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Optimal Economic Operation of Electric Power Systems

M.E.E1-Hawary

G.S.Christensen

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Optimal Economic Operation of Electric Power Systems

M. E. EL-HAWARY

Faculty of Engineering and Applied Science
Memorial University of Newfoundland
St. John's, Newfoundland
Canada

G. S. CHRISTENSEN

Department of Electrical Engineering
University of Alberta
Edmonton, Alberta
Canada

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To our wives,
Ferial and Penelope

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Contents

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Preface

This book treats problems of optimal operation in electric power systems. One of our purposes is to introduce and review a wide range of problems and algorithms in this area. The second is to underline the importance of certain developments in mathematical optimization and computational techniques and their influence on the field of power systems engineering. Recently developed optimal strategies for a wide class of problems are stressed. These rely on the minimum norm approach within the framework of functional analytic optimization methods.

Our audience consists mainly of graduate engineers and applied scientists concerned with the applications of mathematical techniques to large scale systems. The book will be of particular interest to utility engineers involved in planning the operation of electric power systems. Parts of the book may be adopted for a senior or graduate course in electric power systems engineering or optimal control applications.

Chapter 1 reviews briefly the historical developments in the field. The influence of powerful mathematical optimization and computational tools on the advances made in the subject is emphasized. Modeling of various parts of the electric power system for optimal operation studies is presented in Chapter 2. Here we first discuss energy source models. The representation of the performance and operating constraints of thermal and hydro plants is outlined. This is followed by a review of models of the electric network components. Emphasis is given to models adequate for the complexity of the problem. Various approaches to electric and hydro network overall modeling are then presented. Optimality criteria are introduced and their ramifications discussed.

Chapter 3 addresses certain mathematical optimization and computational techniques and concepts that are the basis of many successful results in optimal operation of power systems. We review briefly some but not all of the techniques. Results from variational calculus, dynamic programming, and the maximum principle are offered. Recent results obtained by the authors are based on the powerful minimum norm approach, which is also reviewed in this chapter.

A basic problem in optimal operation of power systems involves an all-thermal system. This is the subject of Chapter 4. Classical results pertaining to a system whose electric network is modeled by the active power balance equation are given. This leads to the recently developed optimal load flow problem whereby the exact electric network model is considered in the formulation.

The inclusion of hydro plants of a system in the formulation is considered next. We treat the case of systems with hydraulically uncoupled plants in Chapter 5. A problem involving fixed-head hydro plants is dealt with first. Three approaches are discussed: the classical approach involving variational calculus principles, the dynamic programming approach, and a minimum norm approach. A similar treatment of the case of variable-head hydro plants is given.

In Chapter 6 formulations involving coupled hydro plants in the system are considered. An approach based on the maximum principle is illustrated, followed by minimum norm approach solutions to two system problems involving coupled plants, that is, systems with plants on the same stream and systems with multiple chains of hydro plants. A computational example of an actual solution leading to an optimal operational strategy implementation is given.

Chapter 7 is devoted to the optimal hydro–thermal load flow problem. Here we discuss the cases of fixed-head hydro plants, multichains of hydro plants when the electric network is represented by the load flow equations, and the more realistic models of trapezoidal reservoirs and variable efficiency. The final chapter is a concluding one in which we summarize the results obtained and offer suggestions for future work in the field.

Each chapter in this book is concluded by a Comments and References section, where the reader's attention is directed to further readings and the merits of the various approaches are discussed. Each of Chapters 4–7 has been designed such that it can be studied independently of the others once the prerequisite relevant material from Chapters 2 and 3 has been covered.

The work reported in this book is an outgrowth of several years of research and teaching in this area. Some of the work has been class tested on many occasions in senior-undergraduate, graduate, and continuing education courses taught both in Canada and Brazil.

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Optimal Economic Operation of Electric Power Systems

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CHAPTER

1

Introduction

1.1 THE OPTIMAL OPERATION PROBLEM IN ELECTRIC POWER SYSTEMS

The electric power systems engineer is faced with the challenging task of planning and operating successfully one of the most complex systems of today's civilization. This demands a knowledge, first of all, of the priorities and objectives involved. The basic requirement is to meet the demand for electric energy by the area served by the system. A prime objective here is to perform the service at the lowest possible cost. Of equal importance is the objective of minimizing the environmental impact of the operation. Continuity of service and reliability are major considerations. Safety for both personnel and equipment is a factor that may override some of the other objectives.

It may be tempting to order the above objectives on a priority list. However, these objectives interact, which makes it difficult for one to come up with a general clear-cut ordering. Not only does this vary with the system, but also with the times and socioeconomic factors.

Economic dispatch ranks high among the major economy–security functions in power systems operation. This is a procedure for the distribution of total thermal generation requirements among alternative sources for optimal system economy with due consideration of generating costs, transmission losses, and several recognized constraints imposed by the requirements of reliable service and equipment limitations. Conventional economic dispatch is a static optimization procedure to dispatch preselected generating units. When excess generation is available in a system such that an economic choice of units can be made, the set of units to be dispatched is normally determined by a unit commitment program.

The objective in minimal emission dispatch is to minimize certain contaminants for the system as a whole or for a certain geographical location while satisfying the system's energy demands. Minimal emission dispatch is closely linked to economic dispatch. As a result a dispatch procedure with multiple generator representations may be used to perform both functions with little changes.

The hydrothermal scheduling problem is different from the all-thermal one. The former involves the planning of the usage of a limited resource over a period of time. The resource is the water available for hydro generation. The short-term optimal economic scheduling problem in a hydro–thermal system consists in allocating the available hydraulic and thermal resources to the various time intervals (hour by hour) of the period under consideration (defined to be one day to one week). The allocation is performed so that the total system production cost is minimized within the limits permitted by the constraints representing reliability, environmental, and other requirements. The location and special operating characteristics of hydro plants are important considerations in hydro–thermal scheduling. The problem is quite different if the hydro stations are located on the same stream or on different ones. In the former case, the water transport delay may be of great importance. An upstream station will highly influence the operation of the next downstream station. The latter will also influence the upstream plant as well. Close hydraulic coupling of stations adds an interesting dimension to the problem.

One of the important functions in power system operation is predispatch, which is one of the many terms used to define the before-the-fact economic hourly scheduling of all available system resources. The results of a predispatch (or preschedule) run provide necessary input to real-time economic dispatch and/or hydro–thermal procedures. Predispatch may also be used as an aid in system planning where evaluation of alternatives for system expansion is performed on the basis of the respective optimal operational modes.

1.2 A BRIEF HISTORICAL SURVEY

In a bibliography published in 1963, Noakes and Arismunandar list 436 papers, books, and other material in the field of optimal operation of power systems and related areas published in North America. This effort includes material published from 1919, when, presumably, power system engineers began to take active interest in the economic allocation of generation among available units. One of the earliest methods to be adopted is the "base load procedure," whereby the most efficient unit is loaded to its maximum capability and then the second most efficient unit is loaded. Another popular method is "best point loading," whereby units are successively loaded to their lowest heat rate point beginning with the most efficient unit.

The incremental method was first formally derived by Steinberg and Smith in 1934, even though it was recognized as early as 1930. The idea is that the next increment in the load should be picked up by the unit with lowest incremental cost. The understanding and appreciation of the method was enhanced further by the publication of Steinberg and Smith's classic book, "Economy Loading of Power Plants and Electric Systems," in 1943.

Loss inclusion in the formulation and solution received a boost in 1943 by the publication of George's derivation of a loss formula. Kron's work represents a concise derivation of the network and loss models. The classic coordination equations were discovered by Kirchmayer and Stagg in 1951. These results form the backbone of our present-day economic operation methodology. Kirchmayer's classic book, entitled "Economic Operation of Power Systems" and published in 1958, gives comprehensive treatments of the loss formula derivations and the conventional economic dispatch problem.

The importance of the hydro–thermal scheduling problem was recognized for a number of years before the advent of Ricard's equations in 1940. This work was adopted and continued by Chandler *et al.* in their pioneering 1953 paper. Their development of the first coordination-type equations was associated with studies for the system of the then Hydro-Electric Power Commission of Ontario. The formulation recognizes losses but assumes constant head. The solution of the variable head case in 1958 is due to Glimn and Kirchmayer. The foregoing results are contained in the very comprehensive treatise by Kirchmayer entitled "Economic Control of Inter-Connected Systems," published in 1959.

Attempts to improve upon the loss formula models date back to the late 1950s. This coincided with the appearance of the first load flow solution algorithms. As a result a start toward the optimal load flow was made in terms of the pioneering work of Squires in 1961 and Carpentier in 1962. Detailed reference to the literature will be made in the text.

The developments cited so far find their mathematical background in the classical optimization results employing classical theory of maxima and minima in the static case and variational calculus in the dynamic case. The solution of further problems has been enhanced by the development of powerful optimization and computational techniques. We will briefly discuss this next.

1.3 THE ROLE OF OPTIMIZATION AND COMPUTATIONAL MATHEMATICS

The general field of power systems engineering and the particular area of optimal economic operation are continually dependent in their development on a balanced blend of engineering operating experience and the skilled adoption of mathematical advances and discoveries. In this regard they are no different from other disciplines of engineering and applied science. The strong influence of newly developed optimization and computational techniques in the area can be partly judged by inspection of titles of published work over the past few decades.

The advent of Bellman's principle of optimality and dynamic programming in the 1950s and the publication of his two early classical books gave quite an impetus to problems involving discrete and discontinuous variables. Examples of this include valve-point loading, which is treated using dynamic programming in Ringlee and Williams' 1962 work. Important hydro–thermal scheduling and unit commitment procedures are also based on this powerful technique. Among the early results we have those of Bernholtz and Graham (1960).

The introduction of the Kuhn–Tucker theorem in 1951 to the optimization literature contributed to the advances in formulating problems including inequality constraints. Indeed, the varied powerful nonlinear programming procedures continue to influence developments in the area of optimal operation of power systems.

The awareness of the Pontryagin's maximum principle in the late 1950s and early 1960s prompted many landmark solutions to important hydro–thermal scheduling problems. Notable among these is Dahlin and Shen's 1966 work. As is the case with dynamic, linear, and nonlinear programming, the maximum principle continues to be one of the essential tools of optimization used in our type of problem.

The developments in adapting functional analysis concepts to optimization, leading to the emergence of the abstract minimum norm problems, can be traced back to the late 1950s. Application of these powerful techniques to many optimal operation problems dates to the early part of this decade.

The strong influence of advances in computational mathematics on the development and utility acceptance of optimal operational strategies is clearly evident. The need continues to exist for reliable, fast, and efficient numerical algorithms for digital implementations. Indeed, the improved convergence characteristics of the Newton–Raphson method over the Gauss–Seidel method prompted the adoption of many load flow algorithms. Advances in sparsity oriented procedures continue to improve both speed and cost of implementation. Many investigations into the relative merits of newly reported numerical algorithms continue to be reported in the literature.

1.4 OUTLINE OF THE BOOK

The book can be divided into two parts: background preparation and problem presentation and discussion. The first part represents the background preparation phase and includes Chapters 2 and 3. It presents us, with the tools necessary to clearly define optimal operation problems in terms of modeling and mathematical prerequisites. Part II comprises four chapters and provides us with statements, formulations, solutions, and implementation procedures for specific problems in optimal operation of power systems. The format of the chapters in this part is essentially the same. What we shall do now is to review the six following chapters.

Chapter 2 provides a general setting for the problems dealt with in this book. Here we discuss modeling of power systems, devices, and associated phenomena from an optimal operation point of view. The organization of the chapter follows closely the pattern of the recognized subsystems involved.

We first present the energy source models for both thermal and hydro plant performance. Emphasis here is on the diversity of models in common use for our type of problem. The next section deals with electric network components modeling. We outline the basis of generator, transmission line, and transformer models. Our presentation is very brief since detailed analyses can be found in many of the available power systems textbooks. This section concludes with a treatment of the important subject of load modeling. Activities in this area can be divided into two categories: short-term load forecasting and modeling of load response. We give a concise summary of advances made in the sister area of transient stability. Load modeling has been gaining more attention recently in the optimal operation area for the potential savings associated with resulting accurate models.

The next section discusses electric network models. We begin with the active power balance models including the active loss formula. We give a simple derivation which serves to motivate the following discussion of the active–reactive power balance. The more rigorous model given by the exact load flow equations (also called power flow) and some approximations are treated at the end of this section.

Following the discussion of electric network models a section is devoted to hydro network modeling. This may in certain systems be of equal importance. We deal with river flow modeling and in this case present a state space approach due to Dahlin and then outline the transport delay approach. This section concludes with a discussion of aspects of reservoir inflow forecasting. The material in this section finds application in Chapter 6.

The last topic discussed in Chapter 2 concerns objective functionals for optimal operation. Our format includes a concluding section wherein comments and reference to the literature are made.

Chapter 3 considers the mathematical basis for studies and results to follow. The first important topic reviewed is certain linear algebraic concepts. Here concepts of vectors, matrices, partitioning, and quadratic forms are briefly reviewed. Our purpose is simply to provide essential tools used in the development. Consequently, nothing is proved in the entire chapter.

The second major topic is that of static optimization results. Pertinent conclusions for optimizing unconstrained, equality constrained, and inequality constrained problems are treated.

The next section describes the dynamic programming approach and the principle of optimality. The following two sections deal with the calculus of variations and the Pontryagin's maximum principle. As is the case throughout the chapter we only cite results which provide a background for material in the work of the second part.

We devote the following section to the functional analytic optimization technique. Here we begin by reviewing the rudiments of functional analysis and follow that with a statement of a minimum norm problem and its solution. This is important because this result is used in Chapters 5, 6, and 7 to specify optimality conditions for a number of optimal operation problems.

