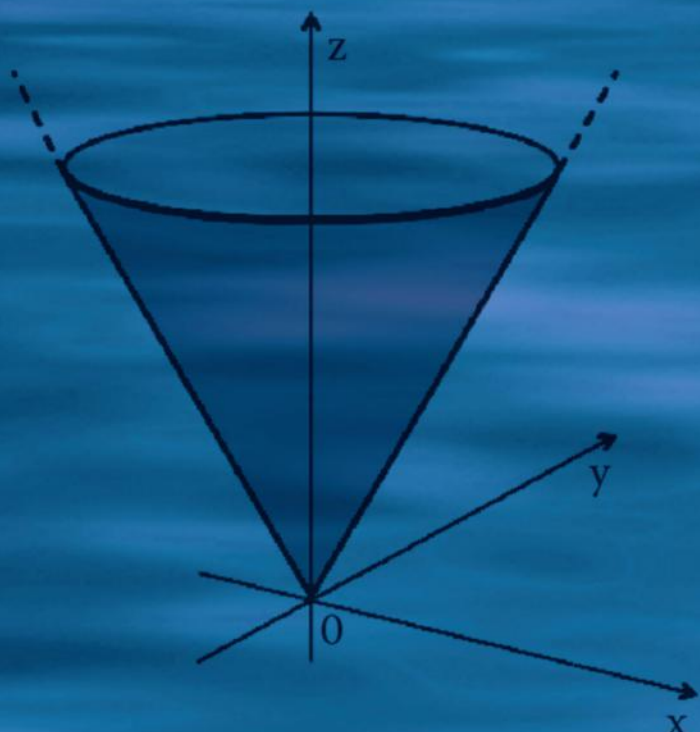
FIXED POINT THEORY, VARIATIONAL ANALYSIS, AND OPTIMIZATION



Edited by

Saleh A. R. Al-Mezel, Falleh R. M. Al-Solamy, and Qamrul H. Ansari

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Preface

Nonlinear analysis is one of the most interesting and fascinating branches of pure and applied mathematics. During the last five decades, several branches of nonlinear analysis have been developed and extensively studied. The main aim of this volume is to include those branches of nonlinear analysis which have different applications in different areas. We divide this volume into three parts: Fixed-Point Theory, Convex Analysis and Variational Analysis, and Vector Optimization.

The first part consists of the first three chapters.

Chapter 1 is devoted to the study of Mann-type iterations for nonlinear mappings on some classes of a metric space. This is achieved through the convex structure introduced by W. Takahashi. The common fixed-point results for asymptotically (quasi-) nonexpansive mappings through their explicit and implicit iterative schemes on nonlinear domains such as $CAT(0)$ spaces, hyperbolic spaces, and convex metric spaces are presented, which provide metric space version of the corresponding well-known results in Banach spaces.

Chapter 2 provides an outline of the recent results in fixed-point theory in modular function spaces. Modular function spaces are natural generalizations of both function and sequence variants of many important (from an applications perspective) spaces such as Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii, and many others. In the context of fixed-point theory, the foundations of the geometry of modular function spaces and other important techniques like extensions of the Opial property to modular spaces are discussed. A series of existence theorems of fixed points for nonlinear mappings, and of common fixed points for semigroups of mappings, is presented.

Chapter 3 discusses key results on the existence of continuous approximations and selections for set-valued maps with an emphasis on the non-convex case (non-convex domains, co-domains, and non-convex values) and in a general and generic framework allowing the passage, by approximation, from simple domains to more elaborate ones. Applications of the approximation and selection results to topological fixed-point and coincidence theory for set-valued maps are also presented.

The second part of the volume consists of Chapters 4, 5, and 6.

Chapter 4 contains the basic definitions, properties, and characterizations of convex, quasiconvex, and pseudoconvex functions, and of their strict counterparts. The aim of this chapter is to present the basic techniques that will help the reader in his/her further reading, so we include almost all proofs of the results presented. At the same time, the definitions of some classes of generalized monotone operators are recalled; it is shown how they are related to corresponding classes of generalized convex functions. Finally, some of the many applications of generalized convex functions and generalized monotone operators are given, which are related to optimization and microeconomics, and especially to consumer theory.

After the huge development of convex optimization during several decades, quasiconvex optimization, or optimization problems involving a quasiconvex objective function, can be considered as a new step to embrace a larger class of problem with powerful mathematical tools. The main aim of Chapter 5 is to show that, using some adapted tools, a sharp and powerful first-order analysis can be developed for quasiconvex optimization.

Chapter 6 gives an introduction to the theory of variational-like inequalities. Some relations between a nonconvex optimization problem and a variational-like inequality problem are provided. Some existence results for a solution of variational-like inequalities are presented under different kinds of assumptions. Two solution methods—auxiliary principle method and the proximal method—for finding the approximate solutions of variational-like inequalities are discussed.

The last part is devoted to vector optimization.

Chapter 7 presents basic concepts of vector optimization, starting with partial orders in a vector space with respect to which optimality is defined. Some criteria for existence of maximal elements of a set in a partially ordered space by using coverings of a set and monotone functions are discussed. For a vector optimization problem with equality and inequality constraints, one can express optimality conditions in terms of derivatives when the data of the problem are differentiable, or in terms of subdifferentials when the data are nonsmooth. Finally, three methods for solving nonconvex vector optimization problems are presented. Two of them are well-known and the other one is more recent, but both are interesting from mathematical and practical points of view.

The last chapter is devoted to multi-objective combinatorial optimization (MOCO) problems, which are integer programs with multiple objectives. The goal in solving a MOCO problem is to find efficient (or Pareto optimal) solutions and their counterparts in objective space, called non-dominated points. Various types of efficient solutions and non-dominated points as well as lexicographic optima are defined. It is shown that MOCO problems are usually NP-hard, #P-hard, and intractable, that is, they can have an exponential number of non-dominated points. The multi-objective shortest-path and spanning-tree problems are presented as examples of MOCO problems for which single objective algorithms can be extended. The two-phase method is an effective tool for problems that are polynomially solvable in the single objective case and for which efficient ranking algorithms to find r-best solutions exist. For problems for which the two-phase approach is not computationally effective, one must resort to general scalarization techniques or adapt general integer programming techniques, such as branch and bound, to deal with multiple objectives. Some popular scalarization methods in the context of a general formulation are presented. Bound sets for non-dominated points are natural multi-objective counterparts of lower and upper bounds that enable multi-objective branch and bound algorithms and several examples are cited. Exact algorithms based on integer programming techniques and scalarization can easily result in prohibitive computation times even for relatively small-sized problems. Metaheuristics may be applied in this case. The main concepts of metaheuristics that have been applied to MOCO problems are introduced and their evolution over time is illustrated.

We thank our friends and colleagues, whose encouragement and help influenced the development of this volume. We mainly are grateful to the Rector of the University of Tabuk Dr. Abdulaziz S. Al-Enazi, for his support in organizing an International Workshop on Nonlinear Analysis and Optimization, March 18–19, 2013 at University of Tabuk, Tabuk, Saudi Arabia. Most of the contributors attended this workshop and agreed to be a part of this volume.

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October 2013

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Part I

Fixed Point Theory

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Chapter 1

Iterative Construction of Common Fixed Points in Convex Metric Spaces

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1.1 Introduction

The Banach Contraction Principle (BCP) asserts that a contraction on a complete metric space has a unique fixed point and its proof hinges on "Picard iterations." This principle is applicable to a variety of subjects such as integral equations, partial differential equations and image processing. This principle breaks down for nonexpansive mappings on metric spaces. This led to the introduction of Mann iterations in a Banach space [33]. Our aim is to study Mann-type iterations for some classes of nonlinear mappings in a metric space. We achieve it through the convex structure introduced by Takahashi [45]. In this chapter, iterative construction of common fixed points of asymptotically (quasi-) nonexpansive mappings [11] by using their explicit and implicit schemes on nonlinear domains such as $CAT(0)$ spaces, hyperbolic spaces, and convex metric spaces[1, 7, 22, 24, 45] will be presented. The new results provide a metric space version of the corresponding known results in Banach spaces and $CAT(0)$ spaces (for example, [12, 25, 32, 26, 47]).

1.2 Preliminaries

Let C be a nonempty subset of a metric space (X, d) and T be a mapping on C. Denote the set of fixed points of T by $F = \{x \in C : Tx = x\}$. The mapping T is said to be:

- *contraction* if there exists a constant $k \in [0, 1)$ such that $d(T(x), T(y)) \leq k\ d(x, y)$, for all $x, y \in C$;
- *uniformly L-Lipschitzian* if $d(T^n(x), T^n(y)) \leq L\ d(x, y)$, for some $L > 0$, $x, y \in C$, $n \geq 1$;
- *asymptotically nonexpansive* [11] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n(x), T^n(y)) \leq k_n\ d(x, y)$, for all $x, y \in C$ and $n \geq 1$;
- *asymptotically quasi-nonexpansive* if $F \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n(x), p) \leq k_n\ d(x, p)$, for all $x \in C$, $p \in F$ and $n \geq 1$.

For $n = 1$, the uniform L-Lipschitzian mapping is known as *L-Lipschitzian.* For $k_n = 1$ for $n \geq 1$ in the definitions above, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping becomes *nonexpansive mapping* and *quasi-nonexpansive mapping*, respectively.

A nonlinear mapping may or may not have a fixed point.

Example 1.1. Define $T : [1, \infty) \to [1, \infty)$ by

(a) $T(x) = \frac{x}{2} + 3$. Then, T is a contraction with $k = \frac{1}{2}$ and $F = \{6\}$,

(b) $T(x) = \frac{25}{26}(x + \frac{1}{x})$. Then, T is a contraction with $k = \frac{25}{26}$ and $F = \{5\}$,

(c) $T(x) = x + \frac{1}{x}$. Then, T is not a contraction and so by the Banach contraction principle, it has no fixed point.

The concept of quasi-nonexpansiveness is more general than that of nonexpansiveness. A nonexpansive mapping with at least one fixed point is quasi-nonexpansive. There are quasi-nonexpansive mappings which are not nonexpansive.

Example 1.2. (a) [36] Let $X = \mathbb{R}$ (the set of real numbers). Define $T_1 : \mathbb{R} \to \mathbb{R}$ by $T_1(x) = \frac{x}{2} \sin \frac{1}{x}$ with $T_1(0) = 0$. The only fixed point of T_1 is 0 as follows: if $x \neq 0$ and $T_1(x) = x$, then $x = \frac{x}{2} \sin \frac{1}{x}$ or $2 = \sin \frac{1}{x}$, which is impossible. T_1 is quasi-nonexpansive because $|T_1(x) - 0| = \left|\frac{x}{2}\right| \left|\sin \frac{1}{x}\right| \leq \frac{|x|}{2} < |x - 0|$ for all $x \in X$. However, T_1 is not nonexpansive mapping. This can be verified by choosing $x = \frac{2}{\pi}$, $y = \frac{2}{3\pi}$; $|T(x) - T(y)| = \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{2}{3\pi} \sin \frac{3\pi}{2} = \frac{8}{3\pi}$ and $|x - y| = \frac{4}{3\pi}$.

(b) [11] Let B be the unit ball in the Hilbert space l^2 and $\{a_i\}$ be a sequence of numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Define $T : B \to B$ by $T(x_1, x_2, x_3, \ldots) = \left(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\right)$. Then,

$$\|T^n(x) - T^n(y)\| \leq 2 \prod_{i=2}^{n} a_i \|x - y\| \text{ for } n \geq 2 \quad \text{and}$$
$$\|T(x) - T(y)\| \leq 2 \|x - y\|$$

give that T is asymptotically nonexpansive but not nonexpansive.

Remark 1.1. (a) The linear quasi-nonexpansive mappings are nonexpansive, but it is easy to verify that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive; for example, the above T_1.

(b) It is obvious that, if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

If a fixed point of a certain mapping exists and its exact value is not known, then we use an iterative procedure to find it. Here is an example : Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = \left(\frac{\sin y}{4}, \frac{\sin z}{3} + 1, \frac{\sin x}{5} + 2\right)$. Then, T is a contraction with $k = \frac{1}{3}$. By (BCP), T has a unique fixed point p. Its exact value is not known. Using the method of proof of (BCP), we can find an approximate value of p with a required accuracy. An answer is $p = (x_6, y_6, z_6) = (0.2406, 1.2961, 2.0477)$ within the accuracy of 0.001 to the fixed point (by measuring it through Euclidean distance).

The reader interested in the iterative approximation of fixed points(common fixed points) for various classes of mappings in the context of Banach spaces and metric spaces is referred to Berinde [3].

Many problems in science and engineering are nonlinear. So translating a linear version of known problems (usually in Banach spaces) into an equivalent nonlinear version (metric spaces) has great importance. This basic problem is usually considered in a $CAT(0)$ space. We include a brief description of a $CAT(0)$ space.

Let (X, d) be a metric space. A *geodesic* from x to y in X is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic (or metric) segment* joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the *segment* joining x to y.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Such a triangle always exists [5].

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \le d(\bar{x}, \bar{y}).$$

A complete $CAT(0)$ space is often called a *Hadamard space* (see [28]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies:

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \le \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This inequality is the (CN)-inequality of Bruhat and Titz [6]. The above inequality has been extended by Khamsi and Kirk [18] as follows:

$$\begin{aligned} d(z, \lambda x \oplus (1-\lambda)y)^2 \le \lambda d(z, x)^2 + (1-\lambda) d(z, y)^2 \\ - \lambda(1-\lambda) d(x, y)^2, \end{aligned} \tag{CN*}$$

for any $\lambda \in [0, 1] = I$ and $x, y, z \in X$.

In 1970, Takahashi [45] introduced a concept of convex structure in a metric space (X, d) as a mapping $W : X^2 \times I \to X$ satisfying

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in I$.

A metric space (X, d) together with a convex structure W is a *convex metric space* (X, d, W), which will be denoted by X for simplicity. A nonempty subset C of a convex metric space X is *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in I$.

If X is a $CAT(0)$ space and $x, y \in X$, then for any $\lambda \in I$, there exists a unique point $\lambda x \oplus (1-\lambda)y \in [x, y] = \{\lambda x \oplus (1-\lambda)y : \lambda \in I\}$ such that

$$d(z, \lambda x \oplus (1-\lambda)y) \le \lambda d(z, x) + (1-\lambda) d(z, y)$$

for any $z \in X$ (see [9] for details).

In view of the above inequality, a $CAT(0)$ space has Takahashi's convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y$.

We now give examples of convex metric spaces which cannot be Banach spaces [45].

Example 1.3. (a) Let $X = \{[a_i, b_i] : 0 \leq a_i \leq b_i \leq 1\}$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$, and $\lambda \in I$, we define

$$W(I_i, I_j, \lambda) = [\lambda a_i + (1-\lambda) a_j, \lambda b_i + (1-\lambda) b_j],$$

$$d(I_i, I_j) = \sup_{\alpha \in I} \left\{ \left| \inf_{b \in I_i} \{|a-b|\} - \inf_{c \in I_j} \{|a-c|\} \right| \right\}, \text{Hausdorff distance on } X.$$

(b) A linear space X with a metric d on it and satisfying the properties: (i) $d(x,y) = d(x-y, 0)$, (ii) $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0)$ for $x, y \in X$ and $\lambda \in I$, is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$.

Recently, Kohlenbach [27] defined the concept of a hyperbolic space by including the following additional conditions in the definition of a convex metric space X:

$$\begin{array}{ll} (1) & d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2| \, d(x,y), \\ (2) & W(x,y,\lambda) = W(y,x,1-\lambda), \\ (3) & d(W(x,z,\lambda), W(y,w,\lambda)) \leq \lambda d(x,y) + (1-\lambda) \, d(z,w), \end{array}$$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in I$.

If $X = \mathbb{R}$, $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, and $d(x,y) = \frac{|x-y|}{1+|x-y|}$, for all $x, y \in \mathbb{R}$, then X is a convex metric space but not a hyperbolic space (the above condition (1) does not hold). In fact, every normed space and its convex subsets are hyperbolic spaces, but the converse is not true, in general.

Some other notions of hyperbolic space have been introduced and studied by Goebel and Reich [13], Khamsi and Khan [17], and Reich and Shafrir [39].

Now we prove some elementary properties of a convex metric space.

Lemma 1.1. Let X be a convex metric space. Then, for all $x, y \in X$ and $\lambda \in I$, we have the following:

(a) $W(x,y,1) = x$ and $W(x,y,0) = y$;

(b) $W(x,x,\lambda) = x$;

(c) $d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) = d(x,y)$;

(d) the open sphere $S_r(x) = \{y \in X : d(y,x) < r\}$ and the closed sphere $S_r[x] = \{y \in X : d(y,x) \leq r\}$ are convex subsets of X; and

(e) the intersection of convex subsets of X is convex.

Proof. (a) and (b) follow easily from the definition of W.

(c) Since X is a convex metric space, we obtain

$$\begin{aligned} d(x,y) &\leq d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) \\ &\leq \lambda d(x,x) + (1-\lambda) d(x,y) + \lambda d(x,y) + (1-\lambda) d(y,y) \\ &= (1-\lambda) d(x,y) + \lambda d(x,y) \\ &= d(x,y), \end{aligned}$$

for all $x, y \in X$ and $\lambda \in I$.

It follows by the above inequalities that

$$d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y).$$

(d) Since X is a convex metric space,

$$\begin{aligned} d(x, W(y,z,\lambda)) &\leq \lambda d(x,y) + (1-\lambda)d(x,z) \\ &< \lambda r + (1-\lambda)r = r, \end{aligned}$$

for any $y, z \in S_r(x)$ and $\lambda \in I$. Hence, $W(y,z,\lambda) \in S_r(x)$. This proves that $S_r(x)$ is a convex subset of X. Similarly, we can prove that $S_r[x]$ is a convex subset of X.

(e) Follows by routine calculations. □

A convex metric space X is said to satisfy *Property* (G) [46]: whenever $w \in X$ and there is $(x,y,\lambda) \in X^2 \times I$ for which

$$d(z,w) \leq \lambda d(z,x) + (1-\lambda)d(z,y), \text{ for every } z \in X,$$

then $w = W(x,y,\lambda)$.

The Property (G) holds in the Euclidean plane equipped with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$.

The proof of next lemma depends on the Property (G).

Lemma 1.2. Let X be a convex metric space satisfying the Property (G). Then, we have the following assertions:

(a) $W(W(x,y,\lambda_1), y, \lambda_2) = W(x,y,\lambda_1\lambda_2)$, for every $x, y \in X$ and $\lambda_1, \lambda_2 \in I$;

(b) The function $f(\lambda) = W(x,y,\lambda)$ is an embedding (one-to-one function) of I into X, for every pair $x, y \in X$ with $x \neq y$.

Proof. (a) Let $z \in X$. Then

$$\begin{aligned} d(z, W(W(x,y,\lambda_1), y, \lambda_2)) &\leq \lambda_2 d(z, W(x,y,\lambda_1)) + (1-\lambda_2) d(z,y) \\ &\leq \lambda_2 [\lambda_1 d(z,x) + (1-\lambda_1) d(z,y)] \\ &\quad + (1-\lambda_2) d(z,y) \\ &\leq \lambda_2\lambda_1 d(z,x) + (1-\lambda_2\lambda_1) d(z,y). \end{aligned}$$

Hence, by Property (G), $W(W(x,y,\lambda_1), y, \lambda_2) = W(x,y,\lambda_1\lambda_2)$.

(b) Let $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 \neq \lambda_2$. Assume, without loss of generality, that $\lambda_1 < \lambda_2$. Then,

$$\begin{aligned} d\left(f\left(\lambda_1\right), f\left(\lambda_2\right)\right) &= d\left(W\left(x, y, \lambda_1\right), W\left(x, y, \lambda_2\right)\right) \\ &= d\left(W\left(x, y, \lambda_2\left(\frac{\lambda_1}{\lambda_2}\right)\right), W\left(x, y, \lambda_2\right)\right) \\ &= d\left(W\left(W\left(x, y, \lambda_2\right), y, \left(\frac{\lambda_1}{\lambda_2}\right)\right), W\left(x, y, \lambda_2\right)\right) \\ &= \left[1-\left(\frac{\lambda_1}{\lambda_2}\right)\right] d\left(W\left(x, y, \lambda_2\right), y\right) \\ &= \left(\lambda_2-\lambda_1\right) d\left(x, y\right) > 0. \end{aligned}$$

That is, $f\left(\lambda_1\right) \neq f\left(\lambda_2\right)$ for $\lambda_1 \neq \lambda_2$. This proves that the function f is an embedding of I into X for every pair $x, y \in X$ with $x \neq y$. □

The argument in the proof of Lemma 1.2 (b) shows that the mapping $W(x,y,\lambda) \mapsto \lambda d(x,y)$ is an isometry of the subspace $\{W(x,y,\lambda) : \lambda \in I\}$ of X onto the closed interval $[0, d(x,y)]$. In particular, $\{W(x,y,\lambda) : \lambda \in I\}$ is homeomorphic with I if $x \neq y$ and, is a singleton if $x = y$. It is not clear whether a convex structure W satisfying the Property (G) is necessarily a continuous function.

However, we have the following result:

Lemma 1.3. Let X be a convex metric space. Then, W is continuous at each point (x,x,λ) of $X^2 \times I$.

Proof. Let $\{(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ be a sequence in $X^2 \times I$ that converges to (x,x,λ). Because $W\left(x,x,\lambda\right) = x$, it suffices to show that $\{W(x_n,y_n,\lambda_n)\}_{n=1}^{\infty}$ converges to x. This is immediate as both the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge to x, and hence the definition of W yields

$$d\left(x, W(x_n, y_n, \lambda_n)\right) \leq \lambda_n d\left(x, x_n\right) + \left(1-\lambda_n\right) d\left(x, y_n\right), \quad \text{for each } n \geq 1.$$

Thus, $d\left(x, x_n\right) \to 0$, $d\left(x, y_n\right) \to 0$, and $\lambda_n \to \lambda$ imply the conclusion. □

The difficulty in obtaining continuity of W as a mapping from the product lies in the fact that there seems to be no way to guarantee that the sequence $\{W(x_n,y_n,\lambda_n)\}_{n=1}^{\infty}$ will converge when $\{(x_n,y_n,\lambda_n)\}_{n=1}^{\infty}$ converges to (x,y,λ) with $x \neq y$. When X is compact, manage this difficulty as follows:

Lemma 1.4. Let X be a compact convex metric space satisfying the Property (G). Then, W is a continuous function.

Proof. Let $\{(x_n,y_n,\lambda_n)\}_{n=1}^{\infty}$ be a sequence in $X^2 \times I$ that converges to (x,y,λ), and let w be a limit point of the sequence $\{W(x_n,y_n,\lambda_n)\}_{n=1}^{\infty}$. Select a subsequence $\{W(x_{n_k},y_{n_k},\lambda_{n_k})\}_{k=1}^{\infty}$ that converges to w. Then, for any $z \in X$, we have $d\left(z, W(x_{n_k},y_{n_k},\lambda_{n_k})\right) \leq \lambda_{n_k} d\left(z, x_{n_k}\right) + \left(1-\lambda_{n_k}\right) d\left(z, y_{n_k}\right)$ for $k \geq 1$. By continuity of d, we conclude that $d(z,w) \leq \lambda d(z,x) + (1-\lambda) d(z,y)$. The

Property (G) now guarantees that $w = W(x, y, \lambda)$. Hence, it follows that $W(x, y, \lambda)$ is the only limit point of the sequence $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$. Since X is compact, $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ must converge to $W(x, y, \lambda)$ and we are done. □

Next we define two geometric structures in a convex metric space and present their basic properties.

A convex metric (hyperbolic) space X is strictly convex [45] if for any $x, y \in X$ and $\lambda \in I$, there exists a unique element $z \in X$ such that $d(z, x) = \lambda d(x, y)$ and $d(z, y) = (1-\lambda)d(x, y)$, and uniformly convex [43] if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r\,(1-\alpha) < r$ for all $r > 0$ and $x, y, z \in X$ with $d\,(z, x) \leq r, d\,(z, y) \leq r$ and $d\,(x, y) \geq r\varepsilon$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ that provides such an $\alpha = \eta(r, \varepsilon)$ for $u, x, y \in X$, $r > 0$, and $\varepsilon \in (0, 2]$, is called *modulus of uniform convexity* [24] of X. We call η monotone if it decreases with respect to r (for a fixed ε).

Example 1.4. Let H be a Hilbert space and $C = \{x \in H : \|x\| = 1\}$. If $x, y \in C$ and $a, b \in I$ with $a + b = 1$, then $\frac{ax+by}{\|ax+bk\|} \in C$ and $\delta(C) \leq \sqrt{2}/2$, where $\delta(C)$ denotes the diameter of C. Let $d(x, y) = \cos^{-1}\{\langle x, y\rangle\}$ for every $x, y \in C$, where $\langle ., . \rangle$ is the inner product of H. Then, C is uniformly convex under $W\,(x, y, \lambda) = \lambda x + (1-\lambda)\,y$.

Now we present some basic properties of a uniformly convex metric space.

Lemma 1.5. Let X be a uniformly convex metric space. Then, we have the following assertions:

(a) X is strictly convex.

(b) If $d\,(x, z) + d\,(z, y) = d\,(x, y)$ for all $x, y, z \in X$, then $z \in \{W\,(x, y, \lambda) : \lambda \in I\}$.

(c) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|\, d(x, y)$, for all $x, y \in X$ and $\lambda_1, \lambda_2 \in I$.

(d) $W(x, y, \lambda) = W(y, x, 1-\lambda)$, for all $x, y \in X$ and $\lambda \in I$.

Proof. (a) Assume that X is not strictly convex. If $x, y \in X$ and $\lambda \in I$, then there exist z_1, z_2 in X such that $z_1 \neq z_2$ and

$$d(z_1, x) = \lambda d(x, y) = d(z_2, x), d(z_1, y) = (1-\lambda)d(x, y) = d(z_2, y).$$

It follows by $z_1 \neq z_2$ and the above identities that $x \neq y$ and $\lambda \in (0, 1)$. Let $r_1 = \lambda d(x, y) > 0, r_2 = (1-\lambda)d(x, y) > 0$. Obviously, $\varepsilon_1 = \frac{d(z_1,z_2)}{r_1} > 0$ and $\varepsilon_2 = \frac{d(z_1,z_2)}{r_2} > 0$. Since X is uniformly convex, we have

$$d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_1\,(1-\alpha_1) < r_1$$

and

$$d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_2\left(1-\alpha_2\right) < r_2.$$

Consider

$$\begin{aligned} d(x,y) &\leq d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) + d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq r_1\left(1-\alpha_1\right) + r_2\left(1-\alpha_2\right) \\ &< r_1 + r_2 \\ &= \lambda d(x,y) + (1-\lambda) d(x,y) \\ &= d(x,y), \end{aligned}$$

a contradiction to the reflexive property of real numbers.

(b) Let $x, y, z \in X$ be such that

$$d(x,z) + d(z,y) = d(x,y). \tag{1.1}$$

Let $u \in \{W(x,y,\lambda) : \lambda \in I\}$ be such that $d(x,u) = d(x,z)$. Then, by Lemma 1.1 (c),

$$d(x,u) + d(u,y) = d(x,y). \tag{1.2}$$

Comparing (1.1) and (1.2), we have that $d(z,y) = d(u,y)$. Now, we show that $z = u$. Assume instead that $z \neq u$. Let $v = W\left(x, y, \frac{1}{2}\right)$ and $r = d(x,u) = d(x,z)$. Since $d(z,u) > 0$, choose $\varepsilon > 0$ so that $d(z,u) > r\varepsilon$. By the uniform convexity of X, there exists $\alpha > 0$ such that

$$d(x,v) \leq r(1-\alpha) < r = d(x,z).$$

Similarly, we can show that $d(y,v) < d(y,z)$.

Therefore,

$$d(x,y) \leq d(x,v) + d(y,v) < d(x,z) + d(y,z) = d(x,y).$$

This is a contradiction to the reflexive property of real numbers. Hence, $z = u \in \{W(x,y,\lambda) : \lambda \in I\}$.

(c) Note that the conclusion holds if $\lambda_1 = 0$ or $\lambda_2 = 0$. Let $x, y \in X, \lambda_1, \lambda_2 \in (0,1]$, $u = W(y,x,\lambda_1)$, and $z = W(y,x,\lambda_2)$. Without loss of generality, we may assume that $\lambda_1 < \lambda_2$. Let $v = W\left(z, x, \frac{\lambda_1}{\lambda_2}\right)$. Then,

$$d(x,v) = \frac{\lambda_1}{\lambda_2} d(x,z) = \lambda_1 d(x,y),$$

and

$$d(v,y) \leq \left(1 - \frac{\lambda_1}{\lambda_2}\right) d(x,y) + \frac{\lambda_1}{\lambda_2} d(z,y) = \left(1-\lambda_1\right) d(x,y).$$

If $u \neq v$, let $w = W\left(u, v, \frac{1}{2}\right)$. By the uniform convexity of X, we can prove that $d(x, w) < d(x, u)$ and $d(y, w) < d(y, u)$. Therefore,

$$d(x, y) < d(x, u) + d(u, y) = d(x, y).$$

This contradicts the reflexive property of real numbers. Hence, $u = v$.

Now, it follows that

$$d(z, u) = d(z, v) = \left(1 - \frac{\lambda_1}{\lambda_2}\right) d(x, z) = |\lambda_2 - \lambda_1|\, d(x, y).$$

(d) Let $x, y \in X$ and $\lambda \in I$. Obviously, the conclusion holds if $\lambda = 0$ or $\lambda = 1$. By the definition of W, we have

$$d(x, W(x, y, \lambda)) = (1 - \lambda)\, d(x, y), \quad d(y, W(x, y, \lambda)) = \lambda d(x, y),$$

and

$$d(x, W(y, x, 1 - \lambda)) = (1 - \lambda)\, d(x, y), \quad d(y, W(y, x, 1 - \lambda)) = \lambda d(x, y).$$

Suppose that $W(x, y, \lambda) = z_1 \neq z_2 = W(y, x, 1 - \lambda)$.

Let $r_1 = (1 - \lambda)\, d(x, y) > 0$, $r_2 = \lambda d(x, y) > 0$, $\varepsilon_1 = \frac{d(z_1, z_2)}{r_1}$, and $\varepsilon_2 = \frac{d(z_1, z_2)}{r_2}$. Obviously $\varepsilon_1, \varepsilon_2 > 0$.

By uniform convexity of X, we have

$$d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_1(1 - \alpha_1) < r_1;$$

$$d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_2(1 - \alpha_2) < r_2.$$

Since $\lambda \in (0, 1)$, we get $x \neq y$.

Finally,

$$\begin{aligned} d(x, y) &\leq d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) + d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq r_1(1 - \alpha_1) + r_2(1 - \alpha_2) \\ &< r_1 + r_2 = d(x, y), \end{aligned}$$

which is against the reflexivity of reals. Therefore, $W(x, y, \lambda) = W(y, x, 1 - \lambda)$. ☐

A convex metric space X is said to satisfy the Property (H) [10] if

$$d(W(x, y, \lambda), W(z, y, \lambda)) \leq \lambda d(x, z) \quad \text{for all } x, y, z \in X \text{ and } \lambda \in I.$$

Lemma 1.6. Let X be a uniformly convex metric space satisfying the Property (H). Then, X is a uniformly hyperbolic space.

Proof. In the light of Lemma 1.5 (c)–(d), it is sufficient to show that

$$d(W(x,z,\lambda),W(y,w,\lambda)) \leq \lambda d(x,y) + (1-\lambda)\,d(z,w),$$

for all $x,y,z,w \in X$, $\lambda \in I$. Using the triangle inequality, Lemma 1.5 (d), and the Property (H), we have

$$\begin{aligned} d(W(x,z,\lambda),W(y,w,\lambda)) &\leq d(W(x,z,\lambda),W(x,w,\lambda)) \\ &\quad +d(W(x,w,\lambda),W(y,w,\lambda)) \\ &= d(W(z,x,1-\lambda),W(w,x,1-\lambda)) \\ &\quad +d(W(x,w,\lambda),W(y,w,\lambda)) \\ &\leq (1-\lambda)\,d\,(z,w) + \lambda d\,(x,y) \\ &= \lambda d\,(x,y) + (1-\lambda)\,d\,(z,w)\,. \end{aligned}$$

□

Lemma 1.7. Let X be a uniformly convex metric space satisfying the Property (H). Then, the convex structure W is continuous.

Proof. It has been shown in Lemma 1.6 that

$$d(W(x,z,\lambda),W(y,w,\lambda)) \leq \lambda d\,(x,y) + (1-\lambda)\,d\,(z,w)\,,$$

for all $x,y,z,w \in X$ and $\lambda \in I$.

Let $\{(x_n,y_n,\lambda_n)\}$ be any sequence in $X^2 \times I$ such that $(x_n,y_n,\lambda_n) \to (x,y,\lambda)$ for all $x,y \in X$ and $\lambda \in I$. We show that $W(x_n,y_n,\lambda_n) \to W(x,y,\lambda)$.

An application of Lemma 1.5 (c) and Lemma 1.6 provide:

$$\begin{aligned} d\,(W(x_n,y_n,\lambda_n),W(x,y,\lambda)) &\leq d\,(W(x_n,y_n,\lambda_n),W(x,y,\lambda_n)) \\ &\quad +d\,(W(x,y,\lambda_n),W(x,y,\lambda)) \\ &\leq \lambda_n d\,(x_n,x) + (1-\lambda_n)\,d\,(y_n,y) \\ &\quad +|\lambda_n-\lambda|\,d\,(x,y)\,. \end{aligned}$$

Since $d\,(x_n,x) \to 0$, $d\,(\lambda_n,\lambda) \to 0$ and $|\lambda_n-\lambda| \to 0$, therefore $W(x_n,y_n,\lambda_n) \to W(x,y,\lambda)$. □

Lemma 1.8. Let X be a uniformly convex metric space with modulus of uniform convexity α(decreases for a fixed ε). If $d(x,z) \leq r, d(y,z) \leq r$, and $d\left(z,W\left(x,y,\frac{1}{2}\right)\right) \geq h > 0$ for all $x,y,z \in X$, then $d(x,y) \leq r\eta\left(\frac{r-h}{r}\right)$ where η is the inverse of α.

Proof. Let $d(x,z) \leq r, d(y,z) \leq r$ and $d\left(z,W\left(x,y,\frac{1}{2}\right)\right) \geq h > 0$ for all $x,y,z \in X$. To show that $d(x,y) \leq r\eta\left(\frac{r-h}{r}\right)$, we assume instead that $d(x,y) > r\eta\left(\frac{r-h}{r}\right)$. Take $\frac{r-h}{r} < \varepsilon_1$ such that $d(x,y) \geq r\,\eta\left(\frac{r-h}{r}\right)$. Now using the uniform convexity of X, we have

$$\begin{aligned} d\left(z, W\left(x, y, \frac{1}{2}\right)\right) &\leq (1-\alpha(\eta(\varepsilon_1)))\, r \\ &= (1-\varepsilon_1)\, r \\ &< \left(1 - \frac{r-h}{r}\right) r \\ &= h. \end{aligned}$$

That is,

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) < h,$$

a contradiction to a given inequality. □

Lemma 1.9. Let X be a uniformly convex metric space with modulus of uniform convexity α (decreases for a fixed ε) and satisfies the Property (H). Let $x_1, x_2, x_3 \in B_r[u] \subset X$ and satisfy $d(x_1, x_2) \geq d(x_2, x_3) \geq l > 0$. If

$$d(u, x_2) \geq \left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) r, \tag{1.3}$$

then

$$d(x_1, x_3) \leq \eta\left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) d(x_1, x_2),$$

where η is the inverse of α.

Proof. Denote $z_1 = W\left(x_1, x_2, \frac{1}{2}\right)$, $z_2 = W\left(x_3, x_2, \frac{1}{2}\right)$, and $z = W\left(z_1, z_2, \frac{1}{2}\right)$. By the uniform convexity of X, we have

$$\begin{aligned} d(u, z) &= d\left(u, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq \frac{1}{2} d(u, z_1) + \frac{1}{2} d(u, z_2) \\ &= \frac{1}{2} d\left(u, W\left(x_1, x_2, \frac{1}{2}\right)\right) + \frac{1}{2} d\left(u, W\left(x_3, x_2, \frac{1}{2}\right)\right) \\ &\leq \left(1 - \alpha\left(\frac{l}{r}\right)\right) r. \end{aligned} \tag{1.4}$$

Using (1.4) in (1.3), we get

$$\begin{aligned} d(u, x_2) &\geq \left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) r \\ &= \left(1 - \alpha\left(\frac{l}{r}\right)\right) r + \frac{1}{2}\alpha\left(\frac{l}{r}\right) r \\ &\geq d(u, z) + \frac{1}{2}\alpha\left(\frac{l}{r}\right) r. \end{aligned}$$

That is,

$$\begin{aligned} \frac{1}{2}\alpha\left(\frac{l}{r}\right)r &\leq d(u,x_2)-d(u,z) \\ &\leq d(x_2,z). \end{aligned} \tag{1.5}$$

Since $d(x_2,z_i) \leq \frac{1}{2}d(x_1,x_2)$ for $i = 1,2$, and $d(z_1,z_2) \geq \frac{1}{2}d(x_1,x_2)$, therefore by uniform convexity of X (with $r = \frac{1}{2}d(x_1,x_2), \varepsilon = 1$), Property (H), and (1.5), we have

$$\begin{aligned} \frac{1}{2}\alpha\left(\frac{l}{r}\right)r &\leq d(x_2,z) \\ &= d\left(x_2, W\left(z_1,z_2,\frac{1}{2}\right)\right) \\ &\leq (1-\alpha(1))\,r \\ &\leq \left(1-\alpha\left(\frac{d(z_1,z_2)}{\frac{1}{2}d\,(x_1,x_2)}\right)\right)r \\ &\leq \left(1-\alpha\left(\frac{\frac{1}{2}d(x_1,x_3)}{\frac{1}{2}d(x_1,x_2)}\right)\right)r. \end{aligned}$$

That is,

$$\frac{1}{2}\alpha\left(\frac{l}{r}\right) \leq 1-\alpha\left(\frac{d(x_1,x_3)}{d(x_1,x_2)}\right).$$

Therefore,

$$d(x_1,x_3) \leq \eta\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)d(x_1,x_2),$$

where η is the inverse of α. □

The condition (1.3) in the above lemma holds as indicated by the following example with $\alpha(n) = \frac{n}{2}$.

Example 1.5. Define $d(x,y) = |x-y|$ on $B_1[0] = [-1,1] \subset \mathbb{R}$. Let $u = 0, x_1 = 0.1, x_2 = 0.99$, and $x_3 = 0.3$. Note that $d(x_1,x_2) \geq d(x_2,x_3) \geq 0.2 = l > 0$, $\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)r = 0.95$, and $d(u,x_2) \leq \left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)r$. All the conditions of Lemma 1.9 are satisfied. Moreover, $d(x_1,x_3) \leq \eta\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)d(x_1,x_2)$, where η is the inverse of α.

1.3 Ishikawa Iterative Scheme

Mann [33] and Ishikawa [15] iterative schemes for nonexpansive and quasi-nonexpansive mappings have been extensively studied in a uniformly convex

Banach space. Senter and Dotson [41] established convergence of the Mann iterative scheme of quasi-nonexpansive mappings satisfying two special conditions in a uniformly convex Banach space. A mapping T on a nonempty set C is a *generalized nonexpansive* [4] if

$$d(T(x),T(y)) \leq a\, d(x,y)+b\{d(x,T(x))+d(y,T(y))\}+c\{d(x,T(y))+d(y,T(x))\}, \tag{1.6}$$

for all $x, y \in C$, where $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$.

In 1973, Goebel et al. [12] proved that a generalized nonexpansive mapping has a fixed point in a uniformly convex Banach space. Based on their work, Bose and Mukerjee [4] proved convergence theorems for the Mann iterative scheme of generalized nonexpansive mapping and got the result obtained by Kannan [16] under relaxed conditions. Maiti and Ghosh [32] generalized the results of Bose and Mukerjee [4] for the Ishikawa iterative scheme using a modified version of the conditions of Senter and Dotson [41].

Based on Lemma 1.8 and Lemma 1.9, Fukhar-ud-din et al. [10] have obtained the following fixed point theorem for a continuous mapping satisfying (1.6) in a uniformly convex metric space.

Theorem 1.1. *Let C be a nonempty, closed, convex, and bounded subset of a complete and uniformly convex metric space X satisfying the Property (H). If T is a continuous mapping on C satisfying (1.6), then T has a fixed point in C.*

In this section, we approximate the fixed point of this continuous mapping satisfying (1.6). We assume that C is a nonempty, closed, and convex subset of a convex metric space X, and T is a mapping on C. For an initial value $x_1 \in C$, we define the *Ishikawa iterative scheme* in C as follows:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= W(T(y_n), x_n, \alpha_n), \\ y_n &= W(T(x_n), x_n, \beta_n), \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $\alpha_n, \beta_n \in I$.

If we choose $\beta_n = 0$, then (1.7) reduces to the following *Mann iterative scheme*:

$$x_1 \in C, \quad x_{n+1} = W(T(x_n), x_n, \alpha_n), \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\} \in I$.

On a convex subset C of a linear space X, $W(x, y, \lambda) = \lambda x + (1-\lambda)y$ is a convex structure on X; (1.7) and (1.8), respectively, become Ishikawa [15] and Mann [33] schemes:

$$\begin{aligned} &x_1 \in C, \; x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n), \\ &y_n = (1-\beta_n)x_n + \beta_n T(x_n), \quad n \geq 1, \end{aligned} \tag{1.9}$$

and

$$x_1 \in C, \; x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n), \quad n \geq 1, \tag{1.10}$$

where $\alpha_n, \beta_n \in I$.

In 1989, Maiti and Ghosh [32] generalized the two conditions due to Senter and Dotson [41]. We state all these conditions in a convex metric space:

Condition 1. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that $d(x, T(x)) \geq f(d(x, F))$, for all $x \in C$.

Condition 2. If there exists a real number $k > 0$ such that $d(x, T(x)) \geq k\ d(x, F)$, for all $x \in C$.

Condition 3. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that $d(x, T(y)) \geq f(d(x, F))$, for all $x \in C$ and all corresponding $y = W(T(x), x, \lambda)$, where $0 \leq \lambda \leq \beta < 1$.

Condition 4. If there exists a real number $k > 0$ such that $d(x, T(y)) \geq kd(x, F)$, for all $x \in C$ and all corresponding $y = W(T(x), x, \lambda)$, where $0 \leq \lambda \leq \beta < 1$.

Note that if T satisfies Condition 1 (respectively, 3), then it satisfies Condition 2 (respectively, 4).

We also note that Conditions 1 and 2 become Conditions A and B, respectively, of Senter and Dotson [41], while Conditions 3 and 4 become Conditions I and II, respectively, of Maiti and Ghosh [32] in a normed space. Further, Conditions 3 and 4 reduce to Conditions 1 and 2, respectively, when $\lambda = 0$.

In this section, we present results under relaxed control conditions which generalize the corresponding results of Kannan [16], Bose and Mukerjee [4], and Maiti and Ghosh [32] from uniformly convex Banach spaces to uniformly convex metric spaces. We present sufficient conditions for the convergence of the Ishikawa iterative scheme of L-Lipschitzian mappings to their fixed points in convex metric spaces and improve Lemma 2 in [8]. A necessary and sufficient condition is obtained for the convergence of an arbitrary sequence to a fixed point of a generalized nonexpansive mapping in metric spaces.

The following theorem of Shimizu is needed in the proof of the next result.

Theorem 1.2. [42] Let X be a uniformly convex metric space with a continuous convex structure $W : X^2 \times I \to X$. Then, for arbitrary positive numbers ε and r, there exists $\alpha > 0$ such that

$$d\left(z, W\left(x, y, \lambda\right)\right) \leq r\left(1 - 2\min\{\lambda, 1-\lambda\}\alpha\right), \quad \text{for all } x, y, z \in X,$$

$d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\lambda \in I$.

Lemma 1.10. Let X be a uniformly convex metric space with a continuous convex structure W. Let C be a nonempty, closed, and convex subset of X, T a quasi-nonexpansive mapping on C, and $\{x_n\}$ as in (1.7). If $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $0 \leq \beta_n \leq \beta < 1$, then $\liminf\limits_{n\to\infty} d(x_n, T(y_n)) = 0$.

Proof. For $p \in F$, we consider

$$\begin{aligned} d(x_{n+1}, p) &= d(p, W(T(y_n), x_n, \alpha_n) \\ &\leq \alpha_n d(p, T(y_n)) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n d(p, y_n) + (1-\alpha_n) d(p, x_n) \\ &= \alpha_n d(p, W(T(x_n), x_n, \beta_n)) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n \beta_n d(p, T(x_n)) + \alpha_n (1-\beta_n) d(p, x_n) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n \beta_n d(p, x_n) + \alpha_n (1-\beta_n) d(p, x_n) + (1-\alpha_n) d(p, x_n) \\ &= d(x_n, p). \end{aligned}$$

Since the sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below, therefore $\lim_{n\to\infty} d(x_n, p)$ exists. Assume that $c = \lim_{n\to\infty} d(x_n, p) > 0$.

For any $p \in F$, we have that

$$\begin{aligned} d(x_n, Ty_n) &\leq d(x_n, p) + d(T(y_n), p) \\ &\leq d(x_n, p) + d(y_n, p) \\ &= d(x_n, p) + d(p, W(T(x_n), x_n, \beta_n)) \\ &\leq d(x_n, p) + \beta_n d(T(x_n), p) + (1-\beta_n) d(x_n, p) \\ &\leq d(x_n, p) + \beta_n d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &= 2d(x_n, p). \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, so $d(x_n, T(y_n))$ is bounded, and hence $\inf_{n\geq 1} d(x_n, T(y_n))$ exists. We show that $\inf_{n\geq 1} d(x_n, T(y_n)) = 0$.

Assume that $\inf_{n\geq 1} d(x_n, T(y_n)) = \sigma > 0$. By the definition of infimum, we have

$$\begin{aligned} d(x_n, T(y_n)) &\geq d(x_n, p) \frac{\sigma}{d(x_n, p)} \\ &\geq d(x_n, p) \frac{\sigma}{d(x_1, p)}. \end{aligned}$$

Hence, by Theorem 1.2, there exists $\alpha\left(\frac{\sigma}{d(x_1,p)}\right) > 0$ such that

$$\begin{aligned} d(x_{n+1,}p) &= d(W(T(y_n), x_n, \alpha_n, p) \\ &\leq d(x_n, p)\left(1 - 2\min\{\alpha_n, 1-\alpha_n\}\alpha\right) \\ &\leq d(x_n, p)\left[1 - 2\alpha_n(1-\alpha_n)\alpha\right]. \end{aligned}$$

That is,

$$2c\alpha_n(1-\alpha_n)\alpha \leq \ d(x_n, p) - d(x_{n+1}, p).$$

Taking $m \geqslant 1$ and summing up the $(m+1)$ terms on both sides in the above inequality, we have

$$2c\alpha \sum_{n=1}^{m} \alpha_n (1-\alpha_n) \leq d(p, x_1) - d(p, x_m), \quad \text{for all } m \geq 1.$$

Let $m \to \infty$. Then, we have

$$\infty \leq d(p, x_1) < \infty,$$

a contradiction, and hence $\inf_{n \geq 1} d(x_n, T(y_n)) = 0$.

By the definition of infimum, we can construct subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $\lim_{i \to \infty} d(x_{n_i}, T(y_{n_i})) = 0$, and hence $\liminf_{n \to \infty} d(x_n, T(y_n)) = 0$. □

We state and prove Ishikawa-type convergence results in a uniformly convex metric space.

Theorem 1.3. Let X be a complete uniformly convex metric space with continuous convex structure W, and C be its nonempty closed convex subset. Let T be a continuous quasi-nonexpansive mapping on C satisfying Condition 3. If $\{x_n\}$ is as in (1.7), where $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $0 \leq \beta_n \leq \beta < 1$, then $\{x_n\}$ converges to a fixed point of T.

Proof. In the proof of Lemma 1.10, we have shown that $d(x_{n+1,}p) \leq d(x_n, p)$. Therefore, $d(x_{n+1}, F) \leq d(x_n, F)$. This implies that the sequence $\{d(x_n, F)\}$ is nonincreasing and bounded below. Thus, $\lim_{n \to \infty} d(x_n, F)$ exists. By Condition 3, we have

$$\liminf_{n \to \infty} f\left(d\left(x_n, F\right)\right) \leq \liminf_{n \to \infty} d(T(y_n), x_n) = 0.$$

Using the properties of f, we have $\liminf_{n \to \infty} d\left(x_n, F\right) = 0$. As $\lim_{n \to \infty} d(x_n, F)$ exists, therefore $\lim_{n \to \infty} d(x_n, F) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence. For $\varepsilon > 0$, there exists a constant n_0 such that for all $n \geq n_{0,}$ we have $d(x_n, F) < \frac{\varepsilon}{4}$. In particular, $d(x_{n_0}, F) < \frac{\varepsilon}{4}$. That is, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{4}$. There must exist $p^* \in F$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{2}$. For $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2d(x_{n_0}, p^*) \\ &< \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of a complete metric space X, therefore it must converge to a point q in C.

Finally, we prove that q is a fixed point of T. Since

$$d(q, F) \leq d(q, x_n) + d(x_n, F),$$

$d(q, F) = 0$. As F is closed, so $q \in F$. □

If we choose $\beta_n = 0$ for $n \geq 1$, in the above theorem, it reduces to the following Mann-type convergence result.

Theorem 1.4. Let X be a complete uniformly convex metric space with continuous convex structure W, and C be its nonempty closed convex subset. Let T be a continuous quasi-nonexpansive mapping on C satisfying Condition 1. If $\{x_n\}$ is as in (1.8), where $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\{x_n\}$ converges to a fixed point of T.

Next we establish strong convergence of the Ishikawa iterative scheme of a generalized nonexpansive mapping.

Theorem 1.5. Let C be a nonempty, closed, convex, and bounded subset of a complete uniformly convex metric space X satisfying Property (H). Let T be a continuous generalized nonexpansive mapping on C satisfying (1.6). If $\{x_n\}$ is as in (1.7), where $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $0 \leq \beta_n \leq \beta < 1$, then $\{x_n\}$ converges to a fixed point of T.

Proof. By Theorem 1.1, T has a fixed point p. Setting $y = p$ in (1.6), we have

$$\begin{aligned} d(T(x),p) &\leq (a+c)d(x,p) + b\ d(x,T(x)) + c\ d(T(x),p) \\ &\leq (a+b+c)d(x,p) + (b+c)d(T(x),p), \end{aligned}$$

which implies

$$d(T(x),p) \leq \frac{a+b+c}{1-b-c} d(x,p) \leq d(x,p).$$

Thus, T is quasi-nonexpansive.

For any $y \in C$, we also observe that

$$d(T(y),p) \leq (a+c)d(y,p) + b\ d(y,T(y)) + c\ d(T(y),p). \tag{1.11}$$

If $y = W(T(x),x,\lambda)$, where $0 \leq \lambda \leq \beta < 1$, then

$$\begin{aligned} d(y,p) &= d(W(T(x),x,\lambda),p) \\ &\leq \lambda d(T(x),p) + (1-\lambda)d(x,p) \\ &\leq d(x,p), \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} d(y,x) &= d(W(T(x),x,\lambda),x) \\ &\leq \lambda d(x,T(x)) + (1-\lambda)d(x,x) \\ &= \lambda d(x,T(x)) \\ &\leq \lambda[d(x,p) + d(T(x),p)] \\ &\leq 2\lambda d(x,p). \end{aligned} \tag{1.13}$$

Using (1.12) in (1.11), we have

$$\begin{aligned} d(T(y),p) &\leq (a+c)d(y,p) + b\ d(y,T(y)) + c\ d(T(y),p) \\ &\leq (a+c)d(y,p) + c\{d(x,p) + d(x,T(y))\} + b\{d(x,y) + d(x,T(y))\} \\ &\leq (a+2c)d(x,p) + b\ d(x,y) + (b+c)d(x,T(y)). \end{aligned} \tag{1.14}$$

Also, it is obvious that

$$d(T(y),p) \geq d(x,p) - d(x,T(y)). \tag{1.15}$$

Combining (1.14) and (1.15), we get

$$\begin{aligned} b\ d(x,y) + (1+b+c)d(x,T(y)) &\geq (1-a-2c)d(x,p) \\ &\geq 2b\ d(x,p). \end{aligned} \tag{1.16}$$

By inserting (1.13) in (1.16), we derive

$$\begin{aligned} (1+b+c)d(x,T(y)) &\geq 2b\ d(x,p) - b\ d(x,y) \\ &\geq 2b(1-\lambda)d(x,p). \end{aligned}$$

That is,

$$d(x,T(y)) \geq \frac{2b(1-\lambda)}{1+b+c}d(x,p) \geq \frac{2b(1-\beta)}{1+b+c}d(x,p), \tag{1.17}$$

where $\frac{2b(1-\lambda)}{1+b+c} > 0$, Thus, T satisfies Condition 4 (and hence Condition 3). The result now follows from Theorem 1.3. □

Remark 1.2. In Theorem 1.5, we assumed that the generalized nonexpansive mapping T has a fixed point. It remains an open question: What conditions on a, b, and c in (1.6) are sufficient to guarantee the existence of a fixed point of T even in the setting of a metric space?

Choose $\beta_n = 0$ for $n \geq 1$ in Theorem 1.5 to get the following Mann-type convergence result.

Theorem 1.6. Let X, C, and T be as in Theorem 1.5. If $\{x_n\}$ is as in (1.8), where $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\{x_n\}$ converges to a fixed point of T.

Proof. For $\beta_n = 0$ for $n \geq 1$, $y = W(T(x), x, 0) = x$, the inequality (1.17) in the proof of Theorem 1.5 becomes:

$$d(x,T(x)) \geq \frac{2b}{1+b+c}d(x,p).$$

Thus, T satisfies Condition 2 (and hence Condition 1), and so the result follows from Theorem 1.4. □

An analogue of the Bose and Mukherjee result ([4], Theorem 6) in a uniformly convex metric space can be deduced from Theorem 1.6 (by taking $a = c = 0$, $b = \frac{1}{2}$, and $\alpha_n = \frac{1}{2}$ for $n \geq 1$) as follows:

Theorem 1.7. Let X be a complete uniformly convex metric space with continuous convex structure, and C be its nonempty closed convex subset. Let T be a continuous mapping on C with at least one fixed point such that $d(T(x),T(y)) \leq \frac{1}{2}d(x,T(x)) + \frac{1}{2}d(y,T(y))$, for all $x, y \in C$. Then, the sequence $\{x_n\}$, where $x_1 \in C$ and $x_{n+1} = W(Tx_n, x_n, \frac{1}{2})$, converges to a fixed point of T.

We give sufficient conditions for the existence of a fixed point of an L-Lipschitzian mapping in terms of its Ishikawa iterates.

Theorem 1.8. Let (X, d) be a convex metric space and let C be its nonempty convex subset. Let T be an L-Lipschitzian mapping on C. Let $\{x_n\}$ be the sequence as in (1.7), where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \alpha_n, \beta_n \leq 1$ for $n \geq 1$,

(ii) $\liminf_{n\to\infty} \alpha_n > 0$,

(iii) $\liminf_{n\to\infty} \beta_n < L^{-1}$.

If $d(x_{n+1}, x_n) = \alpha_n d(x_n, T(y_n))$ and $x_n \to p$, then p is a fixed point of T.

Proof. Let $p \in C$. Then,

$$\begin{aligned}
d(p, T(p)) &\leq d(x_n, p) + d(x_n, T(y_n)) + d(T(y_n), T(p)) \\
&= d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n) + d(T(y_n), T(p)) \\
&\leq d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n) + L\ d(y_n, p) \\
&= d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n) + L\ d(W(T(x_n), x_n, \beta_n), p) \\
&\leq d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n) + L\{\beta_n d(T(x_n), p) \\
&\quad +(1-\beta_n) d(x_n, p)\} \\
&\leq d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n) + k\beta_n\{d(T(x_n), T(p)) + d(p, T(p))\} \\
&\quad +L(1-\beta_n) d(x_n, p)\}.
\end{aligned}$$

That is,

$$(1 - L\beta_n) d(p, T(p)) \leq (1 + L^2\beta_n + L(1-\beta_n)) d(x_n, p) + \frac{1}{\alpha_n} d(x_{n+1}, x_n).$$

Since $\liminf \alpha_n > 0$, therefore there exists $\alpha > 0$ such that $\alpha_n > \alpha$, for all $n \geq 1$. This implies that

$$(1 - L\beta_n) d(p, T(p)) \leq (1 + k^2\beta_n + k(1-\beta_n)) d(x_n, p) + \frac{1}{\alpha} d(x_{n+1}, x_n).$$

Taking lim sup on both sides in the above inequality and using the condition $\liminf_{n\to\infty} \beta_n < L^{-1}$, we have $d(p, T(p)) = 0$. □

Finally, using a generalized nonexpansive map T on a metric space X, we provide a necessary and sufficient condition for the convergence of an arbitrary sequence $\{x_n\}$ in X to a fixed point of T in terms of the approximating sequence $\{d(x_n, T(x_n))\}$.

Theorem 1.9. Suppose that C is a closed subset of a complete metric space X and T is a continuous mapping on C such that for some $a, b, c \geq 0$, $a + 2c < 1$, the following inequality holds:

$$d(T(x), T(y)) \leq a\, d(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + c\{d(x, T(y)) + d(y, T(x))\},$$

for all $x, y \in C$. Then, an arbitrary sequence $\{x_n\}$ in C converges to a fixed point of T if and only if $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

Proof. Suppose that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. First we show that $\{x_n\}$ is a Cauchy sequence in C.

To achieve this goal, we consider the following estimate:

$$\begin{aligned} d(T(x_n), T(x_m)) &\leq a\, d(x_n, x_m) + b\{d(x_n, T(x_n)) + d(x_m, T(x_m))\} \\ &\quad + c\{d(x_n, T(x_m)) + d(x_m, T(x_n))\} \\ &\leq a\, d(x_n, x_m) + b\{d(x_n, T(x_n)) + d(x_m, T(x_m))\} \\ &\quad + c\{d((x_n, x_m) + d(x_m, T(x_m)) + d(x_m, x_n) \\ &\quad + d(x_n, T(x_n)\} \\ &= (a + 2c)d(x_n, x_m) \\ &\quad + (b + c)\{d(x_n, T(x_n)) + d(x_m, T(x_m))\} \\ &\leq (a + b + 3c)\{d(x_n, T(x_n)) + d(x_m, T(x_m))\} \\ &\quad + (a + 2c)d(Tx_n, T(x_m)). \end{aligned}$$

That is,

$$(1 - a - 2c)d(T(x_n), T(x_m)) \leq (a + b + 3c)\{d(x_n, T(x_n)) + d(x_m, T(x_m))\}.$$

Since $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $a + 2c < 1$, therefore from the above inequality, it follows that $\{T(x_n)\}$ is a Cauchy sequence in C. In view of the closedness of C, this sequence converges to an element p of C. Also, $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ gives that $\lim_{n\to\infty} x_n = p$. By using the continuity of T, we have $T(p) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = p$. Hence, p is a fixed point of T.

Conversely, suppose that $\{x_n\}$ converges to a fixed point p of T. Using the continuity of T, we have that $\lim_{n\to\infty} T(x_n) = p$. Thus, $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. □

Remark 1.3. Theorem 1.8 improves Lemma 2 in [8] from the real line to the convex metric space setting. Theorem 1.9 is an extension of Theorem 4 in [38] to metric spaces. If we choose $c = 0$ in Theorem 1.9, it is still an improvement of ([38], Theorem 4).

Remark 1.4. In this section, we have established results in a convex metric space. All these results, hold, in particular, in Banach spaces($CAT(0)$ spaces) with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ ($W(x, y, \lambda) = \lambda x \oplus (1 - \lambda)y$).

Remark 1.5. By Lemma 1.7, W satisfying Property (H) is continuous. Hence, Theorem 1.2 holds well for a convex metric space satisfying Property (H). Therefore, all the results proved for continuous convex structure W also hold in a convex metric space satisfying Property (H).

1.4 Multistep Iterative Scheme

Throughout this section, we set $I_0 = \{1, 2, \ldots, r\}$. Suppose that $a_{in} \in I$, $n \geq 1$ and $i \in I_0$. Let $\{T_i : i \in I_0\}$ be a family of asymptotically quasi- nonexpansive mappings on a convex subset C of a Banach space. For an arbitrary $x_1 \in C$, the following scheme has been introduced by Khan et al. [25]:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{rn})x_n + \alpha_{rn}T_k^n(y_{(r-1)n}), \\ y_{(r-1)n} &= (1 - \alpha_{(r-1)n})x_n + \alpha_{(r-1)n}T_{r-1}^n(y_{(r-2)n}), \\ y_{(r-2)n} &= (1 - \alpha_{(r-2)n})x_n + \alpha_{(r-2)n}T_{r-2}^n(y_{(r-3)n}), \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n(y_{1n}), \\ y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n(y_{0n}), \end{aligned} \tag{1.18}$$

where $y_{0n} = x_n$ for all $n \geq 1$.

Very recently, inspired by the scheme (1.18) and the work in [19], Xiao et al. [48] have introduced an $(r + 1)$ step iterative scheme with error terms and studied its strong convergence under weaker boundary conditions.

The existence of fixed (common fixed) points of one mapping (two mappings or a family of mappings) are not known in many situations. So the approximation of a fixed (common fixed) point of one or more nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive mappings by various iterative schemes has been extensively studied in Banach spaces, convex metric spaces and $CAT(0)$ spaces (see [14, 21, 23, 26, 30, 34, 35, 37, 40, 44]).

We translate the scheme (1.18) from the normed space setting to the more general setup of convex metric spaces as follows:

$$x_{n+1} = U_{n(r)}x_n, \quad n \geq 1, \tag{1.19}$$

where

$$\begin{aligned}
U_{n(0)} &= I \text{ (the identity mapping)},\\
U_{n(1)}(x) &= W(T_1^n U_{n(0)}(x), x, a_{n(1)}),\\
U_{n(2)}(x) &= W(T_2^n U_{n(1)}(x), x, a_{n(2)}),\\
\vdots \quad &\quad \vdots\\
U_{n(r-1)}(x) &= W(T_{r-1}^n U_{n(r-2)}(x), x, a_{n(r-1)})\\
U_{n(r)}(x) &= W(T_r^n U_{n(r-1)}(x), x, a_{n(r)}),
\end{aligned}$$

where $0 \le a_{n(i)} \le 1$, for all $i \in I_0$.

In a convex metric space, the scheme (1.19) provides an analogue of:

(a) the scheme (1.8) if $r = 1$ and $T_1 = T$,

(b) the scheme (1.7) if $r = 2, T_1 = T_2 = T$, and

(c) the Xu and Noor [49] scheme if $r = 3, T_1 = T_2 = T_3 = T$.

In this section, it is assumed that $F_1 = \bigcap_{i \in I_0} F(T_i) \neq \emptyset$.

We begin with a technical result.

Lemma 1.11. Let C be a nonempty convex subset of a convex metric space X and $\{T_i : i \in I_0\}$ be a finite family of asymptotically quasi-nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I_0$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then, for the sequence $\{x_n\}$ in (1.19), we have

(a) $d(x_{n+1}, p) \le k_n^r d(x_n, p)$, where $k_n = \max_{1 \le i \le r} k_n(i)$;

(b) $d(x_{n+m}, p) \le s\, d(x_n, p)$, for all $m \ge 1$, $n \ge 1$, $p \in F_1$, and for some $s > 0$.

Proof. (a) It is clear that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ if and only if $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$.

Now for any $p \in F_1$, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(T_r^n U_{n(r-1)}(x_n), x_n, a_{n(r)}), p)\\
&\le a_{n(r)} d(T_r^n U_{n(r-1)}(x_n), p) + (1 - a_{n(r)}) d(x_n, p)\\
&\le a_{n(r)} k_n d(U_{n(r-1)}(x_n), p) + (1 - a_{n(r)}) d(x_n, p)\\
&\le a_{n(r)} a_{n(r-1)} k_n^2 d(U_{n(r-2)}(x_n), p) + (1 - a_{n(r)}) d(x_n, p)\\
&\quad + a_{n(r)}(1 - a_{n(r-1)}) d(x_n, p)\\
&\le a_{n(r)} a_{n(r-1)} k_n^2 d(U_{n(r-2)}(x_n), p) + (1 - a_{n(r)}) d(x_n, p)\\
&\quad + a_{n(r)}(1 - a_{n(r-1)}) k_n^2\ d(x_n, p)\\
&= a_{n(r)} a_{n(r-1)} k_n^2\ d(U_{n(r-2)}(x_n), p)\\
&\quad + (1 - a_{n(r)} a_{n(r-1)}) k_n^2 d(x_n, p)\\
&\quad \cdots\cdots\cdots\\
&\le a_{n(r)} a_{n(r-1)} a_{n(r-2)} \ldots a_{n(1)} {k_n}^r d(p, U_{n(0)}(x_n))\\
&\quad + (1 - a_{n(r)} a_{n(r-1)} a_{n(r-2)} \ldots a_{n(1)}) k_n^r d(x_n, p).
\end{aligned}$$

That is,

$$d(x_{n+1}, p) \leq k_n^r d(x_n, p). \tag{1.20}$$

(b) If $x \geq 1$, then $x \leq \exp(x-1)$. Therefore, it follows from (1.20) that

$$\begin{aligned}
d(x_{n+m}, p) &\leq k_{n+m-1}^r d(x_{n+m-1}, p) \\
&\leq \exp\left((rk_{n+m-1} - r) d(x_{n+m-1}, p)\right) \\
&\leq \exp\left((rk_{n+m-1} - r)[k_{n+m-2}^r d(x_{n+m-2}, p)]\right) \\
&\leq \exp\left((rk_{n+m-1} + rk_{n+m-2} - 2r) d(x_{n+m-2}, p)\right) \\
&\quad \cdots\cdots\cdots \\
&\leq \exp\left(r \sum_{i=n}^{n+m-1} k_i - mr\right) d(x_n, p) \\
&\leq \exp\left(r \sum_{i=n}^{\infty} k_i - r\right) d(x_n, p) \\
&\leq s\ d(x_n, p),
\end{aligned}$$

where $s = \exp\left(r \sum_{i=1}^{\infty} k_i - r\right)$. That is,

$$d(x_{n+m}, p) \leq s\ d(x_n, p), \quad \text{for all } m \geq 1,\ n \geq 1,\ p \in F_1 \text{ and for some } s > 0. \tag{1.21}$$

□

We need the following lemma for further development.

Lemma 1.12. [25, Lemma 1.1] Let $\{a_n\}$ and $\{u_n\}$ be positive sequences of real numbers such that $a_{n+1} \leq (1+u_n)a_n$ and $\sum_{n=1}^{\infty} u_n < +\infty$. Then,

(a) $\lim_{n\to\infty} a_n$ exists,

(b) if $\lim\inf_{n\to\infty} a_n = 0$, then from (a), we get $\lim_{n\to\infty} a_n = 0$.

Theorem 1.10. Let C be a nonempty, closed, and convex subset of a complete convex metric space X, and $\{T_i : i \in I_0\}$ be a finite family of asymptotically quasi-nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I_0$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then, the sequence $\{x_n\}$ in (1.19) converges to $p \in F_1$ if and only if $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Proof. If $\{x_n\}$ converges to $p \in F_1$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F_1) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From (1.20), we have

$$d(x_{n+1}, F_1) \leq k_n^r\ d(x_n, F_1).$$

As $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, so $\lim_{n\to\infty} d(x_n, F_1)$ exists by Lemma 1.12. Now $\liminf_{n\to\infty} d(x_n, F_1) = 0$ reveals that $\lim_{n\to\infty} d(x_n, F_1) = 0$. Hereafter, we show that

$\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F_1) = 0$, there exists an integer n_0 such that

$$d(x_n, F_1) < \frac{\varepsilon}{3s} \quad \text{for all } n \geq n_0,$$

where s is as in Lemma 1.11 (b). In particular,

$$d(x_{n_0}, F_1) < \frac{\varepsilon}{3s}.$$

That is,

$$\inf\{d(x_{n_0}, p) : p \in F_1\} < \frac{\varepsilon}{3s}.$$

So, there must exist $p^* \in F_1$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{2s}.$$

Now, for $n \geq n_0$, we have from the inequality (1.21) that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2s\ d(x_{n_0}, p^*) < 2s\ \frac{\varepsilon}{2s} = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in X. As X is complete and C is closed, $\{x_n\}$ must converge to a point $q \in C$. We claim that $q \in F_1$. Indeed, let $\varepsilon' > 0$. Since $\lim_{n\to\infty} x_n = q$, there exists an integer $n_1 \geq 1$ such that

$$d(x_n, q) < \frac{\varepsilon'}{2k_1}, \quad \text{for all } n \geq n_1. \tag{1.22}$$

Also, $\lim_{n\to\infty} d(x_n, F_1) = 0$ implies that there exists an integer $n_2 \geq 1$ such that

$$d(x_n, F_1) < \frac{\varepsilon'}{7k_1}, \quad \text{for all } n \geq n_2.$$

Hence, there exists $p' \in F_1$ such that

$$d(x_{n_j}, p') < \frac{\varepsilon'}{6k_1}. \tag{1.23}$$

Using (1.22) and (1.23), we have, for any fixed $i \in I_0$,

$$\begin{aligned} d(T_i(q), q) &\leq d(T_i(q), p') + d(p', T_i(x_{n_j})) + d(T_i(x_{n_j}), p') \\ &\quad + d(x_{n_j}, p') + d(x_{n_j}, q) \\ &\leq k_1 d(q, p') + 2k_1 d(x_{n_j}, p') + d(x_{n_j}, q) \\ &\leq k_1 d(q, x_{n_j}) + k_1 d(x_{n_j}, p') \\ &\quad + 2k_1 d(x_{n_j}, p') + d(x_{n_j}, q) \\ &< k_1 \frac{\varepsilon'}{2k_1} + 3k_1 \frac{\varepsilon'}{6k_1} = \varepsilon'. \end{aligned}$$

That is, $d(T_i(q), q) < \varepsilon'$, for any arbitrary ε'. Therefore, we have $d(T_i(q), q) = 0$. Hence, q is a common fixed point of $\{T_i : i \in I_0\}$. □

Note that every quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, so we have:

Corollary 1.1. Let C be a nonempty, closed, and convex subset of a complete convex metric space X, and $\{T_i : i \in I_0\}$ be a finite family of quasi-nonexpansive mappings on C. Define the iteration scheme $\{x_n\}$ as:

$$x_1 \in C, \quad x_{n+1} = U_{n(r)}(x_n), \quad n \geq 1,$$

where,

$$\begin{aligned} U_{n(0)} &= I \text{ (the identity mapping)}, \\ U_{n(1)}(x) &= W(T_1 U_{n(0)}(x), x, a_{n(1)}), \\ U_{n(2)}(x) &= W(T_2 U_{n(1)}(x), x, a_{n(2)}), \\ &\cdots\cdots\cdots \\ U_{n(r-1)}(x) &= W(T_{r-1} U_{n(r-2)}(x), x, a_{n(r-1)}), \\ U_{n(r)}(x) &= W(T_r U_{n(r-1)}(x), x, a_{n(r)}), \end{aligned}$$

where $0 \leq a_{n(i)} \leq 1$, for all $i \in I_0$. Then, the sequence $\{x_n\}$ converges to $p \in F_1$ if and only if $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Since asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive, we get the following extension of Theorem 2.5 in [48].

Corollary 1.2. Let C be a nonempty, closed, and convex subset of a complete convex metric space X, and $\{T_i : i \in I_0\}$ be a finite family of asymptotically nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I_0$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then, the sequence $\{x_n\}$ in (1.19) converges to $p \in F_1$ if and only if $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Our next result is based on the following definition:

Let $\{x_n\}$ be any bounded sequence in a subset C of a metric space X and T be a mapping on C such that $d(x_n, T(x_n)) \to 0$ as $n \to \infty$. Then, T is *semi-compact* if $\{x_n\}$ has a convergent subsequence.

Theorem 1.11. Let C be a nonempty, closed, and convex subset of a complete convex metric space X, and $\{T_i : i \in I_0\}$ be a finite family of asymptotically nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I_0$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then, $\{x_n\}$ in (1.19) converges to $p \in F_1$ if $\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0$, for each $i \in I_0$, and one member of the family $\{T_i : i \in I_0\}$ is semi-compact.

Proof. Without loss of generality, we assume that T_1 is semi-compact. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$. Hence, for any $i \in I_0$, we have

$$\begin{aligned} d(q, T_i(q)) &\leq d(q, x_{n_j}) + d(x_{n_j}, T_i(x_{n_j})) + d(T_i(x_{n_j}), T_i(q)) \\ &\leq (1 + k_{n_j}) d(q, x_{n_j}) + d(x_{n_j}, T_i(x_{n_j})) \to 0. \end{aligned}$$

Thus, $q \in F_1$. By Lemma 1.12, $x_n \to q$. ∎

The scheme (1.18) in $CAT(0)$ spaces is translated as follows:

$$x_1 \in C, \quad x_{n+1} = U_{n(r)}(x_n), \quad n \geq 1, \tag{1.24}$$

where,

$$\begin{aligned}
U_{n(0)} &= I, \text{the identity mapping,} \\
U_{n(1)}(x) &= a_{n(1)} T_1^n U_{n(0)}(x) \oplus (1 - a_{n(1)})x, \\
U_{n(2)}(x) &= a_{n(2)} T_2^n U_{n(1)}(x) \oplus (1 - a_{n(2)})x, \\
& \cdots\cdots\cdots \\
U_{n(r-1)}(x) &= a_{n(r-1)} T_{r-1}^n U_{n(r-2)}(x) \oplus (1 - a_{n(r-1)})x, \\
U_{n(r)}(x) &= a_{n(r)} T_r^n U_{n(r-1)}(x) \oplus (1 - a_{n(r)})x,
\end{aligned}$$

where $0 \leq a_{n(i)} \leq 1$ for each $i \in I_0$.

We prove some lemmas needed in the sequel.

Lemma 1.13. Let C be a nonempty, bounded, closed, and convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I_0\}$ be a family of uniformly L-Lipschitzian mappings on C. Then, for $\{x_n\}$ in (1.24) with $\lim_{n\to\infty} d(x_n, T_i^n(x_n)) = 0$, we have

$$\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0, \quad \text{for each } i \in I_0.$$

Proof. Denote $d(x_n, T_i^n(x_n))$ by $c_n^{(i)}$ for each $i \in I_0$. Then,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, U_{n(r)}(x_n)) \\
&= d(x_n, a_{n(r)} T_r^n U_{n(r-1)}(x_n) \oplus (1 - a_{n(r)})x_n) \\
&\leq d(x_n, T_r^n(x_n)) + d(T_r^n(x_n), T_r^n U_{n(r-1)}(x_n)) \\
&\leq c_n^{(r)} + L\ d(x_n, U_{n(r-1)}(x_n)) \\
&\leq c_n^{(r)} + L\big\{a_{n(r-1)} d(x_n, T_{r-1}^n U_{n(r-2)}(x_n)) \\
&\quad + (1 - a_{n(r-1)}) d(x_n, x_n)\big\} \\
&\leq c_n^{(r)} + L\ a_{n(r-1)} d(x_n, T_{r-1}^n U_{n(r-2)}(x_n)) \\
&\leq c_n^{(r)} + L\ a_{n(r-1)} \big\{d(x_n, T_{r-1}^n(x_n)) \\
&\quad + d(T_{r-1}^n(x_n), T_{r-1}^n U_{n(r-2)}(x_n))\big\} \\
&\leq c_n^{(r)} + L\ c_n^{(r-1)} + L^2 d(x_n, U_{n(r-2)}(x_n)).
\end{aligned}$$

Continuing in this way, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq c_n^{(r)} + L c_n^{(r-1)} + L^2 c_n^{(r-2)} + \cdots + L^r d(x_n, T_1^n(x_n)) \\
&\leq c_n^{(r)} + L c_n^{(r-1)} + L^2 c_n^{(r-2)} + \cdots + L^r c_n^{(1)}.
\end{aligned}$$

Taking $\limsup$ on both sides, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.25}$$

Further, observe that

$$\begin{aligned} d(x_n, T_i(x_n)) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1}(x_{n+1})) \\ &+ d(T_i^{n+1}x_{n+1}, T_i^{n+1}(x_n)) + d(T_i^{n+1}(x_n), T_i(x_n)) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1}(x_{n+1})) + L\ d(x_{n+1}, x_n) \quad (1.26) \\ &+ L\ d(x_n, T_i^n(x_n)) \\ &= (1+L)d(x_n, x_{n+1}) + c_{n+1}^{(i)} + Lc_n^{(i)}. \end{aligned}$$

Taking lim sup on both sides in (1.26) and using (1.25) and $\lim_{n\to\infty} c_n^{(i)} = 0$, we get

$$\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0, \quad \text{for each } i \in I_0.$$

□

Lemma 1.14. Let C be a nonempty, bounded, closed, and convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I_0\}$ be a family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$ for each $i \in I_0$. Then, for the sequence $\{x_n\}$ in (1.24) with $0 < \delta \le a_{n(i)} \le 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \text{for each } i \in I_0.$$

Proof. Take $p \in F_1$ and apply the inequality (CN*) to the scheme (1.24) to get:

$$\begin{aligned} d(x_{n+1}, p)^2 &= d(a_{n(r)}T_r^n U_{n(r-1)}(x_n) \oplus (1 - a_{n(r)})x_n), p)^2 \\ &\le a_{n(r)} d(T_r^n U_{n(r-1)}(x_n), p)^2 + (1 - a_{n(r)})d(x_n, p)^2 \\ &\quad -a_{n(r)}(1 - a_{n(r)})d(x_n, T_r^n U_{n(r-1)}(x_n))^2 \\ &\le a_{n(r)}k_n^2\ d(U_{n(r-1)}(x_n), p)^2 + (1 - a_{n(r)})d(x_n, p)^2 \\ &\quad -a_{n(r)}(1 - a_{n(r)})d(x_n, T_r^n U_{n(r-1)}(x_n))^2 \\ &= a_{n(r)}k_n^2\ d(a_{n(r-1)}T_{r-1}^n U_{n(r-2)}(x_n) \oplus (1 - a_{n(r-1)})x_n), p)^2 \\ &\quad +(1 - a_{n(r)})d(x_n, p)^2 \\ &\quad -a_{n(r)}(1 - a_{n(r)})d(x_n, T_r^n U_{n(r-1)}(x_n))^2 \\ &\le a_{n(r)}k_n^2\ [a_{n(r-1)}d(p, T_{r-1}^n U_{n(r-2)}(x_n))^2 \\ &\quad +(1 - a_{n(r-1)})d(p, x_n)^2 \\ &\quad -a_{n(r-1)}(1 - a_{n(r-1)})d(x_n, T_{r-1}^n U_{n(r-2)}(x_n))^2] \\ &\quad +(1 - a_{n(r)})d(x_n, p)^2 \\ &\quad -a_{n(r)}(1 - a_{n(r)})d(x_n, T_r^n U_{n(r-1)}(x_n))^2. \end{aligned}$$

That is,

$$\begin{aligned} d(x_{n+1},p)^2 \quad \leq \quad & a_{n(r)}a_{n(r-1)}\left(k_n^2\right)^2 d(U_{n(r-2)}(x_n),p)^2 \\ & + \left[a_{n(r)}(1-a_{n(r-1)})k_n^2 + (1-a_{n(r)})\right] d(x_n,p)^2 \\ & - a_{n(r)}a_{n(r-1)}(1-a_{n(r-1)})d(x_n, T_{r-1}^n U_{n(r-2)}(x_n))^2 \\ & - a_{n(r)}(1-a_{n(r)})d(x_n, T_r^n U_{n(r-1)}(x_n))^2. \end{aligned}$$

Applying the inequality (CN*) to the scheme (1.24) r-times, we get:

$$\begin{aligned} d(x_{n+1},p)^2 \quad \leq \quad & \Big[\Pi_{i=1}^r a_{n(i)} + \left\{\Pi_{i=2}^r a_{n(i)} - \Pi_{i=1}^r a_{n(i)}\right\} \\ & + \left\{\Pi_{i=3}^r a_{n(i)} - \Pi_{i=2}^r a_{n(i)}\right\} \\ & + \cdots + \left\{a_{n(r)} - a_{n(r)}a_{n(r-1)})\right\}\Big] \left(k_n^2\right)^r d(x_n,p)^2 \\ & - (1-a_{n(1)})\Pi_{i=1}^r a_{n(i)} d(x_n, T_1^n x_n)^2 \\ & - (1-a_{n(2)})\Pi_{i=2}^r a_{n(i)} d(x_n, T_2^n U_{n(1)} x_n)^2 \\ & \cdots\cdots\cdots \\ & \cdots\cdots\cdots \\ & - (1-a_{n(r)})a_{n(r)} d(x_n, T_r^n U_{n(r-1)} x_n)^2. \end{aligned}$$

From the above computation, we have the following r inequalities:

$$d(x_{n+1},p)^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - (1-a_{n(1)})\Pi_{i=1}^r a_{n(i)} d(x_n, T_1^n x_n)^2 \qquad (1)$$

$$d(x_{n+1},p)^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - (1-a_{n(2)})\Pi_{i=2}^r a_{n(i)} d(x_n, T_2^n U_{n(1)}(x_n))^2 \qquad (2)$$

.....

$$\begin{aligned} d(x_{n+1},p)^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - a_{n(r)}a_{n(r-1)}(1-a_{n(r-1)}) \\ \times d(x_n, T_{r-1}^n U_{n(r-2)}(x_n))^2. \end{aligned} \qquad \text{(r-1)}$$

$$d(x_{n+1},p)^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - a_{n(r)}(1-a_{n(r)}) d(x_n, T_r^n U_{n(r-1)}(x_n))^2. \qquad \text{(r)}$$

Using $\delta \leq a_{n(i)} \leq 1-\delta$, in the above (1) – (r) inequalities and then rearranging the terms, we have:

$$\delta^{r+1} d(x_n, T_1^n(x_n))^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - d(x_{n+1},p)^2 \qquad (1^*)$$

$$\delta^r d(x_n, T_2^n U_{n(1)}(x_n))^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - d(x_{n+1},p)^2 \qquad (2^*)$$

...

$$\delta^2 d(x_n, T_r^n U_{n(r-1)}(x_n))^2 \leq \left(k_n^2\right)^r d(x_n,p)^2 - d(x_{n+1},p)^2. \qquad (r^*)$$

Since the sequence $\{d(x_n,p)\}$ is convergent and $k_n \to 1$, therefore from the inequalities (1*) – (r*), we deduce:

$$\lim_{n\to\infty} d(x_n, T_i^n U_{n(i-1)}(x_n)) = 0, \quad \text{for all } i \in I_0. \qquad (1.27)$$

Further,

$$\begin{aligned} d(x_n, T_2^n(x_n)) &\le d(x_n, T_2^n U_{n(1)}(x_n)) + d(T_2^n U_{n(1)}(x_n), T_2^n(x_n)) \\ &\le d(x_n, T_2^n U_{n(1)}(x_n)) + L\ d(a_{n(1)} T_1^n(x_n) \oplus (1 - a_{n(1)}) x_n, x_n) \\ &\le d(x_n, T_2^n U_{n(1)}(x_n)) + L\ a_{n(1)} d(T_1^n(x_n), x_n), \end{aligned}$$

together with (1.27)(for $i = 2$) gives that

$$\lim_{n\to\infty} d(x_n, T_2^n(x_n)) = 0.$$

Similar computations show that

$$\lim_{n\to\infty} d(x_n, T_i^n(x_n)) = 0, \quad \text{for each } i \in I_0.$$

Finally, by Lemma 1.13, we get:

$$\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0, \quad \text{for each } i \in I_0.$$

□

For further analysis, we need the following concept:

A family of mappings $\{T_i : i \in I_0\}$ on a subset C of a metric space X with at least one common fixed point is said to satisfy Condition (AV) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F_1)) \le \frac{1}{r} \sum_{i=1}^{r} d(x, T_i(x)), \quad \text{for all } x \in C.$$

Theorem 1.12. Let C be a nonempty, closed, and convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I_0\}$ be a family of uniformly $L-$Lipschitzian asymptotically quasi-nonexpansive mappings on C with sequences $\{k_n(i)\} \subset [1, \infty)$, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$ for each $i \in I_0$, respectively. If $\{T_i : i \in I_0\}$ satisfies the Condition (AV), then the sequence $\{x_n\}$ in (1.24) with $0 < \delta \le a_{n(i)} \le 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$, converges to a common fixed point of $\{T_i : i \in I_0\}$.

Proof. Immediate from Lemma 1.14 and Theorem 1.10. □

1.5 One-Step Implicit Iterative Scheme

In order to reduce computational cost of a two-step iterative scheme for two finite families $\{S_n : n \in I_0\}$ and $\{T_n : n \in I_0\}$ of nonexpansive mappings on a convex subset C of a Banach space, Khan et al. [20] introduced and analyzed the following one-step iterative scheme

$$x_0 \in K,\ x_n = \alpha_n x_{n-1} + \beta_n T_n(x_n) + \gamma_n S_n x_n, \tag{1.28}$$

where $S_n = S_{n(modN)}$ and $T_n = T_{n(modN)}, 0 \leq \alpha_n,\ \beta_n,\ \gamma_n \leq 1$ and satisfy $\alpha_n + \beta_n + \gamma_n = 1$. For a convex subset C of a convex metric space X, we translate the iterative scheme (1.28) as under:

$$x_n = W\left(T_n(x_n), W\left(S_n x_n, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{1.29}$$

where $S_n = S_{n(modN)}$ and $T_n = T_{n(modN)}$, $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ and satisfy $\alpha_n + \beta_n < 1$.

Obviously, (1.29) is equivalent to (1.28) in the Banach space setting.

The following concepts are needed in the sequel.

A sequence $\{x_n\}$ in X is *Fejér monotone* with respect to a subset C of a metric space X if $d(x_n, x) \leq d(x_{n-1}, x)$, for all $x \in C$.

For a bounded sequence $\{x_n\}$ in a metric space X, we define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by $r(x, \{x_n\}) = \limsup\limits_{n\to\infty} d(x, x_n)$, for all $x \in X$. The *asymptotic radius* of $\{x_n\}$ with respect to $C \subseteq X$ is defined as

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

A point $y \in C$ is called the *asymptotic center* of $\{x_n\}$ with respect to $C \subseteq X$ if $r(y, \{x_n\} \leq r(x, \{x_n\})$ for all $x \in C$. The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

The notion of $\triangle-$convergence has been introduced by Lim [31] and it has been adapted in $CAT(0)$ spaces by Kirk and Panyanak [29]:

A sequence $\{x_n\}$ in X is said to $\triangle$-*converge* to $x \in X$ if x is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write x as $\triangle$-limit of $\{x_n\}$, that is, $\triangle - \lim_n x_n = x$.

Let H be a mapping on C defined by

$$H(x) = W\left(T_1(x), W\left(S_1 x, x_0, \frac{\beta_1}{1-\alpha_1}\right), \alpha_1\right).$$

For a given $x_0 \in C$, the existence of $x_1 = W\left(T_1(x_1), W\left(S_1 x_1, x_0, \frac{\beta_1}{1-\alpha_1}\right), \alpha_1\right)$ is guaranteed if H has a fixed point. For any $u, v \in C$ and using property (3) in the definition of hyperbolic space, we have:

$$\begin{aligned} d(H(u), H(v)) &\leq (1-\alpha_1)\, d\Bigg(W\left(S_1(u), x_0, \frac{\beta_1}{1-\alpha_1}\right), \\ &\qquad W\left(S_1(v), x_0, \frac{\beta_1}{1-\alpha_1}\right)\Bigg) + \alpha_1 d(T_1(u), T_1(v)) \\ &\leq \alpha_1 d(T_1(u), T_1(v)) + \beta_1 d(S_1(u), S_1(v)) \\ &\leq (\alpha_1 + \beta_1)\, d(u, v). \end{aligned}$$

Since $\alpha_1 + \beta_1 \in (0,1)$, H is a contraction. By (BCP), H has a unique fixed point. Thus the existence of x_1 is established. Continuing in this way, the existence of $x_2, x_3, \ldots$ is guaranteed. Hence, (1.29) is well-defined.

Some needed key results are listed below for further development.

Lemma 1.15. [24] Let C be a nonempty, closed, and convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in C such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$ (a real number), then $\lim_{m\to\infty} y_m = y$.

Lemma 1.16. [24] Let X be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{a_n\}$ be in $[b,c]$ for some $b, c \in (0,1)$. If $\{w_n\}$ and $\{z_n\}$ are in X such that

$$\limsup_{n\longrightarrow\infty} d(w_n, x) \le r, \quad \limsup_{n\longrightarrow\infty} d(z_n, x) \le r,$$

and

$$\lim_{n\longrightarrow\infty} d(W(w_n, z_n, a_n), x) = r, \quad \text{for some } r \ge 0,$$

then,

$$\lim_{n\to\infty} d(w_n, z_n) = 0.$$

Lemma 1.17. [2] Let C be a nonempty closed subset of a complete metric space X, and $\{x_n\}$ be a Fejér monotone sequence with respect to C. Then, $\{x_n\}$ converges to some $p \in C$ if and only if $\lim_{n\to\infty} d(x_n, C) = 0$. From now onward, for finite families $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ of nonexpansive mappings on C, we set $F_2 = \bigcap_{i\in I_0} F(T_i) \cap F(S_i)$.

We now prove a pair of technical results.

Lemma 1.18. Let C be a closed and convex subset of a convex metric space X, and $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be two finite families of nonexpansive mappings on C such that $F_2 \neq \emptyset$. Then, for the sequence $\{x_n\}$ in (1.29), we have

(a) $\{x_n\}$ is Fejér monotone with respect to F_2,

(b) $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F_2$.

Proof. For any $p \in F_2$, it follows from (1.29) that

$$\begin{aligned} d(x_n, p) &= d\left(W\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\le \alpha_n d(T_n(x_n), p) + (1-\alpha_n) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\le \alpha_n d(T_n(x_n), p) + \beta_n d(S_n(x_n), p) + (1-\alpha_n-\beta_n) d(x_{n-1}, p) \\ &\le \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n) d(x_{n-1}, p). \end{aligned}$$

This implies $(1-\alpha_n-\beta_n)d(x_n,p) \leq (1-\alpha_n-\beta_n)d(x_{n-1},p)$, and so, we get $d(x_n,p) \leq d(x_{n-1},p)$, which gives that (a) $\{x_n\}$ is Fejér monotone with respect to F_2 and (b) $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F_2$. $\square$

Lemma 1.19. Let C be a closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be two finite families of nonexpansive mappings on C such that $F_2 \neq \emptyset$. Then, for the sequence $\{x_n\}$ in (1.29), we have

$$\lim_{n\to\infty} d\left(x_n, S_l(x_n)\right) = 0 = \lim_{n\to\infty} d\left(x_n, T_l(x_n)\right), \quad \text{for each } l \in I_0.$$

Proof. It follows from Lemma 1.18 that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F_2$. Assume that $\lim_{n\to\infty} d(x_n,p) = c$. The result is trivial if $c = 0$. If $c > 0$, then $\lim_{n\to\infty} d(x_n,p) = c$ can be written as

$$\lim_{n\to\infty} d\left(W\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{1.30}$$

Since T_n is nonexpansive, we have $d(T_n(x_n),p) \leq d(x_n,p)$ for each $p \in F_2$. Taking lim sup on both sides, we obtain $\limsup_{n\to\infty} d(T_n(x_n),p) \leq c$.

Since $d(x_n,p) \leq d(x_{n-1},p)$, it follows that

$$\begin{aligned}
& d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
\leq\ & \frac{\beta_n}{1-\alpha_n} d(S_n(x_n),p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) \\
\leq\ & \frac{\beta_n}{1-\alpha_n} d(x_n,p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) \\
\leq\ & \frac{\beta_n}{1-\alpha_n} d(x_{n-1},p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) \\
=\ & d(x_{n-1},p).
\end{aligned}$$

That is,

$$\limsup_{n\to\infty} d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{1.31}$$

Taking $x = p$, $r = c$, $a_n = \alpha_n$, $w_n = T_n(x_n)$, and $z_n = W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right)$ in Lemma 1.16 and using (1.30) and (1.31), we get

$$\lim_{n\to\infty} d\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{1.32}$$

Observe that

$$\begin{aligned} d(x_n, T_n(x_n)) &\leq d\left(W\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), T_n(x_n)\right) \\ &\leq \alpha_n d\left(T_n(x_n), T_n(x_n)\right) \\ &\quad +(1-\alpha_n) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), T_n(x_n)\right) \\ &= (1-\alpha_n) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), T_n(x_n)\right) \\ &\leq (1-a) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), T_n(x_n)\right). \end{aligned}$$

Taking lim sup on both sides in the above inequality and using (1.32), we have

$$\lim_{n\to\infty} d(x_n, T_n(x_n)) = 0. \tag{1.33}$$

Moreover,

$$\begin{aligned} d(x_n, p) &= d\left(W\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_n(x_n), p) + (1-\alpha_n) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right). \end{aligned}$$

Hence, we obtain

$$c \leq \liminf_{n\to\infty} d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{1.34}$$

Combining (1.31) and (1.34), we get

$$\lim_{n\to\infty} d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) = c. \tag{1.35}$$

Again, by Lemma 1.16 (with $x = p$, $r = c$, $a_n = \frac{\beta_n}{1-\alpha_n}$, $w_n = S_n x_n$, $z_n = x_{n-1}$), we get

$$\lim_{n\to\infty} d(x_{n-1}, S_n(x_n)) = 0. \tag{1.36}$$

Observe that

$$\begin{aligned} d(x_n, x_{n-1}) &\leq d\left(W\left(T_n(x_n), W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_{n-1}\right) \\ &\leq \alpha_n d(T_n(x_n), x_{n-1}) \\ &\quad + (1-\alpha_n)\, d\left(W\left(S_n(x_n), x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), x_{n-1}\right) \\ &\leq \alpha_n \left\{d\left(T_n(x_n), x_n\right) + d\left(x_n, x_{n-1}\right)\right\} + \beta_n d\left(x_{n-1}, S_n(x_n)\right). \end{aligned}$$

Re-arranging the terms in the above inequality, we have

$$\begin{aligned} d(x_n, x_{n-1}) &\le \frac{\alpha_n}{1-\alpha_n} d\left(T_n(x_n), x_n\right) + \beta_n d\left(x_{n-1}, S_n(x_n)\right) \\ &\le \frac{b}{1-b} d\left(T_n(x_n), x_n\right) + b\, d\left(x_{n-1}, S_n(x_n)\right). \end{aligned}$$

Taking lim sup on both the sides in the above inequality and then using (1.33) and (1.36), we have

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0.$$

For each $l < r$, we have

$$d(x_n, x_{n+l}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+l-1}, x_{n+l}).$$

Therefore, we get

$$\lim_{n\to\infty} d(x_n, x_{n+l}) = 0, \quad \text{for each } l < r. \tag{1.37}$$

Since $d(x_n, S_n(x_n)) \le d(x_n, x_{n-1}) + d(x_{n-1}, S_n(x_n))$, it follows that

$$\lim_{n\to\infty} d(x_n, S_n(x_n)) = 0. \tag{1.38}$$

Also note that

$$\begin{aligned} d(x_n, S_{n+l}(x_n)) &\le d(x_n, x_{n+l}) + d(x_{n+l}, S_{n+l}(x_{n+l})) + d(S_{n+l}(x_{n+l}), S_{n+l}(x_n)) \\ &\le 2d(x_n, x_{n+l}) + d(x_{n+l}, S_{n+l}(x_{n+l})). \end{aligned}$$

Taking lim sup on both sides in the above inequality and then using (1.37) and (1.38), we get

$$\lim_{n\to\infty} d(x_n, S_{n+l}(x_n)) = 0, \quad \text{for each } l \in I_0.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0, \quad \text{for each } l \in I_0.$$

Since for each $l \in I_0$, the sequence $\{d(x_n, S_l(x_n))\}$ is a subsequence of $\bigcup_{l=1}^{r}\{d(x_n, S_{n+l}(x_n))\}$ and $\lim_{n\to\infty} d(x_n, S_{n+l}(x_n)) = 0$, for each $l \in I_0$, therefore,

$$\lim_{n\to\infty} d(x_n, S_l(x_n)) = 0 = \lim_{n\to\infty} d(x_n, T_l(x_n)), \quad \text{for each } l \in I_0.$$

□

Now we state a result concerning $\triangle-$convergence for the iterative scheme (1.29); the method of its proof closely follows ([24], Theorem 3.1).

Theorem 1.13. Let C be a closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be two finite families of nonexpansive mappings on C such that $F_2 \neq \emptyset$. Then, the sequence $\{x_n\}$ in (1.29) $\triangle$-converges to an element of F_2.

Proof. As in the proof of Lemma 1.18, $\{x_n\}$ is bounded. Therefore, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then, by Lemma 1.19, we have

$$\lim_{n\to\infty} d(u_n, T_l(u_n)) = 0 = \lim_{n\to\infty} d(u_n, S_l(u_n)), \quad \text{for each } l \in I_0.$$

We claim that u is the common fixed point of $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$. We define a sequence $\{z_m\}$ in C by $z_m = T_m(u)$, where $T_m = T_{m(modN)}$. Observe that

$$\begin{aligned} d(z_m, u_n) &\leq d(T_m(u), T_m(u_n)) + d(T_m(u_n), T_{m-1}(u_n)) + \cdots + d(T(u_n), u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i(u_n)). \end{aligned}$$

Therefore, we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.15 that $T_m(u) = u$. Hence, u is the common fixed point of $\{T_i : i \in I_0\}$. Similarly, we can show that u is the common fixed point of $\{S_i : i \in I\}$. Hence, $u \in F_2$. Suppose $x \neq u$. Since $\lim_{n\to\infty} d(x_n, u)$ exists (by Lemma 1.18), the uniqueness of the asymptotic center gives

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

which is a contradiction. Hence, $x = u$. Therefore, $A(\{u_n\}) = \{u\}$, for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to a common fixed point of $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$. ☐

A necessary and sufficient condition for strong convergence of the iterative scheme (1.29) is given in the following result.

Theorem 1.14. Let C be a closed and convex subset of a complete convex metric space X, and $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be two finite families of nonexpansive mappings on C such that $F_2 \neq \emptyset$. Then, the sequence $\{x_n\}$ in (1.29), converges strongly to $p \in F_2$ if and only if $\lim \inf_{n\to\infty} d(x_n, F_2) = 0$.

Proof. It follows from Lemma 1.18 that $\{x_n\}$ is Fejér monotone with respect to F_2 and $\lim_{n\to\infty} d(x_n, F_2)$ exists. Moreover, F_2 is a closed subset of X. Hence, the result follows from Lemma 1.17. □

Let f be a nondecreasing mapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$. Let $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be finite families of nonexpansive mappings on C with $F_2 \neq \emptyset$. Then, the two families are said to satisfy *condition (D)* if

$$d(x, T(x)) \geq f(d(x, F_2)) \text{ or } d(x, S(x)) \geq f(d(x, F_2)), \quad \text{for all } x \in C,$$

holds for at least for one $T \in \{T_i : i \in I_0\}$ or one $S \in \{S_i : i \in I_0\}$.

The statement in the following remark follows easily from Lemma 1.19.

Remark 1.6. Let C be a closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ be two finite families of nonexpansive mappings on C such that $F_2 \neq \emptyset$. If either the two families $\{T_i : i \in I_0\}$ and $\{S_i : i \in I_0\}$ satisfy condition (D) or one of the mappings in $\{T_i : i \in I_0\}$ or $\{S_i : i \in I_0\}$ is semi-compact, then the sequence $\{x_n\}$ in (1.29) converges to an element of F_2.

Remark 1.7. By Lemma 1.6, a uniformly convex metric space satisfying the Property (H) is a uniformly convex hyperbolic space. Therefore, all the results of this section hold well in uniformly convex metric spaces admitting the Property (H).

Remark 1.8. The reader interested in approximation of common fixed points of a countable family of nonexpansive mappings in a uniformly convex metric space is referred to Phuengrattana and Suantai [37].

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Chapter 2

Fixed Points of Nonlinear Semigroups in Modular Function Spaces

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2.1 Introduction

Nonlinear semigroup theory is not only of intrinsic interest, but is also important in the study of evolution problems. In the last forty years, the general theory of flows of holomorphic mappings has been of great interest in the theory of Markov stochastic branching processes, the theory of composition operators, control theory, and optimization. It transpires that the asymptotic behavior of solutions to evolution equations is applicable to the study of the geometry of certain domains in complex spaces. In this chapter, we will discuss the existence of common fixed points of nonlinear semigroups acting in modular function spaces. For the theory of a common fixed point of mappings, we refer to the excellent papers [5, 6, 10, 12, 41].

Modular function spaces are natural generalizations of both function and sequence variants of many important, from an applications perspective, spaces like Calderon-Lozanovskii, Lebesgue, Lorentz, Musielak-Orlicz, Orlicz-Lorentz spaces, and many others. In recent years, modular function spaces have seen a surge of interest due to electrorheological fluids (sometimes referred to as "smart fluids"). Indeed, most materials can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneities this is not adequate, but rather the exponent p should be able to vary. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids (for instance, lithium polymethacrylate) was due to Willis Winslow in 1949. Electrorheological fluids have been used in robotics and space technology. For more information on properties, modeling, and the application of variable exponent spaces to these fluids we refer the reader to [2, 11, 13, 49].

