It also illustrates applications to science and engineering problems.

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