With the above coverage of results from optimization theory, we turn our attention to the numerical implementation aspects. We begin with certain important results from linear dynamic system theory and deal with the concept of a state transition matrix. This sets the stage for citing an important result pertaining to nonlinear two-point value problem representation. Such a problem arises naturally in problems of interest in our work. The chapter concludes with a section on iterative solution methods for nonlinear systems. Included in the presentation are powerful techniques such as the modified contraction mapping, Newton's procedure, and composite and continuation procedures. We refrain from presenting convergence results and restrict our treatment to outlining the salient features of each method discussed.

Chapter 4 presents problems of all-thermal scheduling. We begin with dispatch using power balance models. In this section problems are ordered so that the complexity increases as we progress through the development.

We include active power balance dispatch neglecting losses and present some interesting results pertaining to equivalent machine representations and loss of economy due to inaccuracies in cost models. This is followed by the case including losses where penalty factors are introduced. The fundamental problem of active–reactive dispatch is treated next. This section concludes with a discussion of dynamic programming procedure for dealing with valve points based on the cost characteristics. The chapter concludes with an outline of some optimal load flow representative formulations. We include the classic Carpentier–Siroux treatment and the Dommel–Tinney approach. Due to space limitations we do not attempt to give coverage to many excellent contributions to this important problem. Instead, adequate reference to the literature is made in the final section of the chapter.

In Chapter 5 we treat two major problems of hydro–thermal scheduling in which hydro plants are hydraulically isolated. We begin with the case of fixed-head hydro plants. Here the classical dispatch solution is first developed employing variational calculus principles. This is followed by a dynamic programming approach. We then give a functional analytic solution to a problem with predefined models and cite some computational results. The second major problem in this chapter concerns systems with variable-head plants. As in the earlier case we offer three alternative methods, beginning with the variational approach and followed this time by a maximum principle approach. Finally a minimum norm solution is offered.

In Chapter 6 our interest is focused on hydro–thermal scheduling of systems with hydraulic coupling. We begin with a system with one set of hydro plants on the same stream. With the detailed river flow model established in Chapter 2, we give a maximum principle approach to this fundamental problem. We then use a transport delay approximation and deal with a minimum norm procedure for optimizing the system. The following section generalizes the treatment to systems with multichains of hydro plants. Implementation aspects are discussed and numerical results highlight the performance of some proposed iterative algorithms.

In Chapter 7 we treat two problems in optimal load flow in hydro–thermal systems. The first problem addresses the case of systems with hydraulically isolated plants. The second includes more of the complicating effects of multichain hydro subsystems with a detailed hydro model recognizing efficiency and the more realistic trapezoidal form for the reservoir representation. The problems in this chapter are solved using the minimum norm formulation.

Continuing changes in power system operating conditions require reevaluation of the methods used. Opportunities for improvements continue to exist for efforts to assure security and provide adequate control in the system. Chapter 8 is devoted to a summary and a brief outline of certain directions for future research needs to develop advanced methods for reducing the cost of power system operation while retaining acceptable security and control.

1.5 COMMENTS AND REFERENCES

In this introductory chapter our reference list is very short and includes mainly sources referred to in the previous sections. Further and detailed reference lists will be given at the end of each chapter.

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CHAPTER

2

Modeling for Optimal Operation

2.1 INTRODUCTION

This chapter is intended to cover the fundamental aspects of modeling the various parts of the electric power system and phenomena that bear on optimal economic operation of the system. The system may be viewed as made of subsystems each of which involves a number of components. The degree of detail in component and subsystem models varies with the desired accuracy and relevance to the problem considered. The electric power system also responds to and is affected by phenomena which can best be described as time series. Such a phenomena include power demand variations and water resource availability.

Our starting point in this chapter is the subject of modeling the energy source. In our case we deal with thermal plant as well as hydro plant models. This is followed by a treatment of the electric network components and load models. The modeling of the overall network subsystem is discussed next. This is followed by a treatment of hydro network modeling including river flow dynamics and reservoir inflow forecast methodology. The chapter concludes with a summary of the objectives of optimal economic operation studies.

2.2 THE ENERGY SOURCE

Prime energy sources in use for electric power generation can be broadly classified as renewable and nonrenewable resource based. Hydrocarbon fossil fuels such as oil, natural gas, coal, and nuclear fuel are nonrenewable resources which are used for electric power generation in thermal plants. The most widely used renewable resource for electric power generation is hydro power. The future promises exciting developments with other renewable sources such as wind power, solar energy, tidal power, etc.

The purpose of this section is to briefly outline models for thermal and hydroelectric generation in general use for economic operational purposes. We start with a discussion of modeling the fuel cost variations with the active power generation for thermal generating plants. This is followed by a treatment of hydro plant performance modeling.

2.2.1 Thermal Plant Cost Modeling

In a thermal plant electric power is generated as a result of mechanical rotational energy produced by either steam turbines or combustion turbines. The medium of heat energy transfer to the turbines is steam produced in the boiler or nuclear reactor. Combustion turbines burn liquid or gaseous fuels, mostly light distillate oil or natural gas. No intermediate steps are needed.

In hydrocarbon fossil fuel steam units, fuel is burnt and energy is released in the form of heat in the boiler. Steam at high temperature and pressure is produced as a result. The steam is led via the drum to the turbines, where part of the thermal energy is transformed into mechanical form. The steam turbine drives the electric generator (alternator). The exhaust of the turbine is cooled in the condenser and the resulting water is pumped back to the boiler. A detailed study of the plant dynamic modeling is beyond the scope of our treatment. From an economic operational point of view, our interest is in an input–output model. The input in this case is the fuel cost and the output is the active power generation of the unit. This is essentially an efficiency-type model and we discuss first various factors affecting our modeling efforts.

Boiler efficiency depends on losses that may be classified as

(1) stack heat losses, including those in the flue gas, moisture in the gas, and incomplete combustion of CO;
(2) heat loss in the ash (unburnt carbon);
(3) heat in steam for sootblowing;
(4) heat in mill rejects (for coal fired boilers); and
(5) auxiliary power for mill groups, fans, and ash disposal.

These losses are determined by variables related to the fuel, air, and feed water inputs to the boiler. Certain variables are uncontrollable such as air and water temperature, moisture in the air and fuel (for coal-fired) as well as chemical analysis and calorific value of fuel. Controllable variables affecting the losses include combustion air quantity, pressure, and distribution in addition to fuel preparation. The condition of the boiler is an important factor which depends on the maintenance procedures.

Thermal power plant performance and generating costs are also affected by turbo-alternator efficiency, which is not subject to such random changes as boiler efficiency. This depends on thermal variables related to the Rankine cycle efficiency including temperature and pressure of steam at the turbine stop valve, high pressure, low pressure, cylinder exhaust, condenser, and low pressure and high pressure heaters. The internal condition of the turbine is a major factor for consideration. Prime causes for reduction in turbine internal efficiency include among other things deposition of solids onto blades and blade erosion. It is thus evident that periodic updating of the model is necessary.

The input to the thermal plant is generally measured in MJ/hr in S.I. units (traditionally MBtu/hr or kcal/hr), while the output is measured in megawatts (MW). Although initially prepared on the basis of input versus main unit output, the input–output relationship must be converted to input versus net plant sendout. Auxiliary power requirements that depend on unit loading must be accounted for. The total cost of operation includes the fuel cost, cost of labor, supplies, and maintenance. However, no methods are presently available for expressing the latter as a function of the output. Arbitrary methods for determining these costs are used. The most common one is to assume the cost of labor, supplies, and maintenance to be a fixed percentage of the incoming fuel costs.

Let us introduce some terms used in connection with input–output models for thermal plants. For this purpose we refer to Fig. 2.1, which shows a typical thermal input–output curve (F–P). The average heat rate characteristic is obtained by dividing the input by the corresponding output. This is shown in Fig. 2.2 ($[F/P]$–P). The incremental heat rate characteristic is simply a plot of $\partial F/\partial P$ versus P and a typical curve is shown in Fig. 2.3. Heat rate units are MJ/KW hr.

The discontinuities in the cost curves eminently shown in the incremental heat rate curve of Fig. 2.3 are due to sharp increases in throttle losses due to wire drawing effects occurring at valve points. These are loading (output) levels at which a new steam admission valve is being opened. As the valve is gradually lifted, the losses decrease until the valve is completely open. The shape of the input–output curve in the neighborhood of the valve points is difficult to determine by actual testing. Most utility systems find it satisfactory to represent the input–output characteristic by a smooth curve which can be defined by a polynomial.

Heat rate tests carried out in accordance with the American Society of Mechanical Engineers' standards provide accurate means of measuring tur-

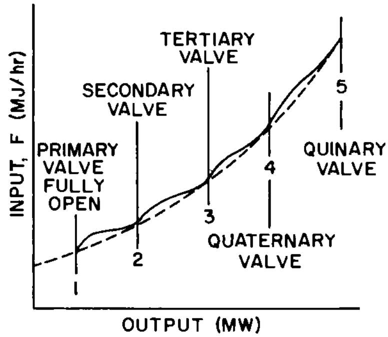

Fig. 2.1 Thermal unit input–output curve.

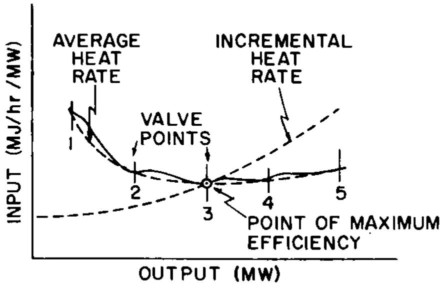

Fig. 2.2 Typical thermal unit heat rate curve.

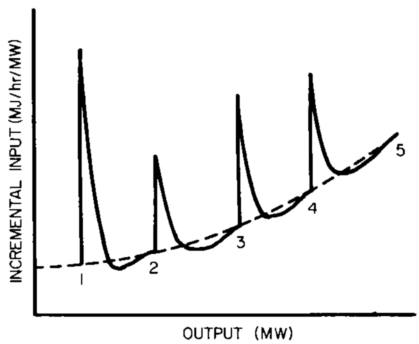

Fig. 2.3 Typical thermal unit incremental heat rate curve.

bine heat rate but they are costly in time and manpower. The cost increases with the unit size. Such tests may be carried out on an annual basis for each unit and pre- and postmajor overhaul tests on selected units. Heat rate evaluation on the basis of operating records does not involve unit outages and is therefore preferable from an economic point of view. A third means is the manufacturer guaranteed performance data adjusted to actual operating conditions. Updating performance characteristics through the performance correction factors is done at regular intervals. This is effected by computing the fuel required based on the established input–output curve and comparing the computed value after adjustments with the actual fuel consumption. It is important to note here that ASME's standard testing procedures call for boiler–turbine–generator to be in steady state operation for a number of hours before recording measurements for heat rate evaluation. Under day-to-day operating conditions this is never the case and thus operational heat rates deviate somewhat from the measured ones. It is therefore deemed necessary to perform the regular update mentioned earlier.

TABLE 2.1

Typical Fossil Generation Unit Net Heat Rates[a]

Fossil fuel	Unit rating	Output (MJ/kW hr)				
		100%	80%	60%	40%	25%
Coal	50	11.59	11.69	12.05	12.82	14.13
Oil	50	12.12	12.22	12.59	13.41	14.78
Gas	50	12.33	12.43	12.81	13.64	15.03
Coal	200	10.01	10.09	10.41	11.07	12.21
Oil	200	10.43	10.52	10.84	11.54	12.72
Gas	200	10.59	10.68	11.01	11.72	12.91
Coal	400	9.49	9.53	9.75	10.31	11.25
Oil	400	9.91	9.96	10.18	10.77	11.75
Gas	400	10.01	10.06	10.29	10.88	11.88
Coal	600	9.38	9.47	9.77	10.37	11.40
Oil	600	9.80	9.90	10.20	10.84	11.91
Gas	600	9.91	10.01	10.31	10.96	12.04
Coal	800/1200	9.22	9.28	9.54	10.14	
Oil	800/1200	9.59	9.65	9.92	10.55	
Gas	800/1200	9.70	9.75	10.03	10.67	

[a] For conversion: 1 Btu = 1054 J.

Typical heat rate data for sample unit sizes for steam units using coal, oil, and gas as primary sources of energy are given in Table 2.1. We remark here that comparison of heat rates for units of the same size shows that coal-fired is the least expensive (heatwise). This is followed by oil-fired. Gas-fired plants are the most expensive. Moreover, the cost decreases as unit size increases.

For economy operation problems treated in this book the fuel cost curve is modeled as a quadratic in the active power generation. This we express as

$$F(P_s) = \alpha + \beta P_s + \gamma P_s^2 \quad \text{GJ/hr} \tag{2.2.1}$$

TABLE 2.2

Typical Cost Coefficients

Unit size (MW)	Coal			Oil			Gas		
	α	β	γ	α	β	γ	α	β	γ
50	49.92	10.06	0.0103	52.87	10.47	0.0116	53.62	10.66	0.0117
200	173.61	8.67	0.0023	180.68	9.039	0.00238	182.62	9.19	0.00235
400	300.84	8.14	0.0015	312.35	8.52	0.00150	316.45	8.61	0.00150
600	462.28	8.28	0.00053	483.44	8.65	0.00056	490.02	8.73	0.00059
800	751.39	7.48	0.00099	793.22	7.74	0.00107	824.4	7.73	0.00117
1200	1130.8	7.47	0.00067	1194.6	7.72	0.00072	1240.32	7.72	0.00078

Typical values of the coefficients α, β, and γ obtained by simple least square estimation for the typical heat rates of Table 2.1 are given in Table 2.2.