2.2 Basic Definitions and Properties

First attempts to generalize the classical function spaces of the Lebesgue type L^p were made in the early 1930s by Orlicz and Birnbaum in connection with orthogonal expansions. Their approach consisted in considering spaces of functions with some growth properties different from the power-type growth control provided by the L^p-norms. Namely, they considered the function spaces defined as follows:

$$L^\varphi = \left\{ f : \mathbb{R} \to \mathbb{R};\ \exists \lambda > 0\ :\ \int_{\mathbb{R}} \varphi\big(\lambda |f(x)|\big)\, dx < \infty \right\},$$

where $\varphi : [0, \infty] \to [0, \infty]$ was assumed to be a convex function increasing to infinity, that is, a function that to some extent behaves similarly to power functions $\varphi(t) = t^p$. Later on, the assumption of convexity for Orlicz functions φ was frequently omitted. Let us mention two typical examples of such functions:

$$\varphi(t) = e^t - 1\ , \qquad \varphi(t) = \ln(1 + t)\ .$$

The possibility of introducing the structure of a linear metric in L^φ as well as the interesting properties of these spaces and many applications to differential and integral equations with kernels of non-power types were among the reasons for the development of the theory of Orlicz spaces, their applications and generalizations for more than half of the century.

We observe two principal directions of further development. The first one is a theory of Banach function spaces initiated in 1955 by Luxemburg and then developed in a series of joint papers with Zaanen. The main idea of that theory consists of considering a function space L of all functions $f : X \to \mathbb{R}$, $f \in M(X, \mathbb{R})$, such that $||f|| < \infty$, where (X, Σ, μ) is a measure space,

$M(X, S)$ denotes the space of all strongly measurable functions acting from X into a Banach space S and $||.||$ is a function norm that satisfies

$$||f|| \leq ||g|| \quad \text{whenever } |f(x)| \leq |g(x)| \ \mu\text{-a.e.}$$

Another approach, also inspired by the successful theory of Orlicz spaces, is based on replacing the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. This idea was the basis of the theory of modular spaces initiated by Nakano [45] in connection with the theory of order spaces and redefined and generalized by Luxemburg and Orlicz in 1959. Let us give a brief account of some basic facts of their theory.

Definition 2.1. [45] Let $\mathcal{X}$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a *modular*, if for arbitrary f and g, in $\mathcal{X}$, the following holds:

(1) $\rho(f) = 0$ if and only if $f = 0$,

(2) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$, and

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha,\ \beta \in [0, 1]$ such that $\alpha + \beta = 1$.

The modular ρ is called *convex* if and only if

$$\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g),$$

whenever $\alpha,\ \beta \in [0, 1]$ such that $\alpha + \beta = 1$.

If ρ is a modular in $\mathcal{X}$, then the set defined by

$$\mathcal{X}_\rho = \{h \in \mathcal{X}\ ;\ \lim_{\lambda \to 0} \rho(\lambda h) = 0\}$$

is called a *modular space.* $\mathcal{X}_\rho$ is a vector subspace of $\mathcal{X}$. For a modular ρ in $\mathcal{X}$, we may define an F-norm by the formula :

$$||f||_\rho = \inf \left\{ t > 0\ ;\ \rho\left(\frac{f}{t}\right) \leq t \right\}.$$

If ρ is a convex modular, then the functional given by

$$||f||_\rho = \inf \left\{ t > 0\ ;\ \rho\left(\frac{f}{t}\right) \leq 1 \right\}$$

is a norm. One can check that

$$||f_n - f||_\rho \to 0 \ \text{ is equivalent to } \ \rho\big(\lambda(f_n - f)\big) \to 0, \ \text{ for all } \lambda > 0.$$

In this way, any Orlicz space becomes a modular space, where $\mathcal{X} = M(X, \mathbb{R})$ and the modular ρ is defined by

$$\rho(f) = \int_{\mathbb{R}} \varphi\left(|f(x)|\right) dx.$$

Within the theory of modular spaces, Musielak and Orlicz developed in 1959 a theory of Musielak-Orlicz spaces, that is, the modular spaces induced by modulars of the following form:

$$\rho(f) = \int_{\mathbb{R}} \varphi\left(x, |f(x)|\right) dx,$$

where $\varphi : X \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function, continuous and increasing to infinity in the second variable, and is measurable in the first one. Such spaces have been studied for almost forty years and there is a large set of applications of such spaces that is known in variuous parts of analysis. Such spaces have many applications in probability and mathematical statistics.

We may observe, however, the situation where on the one hand we have a very abstract general theory of modular spaces that cannot give proper answers to many interesting questions and, on the other hand, spaces constructed on the image of Musielak-Orlicz spaces. In the latter case, the concepts from Musielak-Orlicz theory do not fit the new demands. Another common difficulty consists of the fact that the theory of Musielak-Orlicz spaces, though very useful, is not structural, in the sense that many operations, like taking sums or passing to equivalent modulars, lead beyond the class of Musielak-Orlicz spaces.

In this chapter, we will consider modular spaces that lie somewhere in between, that is, class of modular spaces given by modulars not of any particular form, but nevertheless, having much more convenient properties than the abstract modulars can possess. In other words, we present a useful tool for applications whenever there is a need to introduce a function space by means of functionals that have some reasonable properties but are far from being norms or F-norms.

Let us introduce basic notions related to modular function spaces and related notations that will be used in this chapter. For further detail, we refer the reader to preliminary sections of the recent articles [8, 24, 25] or to the survey articles [38, 39]; see also [34, 35, 36] for the standard framework of modular function spaces.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω such that $E \cap A \in \mathcal{P}$, for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$, for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.2. [36] Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex, and even function. We say that ρ is a *regular convex function pseudomodular* if:

(i) $\rho(0) = 0$;

(ii) ρ is monotone, that is, $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(iii) ρ is orthogonally subadditive, that is, $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;

(iv) ρ has the Fatou property, that is, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$; and

(v) ρ is order continuous in $\mathcal{E}$, that is, $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly, as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind, we define $\mathcal{M} = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \ \rho - a.e\}$, where each element is actually an equivalence class of functions equal to ρ-a.e. rather than an individual function.

Definition 2.3. [36] We say that a regular function pseudomodular ρ is a *regular convex function modular* if $\rho(f) = 0$ implies $f = 0 \ \rho - a.e.$. The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Throughout this chapter, we only consider convex function modulars.

Definition 2.4. [34, 35, 36] Let ρ be a convex function modular. A modular function space is the vector space $L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}$. In the vector space L_ρ, the following formula

$$\|f\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm, frequently called the *Luxemburg norm.*

Note that the monographic exposition of the theory of Orlicz spaces may be found in Krasnosel'skii and Rutickii [40]. For a current review of the theory of Musielak-Orlicz spaces and modular spaces the reader is referred to Musielak [44] and Kozlowski [36].

The following definitions will be needed throughout this chapter.

Definition 2.5. [36]

(a) The sequence $\{f_n\} \subset L_\rho$ is said to be *ρ-convergent* to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(b) The sequence $\{f_n\} \subset L_\rho$ is said to be *ρ-Cauchy* if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.

(c) We say that L_ρ is ρ-*complete* if and only if any ρ-Cauchy sequence in L_ρ is ρ-convergent.

(d) A subset C of L_ρ is called ρ-*closed* if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(e) A subset C of L_ρ is called ρ-*bounded* if

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty.$$

(f) Let $f \in L_\rho$ and $C \subset L_\rho$. Define the ρ-*distance* between f and C as:

$$d_\rho(f, C) = \inf\{\rho(f-g); g \in C\}.$$

Warning: The above terminology is used because of its formal similarity to the metric case. Since ρ does not behave in general as a distance, one should be very careful when dealing with these notions. In particular, ρ-convergence does not imply ρ-Cauchy, since ρ does not satisfy the triangle inequality.

The following proposition brings together a few facts that will often be used in the proofs.

Proposition 2.1. [36] Let $\rho \in \Re$.

(a) L_ρ is ρ-complete.

(b) ρ-balls $B_\rho(f, r) = \{g \in L_\rho; \rho(f-g) \leq r\}$ are ρ-closed.

(c) If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0 \ \rho - a.e.$

(d) $\rho(f) \leq \liminf_{n\to\infty} \rho(f_n)$ whenever $f_n \to f \ \rho - a.e.$ (Note: this property is equivalent to the Fatou property).

(e) Consider the sets

$$L_\rho^0 = \{f \in L_\rho; \rho(f, .) \text{ is order continuous}\}$$

and

$$E_\rho = \{f \in L_\rho; \lambda f \in L_\rho^0 \text{ for any } \lambda > 0\}.$$

Then, we have

(i) $E_\rho \subset L_\rho^0 \subset L_\rho$;

(ii) E_ρ has the Lebesgue property, that is, $\rho(\lambda f, D_n) \to 0$, for $\lambda > 0$, $f \in E_\rho$, and $D_n \downarrow \emptyset$.

We already pointed out that since ρ may not satisfy the triangle inequality, then the modular convergence and F-norm convergence may not be the same. This will only happen if ρ satisfies the so-called Δ_2-*condition.*

Definition 2.6. [36] The modular function ρ is said to satisfy the Δ_2-*condition* if $\rho(2f_n) \to 0$ as $n \to \infty$, whenever $\rho(f_n) \to 0$ as $n \to \infty$.

We have the following proposition.

Proposition 2.2. [36] The following assertions are equivalent:

(a) ρ satisfies the Δ_2-condition;

(b) $\rho(f_n - f) \to 0$ is equivalent to $\rho(\lambda(f_n - f)) \to 0$ for all $\lambda > 0$.

Let us recall the definition of different mappings acting in a modular function space. We start with the concept of Lipschitzian mappings.

Definition 2.7. Let $\rho \in \Re$ and let $C \subset L_\rho$ be a nonempty subset. A mapping $T : C \to L_\rho$ is called *ρ-Lipschitzian mapping* if there exists a constant $L \geq 0$ such that

$$\rho(T(f) - T(g)) \leq L\, \rho(f - g), \quad \text{for any } f, g \in C.$$

When $L < 1$, then T is called *ρ-contraction mapping.* Moreover, if $L \leq 1$, then T is called *ρ-nonexpansive mapping.*

As mentioned before, one of the reasons for our interest in ρ-behavior of mappings is that the F-norm associated with the function modular is defined in an indirect way and consequently harder to handle than the function modular. Therefore, one may ask what the relationship is, if any, between the F-norm nonexpansiveness and the ρ-nonexpansiveness. The following example gives a partial answer.

Example 2.1. [26] Let $X = (0, \infty)$, and let Σ be the σ-algebra of all Lebesgue measurable subsets of X. Let $\mathcal{P}$ denote the δ-ring of subsets of finite measure. Define a function modular by

$$\rho(f) = \frac{1}{e^2} \int_0^\infty |f(x)|^{x+1} dm(x).$$

Let B be the set of all measurable functions $f : (0, \infty) \to \mathbb{R}$ such that $0 \leq f(x) \leq \frac{1}{2}$. Define the linear operator T by the formula

$$T(f)(x) = \begin{cases} f(x-1), & \text{for} \quad x \geq 1, \\ 0, & \text{for} \quad x \in [0, 1]. \end{cases}$$

Clearly, $T(B) \subset B$. We claim that, for every $\lambda \leq 1$ and for all $f, g \in B$, we have

$$\rho\left(\lambda(T(f) - T(g))\right) \leq \lambda \rho\left(\lambda(f - g)\right).$$

Indeed,

$$\begin{aligned}
\rho\left(\lambda(T(f)-T(g))\right) &= e^{-2}\int_0^\infty \lambda^{x+1}|T(f(x))-T(g(x))|^{x+1}dm(x)\\
&= e^{-2}\int_1^\infty \lambda^{x+1}|f(x-1)-g(x-1)|^{x+1}dm(x)\\
&= \lambda e^{-2}\int_0^\infty \lambda^{x+1}|f(x)-g(x)|^{x+1}|f(x)-g(x)|dm(x)\\
&= \lambda e^{-2}\int_0^\infty \lambda^{x+1}|f(x)-g(x)|^{x+1}dm(x)\\
&= \lambda\rho\left(\lambda(f-g)\right),
\end{aligned}$$

which implies that T is ρ-nonexpansive. On the other hand, if we take $f = 1_{[0,1]}$, then

$$\|T(f)\|_\rho > e \geq \|f\|_\rho,$$

which clearly implies that T is not $\|.\|$-nonexpansive.

In the metric setting, Kirk [31, 32] extended the definition of Lipschitzian mappings to a new class of mappings, called pointwise Lipschitzian mappings. Here we give the modular analogue to these new mappings.

Definition 2.8. Let $\rho \in \Re$ and let $C \subset L_\rho$ be a nonempty subset. A mapping $T : C \to C$ is called *ρ-pointwise Lipschitzian mapping* if there exists a function $\alpha : C \to [0, \infty)$ such that

$$\rho(T(x) - T(y)) \leq \alpha(x)\rho(x - y), \quad \text{for all } x, y \in C.$$

If the function $\alpha(x) < 1$, for every $x \in C$, then we say that T is *ρ-pointwise contraction*. Similarly, if $\alpha(x) \leq 1$ for every $x \in C$, then T is said to be *ρ-pointwise nonexpansive mapping*.

Similarly, we have the following extension of asymptotic pointwise mappings from the metric setting to the modular setting [18].

Definition 2.9. Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a *ρ-asymptotic pointwise Lipschitzian mapping* if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\rho(T^n(f) - T^n(g)) \leq \alpha_n(f)\rho(f - g), \quad \text{for any } f, g \in L_\rho.$$

- If $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$, then T is called *ρ-asymptotic pointwise contraction.*
- If $\limsup_{n\to\infty} \alpha_n(f) \leq 1$ for any $f \in L_\rho$, then T is called *ρ-asymptotic pointwise nonexpansive.*

- If $\limsup_{n\to\infty} \alpha_n(f) \le k$ for any $f \in L_\rho$ with $0 < k < 1$, then T is called *ρ-strongly asymptotic pointwise contraction.*

A point $f \in C$ is called a *fixed point* of T whenever $T(f) = f$. The set of fixed points of T will be denoted by $F(T)$.

2.3 Some Geometric Properties of Modular Function Spaces

This section is devoted to the discussion of the modular equivalents of uniform convexity of ρ. As pointed out in [25], one concept of uniform convexity in normed spaces generates several different types of uniform convexity in modular function spaces. In this chapter, we mainly focus on one form of uniform convexity. For more details, the reader may consult [25].

Definition 2.10. Let $\rho \in \Re$. We define the following *uniform convexity (UC)-type properties* of the function modular ρ:

- Let $r > 0$, $\varepsilon > 0$. Define

$$D(r,\varepsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f-g) \ge \varepsilon r\}.$$

  Let

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\Big(\frac{f+g}{2}\Big) : (f,g) \in D(r,\varepsilon)\right\}, \text{ if } D(r,\varepsilon) \neq \emptyset,$$

  and $\delta(r,\varepsilon) = 1$, if $D(r,\varepsilon) = \emptyset$. We say that ρ satisfies (UC) if for every $r > 0$, and $\varepsilon > 0$, we have $\delta(r,\varepsilon) > 0$. Note that for every $r > 0$, $D(r,\varepsilon) \neq \emptyset$, when $\varepsilon > 0$ is small enough.

- We say that ρ satisfies (UUC) if there exists $\eta(s,\varepsilon) > 0$, for every $s \ge 0$, and $\varepsilon > 0$ such that

$$\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0, \text{ for } r > s.$$

It is known that for a wide class of modular function spaces with the Δ_2 property, the uniform convexity of the Luxemburg norm is equivalent to (UC). For example, in Orlicz spaces, this result can be traced to early papers by Luxemburg [42], Milnes [43], Akimovic [3], and Kaminska [19]. It is also known that, under suitable assumptions, (UC) in Orlicz spaces is equivalent to the very convexity of the Orlicz function [27] and that the uniform convexity of the Orlicz function implies (UC) [19]. Typical examples of Orlicz functions that do not satisfy the Δ_2 condition, but are uniformly convex (and hence very convex), are [40, 43]:

$$\varphi_1(t) = e^{|t|} - |t| - 1, \text{ and } \varphi_2(t) = e^{t^2} - 1.$$

See also [17] for a discussion of some geometric properties of Calderon-Lozanovskii and Orlicz-Lorentz spaces. As for Banach spaces, uniform convexity in the modular sense implies strict convexity, that is, for every $f, g \in L_\rho$ such that $\rho(f) = \rho(g)$ and $\rho(\alpha f + (1-\alpha)g) = \alpha\rho(f) + (1-\alpha)\rho(g)$, for some $\alpha \in (0,1)$, there holds $f = g$.

In the next theorem, a relationship between the uniform convexity of function modulars and the unique best approximant property is established. This is very useful as its analogue in the case of Banach spaces. For example, it allows us to prove the property (R) (see Definition 2.11 below). For other results on best approximation in modular function spaces, see for example [28].

Theorem 2.1. [25] Assume $\rho \in \Re$ is (UUC). Let $C \subset L_\rho$ be nonempty, convex, and ρ-closed. Let $f \in L_\rho$ be such that

$$d_\rho(f, C) = \inf\{\rho(f-g); g \in C\} < \infty.$$

Then, there exists a unique best ρ-approximant of f in C, that is, a unique $g_0 \in C$ such that

$$\rho(f - g_0) = d_\rho(f, C).$$

Proof. Uniqueness follows from the strict convexity of ρ. Let us prove the existence of the ρ-approximant. Since C is ρ-closed, we may assume, without loss of any generality, that $d = d_\rho(f, C) > 0$. Clearly there exists a sequence $\{f_n\} \in C$ such that

$$\rho(f - f_n) \le d\left(1 + \frac{1}{n}\right).$$

We claim that $\{f_n\}$ is ρ-Cauchy. Assume to the contrary that this is not the case. Then, there exists an $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\rho\left(f_{n_k} - f_{n_p}\right) \ge \varepsilon_0,$$

for any $p, k \ge 1$. Since ρ is uniformly convex, we get

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \le \left(1 - \delta\left(d(k,p), \frac{\varepsilon_0}{d(k,p)}\right)\right) d(k,p),$$

where $d(k,p) = \left(1 + \dfrac{1}{\min(n_p, n_k)}\right) d$. For $p, k \ge 1$, we have $d(k,p) \le 2d$. Hence,

$$\delta\left(d(k,p), \frac{\varepsilon_0}{d(k,p)}\right) \ge \delta\left(d(k,p), \frac{\varepsilon_0}{2d}\right).$$

Since ρ is (UUC), then there exists $\eta > 0$ such that

$$\delta\left(r, \frac{\varepsilon_0}{2d}\right) \geq \eta, \quad \text{for any } r > d/3.$$

Since $d(k,p) \geq d > d/3$, we get

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leq (1-\eta)\, d(k,p), \quad \text{for any } k, p \geq 1.$$

By the convexity of C, $\frac{f_{n_k} + f_{n_p}}{2} \in C$. Using the definition of d, we get

$$d \leq \rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leq (1-\eta)\, d(k,p), \quad \text{for any } k, p \geq 1.$$

If we let k, p go to infinity, we get $d \leq (1-\eta)d$, which is impossible. Hence, $\{f_n\}$ is ρ-Cauchy. Since L_ρ is ρ-complete, $\{f_n\}$ ρ-converges to a $g \in L_\rho$. Since C is ρ-closed, we get $g \in C$. By the Fatou property, we get

$$\rho(f-g) \leq \liminf_{n\to\infty} \rho\,(f - f_n) \leq d.$$

Hence, $\rho(f-g) \leq d$. Since $g \in C$, we get $d \leq \rho(f-g)$. Therefore, $\rho(f-g) = d$. In other words, $g_0 = g$ is the ρ-approximant of f in C. □

So far, this result was used to prove some geometric properties in modular function spaces. But there is another application that did not attract much attention. This is known as the *Dirichlet energy problem*, which we discuss in the next example.

Example 2.2. Let $\Omega \subset \mathbb{R}$ be an open set and let $p : \Omega \to [1, \infty)$ be a measurable function (called the variable exponent on Ω). We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\rho(\lambda f) = \int_\Omega |\lambda f(x)|^{p(x)} dx < \infty,$$

for some $\lambda > 0$. The functional ρ is called the modular of the space $L^{p(\cdot)}(\Omega)$. The Luxemburg norm on this space is given by the formula

$$\|f\| = \inf\{\lambda > 0; \rho(\lambda f) < \infty\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that f and the distributional derivative f' are in $L^{p(\cdot)}(\Omega)$. The function

$$\rho_1(f) = \rho(f) + \rho(f')$$

defines a module on $W^{1,p(\cdot)}(\Omega)$. Define $W_0^{1,p(\cdot)}(\Omega)$ as the set of $f \in W^{1,p(\cdot)}(\Omega)$, which can be continued by 0 outside Ω. The energy operator corresponding to the boundary value function g acting on the space

$$\{f \in W^{1,p(.)}(\Omega); f - g \in W_0^{1,p(.)}(\Omega)\}$$

is defined by

$$I_g(f) = \int_\Omega |f'(x)|^{p(x)} dx = \rho(f').$$

The *general Dirichlet energy problem* is to find a function that minimizes values of the operator $I_g(.)$. Note that

$$\min\{I_g(g - f); f \in W_0^{1,p(.)}(\Omega)\} = d_\rho\Big(g, W_0^{1,p(.)}(\Omega)\Big).$$

For more information on the Dirichlet energy integral problem, we refer to [2, 15, 16, 50].

The following technical lemma is very useful (see [25] for more details).

Lemma 2.1. Let $\rho \in \Re$ be (UUC). Let $R > 0$. Assume that $\{f_n\}$ and $\{g_n\}$ are in L_ρ such that

$$\limsup_{n\to\infty} \rho(f_n) \leq R; \;\; \limsup_{n\to\infty} \rho(g_n) \leq R; \;\; \text{and} \;\; \lim_{n\to\infty} \rho\left(\frac{f_n + g_n}{2}\right) = R.$$

Then, we must have $\lim_{n\to\infty} \rho(f_n - g_n) = 0$.

The following property plays a role in the theory of modular function spaces, a role similar to reflexivity in Banach spaces (see, for example [27]).

Definition 2.11. We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has a nonempty intersection.

The following theorem is the modular analogue to the well-known result that states that uniformly convex Banach spaces are reflexive.

Theorem 2.2. [25] Let $\rho \in \Re$ be (UUC). Then, L_ρ has property (R).

Proof. Let $\{C_n\}$ be a nonincreasing sequence of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ. Let $f \in L_\rho$, then we have $\sup_{n\geq 1} d_\rho(f, C_n) < \infty$. Let us prove that $\bigcap_{n\geq 1} C_n \neq \emptyset$. Using the proximinality of ρ-closed convex subsets of L_ρ (Theorem 2.1), for every $n \geq 1$ there exists $f_n \in C_n$ such that $\rho(f - f_n) = d_\rho(f, C_n)$. It is easy to show that $\{d_\rho(f, C_n)\}$ is nondecreasing and bounded. Hence, $\lim_{n\to\infty} d_\rho(f, C_n) = d$ exists. If $d = 0$, then $d_\rho(f, C_n) = 0$, for any $n \geq 1$. Since $\{C_n\}$ are ρ-closed, we get $f \in C_n$ for any $n \geq 1$, which implies $\bigcap_{n\geq 1} C_n \neq \emptyset$. Therefore, we can assume $d > 0$. In this case we claim that $\{f_n\}$ is ρ-Cauchy. Indeed, if we assume not, then there exists an $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\rho\left(f_{n_k} - f_{n_p}\right) \geq \varepsilon_0$, for any $p, k \geq 1$. Since ρ is (UUC), we get

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leq \left(1 - \delta\left(d, \frac{\varepsilon_0}{d}\right)\right) d, \quad \text{for any } p, k \geq 1.$$

So

$$d_\rho(f, C_{\min(n_p, n_k)}) \leq \rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leq \left(1 - \delta\left(d, \frac{\varepsilon_0}{d}\right)\right) d,$$

for any $p, k \geq 1$. If we let $p, k \to \infty$, we will get

$$d \leq \left(1 - \delta\left(d, \frac{\varepsilon_0}{d}\right)\right) d,$$

which is a contradiction because $\delta\left(d, \frac{\varepsilon_0}{d}\right) > 0$. Hence, $\{f_n\}$ is ρ-Cauchy and it ρ-converges to some $g \in L_\rho$. Let us prove that $g \in C_n$, for any $n \geq 1$. Indeed, we have $f_k \in C_n$, for any $k \geq n$. Fix $n \geq 1$. Since $\{f_k\}_{k\geq n}$ ρ-converges to g as $k \to \infty$, and C_n is ρ-closed, then $g \in C_n$, for any $n \geq 1$. Hence, $\bigcap_{n\geq 1} C_n \neq \emptyset$. $\square$

Another geometric property that plays a major role in establishing some fixed-point results in modular function spaces (as it did in Banach spaces) is the Opial property.

Definition 2.12. We will say that L_ρ satisfies the ρ-a.e.-Opial property if for every $\{f_n\} \subset L_\rho$ ρ-a.e.-convergent to 0 such that there exists $k > 1$ for which

$$\sup_n \ \rho(kf_n) = M < \infty,$$

then for every $f \in E_\rho$ not equal to 0, we have

$$\liminf_{n\to\infty} \ \rho(f_n) < \liminf_{n\to\infty} \ \rho(f_n + f).$$

We will say that L_ρ satisfies the *ρ-a.e.-uniform Opial property* if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every $\{f_n\} \subset L_\rho$ ρ-a.e.-convergent to 0 such that there exists $k > 1$ for which

$$\sup_n \ \rho(kf_n) = M < \infty,$$

then for every $f \in E_\rho$ such that $\rho(f) \geq \epsilon$, we have

$$\liminf_{n\to\infty} \ \rho(f_n) + \eta \leq \liminf_{n\to\infty} \ \rho(f_n + f).$$

Note that in [21], it was proved that every convex, orthogonally additive function modular ρ has the strong Opial property, which allowed the author to prove a new fixed-point result. Let us recall that ρ is called *orthogonally additive* if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$, whenever $A \cap B = \emptyset$.

Definition 2.13. We will say that L_ρ satisfies the *ρ-a.e.-Strong Opial property* (or *SO*-Property) if for every $\{f_n\} \in L_\rho$ that is ρ-a.e.-convergent to 0 such that there exists a $\beta > 1$ for which

$$\sup\{\rho(\beta f_n)\} < \infty, \tag{2.1}$$

the following equality holds for any $g \in E_\rho$

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g). \tag{2.2}$$

Note that the Opial property in the norm sense does not necessarily hold for several classical Banach function spaces. For instance, Opial, in his original paper [47], showed that L^p spaces, for $p \geq 1$ except for $p = 2$, fails the Opial property for the weak topology, while the modular Strong Opial property holds in L^p, for all $p \geq 1$. The ρ-a.e.-Strong Opial property can also be defined and proved for nonconvex regular function modulars, for example, for some s-convex modulars, like L^s for $0 < s < 1$, [9, 21].

A typical method of proof for fixed-point theorems is to construct a fixed point by finding an element on which a specific type of function attains its minimum. To be able to proceed with this method, one has to know that such an element indeed exists. In modular function spaces, the ρ-types are not in general lower semicontinuous in any strong or weak sense, and therefore one needs additional assumptions to ensure that ρ-types attain their minima. It turns out that, for ρ-a.e.-compact sets C, the Strong Opial property can be such a convenient additional assumption.

Recall the definition of a ρ-type, a powerful technical tool that is used in the proofs of many fixed point results in modular function spaces.

Definition 2.14. Let $C \subset L_\rho$ be a nonempty subset. A function $\tau : C \to [0, \infty]$ is called a *ρ-type* (or a *type*) if there exists a sequence $\{x_n\}$ of elements of C such that for any $x \in C$, the following holds:

$$\tau(x) = \limsup_{n\to\infty} \rho(x_n - x).$$

We have the following result.

Theorem 2.3. [37] Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e.-Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-bounded, and ρ-a.e.-compact convex set. Then, any ρ-type $\tau : C \to [0, \infty]$ attains its minimum in C, that is, there exists $x_0 \in C$ such that

$$\tau(x_0) = \inf\{\tau(x); x \in C\}.$$

2.4 Some Fixed-Point Theorems in Modular Spaces

In this section, we discuss the existence of fixed points for mappings that are nonexpansive or contractive in the modular sense [23]. Certainly, one can also consider mappings that are contractive with respect to the F-norm induced by the modular. We should like to mention that, generally speaking, there is no natural relation between the two kinds of nonexpansiveness. Once again, we would like to emphasize our philosophy that all the results expressed in terms of modulars are more convenient in the sense that their assumptions are much easier to verify.

The first fixed-point result in modular function spaces is the analog-to-Banach contraction principle.

Theorem 2.4. [26] Let C be a ρ-complete ρ-bounded subset of L_ρ, and $T : C \to C$ be a ρ-strict contraction. Then, T has a unique fixed point $z \in C$. Moreover, z is the ρ-limit of the iterate of any point in C under the action of T.

We may relax the assumption regarding the boundedness of C and assume there exists a bounded orbit instead. In this case, the uniqueness of the fixed point is dropped and replaced by: *"if f and g are two fixed points of T such that $\rho(f-g) < \infty$, then $f = g$."*

Next we present some fixed-point theorems for pointwise and asymptotic pointwise contractions in modular function spaces. Theorem 2.5 assumes uniform continuity of ρ.

Definition 2.15. [33] We will say that the function modular ρ is *uniformly continuous* if for every $\epsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h+g)| \leq \epsilon, \quad \text{whenever } \rho(h) \leq \delta \text{ and } \rho(g) \leq L.$$

Let us mention that uniform continuity holds for a large class of function modulars. For instance, it can be proved that in Orlicz spaces over a finite atomless measure [51] or in sequence Orlicz spaces [19] the uniform continuity of the Orlicz modular is equivalent to the Δ_2-type condition.

Theorem 2.5. Let us assume that $\rho \in \Re$ is uniformly continuous and has Property (R) . Let $K \subset L_\rho$ be nonempty, convex, ρ-closed, and ρ-bounded. Let $T : K \to K$ be a pointwise ρ-contraction. Then, T has a unique fixed point $x_0 \in K$. Moreover, the orbit $\{T^n(x)\}$ ρ-converges to x_0, for any $x \in K$.

By using the ρ-a.e.-Strong Opial property of the function modular, the authors [24] proved the next fixed-point theorem, which does not assume uniform continuity of ρ.

Theorem 2.6. Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e.-Strong Opial property. Let $K \subset E_\rho$ be a nonempty, ρ-a.e.-compact convex subset such that there exists $\beta > 1$ such that $\delta_\rho(\beta\ K) = \sup\{\rho(\beta(x-y)); x, y \in K\} < \infty$. Then, any $T : K \to K$ pointwise ρ-contraction has a unique fixed point $x_0 \in K$. Moreover, the orbit $\{T^n(x)\}$ ρ-converges to x_0, for any $x \in K$.

In the next two theorems, the authors [24] deal with asymptotic pointwise contractions in modular function spaces.

Theorem 2.7. Let us assume that $\rho \in \Re$ is uniformly continuous and has property (R) . Let $K \subset L_\rho$ be nonempty, convex, ρ-closed, and ρ-bounded. Let $T : K \to K$ be an asymptotic pointwise ρ-contraction. Then, T has a unique fixed point $x_0 \in K$. Moreover, the orbit $\{T^n(x)\}$ ρ-converges to x_0, for any $x \in K$.

Similarly as it was done in the case of pointwise contractions, the ρ-a.e.-Strong Opial property of the function modular is assumed to prove the next fixed-point theorem for asymptotic pointwise contractions.

Theorem 2.8. Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e.-Strong Opial property. Let $K \subset E_\rho$ be a nonempty, ρ-a.e.-compact convex subset such that there exists $\beta > 1$ such that $\delta_\rho(\beta\ K) = \sup\{\rho(\beta(x-y)); x, y \in K\} < \infty$. Then, any $T : K \to K$ asymptotic pointwise ρ-contraction has a unique fixed point $x_0 \in K$. Moreover, the orbit $\{T^n(x)\}$ ρ-converges to x_0, for any $x \in K$.

Next we discuss the case of ρ-nonexpansive mappings. Before we state the first result, we need the following definitions [26].

Definition 2.16. The *growth function* w_ρ of a function modular ρ is defined as follows:

$$w_\rho(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)}; f \in L_\rho, 0 < \rho(f) < \infty\right\}, \quad t \geq 0.$$

Observe that $w_\rho(t) \leq 1$, for all $t \in [0,1]$. We say that ρ satisfies the *regular growth condition* if $w_\rho(t) < 1$ for all $t \in [0,1)$.

The class of function modulars that satisfies the regular growth condition is quite large. For instance, if ρ is convex, $\rho(tf) \leq t\rho(f)$ for $t \in [0,1]$, and consequently $w_\rho(t) \leq t < 1$ for $t < 1$. Thus, all convex function modulars satisfy the regular growth condition. It is not hard to prove that if ρ is an Orlicz modular, then in the case of a finite measure, ρ satisfies the regular growth condition if and only if

$$\limsup_{s\to\infty} \frac{\varphi(ts)}{\varphi(s)} < 1\ , \quad \text{for all } t \in [0,1),$$

where φ denotes the Orlicz function associated with ρ. If there exists a constant $K > 0$ such that $\rho(2f) \leq K\rho(f)$ for all $f \in L_\rho$, then w_ρ is submultiplicative and hence there exists $p > 1$ such that $w_\rho(t) \leq t^p$ for $t < 1$. Consequently, such function modulars also satisfy the regular growth condition.

Recall that a set $B \subset L_\rho$ is said to be *star-shaped* if there exists $f \in B$ such that $f + \lambda(g - f) \in B$, whenever $\lambda \in [0,1]$ and $g \in B$.

Theorem 2.9. [26] Assume that ρ has the Fatou property and satisfies the regular growth condition. Let B be a star-shaped ρ-bounded and ρ-a.e.-compact subset of L_ρ such that $B - B \subset L^0_\rho$. Assume in addition that for every sequence of functions $f_n \in B$ such that $f_n \to f$ ρ-a.e. with $f \in B$ and for every sequence of sets $G_k \downarrow \emptyset$,

$$\lim_{k\to\infty} \left(\sup_n \rho(f_n - f, G_k) \right) = 0.$$

If $T : B \to B$ is ρ-nonexpansive, then T has a fixed point.

The classic Kirk's fixed-point theorem for nonexpansive mappings [29] in Banach spaces relies heavily on the weak topological compactness assumption. In order to prove an analogue to Kirk's fixed-point theorem in the modular setting, one major difficulty is to overcome such topology. But most of the compactness assumptions made in the modular case were of sequential nature. Since these sequential compactness assumptions are not generated from a topology, many authors had to use the constructive proof developed by Kirk [30]. It is worth mentioning that Kirk's fixed-point theorem in modular spaces is the first example where the constructive proof was used.

Theorem 2.10. [29] Let $\rho \in \Re$. Assume ρ has the Fatou property and let $C \subset L_\rho$ be a nonempty, ρ-bounded subset of L_ρ. Assume that B has ρ-normal structure and is (ρ-a.e.)-compact. If $T : B \to B$ is ρ-nonexpansive, then T has a fixed point.

As for the asymptotic pointwise nonexpansive mappings, we have the following result.

Theorem 2.11. [25] Assume $\rho \in \Re$ is (UUC). Let C be a ρ-closed, ρ-bounded, convex nonempty subset of L_ρ. Then, any $T : C \to C$ asymptotic pointwise nonexpansive has a fixed point. Moreover, the set of all fixed points $Fix(T)$ is ρ-closed.

2.5 Semigroups in Modular Function Spaces

Semigroups of mappings are quite typical objects that used in mathematics and applications. For instance, in the theory of dynamical systems, the modular function space L_ρ would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system.

The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question of whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes, what the structure of a set of such points may look like. In the setting of this chapter, the state space may be an infinite dimensional space. Therefore, it is natural to apply these results not only to deterministic dynamical systems but also to stochastic dynamical systems.

Let us start with the formal definition of semigroups of mappings.

Definition 2.17. Let $\rho \in \Re$ and $C \subset L_\rho$ be a nonempty subset. A one-parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ of mappings from C into itself is said to be a *ρ-Lipschitzian* (respectively, *ρ-nonexpansive*) semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$, for $x \in C$ and $t, s \geq 0$;

(iii) for each $t \geq 0$, T_t is ρ-Lipschitzian (respectively, ρ-nonexpansive).

A point $x \in C$ is called a *fixed point* of $\mathcal{F}$ if and only if $T_t(x) = x$, for any $t \geq 0$. The set of fixed points of $\mathcal{F}$ will be denoted by $F(\mathcal{F})$. This set is also called the set of common fixed points of $\mathcal{F}$.

This definition is extended to the case of asymptotic pointwise nonexpansive setting.

Definition 2.18. Let $\rho \in \Re$ and $C \subset L_\rho$ be a nonempty subset. A one-parameter family $\mathcal{F} = \{T_t : t \geq 0\}$ of mappings from C into itself is said to be *ρ-asymptotic pointwise nonexpansive semigroup* on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(f) = f$ for $f \in C$;

(ii) $T_{t+s}(f) = T_t(T_s(f))$, for $f \in C$ and $t, s \in [0, \infty)$;

(iii) for each $t \geq 0$, T_t is ρ-asymptotic pointwise nonexpansive mapping, i.e., there exists a function $\alpha_t : C \to [0, \infty)$ such that

$$\rho(T_t(f) - T_t(g)) \leq \alpha_t(f)\rho(f - g), \quad \text{for all } f, g \in C,$$

such that $\limsup_{t\to\infty} \alpha_t(f) \leq 1$, for every $f \in C$;

(iv) for each $f \in C$, the mapping $t \to T_t(f)$ is ρ-continuous.

Note that without loss of generality we may assume $\alpha_t(f) \geq 1$ for any $t \geq 0$ and $f \in C$ and $\limsup_{t\to\infty} \alpha_t(f) = \lim_{t\to\infty} \alpha_t(f) = 1$.

The existence and the behavior of nonlinear semigroups in modular spaces was first investigated in [22]. The construction of such semigroups mirrored the work done in Banach spaces. In particular the author obtained an existence result of semigroups generated by the mapping $A = I - T$, where T is ρ-nonexpansive acting within a modular space. Note that ρ does not have to satisfy the Δ_2-condition. Even when ρ satisfies the Δ_2-condition, this existence result was unknown. Most of the work in [22] was done in the case where the modular space L_ρ is a Musielak-Orlicz space. The first result obtained is the following theorem.

Theorem 2.12. Let C be a ρ-closed, ρ-bounded convex subset of a Musielak-Orlicz modular space L_ρ. Let $T : C \to C$ be ρ-nonexpansive and norm-continuous. Let $f \in C$ be fixed and consider the recurrent sequence defined by

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = \exp(-t)f + \displaystyle\int_0^t \exp(s-t)T(u_n(s))ds \end{cases}$$

for $t \in [0, A]$, where A is a fixed positive number. Then, the sequence $\{u_n(t)\}$ is ρ-Cauchy for any $t \in [0, A]$. Therefore, it converges with respect to ρ, to $u(t) \in C$ for any $t \in [0, A]$.

It is not clear if the assumptions on C and T are enough to imply any good behavior of $u(t)$ on $[0, A]$ such as norm continuity, for example. But if ρ satisfies the Δ_2-condition, then $u(t)$ is indeed continuous.

Theorem 2.13. Under the assumptions of Theorem 2.12, if ρ satisfies the Δ_2-condition, then $u(t)$ is a solution of the following initial value problem:

$$\begin{cases} u'(t) + (I - T)u(t) = 0 \\ u(0) = f. \end{cases}$$

Note that when ρ satisfies the Δ_2-condition, there is no reason for T to be norm-nonexpansive. So the classical theorems related to the existence of solutions to the initial value problem won't apply.

Now we are ready to state the existence of semigroups in modular function spaces.

Theorem 2.14. Let C and T be as stated in Theorem 2.12. For any $f \in C$, consider $u_f(t) \in C$ for $t \in [0, \infty)$. Define the one parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ by

$$T_t(f) = u_f(t).$$

Then, $\mathcal{F}$ defines a ρ-nonexpansive semigroup. Moreover, we have the following fixed-point result:

$$F(\mathcal{F}) = \{f \in C; T_t(f) = f \text{ for any } t \geq 0\} = \{f \in C; T(f) = f\} = F(T).$$

These results were extended to systems where T is a ρ-Lipschitz operator, and applied to the perturbed integral equations in modular function spaces [1, 14].

Extensive literature exists on the question of representation of some types of semigroups of nonlinear mappings acting in Banach spaces (see for example [20, 46, 48]). It would be interesting to consider similar representation questions in modular function spaces.

2.6 Fixed Points of Semigroup of Mappings

The ρ-type concept is a powerful technical tool that is used in the proofs of many fixed-point results. The definition of a ρ-type is based on a given sequence. In order to adapt this powerful tool to the case of the semigroup, we need to generalize the ρ-type definition to a one-parameter family of mappings.

Definition 2.19. Let $K \subset L_\rho$ be a nonempty subset.

- A function $\tau : K \to [0, \infty]$ is called a *ρ-type* (or a *type*) if there exists a one-parameter family $\{h_t\}_{t\geq 0}$ of elements of K such that for any $f \in K$ the following holds:

$$\tau(f) = \inf_{M>0} \left(\sup_{t\geq M} \rho(h_t - f) \right).$$

- Let τ be a type. A sequence $\{g_n\}$ is called a *minimizing sequence* of τ if

$$\lim_{n\to\infty} \tau(g_n) = \inf\{\tau(f) : f \in K\}.$$

The next Lemma is the generalization of the minimizing sequence property for types defined by sequences in Lemma 4.3 in [24] to the one-parameter semigroup case. This technical lemma is very useful, which is the main reason why we decided to include its proof in this chapter.

Lemma 2.2. [7] Assume that $\rho \in \Re$ is (UUC). Let C be a nonempty, ρ-bounded, ρ-closed, and convex subset of L_ρ. Let τ be a type defined by a one-parameter family $\{h_t\}_{t\geq 0}$ in C.

(a) If $\tau(f_1) = \tau(f_2) = \inf_{f\in C} \tau(f)$, then $f_1 = f_2$.

(b) Any minimizing sequence $\{f_n\}$ of τ is ρ-convergent. Moreover, the ρ-limit of $\{f_n\}$ is independent of the minimizing sequence.

Proof. First let us prove (a). Let $f_1, f_2 \in C$ such that $\tau(f_1) = \tau(f_2) = \inf_{f\in C} \tau(f)$. Let us consider two cases:

(1) Assume $\inf_{f\in C} \tau(f) = 0$. Since

$$\rho\left(\frac{f_1 - f_2}{2}\right) = \rho\left(\frac{f_1 - h_t + h_t - f_2}{2}\right) \le \rho(f_1 - h_t) + \rho(h_t - f_2),$$

for any $t \ge 0$, we get

$$\rho\left(\frac{f_1 - f_2}{2}\right) \le \sup_{t\ge M} \rho(f_1 - h_t) + \sup_{t\ge M} \rho(h_t - f_2),$$

for any $M > 0$. Since

$$\tau(f) = \inf_{M>0}\Big(\sup_{t\ge M} \rho(f - h_t)\Big) = \lim_{M\to\infty} \sup_{t\ge M} \rho(f - h_t),$$

for any $f \in C$, we get

$$\rho\left(\frac{f_1 - f_2}{2}\right) \le \tau(f_1) + \tau(f_2) = 0,$$

which implies $f_1 = f_2$ as claimed.

(2) Assume $\inf_{f\in C} \tau(f) > 0$. Assume to the contrary that$f_1 \neq f_2$. Set

$$R = \inf_{f\in C} \tau(f) \ \text{ and } \ \varepsilon = \frac{\rho(f_1 - f_2)}{2R}.$$

Let $\nu \in (0, R)$. Then, $\rho(f_1 - f_2) = 2R\varepsilon \ge (R + \nu)\varepsilon$. Using the definition of τ, we deduce that there exists $M_\nu > 0$ such that

$$\sup_{t\ge M_\nu} \rho(f_1 - h_t) \le \tau(f_1) + \nu = R + \nu,$$

and

$$\sup_{t\ge M_\nu} \rho(f_2 - h_t) \le \tau(f_2) + \nu = R + \nu.$$

Since ρ is (UUC), there exists $\eta(R, \varepsilon) > 0$ such that

$$\delta(R + \nu, \varepsilon) \ge \eta(R, \varepsilon)$$

for any $\nu \in (0, R)$. So for any $t \ge M_\nu$, we have

$$\rho\left(\frac{f_1 + f_2}{2} - h_t\right) \le (R + \nu)\Big(1 - \delta(R + \nu, \varepsilon)\Big) \le (R + \nu)\Big(1 - \eta(R, \varepsilon)\Big).$$

Hence,

$$\tau\left(\frac{f_1 + f_2}{2}\right) \le \sup_{t\ge M_\nu} \rho\left(\frac{f_1 + f_2}{2} - h_t\right) \le (R + \nu)\Big(1 - \eta(R, \varepsilon)\Big).$$

Since C is convex, we get

$$R \leq \tau\left(\frac{f_1+f_2}{2}\right) \leq (R+\nu)\Big(1-\eta(R,\varepsilon)\Big).$$

If we let $\nu \to 0$, we will get

$$R \leq R\Big(1-\eta(R,\varepsilon)\Big),$$

which is impossible since $R > 0$ and $\eta(R,\varepsilon) > 0$. Therefore, we must have $f_1 = f_2$.

Next we prove (b). Denote $R = \inf_{g\in C} \tau(g)$. For any $n \geq 1$, let us set

$$K_n = \overline{conv}_\rho\Big\{h_t; t \geq n\Big\},$$

where $\overline{conv}_\rho(A)$ is the intersection of all ρ-closed convex subsets of C that contain $A \subset C$. Since C is itself ρ-closed and convex, we get $K_n \subset C$ for any $n \geq 1$. Property (R) will then imply $\cap K_n \neq \emptyset$. Let us fix arbitrary $f \in \cap K_n$, $g \in C$, and $\varepsilon > 0$. By definition of $\tau(g)$, there exists $M_\varepsilon > 0$ such that $\sup_{t\geq M_\varepsilon} \rho(g-h_t) \leq \tau(g)+\varepsilon$. Let $n \geq M_\varepsilon$. Then for any $t \geq n$, we have $\rho(g-h_t) \leq \tau(g)+\varepsilon$, that is, $h_t \in B_\rho(g,\tau(g)+\varepsilon)$. Since $B_\rho(g,\tau(g)+\varepsilon)$ is ρ-closed and convex, we get $K_n \subset B_\rho(g,\tau(g)+\varepsilon)$. Hence, $f \in B_\rho(g,\tau(g)+\varepsilon)$, that is,

$$\rho(g-f) \leq \tau(g)+\varepsilon. \tag{2.3}$$

Since ε was taken arbitrarily greater than 0, we get $\rho(g-f) \leq \tau(g)$, for any $g \in C$. Let $\{f_n\}$ be a minimizing sequence for τ. If $R = 0$ then, since $\{f_n\}$ is a minimizing sequence, we get $\lim_{n\to\infty} \tau(f_n) = R = 0$. Using (2.3) we can see that $\rho(f_n - f) \leq \tau(f_n)$, for any $n \geq 1$. Hence, $\{f_n\}$ is ρ-convergent to f. Since selection of f was independent of $\{f_n\}$, it follows that any minimizing sequence is ρ-convergent to f if $R = 0$. We can assume therefore that $R > 0$. For any $n \geq 1$, let us set

$$d_n = \sup_{i,j\geq n} \rho(f_i - f_j). \tag{2.4}$$

We claim that $\{f_n\}$ is ρ-Cauchy. Assume to the contrary that this is not the case. Since the sequence $\{d_n\}$ is decreasing and $\{f_n\}$ is not ρ-Cauchy, we get $d := \inf_{n\geq 1} d_n > 0$. Set $\varepsilon = \dfrac{d}{4R} > 0$. Let us fix arbitrary $\nu \in (0,R)$. Since $\lim_{n\to\infty} \tau(f_n) = R$, there exists $n_0 \geq 1$ such that for any $n \geq n_0$, we have

$$\tau(f_n) \leq R + \frac{\nu}{2}. \tag{2.5}$$

Let $n \geq n_0$. By (2.4), there exists $i_n, j_n \geq 1$ such that

$$\rho(f_{i_n} - f_{j_n}) > d_n - \frac{d}{2} \geq \frac{d}{2} = 2R\varepsilon > (R+\nu)\varepsilon.$$

Using the definition of τ and (2.5), we deduce the existence of $M > 0$ such that

$$\sup_{t \geq M} \rho(f_{i_n} - h_t) \leq \tau(f_{i_n}) + \frac{\nu}{2} \leq R + \nu,$$

and

$$\sup_{t \geq M} \rho(f_{j_n} - h_t) \leq \tau(f_{j_n}) + \frac{\nu}{2} \leq R + \nu.$$

Hence,

$$\rho\left(\frac{f_{i_n} + f_{j_n}}{2} - h_t\right) \leq (R+\nu)\Big(1 - \delta(R+\nu, \varepsilon)\Big),$$

for any $t \geq M$. Since ρ is (UUC), there exists $\eta_1(R,\varepsilon) > 0$ such that $\delta_1(R + \nu, \varepsilon) \geq \eta_1(R,\varepsilon)$. Hence,

$$\rho\left(\frac{f_{i_n} + f_{j_n}}{2} - h_t\right) \leq (R+\nu)\Big(1 - \eta(R,\varepsilon)\Big), \quad \text{for any } t \geq M.$$

Hence,

$$R \leq \tau\left(\frac{f_{i_n} + f_{j_n}}{2}\right) \leq \sup_{t \geq M} \rho\left(\frac{f_{j_n} + f_{j_n}}{2} - h_t\right) \leq (R+\nu)\Big(1 - \eta(R,\varepsilon)\Big) < R.$$

Using the definition of R, we get

$$R \leq (R+\nu)\Big(1 - \eta(R,\varepsilon)\Big),$$

for any $\nu \in (0,R)$. If we let $\nu \to 0$, we get $R \leq R(1 - \eta_1(R,\varepsilon))$. This contradiction implies that $\{f_n\}$ is ρ-Cauchy. Since L_ρ is ρ-complete, we deduce that $\{f_n\}$ is ρ-convergent as claimed. In order to finish the proof of (b), let us show that the ρ-limit of $\{f_n\}$ is independent of the minimizing sequence. Indeed, let $\{g_n\}$ be another minimizing sequence of τ. The previous proof will show that $\{g_n\}$ is also ρ-convergent. In order to prove that the ρ-limits of $\{f_n\}$ and $\{g_n\}$ are equal let us show that $\lim\limits_{n\to\infty} \rho(f_n - g_n) = 0$. Assume not, that is, $\lim\limits_{n\to\infty} \rho(f_n - g_n) \neq 0$. Without loss of generality we may assume that there exists $d > 0$ such that $\rho(f_n - g_n) \geq d$, for any $n \geq 1$. Set $\varepsilon = \dfrac{d}{2R} > 0$. Let $\nu \in (0,R)$. Since $\lim\limits_{n\to\infty} \tau(f_n) = \lim\limits_{n\to\infty} \tau(g_n) = R$, there exists $n_0 \geq 1$ such that for any $n \geq 1$, we have $\tau(f_n) \leq R + \dfrac{\nu}{2}$, and $\tau(g_n) \leq R + \dfrac{\nu}{2}$. Fix $n \geq n_0$. Then,

$$\rho(f_n - g_n) \geq d = 2R\varepsilon > (R+\nu)\varepsilon.$$

Using the definition of τ, we deduce the existence of $M > 0$ such that

$$\sup_{t \geq M} \rho(f_n - h_t) \leq \tau(f_n) + \frac{\nu}{2} \leq R + \nu,$$

and

$$\sup_{t \geq M} \rho(g_n - h_t) \leq \tau(g_n) + \frac{\nu}{2} \leq R + \nu.$$

Hence,

$$\rho\left(\frac{f_n + g_n}{2} - h_t\right) \leq (R + \nu)\Big(1 - \delta(R + \nu, \varepsilon)\Big),$$

for any $t \geq M$. Since ρ is (UUC), there exists $\eta(R, \varepsilon) > 0$ such that $\delta(R + \nu, \varepsilon) \geq \eta(R, \varepsilon)$, for any $\nu > 0$. Hence,

$$\rho\left(\frac{f_n + g_n}{2} - h_t\right) \leq (R + \nu)\Big(1 - \eta(R, \varepsilon)\Big), \quad \text{for any } t \geq M.$$

So,

$$\tau\left(\frac{f_n + g_n}{2}\right) \leq \sup_{t \geq M} \rho\left(\frac{f_n + g_n}{2} - h_t\right) \leq (R + \nu)\Big(1 - \eta(R, \varepsilon)\Big).$$

Using the definition of R, we get

$$R \leq (R + \nu)\Big(1 - \eta(R, \varepsilon)\Big), \quad \text{for any } \nu \in (0, R).$$

If we let $\nu \to 0$, we get $R \leq R(1 - \eta(R, \varepsilon))$. This contradiction implies that $\lim_{n \to \infty} \rho(f_n - g_n) = 0$. The Fatou property will finally imply that

$$\rho(f - g) \leq \liminf_{n \to \infty} \rho(f_n - g_n),$$

where f is the ρ-limit of $\{f_n\}$ and g is the ρ-limit of $\{g_n\}$. Hence $\rho(f - g) = 0$, that is, $f = g$. □

The following result allowed the author in [37] to prove the existence of a common fixed point for contractive semigroups in modular function spaces.

Theorem 2.15. [37] Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e.-Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\delta_\rho(\beta\ C) = \sup\{\rho(\beta(x - y)); x, y \in C\} < \infty$, for some $\beta > 1$. Let $\mathcal{F}$ be a ρ-contractive semigroup on C. Then, $\mathcal{F}$ has a unique common fixed point $z \in C$ and for each $u \in C$, $\rho(T_t(u) - z) \to 0$ as $t \to \infty$.

Remark 2.1. Note that the existence and uniqueness part in this theorem is easy to obtain. Indeed, if $\mathcal{F}$ is a family of commutative mappings such that one map $T \in \mathcal{F}$ has one single fixed point, then $\mathcal{F}$ has a single fixed point and we have $F(\mathcal{F}) = F(T)$. The novelty in this theorem is the ρ-convergence of the generalized orbit to the fixed point.

Using Lemma 2.2, a fixed point result for asymptotic pointwise nonexpansive semigroups is proved.

Theorem 2.16. [7] Assume $\rho \in \Re$ is (UUC). Let C be a ρ-closed, ρ-bounded, convex nonempty subset. Let $\mathcal{F} = \{T_t : t \geq 0\}$ be ρ-asymptotic pointwise nonexpansive semigroup on C. Then, $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is ρ-closed and convex.

Proof. Let us fix $f \in C$ and define the function

$$\tau(g) = \inf_{M>0} \Big(\sup_{t \geq M} \rho(T_t(f) - g) \Big).$$

Since C is ρ-bounded, we have $\tau(g) \leq diam_\rho(C) < +\infty$, for any $g \in C$. Hence, $\tau_0 = \inf\{\tau(g) : g \in C\}$ exists and is finite. For any $n \geq 1$, there exists $g_n \in C$ such that

$$\tau_0 \leq \tau(g_n) < \tau_0 + \frac{1}{n}.$$

Therefore, $\lim_{n\to\infty} \tau(z_n) = \tau_0$; that is, $\{g_n\}$ is a minimizing sequence for τ. By Lemma 2.2 there exists $g \in C$ such that $\{g_n\}$ ρ-converges to g. Let us now prove that $g \in F(\mathcal{F})$. Note that

$$\rho(T_{s+t}(f) - T_s(h)) \leq \alpha_s(h)\rho(T_t(f) - h), \quad \text{for all } s, t > 0, \text{ and } h \in C.$$

Using the definition of τ, we get

$$\tau(T_s(h)) \leq \sup_{t+s \geq M} \rho(T_{s+t}(f) - T_s(h)) \leq \alpha_s(h) \sup_{t \geq M-s} \rho(T_t(f) - h),$$

for any $M > s$, which implies that

$$\tau(T_s(h)) \leq \alpha_s(h)\tau(h). \tag{2.6}$$

Since $\lim_{s\to\infty} \alpha_s(g_1) = 1$, there exists $s_1 > 0$ such that for any $s \geq s_1$, we have $\alpha_s(g_1) < 1 + 1$. Repeating this argument, one will find that $s_2 > s_1 + 1$, such that for any $s \geq s_2$, we have $\alpha_s(g_2) < 1 + \frac{1}{2}$. By induction, we will construct a sequence $\{s_n\}$ of positive numbers such that $s_{n+1} < s_n + \frac{1}{n}$, and for any $s \geq s_n$ we have $\alpha_s(g_n) < 1 + \frac{1}{n}$. Let us fix $t \geq 0$. The inequality (2.6) will then imply

$$\tau(T_{s_n+t}(g_n)) \leq \alpha_{s_n+t}(g_n)\tau(g_n) \leq \left(1 + \frac{1}{n}\right)\tau(g_n), \quad \text{for any } n \geq 1.$$

In particular, we find that $\{T_{s_n+t}(g_n)\}$ is a minimizing sequence of τ. Therefore, Lemma 2.2 implies that $\{T_{s_n+t}(g_n)\}$ ρ-converges to g, for any $t \geq 0$. In particular, we have that $\{T_{s_n}(g_n)\}$ ρ-converges to g. Since

$$\rho\Big(T_{s_n+t}(g_n) - T_t(g)\Big) \leq \alpha_t(g)\rho(T_{s_n}(g_n) - g),$$

we find that $\{T_{s_n+t}(g_n)\}$ ρ-converges to $T_t(g)$. Finally, using

$$\rho\left(\frac{T_t(g)-g}{2}\right) \leq \rho\Big(T_t(g) - T_{s_n+t}(g_n)\Big) + \rho\Big(T_{s_n+t}(g_n) - g\Big),$$

we find that $T_t(g) = g$. Since t was arbitrarily positive, we get $g \in F(\mathcal{F})$, that is, $F(\mathcal{F})$ is not empty. Next let us prove that $F(\mathcal{F})$ is ρ-closed. Let $\{f_n\}$ in $F(\mathcal{F})$ be ρ-convergent to f. Since

$$\rho(T_s(f_n) - T_s(f)) \leq \alpha_s(f)\rho(f_n - f), \quad \text{for any } n \geq 1 \text{ and } s > 0,$$

we find that $\{T_s(f_n)\}$ ρ-converges to $T_s(f)$. Since $f_n \in F(\mathcal{F})$, we get $\{T_s(f_n)\} = \{f_n\}$. In other words, $\{f_n\}$ ρ-converges to $T_s(f)$ and f. The uniqueness of the ρ-limit implies that $T_s(f) = f$, for any $s \geq 0$, that is, $f \in F(\mathcal{F})$. Therefore, $F(\mathcal{F})$ is ρ-closed. Let us finish the proof of Theorem 2.16 by showing that $F(\mathcal{F})$ is convex. It is sufficient to show that

$$h = \frac{f+g}{2} \in F(\mathcal{F}), \quad \text{for any } f, g \in F(\mathcal{F}).$$

Without loss of generality, we will assume that $f \neq g$. Let $s > 0$. We have

$$\rho(f - T_s(h)) = \rho(T_s(f) - T_s(h)) \leq \alpha_s(f)\rho\left(f - h\right),$$

and

$$\rho(g - T_s(h)) = \rho(T_s(g) - T_s(h)) \leq \alpha_s(g)\rho\left(g - h\right).$$

Since $\rho(f-h) = \rho(g-h) = \rho\left(\frac{f-g}{2}\right)$, and

$$\rho\left(\frac{f-g}{2}\right) \leq \frac{1}{2}\rho(f - T_s(h)) + \frac{1}{2}\rho(g - T_s(h)),$$

we conclude that

$$\lim_{s\to\infty} \rho(f - T_s(h)) = \lim_{s\to\infty} \rho(g - T_s(h)) = \rho\left(\frac{f-g}{2}\right).$$

Similarly, we have

$$\rho\left(f - \frac{h+T_s(h)}{2}\right) \leq \frac{1}{2}\rho\left(f-h\right) + \frac{1}{2}\rho(f - T_s(h)),$$

and

$$\rho\left(g - \frac{h+T_s(h)}{2}\right) \leq \frac{1}{2}\rho\left(g-h\right) + \frac{1}{2}\rho(g - T_s(h)).$$

Since

$$\rho\left(\frac{f-g}{2}\right) \leq \frac{1}{2}\rho\left(f - \frac{h+T_s(h)}{2}\right) + \frac{1}{2}\rho\left(g - \frac{h+T_s(h)}{2}\right),$$

we conclude that

$$\lim_{s\to\infty} \rho\left(f - \frac{h + T_s(h)}{2}\right) = \lim_{s\to\infty} \rho\left(g - \frac{h + T_s(h)}{2}\right) = \rho\left(\frac{f-g}{2}\right).$$

Therefore, we have

$$\lim_{s\to\infty} \rho(f - T_s(h)) = \lim_{s\to\infty} \rho\left(f - \frac{h + T_s(h)}{2}\right) = \rho(f - h).$$

Lemma 2.1 applied to $A_t = f - T_s(h)$ and $B_t = T_s(h) - g$ implies that $\rho(A_t - B_t) \to 0$. Hence,

$$\lim_{s\to\infty} \rho(h - T_s(h)) = \lim_{s\to\infty} \rho\Big(\frac{A_t - B_t}{2}\Big) \le \lim_{s\to\infty} \rho(A_t - B_t) = 0.$$

Clearly, we will get $\lim\limits_{s\to\infty} \rho(h - T_{s+t}(h)) = 0$, for any $t \ge 0$. Since

$$\rho(T_t(h) - T_{s+t}(h)) \le \alpha_t(h)\rho(h - T_s(h)),$$

we get $\lim\limits_{s\to\infty} \rho(T_t(h) - T_{s+t}(h)) = 0$. Finally, using the inequality

$$\rho\left(\frac{h - T_t(h)}{2}\right) \le \frac{1}{2}\rho(h - T_{s+t}(h)) + \frac{1}{2}\rho(T_t(h) - T_{s+t}(h)),$$

by letting $s \to \infty$, we get $T_t(h) = h$, for any $t \ge 0$; that is, $h \in F(\mathcal{F})$. □

Remark 2.2. Theorem 2.16 may be seen as a generalization of Theorem 2.11. When Theorem 2.11 was published in [25], the fixed point set was only known to be ρ-closed. Using the above conclusion, we know now that it is ρ-closed and convex.

As a corollary to Theorem 2.16, we obtain the following result.

Corollary 2.1. Assume $\rho \in \Re$ is (UUC). Let C be a ρ-closed, ρ-bounded, convex nonempty subset. Let $\mathcal{F}$ be a nonexpansive semigroup on C. Then, the set $F(\mathcal{F})$ of common fixed points is nonempty, ρ-closed, and convex.

Recently, Al-Mezel et al. [4] proved a partial generalization of Corollary 2.1 for the case of any commutative family of ρ-nonexpansive mappings.

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Chapter 3

Approximation and Selection Methods for Set-Valued Maps and Fixed Point Theory

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3.1 Introduction

A number of solvability problems in analysis are modeled by inclusions involving set-valued maps, for example, of the fixed-point type $\bar{x} \in \Phi(\bar{x})$ or the equilibrium type $0 \in \Phi(\bar{x})$, or as a differential inclusion $x\prime(t) \in \Phi(t, x(t))$, etc. Such problems are often treated by reduction to the *single-valued case* via a single-valued selection $s(x) \in \Phi(x)$, or an approximative selection $s(x) \in_{\approx} \Phi(x)$ in a sense determined by the nature of the problem.

This paper discusses fundamental and new results on the existence of continuous approximations and selections for set-valued maps with a focus on the non-convex case (non-convex domains and non-convex values) and in a general and generic framework allowing the passage from "elementary" domains to more elaborate ones. It expands on an earlier paper by the author [13]. Applications of the approximation and selection results to topological fixed-point and coincidence theory for set-valued maps are also discussed.

Before proceeding any further, let us fix basic notations and terminology about spaces and mappings used throughout the paper.

In what follows, a *space* is a Hausdorff topological space. The *closure, interior,* and *boundary* of a subspace A of a space X are, as usual, denoted by $\overline{A}$, $\mathring{A}$, and ∂A. Denote by $\mathcal{N}_X(x)$ a basis of open neighborhoods of an element x in a space X.

Given a subset K of a space X, $Cov_X(K)$ denotes the collection of all covers of K by open subsets of X; $Cov(X) := Cov_X(X)$. We write $\bigcup \omega := \bigcup_{W \in \omega} W$. Given two covers $\omega, \omega' \in Cov_X(K)$, $\omega' \preccurlyeq \omega$ means that ω' is a refinement of

ω. The *star* of a subset $A \subset X$ with respect to a cover $\omega \in Cov(X)$ is the set $\bigcup\{W \in \omega : A \cap W \neq \emptyset\}$. A cover ω' is a *barycentric refinement* of a cover ω if the cover $\{St(x, \omega') : x \in X\}$ refines ω. A cover ω' is a *star refinement* of a cover ω if the cover $\{St(W', \omega') : W' \in \omega'\}$ refines ω.

Given a uniform space $(X, \mathcal{U})$ and an entourage $U \in \mathcal{U}$, the *ball* of radius U around a given element $x \in X$ is the set $U[x] := \{x' \in X : (x, x') \in U\}$. The (tubular) *neighborhood* of radius U around a given subset $A \subset X$ is the set $U[A] := \bigcup_{x \in A} U[x]$.

Given a set X, a space Y, and $\omega \in Cov(Y)$, two single-valued maps $f, g : X \to Y$ are said to be:

- $\omega-$*near* (written $f =_\omega g$) if for each $x \in X, \{f(x), g(x)\} \subset W$ for some member W of ω.

- $\omega-$*homotopic* (written $f \sim_\omega g$) if there exists a homotopy $h : X \times [0,1] \longrightarrow Y$ joining f and g and such that for each $x \in X, h(\{x\} \times [0,1]) \subset W$ for some member W of ω (such a homotopy is called an $\omega-$homotopy between f and g).

A *polyhedron* is a topological space P homeomorphic to the space $|K|$ of a simplicial complex K. The pair $T = (K, \tau)$ where $\tau : |K| \to P$ is this homeomorphism is a *triangulation* of P and the space $|K|$ is the *geometric realization* of P; it has vertices at the unit points in a linear space equipped with the topology induced by the Euclidean topology of its finite dimensional flats. We often do not distinguish between P and $|K|$. The set of all vertices of (the geometric realization of) a polyhedron P is denoted by P^0. A polyhedron is *finite* if P^0 is a finite set (such a polyhedron is a compact space). A polyhedron is *locally finite* if each vertex of its geometric realization belongs to a finite number of simplexes (such a polyhedron is a paracompact and locally compact space).

Let $\omega = \{W_i : i \in I\} \in Cov(X)$ be any cover of a space X. Define the *nerve* of ω as the complex P consisting of all simplexes $\sigma = (u_0, \ldots, u_p)$ for which $W_{i_0} \cap \cdots \cap W_{i_p} \neq \emptyset$. The *geometric nerv*e of ω is the geometric realization $|N(\omega)|$ of P (see Dugundji [36] or Brown [27]).

A *finite convex polyhedron,* called a *polytope*, can be viewed as the convex hull, in a linear space, of a finite set of vectors.

The *convex hull* of a subset A of a vector space is denoted by $conv(A)$.

The *identity mapping* on a set X is denoted by id_X.

The class of *continuous* single-valued mappings from a space X into a space Y is denoted by $\mathbf{c}(X, Y)$.

3.2 Approximative Neighborhood Retracts, Extensors, and Space Approximation

The first section of this chapter aims at familiarizing the reader with key topological concepts and facts about spaces and domains crucial in topological fixed-point theory and its applications. The focus is on topological spaces that are far-reaching topological extensions of convex sets in normed spaces, or more precisely, approximative absolute neighborhood retracts and extensors, and on some topological notions of convexity. Special attention is given to the concepts of domination and approximation of spaces, allowing for the consideration of large classes of domains for the solvability of nonlinear equations and inclusions. For detailed discussion on the theory of retracts and their generalizations, we refer to Borsuk [23], Clapp [30], Gauthier [42], Granas [52], and Hu [59]. A quick pedestrian exposition on retracts is provided in the online *Encyclopedia of Mathematics* [4].

3.2.1 Approximative Neighborhood Retracts and Extensors

For the sake of completeness, let us backtrack to the basic definitions of absolute (neighborhood) retracts and extensors.

Definition 3.1. (i) A subspace A of a space X is a *neighborhood retract* of X if there exist an open neighborhood V of A in X and a mapping $r \in \mathbf{c}(V, A)$ such that the diagram

$$\begin{array}{ccc} & A & \\ id_A & \nearrow \nwarrow & r \\ A & \underset{i}{\hookrightarrow} & V \end{array}$$

commutes, that is, $r(a) = a$ for all $a \in A$.

If $V = X$, A is simply said to be a retract of X. (Note that since spaces are assumed to be Hausdorff, if A is a neighborhood retract of a space X, then A is a closed subspace of X.)

(ii) A space A is an *absolute* (respectively, *neighborhood*) retract for a given class $\mathcal{Q}$ of spaces, written $A \in AR(\mathcal{Q})$, $(ANR(\mathcal{Q})$ resp.), if and only if:

(a) $A \in \mathcal{Q}$, and

(b) for every closed imbedding h of A into a space $X \in \mathcal{Q}$, $h(A)$ is a (neighborhood) retract of X.

Obviously, $ANR(\mathcal{Q})$ contains the class $AR(\mathcal{Q})$.

If $\mathcal{M}$ denotes the class of metric spaces, then $AR(\mathcal{M})$ is precisely the well-known class AR of *absolute retracts*, and $ANR(\mathcal{M})$ is simply the class ANR of *absolute neighborhood retracts.*

Clearly, if A is a retract of a space X with retraction $r : X \to A$, then any continuous mapping $f_0 : A \to Y$ into any space Y, extends to the continuous mapping $f = r \circ f_0 : X \to A \to Y$. Retracts and neighborhood retracts are characterized by extension properties as follows.

Definition 3.2. (i) A space X is an extension space for a class of spaces $\mathcal{Q}$, written $X \in \mathrm{ES}(\mathcal{Q})$, if and only if $\forall Y \in \mathcal{Q}, \forall K$ closed in $Y, \forall f_0 \in \mathbf{c}(K, X), \exists f \in \mathbf{c}(Y, X)$ such that the diagram

$$\begin{array}{ccc} & X & \\ f_0 \nearrow & & \nwarrow f \\ K & \underset{i}{\hookrightarrow} & Y \end{array}$$

commutes.

(ii) A space X is a neighborhood extension space for a class of spaces $\mathcal{Q}$, written $X \in \mathrm{NES}(\mathcal{Q})$, if and only if $\forall Y \in \mathcal{Q}, \forall K$ and is closed in $Y, \forall f_0 \in \mathbf{c}(K, X), \exists V$ open neighborhood of K in $Y, \exists f \in \mathbf{c}(V, X)$ such that the diagram

$$\begin{array}{ccc} & X & \\ f_0 \nearrow & & \nwarrow f \\ K & \underset{i}{\hookrightarrow} & V \end{array}$$

commutes.

We have the characterizations:

Proposition 3.1. (a) If $\mathcal{Q}$ is a class of normal spaces, then $\mathrm{AR}(\mathcal{Q}) = \mathcal{Q} \cap \mathrm{ES}(\mathcal{Q})$, $\mathrm{ANR}(\mathcal{Q}) = \mathcal{Q} \cap \mathrm{NES}(\mathcal{Q})$.

(b) If $\mathcal{BN}$ is the class of binormal spaces,[1] then each space $X \in \mathrm{AR}(\mathcal{BN})$ is contractible in each of its points. Similarly, each space in $\mathrm{ANR}(\mathcal{BN})$ is locally contractible (in each of its points).

Proof. We refer to [23, 52, 59] for the proof of (a).

A well-known fact of point-set topology is that the normality of $X \times [0, 1]$ is in fact equivalent to that of $X \times Y$ for any compact metric space Y (see Engelking [39]). Hence, $X \times [0, 1] \times [0, 1]$ is also normal; that is, $X \times [0, 1] \in \mathcal{BN}$ whenever $X \in \mathcal{BN}$. Assuming that $X \in AR\ (\mathcal{BN}) = \mathcal{BN} \cap ES\ (\mathcal{BN})$, let x_0 be a fixed point in X and define $f : X \times \{0\} \cup X \times \{1\} \to X$ by:

$$f(x, 0) = x_0 \text{ and } f(x, 1) = x, \quad \forall x \in X.$$

The mapping f is clearly continuous and extends to a continuous mapping $h : X \times [0, 1] \to X$, a homotopy contracting the space X into the point x_0. The proof of the second statement is exactly the same, with a local homotopy. □

[1] A space X is said to be *binormal* if it is normal and $X \times [0, 1]$ is also normal. In 1951, C. H. Dowker characterized such spaces as being *normal and countably paracompact* (see [39]).

Prototypes of *ARs* and *AANRs* are, respectively, balls and spheres in Euclidean spaces.

Clearly, every retract of an absolute retract is again an absolute retract.

Proposition 3.1 implies that each absolute retract is contractible in itself and is locally contractible. It is moreover well-known that all homology, cohomology, homotopy, and cohomotopy groups of an absolute retract are trivial (see [23, 27, 59].

Absolute neighborhood retracts are characterized as retracts of open subsets of convex subspaces of normed linear spaces. They include all compact polyhedra. An important property of them is their local contractibility.

Example 3.1. (a) The *Tietze-Uryshon extension theorem* [36, 39, 88] states that, for any index set J, the Tychonoff cube $[0,1]^J \in \mathrm{ES}(\mathcal{N})$, where $\mathcal{N}$ is the class of normal (T_4) spaces.

(b) If a metric space $X \in AR$, then X is a retract of some convex subspace of a normed linear space. Conversely, the *Dugundji extension theorem* [34] asserts that if $\mathcal{C}$ is the class of convex sets in locally convex spaces, and $\mathcal{M}$ is the class of metric spaces, then $\mathcal{C} \subset \mathrm{ES}(\mathcal{M})$. Hence, any metrizable retract of a convex subset of a locally convex topological linear space is an *AR.*

(c) Every infinite polyhedron endowed with a metrizable topology is an *AR.*

(d) Every Fréchet manifold is an *ANR.*

(e) The union $C := \bigcup_{i=1}^n C_i$ of overlapping closed convex subsets $C_1, \ldots, C_n$ in a locally convex space is an *ANR* provided it is metrizable (see Van Mill [88]).

We include an approximation aspect in the definition of (neighborhood) retracts to obtain larger classes of spaces.

Definition 3.3. (i) A closed subset $A \overset{i}{\hookrightarrow} X$ is an approximative neighborhood retract of a space X if for any $\omega \in Cov_X(A)$, there exist an open neighborhood V of A in X and a mapping $r \in \mathbf{c}(V, A)$ such that $r \circ i$ and id_A in the following diagram

$$\begin{array}{ccc} & A & \\ id_A & \nearrow\nwarrow & r \\ A & \underset{i}{\hookrightarrow} & V \end{array}$$

are $\omega-$near.

(ii) A space A is said to be an *approximative absolute neighborhood retract* for a class of spaces $\mathcal{Q}$, written $A \in$AANR($\mathcal{Q}$), if and only if:

(a) $A \in \mathcal{Q}$, and

(b) for every closed imbedding h of A in a space $X \in \mathcal{Q}$, $h(A)$ is an approximative neighborhood retract of X.

The class $A_H ANR(\mathcal{Q})$ is defined in a similar way with "$\omega-$near" replaced by "$\omega-$homotopic." Obviously, $AANR(\mathcal{Q})$ contains $A_H ANR(\mathcal{Q})$ and $ANR(\mathcal{Q})$.

For the class $\mathcal{M}$ of metric spaces, $AANR(\mathcal{M})$—written $AANR$ for short—is the class of *approximative absolute neighborhood retracts.* One can characterize $AANRs$ as metrizable spaces that are homeomorphic to approximative neighborhood retracts of normed spaces.

Obviously, $ANR \subset AANR$, the inclusion being strict. Indeed, the set

$$\Gamma := \{(x, \sin(\frac{1}{x})) \in \mathbf{R}^2 : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

is an $AANR$. However, because Γ is not locally contractible, it cannot be an ANR.

If $\mathcal{K}$ is the class of compact spaces, then $AANR \cap \mathcal{K} \subset AANR(\mathcal{K})$ (see Gauthier [42]).

Definition 3.4. (i) A space X is an approximative extension space for a class of spaces $\mathcal{Q}$, written $X \in$ AES($\mathcal{Q}$), if and only if $\forall \omega \in Cov(X), \forall Y \in \mathcal{Q}, \forall K$ closed in $Y, \forall f_0 \in \mathbf{c}(K, X), \exists f \in \mathbf{c}(Y, X)$, such that $f \circ i$ and f_0 are $\omega-$near.

(ii) A space X is an approximative neighborhood extension space for a class of spaces $\mathcal{Q}$, written $X \in$ ANES($\mathcal{Q}$), if and only if $\forall \omega \in Cov(X), \forall Y \in \mathcal{Q}, \forall K$ closed in $Y, \forall f_0 \in \mathbf{c}(K, X), \exists V$ open neighborhood of K in $Y, \exists f \in \mathbf{c}(V, X)$, such that $f \circ i$ and f_0 are $\omega-$near.

Clearly,

$$ES(\mathcal{Q}) \subset AES(\mathcal{Q}) \cap NES(\mathcal{Q}) \subset ANES(\mathcal{Q}).$$

The classes $A_H ES(\mathcal{Q})$ and $A_H NES(\mathcal{Q})$ of approximative $H-$(neighborhood) extension spaces for $\mathcal{Q}$, are defined in a similar way with "$\omega-$near" replaced by "$\omega-$homotopic."

Proposition 3.2. If $\mathcal{Q}$ is a class of normal spaces, then AANR($\mathcal{Q}$) $= \mathcal{Q} \cap$ ANES($\mathcal{Q}$), and A_HANR($\mathcal{Q}$) $= \mathcal{Q} \cap \mathrm{A}_H$NES($\mathcal{Q}$).

We conclude from this proposition, for the class $\mathcal{K}$ of compact topological spaces, the inclusion

$$AANR \cap \mathcal{K} \subset AANR(\mathcal{K}),$$

which we shall use later on.

It is well known that every ANR Y is homeomorphic to a neighborhood retract of a normed space (namely, the Banach space of all bounded continuous real-valued functions on Y; see Example 2.2 in [52]). But normed spaces are $ES(\mathcal{K})$ for the class $\mathcal{K}$ of compact spaces. Moreover, for any class of spaces $\mathcal{Q}$, neighborhood retracts of an $NES(\mathcal{Q})$ are also $NES(\mathcal{Q})$. Hence, we have:

Proposition 3.3. ANR $\subset$ NES($\mathcal{K}$) for the class $\mathcal{K}$ of compact spaces.

The fact that *ANRs* can be imbedded as neighborhood retracts of normed spaces (which are of course locally convex topological vector spaces) could be used to construct linear homotopies between mappings that are close. More precisely, we have the crucial observation:

Proposition 3.4. [34, Lemma 7.2] Given a metric space X, an ANR Y, and a cover $\omega \in Cov(Y)$, there exists $\omega' \in Cov(Y), \omega' \preceq \omega$, such that any two mappings $f, g : X \to Y$ that are $\omega'-$near are $\omega-$homotopic.

This property can be extended to non-metrizable $NES(\mathcal{K})$ spaces by using a generalization of the "controlled" Borsuk homotopy extension theorem. We can indeed show the following:

Proposition 3.5. Compact subspaces of NES($\mathcal{K}$) spaces, where $\mathcal{K}$ is the class of compact spaces, are uniformly locally contractible.[2] Consequently, for any open cover ω of Z, any two continuous mappings with values in a compact subspace Z of an NES($\mathcal{K}$) space that are close enough are $\omega-$homotopic.

3.2.2 Contractibility and Connectedness

Contractibility (or the continuous deformation of a space onto one of its points) is perhaps the most natural generalization of convexity. We discuss below the topological notions of contractibility and proximal contractibility.

3.2.2.1 Contractible Spaces

A space X is *contractible* (in itself) if there exists a mapping $h \in \mathbf{c}(X \times [0,1], X)$ such that $h(x,0) = x$ and $h(x,1) = \bar{x}$ where $\bar{x}$ is a given point in X (h is said to be a contracting homotopy).

Clearly, every star-shaped set is contractible (the deformation being linear: $h(x,t) = tx + (1-t)\bar{x}$). In particular, every convex set is contractible.

As we have seen above, a topological property can be expressed as (or implies) an *extension property*, which turns out to be the key technical and operational feature. Here, it holds that contractibility implies *$n-$connectedness.*

[2] Let Z be a subspace of a topological space Y and assume that Z has a uniform structure $\mathcal{V}$. Z is said to be *$\omega-$uniformly contractible in* Y ($\omega - ULC$ in Y, for short) for a given open cover $\omega \in Cov_Y(Z)$, if there is a member $V \in \mathcal{V}$ and a continuous mapping $\xi : V \times [0,1] \to Y$ such that:

$$\forall z, z' \in V, \forall t \in [0,1],\ \xi(z,z',0) = \xi(z',z,1) = z, \xi(z,z,t) = z,$$
$$\text{and } \xi((z,z') \times [0,1]) \text{ is contained in a member } W \text{ of } \omega.$$

Z is said to be *uniformly locally contractible* in Y (ULC in Y, for short) if it is $\omega - ULC$ in Y, for any $\omega \in Cov_Y(Z)$. If Z is $\omega - ULC$ in Y, then there exists $V \in \mathcal{V}$ such that any two continuous mappings $f, g : X \to Z$ defined on a space X that are $V-$near are $\omega-$homotopic (take the homotopy $h(x,t) = \xi(f(x), g(x), t)$).

To be more precise, for a given positive integer n, let Δ^n denote the standard $n-$simplex whose vertices $\{e_0, \dots, e_n\}$ form a canonical basis for $\mathbf{R}^{n+1}$and let $\partial\Delta^n$ be its boundary. Let us define:

Definition 3.5. A space X is said to be $n-$*connected* if and only if $\forall f \in \mathbf{c}(\partial\Delta^n, X), \exists \hat{f} \in \mathbf{c}(\Delta^n, X)$ such that the diagram

$$\begin{array}{ccc} & X & \\ f & \nearrow \nwarrow & \hat{f} \\ \partial\Delta^n & \underset{i}{\hookrightarrow} & \Delta^n \end{array}$$

commutes. The space X is said to be *infinitely connected* (C^∞ for short), if X is $n-$connected for each positive integer n.

Proposition 3.6. Every contractible space is indeed C^∞.

Proof. To see this, let $\bar{z}$ be the barycenter of the $n-$simplex Δ^n and write every $z \in \Delta^n \setminus \{\bar{z}\}$ in polar coordinates form $z = t(z)\bar{z} + (1-t(z))v(z)$ where $v(z)$ is the radial projection of z onto $\partial\Delta^n, t(z) \in [0,1)$. Let $h \in \mathbf{c}(X \times [0,1], X)$ be the homotopy contracting X onto a given point $\bar{x} \in X$ and let $f \in \mathbf{c}(\partial\Delta^n, X)$ be given. One readily verifies that the mapping

$$\hat{f}(z) := \begin{cases} h((f \circ v)(z), t(z)) & \text{if} \quad z \neq \bar{z} \\ \bar{x} & \text{if} \quad z = \bar{z} \end{cases}$$

is a continuous extension of f. □

3.2.2.2 Proximal Connectedness

A landmark result of Aronszajn [5] establishes the qualitative structure of the solution sets of a Cauchy initial value problem as a so-called R_δ set, which is the intersection of a countable decreasing sequence of compact contractible metric spaces. Such similar qualitative results were extended to solution sets of differential inclusions or integral equations (see, for example, Andres et al. [3], Bader and Kryszewski [9, 10], Bressan [24], Deimling [32], Górniewicz et al. [50, 49, 47], Plaskacz [82]). Being a limit of compact contractible spaces, an R_δ set A in a space X is contractible in each of its open neighborhoods, that is, for any open neighborhood U of A in X, there exists a continuous mapping $h_U : A \times [0,1] \longrightarrow U$ such that $h(x,0) = x$ and $h(x,1) = x_U, \forall x \in A$, where x_U is a given fixed point in U. This leads us to formulate the following definition. This approximate contractibility is known as the *trivial shape* property (see, for example, Van Mill [88]). Obviously, if $X \supset A$ is contractible, then it has trivial shape in X. However, the set

$$\Gamma := \{(t, \sin(\frac{1}{t})) \in \mathbf{R}^2 : 0 < t \leq 1\} \cup \{(0, v) : -1 \leq v \leq 1\},$$

is not contractible but has trivial shape in $\mathbf{R}^2$. Indeed, $\Gamma := \bigcap_{n\geq 1}\Gamma_n$ where $\{\Gamma_n\}_{n\geq 1}$ is the decreasing sequence of compact contractible sets

$$\Gamma_n := \{(t, \sin(\frac{1}{t})) \in \mathbf{R}^2 : 1/n \leq t \leq \alpha\} \cup \{[0,1/n]\times[-1,1]\}, n \geq 1,$$

that is, Γ is an $R_\delta-$set and has trivial shape in $\mathbf{R}^2$ (see Girolo [45]).

The Borsuk homotopy extension theorem (see [88]) implies that if X is an ANR and if $X \supset A$ has trivial shape in X, then for each open neighborhood U of A in X, there exists an open neighborhood V of A in X contained and contractible in U. This brings to light the following:

Definition 3.6 (Dugundji)**.** [37] A subspace Z of a space X is said to be $\infty-$*proximally connected in* X *(*PC_X^∞ for short) if for each open neighborhood U of Z in X, there exists an open neighborhood V of Z in X such that for every non-negative integer n, every continuous mapping defined on the boundary of the $n-$dimensional standard simplex Δ^n with values in V extends continuously to a mapping of Δ^n into U.

Clearly, if $\{Z_i\}_{i=1}^\infty$ is a decreasing sequence of compact spaces having trivial shape in an ANR X, then $Z = \bigcap_{i=1}^\infty Z_i$ is PC_X^∞. In particular, every R_δ set in an ANR X is PC_X^∞.

Assume that $X \in NES(\mathcal{K})$, where $\mathcal{K}$ is the class of compact spaces, and assume that A is a closed subspace of X. Let U be an open neighborhood of A in X and let $h : A \times [0,1] \to U$ be a homotopy joining $id_X|A = id_A$ and a constant mapping $\bar{a} \in U$. Since U is itself an $NES(\mathcal{K})$ space,[3] we show that the compact homotopy h extends to a homotopy $\hat{h} : V \times [0,1] \longrightarrow U$ defined on an open neighborhood V of A in U and joining id_V to the constant $\bar{a}$. Consequently, in both cases: (A closed $\subset X \in ANR$) or (A closed $\subset X \in NES(\mathcal{K})$), for every positive integer n, every mapping in $\mathbf{c}(\partial\Delta^n, V)$ extends continuously to a mapping in $\mathbf{c}(\Delta^n, U)$, that is, A is PC_X^∞.

3.2.3 Convexity Structures

There has been a growing interest in abstract convexities in recent years. Some are related to metrizability and the existence of minimal sets as well as to fixed-point theory for non-expansive mappings (see Khamsi et al. [63] and references there); others are purely topological and relate to fixed point for trees, and others stem from problems in computer science. Convexity concepts and arguments can be adapted to settings lacking underlying linearity structures through the consideration of abstract convexities of a topological, metric, or even set-theoretical nature. This section reviews key notions and examples of spaces equipped with convexity structures. We refer the reader to Van de Vel [87], Bielawski [21], Horvath [55, 56, 57], Horvath and Llinares [58], Llinares [66], Park and Kim [79], Park [80, 81], Saveliev [83], and Ben-El-Mechaiekh et al. [14, 12, 13] for more on abstract convexities.

[3] An open subspace of an $NES(\mathcal{Q})$ is itself $NES(\mathcal{Q})$.

Definition 3.7. A *convexity structure* on a non-empty set E is a collection $\mathcal{C}$ of subsets of E containing $\emptyset$ and E and closed for arbitrary intersections.

Let us denote by $\langle E \rangle$ the collection of all finite subsets of a given set E and let $\Gamma : \langle E \rangle \rightrightarrows E$ be a set-valued map verifying $\Gamma(\emptyset) = \emptyset$.

Definition 3.8. (i) A subset X of E is said to be $\Gamma-$*convex* (or simply *convex*) in E if and only if it is invariant under Γ, that is, $\Gamma(A) \subseteq X$ for every $A \in \langle X \rangle$.

(ii) Define the Γ-*convex envelope* of X in E by $conv_\Gamma(X) := \bigcap\{C$ convex in $E : X \subseteq C\}$ for any given $X \subseteq E$.

(iii) The pair (E, Γ) is called a *convex pair.*

Clearly, the collection $\mathcal{C}$ of all $\Gamma-$convex subsets of E forms a convexity structure on E.

Remark 3.1. Clearly, a subset X is convex if and only if $X = conv_\Gamma(X)$.

Note that here, the set-valued map Γ actually generates a convexity structure thus justifying the terminology.

Definition 3.9. A uniform space $(E, \mathcal{V})$ with a convexity structure $\mathcal{C}$ is said to be *locally* $\mathcal{C}-$*convex* if

$$A \in \mathcal{C} \Rightarrow V[A] \in \mathcal{C}, \forall V \in \mathcal{V}.$$

C. D. Horvath was the first to define a topological convexity structure (as a substitute to traditional linear convexity) in terms of a set-valued map in the mid-1980s in his Ph.D. thesis (see references in [56]) as follows.

Definition 3.10. (i) A $c-$structure on E is an isotone[4] map $\Gamma : \langle E \rangle \rightrightarrows E$ with non-empty contractible values. The pair (E, Γ) is called a $c-$*space*.

(ii) A subset X of E is a $\Gamma-$*set* if it is invariant under Γ, that is, $\Gamma(A) \subset X, \forall A \in \langle X \rangle$.

(iii) A $c-$space (E, Γ) is said to be an $lc-$*space* if E is a uniform space and if there exists a basis $\{V_i\}_{i \in I}$ for the uniformity such that $\forall i \in I$, the V_i-neighborhood of X, $V_i[X] := \{y \in E : X \cap V_i[y]) \neq \emptyset\}$ is a $\Gamma-$set whenever $X \subset E$ is a $\Gamma-$set.

It is clear that an intersection of $\Gamma-$sets is also a $\Gamma-$set and that both $\emptyset$ and E are $\Gamma-$sets. Hence, the collection of all $\Gamma-$sets forms a convexity structure.

The following examples of $c-$spaces are given in Horvath [55, 56].

[4]Meaning $A \subseteq B \Rightarrow \Gamma(A) \subseteq \Gamma(B)$.

Example 3.2. Let E be a space and $\alpha : E \times E \times [0,1] \longrightarrow E$ be a mapping such that:

$$\forall (x,y) \in E \times E, \alpha(x,y,0) = \alpha(y,x,1) = x.$$

A subset X of E is said to be an $\alpha-$*set* if $\alpha(X \times X \times [0,1]) \subseteq X$. For any $A \in \langle E \rangle$, let $\Gamma(A) := \{X : A \subseteq X \text{ and } X \text{ is an } \alpha-\text{set}\}$ and suppose that there exists $a_0 \in \Gamma(A)$ such that the mapping $(x,t) \to \alpha(x,a_0,t)$ is continuous on $\Gamma(A) \times [0,1]$. Then (E,Γ) is a $c-$space.

In particular, given a contractible semigroup $(G,*)$ with unit e and contractibility mapping $\theta : G \times [0,1] \to G, \theta(x,1) = x$ and $\theta(x,0) = e, x \in G$, let $\alpha(x,y,t) := \theta(x,1-t) * \theta(y,t), x,y \in G, t \in [0,1]$. Then, for Γ defined as above, the pair (G,Γ) is a $c-$space.

Example 3.3. Let $E := L^1_\mu(\Omega, F)$ be the space of $\mu-$integrable functions from a measurable space Ω with non-atomic probability measure μ into a Banach space F. Given $A := \{\phi_1, \ldots, \phi_n\} \subset E$, let:

$$\Gamma(A) := \{\sum_{i=0}^{n} 1_{\Omega_i}\phi_i : \{\Omega_i\} \text{ being a partition of } \Omega \text{ into } \mu - \text{measurable sets}\}.$$

If F is separable, then $\Gamma(A)$ is a retract of E and hence contractible. The pair (E,Γ) is a $c-$space.

This example is related to the concept of the *decomposable set* of Bressan and Colombo [23]. The following list of examples outlines noteworthy instances of abstract convex spaces.

Example 3.4. (Lassonde Convex Spaces) A *convex space* $(X,D) = (X,D;\Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $conv(D) \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology.

Example 3.5 (Bielawski Simplicial Convexity)**.** Following [14], let us call a $B'-$*simplicial convexity* on a topological space E any family of continuous functions $\phi^{[y_0,\ldots,y_n]} : \Delta^n \to E$, defined for each positive integer n and each finite subset $\{y_0,\ldots,y_n\} \subset E$, and satisfying $\forall n \geq 1, \forall \{y_0,\ldots,y_n\} \subset E$,

$$\lambda = \sum_{k=0}^{p} \lambda_{i_k} e_{i_k} \Rightarrow \phi^{[y_{i_0},\ldots,y_{i_p}]}(\lambda) = \phi^{[y_0,\ldots,y_n]}(\sum_{k=0}^{p} \lambda_{i_k} e_k).$$

The set-valued map $\Gamma : \langle E \rangle \rightrightarrows E$ defined by

$$\Gamma(A) := \{\phi^{[y_0,\ldots,y_n]}(\lambda) : \{y_0,\ldots,y_n\} \subset A, \lambda \in \Delta^n\}, A \in \langle E \rangle,$$

defines an $L-$structure, in the sense of Definition 3.11, on Y.

Any $B-$simplicial convexity in the sense of Bielawski [19] is a $B'-$simplicial convexity with the additional property: $\phi^{[y]}(1) = y$.

A convexity structure deriving from a $B'-$simplicial convexity is called a $B'-$convexity.

It is proven in [14] that every $c-$space in the sense of Horvath admits a $B'-$simplicial convexity.

Example 3.6 (Park and Kim $G-$Convex Spaces)**.** [79] A *generalized convex space* or a *$G-$convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$.

The following abstract topological convexity was defined in [14]. It encompasses a variety of topological convexities studied in the literature.

Definition 3.11. An $L-$structure on a topological space Y is a set-valued map $\Gamma : \langle Y \rangle \rightrightarrows Y$ defined on the family $\langle Y \rangle$ of all nonempty finite subsets of Y verifying:

$$\forall A = \{y_0, \ldots, y_n\} \in \langle Y \rangle, \exists f^A \in \mathbf{c}(\Delta^n, \Gamma(A)) \text{ such that}$$
$$f^A(\Delta_J) \subset \Gamma(\{y_i : i \in J\}), \forall J \subset \{0, \ldots, n\},$$

where for $J \subset \{0, \ldots, n\}, \Delta_J = conv(\{e_i : i \in J\})$.

Example 3.7. The pair (Y, Γ) is called an *$L-$space* and a subset $C \subseteq Y$ is said to be convex for the $L-$structure if and only if $\Gamma(A) \subset C, \forall A \in \langle C \rangle$.

The collection of all convex subsets of Y form a convexity structure $\mathcal{C}$ on Y called an *$L-$convexity.*

The concept of an $L-$space is actually a small refinement of that of a $c-$ space of Horvath. The original $G-$convex spaces of Park and Kim [79] are $L-$spaces (with an auxiliary set D much as in the way of Lassonde convex spaces) where, in addition, the $L-$structure Γ is isotone. Later, Park removed the isotony condition in his definition of $G-$convexity. Simplicial convexities also give rise to $L-$structures (see [14]).

It turns out, as one would expect, that $\mathcal{C}-$convex sets are extension spaces for metric spaces:

Example 3.8. [57, Horvath] Any $lc-$space E or any $c-$convex subset of an $lc-$space is an ES($\mathcal{M}$) where $\mathcal{M}$ is the class of metric spaces. Consequently, any metrizable $lc-$space E or any $c-$convex subset of a metrizable $lc-$space is an AR.

We have the remarkable characterization of ARs (Theorem 2.1 and Corollary 2.2 in [21]) as follows:

Example 3.9. Any topological space E equipped with a local $B-$simplicial convexity is an ES($\mathcal{M}$) where $\mathcal{M}$ is the class of metric spaces. In fact, a metrizable space E is an AR if and only if E can be equipped with a local $B-$simplicial convexity.

3.2.4 Space Approximation

The consideration of the class $\mathcal{A}(\mathcal{K};\mathcal{P})$ of spaces by the author in [18] was prompted by a fundamental result of V. Klee [64] on the existence of arbitrarily close displacements of compact subsets of convex sets in locally convex spaces into finite polyhedra. This property, known as the *admissibility of Klee*, suggests a general framework for the approximation of spaces that could be applied to other situations.

3.2.4.1 The Property $\mathcal{A}(\mathcal{K};\mathcal{P})$ for Spaces

Definition 3.12. Let $\mathcal{K},\mathcal{P}$ be two classes of spaces. A space X is said to have the $(\mathcal{K};\mathcal{P})-$approximation property, (respectively the $(\mathcal{K};\mathcal{P})_H-$approximation property) written $X\in\mathcal{A}(\mathcal{K};\mathcal{P})$ ($X\in\mathcal{A}_H(\mathcal{K};\mathcal{P})$ respectively), if and only if:

$$\begin{cases} \forall K\in\mathcal{K} \text{ with } K \underset{closed}{\subseteq} X, \forall\omega\in Cov_X(K), \\ \exists\omega'\in Cov_X(K), \omega'\preccurlyeq\omega, \exists P\in\mathcal{P}, \\ \exists \text{ a pair of continuous functions } s:\bigcup\omega'\to P, \\ r:P\to\bigcup\omega' \text{ such that } r\circ s \text{ and the identity } id_{\bigcup\omega'} \\ \text{are } \omega-\text{near } (\omega-\text{homotopic, respectively}). \end{cases}$$

Clearly, $\mathcal{A}_H(\mathcal{K};\mathcal{P})\subset\mathcal{A}(\mathcal{K};\mathcal{P})$.

Klee's admissibility tells us that convex subsets of locally convex spaces have the following approximation property:

Theorem 3.1. Let $\mathcal{K}$ be the class of compact spaces, $\mathcal{P}$ the class of finite polyhedra, and let X be a non-empty convex subset of a locally convex topological vector space E. Then $X\in\mathcal{A}_H(\mathcal{K};\mathcal{P})$.

Proof. Let K be a compact subset of X and $\omega\in Cov_X(K)$. Let U a convex symmetric open neighborhood of the origin in E such that $\omega':=\{(x_i+U)\cap X\}_{i=1}^n\in cov_X(K)$ is a finite refinement of ω. The Schauder projection associated with U, $s:\bigcup\omega'\longrightarrow X$ is defined by:

$$s(x):=\frac{1}{\sum_{i=1}^n\mu_i(x)}\sum_{i=1}^n\mu_i(x)x_i,$$

where $\mu_i(x):=\max\{0,1-p_U(x-x_i)\}, p_U$ being the Minkowski functional associated to U. It verifies

$$\forall x\in\bigcup\omega', s(x)-x\in U, \text{ and } s(x)\in P,$$

where P is a finite polyhedron, and a copy of the geometric realization of the cover ω'. If r denotes the inclusion $P\hookrightarrow X$, then $s\circ r$ and $id_{\bigcup\omega'}$ are $\omega-$near, that is, $X\in\mathcal{A}(\mathcal{K};\mathcal{P})$. Since U is convex, for each $x\in\bigcup\omega'$, the line segment joining $s(x)$ to x belongs to $(x+U)\cap X$. Hence, $s\circ r$ and $id_{\bigcup\omega'}$ are $\omega-$homotopic through the linear homotopy $h(x,t):=t((s\circ r)(x))+(1-t)x, t\in[0,1]$. Therefore, $X\in\mathcal{A}_H(\mathcal{K};\mathcal{P})$. □

The following observation is worth pointing out.

Remark 3.2. The finite polyhedron P in the proof of Theorem 3.1 is not necessarily convex. The refinement ω' can be chosen so that $P \subset \bigcup \omega$ (to be precise, $P \subset \bigcup_{i=1}^{n} \{(x_i + 2U) \cap X\}$). Obviously, the proof shows that P is contained in the convex hull of the finite set $\{x_1, \ldots, x_n\}$, so that in effect, convex subsets of locally convex spaces are in the smaller class $\mathcal{A}(\textit{compact spaces}; \textit{convex finite polyhedra})$. This inclusion is precisely the *admissibility in the sense of Klee later extended (by Klee himself)* to some non-locally convex spaces (for example, $\ell^p, 0 < p < 1$).

We shall prove next that generalized convex subsets of spaces equipped with some abstract topological convexity are also admissible in the sense of Klee.