2.2.2 Hydro Plant Performance Modeling

In a hydro plant, turbines convert the water potential energy into kinetic energy, which in turn is converted into electric form through the generators. Hydroelectric installations are classified into two types—conventional and pumped storage. The conventional type is further classified into two classes, storage and run-of-river.

A. Pumped-Storage Plants

A pumped-storage hydro plant consists of an upper and a lower reservoir. During light load periods, water is pumped from the lower to the upper reservoirs using the most economic energy available as surplus from other sources in the system. During peak load periods, water stored in the upper reservoir is released to generate power, displacing high cost fossil generation. Economic dispatch of systems with pumped-storage hydroelectric power presents a very challenging problem.

B. Run-of-River Plants

These plants have little storage capacity and use water as it becomes available. Water not utilized is spilled over. The power station can be located in the stream or alongside. The latter is commonly referred to as a canal-type power station. It is clear that in view of the unregulated flow, the generated electric power of a run-of-river plant is not a controllable variable. In economy dispatch studies for systems with run-of-river plants, the output of such a plant is considered as a negative load in arriving at the system load demand.

C. Storage Plants

Plants with reservoirs of significant storage capacity are termed storage plants. In periods with low power requirements water can be stored and utilized when the demand is high. Economy dispatch of systems with storage-type hydro plants is the subject of further discussion in later chapters of this book.

D. Plant Layout

The main elements of a typical hydroelectric plant include an upper level reservoir, usually formed by building a diversion dam across a river. Figure 2.4 shows a typical hydroelectric plant layout. The intake carries the water directly to low-head turbines or to the pressure conduit in the case of medium- and high-head turbines. The intake is a canal or a concrete passageway. The pressure conduit carries the water under pressure to the turbines. A closed pressure conduit is termed a penstock. To provide for pressure regulation a surge tank is installed along the penstock. This will prevent excessive pressure rises and drops during sudden load changes. Trash racks are provided at the inlet to the intake or pressure conduit to protect the turbine against floating and other foreign objects. Water flowing through the hydroturbine runners is taken through draft tubes to the tailrace and tailrace reservoir. The tailrace is used to conduct the water from the draft tube to the tailwater reservoir, which is usually part of the original river at an elevation lower than the upper reservoir.

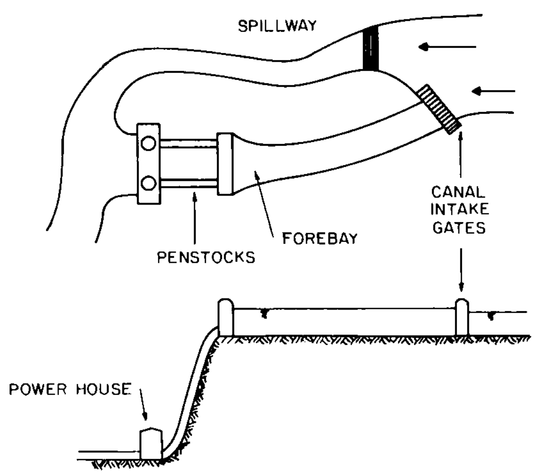

Fig. 2.4 A typical medium-head hydro power development.

E. Hydro Turbine Modeling

There are two general types of hydro turbines, reaction-type and impulse-type. In the reaction-type, water under pressure is only partly converted into velocity before it enters the turbine runner. The Francis wheel, the Kaplan wheel, and the propeller wheel are the most commonly used turbines of the reaction-type. In the impulse-type turbine, water under pressure is entirely converted into velocity before entering the turbine. The Kaplan turbine is characterized by adjustable rotor and stator blades whereas the Francis turbine has adjustable stator blades.

The output power of hydro-turbo generation is a function of both the net hydraulic head and the rate of water discharge. The output power (in MW) is given by

$$P_h = (qh/102)\eta_t\eta_G \tag{2.2.2}$$

where q is the rate of water discharge (m^3/sec), h is the effective water head (m), η_t is the turbine efficiency, and η_G is the generator efficiency. An alternative form of writing Eq. (2.2.2) can be obtained if we define an efficiency variable G, given by

$$G = (\eta_t\eta_G/102)^{-1} \tag{2.2.3}$$

The result is

$$P_hG = qh \tag{2.2.4}$$

The efficiency depends on both discharge and effective head. Typical performance curves are shown in Fig. 2.5. The Kaplan turbine shows a superior performance characteristic as compared to the Francis-type. In the former there is a fairly flat efficiency curve over a range of discharge values above and below the discharge for maximum efficiency. In the Francis turbine efficiency falls off rather sharply both above and below the point of maximum efficiency.

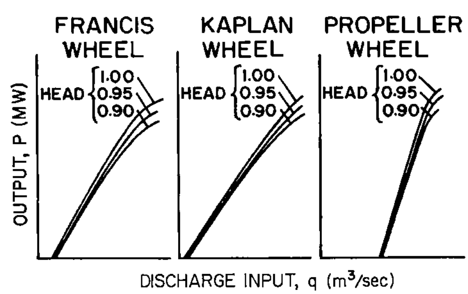

Fig. 2.5 Performance characteristics of hydro turbines. Head values are per unit.

The efficiency variation with active power generation can be effectively modeled using the following quadratic expression:

$$\alpha_g G^2 + \beta_g G + \gamma_g P_h^2 + \delta_g P_h + \theta_g = 0 \tag{2.2.5}$$

This provides a satisfactory approximation.

In addition to the model given by Eq. (2.2.4) and Eq. (2.2.5), many other models for hydro plant performance exist. These are based in one way or the other on Eq. (2.2.2). The reason for the numerous models is simply the diversity of installation characteristics. Probably the most used model is Glimn–Kirchmayer's, giving the variation of rate of discharge as a biquadratic function of effective head and active power generation:

$$q = K\psi(h)\phi(P_h) \tag{2.2.6}$$

$$\psi(h) = a_0 + a_1 h + a_2 h^2 \tag{2.2.7}$$

$$\phi(P_h) = b_0 + b_1 P_h + b_2 P_h^2 \tag{2.2.8}$$

In Eq. (2.2.6) K is a constant of proportionality. A more generalized form of a model is Hildebrand's, which takes the form

$$q = \sum_{i=0}^{k} \sum_{j=0}^{1} C_{ij} P_h^i h^j \tag{2.2.9}$$

More recently a model has been proposed which we refer to as the Hamilton–Lamont model. This takes the form

$$q = (a_0 + a_1 P + a_3 P^3)(b_0 + b_1 h + b_2 h^2)/h \tag{2.2.10}$$

It is noted here that the models of Eq. (2.2.6), Eq. (2.2.9), and Eq. (2.2.10) describe the rate of water discharge q as a function of two variables, namely, effective head and power generation. All of these various models can be interpreted as a consequence of Taylor expansion for a function of several variables.

An exponential variation of head with reservoir storage leads to the Arvanitidis–Rosing model given by

$$P_h = qh(\beta - e^{-\alpha s}) \tag{2.2.11}$$

It is noted here that the model will be completed once the storage–effective head relationship is established. This we discuss in the following:

F. Effective Head Models

The effective head is established by first obtaining the gross head h_g which is defined as difference of forebay elevation y and tailrace elevation y_T

$$h_g = y - y_T \tag{2.2.12}$$

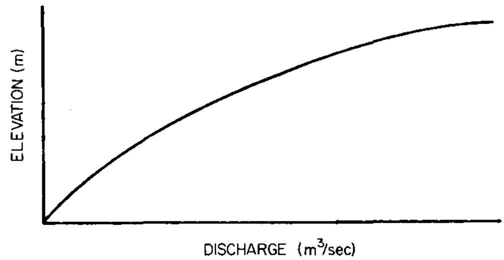

Fig. 2.6 Typical tailrace elevation versus discharge characteristic.

The tailrace elevation is a function of the discharge as well as spillage. This is shown in Fig. 2.6. We can express the tailrace elevation by the following relationship:

$$y_T = y_{t_0} + \alpha_t(q + \sigma) \tag{2.2.13}$$

This assumes a linear relationship between the two variables of interest. The forebay elevation is a function of the reservoir geometry, natural water inflow, spillage, and water discharge. It is thus necessary to consider reservoir modeling in the case of variable head hydro plants.

Once the gross head is obtained, the head losses in the penstock are subtracted from the gross head to obtain the effective head

$$h = h_g - h_p \tag{2.2.14}$$

A typical head loss characteristic variation with the discharge is shown in Fig. 2.7. Again linear variation may be assumed:

$$h_p = h_{p_0} + \alpha_p q \tag{2.2.15}$$

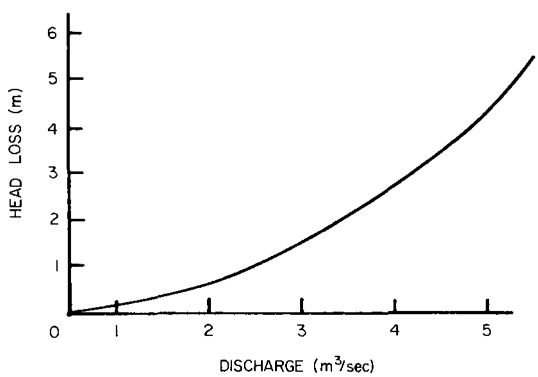

Fig. 2.7 Typical penstock head loss variation with discharge.

The resulting expression for effective head is thus

$$h(t) = y(t) - [y_{T_0} + \alpha_t \sigma(t) + \alpha_T q(t)] \tag{2.2.16}$$

where

$$y_{T_0} = y_{t_0} + h_{p_0} \qquad \text{and} \qquad \alpha_T = \alpha_t + \alpha_p$$

G. Reservoir Models

A reservoir model of interest in an economy operation study is a realistic one that relates the plant's forebay elevation y to the discharge q. These determine the active power generation available from the hydro plants. The reservoir dynamics may be adequately described by the continuity-type equation

$$ds(t)/dt = i(t) - q(t) - \sigma(t) \tag{2.2.17}$$

where $s(t)$ is the reservoir storage, $i(t)$ the rate of natural inflow adjusted for evaporation and seepage losses, $q(t)$ the rate of water release through the hydro plant, and $\sigma(t)$ the rate of water spillage.

It should be noted that the evaporation and seepage losses are normally estimated by multiplying the storage by the appropriate coefficient. For simplicity these are assumed to be independent of the storage. Moreover, $\sigma(t)$, the rate of water spillage, should be assumed zero unless certain overriding constraints are violated.

The variation of storage (or capacity) of a reservoir of regular shape with the elevation can be computed with the formulas for the volumes of solids. This for reservoirs on natural sites is determined from the elevation–storage curve. An area–elevation curve is constructed using the planimeter to find the area enclosed within each contour within the reservoir site. The integral of the area–elevation curve is the elevation–storage curve. A typical elevation–storage curve is shown in Fig. 2.8. It is important to note that natural factors will change the reservoir configuration over time. An example is sediment accumulation. It is thus important to update the reservoir model periodically.

A mathematical model may be obtained by using Taylor's expansion. If we choose the first two terms, this will result in a linear relationship which corresponds to the mathematical artifice of a vertical-sided reservoir model. A trapezoidal reservoir model results if we include the second-order term in the expansion. For a partially filled hemisphere model a third-order equation results. Accordingly the general reservoir model is given by

$$s = \sum_{i=0}^{N} \alpha_i y^i \tag{2.2.18}$$

where N is the highest order of the approximation.

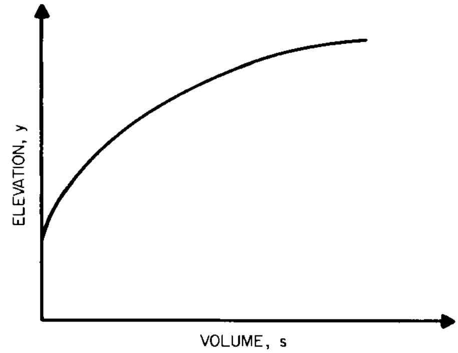

Fig. 2.8 Typical reservoir elevation–storage curve.

H. Operational Constraints

Many dams and associated reservoirs are multipurpose developments. Of interest in our discussion are the requirements of each component as it affects the operation of the hydroelectric plant. Irrigation, flood control, water supply, stream flow augmentation, navigation, and recreation usages are among the possible purposes of a water resource development. For these purposes the reservoir is regulated so that full requirements for each element are available under the design drought conditions.

The operational reservoir constraints are

$$s_{\min} \le s(t) \le s_{\max}$$
$$q_{\min} \le q(t) \le q_{\max}$$
$$q_{\sigma_{\min}}(t) \le q(t) + \sigma(t) \le q_{\sigma_{\max}}(t)$$

The first set of inequality constraints simply states that the reservoir storage (or elevation) may not exceed a maximum level, nor be lower than a minimum level. For the former this is determined by the elevation of the spillway crest or the top of the spillway gates. The minimum level may be fixed by the elevation of the lowest outlet in the dam or by conditions of operating efficiency for the turbines. The second set is solely determined by the discharge capacity of the power plant as well as its efficiency. The last inequality constraint would be imposed by irrigational, navigational, recreational, and flood prevention considerations.

I. A Simple Overall Model

In Chapters 5 and 6 we will find it necessary to adopt a simple overall model for the plant and reservoir dynamics. This is obtained by assuming constant efficiency, no spillage, invariant tailrace elevation, negligible penstock losses, and a vertical-sided reservoir. Equations (2.2.4) and (2.2.16)–

(2.2.18) may thus be combined to give

$$P_h(t) + A(t)q(t) + Bq(t)Q(t) = 0 \tag{2.2.19}$$

where

$$-A(t) = [h(0)/G] + B \int_0^t i(s)\,ds, \qquad B = B_y G$$

with B_y being inverse surface area of the reservoir. In Eq. (2.2.19), the new variable $Q(t)$ is the volume of water discharged up to time t.