Theorem 3.2. Let $\mathcal{K}$ be the class of compact spaces and $\mathcal{P}$ the class of convex finite polyhedra. Assume that X is a non-empty $\mathcal{C}-$ convex subset of a locally $\mathcal{C}-$convex space E, where $\mathcal{C}$ is

(i) a convexity structure in the sense of Horvath, or

(ii) a $B'-$convexity, or

(iii) an $L-$convexity.

Then

$$X \in \mathcal{A}(\mathcal{K}; \mathcal{P}).$$

If in (i) and (ii) the space E is also metrizable, then $X \in \mathcal{A}_H(\mathcal{K}; \mathcal{P})$.

Proof. We only prove (i); the proof of (ii) and (iii) are simpler and are left to the reader. We start by showing that $X \in \mathcal{A}(\mathcal{K}; \mathcal{P})$.

Let K be a compact subset of X and let $\omega \in Cov_X(K)$. The cover ω has a suitable uniform refinement ω' of the form $\{V[y_i] \cap X : y_i \in K, i = 0, ..., n\}$ of open subsets of X, where $V \in \mathcal{V}$, the uniform structure of E.

Let κ be the Kuratowski barycentric mapping that transforms the set $K_V := \bigcup_{i=0}^{n} V[y_i]$ into a subset of the closure $\overline{\sigma}$ of the open $n-$simplex $\sigma = 0...n$, (see Kuratowski [65], Dugundji [36], or Brown [27]). The mapping κ verifies that

$$\text{for each simplex } i_0 \ldots i_k,\ \kappa^{-1}(i_0 \ldots i_k) = V[y_{i_0}] \cap \cdots \cap V[y_{i_k}] \setminus \bigcup\nolimits_{i \neq i_j} V[y_i].$$

We shall now show the existence of a continuous mapping $f : \overline{\sigma} \to X$ such that

$$(f \circ \kappa)(x) \in \Gamma(\{y_i : x \in V[y_i]\}) \text{ for all } x \in K_V.$$

To do this, denote by $\sigma^{(k)}$ the complex consisting of all $k-$dimensional faces of $\sigma, k = 0, \ldots, n, \sigma^{(n)} = \sigma$. We construct a finite sequence of mappings $f^{(k)} : \overline{\sigma^{(k)}} \to X, k = 0, \ldots, n$, with the property $f^{(k)}(\overline{i_0 \ldots i_k}) \subset \Gamma(\{y_{i_j} : j = 0, \ldots, k\})$ for every $k-$dimensional face $i_0 \ldots i_k$ of σ.

If $k = 0$, and let $f^{(0)} : \{0, 1, \ldots, n\} \to X$ be the mapping

$$f^{(0)}(i) = z_i, \text{ where } z_i \text{ is an arbitrary point in } \Gamma(\{y_i\}),\ i = 0, 1, \ldots, n.$$

This mapping is obviously continuous on the set of vertices $\sigma^{(0)}$ of σ equipped with the discrete topology. Assume that for some $k \in \{1, \ldots, n-1\}$ such a function $f^{(k)}$ has been constructed, and let $i_0 \ldots i_{k+1}$ be an arbitrary $(k+1)-$dimensional face of σ. The mapping $f^{(k)}$ maps the boundary of $i_0 ... i_{k+1}$ into the set $\bigcup_{l=0}^{k+1} \Gamma(\{y_{i_j} : j = 0, \ldots, k+1, j \neq l\})$, which by monotonicity of the $c-$structure Γ, is contained in the contractible (thus $(k+1)-$connected) set $\Gamma(\{y_{i_0}, \ldots, y_{i_{k+1}}\})$. Therefore, the restriction of $f^{(k)}$ to the boundary of $i_0 \ldots i_{k+1}$ extends continuously into a partial mapping $f^{(k+1)}_{i_0 \ldots i_{k+1}}$ of the entire simplex $i_0 \ldots i_{k+1}$ into the set $\Gamma(\{y_{i_0}, \ldots, y_{i_{k+1}}\})$. Keeping in mind that $\overline{\sigma}$ is equipped with a $CW-$complex topology, one can certainly piece together these partial mappings to form a continuous mapping $f^{(k+1)} : \overline{\sigma^{(k+1)}} \to X$ verifying that $f^{(k+1)}(\overline{i_0 \ldots i_{k+1}}) \subset \Gamma(\{y_{i_j} : j = 0, \ldots, k+1\})$ for every $(k+1)-$dimensional face $i_0 \ldots i_{k+1}$ of σ. By the property of κ, the mapping $f := f^{(n)}$ has the desired property. It is important to observe that given $x \in K_V$, if $x \in V[y_i]$, then $y_i \in V[x]$, which is a $\mathcal{C}-$convex set. Hence,

$$f(\kappa(x)) \in \Gamma(\{y_i : x \in V[y_i]\}) \subset V[x];$$

that is, the mapping $f \circ \kappa$ and the inclusion $K_V \hookrightarrow X$ are $V-$near, hence $X \in \mathcal{A}(\mathcal{K}; \mathcal{P})$.

In cases (i) and (ii), X is an AR, and hence an ANR. The entourage V in the begining of the proof can be chosen so that $\omega' := \{V[y_i] : i = 0, \ldots, n\}$ verifies Proposition 3.4. Thus, $f \circ \kappa$ and the inclusion $K_V \hookrightarrow X$ are $\omega-$homotopic. ☐

Remark 3.3. (a) Obviously, (iii) is more general than (i) and (ii). The proof of (iii) is, however, simpler in that the mapping f constructed in the proof is readily provided in the definition of an $L-$structure (Definition 3.11).

(b) We conjecture that the metrizability of E is not necessary for the inclusion $X \in \mathcal{A}_H(\mathcal{K}; \mathcal{P})$ to hold in cases (i) and (ii).

(c) It would be interesting to determine sufficient conditions on an abstract convexity structure $\mathcal{C}$ for $\mathcal{C}-$convex sets to be in $\mathcal{A}_H(\mathcal{K}; \mathcal{P})$.

The property $\mathcal{A}(\mathcal{K}; \mathcal{P})$ for a space is closely related to classical properties of "domination of spaces through approximations" widely used in topology (particularly in homotopy theory) which we describe next.

3.2.4.2 Domination of Domain

Let X, P be two spaces. If there are continuous mappings $s \in \mathbf{c}(X, P)$ and $r \in \mathbf{c}(P, X)$ such that the diagram $id_X \circlearrowright X \underset{r}{\overset{s}{\rightleftarrows}} P$ commutes (that is, $r(s(x)) = x$

for all $x \in X$), the space P is said to *dominate* the space X. The class of spaces that are dominated by members of a given class $\mathcal{P}$ of spaces is defined as:

$$X \in R(\mathcal{P}) \Leftrightarrow X \text{ is dominated by some } P \in \mathcal{P}.$$

Given a cover $\omega \in cov(X)$, we say that the space P $\omega-$*dominates* the space X (ω_H-*dominates* X, respectively) if there are mappings $s \in \mathbf{c}(X, P)$ and $r \in \mathbf{c}(P, X)$ such that $r \circ s$ and id_X are $\omega-$near ($\omega-$homotopic, respectively). The classes $D(\mathcal{P})$ and $D_H(\mathcal{P})$ are defined by Definition 3.11.

Definition 3.13.

$$\begin{array}{c} X \in D(\mathcal{P}) \\ (X \in D_H(\mathcal{P}),\ \text{respectively}) \end{array} \Leftrightarrow \left\{ \begin{array}{l} \forall \omega \in cov(X), \exists P \in \mathcal{P}, \\ \text{such that } P\ \omega - \text{dominates } X \\ (\omega_H - \text{dominates } X,\ \text{respectively}). \end{array} \right.$$

It is clear that for any class of spaces $\mathcal{P}$, we have $\mathcal{P} \subset R(\mathcal{P}) \subset D_H(\mathcal{P}) \subset D(\mathcal{P})$.

The enlargements of classes of spaces $\mathcal{P} \to D(\mathcal{P})$ or $\mathcal{P} \to D_H(\mathcal{P})$ preserve certain fundamental topological properties, hence their importance in topology. For example, the passage from finite to infinite products of spaces can be described by the passage $\mathcal{P} \to D(\mathcal{P})$.

Indeed, given an arbitrary family of spaces $\{X_i : i \in I\}$, let $X = \prod_{i \in I} X_i$ be their Cartesian product, and given $J \in \mathcal{J} = \{J \subset I : J \text{ is finite}\}$, let us denote $X_J = \prod_{i \in J} X_i$. Then we have:

Proposition 3.7. If X is paracompact and $\mathcal{P} = \{X_J : J \in \mathcal{J}\}$, then $X \in D(\mathcal{P})$.

Proof. The proof is identical to that of Proposition 4.1 in Granas [52] where the compact case is treated.

For each $i \in I$, let x_i^0 be a base point in the space X_i and for $J \in \mathcal{J}$, let $\tilde{X}_J \subset X$ be the product $X_J \times (x_i^0)_{i \notin J}$, that is,

$$(x_i) \in \tilde{X}_J \Leftrightarrow \left\{ \begin{array}{lll} x_i \in X_i & if & i \in J \\ x_i = x_i^0 & if & i \notin J \end{array} \right. .$$

Clearly, $\tilde{X}_J$ can be identified with X_J.

Consider the subcollection $\mathcal{C}$ of $Cov(X)$ defined as follows:

$$\omega \in \mathcal{C} \Leftrightarrow \left\{ \begin{array}{l} \omega \text{ is a neighborhood-finite cover of } X \text{ by open sets} \\ \quad \text{of the form } \prod_{i \in I} U_i \text{ where, for each } i \in I, \\ \quad U_i \text{ is an open subset of } X_i, \text{ and } U_i = X_i \\ \quad \text{except for at most finitely many indices } i. \end{array} \right.$$

Since X is paracompact, by the very definition of the Cartesian product topology, it follows that $\mathcal{C}$ is cofinal in $Cov(X)$. Let $\omega = \{\omega_\lambda\}_{\lambda \in \Lambda} \in \mathcal{C}$ be arbitrary and let Λ_0 be the finite set of those indices λ for which $x_0 \in \omega_\lambda$. By definition, each $\omega_\lambda, \lambda \in \Lambda_0$, is of the form $\prod_{i\in I} U_i^\lambda$ where, for each $i \in I$, U_i^λ is an open subset of X_i, and $U_i^\lambda = X_i$ except for some essential indices forming a finite set $I(\lambda) := \{i \in I : U_i^\lambda \neq X_i\}$. The set $J(\omega) := \bigcup_{\lambda\in\Lambda_0} I(\lambda)$ is clearly finite and $i \notin J(\omega) \Leftrightarrow U_i^\lambda = X_i$ for all $\lambda \in \Lambda_0$.

Let $s : X \longrightarrow X_{J(\omega)}$ be the projection and $r : \tilde{X}_{J(\omega)} \longrightarrow X$ be the natural imbedding. One readily verifies that $r \circ s$ and id_X are $\omega-$near. This completes the proof. □

Domination, $\omega-$domination, and ω_H-domination can be used to describe the passage from basic types of spaces to more elaborate ones. For instance:

Example 3.10. (a) If $\mathcal{P}$ is the class of all normed spaces, then $R(\mathcal{P})$ is the class AR of absolute retracts.

(b) If $\mathcal{P}$ is the class of all open subsets of normed spaces, then $R(\mathcal{P})$ is the class *ANR* of absolute neighborhood retracts.

(c) If $\mathcal{P}$ is the class of all finite (respectively, locally finite) polyhedra endowed with the CW-topology, then $D_H(\mathcal{P})$ contains the class of compact (respectively, arbitrary) ANRs.

(d) The extension properties in Examples 3.8 and 3.9 together with (c) above yield to a refinement of Theorem 3.2 : $\mathcal{C}-$convex subsets of locally $\mathcal{C}-$convex metrizable spaces in the sense of Horvath or in the sense of Bielawski are in $D_H(\mathcal{P})$, where $\mathcal{P}$ is the class of all finite polyhedra.

The following remark is important for the sequel:

Remark 3.4. One should keep in mind that in the proof of inclusion (c) in Example 3.10, for a given $\omega \in Cov(X)$, the polyhedron P that ω_H-dominates an ANR X is precisely the geometric nerve $|N(\omega)|$ of the cover ω. Moreover, the mappings s and r involved in Definition 3.13 are, respectively, essentially the Kuratowski barycentric mapping κ (as in the proof of Theorem 3.2) and the neighborhood retraction involved in the definition of an ANR.

Theorem 3.3. The property, for a paracompact space X, of being ω_H-dominated is "transitive" in the following sense:

$$\text{If } X \in D_H(\mathcal{P}) \text{ and } \mathcal{P} \subset D_H(\mathcal{P}'),\ \text{then } X \in D_H(\mathcal{P}').$$

Proof. Indeed, given $\omega \in Cov(X)$, let $\omega^* \in Cov(X)$ be a star refinement of ω and let $P \in \mathcal{P}, s \in \mathbf{c}(X,P), r \in \mathbf{c}(P,X)$ be such that $r \circ s$ and id_X are ω^*-homotopic. Let $\alpha^* = r^{-1}(\omega^*) \in Cov(P)$ and let $P' \in \mathcal{P}', s' \in \mathbf{c}(P,P'), r' \in \mathbf{c}(P',P)$ be such that $r' \circ s'$ and id_P are α^*-homotopic (through a homotopy $h : P \times [0,1] \to P$). For any $x \in X$, $\{(r' \circ s' \circ s)(x), s(x)\} \subset A^* = r^{-1}(W^*)$, for some $W^* \in \omega^*$, that is, $\{(r \circ r' \circ s' \circ s)(x), (r \circ s)(x)\} \subset W^*$. But $\{(r\circ s)(x), x\} \subset W^{*\prime}$ for some $W^{*\prime} \in \omega^*$. Thus $\{(r\circ r'\circ s'\circ s)(x), (r\circ s)(x), x\} \subset St(W^*, \omega^*) \subset W$ for some $W \in \omega$. This means that P' $\omega-$dominates X. Now, observe that $r \circ r' \circ s' \circ s$ and $r \circ s$ are homotopic through the homotopy $r \circ h(s(.), .)$. But $r \circ s$ is homotopic to id_X. Thus, $r \circ r' \circ s' \circ s$ and id_X are homotopic (one can choose ω^* in such a way that the latter homotopy is controlled by ω). □

Note that in view of Remark 3.4, if P is an *ANR*, P' can be chosen to be the nerve $|N(\alpha^*)|$ of the cover α^*, and that this nerve is a subpolyhedron of the nerve of ω^* (both nerves having the same vertices).

These remarks lead to Theorem 3.4.

Theorem 3.4. Assume that $X \in D_H(\mathcal{P})$, where $\mathcal{P}$ is the class of finite polyhedra. If X has non-trivial Euler-Poincaré characteristic $\chi(X)$,[5] then, $\forall \omega \in Cov(X), \exists P \in \mathcal{P}$ such that:

$$\text{(i) } P\ \omega - H - \text{dominates } X, \text{ and (ii) } \chi(P) \neq 0.$$

Proof. Let $\omega \in Cov(X)$ be arbitrary but fixed, and let $|N(\omega)|$ be the geometric nerve of ω. By Proposition 3.4, and since $|N(\omega)|$ is an *ANR*, there exists an open cover $\alpha \in Cov(|N(\omega)|)$, such that for any metric space Z, any two mappings $f, g \in \mathbf{c}(Z, |N(\omega)|)$ that are $\alpha-$near are homotopic. Consider now a triangulation τ of $|N(\omega)|$ finer than the cover α. Let us choose the (possibly iterated) star refinement ω^* of ω, the polyhedron P, the cover $\alpha^* = r^{-1}(\omega^*)$ in the proof of Theorem 3.3 in such a way that $|N(\omega^*)|$ is a subpolyhedron of $(|N(\omega)|, \tau)$, and that the cover $\alpha' = r'^{-1}(\alpha^*)$ of $|N(\alpha^*)|$ refines the trace of the cover α on $|N(\alpha^*)|$ (keep in mind that $|N(\alpha^*)|$ is a subpolyhedron of $|N(\omega^*)|$, thus $|N(\alpha^*)|$ is a subpolyhedron of $|N(\omega)|$). Denote $|N(\alpha^*)|$ again by P. Obviously, $P\ \omega - H-$dominates X, and the mappings $id_P, s' \circ s \circ r \circ r' : P \to P$ are $\alpha'-$near. Hence, id_P and $s' \circ s \circ r \circ r'$ are homotopic.

By the homotopy invariance of the Lefschetz number, $\chi(X) = \lambda(r \circ r' \circ s' \circ s)$ and $\chi(P) = \lambda(s' \circ s \circ r \circ r')$. It is well known (for example, see Granas [52] and Lemma 1 III.C in Brown [27]) that, when defined for a pair of mappings f and g, the Lefschetz numbers $\lambda(f \circ g)$ and $\lambda(g \circ f)$ are equal. Hence $\chi(X) = \lambda(r \circ r' \circ s' \circ s) = \lambda(s' \circ s \circ r \circ r') = \chi(P)$. □

3.2.4.3 Domination, Extension, and Approximation

In comparing the concepts of approximation and domination, we clearly have, for any given class of spaces $\mathcal{K}$:

$$\mathcal{K} \cap \mathcal{A}(\mathcal{K}; \mathcal{P}) \subseteq D(\mathcal{P}) \text{ and } \mathcal{K} \cap \mathcal{A}_H(\mathcal{K}; \mathcal{P}) \subseteq D_H(\mathcal{P}).$$

We settle now the opposite inclusions.

[5] Recall that for a spherical complex X, the Čech cohomology graded linear space $\{H^q(X; \mathbf{Q})\}$ is of finite type. The *Euler-Poincaré characteristic* of X is defined to be the Lefschetz number of the identity mapping, namely, $\chi(X)$ is the finite sum $\lambda(id_X) := \sum_{q \geq 0} (-1)^q \beta_q$ where $\beta'_q = \dim H^q(X; \mathbf{Q})$ is the $qth-$Betti number of X (see [27, 52]).

Proposition 3.8. $D(\mathcal{P}) \subset \mathcal{A}(\mathcal{K};\mathcal{P})$ and $D_H(\mathcal{P}) \subset \mathcal{A}_H(\mathcal{K};\mathcal{P})$ for any given two classes of spaces $\mathcal{P}$, $\mathcal{K}$.

Proof. We only prove the first inclusion; the proof of the second one is identical. Let $X \in D(\mathcal{P}), K \in \mathcal{K}$ be a closed subspace of X, and let $\omega \in Cov_X(K)$. By hypothesis, $\exists P \in \mathcal{P}, \exists s \in \mathbf{c}(X,P), \exists r \in \mathbf{c}(P,X)$ such that $r \circ s$ and id_X are $\hat{\omega}-$near, where $\hat{\omega} := \omega \cup \{X \setminus K\}$ is an open cover of X. For any $x \in K$, the member W_x of $\hat{\omega}$ containing $\{x, (r \circ s)(x)\}$ cannot be $X \setminus K$. The open set $U_x := (r \circ s)^{-1}(W_x)$ is a neighborhood of x in X and the family $\omega' := \{W_x \cap U_x : x \in K\}$ is an open cover of K that refines ω. The mappings $s|_{\bigcup \omega'} \circ r$ and $id_{\bigcup \omega'}$ are $\omega'-$near. □

We prove now a more general result than Example 3.10 (c); namely, that the larger class of approximative neighborhood extension spaces for compacts spaces (see Section 3.2 above for the definitions of the classes *ES, NES,* and *ANES*) have the approximation property in Definition 3.12.

Theorem 3.5. Let $\mathcal{K}$ be the class of compact spaces, $\mathcal{P}$ the class of finite polyhedra, and $\mathcal{N}$ the class of normal spaces. Then:

$$\mathcal{N} \cap \text{ ANES}(\mathcal{K}) \subset \mathcal{A}(\mathcal{K};\mathcal{P}) \text{ and } \mathcal{N} \cap \text{ A}_H\text{NES}(\mathcal{K}) \subset \mathcal{A}_H(\mathcal{K};\mathcal{P}).$$

Proof. We only provide the proof of the first inclusion; the second one is a mere adaptation of the homotopical case. Let $X \in ANES(\mathcal{K})$ be a normal space, let $K \hookrightarrow X$ be compact, and let $\omega \in Cov_X(K)$. The Tychonoff imbedding theorem asserts that, as a compact space, K is homeomorphic to a closed subset $\hat{K}$ of a Tychonoff cube T (that is, a Cartesian product of copies of the unit interval imbedded in a normed space E). Let h be this homeomorphism and let $\omega' \in Cov_X(K)$ be a barycentric refinement of ω.

By the definition of an *ANES* space (Definition 3.4 (ii)), there exist an open subset V of $T, V \supset \hat{K}$, and $f \in \mathbf{c}(V,X)$ such that h^{-1} and $f|_{\hat{K}}$ in the following diagram

$$\begin{array}{ccccc} & & X & & \\ h^{-1} & \nearrow & \uparrow f & & \\ \hat{K} & \hookrightarrow & V & \hookrightarrow & T \end{array}$$

are $\omega'-$near.

Due to the compactness of $\hat{K}$, there exist a convex open neighborhood of the origin U in the normed space E containing T and a finite set $\{\hat{x}_i\}_{i=1}^n \subset \hat{K}$ such that $\hat{\omega} := \{(\hat{x}_i + U) \cap T\}_{i=1}^n \in Cov_T(\hat{K})$ and $\bigcup_{i=1}^n \{(\hat{x}_i + 2U) \cap T\} \subset V$.

The Schauder projection $\pi : \bigcup \hat{\omega} \longrightarrow P$ onto a finite polyhedron $P \subset \bigcup_{i=1}^n \{(\hat{x}_i + 2U) \cap T\} \subset V$ and the identity on $\hat{K}$ are $\hat{\omega}-$near (this follows from Theorem 3.1 and Remark 3.2).

The Tietze's extension theorem implies that every Tychonoff cube is an extension space for normal spaces, that is, $T \in ES(\mathcal{N})$. Moreover, it is known that for a given class of normal spaces $\mathcal{Q}$, every open subspace of an $NES(\mathcal{Q})$ space is also $NES(\mathcal{Q})$ (see Théorème 2.2 in Granas [52]). Since $ES(\mathcal{N}) \subset$

$NES(\mathcal{N})$, then $\bigcup \hat{\omega} \in NES(\mathcal{N})$. Consequently, there exists $\tilde{h} \in \mathbf{c}(\bigcup \omega', \bigcup \hat{\omega})$ such that the diagram

$$\begin{array}{ccccc} & & \bigcup \hat{\omega} & \hookrightarrow T \\ h \nearrow & & \uparrow \tilde{h} & \\ K \hookrightarrow & & \bigcup \omega' & \hookrightarrow X \end{array}$$

commutes. The mappings $s = \pi \circ \tilde{h}$ and $r = f \circ j$ in the following diagram

$$\begin{array}{ccc} \bigcup \hat{\omega} \xrightarrow{\pi} P \overset{j}{\hookrightarrow} V \\ \tilde{h} \uparrow & & \downarrow f \\ \bigcup \omega' & \hookrightarrow & X \end{array}$$

are so that $r \circ s$ and $id_{\bigcup \omega'}$ are $\omega-$near, that is, $X \in \mathcal{A}(\mathcal{K}; \mathcal{P})$. □

By Proposition 3.2, the class $AANR(\mathcal{K})$ of approximative *ANRs* for compact spaces is precisely $\mathcal{K} \cap$ $ANES(\mathcal{K})$, which is included in $\mathcal{K} \cap \mathcal{A}(\mathcal{K}; \mathcal{P})$, which is in turn contained in $D(\mathcal{P})$; hence we have:

Corollary 3.1. If $\mathcal{K}$ is the class of compact spaces and $\mathcal{P}_F$ is the class of finite polyhedra, then:

$$\begin{array}{ccc} \mathrm{AANR}(\mathcal{K}) & \subset & D(\mathcal{P}_F) \\ \cup & & \cup \\ \mathrm{A}_H\mathrm{ANR}(\mathcal{K}) & \subset & D_H(\mathcal{P}_F) \end{array}.$$

Since compact *AANRs* (in particular, compact *ANRs*) are in $AANR(\mathcal{K})$ for the class $\mathcal{K}$ of compact spaces (see Gauthier [42]), it follows that:

Corollary 3.2. Every compact AANR (in particular, compact ANR) is in $D(\mathcal{P})$, where $\mathcal{P}$ is the class of finite polyhedra.

3.3 Set-Valued Maps, Continuous Selections, and Approximations

We now turn our attention to the fundamental techniques of continuous selections and approximative selections for set-valued maps.

A set-valued map, simply called a *map,* from a set X into a set Y is a relation denoted Φ: $X \rightrightarrows Y$, that assigns to each element $x \in X$, a subset $\Phi(x)$ of Y. The pre-images $\Phi^{-1}(y) := \{x \in X : y \in \Phi(x)\}$ are called the *fibers* of Φ. The *graph* of Φ is the set $graph(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}$.

Given an abstract class $\mathbf{M}$ of maps, denote by $\mathbf{M}_c$ the class of finite compositions of $M-$maps, that is,

$$\mathbf{M}_c := \{\Phi = \Phi_n \circ \cdots \circ \Phi_1 : \Phi_i \in \mathbf{M}, i = 1, \ldots, n\}.$$

Upper and lower semicontinuity are fundamental regularity requirements in set-valued analysis. We briefly recall them before discussing, in some detail, key selection and approximation properties for convex and non-convex maps. Results on the selection and approximate selection properties for non-convex maps can be found in Gel'man [43], Ginchev and Hoffmann [44], McClendon [71], McLennan [73], Saveliev [83], Repovs-Semenov-Scepin [84], the book of Repovs-Semenov [85], Deguire and the author [15], and in a series of papers by the author [20, 17, 19].

3.3.1 Semicontinuity Concepts

Definition 3.14. Given two spaces X and Y, the map Φ is said to be *upper semicontinuous* (*u.s.c.* for short) at the point $x_0 \in X$ if for any open neighborhood V of $\Phi(x_0)$ in Y, there exists an open neighborhood U of x_0 in X such that $\Phi(U) \subset V$. The map Φ is said to be *u.s.c.* on X if it is *u.s.c.* at every point of X.

Note that Φ is *u.s.c.* on X if and only if the upper inverse image $\Phi^+(V) = \{x \in X; \Phi(x) \subset V\}$ is open in X for any open subspace V of Y. Let us put

$$\mathbf{USC}(X,Y) := \{\Phi : X \rightrightarrows Y : \Phi \text{ is } u.s.c. \text{ with non-empty values}\}.$$

In case of single-valued maps, upper semicontinuity amounts to the usual continuity; thus, the inclusion $\mathbf{c}(X,Y) \subset \mathbf{USC}(X,Y)$.

Let us denote:

$$\begin{aligned} \mathbf{USCL}(X,Y) \quad &: \quad = \{\Phi \in \mathbf{USC}(X,Y) : \Phi(x) \text{ is closed}, \forall x \in X\}. \\ \mathbf{USCO}(X,Y) \quad &: \quad = \{\Phi \in \mathbf{USC}(X,Y) : \Phi(x) \text{ is compact}, \forall x \in X\}. \end{aligned}$$

Suppose now that the space Y is equipped with a uniform structure $\mathcal{V}$ that is compatible with its topology.

Definition 3.15. Given an entourage $V \in \mathcal{V}$, the map Φ is said to be $V-$*upper semicontinuous* ($V-$u.s.c.) at a point $x_0 \in X$ if there exists an open neighborhood U of x_0 in X such that $\Phi(U) \subset V[\Phi(x_0)]$; here, $V[\Phi(x_0)] = \bigcup_{y \in \Phi(x_0)} V[y]$ and $V[y] = \{y' \in X; (y, y') \in V\}$.

The map Φ is $V-$*u.s.c. on* X if it is so at every point of X. It is said to be $\mathcal{V}-$*u.s.c.* if it is $V-$*u.s.c.* for every $V \in \mathcal{V}$.

We denote $\mathcal{V}-\mathbf{USC}(X,Y)$ as the class of all $\mathcal{V}-$*u.s.c.* maps from X into Y. Clearly, if the map is *u.s.c.* on X in the ordinary sense, then it is $\mathcal{V}-$*u.s.c* on X, i.e., $\mathbf{USC}(X,Y) \subset \mathcal{V}-\mathbf{USC}(X,Y)$ The converse holds true in the case where Φ is compact-valued. In the case where Y is a subset of a topological vector space F, the concept of $\mathcal{V}-$upper semicontinuity ($\mathcal{V}$ being the uniformity generated by a fundamental basis of neighborhoods of the origin in F) is known as *Hausdorff upper semicontinuity* (see De Blasi-Myjak [31]).

Definition 3.16. A map $\Phi : X \rightrightarrows Y$ between two spaces X and Y is said to be *lower semicontinuous* (*l.s.c.* for short) at the point $x_0 \in X$ if for any open subspace V of Y such that $V \cap \Phi(x_0) \neq \emptyset$, there exists an open neighborhood U of x_0 in X such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U$. The map Φ is said to be *l.s.c.* on X if it is *l.s.c.* at every point of X.

Note that Φ is *l.s.c.* on X if and only if the lower inverse image $\Phi^-(V) = \{x \in X; \Phi(x) \cap V \neq \emptyset\}$ is open in X for any open subspace V of Y. Lower semicontinuity at a point $x_0 \in X$ can be expressed in terms of nets as follows: for any $y \in \Phi(x_0)$ and for any net x_i in X converging to x_0, there exists a net $y_i \in \Phi(x_i)$ converging to y. It follows from this characterization of lower semicontinuity that if the map Φ is *l.s.c.* on X, then $cl(\Phi^+(Z)) \subseteq \Phi^+(cl(Z))$ for any subspace Z of Y. Denote:

$$\mathbf{LSC}(X, Y) := \{\Phi : X \rightrightarrows Y : \Phi \text{ is } l.s.c. \text{ with non-empty values}\}.$$

Remark 3.5. It is easy to verify that if a map $\Phi : X \rightrightarrows Y$ between two spaces X and Y has open fibers $\Phi^{-1}(y)$ in X for every $y \in Y$ (more generally, if it has an open graph in the product $X \times Y$), then it is *l.s.c.* on X.

Definition 3.17. A map Φ between two topological spaces X and Y is said to be *continuous* at a point x (on X, respectively) if it is both *u.s.c.* and *l.s.c.* at x (on X, respectively).

3.3.2 USC Approachable Maps and Their Properties

A systematic study of the graph approximation property of set-valued maps by continuous single-valued mappings was undertaken by the author in [13, 14, 16]. The class of $\mathbf{A}-$maps is defined as follows.

Definition 3.18. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces and let $\Phi : X \rightrightarrows Y$ be a map.

(i) Given $W \in \mathcal{U} \times \mathcal{V}$, a single-valued map $s : X \to Y$ is said to be a $W-$*approximative selection* of Φ if and only if:

$$graph(s) \subset W[graph(\Phi)].$$

Denote by $\mathbf{a}(\Phi; W) := \{\ s \in \mathbf{c}(X, Y) : s \text{ a } W-\text{approximative selection of } \Phi\}$.

(ii) The map Φ is said to be *approachable* if and only if it has non-empty values and

$$\forall W \in \mathcal{U} \times \mathcal{V}, \mathbf{a}(\Phi; W) \neq \emptyset.$$

Let the class of $\mathbf{A}-$maps from X into Y be

$$\mathbf{A}(X, Y) := \{\Phi \in \mathcal{V} - \mathbf{USC}(X, Y) : \Phi \text{ is approachable}\}.$$

We write $\mathbf{A}(X)$ for $\mathbf{A}(X, X)$.

Clearly,

$$\Phi \in \mathbf{A}(X,Y) \Leftrightarrow \begin{cases} \Phi \in \mathcal{V}-\mathbf{USC}(X,Y) \text{ and} \\ \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \exists s \in \mathbf{c}(X,Y) \text{ such that} \\ \forall x \in X, \exists x' \in U[x] \text{ with } s(x) \in V[\Phi(x')]. \end{cases}$$

Such a mapping s is said to be a continuous $(U,V)-$approximative selection of Φ. Denote by $\mathbf{a}(\Phi;U,V)$ the collection of all such approximations. The next section describes how the class $\mathbf{A}$ is stable.

3.3.2.1 Conservation of Approachability

In this section, unless specified otherwise, X and Y are spaces with respective uniform structures $\mathcal{U}$ and $\mathcal{V}$.

It is clear from Definition 3.18 that the continuous graph approximation problem for a given map Φ amounts to a continuous selection problem for any open neighborhood of the graph of Φ.

Remark 3.6. Evidently, if for two uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, a map $\Phi \in \mathbf{A}(X,Y)$, then the collection of set-valued maps $\{\Phi_{U,V}(.) := V[\Phi(U[.])] : (U,V) \in \mathcal{U} \times \mathcal{V}\}$ is an upper approximating family[6] of admitting continuous selections for the map Φ.

The next proposition is even more precise: under reasonable conditions, every *u.s.c.* map admits an open-graph majorant.

Proposition 3.9 (Open Graph Majorant)**.** Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces with X paracompact, and let $\Phi \in V-\mathbf{USC}(X,Y)$. Then, for any pair of entourages $(U,V) \in \mathcal{U} \times \mathcal{V}$, there exists an open-graph map $\Psi = \Psi_{U,V} : X \rightrightarrows Y$ such that:

$$\Phi(x) \subseteq \Psi(x) \subseteq V[\Phi(U[x])], \forall x \in X.$$

Proof. Let $(U,V) \in \mathcal{U} \times \mathcal{V}$ be given. By upper semicontinuity of Φ, for each $x \in X$, there is an open neighborhood U_x of x in X contained in $U[x]$ such that $\Phi(U_x) \subset V[\Phi(x)]\}$. Let $\{O_i\}_{i\in I}$ be a point-finite open refinement of the cover $\{U_x\}_{x\in X}$, i.e., for every $i \in I$, $O_i \subset U_{x_i}$ for some $x_i \in X$, and the set $I(x) := \{i \in I : x \in O_i\}$ is finite. Define the map $\Psi = \Psi_{U,V} : X \rightrightarrows Y$ by putting

$$\Psi(x) := \bigcap_{i\in I(x)} V[\Phi(x_i)], x \in X.$$

Clearly, $\Phi(x) \subseteq \Psi(x)$ for all $x \in X$. Moreover, for every $x \in X$ and every $i \in I(x)$ we have $\Psi(x) \subseteq V[\Phi(x_i)] \subseteq V[\Phi(U[x])]$. If $x' \in \bigcap_{i\in I(x)} U_{x_i}$ then $I(x) \subseteq I(x')$; consequently, $\Psi(x) \subseteq \Psi(\bigcap_{i\in I(x)} U_{x_i}) \subseteq \Psi(x)$. Finally, for any given $x \in X$, the open set $\bigcap_{i\in I(x)} O_i \times \Psi(x)$ is an open set around $\{x\} \times \Psi(x)$ in $X \times Y$, that is, Ψ has an open graph. ☐

[6] See the definition before Proposition 3.16.

Being an open graph, the majorant Ψ has open fibers, and is therefore lower semicontinuous. Any of its continuous selections is an approximative selection of Φ. If Y is a convex subset of a locally convex space F and $\Phi \in \mathbf{K}(X,Y)$, the class of *u.s.c.* maps with non-empty convex values, then the majorant Ψ also has convex values (indeed, the uniformity $\mathcal{V}$ is determined by a basis of convex neighborhoods of the origin in F, so that the neighborhood $V[\Phi(x_i)]$ of the convex $\Phi(x_i)$ remains convex, for every $i \in I(x)$). But such maps are well known to admit continuous selections (in fact, the opennes of the fibers is sufficient; see [11]) when the domain X is paracompact. Hence, the Cellina approximation theorem holds: $\mathbf{K}(X,Y) \subset \mathbf{A}(X,Y)$ whenever X is paracompact.

It is interesting to point out the fact that if an USCO map Φ of a compact domain is contained in an open graph map Ψ, then a tubular neighborhood of the graph of Φ is also contained in Ψ; more precisely:

Remark 3.7. Let $\Phi, \Psi : X \rightrightarrows Y$ be two maps of a compact uniform space $(X,\mathcal{U})$ into a uniform space $(Y,\mathcal{V})$ such that:

(i) Φ is a multi-selection of Ψ;

(ii) $\Phi \in \mathbf{USCO}(X,Y)$; and

(iii) Ψ has open graph.

Then $\exists (U,V) \in \mathcal{U} \times \mathcal{V}$ such that $V[\Phi(U[x])] \subseteq \Psi(x), \forall x \in X$.

Graph approximation is also conserved under suitable restrictions and Cartesian products.

Proposition 3.10 (Restriction to a Compact Subspace)**.** Assume $\Phi \in \mathbf{A}(X,Y)$ and let K be a compact subspace of X. Then for any $(U,V) \in \mathcal{U}\times\mathcal{V}$, and any $V' \in \mathcal{V}$ with $V'^2 \subset V$, there exist a finite subset N of K and an entourage $U' \in \mathcal{U}, U'^2 \subset U$, such that for any subset D of X containing N, the restriction $s|_{K\cap D} \in \mathbf{a}(\Phi|_{K\cap D}; U, V)$ whenever $s \in \mathbf{a}(\Phi; U', V')$.

In particular, $\Phi \in \mathbf{A}(X,Y) \Longrightarrow \Phi|_K \in \mathbf{A}(K,Y)$ for any compact subspace K of X.

Proof. We show first that given any pair of entourages $(U,V) \in \mathcal{U}\times\mathcal{V}$, the restriction of Φ to K is locally $V-$determined by its values on a finite subset N of K. Indeed, by semicontinuity, for every $x \in X$, there exists an open neighborhood of x in X of the form $U_x[x], U_x \in \mathcal{U}, U_x^2 \subset U$, such that $\Phi(U_x[x]) \subset V'[\Phi(x)]$ for some $V' \in \mathcal{V}, V'^2 \subset V$. Let $N = \{x_1, \ldots, x_n\}$ be a finite subset of K such that $\{U_{x_i}[x_i]\}_{i=1}^n \in Cov_X(K)$ and let $U' \in \mathcal{U}, U' \subset \bigcap_{i=1}^n U_{x_i}$. Each $x \in K$ belongs to some $U_{x_i}[x_i]$. The following inclusions are thus satisfied:

$$\Phi(U'[x]) \subset \Phi(U_{x_i}[x_i]) \subset V'[\Phi(x_i)].$$

Let D be any subset of X containing N and let $s \in \mathbf{a}(\Phi; U', V')$ be a continuous $(U', V')-$approximative selection of Φ. If $x \in K \cap D$, then $x_i \in U_{x_i}[x] \cap K \cap D$ for some $x_i \in N$. Hence, $s(x) \in V'[\Phi(U'[x])] \subset V'^2[\Phi(x_i)] \subset V[\Phi(U_{x_i}[x] \cap K \cap D)] \subset V[\Phi(U[x] \cap K \cap D)]$. □

Proposition 3.11 (Cartesian Product)**.** Let $\{\Phi_i : (X_i, \mathcal{U}_i) \rightrightarrows (Y_i, \mathcal{V}_i)\}_{i \in I}$ be a family of maps with $\Phi_i \in \mathbf{A}(X_i, Y_i), i \in I$ an arbitrary indexing set. Then the product map $\Phi := \prod_{i \in I} \Phi_i$ defined by $\Phi(x) = (\Phi_i(x_i))_{i \in I}, x = (x_i)_{i \in I}$, is in $\mathbf{A}(X, Y)$ for $X = \prod_{i \in I} X_i, Y = \prod_{i \in I} Y_i$.

The proof is left to the interested reader.

Proposition 3.12 (Composition Product 1)**.** Given three uniform spaces $(X, \mathcal{U}), (Y, \mathcal{V})$, and $(Z, \mathcal{W})$, let $\Phi \in \mathbf{A}(X, Y)$ and $\Psi \in \mathbf{A}(Z, X)$. If the space Z is compact and Ψ has compact values, then the composition product $\Phi \circ \Psi \in \mathbf{A}(Z, Y)$.

Proof. Let $W \in \mathcal{W}$ and $V \in \mathcal{V}$ be arbitrary but fixed. By upper semicontinuity of Φ, for each fixed $z \in Z$, and each fixed $x_z \in \Psi(z)$, there exists an entourage $U_{x_z} \in \mathcal{U}$ such that $\Phi(U_{x_z}[x_z]) \subset V'[\Phi(x_z)]$, where V' is a member of $\mathcal{V}$ satisfying $V' \circ V' \subset V$.

For each x_z, let U'_{x_z} be a member of $\mathcal{U}$ such that $U'_{x_z} \circ U'_{x_z} \subset U_{x_z}$, let $\{U'_{x_z^i}[x_z^i]\}_{i=1}^{n_z}$ be an open cover of $\Psi(z)$, and let $U_z = \bigcap_{i=1}^{n_z} U'_{x_z}$. Since Ψ is upper semicontinuous, there exists an entourage $W_z \in \mathcal{W}$ contained in W such that $\Psi(W_z[z]) \subset U'_z[\Psi(z)]$, where U'_z is a member of $\mathcal{U}$ satisfying $U'_z \circ U'_z \subset U_z$.

Let $\{W'_{z_j}[z_j]\}_{j=1}^m$ be an open cover of Z, with $W'_{z_j} \in \mathcal{W}$ satisfying $W'_{z_j} \circ W'_{z_j} \subset W_{z_j}$ for each j. Let W' be a member of $\mathcal{W}$ contained in $\bigcap_{j=1}^m W'_{z_j}$, let U be a member of $\mathcal{U}$ contained in $\bigcap_{j=1}^m U'_{z_j}$, and let $U' \in \mathcal{U}$ be so that $U' \circ U' \subset U$.

Now let s_1 be a $(W', U')-$approximative selection of Ψ, and let s_2 be a $(U', V')-$approximative selection of Φ. For any $z \in Z$, we have:

$$s_2 s_1(z) \in V'[\Phi(U'[s_1(z)])] \subset V'[\Phi(U[\Psi(W'[z])])].$$

Since z belongs to some $W'_{z_j}[z_j]$, it follows that

$$U[\Psi(W'[z])] \subset U'_{z_j}[\Psi(W_{z_j})] \subset U_{z_j}[\Psi(z_j)].$$

Hence, $s_2 s_1(z) \in V'[\Phi(U_{z_j}[\Psi(z_j)])]$ for some j. Finally, there exists an element $x_{z_j}^i \in \Psi(z_j)$ such that

$$s_2 s_1(z) \in V[\Phi(x_{z_j}^i)] \subset V[\Phi\Psi(z_j)] \subset V[\Phi\Psi(W[z])].$$

□

It follows from this result that

$$\mathbf{A}_c(X, Y) = \mathbf{A}(X, Y), \text{ provided } X \text{ is compact.}$$

Remark 3.8. It is not known whether the proposition holds true when the compactness of Z is replaced by that of X or Y (even if one assumes that in addition, Z is paracompact).

However, with uniform upper semicontinuity, compactness is not needed at all.

Definition 3.19. A map $\Phi : X \rightrightarrows Y$ of uniform spaces $(X,\mathcal{U}), (Y,\mathcal{V})$ is said to be *uniformly* $\mathcal{V} - u.s.c.$ *on* X if and only if, given any entourage $V \in \mathcal{V}$, there exists an entourage $U \in \mathcal{U}$ such that $\Phi(U[x]) \subset V[\Phi(x)]$ for any point $x \in X$.

One can readily verify that a $\mathcal{V}-u.s.c.$ map on a compact space is uniformly $\mathcal{V} - u.s.c.$

Proposition 3.13 (Composition Product 2)**.** Given three uniform spaces $(X,\mathcal{U}), (Y,\mathcal{V})$, and $(Z,\mathcal{W})$, let $\Phi \in \mathbf{A}(X,Y)$ be uniformly $\mathcal{V} - u.s.c.$ with non-empty values, and let $\Psi \in \mathbf{A}(Z,X)$. Then the composition product $\Phi\Psi \in \mathbf{A}(Z,Y)$ (and is $\mathcal{V} - u.s.c.$).

Proof. Let $W \in \mathcal{W}$ and $V \in \mathcal{V}$ be arbitrary but fixed. By hypotheses,

$$\forall W' \in \mathcal{W}, \forall U' \in \mathcal{U}, \exists s_1 \in \mathbf{c}(Z,X), \text{ with } s_1(z) \in U'[\Psi(W'[z])], \forall z \in Z, \text{ and}$$
$$\forall U" \in \mathcal{U}, \forall V' \in \mathcal{V}, \exists s_2 \in \mathbf{c}(X,Y) \text{ with } s_2(x) \in V'[\Phi(U"[x])], \forall x \in X.$$

Consequently,

$$s(z) := (s_2 \circ s_1)(z) \in V'[\Phi(U"[U'[\Psi(W'[z])]])], \forall z \in Z.$$

Now, if the entourages $U"$ and U' are chosen so that $U" \circ U' \subset U$, for some arbitrarily prescribed $U \in \mathcal{U}$, it would follow that

$$\forall W' \in \mathcal{W}, \forall U \in \mathcal{U}, \forall V' \in \mathcal{V}, \exists s \in \mathbf{c}(Z,Y) \text{ with}$$
$$s(z) \in V'[\Phi(U[\Psi(W'[z])])], \forall z \in Z.$$

It suffices to show that given V and W, one can find W', U, and V' such that

$$V'[\Phi(U[\Psi(W'[z])])] \subseteq V[\Phi(\Psi(W[z]))], \forall z \in Z.$$

This containment is equivalent to

$$\Phi(U[\Psi(W'[z])]) \subseteq V[\Phi(\Psi(W[z]))], \forall z \in Z.$$

The left-hand side of this inclusion is precisely

$$\Phi(U[\Psi(W'[z])]) = \bigcup_{z' \in W'[z]} \bigcup_{x \in \Psi(z')} \Phi(U[x]),$$

and the right-hand side is

$$V[\Phi(\Psi(W[z]))] = \bigcup_{z' \in W[z]} \bigcup_{x \in \Psi(z')} V[\Phi(x)].$$

Since Φ is uniformly $\mathcal{V}-u.s.c.$ on X, for the arbitrary given V, choose U such that

$$\Phi(U[x]) \subset V[\Phi(x)], \forall x \in X.$$

Finally, choose any $W' \subset W$ to obtain the desired inclusion. This completes the proof. □

Proposition 3.14 (From Polyhedra to Compact Sets)**.** Let X be a convex subset of a Hausdorff locally convex space E, $(Y, \mathcal{V})$ be a uniform space and $\Phi : X \rightrightarrows Y$ be a $V - u.s.c.$ map. If the restriction $\Phi|_P \in \mathbf{A}(P, Y)$ for any finite polyhedron $P \subset X$, then the restriction $\Phi|_K \in \mathbf{A}(K, Y)$ for any compact subset $K \subset X$.

Proof. Let U be an arbitrary but fixed symmetric open neighborhood of the origin in E, and let $V \in \mathcal{V}$ also be arbitrary but fixed. Let $V' \in \mathcal{V}$ be such that $V' \circ V' \subset V$. By upper semicontinuity, for every $x \in X$, there exists an open neighborhood of the origin U_x in E such that $U_x \subset U$ and $\Phi((x + U_x) \cap X) \subset V'[\Phi(x)]$.

Given a compact subset $K \subset X$, let $\{x_i\}_{i=1}^n \subset K$ be such that $\{(x_i + \frac{1}{2}U_{x_i}) \cap X\}_{i=1}^n \in Cov_X(K)$. Put $U' = \bigcap_{i=1}^n \frac{1}{2}U_{x_i}$ and let π_U be the continuous Schauder projection from K into a finite polyhedron P_U contained in X (see Theorem 3.1 above) and satisfying

$$\pi_U(x) \in (x + \frac{1}{2}U') \cap P_U, \forall x \in K.$$

By hypothesis, the restriction $\Phi|_{P_U}$ of Φ to P_U is approachable, and the composition product $\Phi_U : K \overset{\pi_U}{\to} P_U \overset{\Phi|_{P_U}}{\to} \mathcal{P}(Y)$ is also approachable (by Proposition 3.12). Hence, there exists a continuous function $s : K \to Y$ such that for any given $x \in K$, there exists $\widehat{x} \in (x + \frac{1}{2}U') \cap K$ with $s(x) \in V'[\Phi_U(\widehat{x})]$. That is, $s(x) \in V'[\Phi\pi_U(\widehat{x})] \subseteq V'[\Phi((\widehat{x} + \frac{1}{2}U') \cap P_U)] \subseteq V'[\Phi((x + U') \cap P_U)]$. Since x belongs to some $x_i + \frac{1}{2}U_{x_i}$, then $s(x) \in V'[\Phi((x_i + U_{x_i}) \cap X)] \subset V[\Phi(x_i)] \subset V[\Phi((x + U) \cap K)]$ and the proof is complete. □

Recall that a map $\Phi : X \rightrightarrows Y$ is said to be compact, if $\Phi(X) \subseteq K$ compact $\subseteq Y$.

Proposition 3.15 (Finite-Type Approximation)**.** Assume that Y is a nonempty subset of a locally convex topological vector space F and let $\Phi \in \mathbf{A}(X, Y)$ be a compact $\mathcal{V} - u.s.c.$ map. Given an open neighborhood of the origin V in F, there exists a map $\Phi_V \in \mathbf{A}(X, P_V)$ where $P_V = conv(N_V)$ is a convex finite polytope with N_V a finite subset of Y, verifying $\Phi_V(x) \subset \Phi(x) + V, \forall x \in X$.

Proof. Let V be an arbitrary open convex and symmetric neighborhood of the origin in F, and let $N_V := \{y_1, ..., y_n\}$ be a finite subset of Y such that the collection $\{y_i + \frac{1}{6}V : i = 1, ..., n\}$ forms an open cover of the compact set $\overline{\Phi(X)}$.

Consider the Schauder projection $\pi_V : \bigcup_{i=1}^n \{y_i + \frac{1}{3}V\} \to P_V = conv(N_V)$ defined by

$$\pi_V(y) := \frac{1}{\sum_{i=1}^n \mu_i(y)} \sum\nolimits_{i=1}^n \mu_i(y) y_i, \text{ for all } y \in \bigcup\nolimits_{i=1}^n \{y_i + \frac{1}{3}V\},$$

where for $i = 1, \ldots, n$, $\mu_i(y) := \max\{0, 1 - p_{\frac{1}{3}V}(y - y_i)\}$ and $p_{\frac{1}{3}V}$ is the Minkowski functional of the set $\frac{1}{3}V$. Clearly,

$$\pi_V(y) - y \in \frac{1}{3}V \text{ for all } y \in \bigcup\nolimits_{i=1}^n \{y_i + \frac{1}{3}V\}.$$

Let $\Phi' : X \rightrightarrows \bigcup_{i=1}^n \{y_i + \frac{1}{3}V\}$ be the "compression" of Φ to the set $\bigcup_{i=1}^n \{y_i + \frac{1}{3}V\}$ defined by

$$\Phi'(x) := \Phi(x) \cap \bigcup\nolimits_{i=1}^n \{y_i + \frac{1}{3}V\} \text{ for all } x \in X.$$

(Note that $\Phi'(x) = \Phi(x)$ if Y is convex.) Define the set-valued map $\Phi_V : X \rightrightarrows P_V$ as the composition product $\Phi_V := \pi_V \circ \Phi'$. Clearly, Φ_V is *u.s.c.* and it has compact values whenever Φ does. Moreover, the following two properties hold:

$$\Phi_V(x) \subset \Phi(x) + V \text{ for all } x \in X.$$

In addition, if for a given member U of the uniformity $\mathcal{U}$ on X, $s \in \mathbf{a}(\Phi; U \times \frac{1}{6}V)$, i.e., s is continuous and $s(x) \in (\Phi(U[x]) + \frac{1}{6}V) \cap Y$ for all $x \in X$, then $\pi_V s : X \to P_V$ is well-defined, continuous, and verifies

$$\pi_V s(x) \in (\Phi_V(U[x]) + V) \cap P_V \text{ for all } x \in X,$$

that is $\Phi_V \in \mathbf{A}(X, P_V)$. □

Following Ionescu-Tulcea [58], recall that given a space X and a uniform space $(Y, \mathcal{V})$, a downward directed collection $\{\Phi_i : X \rightrightarrows Y\}_{i \in I}$ of maps is said to be *aimed* at a map $\Phi : X \rightrightarrows Y$ if for each $x \in X$, for each entourage $V \in \mathcal{V}$, there exists an index $i_{x,V} \in I$ such that $\Phi_{i_{x,V}}(x) \subset V[\Phi(x)]$. The map Φ is an *upper limit* for the collection $\{\Phi_i\}_{i \in I}$ if this collection is aimed at Φ, and if in addition, for every $x \in X$, $\Phi(x)$ is contained in the set $\bigcap_{i \in I} \Phi_i(x)$.

Lemma 3.1. Given two uniform spaces $(X, \mathcal{U}), (Y, \mathcal{V})$, and a map $\Phi : X \rightrightarrows Y$, assume that a downward directed collection $\{\Phi_i : X \rightrightarrows Y\}_{i \in I}$ of $\mathcal{V} - u.s.c.$ maps is aimed at Φ. Then, given any pair of entourages $U \in \mathcal{U}$ and $V \in \mathcal{V}$, there exist an index $i_0 \in I$, and a pair of entourages $U' \in \mathcal{U}$ and $V' \in \mathcal{V}$, with $U' \subset U, V' \circ V' \circ V' \circ V' \subset V$,such that for each index $h \in I$ with $h \succeq i_0$, every $(U', V')-$approximative selection of Φ_h is a $(U, V)-$approximative selection of Φ, provided the space X is compact.

Proof. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be arbitrary but fixed and let V' be a member of $\mathcal{V}$ satisfying $V' \circ V' \circ V' \circ V' \subset V$. For any $x \in X$, there exists an index $i_{x,V} \in I$ such that $\Phi_h(x) \subset V'[\Phi(x)]$ for each index $h \in I$ with $h \succeq i_{x,V}$. By upper semicontinuity of $\Phi_{i_{x,V}}$, there exists an entourage $U_x \in \mathcal{U}$ contained in U such that for each index $h \in I$ with $h \succeq i_{x,V}$ we have:

$$\Phi_h(U_x[x]) \subset \Phi_{i_{x,V}}(U_x[x]) \subset V'[\Phi_{i_{x,V}}(x)] \subset V' \circ V'[\Phi(x)].$$

Let U'_x be a member of $\mathcal{U}$ satisfying $U'_x \circ U'_x \subset U_x$, and let $\{U'_{x_j}[x_j]\}_{j=1}^m$ be an open cover of X. Let i_0 be an index greater than all the $i_{x_j,V}$'s, $1 \leq j \leq m$, and let U' be a member of $\mathcal{U}$ contained in $\bigcap_{j=1}^m U'_{x_j}$. Now, pick any index $h \in I$ with $h \succeq i_0$, and let s be any $(U', V')-$approximate selection of Φ_h. Since x belongs to some $U'_{x_j}[x_j]$, it follows that:

$$U'[x] \subset U'_{x_j} \circ U'_{x_j}[x_j] \subset U_{x_j}[x_j].$$

Therefore, $s(x) \in V'[\Phi_h(U'[x])] \subset V'[\Phi_h(U_{x_j}[x_j]] \subset V[\Phi(x_j)] \subset V[\Phi(U[x])]$. ☐

As an immediate consequence, one obtains the stability of approachability under upper limits:

Proposition 3.16 (Upper Limits)**.** Assume that a collection of $\mathcal{V}-u.s.c.$ maps $\{\Phi_i : X \rightrightarrows Y\}_{i \in I}$ is aimed at a map $\Phi : X \rightrightarrows Y$ having non-empty closed values. If X is compact and the Φ_i's are eventually approachable, then Φ is also $\mathcal{V}-u.s.c.$ and approachable.

3.3.2.2 Homotopy Approximation, Domination of Domain, and Approachability

The consideration of a subclass $\mathbf{A}_H$ of the class $\mathbf{A}$ of approachable maps is crucial for the development of a degree or an index theory for approachable maps.

Definition 3.20. A map $\Phi \in \mathbf{A}(X, Y)$ is said to have property (H) if and only if:

$$\left(\begin{array}{c} \forall W \in \mathcal{U} \times \mathcal{V}, \exists W' \in \mathcal{U} \times \mathcal{V}, \forall s_1, s_2 \in \mathbf{a}(\Phi; W'), \\ \exists h : X \times [0,1] \longrightarrow Y \text{ such that } h(.,0) = s_1, h(.,1) = s_2, \text{ and} \\ h(.,t) \in \mathbf{a}(\Phi; W), \forall t \in [0,1]. \end{array} \right).$$

Let $\mathbf{A}_H(X, Y)$ be the subclass of $\mathbf{A}(X, Y)$ consisting of those maps having property (H).

Let $\mathcal{K}$ be the class of compact topological spaces. We show that the classes of maps $\mathbf{A}$ and $\mathbf{A}_H$, respectively, are stable under the respective enlargements $\mathcal{K} \rightarrow D(\mathcal{K})$ and $\mathcal{K} \rightarrow D_H(\mathcal{K})$ defined earlier.

Proposition 3.17 (Domination of Domain)**.** Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be uniform spaces with X paracompact. Let $\mathbf{M}$ be an abstract class of maps satisfying the condition:

$$\forall\Phi\in\mathbf{M},\forall s\in\mathbf{c},\ \text{the composition product } \Phi s\in\mathbf{M}.$$

Then,

(a)
$$\begin{pmatrix}\mathbf{M}(P,Y)\subset\mathbf{A}(P,Y)\\ \forall P\in\mathcal{K}\end{pmatrix}\Rightarrow\begin{pmatrix}\mathbf{M}(X,Y)\subset\mathbf{A}(X,Y)\\ \text{provided } X\in D(\mathcal{K}).\end{pmatrix}$$

(b)
$$\begin{pmatrix}\mathbf{M}(P,Y)\subset\mathbf{A}_H(P,Y)\\ \forall P\in\mathcal{K}\end{pmatrix}\Rightarrow\begin{pmatrix}\mathbf{M}(X,Y)\subset\mathbf{A}_H(X,Y)\\ \text{provided } X\in D_H(\mathcal{K}).\end{pmatrix}.$$

Proof. We only provide the proof of (b) and refer to [15] for the proof of (a). Given $\Phi\in\mathbf{M}(X,Y)$, let $(U,V)\in\mathcal{U}\times\mathcal{V}$ be arbitrary but fixed. Choose $\hat{U}\in\mathcal{U}$ such that $\hat{U}\circ\hat{U}\subset U$ and let $\omega:=\{\hat{U}[x]:x\in X\}\in Cov(X)$. By definition of ω_H-domination, there exist $P\in\mathcal{K}$ (with uniformity $\mathcal{R}$), and mappings $s:X\to P,r:P\to X$ such that rs and id_X are $\omega-$homotopic. Let $\tilde{h}$ be that homotopy.

Choose $R\in\mathcal{R}$ in such a way that $(r(p),r(p'))\in\hat{U}$ whenever $(p,p')\in R$.

By hypothesis, Φr belongs to $\mathbf{M}(P,Y)$ and is approachable as a composition of approachable *u.s.c.* maps with compact domain (Proposition 3.12).

Let $U'\in\mathcal{U},U'\subset\hat{U}$, $R'\in\mathcal{R},V'\in\mathcal{V}$ be chosen so that:

- the composites s_1r and $s_2r\in\mathbf{a}(\Phi r;R'\times V')$ for some arbitrarily fixed mappings $s_1,s_2\in\mathbf{a}(\Phi;U'\times V')$; and
- s_1r and s_2r are joined by a homotopy $k:P\times[0,1]\to Y$ such that $k(.,t)\in\mathbf{a}(\Phi r;R\times V),\forall t\in[0,1]$.

The homotopy $h_1:X\times[0,1]\to Y$ defined by:

$$h_1(x,t):=k(s(x),t),\forall(x,t)\in X\times[0,1],$$

joins the functions s_1rs and s_2rs.

Moreover, $\forall t\in[0,1]$, $h_1(.,t)$ is a $(U\times V)-$approximative selection of Φ. Indeed, $\forall(x,t)\in X\times[0,1]$, $k(s(x),t)\in V[\Phi r(R[s(x)])]$, that is, $k(s(x),t)\in V[\Phi r(z)]$ for some $z\in R[s(x)]$. If we choose $R,r(z)\in\hat{U}[rs(x)]\subset\hat{U}\circ\hat{U}[x]\subset U[x]$, $h_1(x,t)\in V[\Phi(U[x])]$.

Define homotopies $h_0,h_2:X\times[0,1]\to Y$ by putting:

$$h_0(x,t):=s_1(\tilde{h}(x,1-t))\text{ and }h_2(x,t):=s_2(\tilde{h}(x,t)),\forall(x,t)\in X\times[0,1].$$

The homotopy $h:X\times[0,1]\to Y$ defined by:

$$h(x,t):=\begin{cases}h_0(x,3t) & 0\le t\le 1/3\\ h_1(x,3t-1) & 1/3\le t\le 2/3.\\ h_2(x,3t-2) & 2/3\le t\le 1\end{cases}$$

is continuous and joins s_1 and s_2. For every $t \in [0,1], h(.,t) \in \mathbf{a}(\Phi; U' \times V')$. Property (H) is thus satisfied by Φ. □

3.3.3 Examples of A−Maps

We are concerned, in this section, with the following classes of maps:

- *Kakutani maps*:

$$\mathbf{K}(X,Y) := \{\Phi \in \mathbf{USC}(X,Y) : \Phi(x) \text{ is convex}, \forall x \in X\},$$

  where X, Y are spaces, and Y is contained in a topological vector space.

- Maps with generalized convex values:

$$\mathbf{C}(X,Y) := \{\Phi \in \mathbf{USC}(X,Y) : \Phi(x) \text{ is } \mathcal{C} - \text{convex}, \forall x \in X\},$$

  where X, Y are spaces and $\mathcal{C}$ is a convexity structure on Y.

- Maps with contractible (or infinitely connected) values:

$$\mathbf{C}^\infty(X,Y) := \{\Phi \in \mathbf{USCO}(X,Y) : \Phi(x) \text{ is contractible}, \forall x \in X\},$$

  where X, Y are spaces.

- *Dugundji maps*:

$$\mathbf{D}(X,Y) := \{\Phi \in \mathbf{USCO}(X,Y) : \Phi(x) \text{ is } PC_Y^\infty, \forall x \in X\}.$$

Clearly, $\mathbf{K} \subset \mathbf{C}$ and, for compact valued maps, $\mathbf{K} \subset \mathbf{C}^\infty \subset \mathbf{D}$.

The classical approximation result of Cellina [28, 29] for Kakutani maps asserts the inclusion:

$$\mathbf{K}(X,Y) \subset \mathbf{A}(X,Y),$$

whenever X is a paracompact space and Y is a convex subset of a locally convex topological vector space. The proof of this classical inclusion can be found in the books by Aubi-Cellina [6], Aubin-Frankowska [7], or Górniewizc [46].

The Cellina's graph approximation property $\mathbf{K} \subset \mathbf{A}$ for Kakutani maps extends to maps with non-convex values in several directions.

We start with a new refinement of Cellina's theorem to possibly non-convex maps with values in a locally convex space.

Theorem 3.6. Let X be a paracompact space equipped with a compatible uniformity $\mathcal{U}$, and let Y be a convex subset of a locally convex topological vector space F. Let $\mathcal{V}$ denote the uniformity induced by the relative topology of Y. Assume that a given map $\Phi : X \rightrightarrows Y$ with non-empty values satisfies the following regularity condition:

$$\begin{gathered}\forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \forall x \in X, \exists U_x \in \mathcal{U} \text{ such that} \\ U_x \subset U, U_x[x] \text{ is an open neighborhood of } x \text{ in } X, \text{ and} \\ conv\left(\Phi\left(U_x[x]\right)\right) \subset V\left[\Phi\left(U[x]\right)\right].\end{gathered}$$

Then $\Phi \in \mathbf{A}(X,Y)$.

Proof. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be fixed. Let $W \in \mathcal{U}$ be such that $W \circ W \subset U$.

By the regularity condition, we may choose, for every $x \in X$, an entourage $W_x \in \mathcal{U}$ such that $W_x[x]$ is an open neighborhood of x in X and:

$$W_x = W_x^{-1} \subset W \text{ and } conv\left(\Phi\left(W_x[x]\right)\right) \subset V\left[\Phi\left(W[x]\right)\right].$$

Let $\omega = \{O_i\}_{i \in I}$ be a locally finite open cover of X that is a barycentric refinement of the open cover $\{W_x[x]\}_{x \in X}$ and, for every $i \in I$, let

$$y_i \in conv\left(\Phi\left(\bigcap\nolimits_{W_z[z] \supset O_i} W_z[z]\right)\right)$$

be arbitrary (we may instead choose $y_i \in \Phi(O_i)$ if we wish). Let $\{\lambda_i\}_{i \in I}$ be a continuous partition of unity subordinated to the cover ω and, for each $x \in X$, let

$$I_x := \{i \in I : \lambda_i(x) \neq 0\}.$$

Since, for every $x \in X$, $St(x,\omega) = \bigcup_{i \in I(x)} O_i \subset W_{\bar{x}}[\bar{x}]$ for some $\bar{x} \in X$, it follows that, for every $i \in I(x)$, $\bigcap_{W_z[z] \supset O_i} W_z[z] \subset W_{\bar{x}}[\bar{x}]$ and, consequently,

$$y_i \in conv\left(\Phi\left(\bigcap\nolimits_{W_z[z] \supset O_i} W_z[z]\right)\right) \subset conv\left(\Phi\left(W_{\bar{x}}[\bar{x}]\right)\right).$$

It follows that, for every $x \in X$,

$$s(x) := \sum\nolimits_{i \in I} \lambda_i(x)\, y_i \in conv\left(\Phi\left(W_{\bar{x}}[\bar{x}]\right)\right) \subset V\left[\Phi\left(W_{\bar{x}}[\bar{x}]\right)\right].$$

Since $x \in W_{\bar{x}}[\bar{x}]$, we have $\bar{x} \in W_{\bar{x}}[x]$ and consequently,

$$W_{\bar{x}}[\bar{x}] \subset W_{\bar{x}}[W_{\bar{x}}[x]] \subset W[W[x]] \subset U[x],$$

which implies that, for every $x \in X$,

$$s(x) \in V\left[\Phi\left(U[x]\right)\right].$$

This completes the proof. □

As a corollary, we obtain a generalization of Cellina's result:

Corollary 3.3 (Convex Case)**.** Let X and Y be as in Theorem 6 and let $\Phi : X \rightrightarrows Y$ be a map with non-empty convex values. If we assume that Φ satisfies the following regularity condition: $\forall U \in \mathcal{U}$, $\forall V \in \mathcal{V}$, $\forall x \in X$, there exists an entourage $U_x \in \mathcal{U}$ such that $U_x \subset U$ and $U_x[x]$ is an open neighborhood of x in X, and there exists $\bar{x} \in U_x[x]$ such that $\Phi(U_x[x]) \subset V[\Phi(\bar{x})]$. Then $\Phi \in \mathbf{A}(X, Y)$.

Proof. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be fixed. Let $V' \in \mathcal{V}$ be such that

$$conv\left(V'[y]\right) \subset V[y], \text{ for all } y \in Y.$$

The hypothesis holds with V replaced by V' and, consequently,

$$\begin{aligned} conv\left(\Phi\left(U_x[x]\right)\right) &\subset conv\left(V'\left[\Phi(\bar{x})\right]\right) \subset V\left[\Phi(\bar{x})\right] \\ &\subset V\left[\Phi\left(U_x[x]\right)\right] \subset V\left[\Phi\left(U[x]\right)\right]. \end{aligned}$$

Hence the hypothesis of Theorem 3.6 is satisfied and the conclusion follows. □

Clearly, Corollary 3.3 contains Cellina's theorem due to the convexity together with upper semicontinuity assumptions in Cellina's result. This simple example illustrates the fact that Theorem 3.6 is strictly stronger than Cellina's result. Here we have of lack of convexity of the values together with a weaker upper semicontinuity that is worth further consideration.

Example 3.11. Let $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$\Phi(x) := \begin{cases} \left[-\frac{1}{|x|}, \frac{1}{|x|}\right] & if \quad x \neq 0 \\ 0 & if \quad x = 0. \end{cases}$$

The assumptions of Theorem 3.6 are satisfied while the assumptions of Corollary 3.3 fail to hold at $x = 0$. Clearly, $s(x) \equiv 0$ is a continuous selection for Φ.

Also, it is easy to construct examples where Φ satisfies the assumptions of Theorem 3.6 but does not have convex values, and hence Cellina's result may not be applied.

For abstract convexities, we have (see [14]):

Theorem 3.7. $\mathbf{C}(X, Y) \subset \mathbf{A}(X, Y)$ provided that X is paracompact and Y is a locally $\mathcal{C}-$convex space with convexity structure $\mathcal{C}$.

Proof. Let $\mathcal{U}$ be a uniformity on X and $\mathcal{V}$ be a uniformity on Y. Given $\Phi \in \mathbf{C}(X, Y)$ and $U \times V \in \mathcal{U} \times \mathcal{V}$, let $\Psi : X \to 2^Y$ be the open-graph majorant of Φ given by Proposition 3.9. By construction, the values of Ψ are finite intersections of $\mathcal{C}-$convex sets (namely, neighborhoods of radius V of the values of Φ) and are also $\mathcal{C}-$convex. By Theorem 2 in [57], the map Ψ admits a continuous selection s satisfying $s(x) \in \Psi(x) \subseteq V[\Phi(U[x])], \forall x \in X$. If the entourages U and V are arbitrary, it follows that $\Phi \in \mathbf{A}(X, Y)$. □

For Dugundji maps with compact $\infty-$proximally connected values (in particular for maps with compact contractible values), the following inclusion was proven in [15], extending an earlier result of Górniewicz et al. in which Y is an ANR [48].

Theorem 3.8. $\mathbf{D}(X,Y) \subset \mathbf{A}(X,Y)$ provided that X is a finite polyhedron and Y is an arbitrary subset of a topological vector space.

We shall show next that the classes $\mathbf{C}^\infty$ and $\mathbf{D}$ defined above are in the smaller class $\mathbf{A}_H$. To this aim, we need preparatory material and results.

Definition 3.21. Let $\mathcal{P}$ be the class of all finite polyhedra and let $(Y, \mathcal{V})$ be a uniform space. A class of set-valued maps $\mathbf{M}$ is said to have the *approximative selection extension property on finite polyhedra (ASEP($\mathcal{P}$))* if and only if:

$$\begin{cases} \text{(i) } \mathbf{M}(P,Y) \subset \mathbf{A}(P,Y), \text{ and} \\ \text{(ii) } \forall P \in \mathcal{P} \text{ with uniform structure } \mathcal{U}, \forall P_0 \text{ sub-} \\ \text{polyhedron of } P \text{ containing the } 0-\text{dimensional} \\ \text{skeleton of } P, \forall (U,V) \in \mathcal{U} \times \mathcal{V}, \exists (U^0,V^0) \in \mathcal{U} \times \mathcal{V}, \\ \text{such that } \forall s_0 \in \mathbf{a}(\Phi|P_0; U^0, V^0), \exists s \in \mathbf{a}(\Phi; U, V), \\ \text{with } s|P_0 = s_0. \end{cases}$$

Theorem 3.9. Let $\mathcal{P}$ be the class of all finite polyhedra, $(Y, \mathcal{V})$ be a uniform space, and let $\mathbf{M}$ be an abstract class of set-valued maps satisfying

$$\forall \Phi \in \mathbf{M}, \forall s \in \mathbf{c}, \text{ the composition product } \Phi \circ s \in \mathbf{M}.$$

If $\mathbf{M}$ has $ASEP(\mathcal{P})$, then $\mathbf{M}(P,Y) \subset \mathbf{A}_H(P,Y), \forall P \in \mathcal{P}$.

Proof. Let $\hat{P} := P \times [0,1]$ and $\hat{P}_0 := (P \times \{0\}) \cup (P \times \{1\})$. Clearly, $\hat{P}$ is a finite polyhedron equipped with a product uniformity $\hat{\mathcal{U}}$, and $\hat{P}_0$ is a subpolyhedron of $\hat{P}$ containing all the vertices of $\hat{P}$. Define $\hat{\Phi} : \hat{P} \rightrightarrows Y$ as

$$\hat{\Phi}(x,t) := \Phi(x), \forall (x,t) \in \hat{P}.$$

The map $\hat{\Phi}$ can be viewed as the composition $\Phi \circ p_1$ where p_1 is the projection of $\hat{P}$ onto P. By hypothesis, $\hat{\Phi} \in \mathbf{M}(\hat{P}, Y) \subset \mathbf{A}(\hat{P}, Y)$.

For an arbitrarily fixed pair $(U,V) \in \mathcal{U} \times \mathcal{V}$, let $\hat{U}$ be an entourage of the diagonal in $\hat{P} \times \hat{P}$ homeomorphic to a product $U \times [0,1]^2$. By hypothesis, $\exists (\hat{U}^0, V^0) \in \hat{\mathcal{U}} \times \mathcal{V}$, such that any $h_0 \in \mathbf{a}(\hat{\Phi}|\hat{P}_0; \hat{U}^0, V^0)$ extends continuously to a mapping $h \in \mathbf{a}(\hat{\Phi}; \hat{U}, V)$. The set $\hat{U}^0$ contains a copy of a product $U^0 \times O^0$ where $U^0 \in \mathcal{U}, U^0 \subset U$, and O^0 is an entourage of the diagonal in $[0,1]^2$. Now given any $s_1, s_2 \in \mathbf{a}(\Phi; U^0, V^0)$, let $h_0 : \hat{P}_0 \to Y$ be

$$h_0(x,0) := s_1(x) \text{ and } h_0(x,1) := s_2(x), \forall x \in P.$$

Clearly, $h_0 \in \mathbf{a}(\hat{\Phi}|\hat{P}_0; \hat{U}^0, V^0)$. Let $h \in \mathbf{a}(\hat{\Phi}; \hat{U}, V)$ be an extension of h_0 to $\hat{P}$. For any pair $(x,t) \in \hat{P}, h(x,t) \in V[\hat{\Phi}(x',t')]$ for some $(x',t') \in \hat{U}[(x,t)]$, that is, $h(x,t) \in V[\Phi(x')]$ for some $x' \in U[x]$. Thus, $h(.,t) \in \mathbf{a}(\Phi; U, V), \forall t \in [0,1]$ and the proof is complete. □

As an immediate consequence and in view of Proposition 3.17, we have:

Theorem 3.10. $\mathbf{D}(P,Y) \subset \mathbf{A}_H(P,Y)$ provided P is a finite polyhedron and $Y \in \text{NES}(\mathcal{K})$ where $\mathcal{K}$ is the class of compact spaces.

Proof. In the metrizable case where Y is an *ANR*, this result is due to [48]. We provide here a shorter proof of this more general case.

In view of the preceding theorem and since any composition $\Phi \circ s$ of a set-valued map $\Phi \in \mathbf{D}$ and a continuous single-valued mapping s is also in $\mathbf{D}$, its suffices to show that the class $\mathbf{D}$ of Dugundji maps has $ASEP(\mathcal{P})$ for the class $\mathcal{P}$ of finite polyhedra.

Let $\Phi \in \mathbf{D}(P,Y)$ where $P \in \mathcal{P}$ and let $\mathcal{U}, \mathcal{V}$ be uniform structures on P and Y, respectively. Since Φ has compact values in an $NES(\mathcal{K})$ space, we have:

$$\begin{cases} \forall x \in P, \forall V \in \mathcal{V}, \exists V_x \in \mathcal{V}, V_x \subset V, \text{ such that} \\ \quad V_x[\Phi(x)] \text{ is contractible in } V[\Phi(x)], \end{cases}$$

which, due to the compactness of P and the upper semicontinuity of Φ, can easily be made uniform in the following sense:

$$\begin{cases} \forall (U,V) \in \mathcal{U} \times \mathcal{V}, \exists (U',V') \in \mathcal{U} \times \mathcal{V}, U' \subset U, V' \subset V, \\ \text{such that } \forall x \in P, \forall n \geq 0, \forall s_0 \in \mathbf{c}(\partial\Delta^n, V'[\Phi(U'[x])]), \\ \exists s \in \mathbf{c}(\Delta^n, V[\Phi(U[x])]) \text{ with } s|\partial\Delta^n = s_0. \end{cases}$$

Assume now that P has dimension $n+1$ and let $(U^{n+1}, V^{n+1}) \in \mathcal{U} \times \mathcal{V}$ be arbitrary but fixed. We define a finite sequence $\{(U^s, V^s) \in \mathcal{U} \times \mathcal{V} : s = n, n-1, \ldots, 0\}$ as follows:

Let (U'^{n+1}, V'^{n+1}) be given by the above property and put $V^n = V'^{n+1}$; $\forall x \in P$, choose $U''^{n+1} \in \mathcal{U}$ such that $\Phi(U''^{n+1}[x]) \subset V^n[\Phi(x)]$, and let $U^n \in \mathcal{U}$ be so that $\{U^n[x] : x \in P\} \preceq \{U'^{n+1}[x] \cap U''^{n+1}[x] : x \in P\}$. We then proceed recursively until $s = 0$.

We show that the pair (U^0, V^0) verifies the property above.

Let P_0 be a subpolyhedron of P containing all the vertices of P and let $s_0 \in \mathbf{a}(\Phi|P_0; U^0, V^0)$. Assume that for some $0 \leq r \leq n$ the function s_0 has been extended to an approximative selection $s_r \in \mathbf{a}(\Phi|P_0 \cup P^r; U^r, V^r)$ where P^r is the $r-$dimensional skeleton of P. It suffices to show that s_r extends to an approximative selection $s_{r+1} \in \mathbf{a}(\Phi|P_0 \cup P^{r+1}; U^{r+1}, V^{r+1})$.

Let (P', P'_0) be a triangulation of (P, P_0) finer than the cover $\{U^0[x] : x \in P\}$ of P and let σ be an arbitrary $(r+1)-$dimensional simplex of P'^{r+1}. By the choice of P', σ is contained in some open set $U^0[x_\sigma], x_\sigma \in P$, which in turn is contained in $U^r[x_\sigma]$. Thus, for each $x \in \partial\sigma, s_r(x) \in V^r[\Phi(U^r[x] \cap (P_0 \cup P^{r+1}))]$. By the choice of (U^r, V^r), s^r extends to a continuous mapping $s_{r+1} : \sigma \to V^{r+1}[\Phi(U^{r+1}[x] \cap (P_0 \cup P^{r+1}))]$. Hence, the class $\mathbf{D}$ has $ASEP(\mathcal{P})$. Theorem 3.9 ends the proof. ☐

Since spaces in $AANR(\mathcal{K})$ ($A_HANR(\mathcal{K})$, respectively) are $\omega-$dominated (resp. $\omega - H-$dominated) by finite polyhedra (see Corollary 1.1), we have:

Corollary 3.4. Assume that X is an AANR($\mathcal{K}$) (respectively A$_H$ANR($\mathcal{K}$)) for the class $\mathcal{K}$ of compact spaces and that $Y \in \text{NES}(\mathcal{K})$. Then

$$\mathbf{D}(X,Y) \subset \mathbf{A}(X,Y), \; (\mathbf{D}(X,Y) \subset \mathbf{A}_H(X,Y), \text{ respectively.}$$

The inclusion $AANR \cap \mathcal{K} \subset AANR(\mathcal{K})$ for the class $\mathcal{K}$ of compact spaces (see Proposition 3.2 above), implies the known result (Górniewicz et al. [48], see Mas Colell [68] for the contractible case):

Corollary 3.5. Assume that X, Y are *ANRs* with X compact. Then,

$$\mathbf{C}^\infty(X,Y) \subset \mathbf{D}(X,Y) \subset \mathbf{A}_H(X,Y).$$

Remark 3.9. It is an open question as to whether the inclusion above holds when the domain X is a locally finite polyhedron (and consequently for a space dominated by locally finite polyhedra such as arbitrary *ANRs*) or not. Such spaces are paracompact and Cellina's theorem suggests that the inclusion should be true with mere paracompactness.

In light of Proposition 3.7, and keeping in mind that an infinite product of *ANRs* is not necessarily an *ANR* we have:

Corollary 3.6. If $X = \prod_{i \in I} X_i$ is an infinite product of compact *ANRs and* Y is a uniform space, then $\mathbf{D}(X,Y) \subset \mathbf{A}(X,Y)$.

Corollary 3.7. Assume that X is a compact subset of a metrizable locally $\mathcal{C}-$convex space E and that Y is a $\mathcal{C}-$convex subset of a metrizable locally $\mathcal{C}-$convex space. Then $\mathbf{C}(X,Y) \subset \mathbf{A}_H(X,Y)$.

Proof. We know by Theorem 3.7 that $\mathbf{C}(X,Y) \subset \mathbf{A}(X,Y)$. According to Theorem 3.2, $E \in \mathcal{A}_H(\mathcal{K};\mathcal{P})$ where $\mathcal{K}$ is the class of compact spaces and $\mathcal{P}$ is the class of finite polyhedra. Since $X \in \mathcal{K}$, the definition of the class $\mathcal{A}_H(\mathcal{K};\mathcal{P})$ implies that $X \in D_H(\mathcal{P})$. Moreover, the set Y as well as all values of any map $\Phi \in \mathbf{C}(X,Y)$ are absolute retracts, and hence contractible spaces. So, in this case, $\mathbf{C}(X,Y) \subset \mathbf{C}^\infty(X,Y)$. Proposition 3.17 ends the proof. □

3.3.4 Continuous Selections for LSC Maps

Keeping in mind that l.s.c. maps are assumed to have non-empty values, let us consider the following classes of maps:

- Lower semicontinuous maps with non-empty closed values:

$$\mathbf{LSCL}(X,Y) := \{\Psi \in \mathbf{LSC}(X,Y) : \Psi(x) \text{ is closed in } Y, \forall x \in X\}.$$

- Maps having continuous selections:

$$\mathbf{S}(X,Y) := \{\Psi : X \rightrightarrows Y : \exists s \in \mathbf{c}(X,Y) \text{ with } s(x) \in \Psi(x), \forall x \in X\}.$$

3.3.4.1 Michael Selections

It is interesting to note, as pointed out in Repovs-Semenov [85], the equivalence of lower semicontinuity with the existence of local selections:

$$\Psi \in \mathbf{LSC}(X,Y) \Longleftrightarrow \begin{cases} \forall(x,y) \in graph(\Phi), \forall V_y \in \mathcal{N}_Y(y), \exists U_x \mathcal{N}_X(x), \\ \exists s : U_x \longrightarrow Y \text{ (not necessarily continuous)} \\ \text{such that } s(x) \in \Psi(x) \cap V_y. \end{cases}$$

A landmark result at the interface of topology and functional analysis relates intimately the concepts of lower semicontinuity, convexity, and paracompactness with the existence of continuous selections for carriers (set-valued maps), namely, the theorem of Ernest Michael on the existence of continuous selectors for what we shall call Michael maps. Define the class of *Michael maps* as

$$\mathbf{M}(X,Y) := \{\Psi \in \mathbf{LSCL}(X,Y) : \Psi \text{ has convex values}\},$$

where Y is a subset of a topological vector space.

If Y is a subset of locally $\mathcal{C}-$convex space (as per Definition 3.9, where $\mathcal{C}$ is an abstract convexity structure), consider:

$$\mathbf{M}_{\mathcal{C}}(X,Y) := \{\Psi \in \mathbf{LSCL}(X,Y) : \Psi(x) \text{ is } \mathcal{C} - \text{convex}, \forall x \in X\}\}.$$

- The Michael selection theorem [74] asserts:

Theorem 3.11. If X is a T_1 topological space and Y is a convex subset of a Fréchet space,[7] then:

$$X \text{ is paracompact } \Leftrightarrow (\mathbf{M}(X,Y) \subset \mathbf{S}(X,Y)).$$

The proof of the necessity of paracompactness ($\Longrightarrow$) can be broken into three steps:

- Step 1: Given $X =$ paracompact space, $Y =$ convex subset of a locally convex space F, $V =$ a convex open neighborhood of the origin in F, any map $\Psi \in \mathbf{M}(X,Y)$ amidst a *continuous exterior* $V-$*approximation*, i.e., a continuous mapping $s_V : X \longrightarrow Y$ such that $s_V(x) \in V[\Psi(x)], \forall x \in X$. The proof of this step follows, in all respects, that of the Cellina's approximation theorem (the closedness of the values of Ψ are not required at this stage). (Note that such an exterior approximation is a particular case of an approximate selection.)

- Step 2: If the locally convex space F in Step 1 is metrizable with a countable basis of open convex neighborhoods $\{V_n\}_{n=1}^{\infty}$ of the origin, with $diam(V_n) \downarrow 0^+$, then there exists a uniform Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ of continuous exterior V_n-approximations to Ψ.

[7] A Fréchet space is a completely metrizable locally convex topological vector space.

- Step 3: If the space F is, in addition, complete (hence Fréchet) and Ψ has closed values, the sequence $\{s_n\}_{n=1}^{\infty}$ is made to converge uniformly to the desired continuous selection of Ψ.

Linear convexity was replaced by abstract convexity as follows (see [14, 57, 83]):

Theorem 3.12. Let $(Y, \mathcal{C})$ be a completely metrizable space with a local convexity structure $\mathcal{C}$ defined by an $L-$convexity, and let X be a paracompact space. Then $\mathbf{M}_{\mathcal{C}}(X, Y) \subset \mathbf{S}(X, Y)$.

It is interesting to point out the intimate relationship between the selection and the extension problems.

The diagram

$$\begin{array}{ccc} & Y & \\ f_0 \nearrow & & \nwarrow f \\ A & \overset{closed}{\hookrightarrow} & X \end{array} \quad \Leftrightarrow f(x) \in \Psi_A(x) = \begin{cases} \{f_0(x)\} & \text{if } x \in A, \\ Y & \text{if } x \in X \setminus A. \end{cases}$$

commutes

Thus, the Michael selection theorem implies a refinement of the Dugundji extension theorem (Example 3.1 (b) above):

Proposition 3.18. If $\mathcal{C}_{\mathcal{B}}$ is the class of convex subsets of Banach spaces, then: $\mathcal{C}_{\mathcal{B}} \subset$ ES (Paracompact) $\subset$ ES(Metric).

There is no known example of a non-locally convex, completely metrizable space that can be substituted for a Banach (Fréchet) space in the Michael selection theorem.

For maps with non-convex values (classical or abstract convexity), a remarkable theorem for l.s.c. maps is also due to Michael [74, II].

Theorem 3.13. Let X be a paracompact zero-dimensional space and Y a completely metrizable space. If $\Psi \in \mathbf{LSCL}(X, Y)$ has an $n-$connected value and $\Psi(X)$ is equi-locally $n-$connected, then $\Psi \in \mathbf{S}(X, Y)$.

The closedness of values is crucial to selection theorems of the Michael type. Filipov has proven the existence of a map $\Psi \in \mathbf{LSC}([0, 1], Y), Y$ a Banach space, with non-closed convex values without a continuous selector (see Repovs and Semenov [85]).

An extension by Oudadess and the author to the abstract convexity structure setting of a result of Michael and Pixley [75] (classical linear convexity) only requires the closedness of the values on a subset of the domains:

Theorem 3.14. [19] Let X and $(Y, \mathcal{C})$ be as in Theorem, $Z \subseteq X$ with $\dim_X(Z) \leq 0, D \subseteq X$, D countable, and let $\Psi \in \mathbf{LSC}(X, Y)$ be such that $\Psi(x)$ is closed for all $x \notin D$ and $\overline{\Psi(x)}$ is $\mathcal{C}-$convex for all $x \notin Z$. Then $\Psi \in \mathbf{S}(X, Y)$.

3.3.4.2 A Hybrid Continuous Approximation-Selection Property

A key step in proving solvability theorems for a non-self map Φ of the USC type, subject to tangency conditions (inwardness or outwardness) $\Phi \cap \Psi \neq \emptyset$ on the boundary, expressed in terms of an LSC tangent cone Ψ, is to prove the existence of a single-valued mapping that is a continuous graph approximation for Φ and a continuous selection for Ψ. The first such hybrid "Cellina-Michael" theorem was established by Kryszewski and the author (see, for example, [13]) for convex valued maps with values in a normed space. It was recently extended to spaces with abstract convexity structures by Ben-El-Mechaiekh et al. [18] with a streamlined proof and applications to the solvability of nonlinear equations.

Theorem 3.15. Let X be a paracompact space with compatible uniformity structure $\mathcal{U}$, and let $(Y, \mathcal{C})$ be a completely metrizable space with a local convexity structure $\mathcal{C}$. Then the following holds:

$$\begin{bmatrix} \Phi \in \mathbf{C}(X,Y), \\ \Psi \in \mathbf{M}_{\mathcal{C}}(X,Y), \\ \Phi(x) \cap \Psi(x) \neq \emptyset, \forall x \in X. \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall U \in \mathcal{U}, \forall \epsilon > 0, \\ \exists s \in \mathbf{c}(X,Y) \text{ such that } \forall x \in X, \\ s(x) \in \Psi(x), \\ s(x) \in B_\epsilon(\Phi(U[x])). \end{bmatrix}.$$

The reader is referred to [18] for the proof.

3.3.4.3 More on Continuous Selections for Non-Convex Maps

The openness of pre-images (or, more generally, the non-emptiness of the interior $int(\Phi^{-1}(y))$ for a map $\Phi : X \rightrightarrows Y$ between spaces), is a stronger form of lower semicontinuity. Selection theorems for such maps do not require completeness of the codomain Y. A key selection property for such strongly LSC maps was implicit in Browder's proof of the Browder-Ky Fan fixed-point theorem for so-called $F-$maps of topological vector spaces. Following the approach in Ben-El-Mechaiekh et al. [16], let us define:

Definition 3.22. Given X, a topological space, and Y, a subset of a vector space, define the class of Ky Fan maps, or $F-$maps, by as:

$$\mathbf{F}(X,Y) := \{\Phi : X \rightrightarrows Y : \emptyset \neq \Phi(x) \text{ convex}, \forall x \in X \text{ and } \Phi^{-1}(y) \text{ open in } X, \forall y \in Y\}.$$

Theorem 3.16. [16] If X is paracompact and Y is convex, then $\mathbf{F}(X,Y) \subset \mathbf{S}(X,Y)$.

If in addition, X is compact, the continuous selection s of an $F-$map has values in a finite dimensional convex polytope; that is, there exists $\{y_1, \cdots, y_n\} \subset Y$ such that $s(X) \subset conv\{y_1, \cdots, y_n\} \subset Y$ and $s(x) \in \Phi(x), \forall x \in X$.

Proof. For each $x \in X$, there exists $y \in \Phi(x)$, i.e., $x \in \Phi^{-1}(y)$. Hence, the collection of open sets $\omega := \{\Phi^{-1}(y) : y \in Y\}$ covers X. Let $\mathcal{O} := \{O_i : i \in I\}$ be a locally finite open refinement of ω and let $\{\lambda_i : i \in I\}$ be a continuous partition of unity subordinated to $\mathcal{O}$. For each $x \in X$, the set of essential indices $I(x) := \{i \in I : \lambda_i(x) \neq 0\}$ is finite. Note that $i \in I(x) \Longrightarrow x \in O_i \subset \Phi^{-1}(y_i)$ for some $y_i \in Y$, and the finite set $\{y_i : i \in I(x)\}$ together with its convex hull is contained in $\Phi(x)$ because the latter is convex. Define a continuous mapping $s : X \to Y$ by:

$$s(x) := \sum_{i\in I} \lambda_i(x) y_i = \sum_{i\in I(x)} \lambda_i(x) y_i, \text{ for all } x \in X.$$

Since $s(x)$ is a convex combination of $\{y_i : i \in I(x)\}$, it follows that $s(x) \in \Phi(x)$.

If X is compact, ω admits a finite subcover $\{\Phi^{-1}(y_i) : i = 1, \cdots, n\}$ and $s(x) = \sum_{i=1}^{n} \lambda_i(x) y_i \in \Phi(x) \cap conv\{_1, \cdots, y_n\} \subset Y$. □

Remark 3.10. A number of authors observed that it was in fact sufficient to weaken the regularity condition (ii) in the definition of an $F-$map above and not ask that the fibers $\Phi^{-1}(y)$ be open, but merely consider covers of the domain X by $\{int(\Phi^{-1}(y)) : y \in Y\}$. One of the first such contributions was done in [16] with the consideration of the class of $\Phi-$maps. Such considerations, including the concept of transfer openess are in fact superfluous as pointed out in Section 7.2.1 of [11].

The next result, proven by the author in [12], extends Theorem 3.16 to maps with non-convex values. Part (a) was essentially formulated in [56], to which the reader is referred for a proof; our formulation is, however, slightly different.

Theorem 3.17. Let $\Phi, \tilde{\Phi} : X \rightrightarrows Y$ be two maps of a paracompact space X into a space Y such that:

(i) $\tilde{\Phi}$ is a selection of Φ;

(ii) $\tilde{\Phi}$ has non-empty values and open preimages.

Then, the following two statements hold:

(a) If, in addition, Φ satisfies the following condition:

 (iii) for each $x \in X$, there exists an open neighborhood U_x of x in X such that for each open neighborhood O of x in X contained in U_x, the set $\bigcap_{z\in O} \Phi(z)$ is $\infty-$connected,

 then $\Phi \in \mathbf{S}(X, Y)$.

(b) If Y is equipped with a compatible uniformity $\mathcal{V}$, and Φ satisfies the additional condition:

(iii)$'$ for each open subset U of X, the set $\bigcap_{x\in U}\Phi(x)$ is empty or $\infty-$proximally connected and compact in Y,

then, for any $V\in\mathcal{V}$, Φ has a continuous exterior $V-$approximation, that is, a mapping $s\in\mathbf{c}(X,Y)$ such that $s(x)\in V[\Phi(x)], \forall x\in X$.

Proof. See [56] for a proof of (a). Of course, if Φ has convex values (that is, Φ is a so-called generalized Ky Fan map in the sense of [16]), then (iii) is true, as the intersection of convex sets is convex.

We only prove (b). Let $\omega=\{U_i : i\in I\}$ be a point-finite open refinement of the open cover $\{\tilde{\Phi}^{-1}(y) : y\in Y\}$ of X, that is, ω covers X, U_i is contained in some $\tilde{\Phi}^{-1}(y_i)$ for each $i\in I$, and for each $x\in X$, the set $I(x)=\{i\in I : x\in U_i\}$ is finite.

Let E be a real vector space equipped with the finite (weak) topology, having a basis $\{e_i : i\in I\}$ in one-to-one correspondence with the index set I. Let $N(\omega)$ be the geometric nerve of the cover ω in E, and denote by $N(\omega)^k$, $k\geq 0$, its $k-$dimensional skeleton. For any simplex σ of $N(\omega)$, let U_σ be the open subset of X defined by $U_\sigma := \bigcap_{e_i\in\ddot{\sigma}} U_i$ where $\ddot{\sigma}$ denotes the set of all vertices of σ. The set $\Phi_\sigma=\bigcap_{z\in U_\sigma}\Phi(z)$ is certainly a non-empty compact $\infty-$proximally connected subset of Y contained in every $\Phi(x)$, $x\in U_\sigma$. We shall prove the following property by induction on k:

$$\begin{cases} \forall k\geq 0, \forall V\in\mathcal{V}, \exists f_k\in\mathbf{c}(N(\omega)^k, Y) \text{ such that} \\ (1)\ f_k(\sigma)\subset V[\Phi_\sigma],\ \text{for any simplex } \sigma\in N(\omega)^k,\ \text{and} \\ (2)\ f_{k+1}|_{N(\omega)^k}=f_k. \end{cases}$$

Let $k=0$. For any $i\in I$, choose $f_0(e_i)$ to be any point in Φ_{e_i}. Suppose that this property holds true for $n=k$ and let $n=k+1$. Let $V\in\mathcal{V}$ be arbitrary and let $\sigma=(e_{i_0},\ldots,e_{i_{k+1}})$ be an arbitrary $(k+1)-$simplex in $N(\omega)^{k+1}$. By (iii) and the characterization of proximal connectedness by an extension property (see Definition 3.6), together with the induction hypothesis, there exists $W_\sigma\in\mathcal{V}$ such that the restriction $f_k|_{\partial\sigma} : \partial\sigma\longrightarrow W_\sigma[\Phi_\sigma]$ extends continuously to a function $f_{k+1,\sigma} : \sigma\longrightarrow V[\Phi_\sigma]$. Clearly, if σ and σ' are distinct $(k+1)-$simplices sharing a common face, then $f_{k+1,\sigma}$ and $f_{k+1,\sigma'}$ coincide on $\sigma\cap\sigma'$. The function $f_{k+1} : N(\omega)^{k+1}\longrightarrow Y$ given by $f_{k+1}(\sigma)=f_{k+1,\sigma}(\sigma)$ is the desired extension of f_k.

Let $f : N(\omega)\longrightarrow Y$ be defined by $f(\sigma)=f_k(\sigma)$ if $\sigma\in N(\omega)^k$ and let $\kappa : X\longrightarrow N(\omega)$,

$$\kappa(x) := \sum\nolimits_{i\in I}\lambda_i(x)e_i, x\in X,$$

where $\{\lambda_i\}$ being a partition of unity subordinated to the cover ω, be the canonical map of the nerve of the cover ω. The composition $s=f\circ\kappa : X\longrightarrow N(\omega)\longrightarrow Y$ verifies $s(x)\in V[\Phi(x)]$, for all $x\in X$. □

Corollary 3.8 follows from part (a) of Theorem 3.17 and Proposition 3.9.

Corollary 3.8. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces with X paracompact and $\Phi \in \mathbf{USC}(X,Y)$ (whose values are not necessarily closed). Assume that for every entourage $V \in \mathcal{V}$ and every finite subset $\{x_1,\ldots,x_n\}$ of X, the set $\bigcap_{i=1}^{n} V[\Phi(x_i)]$ is empty or $\infty-$connected. Then $\Phi \in \mathbf{A}(X,Y)$.

Proof. Let $(U,V) \in \mathcal{U} \times \mathcal{V}$ be arbitrary and let $\Psi = \Psi_{U,V} : X \rightrightarrows Y$ be the open-graph majorant of Φ given by Proposition 3.9 (recall that the values $\Psi(x)$ are non-empty intersections of finite tubular neighborhoods $V[\Phi(x_i)]$, and hence $\infty-$connected). The map Ψ being locally constant, for each $x \in X$, there exists an open neighborhood U_x of x in X such that $\Psi(z) = \Psi(x)$ for all $z \in U_x$. Clearly, $\bigcap_{z\in O} \Psi(z) = \Psi(x)$ for each subset O of U_x. Having (non-empty) $\infty-$connected values, the map Ψ has a continuous selection by Theorem 3.17 (a). Any selection of Ψ is a $(U,V)-$approximate selection of Φ. □

Obviously, if the values of Φ are (generalized) convex and if Y is endowed with a local convexity, we recover Cellina's approximation theorem for convex and generalized convex maps.

It is natural to ask whether, conversely, selection theorems can be derived from approximation theorems for non-convex maps. The next result generalizes a theorem of Mas Colell [69] and provides an answer to this question. This result is in fact in the spirit of a theorem of Michael on the existence of a multiselection in $\mathbf{USCO}(X,Y)$ for a map in $\mathbf{LSCL}(X,Y)$ given X is paracompact and Y is completely metrizable (see [85]).

Proposition 3.19. Let $\Psi : P \rightrightarrows Y$ be an open-graph map with contractible values of a finite (topological) polyhedron P into a compact space Y. Then Ψ has a multiselection $\Phi \in \mathbf{C}^{\infty}(P,Y)$, that is, Φ is u.s.c. with compact contractible values $\Phi(x) \subseteq \Psi(x), x \in X$.

Proof. The proof is made by induction on the dimension of the polyhedron P and is an adaptation of the proof in [69].

If the dimension of P is equal to 0, that is, P is a finite set of points $\{e_0,\ldots,e_n\}$, let $\Phi(e_i)$ be any point in $\Psi(e_i)$. Suppose that the result is true for any $(n-1)-$dimensional polyhedron, and let P be an $n-$dimensional polyhedron.

For any given $x \in P$, let $h_x : \Psi(x) \times [0,1] \to \Psi(x)$ be a continuous deformation onto a given point y_x in $\Psi(x)$:

$$\forall y \in \Psi(x), \begin{cases} h_x(y,0) = y, \\ h_x(y,1) = y_x. \end{cases}$$

Any subset Z of $\Psi(x)$ containing y_x is contained in the set $h_x(Z \times [0,1])$, which is contractible (consider the homotopy $\varphi(z,\tau) = h_x(y,\tau + (1-\tau)t)$ for any $z = h_x(y,t) \in h_x(Z \times [0,1])$). Thus, if V_x is an open neighborhood of y_x in Y with closure $\overline{V}_x$ contained in $\Psi(x)$, then the set $\Theta(x) = h_x(\overline{V}_x \times [0,1])$ is compact and contractible. Hence, there exists an open neighborhood U_x of x in P such that $\Theta(x) \subset \Psi(x')$ for all $x' \in U_x$.

Let $\omega = \{U_{x_1}, \ldots, U_{x_n}\} \in Cov(P)$ be an open cover of P, and let (K', f') be a triangulation of P such that each closed vertex star of K' is contained in some member of ω. Let σ be an arbitrary closed $n-$simplex of K' and let U_{x_i} be a member of ω containing σ.