2.3 ELECTRIC NETWORK COMPONENTS

The basic components of an ac electric power network include the synchronous generator, the power transformer, and the transmission line. It is the objective of the present section to outline the fundamental models of the three basic components as well as the load to which the system must respond. The treatment here is for the case of balanced three-phase ac steady state operation, which is the mode of interest in optimal operation studies considered in this work.

2.3.1 The Generator Model

The first basic component of the electric power system network is the synchronous generator. The degree of detail of a synchronous machine model depends on the area of application in which one is interested. Greater detail is necessary for dynamic power systems studies involving transient analysis such as stability evaluation. It is universal practice in synchronous machine modeling to apply the Park transformation to the three-phase armature coils. The result is a set of algebraic and differential equations relating to two equivalent orthogonal armature coils whose location is determined by the rotor position. The range of forms based on Park's transformation models is tremendous. The primary source of these models has been power system transient stability analysis.

The updating time interval involved in economic operation algorithms and controllers is longer than that for stability studies. Furthermore, the system is assumed to be in a sinusoidal steady state for modeling purposes. The control action in this case is to move the system from a certain steady state level to the next in an optimal way. Less detailed models are thus considered sufficient for economic operation purposes. It should be emphasized here that for shorter updating time intervals (less than five minutes) it may become necessary for us to include the machine, exciter, and governor dynamics in some detail. The order of magnitude of the problem becomes obvious if we note that the dynamical model of a synchronous machine with amortisseur windings, exciter, and turbine governor can involve fourteen differential equations. Other controls such as the stabilizer and boiler offer added complexity.

The synchronous machine model commonly used in optimal economy operation studies is that of the classical voltage source behind the equivalent characteristic reactance (synchronous and armature reaction reactance). This model is a consequence of the equations based on the detailed Park's transformation. The basic assumptions involved are those of balanced loading and sinusoidal steady state operation. We can thus avail ourselves of the conveniences offered by phasor diagram techniques.

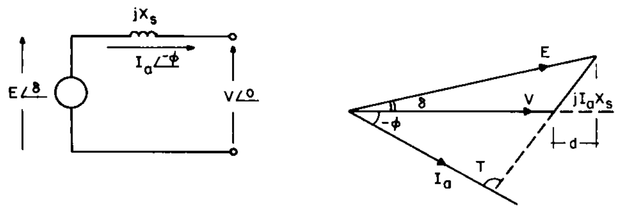

Fig. 2.9 Equivalent circuit and phasor diagram of a synchronous machine.

The simplest equivalent circuit and the associated phasor diagram for a synchronous machine are shown in Fig. 2.9. The armature resistance is neglected in this presentation since our object is to demonstrate the basis for limitations accounted for in our type of study. The machine terminal voltage V is taken as reference with the armature current I_a lagging by an angle ϕ. A fictitious voltage source E represents the voltage required to be generated by the field current I_f. The angle δ is the torque angle. Here j denotes $\sqrt{-1}$ as usual.

From the basic definition of active and reactive power, we have for one phase:

$$P_G = VI_a \cos\phi \tag{2.3.1}$$

$$Q_G = VI_a \sin\phi \tag{2.3.2}$$

From the geometry of the phasor diagram

$$E \sin\delta = I_a X_s \cos\phi \tag{2.3.3}$$

Using Eq. (2.3.3) in (2.3.1) we arrive at

$$P_G = (EV/X_s) \sin\delta \tag{2.3.4}$$

Moreover, we have

$$E\cos\delta = V + I_a X_s \sin\phi \tag{2.3.5}$$

Using Eq. (2.3.5) in Eq. (2.3.2), we obtain

$$Q_G = (V/X_s)(E\cos\delta - V) \tag{2.3.6}$$

In the above X_s denotes the machine's synchronous reactance.

Equations (2.3.4) and (2.3.6) have some important consequences. The first tells us that the generated power is limited to a certain maximum. This maximum is given by

$$P_{max} = EV/X_s$$

In practice the allowable maximum is about half the above value due to system stability considerations. The active generated power is controlled mainly through the torque input from the turbine, which is regulated by the governor system. The reactive power generated as given by Eq. (2.3.6) depends on the difference

$$d = E\cos\delta - V$$

Normal excitation is defined for $d = 0$, or

$$V = E\cos\delta$$

overexcitation is the situation in which the generator is supplying lagging reactive power. In this case $d > 0$, or

$$E\cos\delta > V$$

For underexcitation $d < 0$, or

$$E\cos\delta < V$$

This implies that the generator is supplying leading reactive power. The value of the internal voltage E is directly proportional to the field excitation current I_f for operation in the unsaturated region

$$E = KI_f$$

The allocation of reactive power generations among the system's sources is thus controllable through control of the field excitation.

Loading of the generator is normally carried out according to its capability curve. The curve is frequently called the reactive capability curve since it shows the active power loading versus the reactive power loading and the limits of output for the rating and at various hydrogen gas pressures. Constant power factor lines are also indicated on this curve. A typical capability curve is shown in Fig. 2.10. Limits on the capability curve may be divided into

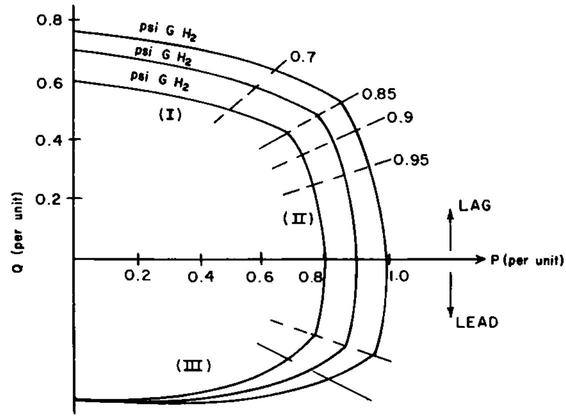

Fig. 2.10 Reactive capability curve.

three regions. In region I, field heating dictates the limits and hence the curve in this region is a constant field current locus. Armature heating is the limiting factor in region II; the curve in this region is a constant apparent power (MVA) locus. Region III of the curve is limited by end heating and stability.

2.3.2 The Transmission Line Model

An electric transmission line has four parameters which affect its performance characteristics. These parameters are the series resistance and inductance, shunt capacitance, and shunt conductance. The line resistance and inductive reactance are of importance in almost all problems. For some studies it is possible to omit the shunt capacitance and conductance and thus simplify the equivalent circuit considerably. The determination of the parameters on the basis of line length, conductors used, and conductor spacing as they are mounted on the supporting structure has been the subject of extensive studies. For our purpose it is assumed that the parameters are available. It is further assumed that the system is operating in the sinusoidal steady state.

The so-called exact transmission line model can be derived by direct application of Kirchhoff's voltage and current laws to an incremental length of the line. The result is two second-order differential equations relating the voltage and current variations along the line. Application of boundary conditions leads to the fundamental two-port network model,

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix} \tag{2.3.7}$$

This is the A, B, C, and D parameter form. The subscripts s and r stand for sending and receiving end variables, respectively. Here the parameters are given by

$$A = \cosh\theta, \qquad B = Z(\sinh\theta)/\theta$$
$$C = Y(\sinh\theta)/\theta, \qquad D = A$$
$$\theta = \sqrt{ZY}$$

The total line series impedance is denoted by Z while the shunt admittance is Y.

An equivalent circuit favored by power system engineers is the π circuit. It is on the basis of Eq. (2.3.7) that we obtain the equivalent π circuit shown in Fig. 2.11.

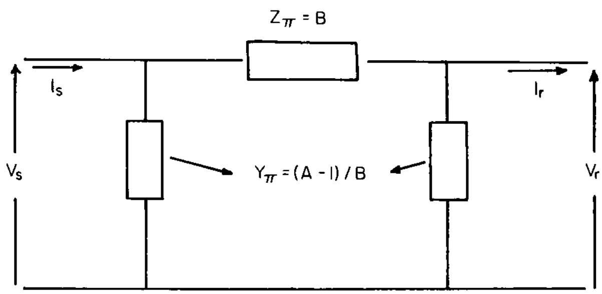

Fig. 2.11 Equivalent π circuit for a transmission line.

Consider the series expansion of the hyperbolic functions defining the A, B, C, and D parameters given by

$$A = 1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \frac{Z^3Y^3}{720} + \cdots$$

$$B = Z\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \frac{Z^3Y^3}{5040} + \cdots\right)$$

$$C = Y\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \frac{Z^3Y^3}{5040} + \cdots\right)$$

The number of terms taken into consideration when applying the above expressions will depend on the required accuracy. Usually no more than three terms are required. For overhead lines less than 500 km in length, the following approximate expressions are satisfactory:

$$A = D = 1 + ZY/2, \qquad B = Z(1 + ZY/6), \qquad C = Y(1 + ZY/6)$$

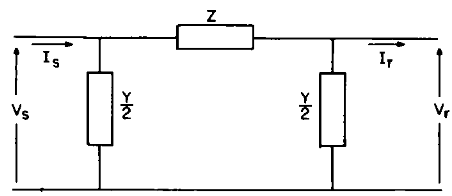

Fig. 2.12 Norminal π equivalent circuit of a transmission line.

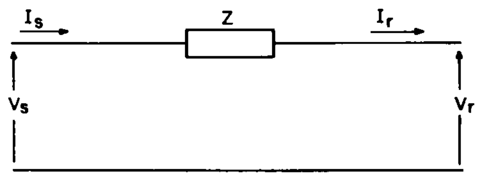

Fig. 2.13 Short line equivalent circuit.

If the first term of the expansion is used only then we obtain

$$B = Z, \qquad (A - 1)/B = Y/2$$

In this case the equivalent π circuit reduces to the nominal π model which is used generally for lines classified as medium lines (up to 250 km). Figure 2.12 shows the nominal π model of a medium transmission line. The result, obtained rigorously, can be obtained by the intuitive assumption that the lines series impedance is lumped and the shunt admittance Y is divided equally with each half placed at an end of the line.

A final model is the short line (up to 80 km) model and in this case the shunt admittance is neglected altogether. The line is thus represented only by its series impedance, as shown in Fig. 2.13.

2.3.3 The Transformer Model

The power transformer is an important component of the electric power network. The need for high power transmission capacity and reduced transmission losses requires high transmission voltages. This is made possible by the use of power transformers. As is the case with the synchronous machine and transmission line, the modeling of the power transformer has been the focus of attention of many investigators.

For our purposes the most basic form of a transformer model is the equivalent T circuit shown in Fig. 2.14. The left and right series impedances are essentially those of the primary and secondary winding. The shunt

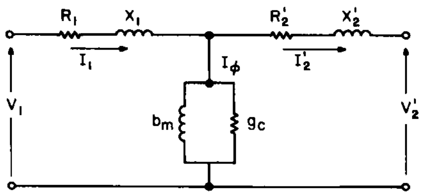

Fig. 2.14 Exact equivalent circuit of an iron-core transformer.

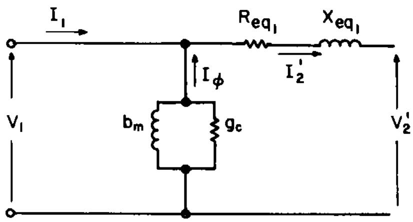

Fig. 2.15 Approximate equivalent circuit of transformer.

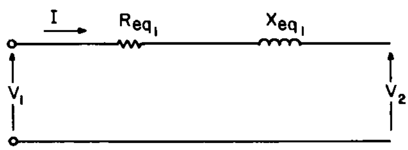

Fig. 2.16 Approximate equivalent circuit of transformer.

admittance relates to the magnetic circuit path (exciting current). If one is interested in obtaining a π representation, then the use of the Y–Δ transformation will facilitate that. However, this is not normally done since for most transformers operated at power frequencies (50 or 60 Hz), the exciting current will be small. This is of the order of 2 to 4% of the current delivered by the transformer to its secondary. Consequently the shunt branch can be moved to either end of the circuit, and the L circuit shown in Fig. 2.15 is obtained. In the figures, subscript 1 denotes primary while 2 denotes secondary quantities. The prime refers to a quantity viewed or referred to another side. Thus V'_2 is the secondary voltage referred to primary. The resistance is denoted by R, and reactance is X. Conductance is symbolized by g_c while susceptance is symbolized by b_m.

For most applications, the shunt branch may be removed altogether. This gives us only one series impedance representing the transformer with which to contend. This is shown in Fig. 2.16.

2.3.4 Load Models

An operating electric utility responds to its existing and prospective loads. The load at various points in the system is determined by the large number of devices connected by the customers to the system as well as distribution and transmission system components. The number and type of devices connected to the power system at any time depend on factors such as the time of day, the weather, the pollution level, special events, customer use patterns, to mention a few. The resulting load exhibits complex behavior which has proven to be a challenging modeling endeavor.

In trying to characterize the nature and behavior of loads, two distinct classes of load models emerge. Models belonging to the first class are referred to as load demand models. With this class one attempts to define the power demand at a given point in the system at some time. The second class is referred to as load response models. With this type of model one attempts to characterize the change in active and reactive power injected at a point in the system if the voltage and frequency at that point are varied. Both classes of models influence the effectiveness of economic operational strategies of interest. For on the one hand, the pattern function of load demand forecast is a basic driving input to which the optimizing algorithm responds. On the other hand, a realistic load response model is a basic requirement for an effective network model for use in developing the optimizing algorithms.