The constant map $\Xi_\sigma : \sigma \rightrightarrows Y$ given by $\Xi_\sigma(u) = \Theta(x_i), u \in \sigma$, is a multiselection of the restriction of Ψ to σ and has compact contractible values. By construction, the compact-valued map $\tilde{\Xi} : P \rightrightarrows Y$ defined by $\tilde{\Xi}(u) = \bigcup\{\Xi_\sigma(u) : \sigma \in P^n \text{ with } u \in \sigma\}$ for all $u \in P$ is a u.s.c. multiselection with compact contractible values of Ψ.

Let P' be the polyhedron determined by the $(n-1)-$dimensional skeleton of K' and let $u \in P \setminus P'$ be arbitrary. Denote by σ_u the (unique) carrier of u. Then $\tilde{\Xi}(u) = \Xi_{\sigma_u}(u)$ is a compact contractible set. By hypothesis, the restriction of Ψ to P' has a u.s.c. multiselection $\hat{\Xi}$ with compact contractible values. Finally, the map $\Phi : P \rightrightarrows Y$ given by:

$$\Phi(u) = \begin{cases} \tilde{\Xi}(u) & \text{if } u \in P \setminus P', \\ \hat{\Xi}(u) & \text{if } u \in P'. \end{cases}$$

is the desired multiselection of Ψ. □

Now, by Remark 3.7, an open tubular neighborhood of the graph of Φ is still a multi-selection, that is, $W[graph(\Phi)] \subseteq graph(\Psi)$. Hence, a $W-$approximative selection of Φ, which exists by Theorem 3.8, is a continuous selection of Ψ. We have thus obtained, in a rather elementary way, a selection property similar to a result of McClendon [72] obtained there from a lifting property for functions defined on a simplicial complex.

Corollary 3.9. Let $\Psi : P \rightrightarrows Y$ be a map with open graph and contractible values of a finite polyhedron P into a compact ANR Y. Then $\Psi \in \mathbf{S}(P, Y)$.

In fact, the main result in [72] garantees the existence of a continuous selection for an open-graph map of a locally finite simplicial complex with contractible values in an absolute retract. The following example suggests that this hypothesis on the domain cannot be drastically improved for the selectionability of open-graph maps with contractible values.

Example 3.12. Let I^∞ be the Hilbert cube and let $\Psi : I^\infty \rightrightarrows I^\infty$be the map defined by $\Psi(x) = I^\infty \setminus \{x\}$ for every $x \in I^\infty$. Then Ψ is an open-graph map with contractible values. Certainly, it cannot be selectable; otherwise, by the generalized Brouwer theorem for compact absolute retracts of Borsuk, it would have a fixed point, which is impossible.

It is therefore interesting to compare Theorem 3.17 with Corollary 3.9 and the selection theorem of [72]. It appears that for a much more general domain (paracompact topological space vs. locally finite simplicial complex), a restrictive condition on the topological structure of the values of the map (condition (iii) in Theorem 3.17) imposes itself.

It is well known that if a map has open values and open preimages, then it is not necessarily open-graph. However, given a continuous map $\Phi : X \rightrightarrows Y$ of a space X into a space Y, if the connected components of X are open (for example, X is locally connected) and Φ has open values and open preimages, then Φ is open-graph (see reference in [67]). Thus, the following result follows directly from the main theorem in [70].

Theorem 3.18. Let X be a locally finite simplicial complex and Y be an AR. Let $\Phi : |K| \rightrightarrows Y$ be a continuous map satisfying the following properties:

(i) the values of Φ are open and contractible;

(ii) the preimages of Φ are open.

Then $\Phi \in \mathbf{S}(X, Y)$.

3.3.4.4 Non-Expansive Selections

Another interesting line of research concerns the existence of non-expansive selections for set-valued maps. We refer to the work by M. A. Khamsi et al. [61] and references there, particularly for the non-convex metrizable case. Recall that a metric space (Y, d) is said to be *hyperconvex* if and only if for every arbitrary family $\{(y_i, r_i) \in Y \times (0, +\infty)\}_{i \in I}$,

$$(d(y_i, y_j) \leq r_i + r_j, \forall i, j \in I) \Longrightarrow \bigcap_{i \in I} \overline{B}(y_i, r_i) \neq \emptyset.$$

It is interesting to note that hyperconvex spaces equipped with the ball convexity are $L-$convex spaces in the sense of Definition 11 above.

A subset $A \subset (Y, d)$ is said to be *externally hyperconvex* if for any $\{(y_i, r_i) \in Y \times (0, +\infty)\}_{i \in I}$ with $d(y_i, A) \leq r_i$ and $d(y_i, y_j) \leq r_i + r_j, \forall i, j \in I$, we have $A \cap \bigcap_{i \in I} \overline{B}(y_i, r_i) \neq \emptyset$.

Khamsi-Kirk-Yanez selection theorem [63] states:

Theorem 3.19. Let X be a set, (Y, d) a hyperconvex space, and $\Phi : X \rightrightarrows Y$ a map with externally convex values. Then Φ has a selection $s : X \to Y$ satisfying

$$d(s(x), s(x\prime)) \leq H(\Phi(x), \Phi(x\prime)), \forall x, x' \in X,$$

where H is the Hausdorff distance.

As an immediate consequence of this result, the authors obtain a non-expansive selection result in hyperconvex spaces: if in addition (X, ρ) is a metric space and $\Phi : X \rightrightarrows Y$ is a non-expansive set-valued map with bounded externally hyperconvex values, then it has a non-expansive selection.

Along the same lines and in case X is a *metric tree* (that is, a metric space in which any two points are joined by a unique arc isometric to a line segment), Aksoy and Khamsi [2] prove the following: a map $\Phi : X \rightrightarrows Y$ with bounded closed convex values admits a selection s such that $d(s(x), s(x\prime)) \leq H(\Phi(x), \Phi(x\prime)), \forall x, x' \in X$.

3.4 Fixed Point and Coincidence Theorems

3.4.1 Generalizations of the Himmelberg Theorem to the Non-Convex Setting

The Himmelberg fixed-point theorem [54] generalizes, to set-valued maps, the Schauder fixed-point theorem. It asserts that every upper semicontinuous compact set-valued map Φ with closed convex values from a convex subset X of a locally convex topological vector space into itself has a fixed point, that is, convex subsets of locally convex spaces have the *fixed-point property (FPP)* for the class $(\mathbf{K} \cap \mathbf{CL})^{\mathcal{K}}$ of compact Kakutani maps with closed values.

In this section, we review extensions of the Himmelberg theorem to the class $\mathbf{A}$ of approachable maps and its subclasses of non-convex maps defined on retracts.

We recall some basic notions first.

- If $X \subseteq Y$, a *fixed point* for a map $\Phi : X \to 2^Y$ is an element $x \in X$ with $x \in \Phi(x)$. If Y is a subset of a vector space, an *equilibrium* for Φ is an element $x \in X$ with $0 \in \Phi(x)$.

- Given a set X and a map $\Phi : X \rightrightarrows X$, $Fix(\Phi) := \{x \in X : x \in \Phi(x)\}$ is the set of all fixed points of Φ.

- Given a set X, a collection $\omega \subset 2^X$, and a set-valued map $\Phi : X \rightrightarrows X$, an element $x \in X$ is said to be a $\omega-$*fixed point* for Φ if both $\{x\}$ and $\Phi(x)$ intersect a common member of ω.

- Given an abstract class of maps $\mathbf{M}$ and an abstract class of spaces $\mathcal{K}$, denote by $\mathbf{M}^{\mathcal{K}}$ those maps in $\mathbf{M}$ with a range contained in a member K of $\mathcal{K}$, that is, $\Phi \in \mathbf{M}^{\mathcal{K}}(X, Y) \iff \Phi \in \mathbf{M}(X, Y)$ and $\Phi(X) \subset K$ for some space $K \in \mathcal{K}$.

- Following Granas [52], we say that X is a *fixed-point space for the class* $\mathbf{M}$ if $Fix(\Phi) \neq \emptyset$ for all $\Phi \in \mathbf{M}(X)$. We write

$$\mathcal{F}_{\mathbf{M}} := \{X : X \text{ is a fixed point space for the class } \mathbf{M}\}.$$

Fixed points for **USCL** set-valued maps are usually obtained as limits of nets of approximate fixed points. The passage from approximate fixed points to fixed points is provided by Lemma 3.2.

Lemma 3.2. [15] Let X be a regular space and $\Phi \in \mathbf{USCL}(X)$. Assume that there exists a cofinal family $\{\omega\}$ in $Cov_X(\overline{\Phi(X)})$ such that Φ has a $\omega-$fixed point for all $\omega \in \{\omega\}$. Then Φ has a fixed point.

Note that in the metrizable setting, the existence of a cofinal family of open covers for a compact set is guaranteed by the Lebesgue number lemma (see, for example, [36]). In a general (that is, not necessarily metrizable) uniform space X—which is always completely regular—open covers of a compact subset A admit refinements of the form $\{U[x] : x \in A\}$, where U is a member of the uniformity. Thus, such refinements form a cofinal family of open covers. So, in proving the existence of fixed points in uniform spaces for **USCL** compact maps, it suffices to prove the existence of approximate fixed points.

The Himmelberg fixed-point theorem on convex subsets of locally convex topological vector spaces was extended by the author to the class $\mathbf{A} \cap \mathbf{CL}$ of approachable closed-valued maps as stated below.[8]

Proposition 3.20. [12] If X is a nonempty convex subset of a locally convex topological vector space (or, more generally, if $X \in \mathcal{F}_{\mathbf{c}}$ has the fixed-point property for continuous single-valued mappings) and $\mathcal{K}$ is the class of compact spaces, then $X \in \mathcal{F}_{(\mathbf{A}\cap\mathbf{CL})^{\mathcal{K}}}$, i.e., every compact approachable set-valued map with closed values from X into itself has a fixed point.

3.4.1.1 Preservation of the FPP from $\mathcal{P}$ to $\mathcal{A}(\mathcal{K};\mathcal{P})$

A primary interest in the passage $\mathcal{P} \to \mathcal{A}(\mathcal{K};\mathcal{P})$ is not only in the preservation, under very mild assumptions on the spaces and maps involved, of the fixed point property, but also in the shifting of compactness from domains to maps. More precisely:

Theorem 3.20. Let $\mathbf{M}$ be an abstract class of set-valued maps and let $\mathcal{K}, \mathcal{P}$ be two classes of topological spaces such that:

(i) $\mathbf{c} \subset \mathbf{M}$;

(ii) $(\Phi \in \mathbf{M}_c(X,Y)$ and $\overline{\Phi(X)} \subset O \underset{open}{\subset} Y) \Rightarrow (\Phi \cap O \in \mathbf{M}_c(X,O))$;

(iii) each space in $\mathcal{A}(\mathcal{K};\mathcal{P})$ is regular.

Then

$$(\mathcal{P} \subset \mathcal{F}_{\mathbf{M}_c}) \Rightarrow (\mathcal{A}(\mathcal{K};\mathcal{P}) \subset \mathcal{F}_{\mathbf{CL}\cap(\mathbf{M}_c^{\mathcal{K}})}).$$

Proof. Let $X \in \mathcal{A}(\mathcal{K};\mathcal{P}), \Phi \in (\mathbf{M}_c^{\mathcal{K}} \cap \mathbf{CL})(X)$ be arbitrary, that is, Φ is a closed-valued finite composition of $\mathbf{M}-$maps and $\Phi(X) \subseteq K \subseteq X, K \in \mathcal{K}$. Let $\omega \in Cov_X(K)$ be arbitrary. By Definition 3.12, there exist a cover $\omega' \in Cov_X(K), \omega' \preceq \omega$, a space $P \in \mathcal{P}$, and a pair of continuous mappings $\bigcup\omega' \xrightarrow{s} P \xrightarrow{r} X$ such that $r \circ s$ and $id_{\bigcup\omega'}$ are $\omega-$near. The diagram

$$\begin{array}{ccccc} & X \overset{\Phi'}{\rightrightarrows} \bigcup\omega' & \overset{s}{\to} & P & \\ r\circ s\circ\Phi' & \upuparrows \quad \swarrow r & & \upuparrows & s\circ\Phi'\circ r \\ & X \underset{\Phi'}{\rightrightarrows} \bigcup\omega' & \underset{s}{\to} & P & \end{array}$$

[8] Let us denote by $\mathbf{CL}(X,Y)$ the class of set-valued maps of topological spaces with closed values.

where $\Phi'(x) := \Phi(x) \cap \bigcup \omega'$, commutes. By (i) and (ii), s, r, and Φ' belong to $\mathbf{M}$, so $s \circ \Phi' \circ r \in \mathbf{M}_c$. Hence, $s \circ \Phi' \circ r$ has a fixed point. It follows that $r \circ s \circ \Phi'$ also has a fixed point $x_\omega \in r(s(\Phi'(x_\omega)))$. Such a fixed point is a $\omega-$fixed point for Φ. Lemma 2 ends the proof. □

Remark 3.11. (a) Condition (ii) is superfluous for many particular examples of set-valued maps. For instance, if $\Phi \in \mathbf{K}$, the class of Kakutani upper semicontinuous maps with convex values, restricting the codomain from Y to an open subset of Y containing the range of Φ does not change the nature of the map Φ. However, in some other cases (for example, for the class $\mathbf{D}$), relevant features depend on the way the values $\Phi(x)$ of Φ are imbedded in Y rather than on their intrinsic topological properties; thus, compressing the codomain without altering the values themselves may disqualify a map from belonging to the original class.

(b) Theorem 3.20 implies, in particular, that if a regular topological space is a fixed-point space for a class of closed-valued upper semicontinuous maps, then so is every retract of the space.

An immediate consequence of Theorem 3.20 together with Propositions 3.7, 3.8, and 3.17, yields:

Corollary 3.10. Let $\mathbf{M}$ be a class of maps as in Theorem 3.20 and let $\{X_i\}_{i\in I}$ be a collection of compact sets such that, for each finite subset I' of I, the product $X_{I'} = \prod\{X_i : i \in I'\} \in \mathcal{F}_{\mathbf{M}_c}$, that is, has the FPP for the class $\mathbf{M}_c$ of finite compositions of $M-$maps. Then the same holds true for the product $X = \prod_{i\in I} X_i$, that is, $X \in \mathcal{F}_{\mathbf{M}_c}$.

Corollary 3.11. Let $\mathcal{K}$ be the class of compact spaces and $\mathcal{P}$ the class of convex finite polyhedra. Then,

$$\mathcal{A}(\mathcal{K};\mathcal{P}) \subset \mathcal{F}_{(\mathbf{A}\cap\mathbf{CL})_c^{\mathcal{K}}},$$

that is, every compact finite composite of approachable maps with closed values of a space $X \in \mathcal{A}(\mathcal{K};\mathcal{P})$ has a fixed point.

Proof. We apply Theorem 3.20 to the class $\mathbf{M} = \mathbf{A} \cap \mathbf{USCO}$. Naturally, every continuous single-valued mapping is approachable (by itself), that is $\mathbf{c} \subset \mathbf{A} \cap \mathbf{USCO}$. From the definition of the class $\mathbf{A}$, one easily verifies that for given uniform spaces X, Y and a map $\Phi \in (\mathbf{A} \cap \mathbf{USCO})(X, Y)$, any open neighborhood O of $\overline{\Phi(X)}$ in Y contains the ranges of $(U, V)-$approximative selections of Φ for all members U, V of the respective uniformities on X, Y that are small enough. This implies that condition (ii) is always satisfied by the class $\mathbf{A} \cap \mathbf{USCO}$.

Since any finite polyhedron P is a compact space, Proposition 3.12 implies that $(\mathbf{A} \cap \mathbf{USCO})_c(P) = (\mathbf{A} \cap \mathbf{USCO})(P)$.

It remains to verify that every convex finite polyhedra has the fixed-point property for u.s.c. closed-valued approachable maps.

But this follows from the generalized Himmelberg theorem (Proposition 3.20), or by the following elementary direct argument. Consider a convex finite polyhedra P imbedded in a Euclidean space E (the topology induced by the Euclidean norm on E is uniformizable) and consider a set-valued map $\Phi \in (\mathbf{A} \cap \mathbf{USCO})(P)$. By definition of the class $\mathbf{A}$ for the metrizable case, for any $\varepsilon > 0$, there exists $s \in \mathbf{c}(P)$ such that

$$\forall x \in P, \exists x' \in B_\varepsilon(x), \text{ with } s(x) \in B_\varepsilon(\Phi(x')).$$

The Brouwer fixed-point theorem guarantees the existence of a fixed point $x_\varepsilon = s(x_\varepsilon)$ of s in P. To a given sequence $\{\varepsilon_n\}$ of positive real numbers converging to 0, there corresponds a sequence $\{(x_n, y_n)\}$ in $P \times P, x_n = x_{\varepsilon_n} = s(x_{\varepsilon_n}) \in B_{\varepsilon_n}(y_n), y_n \in \Phi(B_{\varepsilon_n}(x_n))$. This sequence has a cluster point (x_0, y_0) due to the compactness of P. It is clear that $x_0 = s(x_0) = y_0$. Moreover, since a u.s.c. map with closed values in a compact space has a closed graph and since $\{(x_n, y_n)\} \subset B_{\varepsilon_n}(graph(\Phi))$ in $P \times P$, it follows that $(x_0, x_0) \in graph(\Phi)$. This completes the proof of the inclusion $\mathcal{P} \subset \mathcal{F}_{(\mathbf{A} \cap \mathbf{USCO})_c}$ and the proof of our assertion. □

In view of Remark 3.2 and Theorem 3.2, this last result contains, in addition to the classical fixed-point theorems of Ky Fan [40] and Himmelberg [52], results of the author [12], Deguire and the author [15], Chebbi, Florenzano, Llinares and the author [14], Bielawski [21], Horvath [55, 56, 57], and Park and Kim [79], and many others, for $\mathcal{C}-$convex sets.

Corollary 3.12. Assume that X is a non-empty $\mathcal{C}-$convex subset of a locally $\mathcal{C}-$convex space E where $\mathcal{C}$ is:

(i) a linear convexity structure and E a locally convex space, or

(ii) a convexity structure in the sense of Horvath, or

(iii) a $B-$convexity of Bielawski, or

(iv) an $L-$convexity in the sense of Definition 11.

Assume also that Φ is a compact map belonging to any one of the classes:

(iv) $\mathbf{C}_c(X)$, or

(v) $\mathbf{D}_c(X)$.

Then Φ has a fixed point.

Theorem 3.21. Let $\mathcal{K}$ and $\mathcal{P}$ be the classes of compact spaces and finite polyhedra, respectively. If $X \in \mathcal{A}_H(\mathcal{K}; \mathcal{P})$ is compact and $\chi(X) \neq 0$, then $X \in \mathcal{F}_{\mathbf{A}_c \cap \mathbf{CL}}$, that is, every compact closed-valued finite composition of approachable set-valued maps from X into itself has a fixed point.

Proof. Let $\Phi \in (\mathbf{A}_c \cap \mathbf{CL})(X)$. Remember that $\mathcal{K} \cap \mathcal{A}_H(\mathcal{K};\mathcal{P}) \subset D_H(\mathcal{P})$. By Theorem 3.4, for any given $\omega \in Cov(X)$, there is a finite polyhedron P with non-trivial Euler-Poincaré characteristic that $\omega - H-$dominates X. Such a space has the fixed-point property for finite compositions of approachable maps with closed values. This follows from the fact that for a compact space K, we have the equivalence ($K \in \mathcal{F}_{\mathbf{c}} \Leftrightarrow K \in \mathcal{F}_{\mathbf{A}_c \cap \mathbf{CL}}$) (see Proposition 7.1 in [15]), and from the Lefschetz theorem for finite polyhedra: $P \in \mathcal{F}_{\mathbf{c}}$. This implies the existence of an $\omega-$fixed point for Φ. Lemma 3.2 ends the proof. ☐

Theorem 3.5 and Proposition 3.2 imply:

Corollary 3.13. Let $\mathcal{K}$ be the class of compact spaces and let $X \in \mathrm{A}_H\mathrm{ANR}(\mathcal{K})$. Any set-valued map $\Phi \in \mathbf{D}_c(X)$ has a fixed point provided that the Euler-Poincaré characteristic $\chi(X) \neq 0$.

We adapt an argument of Granas [51, 53] to provide a direct and elementary proof of a Leray-Schauder-type nonlinear alternative. Another interesting proof based on a matching theorem of Ky Fan was provided by Idzik and the author in [17]. Proofs of such alternatives are usually based on a homotopy invariant (degree theory for convex-valued correspondences, index theory for compact *ANRs* [41, 48], topological transversality for acyclic correspondence [46]).

3.4.1.2 A Leray-Schauder Alternative for Approachable Maps

Theorem 3.22. Let X be closed convex subset of a locally convex topological vector space E with boundary ∂X and assume that 0 is an interior point of X. Assume that $\Phi \in (\mathbf{A} \cap \mathbf{CL})^{\mathcal{K}}(X,E)$ where $\mathcal{K}$ is the class of compact spaces. Then one of the following properties holds:

(a) (<u>Fixed point</u>) $\exists x_0 \in X$, with $x_0 \in \Phi(x_0)$;

(b) (<u>Invariant direction</u>) $\exists \hat{x} \in \partial X, \exists \lambda \in (0,1)$, with $\hat{x} \in \lambda\Phi(\hat{x})$.

Proof. The proof is an adaptation of the proof of Theorem 2.2 in Granas [53] (where a similar alternative for convex-valued contractions was presented) with changes relevant to the present context. Let p_X be the Minkowski's functional of the set X, and let $r : E \longrightarrow X$ be the standard retraction of E onto X :

$$r(y) := \begin{cases} y & \text{for } y \in X, \\ \frac{y}{p_X(y)} & \text{for } y \notin X. \end{cases}$$

Let $V \in \mathcal{N}(0_E)$ be arbitrary but fixed. Consider the finite subset N_V of E and the map $\Phi_V \in \mathbf{A}(X, conv(N_V))$ verifying $\Phi_V(x) \subset \Phi(x) + V$, for all $x \in X$, both provided by Proposition 15. Let r_V be the restriction of r to the compact convex set $conv(N_V)$. Now the composition product Θ_V : $conv(N_V) \xrightarrow{r_V} X \overset{\Phi_V}{\rightrightarrows} conv(N_V)$ is an approachable map belonging to the class $\mathbf{A}(conv(N_V), conv(N_V))$ (see Proposition 3.12 above). By the generalized

Himmelberg fixed-point theorem (Proposition 3.20), it has a fixed point $x_V \in \Theta_V(x_V)$. The point $y_V := r_V(x_V)$ of X verifies:

$$y_V \in r\Theta_V(y_V).$$

Indeed, $y_V = r_V(x_V) \in r(\Theta_V(x_V)) = r(\Phi_V r_V(x_V)) = r\Phi_V(y_V)$.

Let $z_V \in \Phi_V(y_V)$ be such that $y_V = r(z_V)$.

Two cases are now possible:

Case 1: $z_V \in X$. In this case, $y_V = r(z_V) = z_V \in \Phi_V(y_V) \subset \Phi(y_V) + V$, that is, y_V is an approximative fixed point for Φ.

Case 2: $z_V \notin X$. Hence, $y_V = r(z_V) = \frac{z_V}{p_X(z_V)} \in \partial X$. Therefore, $y_V = \lambda_V z_V \in \lambda_V \Phi_V(y_V) \subset \lambda_V(\Phi(y_V) + V) \subset \lambda_V \Phi(y_V) + V$ where $\lambda_V := 1/p_X(z_V) \in (0,1)$.

This implies the existence of a point y'_V with:

$$y_V - y'_V \in V \text{ and } y'_V \in \lambda_V \Phi(y_V).$$

We have proven the $V-$alternative:

(1) $_V$ ($V-$ fixed point) $\exists x_V \in X$, with $x_V \in \Phi(x_V)$; or

(2) $_V$ ($V-$ invariant direction) $\exists y_V \in \partial X, \exists y'_V \in \Theta_V(y_V), \exists \lambda \in (0,1)$, with $y'_V \in \lambda\Phi(y_V)$.

A standard argument based on Lemma 3.2 (as in [17]) on the compactness of Φ, its upper semicontinuity, and the closedness of its values ends the proof. □

3.4.2 Coincidence Theorems

We present, in this last section, some abstract coincidence theorems that may prove useful in establishing existence results in the theories of minimax, systems of inequalities, and generalized games in the presence of non-convexity.

First, let us recall that two maps Φ and Ψ of a set X into a set Y are said to have a *coincidence* if $\Phi(x_0) \cap \Psi(x_0) \neq \emptyset$ for some $x_0 \in X$, or equivalently, if their graphs intersect: $graph(\Phi) \cap graph(\Psi) \neq \emptyset$.

A trivial consequence of definitions is the coincidence $(\mathbf{A}, \mathbf{S}^{-1})$ (equivalently $(\mathbf{A}^{-1}, \mathbf{S})$:

Theorem 3.23. Let X, Y be two spaces with either one of them being compact, and with the fixed-point property for continuous mappings and $\Phi, \Psi : X \rightrightarrows Y$ be maps such that:

(i) $\Phi \in (\mathbf{A} \cap \mathbf{CL})_c(X, Y)$;

(ii) $\Psi^{-1} \in \mathbf{S}(Y, X)$;

(iii) either one of the spaces X or Y is compact and is in $\mathcal{F}_\mathbf{c}$ (that is, has the fixed-point property for continuous single-valued mappings).

Then Φ and Ψ have a coincidence.

Proof. Assume that Y is compact and has the fixed point property for continuous mappings, the other case being similar. Let s be a continuous selection of $\Psi^{-1} : Y \rightrightarrows X$. By Proposition 3.12 and since Y is compact, the composition product $\Phi \circ s : Y \rightrightarrows Y$ is a compact-valued approachable u.s.c. map. By the generalized Himmelberg theorem (Proposition 3.20), $\Phi \circ s$ has a fixed point that is a coincidence point for Φ and Ψ. □

As an immediate consequence, and combining Proposition 3.12 on the composition product with the Cellina's approximation theorem, one obtains the following generalization of a coincidence theorem of Browder presented in [16].

Corollary 3.14. Let X, Y be two subsets of two topological vector spaces E, F, respectively, with X convex. Let $\Phi \in \mathbf{K}_c(X, Y)$ be a composition product of a finite number of u.s.c. maps with non-empty closed convex values, and let $\Psi \in \mathbf{F}_c^{-1}(X, Y)$ be a composition product of a finite number of maps whose inverses are Ky Fan maps. If Y is compact or Φ is a compact map, then Φ and Ψ have a coincidence.

Proof. Assume that Φ is compact, that is, $\Phi(X)$ is contained in some compact subset K of Y. By Theorem 3.17, the restriction of Ψ^{-1} to K admits a continuous selection f with an image in a finite dimensional convex polytope C in X. By Cellina's approximation theorem and Proposition 3.10, the restriction of $\Phi|_C$ is a compact-valued u.s.c. approachable map. The polytope C being compact, has the fixed-point property by Brouwer's fixed-point theorem. Theorem 3.23 applies to the restrictions $\Phi|_C$ and $\Psi|_C$, leading to a coincidence between Φ and Ψ. □

A straightforward application of Theorems 3.23, 3.18, and Corollary 3.5 leads to:

Corollary 3.15. Let X be a compact AR and Y be a locally finite simplicial complex. Let $\Phi \in \mathbf{D}(X, |Y|)$ and $\Psi : X \rightrightarrows |Y|$ be a continuous map with open and contractible fibers and open values. Then Φ and Ψ have a coincidence.

Proof. A locally finite simplicial complex being an *ANR*, Corollary 3.5 implies that $\Phi \in \mathbf{A}(X, |Y|)$. On the other hand, Theorem 3.18 asserts that $\Psi^{-1} \in \mathbf{S}(|Y|, X)$. Theorem 3.23 applies as any compact AR has the FPP for continuous mappings (Borsuk theorem). □

Also, Theorems 3.23, 3.17, and Corollary 3.8 imply:

Corollary 3.16. Let X be an arbitrary space and let $Y \in \text{AANR}(\mathcal{K})$ for the class $\mathcal{K}$ of compact spaces with compatible uniformity $\mathcal{V}$. Assume that the Euler characteristic $\chi(Y)$ is non-trivial and let $\Phi, \Psi, \tilde{\Psi} : X \rightrightarrows Y$ be three maps satisfying the following conditions:

(i) $\Phi \in \mathbf{USCL}(X, Y)$;

(ii) $\forall V \in \mathcal{V}, \forall \{x_1, \ldots, x_n\} \subset X$, the set $\bigcap_{i=1}^n V[\Phi(x_i)]$ is empty or contractible;

(iii) $\tilde{\Psi}$ is a multi-selection of Ψ;

(iv) $\tilde{\Psi}$ has open values and non-empty preimages;

(v) $\forall y \in Y, \exists V_y \in \mathcal{V}$ such that the set $\bigcap_{z \in V} \Psi^{-1}(z)$ is contractible for each entourage V of the diagonal in Y with $V \subset V_y$.

Then Φ and Ψ have a coincidence.

Theorem 3.24. Let X be a subset of a topological vector space E, Y be a convex subset of a locally convex topological vector space F, and $\Phi \in (\mathbf{A} \cap \mathbf{CL})(X, Y)$ be a compact map. If for any finite subset N of Y, the restriction of $\Psi^{-1}$to the convex hull of N is a compact-valued approachable u.s.c. map, then Φ and Ψ have a coincidence.

Proof. Given an open neighborhood V of the origin in F, let $N = \{y_i \in \Phi(X), i = 1, \ldots, n\}$ be a finite set such that $\{(y_i + \frac{1}{3}V) \cap K | i = 1, \ldots, n\}$ is an open cover of the compact subset $K = \overline{\Phi(X)}$ of Y. Let $\pi_V : \bigcup_{i=1}^n (y_i + \frac{2}{3}V) \longrightarrow C_V = Conv\{N\}$ be the Schauder projection associated with $(N, \frac{2}{3}V)$. We show that the map $\Phi_V : X \rightrightarrows C_V$ defined by $\Phi_V(x) = (\Phi(x) + \overline{V}) \cap C_V$ is a compact-valued approachable map.

It is indeed clear that Φ_V is a compact-valued u.s.c. map. In order to see that it is approachable, let s be any continuous approximation mapping in $\mathbf{a}(\Phi; U, V\prime)$ where V' is any convex open neighborhood of the origin in V satisfying $V' \subset \frac{1}{3}V$, and let U be an arbitrary open neighborhood of the origin in E. Clearly, $s(X) \subset \bigcup_{i=1}^n (y_i + \frac{2}{3}V)$ and consequently, the function $\pi_V s : X \to C_V$ is well defined and continuous. Moreover, it verifies the containments

$$\begin{aligned}\pi_V s(x) &\in (s(x) + \frac{2}{3}V) \cap C_V \subset (\Phi((x+U) \cap X) + V) \cap C_V \\ &\subset \Phi_V((x+U) \cap X) \text{ for all } x \in X,\end{aligned}$$

that is, $\pi_V s \in \mathbf{a}(\Phi_V; U, W)$ for any open neighborhood of the origin W in F. The set U being arbitrary, it follows that Φ_V is approachable.

By hypothesis, the restriction $\Psi^{-1}|_{C_V}$ is a compact-valued approachable u.s.c. map. By Proposition 3.12, the composition product $\Phi_V \circ \Psi^{-1}|_{C_V} : C_V \rightrightarrows C_V$ is also compact-valued, approachable, and u.s.c. The existence of a fixed point for this product is provided by the generalized Himmelberg theorem (Proposition 3.20). Such a fixed point is an approximate fixed point for the compact map $\Phi \circ \Psi^{-1}$. The usual compactness argument based on Lemma 3.2 ends the proof. □

Remark 3.12. It is clear from the above proof that the local convexity of the space F can be replaced by the admissibility in the sense of Klee (see Section 3.2 for the definition).

As an immediate consequence and in view of Cellina's approximation theorem, we obtain the following generalization of a result of Granas-Liu (see reference in [12]).

Corollary 3.17. Let X be a paracompact subset of a topological vector space E, and let Y be a convex subset of a topological vector space F having sufficiently many linear functionals. Let $\Phi \in \mathbf{K}(X,Y), \Psi^{-1} \in \mathbf{K}(Y,X)$. If Φ is a compact map, then Φ and Ψ have a coincidence.

We illustrate Theorem 3.24 by a non-convex coincidence theorem based on the approachability of upper semicontinuous with $\infty-$proximally connected values (Theorem 3.8 and Corollary 3.4).

Corollary 3.18. Let X be a compact ANR, and Y be a convex subset of a locally convex topological vector space F, $\Phi \in \mathbf{D}(X,Y), \Psi^{-1} \in \mathbf{D}(Y,X)$. Then Φ and Ψ have a coincidence.

We formulate now a coincidence theorem for an infinite collection of approachable maps. We will make use of the fact that an infinite product of compact topological spaces $\{X_i\}_{i\in I}$ has the fixed-point property for continuous functions if and only if, for each finite subset I' of I, the product $X_{I'} = \prod\{X_i : i \in I'\}$ enjoys the same property (see Corollary 11 above).

Theorem 3.25. Let $\{X_i\}_{i\in I}$ be a collection of compact sets each of which is in a uniform space, such that, for each finite subset I' of I, the product $X_{I'} = \prod\{X_i : i \in I'\}$ has the fixed-point property for continuous mappings. For each $i \in I$, let $X^i = \prod\{X_j : j \neq i\}$, and let $\Phi_i \in (\mathbf{A} \cap \mathbf{CL})(X^i, X_i)$. Then $\bigcap_{i\in I} graph(\Phi_i) \neq \emptyset$.

Proof. For any $x \in X = \prod_{i\in I} X_i$, let $x^i = p^i(x)$ be the projection of x onto X^i. It is easy to see that

$$\bigcap\nolimits_{i\in I} graph(\Phi_i) \neq \emptyset \Longleftrightarrow (\exists \bar{x} = (\bar{x}_i) \in X \text{ with } \bar{x}_i \in \Phi_i(\bar{x}^i), \forall i \in I.)$$

By Proposition 3.12 and since X is compact, the composition product $\hat{\Phi}_i = \Phi_i \circ p^i \in (\mathbf{A} \cap \mathbf{CL})(X, X_i)$ for each $i \in I$. Consequently, and by Proposition 3.11, the map $\Phi : X \rightrightarrows X$ defined by $\Phi(x) = \prod_{i\in I} \hat{\Phi}_i(x)$ is in $(\mathbf{A} \cap \mathbf{CL})(X)$. By Corollary 3.11, X also has the FPP for continuous mappings, and Φ has a fixed point, that is, for each $i \in I$, $\bar{x}_i \in \hat{\Phi}_i(\bar{x}) = \Phi_i(\bar{x}^i)$ for some point $\bar{x} \in X$. This completes the proof. □

When the spaces X_i are convex and compact subsets of locally convex spaces, and the maps $\Phi_i \in \mathbf{K}(X^i, X_i)$ are Kakutani maps, we recover a theorem of Granas and Liu (see reference in [12]).

As an illustration, we finally formulate an intersection theorem for non-convex maps based on Corollary 3.6 and on the generalized Schauder's theorem for compact absolute retracts.

Corollary 3.19. Let $\{X_i\}_{i\in I}$ be a collection of compact ARs. For each $i \in I$, let $\Phi_i \in \mathbf{D}(X^i, X_i)$. Then $\bigcap_{i\in I} graph(\Phi_i) \neq \emptyset$.

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Part II

Convex Analysis and Variational Analysis

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Chapter 4

Convexity, Generalized Convexity, and Applications

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4.1 Introduction

The study of convex functions has given rise to one of the most beautiful branches of mathematical analysis, namely convex analysis, which has had a considerable influence on other more abstract areas, such as functional analysis. At the same time, convex functions have been applied to model many problems in engineering, economics, management, etc.

In recent years it has been understood that other, broader classes of functions may be used to provide models for a much more accurate representation of reality. These functions retain some of the characteristics of the convex ones. For instance, in some cases they can guarantee that the Karush-Kuhn-Tucker conditions for the existence of an extremum are not only necessary, but also sufficient for the existence of extrema.

This chapter presents some of the main properties and characterizations of convex and generalized convex functions. Besides the basic definitions, we will also give some examples and treat some properties related to continuity and minimization of such functions. For differentiable functions, we will also provide characterizations of convexity and generalized convexity in terms of monotonicity and generalized monotonicity of the derivatives. Finally, we will present some applications in optimization and economics.

We will not be even remotely exhaustive, since the theory of convexity and generalized convexity is very rich and is described in various comprehensive books. For instance, some basic definitions and properties of convex functions will be presented, but the fundamental notions of subdifferentiability, duality, etc. will not be mentioned. Other important classes of generalized convex functions, as well as vector-valued functions, are omitted. Also, no mention is made of the very important case of nonsmooth functions, to avoid losing the essence of convexity amid technicalities. Many of the ideas and methods related to convexity and generalized convexity are present already in the differentiable case, and it is our aim to present them in this simple framework. We work in general normed spaces, but only occasionally will need some tools of functional analysis, so in most of the cases the reader may suppose that the underlying space is $\mathbb{R}^n$.

4.2 Preliminaries

In what follows, X will be a normed space. When no topology is necessary, as is the case with several (generalized) convexity definitions, one can simply assume that X is a vector space. Given $x, y \in X$, we will denote by $[x, y]$ the line segment $[x, y] = \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$. The segments $(x, y]$, $[x, y)$ and (x, y) are defined analogously. Given a set $K \subseteq X$ and $\lambda > 0$, λK is the set $\{\lambda x : x \in K\}$.

We will denote by X^* the topological dual of X, and use $\langle x^*, x\rangle$ to denote the value of the functional $x^* \in X^*$ at $x \in X$. Given $\rho > 0$ and $x \in X$, $B(x, \rho)$ will be the open ball $\{y \in X : \|y - x\| < \rho\}$ and $\overline{B}(x, \rho)$ the closed ball $\{y \in X : \|y - x\| \leq \rho\}$. A function f defined on $A \subseteq X$ with values in some topological space is called *radially continuous* at an interior point x of A, if its restriction on every line segment through x is continuous at x, that is, for every $v \in X$, $\lim_{t \to 0} f(x + tv) = f(x)$.

Given a set $K \subseteq X$, $\operatorname{int} K$ denotes its interior, $\overline{K}$ its closure, and $\operatorname{co} K$ its convex hull. We assume that the reader is acquainted with the basic properties of convex sets. We only recall that if $y \in \operatorname{int} K$ and $x \in K$ (or $x \in \overline{K}$), then $(x, y) \subseteq \operatorname{int} K$.

The *algebraic interior* of a set K is the set $\operatorname{core} K$ of points $x \in K$ such that for every $e \in X$, there exists $\gamma > 0$ such that $[x - \gamma e, x + \gamma e] \subseteq K$. It is obvious that $\operatorname{int} K \subseteq \operatorname{core} K$, but equality might not hold, even if K is convex. For example, let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in a vector space X that are not equivalent; then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to zero with respect to one of the norms, say $\|x_n\|_1 \to 0$, while $\|x_n\|_2 > 1$. Equip X with $\|\cdot\|_1$ so that it becomes a normed space with the corresponding topology. Set $K = \{x \in X : \|x\|_2 \leq 1\}$. Then K is of course convex, and $0 \in \operatorname{core} K$. However, $0 \notin \operatorname{int} K$ since $x_n \to 0$ but $x_n \notin K$. In Banach spaces, however, we have the following important property, which is a consequence of Baire's theorem.

Proposition 4.1. Let K be a closed convex set in a Banach space X. Then, $\operatorname{int} K = \operatorname{core} K$.

Proof. We have only to prove that $\operatorname{core} K \subseteq \operatorname{int} K$. Let $x_0 \in \operatorname{core} K$. By making a translation, if necessary, we may suppose that $x_0 = 0$. Since $0 \in \operatorname{core} K$, for every $x \in X$ there exists $\gamma > 0$ such that $[-\gamma x, \gamma x] \subseteq K$; thus, there exists $n \in \mathbb{N}$ such that $\frac{1}{n}x \in K$. This means that $X = \bigcup_{n\in\mathbb{N}} (nK)$. By Baire's theorem, there exists $n \in \mathbb{N}$ such that $\operatorname{int}(nK) \neq \emptyset$. It follows easily that $\operatorname{int} K \neq \emptyset$.

Now, fix $y \in \operatorname{int} K$. If $y = 0$, we are done. If $y \neq 0$, using again that $0 \in \operatorname{core} K$, we infer that there exists $\delta > 0$ such that $[-\delta y, \delta y] \subseteq K$. This means that $0 \in (-\delta y, y)$ (just make a drawing!). Since $-\delta y \in K$ and $y \in \operatorname{int} K$, we deduce that $0 \in \operatorname{int} K$. □

Let U be a neighborhood of x and $f : U \to \mathbb{R}$ be a function. f is called *Gâteaux differentiable* at x, if there exists an element $x^* \in X^*$ such that for every $v \in X$,

$$\langle x^*, v\rangle = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}.$$

The element x^* is denoted $\nabla f(x)$ and called the *Gâteaux derivative* of f at x.

4.3 Convex Functions

An elementary definition of the concept of convex function, suitable for first-year calculus, is the following: Given a convex set C in a vector space X, a function $f : C \to \mathbb{R}$ is called *convex* if for every $x, y \in C$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{4.1}$$

In the study of convex functions, it is very convenient to consider functions that can take infinite values. One of the many reasons is that this allows us to consider only functions defined on the whole space, by imposing the value of the function to be $+\infty$ outside the set C. We first give some definitions. The extended real line is the set $\overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty, -\infty\}$. Given a function $f : X \to \overline{\mathbb{R}}$, its *domain* is the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. A function f is called *proper* if $f(x) > -\infty$ for all $x \in X$, and $\operatorname{dom} f \neq \emptyset$. For reasons of simplicity, we will confine ourselves to functions that never take the value $-\infty$.

A function $f : X \to (-\infty, +\infty]$ is called *convex* if, for every $x, y \in X$ and $\lambda \in [0,1]$, relation (4.1) holds. This is equivalent to saying that (4.1) holds for every $x, y \in \operatorname{dom} f$ and $\lambda \in [0,1]$. The function $f \to (-\infty, +\infty]$ is called convex on a set $C \subseteq X$ if its restriction on C satisfies (4.1).

It is clear that the domain of a convex function is a convex set. There are also other sets closely related to the convexity of a function:

Definition 4.1. Given a function f, its epigraph is the set

$$\operatorname{epi} f = \{(x,t) \in X \times \mathbb{R} : t \geq f(x)\}.$$

Given $\alpha \in \mathbb{R}$, its level set and strict level set are, respectively, the sets

$$[f \leq \alpha] = \{x \in X : f(x) \leq \alpha\}$$
$$[f < \alpha] = \{x \in X : f(x) < \alpha\}.$$

It is an easy exercise to verify that a function is convex if and only if its epigraph is convex. If f is convex, then both its level and its strict level sets are convex. However, convexity of the level sets does not guarantee that a function is convex. For instance, the function $f(x) = \arctan x$ is not convex, but its level sets

$$[f \leq \alpha] = \begin{cases} \emptyset, & \alpha \leq -\frac{\pi}{2}, \\ (-\infty, \tan\alpha], & \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ \mathbb{R}, & \alpha \geq \frac{\pi}{2}, \end{cases}$$

are all convex.

One can construct convex functions starting from other convex ones. For instance, we have the following.

Proposition 4.2. (a) Let $f_i : X \to (-\infty, +\infty]$ be convex functions and $\lambda_i > 0$, $\forall i = 1, \ldots k$. Then, the function $\sum_{i=1}^{k} \lambda_i f_i$ is convex.

(b) If $f_i : X \to (-\infty, +\infty]$, $i \in I$ is a family of convex functions, then the function f defined by $f(x) = \sup\{f_i(x) : i \in I\}$ is also convex.

(c) Let Y be a vector space, $f : X \times Y \to (-\infty, +\infty]$ be convex and C be a convex subset of Y. If the function $h(x) = \inf\limits_{y \in C} f(x,y)$ is proper, then it is convex.

(d) If $f : X \to \mathbb{R}$ is convex (respectively, concave) and $g : \mathbb{R} \to \mathbb{R}$ is convex and nondecreasing (respectively, nonincreasing) on the range of f, then $g \circ f$ is convex.

Proof. We omit the easy proofs of (a) and (b). We note, however, that (b) can also be shown by noting that whenever $f = \sup\limits_{i \in I} f_i$, one has $\operatorname{epi} f = \bigcap_{i \in I} \operatorname{epi} f_i$.

We now show (c): For every $x_1, x_2 \in X$, $y_1, y_2 \in C$ and $\lambda \in [0,1]$,

$$\begin{aligned} h(\lambda x_1 + (1-\lambda)x_2) &\le f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ &= f(\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2)) \\ &\le \lambda f(x_1,y_1) + (1-\lambda)f(x_2,y_2). \end{aligned}$$

By taking the infimum over $y_1, y_2 \in C$ in the inequality

$$h(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1,y_1) + (1-\lambda)f(x_2,y_2),$$

we deduce that $h(\lambda x_1+(1-\lambda)x_2) \le \lambda h(x_1)+(1-\lambda)h(x_2)$, that is, h is convex.
The proof of (d) is also straightforward and is omitted. □

There are many examples of convex functions. One of the simplest is the indicator function of a convex set: Given a set $C \subseteq X$, its *indicator function* ι_C is the function

$$\iota_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & x \in X \backslash C. \end{cases}$$

It is easy to see that C is convex if and only if ι_C is convex. Another example is the class of sublinear functions:

Definition 4.2. A function $f : X \to (-\infty, +\infty]$ is called *sublinear* if

(i) for all $x, y \in X$, $f(x+y) \le f(x) + f(y)$ (subadditivity)

(ii) for all $x \in X$ and $\lambda > 0$, $f(\lambda x) = \lambda f(x)$ (positive homogeneity).

Sublinear functions are obviously convex. An example of a sublinear function in a normed space X is the norm $\|x\|$. Using Proposition 4.2(d) we deduce that for every $\alpha > 1$ the function $f(x) = \|x\|^\alpha$ is convex, since the function $g(x) = x^\alpha$, $x \in \mathbb{R}_+$, is convex and nondecreasing.

An important aspect of convexity is related to continuity and semicontinuity. First, we recall that a function $f : X \to (-\infty, +\infty]$ is called locally Lipschitz at $x_0 \in \operatorname{dom} f$, if there exist $\rho > 0$ and $L > 0$ such that for each $x, y \in B(x_0, \rho)$,

$$|f(x) - f(y)| \le L\,\|x - y\|\,.$$

The function f is called *locally Lipschitz* on $A \subseteq \operatorname{dom} f$ if it is locally Lipschitz at every $x_0 \in A$. If f is locally Lipschitz on A, then of course it is continuous on A. One can deduce that f is locally Lipschitz from a surprisingly weak assumption:

Proposition 4.3. If a convex function f is bounded from above at a neighborhood of some point, then it is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$.

Proof. We divide the proof in a few easy steps:

Step 1. If f is bounded from above on some neighborhood $B(x_0, \rho)$, then it is also bounded from below. Indeed, by making a translation if necessary, we can assume that $x_0 = 0$ and $f(0) = 0$. Thus, there exists $M \in \mathbb{R}$ such that for all $x \in B(0,\rho)$, $f(x) \leq M$. Now for every $x \in B(0,\rho)$ one has $-x \in B(0,\rho)$ and $0 = \frac{1}{2}x + \frac{1}{2}(-x)$, so

$$0 = f(0) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x).$$

But $f(-x) \leq M$, so we find that $f(x) \geq -M$ and f is bounded from below on $B(0,\rho)$.

STEP 2. If f is bounded from above on some neighborhood $B(x_0,\rho)$, then for each $y \in \operatorname{int}\operatorname{dom} f$, f is bounded from above on some neighborhood of y. Indeed, assume that $f(x) \leq M$ for all $x \in B(x_0,\rho)$ and let $y \in \operatorname{int}\operatorname{dom} f$. Then there exists $x_1 \in \operatorname{dom} f$ such that $y \in (x_1, x_0)$; fix this x_1. Then $y = (1-\lambda)x_1 + \lambda x_0$ for some $\lambda \in (0,1)$. For every $y' \in B(y,\lambda\rho)$ there exists $x' \in X$ such that $y' = (1-\lambda)x_1 + \lambda x'$ (just solve with respect to x'). Then $y' - y = \lambda(x' - x_0)$, so $\|x' - x_0\| = \frac{\|y'-y\|}{\lambda} < \rho$. We deduce that $f(x') < M$ and consequently

$$f(y') \leq (1-\lambda)f(x_1) + \lambda f(x') \leq (1-\lambda)f(x_1) + \lambda M.$$

Thus, f is bounded from above on $B(y,\lambda\rho)$.

STEP 3. f is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$. Indeed, let $x_0 \in \operatorname{int}\operatorname{dom} f$. By using Steps 1 and 2, we deduce the existence of some neighborhood $B(x_0,\rho)$ such that f has an upper bound M and a lower bound m on $B(x_0,\rho)$. For every distinct $x, y \in B(x_0, \frac{\rho}{2})$, take z such that $y \in]x,z[$ and $\|y - z\| = \frac{\rho}{2}$. Then $\|z - x\| > \frac{\rho}{2}$, $z \in B(x_0,\rho)$, and $y = (1-\lambda)x + \lambda z$ for some $\lambda \in (0,1)$. It follows that

$$\begin{aligned} f(y) &\leq (1-\lambda)f(x) + \lambda f(z) \Rightarrow \\ f(y) - f(x) &= \lambda(f(z) - f(x)) \leq \lambda(M - m). \end{aligned} \tag{4.2}$$

But $y - x = \lambda(z - x)$, so $\|y - x\| = \lambda \|z - x\| > \frac{\lambda\rho}{2}$. We deduce from (4.2) that $f(y) - f(x) \leq 2(M-m)\|y - x\|/\rho$. The same inequality holds for $f(x) - f(y)$, so finally, $|f(y) - f(x)| \leq 2(M - m)\|y - x\|/\rho$ for all $x, y \in B(x_0, \frac{\rho}{2})$ and f is locally Lipschitz at x_0. □

In a finite dimensional space, Proposition 4.3 implies continuity on $\operatorname{int}\operatorname{dom} f$:

Corollary 4.1. Every proper convex function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is continuous on $\operatorname{int}\operatorname{dom} f$.

Proof. Let $x_0 \in \operatorname{int}\operatorname{dom} f$. Then there exist a finite number of points $x_1, \dots x_k$ in $\operatorname{dom} f$ such that $x_0 \in \operatorname{int}\operatorname{co}\{x_1, \dots x_k\}$; for instance, one can consider the vertices of the ball $\{x \in X : \|x - x_0\|_\infty < \varepsilon\}$ with respect to the sup norm

$$\|x\| = \max\{|\alpha_1|, \dots, |\alpha_n|\}, \quad x = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n.$$

Every $x \in \operatorname{co}\{x_1, \dots x_k\}$ can be written as $x = \sum_{i=1}^k \lambda_i x_i$ with $\lambda_i \in [0,1]$, so $f(x) \le \sum_{i=1}^k \lambda_i f(x_i)$ and f is bounded from above in a neighborhood of x_0. According to Proposition 4.3, this implies that f is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$. □

A more general kind of continuity, is lower semicontinuity. We recall that a function $f : X \to (-\infty, +\infty]$ is called *lower semicontinuous* (lsc) at x_0 if $f(x_0) \le \liminf\limits_{x \to x_0} f(x)$. Equivalently, f is lsc at x_0 if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x_0$ and the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges, one has $f(x_0) \le \lim f(x_n)$. The function f is called lsc if it is lsc at every point $x \in X$. Just as the convexity of the epigraph characterizes the convexity of a function f, its closedness characterizes the lower semicontinuity of f. In contrast with convexity, lower semicontinuity is also characterized by the closedness of the level sets:

Proposition 4.4. Let $f : X \to (-\infty, +\infty]$ be a function. The following statements are equivalent:

(a) f is lsc;

(b) $\operatorname{epi} f$ is a closed set in $X \times \mathbb{R}^n$;

(c) $[f \le \alpha]$ is closed in X for every $\alpha \in \mathbb{R}$.

Proof. (a) ⇒ (b): Let $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ be a sequence in $\operatorname{epi} f$ such that $(x_n, t_n) \to (x_0, t_0)$. By taking a subsequence if necessary, we may assume that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges. From $f(x_n) \le t_n$ we find

$$f(x_0) \le \lim f(x_n) \le \lim t_n = t_0.$$

This means that $(x_0, t_0) \in \operatorname{epi} f$ and $\operatorname{epi} f$ is closed.

(b) ⇒ (a): Let $\{x_n\}_{n\in\mathbb{N}}$ be such that $x_n \to x_0$ and $\{f(x_n)\}_{n\in\mathbb{N}}$ converges. We want to show that $f(x_0) \le \lim f(x_n)$, so we may assume that $\lim f(x_n) \in \mathbb{R}$. Then $(x_n, f(x_n)) \in \operatorname{epi} f$, so by closedness $(x_0, \lim f(x_n)) \in \operatorname{epi} f$, that is, $f(x_0) \le \lim f(x_n)$. Since this is true for every such sequence $\{x_n\}_{n\in N}$, we deduce that f is lsc.

(a) ⇒ (c) and (c) ⇒ (a) can be proven similarly. □

In infinite dimensional spaces, we do not have the analog of Corollary 4.1. But we do have the following.

Proposition 4.5. Let X be a Banach space. If $f : X \to (-\infty, +\infty]$ is convex and lsc, then it is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$.

The proof of the proposition is postponed until the next section, after Proposition 4.13. We will also see that the assumption that X is a Banach space is essential.

For differentiable functions f, one can characterize their convexity by using their derivative.

Proposition 4.6. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is convex.

(b) For every $x, y \in \operatorname{dom} f$, $\langle \nabla f(x), y - x\rangle \le f(y) - f(x)$.

(c) For every $x, y \in \operatorname{dom} f$, $\langle \nabla f(y) - \nabla f(x), y - x\rangle \ge 0$.

Before providing the proof, let us note that convexity, and all notions of generalized convexity to be encountered later, are defined as an essentially one-dimensional property: It says something about the behavior of a function on straight lines. A function f is convex if and only if its restriction on any line segment is convex. More formally, f is convex if and only if for every $x_0 \in X$ and every unit vector $e \in X$, the function $g(t) := f(x_0 + te)$ is convex. The domain of such a function g is an open interval in $\mathbb{R}$, possibly empty. Its derivative is $g'(t) = \langle \nabla f(x_0 + te), e\rangle$. If x and y in $\operatorname{dom} f$ belong to the line $\{x_0 + te : t \in \mathbb{R}\}$, say $x = x_0 + te$ and $y = x_0 + se$, then property (b) says that $g'(t)(s - t) \le g(s) - g(t)$ and property (c) says that $(g'(s) - g'(t))\,(s - t) \ge 0$. Thus, in order to prove the proposition, it is enough to prove it for functions $g : \mathbb{R} \to (-\infty, +\infty]$.

Proof. We assume without loss of generality that $f : \mathbb{R} \to (-\infty, +\infty]$.

(a) $\Rightarrow$ (b): We want to show that $f'(t)(s - t) \le f(s) - f(t)$ for all $t, s \in \operatorname{dom} f$. Let $s \ne t$. For every r in the open interval with endpoints s and t, r can be written as $r = (1 - \lambda)t + \lambda s$ for some $\lambda \in (0, 1)$. Then by using convexity of f and $r - t = \lambda(s - t)$, we find immediately

$$\frac{f(r) - f(t)}{r - t}(s - t) \le f(s) - f(t).$$

Thus,

$$f'(t)(s - t) = (s - t)\lim_{r\to t}\frac{f(r) - f(t)}{r - t} \le f(s) - f(t).$$

(b) $\Rightarrow$ (c): For $s, t \in \operatorname{dom} f$ we have $f'(t)(s - t) \le f(s) - f(t)$ and (interchanging s and t), $f'(s)(t - s) \le f(t) - f(s)$. By adding these two inequalities we obtain $(f'(t) - f'(s))(t - s) \ge 0$.

(c) $\Rightarrow$(a): Condition (c), that is, $(f'(t) - f'(s))(t - s) \ge 0$ for all $s, t \in \operatorname{dom} f$, means simply that f' is nondecreasing. Let $s, t \in \operatorname{dom} f$ be given with, say, $t < s$, and $r = (1 - \lambda)t + \lambda s$ for some $\lambda \in (0, 1)$. By using the mean value theorem twice, we deduce that there exist u, v such that $t < u < r < v < s$ such that

$$f'(u) = \frac{f(r) - f(t)}{r - t}, \qquad f'(v) = \frac{f(s) - f(r)}{s - r}. \tag{4.3}$$

Since f' is nondecreasing, $f'(u) \leq f'(v)$. Using (4.3), $r - t = \lambda(s - t)$ and $s - r = (1 - \lambda)(s - t)$, we immediately obtain that $f(r) \leq (1 - \lambda)f(t) + \lambda f(s)$, that is, f is convex. □

We recall that a function $f : X \to (-\infty, +\infty]$ is called *strictly convex* if for all $x, y \in \operatorname{dom} f$ with $x \neq y$, and all $\lambda \in (0, 1)$, one has

$$f(x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

A convex function is strictly convex if and only if its graph does not contain line segments, that is, if $x, y \in \operatorname{dom} f$, $x \neq y$, then the line segment $[(x, f(x)), (y, f(y))]$ is not included in the graph of f. For instance, $f(x) = \|x\|$ is convex but not strictly convex.

We have a characterization of differentiable strictly convex functions, as in the convex case:

Proposition 4.7. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is strictly convex.

(b) For every distinct $x, y \in \operatorname{dom} f$, $\langle \nabla f(x), y - x \rangle < f(y) - f(x)$.

(c) For every distinct $x, y \in \operatorname{dom} f$, $\langle \nabla f(y) - \nabla f(x), y - x \rangle > 0$.

Proof. (a) ⇒ (b): Since f is strictly convex, it is convex. For $x, y \in \operatorname{dom} f$, $x \neq y$, set $z = \frac{1}{2}x + \frac{1}{2}y$. Using Proposition 4.6 we find

$$\langle \nabla f(x), y - x \rangle = 2 \langle \nabla f(x), z - x \rangle \leq 2(f(z) - f(x)) < f(y) - f(x).$$

The implications (b) ⇒ (c) and (c) ⇒ (a) can be shown with the same arguments as the corresponding implications in Proposition 4.6. □

In Propositions 4.6(c) and 4.7(c) the Gâteaux derivative exhibits a property that is characteristic of monotonicity:

Definition 4.3. Let $C \subseteq X$ and $T : C \to X^*$ be an operator. The operator T is called

- *monotone* if for every $x, y \in C$, $\langle T(y) - T(x), y - x \rangle \geq 0$;
- *strictly monotone* if for every distinct $x, y \in C$, $\langle T(y) - T(x), y - x \rangle > 0$.

Thus, a differentiable function f is convex (strictly convex) if and only if its derivative is a monotone (strictly monotone) operator. This property can be extended, as we will see, to generalized convex functions.

4.4 Quasiconvex Functions

A function $f : X \to (-\infty, +\infty]$ is called *quasiconvex* if for every $x, y \in X$ and $z \in [x, y]$,

$$f(z) \leq \max\{f(x), f(y)\}. \tag{4.4}$$

Every convex function is obviously quasiconvex. The converse is not true. For instance, the function $\arctan x$ is quasiconvex without being convex.

It is clear that the domain of a quasiconvex function is convex. Its level sets are also convex. In fact, this property characterizes quasiconvexity, as shown by the following proposition whose easy proof is left to the reader:

Proposition 4.8. Let $f : X \to (-\infty, +\infty]$ be a function. The following statements are equivalent:

(a) f is quasiconvex.

(b) The level sets $[f \leq \alpha]$ are convex for all $\alpha \in \mathbb{R}$.

(c) The strict level sets $[f < \alpha]$ are convex for all $\alpha \in \mathbb{R}$.

As for convex functions, we note that the definition of quasiconvexity through (4.4) is in some sense one-dimensional; a function f is quasiconvex if and only if its restriction on straight lines is quasiconvex. More precisely, $f : X \to (-\infty, +\infty]$ is quasiconvex if and only if for every $x \in X$ and every unit vector $e \in X$, the function $g : \mathbb{R} \to (-\infty, +\infty]$ defined by $g(t) = f(x+te)$ is quasiconvex. For this reason, functions defined on $\mathbb{R}$ are important in order to understand the structure of quasiconvex functions on higher dimensional spaces. We have the following proposition, whose easy proof is also omitted:

Proposition 4.9. A function $\mathbb{R} \to (-\infty, +\infty]$ is quasiconvex if and only if there exists an interval I of the form $(-\infty, t)$ or $(-\infty, t]$, where $t \in [-\infty, +\infty]$, such that f is nonincreasing on I and nondecreasing on its complement.

Note that I or its complement might be empty, that is, the function may be nondecreasing or nonincreasing throughout $\mathbb{R}$. The function $-|x|$ is not quasiconvex since it is not of this form. On the other hand, a function such as

$$f(x) = \begin{cases} 0, & |x| \leq 1, \\ x^2, & 1 < |x| \leq 2, \\ 5, & |x| > 2, \end{cases} \tag{4.5}$$

is quasiconvex since it is nonincreasing in $(-\infty, 0]$ and nondecreasing in $(0, \infty)$.

If C is a convex subset of X, a function $f : C \to \mathbb{R}$ is called *quasiconvex on* C if (4.4) holds for every $x, y \in C$ and $z \in [x, y]$. A function $f : C \to \mathbb{R}$ is called *quasiconcave on* C if $-f$ is quasiconvex on C. It is called *quasilinear on* C if f is quasiconvex and quasiconcave on C.

The following easy proposition gives some operations that preserve quasiconvexity.

Proposition 4.10. (a) If $f_i : X \to (-\infty, +\infty]$, $i \in I$, is a family of quasiconvex functions, then the function f defined by $f(x) = \sup\{f_i(x) : i \in I\}$ is also quasiconvex.

(b) If $f : C \to \mathbb{R}$ is quasiconvex and $g : f(C) \to \mathbb{R}$ is nondecreasing, then $g \circ f$ is quasiconvex.

(c) Let Y be a vector space, $f : X \times Y \to (-\infty, +\infty]$ be quasiconvex, and $C \subseteq Y$ be convex. If the function $h(x) = \inf_{y \in C} f(x, y)$ is proper, then it is quasiconvex.

The proofs are easy and are omitted. For part (c), one can use the same arguments as in the proof of Proposition 4.2 (c).

The sum of two quasiconvex functions need not be quasiconvex. Even if we add a linear function to a quasiconvex one, the result might not be quasiconvex. For instance, $-\frac{1}{x}$ and $-x$ are quasiconvex on $(0, +\infty)$, but $-\frac{1}{x} - x$ is not. In fact, among all functions, only the convex ones have this property [10]:

Proposition 4.11. Let $f : X \to (-\infty, +\infty]$ be a function. If $f + x^*$ is quasiconvex for every $x^* \in X^*$, then f is convex.

Proof. Given x, $y \in \operatorname{dom} f$, choose $x^* \in X^*$ such that

$$\langle x^*, x - y \rangle = f(y) - f(x). \tag{4.6}$$

This is always possible due to the Hahn-Banach theorem. Note that (4.6) implies that $f(x) + \langle x^*, x \rangle = f(y) + \langle x^*, y \rangle$, thus the quasiconvex function $f + x^*$ takes the same value at x and y. It follows that for every $z = \lambda x + (1 - \lambda)y$, $\lambda \in [0, 1]$,

$$\begin{gathered} f(z) + \langle x^*, z \rangle \le f(x) + \langle x^*, x \rangle \Rightarrow \\ f(z) + \lambda \langle x^*, x \rangle + (1 - \lambda) \langle x^*, y \rangle \le f(x) + \langle x^*, x \rangle \Rightarrow \\ f(z) \le f(x) + (1 - \lambda) \langle x^*, x - y \rangle = \lambda f(x) + (1 - \lambda) f(y). \end{gathered}$$

Hence, f is convex. ∎

For differentiable functions, quasiconvexity can be characterized as follows.

Proposition 4.12. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is quasiconvex.

(b) For every $x, y \in \operatorname{dom} f$,

$$\langle \nabla f(x), y - x \rangle > 0 \Rightarrow f(y) > f(x).$$

(b′) For every $x, y \in \operatorname{dom} f$,

$$\langle \nabla f(x), y - x \rangle > 0 \Rightarrow f(y) \geq f(x).$$

(c) For every $x, y \in \operatorname{dom} f$,

$$\langle \nabla f(x), y - x \rangle > 0 \Rightarrow \langle \nabla f(y), y - x \rangle \geq 0.$$

Proof. As in the proof of Proposition 4.6, we may suppose without loss of generality that $X = \mathbb{R}$.

(a) ⇒ (b): We show the contrapositive, that is, $f(s) \leq f(t) \Rightarrow f'(t)(s-t) \leq 0$. For every $r = \lambda t + (1-\lambda)s$, $\lambda \in (0,1)$, we have $f(r) \leq \max\{f(s), f(t)\} = f(t)$, so

$$\frac{f(r) - f(t)}{r - t}(s - t) = \frac{f(r) - f(t)}{1 - \lambda} \leq 0.$$

Thus,

$$f'(t)(s - t) = \lim_{r \to t} \frac{f(r) - f(t)}{r - t}(s - t) \leq 0.$$

(b) ⇒ (b′) is trivial.

(b′) ⇒ (b): Assume that (b′) holds. Let $s, t \in \operatorname{dom} f$ be such that $f'(t)(s-t) > 0$. If, say, $s > t$, then $f'(t) > 0$, so by using the definition of the derivative, $f(r) > f(t)$ for all $r \in (t, s)$ sufficiently close to t. Choose such an r. By the mean value theorem, there exists $w \in (t, r)$ such that $f'(w)(r - t) = f(r) - f(t) > 0$. Hence $f'(w)(s - w) > 0$. Condition (b′) entails $f(s) \geq f(w) > f(t)$. The case $s < t$ is similar.

(b) ⇒ (c): If (c) is not true, then it would be possible to have $f'(t)(s-t) > 0$ and $f'(s)(s-t) < 0$ for some $s, t \in \operatorname{dom} f$. However, the first inequality implies $f(s) > f(t)$ and the second $f(t) > f(s)$, a contradiction.

(c) ⇒ (a): Assume that f is not quasiconvex. Then there would exist $t, s \in \operatorname{dom} f$, $t < s$ and $r = \lambda t + (1-\lambda)s$, $\lambda \in (0,1)$ such that $f(r) > \max\{f(t), f(s)\}$. Then it is clear by the mean value theorem, that there exist $u \in (t, r)$ and $v \in (r, s)$ such that $f'(u) > 0$ and $f'(v) < 0$. This means that $f'(u)(v-u) > 0$ and $f'(v)(v - u) < 0$, which contradicts (c). □

Quasiconvexity does not imply continuity under assumptions as weak as in the case of convex functions (see Proposition 4.3 and Corollary 4.1). However, there is still some connection to continuity and boundedness as shown below.

Proposition 4.13. [1] Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be lsc and quasiconvex. Then, for each $x_0 \in \operatorname{int} \operatorname{dom} f$, f is bounded from above in a neighborhood of x_0.

Proof. Let $\varepsilon > 0$ be such that $\overline{B}(x_0, \varepsilon) \subseteq \operatorname{dom} f$. Set $S_n = \overline{B}(x_0, \varepsilon) \cap [f \leq n]$. Since f is quasiconvex and lsc, S_n are convex and closed. In addition, $\bigcup_{n \in \mathbb{N}} S_n = \overline{B}(x_0, \varepsilon)$. By Baire's theorem, there exists $n \in \mathbb{N}$ such that $\operatorname{int} S_n \neq \emptyset$. Fix any $x_1 \in \operatorname{int} S_n$ and any $x_2 \neq x_0$ such that $x_2 \in \overline{B}(x_0, \varepsilon)$ and $x_0 \in [x_1, x_2]$. Choose $n_1 > \max\{n, f(x_2)\}$. Then $x_2 \in [f \leq n_1]$ so $x_2 \in S_{n_1}$. Since $x_1 \in \operatorname{int} S_{n_1}$, we deduce that $x_0 \in \operatorname{int} S_{n_1}$. Accordingly, $\operatorname{int} S_{n_1}$ is a neighborhood of x_0, and$f$ is bounded by n_1 on this neighborhood. □

Note that in a finite dimensional space, Proposition 4.13 can be shown by a constructive proof, analogous to the proof of Corollary 4.1.

Having this result, we can now provide a proof for Proposition 4.5.

Proof. (*of Proposition 4.5*). Since f is convex, it is also quasiconvex. By Proposition 4.13, it is bounded from above at a neighborhood of every point of $\operatorname{int} \operatorname{dom} f$. By Proposition 4.3, it is locally Lipschitz. □

An lsc quasiconvex function is not necessarily continuous, even in finite dimensions. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$ is quasiconvex, lsc, but not continuous. We have the following result, which is compared with Proposition 4.5 [17].

Proposition 4.14. Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be quasiconvex and lsc. If f is radially continuous at some point $x_0 \in \operatorname{dom} f$, then it is continuous at x_0.

Proof. Since f is lsc at x_0, it is enough to show that it is also upper semicontinuous at x_0. This means that given $\varepsilon > 0$, we have to show that there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ one has $f(x) < f(x_0) + \varepsilon$. Indeed, set $\alpha = f(x_0) + \varepsilon$. The set $[f \leq \alpha]$ is convex and closed. In addition, since f is radially continuous and $\alpha > f(x_0)$, for every $e \in X$, one can find $\gamma > 0$ such that the open line segment $(x - \gamma e, x + \gamma e)$ is in $[f \leq \alpha]$. Hence, x_0 is in the algebraic interior of $[f \leq \alpha]$. By Proposition 4.1, the interior and the algebraic interior of a closed convex set in a Banach space coincide; hence, $x_0 \in \operatorname{int}[f \leq \alpha]$. This means exactly what we wanted to prove, that is, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ one has $f(x) < f(x_0) + \varepsilon$. □

For general functions, it is well known that even in $\mathbb{R}^2$, Gâteaux differentiability does not imply continuity or even semicontinuity. However, for quasiconvex functions, there is a connection [17]:

Proposition 4.15. Assume that $f : X \to (-\infty, +\infty]$ is quasiconvex and Gâteaux differentiable on $\operatorname{dom} f$. Then, f is lsc on $\operatorname{dom} f$. If in addition X is a Banach space, then f is continuous on $\operatorname{dom} f$.

Proof. We have to prove that $[f \leq \alpha]$ is closed as a subset of $\operatorname{dom} f$, for every $\alpha \in \mathbb{R}$. If this is not the case, then there exists a sequence $x_k \in [f \leq \alpha]$, $k \in \mathbb{N}$ converging to $x \in \operatorname{dom} f$, $x \notin [f \leq \alpha]$. For every $y \in \operatorname{dom} f \backslash [f \leq \alpha]$, one has

$f(x_k) < f(y)$, so by Proposition 4.12, $\langle \nabla f(y), x_k - y\rangle \le 0$. Taking the limit we obtain

$$\langle \nabla f(y), x - y\rangle \le 0, \quad \forall y \in \operatorname{dom} f \backslash [f \le \alpha]. \tag{4.7}$$

Fix $z \in [f \le \alpha]$ and set $z_t = tx + (1-t)z$ and $g(t) = f(z_t)$, $t \in [0,1]$. Define

$$t_0 = \sup\{t \in [0,1] : g(t) \le \alpha\}.$$

Note that g is differentiable, and thus continuous, so $g(t_0) \le \alpha$. For every $t \in (t_0, 1)$, one has $f(z_t) = g(t) > a$. Using (4.7) we find

$$g'(t) = \langle \nabla f(z_t), x - z\rangle = \frac{1}{1-t}\langle \nabla f(z_t), x - z_t\rangle \le 0.$$

This means that g is nonincreasing on $[t_0, 1]$, so $g(t_0) \ge g(1) = f(x) > \alpha$, which is a contradiction.

The second part follows from the previous proposition and the obvious fact that a Gâteaux differentiable function is radially continuous. □

Propositions 4.5, 4.13, and 4.14 do not hold in general normed spaces, as seen in the following example.

Example 4.1. Let X be the space of real continuous functions on $[0,1]$. We equip X with the norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$. Note that X is not a Banach space. Define the function $g : X \to \mathbb{R}$ by

$$g(f) = \|f\|_2 = \left(\int_0^1 f^2(x)dx\right)^{1/2}.$$

Since g is a norm, it is sublinear, and thus convex. For every $h \in X$ and $f \in X\backslash\{0\}$, an easy calculation gives

$$\lim_{t\to 0} \frac{g(f+th) - g(f)}{t} = \frac{1}{\|f\|_2}\int_0^1 f(x)h(x)dx.$$

Given $f \in X\backslash\{0\}$, the functional $F(h) = \frac{1}{\|f\|_2}\int_0^1 f(x)h(x)dx$ is linear and continuous on X, since

$$|F(h)| \le \frac{\max_{x\in[0,1]} f(x)}{\|f\|_2}\|h\|_1 .$$

Hence, g is Gâteaux differentiable, so it is radially continuous. It is also lsc by Proposition 4.15. However, g is nowhere continuous since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent. Indeed, one can easily construct a sequence $h_k \in X$ such that $\|h_k\|_1 \to 0$ (thus $h_k \to 0$ with our choice of norm on X) and $\|h_k\|_2 \to +\infty$. It follows that g is not continuous at 0, and not bounded from above in a neighborhood of 0. The same is true at every $f \in X$ since $f + h_k \to f$ but

$$g(f+h_k) = \|f+h_k\|_2 \ge \|h_k\|_2 - \|f\|_2 \to +\infty.$$

Thus, the assumption that X is a Banach space, cannot be dropped from Propositions 4.5, 4.13, and 4.14.

As for convex functions, there is a notion of strict quasiconvexity. There is also another, weaker notion:

Definition 4.4. A function $f : X \to (-\infty, +\infty]$ is called

- *strictly quasiconvex* if for all $x, y \in \operatorname{dom} f$ and all $z \in (x, y)$ one has
$$f(z) < \max\{f(x), f(y)\};$$
- *semistrictly quasiconvex* if $\operatorname{dom} f$ is convex and for all $x, y \in \operatorname{dom} f$ and all $z \in (x, y)$ the following implication holds:
$$f(x) < f(y) \Rightarrow f(z) < f(y).$$

A function f is called *strictly* (respectively, *semistrictly*) *quasiconcave* if $-f$ is strictly (respectively, semistrictly) quasiconvex. It is called *strictly* (respectively, *semistrictly*) *quasilinear* if both f and $-f$ are strictly (resp., semistrictly) quasiconvex.

Obviously, a strictly quasiconvex function is semistrictly quasiconvex and quasiconvex. In general, a semistrictly quasiconvex function might not be quasiconvex; the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$, is such an example. However, this assertion is true under lower semicontinuity:

Proposition 4.16. Let f be a semistrictly quasiconvex function. If f is lsc (or radially continuous) on $\operatorname{dom} f$, then f is quasiconvex.

Proof. Assume that f is not quasiconvex, so there exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(z) > \max\{f(x), f(y)\}$. Since f is semistrictly quasiconvex, $\operatorname{dom} f$ is convex, so $z \in \operatorname{dom} f$. Choose α such that $f(z) > \alpha > \max\{f(x), f(y)\}$. Since f is lsc at z, we can find a neighborhood $B(z, \rho)$ such that for every $w \in B(z, \rho)$ one has $f(w) > \alpha$. Choose $w \in B(z, \rho) \cap (x, z)$. Then $z \in (w, y)$ and $f(y) < \alpha < f(w)$ thus by semistrict quasiconvexity $f(z) < f(w)$. On the other hand, $w \in (x, z)$ and $f(x) < f(z)$ entail $f(w) < f(z)$, which is a contradiction. The case of radial continuity is similar. □

The relation between the three notions is made clear by the following proposition. Let us first show a simple lemma.

Lemma 4.1. Let f be quasiconvex. If for some $x, y \in X$ and $z \in (x, y)$ one has $f(x) \leq f(z) = f(y)$, then f is constant on $[x, z]$ or on $[z, y]$.

Proof. If this is not true, then quasiconvexity of f implies that there would exist $u \in [x, z]$ and $v \in [z, y]$ such that $f(u) < f(z)$ and $f(v) < f(z)$. But $z \in [u, v]$, so we should have $f(z) \leq \max\{f(u), f(v)\}$, a contradiction. □

Proposition 4.17. Let f be quasiconvex.

(a) f is strictly quasiconvex if and only if it is not constant on any line segment in $\operatorname{dom} f$.

(b) f is semistrictly quasiconvex if and only if for every distinct $x, y \in \operatorname{dom} f$, if f is constant on $[x, y]$, then

$$f(z) \geq f(x), \quad \forall z = \lambda x + (1-\lambda)y, \quad \lambda \in \mathbb{R}.$$

Proof. (a) It is clear that if f is strictly quasiconvex, then it cannot be constant on any line segment $[x, y]$, $x \neq y$, $x, y \in \operatorname{dom} f$. Conversely, assume that f is not strictly quasiconvex. Then there exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(x) \leq f(y)$ and $f(z) = f(y)$. By Lemma 4.1, f is constant on $[x, z]$ or $[z, y]$.

(b) Assume that f is semistrictly quasiconvex. If f is constant on some line segment $[x, y]$, $x \neq y$, $x, y \in \operatorname{dom} f$, then for every $z = \lambda x + (1-\lambda)y$, one of these three possibilities holds: $\lambda < 0$, $\lambda \in [0, 1]$, or $\lambda > 1$. If $\lambda \in [0, 1]$, then $f(z) = f(x)$ by assumption. If $\lambda < 0$, then $x \in (z, y)$. If we assume that $f(z) < f(x)$, then $z \in \operatorname{dom} f$; semistrict quasiconvexity implies that $f(x) < f(y)$, a contradiction. Thus, $f(z) \geq f(x)$. The case $\lambda > 1$ is similar.

Conversely, if f is not semistrictly quasiconvex, then there would exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(x) < f(z) = f(y)$. Lemma 4.1 then implies that f is constant on $[z, y]$, while $f(x) < f(z)$ with $x = \lambda z + (1-\lambda)y$ for some $\lambda \in \mathbb{R}$. □

A function f that is quasiconvex and semistrictly quasiconvex is often called *explicitly quasiconvex.*

Proposition 4.18. A function f is explicitly quasiconvex if and only if for every $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ the following implication holds:

$$f(z) \geq f(x) \Rightarrow f(y) \geq f(z). \tag{4.8}$$

Proof. It is easy to see that (4.8) implies that f is quasiconvex and semistrictly quasiconvex. Conversely, assume that f is quasiconvex and semistrictly quasiconvex, and assume that for some $x, y \in \operatorname{dom} f$ and $z \in (x, y)$, $f(z) \geq f(x)$. Assume that $f(y) < f(z)$. Since f is quasiconvex, $f(z) \leq \max\{f(x), f(y)\}$, so necessarily $f(z) = f(x)$. Then $f(y) < f(x)$, so semistrict quasiconvexity implies that $f(z) < f(x)$, a contradiction. □

Based on Propositions 4.9 and 4.17, one can see immediately which functions defined on $\mathbb{R}$ are quasiconvex, strictly quasiconvex, or semistrictly quasiconvex. For instance, the quasiconvex function defined in (4.5) is neither strictly quasiconvex, nor semistrictly quasiconvex; compare also with Propositions 4.28 and 4.29 below. The function

$$f(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1 - \frac{1}{1+(|x|-1)^2}, & \text{if } |x| > 1, \end{cases} \tag{4.9}$$

is semistrictly quasiconvex, but not strictly quasiconvex.

In higher dimension, Propositions 4.9 and 4.17 may help to prove or disprove that a function is (strictly, semistrictly) quasiconvex, but in general this is not so immediate. An important example of a quasiconvex function in a subset of $\mathbb{R}^n$ is the function f defined by the fraction of two affine functions

$$f(x) = \frac{\langle a, x\rangle + k}{\langle b, x\rangle + l}.$$

Restricted on the set $\{x \in \mathbb{R}^n : \langle b, x\rangle + l > 0\}$, the function f is semistrictly quasilinear [6].

Strictly and semistrictly quasiconvex differentiable functions have a first-order characterization in terms of generalized monotone operators.

Definition 4.5. Let $C \subseteq X$ be convex and $T : C \to X^*$ be an operator. T is called

- *quasimonotone* if for every $x, y \in C$ the following implication holds:

$$\langle T(x), y - x\rangle > 0 \Rightarrow \langle T(y), y - x\rangle \geq 0;$$

- *strictly quasimonotone* if T is quasimonotone and for every distinct $x, y \in C$, there exists $z \in (x, y)$ such that

$$\langle T(z), y - x\rangle \neq 0;$$

- *semistrictly quasimonotone* if T is quasimonotone and for every $x, y \in C$ the following implication holds:

$$\langle T(x), y - x\rangle > 0 \Rightarrow \forall w \in (x, y), \exists z \in (w, y) : \langle T(z), y - x\rangle > 0. \quad (4.10)$$

If T is radially continuous, then the assumption of quasimonotonicity is not necessary in the definition of semistrict quasimonotonicity:

Proposition 4.19. Let $C \subseteq X$ be convex. If T satisfies implication (4.10) and is radially continuous, then it is semistrictly quasimonotone.

Proof. We have to show that T is quasimonotone. Assume that $\langle T(x), y - x\rangle \geq 0$ for some $x, y \in C$. We construct a sequence z_k, $k \in \mathbb{N}$ such that

$$\langle T(z_k), y - x\rangle > 0 \quad (4.11)$$

and $z_k \to y$, as follows. Choose $z_1 \in \left(\frac{x+y}{2}, y\right)$ such that $\langle T(z_1), y - x\rangle > 0$; this implies $\langle T(z_1), y - z_1\rangle > 0$. Then we choose $z_2 \in \left(\frac{z_1+y}{2}, y\right)$ such that $\langle T(z_2), y - z_1\rangle > 0$, that is, $\langle T(z_2), y - x\rangle > 0$, etc. By taking the limit at (4.11), we find that $\langle T(y), y - x\rangle \geq 0$, that is, T is quasimonotone. ◻

Proposition 4.20. If T is strictly quasimonotone, then it is semistrictly quasimonotone.

Proof. Let T be strictly quasimonotone and $x, y \in C$ be such that $\langle T(x), y - x\rangle > 0$. Let $w \in (x, y)$. By assumption, there exists $z \in (w, y)$ such that $\langle T(z), y - w\rangle \neq 0$. This implies that $\langle T(z), z - x\rangle \neq 0$. Since we also have $\langle T(x), z - x\rangle > 0$, quasimonotonicity implies that $\langle T(z), z - x\rangle \geq 0$, hence $\langle T(z), z - x\rangle > 0$. Finally, this gives $\langle T(z), y - x\rangle > 0$. □

According to Proposition 4.12, a differentiable function f is quasiconvex if and only if ∇f is quasimonotone. We now have the required characterizations of strict and semistrict quasiconvexity.

Proposition 4.21. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is strictly quasiconvex.

(b) ∇f is strictly quasimonotone.

Proof. As in Proposition 4.6, it is enough to prove the proposition by assuming that $X = \mathbb{R}$. If f is strictly quasiconvex, then by Proposition 4.12, f' is quasimonotone. In addition, by Proposition 4.17, f is not constant on any interval $[s, t]$ in $\operatorname{dom} f$. Thus, there exists $r \in [s, t]$ such that $f'(r) \neq 0$. This means that (b) holds. The converse can be proved in the same way. □

Proposition 4.22. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is semistrictly quasiconvex.

(b) For every $x, y \in \operatorname{dom} f$, if $\langle \nabla f(x), y - x\rangle > 0$, then $f(z) < f(y)$ for every $z \in [x, y)$.

(b′) For every $x, y \in \operatorname{dom} f$, if $\langle \nabla f(x), y - x\rangle > 0$, then f is increasing along the segment $[x, y]$, that is, if $z \in [x, y]$ and $w \in (z, y]$ then $f(z) < f(w)$.

(c) ∇f is semistrictly quasimonotone.

Proof. (a) ⇒ (b): Assume that $\langle \nabla f(x), y - x\rangle > 0$ for some $x, y \in \operatorname{dom} f$. Since f is radially continuous on $\operatorname{dom} f$, it is quasiconvex. Thus by Proposition 4.12, $f(y) > f(x)$. Semistrict quasiconvexity then implies that for every $z \in [x, y)$, $f(z) < f(y)$.

(b) ⇒ (b′): If $\langle \nabla f(x), y - x\rangle > 0$ for $x, y \in \operatorname{dom} f$, then for $z \in [x, y]$ and $w \in (z, y]$ one has $\langle \nabla f(x), w - x\rangle > 0$, so $f(z) < f(w)$ in virtue of (b).

(b′) ⇒ (b) is obvious.

(b′) ⇒ (c): Let $\langle \nabla f(x), y - x\rangle > 0$ for $x, y \in \operatorname{dom} f$. By (b′), f is increasing along $[x, y]$, so obviously

$$\langle \nabla f(y), y - x\rangle = (f(y + t(y - x))'_{t=0} \geq 0.$$

Hence ∇f is quasimonotone. Also by (b), for every $w \in (x, y)$, $f(w) < f(y)$. By the mean value theorem, there exists $z \in (w, y)$ such that $\langle \nabla f(z), y - w \rangle = f(y) - f(w) > 0$. Thus, $\langle \nabla f(z), y - x \rangle > 0$ and ∇f is semistrictly quasimonotone.

(c) $\Rightarrow$ (a): Since ∇f is quasimonotone, f is quasiconvex. Assume that it is not semistrictly quasiconvex, so there exist $x, y \in C$ and $w \in (x, y)$ such that $f(x) < f(w) = f(y)$. By Lemma 4.1, f is constant on $[w, y]$. It follows that the function $g(t) = f(w + t(y - w))$ is constant on $[0, 1]$. Hence, $g' = 0$, which implies that $\langle \nabla f(z), y - w \rangle = 0$ for every $z \in (w, y)$.

Using the mean value theorem on $[x, w]$ we deduce that there exists $x_1 \in (x, w)$ such that $\langle \nabla f(x_1), w - x \rangle = f(w) - f(x) > 0$; thus $\langle \nabla f(x_1), y - x_1 \rangle > 0$. Since $\langle \nabla f(z), y - x_1 \rangle = 0$ for every $z \in (w, y)$, ∇f is not semistrictly quasimonotone. □

4.5 Pseudoconvex Functions

Pseudoconvex functions can be defined without any assumption of differentiability or even subdifferentiability [31], but usually they are assumed to be differentiable.

Definition 4.6. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. The function f is called

- *pseudoconvex* if for all $x, y \in \operatorname{dom} f$, the following implication holds:

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x), \text{ and}$$

- *strictly pseudoconvex* if for all distinct $x, y \in \operatorname{dom} f$, the following implication holds:

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) > f(x).$$

So far, we have given definitions of seven notions of generalized convexity. There are some relations between them; for instance, a differentiable convex function is pseudoconvex, due to Proposition 4.6. Let us show some more.

Proposition 4.23. (a) A differentiable strictly pseudoconvex function is strictly quasiconvex.

(b) A differentiable pseudoconvex function is semistrictly quasiconvex.

Proof. (a) Let f be strictly pseudoconvex. Then by Proposition 4.12, f is quasiconvex. Assume that it is not strictly quasiconvex. Then there should exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(x) \leq f(z) = f(y)$. By Lemma 4.1, f is constant on $[x, z]$ or on $[z, y]$. In the first case, $\langle \nabla f(z), x - z \rangle = 0$

(consider the function $g(t) = f(tz + (1-t)x)$, which is constant on $[0,1]$). Since f is strictly pseudoconvex, $f(x) > f(z)$, which is a contradiction. The second case is similar.

(b) Let f be pseudoconvex. Since (b′) of Proposition 4.12 holds, f is quasiconvex. If f is not semistrictly quasiconvex, then there exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(x) < f(z) = f(y)$. As in the proof of (a), we deduce that f is constant on $[z, y]$ and $\langle \nabla f(z), y - z\rangle = 0$. This implies that $\langle \nabla f(z), x - z\rangle = 0$. By pseudoconvexity, $f(x) \geq f(z)$, a contradiction. □

The function $f(x) = x^3$, $x \in \mathbb{R}$ can be easily seen to be strictly quasiconvex without being pseudoconvex, since $f'(0) = 0$ but $f(x) < 0$ for $x < 0$ (compare with Proposition 4.30 below). The function f defined in relation (4.9) is pseudoconvex, but not convex or strictly pseudoconvex. Finally, $\arctan x$ is strictly pseudoconvex without being convex.

Other implications between the generalized convexity notions, besides those given in various propositions above, are trivial. Assuming differentiability when pseudoconvexivity is involved, only the following implications hold. We use the abbreviations "s.," "ss.," and "expl.," for "strictly," "semistrictly," and "explicitly," respectively:

$$\begin{array}{ccc}
\text{s. convex} & \Longrightarrow & \text{convex} \\
\Downarrow & & \Downarrow \\
\text{s. pseudoconvex} & \Longrightarrow & \text{pseudoconvex} \\
\Downarrow & & \Downarrow \\
\text{s. quasiconvex} & \Longrightarrow & \text{expl. quasiconvex (=ss. quasiconvex if lsc)} \\
 & & \Downarrow \\
 & & \text{quasiconvex}
\end{array}$$

Pseudoconvex and strictly pseudoconvex functions are characterized by a corresponding generalized monotonicity of their Gâteaux derivative. We introduce the relevant notions.

Definition 4.7. Let $C \subseteq X$ be convex and $T : C \to X^*$ be an operator. T is called

- *pseudomonotone* if for every $x, y \in C$ the following implication holds:

$$\langle T(x), y - x\rangle \geq 0 \Rightarrow \langle T(y), y - x\rangle \geq 0;$$

- *strictly pseudomonotone* if for every distinct $x, y \in C$, the following implication holds:

$$\langle T(x), y - x\rangle \geq 0 \Rightarrow \langle T(y), y - x\rangle > 0.$$

By taking the contrapositive of (a), we find that an operator T is pseudomonotone if and only if for every $x, y \in C$,

$$\langle T(x), y - x\rangle > 0 \Rightarrow \langle T(y), y - x\rangle > 0. \tag{4.12}$$

Proposition 4.24. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is pseudoconvex.

(b) ∇f is pseudomonotone.

Proof. (a) $\Rightarrow$ (b): Let f be pseudoconvex and $\langle \nabla f(x), y - x \rangle \geq 0$. By pseudoconvexity, $f(y) \geq f(x)$. Assume that $\langle \nabla f(y), y - x \rangle < 0$, that is, $\langle \nabla f(y), x - y \rangle > 0$. Since f is also quasiconvex, by Proposition 4.12 we have $f(x) > f(y)$, a contradiction.

(b) $\Rightarrow$ (a): Let ∇f be pseudomonotone. If f is not pseudoconvex, then there exist $x, y \in \operatorname{dom} f$ such that $\langle \nabla f(x), y - x \rangle \geq 0$ and $f(y) < f(x)$. By the mean value theorem, there exists $z \in (x, y)$ such that $\langle \nabla f(z), x - y \rangle = f(x) - f(y) > 0$. Then $\langle \nabla f(z), x - z \rangle > 0$. By implication (4.12), $\langle \nabla f(x), x - z \rangle > 0$. This implies $\langle \nabla f(x), y - x \rangle < 0$, a contradiction. □

Likewise, we have a characterization of strict pseudoconvexity:

Proposition 4.25. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$. Then, the following statements are equivalent:

(a) f is strictly pseudoconvex.

(b) ∇f is strictly pseudomonotone.

The proof follows the same lines as the proof of Proposition 4.24, so it is omitted.

Besides the notions of generalized convexity, we also recalled some corresponding notions of generalized monotonicity, and we showed that a differentiable function is convex (resp., strictly convex, quasiconvex, strictly quasiconvex, semistrictly quasiconvex, pseudoconvex, strictly pseudoconvex) if and only if its derivative is monotone (resp, strictly monotone, quasimonotone, strictly quasimonotone, semistrictly quasimonotone, pseudomonotone, strictly pseudomonotone). Therefore, *only* the following implications hold among the various notions of generalized monotonicity:

$$\begin{array}{ccc}
\text{s. monotone} & \Longrightarrow & \text{monotone} \\
\Downarrow & & \Downarrow \\
\text{s. pseudomonotone} & \Longrightarrow & \text{pseudomonotone} \\
\Downarrow & & \Downarrow \\
\text{s. quasimonotone} & \Longrightarrow & \text{ss. quasimonotone} \\
& & \Downarrow \\
& & \text{quasimonotone}
\end{array}$$

All the above implications are trivial, except possibly the ones contained in the following proposition.

Proposition 4.26. Let $C \subseteq X$ be convex and $T : C \to X^*$ be an operator.

(a) If T is strictly pseudomonotone, then it is strictly quasimonotone.

(b) If T is pseudomonotone, then it is semistrictly quasimonotone.

Proof. (a) Since T is strictly pseudomonotone, it is obviously quasimonotone. For any $x, y \in C$, choose $z \in (x, y)$. If $\langle T(z), y - x\rangle \neq 0$, we are done. If $\langle T(z), y - x\rangle = 0$, then for every $w \in (z, y)$ one has $\langle T(z), w - z\rangle = 0$. By strict pseudomonotonicity, $\langle T(w), w - z\rangle > 0$, so finally $\langle T(w), y - x\rangle \neq 0$.

(b) Since T is pseudomonotone, it is quasimonotone. Assume that $\langle T(x), y - x\rangle > 0$ for some $x, y \in C$. Then for every $z \in (x, y)$, $\langle T(x), z - x\rangle > 0$, so from (4.12) we find $\langle T(z), z - x\rangle > 0$, that is, $\langle T(z), y - x\rangle > 0$. Thus, T is semistrictly quasimonotone. □

We finish this section with some remarks on the relation between pseudomonotone and quasimonotone operators. Every pseudomonotone operator is quasimonotone, but the converse also holds under some simple assumptions. We call an operator T *hemicontinuous* if its restriction on line segments is continuous with respect to the weak* topology in X^*.

Proposition 4.27. Let $C \subseteq X$ be open and convex and $T : C \to X^*$ be a hemicontinuous quasimonotone operator such that $T(x) \neq 0$, for all $x \in C$. Then, T is pseudomonotone.

Proof. Let $x, y \in C$ be such that $\langle T(x), y - x\rangle \geq 0$. Since $T(x) \neq 0$ by assumption, we can choose $v \in X$ such that $\langle T(x), v\rangle > 0$. For $t > 0$ sufficiently small, $y + tv \in C$ and $\langle T(x), y + tv - x\rangle \geq t\,\langle T(x), v\rangle > 0$. Since T is quasimonotone,

$$\langle T(y + tv), y + tv - x\rangle \geq 0. \tag{4.13}$$

By taking the limit in (4.13) as $t \downarrow 0$, we find $\langle T(y), y - x\rangle \geq 0$. This means that T is pseudomonotone. □

If C is not open, the above proposition does not hold. For instance, if $X = \mathbb{R}^2$, and $C = [0, 1] \times \{0\}$, define T on C by $T(t, 0) = (-t, 1)$. The operator T is continuous and quasimonotone, $T(t, 0) \neq 0$ for all $(t, 0) \in C$, but it is not pseudomonotone, as one can see by taking $x = (1, 0)$ and $y = (0, 0)$. The failure of pseudomonotonicity is due to the fact that the set C is lower-dimensional than X, so we can construct the example by taking T to be perpendicular to C at some point; then the projection of $T(x)$ on C is zero. Thus, in some sense, we cheated with the assumption $T(x) \neq 0$. A less trivial example where the dimension of C is the dimension of the space is the following.

Example 4.2. Let $X = \mathbb{R}^2$, $C = [0, 1] \times [0, 1]$ and define T as follows. For every $(x_1, x_2) \in C$ with $(x_1, x_2) \neq (0, 0)$, define t to be the unique positive root of $t^2 - x_1 t - x_2 = 0$, that is,

$$t = \frac{x_1 + \sqrt{x_1^2 + 4x_2}}{2}.$$

Then, define $T(x_1, x_2)$ by

$$T(x_1, x_2) = \left(\frac{-t}{t+1}, \frac{-1}{t+1} \right).$$

Note that the point (x_1, x_2) belongs to the segment joining $(t, 0)$ and $(0, t^2)$, while $T(x_1, x_2)$ is perpendicular to this segment. Finally, define $T(0,0) = (0, -1)$. Then T is continuous, $T(x_1, x_2) \neq (0, 0)$ for all $(x_1, x_2) \in C$, and it is not pseudomonotone as one can see by considering the points $x = (0, 0)$ and $y = (1, 0)$. However, it is quasimonotone, as can be seen by inspecting the above segments, or by a direct calculation (see [21])

4.6 On the Minima of Generalized Convex Functions

Generalized convex functions are frequently encountered in optimization problems, and the properties of their minima are very important. Here are some properties whose proofs are easily derived from the definitions.

Proposition 4.28. Let $f : X \to (-\infty, +\infty]$ be a proper function.

(a) If f is quasiconvex, then the set of its global minima is convex.

(b) If f is strictly quasiconvex, then it has at most one global minimum.

Semistrictly quasiconvex functions are characterized by the equality of local and global minima [28]:

Proposition 4.29. Let $f : X \to (-\infty, +\infty]$ be a proper function. If f is semistrictly quasiconvex, then every local minimum of f is global. Conversely, assume that $\operatorname{dom} f$ is open, and f is a continuous quasiconvex function. If every local minimum of f is global, then f is semistrictly quasiconvex.

Proof. Let f be semistrictly quasiconvex, and assume that $x_0 \in \operatorname{dom} f$ is a local minimum of f. Then there exists $\varepsilon > 0$ such that $f(x) \geq f(x_0)$ for every $x \in B(x_0, \varepsilon)$. Assume that x_0 is not a global minimum; then there exists $x_1 \in X$ such that $f(x_1) < f(x_0)$. Choose $y \in (x_1, x_0)$ close enough to x_0 so that $y \in B(x_0, \varepsilon)$. Then $f(y) \geq f(x_0)$ since x_0 is a minimum on $B(x_0, \varepsilon)$. Since f is semistrictly quasimonotone, $f(x_1) < f(x_0)$ implies $f(y) < f(x_0)$, a contradiction.

To show the converse, assume that there exist $x, y \in \operatorname{dom} f$ and $z \in (x, y)$ such that $f(x) < f(z) = f(y)$. Choose $w \in (z, y)$. Lemma 4.1 implies that f is constant on $[z, y]$, so $f(w) = f(z)$.

Since f is continuous, there exists $\varepsilon > 0$ such that $f(u) < f(z)$ for all $u \in B(x, \varepsilon)$. Consider the set

$$K = z + \bigcup_{\lambda > 0} \lambda(z - B(x, \varepsilon)). \tag{4.14}$$

K is open (actually it is an open cone with apex at z). Since $z \in (x, w)$, it is easy to see that $w = z + \lambda(z - x)$ for some $\lambda > 0$, that is, $w \in K$. Hence, K is a neighborhood of w. For every $v \in K$, according to (4.14), we can choose $\lambda > 0$ and $u \in B(x, \varepsilon)$ such that $v = z + \lambda(z - u)$. Then, $f(u) < f(z)$, and

$$z = \frac{\lambda}{1+\lambda}u + \frac{1}{1+\lambda}v \in (v, u),$$

so $f(z) \le \max\{f(u), f(v)\}$. This implies that $f(w) = f(z) \le f(v)$ for every $v \in K$. Thus, w is a local minimum of f. By assumption, it is also a global minimum. But this contradicts $f(x) < f(w)$. □

It is a well-known property of differentiable convex functions that the condition $\nabla f(x) = 0$ is not only necessary, but also sufficient for f to have a global minimum at x. This is trivially true for pseudoconvex functions too. Actually, pseudoconvex functions are exactly the right class of functions to have this property [13]:

Proposition 4.30. Let $f : X \to (-\infty, +\infty]$ be such that $\operatorname{dom} f$ is an open convex set and f is Gâteaux differentiable on $\operatorname{dom} f$.

(a) If f is pseudoconvex and $\nabla f(x) = 0$, then x is a global minimum of f.

(b) If f is quasiconvex and has a local minimum at every point x such that $\nabla f(x) = 0$, then f is pseudoconvex.

Proof. (a) is obvious, so we prove only (b). Assume that f is not pseudoconvex; then there exist $x, y \in \operatorname{dom} f$ such that $\langle \nabla f(x), y - x\rangle \ge 0$ but $f(y) < f(x)$. Since f is quasiconvex, Proposition 4.12 implies that $\langle \nabla f(x), y - x\rangle = 0$. Also, for every $z \in [x, y]$, $f(z) \le f(x)$. Set

$$t_0 = \sup\{t \in [0, 1] : f(x + t(y - x)) = f(x),$$

and $z_0 = x + t_0(y - x)$. Then, $f(x) = f(z_0)$ and $f(z) < f(z_0)$ for every $z \in (z_0, y)$. By Lemma 4.1, f is constant on $[x, z_0]$, hence $\langle \nabla f(z_0), y - z_0\rangle = 0$. We show that $\nabla f(z_0) = 0$. If this is not the case, then we can choose $v \in X$ such that $\langle \nabla f(z_0), v\rangle > 0$. For $t > 0$ small enough so that $y + tv \in \operatorname{dom} f$, one has

$$\langle \nabla f(z_0), y + tv - z_0\rangle = \langle \nabla f(z_0), y - z_0\rangle + t\,\langle \nabla f(z_0), v\rangle > 0,$$

so again by Proposition 4.12, $f(y + tv) \ge f(z_0)$. Taking the limit at $t \downarrow 0$ we deduce $f(y) \ge f(z_0)$, a contradiction. Accordingly, $\nabla f(z_0) = 0$. By assumption, z_0 is a local minimum of f. But this is not possible, since $f(z) < f(z_0)$ for every $z \in (z_0, y)$. We have arrived at a contradiction, thus f is pseudoconvex. □

4.7 Applications

The range of applications of generalized convex functions and generalized monotone operators is very wide. Here we give just a few.

4.7.1 Sufficiency of the KKT Conditions

The Karush-Kuhn-Tucker conditions for the existence of a solution to the following program are central features of optimization theory:

$$\begin{cases} \min f(x), \\ x \in C, \\ g_i(x) \leq 0, \quad i = 1, \ldots, m, \\ h_j(x) = 0, \quad j = 1, \ldots, k. \end{cases} \tag{P}$$

Here, C is an open subset of $\mathbb{R}^n$; f, g_i, and h_j are differentiable functions on C. It is well known that if some regularity conditions are met (the so-called *constraint qualification conditions*), then for every solution $\bar{x}$ of (P) there exist λ_i, $i = 1, \ldots m$ and μ_j, $j = 1, \ldots k$ satisfying the following *Karush-Kuhn-Tucker (KKT) conditions*:

$$\begin{cases} \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{k} \mu_j \nabla h_j(\bar{x}) = 0, \\ \lambda_i \geq 0, \quad i = 1, \ldots, m, \\ \lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m. \end{cases} \tag{KKT}$$

It is also well known that the KKT conditions are not sufficient to guarantee that $\bar{x}$ is a solution of (P). This becomes true under some generalized convexity conditions for the functions involved. We recall that $\bar{x} \in C$ is called a feasible solution of (P) if $g_i(\bar{x}) \leq 0$ and $h_j(\bar{x}) = 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, k$.

Theorem 4.1. Assume that f is pseudoconvex, the functions g_i are quasiconvex, and the functions h_j are quasilinear. If $\bar{x}$ is a feasible solution of (P) and there exist $\lambda_i, \mu_j \in \mathbb{R}$ satisfying the KKT conditions, then $\bar{x}$ is an optimal solution.

Proof. We want to show that for every feasible solution $x \in C$, $f(x) \geq f(\bar{x})$. Since f is pseudoconvex, it is enough to show that

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0$$

or, using (KKT),

$$\left\langle \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{k} \mu_j \nabla h_j(\bar{x}), x - \bar{x} \right\rangle \leq 0.$$

Again, it is enough to show that for all $i = 1, \dots m$, $j = 1, \dots k$,

$$\lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \leq 0, \tag{4.15}$$

$$\langle \nabla h_j(\bar{x}), x - \bar{x} \rangle = 0. \tag{4.16}$$

If $\lambda_i = 0$, then of course (4.15) holds. If $\lambda_i > 0$, then from (KKT) we get $g_i(\bar{x}) = 0$; hence $g_i(x) \leq g_i(\bar{x})$. Since g_i is quasiconvex, Proposition (4.12) implies that $\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \leq 0$, so inequalities (4.15) hold in all cases.

We also note that $h_j(\bar{x}) = h_j(x)$ by assumption. Since both h_j and $-h_j$ are quasiconvex, Proposition (4.12) implies that $\langle \nabla h_j(\bar{x}), x - \bar{x} \rangle > 0$ and $\langle \nabla h_j(\bar{x}), x - \bar{x} \rangle < 0$ are not possible. Thus one has necessarily $\langle \nabla h_j(\bar{x}), x - \bar{x} \rangle = 0$, so equalities (4.16) hold. □

4.7.2 Applications in Economics

Mathematical economics is one of the fields where generalized convexity and generalized monotonicity have been mostly applied. In fact, in many cases some assumptions based on empirical arguments led economists to introduce notions that correspond exactly to generalized convexity or monotonicity concepts, well before these concepts were introduced in mathematics.

In consumer theory, one supposes that there exist n commodities. A *commodity bundle* is just a vector $x = (x_1, \dots x_n) \in \mathbb{R}^n_+$. The numbers x_i represent a certain amount of a commodity i, $i = 1, \dots n$. It is supposed that each consumer has a *preference relation* $\succeq$ defined on $\mathbb{R}^n_+$; $y \succeq x$ means that the consumer considers y to be at least as good as x. Usually, the preference relation is defined through a *utility function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ as follows: $u(y) \geq u(x)$ if and only if $y \succeq x$. The existence of such a function is not only very convenient, but it can also be deduced from reasonable economic assumptions (see [29]). One of the main assumptions imposed on u is that it is quasiconcave, that is, $u(\lambda x + (1 - \lambda y) \geq \min\{u(x), u(y)\}$ for every commodity bundle x and y and every $\lambda \in [0, 1]$. In terms of the preference relation, this means that either $\lambda x + (1 - \lambda)y \succeq x$ or $\lambda x + (1 - \lambda)y \succeq y$, and can be seen as an expression of the consumer's inclination for diversify; $\lambda x + (1 - \lambda)y$ cannot be worse than both x and y.

A *price vector* $p = (p_1, \dots p_n)$ is by definition an element of $\operatorname{int} \mathbb{R}^n_+$; p_i is the price per unit of the commodity x_i. Given a price vector p, the cost of the consumption bundle x is $\sum_{i=1}^{n} p_i x_i = \langle p, x \rangle$. The *budget* of the consumer is given by a number $w > 0$. For a given budget w and price vector p, the set of commodity bundles that are affordable by the consumer is the so-called *budget set* $\{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq w\}$. The budget set does not change if we multiply w and p by the same positive number λ. Thus, by dividing both w and p by w, we may suppose with no loss of generality that $w = 1$ and define the budget set by

$$B(p) = \{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq 1\}.$$

It is expected that a consumer will choose the commodity bundle that is the best possible for him among the commodity bundles that he can afford. This leads to a problem of maximization of the utility u on the budget set $B(p)$:

$$\text{maximize } u(x) \text{ s.t. } x \in B(p).$$

The map X that gives the solution set of the above problem is the so-called *demand correspondence*. It is a multivalued map defined by

$$X(p) = \{x \in B(p) : u(x) \geq u(y), \ \forall y \in B(p)\}.$$

One also defines the *indirect utility function* v on $\operatorname{int} \mathbb{R}^n_+$, which for every price vector $p \in \operatorname{int} \mathbb{R}^n_+$ gives the maximum utility that the consumer can achieve:

$$v(p) = \sup\{u(x) : x \in \mathbb{R}^n_+, \langle p, x \rangle \leq 1\}. \tag{4.17}$$

It is easy to see that the indirect utility has a generalized convexity property:

Proposition 4.31. The function v defined by (4.17) is quasiconvex.

Proof. It is enough to show that for every $\alpha \in \mathbb{R}$, $[v \leq \alpha]$ is convex. Indeed, let $p = \lambda p_1 + (1 - \lambda) p_2$ where $p_1, p_2 \in [v \leq \alpha]$. For every $x \in \mathbb{R}^n_+$ such that $\langle p, x \rangle \leq 1$, one has $\lambda \langle p_1, x \rangle + (1 - \lambda) \langle p_2, x \rangle \leq 1$. It follows that at least one of the inequalities $\langle p_1, x \rangle \leq 1$ and $\langle p_2, x \rangle \leq 1$ holds. If, say, $\langle p_1, x \rangle \leq 1$, then (4.17) implies that $u(x) \leq v(p_1) \leq \alpha$, so $u(x) \leq \alpha$ for every $x \in \mathbb{R}^n_+$ such that $\langle p, x \rangle \leq 1$. Applying (4.17) again we obtain $v(p) \leq \alpha$ and v is quasiconvex. □

Actually, one can show that v has other interesting properties, and there is a duality between the direct and indirect utility functions, that permit us to construct u by using v. The interested reader may consult, for instance, [30] for a general result and the related bibliography.

Generalized monotonicity also plays a crucial role. One of the usual assumptions imposed on the utility u is nonsatiation. A utility function u is called *nonsatiated* if for every $p \in \operatorname{int} \mathbb{R}^n_+$, whenever $x \in \mathbb{R}^n_+$ is such that $\langle p, x \rangle < 1$, there exists $x' \in B(p)$ such that $u(x') > u(x)$. It is easy to see that u is nonsatiated if and only if for every $x \in X(p)$, $\langle p, x \rangle = 1$ holds.

Proposition 4.32. Assume that u is nonsatiated. Then for every $p_1, p_2 \in \operatorname{int} \mathbb{R}^n_+$ and $x_1 \in X(p_1)$, $x_2 \in X(p_2)$, the following implication holds:

$$\langle p_1, x_2 - x_1 \rangle \leq 0 \quad \Rightarrow \quad \langle p_2, x_2 - x_1 \rangle \leq 0. \tag{4.18}$$

Proof. If $\langle p_1, x_2 - x_1 \rangle \leq 0$, then $\langle p_1, x_2 \rangle \leq \langle p_1, x_1 \rangle$. Since $x_1 \in X(p_1)$, we deduce that $\langle p_1, x_2 \rangle \leq 1$, so $x_2 \in B(p_1)$. Again by the definition of x_1 we obtain $u(x_2) \leq u(x_1)$. It follows that $\langle p_2, x_2 - x_1 \rangle > 0$ is not possible, otherwise we would have $\langle p_2, x_1 \rangle < \langle p_2, x_2 \rangle = 1$, so again by nonsatiation $u(x_1) < u(x_2)$, a contradiction. Thus, $\langle p_2, x_2 - x_1 \rangle \leq 0$ and (4.18) holds. □

Property (4.18) is a version of the *Axiom of Revealed Preference* in economics. The reader might observe that property (4.18) is related to pseudomonotonicity. To see this more clearly, let us assume (as is often done) that the demand correspondence is single-valued, i.e., there is a map $x(\cdot)$: $\operatorname{int}\mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $X(p) = \{x(p)\}$. Then (4.18) can be rewritten as

$$\langle p_1, x(p_2) - x(p_1)\rangle \leq 0 \quad \Rightarrow \quad \langle p_2, x(p_2) - x(p_1)\rangle \leq 0. \tag{4.19}$$

If we again assume nonsatiation, then $\langle p_1, x(p_1)\rangle = \langle p_2, x(p_2)\rangle = 1$, so we can rewrite (4.19) as

$$\langle x(p_2), p_1 - p_2\rangle \leq 0 \quad \Rightarrow \quad \langle x(p_1), p_1 - p_2\rangle \leq 0$$

By interchanging the indices 1 and 2, we deduce that for every $p_1, p_2 \in \operatorname{int}\mathbb{R}^n_+$,

$$\langle x(p_1), p_2 - p_1\rangle \leq 0 \quad \Rightarrow \quad \langle x(p_2), p_2 - p_1\rangle \leq 0.$$

Thus, the version of the axiom of revealed preference that we presented says that $-x(\cdot)$ is pseudomonotone. Other versions (for instance, the weak axiom of revealed preference, the strong axiom of revealed preference, etc.) are related to other notions of generalized monotonicity. The reader is referred to Chapter 14 in [22] for additional information. By assuming a plausible generalized monotonicity assumption on an operator $x(\cdot)$ defined on $\operatorname{int}\mathbb{R}^n_+$, it has been shown that $x(\cdot)$ can be seen as the demand map resulting from a utility function u. See for example [14].

4.8 Further Reading

The basic reference for the study of convexity in $\mathbb{R}^n$ is the seminal book of Rockafellar [35]. Convexity in more general spaces is studied in several more recent books, such as the books of Magaril-Il'yaev and Tikhomirov [27], Lucchetti [26], Borwein and Wanderwerff [8], and Zalinescu [37]. Generalized convexity in $\mathbb{R}^n$ is presented, together with many examples and applications, in the books of Avriel, Diewert, Schaible, and Zang [6] and Cambini and Martein [9]. For spaces of finite or infinite dimension, there is the handbook edited by Hadjisavvas, Komlosi, and Schaible [22]. Additional information on quasiconvex functions may be found in Chapter 5 (by Aussel) of this volume.

There are many papers (probably more than one thousand) studying the properties, characterizations and applications of generalized convex functions and generalized monotone operators. Here we mention just a few, referring the reader to their bibliographies for further references. Continuity and differentiability properties of quasiconvex functions have been studied in detail in [11, 12]. Characterizations of generalized convexity and generalized monotonicity characteristics are presented in [23, 20] in the smooth case, and [3, 4, 15, 32, 33] in the nonsmooth case. For nonsmooth generalized convexities and monotonicities, we refer to [2]. Also, some continuity and maximality properties of generalized monotone operators are studied in [18]. Applications in variational inequalities, equilibrium problems, economics, etc. may be found, in addition to the above-mentioned books, in [5, 7, 16, 21, 25, 36]. A lot of papers may also be found in the list contained in [34] (the list contains papers up to 1995) and at the website http://www.genconv.org/aigaion/ .

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Chapter 5

New Developments in Quasiconvex Optimization

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5.1 Introduction

It is well known that convex functions form an ideal class of functions for minimization purposes. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function, then the global minimizers of f over $\mathbb{R}^n$, that is, the solutions of the unconstrained optimization problem

$$\text{(P)} \qquad \inf_{x \in \mathbb{R}^n} f(x),$$

are simply the stationary point of f, that is, the point of $\mathbb{R}^n$ for which $\nabla f(x)$, the gradient of f, is the null vector. This is thus an perfect situation since, usually, this last condition "$\nabla f(x) = 0$" is only a necessary one for the point x to be a local minimizer of f.

And even if some convex constraint set $C \subset \mathbb{R}^n$ is considered, the constraint optimization problem becomes

(P$_C$)
$$(P_C) \qquad \inf_{x \in C} f(x),$$

and, again, a necessary and sufficient optimality condition for a point x to be a global solution of this problem (P_C) simply expressed as

$$\langle \nabla f(x), y - x \rangle \geq 0, \quad \forall\, y \in C. \tag{5.1}$$

Let us remark that formulation (5.1) is called the variational inequality defined by ∇f and C. It will be extensively discussed in generalized forms in this chapter.

But it is also well known that, if f is not convex, and even if the subset C is convex, the condition (5.1) is only a necessary optimality condition as it can be seen by simply considering the real function $f(t) = t^3$ and $C = [-1, 1]$[1].

Our aim, in this chapter, is to show that there is actually another class, a broader class, of functions for which powerful properties, such as necessary and sufficient optimality conditions, can be easily obtained. It is the class of *quasiconvex functions*. Though the convexity of a function is totally characterized by the convexity of its epigraph, the quasiconvexity of a function $f : \mathbb{R}^n \to \mathbb{R}$ is described by the convexity of its *sublevel sets*

$$f \text{ is quasiconvex} \quad \Leftrightarrow \quad \forall\, \lambda \in \mathbb{R}, \;\; S_\lambda(f) \text{ is a convex subset},$$

where the sublevel set $S_\lambda(f)$ of f of level λ is defined by $S_\lambda(f) = \{x \in \mathbb{R}^n \; : \; f(x) \leq \lambda\}$. A typical example of quasiconvex function is given in Figures 5.1 and 5.2.

As seen in Figure 5.2 the sublevel sets are convex, but more important is that typical quasiconvex functions can have "flat parts." As will be seen in Section 5.5 of the reformulation (5.1) can provide a necessary and sufficient optimality condition for this broad class of quasiconvex functions if one replaces the gradient of the function by the so-called *normal operator* of the function.

At first glance, the definition of this class of quasiconvex functions could be simply considered as "another" generalization of convexity, but that is not at all the case. First, the class of quasiconvex functions inherits powerful properties (see Section 5.3). Second, quasiconvexity plays a fundamental role in mathematical economics. For example, it describes important features of the utility functions of agents in economic games and is also one of the main hypotheses of existence results for equilibrium problems; see, for example, the pioneering work of Arrow and Debreu [2] concerning Nash equilibrium.

[1]Indeed, for $x = 0$, one has $\nabla f(0) = 0$ and thus x is a solution of (5.1) but clearly x is not a local minimum of f on C.

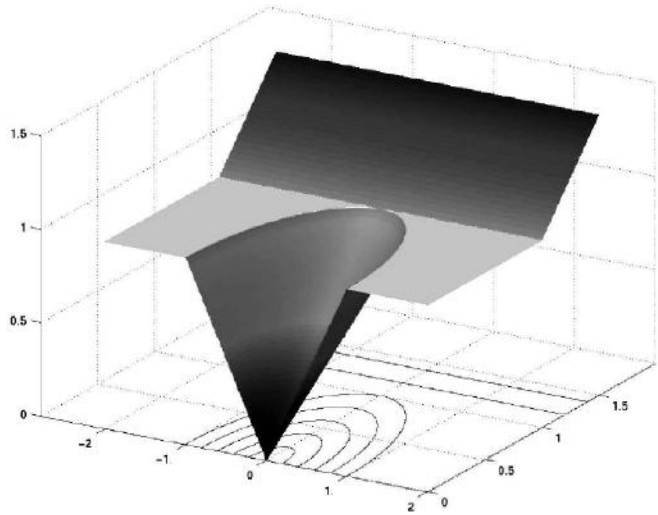


FIGURE 5.1: A quasiconvex function.

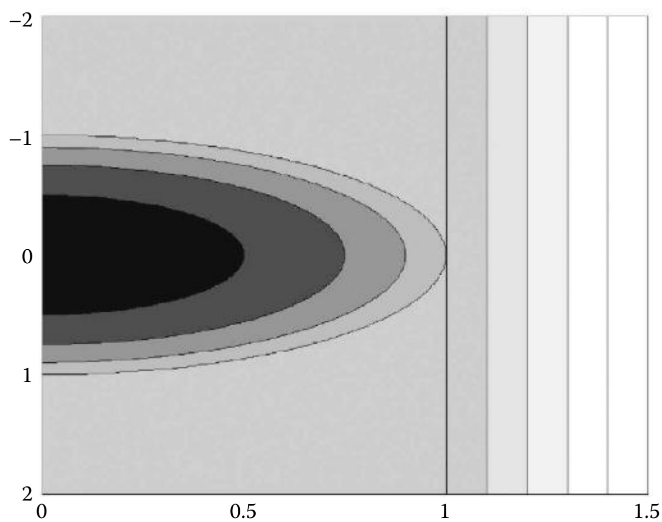


FIGURE 5.2: Its sublevel sets.

5.2 Notations

In this chapter, we assume that all spaces $X \neq \{0\}$ are Banach spaces with topological dual X^*, equipped with the weak* topology (denoted by w^* in the sequel) and $\langle \cdot, \cdot \rangle$ stands for the duality pairing. Nevertheless, if the reader is not confident with the possible infinite dimensional setting, in most of the results, X can be identified with $\mathbb{R}^n$.

As usual, the norm on X induces a distance function defined as $d(x, y) = \|y - x\|$. For $x \in X$ and $\rho > 0$, we denote by $B(x, \rho)$, $\overline{B}(x, \rho)$, and $S(x, \rho)$, respectively, the open ball, the closed ball, and the sphere of center x and radius ρ, whereas for the equivalent balls and sphere of the dual space X^*, we will use the notations $B^*(x, \rho)$, $\overline{B}^*(x, \rho)$, and $S^*(x, \rho)$.

For $x, x' \in X$, we denote by $[x, x']$ the closed segment

$$\{tx + (1 - t)x' \ : \ t \in [0, 1]\}.$$

The segments $]x, x'[$, $]x, x']$, $[x, x'[$ are defined analogously. For any element x^* of X^*, the rays defined by x^* are described by

$$\begin{aligned} \mathbb{R}_+\{x^*\} \quad &= \{tx^* \in X^* \ : \ t \geq 0\} \\ \mathbb{R}_{++}\{x^*\} \quad &= \{tx^* \in X^* \ : \ t > 0\}. \end{aligned}$$

The topological closure, the interior, the boundary, the conical hull, and the convex hull of a set $A \subset X$ will be denoted, respectively, by

$$\begin{aligned} \mathrm{cl}A \quad &= \{x \in A \ : \ B(x, \varepsilon) \cap A \neq \emptyset, \text{ for any } \varepsilon > 0\} \\ \mathrm{int}A \quad &= \{x \in A \ : \ B(x, \varepsilon) \subset A, \text{for some } \varepsilon > 0\} \\ \mathrm{bd}\, A \quad &= \mathrm{cl}A \setminus \mathrm{int}A \\ \mathrm{co}A \quad &= \{\textstyle\sum_{i=1}^n \lambda_i a_i \ : \ \sum_{i=1}^n \lambda_i = 1; \ \lambda_i \in [0, 1], \ a_i \in A, \ i = 1, \ldots, n \in \mathbb{N}\} \\ \mathrm{cone}A \quad &= \mathbb{R}_+(A) = \{\lambda.x \ : \ x \in A\}. \end{aligned}$$

A set $B \subset X^*$ is called a *base of a cone* C of X^* if and only if $0 \notin \overline{B}^{w*}$ and $C = \mathrm{cone}(B)$, and for any convex subset A, $N_A(x)$ stands for the normal cone to A at x, that is,

$$N_A(x) = \big\{x^* \in X^* \ : \langle x^*, u - x\rangle \leq 0, \forall\, u \in A\big\}.$$

Since our aim is to deal with possibly nonsmooth functions, we will naturally manipulate set-valued maps, that is, applications $T : X \to 2^Y$, where 2^Y is the set of subsets of the topological vector space Y. This means that $T(x)$ is a subset of Y. Of course, whenever the set-valued map is single valued, that is, $T(x)$ is a singleton for any x, then T is simply a function. The classical following notations for a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a set-valued maps $T : X \to 2^Y$ will be used:

$$\begin{aligned} \mathrm{dom}\, T \quad &= \{x \in X \ : \ T(x) \neq \emptyset\} \\ \mathrm{dom}\, f \quad &= \{x \in X \ : \ f(x) < +\infty\} \\ \mathrm{Gr}T \quad &= \{(x, y) \in X \times Y \ : \ y \in T(x)\} \\ \arg\min_X f \quad &= \{x \in X \ : \ f(x) = \min_X f\}. \end{aligned}$$

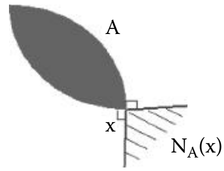


FIGURE 5.3: Example of normal cones.

Let us now recall the well-known continuity concepts for application. The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be:

- *continuous at* $x \in \operatorname{dom} f$ if for any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(u) \in V$, for any $u \in U$;
- *lower semicontinuous at* $x \in \operatorname{dom} f$ if for any sequence $(x_n)_n \subset \operatorname{dom} f$ converging to x, one has $f(x) \leq \liminf_{n\to\infty} f(x_n)$;
- *upper semicontinuous at* $x \in \operatorname{dom} f$ if for any sequence $(x_n)_n \subset \operatorname{dom} f$ converging to x, one has $f(x) \geq \limsup_{n\to\infty} f(x_n)$.

On the other hand, classical continuity notions for a set-valued operator $T : X \to 2^Y$, where Y is a topological vector space, are as follows: T is said to be

- *lower semicontinuous* at x_0 in $\operatorname{dom} T$ if for any sequence $(x_n)_n$ of X converging to x_0 and any element y_0 of $T(x_0)$, there exists a sequence $(y_n)_n$ of Y converging to y_0, with respect to the considered topology on Y, and such that $y_n \in T(x_n)$, for any n;
- *upper semicontinuous* at $x_0 \in \operatorname{dom} T$ if for any neighborhood V of $T(x_0)$, there exists a neighborhood U of x_0 such that $T(U) \subset V$;
- *closed* at $x_0 \in \operatorname{dom} T$ if for any sequence $((x_n, y_n))_n \subset \operatorname{Gr}T$ converging to (x_0, y_0), one has $(x_0, y_0) \in \operatorname{Gr}T$.

Whenever the image set Y of a set-valued map $T : X \to 2^Y$ is equipped with a weak topology, the use of sequences in the above definition will be replaced by the use of nets (see, for example, [26, Proposition 3.2.14]): T is said to be

- *weakly closed* iff for any net $(x_\alpha, y_\alpha)_{\alpha\in\Lambda} \subset \operatorname{Gr}T$ such that $(x_\alpha)_\alpha$ weakly converges to $\bar{x}$ and $(y_\alpha)_\alpha$ weakly converges to $\bar{y}$, we have $\bar{y} \in T(\bar{x})$;
- *weakly lower semicontinuous* iff for any net $(x_\alpha)_{\alpha\in\Lambda}$ weakly converging to $\bar{x}$ and any $\bar{y} \in T(\bar{x})$, there exists a net $(y_\alpha)_\alpha$ weakly converging to $\bar{y}$ and such that $y_\alpha \in T(x_\alpha)$, for any α.

In the particular case where $Y = X^*$, the considered weak topology is the weak* topology and thus, in both above definitions, the weak convergence of the net $(y_\alpha)_\alpha$ corresponds to its w^*-convergence.

5.3 The Class of Quasiconvex Functions

In the following we shall deal with *proper* functions $f : X \to \mathbb{R} \cup \{+\infty\}$ (that is, functions for which $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ is nonempty). So let us now define the different notions of quasiconvexity.

Definition 5.1. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be

- *quasiconvex* on a convex subset $C \subset \operatorname{dom} f$ if for any $x, y \in C$ and any $t \in [0, 1]$,
$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\};$$
- *semistrictly quasiconvex* on a convex subset $C \subset \operatorname{dom} f$ if f is quasiconvex on C and for any $x, y \in C$,
$$f(x) < f(y) \Rightarrow f(z) < f(y) \quad \forall\, z \in [x, y[;$$
- *strictly quasiconvex* on a convex subset $C \subset \operatorname{dom} f$ if for any $x, y \in C$ and any $t \in [0, 1]$,
$$f(tx + (1-t)y) < \max\{f(x), f(y)\}.$$

It is clear that strict quasiconvexity implies semistrict quasiconvexity, which induces, by definition, quasiconvexity.

Let us denote, for any $\lambda \in \mathbb{R}$, by $S_\lambda(f)$ and $S_\lambda^<(f)$ the sublevel set and the strict sublevel set, respectively, associated with f and λ :

$$S_\lambda(f) = \{x \in X \ : \ f(x) \leq \lambda\} \quad \text{and} \quad S_\alpha^<(f) = \{x \in X \ : \ f(x) < \lambda\}.$$

Where no confusion can occur, we will use, for any $x \in \operatorname{dom} f$, the simplified notation $S_{f(x)}$ and $S_{f(x)}^<$ instead of $S_{f(x)}(f)$ and $S_{f(x)}^<(f)$.

Now, using the concept of sublevel sets, it is thus immediate to give equivalent definitions of quasiconvexity: if $\operatorname{dom} f$ is convex, then

$$\begin{aligned} f \text{ is quasiconvex on } \operatorname{dom} f \quad &\Leftrightarrow \quad \forall\, \lambda \in \mathbb{R}, \quad S_\lambda(f) \text{ is a convex subset} \\ &\Leftrightarrow \quad \forall\, \lambda \in \mathbb{R}, \quad S_\lambda^<(f) \text{ is a convex subset.} \end{aligned}$$

Analogously, it is easy to check that any lower semicontinuous function f, semistrictly quasiconvex on its domain $\operatorname{dom} f$, satisfies the following property:

$$\forall\, \lambda > \inf_X f, \qquad \operatorname{cl}\big(S_\lambda^<(f)\big) = S_\lambda(f). \tag{5.2}$$

Roughly speaking, this means that a lower semicontinuous semistrictly quasiconvex function f does not have any "flat part" with nonempty interior on $\operatorname{dom} f \setminus \operatorname{argmin}_X f$.

In the case of a function defined on $X = \mathbb{R}$, the quasiconvexity can be equivalently described in a very elementary way.

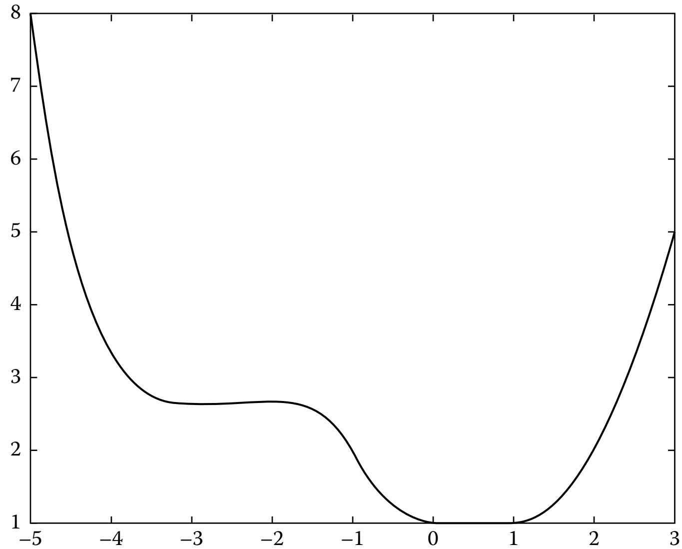


FIGURE 5.4: Quasiconvex.

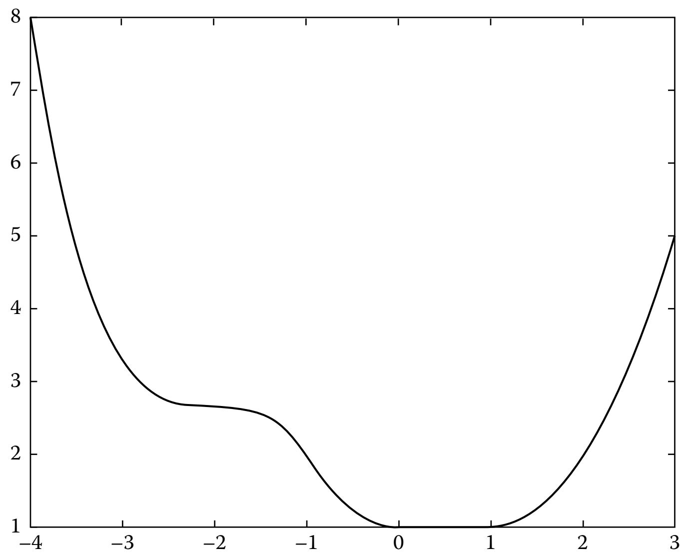


FIGURE 5.5: Semistrictly quasiconvex.

Proposition 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then, f is quasiconvex on $\mathbb{R}$ if and only if there exists $t_0 \in \mathbb{R}$ such that

(i) either f is nonincreasing on $]-\infty, t_0[$ and nondecreasing on $[t_0, +\infty[$;

(ii) or f is nonincreasing on $]-\infty, t_0]$ and nondecreasing on $]t_0, +\infty[$

Even when the dimension of X is greater than one, the quasiconvexity of a function defined on X is linked to its quasiconvexity along the lines.

Proposition 5.2. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with a convex domain. Then f is quasiconvex on its domain if and only if, for any x, $d \in X$, the restriction $f_{x,d}$ of f along the line $\Delta_{(x,d)}$ passing through x and of direction d is quasiconvex on $\operatorname{dom} f \cap \Delta_{(x,d)}$, the function $f_{x,d}$ being defined on $\mathbb{R}$ by $f_{x,d}(t) = f(x + td)$.

Let us now describe other ways to construct quasiconvex functions:

- Any convex function is quasiconvex.
- If $f : X \to \mathbb{R}$ is a convex function and $\theta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, then the function $g = f \circ \theta$ is quasiconvex.
- If $f : X \to \mathbb{R}$ is a quasiconvex function and λ is a positive real number, then the function $\lambda.f$ is quasiconvex.
- If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are two quasiconvex functions, then their supremum $h = \max\{g, f\}$, defined by $h(x) = \max\{f(x), g(x)\}$, is quasiconvex.

Example 5.1. Let us consider, for example, the quasiconvex function $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} -t/2 + 5/2, & \text{if } t \leq 1, \\ -t + 3, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(t) = \begin{cases} t/2, & \text{if } t \leq 2, \\ 1, & \text{if } 2 < t < 3, \\ t - 3, & \text{otherwise.} \end{cases}$$

The obtained function $h = \sup\{f, g\}$ is semistrictly quasiconvex on $\mathbb{R}$:

Proposition 5.3. Let $f : X \to \mathbb{R}$ be a quasiconvex function, $\lambda \in \mathbb{R}$ and $x \in X$, such that that $\operatorname{int}(S_\lambda(f)) \neq \emptyset$ and $x \notin \operatorname{cl}(S_\lambda(f))$. Then, there exists an open convex neighborhood U of x and a nonempty open convex cone K such that

$$\forall\, u, v \in U, \quad (v - u \in K \Rightarrow f(u) \leq f(v))\,. \tag{5.3}$$

This property (5.3) is usually called *nondecreasing with respect to K on U* and is illustrated by the following example.

Example 5.2. Let us consider the quasiconvex function f defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} x + y, & \text{if } x + y > 0, \\ \max\{x, y\}, & \text{if } x \leq 0 \text{ and } y \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

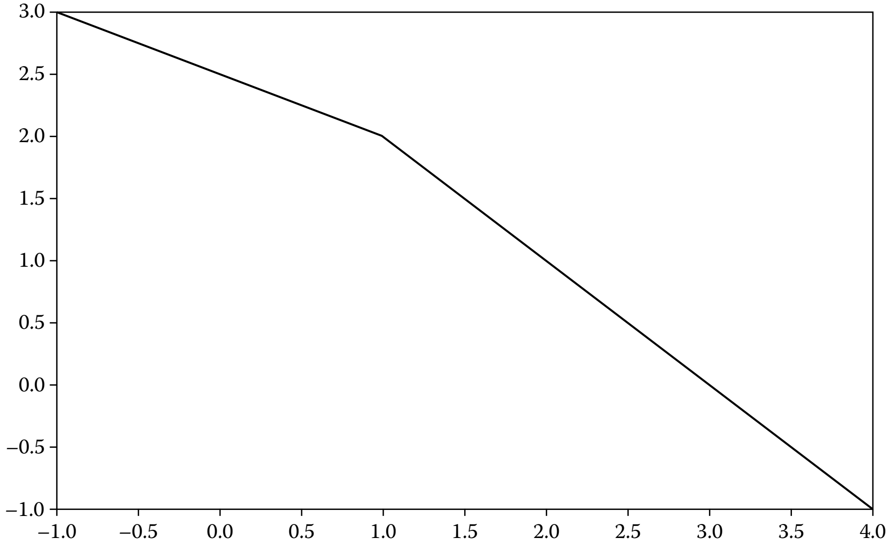


FIGURE 5.6: Function f.

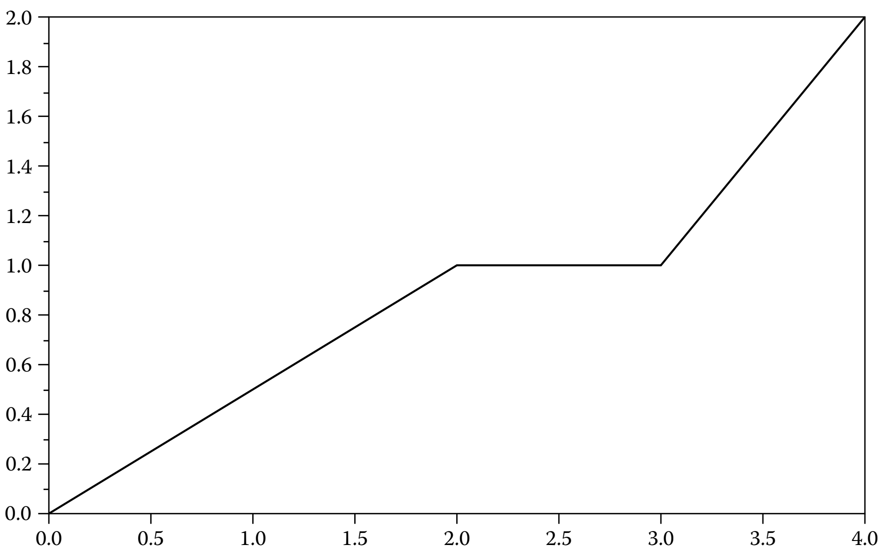


FIGURE 5.7: Function g.

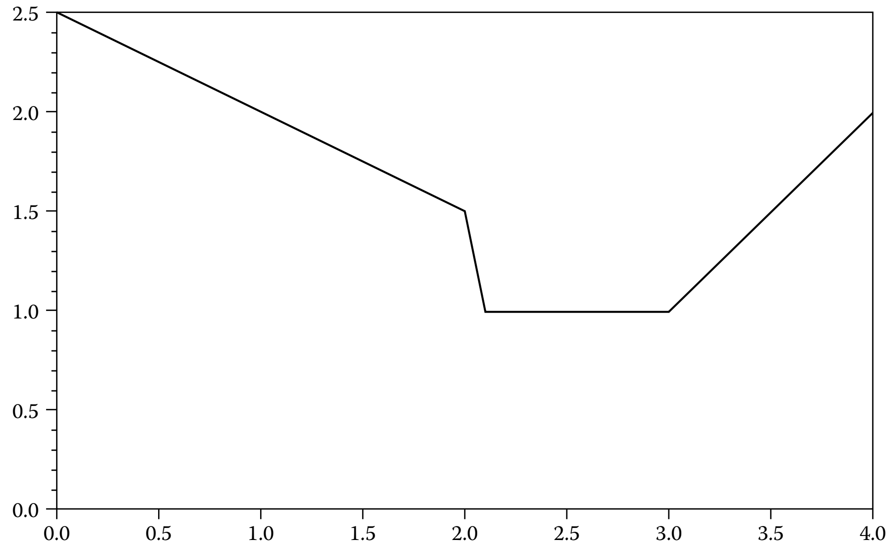
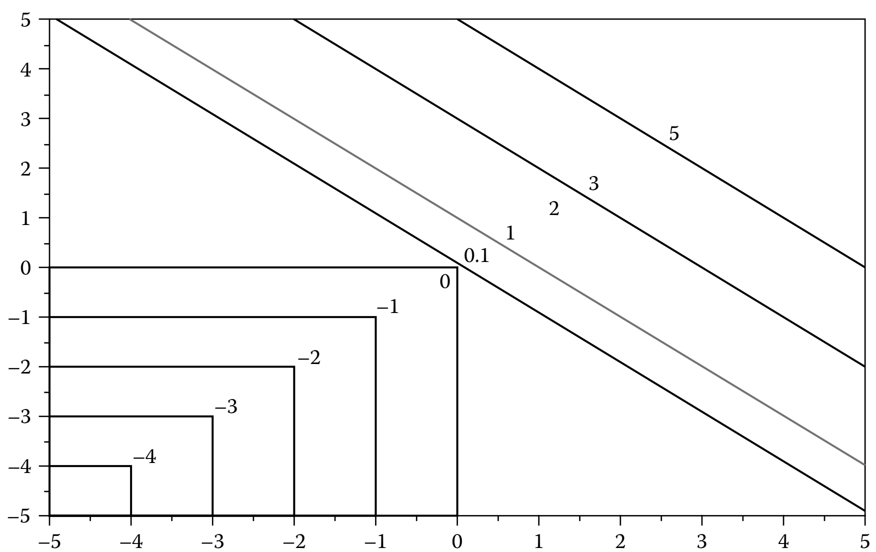


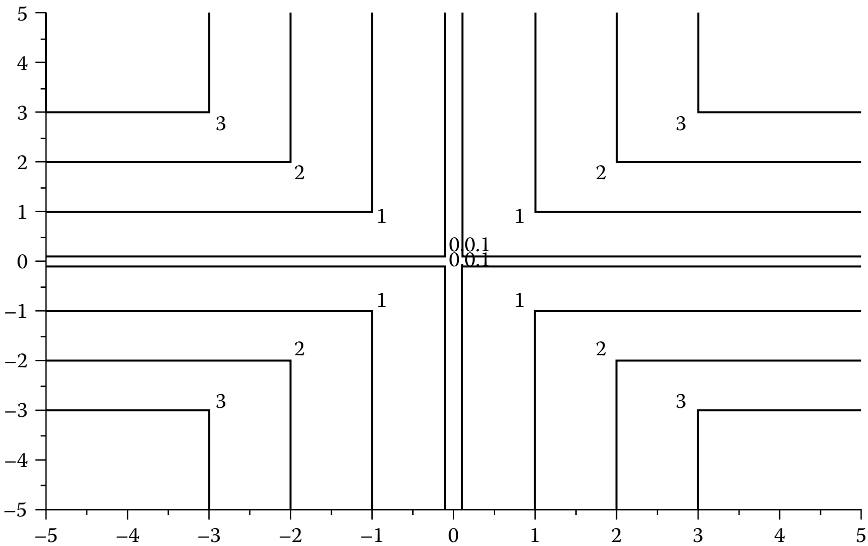
FIGURE 5.8: Function $h = \sup\{f, g\}$.



This quasiconvex function is nondecreasing with respect to the nonnegative orthant $K = (\mathbb{R}^+)^2$ on $U = B((x, y), 1)$ at any point $(x, y) \in \mathbb{R}^2$.

Let us nevertheless observe that the above proposition doesn't provide a characterization of quasiconvexity. Indeed, if one considers the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \min\{|x|, |y|\}$, then at any point (x, y) of $\mathbb{R}^2$, the function

f is nondecreasing with respect to some open convex cone $((\mathbb{R}^+)^2$, $(\mathbb{R}_-)^2$ or $\mathbb{R}^2)$ on some neighborhood of (x, y). But, as can be seen in the following figure of its sublevel sets



the function f is clearly not quasiconvex.

5.3.1 Continuity Properties of Quasiconvex Functions

In Proposition 5.2, we have seen that the quasiconvexity of a function is characterized by the quasiconvexity of its restriction along the lines. The same phenomenon occurs for the continuity properties of a quasiconvex function. The phenomenon is summarized in the following results, using the same notations as in Proposition 5.2.

Proposition 5.4. Let $f : X \to \mathbb{R}$ be a function and x be an element of the domain of f. Let U be a convex neighborhood x, K be an open convex cone, and $d \in K$. If f is nondecreasing with respect to K on U, then f is lower (upper) semicontinuous at x if and only if, for all $d \in X$, $f_{x,d}$ is lower (upper) semicontinuous at 0.

Combining the above proposition with Proposition 5.3, the semicontinuity of a quasiconvex function turns out to be equivalent to the semicontinuity of the restriction of the function on lines.

Proposition 5.5. Assume that $f : X \to \mathbb{R}$ is a quasiconvex function and that x, $y \in X$ are such that $f(y) < f(x)$ and f is upper semicontinuous at y. Set $d = y - x$. Then, f is lower semicontinuous (upper semicontinuous) at x if and only if $f_{x,d}$ is lower semicontinuous (upper semicontinuous) at 0.

Let us note that those semicontinuities along the lines are called lower and upper hemicontinuity by some authors.

Theorem 5.1. Assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex on $\mathbb{R}^n$ and that $x \in \operatorname{dom} f$. Then, f is lower semicontinuous (upper semicontinuous) at x if and only if f is lower semicontinuous (upper semicontinuous) along the lines at x.

5.3.2 Differentiability Properties of Quasiconvex Functions

Let us first recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be

- *Gâteaux-differentiable et* x if there exists a continuous linear function φ such that
$$\forall\, d \in X, \quad \lim_{t \to 0} \frac{f(x+td) - f(x)}{t} = \varphi(d);$$
- *Fréchet-differentiable et* x if there exists a continuous linear function $\varphi : X \to \mathbb{R}$ such that
$$\lim_{y \to x} \frac{f(y) - f(x) - \varphi(y-x)}{\|y-x\|} = 0.$$

Clearly, Fréchet differentiability is a stronger property than Gâteaux differentiability. Both concepts coincide if the considered function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is *locally Lipschitz* at some point x, that is, there exists $L \geq 0$ and a neighborhood U of x such that

$$|f(u) - f(v)| \leq L\|v-u\|, \text{ for any } u, v \in U.$$

But, as will be seen in the following propositions, Gâteaux and Fréchet differentiability coincide, in the finite dimensional setting, whenever the function is decreasing with respect to a cone, and in particular, whenever the function is quasiconvex.

Proposition 5.6 (Chabrillac-Crouzeix)**.** [2]) Let K be a nonempty open convex cone. Assume f is nondecreasing with respect to K and is Gâteaux-differentiable at $x \in \operatorname{dom} f$. Then, f is Fréchet-differentiable at x.

Corollary 5.1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function and $x \in \operatorname{dom} f$. If f is Gâteaux-differentiable at x, then it is Fréchet-differentiable at x.

Proposition 5.7 (Chabrillac-Crouzeix)**.** [17] Let K be an open nonempty convex cone of $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing with respect to K. Then, f is almost everywhere Fréchet-differentiable.

Combining this result with Proposition 5.3, we can prove that Fréchet differentiability of a quasiconvex function holds almost everywhere. Nevertheless a direct proof is given in Crouzeix [20].

Remark 5.1. Since the continuity and differentiability results described in Subsections 5.3.1 and 5.3.2 are well known, the proofs are not included. The interested reader can refer to Crouzeix [18, 20, 21] for precise proofs.

5.3.3 Associated Monotonicities

As for many properties of functions, convexity and generalized convexity are linked to first-order properties, that is, properties of the derivative, if it exists, or of generalizations of the derivative in case of nonsmooth functions.

Let us recall that a set-valued map $T : X \to 2^{X^*}$ is said to be

- *monotone* if for every (x, x^*), $(y, y^*) \in \mathrm{Gr}T$, the following implication holds
$$\langle y^* - x^*, y - x\rangle \geq 0;$$
- *quasimonotone* if for every (x, x^*), $(y, y^*) \in \mathrm{Gr}T$, the following implication holds
$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$
- *cyclically quasimonotone* if for every $(x_i, x_i^*) \in \mathrm{Gr}T$, $i = 1, 2, \ldots, n$, the following implication holds
$$\langle x_i^*, x_{i+1} - x_i\rangle > 0, \forall i = 1, 2, \ldots n-1 \Rightarrow \langle x_n^*, x_{n+1} - x_n\rangle \leq 0$$
where $x_{n+1} = x_1$.

Clearly, any monotone map is also quasimonotone and every cyclically quasimonotone operator is also quasimonotone.

It is well known that a differentiable function $f : X \to \mathbb{R}$ is convex if and only if its derivative f' of f is monotone. A similar link exists for quasiconvexity. More precisely, if $f : X \to \mathbb{R}$ is differentiable on X, then

$$f \text{ is quasiconvex } \Leftrightarrow f' \text{ is quasimonotone.}$$

The above link between quasiconvexity and quasimonotonicity can be extended to nonsmooth function thanks to the use of the subdifferential. Many subdifferentials exist, most of them being included in a concept of an abstract subdifferentials developed in [6, 7]. In the following result we will limit ourselves to the case of the *lower Dini subdifferential*

$$\partial f(x) = \left\{ x^* \in X^* \ : \ \langle x^*, d\rangle \leq \liminf_{t \searrow 0} \frac{f(x+td) - f(x)}{t}, \ \ \forall\, d \in X \right\}.$$

For different kinds of generalized directional derivatives and subdifferentials, we refer [1].

Proposition 5.8. Let X be a Banach space that admits Gâteaux-differentiable renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then, the following assertions are equivalent:

(a) f is quasiconvex;

(b) $\exists\, x^* \in \partial f(x) \;:\; \langle x^*, y - x\rangle > 0 \Rightarrow f(z) \leq f(y),\ \forall\, z \in [x, y]$;

(c) ∂f is quasimonotone.

Actually, as shown in the next section, the derivative/subdifferential approach is not really adapted to the quasiconvex setting. We will instead develop the so-called *normal approach* based on the concept of the "normal operator."

5.4 Normal Operator: A Natural Tool for Quasiconvex Functions

In the Introduction of this chapter, we have seen that, whenever the objective function and the constraint set of the minimization problem are convex, then the problem

$$\text{find} \;\; \bar{x} \in C \;\; \text{such that} \;\; \langle \nabla f(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \forall\, y \in C, \tag{5.4}$$

is actually a necessary and sufficient optimality condition, that is, each solution of the above problem is a global minimizer of f over C and vice versa. This problem (5.4) is called the *Stampacchia variational inequality* and is defined by the function ∇f and the subset C. Existence results for this kind of variational inequality will be discussed in Section 5.6.

However when the objective function is not convex but quasiconvex, then problem (5.4) is no longer a sufficient optimality condition. Indeed, if we consider, for example, the quasiconvex function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ and the constraint set $K = [-1, 1]$, then clearly the unique solution of problem (5.4) is the real number $\bar{x}_1 = 0$ while the unique minimizer of f over C is $\bar{x}_2 = -1$.

The above example shows that problem (5.4) is not adapted to provide reformulation, that is, necessary and sufficient conditions, in the case of quasiconvex optimization. And it is not really surprising if we try to geometrically interpret the situation. Indeed, let us observe that from the point of view of optimization, there is no difference between the following two problems:

$$(P1) \qquad \inf_{x \in [-1,1]} x \quad \text{and} \quad (P2) \quad \inf_{x \in [-1,1]} x^3,$$

that is, $(P1)$ and $(P2)$ share the same solution set $\mathcal{S} = \{-1\}$. But from the point of view of problem 5.4, there is an important difference because the derivative of the objective function of $(P1)$ is constant and equal to 1- while for problem $(P2)$, the derivative of the objective function f_2 is given by $f_2'(x) = 3x^2$.

In the sequel, we will mainly deal with operators whose values are convex cones. For such operators, since the values are unbounded, we have to consider a modified definition of upper semicontinuity. We first recall that a convex subset C of a convex cone L in X^* is called a *base of* C if $0 \notin \overline{C}^*$ and $L = \bigcup_{t \geq 0} tC$.

Definition 5.2. An operator $T : X \to 2^{X^*}$ whose values are convex cones is called norm-to-w^* *cone-upper semicontinuous* at $x \in \operatorname{dom} T$ if there exists a neighborhood U of x and a base $C(u)$ of $T(u)$ for each $u \in U$, such that $u \to C(u)$ is norm-to-w^* upper semicontinuous at x.

It turns out that we may always suppose that, locally, the base $C(u)$ is the intersection of $T(u)$ with a fixed hyperplane. To see this, we first define a conic w^*-neighborhood of a cone L in X^* to be a w^*-open cone M (i.e., a w^*-open set such that $tM \subseteq M$ for all $t > 0$) such that $L \subseteq M \cup \{0\}$.

Proposition 5.9. Let $T : X \to 2^{X^*}$ be a set-valued operator whose values are convex cones that are different from $\{0\}$. Given $x \in \operatorname{dom} T$, the following statements are equivalent:

(a) T is norm-to-w^* cone-upper semicontinuous at x.

(b) $T(x)$ has a base, and for every conic w^*-neighborhood M of $T(x)$ there exists a neighborhood U of x such that $T(u) \subseteq M \cup \{0\}$ for all $u \in U$.

c) There exists a w^*-closed hyperplane A of X^* and a neighborhood U of x such that $\forall u \in U, D(u) = T(u) \cap A$ is a base of $T(u)$ and the operator D is norm-to-w^* upper semicontinuous at x.

A definition of upper semicontinuity suitable for cone-valued operators, similar to property (b) in the proposition above, was given in [25] (where continuity was taken with respect to the norm topology) and in [16] (where the definition was given in a finite-dimensional setting), the main difference being that in these papers no reference to bases was made.

5.4.1 The Semistrictly Quasiconvex Case

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Set the normal operators by

$$N(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \quad \forall y \in S_{f(x)}\right\},$$
$$N^<(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \quad \forall y \in S^<_{f(x)}\right\},$$

for every $x \in \operatorname{dom} f$, while we set $N(x) = N^{<}(x) = \emptyset$ for $x \notin \operatorname{dom} f$. Equivalently, $x^* \in N(x)$ if and only if the following implication holds:

$$\langle x^*, y - x \rangle > 0 \Rightarrow f(y) > f(x);$$

also, $x^* \in N^{<}(x)$ if and only if

$$\langle x^*, y - x \rangle > 0 \Rightarrow f(y) \geq f(x).$$

These "normal operators" were studied in [16] for functions defined on $\mathbb{R}^n$. They have interesting properties:

Proposition 5.10. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper quasiconvex function. Then,

(a) the operator N is cyclically quasimonotone;

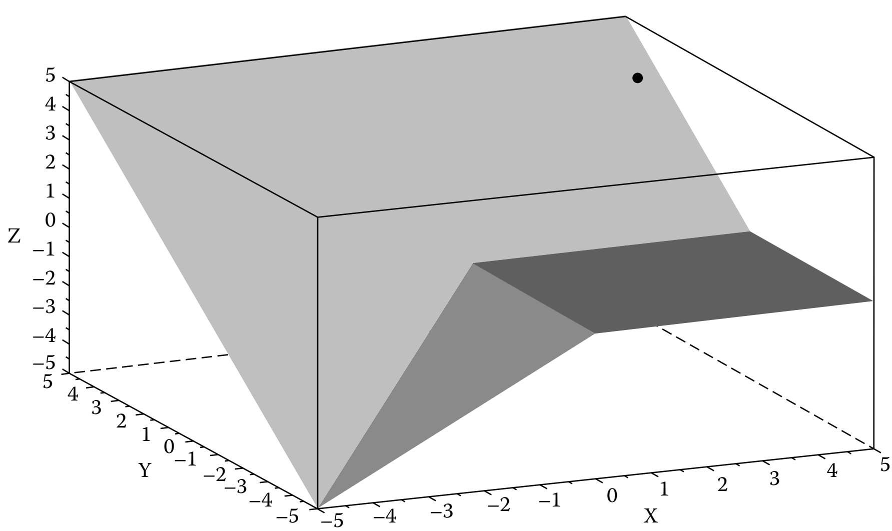
(b) the operator $N^{<}$ is cone-upper semicontinuous at every point x where f is lower semicontinuous, provided that there exists $\lambda < f(x)$ such that $\operatorname{int} S_\lambda \neq \emptyset$.

However, the two operators N and $N^{<}$ are essentially adapted to the class of quasiconvex functions such that any local minimum is a global minimum, in particular, semistrictly quasiconvex functions. Indeed, as observed above, in this case, for each $x \in \operatorname{dom} f \setminus \arg\min f$, the sets $S_{f(x)}$ and $S^{<}_{f(x)}$ have the same closure and thus N and $N^{<}$ coincide at x, that is $N(x) = N^{<}(x)$.

But for general quasiconvex functions, the operator N can fail to be closed. This can easily be seen in the following simple example taken from [16].

Example 5.3. Let us consider the function f, defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} \max\{x, y\}, & \text{if } x < 0 \text{ and } y < 0, \\ 0, & \text{if } (x \geq 0 \text{ and } y < 0) \text{ or if } y = 0, \\ y, & \text{otherwise.} \end{cases}.$$

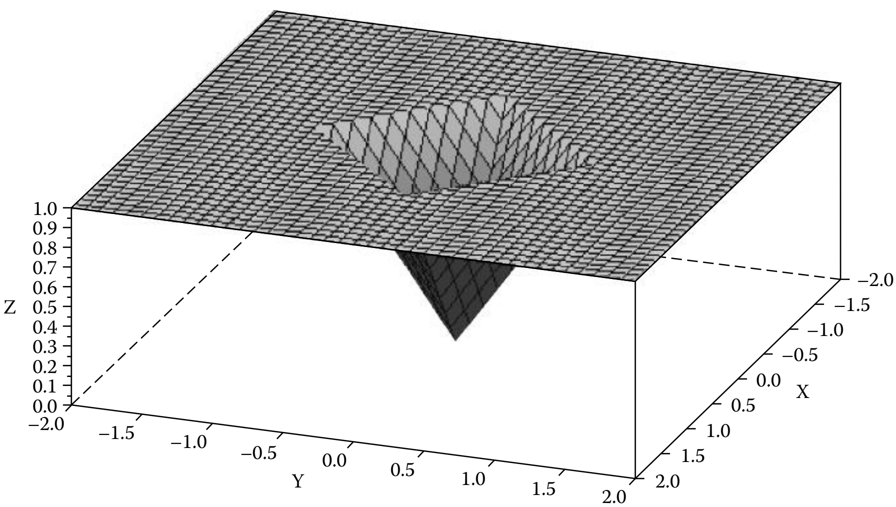


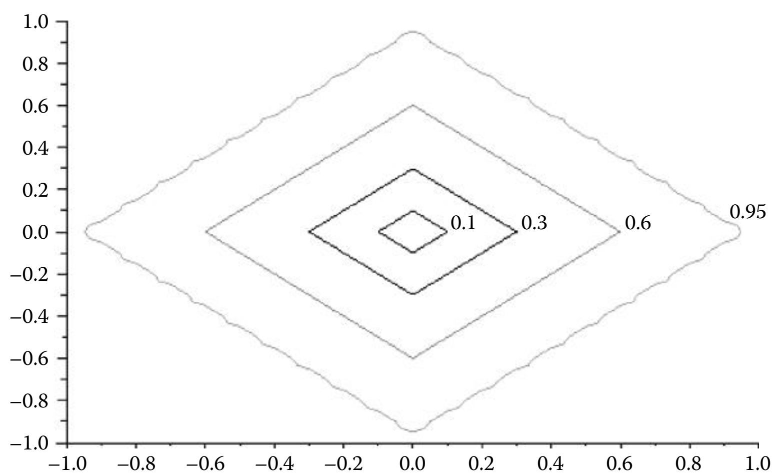
Then, $N((0,0)) = \{(0,v) \ : \ v \geq 0\}$, while for any $x > 0$, $N((x,x)) = (\mathbb{R}^+)^2$, thus showing that the operator N is not closed.

On the other hand, even if the operator $N^<$ is cone-upper semicontinuous under mild assumptions, $N^<$ is not quasimonotone in general as shown in the example below.

Example 5.4. Define the quasiconvex function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a,b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1, \\ 1, & \text{if } |a| + |b| > 1. \end{cases} .$$





Then, f is quasiconvex. Consider $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$, and $y^* = (2, 1)$. We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0)$, while $\langle x^*, y - x\rangle > 0$ and $\langle y^*, y - x\rangle < 0$. Hence $N^<$ is not quasimonotone.

5.4.2 The Adjusted Sublevel Set and Adjusted Normal Operator

We saw, in the previous section, that, even if each of the normal operators N and $N^<$ separately inherit either the quasimonotonicity or cone-upper semicontinuity, none of them have both properties for a general quasiconvex function. But as we will see in Section 5.7, it is important, in order to obtain existence result for optimization problems, to have a first-order tool satisfying both properties. We already know that it is not the case for derivative, subdifferential operators N and $N^<$.

In what follows, we will define an operator, called the *adjusted normal operator*, that has both these properties (cone-upper semicontinuity and quasimonotonicity).

5.4.2.1 Adjusted Normal Operator: Definitions

For any $x \in \operatorname{dom} f \setminus \arg\min f$, let us denote by ρ_x the positive real number $\rho_x = dist(x, S^<_{f(x)})$.

Definition 5.3. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function. To any element $x \in \operatorname{dom} f$, we associate the *adjusted sublevel set* $S^a_f(x)$ defined by

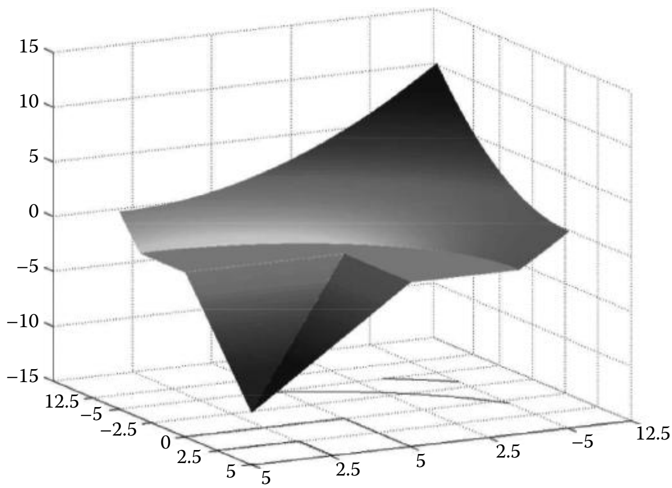
$$S^a_f(x) = S_{f(x)} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right),$$

if $x \notin \arg\min f$, and $S^a_f(x) = S_{f(x)}$ otherwise.

Clearly x is always an element of $S^a_f(x)$. If $x \in \operatorname{dom} f \setminus \arg\min f$ is such that $\rho_x = 0$, then $S^a_f(x) = S_{f(x)} \cap \overline{S^<_{f(x)}}$; if, moreover, f is lower semicontinuous on $\operatorname{dom} f$, then $S^a_f(x) = \overline{S^<_{f(x)}}$.

Example 5.5. Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \min\{x, y\}, & \text{if } x < 0 \text{ and } y < 0, \\ x^1 + y^2, & \text{if } x^1 + y^2 \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$



Then the sublevel set and adjusted sublevel set are given in Figures 5.9 and 5.10.

It is important to notice that one always has the following inclusions:

$$S_f^<(x) \subseteq S^a(x) \subseteq S_f(x), \quad \text{for all } x \in \operatorname{dom} f.$$

The convexity of the sublevel sets $S_{f(x)}$ (resp. strict sublevel sets $S_{f(x)}^<$) characterizes the quasiconvexity of the function. This still holds true for the adjusted sublevel sets.

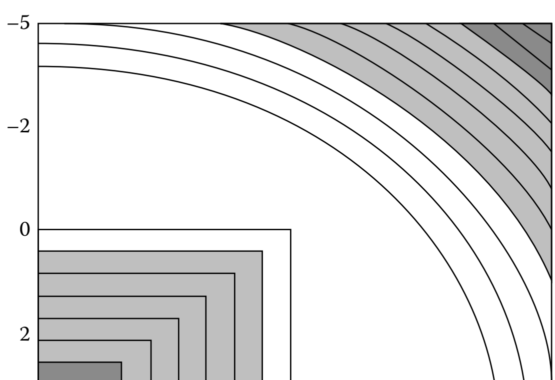


FIGURE 5.9: The sublevel sets.

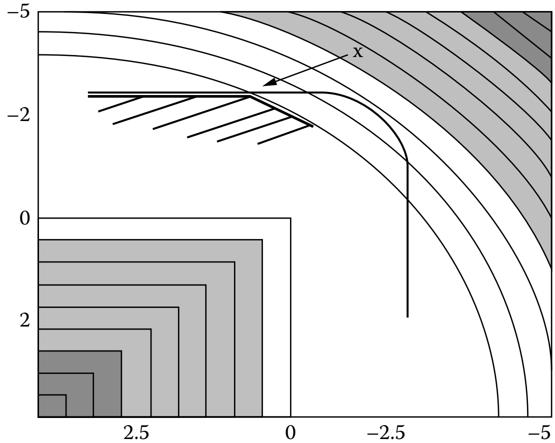


FIGURE 5.10: The adjusted sublevel sets at point x.

Proposition 5.11. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain $\operatorname{dom} f$. Then

$$f \text{ is quasiconvex} \quad \Leftrightarrow \quad S_f^a(x) \text{ is convex}, \quad \forall\, x \in \operatorname{dom} f.$$

Proof. Let us suppose that $S_f^a(u)$ is convex for every $u \in \operatorname{dom} f$. We will show that for any $x \in \operatorname{dom} f$, $S_{f(x)}$ is convex. If $x \in \arg\min f$, then $S_{f(x)} = S_f^a(x)$ is convex by assumption. Assume now that $x \notin \arg\min f$ and take $y, z \in S_{f(x)}$.

If both y and z belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then $y, z \in S_f^a(x)$, and thus $[y, z] \subseteq S_f^a(x) \subseteq S_{f(x)}$. Now, if both y and z do not belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then $f(x) = f(y) = f(z)$, $\overline{S^<_{f(z)}} = \overline{S^<_{f(y)}} = \overline{S^<_{f(x)}}$ and ρ_y, ρ_z are positive. If, say, $\rho_y \geq \rho_z$, then $y, z \in \overline{B}\left(\overline{S^<_{f(y)}}, \rho_y\right)$ and thus $y, z \in S_f^a(y)$ and $[y, z] \subseteq S_f^a(y) \subseteq S_{f(y)} = S_{f(x)}$.

Finally, suppose that only one of y, z, say z, belongs to $\overline{B}(S^<_{f(x)}, \rho_x)$ while $y \notin \overline{B}(S^<_{f(x)}, \rho_x)$. Then $f(x) = f(y)$, $\overline{S^<_{f(y)}} = \overline{S^<_{f(x)}}$ and $\rho_y > \rho_x$, so we have $z \in \overline{B}\left(S^<_{f(x)}, \rho_x\right) \subseteq \overline{B}\left(\overline{S^<_{f(y)}}, \rho_y\right)$ and we deduce as before that $[y, z] \subseteq S_f^a(y) \subseteq S_{f(y)} = S_{f(x)}$. The other implication is straightforward. □

Definition 5.4. To any function $f : X \to \mathbb{R} \cup \{+\infty\}$ we associate the set-valued map $N_f^a : \operatorname{dom} f \to 2^{X^*}$, called the *adjusted normal operator of* f, defined for any $x \in \operatorname{dom} f$ as the normal cone to the adjusted sublevel set $S_f^a(x)$ at x, that is,

$$N_f^a(x) = \left\{x^* \in X^* : \langle x^*, y - x \rangle \leq 0,\, \forall y \in S_f^a(x)\right\}.$$

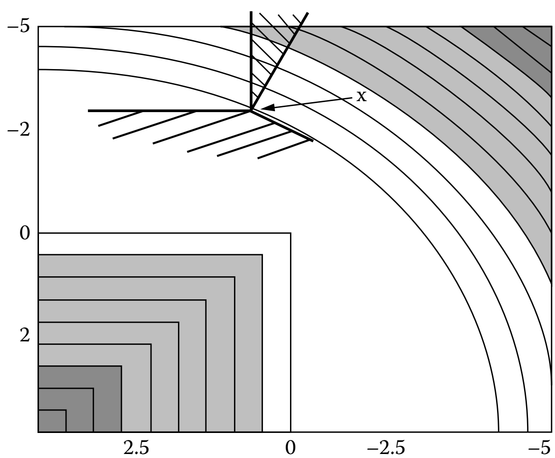
Note that since $S^<_{f(x)} \subseteq S^a_f(x) \subseteq S_{f(x)}$, one immediately have

$$N(x) \subseteq N^a_f(x) \subseteq N^<(x), \text{ for all } x \in \operatorname{dom} f. \tag{5.5}$$

Example 5.6. Using the function given in Example 5.5, the different normal operators are

$$\text{if } (x,y) = (-1,1) \text{ then } \begin{cases} N_f(x) = \{(0,0)\} \\ N^a_f(x) = \operatorname{cone}\{(0,1)\} \\ N^<_f(x) = \operatorname{cone}(\operatorname{co}\{(0,1),(1,1)\}), \end{cases}$$

whereas if $(x,y) = (0,1)$, then $N_f(x) = \operatorname{cone}\{(1,1)\}$ and $N^<_f(x) = \operatorname{cone}(\operatorname{co}\{(0,1),(1,0)\})$ and $N^a_f(x) = \operatorname{cone}(\operatorname{co}\{(0,1),(1,1)\})$ thus providing an example of strict inclusions in relation (5.5). Finally, the adjusted normal operator of another point x is described in the figure below.



In the next subsection, we investigate properties of the normal operator N^a_f for quasiconvex functions: equivalent definition, nonemptiness, quasimonotonicity, and cone upper semicontinuity are considered.

5.4.2.2 Some Properties of the Adjusted Normal Operator

We will give, for quasiconvex functions, an equivalent definition of N^a_f which clearly suggests that this operator corresponds to a refined version of the operator N. Let us first define for any $x \in \operatorname{dom} f$ the *extended normal cone* of f at x as follows. For every $x \in \operatorname{dom} f \setminus \arg\min f$ we set

$$EN(x) = \left\{ x^* \in X^* \ : \ \langle x^*, y\rangle \le \langle x^*, z\rangle, \ \forall\, y \in S^<_{f(x)}, \ \forall\, z \in \overline{B}(x, \rho_x) \right\},$$

while for $x \in \arg\min f$ we set $EN(x) = \{0\}$. Note that $EN(x)$ is a closed convex cone. In fact, for $x \in \operatorname{dom} f \setminus \arg\min f$ it is the normal cone at x to the set $S^<_{f(x)} + \overline{B}(0, \rho_x)$ or, equivalently, to its closure $\overline{B}(S^<_{f(x)}, \rho_x)$. In addition, x^* is an element of $EN(x)$ if and only if for all $y \in S^<_{f(x)}$ and all $v \in \overline{B}(0,1)$ one has $\langle x^*, x-y\rangle \geq -\rho_x\langle x^*, v\rangle$. Consequently, for any $x \in \operatorname{dom} f \setminus \arg\min f$, $EN(x)$ admits the following equivalent definition

$$x^* \in EN(x) \Leftrightarrow \langle x^*, x-y\rangle \geq \rho_x\|x^*\|, \quad \forall\, y \in S^<_{f(x)}. \tag{5.6}$$

Proposition 5.12. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be quasiconvex. Then for each $x \in \operatorname{dom} f$,

$$N^a_f(x) = N(x) + EN(x) = \operatorname{co}\left(N(x) \cup EN(x)\right). \tag{5.7}$$

Before proving Proposition 5.12, let us state the following well-known basic lemma, which can be found in [12].

Lemma 5.1. Let A, B be convex subsets of X. If $A \cap \operatorname{int} B \neq \emptyset$, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Proof. (of Proposition 5.12) If $x \in \arg\min f$, the equality is obvious. Assume that $x \notin \arg\min f$. We consider two cases. If $\rho_x = 0$, then $S^a_f(x) = \overline{S^<_{f(x)}} \cap S_{f(x)}$, thus $S^<_{f(x)} \subseteq S^a_f(x) \subseteq \overline{S^<_{f(x)}}$. It follows that $N^a_f(x) = N^<(x) = EN(x)$. Since $N(x) \subseteq N^<(x)$, we have $N(x) + EN(x) = EN(x) = N^a_f(x)$.

Now assume that $\rho_x > 0$. Obviously, $N^a_f(x)$ is the normal cone to the set $\overline{S_{f(x)} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right)}$ at x. However,

$$S_{f(x)} \cap \operatorname{int}\overline{B}\left(S^<_{f(x)}, \rho_x\right) \supseteq S^<_{f(x)} \neq \emptyset, \tag{5.8}$$

hence by Lemma 5.1,

$$\overline{S_{f(x)} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right)} = \overline{S_{f(x)}} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right).$$

Therefore, $N^a_f(x)$ is the normal cone to $\overline{S_{f(x)}} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right)$ at x. From (5.8) and using [3, Th. 4.1.16] we deduce that $N^a_f(x) = N(x) + EN(x)$. The second equality is obvious. □

Using the above reformulation, it is now easy to prove some monotonicity properties for the adjusted normal operator.

Proposition 5.13. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function. Then,

(a) $EN \cap S^*(0,1)$ is cyclically monotone on any nonempty level subset $S^=_a = \{x \in X \ : \ f(x) = a\}$.

(b) N^a_f is cyclically quasimonotone.

Proof. (a) Let us consider $x_1, x_2, \ldots, x_n \in S_a^=$. We assume that $x_i \notin \arg\min f$, since otherwise $EN(x_i) \cap S^*(0,1)$ is empty. Set $x_{n+1} = x_1$ and take $x_i^* \in EN(x_i) \cap S^*(0,1)$, $i = 1, 2, \ldots n$. According to (5.6), for any $y \in S^<_{f(x_i)}$, $\langle x_i^*, x_i - y\rangle \geq \rho_{x_i}$. This yields

$$\|x_{i+1} - y\| \geq \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_i^*, x_i - y\rangle \geq \rho_{x_i} + \langle x_i^*, x_{i+1} - x_i\rangle,$$

from which, remembering that $f(x_i) = a$, we get

$$\forall i = 1, 2, \ldots n, \quad \rho_{x_{i+1}} = d(x_{i+1}, S^<_{f(x_i)}) \geq \rho_{x_i} + \langle x_i^*, x_{i+1} - x_i\rangle.$$

Adding the inequalities for all i's we obtain

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0;$$

that is, $EN \cap S^*(0,1)$ is cyclically monotone on $S_a^=$.

(b) If N_f^a is not cyclically quasimonotone, then there exist $x_i \in \mathrm{dom}\,(f)$, $x_i^* \in N_f^a(x_i)$, $i = 1, 2, \ldots, n$ such that

$$\langle x_i^*, x_{i+1} - x_i\rangle > 0, \quad i = 1, 2, \ldots, n, \tag{5.9}$$

where $x_{n+1} = x_1$.

Since $N_f^a(x_i) \subseteq N^<(x_i)$, (5.9) implies that for all $i = 1, 2, \ldots, n$, $f(x_i) \leq f(x_{i+1})$. Consequently, $f(x_1) = f(x_2) = \ldots = f(x_n)$. This means that $S^<_{f(x_i)}$ is the same for all i. We denote this set by A. From (5.9) and $x_i^* \in N_f^a(x_i)$, it also follows that $x_{i+1} \notin S_{f(x_i)} \cap \overline{B}(A, \rho_{x_i})$. Since $f(x_{i+1}) = f(x_i)$, we have $x_{i+1} \in S_{f(x_i)}$. Hence, $x_{i+1} \notin \overline{B}(A, \rho_{x_i})$ for all $i = 1, 2, \ldots, n$. It follows that $\rho_{x_{i+1}} > \rho_{x_i}$ for all $i = 1, 2, \ldots, n$. This easily leads to $\rho_{x_{n+1}} > \rho_{x_1}$, a contradiction. □

According to the preceding proposition, the operator N_f^a is always quasimonotone. This is thus an interesting point since it means that the quasimonotonicity of the operator N_f^a will not, as quasimonotonicity of the derivative, characterize the quasiconvexity of the function. Actually, the quasiconvexity of the associated function is characterized by the non-emptyness of the values of N_f^a on a dense subset of $\mathrm{dom}\,(f)$.

Proposition 5.14. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that either f is radially continuous, or $\mathrm{dom}\,(f)$ is convex and $\mathrm{int}(S_a) \neq \emptyset$, $\forall\, a > \inf_X f$. Then,

(a) if $N_f^a(x) \setminus \{0\}$ is nonempty on a dense subset of $\mathrm{dom}\,(f) \setminus \arg\min f$, then f is quasiconvex;

(b) if f is quasiconvex, then $N_f^a(x) \setminus \{0\} \neq \emptyset$, $\forall\, x \in \mathrm{dom}\,(f) \setminus \arg\min f$;

(c) f is quasiconvex if and only if $\mathrm{dom}\,(N_f^a \setminus \{0\}\,)$ is dense in $\mathrm{dom}\,(f) \setminus \arg\min f$.

Proof. (a) Looking closely into the proof of Proposition 11 of [22], one can observe that, under the assumptions of the present proposition, the function f is quasiconvex provided that the domain of $N^< \setminus \{0\}$ is dense in $\mathrm{dom}\, f \setminus \arg\min f$. Since $N_f^a(x) \setminus \{0\} \subseteq N^<(x) \setminus \{0\}$, the assertion follows.

(b) For every $x \in \mathrm{dom}\,(f) \setminus \arg\min f$, one has $x \notin S^<_{f(x)}$. It is known that a quasiconvex, lsc, and radially continuous function is continuous [22, Prop. 9]. Thus, our assumptions imply that $\mathrm{int}(S^<_{f(x)}) \neq \emptyset$. Hence there exists $x^* \in X^* \setminus \{0\}$ such that

$$\forall\, y \in S^<_{f(x)}, \forall\, z \in \overline{B}(x, \rho_x), \quad \langle x^*, y\rangle \leq \langle x^*, z\rangle.$$

Therefore, $x^* \in EN(x)$ and from Proposition 5.12 follows that $N_f^a(x)\setminus\{0\} \neq \emptyset$. Finally, assertion (c) corresponds to the conjunction of (a) and (b). □

Proposition 5.15. Let f be quasiconvex such that $intS_a \neq \emptyset$ for all $a > \inf f$. If f is lsc at $x \in \mathrm{dom}\,(f) \setminus \arg\min f$, then N_f^a is norm-to-w^* cone-upper semicontinuous at x.

Before proving Proposition 5.15, we establish the following lemma. For any set $U \subseteq X$, $N^<(U)$ denotes as usual the set $\cup_{x\in U} N^<(x)$.

Lemma 5.2. Let f be quasiconvex such that $intS_a \neq \emptyset$ for all $a > \inf f$. If f is lsc at $x \in \mathrm{dom}\, f \setminus \arg\min f$, then there exists a neighborhood U of x and an element $z \in X\setminus\{0\}$ such that the set $N^<(U) \cap A$, with $A = \{x^* \in X^* : \langle x^*, z\rangle = 1\}$, is a bounded base for the cone $N^<(U)$.

Proof. Choose $y_0 \in X$ and $\delta > 0$ such that $y_0 \in \mathrm{int}\, S^<_{f(x)-\delta}$. There exists $\varepsilon > 0$ such that

$$\forall z \in B\,(0,1)\,, f\,(y_0 + \varepsilon z) < f(x) - \delta.$$

Since f is lsc at x, we can choose $\varepsilon_1 > 0$ such that for every $u \in x + \varepsilon_1 B\,(0,1)$, $f\,(u) > f\,(x) - \delta$. Thus,

$$\forall u \in x + \varepsilon_1 B\,(0,1)\,,\; y_0 + \varepsilon B\,(0,1) \subseteq S^<_{f(u)}. \tag{5.10}$$

Set $\varepsilon_2 = \min\{\varepsilon/2, \varepsilon_1\}$, $U = x + \varepsilon_2 B\,(0,1)$. For every $u \in U$, from (5.10) we deduce that $f\,(y_0 + \varepsilon w) < f\,(u)$ for all $w \in B\,(0,1)$ and thus, for every $x^* \in N^<(u)$ we obtain:

$$\forall w \in B\,(0,1) \quad , \langle x^*, y_0 + \varepsilon w - u\rangle \leq 0.$$

It follows that

$$\begin{aligned} \varepsilon\,\|x^*\| &= \sup_{w\in B(0,1)} \langle x^*, \varepsilon w\rangle \leq \langle x^*, u - y_0\rangle \\ &= \langle x^*, x - y_0\rangle + \langle x^*, u - x\rangle \leq \langle x^*, x - y_0\rangle + \|x^*\|\,\frac{\varepsilon}{2}. \end{aligned}$$

Thus,

$$\forall u \in U, \ \ \forall x^* \in N^< (u) \ \ , \langle x^*, x - y_0 \rangle \geq (\varepsilon/2) \|x^*\| . \tag{5.11}$$

In particular, $\langle x^*, x - y_0 \rangle > 0$ whenever $x^* \in N^< (u) \backslash \{0\}$. Now set $A = \{x^* \in X^* : \langle x^*, x - y_0 \rangle = 1\}$. Obviously, for every $u \in U$ and $x^* \in N^< (u) \cap A$, one has $\|x^*\| \leq 2/\varepsilon$, i.e., $N^< (U) \cap A$ is bounded. □

Proof of Proposition 5.15. Let U and A be the neighborhood and hyperplane given by Lemma 5.2. Define $C(u) = N_f^a(u) \cap A$, $u \in U$. Obviously, $C(u)$ is a convex, w^*-compact base of $N_f^a(u)$. We have to show that C is norm-to-w^* usc at x. Define $D(u) = (N(u) \cup EN(u)) \cap A$, $u \in U$. We first show that D is norm-to-w^* usc. According to [24, Prop. 1.2.23] it is sufficient to show that if $(x_i, x_i^*)_{i \in I}$ is a net in $\operatorname{gr} D$ such that $x_i \to x$ in norm and $x_i^* \overset{w^*}{\rightharpoonup} x^*$, then $x^* \in D(x)$. Since obviously $x^* \in A$, we have to show that $x^* \in EN(x) \cup N(x)$. Since $x_i^* \in EN(x_i) \cup N(x_i)$, we may consider, without loss of generality, that either $x_i^* \in N(x_i)$ for all $i \in I$ or $x_i^* \in EN(x_i)$ for all $i \in I$.

Suppose first that $x_i^* \in N(x_i)$. For every $y \in S_{f(x)}^<$, there exists i_0 such that for all $i > i_0$, $f(y) < f(x_i)$. Thus, $\langle x_i^*, x_i - y \rangle \geq 0$. Taking into account that x_i^* are bounded as they belong to $N^< (U) \cap A$, we obtain at the limit $\langle x^*, x - y \rangle \geq 0$. This means that $x^* \in N^<(x)$. If x is not a local minimum, then $\rho_x = 0$ and hence $N^< (x) = EN(x)$, so that $x^* \in EN(x) \cup N(x)$ and we are done. If x is a local minimum, then for i sufficiently large, $f(x_i) \geq f(x)$. Hence, for every $y \in S_{f(x)}$ we have $y \in S_{f(x_i)}$. Consequently, $\langle x_i^*, x_i - y \rangle \geq 0$, thus implying that $\langle x^*, x - y \rangle \geq 0$ for all $y \in S_{f(x)}$. It follows that $x^* \in N(x) \subseteq EN(x) \cup N(x)$.

Now suppose that $x_i^* \in EN(x_i)$. Without loss of generality, we may assume that for all i's we have either $f(x_i) > f(x)$ or $f(x_i) \leq f(x)$. If $f(x_i) > f(x)$ holds, then $S_{f(x)} \subseteq S_{f(x_i)}^<$. Thus,

$$\forall y \in S_{f(x)}, \quad \langle x_i^*, x_i - y \rangle \geq 0$$

and at the limit $\langle x^*, x - y \rangle \geq 0$ for all $\forall y \in S_{f(x)}$, which shows that $x^* \in N(x)$. If on the contrary $f(x_i) \leq f(x)$ holds, then $S_{f(x_i)}^< \subseteq S_{f(x)}^<$; thus

$$\liminf \rho_{x_i} = \liminf \operatorname{dist}\left(x_i, S_{f(x_i)}^<\right) \geq \lim \operatorname{dist}\left(x_i, S_{f(x)}^<\right) = \rho_x. \tag{5.12}$$

Now, for each $y \in S_{f(x)}^<$ there exists $i_0 \in I$ such that for all $i > i_0$, $f(x_i) > f(y)$. Thus, $y \in S_{f(x_i)}^<$ and

$$\langle x_i^*, x_i - y \rangle \geq \rho_{x_i} \|x_i^*\| .$$

Using (5.12) and lower semicontinuity of $\|\cdot\|$ at x^*, we find

$$\forall y \in S_{f(x)}^<, \quad \langle x^*, x - y \rangle \geq \rho_x \|x^*\| ,$$

which means that $x^* \in EN(x)$. Thus, in all cases $x^* \in EN(x) \cup N(x)$. This shows that D is norm-to-w^* usc at x, as desired.

To show that C is norm-to-w^* usc at x, it is again sufficient to show that if $(x_i, x_i^*)_{i\in I}$ is a net in $\operatorname{gr} C$ such that $x_i \to x$ in norm and $x_i^* \overset{w^*}{\rightharpoonup} x^*$, then $x^* \in C(x)$. Note that in view of Proposition 5.12,

$$C(x_i) = \operatorname{co}((N(x_i) \cap A) \cup (EN(x_i) \cap A));$$

hence, each x_i^* can be written in the form $x_i^* = \lambda_i y_i^* + (1-\lambda_i) z_i^*$ where $y_i^* \in N(x_i) \cap A$, $z_i^* \in EN(x_i) \cap A$ and $\lambda_i \in [0,1]$. Since y_i^* and z_i^* are bounded (as they belong to $N^<(U) \cap A$), by considering subnets if necessary, we may assume that $y_i^* \overset{w^*}{\rightharpoonup} y^*$, $z_i^* \overset{w^*}{\rightharpoonup} z^*$ and $\lambda_i \to \lambda$. By the norm-to-w^* upper semicontinuity of D, we know that $y^*, z^* \in D(x)$; hence, $x^* \in C(x)$ and C is norm-to-w^* usc at x.

5.5 Optimality Conditions for Quasiconvex Programming

Variational inequalities provide a perfect formulation to express optimality conditions. They have thus been extensively studied during the last decades. One of the most classical variational inequalities is the so-called *Stampacchia variational inequalities* described as follows: given a nonempty subset C of X and a set-valued map $T : X \to 2^{X^*}$,

$S(T,C)$ find $\bar{x} \in C$ such that $\langle x^*, y - x\rangle \geq 0$, for some $x^* \in T(\bar{x})$.

The set of solutions of this problem will also be denoted by $S(T,C)$.

We are now in a position to state, in a very elementary way, a sufficient optimality condition for quasiconvex programming.

Proposition 5.16. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, radially continuous on $\operatorname{dom} f$, and C be a nonempty subset of $\operatorname{dom} f$.

If $C \subseteq \operatorname{int}(\operatorname{dom} f)$, then any solution of the Stampacchia variational inequality defined by the operator $N_f^a \setminus \{0\}$ on C is a global minimizer of f over C, that is,

$$\bar{x} \in S(N_f^a \setminus \{0\}, C) \quad \Rightarrow \quad f(\bar{x}) = \inf_C f.$$

It is important to emphasize that the above sufficient optimality condition is obtained without assuming that the constraint set C is convex. This allows us to use this result for quasiconvex mathematical problem with equilibrium constraints (MPEC) or even quasiconvex bilevel problems as in [14, 15].

Proof. Indeed, let $x, y \in \text{int}(\text{dom} f)$. According to Proposition 5.14, the quasiconvexity of f implies that $N_f^a(x) \setminus \{0\}$ is nonempty. Let us suppose that $\langle x^*, y - x \rangle \geq 0$. Again, one can construct a sequence $\{y_n\}_n$ of $\text{dom} f$ converging (radially) to y such that $\langle x^*, y_n - x \rangle > 0$ for any n, which implies that $y_n \notin S_f^<(x)$ since $x^* \in N_f^a(x) \subset N_f^<(x)$. It follows by the radial continuity of f that $f(y) \geq f(x)$. □

Remark 5.2. (a) From the proof of Proposition 5.16, one can easily see that the sufficient optimality condition still holds if the operator N_f^a is replaced by $N_f^<$.

(b) As shown by the following simple example, the assumption "$C \subseteq \text{int}(\text{dom} f)$" cannot be easily omitted. Indeed, if one considers the quasiconvex function $f(x_1, x_2) = x_2$ if $x_1 \geq 0$ and $+\infty$ otherwise with the set $C = \{0\} \times \mathbb{R}$, then any $x = (0, x_2) \in S(N_f^a \setminus \{0\}, C)$, while none of them is a minimum of f over C. Nevertheless, for some applications, assuming that the constraint set is included in the interior of the domain is somehow too restrictive. An alternative hypothesis was investigated in [12, Prop. 4.1].

Let us now concentrate on necessary optimality conditions for quasiconvex programming. We will first state a very general necessary condition in which the constraint set is not supposed to be convex, but only locally starshaped. Recall that a set is said to be

- *starshaped* at $\bar{x} \in C$ if $[\bar{x}, y] \subseteq C$ for any $y \in C$;
- *locally starshaped* at $\bar{x} \in C$ if there exists a positive real δ such that $C \cap B(\bar{x}, \delta)$ is starshaped at $\bar{x}$;
- *locally starshaped* if it is locally starshaped at any element $\bar{x}$ of C.

This concept of a locally starshaped set is very general. In order to have some picture in mind, the reader can think of a locally starshaped set as a locally finite union of convex sets.

Proposition 5.17. Let $f : X \to \mathbb{R}$ be a continuous semistrictly quasiconvex function and C be a locally starshaped subset of X. If $\bar{x} \in X$ is a local minimizer of f over C and $f(x) > \inf_X f$, then there exists $r > 0$ such that $\text{int}(S_{f(\bar{x})}) \cap C \cap B(\bar{x}, r) = \emptyset$.

Proof. Since f is semistrictly quasiconvex and $f(x) > \inf_X f$, $\overline{S_{f(\bar{x})}^<} = S_{f(\bar{x})}$ and, due to the continuity of f, $\text{int}(S_{f(\bar{x})}) = S_{f(\bar{x})}^<$. Let $r > 0$ be such that $B(\bar{x}, r) \cap C$ is starshaped at $\bar{x}$ and $f(u) \geq f(\bar{x})$ for any $u \in B(\bar{x}, r) \cap C$. Then $\text{int}(S_{f(\bar{x})}) \cap C \cap B(\bar{x}, r) = \emptyset$. Indeed, if there exists $\tilde{x} \in \text{int}(S_{f(\bar{x})}) \cap C \cap B(\bar{x}, r) = S_{f(\bar{x})}^< \cap C \cap B(\bar{x}, r)$, by semistrict quasiconvexity of f, one has $f(u) < \max\{f(\tilde{x}), f(\bar{x})\}$, for any $u \in]\tilde{x}, \bar{x}[$. But this is impossible since $]\tilde{x}, \bar{x}[\subseteq B(\bar{x}, r) \cap C$ and $\bar{x}$ is a minimizer of f over $B(\bar{x}, r) \cap C$. □

For the above property, a necessary optimality condition can be proved under a very weak assumption if X is an Asplund space; see [14]. Here we will state the result in finite dimensions.

Theorem 5.2. [14] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quasiconvex function and let C be a closed locally starshaped subset of $\mathbb{R}^n$. If $\bar{x} \in C$ is a local minimizer of f over C and one of the following hypotheses holds:

(i) f is upper semicontinuous, $f(\bar{x}) > \inf_X f$ and there exists $r > 0$ such that $\text{int}(S_{f(\bar{x})}) \cap C \cap B(\bar{x}, r) = \emptyset$, or

(ii) f is lower semicontinuous and strictly quasiconvex,

then there exists $\bar{x}^* \in N_f^a(\bar{x}) \setminus \{0\})$ such that $-\bar{x}^* \in N^L(C, \bar{x})$.

Here, $N^L(C, x)$ stands for the *limiting normal cone* to C at x. See, for example, [14] for the definition of this normal cone concept. Let us emphasize that such a "nonconvex" notion of the normal cone is needed since the above constraint set is only supposed to be locally starshaped. But of course, if C turns out to be a convex set, then $N^L(C, x)$ is simply the classical polar cone to C at the point x of C.

Combining Proposition 5.17 with a separation theorem, we immediately obtain the following necessary optimality condition.

Corollary 5.2. Let C be a nonempty convex subset of X, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be a continuous semistrictly quasiconvex function such that $\text{int}(S_{f(\bar{x})}) \neq \emptyset$ and $f(\bar{x}) > \inf_X f$. If $\bar{x}$ is a local minimum of f on C, then

$$0 \in N_f^a(\bar{x}) + N(C, \bar{x}).$$

Finally, if the function is semistrictly quasiconvex and the constraint set is convex, then combining Proposition 5.16 and Corollary 5.2, the sufficient condition given in Proposition 5.16 turns out to be necessary and sufficient.

Proposition 5.18. Let C be a convex subset of X, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be a continuous semistrictly quasiconvex function such that $\text{int}(S_{f(\bar{x})}) \neq \emptyset$ and $f(\bar{x}) > \inf_X f$. Then the following assertions are equivalent:

(a) $f(\bar{x}) = \min_C f$.

(b) $\bar{x} \in S(N_f^a \setminus \{0\}, C)$.

(c) $0 \in N_f^a(\bar{x}) \setminus \{0\} + N(C, \bar{x})$.

5.6 Stampacchia Variational Inequalities

In the previous section, we showed that the optimality conditions are expressed in terms of variational inequalities. It is thus of main importance to prove now some existence results for those problems under the weakest hypothesis. In the next two subsections, we distinguish between the finite dimensions case and the infinite dimensional setting.

5.6.1 Existence Results: The Finite Dimensions Case

One of the cornerstones in the history of the existence results for variational inequalities is the following theorem.

Theorem 5.3 (Stampacchia 1966)**.** Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ be a continuous function. Then, $S(f,C)$ is nonempty, that is,

$$\text{there exists } \; \bar{x} \in C \; \text{ such that } \; \langle f(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \forall\, y \in C.$$

Proof. The proof is essentially based on the basic properties of the projection of a point on a convex set. Indeed, let us consider the application $\psi : C \to C$ defined by

$$\psi(x) = P_C \circ (Id - f)(x) = P_C(x - f(x)),$$

where P_C stands for the projection map on the subset C. The function ψ is thus continuous on the convex compact set C, and therefore, according to the Brouwer fixed point theorem, there exists a point $\bar{x} \in C$ such that $\bar{x} = \psi(\bar{x})$, a fixed point of ψ on C. But this means that $\bar{x} = P_C(\bar{x} - f(\bar{x}))$, and therefore, combined with the characterization of the projection of a point on a convex subset C,

$$\langle \bar{x} - f(\bar{x}) - \bar{x}, y - \bar{x}\rangle \leq 0, \quad \forall\, y \in C,$$

thus proving that $\bar{x}$ is a solution of the Stampacchia variational inequality defined by f and C. □

The compactness assumptions can be easily replaced by a "coercivity condition":

Proposition 5.19. Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ be a continuous function. If the following condition holds

$$\exists\, r > 0 \text{ such that } S(f, C \cap \overline{B}(0,r)) \cap B(0,r) = \emptyset,$$

then $S(f,C)$ is nonempty.

Again, using a fixed point theorem, Proposition 5.3 can be adapted to Stampacchia variational inequalities defined by set-valued map. Observe that the theorem below still concerns the finite dimensional case.

Theorem 5.4. Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $T : C \to 2^{\mathbb{R}^n}$ be an upper semicontinuous map with convex compact values. Then, $S(T, C)$ is nonempty, that is,

$$\text{there exist } \bar{x} \in C \text{ and } \bar{x}^* \in T\bar{x}) \text{ such that } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

The proof is based on the classical Kakutani fixed point theorem (see, for example, [4]):

> *If C is a nonempty convex compact subset of $\mathbb{R}^n$ and $T : C \to 2^C$ is an upper semicontinuous map with nonempty convex and compact values map, then T admits at least a fixed point.*

Proof. (of Theorem 5.4) Since T is upper semicontinuous with compact values, then (see, for example, [4]) its range $T(C)$ is compact. Combining this with the continuity of the projection map P_C, one can then prove that the set-valued map $F : C \to 2^C$ defined by $F(x) = (Id - T) \circ P_C(x)$ is also upper semicontinuous with nonempty convex compact values.

Now, one can observe that $F(C - T(C)) = C - T(C)$. Therefore, F has a fixed point (say x_0) on $C - T(C)$, that is, $x_0 \in F(x_0) = (Id - T) \circ P_C(x_0)$. Now, if we set $\bar{x} = P_C(x_0)$ and $\bar{x}^* = \bar{x} - x_0 \in T(\bar{x})$, from the characterization of the projection, one has

$$\langle \bar{x}^*, y - \bar{x} \rangle \leq 0, \quad \forall\, y \in C$$

and thus $\bar{x}$ solves the Stampacchia variational inequality $S(T, C)$. □

Actually, Theorem 5.4 is equivalent to the Kakutani fixed point theorem. Indeed, let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $T : C \to 2^C$ be an upper semicontinuous map with nonempty convex compact values. Observing that the set-valued map $F = Id - T$ is also upper semicontinuous with nonempty convex compact values, Theorem 5.4 implies the solvability of the set-valued Stampacchia variational inequality defined by F and C, that is, there exists $\bar{x} \in C$ such that

$$\begin{aligned} \bar{x} \in S(Id - T, C) \quad &\Leftrightarrow \quad \begin{cases} \exists\, u^* \in (Id - T)(\bar{x}) \text{ such that} \\ \langle u^*, y - \bar{x} \rangle \geq 0, \ \forall\, y \in C \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \exists\, \bar{x}^* \in T(\bar{x}) \text{ such that} \\ \langle \bar{x} - \bar{x}^*, y - \bar{x} \rangle \geq 0, \ \forall\, y \in C. \end{cases} \end{aligned}$$

Taking $y = \bar{x}^*$, one has $\langle \bar{x} - \bar{x}^*, \bar{x}^* - \bar{x} \rangle \geq 0$, which shows that $\bar{x}$ is a fixed point of T.

For further reading on variational inequalities and nonsmooth optimization in the setting of finite dimensional spaces, we refer the reader to [1].

5.6.2 Existence Results: The Infinite Dimensional Case

We have seen, in the previous subsection, that if $X = \mathbb{R}^n$, then the compactness of the constraint set and the semicontinuity of the map are sufficient to ensure the existence of a solution of the Stampacchia variational inequality. This is no longer the case if X is an infinite dimensional space.

Either some special structure of the variational inequality will be needed, as in the classic Theorem 5.5 of Stampacchia, or some monotonicity assumption, namely some quasimonotonicity, as in Theorem 5.7.

Theorem 5.5 (Stampacchia 64)**.** Let H be a Hilbert space, f an element of $H^* = H$, and $a : H \times H \to H$ a bilinear application. Let C be a nonempty closed convex subset of H. If a is continuous and satisfies the following property

$$\exists\, \alpha > 0 \text{ such that } a(v,v) \geq \alpha\|v\|^2, \quad \forall\, v \in H,$$

then there exists $\bar{x} \in C$ such that $a(\bar{x}, y - \bar{x}) \geq \langle f, y - \bar{x}\rangle, \quad \forall\, y \in C.$

From the above existence theorem, one can easily deduce the famous Lax-Milgram theorem.

Theorem 5.6. If a is continuous and satisfies the following property

$$\exists\, \alpha > 0 \text{ such that } a(v,v) \geq \alpha\|v\|^2, \quad \forall\, v \in H,$$

then there exists a unique $\bar{x} \in H$ such that $a(\bar{x}, u) = \langle f, u\rangle$, for any $u \in H$.

Proof. From Theorem 5.5, there exists $u \in H$ such that $a(u, v-u) \geq \langle f, v-u\rangle$, for any $v \in H$, or in other words, $\varphi_u(v) \geq \varphi_u(u)$, for any $v \in H$ where $\varphi_u(v) = a(u,v) - \langle f, v\rangle$. But φ_u is linear and therefore $\varphi_u \equiv 0$. □

For general set-valued Stampacchia variational inequalities, several results have been proved assuming that the operator involved is monotone, and intense efforts have been made to weaken this monotonicity hypothesis. The strongest result is the following.

Theorem 5.7. If C is a nonempty compact convex subset of X, and $T : C \to 2^{X^*}$ is an upper hemi-continuous quasimonotone set-valued map with nonempty convex w*-compact values, then $S(T, C) \neq \emptyset$.

The proof of Theorem 5.7 is based on the famous Knaster-Kuratowski-Mazurkiewicz lemma stated in 1929:

> Assume that C is a nonempty closed convex subset of X and $\varphi : C \times C \to \mathbb{R}$ is a KKM-application with $\varphi(x, .)$ being upper semicontinuous quasiconcave, for any $x \in C$. If there exists $\tilde{x} \in C$ such that $\{y \in C \ : \ \varphi(\tilde{x}, y) \geq 0\}$ is compact, then there exists $\bar{y} \in C$ such that $\varphi(x, \bar{y}) \geq 0$, for all $x \in C$.

Let us recall that a bifunction $\varphi : C \times C \to \mathbb{R}$ is said to be a *KKM-application* if $\forall\, x_1, \dots, x_n \in C$, $\forall\, x \in \text{co}\{x_1, \dots, x_n\}$ $\exists\, i \in 1, \dots, n$ such that $\varphi(x_i, x) \geq 0$.

Proof of Theorem 5.7. Let us define the bifunction $\varphi : C \times C \to \mathbb{R}$ by $\varphi(x,y) = \inf_{x^* \in T(x)} \langle x^*, x - y\rangle$. This bifunction φ is continuous concave with respect to y and the subset $\{y \in C \ : \ \varphi(x,y) \geq 0\}$ is compact, for all $x \in C$.

If φ is a KKM map, then from the Knaster-Kuratowski-Mazurkiewicz lemma, there exists $\bar{y} \in C$ such that

$$\varphi(x, \bar{y}) \geq 0, \quad \forall\, x \in C. \tag{5.13}$$

Now if φ is not a KKM map, then one can find $x_1, \ldots, x_n \in C$ and $\bar{y} \in co\{x_1, \ldots, x_n\}$ such that $\varphi(x_i, \bar{y}) < 0$, for all $i = 1, \ldots, n$, that is,

$$\forall\, i = 1, \ldots, n, \quad \exists\, x_i^* \in T(x_i) \ : \ \langle x_i^*, \bar{y} - x_i\rangle > 0$$

and, for some $\rho > 0$, all $z \in B(\bar{y}, \rho) \cap C$ and all $\forall\, i$, there exists $x_i^* \in T(x_i)$ such that $\langle x_i^*, z - x_i\rangle > 0$. Now, by quasimonotonicity of T, one immediately obtains

$$\forall\, z \in B(\bar{y}, \rho) \cap C, \quad \forall\, z^* \in T(z), \quad \forall\, i = 1, \ldots, n, \langle z^*, z - x_i\rangle \geq 0,$$

which corresponds to a local version of (5.13). In both cases, the point $\bar{y}$ satisfies

$$\forall\, z \in B(\bar{y}, \rho) \cap C, \ \ \forall\, z^* \in T(z), \quad \langle z^*, z - \bar{y}\rangle \geq 0. \tag{5.14}$$

Let us prove that this point $\bar{y}$ is actually a solution of the Stampacchia variational inequality $S(T, C)$. Let x be any point of $C \setminus \{\bar{y}\}$. There exists $t_0 > 0$ such that $z_t = \bar{y} + t(x - \bar{y}) \in B(\bar{y}, \rho) \cap C$, for all $t \in]0, t_0[$. From (5.14), we have

$$\langle z_t^*, x - \bar{y}\rangle \geq 0, \quad \forall\, t \in]0, t_0[, \ \ \forall\, z_t^* \in T(z_t). \tag{5.15}$$

Let us now define the open subset $W = \{y^* \in X^* \ : \ \langle y^*, x - \bar{y}\rangle < 0\}$. If $T(\bar{y}) \subset W$, by upper hemicontinuity, $T(z_t) \subset W$, for any t sufficiently closed to 0, which is a contradiction of (5.15). Thus, $T(\bar{y}) \cap W$ is nonempty, and we can therefore find $\bar{y}^* \in T(\bar{y})$ satisfying $\langle \bar{y}^*, x - \bar{y}\rangle \geq 0$ or, equivalently,

$$\inf_{x \in C} \sup_{y^* \in T(\bar{y})} \langle y^*, x - \bar{y}\rangle \geq 0.$$

thus showing that $\bar{y} \in S(T, C)$, by a classic Sion minimax theorem.