The class of load demand models can be partitioned into two subclasses. Over the short range, changes in load result from user decisions as to use of equipment. Over the longer range, beyond some time threshold, the situation is different. In this case the major changes are due to load growth which results from user decisions as to purchase and use of new equipment, e.g., air conditioners, electric automobiles, or whatever. The impact of both of these decision processes on the electric energy system constitutes a stochastic process. The underlying decision processes, and therefore the statistical characteristics of the respective processes, are very different. The focus of our attention is on the former class: near-term load demand models. These form the basis for short-term load forecasting.

A. Short-Term Load Forecasting

The consumption pattern in a power system is influenced by a number of factors which are essential in the development of load models for forecasting purposes. Briefly, these are classified as time and weather factors. Time factors include the effect of weak periodicity, seasonal variations, holidays, year-by-year growth or decline. This last factor is becoming more and more difficult to forecast. Weather factors include temperature, humidity, visibility, and wind speed. In addition to these, random effects are reported to present variation in the load. These include a reflection of the inherent statistical nature of the load, special events and unannounced switchings of bulk power equipment.

The most apparent difference in the available techniques for short-term load forecasting is in the effects which are considered by the hypothesized model. These effects lead to the following classifications:

I. Type 1 Models. Weather sensitive models which do not rely on the latest load behavior for the forecast. These are usually of the form

$$P_D(t) = P_B(t) + F[W(t)]$$

The base load $P_B(t)$ is sometimes assumed to be a weather insensitive function to which a weather dependent function $F(W)$ is added to define the load. In our notation subscript D refers to demand quantities.

II. Type 2 Models. Models including the time of day and latest load behavior as well as weather effects. The load is the so-called time series-type model and is assumed to be represented by

$$P_D(t) = \sum_{i=1}^{m} a_i f_i(t) + Z_0(t)$$

In this formulation $Z_0(t)$ represents the long-term trends of load behavior. The functions f_i are explicit functions of time. They can be a priori fixed by exploiting the semiperiodic weekly load behavior, in which case they are normally defined as sine curves with a basic period. Other a priori chosen time functions such as polynomials in t are also possible. A more mathematically sophisticated approach uses the autocorrelation function of $P_D(t) - Z_0(t)$ (experimentally determined) to obtain a Karhunen–Loève expansion of this process.

The basic idea behind time series models is that the load curve has a relatively recognizable shape and an attempt is made to fit a time function to it. In this form a time series model has no way of "remembering" and exploiting the immediate load's past. This important feature can, however, be incorporated by assuming the coefficients a_i to be random variables and by continuously reestimating their values according to the latest load data. This can be done by a simple estimating procedure also referred to as exponential smoothing.

Another type of model is referred to as the dynamic-type. An expression for the time variation of load is assumed to be

$$P_D(t) = \sum_{i=1}^{m} a_i f_i(t) + Z_0(t) + y(t)$$

The parameters a_i are considered constant or slowly varying. The term $y(t)$ describes the correlated load nature; that is to say, random load changes have a rather limited spectrum and take place slowly. Models for $y(t)$, which define the present trend of the load, take various forms, the simplest of which is the autoregressive moving average model:

$$y(t) = \sum_{k=1}^{n} d_k y(t-k) + \sum_{j=1}^{m} e_j u(t-j) + w(t)$$

In the above, d_k, e_j are assumed constant. The variable $u(t)$ is assumed to be a function of the weather variables at t. The function $w(t)$ is a white mean process whose variance is assumed to have the constant value Q. Dynamic load models have the property that they explicitly remember the past load behavior. The expected accuracy decreases as the prediction lead time decreases.

B. Load Response Models

The typical system load bus will supply active and reactive power to a variety of load components, each with different characteristics. Variations in voltage and frequency at a load bus will in general affect the active and reactive power requirements of the components connected to that bus. Many optimal economic operation problems include no voltage and frequency control in the formulation. In this case no accounting for the functional dependence of the load on the voltage and frequency is needed. On the other hand, when the load bus voltage is a control function accounting for the load dependence on this control action is necessary.

The primary source of load response models has been power system transient stability analysis. The first load voltage response model to be used was the constant impedance model. This results in the real and reactive load power being proportional to the square of the voltage. This was adopted initially because of its ease of implementation. Other basic models include constant current models, in which real and reactive power are proportional to voltage, and constant power models, which are independent of voltage. Loads have been modeled as linear combinations of these models, resulting in real and reactive power being polynominal functions of voltage. Other models have been proposed which express real and reactive load power as a fractional power of voltage and frequency. A general expression for the dependence of active power P_{D_i} and reactive power Q_{D_i} at bus i may thus be written as

$$P_{D_i} = \sum_{j=1}^{N_p} C_{p_j} V_i^{K_{p_j}}, \qquad Q_{D_i} = \sum_{j=1}^{N_q} C_{q_j} V_i^{K_{q_j}}$$

In the above, C_{p_j}, C_{q_j}, K_{p_j}, and K_{q_j} are constants determined by either physically based modeling procedures and/or verified by field data.

2.4 ELECTRIC NETWORK MODELS

The previous section dealt with models of the individual components of the electric power system network. It is the purpose of this section to study models of the overall network, which is made of an interconnection of the various components. As is the case with all modeling efforts, various representations are possible and these vary in sophistication depending on the requirements imposed by the scope of the study. Our approach is to present models that increase in detail and sophistication. We thus start with models that account for active power balance only. From there we proceed to introduce the idea of reactive power balance inclusion. The culmination of the section is the study of the exact load flow equations approach and the various approximations based on this precise model.

2.4.1 Active Power Balance Equations

The basic idea behind active (real) power balance equation models is the principle of continuity of power flow. The electric network is thus viewed as a medium of active power transfer from the generating nodes (buses) to the load nodes. In this approach only one equation is needed to model the electric network. The most attractive feature of this type of model is the ease of implementation. There are two basic models that account for active power losses in the electric network.

Let the active power generation at a generator bus be denoted by P_{G_i}, and the active power demand at a load bus by P_{D_i}. Generator buses are members of the set R_G while load (demand) buses belong to the set R_D. We define a system demand P_D as

$$P_D = \sum_{i \in R_D} P_{D_i} \tag{2.4.1}$$

The first active power balance equation model neglects transmission losses and hence we can write

$$P_D = \sum_{i \in R_G} P_{G_i} \tag{2.4.2}$$

This model is useful in the treatment of parallel generating units at the same plant since in this case the assumption of negligible transmission losses is valid.

Including active power transmission losses in the model of Eq. (2.4.2) leads to requiring

$$P_D = \sum_{i \in R_G} P_{G_i} - P_L(\mathbf{P}_G) \tag{2.4.3}$$

In this model the loss P_L is considered a function of the active power generations vector $\mathbf{P}_G$ and is conventionally known as the loss formula. This we treat next.

A. The Transmission Loss Formula

Fundamental to the economic dispatch of real power is the ability to include active power transmission losses. When it is necessary to transmit electric energy over large distances, or in the case of a relative low load density vast area, the transmission losses are a major factor to be considered. The active power transmission losses may amount to 20 to 30% of the total load demand.

Ideally, the exact power flow equations should be used to obtain the active power transmission losses in the power system. However, the electric power system engineer may opt for expressing the losses in terms of active power generations only. This expression is commonly referred to as the loss formula, or B-coefficient method. The simplest form of the equation is George's formula, given by

$$P_L = \sum_{i \in R_G} \sum_{j \in R_G} P_i B_{ij} P_j \tag{2.4.4}$$

The parameters B_{ij} are commonly referred to as the loss coefficients.

Attempts to obtain a more accurate expression for power system losses have been made. These result in adding a linear term and a constant to the original quadratic expression. The resulting expression, frequently called Kron's loss formula, is

$$P_L = K_{L0} + \sum_{i \in R_G} B_{i0} P_i + \sum_{i \in R_G} \sum_{j \in R_G} P_i B_{ij} P_j \tag{2.4.5}$$

The set of generating plants is denoted by R_G.

We give a simple derivation based on the power flow equations which will shed some light on the simplifying assumptions involved.

B. Derivation of the Loss Formula

We consider a system with buses in the set R_B. The system losses can be obtained by adding all powers into the buses. Thus with S denoting apparent power we have

$$S_L = \sum_{i \in R_B} S_i \tag{2.4.6}$$

In terms of bus voltages and currents we can write for the apparent power (MVA) losses

$$S_L = \mathbf{V}_B^T \mathbf{I}_B^* \tag{2.4.7}$$

The asterisk * denotes, in our case, the conjugates of the vector components. We use the bus impedance equality relating voltages and currents given by

$$\mathbf{V}_{\mathrm{B}} = \mathbf{Z}_{\mathrm{B}}\mathbf{I}_{\mathrm{B}} \tag{2.4.8}$$

Let the bus impedance matrix $\mathbf{Z}_{\mathrm{B}}$ be expressed as the sum

$$\mathbf{Z}_{\mathrm{B}} = \mathbf{R} + j\mathbf{X} \tag{2.4.9}$$

Further, we express $\mathbf{I}_{\mathrm{B}}$ in rectangular form

$$\mathbf{I}_{\mathrm{B}} = \mathbf{I}_p + j\mathbf{I}_q \tag{2.4.10}$$

The active and reactive power losses P_{L} and Q_{L} are then obtained as the real and imaginary components of S_{L} given by Eq. (2.4.7) with substitutions from Eqs. (2.4.8)–(2.4.10). The result is

$$P_{\mathrm{L}} = \mathbf{I}_p^{\mathrm{T}}\mathbf{R}\mathbf{I}_p + \mathbf{I}_q^{\mathrm{T}}\mathbf{R}\mathbf{I}_q \tag{2.4.11}$$

$$Q_{\mathrm{L}} = \mathbf{I}_p^{\mathrm{T}}\mathbf{X}\mathbf{I}_p + \mathbf{I}_q^{\mathrm{T}}\mathbf{X}\mathbf{I}_q \tag{2.4.12}$$

It is desirable at this stage to eliminate the current variables. Here we use the expression for net apparent power,

$$P_i + jQ_i = V_i I_i^*$$

We also express the bus voltages and currents as

$$V_i = |V_i|(\cos\delta_i + j\sin\delta_i) \tag{2.4.13}$$

$$I_i = I_{p_i} + jI_{q_i} \tag{2.4.14}$$

The resulting expressions for the current components obtained are

$$I_{p_i} = (1/|V_i|)[P_i(\cos\delta_i) + Q_i(\sin\delta_i)] \tag{2.4.15}$$

$$I_{q_i} = (1/|V_i|)[P_i(\sin\delta_i) - Q_i(\cos\delta_i)] \tag{2.4.16}$$

This in vector form is

$$\mathbf{I}_p = \mathbf{CP} + \mathbf{DQ} \tag{2.4.17}$$

$$\mathbf{I}_q = \mathbf{DP} - \mathbf{CQ} \tag{2.4.18}$$

where

$$\mathbf{C} = \mathrm{diag}(\cos\delta_i/|V_i|) \tag{2.4.19}$$

$$\mathbf{D} = \mathrm{diag}(\sin\delta_i/|V_i|) \tag{2.4.20}$$

The active power loss is obtained as a result of eliminating the current vectors.

$$P_{\mathrm{L}} = [\mathbf{P}^{\mathrm{T}} \quad \mathbf{Q}^{\mathrm{T}}]\begin{bmatrix} \mathbf{A}_p & -\mathbf{B}_p \\ \mathbf{B}_p & \mathbf{A}_p \end{bmatrix}\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \tag{2.4.21}$$

Elements of the newly defined matrices $\mathbf{A}_p$ and $\mathbf{B}_p$ are given by

$$a_{ij_p} = [r_{ij}\cos(\delta_i - \delta_j)]/|V_i||V_j| \tag{2.4.22}$$

$$b_{ij_p} = [r_{ij}\sin(\delta_i - \delta_j)]/|V_i||V_j| \tag{2.4.23}$$

In our development $\mathbf{P}$ and $\mathbf{Q}$ are vectors of net active and reactive powers *into* the system. We now partition these vectors in terms of net generation and net demand values at each bus as

$$\mathbf{P} = [\mathbf{P}_\mathrm{G}^\mathrm{T} | -\mathbf{P}_\mathrm{D}^\mathrm{T}] \tag{2.4.24}$$

$$\mathbf{Q} = [\mathbf{Q}_\mathrm{G}^\mathrm{T} | -\mathbf{Q}_\mathrm{D}^\mathrm{T}] \tag{2.4.25}$$

The active power loss expression is thus obtained in terms of the generated active and reactive powers as

$$P_\mathrm{L} = [\mathbf{P}_\mathrm{G}^\mathrm{T} \quad \mathbf{Q}_\mathrm{G}^\mathrm{T}]\begin{bmatrix} \mathbf{A}_{p\mathrm{GG}} & -\mathbf{B}_{p\mathrm{GG}} \\ \mathbf{B}_{p\mathrm{GG}} & \mathbf{A}_{p\mathrm{GG}} \end{bmatrix}\begin{bmatrix} \mathbf{P}_\mathrm{G} \\ \mathbf{Q}_\mathrm{G} \end{bmatrix} + [\mathbf{E}_{pp}^\mathrm{T} \quad \mathbf{E}_{pq}^\mathrm{T}]\begin{bmatrix} \mathbf{P}_\mathrm{G} \\ \mathbf{Q}_\mathrm{G} \end{bmatrix} + \tilde{K}_{\mathrm{L}0_p} \tag{2.4.26}$$