A more general existence results has been obtained in [12].

Proposition 5.20. Let C be a convex subset of X such that $C \cap \overline{B}(0, n)$ is weakly compact for every $n \in \mathbb{N}$. Let $T : C \to 2^{X^*} \setminus \{\emptyset\}$ be a quasimonotone operator such that the following coercivity condition holds:

$$\begin{array}{l} \exists\, n \in \mathbb{N}, \ \forall\, x \in C \setminus \overline{B}(0, n), \ \exists\, y \in C \text{ with } \|y\| < \|x\| \\ \text{such that } \forall\, x^* \in T(x), \langle x^*, x - y\rangle \geq 0. \end{array} \tag{5.16}$$

Suppose, moreover, that for every $x \in C$ there exists a neighborhood V_x of x and an upper sign-continuous operator $S_x : V_x \cap C \to 2^{X^*} \setminus \{\emptyset\}$ with convex, w^*-compact values satisfying $S_x(y) \subseteq T(y)$, $\forall\, y \in V_x \cap C$. Then the Stampacchia variational inequality $S(T, K)$ admits at least one solution.

5.7 Existence Result for Quasiconvex Programming

Now, combining the sufficient optimality condition obtained in Section 5.5 with the existence results of the previous section, we can now establish a very general existence for the following constrained optimization problem

$$\inf_{x \in C} f(x),$$

where f is a lower semicontinuous quasiconvex function and C is a possibly nonconvex subset. Of course, it is clear that C is compact, then the existence of a solution for this optimization problem comes directly from the classical Wierstarss theorem. But our aim is to treat the more general case of a non-compact constraint set. Thus a coercivity condition will be needed for the adjusted normal operator:

$$\begin{array}{l} \exists\, n \in \mathbb{N},\ \forall\, x \in C \setminus \overline{B}(0,n),\ \exists\, y \in K \text{ with } \|y\| < \|x\| \\ \text{such that } \forall\, x^* \in N_f^a(x), \langle x^*, x-y\rangle \geq 0. \end{array} \tag{5.17}$$

Theorem 5.8. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc quasiconvex function, radially continuous on $\operatorname{dom}(f)$. Assume that for every $\lambda > \inf_X f$, $\operatorname{int}(S_\lambda) \neq \emptyset$. Let $K \subseteq \operatorname{dom}(f)$ be convex with $K^\perp = \{0\}$, such that $K \cap \overline{B}(0,n)$ is weakly compact for every $n \in \mathbb{N}$. If condition (5.17) holds with $T = N_f^a$, then there exists $x_0 \in K$ such that

$$\forall\, x \in K, \quad f(x) \geq f(x_0).$$

Proof. If $\arg\min f \cap K \neq \emptyset$, we have nothing to prove. Suppose that $\arg\min f \cap K = \emptyset$. According to Proposition 5.13, N_f^a is quasimonotone. Further, according to Proposition 5.15, it is norm-to-w^* cone-upper semicontinuous on K. Thus, all assumptions of Proposition 5.20 hold for the operator $N_f^a \setminus \{0\}$, so $S\left(N_f^a \setminus \{0\}, K\right) \neq \emptyset$. Finally, using Proposition 5.16 we infer that f has a global minimum on K. □

Corollary 5.3. Given the assumptions on f and K as in Theorem 5.8, assume that there exists $n \in \mathbb{N}$ such that for all $x \in K$, $\|x\| > n$, there exists $y \in K$, $\|y\| < \|x\|$ such that $f(y) < f(x)$. Then, there exists $x_0 \in K$ such that

$$\forall\, x \in K, \quad f(x) \geq f(x_0).$$

Proof. If $f(y) < f(x)$, then for every $x^* \in N_f^a(x) \subseteq N^<(x)$, $\langle x^*, y - x\rangle \leq 0$. Hence, coercivity condition (5.17) with $T = N_f^a$ holds. The corollary follows from Theorem 5.8. □

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Chapter 6

An Introduction to Variational-like Inequalities

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6.1 Introduction

The theory of variational inequalities started with the pioneer work of Fichera [26] and Stampacchia [64], independently, related to Signori's contact problem. Variational inequalities arise in models for a wide class of problems from science, social sciences, engineering, management, etc.; see, for example [2, 7, 8, 10, 13, 23, 30, 29, 32, 35, 36, 37, 41, 42, 43, 46, 47, 52, 58, 74] and the references therein. Because of its applications in branches of sciences, engineering, optimization, economics, equilibrium theory, etc. it has been extended and generalized in many different directions. It has been used as a tool to study different aspects of optimization problems; see, for example, [2, 23, 30, 28] and the references therein. Motivated by the application of variational inequalities to optimization problems, and the concept of invexity, Parida et al. [57] and Yang and Chen [72] independently replaced the linear term $y - x$ appearing in the formulation of variational inequalities by a vector-valued term $\eta(y, x)$, where η

is a vector-valued bifunction. Such variational inequality is called *variational-like inequality* or *pre-variational inequality*. Parida et al. [57] established some existence results for a solution of variational-like inequalities in the setting of the finite dimensional Euclidean space $\mathbb{R}^n$ by using the Kakutani fixed-point theorem. They studied the relation between a variational-like inequality with a mathematical programming problem. Yang and Chen [72] introduced a new class of non-convex and non-smooth functions, called *semi-preinvex functions*. They derived the Fritz-John condition by using an alternative theorem for semi-preinvex program and studied the variational-like inequality. They also derived a necessary condition for an optimal solution of an optimization problem. Some existence theorems for solutions of a variational-like inequality were also proved. In 1994, Siddiqi et al. [62] and Ansari and Yao [3] studied variational-like inequalities in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions. Noor [53, 55] studied the relationship between a variational-like inequality problem and an optimization problem. He proved that the minimum of the arcwise directional differentiable semi-invex functions can be characterized by the class of variational-like inequalities. He also established an existence result for a solution of a variational-like inequality problem in the setting of Hilbert spaces and under the strong monotonicity and Lipschitz continuity assumptions of the underlying mappings.

The aim of this chapter is to give an introduction to the theory of variational-like inequalities. We provide some relations between a nonconvex optimization problem and a variational-like inequality problem. Some existence results for a solution of variational-like inequalities are presented under different kinds of assumptions. We discuss two solution methods, namely, the auxiliary principle method and the proximal method, for finding the approximate solutions of variational-like inequalities.

For further details and study, we refer the reader to [1, 3, 4, 6, 9, 15, 16, 18, 19, 20, 21, 25, 38, 45, 53, 54, 55, 57, 62, 71, 72, 75, 77] and the references therein.

6.2 Formulations of Variational-like Inequalities

Let $\langle X, X^* \rangle$ be a dual system of locally convex spaces and K be a nonempty subset of X. Given two mappings $F : K \to X^*$ and $\eta(.,.) : K \times K \to X$, the *variational-like inequality problem* (VLIP) is to find $\bar{x} \in K$ such that

$$\langle F(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0, \quad \text{for all } y \in K. \tag{6.1}$$

The inequality (6.1) is called a *variational-like inequality*. A large number of research papers have appeared in the literature on different aspects of variational-like inequalities; see, for example, [1, 3, 4, 6, 9, 15, 16, 18, 19, 20, 21, 25, 38, 45, 53, 54, 55, 57, 62, 71, 72, 75, 77] and the references therein.

When $\eta(y,x) = y - x$, for all $x, y \in K$, the variational-like inequality (6.1) is known as a *variational inequality*. For further details on variational inequalities, we refer the reader to Chapter 5, [2, 10, 13, 23, 30, 29, 32, 35, 36, 37, 41, 42, 43, 52, 58] and the references therein.

Another problem that is closely related to the variational-like inequality problem is known as the *Minty variational-like inequality problem* (MVLIP), which is defined as follows:

$$\text{MVLIP} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \langle F(y), \eta(y, \bar{x}) \rangle \geq 0, \quad \text{for all } y \in K. \end{cases} \tag{6.2}$$

When $\eta(y,x) = y - x$, for all $x, y \in K$, the Minty variational-like inequality (6.2) is called a *Minty variational inequality* [50].

The Minty variational-like inequality problem (MVLIP) plays an important role in establishing the existence of a solution of a variational-like inequality problem (VLIP). Basically, we first establish some relations between VLIP and MVLIP and then we apply these results to study the existence of a solution of VLIP. We need the following definitions of important monotonicities to study a solution of variational-like inequalities.

Definition 6.1. Let X be a Banach space with its dual X^*, K be a nonempty subset of X, and $\eta : K \times K \to X$ be a mapping. A mapping $F : K \to X^*$ is said to be

- *η-monotone* if for all $x, y \in K$, $x \neq y$,

$$\langle F(y) - F(x), \eta(y,x) \rangle \geq 0;$$

- *strictly η-monotone* if for all $x, y \in K$, $x \neq y$,

$$\langle F(y) - F(x), \eta(y,x) \rangle > 0;$$

- *η-pseudomonotone* if for all $x, y \in K$, $x \neq y$,

$$\langle F(x), \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad \langle F(y), \eta(y,x) \rangle \geq 0;$$

- *strictly η-pseudomonotone* if for all $x, y \in K$, $x \neq y$,

$$\langle F(x), \eta(y,x) \rangle \geq 0 \quad \Rightarrow \quad \langle F(y), \eta(y,x) \rangle > 0.$$

Definition 6.2. Let K be a nonempty convex subset of a Banach space X and $\eta : K \times K \to X$ be a mapping. A mapping $F : K \to X^*$ is said to be *hemicontinuous* if for any fixed $x, y \in K$, the mapping $\lambda \mapsto \langle F(x + \lambda(y - x)), \eta(y,x) \rangle$ defined on $[0,1]$ is continuous.

The following lemma provides the relation between the variational-like inequality problem (6.1) and the Minty variational-like inequality problem (6.2).

Lemma 6.1 (Minty Lemma)**.** Let K be a nonempty subset of a Banach space X, and let $F : K \to X^*$ and $\eta : K \times K \to X$ be mappings such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. The following assertions hold.

(a) If F is η-pseudomonotone, then every solution of VLIP (6.1) is a solution of MVLIP (6.2).

(b) If K is convex and F is hemicontinuous, such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex for all $x, z \in K$, then every solution of MVLIP (6.2) is a solution of VLIP (6.1).

Proof. (a) By η-pseudomonotonicity of F, it is obvious that every solution of VLIP (6.1) is a solution of MVLIP (6.2).

(b) Let $\bar{x} \in K$ be a solution of MVLIP (6.2). Then, for all $y \in K$ and $\lambda \in\ ]0,1]$, $z_\lambda := \lambda y + (1-\lambda)\bar{x} \in K$, and hence,

$$\langle F(z_\lambda), \eta(z_\lambda, \bar{x})\rangle \geq 0, \quad \text{for all } \lambda \in\ ]0,1].$$

Since $\eta(x,x) = \mathbf{0}$ and the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex, we have

$$\begin{aligned} 0 &\leq \langle F(z_\lambda), \eta(z_\lambda, \bar{x})\rangle \\ &\leq \lambda\langle F(z_\lambda), \eta(y, \bar{x})\rangle + (1-\lambda)\langle F(z_\lambda), \eta(\bar{x}, \bar{x})\rangle \\ &= \lambda\langle F(z_\lambda), \eta(y, \bar{x})\rangle, \end{aligned}$$

that is,

$$\langle F(z_\lambda), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } \lambda \in\ ]0,1].$$

By the hemicontinuity of F, we have

$$\langle F(\bar{x}), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of VLIP (6.1). □

Lemma 6.2. Let K be a nonempty closed convex subset of a Banach space X and $\eta : K \times K \to X$ be mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex. If the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is concave, then the solution set of VLIP (6.1) is convex. Further, if the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous, then the solution set of VLIP (6.1) is closed.

Proof. In view of above lemma, the solution sets of VLIP (6.1) and MVLIP (6.2) are same. Therefore, it is sufficient to show that the solution set of MVLIP (6.2) is closed and convex.

Let $\bar{x}$ and $\hat{x}$ be any two solutions of MVLIP (6.2). Then, for all $y \in K$,

$$\langle F(y), \eta(y, \bar{x}) \rangle \geq 0 \quad \text{and} \quad \langle F(y), \eta(y, \hat{x}) \rangle \geq 0.$$

Multiplying the first inequality by $\lambda \in [0, 1]$ and the second inequality by $1 - \lambda$, and then adding the results we get

$$\begin{aligned} 0 &\leq \langle F(y), \lambda\eta(y, \bar{x}) \rangle + \langle F(y), (1-\lambda)\eta(y, \hat{x}) \rangle \\ &= \langle F(y), \lambda\eta(y, \bar{x}) + (1-\lambda)\eta(y, \hat{x}) \rangle \\ &\leq \langle F(y), \eta(y, \lambda\bar{x} + (1-\lambda)\hat{x}) \rangle \end{aligned}$$

where the last inequality is due to the concavity of the mapping $x \mapsto \langle F(y), \eta(y, x) \rangle$. Hence, $\lambda\bar{x} + (1-\lambda)\hat{x}$ is a solution of MVLIP (6.2). Thus, the solution set of MVLIP (6.2) is convex.

Let $\{x_m\}$ be a sequence in the solution set of MVLIP (6.2) such that $x_m \to \bar{x}$ as $m \to \infty$. Then, for all $y \in K$,

$$\langle F(y), \eta(y, x_m) \rangle \geq 0, \quad \text{for all } m.$$

By the upper semicontinuity of the mapping $x \mapsto \langle F(y), \eta(y, x) \rangle$, we have

$$\langle F(y), \eta(y, \bar{x}) \rangle \geq \limsup_{m \to \infty} \langle F(y), \eta(y, x_m) \rangle \geq 0,$$

and thus, $\bar{x}$ is a solution of MVLIP (6.2). Therefore, the solution set of MVLIP (6.2) is closed. □

Lemma 6.3. If VLIP (6.1) is solvable, F is strictly η-monotone and $\eta(y, x) + \eta(x, y) = \mathbf{0}$, then the solution of VLIP (6.1) is unique.

Proof. Let $\bar{x}$ and $\hat{x}$ be two distinct solutions of VLIP (6.1). Then, for all $y \in K$, we have

$$\langle F(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0, \tag{6.3}$$

and

$$\langle F(\hat{x}), \eta(y, \hat{x}) \rangle \geq 0. \tag{6.4}$$

Putting $y = \hat{x}$ in (6.3) and $y = \bar{x}$ in (6.4) and then adding the results, we obtain

$$\langle F(\bar{x}), \eta(\hat{x}, \bar{x}) \rangle + \langle F(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle \geq 0.$$

Since $\eta(y, x) + \eta(x, y) = \mathbf{0}$, we have

$$\langle F(\bar{x}) - F(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle \leq 0.$$

Since F is strictly η-monotone, we have

$$\langle F(\bar{x}) - F(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle > 0.$$

From the above two inequalities, we get

$$\langle F(\bar{x}) - F(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle = 0,$$

which implies that $\bar{x} = \hat{x}$ by the strictly η-monotonicity of F. □

6.3 Variational-like Inequalities and Optimization Problems

6.3.1 Invexity

We adopt the following definition of affine mapping.

Let K be a nonempty convex subset of a vector space X. A mapping $g : K \to X$ is said to be *affine* if for all $x_1, x_2, \ldots, x_m \in K$ and $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$g\left(\sum_{i=1}^{m} \lambda_i x_i\right) = \sum_{i=1}^{m} \lambda_i g(x_i).$$

Definition 6.3. Let K be a nonempty subset of a vector space X and $\eta : K \times K \to X$ be a mapping. The set K is said to be *invex* with respect to (wrt). η if for all $x, y \in K$ and all $\lambda \in [0, 1]$, we have $x + \lambda\eta(y, x) \in K$.

Remark 6.1. It can be easily seen that any subset of X is invex wrt $\eta(y, x) = \mathbf{0}$, for all $x, y \in X$, where $\mathbf{0}$ is the zero vector of the vector space X. Mohan and Neogy [51] pointed out that the definition of an invex set essentially says that there is a path starting from x that is contained in K. It is not required that y be one of the end points of the path. However, if we demand that x be an end point of the path for every pair x, y, then $\eta(y, x) = y - x$, and reduces to convexity.

We say that the mapping η is *skew* if for all $x, y \in K$,

$$\eta(y, x) + \eta(x, y) = \mathbf{0}.$$

Condition C. [51] Let X be a vector space and $K \subseteq X$ be an invex set wrt $\eta : K \times K \to X$. Then, for all $x, y \in K$, $\lambda \in [0, 1]$,

(a) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$,

(b) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$.

Obviously, the mapping $\eta(y, x) = y - x$ satisfies Condition C. The examples of the mapping η that satisfy Condition C are given in [68, 69].

Remark 6.2. It is shown in [70] that Condition C implies that

$$\eta(x + s\eta(y, x), x) = s\eta(y, x), \quad \text{for all } s \in [0, 1].$$

The following example shows that a mapping η that satisfies Condition C may not be affine in the first argument and may not be skew and vice versa.

Example 6.1. [70] Let $K \subseteq \mathbb{R}$ be a nonempty set. Consider the mapping $\eta : K \times K \to \mathbb{R}$ defined by

$$\eta(y,x) = \begin{cases} y - x, & \text{if } x \geq 0,\ y \geq 0, \\ y - x, & \text{if } x \leq 0,\ y \leq 0, \\ -2 - x, & \text{if } x \leq 0,\ y > 0, \\ 2 - x, & \text{if } x > 0,\ y \leq 0. \end{cases}$$

Then, it is easy to see that η satisfies Condition C, but it is not affine in the first argument and not skew.

Consider the mapping $\eta : K \times K \to \mathbb{R}$ defined by $\eta(y,x) = 3(y-x)$, for all $x, y \in K \subseteq \mathbb{R}$. Then, η is affine in the first argument and skew, but does not satisfy Condition C.

The following lemma can be easily proved.

Lemma 6.4. [59] Let K be a nonempty convex subset of a vector space X and $\eta : K \times K \to X$ be a mapping. If η is affine in the first argument and skew, then it is also affine in the second argument.

The following definition of invex functions, pseudoinvex functions, strictly pseudoinvex functions, and quasi-invex functions were introduced and studied in [11, 31, 40].

Definition 6.4. Let K be a nonempty open subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be a mapping. A differentiable function $f : K \to \mathbb{R}$ is said to be

- *invex* wrt η if for all $x, y \in K$,

$$f(y) - f(x) \geq \langle \nabla f(x), \eta(y,x) \rangle;$$

- *pseudoinvex* wrt η if for all $x, y \in K$,

$$\langle \nabla f(x), \eta(y,x) \rangle \geq 0 \quad \text{implies} \quad f(y) \geq f(x),$$

  equivalently,

$$f(y) < f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle < 0;$$

- *strictly pseudoinvex* wrt η if for all $x, y \in K$, $x \neq y$,

$$\langle \nabla f(x), \eta(y,x) \rangle \geq 0 \quad \text{implies} \quad f(y) > f(x),$$

  equivalently,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle < 0;$$

- *quasi-invex* wrt η if for all $x, y \in K$,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle \leq 0.$$

The term "invex" was given by Craven [14] and stands for "invariant convex".

Definition 6.5. Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set wrt $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be *pre-quasi-invex* wrt η if

$$f(x + \lambda\eta(y,x)) \leq \max\{f(x), f(y)\}, \quad \text{for all } x, y \in K \text{ and } \lambda \in [0,1].$$

Lemma 6.5. [51] Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set wrt $\eta : K \times K \to \mathbb{R}^n$ such that η satisfies Condition C. A differentiable function $f : K \to \mathbb{R}$ is quasi-invex wrt η if and only if it is pre-quasi-invex wrt the same η.

Condition A. [68] Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$, and let $f : K \to \mathbb{R}$ be a function. Then,

$$f(x + \eta(y,x)) \leq f(y), \quad \text{for all } x, y \in K.$$

Lemma 6.6. [27] Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set wrt $\eta : K \times K \to \mathbb{R}^n$. Let $f : K \to \mathbb{R}$ be differentiable pseudoinvex wrt η, and f and η satisfy Condition A and C, respectively. Then, f is pre-quasi-invex wrt the same η.

For further details on invexity and invex functions, we refer to [11, 14, 27, 31, 40, 48, 49, 51, 51, 54, 55, 59, 60, 61, 65, 66, 67, 68, 69, 70] and the references therein.

6.3.2 Relations between Variational-like Inequalities and an Optimization Problem

Let K be a nonempty subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a function. The *minimization problem* (MP) is defined as follows:

$$\text{Minimize } f(x), \text{ subject to } x \in K. \tag{6.5}$$

Throughout this subsection, we assume that $f : K \to \mathbb{R}$ is a differentiable function on an open subset of K.

We consider the following variational-like inequality problem where $F(x) = \nabla f(x)$.

$$\text{VLIP} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \langle \nabla f(\bar{x}), \eta(y,\bar{x}) \rangle \geq 0, \quad \text{for all } y \in K, \end{cases} \tag{6.6}$$

where $\eta : K \times K \to \mathbb{R}^n$ is a mapping.

The following results provide the relations between the MP (6.5) and the VLIP (6.6).

Theorem 6.1. Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set wrt $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a function. Then, every solution of the MP (6.5) is a solution of the VLIP (6.6).

Proof. Let $\bar{x}$ be a solution of the MP (6.5). Then,

$$f(\bar{x}) \leq f(y), \quad \text{for all } y \in K.$$

Since K is invex, we have $\bar{x} + \lambda\eta(y, \bar{x}) \in K$ for all $\lambda \in [0, 1]$; thus,

$$\frac{f(\bar{x} + \lambda\eta(y, \bar{x})) - f(\bar{x})}{\lambda} \geq 0, \quad \text{for all } \lambda \in \ ]0, 1].$$

Taking the limit as $\lambda \to 0$, we obtain

$$\langle \nabla f(\bar{x}), \eta(y, \bar{x}) \rangle = \lim_{\lambda \to 0} \frac{f(\bar{x} + \lambda\eta(y, \bar{x})) - f(\bar{x})}{\lambda} \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of the VLIP (6.6). □

Theorem 6.2. Let $K \subseteq \mathbb{R}^n$ be a nonempty open set, $\eta : K \times K \to \mathbb{R}^n$ be a mapping, and $f : K \to \mathbb{R}$ be a differentiable function. If f is pseudoinvex wrt η, then every solution of VLIP (6.6) is a solution of the MP (6.5).

Proof. Assume that $\bar{x} \in K$ is a solution of VLIP (6.6), but not a solution of the MP (6.5). Then, there exists $y \in K$ such that

$$f(\bar{x}) > f(y). \tag{6.7}$$

By the pseudoinvexity of f, we obtain

$$\langle \nabla f(\bar{x}), \eta(y, \bar{x}) \rangle < 0.$$

Thus, $\bar{x}$ is not a solution of VLIP (6.6), a contradiction to our assumption. Hence, $\bar{x} \in K$ is a solution of the MP (6.5). □

Corollary 6.1. Let $K \subseteq \mathbb{R}^n$ be a nonempty open set, $\eta : K \times K \to \mathbb{R}^n$ be a mapping, and $f : K \to \mathbb{R}$ be a differentiable invex function wrt η. Then, every solution of the VLIP (6.6) is a solution of the MP (6.5).

Proof. Since every invex function is pseudoinvex, we obtain the desired result from Theorem 6.2. □

Theorem 6.3. Let $K \subseteq \mathbb{R}^n$ be a nonempty open set, $\eta : K \times K \to \mathbb{R}^n$ be a mapping, and $f : K \to \mathbb{R}$ be a differentiable function such that $-f$ is invex wrt η, that is, $f(y) - f(x) \leq \langle \nabla f(x), \eta(y, x) \rangle$, for all $x, y \in K$. Then, every solution of the MP (6.5) is a solution of the VLIP (6.6).

Proof. Assume that $\bar{x}$ is a solution of the MP (6.5), but not a solution of the VLIP (6.6). Then, there exists $y \in K$ such that

$$\langle \nabla f(\bar{x}), \eta(y, \bar{x}) \rangle < 0. \tag{6.8}$$

Since $-f$ is invex wrt η, we have

$$f(y) - f(\bar{x}) \leq \langle \nabla f(\bar{x}), \eta(y, \bar{x}) \rangle. \tag{6.9}$$

Combining (6.8) and (6.9), we obtain

$$f(\bar{x}) > f(y),$$

a contradiction of our assumption that $\bar{x}$ is a solution of the MP (6.5). Hence, $\bar{x}$ is a solution of the VLIP (6.6). □

Definition 6.6. Let X be a Banach space and $K \subseteq X$ be a nonempty invex set wrt $\eta : K \times K \to X$. A mapping $F : K \to X^*$ is said to be *η-hemicontinuous* if for any fixed $x, y \in K$, the mapping $\lambda \mapsto \langle F(x + \lambda\eta(y, x)), \eta(y, x)\rangle$ defined on $[0, 1]$ is continuous.

Lemma 6.7. Let K be a nonempty subset of a Banach space X, $F : K \to X^*$ be mapping, and $\eta : K \times K \to X$ be skew such that Condition C holds. If K is invex wrt η and F is η-hemicontinuous, then every solution of MVLIP (6.2) is a solution of VLIP (6.1).

Proof. Let $\bar{x} \in K$ be a solution of MVLIP (6.2). Then, for all $y \in K$ and $\lambda \in \ ]0, 1]$, $z_\lambda := \bar{x} + \lambda\eta(y, \bar{x}) \in K$, and hence,

$$\langle F(z_\lambda), \eta(z_\lambda, \bar{x})\rangle \geq 0, \quad \text{for all } \lambda \in \ ]0, 1].$$

Since η is skew, we have

$$\langle F(z_\lambda), \eta(\bar{x}, z_\lambda)\rangle \leq 0, \quad \text{for all } \lambda \in \ ]0, 1],$$

and $\eta(x, x) = \mathbf{0}$, for all $x \in K$. Therefore, we have

$$0 \geq \langle F(z_\lambda), \eta(\bar{x}, z_\lambda)\rangle = -\lambda\langle F(z_\lambda), \eta(y, \bar{x})\rangle,$$

that is,

$$\langle F(z_\lambda), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } \lambda \in \ ]0, 1].$$

By the η-hemicontinuity of F, we have

$$\langle F(\bar{x}), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of VLIP (6.1). □

Theorem 6.4. Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : K \times K \to \mathbb{R}^n$ be a skew mapping, and $f : K \to \mathbb{R}$ be strictly pseudoinvex wrt η. Then, every solution of the MP (6.5) is a solution of MVLIP (6.2) with $F(y) = \nabla f(y)$.

Proof. Assume that $\bar{x} \in K$ is a solution of MP (6.5), but not a solution of MVLIP (6.2) with $F(y) = \nabla f(y)$. Then, there exists $y \in K$ such that

$$\langle \nabla f(y), \eta(y, \bar{x})\rangle < 0.$$

By skewness of η, we have

$$\langle \nabla f(y), \eta(\bar{x}, y)\rangle > 0.$$

The strict pseudoinvexity of f wrt η implies

$$f(\bar{x}) > f(y),$$

a contradiction of our assumption that $\bar{x}$ is a solution of MP (6.5). □

Theorem 6.5. Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew and satisfies Condition C. Let $f : K \to \mathbb{R}$ be pseudoinvex wrt η such that Condition A is satisfied. Then, $\bar{x} \in K$ is a solution of the MVLIP (6.2) with $F(y) = \nabla f(y)$ if and only if it is a solution of MP (6.5).

Proof. Let $\bar{x} \in K$ be a solution of MVLIP (6.2) with $F(y) = \nabla f(y)$, but not a solution of MP (6.5). Then, there exists $y \in K$ such that

$$f(\bar{x}) > f(y). \tag{6.10}$$

Since f is pseudoinvex wrt η, it follows from Lemma 6.6 that f is pre-quasi-invex wrt the same η. Therefore, for $x_\lambda = \bar{x} + \lambda\eta(y, x)$, we have

$$f(x_\lambda) < f(\bar{x}), \quad \text{for all } \lambda \in \ ]0, 1[. \tag{6.11}$$

By the mean-value theorem, there exists $\alpha \in \ ]0, \lambda[$ such that

$$\begin{aligned} f(\bar{x} + \lambda\eta(y, \bar{x})) - f(\bar{x}) &= \langle \nabla f(\bar{x} + \alpha\eta(y, \bar{x})), \bar{x} + \lambda\eta(y, \bar{x}) - \bar{x} \rangle \\ &= \lambda \langle \nabla f(\bar{x} + \alpha\eta(y, \bar{x})), \eta(y, \bar{x}) \rangle, \end{aligned}$$

that is,

$$\langle \nabla f(\bar{x} + \alpha\eta(y, \bar{x})), \eta(y, \bar{x}) \rangle < 0.$$

By Remark 6.2, we have $\eta(\bar{x} + \alpha\eta(y, \bar{x}), \bar{x}) = \alpha\eta(y, x)$, and therefore,

$$\langle \nabla f(\bar{x} + \alpha\eta(y, \bar{x})), \eta(\bar{x} + \alpha\eta(y, \bar{x}), \bar{x}) \rangle < 0,$$

that is, $\bar{x}$ is not a solution of MVLIP (6.2) with $F(x_\lambda) = \nabla f(x_\lambda)$, a contradiction to our assumption. Hence, $\bar{x}$ is a solution of MP (6.5).

Conversely, suppose that $\bar{x} \in K$ is a solution of MP (6.5). Then,

$$f(\bar{x}) \leq f(y), \quad \text{for all } y \in K.$$

By Lemmas 6.6 and 6.5, f is quasi-invex wrt η, and hence,

$$\langle \nabla f(y), \eta(\bar{x}, y) \rangle \leq 0, \quad \text{for all } y \in K.$$

By skewness of η, we have

$$\langle \nabla f(y), \eta(y, \bar{x}) \rangle \geq 0, \quad \text{for all } y \in K;$$

that is, $\bar{x} \in K$ is a solution of MVLIP (6.2) with $F(y) = \nabla f(y)$. □

6.4 Existence Theory

This section deals with the study of the existence of solutions of variational-like inequalities.

Theorem 6.6. Let K be a nonempty weakly compact convex subset of a Banach space X, and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex and the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous, for all $x, z, y \in K$. Then, VLIP (6.1) has a solution.

Proof. Define two set-valued maps $P, Q : K \to 2^K$ by

$$P(y) = \{x \in K : \langle F(x), \eta(y,x)\rangle \geq 0\}, \quad \text{for all } y \in K,$$

and

$$Q(y) = \{x \in K : \langle F(y), \eta(y,x)\rangle \geq 0\}, \quad \text{for all } y \in K.$$

Then, P is a KKM map. Indeed, let $\{y_1, y_2, \ldots, y_m\}$ be a finite subset of K and $\tilde{x} \in \text{co}(\{y_1, y_2, \ldots, y_m\})$. Then, $\tilde{x} = \sum_{i=1}^m \lambda_i y_i$ for some $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ with $\sum_i^m \lambda_i = 1$. If $\tilde{x} \notin \bigcup_{i=1}^m P(y_i)$, then,

$$\langle F(\tilde{x}), \eta(y_i, \tilde{x})\rangle < 0, \quad \text{for all } i = 1, 2, \ldots, m,$$

and so,

$$\sum_{i=1}^m \lambda_i \langle F(\tilde{x}), \eta(y_i, \tilde{x})\rangle < 0.$$

Therefore, from the hypothesis, we have

$$\begin{aligned} 0 = \langle F(\tilde{x}), \eta(\tilde{x}, \tilde{x})\rangle &= \left\langle F(\tilde{x}), \eta\left(\sum_{i=1}^m \lambda_i y_i, \tilde{x}\right)\right\rangle \\ &\leq \sum_{i=1}^m \lambda_i \langle F(\tilde{x}), \eta(y_i, \tilde{x})\rangle < 0, \end{aligned}$$

a contradiction. Thus, we must have $\text{co}(\{y_1, y_2, \ldots, y_m\}) \subseteq \bigcup_{i=1}^m P(y_i)$, and hence, P is a KKM mapping.

Since F is an η-pseudomonotone map, and $P(y) \subseteq Q(y)$ for all $y \in K$, thus, Q is also a KKM map.

For each $y \in K$, $Q(y)$ is closed. Indeed, let $\{x_m\}$ be a sequence in $Q(y)$ for any fixed $y \in K$ such that x_m converges to $\hat{x} \in K$. Then,

$$\langle F(y), \eta(y, x_m)\rangle \geq 0, \quad \text{for all } m.$$

By the upper semicontinuity of the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$, we have

$$0 \leq \limsup_{m\to\infty} \langle F(y), \eta(y, x_m)\rangle \leq \langle F(y), \eta(y, \hat{x})\rangle$$

and thus, $\hat{x} \in Q(y)$. Therefore, $Q(y)$ is a closed subset of a weakly compact set K, and so it is weakly compact. By Fan-KKM Theorem 6.18, $\bigcap_{y\in K} Q(y) \neq \emptyset$. Hence, there exists $\bar{x} \in K$ such that

$$\langle F(y), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } y \in K.$$

By Lemma 6.1, $\bar{x}$ is a solution of VLIP (6.1). □

Since a Banach space X is reflexive if and only if every closed convex bounded subset of X is weakly compact, we have the following result.

Theorem 6.7. Let K be a nonempty closed convex bounded subset of a reflexive Banach space X and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex and the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous, for all $x, z, y \in K$. Then, VLIP (6.1) has a solution.

Example 6.2. Let $X = \mathbb{R}$, $K = [1, +\infty[$, and $\eta : K \times K \to X$ be defined by $\eta(y,x) = 2(y-x)$, for all $x, y \in K$. Let $F : K \to X^*$ be defined as $F(x) = x$, for all $x \in K$. Since K is not compact, Theorem 6.6 is not applicable. We note that $\eta(x,x) = \mathbf{0}$ for all $x \in K$, and F is η-pseudomonotone and hemicontinuous. However, it is easy to see that $\bar{x} = 1 \in K$ is such that $\langle F(\bar{x}), \eta(y,\bar{x})\rangle = 2\bar{x}(y-\bar{x}) = 2(y-1) \geq 0$ for all $y \in K$, that is, $\bar{x} = 1$ is a solution of VLIP (6.1).

When K is not necessarily bounded, we have the following results.

Theorem 6.8. Let K be a nonempty closed convex subset of a Banach space X and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex and the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous, for all $x, z, y \in K$. Assume that there exist a weakly compact subset C of X and $\tilde{y} \in K \cap C$ such that

$$\langle F(x), \eta(\tilde{y}, x)\rangle < 0, \quad \text{for all } x \in K \setminus C. \tag{6.12}$$

Then, VLIP (6.1) has a solution.

Proof. Let the set-valued maps $P, Q : K \to 2^K$ be the same as in the proof of Theorem 6.6. Let $\tilde{y} \in K$ and the set C be the same as in the hypothesis. Then, we want to show that the weak closure $\overline{P(\tilde{y})}^w$ of $P(\tilde{y})$ is a weakly compact subset of K. If $P(\tilde{y}) \not\subseteq C$, then there exists $x \in P(\tilde{y})$ such that $x \in K \setminus C$. It follows that

$$\langle F(x), \eta(\tilde{y}, x)\rangle \geq 0,$$

which contradicts (6.13). Therefore, we have $P(\tilde{y}) \subseteq C$. Then, the weak closure $\overline{P(\tilde{y})^w}$ of $P(\tilde{y})$ is a weakly compact subset of K. As we have seen in Theorem 6.6, P is a KKM map. Therefore, by Fan-KKM Theorem 6.18, $\bigcap_{y \in K} \overline{P(y)^w} \neq \emptyset$. Since for each $y \in K$, $Q(y)$ is closed convex and $P(y) \subseteq Q(y)$, we have

$$\overline{P(y)^w} \subseteq \overline{Q(y)^w} = Q(y),$$

because a convex subset of a normed space is weakly closed if and only if it is closed. Therefore,

$$\emptyset \neq \bigcap_{y \in K} \overline{P(y)^w} \subseteq \bigcap_{y \in K} Q(y).$$

Thus, there exists $\bar{x} \in K$ such that

$$\langle F(y), \eta(y, \bar{x}) \rangle \geq 0, \quad \text{for all } y \in K.$$

By Lemma 6.1, $\bar{x}$ is a solution of VLIP (6.1). □

Theorem 6.9. Let K be a nonempty closed convex subset of a Banach space X, and $\eta : K \times K \to X$ be a mapping such that $\eta(x, x) = \mathbf{0}$, for all $x, y \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y, x) \rangle$ is convex and the mapping $x \mapsto \langle F(y), \eta(y, x) \rangle$ is upper semicontinuous, for all $x, z, y \in K$. Assume that there exist a weakly compact subset D of X and $\tilde{y} \in K \cap D$ such that

$$\langle F(\tilde{y}), \eta(\tilde{y}, x) \rangle < 0, \quad \text{for all } x \in K \setminus D. \tag{6.13}$$

Then, VIP (6.1) has a solution.

Proof. Let the set-valued maps $P, Q : K \to 2^K$ be the same as in the proof of Theorem 6.6. Let $\tilde{y} \in K$ and the set D be the same as in the hypothesis. By the same argument as in the proof of Theorem 6.6, we derive that Q is a KKM map and, for each $y \in K$, $Q(y)$ is closed.

We show that $Q(\tilde{y}) \subseteq K \cap D$. If $Q(\tilde{y}) \nsubseteq D$, then there exists $x \in Q(\tilde{y})$ such that $x \in K \setminus D$. It follows that

$$\langle F(\tilde{y}), \eta(\tilde{y}, x) \rangle \geq 0,$$

which contradicts (6.13). Hence, $Q(\tilde{y}) \subseteq D$, and thus, $Q(\tilde{y}) \subseteq K \cap D$. Since K is closed convex and D is weakly compact, $K \cap D$ is weakly compact. Therefore, $Q(\tilde{y})$ is a closed subset of a weakly compact set $K \cap D$, and hence, $Q(\tilde{y})$ is weakly compact. Then, by Fan-KKM Theorem 6.18, $\bigcap_{y \in K} Q(y) \neq \emptyset$. The rest of the proof follows on the lines of the proof of Theorem 6.6. □

We establish the following existence result for a solution of VLIP (6.1) by using a Browder-type fixed-point theorem.

Theorem 6.10. Let K be a nonempty convex subset of a Banach space X, and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be hemicontinuous and η-pseudomonotone such that the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ is convex for all $x, z \in K$, and the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous. Assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$, there exists $\tilde{y} \in B$ such that $\langle F(x), \eta(\tilde{y}, x)\rangle < 0$. Then, VLIP (6.1) has a solution.

Proof. For each $x \in K$, define set-valued maps $P, Q : K \to 2^K$ by

$$P(x) = \{y \in K : \langle F(y), \eta(y,x)\rangle < 0\}$$

and

$$Q(x) = \{y \in K : \langle F(x), \eta(y,x)\rangle < 0\}.$$

The convexity of the mapping $y \mapsto \langle F(z), \eta(y,x)\rangle$ implies that the set $Q(x)$ is convex, for each $x \in K$. By η-pseudomonotonicity of F, we have $P(x) \subseteq Q(x)$, and hence, $\mathrm{co}(P(x)) \subseteq \mathrm{co}(Q(x)) = Q(x)$, for all $x \in K$.

For each $y \in K$, the complement of $P^{-1}(y)$ in K is

$$[P^{-1}(y)]^c = \{x \in K : \langle F(y), \eta(y,x)\rangle \geq 0\}$$

and is closed in K by upper semicontinuity of the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$. Thus, $P^{-1}(y)$ is open in K.

Assume that for all $x \in K$, $P(x)$ is nonempty. Then, all the conditions of Theorem 6.19 are satisfied, and therefore, there exists $\hat{x} \in K$ such that $\hat{x} \in Q(\hat{x})$. It follows that

$$0 = \langle F(\tilde{x}), \eta(\tilde{x}, \tilde{x})\rangle < 0,$$

a contradiction. Hence, there exists $\bar{x} \in K$ such that $P(\bar{x}) = \emptyset$. This implies that for all $y \in K$,

$$\langle F(y), \eta(y, \bar{x})\rangle \geq 0,$$

that is, $\bar{x} \in K$ is a solution of MVLIP (6.2). By Lemma 6.1, $\bar{x} \in K$ is a solution of VLIP (6.1). $\square$

Definition 6.7. Let K be a nonempty subset of a Banach space X and $\eta : K \times K \to X$ be a mapping. A mapping $F : K \to X^*$ is said to be *properly η-quasimonotone* if for every $x_1, x_2, \ldots, x_m \in K$ and every $y \in \mathrm{co}(\{x_1, x_2, \ldots, x_m\})$, there exist $i \in \{1, 2, \ldots, m\}$ such that

$$\langle F(x_i), \eta(y, x_i)\rangle \leq 0.$$

Theorem 6.11. Let K be a nonempty compact convex subset of a Banach space X and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be properly η-quasimonotone such that the mapping $x \mapsto \langle F(y), \eta(y,x)\rangle$ is upper semicontinuous. Then, MVLIP (6.2) has a solution.

Proof. Define the set-valued mapping $Q : K \to 2^K$ by

$$Q(y) = \{x \in K : \langle F(y), \eta(y,x)\rangle \geq 0\}, \quad \text{for all } y \in K.$$

For any $y_1, y_2, \ldots, y_m \in K$ and $\tilde{y} \in \text{co}(\{y_1, y_2, \ldots, y_m\})$, proper η-quasimonotonicity of F implies that $\tilde{y} \in \bigcup_{i=1}^m Q(y_i)$. Also, for each $y \in K$, $Q(y)$ is a closed subset of the compact set K, and hence, compact. Therefore, by Fan-KKM Theorem 6.18, it follows that $\bigcap_{y\in K} Q(y) \neq \emptyset$. Thus, any $\bar{x} \in \bigcap_{y\in K} Q(y)$ is a solution of MVLIP (6.2). □

We present the definition of η-pseudomonotonicity according to Brézis [12].

Definition 6.8. Let K be a nonempty subset of a Banach space X and $\eta : K \times K \to X$ be a mapping. A function $F : K \to X^*$ is said to be *B-η-pseudomonotone* if for each $x \in K$ and every sequence $\{x_m\}$ in K converging to x with

$$\liminf_{m\to\infty} \langle F(x_m), \eta(x, x_m)\rangle \geq 0,$$

we have

$$\langle F(x), \eta(y,x)\rangle \geq \limsup_{m\to\infty} \langle F(x_m), \eta(y, x_m)\rangle, \quad \text{for all } y \in K.$$

The following result provides a solution of VLIP (6.1) under the assumption of η-pseudomonotonicity according to Brézis [12].

Theorem 6.12. Let K be a nonempty convex subset of a Banach space X and $\eta : K \times K \to X$ be a mapping such that $\eta(x,x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be B-η-pseudomonotone such that for each finite subset A of K, $x \mapsto \langle F(x), \eta(y,x)\rangle$ is upper semicontinuous on $\text{co}(A)$, and the mapping $y \mapsto \langle F(x), \eta(y,x)\rangle$ is quasiconvex. Assume that there exist a nonempty compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\langle F(x), \eta(\tilde{y}, x)\rangle < 0$. Then, VLIP (6.1) has a solution.

Proof. For each $x \in K$, define a set-valued map $P : K \to 2^K$ by

$$P(x) = \{y \in K : \langle F(x), \eta(y,x)\rangle < 0\}.$$

Then for all $x \in K$, $P(x)$ is convex. Let A be a finite subset of K. Then for all $y \in \text{co}(A)$,

$$[P^{-1}(y)]^c \cap \text{co}(A) = \{x \in \text{co}(A) : \langle F(x), \eta(y,x)\rangle \geq 0\}$$

is closed in $\text{co}(A)$ by upper semicontinuity of the map $x \mapsto \langle F(x), \eta(y,x)\rangle$ on $\text{co}(A)$. Hence, $P^{-1}(y) \cap \text{co}(A)$ is open in $\text{co}(A)$.

Suppose that $x, y \in \text{co}(A)$ and $\{x_m\}$ is a sequence in K converging to x such that

$$\langle F(x_m), \eta(\lambda y + (1-\lambda)x, x_m)\rangle \geq 0, \quad \text{for all } m \in \mathbb{N} \text{ and all } \lambda \in [0,1].$$

For $\lambda = 0$, we have

$$\langle F(x_m), \eta(x, x_m)\rangle \geq 0, \quad \text{for all } m \in \mathbb{N},$$

and therefore,

$$\liminf_{m\to\infty} \langle F(x_m), \eta(x, x_m)\rangle \geq 0.$$

By B-η-pseudomonotonicity of F, we have

$$\langle F(x), \eta(y, x)\rangle \geq \limsup_{m\to\infty} \langle F(x_m), \eta(y, x_m)\rangle. \tag{6.14}$$

For $\lambda = 1$, we have

$$\langle F(x_m), \eta(y, x_m)\rangle \geq 0, \quad \text{for all } m \in \mathbb{N},$$

and therefore,

$$\liminf_{m\to\infty} \langle F(x_m), \eta(y, x_m)\rangle \geq 0. \tag{6.15}$$

From the inequalities (6.14) and (6.15), we obtain

$$\langle F(x), \eta(y, x)\rangle \geq 0,$$

and thus, $y \notin P(x)$.

Assume that for all $x \in D$, $P(x)$ is nonempty. Then, all the conditions of Theorem 6.20 are satisfied. Hence, there exists $\hat{x} \in K$ such that $\hat{x} \in P(\hat{x})$, that is,

$$0 = \langle F(\hat{x}), \eta(\hat{x}, \hat{x})\rangle < 0,$$

a contradiction. Thus, there exists $\bar{x} \in D \subseteq K$ such that $P(\bar{x}) = \emptyset$, that is,

$$\langle F(\bar{x}), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of VLIP (6.1). □

Corollary 6.2. Let K be a nonempty convex subset of a Banach space X and $\eta : K \times K \to X$ be a mapping such that $\eta(x, x) = \mathbf{0}$, for all $x \in K$. Let $F : K \to X^*$ be B-η-pseudomonotone such that for each finite subset A of K, $x \mapsto \langle F(x), \eta(y, x)\rangle$ is upper semicontinuous on $\text{co}(A)$, and the mapping $y \mapsto \langle F(x), \eta(y, x)\rangle$ is quasiconvex. Assume that there exists $\tilde{y} \in K$ such that

$$\lim_{\|x\|\to\infty,\ x\in K} \langle F(x), \eta(\tilde{y}, x)\rangle < 0. \tag{6.16}$$

Then, VLIP (6.1) has a solution.

Proof. Let

$$\alpha = \lim_{\|x\|\to\infty,\ x\in K} \langle F(x), \eta(\tilde{y}, x)\rangle.$$

Then, by the inequality (6.16), $\alpha < 0$. Let $r > 0$ be such that $\|\tilde{y}\| \leq r$ and

$$\langle F(x), \eta(\tilde{y}, x)\rangle < \frac{\alpha}{2}, \quad \text{for all } x \in K \text{ with } \|x\| > r.$$

Let $\mathbb{B}_r = \{x \in K : \|x\| \le r\}$ be a closed unit ball. Then, $\mathbb{B}_r$ is a nonempty and weakly compact subset of K. Note that for any $x \in K \setminus \mathbb{B}_r$, $\langle F(x), \eta(\tilde{y}, x)\rangle < \frac{\alpha}{2} < 0$, and the conclusion follows from Theorem 6.12 by taking $D = \mathbb{B}_r$. □

We now present some existence results for a solution of VLIP (6.1) without any kind of monotonicity.

Theorem 6.13. [57] Let K be a nonempty convex compact subset of a Banach space X, $F : K \to X^*$ and $\eta : K \times K \to X$ be two continuous mappings. Suppose that for each fixed $x \in K$, the mapping $v \mapsto \langle F(x), \eta(v, x)\rangle$ is quasiconvex, and $\langle F(x), \eta(x, x)\rangle = 0$, for all $x \in K$. Then, VLIP (6.1) has a solution.

Proof. For each $x \in K$, define

$$V(x) = \left\{ z \in K : \langle F(x), \eta(z, x)\rangle = \min_{v \in K} \langle F(x), \eta(v, x)\rangle \right\}.$$

Since K is compact and the mapping $v \mapsto \langle F(x), \eta(v, x)\rangle$ is quasiconvex, $V(x)$ is a nonempty, closed, and convex subset of K. It is easy to see that the set-valued map $V : K \to 2^K$ is upper semicontinuous. By Kakutani fixed-point theorem 6.21, we have $\bar{x} \in V(\bar{x})$. Consequently, for all $y \in K$, $\langle F(\bar{x}), \eta(y, \bar{x})\rangle \ge \langle F(\bar{x}, \eta(\bar{x}, \bar{x})\rangle = 0$. □

Consider the set $K_r = \{x \in K : \|x\| \le r\}$ for real $r > 0$. There always exists an $r_0 > 0$ such that K_r is nonempty whenever $r \ge r_0$.

Assumption 6.1. Let K be a closed convex subset of a Banach space X, and $F : K \to X^*$ and $\eta : K \times K \to X$ be two continuous mappings such that

(i) $\langle F(x), \eta(x, x)\rangle = 0$, for all $x \in K$;

(ii) for each fixed $x \in K$, the mapping $y \mapsto \langle F(x), \eta(y, x)$ is convex.

We notice that K_r is compact and convex, and hence, by Theorem 6.13 there exists at least one $x_r \in K_r$ such that

$$\langle F(x_r), \eta(y, x_r)\rangle \ge 0, \quad \text{for all } y \in K \tag{6.17}$$

whenever Assumption 6.1 is satisfied.

Theorem 6.14. [57] Let K, F, and η be such that Assumption 6.1 is satisfied. A necessary and sufficient condition for the existence of a solution to VLIP (6.1) is that there exists an $r > 0$ such that a solution $x_r \in K_r$ of (6.17) satisfies the inequality $\|x_r\| < r$.

Proof. It is clear that if there exists a solution $\bar{x}$ to VLIP (6.1), then $\bar{x}$ is a solution to (6.17) whenever $\|\bar{x}\| < r$.

Suppose that $x_r \in K_r$ is a solution of (6.17) and that $\|x_r\| < x$. Given $y \in K$, we can choose $0 < \lambda < 1$ sufficiently small so that $z = \lambda y + (1-\lambda)x_r \in$

K_r. Consequently, by the convexity of $z \mapsto \langle F(x_r), \eta(z, x_r)\rangle$, $x_r \in K_r \subseteq K$ satisfies

$$\begin{aligned} 0 &\leq \langle F(x_r), \eta(z, x_r)\rangle \\ &\leq \lambda(F(x_r, \eta(y, x_r)\rangle + (1-\lambda)\langle F(x_r), \eta(x_r, x_r)\rangle \\ &= \lambda\langle F(x_r), \eta(y, x_r)\rangle \end{aligned}$$

for all $y \in K$, which implies that x_r is a solution to VLIP (6.1). □

By using Theorem 6.14, Parida et al. [57] provided three other sufficient conditions for the existence of a solution of VLIP (6.1).

6.5 Solution Methods

Because of the involvement of the vector-valued term $\eta(y, x)$ in the formulations of variational-like inequalities, few solution methods are available in the literature to compute approximate solutions of variational-like inequality problems. The auxiliary principle method and the proximal method are the most studied methods for solving variational-like inequalities. Noor [53, 55] proposed the auxiliary principle method for solving variational-like inequality problems and studied the strong convergence of the sequence generated by the proposed method to a unique solution of a variational-like inequality problem under strong monotonicity and Lipschitz continuity of the mappings F and η involved in the formulation of a variational-like inequality. Ding [19, 20] considered the auxiliary principle method for variational-like inequality problems in the setting of Banach spaces. The convergence of the sequence generated by the proposed algorithm was studied. The auxiliary principle method for variational-like inequality problems was further studied in [6, 79] and the references therein. Jie and Li-Ping [38] proposed a bundle-type auxiliary principle method for variational-like inequalities and studied its convergence analysis.

In 2000, Lee at al. [45] and Ding and Luo [21] independently introduced the concepts of η-subdifferential and η-proximal mappings. By using η-subdifferentiability, they suggested a perturbed algorithm for finding the approximate solutions of variational-like inclusions, which are generalizations of variational-like inequalities. The convergence analysis of the method is also studied in these two papers. This method has been further extended and studied in [15, 75] for extended and generalized variational-like inequality problems.

In this section, we mainly discuss the auxiliary principle method and proximal method. Most of the results presented in this section are taken from [6, 45].

6.5.1 Auxiliary Principle Method

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty convex subset of H. Let $F : K \to H$ and $\eta : K \times K \to H$ be mappings and $\Phi : K \to \mathbb{R}$ be a functional. We consider the following *variational-like inequality problem* (VLIP) in the setting of Hilbert spaces:

$$\text{VLIP} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \langle F(\bar{x}), \eta(y, \bar{x}) \rangle + \Phi(y) - \Phi(\bar{x}) \geq 0, \quad \text{for all } y \in K. \end{cases} \tag{6.18}$$

Of course, if $\Phi \equiv 0$, then (6.18) reduces to the variational-like inequality problem (6.1).

Definition 6.9. Let K be a nonempty subset of a Hilbert space H and $\eta : K \times K \to H$ be a mapping. A mapping $F : K \to H$ is said to be:

- *η-co-coercive* if there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), \eta(x, y) \rangle \geq \alpha \|F(x) - F(y)\|^2, \quad \text{for all } x, y \in K;$$

- *η-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle F(x) - F(y), \eta(x, y) \rangle \geq \beta \|x - y\|^2, \quad \text{for all } x, y \in K;$$

- *Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|F(x) - F(y)\| \leq \gamma \|x - y\|, \quad \text{for all } x, y \in K.$$

If $\eta(y, x) = y - x$ for all $x, y \in K$, then η-strong monotonicity is called *strong monotonicity*.

Definition 6.10. A mapping $\eta : K \times K \to H$ is said to be *Lipschitz continuous* if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda \|x - y\|, \quad \text{for all } x, y \in K.$$

It is clear that an η-co-coercive mapping is η-monotone, but the converse is not true in general. We note that every η-co-coercive mapping is Lipschitz continuous provided that η is Lipschitz continuous. Indeed,

$$\begin{aligned} \alpha \|F(x) - F(y)\|^2 &\leq \langle F(x) - F(y), \eta(x, y) \rangle \\ &\leq \|F(x) - F(y)\| \, \|\eta(x, y)\| \\ &\leq \lambda \|F(x) - F(y)\| \, \|x - y\|. \end{aligned}$$

Hence, $\|F(x) - F(y)\| \leq \frac{\lambda}{\alpha} \|x - y\|$, and thus F is Lipschitz continuous with constant $\frac{\lambda}{\alpha}$. But every η-co-coercive mapping need not be η-strongly monotone.

Every η-strongly monotone and Lipschitz continuous mapping is η-co-coercive, and it follows that η-co-coercivity is an intermediate concept that lies between η-monotonicity and η-strong monotonicity. In general, every η-monotone and Lipschitz continuous mapping need not be η-co-coercive.

Definition 6.11. Let K be a nonempty convex subset of a Hilbert space H and $\eta : K \times K \to H$ be a mapping. A Fréchet differentiable function $h : K \to \mathbb{R}$ is called *strongly invex* wrt η if there exists a constant $\mu > 0$ such that

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \geq \frac{\mu}{2}\|x - y\|^2, \quad \text{for all } x, y \in K,$$

where $h'(x)$ denotes the Fréchet derivative of h at x.

The following result can be easily proved.

Proposition 6.1. Let K be a nonempty convex subset of a Hilbert space H, $\eta : K \times K \to H$ be a skew mapping, and $h : K \to \mathbb{R}$ be Fréchet differentiable strongly invex wrt η. Then, h' is η-strongly monotone.

Recall that a mapping $F : K \to \mathbb{R}$ is called *sequentially continuous* at x [44] if $F(x_n) \to F(x)$ for all sequences $x_n \to x$. F is called *sequentially continuous* on K if it is sequentially continuous at each of its point.

Lemma 6.8. Let K be a nonempty convex subset of a Hilbert space H. Let $\eta : K \times K \to H$ and $h : K \to \mathbb{R}$ be Fréchet differentiable such that h' is sequentially continuous from the weak topology to the weak topology and from the weak topology to the strong topology, respectively. Then, the mapping $g : K \to \mathbb{R}$ defined by $g(x) = \langle h'(x), \eta(y, x)\rangle$ for each fixed $y \in K$, is also sequentially continuous from the weak topology to the strong topology.

Proof. Let $\{x_n\}$ be a sequence that converges (in the weak topology) to x, which is denoted by $x_n \rightharpoonup x$. Then,

$$\|h'(x_k) - h'(x)\| \to 0 \quad \text{and} \quad \eta(y, x_k) \rightharpoonup \eta(y, x).$$

Now,

$$\begin{aligned} |g(x_n) - g(x)| &= |\langle h'(x_n), \eta(y, x_n)\rangle - \langle h'(x), \eta(y, x)\rangle| \\ &= |\langle h'(x_n) - h'(x), \eta(y, x_n)\rangle + \langle h'(x), \eta(y, x_n) - \eta(y, x)\rangle| \\ &\leq \|h'(x_n) - h'(x)\| \ \|\eta(y, x_n)\| + |\langle h'(x), \eta(y, x_n) - \eta(y, x)\rangle| . \end{aligned}$$

Since each weakly convergent sequence is bounded, we have $|g(x_n) - g(x)| \to 0$ as $n \to \infty$. Hence, $g(x)$ is sequentially continuous from the weak topology to the strong topology. ☐

We present the following basic algorithm framework for (6.18).

Theorem 6.15. Let K be a nonempty convex and bounded subset of a Hilbert space H and $\Phi : K \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $F : K \to H$ be an η-co-coercive mapping with constant α. Assume that

(i) $\eta : K \times K \to H$ is Lipschitz continuous with constant λ such that

Algorithm 1

Let ρ be a positive parameter and, for a given iterate x_n, consider the auxiliary problem that consists of finding x_{n+1} such that

$$\langle \rho F(x_n) + h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1})\rangle + \rho[\Phi(y) - \Phi(x_{n+1})] \geq 0, \quad (6.19)$$

for all $y \in K$, where $h'(x)$ is the Fréchet derivative of a functional $h : K \to \mathbb{R}$ at x.

(a) $\eta(x, y) + \eta(y, x) = \mathbf{0}$, for all $x, y \in K$;

(b) $\eta(x, y) = \eta(x, z) + \eta(z, y)$, for all $x, y, z \in K$;

(c) $\eta(.,.)$ is affine in the first variable;

(d) for each fixed $y \in K$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;

(ii) $h : K \to \mathbb{R}$ is strongly invex wrt η with constant μ and its derivative h' is sequentially continuous from the weak topology to the strong topology;

(iii) there exists $\gamma > 0$ such that for all $x, y \in K$,

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \leq \gamma\|y - x\|^2.$$

Then, there exists a unique solution $x_{n+1} \in K$ to the auxiliary problem (6.19). If

$$0 < \rho < \frac{2\alpha\mu}{\lambda^2}, \quad (6.20)$$

then the sequence $\{x_n\}$ generated by (6.19) converges to a solution of VLIP (6.18).

Proof. Existence of Solutions of Auxiliary Problem (6.19): For the sake of simplicity, we write (6.19) as follows: Find $\bar{x} \in K$ such that

$$\langle \rho F(x_n) + h'(\bar{x}) - h'(x_n), \eta(y, \bar{x})\rangle + \rho[\Phi(y) - \Phi(\bar{x})] \geq 0, \quad \text{for all } y \in K.$$

For each fixed n and each $y \in K$, define

$$P(y) = \{x \in K : \langle \rho F(x_n) + h'(x) - h'(x_n), \eta(y, x)\rangle + \rho[\Phi(y) - \Phi(x)] \geq 0\}.$$

Note that, for each $y \in K$, $P(y)$ is nonempty, since $y \in P(y)$.

We prove that P is a KKM map. Suppose that there is a finite subset $\{y_1, y_2, \ldots, y_k\}$ of K and $\alpha_i \geq 0$, for all $i = 1, 2, \ldots, k$ with $\sum_{i=1}^k \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^k \alpha_i y_i \notin P(y_i)$ for all i. Then, we have

$$\langle \rho F(x_n) + h'(\hat{x}) - h'(x_n), \eta(y_i, \hat{x})\rangle + \rho[\Phi(y_i) - \Phi(\hat{x})] < 0, \quad \text{for all } i.$$

Therefore,

$$\sum_{i=1}^{k} \alpha_i \langle \rho F(x_n) + h'(\hat{x}) - h'(x_n), \eta(y_i, \hat{x}) \rangle + \rho \sum_{i=1}^{k} \alpha_i [\Phi(y_i) - \Phi(\hat{x})] < 0.$$

From condition (i)(a), we have $\eta(x,x) = \mathbf{0}$, for all $x \in K$. By using the convexity of Φ and assumption (i)(c), we get

$$0 = \langle \rho F(x_n) + h'(\hat{x}) - h'(x_n), \eta(\hat{x}, \hat{x}) \rangle < 0,$$

a contradiction. Hence, P is a KKM map.

Since $\overline{P(y)}^w$, the weak closure of $P(y)$ is a weakly closed subset of a bounded set $K \subseteq H$, it is weakly compact. Hence, by Fan-KKM Theorem 6.18, $\bigcap_{y \in K} \overline{P(y)}^w \neq \emptyset$. Let $\bar{x} \in \bigcap_{y \in K} \overline{P(y)}^w$. Then, there exists a sequence $\{x_m\}$ in $P(y)$ such that $x_m \rightharpoonup \bar{x}$ (See [22, p. 93]). Therefore,

$$\langle \rho F(x_n) + h'(x_m) - h'(x_n), \eta(y, x_m) \rangle \geq \rho [\Phi(x_m) - \Phi(y)],$$

and hence,

$$\lim_{m \to \infty} \langle \rho F(x_n) + h'(x_m) - h'(x_n), \eta(y, x_m) \rangle \geq \rho \lim_{m \to \infty} [\Phi(x_m) - \Phi(y)].$$

Since Φ is convex and lower semicontinuous, it is lower semicontinuous in the weak topology. Using Lemma 6.8, we get

$$\langle \rho F(x_n) + h'(\bar{x}) - h'(x_n), \eta(y, \bar{x}) \rangle \geq \rho [\Phi(\bar{x}) - \Phi(y)].$$

Therefore, $\bar{x} \in K$ is a solution of the auxiliary problem (6.19).

Uniqueness of Solution of the Auxiliary Problem (6.19): Let x_1 and x_2 be two solutions of the auxiliary problem (6.19). Then, for all $y \in K$,

$$\langle \rho F(x_n) + h'(x_1) - h'(x_n), \eta(y, x_1) \rangle + \rho [\Phi(y) - \Phi(x_1)] \geq 0, \tag{6.21}$$

$$\langle \rho F(x_n) + h'(x_2) - h'(x_n), \eta(y, x_2) \rangle + \rho [\Phi(y) - \Phi(x_2)] \geq 0. \tag{6.22}$$

Taking $y = x_2$ in (6.21) and $y = x_1$ in (6.22) and adding the results, we obtain

$$\rho \langle F(x_n), \eta(x_2, x_1) \rangle + \langle h'(x_1) - h'(x_n), \eta(x_2, x_1) \rangle +$$

$$\rho \langle F(x_n), \eta(x_1, x_2) \rangle + \langle h'(x_2) - h'(x_n), \eta(x_1, x_2) \geq 0.$$

Since $\eta(x,y) + \eta(y,x) = \mathbf{0}$ for all $x, y \in K$, we have

$$\langle h'(x_1), \eta(x_2, x_1) \rangle \geq -\langle h'(x_2), \eta(x_1, x_2) \rangle.$$

By using strong invexity w.r.t. η of h, we obtain

$$h(x_2) - h(x_1) - \frac{\mu}{2} ||x_1 - x_2||^2 \geq -h(x_1) + h(x_2) + \frac{\mu}{2} ||x_2 - x_1||^2,$$

and therefore,

$$\mu\|x_1 - x_2\|^2 \leq 0.$$

Since $\mu > 0$, we get $x_1 = x_2$, and hence, the solution of the auxiliary problem (6.19) is unique.

Let x^* be any fixed solution of VLIP (6.18). For each $y \in K$, define a functional

$$\Lambda(y) = h(x^*) - h(y) - \langle h'(y), \eta(x^*, y)\rangle.$$

By the strong invexity w.r.t. η of h, we have

$$\Lambda(y) = h(x^*) - h(y) - \langle h'(y), \eta(x^*, y)\rangle \geq \frac{\mu}{2}||y - x^*||^2. \tag{6.23}$$

From the strong invexity of h, assumption (i)(b) and (6.19) with $y = x^*$, we get

$$\begin{aligned}
\Lambda(x_n) - \Lambda(x_{n+1}) &= h(x_{n+1}) - h(x_n) - \langle h'(x_n), \eta(x^*, x_n)\rangle \\
&\quad + \langle h'(x_{n+1}), \eta(x^*, x_{n+1})\rangle \\
&= h(x_{n+1}) - h(x_n) - \langle h'(x_n), \eta(x^*, x_{n+1})\rangle \\
&\quad - \langle h'(x_n), \eta(x_{n+1}, x_n)\rangle + \langle h'(x_{n+1}), \eta(x^*, x_{n+1})\rangle \\
&= h(x_{n+1}) - h(x_n) - \langle h'(x_n), \eta(x_{n+1}, x_n)\rangle \\
&\quad + \langle h'(x_{n+1}) - h'(x_n), \eta(x^*, x_{n+1})\rangle \\
&\geq \tfrac{\mu}{2}\|x_n - x_{n+1}\|^2 + \langle h'(x_{n+1}) - h'(x_n), \eta(x^*, x_{n+1})\rangle \\
&\geq \tfrac{\mu}{2}\|x_n - x_{n+1}\|^2 - \rho\langle F(x_n), \eta(x^*, x_{n+1})\rangle \\
&\quad - \rho(\Phi(x^*) - \Phi(x_{n+1})] \\
&= \tfrac{\mu}{2}||x_n - x_{n+1}||^2 + \rho\langle F(x_n), \eta(x_{n+1}, x^*)\rangle \\
&\quad + \rho[\Phi(x_{n+1}) - \Phi(x^*)].
\end{aligned} \tag{6.24}$$

We set $y = x_{n+1}$ in (6.18) and combine it with (6.24), and we get

$$\begin{aligned}
\Lambda(x_n) - \Lambda(x_{n+1}) &\geq \frac{\mu}{2}\|x_n - x_{n+1}\|^2 + \rho\langle F(x_n), \eta(x_{n+1}, x^*)\rangle \\
&\quad - \rho\langle F(x^*), \eta(x_{n+1}, x^*)\rangle \\
&= \frac{\mu}{2}\|x_n - x_{n+1}\|^2 + Q.
\end{aligned}$$

Now,

$$\begin{aligned}
Q &= \rho\langle F(x_n) - F(x^*), \eta(x_{n+1}, x^*)\rangle \\
&= \rho\langle F(x_n) - F(x^*), \eta(x_n, x^*)\rangle + \rho\langle F(x_n) - F(x^*), \eta(x_{n+1}, x_n)\rangle \\
&\geq \rho\left[\alpha\|F(x_n) - F(x^*)\|^2 + \langle F(x_n) - F(x^*), \eta(x_{n+1}, x_n)\rangle\right] \\
&\geq \rho\left[-\frac{1}{4\alpha}\|\eta(x_{n+1}, x_n)\|^2\right] \\
&\geq \left(-\frac{\rho\lambda^2}{4\alpha}\right)\|x_{n+1} - x_n\|^2.
\end{aligned}$$

Therefore,

$$\Lambda(x_n) - \Lambda(x_{n+1}) \geq \frac{1}{2}\left(\mu - \frac{\rho\lambda^2}{2\alpha}\right)||x_{n+1} - x_n||^2. \tag{6.25}$$

If $x_{n+1} = x_n$ for some n, then x_n is a solution of (6.18). Otherwise, it follows from (6.20) that $\Lambda(x_n) - \Lambda(x_{n+1})$ is a nonnegative sequence, from which we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Also, from (6.23), we conclude that $\{\Lambda(x_n)\}$ is a decreasing sequence, and hence $\{x_n\}$ must be a bounded sequence. It is easy to see that any limit point of $\{x_n\}$ is a solution of (6.18).

Now, let $\bar{x}$ be any limit point of $\{x_n\}$, and let

$$\bar{\Lambda}(x_n) = h(\bar{x}) - h(x_n) - \langle h'(x_n), \eta(\bar{x}, x_n)\rangle \geq \frac{\mu}{2}\|\bar{x} - x_n\|^2. \tag{6.26}$$

By the above argument, we know that $\{\bar{\Lambda}(x_n)\}$ is also a decreasing sequence and by assumption (iii), we have

$$\bar{\Lambda}(x_n) \leq \gamma\|\bar{x} - x_n\|^2,$$

from which it follows that

$$\lim_{n\to\infty} \bar{\Lambda}(x_n) = 0. \tag{6.27}$$

Combining (6.26) and (6.27), we conclude that the sequence $\{x_n\}$ converges to $\bar{x}$. □

Remark 6.3. When $\eta(y, x) = y - x$ for all $x, y \in K$, Theorem 6.15 can be treated as an infinite dimensional version of Theorem 3.2 in [78].

6.5.2 Proximal Method

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle ., .\rangle$. Given a nonlinear operator $F : H \to H$ and a mapping $\eta : H \times H \to H$, we consider the following *variational-like inclusion problem* (VLIP): Find $\bar{x} \in H$ such that $\bar{x} \in \text{dom } \Phi$ and

$$\langle F(\bar{x}), \eta(y, \bar{x})\rangle \geq \Phi(\bar{x}) - \Phi(y), \quad \text{for all } y \in H, \tag{6.28}$$

where $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ and dom $\Phi = \{z \in H : \Phi(z) < \infty\}$.

If $\Phi \equiv \delta_K$, then the problem (6.28) reduces to the variational-like inequality problem (6.1).

We define the concept of η-subdifferential in a more general setting than given in [73]. This notion of η-subdifferential will be used in the perturbed iterative algorithm for finding the approximate solution of variational-like inclusion (6.28).

Let $\eta : H \times H \to H$ be a mapping and $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a functional. A vector $w \in H$ is called an *η-subgradient* of Φ at $x \in \text{dom } \Phi$ if

$$\langle w, \eta(y, x)\rangle \leq \Phi(y) - \Phi(x), \quad \text{for all } y \in H. \tag{6.29}$$

We can associate each Φ with the *η-subdifferential map* $\partial_\eta \Phi$ defined by

$$\partial_\eta \Phi(x) = \begin{cases} \{w \in H : \langle w, \eta(y,x)\rangle \leq \Phi(y) - \Phi(x), \quad \forall\, y \in H\}, & x \in \text{dom}\ \Phi, \\ \emptyset, & x \notin \text{dom}\ \Phi. \end{cases}$$

For $x \in \text{dom}\ \Phi$, $\partial_\eta \Phi(x)$ is called the *η-subdifferential* of Φ at x.

In the definition of η-subdifferential according to Yang and Craven [73], the function Φ should be local Lipschitz and cannot take the value $+\infty$. The following example shows that our definition of η-subdifferential is more general than that in [73].

Example 6.3. Let $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\Phi(x) = \begin{cases} x, & \text{if } x \leq 0, \\ +\infty, & \text{if } x > 0, \end{cases}$$

and $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\eta(x,y) = x + 2y.$$

Then, we have $\partial_\eta \Phi(x) = [1, \infty)$, $x \in \text{dom}\ \Phi$.

Let $x \in \text{dom}\ \Phi$, that is, $\Phi(x) = x$ and $x \leq 0$. If $w \in \partial_\eta \Phi(x)$, then

$$w(y + 2x) \leq \Phi(y) - x, \quad \text{for all } y \in \mathbb{R}.$$

Since $\Phi(y) = +\infty$ for $y > 0$, we thus have

$$w(y + 2x) \leq y - x, \quad \text{for all } y \leq 0.$$

If $x = 0$, then $w(y) \leq y$ for $y \leq 0$, and thus $w \geq 1$. If $x < 0$, then

$$w \geq \frac{y - x}{y + 2x}, \quad \text{for all } y \leq 0.$$

Letting $y \to -\infty$, we have $w \geq 1$. In any case, we have $w \geq 1$, that is, $\partial_\eta \Phi(x) \subseteq [1, \infty)$ for all $x \leq 0$. Now let $w \in [1, \infty)$. If $w \notin \partial_\eta \Phi(x)$ for some $x_0 \leq 0$, then there exists $y_0 \leq 0$ such that

$$w(y_0 + 2x_0) > y_0 - x_0.$$

It is clear that the case $x_0 = y_0 = 0$ cannot happen. If $x_0 < 0$ and $y_0 = 0$, then we have $w < -\frac{1}{2}$, which is a contradiction. If $x_0 = 0$ and $y_0 < 0$, then we have $w < 1$, which is also a contradiction. Finally, if $x_0 < 0$ and $y_0 < 0$, then

$$1 \leq w < \frac{y_0 - x_0}{y_0 + 2x_0},$$

from which it follows that $x_0 > 0$, and this is also a contradiction. In our claim, we have $[1, \infty) \subseteq \partial_\eta \Phi(x)$ for all $x \leq 0$. Consequently, $\partial_\eta \Phi(x) = [1, \infty)$ for all $x \leq 0$.

The following result directly follows from the definition of η-subdifferential.

Theorem 6.16. Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be nontrivial, that is, dom $\Phi \neq \emptyset$. Then, $\bar{x} \in H$ is a solution of (6.28) if and only if $\bar{x} \in$ dom Φ and $-F(x) \in \partial_\eta \Phi(x)$.

Definition 6.12. A mapping $\eta : H \times H \to H$ is called:

- *monotone* if
$$\langle x - y, \eta(x, y) \rangle \geq 0, \quad \text{for all } x, y \in H; \tag{6.30}$$
- *strictly monotone* if the equality holds in (6.30) only when $x = y$;
- *strongly monotone* if there exists a constant $\sigma \geq 0$ such that
$$\langle x - y, \eta(x, y) \rangle \geq \sigma ||x - y||^2, \quad \text{for all } x, y \in H.$$

Let $Q : H \to 2^H$ be a set-valued map. Then the graph of Q, denoted by Graph(Q), is defined as

$$\text{Graph}(Q) = \{(x, y) \in H \times H : y \in Q(x)\}.$$

Definition 6.13. Let $\eta : H \times H \to H$ be a given map. A set-valued map $Q : H \to 2^H$ is called *η-monotone* if for all $x, y \in H$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \text{for all } u \in Q(x),\ v \in Q(y).$$

Q is called *maximal η-monotone* if and only if it is η-monotone and there is no other η-monotone set-valued map whose graph strictly contains the Graph(Q).

The proof of the following proposition is similar to of the proof of Lemma 3 in [76] and we therefore omit it.

Proposition 6.2. Let $\eta : H \times H \to H$ be a map. A multivalued map Q is maximal η-monotone if and only if Q is η-monotone and it follows from $(x, u) \in H \times H$ and

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \text{for all } (y, v) \in \text{Graph}(Q)$$

that $(x, u) \in$ Graph(Q).

Proposition 6.3. Let $\eta : H \times H \to H$ be strictly monotone and $Q : H \to 2^H$ be an η-monotone set-valued map. If the range of $(I + \lambda Q)$, $R(I + \lambda Q) = H$ for $\lambda > 0$ and I is the identity operator, then Q is maximal η-monotone. Furthermore, the inverse operator $(I + \lambda Q)^{-1} : H \to H$ is single-valued.

Proof. If we suppose that Q is not a maximal η-monotone, then there exists $(x_0, u_0) \notin \text{Graph}(Q)$ such that

$$\langle u_0 - v, \eta(x_0, y)\rangle \geq 0, \quad \text{for all } (y, v) \in \text{Graph}(Q). \tag{6.31}$$

By assumption, $R(I + \lambda Q) = H$, and therefore, there exists $(x_1, u_1) \in \text{Graph}(Q)$ such that

$$x_1 + \lambda u_1 = x_0 + \lambda u_0. \tag{6.32}$$

Since (6.31) is true for all $(y, v) \in \text{Graph}(Q)$, we have

$$\langle u_0 - u_1, \eta(x_0, x_1)\rangle \geq 0.$$

But from (6.31), we have $\lambda(u_0 - u_1) = x_1 - x_0$, and hence,

$$\frac{1}{\lambda}\langle x_1 - x_0, \eta(x_0, x_1)\rangle \geq 0.$$

Multiplying by $\lambda > 0$, we have

$$\langle x_0 - x_1, \eta(x_0, x_1)\rangle \leq 0.$$

Since η is strictly monotone, we have $x_0 = x_1$, and hence, from (6.32), we get $u_1 = u_0$. So, we have a contradiction that $(x_1, u_1) \in \text{Graph}(Q)$ or $(x_0, u_0) \in \text{Graph}(Q)$. Therefore, Q is maximal η-monotone.

Now, let $x, y \in (I + \lambda Q)^{-1}(z)$. Then, $\frac{1}{\lambda}(z - x) \in Q(x)$ and $\frac{1}{\lambda}(z - y) \in Q(y)$. We set $u = \frac{1}{\lambda}(z - x)$ and $v = \frac{1}{\lambda}(z - x)$. Therefore, $z = \lambda u + x$ and $z = \lambda v + y$. By η-monotonicity of Q, we have

$$\begin{aligned}
0 = \langle z - z, \eta(x, y)\rangle &= \langle \lambda u + x - (\lambda v + y), \eta(x, y)\rangle \\
&= \lambda\langle u - v, \eta(x, y)\rangle + \langle x - y, \eta(x, y)\rangle \\
&\geq \langle x - y, \eta(x, y)\rangle.
\end{aligned}$$

Since η is strictly monotone, we have $x = y$, and hence $(I + \lambda Q)^{-1}$ is a single-valued map. □

Problem 6.1 (Open Problem)**.** If Q is maximal η-monotone, then under what conditions is $R(I + \lambda Q) = H$?

The following lemma can be easily proved, and therefore we omit its proof.

Lemma 6.9. Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a functional and $\eta : H \times H \to H$ be a skew mapping. Then, the set-valued map $\partial_\eta \Phi : H \to 2^H$ is η-monotone.

In the remainder of this section, we assume that $\eta : H \times H \to H$ is strictly monotone such that $\eta(y, x) + \eta(x, y) = \mathbf{0}$ for all $x, y \in H$, and $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ is a functional such that $R(I + \lambda\partial_\eta \phi) = H$ for some $\lambda > 0$.

From Proposition 6.2 and Lemma 6.9, we note that the mapping

$$J_\lambda^\Phi(x) := (I + \lambda\partial_\eta \Phi)^{-1}(x), \quad \text{for all } x \in H$$

is single-valued.

Let us transform (6.28) into a fixed-point problem.

Lemma 6.10. The element x is a solution of (6.28) if and only if

$$x = J_\lambda^\phi(x - \lambda F(x)), \tag{6.33}$$

where $\lambda > 0$ is a constant, $J_\lambda^\Phi = (I + \lambda\partial_\eta\Phi)^{-1}$ is the so-called *proximal map*, and I stands for the identity operator on H.

Proof. From the definition of J_λ^Φ, we have

$$x - \lambda F(x) \in x + \lambda\partial_\eta\Phi(x),$$

and hence,

$$-F(x) \in \partial_\eta\Phi(x).$$

By using the definition of η-subdifferential, we have

$$\langle -F(x), \eta(y, x)\rangle \leq \Phi(y) - \Phi(x), \quad \text{for all } y \in H.$$

This implies that x is a solution of (6.28). □

Lemma 6.10 enables us to reformulate (6.28) as a fixed-point problem of solving

$$x = T(x), \tag{6.34}$$

where

$$T(x) = J_\lambda^\Phi(x - \lambda F(x)).$$

On the basis of this observation, we can suggest the following algorithm to find an approximate solution of (6.28).

Algorithm 2

Given $x_0 \in H$, compute x_{n+1} by the rule

$$x_{n+1} = J_\lambda^\Phi(x_n - \lambda F(x_n)), \tag{6.35}$$

where $\lambda > 0$ is a constant.

To perturb scheme (6.35), we first add to the right-hand side of (6.35) an error e_n to take into account a possible inexact computation of the proximal point and we consider another perturbation by replacing Φ by Φ_n in (6.35) where the sequence $\{\Phi_n\}$ approximates Φ. Finally, we obtain the following perturbed algorithm.

Algorithm 3

Generate, from any starting point $x_0 \in H$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_\lambda^{\Phi_n}(x_n - \lambda F(x_n)) + e_n. \tag{6.36}$$

Lemma 6.11. Let $\eta : H \times H \to H$ be skew, strongly monotone, and Lipschitz continuous with constants σ and δ, respectively. Then,

$$\left\|J_\lambda^\Phi(x) - J_\lambda^\Phi(y)\right\| \leq \tau\|x-y\|, \quad \text{for all } x, y \in H,$$

where $\tau = \frac{\delta}{\sigma}$.

Proof. From the definition of J_λ^Φ, we have

$$J_\lambda^\Phi(x) = (I + \lambda\partial_\eta\Phi)^{-1}(x),$$

and hence,

$$\frac{1}{\lambda}\left(x - J_\lambda^\Phi(x)\right) \in \partial_\eta\Phi\left(J_\lambda^\Phi(x)\right)$$

$$\frac{1}{\lambda}\left(y - J_\lambda^\Phi(y)\right) \in \partial_\eta\Phi\left(J_\lambda^\Phi(y)\right), \quad \text{for all } x, y \in H.$$

Since $\partial_\eta\Phi$ is η-monotone, we have

$$\frac{1}{\lambda}\left\langle x - J_\lambda^\Phi(x) - \left(y - J_\lambda^\Phi(y)\right), \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle \geq 0.$$

Multiplying by $\lambda > 0$, we get

$$\left\langle x - y - \left(J_\lambda^\Phi(x) - J_\lambda^\Phi(y)\right), \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle \geq 0,$$

that is,

$$\left\langle x - y, \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle \geq \left\langle J_\lambda^\Phi(x) - J_\lambda^\Phi(y)), \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle. \tag{6.37}$$

Since η is strongly monotone, we have

$$\left\langle J_\lambda^\Phi(x) - J_\lambda^\Phi(y)), \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle \geq \sigma\left\|J_\lambda^\Phi(x) - J_\lambda^\Phi(y)\right\|^2. \tag{6.38}$$

From Lipschitz continuity of η, we get

$$\begin{aligned} \left\langle x - y, \eta\left(J_\lambda^\Phi(x), J_\lambda^\Phi(y)\right)\right\rangle &\leq \|x-y\| \left\|\eta\left(J_\lambda^\Phi(x), J_\lambda^\phi(y)\right)\right\| \\ &\leq \delta\|x-y\| \left\|J_\lambda^\phi(x) - J_\lambda^\phi(y)\right\|. \end{aligned} \tag{6.39}$$

By combining inequalities (6.37), (6.38), and (6.39), we have

$$\left\|J_\lambda^\Phi(x) - J_\lambda^\Phi(y)\right\| \leq \tau\|x-y\|, \quad \text{for all } x, y \in H,$$

where $\tau = \frac{\delta}{\sigma}$. □

Theorem 6.17. Let $F : H \to H$ and $\eta : H \times H \to H$ be skew, Lipschitz continuous, and strongly monotone. For each n, let $\Phi_n : H \to \mathbb{R} \cup \{+\infty\}$ and $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be such that $R(I + \lambda\partial_\eta\Phi_n) = R(I + \lambda\partial_\eta\Phi) = H$ for some $\lambda > 0$. Assume that $\lim_{n\to+\infty}\left\|J_\lambda^{\Phi_n}(z) - J_\lambda^{\Phi}(z)\right\| = 0$ for all $z \in H$, $\{x_n\}$ generated by (6.36) with $\lim_{n\to+\infty}\|e_n\| = 0$ and x is a solution of (6.28), then x_{n+1} converges strongly to x, for

$$\left|\lambda - \frac{\alpha\tau}{\beta^2}\right| < \frac{\sqrt{\alpha^2\tau^2 - \beta^2(\tau^2-1)}}{\beta^2},$$

$$\alpha\tau > \beta\sqrt{\tau^2-1} \text{ and } \tau > 1,$$

where β and δ are Lipschitz constants of F and η, respectively, and α and σ are strong monotonicity constants of F and η, respectively, with $\tau = \frac{\sigma}{\delta}$.

Proof. From Lemma 6.10, we see that $x \in H$ satisfies (6.28) as a solution of (6.33) and vice versa. Thus, we have

$$x = J_\lambda^{\phi}\left(x - \lambda F(x)\right).$$

By setting $h(x) := x - \lambda F(x)$ and from (6.33) and (6.36), we have

$$\begin{aligned} \|x_{n+1} - x\| &\le \left\|J_\lambda^{\Phi_n}\left(x_n - \lambda F(x_n)\right) - J_\lambda^{\Phi}\left(x - \lambda F(x)\right)\right\| + \|e_n\| \\ &= \left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi}(h(x))\right\| + \|e_n\|. \end{aligned} \tag{6.40}$$

By introducing the term $J_\lambda^{\Phi_n}(h(x))$, we get

$$\begin{aligned} \left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi}(h(x))\right\| &\le \left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi_n}(h(x))\right\| \\ &\quad + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\|. \end{aligned}$$

By Lemma 6.11, we have

$$\left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi}(h(x))\right\| \le \tau\|h(x_n) - h(x)\| + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\|.$$

Hence,

$$\begin{aligned} \left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi}(h(x))\right\| &\le \tau\left\|x_n - \lambda F(x_n) - x + \lambda F(x)\right\| \\ &\quad + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\| \\ &\le \tau\left\|x_n - x - \lambda(F(x_n) - F(x))\right\| \\ &\quad + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\|. \end{aligned} \tag{6.41}$$

By using Lipschitz continuity and strong monotonicity of F, we have

$$\begin{aligned} &\|x_n - x - \lambda(F(x_n) - F(x))\|^2 \\ = \;& \|x_n - x\|^2 - 2\lambda\langle x_n - x, F(x_n) - F(x)\rangle + \lambda^2\|F(x_n) - F(x)\|^2 \\ \le \;& \|x_n - x\|^2 - 2\lambda\alpha\|x_n - x\| + \lambda^2\beta^2\|x_n) - x\|^2 \\ = \;& (1 + \lambda^2\beta^2 - 2\lambda\alpha)\|x_n - x\|^2. \end{aligned} \tag{6.42}$$

By combining inequalities 6.41 and 6.42, we get

$$\begin{aligned}\left\|J_\lambda^{\Phi_n}(h(x_n)) - J_\lambda^{\Phi}(h(x))\right\| &\leq \left(\tau\sqrt{(1+\lambda^2\beta^2 - 2\lambda\alpha)}\right)\|x_n - x\| \\ &\quad + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\|.\end{aligned}$$

From the above inequality and by combining (6.40) and (6.41), we get

$$\|x_{n+1} - x\| \leq \theta\|x_n - x\| + \left\|J_\lambda^{\Phi_n}(h(x)) - J_\lambda^{\Phi}(h(x))\right\| + \|e_n\|,$$

where $\theta = \tau\sqrt{(1+\lambda^2\beta^2 - 2\lambda\alpha)} < 1$, for

$$\left|\lambda - \frac{\alpha\tau}{\beta^2}\right| < \frac{\sqrt{\alpha^2\tau^2 - \beta^2(\tau^2-1)}}{\beta^2},$$

$$\alpha\tau > \beta\sqrt{\tau^2 - 1} \text{ and } \tau > 1.$$

By setting

$$\epsilon_n = \left\|J_\lambda^{\phi_n}(h(x)) - J_\lambda^{\phi}(h(x))\right\| + \|e_n\|,$$

we can write

$$\|x_{n+1} - x\| \leq \theta\|x_n - x\| + \epsilon_n.$$

Hence,

$$\|x_{n+1} - x\| \leq \theta^{n+1}\|x_0 - x\| + \sum_{j=0}^{n} \theta^j \epsilon_{n-j}.$$

Since $\lim_{n\to+\infty} \epsilon_n = 0$, from Ortega and Rheinboldt [56], we see that x_n converges strongly to x. □

6.6 Appendix

Let B be a subset of a vector space X. We denote by co$B$ the convex hull of B.

Definition 6.14. Let K be a nonempty convex subset of a vector space X. A set-valued map $P : K \to 2^K$ is said to be a *KKM map* if for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of K,

$$\text{co}\{x_1, x_2, \ldots, x_m\} \subseteq \bigcup_{i=1}^{m} P(x_i),$$

where co$\{x_1, x_2, \ldots, x_m\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_m\}$.

The following Fan-KKM theorem and the Browder-type fixed-point theorem for set-valued maps will be the key tools to establish existence results for solutions of nonsmooth vector variational-like inequalities.

Theorem 6.18 (Fan-KKM Theorem)**.** [24] Let K be a nonempty convex subset of a Hausdorff topological vector space X and $P : K \to 2^K$ be a KKM map such that $P(x)$ is closed for all $x \in K$, and $P(x)$ is compact for at least one $x \in K$. Then, $\bigcap_{x \in K} P(x) \neq \emptyset$.

Theorem 6.19. [5] Let K be a nonempty convex subset of Hausdorff topological vector space X, and $P, Q : K \to 2^K$ be two set-valued maps. Assume that the following conditions hold:

(i) For each $x \in K$, $\mathrm{co}P(x) \subseteq Q(x)$ and $P(x)$ is nonempty.

(ii) For each $y \in K$, $P^{-1}(y) = \{x \in K : y \in P(x)\}$ is open in K.

(iii) If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that $\tilde{y} \in P(x)$.

Then, there exists $\bar{x} \in K$ such that $\bar{x} \in Q(\bar{x})$.

Theorem 6.20. [17] Let K be a nonempty convex subset of Hausdorff topological vector space X and $P : K \to 2^K$ a set-valued map. Assume that the following conditions hold:

(i) For all $x \in K$, $P(x)$ is convex.

(ii) For each finite subset A of K and for all $y \in \mathrm{co}(A)$, $T^{-1}(y) \cap \mathrm{co}(A)$ is open in $\mathrm{co}(A)$.

(iii) For each finite subset A of K and all $x, y \in \mathrm{co}(A)$ and every sequence $\{x_m\}$ in K converging to x such that $\lambda y + (1 - \lambda)x \notin P(x_m)$ for all $m \in \mathbb{N}$ and all $\lambda \in [0, 1]$, we have $y \notin P(x)$.

(iv) There exist a nonempty compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in P(x)$ for all $x \in K \setminus D$.

(v) For all $x \in D$, $P(x)$ is nonempty.

Then, there exists $\hat{x} \in K$ such that $\hat{x} \in P(\hat{x})$.

Theorem 6.21 (Kakutani)**.** [39] Let K be a nonempty compact convex subset of a Banach space X and $P : K \to 2^K$ be a set-valued maps such that for each $x \in K$, $P(x)$ is nonempty, compact, and convex. Then, P has a fixed point, that is, there exists $\bar{x} \in K$ such that $\bar{x} \in P(\bar{x})$.

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Part III

Vector Optimization

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Chapter 7

Vector Optimization: Basic Concepts and Solution Methods

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7.1 Introduction

Mathematical optimization studies the problem of finding a best element from a set of feasible alternatives with regard to a criterion or objective function. It is written in the form

$$\begin{array}{ll} \text{optimize} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where X is a nonempty set, called a feasible set or a set of feasible alternatives, and f is a real function on X, called a criterion or objective function. Here "optimize" stands for either "minimize" or "maximize," which amounts to finding $\bar{x} \in X$ such that either $f(\bar{x}) \leqq f(x)$ for all $x \in X$, or $f(\bar{x}) \geqq f(x)$ for all $x \in X$.

This model offers a general framework for studying a lot of real-world and theoretical problems in the sciences and human activities. However, in many practical situations, we are faced with problems that involve not only one criterion, but a number of criteria, often in conflict with each other. It is impossible, then, to model such problems in the above-mentioned optimization framework. Here are some example situations.

Automobile design. The objective of car design is to determine technical parameters of a vehicle to minimize (1) production cost, (2) fuel consumption, and (3) emissions, while maximizing (4) performance, and (5) crash safety. These criteria are not always compatible; for instance a best performance engine is often subject to a very high production cost, which means that there exists no design that meets all criteria at their best.

House purchase. Buying property is a decision of a lifetime and is often realized with the help of real estate agencies. A buyer is offered a number of houses or apartments that roughly meet his budget and preliminary requirements. In order to make a decision, the buyer employs his criteria to classify the available offers. The final choice should satisfy a minimal cost, minimal maintenance charges, maximal quality and comfort, best environment, etc. It is quite natural that a better-quality house is more expensive, and so without compromise, the best choice is impossible.

Low-cost power. In a system of thermal generators the problem of low-cost power delivery consists of allocating power that is generated by each generator in the system. The aim is not only to satisfy the demand in electricity, but also to fulfill two main criteria: minimize the cost of generation and minimize the emission of gazes. The cost and the emissions are measured in different units, and so it is not possible to combine the two criteria in one.

Queen Dido's city. Queen Dido's famous problem consisted of finding a territory bounded by a line that enclosed the maximum area for a given perimeter.

The solution is known to be the circle by elementary calculus. However, because it is inconceivable to have a city touching the sea without seashore, Queen Dido set another objective for her territory—to have the maximum seashore. As a result, a half-circle partially meets her two objectives.

As we have seen even in the simple situations described above, there exist no alternatives simultaneously satisfying all criteria, which means that the known concepts of optimization do not apply and there is a real need to develop new notions of optimality for problems involving vector objective functions. The inventor of such a concept is V. F. Pareto (1848–1923), an Italian economist who introduced the Pareto optimum as "The optimum allocation of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation." Prior to Pareto, F. Y. Edgeworth (1845–1926), an Irish economist, defined an optimum for the multiutility problem of two consumers P and Q as "a point (x, y) such that in whatever direction we take an infinitely small step, P and Q do not increase together but that, while one increases, the other decreases." According to Pareto's definition, among the feasible alternatives, those that can simultaneously be improved with respect to all criteria cannot be optimal. And an alternative is optimal if any alternative that is better than it, with respect to a certain criterion, is worse with respect to some other criterion; that is, a trade-off takes place when trying to get a better alternative. From a mathematical point of view, if one defines a domination order in the set of feasible alternatives by a set of criteria—an alternative a dominates an alternative b if the value of every criterion function at a is bigger than that at b—then an alternative is optimal in the Pareto sense if it is dominated by no other alternatives. In other words, an alternative is optimal if it is maximal with respect to the above said order. This explains the mathematical origin of the theory of vector optimization which stems from the theory of ordered spaces developed by G. Cantor (1845–1918) and F. Hausdorff (1868–1942).

7.2 Mathematical Backgrounds

Let A be a nonempty set in the k-dimensional Euclidean space $\mathbb{R}^k$. We shall make use of the following standard notations:

- $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and A^c stand for the closure, the interior, and the complement of A, respectively. When A is convex, $\mathrm{ri}(A)$ denotes the relative interior of A.

- $\mathrm{co}(A)$ is the convex hull of A, which consists of convex combinations $\sum_{i=1}^{m} \lambda_i a_i$ with $\lambda_i \geqq 0, a_i \in A, i = 1, \ldots, m, m \in \mathbb{N}$ and $\sum_{i=1}^{m} \lambda_i = 1$.

- pos(A) is the positive hull of A, which consists of positive combinations $\sum_{i=1}^{m} \lambda_i a_i$ with $\lambda_i \geqq 0, a_i \in A, i = 1, \ldots, m, m \in \mathbb{N}$.

- cone(A) is the cone generated by A, that is,

$$\text{cone}(A) := \{tx : t \geq 0\ , \quad x \in A\}.$$

- $B(x, r)$ is the closed ball centered at x with radius $r > 0$.

If C is a cone, its linear part is denoted $\ell(C)$, that is, $\ell(C) = C \cap (-C)$. When the linear part of C is trivial, that is, $\ell(C) = \{0\}$; we say that C is pointed. Elements of $\mathbb{R}^k$ will be written as row vectors. If $x \in \mathbb{R}^k$, its components are denoted $x_1, \ldots, x_k$, that is, $x = (x_1, \ldots, x_k)$, while x^T denotes the transpose of x.

7.2.1 Partial Orders

The real numbers are ordered by the standard order relation "$\leq$" (less than or equal to) in which every pair of elements is related and known as a total order. When not every pair of elements can be related, in particular, pairs of vectors in a high-dimensional space, partial orders are an alternative method to rank them.

Definition 7.1. Let R be a binary relation on $\mathbb{R}^k$, that is, R is a subset of $\mathbb{R}^k \times \mathbb{R}^k$. It is said to be a *partial order* on $\mathbb{R}^k$ if it is

(i) reflexive: $(x, x) \in R$ for every $x \in \mathbb{R}^k$;

(ii) transitive: $(x, y), (y, z) \in R$ imply $(x, z) \in R$;

(iii) antisymmetric: $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$.

In some literature, the antisymmetry is not required, in which case two elements x and y are called equivalent if and only if $(x, y) \in R$ and $(y, x) \in R$. One may then generate an order, denoted $[R]$, on equivalence classes as follows. For $x, y \in \mathbb{R}^k$, denote the class of all elements equivalent to x by $[x]$. Then $([x], [y]) \in [R]$ if and only if $(x', y') \in R$ for some representatives x' and y' of the equivalence classes $[x]$ and $[y]$, respectively. It is easy to check that $[R]$ is a partial order if R is a reflexive and transitive (not necessarily antisymmetric) binary relation.

Being a partial order, R is said to be compatible with the linear structure of the space if $(x, y) \in R$ implies $(x + z, y + z), (tx, ty) \in R$ for all $z \in \mathbb{R}^k$ and $t > 0$. Some authors use the terminology of linear partial orders for total orders, which means that every couple of elements are comparable.

When R is a binary relation, we write $x \succeq_R y$ and say that x dominates y with respect to R if $(x, y) \in R$. Thus, with this notation, the relation "$\succeq_R$" is a partial order if

(i) $x \succeq_R x$ for every $x \in \mathbb{R}^k$;

(ii) $x \succeq_R y, y \succeq_R z$ implies $x \succeq_R z$ for all $x, y, z \in \mathbb{R}^k$;

(iii) $x \succeq_R y, y \succeq_R x$ implies $x = y$ for all $x, y \in \mathbb{R}^k$.

Moreover, it is compatible with the linear structure if for every $x, y \in \mathbb{R}^k$, $x \succeq_R y$ implies $tx + z \succeq_R ty + z$ for every $z \in \mathbb{R}^k$ and $t \geq 0$. Notice that not every partial order is linear.

Example 7.1. Consider the set

$$R = \bigcup_{k,k' \in \mathbb{N}} \{x \in \mathbb{R}^4 : x_1 - x_3 = k, x_2 - x_4 = k'\}.$$

It defines a binary relation in $\mathbb{R}^2$, which is a partial order. This order is not compatible with the linear structure because for $x = (1, 2)$ and $y = (0, 1)$, we have $x \succeq_R y$, but $(1/2)x \not\succeq_R (1/2)y$.

Partial orders that are compatible with the linear structure are characterized by convex cones.

Proposition 7.1. If $R \subseteq \mathbb{R}^k \times \mathbb{R}^k$ is a partial order compatible with the linear structure, then the set $C := \{x \in \mathbb{R}^k : (x, 0) \in R\}$ is a pointed convex cone in $\mathbb{R}^k$. Conversely, if C is a pointed convex cone in $\mathbb{R}^k$, then the relation R is defined by $(x, y) \in R$ if and only if $x - y \in C$ is a partial order compatible with the linear structure in $\mathbb{R}^k$.

Proof. Let R be a partial order compatible with the linear structure in $\mathbb{R}^k$. Let $(x, 0) \in R$. Then one has $(tx, 0) \in R$ for all $t > 0$. Hence, $tx \in C := \{y \in \mathbb{R}^k : (y, 0) \in R\}$ for $t > 0$. When $t = 0$, by reflectivity one has $(0, 0) \in R$, hence $0 \in C$ and C is a cone. This cone is convex because $x, y \in C$ means $(x, 0)$ and $(y, 0)$ belong to R, hence by transitivity $(x+y, 0) \in R$, and therefore $x+y \in C$. Finally, C is pointed because for any $x \in C \cap (-C)$, one has $(x, 0) \in R$ and $(0, x) \in R$, which implies $x = 0$ by the antisymmetry of the order.

Conversely, assume that C is a pointed and convex cone in $\mathbb{R}^k$. Since $0 \in C$ and $x - x = 0$ for all $x \in \mathbb{R}^k$, we have $(x, x) \in R$. This shows that R is reflexive. Moreover, if $x - y \in C$ and $y - z \in C$, then by the convexity of C we obtain $x - z = x - y + y - z \in C$ or equivalently, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$, proving that R is transitive. Furthermore, if $(x, y) \in R$ and $(y, x) \in R$, then we have simultaneously $x - y \in C$ and $y - x \in C$. As the cone C is pointed, $x - y = 0$, by which R is antisymmetric. We conclude that R is a partial order in $\mathbb{R}^k$. It is compatible with the linear structure because $x - y \in C$ implies $t(x - y) \in C$ for $t > 0$ and $(x + z) - (y + z) \in C$ for all $z \in \mathbb{R}^k$, which means $(x, y) \in R$ implies $(tx, ty) \in R$ and $(x + z, y + z) \in R$ for all $t \geq 0$ and $z \in \mathbb{R}^k$. The proof is complete. □

The order determined by a pointed convex cone C is often written as $x \geq_C y$ if and only if $x - y \in C$. Given a convex cone $C \subseteq \mathbb{R}^k$, the positive polar cone and the strictly positive polar cone of C are defined, respectively, by

$$C' := \{\xi \in \mathbb{R}^k : \langle \xi, x\rangle \geq 0 \quad \text{for all} \quad x \in C\}$$
$$C^+ := \{\xi \in \mathbb{R}^k : \langle \xi, x\rangle > 0 \quad \text{for all} \quad x \in C \setminus \{0\}\},$$

where $\mathbb{R}^k$ is identified with the topological dual space of $\mathbb{R}^k$.

Here are some particular partial orders that are frequently used in optimization.

The Pareto order. Let $\mathbb{R}^k_+$ be the positive octant of the n-dimensional Euclidean space $\mathbb{R}^k$. This cone is convex, closed, and pointed. The *Pareto*

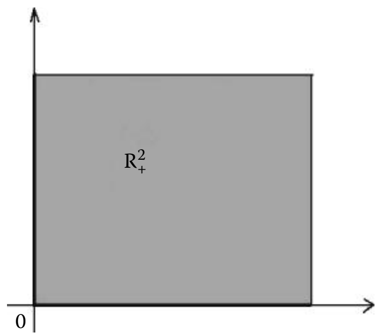

FIGURE 7.1: The Pareto cone in $\mathbb{R}^2$.

order determined by this cone is given as follows: for two vectors $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k)$ in $\mathbb{R}^k$, one has $x \geq_{\mathbb{R}^k_+} y$ if and only if $x_i \geq y_i$, $i = 1, \cdots, k$. The cone $\mathbb{R}^k_+$, is called the *Pareto cone* because the original Pareto optimality is defined by the order generated by this cone. When $k = 1$, the usual order of real numbers is exactly the order $(\geq_{\mathbb{R}_+})$. This order is total in the sense that any two numbers x and y are comparable: either $x \geq y$ or $y \geq x$. When $k > 1$, the order $(\geq_{\mathbb{R}^k_+})$ is not total.

The extended Pareto order. Let $\epsilon > 0$. The *ϵ-extended Pareto cone*, denoted $\mathbb{R}^k_{+\epsilon}$ is given by the system $(\epsilon e + e^i)x^T \geq 0, i = 1, \ldots, k$ where e is the vector of ones and e^i is the ith unit vector. This cone is closed, pointed and convex, and contains the cone $\mathbb{R}^k_+ \setminus \{0\}$ in its interior. The corresponding order is $x \geq_{\mathbb{R}^k_{+\epsilon}} y$ if and only if $\min\{x_i - y_i : i = 1, \ldots, k\} + \epsilon \sum_{i=1}^k (x_i - y_i) \geq 0$. When ϵ is zero, the above order is exactly the Pareto order.

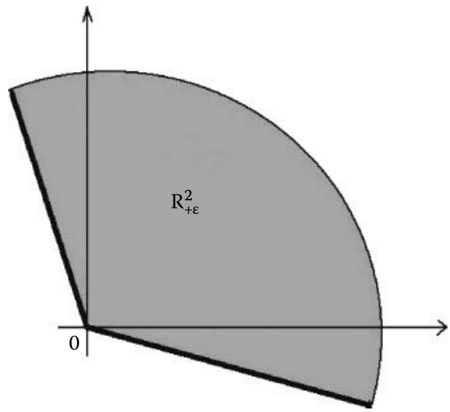

FIGURE 7.2: The ϵ-extended Pareto cone in $\mathbb{R}^2$.

The lexicographic order. The *lexicographic cone*, denoted *Lex*, consists of the origin and all vectors whose first nonzero component is positive. It is convex, pointed, but not closed. Its closure is the half-space of vectors x with $x_1 \geq 0$. The *lexicographic order* generated by it is given as $x \geq_{Lex} y$ if and only if either $x = y$, or there is some $j \in \{1, \ldots, k-1\}$ such that $x_i = y_i$ for $i = 1, \ldots, j$ and $x_{j+1} > y_{j+1}$.

The lexicographic order is total: for every x and y in $\mathbb{R}^k$, either $x \geq_{Lex} y$ or $y \geq_{Lex} x$.

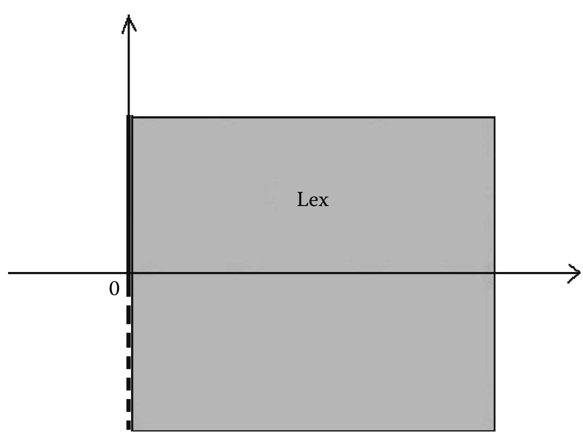

FIGURE 7.3: The lexicographic cone in $\mathbb{R}^2$.

The ubiquitous order. This cone, denoted *Ub*, consists of the origin and all vectors whose last nonzero component is positive. It is convex, pointed, and not closed. Its closure is the half-space of vectors x with $x_k \geq 0$.

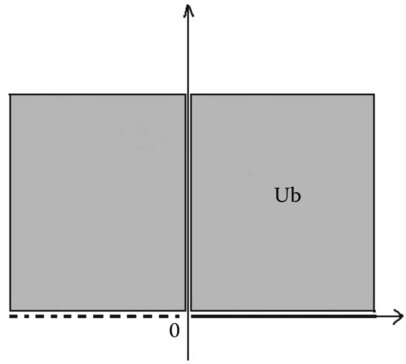

FIGURE 7.4: The ubiquitous cone in $\mathbb{R}^2$.

The ubiquitous order is given by $x \geq_{Ub} y$ if and only if either $x = y$, or there is some $j \in \{1, \ldots, k\}$ such that $x_i = y_i$ for $i = j+1, \ldots, k$ and $x_j > y_j$. The ubiquitous order is total: for every x and y in $\mathbb{R}^k$, either $x \geq_{Ub} y$ or $y \geq_{Ub} x$. The name of this cone comes from the fact that in the space of sequences with finite support ℓ_0 (a sequence of real numbers belongs to ℓ_0 if and only if its terms are all zero except for a finite number of them), equipped with the max-norm, the closure of the ubiquitous cone is the whole space. More precisely, for every element $x \in \ell_0$, there is an element $y \in Ub$ such that the whole half-open interval $(x, y]$ lies in the cone Ub.

The Lorentz order. The *Lorentz cone* (called also the *second-order cone* or the *ice-cream cone*) is given in the form

$$L = \{x \in \mathbb{R}^k : \|x^{-k}\| \leq x_k\},$$

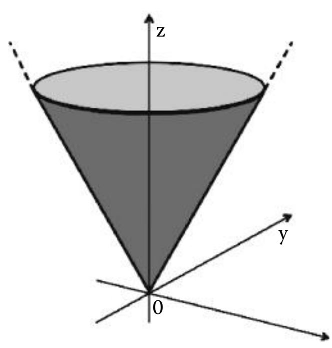

FIGURE 7.5: The Lorentz cone in $\mathbb{R}^3$.

where x^{-k} denotes the vector obtained from x by deleting the last component x_k, and the corresponding order is given as $x \geq_K y$ if and only if $\|x^{-k} - y^{-k}\| \leq x_k - y_k$. This order is used to express constraints in second-order cone programming. Namely, if A is a matrix, b, c are vectors of appropriate dimensions, and d is a real number, then the constraint $\|Ax + b\| \leq c^T x + d$ is equivalent to the inequality $\begin{pmatrix} A \\ c^T \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix} \geq_L 0$.

Conic extended orders. Let C be a convex cone whose closure is pointed. For a positive ϵ, we define the ϵ-*conic neighborhood* of C by

$$C_\epsilon = \{x \in \mathbb{R}^k : d(x, C) \leq \epsilon \|x\|\},$$

where $d(x, C)$ is the distance from x to C. One may prove that for $\epsilon > 0$ sufficiently small, C_ϵ is a pointed, closed and convex cone and contains $C \setminus \{0\}$ in its interior. Moreover, for $\epsilon < \epsilon'$, one has $C_\epsilon \setminus \{0\} \subset \mathrm{int}(C_{\epsilon'})$ and $\mathrm{cl}(C) = \bigcap_{\epsilon > 0} C_\epsilon$. The order generated by C_ϵ is called the ϵ-conic extended order of the order $\geq_C$. In $\mathbb{R}^2$, the δ-conic extended order of the Pareto cone $\mathbb{R}^2_+$ coincides with the ϵ-extended Pareto order, where $\delta = \epsilon / \sqrt{\epsilon^2 + (1 + \epsilon)^2})$.

Correct orders. We say that a cone C in $\mathbb{R}^k$ is correct if $\mathrm{cl}C + C \setminus \ell(C) \subseteq C$ or equivalently $\mathrm{cl}C + C \setminus \ell(C) \subseteq C \setminus \ell(C)$. Let us mention some cases of correct cones:

(a) every closed and convex cone is correct;

(b) if $C \setminus \ell(C)$ is open, then C is correct;

(c) if C consists of the origin and an intersection of half-spaces that are either open or closed, then C is correct.

An order determined by a convex cone C is said to be correct if C is a correct cone. Among the cones we presented above, the Pareto cone, the extended Pareto cone, the Lorentz cone, and the conic extended cone are correct, while the lexicographic cone and the ubiquitous cone are not correct.

7.2.2 Increasing Sequences

Assume that the space $\mathbb{R}^k$ is partially ordered by a pointed convex cone C. The order is denoted $\geq_C$. The associated strict order $>_C$ is defined by

$$x >_C y \text{ if and only if } x \geq_C y \text{ and } y \not\geq_C x.$$

which is equivalent to the inclusion $x - y \in C \setminus \{0\}$. In this case we say that x strictly dominates y.

Definition 7.2. A sequence $\{x^i\}_{i \geq 1}$ of elements in $\mathbb{R}^k$ is said to be *increasing* if $x^{i+1} \geq_C x^i$ for every $i = 1, 2, \ldots$ and it is strictly increasing if the above inequality is strict.

Proposition 7.2. Assume that the order $\geq_C$ is correct. Then, the limit of a convergent strictly increasing sequence strictly dominates the terms of the sequence.

Proof. Let $\{x^i\}_{i\geq 1}$ be a strictly increasing sequence and let x be its limit. By transitivity, we have $x^{i+m} - x^{i+1} \in C \setminus \{0\}$ for all $i \geq 1$ and $m \geq 2$. When m tends to ∞, we obtain $x - x^{i+1} \in cl(C)$. Since C is correct, we deduce

$$x - x^i = (x - x^{i+1}) + (x^{i+1} - x^i) \in cl(C) + C \setminus \{0\} \subset C \setminus \{0\},$$

which shows that x strictly dominates x^i for all $i \geq 1$. $\square$

7.2.3 Monotone Functions

Definition 7.3. A real function $g : \mathbb{R}^k \to \mathbb{R}$ is said to be *increasing* (with respect to the partial order $\geq_C$) if for every a, $b \in \mathbb{R}^k$

$$a >_C b \text{ implies } g(a) > g(b).$$

When C has a nonempty interior, an increasing function with respect to the order $\geq_{C^0}$, where $C^0 = \{0\} \cup \operatorname{int}(C)$ is called weakly increasing with respect to $\geq_C$.

It is clear that an increasing function is weakly increasing. The converse is not true; for instance, the function $g(x, y) = x$ on $\mathbb{R}^2$ equipped with the Pareto order is weakly increasing, but not increasing, for $g(0, 1) = g(0, 2)$ while $(0, 2) >_{\mathbb{R}^2_+} (0, 1)$.

Proposition 7.3. The set of increasing functions is a pointed and convex cone without the origin. Moreover, the sum of an increasing function and a continuous weakly increasing function is increasing.

Proof. Let f and g be increasing functions. For every $t > 0$ and for every a, $b \in \mathbb{R}^k$ with $a >_C b$, one has $f(a) > f(b)$ and $g(a) > g(b)$ and deduces $tf(a) > tf(b)$ and $f(a)+g(a) > f(b)+g(b)$. Hence, tf and $f+g$ are increasing. To prove the second assertion, let f be an increasing function and let g be a continuous weakly increasing function. For $a >_C b$, by choosing a vector v from the interior of C, one has

$$a + tv \geq_{C^0} b, \text{ for all } t > 0,$$

which implies $f(a) > f(b)$, and $g(a + tv) > g(b)$, for every $t > 0$. Since g is continuous, by letting t tend to 0, one has $g(a) \geq g(b)$. Then, $f(a) + g(a) > f(b) + g(b)$, and hence $f + g$ is increasing. $\square$

Proposition 7.4. Let f be a linear function on $\mathbb{R}^k$. It is increasing (respectively weakly increasing) if and only if there is some vector $\xi \in C^+$ (respectively, $\xi \in C' \setminus \{0\}$) such that $f(x) = \langle \xi, x \rangle$ for all $x \in \mathbb{R}^k$.

Proof. Since f is linear, there is a vector $\xi \in \mathbb{R}^k$ such that $f(x) = \langle \xi, x \rangle$, $x \in \mathbb{R}^k$. If f is increasing, one has

$$\langle \xi, x \rangle > \langle \xi, 0 \rangle = 0, \text{ for every } x >_C 0.$$

By definition, $\xi \in C^+$. The converse is clear; that is, if $\xi \in C^+$, then for $a >_C b$, one has $a - b \in C \setminus \{0\}$ and $\langle \xi, a \rangle - \langle \xi, b \rangle = \langle \xi, a - b \rangle > 0$, which shows that $f(x) = \langle \xi, x \rangle$ is increasing. The proof for weakly increasing functions is similar. □

7.2.4 Biggest Weakly Monotone Functions

Let $a \in \mathbb{R}^k$ and $v \in \text{int }(C)$. We define a function $h_{a,v}$ on $\mathbb{R}^k$ by

$$h_{a,v}(x) = \sup\{t \in \mathbb{R} : x \in a + tv + C\}.$$

This function is well defined because for a given x, with $t < 0$ large, $x \in a + tv + C$, and with $t > 0$ large, $x \notin a + tv + C$. It is called a *biggest weakly increasing function* at a because of the properties given in the next proposition.

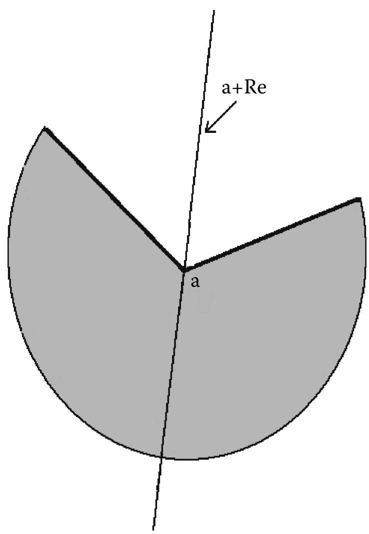

FIGURE 7.6: The lower level set of $h_{a,v}$ at a.

Proposition 7.5. The function $h_{a,v}$ is continuous and weakly increasing. Moreover, for every weakly increasing function g with $g(a) = 0$, we have inclusion of lower-level sets

$$lev_g(a) := \{x \in \mathbb{R}^k : g(x) \leq g(a)\} \subseteq lev_{h_{a,v}}(a) = \{x \in \mathbb{R}^k : x \notin a + \text{int }(C)\}.$$

Proof. The continuity of $h_{a,v}$ is clear. For weakly increasing functions, let $x - y \in \text{int}\,(C)$. There is $\varepsilon > 0$ such that $x - (y + \varepsilon v) \in \text{int}\,(C)$, and by the definition of $h_{a,v}$, one has $y \in a + \left(h_{a,v}(y) - \frac{\varepsilon}{2}\right) v + C$. Consequently,

$$x \in a + \left(h_{a,v}(y) - \frac{\varepsilon}{2}\right) v + C,$$

which implies that

$$h_{a,v}(x) \geq h_{a,v}(y) + \frac{\varepsilon}{2} > h_{a,v}(y),$$

as requested.

Now, let $x \notin lev_{h_{a,v}}(a)$. We have $x >_{C^0} a$. Since g is weakly increasing, $g(x) > g(a) = 0$ and therefore $x \notin lev_g(a)$. This proves the inclusion of the proposition. □

7.3 Pareto Maximality

Throughout this section, we assume that $\mathbb{R}^k$ is partially ordered by a pointed convex cone C. As already mentioned, the strict order associated with the order $\geq_C$ is denoted $>_C$.

Definition 7.4. Let $A \subseteq \mathbb{R}^k$ be a nonempty set. We say that

- a point $a \in A$ is an *ideal* (or *utopia*) *maximal point* of A if $a \geq_C x$ for every $x \in A$. The set of all ideal maximal points of A is denoted by $\text{IMax}(A)$ or $\text{IMax}(A|C)$;
- a point $a \in A$ is a *maximal* (or *Pareto maximal / efficient / nondominated*) *point* of A if whenever $x \geq_C a$ for some $x \in A$ one has $a \geq_C x$. The set of all maximal points of A is denoted by $\text{Max}(A)$ or $\text{Max}(A|C)$.

The sets of all ideal minimal points and minimal points of A are defined in a similar way and denoted, respectively, $\text{IMin}(A)$ and $\text{Min}(A)$. Actually, minimal points with respect to the order $\geq_C$ can be seen as maximal points with respect to the order generated by the cone $(-C)$:

$$\begin{aligned} \text{IMin}(A|C) &= \text{IMax}(A|-C) \\ \text{Min}(A|C) &= \text{Max}(A|-C). \end{aligned}$$

Example 7.2 illustrates Definition 7.4.

Example 7.2. Consider the set A (see Fig. 7.7) defined by

$$A = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \leq 1, 0 \leq x \leq 2, 0 \leq y \leq 1\}.$$

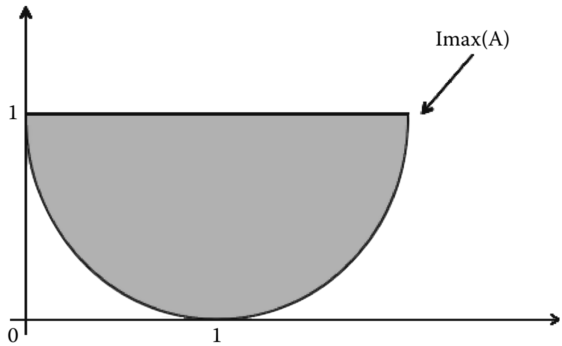

FIGURE 7.7: Ideal maximal points.

We have $\text{IMax}(A|\mathbb{R}^2_+) = \text{Max}(A|\mathbb{R}^2_+) = \{(2,1)\}$. However, if $\mathbb{R}^2$ is equipped with the order by the cone C (see Fig. 7.8)

$$C = \{(x,y) \in \mathbb{R}^2 : x - y \geq 0, x \geq 0, y \geq 0\},$$

then $\text{IMax}(A|C) = \emptyset$ and $\text{Max}(A|C) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = 1, 1 \leq x \leq 2, \frac{1}{2} \leq y \leq 1\}$.

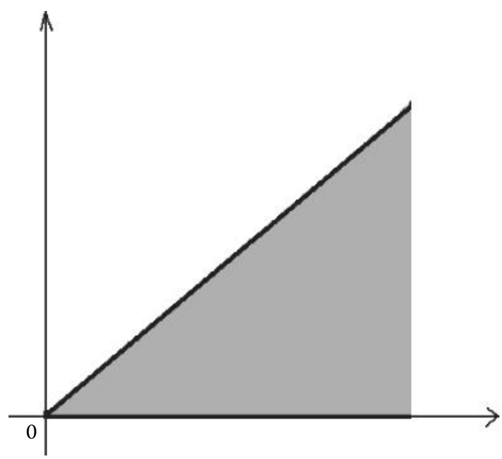

FIGURE 7.8: The cone C of Example 7.2.

Proposition 7.6. Let A be a nonempty set in $\mathbb{R}^k$. Then,

(a) $a \in \text{IMax}(A)$ if and only if $a \in A$ and $A \subseteq a - C$;

(b) $a \in \text{Max}(A)$ if and only if $a \in A$ and $A \cap (a + C \setminus \{0\}) = \emptyset$;

(c) if $\text{IMax}(A|C)$ is nonempty, then it is a singleton and $\text{IMax}(A|C) = \text{Max}(A|C)$.

Proof. The first part of (a) is obvious from the definition of ideal maximal points. For the second assertion, let $a \in \mathrm{Max}(A)$. Then, $a \in A$ by definition. Let $x \in A \cap (a + C \setminus\{0\})$. There is $y \in C\setminus\{0\}$ such that:

$$x = a + y. \tag{7.1}$$

In particular, $x \geq_C a$, and by the maximality of a, we also have $a \geq_C x$, which implies that $x - a = 0$. This contradicts (7.1) with $y \neq 0$. Thus, $A \cap (a + C \setminus\{0\}) = \varnothing$. Conversely, assume that $a \in A$ satisfies the condition $A \cap (a + C \setminus\{0\}) = \varnothing$. Let $x \in A$ with $x \geq_C a$. Then there is some $y \in C$ such that (7.1) holds. It follows that $y \in A \cap (a + C)$. By hypothesis $y = 0$, and hence $a = x$, implying $a \geq_C x$ as well. By this, a is a maximal element of A.

To prove the last assertion, let a and b be two ideal maximal points of A. Then we have $a \geq_C b$ and $b \geq_C a$, which means that both $a-b$ and $b-a$ belong to C. As C is pointed, $a = b$, which proves that IMax(A) is a singleton if it is nonempty. Furthermore, the inclusion $\mathrm{IMax}(A) \subseteq \mathrm{Max}(A)$ is clear. For the converse inclusion, let y be a maximal point of A. Choose any ideal maximal point x of A. Then $x \geq_C y$ by definition. Since y is maximal, one also has $y \geq_C x$. Now, for every element $x' \in A$. It is true that $x \geq_C x'$, which implies $y \geq_C x'$ by transitivity. Hence, y is an ideal maximal point of A. □

7.3.1 Maximality with Respect to Extended Orders

Assume that $\widetilde{C}$ is a pointed convex cone in $\mathbb{R}^k$. The partial order generated by $\widetilde{C}$ is denoted $\geq_{\widetilde{C}}$. We say that the order $\geq_{\widetilde{C}}$ is an extended order of the order $\geq_C$ if $x \geq_C y$ implies $x \geq_{\widetilde{C}} y$. This is equivalent to the inclusion $C \subseteq \widetilde{C}$. We wish to find a relation between maximality with respect to C and maximality with respect to $\widetilde{C}$.

Proposition 7.7. Let A be a nonempty set and $\widetilde{C}$ be a pointed and convex cone containing C. The following assertions hold:

(a) $\mathrm{IMax}(A|C) = \mathrm{IMax}(A|\widetilde{C})$ if $\mathrm{IMax}(A|C)$ is nonempty.

(b) $\mathrm{Max}(A|C) \supseteq \mathrm{Max}(A|\widetilde{C})$.

Proof. If $\mathrm{IMax}(A|C)$ is nonempty, then for $a \in \mathrm{IMax}(A|C)$, in view of Proposition 7.6(a) we have $a \in A$ and $A \subseteq a - C \subseteq a - \widetilde{C}$, which implies that $a \in \mathrm{IMax}(A|\widetilde{C})$. Moreover, as the set $\mathrm{IMax}(A|\widetilde{C})$ is a singleton, equality follows. Now, let $a \in \mathrm{Max}(A|\widetilde{C})$. By Proposition 7.6, $a \in A$ and $A \cap (a + \widetilde{C} \setminus \{0\}) = \emptyset$. Hence, $A \cap (a + C \setminus \{0\}) = \emptyset$, and a is maximal with respect to C. □

We saw in Example 7.2 that $\mathrm{IMax}(A|C)$ may be empty while $\mathrm{IMax}(A|\widetilde{C})$ with $\widetilde{C} = \mathbb{R}^2_+$ is not. Also, the inclusion in the second assertion of Proposition 7.7 is strict in that example.

7.3.2 Maximality of Sections

Let x be a given point and A a nonempty set in $\mathbb{R}^k$. We define a section of A at x by

$$A_x := A \cap (x + C).$$

Thus, A_x consists of all elements of A that dominate x. This section may be empty, but is always nonempty once x lies in A.

Proposition 7.8. For every $x \in \mathbb{R}^k$, we have

$$\text{Max}(A_x) \subseteq \text{Max}(A) = \bigcup_{y \in A} \text{Max}(A_y).$$

Proof. Let a be a maximal element of the section A_x. By Proposition 7.6, we have $a \in A_x$ and $A_x \cap (a + C \setminus \{0\}) = \emptyset$. We show that $A \cap (a + C \setminus \{0\}) = \emptyset$ too. Indeed, let $y \in A$ and $y \in a + C \setminus \{0\}$. By definition, $y \in A_x$. Hence, $y \in A_x \cap (a + C \setminus \{0\})$, which is a contradiction. In view Proposition 7.6, a is a maximal element of A.

Observe further that for $a \in \text{Max}(A)$, the section A_a is the singleton $\{a\}$. Hence, in view of the first part, $\text{Max}(A) = \bigcup_{a \in \text{Max}(A)} \text{Max}(A_a) \subseteq \bigcup_{y \in A} \text{Max}(A_y) \subseteq \text{Max}(A)$ and equality follows. □

7.3.3 Proper Maximality and Weak Maximality

Definition 7.5. Let A be a nonempty set.

- A point $a \in A$ is said to be a *weak maximal point* of A if it is a maximal point of A with respect to the cone

  $$C^0 := \{0\} \cup \text{int}(C),$$

  under the assumption that $\text{int}(C) \neq \varnothing$. The set of all weak maximal points of A is denoted by $\text{WMax}(A)$ or $\text{WMax}(A|C)$.

- A point $a \in A$ is said to be a *proper maximal point* of A if there is a pointed and convex cone $\widetilde{C}$ such that

  $$cl(C)\backslash\{0\} \subseteq \text{int}(\widetilde{C})$$

  and a is a maximal point of A with respect to the order $\geq_{\widetilde{C}}$. The set of all proper maximal points of A is denoted $\text{PrMax}(A)$ or $\text{PrMax}(A|C)$.

When speaking about weak maximal elements, we tacitly consent that the cone C has a nonempty interior.

Example 7.3. We consider the set A given in Example 7.2 and equip $\mathbb{R}^2$ with the cone

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq y,\ x \geq 0,\ y \geq 0\}.$$

The set of proper maximal points is given by:

$$\operatorname{PrMax}(A|C) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = 1,\ \frac{1}{2} < y \leq 1,\ \frac{3}{2} < x \leq 2\}.$$

While the set of weak maximal points is the following:

$$\begin{aligned}\operatorname{WMax}(A|C) = \ & \{(x,y) \in \mathbb{R}^2 : y = 1,\ 0 \leq x \leq 2\} \\ & \cup \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = 1,\ \tfrac{1}{2} \leq y \leq 1,\ \tfrac{3}{2} \leq x \leq 2\}.\end{aligned}$$

Here is a relationship between proper maximality, maximality, and weak maximality.

Proposition 7.9. For a nonempty set A in $\mathbb{R}^k$, one has the inclusions

$$\operatorname{PrMax}(A) \subseteq \operatorname{Max}(A) \subseteq \operatorname{WMax}(A).$$

Proof. We prove first the inclusion $\operatorname{PrMax}(A) \subseteq \operatorname{Max}(A)$. Let $a \in \operatorname{PrMax}(A)$. By definition, there is a pointed convex cone $\widetilde{C}$ as stated in Definition 7.5. In particular, $C \subset \widetilde{C}$. In view of Proposition 7.7, $a \in \operatorname{Max}(A|\widetilde{C}) \subseteq \operatorname{Max}(A|C)$ and the inclusion follows. To prove the second inclusion, we have $C^0 \subseteq C$ when C has a nonempty interior. Again, by Proposition 7.7, $\operatorname{Max}(A|C) \subseteq \operatorname{Max}(A|C^0) = \operatorname{WMax}(A|C)$ as requested. □

Example 7.3 above shows that the inclusions stated in Proposition 7.9 are generally strict. One of the reasons for considering weak maximal elements is the fact that when the set A is closed, the set of weak maximal elements is closed too. In other words, the limit of any convergent sequence of weak maximal points of A is weak maximal if it belongs to A. This property is no longer true for maximal points and proper maximal points; see Example 7.4 below. The limit of a convergent sequence of maximal points is weak maximal, and not necessarily maximal. On the other hand, weak maximal points are sensitive with respect to modification of the order, while proper maximal points are not.

Example 7.4. Consider the set A (see Fig. 7.9) to be the convex hull of the quarter disc $D := \{(x,0,z) \in \mathbb{R}^3_+ : x^2 + z^2 \leq 1\}$ and the point $(0,2,1)$. A point a belongs to A if and only if there is some $(x,0,z) \in D$ and $t \in [0,1]$ such that

$$a = t(x,y,z) + (1-t)(0,2,1) = (tx, 2-2t, tz-t+1).$$

Consider the sequence of vectors $a^\nu = \left(\frac{1}{\nu}, 0, \frac{\sqrt{\nu^2-1}}{\nu}\right)$, $\nu \in \mathbb{N}$. Under the Pareto ordering cone, this is a sequence of maximal elements of A and converges to the point $(0,0,1)$, which is weak maximal, but not maximal. The set $\operatorname{Max}(A)$ is not closed.

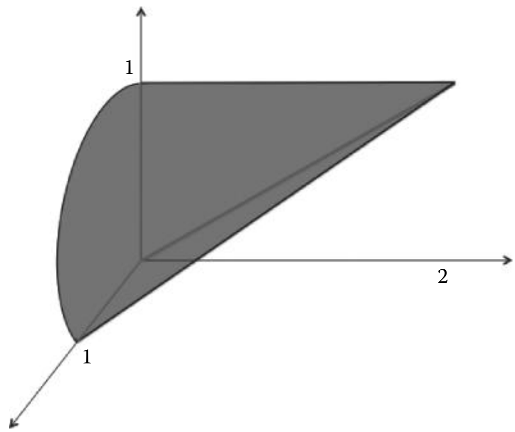

FIGURE 7.9: Max(A) is not closed in $\mathbb{R}^3$.

Example 7.5. Consider the set A (see Fig. 7.10) in $\mathbb{R}^2$ given by

$$0 \leq y \leq 1, \text{ for } 0 \leq x \leq 1,$$

and

$$(x-1)^2 + y^2 \leq 1, \text{ for } 1 < x \leq 2.$$

The space $\mathbb{R}^2$ is partially ordered by the Pareto cone $\mathbb{R}^2_+$. The set WMax(A) consists of the segment $[P, Q]$ and the quarter circumference $\overset{\frown}{QR}$. The set Max(A) consists of the quarter circumference $\overset{\frown}{QR}$ and the set PrMax(A) $=$ Max(A) $\setminus \{Q, R\}$. For every $\varepsilon > 0$, consider the $\varepsilon-$extended Pareto cone $\mathbb{R}^2_{+\varepsilon}$. It is clear that all the points of the segment $[P, Q]$, as well as the point Q, are no longer maximal with respect to the order $\mathbb{R}^2_{+\varepsilon}$. On the other hand, for every proper maximal point $a \in \overset{\frown}{QR} \setminus \{Q, R\}$, say $a = (1 + \cos\alpha, \sin\alpha)$ for some $\alpha \in (0, \frac{\pi}{2})$, we choose $\varepsilon_0 = \min\{\cos\alpha, \sin\alpha\}$. Then a remains maximal with respect to the extended order generated by $\mathbb{R}^2_{+\varepsilon}, \varepsilon \in (0, \varepsilon_0)$.

Note that a set may have weak maximal points without having a maximal point, or it has maximal points without having proper maximal points.

Example 7.6. Consider the set A in $\mathbb{R}^2$ consisting of vectors (x, y) with $y > 0$ and $x \leq 1/y$ (see Fig. 7.11). Under the Pareto ordering cone, the set Max(A) is the graph of the function $\frac{1}{x}, x > 0$. We prove that A has no proper maximal points. Indeed, for every $\varepsilon > 0$ and every $a = (\alpha, \frac{1}{\alpha}) \in$ Max(A) with $\alpha > 0$, we choose

$$\begin{aligned} x_1 &= \frac{\varepsilon}{\alpha(1+\varepsilon)} \\ x_2 &= \frac{\alpha(1+\varepsilon)}{\varepsilon}. \end{aligned}$$

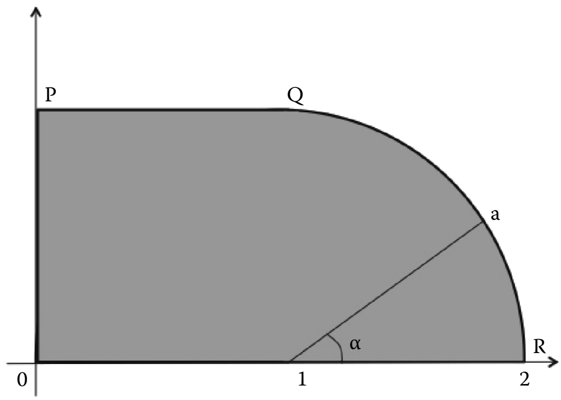

FIGURE 7.10: Maximal, weak maximal, and proper maximal points in $\mathbb{R}^2$.

Then, $(x_1, x_2) \in A$ and $(x_1, x_2) >_{\mathbb{R}^2_{+\varepsilon}} a$, which shows that a is not a proper maximal point of A.

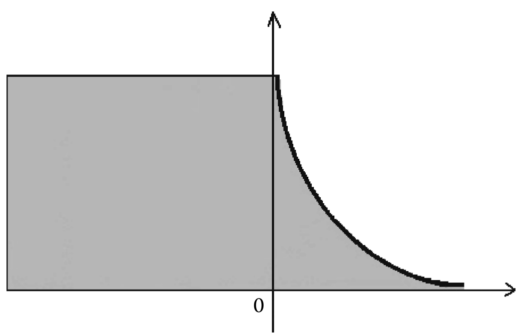

FIGURE 7.11: A set without proper maximal points in $\mathbb{R}^2$.

7.3.4 Maximal Points of Free Disposal Hulls

We say that a set $A \subseteq \mathbb{R}^k$ is *free disposal* if for every $x \in A$, inequality $x' \leq_C x$ implies $x' \in A$. In other words A is free disposal if $A = A - C$. If A is not free disposal, the set $A - C$ is called the free disposal hull of A. It is the smallest free disposal set that contains A.

Proposition 7.10. Let A be a nonempty set. The following assertions hold:

(a) $\text{IMax}(A) = \text{IMax}(A - C)$;

(b) $\text{Max}(A) = \text{Max}(A - C)$;

(c) $\mathrm{PrMax}(A) = \mathrm{Pr}\,\mathrm{Max}(A - C)$;

(d) $\mathrm{WMax}(A) = A \cap \mathrm{WMax}(A - C)$.

Proof. We prove (a). Let $x \in \mathrm{IMax}(A)$. By Proposition 7.6, $A \subseteq x - C$. Since $C + C \subseteq C$, we deduce that $A - C \subseteq x - C - C \subseteq x - C$, by which $x \in \mathrm{IMax}(A - C)$. For the converse, let $x \in \mathrm{IMax}(A - C)$. From the cone C containing the origin we obtain $A \subseteq A - C \subseteq x - C$, which proves that $x \in \mathrm{IMax}(A)$.

For (b), let $x \in \mathrm{Max}(A)$. Again, by Proposition 7.6, $x \in A$ and $A \cap (x + C \setminus \{0\}) = \emptyset$. It follows that $x \in A - C$ and $(A - C) \cap (x + C \setminus \{0\}) = \emptyset$ too. Thus, $x \in \mathrm{Max}(A - C)$. For the converse, let $x \in \mathrm{Max}(A - C)$. We show first that $x \in A$. Indeed, if not, $x = a - c$ for some $a \in A$ and $c \in C \setminus \{0\}$. Since $a \in A - C$ as well, we obtain $a = x + c >_C x$, which is a contradiction. Moreover, as $A \subseteq A - C$, the relation $(A - C) \cap (x + C \setminus \{0\}) = \emptyset$ (because $x \in \mathrm{Max}(A - C)$), yields also $A \cap (x + C \setminus \{0\}) = \emptyset$. By this, $x \in \mathrm{Max}(A)$.

For (c), let $x \in \mathrm{PrMax}(A)$. Let $\widetilde{C}$ be a pointed, convex cone such that *cl* $C \setminus \{0\} \subseteq \mathrm{int}\,\widetilde{C}$ and $x \in \mathrm{Max}(A|\widetilde{C})$. Then, $x \in A \subseteq A - C$ and

$$A \cap (x + \widetilde{C} \setminus \{0\}) = \emptyset. \tag{7.2}$$

We wish to prove that $(A - C) \cap (x + \widetilde{C} \setminus \{0\} = \emptyset$, which implies that $x \in \mathrm{PrMax}(A - C)$. Indeed, if not, say there is some $y \in A - C$ such that $y = x + \widetilde{c}$, for some $\widetilde{c} \in \widetilde{C} \setminus \{0\}$. Let $a \in A$ and $c \in C$ be such that $y = a - c$. Then $a = y + c = x + (c + \widetilde{c})$. The cone $\widetilde{C}$ being pointed, we have $c + \widetilde{c} \in \widetilde{C} \setminus \{0\}$ and arrive at a contradiction with (7.2).

Conversely, let $x \in \mathrm{PrMax}(A - C)$. Let $\widetilde{C}$ be a cone as above such that $x \in \mathrm{Max}(A - C|\widetilde{C})$. We wish to show that $x \in \mathrm{Max}(A|\widetilde{C})$, which proves that $x \in \mathrm{PrMax}(A)$. Indeed, we have $x \in A - C$ and

$$(A - C) \cap (x + \widetilde{C} \setminus \{0\}) = \emptyset \tag{7.3}$$

We can see that $x \in A$ by the same argument we discussed above and the fact that $C \subseteq \widetilde{C}$. Moreover, as already said, $A \subseteq A - C$, so that (7.3) implies (7.2). Consequently $x \in \mathrm{Max}(A|\widetilde{C})$ by Proposition 7.6.

Finally, let $x \in \mathrm{WMax}(A)$. Then, $x \in A$ and also $A \cap (x + \mathrm{int}(C)) = \emptyset$, which implies $(A - C) \cap (x + \mathrm{int}(C)) = \emptyset$, because $C + \mathrm{int}(C) \subseteq \mathrm{int} C$. By this, $x \in A \cap \mathrm{WMax}(A - C)$. Conversely, let $x \in A \cap \mathrm{WMax}(A - C)$. We have $(A - C) \cap (x + \mathrm{int}\ (C)) = \emptyset$, which also implies $A \cap (x + \mathrm{int}\ (C)) = \emptyset$. Hence, $x \in \mathrm{WMax}(A)$. □

7.4 Existence

In this section, we wish to establish some criteria for the existence of maximal elements of a set in the space $\mathbb{R}^k$, which is partially ordered by a pointed convex cone C.

7.4.1 The Main Theorems

We establish equivalent conditions for a point of a set to be maximal by coverings of the set.

Theorem 7.1. Assume that C is closed. The following assertions are equivalent:

(a) A point $a \in A$ is maximal.

(b) The set $A \setminus \{a\}$ is covered by the family $\{(x^\nu + C)^c : \nu \in \mathbb{N}\}$ for every increasing sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ in A converging to a.

(c) The set $A \setminus \{a\}$ is covered by the family $\{(x^\nu + C)^c : \nu \in \mathbb{N}\}$ for some increasing sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ in A converging to a.

Proof. We first establish the implication (a) $\Rightarrow$ (b). Suppose to the contrary that (b) does not hold. There exist an increasing sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ converging to a and an element $b \in A \setminus \{a\}$ such that $b \notin (x^\nu + C)^c$, or equivalently, $b \in x^\nu + C$ for all $\nu \in \mathbb{N}$. By passing to the limit, and due to the closedness of C, we derive $b \in a + C$. This contradicts (a).

The implication (b) $\Rightarrow$ (c) is clear. We finally prove the implication (c) $\Rightarrow$ (a). Suppose to the contrary that a is not maximal. Then one can find some element $b \in A \setminus \{a\}$ such that $b \in a + C$. In view of (c), there is some index $\nu_0 \in \mathbb{N}$ such that $b \in (x^{\nu_0} + C)^c$. This implies that $a \in (x^{\nu_0} + C)^c$, and hence

$$a \notin x^{\nu_0} + C. \tag{7.4}$$

On the other hand, because the sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ is increasing, we have $x^\nu \in x^{\nu_0} + C$ for all $\nu \geq \nu_0$. By passing to the limit in the above inclusion, we obtain $a \in x^{\nu_0} + C$, which contradicts (7.4). The proof is complete. □

Theorem 7.2. Assume that there is a convex cone K not identical to the whole space such that $C \setminus \{0\}$ is contained in $\mathrm{int}(K)$ and A is a nonempty compact set. Then, A has maximal points.

Proof. Let $x \in A$. Consider the section of A at x with respect to $K^0 = \{0\} \cup \mathrm{int}(K)$: $A_x = A \cap (x + K^0)$. This follows from the assumption that the cone K^0 is pointed. We wish to prove that section A_x possesses a maximal point with respect to the cone K^0. Indeed, suppose to the contrary that the set $\mathrm{Max}(A_x|K^0)$ is empty. Consider the set P of chains in A_x with respect to the order $\geq_{K^0}$. One checks easily that the set P equipped with the partial order by inclusion satisfies the hypothesis of Zorn's lemma (every chain in P with respect to inclusion has an upper bound), by which there is a maximal chain, say $p_* \in P$.

We choose a sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ of elements of p_* such that $p_* \subseteq cl\{x^\nu : \nu \in \mathbb{N}\}$ and construct an increasing subsequence $\{y^\nu\}_{\nu\in\mathbb{N}}$ as follows. Set $y^1 = x^1$. For $m > 1$, set $y^m = x^{\nu_m}$, with $\nu_m = \min\{\nu \in \mathbb{N} : x^\nu >_{K^0} y^{m-1}\}$. Such ν_m exists because the chain p_* is maximal. It follows from this construction that for every $\nu \in \mathbb{N}$, one has $x^\nu \leq_{K^0} y^\nu$. Moreover, for every $y \in p_*$ there is $\nu \in \mathbb{N}$ such that $y^\nu \geq_{K^0} y$. In fact, if not, we have $y \geq_{K^0} y^\nu$ and hence $y \geq_{K^0} x^\nu$, for all $\nu \in \mathbb{N}$. Since $\mathrm{Max}(A_x|K^0)$ is empty and p_* is maximal, one finds $z \in p_*$ such that $z >_{K^0} y \geq_{K^0} x^\nu$, for all $\nu \in \mathbb{N}$. As $K^0 \setminus \{0\}$ is open, the above strict inequality proves that $z \notin cl\{x^\nu : \nu \in \mathbb{N}\}$, which is a contradiction. Now we work with the sequence $\{y^\nu\}_{\nu\in\mathbb{N}}$. By the compactness hypothesis, this sequence has an accumulation point, say $y \in A$. We prove that $y >_{K^0} x^\nu$ for all $\nu \in \mathbb{N}$. In fact, for arbitrary x^ν we choose $j(\nu) > \nu$ such that the term $y^{j(\nu)}$ belongs to a subsequence converging to y and obtain $y^{j(\nu)} >_{K^0} y^\nu \geq_{K^0} x^\nu$. Because the sequence $\{y^\nu\}_{\nu\in\mathbb{N}}$ is increasing, we deduce that $y \in y^{j(\nu)} + cl(K)$. Then,

$$y \in y^{j(\nu+1)} - y^{j(\nu)} + y^{j(\nu)} + cl(K) \subset y^{j(\nu)} + cl(K) + \mathrm{int}(K) \subseteq y^{j(\nu)} + \mathrm{int}(K),$$

which shows that $y >_{K^0} y^{j(\nu)} \geq_{K^0} x^\nu$ for every $\nu \in \mathbb{N}$. We may find then $z \in A_x$ such that $z >_{K^0} y$ for every $y \in p_*$ because $\mathrm{Max}(A_x|K^0)$ is empty. This contradicts the maximality of the chain p_*. Thus, the set $\mathrm{Max}(A_x|K^0)$ is nonempty. It remains to apply Proposition 7.6 to see that $\mathrm{Max}(A|K^0)$ is nonempty, and in view of Proposition 7.7, A has maximal points with respect to the ordering cone C. □

When the closure of the cone C is pointed, one may choose the cone K by

$$K = \{0\} \cup \{x \in \mathbb{R}^k : d(x, C) < \epsilon\|x\|\}$$

with $\epsilon > 0$ sufficiently small. Then the closure of K is the ϵ-conic neighborhood of C that we defined in Section 7.2.

7.4.2 Generalization to Order-Complete Sets

Definition 7.6. A set $A \subseteq \mathbb{R}^k$ is said to be $C-$*complete* (respectively, *strongly* $C-$*complete*) if it has no covering of the form

$$\{(x^\nu + \mathrm{cl}\, C)^c : \nu \in \mathbb{N}\} \quad \big(\text{respectively } \{(x^\nu + C)^c : \nu \in \mathbb{N}\}\big),$$

where $\{x^\nu\}_{\nu\in\mathbb{N}}$ is a strictly increasing sequence in A.

We note that every strongly $C-$complete set is $C-$complete. The converse is not always true. When C is a closed cone these two concepts coincide.

Example 7.7. Consider the set A in $\mathbb{R}^2$ consisting of the vectors $(\nu, 0)$, $\nu \in \mathbb{N}$ and equip $\mathbb{R}^2$ with the ubiquitous cone Ub. We show that A is Ub-complete, but not strongly Ub-complete. Indeed, let $\{(\nu_i, 0)\}_{i\in\mathbb{N}}$ be an increasing sequence in A. Then the real sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is increasing in $\mathbb{R}$. For every term $(\nu_i, 0)$, we have

$$((\nu_i, 0) + cl(Ub))^C = \{(x, y) \in \mathbb{R}^2 : y < 0\},$$

which does not meet A. Therefore, the family $\{((\nu_i, 0) + cl(Ub))^C : i \in \mathbb{N}\}$ is not a covering of A, and A is Ub-complete. On the other hand, for every $(\nu, 0) \in A$, we have:

$$((\nu + 1, 0) + Ub)^C = \{(x, y) \in \mathbb{R}^2 : y < 0\} \cup \{(x, 0) \in \mathbb{R}^2 : x < \nu + 1\},$$

which contains $(\nu, 0)$. Consequently, the family $\{((\nu, 0) + Ub)^C : \nu \in \mathbb{N}\}$ is a covering of A, and A is not strongly Ub-complete.

The method of proof of Theorem 7.2 can be developed to establish the following general result, which is applicable to the case when the space is of infinite dimension too.

Theorem 7.3. If the cone C is correct and the set A admits a nonempty C-complete section, then the set A has maximal points.

Corollary 7.1. Assume that the cone C is correct and the set $A - C$ has a nonempty compact section. Then, A has maximal points.

Proof. Let $B = (A - C) \cap (x + C)$ be a compact section of $A - C$. We show that it is strongly C-complete, and hence C-complete as well. Suppose to the contrary that B is not strongly C-complete. There exists a strictly increasing sequence $\{x^\nu\}_{\nu\in\mathbb{N}}$ in B such that the family $\{(x^\nu + C)^C : \nu \in \mathbb{N}\}$ is a covering of B. We may assume that sequence converges to some limit $y \in B$. Then, there is ν_0 such that

$$y \notin x^{\nu_0} + C. \tag{7.5}$$

On the other hand, for every $m \geq 1$, one has $x^{\nu+m} \in x^\nu + C$ because the considered sequence is increasing. By letting m tend to infinity, we deduce $y \in x^{\nu_0} + cl\,(C)$, and so

$$\begin{aligned} y \in x^{\nu_0+1} + cl\,(C) \quad &\subseteq x^{\nu_0+1} - x^{\nu_0} + x^{\nu_0} + cl\,(C) \\ &\subseteq x^{\nu_0} + C \setminus \{0\} + cl\,(C) \subseteq x^{\nu_0} + C, \end{aligned}$$

because C is correct. This contradicts (7.5). Thus, B is strongly C-complete. According to Theorem 7.3, B has maximal points. It remains to apply Propositions 7.8 and 7.10 to complete the proof. □

7.4.3 Existence via Monotone Functions

Theorem 7.4. Assume that the strictly positive polar cone C^+ of C is nonempty. A nonempty set A has a maximal point if and only if it has a compact section.

Proof. If A has a maximal point, then the section at that point is a singleton, and hence a compact set. Assume that $A_x = A \cap (x + C)$ is a compact section of A. Consider the linear increasing function $\langle \xi, \cdot \rangle$ on A_x where ξ is an element of C^+. Since A_x is compact, there is a point $a \in A_x$ maximizing $\langle \xi, \cdot \rangle$ on A_x. We prove that $a \in \mathrm{Max}(A_x|C)$. Indeed, let $y \in A_x$ with $y >_C a$. Then, $\langle \xi, y \rangle > \langle \xi, a \rangle$, which contradicts the maximality of a. It remains to apply Proposition 7.8 to conclude that a is a maximal point of A. □

Notice that C^+ is nonempty if and only if $cl(C)$ is pointed. We derive a useful corollary.

Corollary 7.2. If the closure of the cone C is pointed, then every compact set has maximal points.

Proof. Apply Theorem 7.4. □

We remark that a more subtle proof shows that the conclusion of Corollary 7.2 remains true without any condition on the cone C. This fact is, however, not true in infinite dimension.

Before giving conditions for the existence of maximal points of polyhedral sets and convex sets, we recall the following separation result from convex analysis.

Lemma 7.1. Let C_1 and C_2 be two convex cones with $C_1 \cap C_2 = \{0\}$. Then, there is a nonzero vector ξ such that

$$\langle \xi, c_1 \rangle \geq \langle \xi, c_2 \rangle \text{, for all } c_1 \in C_1 \text{ and } c_2 \in C_2.$$

Moreover, if either C_1 is pointed, closed, and C_2 is closed or $c_1 \setminus \{0\}$ is open, then there is a nonzero vector ξ such that

$$\langle \xi, c_1 \rangle > 0 \geq \langle \xi, c_2 \rangle \text{, for all } c_1 \in C_1 \setminus \{0\} \text{ and } c_2 \in C_2.$$

Proposition 7.11. If A is a polyhedral convex set and C is a polyhedral cone, then $x \in \mathrm{Max}(A|C)$ if and only if there is some $\xi \in C^+$ such that

$$\langle \xi, x \rangle \geq \langle \xi, x' \rangle \text{, for all } x' \in A.$$

Consequently, the set $\mathrm{Max}(A|C)$ consists of faces of A.

Proof. Let x be a maximum of $\langle \xi, \cdot \rangle$ on A. If $y >_C x$, $y \in A$, then $\langle \xi, x \rangle \geq \langle \xi, y \rangle > \langle \xi, x \rangle$, which is a contradiction. Hence, $x \in \mathrm{Max}(A|C)$. Conversely, if $x \in \mathrm{Max}(A|C)$, then $(A - x) \cap C = \{0\}$. Since $A - x$ is a polyhedral set, the set $\mathrm{cone}(A-x)$ is a polyhedral cone and $\mathrm{cone}(A-x)\cap C=\{0\}$. Because the cone C is pointed, convex, and closed and $\mathrm{cone}(A-x)$ is convex and closed, we may apply the separation lemma to find a nonzero vector ξ such that

$$\langle \xi, c\rangle > 0 \geq \langle \xi, x'-x\rangle ,$$

for all $x\in C\setminus\{0\}$ and $x'\in A$. It follows that $\xi\in C^{+}$ and $\langle \xi, x\rangle \geq \langle \xi, x'\rangle$, for every $x'\in A$.

For the last part of the proposition, we know that the set of the points maximizing the linear function $\langle \xi,\cdot\rangle$ on A is a face of A. Hence, $\mathrm{Max}(A|C)$ is composed of some faces of A. □

Proposition 7.12. Let $\xi\in C'\backslash\{0\}$ and let $x\in A$ maximize the linear function $\langle \xi,\cdot\rangle$ on A. Then,

(a) $x\in \mathrm{WMax}(A|C)$;

(b) $x\in \mathrm{Max}(A|C)$ if it is a unique maximum of $\langle \xi,\cdot\rangle$ on A;

(c) $x\in \mathrm{PrMax}(A|C)$ if $\xi\in (cl(C))^{+}$.

Proof. For (a), if $x\notin \mathrm{WMax}(A|C)$, one can find $x'\in A$ and $c\in \mathrm{int}(C)$ such that $x'=x+c$. It follows that $\langle \xi, x'\rangle = \langle \xi, x\rangle + \langle \xi, c\rangle > \langle \xi, x\rangle$, which is a contradiction.

For (b), let $x'\in A$ be such that $x'\geq_C x$. Then, $\langle \xi, x\rangle \geq \langle \xi, x'\rangle \geq \langle \xi, x\rangle$, which yields the equality $\langle \xi, x'\rangle = \langle \xi, x\rangle$. By uniqueness, one has $x=x'$ and hence $x\in \mathrm{Max}(A|C)$. For the last assertion, we consider the cone:

$$\widetilde{C}=\{0\}\cup\{y\in\mathbb{R}^k_+ : \langle \xi, y\rangle > 0\}.$$

Then, $cl(C)\setminus\{0\}\subseteq \mathrm{int}\widetilde{C}$ and $x\in \mathrm{WMax}(A|\widetilde{C})$. Since $\widetilde{C}\backslash\{0\}\subseteq \mathrm{int}(\widetilde{C})$, one sees also that $x\in \mathrm{Max}(A|\widetilde{C})$. By definition, x is a proper maximal element of A (with respect to the order $\geq_C$). □

Proposition 7.13. Assume that $A-C$ is a convex set. The following assertions hold:

(a) $x\in \mathrm{WMax}(A|C)$ if and only if there is some $\xi\in C'\backslash\{0\}$ such that x is a maximum of $\langle \xi,\cdot\rangle$ on A.

(b) $x\in \mathrm{PrMax}(A|C)$ if and only if there is some $\xi\in (cl(C))^{+}$ such that x is a maximum of $\langle \xi,\cdot\rangle$ on A.

(c) If $x\in \mathrm{Max}(A|C)$, then there is some $\xi\in C'\backslash\{0\}$ such that x is a maximum of $\langle \xi,\cdot\rangle$ on A.

Proof. For (a), let x be a weak maximal point of A, then $(A-x)\cap \text{int}\,(C)=\emptyset$, which also implies that $cone(A-x)\cap \text{int}C=\emptyset$. By separation, there is some nonzero vector ξ such that

$$\langle \xi, x'-x\rangle < \langle \xi, c\rangle\,,\ \text{for all } x'\in A \text{ and } c\in \text{int}(C).$$

This implies $\xi \in C'\backslash\{0\}$ and $\langle \xi, x'\rangle \le \langle \xi, x\rangle$, for all $x'\in A$. The converse follows from Proposition 7.12. For (ii), in view of Proposition 7.12, it suffices to prove the "only if" part. Let $x\in \text{PrMax}(A|C)$ and let $\widetilde{C}$ be a convex cone as given in Definition 7.5 such that $x\in \text{Max}(A|\widetilde{C})\subseteq \text{WMax}(A|\widetilde{C}^0)$. In view of (i), there is a nonzero vector $\xi\in(\widetilde{C}^0)'$ such that x maximizes $\langle \xi,\cdot\rangle$ on A. It remains to observe that $\langle \xi, c\rangle>0$, for every $c\in cl(C)\backslash\{0\}$, which means that $\xi\in(cl(C))^+$ because $cl(C)\backslash\{0\}\subseteq \text{int}(\widetilde{C})$ by Definition 7.5. We prove the last assertion. For $x\in\text{Max}(A|C)$, we have $(A-x)\cap C=\{0\}$, which also implies that $\text{cone}(A-C-x)\cap C=\{0\}$. Because the two cones in this intersection is convex by hypothesis, we may separate them by a nonzero vector ξ :

$$\langle \xi, c\rangle \ge \langle \xi, a-x\rangle\,,\ \text{for every } c\in C \text{ and } a\in A.$$

It follows that $\xi\in C'\backslash\{0\}$ and $\langle\xi,\cdot\rangle$ attains its maximum on A at x. □

It is clear that the converse of (c) above is not true. When $\text{int}(C)\neq\emptyset$, it suffices to choose a weak maximal point that is not maximal to obtain a counterexample.

7.5 Vector Optimization Problems

Let X be a nonempty set in $\mathbb{R}^n$ and let f be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^k$. The space $\mathbb{R}^k$ is assumed to be partially ordered by a convex and pointed cone C. In this section, we study the following problem, which is called a vector optimization problem or a multiple objective optimization problem, and is denoted (MOP):

$$\begin{array}{ll}\text{Max} & f(x)\\ \text{subject to} & x\in X,\end{array}$$

which amounts to finding $x\in X$ such that $f(x)$ is a maximal point of the image set $f(X)$. Such solutions are called *maximal* or *efficient solutions* of (MOP). The set of all maximal solutions is denoted $S(X,f)$. The sets of all proper maximal solutions and weak maximal solutions of (MOP) are defined in a similar way and denoted, respectively, by $\text{PrS}(X,f)$ and $\text{WS}(X,f)$. The existence criteria of maximal points we established in the previous section are applied to derive the existence of maximal solutions of (MOP). Below we give a typical one.

Theorem 7.5. Assume that cl(C) is pointed, X is a nonempty compact set, and f is a continuous function. Then, the problem (MOP) has maximal solutions.

Proof. Because the image set $f(X)$ is nonempty compact, we apply Corollary 7.2 to obtain maximal elements of the set $f(X)$, which assures the existence of maximal solutions of (MOP). □

7.5.1 Scalarization

A frequently used method in the study of (MOP) is to convert it into a scalar optimization problem of the form (P)

$$\begin{array}{ll} \max & g \circ f(x) \\ \text{subject to} & x \in X, \end{array}$$

where g is a real-valued function on $f(X)$, called a scalarizing function. In this section, we shall develop a class of scalarizing functions with a property that optimal solutions of (P) furnish efficient solutions of (MOP).

General case. The following result expresses a relationship between the solution set $S(X, g \circ f)$ of problem (P) and that of (MOP).

Proposition 7.14. The following assertions are true:

(a) If g is increasing (respectively, weakly increasing), then every optimal solution of (P) is an efficient solution (respectively, a weakly efficient solution) of (MOP).

(b) Conversely, for every weakly efficient solution x of (MOP), there exists a continuous weakly increasing function g such that x is an optimal solution of (P).

Proof. For (a), let x be an optimal solution of (P). If it is not a maximal solution of (MOP), then there is some $x' \in X$ such that $f(x') >_C f(x)$. Because the function g is increasing, we deduce $g \circ f(x') > g \circ f(x)$, which contradicts the assumption. The case of weak maximal solutions is proven similarly.

For (b) we choose $a = f(x)$ and $v \in \text{int}(C)$, and consider $g(x) = h_{a,v}(x)$ for every $x \in X$. By Proposition 7.5, g is continuous, weakly increasing. Moreover, as $f(x) \cap (a + \text{int } C) = \emptyset$, we deduce that $g(f(x)) \leq 0$ for every $x \in X$, while $g(f(a)) = 0$. Consequently, a is an optimal solution of (P). □

Below are some useful scalarizations.

Linear scalarization. We consider the case of the Pareto ordering cone

$\mathbb{R}^n_+$. The scalarized problem that uses a linear increasing function is written as

$$(P_\xi) \qquad \begin{array}{ll} \max & \sum_{i=1}^k \xi_i f_i(x) \\ \text{subject to} & x \in X \end{array}$$

for some positive vector $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k_+$. The positive numbers $\xi_1, \ldots, \xi_k$ are called weights. Each number ξ_i expresses the importance of the criterion (component) f_i with respect to the others. For instance, by choosing the first component of ξ equal to one and the other components equal to zero, we mean to take only the first criterion f_1 into consideration and neglect the others. The scalar problem (P_ξ) is called the weighted problem of (MOP) associated with the weight vector ξ. Certain generalizations of (P_ξ) are also in use. They are of the form

$$(P'_\xi) \qquad \begin{array}{ll} \max & \sum_{i=1}^k \xi_i [f_i(x)]^\rho \\ \text{subject to} & x \in X \end{array}$$

and

$$(P''_\xi), \qquad \begin{array}{ll} \max & \sum_{i=1}^k [\xi_i f_i(x)]^\rho \\ \text{subject to} & x \in X \end{array}$$

where ρ is a positive number chosen a priori and it is assumed that $f_i(x) \geq 0$ for all $i = 1, \ldots, k$ and $x \in X$.

In the case of a general ordering cone C, the problem (P_ξ) is

$$\begin{array}{ll} \max & \langle \xi, f(x) \rangle \\ \text{subject to} & x \in X. \end{array} \tag{7.6}$$

The objective function $\langle \xi, f(x) \rangle$ is often written as $\xi \circ f(x)$. Linear scalarization is particularly helpful when the vector problem is linear or concave. The problem (MOP) is called linear if the objective function f is linear, the constraint set X is a polyhedral convex set, and the ordering cone C is polyhedral. It is concave when f is concave and the set X is convex. Also, f is said to be concave (with respect to the order $\geq_C$) if for every $x, y \in X$, $t \in [0, 1]$ one has

$$f(tx + (1-t)y) \geq_C tf(x) + (1-t)f(y).$$

Proposition 7.15. Assume that (MOP) is a linear problem. Then, there exists a finite number of weight vectors $\xi^i : i = 1, \ldots, m$ such that the solution set of (MOP) is exactly the union of the solution sets of the scalarized problems $(P_{\xi^i}), i = 1, \ldots, m$.

Proof. If (MOP) is linear, the image set $f(X)$ is a polyhedral convex set. In view of Proposition 7.11, the maximal value set $\operatorname{Max}(f(X)|C)$ of the problem is the union of some faces of $f(X)$. For every efficient face F_i there is some $\xi^i \in C^+$ such that

$$F_i = \arg\max\{\langle \xi^i, y \rangle : y \in f(X)\}.$$

As the number of faces of $f(X)$ is finite, so is the number of efficient faces of $f(X)$. Thus, there exist $\xi^1, \ldots, \xi^m \in C^+$ such that

$$\text{Max}(f(X)|C) = \bigcup_{i=1}^{m} \arg\max\{\langle \xi^i, y\rangle : y \in f(X)\},$$

which implies

$$S(X, f) = \bigcup_{i=1}^{m} \arg\max\{\langle \xi^i, f(x)\rangle : x \in X\}.$$

This set is the union of the solution sets to the problems (P_{ξ^i}), $i = 1, \ldots, m$. □

For a concave problem we have the following result.

Proposition 7.16. Assume that (MOP) is a concave problem. Then,

(a) a point $x \in X$ is a properly efficient solution of (MOP) if and only if it is an optimal solution of (P_ξ) with $\xi \in (\text{cl}(C))^+$;

(b) a point $x \in X$ is a weakly efficient solution of (MOP) if only if it is an optimal solution of (P_ξ) with $\xi \in C^{'} \setminus \{0\}$.

Proof. It is clear that under the convexity hypothesis the set $f(X) - C$ is convex. In view of Proposition 7.10, x is a proper maximal solution if and only if $f(x)$ is a proper maximal element of the set $f(X) - C$. The first assertion is a direct consequence of Propositions 7.12 and 7.14. The second part is proven by a similar argument. □

It should be noted that an efficient solution of a concave problem, which is not proper, cannot be a solution of any scalar problem (P) with $g \in C^+$. Thus, we have the following inclusions for a concave problem

$$\begin{aligned} PrS(X, f) &= \bigcup_{g \in C^+} S(g \circ f, A) \subseteq S(X, f) \\ &\subseteq WS(f, X) = \bigcup_{g \in C^{'} \setminus \{0\}} S(g \circ f, X). \end{aligned}$$

Another important feature of linear scalarization is that for $\xi \in C^{'}$, problem (P_ξ) is concave whenever (MOP) is concave. This allows us to apply convex optimization techniques to solve vector problems.

Scalarization by the biggest weakly increasing functions. With the help of the function $h_{a,v}$ defined in Section 7.2 we obtain the following scalarized problem

$$(P_{h_{a,v}}) \qquad \begin{array}{ll} \max & \sup\{t \in \mathbb{R} : f(x) \in a + tv + C\} \\ \text{subject to} & x \in X \end{array}$$

The usefulness of this scalarization is seen in the next result, which concretizes the second assertion of Proposition 7.14.

Proposition 7.17. Let $v \in \text{int}\, C$ be given. Then, $x \in WS(X, f)$ if and only if x is an optimal solution of the problem $(P_{h_{f(x),v}})$.

Proof. By definition, $x \in WS(X, f)$ if and only if

$$(f(X) - f(x)) \cap \text{int}\ (C) = \emptyset.$$

We have $h_{f(x),v}(f(x)) = 0$, and $h_{f(x),v}(y) \leq 0$ if and only if $y \notin f(x) + \text{int}(C)$. Hence,

$$h_{f(x),v}(f(x')) \leq 0, \text{ for all } x' \in X.$$

By this, $x \in WS(X, f)$ if and only if x maximizes the function $h_{f(x),v} \circ f$ on A. □

7.6 Optimality Conditions

In this section we establish conditions for efficient solutions of (MOP) in terms of derivatives or subdifferentials. The ordering cone C in $\mathbb{R}^k$ is assumed to be the closed, pointed, convex, and having a nonempty interior.

7.6.1 Differentiable Problems

Let us consider the problem (MOP) with the constraints

$$\begin{array}{rl} \text{Max} & f(x) \\ \text{subject to} & g(x) \leq_K 0 \\ & h(x) = 0. \end{array}$$

where the f, g, and h vector, are functions on $\mathbb{R}^n$ with values in $\mathbb{R}^k$, $\mathbb{R}^m$, and $\mathbb{R}^q$, respectively. The space $\mathbb{R}^m$ is equipped with a partial order generated by a closed, pointed, and convex cone K, with $\text{int}(K) \neq \emptyset$. This is often the case when $K = \mathbb{R}^m_+$. Then $g(x) \leq_K 0$ is exactly the system of m inequalities $g_i(x) \leq 0,\ i = 1, \ldots, m$ where $g_1, \ldots, g_m$ are component functions of g.

Theorem 7.6. Assume that f, g, and h are Fréchet differentiable with f' and g' bounded and h' continuous in a neighborhood of x^0. If x^0 is a weakly efficient solution of (MOP), then there exist multipliers $(\xi, \theta, \gamma) \in (C, K, \{0\})' \setminus \{0\}$ such that

$$\xi f'(x^0) + \theta g'(x^0) + \gamma h'(x^0) = 0, \qquad \theta g(x^0) = 0.$$

Proof. Assume first that $h'(x^0)$ is not surjective, i.e., $h'(x^0)(\mathbb{R}^n)$ is a proper subspace of $\mathbb{R}^q$. Then there exists a nonzero functional $\gamma \in \mathbb{R}^q \setminus \{0\}$ such that

$$\langle \gamma, h'(x^0)(u) \rangle \ = 0 \text{ for all } u \in \mathbb{R}^n.$$

This implies that $\gamma h'(x^0) = 0$. Setting $\xi = 0$ and $\theta = 0$ we obtain multipliers (ξ, θ, γ) as requested.

Now consider the case where $h'(x^0)$ is surjective. We want to show that

$$(f'(x^0), g'(x^0), h'(x^0))(\mathbb{R}^n) \cap (-\text{int } (C), -g(x^0) - \text{int } (K), \{0\}) = \emptyset. \quad (7.7)$$

In fact, if this intersection is not empty, then there is a vector $u \in \mathbb{R}^n$ with $||u|| = 1$ such that

$$\begin{array}{rcl} f'(x^0)(u) & \in & -\text{int } (C) \\ g'(x^0)(u) & \in & -g(x^0) - \text{int } (K) \\ h'(x^0)(u) & = & 0. \end{array}$$

Applying Lyusternik's open mapping theorem ("If h is Frechet differentiable with h' continuous at x^0 and if $h'(x^0)$ is surjective, then the tangent cone to the set $M := \{x \in \mathbb{R}^n : h(x) = 0\}$ at $x^0 \in M$ defined by

$$T_M(x^0) := \{v \in K : v = \lim_{x \to \infty} t_i(x^i - x^0), t_i > 0, x^i \to x^0, x^i \in M\},$$

coincides with Ker$h'(x^0)$."), we find $x^i \in M \setminus \{x^0\}$ such that $\{x^i\}$ converges to x^0 and $\{u^i\}$ with $u^i = (x^i - x^0)/||x^i - x^0||$, converges to u. Note that as f' is bounded in a neighborhood of x^0, in view of the mean value theorem (if f is Gateaux differentiable, then for every couple of points a and b, one has $\|f(b) - f(a)\| \leq \sup\{\|f'(c)\| \times \|b - a\| : c \in [a, b]\}$) we have the following estimate:

$$\lim \frac{f(x^i) - f(x^0)}{||x^i - x^0||} = f'(x^0)(u).$$

Hence, for i sufficiently large we obtain

$$f(x^i) - f(x^0) \in \text{int } (C). \quad (7.8)$$

Similarly, for i sufficiently large we have

$$\frac{g(x^i) - g(x^0)}{||x^i - x^0||} = -g(x^0)(u) - \text{int}(K).$$

Since $||x^i - x^0||$ tends to 0 as i tends to ∞, the above implies

$$g(x^i) \in (1 - ||x^i - x^0||)g(x^0) - \text{int } (K) \subseteq -K,$$

for i sufficiently large. This, and the fact that $h(x^i) = 0$ (because $x^i \in M$) together with (7.8), show that x^0 is not a weakly efficient solution of (MOP), which is a contradiction.

Consequently (7.7) is true. We separate the sets in (7.7) by a linear functional $(\xi, \theta, \gamma) \in (\mathbb{R}^k, \mathbb{R}^m, \mathbb{R}^q)' \setminus \{0\}$:

$$\xi f'(x^0)(u) + \theta[g'(x^0)(v) + g(x^0)] + \gamma h'(x^0)(w) \geq \langle \xi, -c \rangle + \langle \theta, -k \rangle$$

for all $u, v, w \in \mathbb{R}^n, c \in C, k \in K$. It follows from the above inequality that $\xi \in C', \theta \in K', \gamma \in \mathbb{R}^q$ and $\theta g(x^0) \geq 0$. Remember that $g(x^0) \in -K$, and hence $\theta g(x^0) = 0$. Moreover, one has

$$\xi f'(0)(u) + \theta g'(x^0)(v) + \gamma h'(x^0)(w) \geq 0$$

for all $u, v, w \in \mathbb{R}^n$, which implies

$$\xi f'(x^0) + \theta g'(x^0) + \gamma h'(x^0) = 0,$$

as required. □

7.6.2 Lipschitz Continuous Problems

In order to derive optimality for problems in which the objective function and the constraint functions are not differentiable, but locally Lipschitz continuous, we will use the concept of Clarke's subdifferential. Clarke's generalized Jacobian of a locally Lipschitz function f from $\mathbb{R}^n$ to $\mathbb{R}^k$ is defined to be the set

$$\partial f(x) := \overline{\text{co}} \left\{ \lim_{x \to \infty} f'(x^i) : x^i \to x, f'(x^i) \text{ exists} \right\},$$

where $\overline{\text{co}}$ denotes the closed convex hull. Here are some properties of generalized Jacobian to be used in the sequent:

(i) $\partial f(x)$ is compact and convex.
(ii) The set-valued map $x \mapsto \partial f(x)$ is upper semi-continuous in the sense that for every $\epsilon > 0$ there is $\delta > 0$ such that $\partial f(x') \subseteq \partial f(x) + B(0, \epsilon)$ for every x' satisfying $\|x' - x\| < \delta$.
(iii) In the case $k = 1$,

$$\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + \partial f_2(x),$$

$\partial(\max_{\alpha \in T} f_\alpha)(x) = \partial f_{\alpha_0}(x)$ if α_0 the unique index where the maximum is attained.
(iv) $0 \in \partial f(x)$ if x is a local minimum of f.
(v) The mean value theorem: for $a, b, \in \mathbb{R}^n$, one has

$$f(b) - f(a) \in \overline{\text{co}}\{M(b - a) : M \in \partial f(c), c \in [a, b]\}.$$

We shall also use Ekeland's variational principle: Let φ be a lower semicontinuous function on $\mathbb{R}^n$. If $\varphi(x^0) \leq \inf \varphi + \zeta$ for $\zeta > 0$, then there is $x^\zeta \in R^n$ such that

$$\begin{aligned} ||x^\zeta - x^0|| &\leq \sqrt{\zeta} \\ \varphi(x^\zeta) &\leq \varphi(x^0) \\ \varphi(x^\zeta) &\leq \varphi(x) + \sqrt{\zeta}||x - x^\zeta)|| \quad \text{for all } x \neq x^\zeta. \end{aligned}$$

Theorem 7.7. Assume that f, g, and h are Lipschitz continuous and x^0 is a weakly efficient solution of (MOP). Then there exist multipliers $(\xi, \theta, \gamma) \in (C, K, \{0\})' \setminus \{0\}$ such that

$$0 \in \partial(\xi f + \theta g + \gamma h)(x^0)$$

$$\theta g(x^0) = 0.$$

Proof. Let $\lambda = (\xi, \theta, \gamma) \in (C, K, \{0\})' \setminus \{0\}$ and $T = \{\lambda : ||\lambda|| = 1\}$. Let $v \in \mathrm{int}(C)$ such that

$$1 = \max\{\langle \xi, v \rangle : \xi \in C', ||\xi|| = 1\}.$$

For $\zeta > 0$ set

$$H_\zeta(x) := (f(x) - f(x^0) + \zeta v, g(x), h(x)),$$

and consider the function

$$F_\zeta(x) := \max_{\lambda \in T} \langle \lambda, H_\zeta(x) \rangle.$$

It is evident that $F_\zeta(x)$ is Lipschitz continuous. We want to apply Ekeland's principle to obtain a point x^ζ that minimizes the function $F_\zeta(x) + \sqrt{\zeta}||x - x^\zeta||$. To this purpose, we prove that $F_\zeta(x) > 0$ for all $x \in \mathbb{R}^n$. Indeed, if not, that is $F_\zeta(x) \leq 0$ for some x, then $g(x) \leq 0, h(x) = 0$ and

$$\langle \xi, f(x) - f(x^0) \rangle \ < 0 \text{ for all } \xi \in C' \setminus \{0\}.$$

This means that x is a feasible solution and satisfies

$$f(x) - f(x^0) \in \mathrm{int}(C),$$

a contradiction to the optimally of x^0. In this way $F_\zeta(x) > 0$. We then obtain

$$F_\zeta(x^0) = \zeta \leq \inf_x F_\zeta(x) + \zeta.$$

According to Ekeland's principle, there is x^ζ such that

$$||x^\zeta - x^0|| \leq \sqrt{\zeta}$$

$$F_\zeta(x^\zeta) < F_\zeta(x) + \sqrt{\zeta}||x - x^\zeta||, \text{ for all } x \neq x^\zeta.$$

In other words, x^ζ is a minimum of the function $F_\zeta(x) + \sqrt{\zeta}||x - x^\zeta||$. Consequently we have

$$0 \in \partial(F_\zeta(x) + \sqrt{\zeta}||x - x^\zeta||)(x^\zeta) \subseteq \partial F_\zeta(x^\zeta) + \sqrt{\zeta}B(0,1), \qquad (7.9)$$

where the ball $B(0,1)$ is Clark's subdifferential of the function $x \mapsto ||x - x^\zeta||$ at x^ζ. To calculate the subdifferential $\partial F_\zeta(x^\zeta)$, we make the following observation: since $F_\zeta(x) > 0$, the vector $H_\zeta(x^\zeta) \neq 0$, hence the linear function

$\lambda \mapsto \langle \lambda, H_\zeta(x^\zeta) \rangle$ attains its maximum at a unique point $\lambda_\zeta \in T$ on T. (This is so because if that function has two distinct minima λ_1 and λ_2 on T, then at $\lambda = (\lambda_1 + \lambda_2)/||\lambda_1 + \lambda_2|| \in T$, one has

$$\langle \lambda, H_\zeta(x^\zeta) \rangle = \frac{2}{||\lambda_1 + \lambda_2||} \langle \lambda_1, H_\zeta(x^\zeta) \rangle > \lambda_2, H_\zeta(x^\zeta) \rangle,$$

because $||\lambda_1 + \lambda_2|| < ||\lambda_1|| + ||\lambda_2|| \leq 2$, which is a contradiction [note that $\lambda_1 + \lambda_2 \neq 0$].) Thus,

$$\partial F_\zeta(x^\zeta) = \partial \langle \lambda_\zeta, H_\zeta(x^\zeta) \rangle = \partial(\xi_\zeta f + \theta_\zeta g + \gamma_\zeta h)(x^\zeta).$$

Observe that in Ekeland's principle, if $\zeta \to 0$, then $x^\zeta \to x^0$. Moreover, as $H_\zeta(x^\zeta) \to (0, g(x^0), h(x^0))$, one has $\lambda_\zeta \to \lambda_0 \in T$ for some λ_0. Further, since $\partial \langle \lambda_\zeta, H_\zeta(x^\zeta) \rangle = \partial \langle \lambda_\zeta, H_0(x^\zeta) \rangle$, the upper semicontinuity of the subdifferential map

$$\langle \lambda, x \rangle \mapsto \langle \lambda, H_0(x^\xi) \rangle$$

and (7.9) show that

$$0 \in \partial \langle \lambda_0, H_0(x^0) \rangle = \partial(\xi f + \theta g + \gamma h)(x^0).$$

Finally, to see $\theta g(x^0) = 0$, it suffices to note that as $F_\zeta(x) > 0$, by letting $\zeta \to 0$, we obtain $\theta g(x^0) \geq 0$. On the other hand, $g(x^0) \in -K$ and $\theta \in K$ imply $\theta g(x^0) \leq 0$. Hence, $\theta g(x^0) = 0$ and the proof is complete. □

Note that the condition presented in the above theorem is useful if the first multiplier $\xi \neq 0$. One can guarantee this by imposing certain constraint qualifications, for instance all the matrices $N \in \partial h(x^0)$, have rank equal to q and there exists $u \in \cap\{\text{Ker}N : N \in \partial h(x^0)\}$ such that $M(u) \in -\text{int}(K)$ for all $M \in \partial g(x^0)$.

7.6.3 Concave Problems

Consider the following concave problem (MOP)

$$\begin{array}{cc} \text{Max} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{array}$$

where f is a concave function from R^n to R^k, and g is a convex function from $\mathbb{R}^n$ to $\mathbb{R}^m$. The ordering cone C is supposed to be convex, closed, and pointed with a nonempty interior, and the ordering cone $K \subseteq R^m$ is supposed to be convex and closed. For a concave problem, we have the following sufficient condition.

Theorem 7.8. Assume that (MOP) is concave and there exist multiplicators $(\xi, \theta) \in (C, K)'$ with $\xi \neq 0$ such that

$$0 \in \partial(\xi f)(x^0) + \partial(\theta g)(x^0)$$

$$\theta g(x^0) = 0.$$

Then, x^0 is a weak maximal solution of (MOP).

Proof. If x^0 is not weakly efficient, then there exists $x \in R^n$ with $g(x) \leq 0$ such that

$$f(x) - f(x^0) \in \text{int } (C).$$

On one hand we have

$$\max_{\lambda \in \partial(\xi f)(x^0)} \langle \lambda, x - x^0 \rangle = (\xi f)'(x^0, x - x^0) \geq \xi f(x) - \xi f(x^0) > 0,$$

because $\xi \in C' \setminus \{0\}$ [note that $\lambda \in \partial(\xi f)(x^0)$ coincides with the convex analysis subdifferential of $\xi \circ f$ at x^0]. On the other hand, for $g(x)$ one has

$$\max_{\lambda \in \partial(\theta g)(x^0)} \langle \lambda, x - x^0 \rangle = (\theta g)'(x^0, x - x^0) \leq \theta g(x) - \theta g(x^0) \leq 0,$$

because $\theta g(x^0) = 0$ and $g(x) \in -K, \theta \in K'$. Then

$$\max_{\lambda \in \partial(\xi f)(x^0) + \partial(\theta g)(x^0)} \langle \lambda, x - x^0 \rangle < 0,$$

which shows $0 \notin \partial(\xi f)(x^0) + \partial(\theta g)(x^0)$, a contradiction. □

7.7 Solution Methods

In vector optimization, the maximal value set of a problem is not a singleton in general. Therefore it is important to find not only one maximal solution, but a subset of maximal solutions that generates the whole maximal value set, or at least a representative part of it. In this section, we present three typical numerical methods that are interesting from mathematical and practical point of view. The two first methods are well known, and the third one is more recent. The ordering cone C is assumed to be the Pareto cone $\mathbb{R}^k_+$. The problem we are considering is the (MOP) introduced in Section 7.5.

7.7.1 Weighting Method

This method consists of choosing weights $p_1, \ldots, p_k \geq 0$, not all zero, and solving the associated scalar problem (P) by known techniques:

$$(P) \qquad \begin{array}{ll} \max & \sum_{i=1}^k p_i f_i(x) \\ \text{subject to} & x \in X. \end{array}$$

Theorem 7.9. For the problems (MOP) and (P) above we have the following

(a) If $p_i > 0, i = 1, \dots, m$, then any optimal solution of (P) is an efficient solution of (MOP).

(b) If $p_i \geq 0, i = 1, \dots, k$ and not all are zero, then any optimal solution of (P) is a weakly efficient solution of (MOP). If, in addition, the set f $(argmin\,(P))$ is a singleton, then it is an efficient solution.

Proof. Apply Proposition 7.12 □

In practice, one chooses a family of weighting vectors $p = (p_1, \dots, p_k)$ and solves the corresponding scalar problems (P), generating a subset of maximal solutions of (MOP). In case (b) of the theorem, in order to obtain an efficient solution, one proceeds as follows: let $p_1 > 0, \dots, p_l > 0$ and $p_{l+1} = \cdots = p_k = 0$. Set $f_i^* = f_i(x^0)$, where x^0 is an optimal solution of (P) and solves a subsidiary problem (P_*):

$$\begin{array}{ll} \max & \sum_{j=l+1}^{k} f_j(x) \\ \text{subject to} & x \in X, f_i(x) = f_i^*, i = 1, \dots, l. \end{array}$$

It is not difficult to see that any solution of (P_*) is an efficient solution of (MOP).

Here is the algorithm.

STEP 1. Choose $\xi > 0$ and $m \geq 1$ such that $\frac{1}{m} \leq \xi$. Choose $\lambda = \frac{1}{m}(m_1, \dots, m_k)$ with $m_i \in \{0, 1, \dots, m\}$ such that $m_1 + \cdots + m_k = m$.

STEP 2. Solve (P_λ). If $m_i > 0$, $i = 1, \dots, k$, store an optimal solution x^λ and its value $f(x^\lambda)$. If $m_i = 0$ for some i, solve (P_λ^*) and store an optimal solution x^λ and its value $f(x^\lambda)$.

STEP 3. Choose another λ in Step 1 and go to Step 2 unless there are no λ left.

In the final result, the method generates a set of maximal solutions and a set of maximal values that corresponds to an ϵ-net of weighting vectors (in the sense that for every $\xi \in \mathbb{R}_+^k$ with $\sum_{i=1}^k \xi_i = 1$, there is λ of that family such that $\|\xi - \lambda\| \leq \epsilon$). We observe, however, that for nonconcave problems and even for linear problems, the generated set of maximal values may be a very small part of the maximal value set of the problem, even if ϵ tends to zero.

Example 7.8. Consider the problem (MOP) with

$$X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + \sqrt{2}x_2 = 1, x_1 \geq 0, x_2 \geq 0\right\},$$

where f is the identity map from $\mathbb{R}^2$ to itself and $\mathbb{R}^2$ is equipped with the

Pareto order. For every $m \geq 1$, the set of weighting vectors to use in the algorithm is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, 1 - \frac{i}{m} \right) : i = 0, 1, \ldots, m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, 1 - \frac{i}{m}\right)$, the problem (P_{λ^i}) is written as

$$\begin{array}{ll} \max & \frac{i}{m} x_1 + \left(1 - \frac{i}{m}\right) x_2 \\ \text{subject to} & x \in X. \end{array}$$

It is clear that $\lambda^i \neq \left(\frac{\sqrt{2}}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}} \right)$, and so the solution set of (P_λ) consists of either the singleton $\{(0, \sqrt{2})\}$ when $\frac{i}{m} < \frac{\sqrt{2}}{1+\sqrt{2}}$ or the singleton $\{(1, 0)\}$ when $\frac{i}{m} > \frac{\sqrt{2}}{1+\sqrt{2}}$. Consequently, for every $m \geq 1$, no maximal solutions of (MOP) between the end points $(0, \sqrt{2})$ and $(1, 0)$ can be approached.

Example 7.9. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-x_1^2 - x_2 + 2, -11x_1^2 + 6x_1x_2 - x_2^2 + 2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\right\}$. For $m \geq 1$, the set of weighting vectors is:

$$\Lambda_m = \left\{ \left(\frac{i}{m}, 1 - \frac{i}{m} \right) : i = 0, \ldots, m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, 1 - \frac{i}{m}\right)$, the problem (P_{λ^i}) is:

$$\begin{array}{ll} \max & \frac{i}{m} \left(-x_1^2 - x_2 + 2\right) + \left(1 - \frac{i}{m}\right) \left(-11x_1^2 + 6x_1x_2 - x_2^2 + 2\right) \\ \text{subject to} & x \in X. \end{array}$$

For every $m \geq 1$, we obtain one maximal solution $x = (0, 0)$ with value $f(x) = (2, 2)$.

Example 7.10. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-x_1^2 - x_2 + 2, -5x_1^2 + 2x_1x_2 - x_2^2 - 3\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2\right\}$. For $m \geq 1$, the set of weighting vectors is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, 1 - \frac{i}{m} \right) : i = 0, \ldots, m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, 1 - \frac{i}{m}\right)$, the problem (P_{λ^i}) is

$$\begin{array}{ll} \max & \frac{i}{m} \left(-x_1^2 - x_2 + 2\right) + \left(1 - \frac{i}{m}\right) \left(-5x_1^2 + 2x_1x_2 - x_2^2 - 3\right) \\ \text{subject to} & x \in X. \end{array}$$

For every $m \geq 1$, we obtain one maximal solution $x = (0, 0)$ with value $f(x) = (2, -3)$.

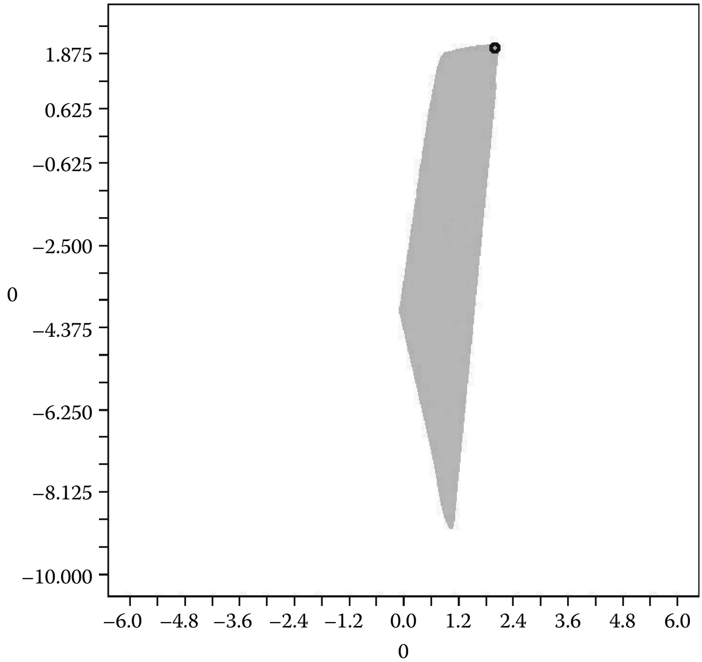

FIGURE 7.12: Figure for Example 7.9, for $m \geq 1$.

Example 7.11. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(1 - x_1^2 - x_2^2, 300x_1 + 400x_2 - x_1^2 - 2x_1x_2 - 2x_2^2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2\right\}$. For $m \geq 1$, the set of weighting vectors is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, 1 - \frac{i}{m} \right) : i = 0, \ldots, m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, 1 - \frac{i}{m}\right)$, the problem (P_{λ^i}) is

$$\begin{array}{ll} \max & \frac{i}{m}\left(1 - x_1^2 - x_2^2\right) + \left(1 - \frac{i}{m}\right)\left(300x_1 + 400x_2 - x_1^2 - 2x_1x_2 - 2x_2^2\right) \\ \text{subject to} & x \in X. \end{array}$$

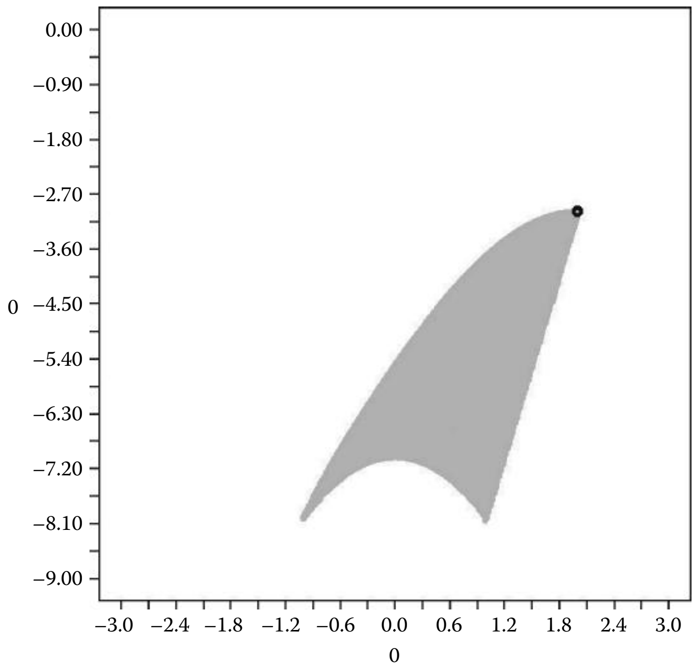

FIGURE 7.13: Figure for Example 7.10, for $m \geq 1$.

We obtain the following results:

m	Maximal solution	Maximal value
1, ..., 20	$(1, 2)$ $(0, 0)$	$(-4, 1087)$ $(1, 0)$
100	$(1, 2)$ $(1, 1.9703)$ $(0, 0)$	$(-4, 1087)$ $(-3.8821, 1.0754e + 003)$ $(1, 0)$
225	$(1, 2)$ $(1, 1.7533)$ $(0.6629, 0.8824)$ $(0, 0)$	$(-4, 1087)$ $(-3.0741, 990.6653)$ $(-0.2181, 548.6634)$ $(1, 0)$
300	$(1, 2)$ $(1, 1.9703)$ $(0.9912, 1.3179)$ $(0.4973, 0.6630)$ $(0, 0)$	$(-4, 1087)$ $(-3.8821, 1.0754e + 003)$ $(-1.7193, 817.4512)$ $(0.3131, 412.6041)$ $(1, 0)$
500	$(1, 2)$ $(1, 1.9703)$ $(1, 1.5794)$ $(0.8929, 1.1870)$ $(0.5969, 0.7945)$ $(0.2994, 0.3985)$ $(0, 0)$	$(-4, 1087)$ $(-3.8821, 1.0754e + 003)$ $(-2.4945, 922.6122)$ $(-1.2062, 736.9350)$ $(0.0125, 494.3028)$ $(0.7516, 248.5741)$ $(1, 0)$

TABLE 7.1: Table for Example 7.11.

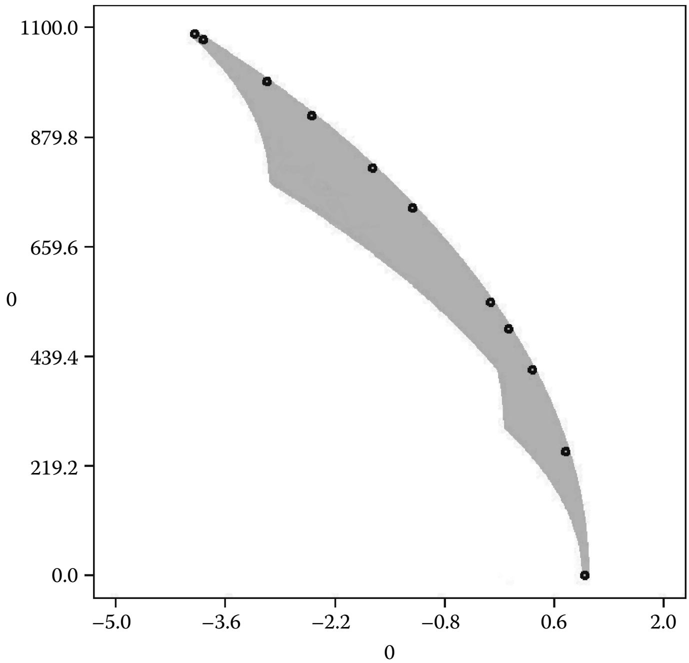

FIGURE 7.14: Figure for Example 7.11, for $m = 500$.

Example 7.12. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \big(-x_1^2 - x_2 + 2x_1 + 2x_1x_2 - x_2^2, -0.1x_1^2 + 2x_1 \\ & \quad -0.1x_2^2 - 3x_2\big) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$. For $m \geq 1$, the set of weighting vectors is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, 1 - \frac{i}{m} \right) : i = 0, \ldots, m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, 1 - \frac{i}{m}\right)$, the problem (P_{λ^i}) is:

$$\begin{array}{ll} \max & \frac{i}{m}\left(-x_1^2 - x_2 + 2x_1 + 2x_1x_2 - x_2^2\right) + \left(1 - \frac{i}{m}\right) \\ & \left(-0.1x_1^2 + 2x_1 - 0.1x_2^2 - 3x_2\right) \\ \text{subject to} & x \in X. \end{array}$$

We obtain the following results:

m	Maximal solution	Maximal value
$1, ..., 4$	$(1, 0)$	$(1, 1.9000)$
	$(1, 0.5)$	$(1.2500, 0.3750)$
5	$(1, 0)$	$(1, 1.9000)$
	$(1, 0.1220)$	$(1.1071, 1.5325)$
	$(1, 0.5)$	$(1.2500, 0.3750)$
10	$(1, 0)$	$(1, 1.9000)$
	$(1, 0.1220)$	$(1.1071, 1.5325)$
	$(1, 0.3297)$	$(1.2210, 0.9000)$
	$(1, 0.5)$	$(1.2500, 0.3750)$
20	$(1, 0)$	$(1, 1.9000)$
	$(1, 0.3297)$	$(1.2210, 0.9000)$
	$(1, 0.1220)$	$(1.1071, 1.5325)$
	$(1, 0.2312)$	$(1.1777, 1.2011)$
	$(1, 0.4188)$	$(1.2434, 0.6261)$
	$(1, 0.5)$	$(1.2500, 0.3750)$
100	$(1, 0)$	$(1, 1.9000)$
	$(1, 0.0255)$	$(1.0248, 1.8234)$
	$(1, 0.0504)$	$(1.0479, 1.7485)$
	$(1, 0.0748)$	$(1.0692, 1.6750)$
	$(1, 0.0986)$	$(1.0889, 1.6032)$
	$(1, 0.1220)$	$(1.1071, 1.5325)$
100	$(1, 0.1448)$	$(1.1238, 1.4635)$
	$(1, 0.1671)$	$(1.1392, 1.3959)$
	$(1, 0.1889)$	$(1.1532, 1.3297)$
	$(1, 0.2103)$	$(1.1661, 1.2647)$
	$(1, 0.2312)$	$(1.1777, 1.2011)$
	$(1, 0.2517)$	$(1.1883, 1.1386)$
	$(1, 0.2718)$	$(1.1979, 1.0772)$
	$(1, 0.2915)$	$(1.2065, 1.0170)$
	$(1, 0.3108)$	$(1.2142, 0.9579)$
	$(1, 0.3297)$	$(1.2210, 0.9000)$
	$(1, 0.3482)$	$(1.2270, 0.8433)$
	$(1, 0.3664)$	$(1.2322, 0.7874)$
	$(1, 0.3842)$	$(1.2366, 0.7326)$
	$(1, 0.4017)$	$(1.2403, 0.6788)$
	$(1, 0.4188)$	$(1.2434, 0.6261)$
	$(1, 0.4357)$	$(1.2459, 0.5739)$
	$(1, 0.4522)$	$(1.2477, 0.5230)$
	$(1, 0.4684)$	$(1.2490, 0.4729)$
	$(1, 0.4844)$	$(1.2498, 0.4233)$
	$(1, 0.5)$	$(1.2500, 0.3750)$

TABLE 7.2: Table for Example 7.12.

Example 7.13. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-3x_1^2 + 8 - 5x_2^2, -x_1^2 - x_2^2, -1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\right\}$. For $m \geq 1$, the set of weighting vectors is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, \frac{j}{m}, 1 - \frac{i+j}{m} \right) : i, j = 0, \ldots, m, 0 \leq i + j \leq m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, \frac{j}{m}, 1 - \frac{i+j}{m}\right)$, the problem (P_{λ^i}) is

$$\begin{array}{ll} \max & \frac{i}{m}(-3x_1^2 + 8 - 5x_2^2) + \frac{j}{m}\left(-x_1^2 - x_2^2\right) + \\ & + \left(1 - \frac{i+j}{m}\right)(-1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2) \\ \text{subject to} & x \in X. \end{array}$$

For every $m \geq 1$, we obtain one maximal solution $x = (0, 0)$ with value $f(x) = (8, 0, 0)$.

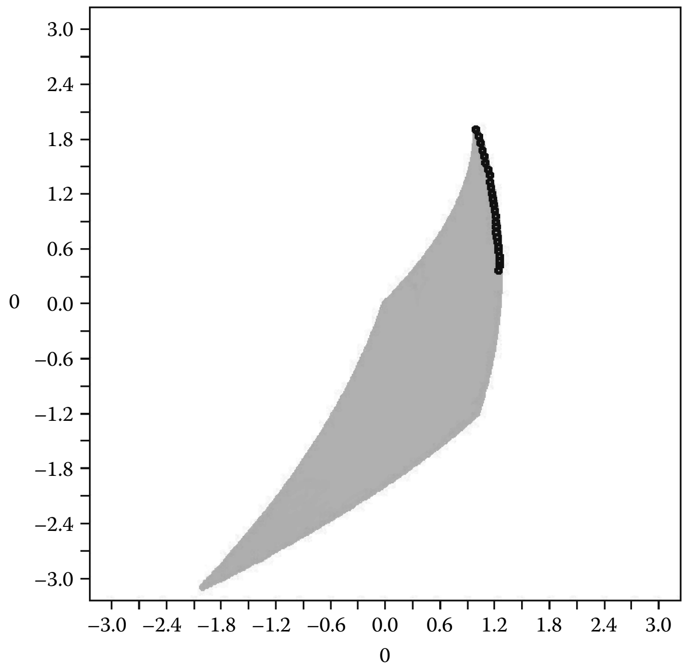

FIGURE 7.15: Figure for Example 7.12, for $m = 100$.