In the above expression the loss formula parameters are

$$\tilde{K}_{\mathrm{L}0_p} = [\mathbf{P}_\mathrm{D}^\mathrm{T} \quad \mathbf{Q}_\mathrm{D}^\mathrm{T}]\begin{bmatrix} \mathbf{A}_{p\mathrm{DD}} & -\mathbf{B}_{p\mathrm{DD}} \\ \mathbf{B}_{p\mathrm{DD}} & \mathbf{A}_{p\mathrm{DD}} \end{bmatrix}\begin{bmatrix} \mathbf{P}_\mathrm{D} \\ \mathbf{Q}_\mathrm{D} \end{bmatrix} \tag{2.4.27}$$

$$\mathbf{E}_{pp} = 2(\mathbf{B}_{p\mathrm{GD}}\mathbf{Q}_\mathrm{D} - \mathbf{A}_{p\mathrm{GD}}\mathbf{P}_\mathrm{D}) \tag{2.4.28}$$

$$\mathbf{E}_{pq} = 2(\mathbf{B}_{p\mathrm{DG}}^\mathrm{T}\mathbf{P}_\mathrm{D} - \mathbf{A}_{p\mathrm{DG}}^\mathrm{T}\mathbf{Q}_\mathrm{D}) \tag{2.4.29}$$

Note that $\mathbf{A}_p$ is symmetric:

$$\mathbf{A}_p = \mathbf{A}_p^\mathrm{T}$$

Also

$$\mathbf{B}_p = -\mathbf{B}_p^\mathrm{T}$$

The dimension of each of these two matrices is equal to the number of buses in the system. In the above expressions partitioning of the following matrices as shown below is implied:

$$\mathbf{A}_p = \begin{bmatrix} \mathbf{A}_{p\mathrm{GG}} & \mathbf{A}_{p\mathrm{GD}} \\ \mathbf{A}_{p\mathrm{DG}} & \mathbf{A}_{p\mathrm{DD}} \end{bmatrix} \tag{2.4.30}$$

$$\mathbf{B}_p = \begin{bmatrix} \mathbf{B}_{p\mathrm{GG}} & \mathbf{B}_{p\mathrm{GD}} \\ \mathbf{B}_{p\mathrm{DG}} & \mathbf{B}_{p\mathrm{DD}} \end{bmatrix} \tag{2.4.31}$$

The active power loss equation derived so far is exact and has time varying coefficients. The general loss formula is obtained by making the following assumptions:

(1) Assume a linear generator reactive characteristic such that

$$Q_{G_i} = Q_{G_{i_0}} + f_i P_{G_i}$$

(2) Assume constant generator angular positions δ_i.
(3) Assume constant generator–bus voltage magnitudes.
(4) Assume a fixed demand pattern.

In terms of the physical variables of the system the loss formula coefficients are defined by

$$\mathbf{B}_L = \mathbf{A}_{pGG} + \mathbf{F}^T\mathbf{A}_{pGG}\mathbf{F} + 2\mathbf{F}^T\mathbf{B}_{pGG} \tag{2.4.32}$$
$$\mathbf{B}_{L0}^T = \mathbf{E}_{pp}^T + 2\mathbf{Q}_{G0}^T(\mathbf{A}_{pGG}\mathbf{F} + \mathbf{B}_{pGG}) + \mathbf{E}_{pq}^T\mathbf{F} \tag{2.4.33}$$
$$K_{L0} = \tilde{K}_{L0p} + \mathbf{Q}_{G0}^T\mathbf{A}_{pGG}\mathbf{Q}_{G0} + \mathbf{E}_{pq}^T\mathbf{Q}_{G0} \tag{2.4.34}$$

Here

$$\mathbf{Q}_{G0} = \mathrm{col}(Q_{G_{i_0}})$$
$$\mathbf{F} = \mathrm{diag}(f_i)$$

The resulting expression is the celebrated general loss formula:

$$P_L = \mathbf{P}_G^T\mathbf{B}_L\mathbf{P}_G + \mathbf{B}_{L0}^T\mathbf{P}_G + K_{L0} \tag{2.4.35}$$

Here $\mathbf{B}_L$ is an $N_G \times N_G$ square symmetric matrix of the loss coefficients B_{ij}. $\mathbf{B}_{L0}^T$ is a $1 \times N_G$ row vector of linear loss coefficients B_{i0}; K_{L0} is a constant. An alternative way of writing the loss formula is

$$P_L = \sum_{i \in R_G} \sum_{j \in R_G} P_{G_i} B_{ij} P_j + \sum_{i \in R_G} B_{i0} P_{G_i} + \tilde{K}_{L0} \tag{2.4.36}$$

Amazingly enough, this loss formula can be obtained by simply using the first three terms of the Taylor expansion of a function of several variables, assuming, of course, that the losses are dependent on the active power generations only. This we discuss next.

C. Taylor Expansion Interpretation

In the case of a multivariable function, the truncated Taylor expansion takes the form

$$f(\mathbf{x} + \boldsymbol{\Delta}) = f(\mathbf{x}) + \boldsymbol{\Delta}^T \nabla f\big|_{\mathbf{x}} + \tfrac{1}{2}\boldsymbol{\Delta}^T\mathbf{H}_{\mathbf{x}}\boldsymbol{\Delta} \tag{2.4.37}$$

Here $\mathbf{x}$ is a nominal vector and $\boldsymbol{\Delta}$ is the increment vector. The gradient of $\mathbf{f}$ is

$$\nabla\mathbf{f} = \mathrm{col}(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n) \tag{2.4.38}$$

The Hessian matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & & \\ \vdots & & & \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{2.4.39}$$

We offer now an interpretation of the generalized active transmission loss formula as a consequence of Taylor expansion.

The transmission losses $P_L(\mathbf{P})$ for an operating vector $\mathbf{P}$ sufficiently near to a nominal $\mathbf{P}_0$ (base case) is

$$P_L(\mathbf{P}) = P_L(\mathbf{P}_0 + \mathbf{\Delta}) \tag{2.4.40}$$

Using the truncated Taylor expansion, we find that

$$P_L(\mathbf{P}) = K_{L0} + \mathbf{B}_0^T\mathbf{P} + \mathbf{P}^T\mathbf{B}_L\mathbf{P} \tag{2.4.41}$$

Here we have

$$\mathbf{B}_L = \mathbf{H} \tag{2.4.42}$$

$$\mathbf{B}_0^T = \nabla\mathbf{P}_L^T - 2\mathbf{P}_0^T\mathbf{B}_L^T \tag{2.4.43}$$

$$K_{L0} = P_L(\mathbf{P}_0) - \mathbf{P}_0^T\nabla\mathbf{P}_L + \mathbf{P}_0^T\mathbf{B}_L\mathbf{P}_0 \tag{2.4.44}$$

The gradient and Hessian are both evaluated at the base case power vector $\mathbf{P}_0$.

The ease of computation possible with the use of the loss formula is highly advantageous, especially if the complexity of calculating these coefficients can be reduced without loss of accuracy. It should be emphasized that there are a number of approximations involved in the loss formula and that it is valid only for a certain range of loadings. It is surprising that in practice this produces such close answers, with errors up to a few percent. Very sophisticated methods of calculating the B constants exist and are being used by utilities in connection with the economy dispatch problem.

2.4.2 Active–Reactive Power Balance Equations (ARPBE)

We have seen by means of the development of the transmission loss formula using the bus impedance formulation that the active power loss P_L given by Eq. (2.4.26) depends on the reactive power generation vector $\mathbf{Q}_G$. We note also that the reactive power loss expression Q_L given by Eq. (2.4.12) is similar to the active loss expression of Eq. (2.4.11). We can reason then that less approximation will be involved if we opt for the following model of the transmission network:

$$P_D = \sum_{i \in R_G} P_{G_i} - P_L(\mathbf{P}_G, \mathbf{Q}_G) \tag{2.4.45}$$

$$Q_D = \sum_{i \in R_G} Q_{G_i} - Q_L(\mathbf{P}_G, \mathbf{Q}_G) \tag{2.4.46}$$

The above relations are called the active–reactive power balance equations (ARPBE).

In the ARPBE approach we take into account the active and reactive power balance of the system. Following the development of transmission loss formula we can write

$$Q_L = [\mathbf{P}_G^T \quad \mathbf{Q}_G^T]\begin{bmatrix} \mathbf{A}_{qGG} & -\mathbf{B}_{qGG} \\ \mathbf{B}_{qGG} & \mathbf{A}_{qGG} \end{bmatrix}\begin{bmatrix} \mathbf{P}_G \\ \mathbf{Q}_G \end{bmatrix} + [\mathbf{E}_{qp}^T \quad \mathbf{E}_{qq}^T]\begin{bmatrix} \mathbf{P}_G \\ \mathbf{Q}_G \end{bmatrix} + K_{L0_Q} \tag{2.4.47}$$

In the above, we have defined the matrices $\mathbf{A}_q$ and $\mathbf{B}_q$. Elements of these are related to elements of $\mathbf{A}_p$ and $\mathbf{B}_p$ by

$$a_{ij_q} = (x_{ij}/r_{ij})a_{ij_p} \tag{2.4.48}$$

$$b_{ij_q} = (x_{ij}/r_{ij})b_{ij_p} \tag{2.4.49}$$

where x_{ij} are the reactive components of the bus impedance matrix. Moreover, we have

$$\mathbf{E}_{qp} = -2(\mathbf{A}_{qGD}\mathbf{P}_D - \mathbf{B}_{qGD}\mathbf{Q}_D) \tag{2.4.50}$$

$$\mathbf{E}_{qq} = 2(\mathbf{B}_{qDG}^T\mathbf{P}_D - \mathbf{A}_{qDG}^T\mathbf{Q}_D) \tag{2.4.51}$$

$$\mathbf{K}_{L0_Q} = [\mathbf{P}_D^T \quad \mathbf{Q}_D^T]\begin{bmatrix} \mathbf{A}_{qDD} & -\mathbf{B}_{qDD} \\ \mathbf{B}_{qDD} & \mathbf{A}_{qDD} \end{bmatrix}\begin{bmatrix} \mathbf{P}_D \\ \mathbf{Q}_D \end{bmatrix} \tag{2.4.52}$$

The interpretation of the ARPBE as a consequence of Taylor's expansion follows immediately if we note that our new assumptions simply require dependence on reactive power generations. The attractive feature of the ARPBE model is that it allows inclusion of the reactive power considerations using a simple model.

2.4.3 Exact Load Flow Models

Load flow models are the direct result of applying the basic Kirchhoff's circuit laws in conjunction with passive component terminal characteristics in the sinusoidal steady state (complex Ohm's law). Network models may be classified into three distinct classes: nodal models, loop models, and branch models. In nodal models (or bus frame of reference) the voltage and current variables are nodal quantities. In loop models (or loop frame of reference) loop variables are used. In branch models, branch currents and voltages are the set of variables. This last class is not in common use. Each of the three classes can further be classified into impedance and admittance subclasses. The independent variables in the former subclass are currents and in the latter are voltages.

The nodal admittance model is almost universally preferred in power system applications. This is described by the system of equations in vector form

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} \tag{2.4.53}$$

where $\mathbf{Y}$ is the nodal admittance matrix. $\mathbf{Y}$ is square, sparce, and symmetrical. The symmetry is not preserved if the network includes phase shifters, or mutual couplings. The vectors $\mathbf{I}$ and $\mathbf{V}$ are nodal current and voltage injection vectors, respectively.

For an n-node system, we can write

$$I_i = \sum_{k \in \alpha_i} Y_{ik} V_k, \qquad i = 1, \ldots, n \tag{2.4.54}$$

α_i is the set of nodes connected to the ith node. Specification of current in load flow type studies is seldom used. Instead, power specifications are used. We can thus write (2.4.54) in the alternative form

$$P_i - jQ_i = V_i^* \sum_{k \in \alpha_i} Y_{ik} V_k, \qquad i = 1, \ldots, n \tag{2.4.55}$$

Use has been made of

$$I_i = S_i^* / V_i^*.$$

The system equations (2.4.55) can further be written as

$$P_i = \mathrm{Re}\left(V_i^* \sum_{k \in \alpha_i} Y_{ik} V_k \right), \qquad i = 1, \ldots, n \tag{2.4.56}$$

$$Q_i = -\mathrm{Im}\left(V_i^* \sum_{k \in \alpha_i} Y_{ik} V_k \right), \qquad i = 1, \ldots, n \tag{2.4.57}$$

In the above, the asterisk denotes complex conjugation, Re denotes taking the real part, and Im denotes taking the imaginary part, of the complex arguments. Note that corresponding to each node there are four variables and two equations. The four variables are active power P_i, reactive power

Q_i, voltage magnitude, and phase angle. It is thus necessary to specify two variables per node to completely solve for the network state.

The load flow (power flow) problem is concerned with the solution of Eqs. (2.4.56) and (2.4.57) for the static operating conditions of the electric power system. Load flow calculations are routinely performed for power system planning, operational planning, and operation control purposes. The static operating state of the system is defined by the constraints on power and/or voltage at the network buses. In conventional load flows, buses are categorized according to the two variables specified.

(1) A load bus (P–Q bus) is one at which $S_i = P_i + jQ_i$ is specified.

(2) A generator bus (P–V bus) is a bus with specified injected active power and a fixed voltage magnitude.

(3) A system slack (swing) bus at which both the magnitude and phase angle of the voltage are specified. This is a fictitious concept which arises because the system active power loss is not known precisely in advance. It is customary to choose one of the available P–V buses as slack and to regard its active power as the unknown.

As we have seen before, each bus is modeled by two equations. In all we have $2n$ equations in $2n$ unknowns. These are $|V|$ and δ at the load buses, Q and δ at generator buses, and P and Q at the slack bus. It is because of this that the problem is nonlinear. Hence iterative methods are used to obtain the solution.