Example 7.14. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \big(-3x_1^2 + 8 - 5x_2^2, -x_1^2 + 11x_1 - x_2^2 - 12, \\ & \qquad -1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2\big) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \big\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\big\}$. For every $m \geq 1$, the set of weighting vectors is

$$\Lambda_m = \left\{ \left(\frac{i}{m}, \frac{j}{m}, 1 - \frac{i+j}{m} \right) : i, j = 0, \ldots, m, 0 \leq i + j \leq m \right\}.$$

With $\lambda^i = \left(\frac{i}{m}, \frac{j}{m}, 1 - \frac{i+j}{m}\right)$, the problem (P_{λ^i}) is:

$$\begin{array}{ll} \max & \frac{i}{m}\left(-3x_1^2 + 8 - 5x_2^2\right) + \frac{j}{m}\left(-x_1^2 + 11x_1 - x_2^2 - 12\right) + \\ & + \left(1 - \frac{i+j}{m}\right)\left(-1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2\right) \\ \text{subject to} & x \in X. \end{array}$$

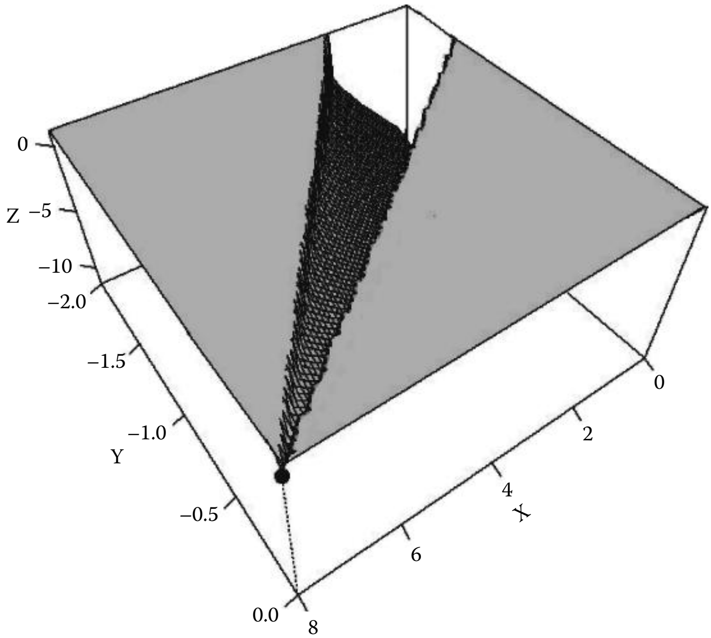

FIGURE 7.16: Figure for Example 7.13, for $m \geq 1$.

We obtain the following results:

m	i	j	Maximal solution	Maximal value
1	0	0	$(0, 0)$	$(8, -12, 0)$
		1	$(1, 0)$	$(5, -2, -1.5000)$
	1	0	$(0, 0)$	$(8, -12, 0)$
2	0	0	$(0, 0)$	$(8, -12, 0)$
		1	$(1, 0)$	$(5, -2, -1.5000)$
		2	$(1, 0)$	$(5, -2, -1.5000)$
	1	0	$(0, 0)$	$(8, -12, 0)$
		1	$(1, 0)$	$(5, -2, -1.5000)$
	2	0	$(0, 0)$	$(8, -12, 0)$

10	0	$0, 1$	$(0, 0)$	$(8, -12, 0)$
		$2, ..., 10$	$(1, 0)$	$(5, -2, -1.5000)$
	1	0	$(0, 0)$	$(8, -12, 0)$
		1	$(0.0536, 0)$	$(7.9914, -11.4133, -0.0703)$
		$2, ..., 9$	$(1, 0)$	$(5, -2, -1.5000)$
	2	0	$(0, 0)$	$(8, -12, 0)$
		1	$(0.1131, 0)$	$(7.9616, -10.7687, -0.1496)$
		2	$(0.7717, 0)$	$(6.2134, -4.1068, -1.1223)$
		$3, ..., 8$	$(1, 0)$	$(5, -2, -1.5000)$

m	i	j	Maximal solution	Maximal value
	3	0	$(0, 0)$	$(8, -12, 0)$
		1	$(0.1429, 0)$	$(7.9387, -10.4485, -0.1899)$
		2	$(0.6458, 0)$	$(6.7488, -5.3133, -0.9230)$
		$3, ..., 7$	$(1, 0)$	$(5, -2, -1.5000)$
	7	0	$(0, 0)$	$(8, -12, 0)$
		1	$(0.1875, 0)$	$(7.8945, -9.9727, -0.2508$
		2	$(0.4461, 0)$	$(7.4030, -7.2919, -0.6197)$
		3	$(0.6875, 0)$	$(6.5820, -4.9102, -0.9883)$
	9	0	$(0, 0)$	$(8, -12, 0)$
		1	$(0.1964, 0)$	$(7.8843, -9.8782, -0.2630)$
	10	0	$(0, 0)$	$(8, -12, 0)$

TABLE 7.3: Table for Example 7.14.

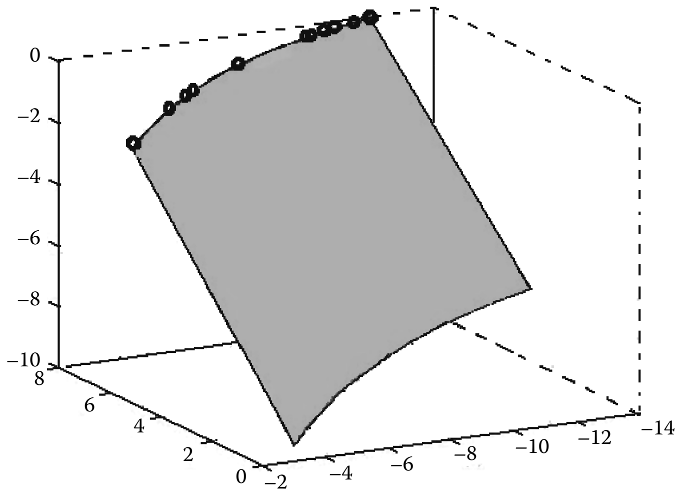

FIGURE 7.17: Figure for Example 7.14, for $m = 10$.

7.7.2 Constraint Method

In this method, one maximizes one objective, while other objectives are considered as constraints. Let us choose $\ell \in \{1, \ldots, k\}, L_j \in R, j = 1, \ldots, k, j \neq \ell$, and solve the scalar problem (P_ℓ):

$$\begin{array}{rc} \text{Max} & f_\ell(x) \\ \text{subject to} & f_j(x) \geq L_j, j = 1, \ldots, k, j \neq \ell \\ & x \in X. \end{array}$$

Note that if L_j are small, then (P_ℓ) may have no feasible solutions, and if L_j are too big, then an optimal solution of (P_ℓ) may be not efficient. We shall say that a constraint $f_j(x) \geq L_j$ is binding if equality $f_j(x) = L_j$ is satisfied at every optimal solution of (P_ℓ).

Theorem 7.10. Let $x^0 \in X$ be given. The following assertions hold:

(a) $x^0 \in WS(X, f)$ if it is an optimal solution of (P_ℓ);

(b) $x^0 \in S(X, f)$ if it is an optimal solution of (P_ℓ) and if it is a unique optimal solution or all constraints of (P_ℓ) are binding;

(c) $x^0 \in S(X, f)$ if and only if it is optimal for all (P_ℓ), $\ell = 1, \ldots, k$ and $L^{-\ell} = (f_1(x^0), \ldots, f_{\ell-1}(x^0), f_{\ell+1}(x^0), \ldots, f_k(x^0))$.

Proof. To establish (a), we assume that x^0 is optimal for (P_ℓ). If it is not a weak efficient solution of (MOP), then there is some $y \in X$ such that $f(y) > f(x)$. It follows that y is feasible for (P_ℓ) because $f_j(y) > f_j(x^0) \geq L_j$, $j \in \{1, \ldots, k\} \backslash \{\ell\}$. Moreover, $f_\ell(y) > f_\ell(x^0)$, which contradicts the optimality of x^0.

For (b), assume that there is $y \in X$ such that $f(y) \geq f(x^0)$. It follows that y is feasible for (P_ℓ). Since x^0 is optimal, we have $f_\ell(y) = f_\ell(x^0)$. If x^0 is unique, then $y = x^0$, and if all the constraints are binding, then $f_j(y) = f_j(x^0) = L_j$, $j \in \{1, \ldots, k\} \backslash \ \{\ell\}$. In both cases, $f(y) = f(x^0)$, by which $x^0 \in S(X, f)$.

For (c), assume that $x^0 \in S(X, f)$. Suppose to the contrary that for some ℓ, x^0 is not optimal for the problem (P_ℓ):

$$\begin{array}{ll} \max & f_\ell(x) \\ \text{subject to} & f_j(x) \geq f_j(x^0), j = 1, \ldots, k, j \neq \ell \\ & x \in X. \end{array}$$

Then, there is some $y \in X$ with $f_j(y) \geq f_j(x^0), j = 1, \ldots, k, j \neq \ell$, such that $f_\ell(y) > f_\ell(x^0)$. It follows that $f(y) > f(x^0)$, which contradicts the assumption. For the converse, we assume that $x^0 \notin S(X, f)$. Then there is some $y \in X$ such that $f(y) > f(x^0)$. There is some $l \in \{1, \ldots, k\}$ such that $f_\ell(y) > f_\ell(x^0)$ and $f_j(y) \geq f_j(x^0)$, for $j = 1, \ldots, k$. It is clear that x^0 cannot be optimal for (P_ℓ) above. The proof is complete. □

The following is the constraint algorithm to solve (MOP):

STEP 1. Solve

$$\begin{array}{ll} \max & f_i(x) \\ \text{subject to} & x \in X \end{array}$$

for $i = 1, \ldots, k$. Let $x^1, \ldots, x^k$ be optimal solutions.

STEP 2. Construct the payoff table

$$\begin{bmatrix} f_1(x^1) & \cdots & f_k(x^1) \\ \vdots & & \vdots \\ f_1(x^k) & \cdots & f_k(x^k) \\ M_1 & \cdots & M_k \\ m_1 & \cdots & m_k \end{bmatrix}$$

where

$$M_i = \max\{f_i(x^1), \ldots, f_i(x^k)\}$$
$$m_i = \min\{f_i(x^1), \ldots, f_i(x^k)\}.$$

STEP 3. Choose $r = 1, 2, \ldots$ and solve (P_ℓ) with

$$L_j = M_j - \frac{t}{r-1}(M_j - m_j), t = 0, \ldots, r-1.$$

If at a solution x^* of (P_ℓ), all the constraints are binding, then this solution is maximal for (MOP). Otherwise, assuming $f_1, \ldots, f_s$ is active, and $f_{s+1}, \ldots, f_k (\neq f_\ell)$ is nonbinding, one solves (P_*):

$$\begin{array}{ll} \max & \sum\limits_{j=s+1,\ldots,k, j\neq\ell} f_j(x) \\ \text{subject to} & x \in X, f_i(x) = f(x^*)_i, i = 1, \ldots, s, i \neq \ell, \end{array}$$

to obtain an efficient solution. Moreover, in the last step for each t the problem (P_ℓ) (or the corresponding (P_*)) provides an efficient solution and hence a maximal value of (MOP). With r large, one may generate a good representative subset of the maximal value set of the problem.

Example 7.15. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-x_1^2 - x_2 + 2, -11x_1^2 + 6x_1x_2 - x_2^2 + 2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = -x_1^2 - x_2 + 2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (0, 0)$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = -11x_1^2 + 6x_1x_2 - x_2^2 + 2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^2 = (0, 0)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) \\ f_1(x^2) & f_2(x^2) \\ M_1 & M_2 \\ m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix},$$

where $M_1 = \max\left\{f_1(x^1), f_1(x^2)\right\} = 2$, $M_2 = \max\left\{f_2(x^1), f_2(x^2)\right\} = 2$, $m_1 = \min\left\{f_1(x^1), f_1(x^2)\right\} = 2$, and $m_2 = \min\left\{f_2(x^1), f_2(x^2)\right\} = 2$.

For every $r \geq 1$, we obtain one maximal solution $x = (0, 0)$ with value $f(x) = (2, 2)$. (See Figure 7.12).

Example 7.16. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-x_1^2 - x_2 + 2, -5x_1^2 + 2x_1x_2 - x_2^2 - 3\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = -x_1^2 - x_2 + 2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (0,0)$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = -5x_1^2 + 2x_1x_2 - x_2^2 - 3 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^2 = (0,0)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) \\ f_1(x^2) & f_2(x^2) \\ M_1 & M_2 \\ m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -3 \\ 2 & -3 \\ 2 & -3 \end{bmatrix},$$

where $M_1 = \max\left\{f_1(x^1), f_1(x^2)\right\} = 2$, $M_2 = \max\left\{f_2(x^1), f_2(x^2)\right\} = -3$, $m_1 = \min\left\{f_1(x^1), f_1(x^2)\right\} = 2$, and $m_2 = \min\left\{f_2(x^1), f_2(x^2)\right\} = -3$.

For every $r \geq 1$, we obtain one maximal solution $x = (0,0)$ with value $f(x) = (2, -3)$. (See Figure 7.13).

Example 7.17. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(1 - x_1^2 - x_2^2, 300x_1 + 400x_2 - x_1^2 - 2x_1x_2 - 2x_2^2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = 1 - x_1^2 - x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (0,0)$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = 300x_1 + 400x_2 - x_1^2 - 2x_1x_2 - 2x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^2 = (1,2)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) \\ f_1(x^2) & f_2(x^2) \\ M_1 & M_2 \\ m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1087 \\ 1 & 1087 \\ -4 & 0 \end{bmatrix},$$

where $M_1 = \max\left\{f_1(x^1), f_1(x^2)\right\} = 1$, $M_2 = \max\left\{f_2(x^1), f_2(x^2)\right\} = 1087$, $m_1 = \min\left\{f_1(x^1), f_1(x^2)\right\} = -4$, and $m_2 = \min\left\{f_2(x^1), f_2(x^2)\right\} = 0$.

We obtain the following results:

l	j	r	t	L_j	Maximal solution	Maximal value
1	2	1	0	1087	$(1, 2)$	$(-4, 1087)$
1	2	2	1	0	$(0, 0)$	$(1, 0)$
1	2	3	1	543.5	$(0.6567, 0.8740)$	$(-0.1951, 543.5031)$
1	2	4	1	724.66	$(0.8776, 1.1673)$	$(-1.1328, 724.6558)$
1	2	4	2	362.33	$(0.4365, 0.5819)$	$(0.4709, 362.3343)$
2	1	1	0	1	$(0, 0)$	$(1, 0)$
2	1	2	1	-4	$(1, 2)$	$(-4, 1087)$
2	1	3	1	$-3/2$	$(0.9503, 1.2637)$	$(-1.5000, 784.0713)$
2	1	4	1	$-2/3$	$(0.7757, 1.0320)$	$(-0.6667, 641.1772)$
2	1	4	2	$-7/3$	$(1, 1.5275)$	$(-2.3333, 902.2785)$

TABLE 7.4: Table for Example 7.17.

Example 7.18. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \big(-x_1^2 - x_2 + 2x_1 + 2x_1x_2 - x_2^2, \\ & \quad -0.1x_1^2 + 2x_1 - 0.1x_2^2 - 3x_2\big) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 1\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = -x_1^2 - x_2 + 2x_1 + 2x_1x_2 - x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (1, 0.5)$.

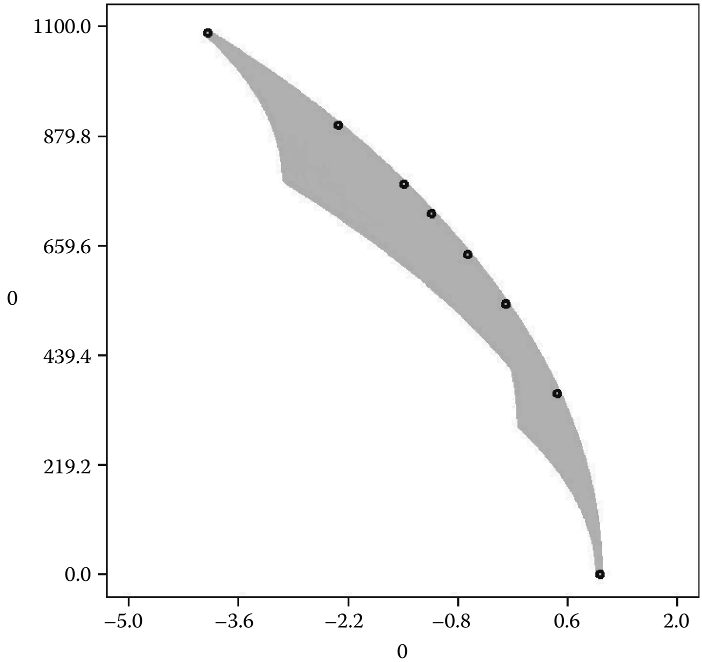

FIGURE 7.18: Figure for Example 7.17, for $r = 4$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = -0.1x_1^2 + 2x_1 - 0.1x_2^2 - 3x_2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^2 = (1, 1)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) \\ f_1(x^2) & f_2(x^2) \\ M_1 & M_2 \\ m_1 & m_2 \end{bmatrix} = \begin{bmatrix} 1.25 & 0.375 \\ 1 & -1.2 \\ 1.25 & 0.375 \\ 1 & -1.2 \end{bmatrix},$$

where $M_1 = \max\{f_1(x^1), f_1(x^2)\} = 1.25$, $M_2 = \max\{f_2(x^1), f_2(x^2)\} = 0.375$, $m_1 = \min\{f_1(x^1), f_1(x^2)\} = 1$, and $m_2 = \min\{f_2(x^1), f_2(x^2)\} = -1.2$.

We obtain the following results:

l	j	r	t	L_j	Maximal solution	Maximal value
1	2	1	0	0.375	$(1, 0.5)$	$(1.2500, 0.3750)$
1	2	2	1	-1.2	$(1, 0.5)$	$(1.2500, 0.3750)$
1	2	3	1	-0.4125	$(1, 0.5)$	$(1.2500, 0.3750)$
1	2	4	1	-0.15	$(1, 0.5)$	$(1.2500, 0.3750)$
1	2	4	2	-0.675	$(1, 0.5)$	$(1.2500, 0.3750)$
2	1	1	0	1.25	$(1, 0.4986)$	$(1.2500, 0.3793)$
2	1	2	1	1	$(1, 0)$	$(1, 1.9000)$
2	1	3	1	1.125	$(1, 0.1464)$	$(1.1250, 1.4587)$
2	1	4	1	1.1666	$(1, 0.2112)$	$(1.1666, 1.2619)$
2	1	4	2	1.0833	$(1, 0.0917)$	$(1.0833, 1.6241)$

TABLE 7.5: Table for Example 7.18.

Example 7.19. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \left(-3x_1^2 + 8 - 5x_2^2, -x_1^2 - x_2^2, -1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2\right) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 1\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = -3x_1^2 + 8 - 5x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (0, 0)$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = -x_1^2 - x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

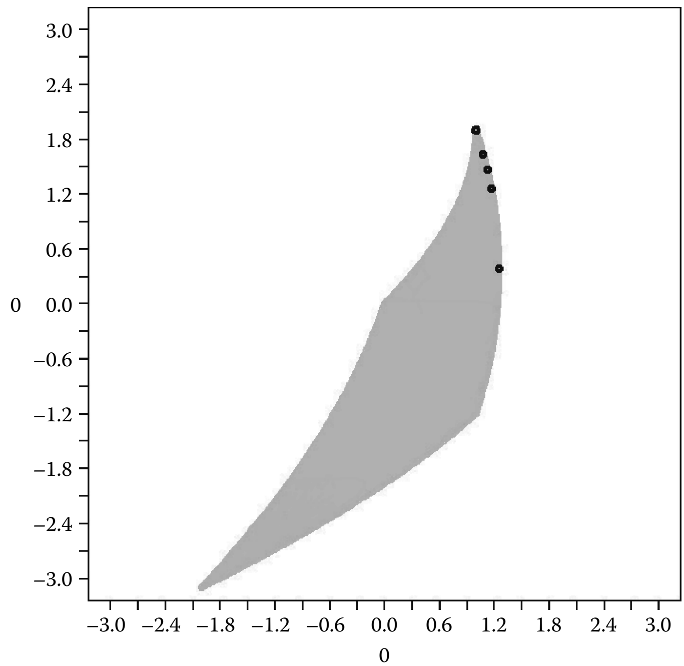

FIGURE 7.19: Figure for Example 7.18, for $r = 4$.

the optimal solution is $x^2 = (0, 0)$.

For the problem

$$\begin{array}{ll} \max & f_3(x) = -1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^3 = (0, 0)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) & f_3(x^1) \\ f_1(x^2) & f_2(x^2) & f_3(x^2) \\ f_1(x^3) & f_2(x^3) & f_3(x^3) \\ M_1 & M_2 & M_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 8 & 0 & 0 \\ 8 & 0 & 0 \\ 8 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix},$$

where $M_1 = \max\{f_1(x^1), f_1(x^2), f_1(x^3)\} = 8$, $M_2 = \max\{f_2(x^1), f_2(x^2), f_2(x^3)\} = 0$, $M_3 = \max\{f_3(x^1), f_3(x^2), f_3(x^3)\} = 0$, $m_1 = \min\{f_1(x^1), f_1(x^2), f_1(x^3)\} = 8$, $m_2 = \min\{f_2(x^1), f_2(x^2), f_2(x^3)\} = 0$, and $m_3 = \min\{f_3(x^1), f_3(x^2), f_3(x^3)\} = 0$.

For every $r \geq 1$, we obtain one maximal solution $x = (0, 0)$ with value $f(x) = (8, 0, 0)$. (See Figure 7.16).

Example 7.20. Consider the concave problem

$$\begin{array}{ll} \max & f(x) = \big(-3x_1^2 + 8 - 5x_2^2, -x_1^2 + 11x_1 - x_2^2 - 12, -1.3x_1 - 0.2x_1^2 \\ & -x_1x_2 - 7x_2^2 \big) \\ \text{subject to} & x \in X, \end{array}$$

where $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\right\}$.

For the problem

$$\begin{array}{ll} \max & f_1(x) = -3x_1^2 + 8 - 5x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^1 = (0, 0)$.

For the problem

$$\begin{array}{ll} \max & f_2(x) = -x_1^2 + 11x_1 - x_2^2 - 12 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^2 = (1, 0)$.

For the problem

$$\begin{array}{ll} \max & f_3(x) = -1.3x_1 - 0.2x_1^2 - x_1x_2 - 7x_2^2 \\ \text{subject to} & x \in X, \end{array}$$

the optimal solution is $x^3 = (0, 0)$.

The payoff table is

$$\begin{bmatrix} f_1(x^1) & f_2(x^1) & f_3(x^1) \\ f_1(x^2) & f_2(x^2) & f_3(x^2) \\ f_1(x^3) & f_2(x^3) & f_3(x^3) \\ M_1 & M_2 & M_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 0 \\ 5 & -2 & -1.5 \\ 8 & -12 & 0 \\ 8 & -2 & 0 \\ 5 & -12 & -1.5 \end{bmatrix},$$

where $M_1 = \max\{f_1(x^1), f_1(x^2), f_1(x^3)\} = 8$, $M_2 = \max\{f_2(x^1), f_2(x^2), f_2(x^3)\} = -2$, $M_3 = \max\{f_3(x^1), f_3(x^2), f_3(x^3)\} = 0$, $m_1 = \min\{f_1(x^1), f_1(x^2), f_1(x^3)\} = 5$, $m_2 = \min\{f_2(x^1), f_2(x^2), f_2(x^3)\} = -12$, and $m_3 = \min\{f_3(x^1), f_3(x^2), f_3(x^3)\} = -1.5$

We obtain the following results:

l	j	r	t	L_j	Maximal solution	Maximal value
1	2	$1, 2$	0	-2	$(1, 0)$	$(5, -2, -1.5)$
1	2	2	1	-12	$(0, 0)$	$(8, -12, 0)$
1	3	$1, 2$	0	0	$(0, 0)$	$(8, -12, 0)$

l	j	r	t	L_j	Maximal solution	Maximal value
1	3	2	1	-1.5	$(0, 0)$	$(8, -12, 0)$
2	1	$1, 2$	0	8	$(0, 0)$	$(8, -12, 0)$
2	1	2	1	5	$(1, 0)$	$(5, -2, -1.5)$
2	3	$1, 2$	0	0	$(0, 0)$	$(8, -12, 0)$
2	3	2	1	-1.5	$(1, 0)$	$(5, -2, -1.5)$
3	1	$1, 2$	0	8	$(0, 0)$	$(8, -12, 0)$
3	1	2	1	5	$(1, 0)$	$(5, -2, -1.5)$
3	2	$1, 2$	0	-2	$(1, 0)$	$(5, -2, -1.5)$
3	2	2	1	-12	$(0, 0)$	$(8, -12, 0)$

TABLE 7.6: Table for Example 7.20.

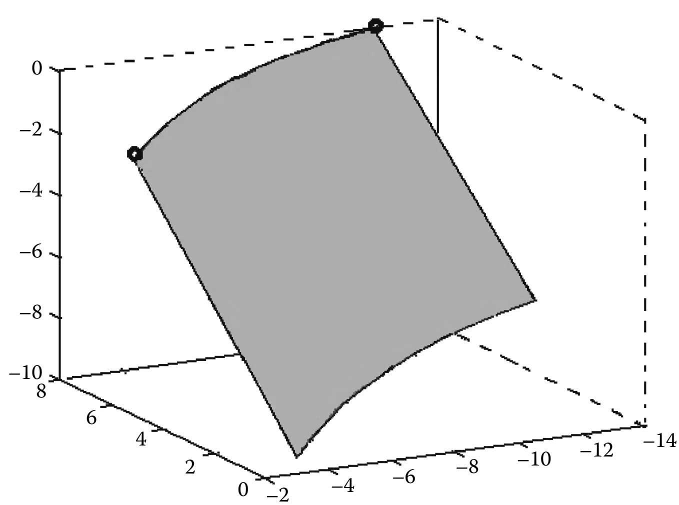

FIGURE 7.20: Figure for Example 7.20, for $r = 2$.

7.7.3 Outer Approximation Method

Assume that $f(X)$ is a compact set. Without loss of generality, we may assume that it is contained in the interior of the Pareto cone. The set

$$f(X)^{\diamondsuit} := \left(f(X) - \mathbb{R}^k_+\right) \cap \mathbb{R}^k_+,$$

is known as the Edgeworth-Pareto hull (E-P hull for short) of $f(X)$. It is said to be finitely generated if there are a finite number of vectors $v^1, \dots, v^m$ such that $f(X)^{\diamond}$ coincides with $\{v^1, \dots, v^m\}^{\diamond}$. We will need the following properties of $f(X)^{\diamondsuit}$.

Proposition 7.18. The following assertions hold:

(a) $\operatorname{Max}(f(X)) = \operatorname{Max}\left(f(X)^{\diamondsuit}\right)$;

(b) $\operatorname{WMax}(f(X)) \subseteq \operatorname{WMax}\left(f(X)^{\diamondsuit}\right)$;

(c) $f(X)^{\diamondsuit} = [\operatorname{Max}(f(X))]^{\diamondsuit} = [\operatorname{WMax}(f(X))]^{\diamondsuit}$.

Proof. The two first assertions follow from Proposition 7.10 because $f(X) - \mathbb{R}^k_+ = f(X)^{\diamondsuit} - \mathbb{R}^k_+$. For the last assertion, we observe that

$$\operatorname{Max}(f(X)) \subseteq \operatorname{WMax}(f(X)) \subseteq f(X),$$

which implies

$$[\operatorname{Max}(f(X))]^{\diamondsuit} \subseteq [\operatorname{WMax}(f(X))]^{\diamondsuit} \subseteq f(X)^{\diamondsuit}.$$

For the converse, given $a \in f(X)^{\diamondsuit}$, there are $x \in X$ and $u \in \mathbb{R}^k_+$ such that

$$a = f(x) - u \in \mathbb{R}^k_+.$$

Consider the section of $f(X)$ at $f(x)$:

$$A_{f(x)} = f(X) \cap \left(f(x) + \mathbb{R}^k_+\right).$$

It is a compact set, hence it has a maximal point, say $y \in X$ with $f(y) \in \operatorname{Max}(A_{f(x)}) \subseteq \operatorname{Max}(f(X))$. Then $a = f(y) + (f(x) - f(y)) - u \in \operatorname{Max}(f(X)) - \mathbb{R}^k_+$, which implies $a \in [\operatorname{Max}(f(X))]^{\diamondsuit}$ and the equalities of (iii) follow. □

Here is the algorithm.

STEP 1. For $i = 1, \dots, k$ solve

$$(P_0) \qquad q_i^0 = \max_{x \in X} f_i(x).$$

Put $\ell = 1$, $W_1 = \{(q_1^0, \dots, q_k^0)\}$, $E_0 = \emptyset$, $V_0 = W_1$, $E = \emptyset$, $S = \emptyset$.

STEP 2. For $q \in W_\ell \setminus E_{\ell-1}$. Solve

$$(P_{q,e}) \qquad t_q = \max h_{q,e}(f(x)), \text{ subject to } x \in X.$$

where $e = (1, \dots, 1) \in \mathbb{R}^k$.
Compute

$$\begin{aligned} E_\ell &= E_{\ell-1} \cup \{q \in W_\ell \setminus E_{\ell-1} : t_q = 0\} \\ V_\ell &= W_\ell \setminus E_\ell \end{aligned}$$

and set

$$\begin{aligned} S &= S \cup \{x \in X : f(x) = q, q \in E_\ell\} \\ E &= E \cup E_\ell. \end{aligned}$$

STEP 3. If $V_\ell = \emptyset$, stop. Otherwise for $q \in V_\ell$ solve

$$(SP_q) \qquad \begin{array}{ll} \max & \sum_{i=1}^k f_i(x) \\ \text{subject to} & x \in X, f(x) \geq q + t_q e. \end{array}$$

Set

$$\begin{aligned} S &= S \cup \{x \in X : x \text{ solves } (SP_q)\} \\ E &= E \cup \{f(x) : x \text{ solves } (SP_q)\}. \end{aligned}$$

STEP 4. Determine $W_{\ell+1}$ by

$$W_{\ell+1} = \mathrm{Max}(A_{\ell+1})$$

with $A_{\ell+1} = W_\ell^\diamond \cap \{y \in \mathbb{R}_+^k : h_{q,e}(y) \leq t_q, q \in V_\ell\}$.
Put $\ell = \ell + 1$ and return to STEP 2.

We give below some explanations of the algorithm. The set $W_1^\diamond$ (see Step 1) is the E-P hull of the vertex $(q_1^0, \dots, q_k^0) \in \mathbb{R}^k$. It is the first finitely generated E-P hull outer approximation of the set $f(X)^\diamond$. The aim of the subsequent steps is to generate a decreasing (by inclusion) sequence of finitely generated E-P approximations of $f(X)^\diamond$ while generating a set of maximal values of (MOP). In Step 2, one considers a vertex q of W_l. If $q \in [f(X)]^\diamond$, which corresponds to the case $t_q = 0$, then it is a maximal value of (MOP). If q is outside of $[f(X)]^\diamond$, the problem (P_q) will provide a new maximal value to add to the collection E. If all vertices of W_l belong to $[f(X)]^\diamond$, then the E-P hull $W_l^\diamond$ coincides with $[f(X)]^\diamond$ and the algorithm stops. Otherwise, one determines new vertices for a smaller approximation of $f(X)^\diamond$ in Step 4. One may prove that the sequence $(W_l)^\diamond$ approaches the set $[f(X)]^\diamond$ as l tends to ∞, and the collection E approaches the closure of the maximal value set of (MOP).

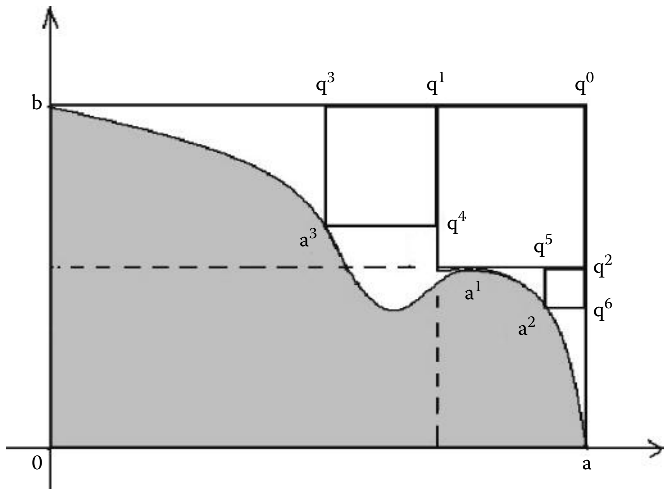

FIGURE 7.21: Construction of A_1, A_2, and A_3.

Example 7.21. Consider the problem (MOP) in which the objective function is the identity function on $\mathbb{R}^2$, and the constraint set is the set A given in Fig. 7.21. We wish to generate the maximal value set, that is, the set of maximal points of A.

The first (nonconvex) polyhedron A_1 approximating A is the box $\left[0aq^0b\right]$, where q^0 is found by solving (P_0). At Step 2, we solve (P_{q^0}), which gives us t_{q^0} and $d = q^0 + t_{q^0}e$, and obtain the nonconvex polyhedron A_2 generated by $W_2 = \left\{q^1, q^2\right\}$. The third nonconvex polyhedron A_3 is the E-P hull of the set $W_3 = \{q^3, q^4, q^5, q^6\}$. The collection E contains the maximal points a^1 after the first iteration, and a^1, a^2, and a^3 after the second iteration.

Reading. For more complete and advanced results on theoretical aspects, applications in real-world situations, and numerical methods of vector optimization, interested readers are referred to the textbooks and monographs cited in the Bibliography.

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Chapter 8

Multi-objective Combinatorial Optimization: Concepts, Exact Algorithms, and Metaheuristics

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8.1 Introduction

Mathematical optimization and operations research have, over a period of almost 75 years, contributed tremendously to progress in engineering, finance, medicine, transportation, to name just a few. Combinatorial optimization, which deals with optimization over discrete sets of objects, has been particularly successful in solving many difficult and large-scale planning problems in real-world applications. However, in many real-world decision-making situations, people have to deal with conflicting objectives that need to be optimized simultaneously. It is therefore evident that multi-objective combinatorial optimization has tremendous potential for contributing to further development of decision support systems built on optimization tools.

But the area of multi-objective optimization is much younger than mathematical optimization in general. The first traces can be found in the 1950s and in the 1970s for multi-objective combinatorial optimization (MOCO). However, progress in MOCO was slow until the 1990s, possibly due to the lack of adequate computing facilities. However, in the last 20 years, the area has seen considerable and increasing growth. It is therefore the right time to provide a text that can serve as an introduction to the field and provide a guideline to the literature for researchers entering the field that will take them from basic concepts to latest developments. This is the aim of this chapter.

The text is organized as follows: Section 8.2 covers basic definitions and properties of multi-objective combinatorial optimization problems. The following sections focus on solution approaches and are organized by problem categories, as the difficulty of their (exact) solution increases. In Section 8.3 we consider problems where algorithms for the solution of single objective optimization can be adapted to the multi-objective case. Next, Section 8.4 covers problems where the extensive use of efficient single-objective algorithms is key to success. For the problems in Section 8.5, single-objective variants may already be complex to solve, and scalarizations of the multi-objective optimization problems may require the use of general integer programming techniques, such as branch and bound. Finally, in Section 8.6 we touch on problems that are currently beyond the capabilities of exact solution methods and present metaheuristic techniques. We conclude with some suggestions for future research in Section 8.7.

Throughout the chapter, the emphasis is on understanding concepts, rather than a rigorous mathematical development. We provide graphics to illustrate the main ideas and list many references that give a starting point for further reading.

8.2 Definitions and Properties

Formally, a multi-objective combinatorial optimization problem can be written as a linear integer program

$$\min\{z(x) = Cx : Ax = b, x \in \{0,1\}^n\}, \tag{8.1}$$

where $x \in \{0,1\}^n$ is a (column) vector of n binary variables $x_j, j = 1, \ldots, n$, $C \in \mathbb{Z}^{p \times n}$ contains the rows c^k of coefficients of p linear objective functions $z_k(x) = c^k x, k = 1, \ldots, p$, and $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ describe m constraints $a^i x = b_i, i = 1, \ldots, m$, where $a^i, i = 1, \ldots, m$ are m row vectors from $\mathbb{R}^n$. The constraints define combinatorial structures such as paths, trees, or cycles in a network or partitions of a set, etc., and it will be convenient to assume that all coefficients, that is, all entries of A, b, and c are integers.

The set $X = \{x \in \{0,1\}^n : Ax = b\}$ is called a feasible set in decision space $\mathbb{R}^n$ and $Y = z(X) = \{Cx : x \in X\}$ is the feasible set in objective space $\mathbb{R}^p$. The set of points dominated by any point on the "lower left" boundary of the convex hull of Y, sometimes called the Edgeworth-Pareto hull of Y, and defined as $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$, is very important. Figure 8.1 shows both Y as a finite set of circles and $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$ as a shaded area for a MOCO problem with two objectives.

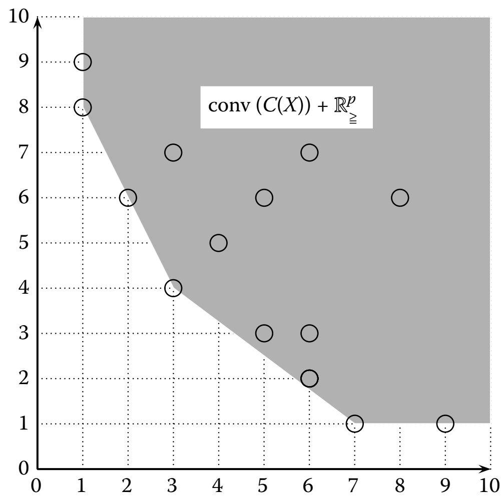


FIGURE 8.1: Feasible set and Edgeworth-Pareto hull.

For two vectors $y^1, y^2 \in \mathbb{R}^p$ we use the notation $y^1 \leqq y^2$ if $y^1_k \leqq y^2_k$ for $k = 1, \ldots, p$; $y^1 \leq y^2$ if $y^1 \leqq y^2$ but $y^1 \neq y^2$, and $y^1 < y^2$ if $y^1_k < y^2_k$ for $k = 1, \ldots, p$. Because MOCO problem (8.1) is an optimization problem over vectors, the meaning of the min operator needs to be defined. The individual minimizers of the p objective functions $z_k(x)$, that is, feasible solutions $\hat{x} \in X$ such that $z_k(\hat{x}) \leqq z_k(x)$ for all $x \in X$ for some $k \in \{1, \ldots, p\}$, do only minimize a single objective and may in fact be bad choices as solutions to MOCO problem (8.1), in particular if they are not unique; see Figure 8.2. More important are the lexicographic optima, that is, feasible solutions $\hat{x}$ such that $z(\hat{x}) \leqq_{lex} z(x)$ for all $x \in X$, and more generally, $z^\pi(\hat{x}) \leqq_{lex} z^\pi(x)$ for all $x \in X$ and some permutation $z^\pi = (z_{\pi(1)}, \ldots, z_{\pi(p)})$ of the components of the vector-valued function $z = (z_1, \ldots, z_p)$. Figure 8.2 shows non-unique individual minima for z_1 and z_2 (filled circles). Among those, $\hat{y}^1 = (1,8)^T$ and $\hat{y}^2 = (7,1)^T$ are lexicographically minimal for the permutation of objectives (z_1, z_2) and (z_2, z_1), respectively. Note that y^1 is lexicographically smaller than y^2 with respect to permutation π if there is some $k \in \{1, \ldots, p\}$ such that $y^1_{\pi(i)} = y^2_{\pi(i)}$ for $i = 1, \ldots k-1$ and $y^1_{\pi(k)} < y^2_{\pi(k)}$. It is also clear from Figure 8.2 that a point combining the minimal values for all objectives, called the ideal point and denoted y^I, that is, $(1,1)^T$ in Figure 8.2, does usually not belong to Y or even $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. Hence, an ideal solution x^I such that $z_k(x^I) \leqq z_k(x)$ for all $x \in X$ and all $k = 1, \ldots, p$ does not exist.

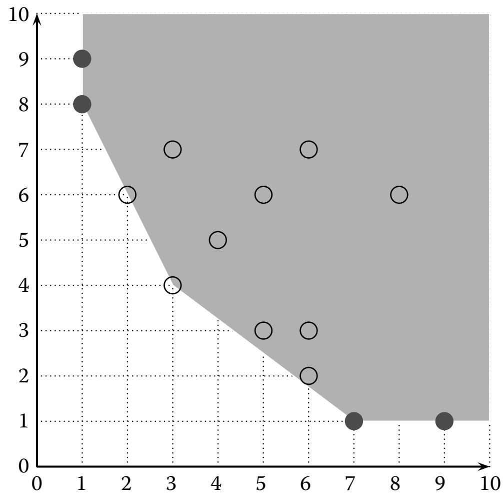


FIGURE 8.2: Individual and lexicographic minima.

While individual and lexicographic optima refer to total (pre)orders on $\mathbb{R}^p$, multi-objective optimization is based on the concept of efficiency or Pareto optimality. It is defined using partial (pre)orders based on componentwise comparison of vectors. A feasible solution $\hat{x} \in X$ belongs to the set of weakly efficient solutions X_{wE} if there is no $x \in X$ with $z(x) < z(\hat{x})$. In that case, $z(\hat{x})$ is called weakly non-dominated. We denote $Y_{wN} := z(X_{wN})$ as the set of all weakly non-dominated points. Finally, $\hat{x} \in X_E$, the set of efficient solutions, if there is no feasible x with $z(x) \leq z(\hat{x})$. The objective vector $z(\hat{x})$ is non-dominated and $Y_N := z(X_E)$ is the set of non-dominated points.

An efficient solution is supported if there is some $\lambda > 0$ such that $\lambda^T C\hat{x} \leqq \lambda^T Cx$ for all $x \in X$. If $C\hat{x}$ is an extreme point of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$, it is an extreme efficient solution, and $C\hat{x}$ is an extreme non-dominated point. We let X_{SE1} be the set of extreme efficient solutions and let X_{SE2} denote the set of supported efficient solutions such that $C\hat{x}$ is in the relative interior of a face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. All supported efficient solutions are $X_{SE} = X_{SE1} \cup X_{SE2}$. Finally, $X_{NE} = X_E \setminus X_{SE}$ is the set of non-supported efficient solutions, i.e., $\hat{x}$ such that $C\hat{x}$ is in the interior of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. It is very easy to construct small examples that show that even the bi-objective shortest path, spanning tree, and assignment problems have non-supported efficient solutions. Figure 8.3 shows (weakly)

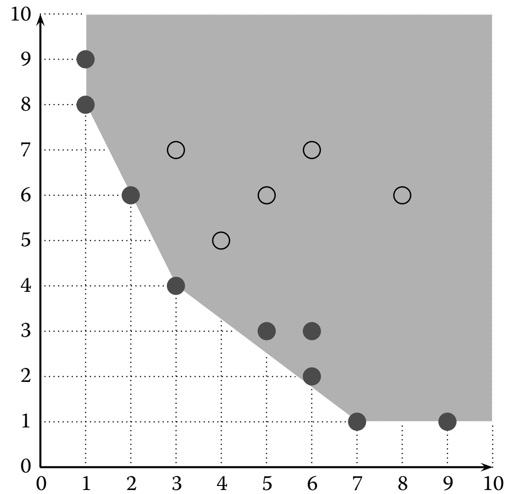


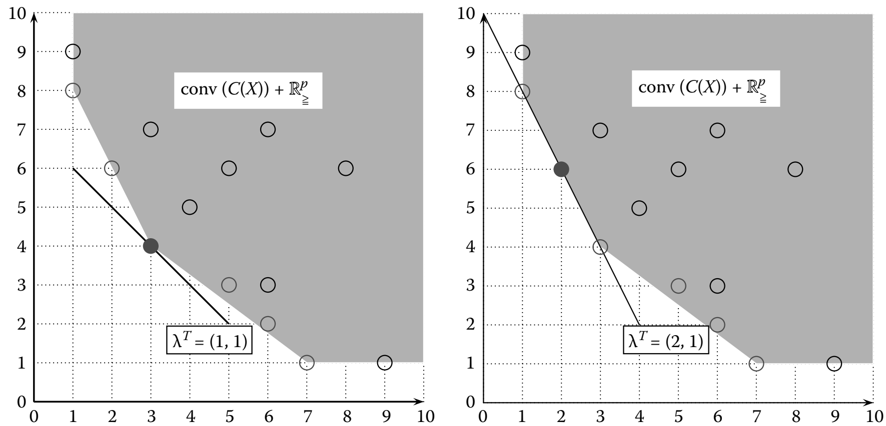
FIGURE 8.3: (Weakly) non-dominated points.

non-dominated points as filled circles. Note that $(1,9)^T$, $(6,3)^T$, and $(9,1)^T$ are weakly non-dominated but not non-dominated. Moreover, $(5,3)^T$ and $(6,2)^T$ are non-supported. Figure 8.4 illustrates supported non-dominated points. Figure 8.4(a) shows that $(3,4)^T$ is an extreme non-dominated point and Figure 8.4(b) that $(2,6)^T$ is a supported non-dominated point in the relative interior of a face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$.

Following [31] we call $x^1, x^2 \in X_E$ equivalent if $Cx^1 = Cx^2$. A complete set of efficient solutions is $\hat{X} \subset X_E$ such that for all $y \in Y_N$ there is some $x \in \hat{X}$ with $z(x) = y$. A minimal complete set contains no equivalent solutions; the maximal complete set X_E contains all equivalent solutions. We can now speak about, for example, a minimal complete set of extreme efficient solutions. This classification allows us to precisely describe what is to be understood by statements that a certain algorithm "solves" a certain MOCO problem. We shall always understand solving a MOCO problem as finding all non-dominated points, and for each $y \in Y_{ND}$, one x such that $Cx = y$, that is, finding a minimal complete set.

Considering computational complexity, MOCO problems are hard. The decision problem related to optimization problem (8.1) is: Given $d \in \mathbb{Z}^p$, does there exist $x \in X$ such that $Cx \leqq d$? and the associated counting problem is: Given $d \in \mathbb{Z}^p$, how many $x \in X$ satisfy $Cx \leqq d$? We are also interested in knowing how many efficient solutions (non-dominated points) may exist in the worst case.

We present an argument of [11] and use the binary knapsack problem (Given $a^1, a^2 \in \mathbb{Z}^n$ and $b_1, b_2 \in \mathbb{Z}$, does there exist $x \in \{0,1\}^n$ such that $(a^1)^T x \leqq b_1$ and $(a^2)^T x \geqq b_2$?), which is known to be NP-complete and



(a) Extreme non-dominated point for $\lambda^T = (1, 1)$.

(b) Supported non-dominated point in the relative interior of a face for $\lambda^T = (2, 1)$.

FIGURE 8.4: Supported non-dominated points.

#P-complete [36, 81] to show that the most basic MOCO problem, called the unconstrained MOCO problem, is hard. The unconstrained bi-objective combinatorial optimization problem is

$$\min \left\{ \sum_{i=1}^{n} c_i^k x_i \text{ for } k = 1, 2 : x_i \in \{0, 1\} \text{ for } i = 1, \dots, n \right\}. \tag{8.2}$$

The decision problem "Does there exist $x \in \{0, 1\}^n$ such that $c^1 x \leqq d_1$ and $c^2 x \leqq d_2$?" has a parsimonious transformation to the binary knapsack problem, which is defined by $c^1 := a^1$, $d_1 = b_1$, $c^2 := -a^2$, $d_2 := -b_2$, showing NP- and #P-completeness of the unconstrained MOCO problem in Equation (8.2). Furthermore, setting $c_i^k := (-1)^k 2^{i-1}$, it holds that $Y = Y_N$, showing that an exponential number of non-dominated points may exist.

Similar results are known for the bi-objective shortest path problem [31, 70], the assignment problem [50, 70], the spanning tree problem [29]c, and the network flow problem [66]. Although intractability implies that X_E and even Y_{SN} can be exponential in the size of the instance, numerical tests reveal that the number of non-dominated points is often small, particularly in real-world applications. This has been observed by [63] and [48] for randomly generated and real-world instances of the shortest path problem and by [58] for randomly generated instances of the bi-objective assignment problem.

Empirically, it is also often observed that $|X_{NE}|$ grows exponentially with instance size, whereas $|X_{SE}|$ grows polynomially with instance size. It is clear that the numerical values of objective function coefficients in C play an important role, as does the structure of the instance; for example, in the multi-objective shortest path problem, grid graphs have many more efficient paths than NETGEN [39] generated ones [63]. This issue is poorly understood today.

8.3 Two Easy Problems: Multi-objective Shortest Path and Spanning Tree

In this section, we shall discuss algorithms to solve a first class of multi-objective combinatorial optimization problems, which we shall call easy problems. For such problems, efficient polynomial time algorithms exist to solve their single objective versions, and it is possible to extend these algorithms to deal with the multi-objective versions. The prime examples are the multi-objective shortest path and spanning tree problems. The algorithms to solve these problems show that dynamic programming and greedy algorithms can be applied to multi-objective optimization problems with any number of objectives.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a directed graph defined by node set $\mathcal{V}$ and arc set $\mathcal{A}$ with p arc costs $c_{ij}^k, k = 1, \ldots, p$ on arcs $(i, j) \in \mathcal{A}$. The multi-objective shortest path problem is to find efficient paths from an origin node s to a destination node t or to all other nodes of $\mathcal{G}$. In the single objective case $p = 1$, label-setting algorithms, such as Dijkstra's algorithm [8], or label-correcting algorithms, such as Bellman's algorithm [3], are well-known polynomial time algorithms. We will provide generalizations of these for multi-objective shortest path problems.

Label-setting algorithms rely on the following fact. Assuming that all $c_{ij}^k \geqq 0$, let P_{st} be an efficient path from s to t. Then any subpath P_{uv} from u to v, where u and v are vertices on P_{st}, is an efficient path from u to v. Notice that, on the other hand, concatenations of efficient paths need not be efficient. This principle of optimality shows that the multi-objective shortest path problem is an example of multi-objective dynamic programming and implies that generalizations of both Dijkstra's and Bellman's algorithms are possible. Such algorithms have vector-valued labels and therefore, due to the partial orders used, need to maintain sets of non-dominated labels at each node rather than single labels.

For a label-setting algorithm, lists of permanent and temporary labels are stored and it is necessary to ensure that a permanent label defines an efficient path from s to the labeled node. This can be done by selecting the lexicographically smallest label from the temporary list to become permanent. A label-setting algorithm then follows the same steps as in the single-objective case, except that newly created labels need to be compared with label sets.

New labels dominating existing labels lead to the deletion of those dominated labels. An existing label dominating a new label results in the new label being discarded, otherwise the new label is added to the label list. An interesting feature of the multi-objective shortest path problem is that, in contrast to the single-objective one, it is not possible to terminate a multi-objective label-setting algorithm once the destination node t has been reached, since further unprocessed temporary labels at nodes in $\mathcal{V} \setminus \{s, t\}$ may lead to the detection of more efficient paths from s to t, another effect of the partial order. Nevertheless, any label at t can serve as an upper bound and temporary labels at any intermediate node dominated by such an upper bound can be eliminated.

Just as in the single objective case, label-setting algorithms fail if negative arc lengths are permitted. In the multi-objective case, the following situations may occur: If there is a cycle C with $\sum_{(i,j)\in C} c_{ij} \leq 0$, there is no efficient path; if there is a cycle C with $\sum_{(i,j)\in C} c_{ij}^k < 0$ and $\sum_{(i,j)\in C} c_{ij}^l > 0$ for some k and some $l \neq k$, there are infinitely many efficient paths as every pass of the cycle reduces one objective and increases another, thereby creating one more efficient path every time. In the presence of negative-arc-length label-correcting algorithms are needed. Once again, one proceeds by processing the labels as in the single-objective case, keeping in mind that all newly created labels need to be compared with existing label sets so that all dominated labels can be eliminated. Of course, no label is permanent until termination of the algorithm. For details on a variety of multi-objective shortest-path problems, including pseudocode and numerical results, the reader is referred to [63].

Another problem that we discuss is the multi-objective spanning-tree problem. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with edge set E and edge costs $c_{ij}^k, k = 1, \ldots, p$ for all edges $[i, j] \in \mathcal{E}$, its aim is to find efficient spanning trees of $\mathcal{G}$, that is, spanning trees for which the cost vectors are non-dominated. The following theorem is the foundation for greedy algorithms that generalize Prim's [55] and Kruskal's [40] algorithms.

Theorem 8.1. [29] Let T be an efficient spanning tree of $\mathcal{G}$. The following assertions hold.

(a) Let $e \in \mathcal{E}(T)$ be an edge of T. Let $(\mathcal{V}(T_1), \mathcal{E}(T_1))$, and $(\mathcal{V}(T_2), \mathcal{E}(T_2))$ be the two connected components of $\mathcal{G} \setminus \{e\}$. Let $C(e) := \{f = [i, j] \in \mathcal{E} : i \in \mathcal{V}(T_1), j \in \mathcal{V}(T_2)\}$ be the cut defined by deleting e. Then, $c(e) \in \min\{c(f) : f \in C(e)\}$.

(b) Let $f \in \mathcal{E} \setminus \mathcal{E}(T)$ and let $P(f)$ be the unique path in T connecting the end nodes of f. Then, $c(f) \leq c(e)$ does not hold for any $e \in P(f)$.

The first statement of Theorem 8.1 shows that starting from a single node and adding efficient edges between nodes that are already included and nodes that a not yet included in the tree will eventually construct all efficient trees. Notice that several such edges might exist, and hence in every iteration there will be a set of efficient partial trees. As in the multi-objective shortest-path problem, adding edges to an efficient partial tree does not necessarily lead to another efficient partial tree, but adding efficient edges to a dominated partial tree may yield an efficient partial tree. It is therefore necessary to filter out dominated trees at termination of the algorithm. In a similar way, the second statement provides a justification for a Kruskal-like algorithm for the multi-objective spanning-tree problem.

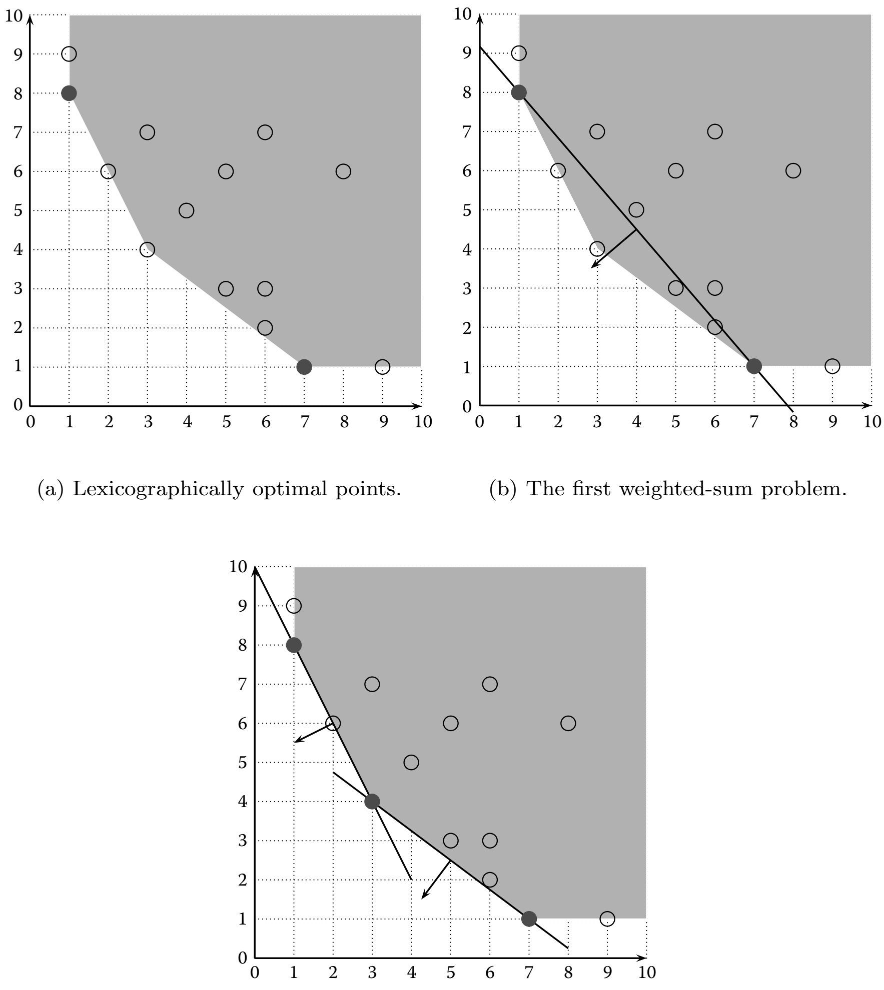
8.4 Nice Problems: The Two-Phase Method

In this section we consider a class of multi-objective optimization problems that we shall call nice. As in the previous section, these are problems for which efficient polynomial time algorithms to solve the single-objective version are known. Considering the hardness of even the "simplest" bi-objective combinatorial optimization problems as discussed in Section 8.2, this indicates that there is benefit in solving the single-objective version as a subproblem even if that has to be done often, so that the polynomial time algorithms can be exploited as much as possible. We shall describe the two-phase method, originally proposed by [76], as a general framework to do this. A detailed exposition of the method can be found in [61].

8.4.1 The Two-Phase Method for Two Objectives

We first explain the two-phase method for MOCO problems with two objectives. In Phase 1, at least a (minimal) complete set of extreme efficient solutions X_{SE1} is found. This is typically done starting from two lexicographically optimal solutions. The method then recursively calculates a weight vector $\lambda^T = (\lambda_1, \lambda_2) > 0$ as a normal to the line connecting two currently known non-dominated points and solves a weighted-sum problem $\min\{\lambda^T Cx : x \in X\}$. In Figure 8.5(a), lexicographic minima $(1,8)^T$ and $(7,1)^T$ are identified and $\lambda^T = (8-1, 7-1) = (7,6)$ is defined. The corresponding weighted-sum problem yields a non-dominated point $(3,4)^T$ (Figure 8.5(b)) and allows us to divide the problem in two: The first weighted-sum problem with $\lambda^T = (8-4, 3-1) = (4,2)$ looks for new non-dominated extreme points between $(1,8)^T$ and $(3,4)^T$. The second, with $\lambda^T = (4-1, 7-3) = (3,4)$, is used to explore the area between $(3,4)^T$ and $(7,1)^T$. Neither finds any further non-dominated extreme points in the example. The first, however, may or may not find supported (but non-extreme) non-dominated points $(2,6)^T$, depending on the solver (Figure 8.5(c)).

In Phase 2, any other efficient solutions (non-dominated points) are determined, in particular, non-supported efficient solutions X_{NE}. After Phase 1, it is possible to restrict the search for non-dominated points to triangles defined by consecutive non-dominated extreme points. In the literature, several methods have been proposed to achieve that. Neighborhood search, as suggested,



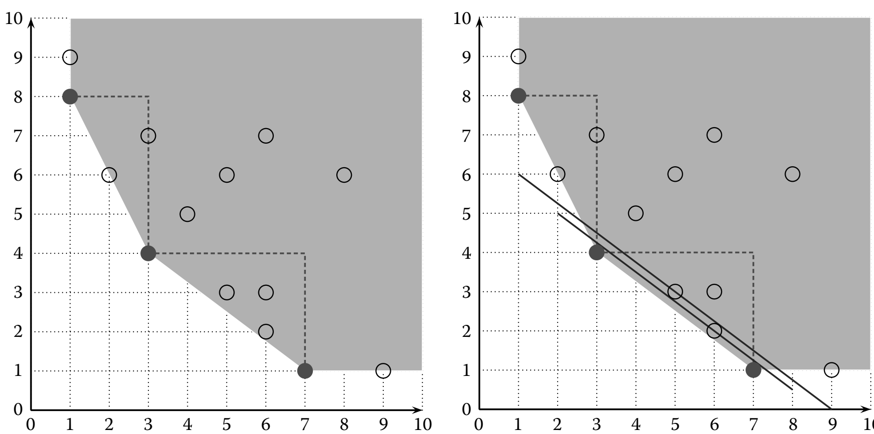
(a) Lexicographically optimal points.

(b) The first weighted-sum problem.

(c) Dividing the problem.

FIGURE 8.5: Phase 1 of the two-phase method.

for example, in [42, 69], is in general wrong, because efficient solutions are in general not connected via a neighborhood structure [15]. One can apply constraints to restrict objective values to triangles and modify those constraints as further points are discovered. However, the computational effort may be high, in particular because problems tend to get harder when adding further (knapsack type) constraints. This idea is therefore contrary to the spirit of the two-phase method. Variable fixing strategies as suggested by [76] can be a reasonable alternative. However, currently the best-performing two-phase algorithms are those that exploit a ranking algorithm that generates solutions of single objective (weighted sum) problems in order of their objective value (Figure 8.6(a)). The ranking method has been successfully applied for a number of bi-objective problems, namely bi-objective assignment [58], multi-modal assignment [54], spanning tree [73], shortest path [63], network flow [64], and binary knapsack [35]. It is important to note that for all these problems efficient ranking algorithms to find r-best solutions of the single-objective version of the problem are available, references to which can be found in the original papers.



(a) The triangles where non-supported non-dominated points may be located.

(b) Ranking non-supported non-dominated points.

FIGURE 8.6: Phase 2 of the two-phase method.

The ranking algorithm is applied to weighted-sum problems corresponding to each triangle (see Figure 8.6(a)). Let $\lambda > 0$ be a weight vector defined as the normal to the line connecting two consecutive non-dominated extreme points, for example, $\lambda^T = (4, 2)$ (left triangle) and $\lambda^T = (3, 4)$ (right triangle) in Figure 8.6(a). The ranking algorithm then finds second, third, etc. best solutions for the problem of minimizing $\lambda^T Cx$, in increasing order of objective values for $\lambda^T Cx$. The two non-dominated extreme points defining the triangle (and λ) are clearly optimal solutions for this problem.

This process stops once a solution the (weighted sum) objective function value of which is worse than the value of the third corner of the triangle. In Figure 8.6(b), with $\lambda^T = (3, 4)$, the algorithm finds $(6, 2)^T$ with objective value 26, then $(5, 3)^T$ with objective value 27. Both points are non-dominated. The 4th best point is $(6, 3)^T$, which is dominated by both previous points. The algorithm can now be stopped, because any further point would also be dominated. This indicates that the enumeration of solutions can be stopped even before the search has left the triangle.

In order to facilitate this process, upper bounds on the weighted sum objective value for any efficient solution in the triangle can be derived. Let $\{x^i : 0 \leqq i \leqq q\}$ be feasible solutions with $z(x^i) \in \Delta(x^0, x^q)$, the triangle defined by $z(x^0)$ and $z(x^q)$, sorted by increasing value of z_1, where $z(x^0)$ and $z(x^q)$ are two consecutive non-dominated extreme points of conv(Y). Let

$$\beta_0 := \max_{i=0}^{q-1}\{\lambda_1 z_1(x^{i+1}) + \lambda_2 z_2(x^i)\}, \tag{8.3}$$

$$\beta_1 := \max\left\{\max_{i=1}^{q-1}\{\lambda^1 z^1(x^i) + \lambda^2 z^2(x^i)\},\right.$$
$$\left.\max_{i=0}^{q}\{\lambda^1(z^1(x^i) - 1) + \lambda^2(z^2(x^{i-1}) - 1)\}\right\}, \tag{8.4}$$

$$\beta_2 := \max_{i=0}^{q}\{\lambda^1(z^1(x^i) - 1) + \lambda^2(z^2(x^{i-1}) - 1)\}. \tag{8.5}$$

Then $\beta_0 \geqq \beta_1 \geqq \beta_2$ are upper bounds for the weighted sum objective value of any non-dominated point in $\Delta(x^0, x^q)$. The ranking process can be stopped as soon as bound β_j is reached. The use of β_0 and β_1 from Equations (8.3) and (8.4) allows computation of the maximal complete set, whereas β_2 guarantees a minimal complete set only. Notice that these bounds will be improved every time a new efficient solution is found. In Figure 8.6(b), the initial bound β_2 using Equation (8.5) with $\{y^0 = (3, 4)^T, y^1 = (7, 1)^T\}$ is 30, but once $(6, 2)^T$ is found, this improves to 27, and with $(5, 3)^T$ also discovered, to 24. Clearly no additional feasible point in the triangle can be better than that bound.

The two-phase method involves the solution of enumeration problems. In order to find a maximal complete set one must, of course, enumerate all optimal solutions of $\min_{x\in X} \lambda^T Cx$ for all weighted-sum problems solved in Phase 1 and enumerate all $x \in X_{NE}$ with $Cx = y \in Y_{ND}$ for all non-supported non-dominated points y. But even to compute a minimal complete set, enumeration is necessary to find X_{SE2}. There can indeed be many optimal solutions of $\min_{x\in X} \lambda^T Cx$ that are non-extreme and not equivalent to one another (see, e.g., points $(1, 8)^T, (2, 6)^T, (3, 4)^T$ in Figure 8.5(c)).

As an example of a very efficient implementation of a two-phase method, [58] developed a two-phase algorithm for the bi-objective assignment problem using the Hungarian method [41] to solve weighted-sum assignment problems, an enumeration algorithm by [19] to enumerate all optimal solutions of these problems, and a ranking of (non-optimal) solutions of $\min_{x \in X} \lambda^T Cx$ by [4]. The algorithm outperformed a two-phase method using variable fixing, a two-phase method using a heuristic to find a good feasible solution before Phase 2, general exact MOCO algorithms, and CPLEX using constraints on objectives. An explanation for the good performance of the method is given by the distribution of objective function values for randomly generated instances. The objective values of an AP with $c_{ij} \in \{0, \dots, r-1\}$ are asymptotically normally distributed with $\mu = \frac{n(r-1)}{2}, \sigma^2 = \frac{n(r^2-1)}{12}$, as shown in [56].

8.4.2 The Two-Phase Method for Three Objectives

Extending the two-phase method to three objectives is not trivial. In Phase 1, a weight vector defines the normal to a plane. While three non-dominated points are sufficient to calculate such a weight vector, there may be up to six different lexicographically optimal points, so it is unclear which points to choose for calculating weights to start with. Moreover, even if the minimizers of the three objective functions are unique, the normal to the plane defined by three non-dominated points may not be positive. In this case no further non-dominated points would be calculated; see an example presented in [59].

Hence, a direct generalization of the two-phase method already fails with the initialization. Two generalizations of Phase 1 have been proposed by [59] and [52]. We present the ideas of [59], where Phase 1 relies on decomposition of the simplex of all non-negative normalized weights

$$W^0 := \left\{ \lambda : \lambda_1 > 0, \dots, \lambda_p > 0, \lambda_p = 1 - \sum_{k=1}^{p-1} \lambda_k \right\} \tag{8.6}$$

into subsets

$$W^0(y) := \{\lambda \in W^0 : \lambda^T y \leqq \lambda^t Y' \text{ for all } y' \in Y\} \tag{8.7}$$

of W^0 consisting of all weight vectors λ such that y is a point minimizing $\lambda^T y$ over Y. It turns out that y is a non-dominated extreme point if and only if $W^0(y)$ has dimension $p-1$. This allows us to define adjacency of non-dominated extreme points as follows. Non-dominated extreme points y^1 and y^2 are adjacent if and only if $W^0(y^1) \cap W^0(y^2)$ is a polytope of dimension $p-2$, which then makes it possible to derive the optimality condition of Theorem 8.2.

Theorem 8.2. [59] If S is a set of supported non-dominated points, then

$$Y_{SN1} \subseteq S \quad \Leftrightarrow \quad W^0 = \bigcup_{y \in S} W^0(y). \tag{8.8}$$

The results above lead to a new Phase 1 algorithm. Let S be a set of supported non-dominated points. Let $W_p^0(y) = \{\lambda \in W^0 : \lambda^T y \leqq \lambda^T y^* \text{ for all } y^* \in S\}$. Then $W^0(y) \subseteq W_p^0(y)$ for all $y \in S$ and $W^0 = \bigcup_{y \in S} W_p^0(y)$. The algorithm initializes S with the lexicographically optimal points. While S is not empty, it chooses $\hat{y} \in S$, computes $W_p^0(\hat{y})$ and investigates all facets F of $W_p^0(\hat{y})$ defined by $\lambda^T \hat{y} = \lambda^T y'$ for $y' \in S$ to determine whether F is also a facet of $W^0(\hat{y})$. If $\hat{y}$ minimizes $\lambda^T y$ for all $\lambda \in F$ then $\hat{y}$ and y' are adjacent and F is the common face of $W^0(\hat{y})$ and $W^0(y')$. If there are $y^* \in Y$ and $\lambda \in F$ such that $\lambda^T y^* < \lambda^T y$, then $W^0(\hat{y})$ is a proper subset of $W_p^0(\hat{y})$, y^* is added to S and $W_p^0(\hat{y})$ is updated.

At the end of Phase 1, all non-dominated extreme points of Y and a complete set of extreme efficient solutions is known. Any other supported efficient solutions must be optimal solutions to weighted sum problems with λ belonging to 0- and 1-dimensional faces of some $W^0(y)$ of some non-dominated extreme point y. To find these, it is enough to find all optimal solutions of weighted-sum problems with one λ chosen from each common boundary of two weight sets $W^0(y^1)$ and $W^0(y^2)$ plus the intersection points of three weight sets. To find non-supported non-dominated points, one can once again employ a ranking algorithm for weighted-sum problems, where λ is chosen to be a normal to a facet defining hyperplane of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$. Note that this is analogous to the bi-objective case. The difficult part in the completion of the Phase 2 algorithm is the computation of good upper bounds and the selection of weights in a way that keeps the ranking of solutions as limited as possible. Here the difficulty arises from the fact that unlike in the case of two objectives, the area where non-supported non-dominated points can be found does not decompose into disjoint subsets. Details about Phase 2 for multi-objective combinatorial optimization problems can be found in [60], where its generalization to more than three objectives via a recursive scheme is also discussed.

Numerical results in [57] show that the method outperforms three general methods to solve MOCO problems by a factor of up to 1000 on the three-objective assignment problem. The biggest advantage of the two-phase method is that it respects problem structure, thereby enabling use of efficient algorithms for single-objective problems as much as possible.

8.5 Difficult Problems: Scalarization and Branch and Bound

In Sections 8.3 and 8.4 we considered MOCO problems for which the case $p = 1$ is a polynomially solvable single-objective optimization problem. It is now time to give up this assumption and consider problems whose single-objective counterpart is NP-hard. Since many such problems are solvable in practice despite their theoretical NP-hardness, that is, they are solvable in reasonable time, in particular when real-world applications are involved, it makes sense to first consider how far we can get when considering scalarized versions. In doing so, we give up the idea of considering only scalarizations that yield single-objective problems of the same structure as the MOCO problem for $p = 1$.

8.5.1 Scalarization

The idea of scalarization is to convert a multi-objective optimization problem to a (parameterized) single-objective problem that is solved repeatedly with different parameter values. We are interested in the following desirable properties of scalarizations [82]. *Correctness* requires that an optimal solution of the scalarized problem is (at least weakly) efficient. A scalarization method is *complete* if all efficient solutions can be found by solving a scalarized problem with appropriately chosen parameters. For *computability* it is important that the scalarization is not harder than a single-objective version of the MOCO problem. This relates to theoretical computational complexity as well as computation time in practice. Furthermore, since MOCO problem (8.1) has linear constraints and objectives, the scalarization should have a linear formulation.

The scalarization techniques that are most often applied in MOCO, illustrated in Figure 8.7, are the weighted sum method (Figure 8.7(a))

$$\min \left\{ \lambda^T z(x) : x \in X \right\}, \tag{8.9}$$

the ε-constraint method (Figure 8.7(b))

$$\min \left\{ z_l(x) : z_k(x) \leq \varepsilon_k, k \neq l, x \in X \right\}, \tag{8.10}$$

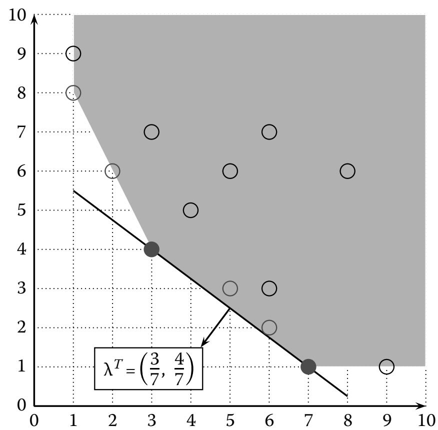
and the weighted Chebychev method (Figure 8.7(c))

$$\min \left\{ \max_{k=1,\dots,p} \nu_k (z_k(x) - y_k^I) : x \in X \right\}. \tag{8.11}$$

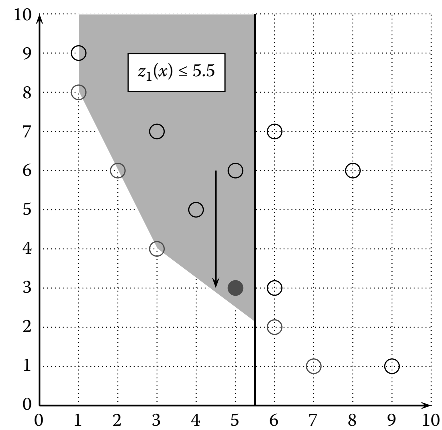
In Table 8.1 we summarize the properties of the methods listed in Equations (8.9)–(8.11).

TABLE 8.1: Properties of popular scalarization methods.

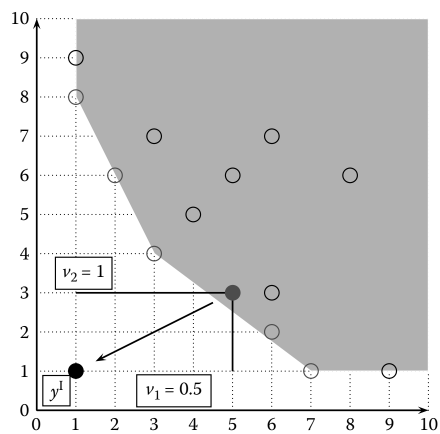
Scalarization	Correct	Complete	Computable	Linear
Weighted sum	+	−	+	+
ε-constraint	+	+	−	+
Chebychev	+	+	−	+



(a) The weighted-sum scalarization.



(b) The ε-constraint scalarization.



(c) The Chebychev scalarization.

FIGURE 8.7: Popular scalarization methods.

All three scalarized problems clearly have a linear formulation (with integer variables, of course). They do also all compute (weakly) efficient solutions, see, for example, [11] for proofs. As for completeness, it follows from the definition of non-supported non-dominated points, that the weighted-sum scalarization cannot compute any non-supported efficient solution. On the other hand, both the ε-constraint scalarization and the Chebychev scalarization can be tuned to find all efficient solutions. For proofs, we again refer the reader to, for example, [11]. In terms of computability, we have seen before that the weighted-sum scalarization (8.9) does maintain the structure of the single-objective variant of MOCO problem (8.1), and it is therefore solvable with the same computational effort. On the other hand, the ε-constraint method (8.10) and the (linearized version of the) Chebychev scalarization (8.11) both include bounds on objective function values. Such additional constraints usually make single-objective versions of MOCO problem (8.1) harder to solve, because they destroy the structure of the problem, which is exploited in efficient algorithms for their solution.

In [10] it was shown that all three scalarizations (and several others) are special cases of the more general formulation

$$\min_{x \in X} \left\{ \max_{k=1}^{p} \left[\nu_k (c_k x - \rho_k)\right] + \sum_{k=1}^{p} \left[\lambda_k (c_k x - \rho_k)\right] : c_k x \leqq \varepsilon_k, k = 1, \dots, p \right\}, \tag{8.12}$$

where ν and λ denote (non-negative) weight vectors in $\mathbb{R}^p$, $\rho \in \mathbb{R}^p$ is a reference point, and scalars ε_k represent bounds on objective function values. To see this, set $\nu_k = 0, \rho_k = 0$, and $\varepsilon_k = M$ for all $k = 1, \dots, p$ and sufficiently large M to obtain the weighted-sum problem (8.9); $\nu_k = 0$ for $k = 1, \dots, p$, $\lambda_l = 1, \lambda_k = 0$ for all $k \neq l$ and $\rho_k = 0$ for all $k = 1, \dots, p$ for the ε-constraint scalarization (8.10); and finally, $\rho = y^I, \lambda = 0$, $\varepsilon_k = M$ for the Chebychev scalarization (8.11). With regard to the general scalarization in Equation (8.12), we cite the following result.

Theorem 8.3. [10]

(a) The general scalarization (8.12) is correct, complete, and NP-hard.

(b) An optimal solution of the Lagrangean dual of the linearized general scalarization is a supported efficient solution of the MOCO problem (8.1).

Theorem 8.3 shows that the general scalarization will be difficult to solve and that solving it by Lagrangian relaxation is not useful. Moreover, Table 8.1 indicates that complete scalarizations are not computable, whereas computable ones are not complete. To resolve this dilemma, one may ask whether it is possible to come up with a compromise, that is, a scalarization that falls somewhere between the weighted-sum scalarization and the ε-constraint/Chebychev scalarization, combining their strengths and eliminating their weaknesses. This is indeed possible with the elastic constraint scalarization. This scalarization is derived from the ε-constraint scalarization, but allows the constraints on objective function values to be violated, with the violation penalized in the objective function. Formally, the elastic constraint scalarization is defined as in Equation (8.13):

$$\min \left\{ c_l x + \sum_{k \neq l} \mu_k w_k : c_k x + v_k - w_k = \varepsilon_k, k \neq l; \; x \in X; \; v_k, w_k \geq 0, k \neq l \right\}. \tag{8.13}$$

The constraints on objective values in the ε-constraint scalarization are turned into equality constraints by means of slack and surplus variables v_k and w_k. Positive values of w_k indicate constraint violations in the ε-constraint scalarization and are penalized with a contribution of penalty parameter μ_k in the objective function. Figure 8.8 compares these two scalarizations. Figure 8.8(a) repeats Figure 8.7(b), where the vertical line indicates a hard constraint on $z_1(x) \leq 5.5$ and the arrow indicates minimization of z_2. The optimal point is indicated by a filled circle. Figure 8.8(b) shows that points to the right of the vertical line are now feasible, with lighter shading indicating greater violation of the limit of ε on z_1. To the right of the vertical line, for every feasible point of the MOCO problem, an arrow indicates the objective value of the same point in Equation (8.13). Note that the optimal point for Equation (8.13) in this example is now to the right of the vertical line. Theorem 8.4 summarizes the properties of the elastic constraint scalarization.

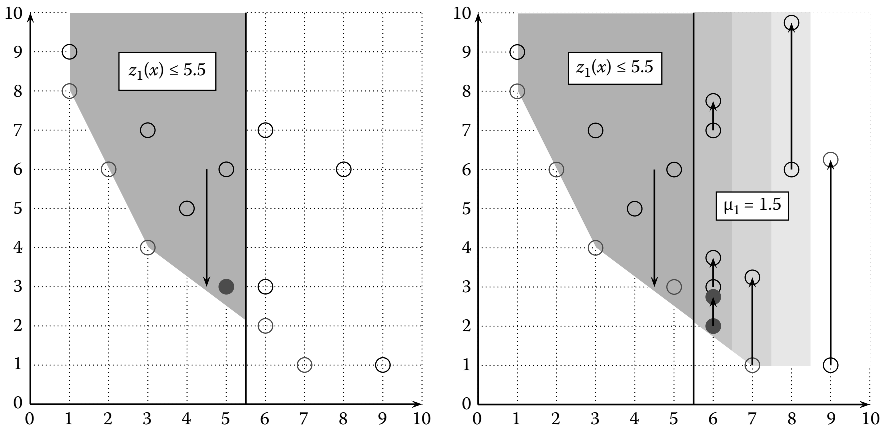
Theorem 8.4. [17] The method of elastic constraints is correct and complete. It contains the weighted-sum and ε-constraint methods as special cases.

For a proof of the first part we refer to [17]. The second part follows by first setting $\varepsilon_k < \min_{x \in X} c^k x$ for $k \neq l$. Then $v_k = 0$ for all feasible solutions $x \in X$ and hence $w_k = c^k x - \varepsilon_k$, and the problem reduces to the weighted-sum problem. Secondly, we can set $\mu_k = M$ for a sufficiently large number M. Then any feasible point with $y_k > \varepsilon_k$ will contribute so much penalty to the objective function of elastic constraint scalarization (8.13) that it is not optimal. Hence, scalarizations (8.13) and (8.10) will have the same set of optimal solutions.

Although the elastic constraint scalarization (8.13) is NP-hard, it is often solvable in practice reasonable time because it respects problem structure better than the ε-constraints of Equation (8.10). It limits the damage done by adding hard ε-constraints to the model. A successful implementation of the method for an application to bi-objective set partitioning problems has been reported in [16].

8.5.2 Multi-objective Branch and Bound

Branch and bound is a standard method to solve single-objective combinatorial optimization problems and is contained in any textbook on combinato-



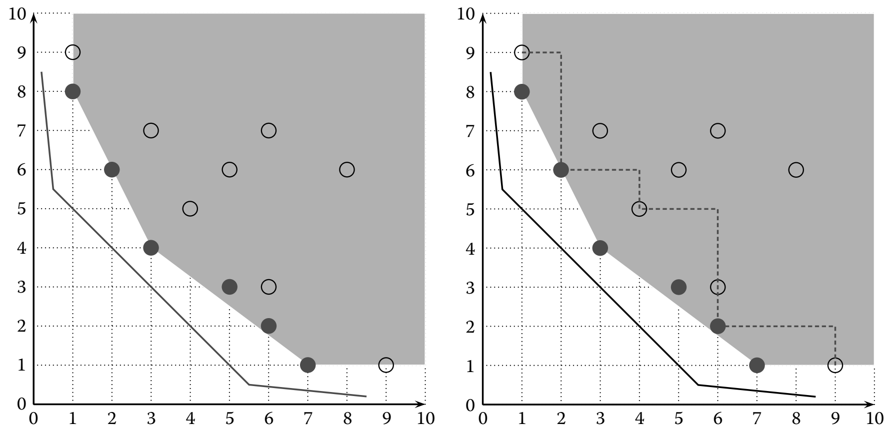
(a) The ε-constraint scalarization.

(b) The elastic constraint scalarization.

FIGURE 8.8: Comparison of the ε- and elastic constraint scalarizations.

rial optimization; see for example [49]. In order to apply it to multi-objective combinatorial optimization problems, we need to define the branching and the bounding parts of the algorithm. We first notice that the branching is done with respect to variables, i.e., the feasible set at the subproblem at any node of the branch tree is subdivided according to some branching rule. Since the feasible set is not affected by the presence of one or several objective functions, any branching strategy used in single-objective combinatorial optimization can also be used in the multi-objective case. The focus must therefore be on the bounding strategy. An effective branch and bound algorithm requires good lower and upper bounds on the objective function values at each node of the branch-and-bound tree in order to prune as many nodes as possible. Because a MOCO problem (8.1) has a set of non-dominated points rather than a single optimal value, it seems to be intuitive to consider lower- and upper-bound sets rather than numbers. Reference [13] defines lower- and upper-bound sets as follows.

A lower-bound set L is a $\mathbb{R}^p_{\geqq}$-closed, $\mathbb{R}^p_{\geqq}$-bounded set such that $Y_N \subset L + \mathbb{R}^p_{\geqq}$ and $L \subset \left(L + \mathbb{R}^p_{\geqq}\right)_N$. An upper-bound set U is a $\mathbb{R}^p_{\geqq}$-closed, $\mathbb{R}^p_{\geqq}$-bounded set such that $Y_N \in \text{cl}\left[\left(U + \mathbb{R}^p_{\geqq}\right)^c\right]$ and $U \subset \left(U + \mathbb{R}^p_{\geqq}\right)_N$, where ^{c} indicates the complement of a set in $\mathbb{R}^p$. Figure 8.9(a) shows a lower-bound set consisting of three line segments that could represent the non-dominated set of the linear programming relaxation of a MOCO problem (8.1). Figure 8.9(b)



(a) A lower-bound set.

(b) An upper-bound set defined by feasible points.

FIGURE 8.9: Lower- and upper-bound sets.

illustrates how six feasible points $U = \{(1,9)^T, (3,6)^T, (4,5)^T, (6,2)^T, (9,1)^T\}$ define an upper-bound set. In this way, any multi-objective relaxation of MOCO problem (8.1) defines a lower-bound set and any set of feasible points defines an upper-bound set, preserving this important property from single-objective branch and bound. A node of the branch-and-bound tree can be pruned whenever its lower-bound set is dominated by the upper-bound set of another node. The arguably simplest and most often used lower-bound set is $U = \{y^I\}$, which is the set consisting of the ideal point y^I defined by $y^I_k := \min\{c^k x : x \in X\}$. Similarly, an upper-bound set can be obtained by the anti-ideal point y^{AI} with $y^{AI}_k := \max\{c^k x : x \in X\}$ or the nadir point y^N consisting of the worst objective values over the efficient set $y^N_k := \max\{c^k x : x \in X_E\}$. Note that the nadir point is hard to compute for $p \geqq 3$, even for linear problems; see [18].

Because the branching rules are often problem specific, it is not possible to describe specific branch and bound algorithms within the space of this chapter. We simply summarize some literature proposing branch and bound algorithms for specific MOCO problems in Table 8.2.

TABLE 8.2: Multi-objective branch and bound algorithms.

Reference	MOCO problem
[77]	Multi-objective knapsack problem
[80]	Bi-objective knapsack problem
[20]	Multi-objective knapsack problem
	Multi-objective spanning-tree problem
[72]	Bi-objective spanning-tree problem
[46]	Bi-objective flow-shop problem
[44, 45]	Multi-objective mixed 0-1 linear programming
[38]	Multi-objective 0-1 linear programming

8.6 Challenging Problems: Metaheuristics

At this point of our discussion, we have reached the current limits of exact algorithms for the solution of MOCO problems. For problems of the difficult class in Section 8.5, exact algorithms can consume prohibitive computation time as instance size grows. For other problems, this may even occur for rather small instances. In such cases, only heuristic approaches remain as solution methods.

A heuristic is an algorithm that seeks near-optimal (or rather "near-efficient" in the MOCO context) solutions at a reasonable computational cost without being able to guarantee optimality. Heuristics are often problem specific [65]. Examples of general heuristic principles for MOCO problems are multi-objective greedy and local search algorithms [12, 53]. On the other hand, a metaheuristic is an iterative master strategy that guides and modifies the operations of subordinate heuristics by combining different concepts for exploring the search area and exploiting its structure. A metaheuristic is applicable to a large number of problems, as pointed out by [26, 51]. In this section we will summarize the ideas of metaheuristics that have been applied to MOCO problems. The main examples are multi-objective evolutionary algorithms, multi-objective tabu search, and multi-objective simulated annealing. Because heuristics and metaheuristics do not, in general, not guarantee the optimality or approximation quality (or even feasibility) of the solutions they find, the fundamental question with the use of metaheuristics to generate solutions to multi-objective combinatorial optimization problems is how to measure the quality of the produced solutions.

Figure 8.10 shows the (in general unknown) non-dominated set of a MOCO problem in green. The dots along the three broken lines represent objective vectors of solutions found by three different heuristics. Which of those three sets of solutions is best? How is it possible to decide which heuristic performed better in this example, even with knowledge of the exact non-dominated set?

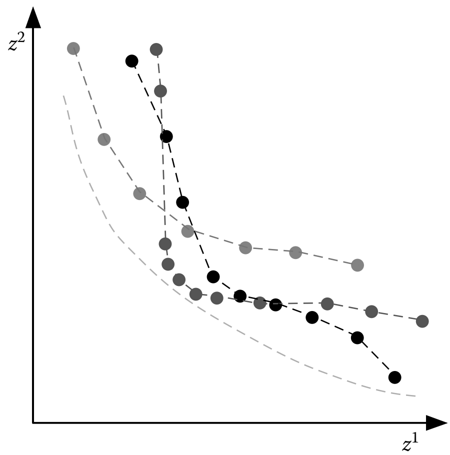


FIGURE 8.10: Which approximation is best? (From [14])

The general consensus is, that the objective function vectors of the set of solutions found by a heuristic should be uniformly distributed along the entire true non-dominated set. These are, of course, conflicting goals, and there is no single universally accepted quality criterion. Quite to the contrary, many different measures have been proposed to capture the two essential features of approximate solution sets. We mention just a few of those, such as the cardinal and geometric measures of [34]; the hypervolume indicator [84]; coverage, uniformity, cardinality as defined in [67]; distance-based measures [79]; integrated convex preference [37]; and volume-based measures [75]. An analysis and review can be found in [85].

Metaheuristics for MOCO problems can be subdivided into two main classes, namely multi-objective evolutionary algorithms and neighborhood search algorithms. The former are based on three main principles. They operate with a population of solutions instead of a single one and combine self-adaptation (the independent change of individual solutions) and cooperation (the exchange of information between individual solutions). They address the problem of convergence to the non-dominated set through fitness assignment by ranking (solutions that are not dominated by any other solution in the same population have a better rank), and selection with elitism (solutions with better rank are preferred for survival and for selection). To obtain a uniform distribution of solutions along the whole non-dominated set, they employ mechanisms such as niching, to enhance the generation of solutions with objective vectors where few feasible points exist so far. While there is a huge number of publications regarding evolutionary multi-objective optimization, relatively few evolutionary algorithms have been specifically designed to address combinatorial problems with multiple objectives. A framework for a simple evolutionary algorithm for multi-objective optimization is presented in

Algorithm 4. It works with a population of solutions, and in each iteration selects individuals from the solution that contribute to modification, such as mutation and crossover. The population for the next generation is then chosen from the original and modified members of the original "parent" generation and the modified "offspring." The output are those members of the final population that are not dominated by any other member of that population.

Algorithm 4 Multi-objective evolutionary algorithm.

input: Population size, number of generations, and other parameters
Generate initial population P_1
while stopping criterion not met **do**
    Evaluate P_t
    Select individuals from P_t
    Perform genetic operators to modify P_t
    Choose P_{t+1} from original and modified P_t
    $E :=$ `filter`(P_t), t:=t+1
end while
output: approximate solution set E

The main multi-objective neighborhood search algorithms are multi-objective simulated annealing and multi-objective tabu search. The first such methods built on single-objective counterparts, essentially using scalarization functions (see Section 8.5.1) and running several independent tabu searches or simulated annealing procedures. A simple scheme for a multi-objective tabu search algorithm is presented in Algorithm 5. The algorithm receives a set of parameters for the scalarizing function s, the length of the tabu list, and a tabu threshold as input. It then runs a tabu search to minimize s for each parameter $\lambda \in \Lambda$. In every iteration of the tabu search, the best solution $\hat{x}$ in the neighborhood of current solution x^n and the best non-tabu neighbor $\bar{x}$ are selected. If $\hat{x}$ is at least Δ better than $\bar{x}$, its tabu status is overwritten and it becomes the next current solution, otherwise the next solution is $\bar{x}$. Finally, x^{n+1} is added to E and dominated solutions are filtered out (`merge`).

A simple simulated annealing procedure is presented in Algorithm 6. Similar to Algorithm 5, the algorithm receives a set of parameters for a scalarizing function as input and runs a simulated annealing heuristic for each choice of parameter. In each iteration of the simulated annealing, a neighbor x of x^n is selected. x is accepted over x^n if either $s(x, \lambda) < s(x^n, \lambda)$ or, if it is worse, with a probability that depends on the current "temperature" and on the difference between $s(x, \lambda)$ and $s(x^n, \lambda)$ according to the rules of simulated annealing algorithms (procedure `isaccepted`).

Since both multi-objective evolutionary algorithms and multi-objective neighborhood search methods have their strengths and weaknesses, it did not take long until researchers started investigating hybrid algorithms, that is, algorithms combining features of several metaheuristics or of heuristics and exact algorithms. One can distinguish several types of hybridization.

Algorithm 5 Multi-objective tabu search.

input: set Λ of parameters for s, tabu list length l, tabu threshold Δ
$E = \emptyset$
while Λ is non-empty **do**
  $T := \emptyset$, remove λ from Λ, $x^1 := \texttt{select}(X)$
  while stopping criterion not met **do**
    $\bar{x} = \texttt{selectbest}(N(x^n) \setminus \{x : \texttt{move}(x^n, x) \in T\})$
    $\hat{x} = \texttt{selectbest}(N(x_n))$
    if $s(\hat{x}, \lambda) < s(\bar{x}, \lambda) - \Delta$ **then**
      $x^{n+1} := \hat{x}$
    else
      $x^{n+1} := \bar{x}$
    end if
    $E := \texttt{merge}(E, x^{n+1})$, $n = n + 1$
    if $|T| < l$ **then**
      $T := T \cup \{\texttt{move}(x^n, \bar{x})\}$
    end if
  end while
end while
output: approximate solution set E

Some authors use components of evolutionary algorithms in neighborhood search algorithms to improve the coverage of the non-dominated set. For instance, Pareto simulated annealing [6] uses a set of starting solutions, and the simulated annealing component makes use of information obtained from this set of solutions. TAMOCO [30] uses a set of starting solutions in a tabu search, where the tabu search exploits information from a set of solutions. Conversely, neighborhood search strategies can be used inside evolutionary algorithms to improve convergence. Examples of this idea are the MGK algorithm [23], which uses local search to improve the solutions in an evolutionary algorithm. MOGLS [33] does the same in a genetic algorithm.

Other hybridization ideas are alternations between evolutionary and neighborhood search algorithms. For example, [1] use a genetic algorithm to find a first approximation of the non-dominated set and then apply tabu search to improve the results of the genetic algorithm; [7] uses a greedy randomized search procedure (GRASP) to compute a first approximation and the strength Pareto evolutionary algorithm (SPEA) to improve the results of GRASP. References [23, 24] combine evolutionary methods (crossover and a population of solutions) and neighborhood search (a strategy called path relinking) with problem-dependent components, namely, the exact calculation of a minimal complete set of supported efficient solutions and bound sets. An example for the combination of heuristics and exact algorithms is given in [58], where the two-phase method (see Section 8.4.1) as an exact algorithm is combined with a heuristic to reduce the search area.

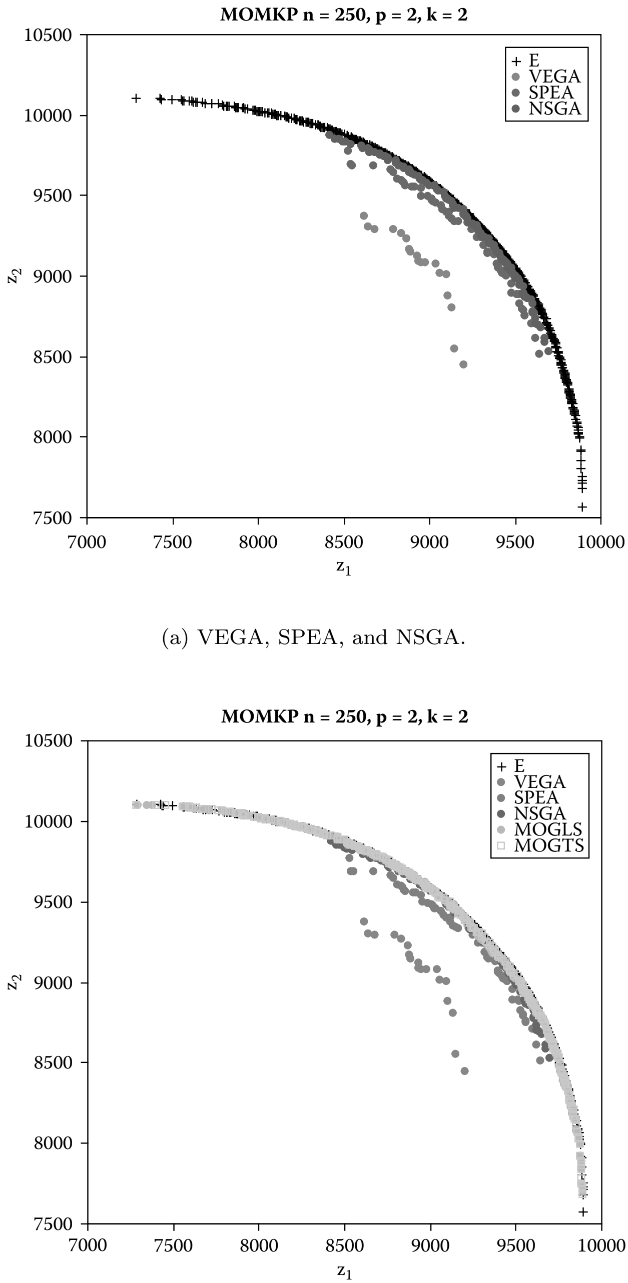
Algorithm 6 Multi-objective simulated annealing.

```
input: set Λ of parameters for s, annealing schedule
E = ∅
while Λ is non-empty do
  remove λ from Λ, x^1 := select(X), E_λ := {x^1}
  while stopping criterion not met do
    x̄ = select(N(x^n)
    if is accepted(x, x^n) then
      merge(E_λ, x), x^{n+1} := x
    else
      x^{n+1} := x^n
    end if
    n := n + 1
  end while
  E := merge(E, E_λ)
end while
output: approximate solution set E
```

To conclude this section, Table 8.3 gives an overview of the introduction of important paradigms of metaheuristics in the field of multi-objective optimization, although not all of the references are concerned with combinatorial problems. It is worth noting that the given dates are those of refereed publications, wherever available. In some cases the algorithms were presented much earlier at scientific conferences or in technical reports. Much more detailed information on metaheuristics for multi-objective (combinatorial) optimization can be found in some of our earlier survey papers [12, 14, 21]. Finally, we present a glimpse of the evolution of metaheuristic techniques by showing the approximate non-dominated sets found by six different heuristics on the same instance of a bi-objective knapsack problem with 250 items. For ease of comparison, Figures 8.11(b) and 8.11(c) are overlaid on the earlier pictures.

TABLE 8.3: A timeline for multi-objective metaheuristics.

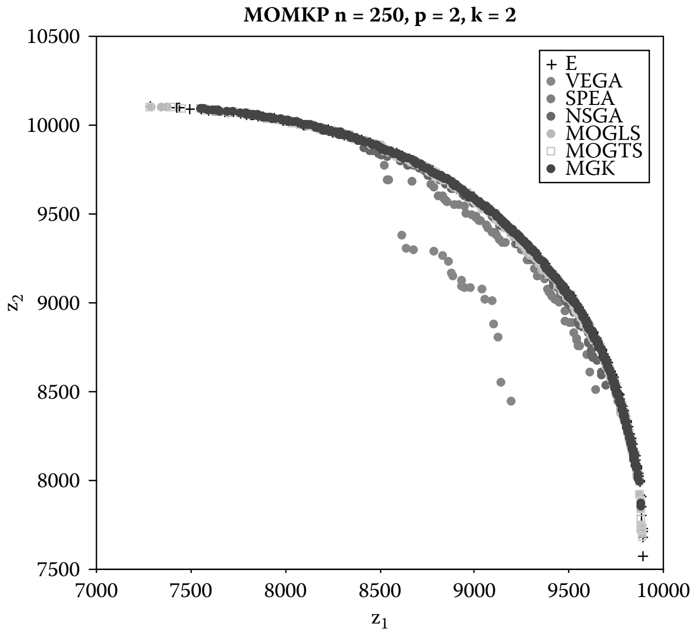
Technique	Year	Reference
Evolutionary algorithms	1984	[68]
Artificial neural networks	1990	[43]
Simulated annealing	1992, 1999	[71, 78]
Tabu search	1997, 2000	[22, 74]
Greedy randomized adaptive search procedure	1998	[25]
Ant colony optimization	2001, 2002	[32, 28, 9]
Scatter search	2001, 2007	[2, 27]
Particle swarm optimization	2002–2007	[5, 47, 62, 83]



(a) VEGA, SPEA, and NSGA.

(b) MOGLS and MOGTS.

FIGURE 8.11: Development of multi-objective metaheuristics. (Adapted from [14] and [21].)



(c) MGK.

FIGURE 8.11: (*Continued.*)

8.7 Conclusion

In this chapter, we provided an overview of multi-objective combinatorial optimization. We first gave definitions and important properties to set the scene. The main focus of the chapter is on solution algorithms. We have grouped MOCO problems into several categories, based on which solution approach works effectively. "Easy" problems allow the extensions of single-objective algorithms to the multi-objective case. Problems in this category are notably the multi-objective shortest-path and spanning-tree problems. For "nice" problems (those that are polynomially solvable in the single-objective case and have efficient ranking algorithms), we described the two-phase method as a general effective framework. We pointed out that if a fast ranking algorithm to find r-best solutions to the single-objective problem is available, the two-phase method is currently the algorithm of choice in practice. As problems go into the realm of NP-hard problems (in the single-objective case), problems we term "difficult," but that can be solved in the single-objective case using general integer programming techniques, it may still be worth exploiting algorithms that solve those in reasonable time. Hence, we first discussed scalarization techniques before describing a generalization of branch and bound for multi-objective optimization. Finally, challenging problems, for which general integer programming techniques are currently too slow, are still amenable to heuristics. Therefore, we outlined principles of some metaheuristic schemes for multi-objective optimization.

There are areas that we did not discuss. We mention in particular the area of approximation algorithms, that is algorithms that find approximate non-dominated sets with a performance guarantee. We also could not cover applications in any detail. Much of the research in multi-objective combinatorial optimization is driven by the practical need to solve MOCO problems arising in real-world applications. Naturally, many challenges exist in which progress can be made. Let us just mention two. Empirically, it has been observed that objective function coefficients have enormous impact on the number of efficient solutions and on the run time of algorithms. This provides areas of study in the analysis of algorithms as well as the structure of problems. For example, the influence of objective functions on the polyhedral theory of combinatorial optimization problems that arise from the scalarization of MOCO problems has, to the best of our knowledge, not been studied at all. Progress in theoretical and algorithmic analysis should then help attack what appears to be the ultimate goal of research in multi-objective combinatorial optimization—a toolbox to build exact algorithms for challenging problems that can today only be addressed with metaheuristics.

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Mathematics

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