The fundamental nodal admittance matrix load flow model is given by Eqs. (2.4.56) and (2.4.57). The right-hand side of both equations involves a complex expression in the phasor voltages and admittances. Expressing these quantities in polar form is called the polar formulation. In this case the following notation is introduced:

$$V_i = |V_i| \exp(j\delta_i) \tag{2.4.58}$$

$$Y_{ik} = |Y_{ik}| \exp(-j\theta_{ik}) \tag{2.4.59}$$

The result is

$$P_i = |V_i| \sum_{k \in \alpha_i} |Y_{ik}| |V_k| \cos(\delta_i - \delta_k + \theta_{ik}) \tag{2.4.60}$$

$$Q_i = |V_i| \sum_{k \in \alpha_i} |Y_{ik}| |V_k| \sin(\delta_i - \delta_k + \theta_{ik}) \tag{2.4.61}$$

In the rectangular form, the following notation is introduced:

$$V_i = e_i + jf_i \tag{2.4.62}$$

$$Y_{ik} = G_{ik} - jB_{ik} \tag{2.4.63}$$

The rectangular form of the load flow equations thus obtained is

$$P_i = \sum_{k \in \alpha_i} [e_i(e_k G_{ik} + f_k B_{ik}) + f_i(f_k G_{ik} - e_k B_{ik})] \qquad (2.4.64)$$

$$Q_i = \sum_{k \in \alpha_i} [f_i(e_k G_{ik} + f_k B_{ik}) - e_i(f_k G_{ik} - e_k B_{ik})] \qquad (2.4.65)$$

A third form, frequently called the hybrid form, results if we express the voltage in polar form and admittance in rectangular form. Thus we use (2.4.58) and (2.4.63) in (2.4.56) and (2.4.57). The result is

$$P_i = |V_i| \sum_{k \in \alpha_i} |V_k|[G_{ik}\cos(\delta_{ik}) - B_{ik}\sin(\delta_{ik})] \qquad (2.4.66)$$

$$Q_i = |V_i| \sum_{k \in \alpha_i} |V_k|[G_{ik}\sin(\delta_{ik}) + B_{ik}\cos(\delta_{ik})] \qquad (2.4.67)$$

where

$$\delta_{ik} = \delta_i - \delta_k \qquad (2.4.68)$$

In optimal load flow studies the specified variables will be less than those specified in a conventional load flow study. The control variables may include active power generation at generator buses, reactive power generation, voltage magnitudes, transformer taps, and phase shift of the phase shifting transformer. In such applications the rectangular form, (2.4.64) and (2.4.65), and the hybrid form, (2.4.66) and (2.4.67), are preferred.

2.4.4 Approximate Load Flow Models

These models are derived from the fundamental load flow equations with the help of certain simplifying assumptions. Primarily derived for steady state security monitoring by fast approximate load flow solutions, the models nevertheless hold out promise of replacing the active power balance equation in optimal operation studies.

A basic variety of approximate models is a consequence of the Taylor expansion and advances in the application of the Newton–Raphson method for load flow solutions. Assume that the system state for a base load flow is known. Thus $P_i^{(0)}, Q_i^{(0)}, V_i^{(0)}, \delta_i^{(0)}$ are available. The performance of the system in the neighbourhood of the base case can be described approximately, using the hybrid form (2.4.66) and (2.4.67), by

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{(0)} \\ \mathbf{Q}^{(0)} \end{bmatrix} + \begin{bmatrix} \mathbf{H} & \mathbf{N} \\ \mathbf{J} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\delta} \\ \left(\dfrac{\Delta\mathbf{V}}{\mathbf{V}}\right) \end{bmatrix} \qquad (2.4.69)$$

The Jacobian matrix of the right-hand side of the above equations has element submatrices defined by

$$\begin{aligned}
H_{ik} &= |V_i||V_k|(G_{ik}\sin\delta_{ik} + B_{ik}\cos\delta_{ik}), & H_{ii} &= -Q_i + B_{ii}|V_i|^2,\\
N_{ik} &= |V_i||V_k|(G_{ik}\cos\delta_{ik} - B_{ik}\sin\delta_{ik}), & N_{ii} &= P_i + G_{ii}|V_i|^2,\\
L_{ik} &= H_{ik}, & L_{ii} &= Q_i + B_i|V_i|^2,\\
J_{ik} &= -N_{ik}, & J_{ii} &= P_i - G_{ii}|V_i|^2.
\end{aligned}$$

The right-hand sides of the above are evaluated at the base case. The same approach can be utilized in the rectangular form, (2.4.64) and (2.4.65).

Carpentier's decoupled load flow recognizes the strong coupling between the active power and phase angles, and between reactive power and voltage magnitudes. In this case the submatrices **N** and **J** in (2.4.69) are neglected. The resulting decoupled linear load flow model is thus given by

$$\mathbf{P} = \mathbf{P}^{(0)} + \mathbf{H}\,\Delta\boldsymbol{\delta} \tag{2.4.70}$$

$$\mathbf{Q} = \mathbf{Q}^{(0)} + \mathbf{L}\left(\frac{\Delta\mathbf{V}}{\mathbf{V}}\right) \tag{2.4.71}$$

These decoupled load flow equations are the bases for many fast and efficient algorithms for steady state security monitoring.

2.5 HYDRO NETWORK MODELS

Performance modeling for hydro plants including turbine and reservoir characteristics and the attendant operational constraints have been considered in Section 2.2. Two aspects related to the hydro subsystem are further dealt with in the present section.

The first arises as a result of the location of more than one plant on the same stream. This may necessitate the inclusion of the river flow dynamics in the modeling effort. Here we discuss two approaches for modeling purposes. The first is the elaborate Dahlin's state space model and is followed by the less sophisticated transport delay approach.

The availability of accurate reservoir inflow forecasts is of fundamental importance in economy operation studies. We discuss briefly the basis of such forecasts to conclude this section.

2.5.1 River Flow Modeling

The hydraulic coupling effects between hydro plants on the same stream may prove in certain cases to be an important factor in economy dispatch problems. The forebay elevation at a downstream plant influences the head and thereby the active generation of this plant. This also affects the upstream plant's tailrace elevation. On the other hand, the water discharge at the upstream plant affects the forebay at the downstream plant. The propagation characteristics of the river sections between plants (transport delay) are thus of importance in some optimal dispatch problems. We outline aspects of river flow dynamics modeling which pertain to our type of study.

A. A State Space Model

Consider two hydro plants on the same stream as shown in Fig. 2.17. Plant 1 is upstream to plant 2. The river is divided into I elements, each one with a width equal to the average for the corresponding river section. Each element is considered friction-free, and all friction appears as a head loss between the elements. The water surface in the model is therefore discontinuous at the junction of the elements. Any number of elements can be chosen to represent a certain segment of a river, depending upon the accuracy needed. It is convenient to select the lengths of the elements so that the wave propagation time across the element is equal for all. Since the wave velocity is a function of average depth, such a model will in general not have elements of equal length.

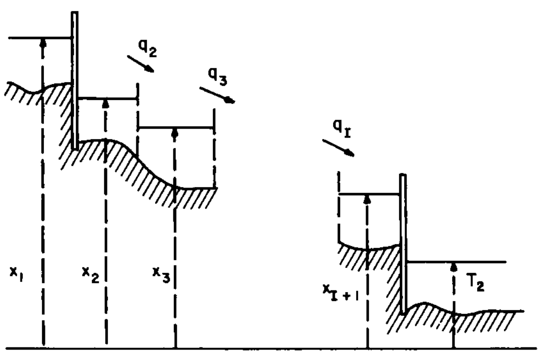

Fig. 2.17 River flow elements.

Let x_1 be the upstream plant's forebay elevation; x_i is the elevation of the ith element. The head loss between two consecutive elements is expressed as

$$x_i(t) - x_{i+1}(t) = (K_i/E_i^2)q_i^2(t), \qquad i = 2, \ldots, I \tag{2.5.1}$$

The average cross-sectional area of the river element is E_i, and K_i is a constant friction coefficient for each element. The continuity of flow equations is given by

$$\dot{x}_1(t) = \beta_{y_1}[i_1(t) - q_1(t)] \tag{2.5.2}$$

$$\dot{x}_i(t) = \beta_{y_i}[q_{i-1}(t) - q_i(t)], \qquad i = 2, 3, \ldots, I \tag{2.5.3}$$

$$\dot{x}_{I+1}(t) = \beta_{y_{I+1}}[i_2(t) + q_I(t) - q_{I+1}(t)] \tag{2.5.4}$$

The inverse of element water surface area is denoted β_{y_i} and the natural reservoir inflows for the upstream and downstream plants are $i_1(t)$ and $i_2(t)$, respectively. Boundary conditions are

$$x_i(0) = x_{i0} \tag{2.5.5}$$

$$x_1(T_f) = b_1 \tag{2.5.6}$$

$$\sum_{i=2}^{I+1} x_i(T_f) = b_t \tag{2.5.7}$$

x_{i0}, b_1, and b_t are specified constants.

It is sometimes more convenient to eliminate the intermediate discharges from the continuity equations. This can be done using Eq. (2.5.1). The resulting expressions are nonlinear due to the presence of square root terms of the elevation differences. It is practical to linearize these equations around reference values x_{i0} corresponding to steady state flow in the river section considered. The linearized equations are given by

$$\dot{y}_1(t) = \beta_{y_1}[i(t) - q_1(t)] \tag{2.5.8}$$

$$\dot{y}_2(t) = \beta_{y_2}[q_1(t) - w_2(y_2 - y_3) - \sigma_2] \tag{2.5.9}$$

$$\dot{y}_i(t) = \beta_{y_i}[w_{i-1}(y_{i-1} - y_i) - w_i(y_i - y_{i+1})], \qquad i = 3, \ldots, I \tag{2.5.10}$$

$$\dot{y}_{I+1}(t) = \beta_{y_{I+1}}[w_I(y_I - y_{I+1}) - q_{I+1}(t) + \sigma_I] \tag{2.5.11}$$

The new variables y_i are incremental changes from the reference values x_{i0}:

$$y_i = x_i - x_{i0} \tag{2.5.12}$$

The new parameters w_i and σ are defined by

$$w_i = (E_i/2)[K_i(x_{i0} - x_{i+1,0})]^{1/2} \tag{2.5.13}$$

$$\sigma_i = E_i[(x_{i0} - x_{i+1,0})/K_i]^{1/2} \tag{2.5.14}$$

Note that for steady state flow all flow variables are equal and thus

$$q_i(t) = q(t), \qquad i = 2, \ldots, I$$

$$(E_i^2/K_i)(x_{i0} - x_{i+1,0}) = q(t), \qquad i = 2, \ldots, I$$

This accounts for the absence of the σ terms in Eq. (2.5.10). The utilization of this river flow model will be shown in Chapter 6.

Approximations. The rate of change of outflow from the ith element can be expressed as

$$dq_i/dt = (dq_i/ds_i)(ds_i/dt) \tag{2.5.15}$$

with s_i being the storage volume. In the terminology of Eq. (2.5.3) we have

$$ds_i/dt = (1/\beta_{y_i})\dot{x}_i \tag{2.5.16}$$

The continuity equations become

$$q_{i-1}(t) - q_i(t) = ds_i/dt \tag{2.5.17}$$

A common hydrological assumption relating the outflow q_i and storage is given by

$$q_i = \alpha_i s_i^{\eta_i} - \sum_{j=1}^{\infty} r_i^j (d^j s_i/dt^j) \tag{2.5.18}$$

with α_i, η_i, and r_i being constants. It is customary to discard all terms except for the first two to give

$$q_i = \alpha_i s_i^{\eta_i} - r_i(ds_i/dt) \tag{2.5.19}$$

Solving the above for (ds_i/dt) and substituting in the continuity equation (2.5.17) results in

$$s_i = (1/\alpha_i)^{1/\eta_i}[r_i q_{i-1}(t) + q_i(t)(1 - r_i)]^{1/\eta_i} \tag{2.5.20}$$

In the hydrologist's standard notation we let

$$k_i = (1/\alpha_i)^{1/\eta_i}$$

The exponent η_i is normally taken to be unity. Thus

$$s_i(t) = k_i[r_i q_{i-1}(t) + q_i(t)(1 - r_i)] \tag{2.5.21a}$$

This result is commonly known as the Muskingum's storage equation.

A special case of the Muskingum's equation results from setting $r_i = 0$. In this case we have Zoch's linear reservoir:

$$s_i(t) = k_i q_i(t) \tag{2.5.21b}$$

The coefficient k_i is dimensionally equal to time and we set

$$T_{s_i} = k_i \tag{2.5.22}$$

where T_{s_i} is called time of storage in the element. As a result we have

$$ds_i/dt = T_{s_i}(dq_i/dt) \tag{2.5.23}$$

Thus we can rewrite an approximation to Eq. (2.5.17) as

$$q_{i-1}(t) - q_i(t) = T_{s_i}(dq_i/dt) \tag{2.5.24}$$

In terms of the Laplace transform we can now write

$$q_i(s) = \frac{q_{i-1}(s) + T_{s_i} q_i(0)}{(1 + sT_{s_i})}, \qquad i = 2, 3, \ldots, I \tag{2.5.25}$$

Equation (2.5.25) enables us now to write an input–output relationship for the river flow without appealing to the intermediate variables associated with the chosen elements. The resulting expression with the further assumption of equal element storage times is

$$q_I(s) = \frac{q_1(s)}{(1 + sT_s)^{I-1}} + \sum_{i=2}^{I} \frac{T_s q_i(0)}{(1 + sT_s)^{I-i+1}} \tag{2.5.26}$$

This model is sometimes referred to as Nash's model for routing. Defining the time domain expression for the upstream discharge $q_i(t)$ enables an evaluation of the resulting inflow to the downstream reservoir $q_I(t)$ through an inverse Laplace transformation.

B. Transport Delay Approach

The state space model of river flow dynamics outlined above looks attractive from a theoretical point of view. The major drawback of this method is that it offers a problem of high dimension, which becomes more prominent when, in particular, a large number of coupled hydro plants is present in the system. The difficulties may be alleviated if we appeal to the approximate expression (2.5.26).

An alternative is the transport-delay approach. The basis here is that as the discharge wave moves downstream, its shape is changed by the addition of flow from tributaries and also because velocities at various points along the wave are not the same. Without additional inflow the modification consists of an attenuation or lengthening of the time base of the wave and a lowering of the peak flow. With additional flow the attenuation effect is still present but the increase in total volume makes it less obvious. On this basis we assume that the travel time to reach the downstream plant is τ. With this reasoning the dynamics of a downstream reservoir turn out to be described by

$$\dot{s}_2(t) = i_2(t) + e^{-\alpha d_{12}} q_1(t - \tau_{12}) - q_2(t) - \sigma_2(t) \tag{2.5.27}$$

A coarser approximation would employ the Taylor expansion. In this case we have

$$q_1(t - \tau_{12}) = q_1(t) - \tau_{12} \left.\frac{\partial q_1}{\partial t}\right|_t + \frac{\tau_{12}{}^2}{2} \left.\frac{\partial^2 q_1}{\partial t^2}\right|_t + \cdots \tag{2.5.28}$$

If the attenuation factor α is negligible, then Eq. (2.5.27) reduces to

$$\dot{s}_2(t) = i_2(t) + q_1(t - \tau_{12}) - q_2(t) - \sigma_2(t) \tag{2.5.29}$$

We will use the approximation of Eq. (2.5.29) in Chapter 6. The modification of the procedure to account for the attenuation is straightforward.

2.5.2 Reservoir Inflow Forecasting

A prediction of the future rate of natural water inflow to reservoirs and river sections between cascading plants is a basic required input for economy operation purposes. This is in the domain of hydrology. This discipline deals with the waters of the earth and associated phenomena. In the hydrologic cycle, stream flows are the result of precipitation. Precipitation occurs in the liquid form or the frozen form. Rainfall and snow-melt runoff result in direct stream flow and surface runoff. Other parts are also diverted into paths such as interception, evaporation, depression storage, and infiltration which results in ground water flows. Stream flows are the result of contributions from surface runoff, rainfall, and groundwater infiltration. Surface runoff produces the largest contribution. Rainfall produces flow almost instantly. Groundwater infiltration is a slower process. Precipitation and associated losses vary with many environmental conditions. These include geologic, topographic, vegetative, and man-made features. Temporal, seasonal, and hence varying climatic factors such as temperature, wind speed, and vapor pressure influence precipitation and associated losses.

Many complex factors are involved in the process with input as precipitation and output as runoff. Many empirical and approximate models have been developed as well as ones based on fundamental physical concepts and relations. A common approach follows the linear relationship

$$Q_r = (1/s_r)(P_r - P_{r_b}) \tag{2.5.30}$$

Where Q_r is the runoff discharge and P_r is the precipitation rate. The constants s_r and P_{r_b} are obtained by least square fitting. Improvements in the correlation are reported with the inclusion of other parameters such as antecedent precipitation, soil moisture, and season as well. Antecedent precipitation indexes (API) have received widest use. The Kohler–Linsley form expressing the API, denoted by P_a, in terms of past precipitation is

$$P_a(I) = \sum_{i=0}^{N} K^i P_r(I - i) \tag{2.5.31}$$

The constant K is commonly known as the recession constant.

The time distribution of the runoff is called a hydrograph and is essentially a graph of the time variation of discharge passing a particular point. The measured hydrograph is frequently obtained as stage (depth of flow) versus time. A runoff hydrograph is obtained by subtracting the baseflow (groundwater contribution) from the measured hydrograph. The baseflow curve is frequently assumed to be modeled by

$$Q_b(t + \Delta t) = Q_b(t)K_b^{\Delta t} \qquad K_b < 1 \tag{2.5.32}$$

Efforts to develop a storm runoff hydrograph equation have resulted in many models. A representative is given by

$$\begin{aligned} Q_p(t) &= Bt^{\alpha}e^{-\beta t}, \qquad & 0 < t < t_0 \\ Q_r(t) &= q_0 e^{-(t-t_0)/d}, \qquad & t > t_0 \end{aligned} \tag{2.5.33}$$

The recession time t_0, the corresponding discharge q_0, and the constant' d are obtained from a semilogarithmic plot of the runoff hydrograph. The peak time t_p, corresponding discharge q_p, and recession volume $\tilde{Q}_{re}$ are obtained from the hydrograph. The exponent α can be shown to be given by

$$\alpha = \ln(\tilde{Q}_{re}/dq_p)/(K + \ln K)$$

with

$$K = t_0/t_p$$

Thus the parameter β is obtained from

$$\beta = \alpha/t_p$$

Finally the constant B is given by

$$B = q_p e^{\alpha}/t_p^{\alpha}$$

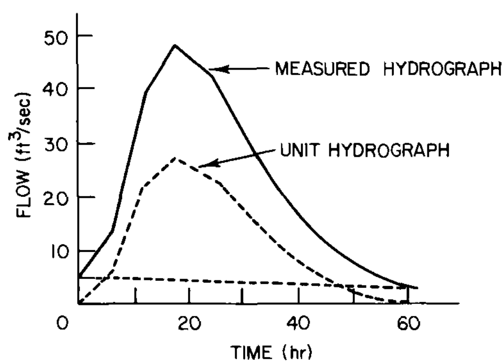

Fig. 2.18 Measured and unit hydrographs.

The concept of a unit hydrograph is of considerable utility in many areas of hydrologic design. A unit hydrograph is defined as the hydrograph that would result from 1 in. of excess rainfall occurring during a storm of particular duration. The unit hydrograph can be obtained from a runoff hydrograph by simply dividing the runoff coordinates by the average depth of rainfall excess. The latter is obtained as a result of dividing the volume of surface runoff through the watershed area. It is common practice to obtain an average unit hydrograph for the area under consideration. With a unit hydrograph available construction of hydrographs for storms of other durations and with other depth of rainfall excess is possible. Figure 2.18 shows a typical measured hydrograph and the associated unit hydrograph.

Synthetic unit hydrographs are based on empirical relationships. Typical of these is Snyder's unit hydrograph and the triangular hydrograph. The relationships for these are given for the time in hours to hydrographic peak, peak rate of discharge, and base time.

Storm runoff hydrographs can be used to provide a complete forecast of water inflows to reservoirs and river sections. Given a forecast of a sequence of events in the time interval considered, a corresponding number of hydrographs may be superimposed to provide the required forecast.

Many digital simulation packages based on a comprehensive structure of the hydrologic cycle have been developed. The packages can provide information on stream flows, provided that meteorological, geographical, and other input functions are given.

2.6 OBJECTIVE FUNCTIONALS FOR OPTIMAL OPERATION

The traditional objective in the economic operation of a power system is to minimize the cost of meeting the energy requirements of the system over some appropriate period of time and in a manner consistent with reliable service. Obviously, the aim in the utilization of limited energy resources is to realize the greatest possible value during the operation in terms of fuel replacement at those plants where the available fuel supply is not a limiting factor.

The objective in the operation of thermal or combined thermal–hydro systems would obviously be the minimization of fuel costs. These are obtained for each thermal plant on the basis of the unit input–output curve. According to the previous discussion of thermal cost models, the basic objective functional for minimization is

$$I_0 = \sum_{i \in R_s} \alpha_i + \beta_i P_{s_i} + \gamma_i P_{s_i}^2 \tag{2.6.1}$$

for the commonly accepted fuel cost representation. The set of thermal plants is denoted by R_s. If the optimization study is carried over an interval of time $[0, T_f]$, we write the objective functional as

$$J_0 = \int_0^{T_f} \left[\sum_{i \in R_s} \alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t) \right] dt \qquad (2.6.2)$$

Apart from the fuel cost objective, other objectives may be used. Among the possible objectives we have:

(1) *Minimum loss dispatch.* Here the electric power system's engineer is enquiring about the levels of generation that will minimize the loss expression or the losses expressed directly from the power flow equations. In the former case the objective functional for minimization using our previously defined terminology is

$$J_1 = \int_0^{T_f} [\mathbf{P}_G^T(t)\mathbf{B}_L \mathbf{P}_G(t) + \mathbf{B}_0^T \mathbf{P}_G(t) + K_{L0}]\, dt \qquad (2.6.3)$$

(2) *Best efficiency dispatch.* The purpose here would be to operate the hydro turbines at a discharge rate at or near the point of maximum efficiency. Thus the object would be to minimize

$$J_2 = \int_0^{T_f} \sum_{i \in R_h} [P_{h_i}(t) - P_{h_i}^M]^2\, dt \qquad (2.6.4)$$

Here $P_{h_i}^M$ denotes the best efficiency generation.

(3) *Replacement cost dispatch.* In the case of hydro generation we express the basic performance curves in terms of water input (in m^3) versus power output (in MW). The appropriate price per unit has to be predetermined to convert the performance curves to a cost basis.

Dispatch objectives other than economy ones are worth mentioning here as well. In a later chapter we will use an objective functional which attempts to achieve a minimally proportioned reactive power generation allocation. The purpose of such an objective is to ensure that reactive power loadings on the generating equipment remain within tolerable limits as set by the capability curve discussed earlier in this chapter. The objective takes on the form

$$J_Q = \int_0^{T_f} [\mathbf{Q}_G(t) - \mathbf{Q}_d(t)]^T \mathbf{K} [\mathbf{Q}_G(t) - \mathbf{Q}_d(t)]\, dt \qquad (2.6.5)$$

In the above, $\mathbf{Q}_d(t)$ is a desired reactive generation profile vector. The matrix $\mathbf{K}$ is a weighting matrix.

Recent concerns about the environmental impact of emissions from fossil fuel-fired thermal plants provided the impetus for the minimal emission dispatch problem. The objective in NOX dispatch is to minimize the total emissions of nitrous oxides from the system's thermal generating sources. An objective functional Y_{NOX_i} may be formed using expressions for NOX emitted per hour. A model describing this is given by

$$Y_{NOX_i} = a_{N_i} + b_{N_i} P_{s_i} + c_{N_i} \exp(d_{N_i} P_{s_i}) \tag{2.6.6}$$

In which a_{N_i}, b_{N_i}, c_{N_i}, and d_{N_i} are constants and P_{s_i} is the active power output of the ith thermal plant. The above model can be modified to fit into the quadratic category. In this case we write

$$Y_{NOX_i} = \alpha_{N_i} + \beta_{N_i} P_{s_i} + \gamma_{N_i} P_{s_i}^2 \tag{2.6.7}$$

The overall emission dispatch objective would be

$$J_{NOX} = \int_0^{T_f} \sum_{i \in R_G} Y_{NOX_i}[P_{s_i}(t)]\, dt \tag{2.6.8}$$

It should be noted that the above procedure can be extended to include sulfur dioxide emission models as well.

2.7 COMMENTS AND REFERENCES

Section 2.2

Detailed analysis of thermal plant performance can be found in the comprehensive Central Electricity Generating Board (CEGB) volume on operation and efficiency (1971). The Institute of Electrical and Electronic Engineers (IEEE) Committee report (1971) discusses many aspects pertinent to input–output modeling of thermal and hydro plants. A very readable treatment of the subject is given in the classic by Kirchmayer (1958). The data contained in this section for thermal unit costs are based on material published in the report (1977) sponsored by the Electric Power Research Institute (EPRI). For a discussion of input–output modeling see Happ (1975) and accompanying discussions.

The material on hydro plant types, layout, turbine types, and characteristics is standard in many water resources references. An excellent source is the text by Linsley and Franzini (1972). Among handbooks treating this subject we mention Brown (1958), Creager and Justin (1950), and Davis and Sorenson (1969). Fink and Carroll (1968) is also recommended for a general treatment of thermal as well as hydro power plants.

The efficiency model Eq. (2.2.5) is due to Christensen and El-Hawary (1973). The Glimn–Kirchmayer model is introduced in the classic paper by Glimn and Kirchmayer (1958). Hildebrand's work (1960) introduces in addition to the model Eq. (2.2.9) a reservoir routing model based on the concept of cascades of linear reservoirs. In discussing realistic modeling effects, Hamilton and Lamont (1977) derive the model Eq. (2.2.10). For a practical discussion of the preparation of performance curves the recent work by Alley (1977) is recommended.

SECTION 2.3

Most of this section is standard electric power systems and machines textbook material. A partial listing in the area of power includes Elgerd (1971, 1977), Neuenswander (1971), Stevenson (1975), and Weedy (1972). Discussions of the origin and interpretations of generator reactive capability curves are given in Carter and Gorman (1956) and Jackson (1971).

The treatment of the subject of load models has been gaining attention recently. For a comprehensive survey of the load forecasting aspects, see Galiana (1975). The earliest attempt to correlate weather and load trends is due to Dryar (1944). Davies (1958), Heineman *et al.* (1966), Matthewman and Nicholson (1968), Stanton and Gupta (1970) are specifically directed toward peak load forecasting with nonlinear weather dependent models. The development of time series model is the subject treated by Farmer and Patton (1966, 1968), Lijsen and Rosing (1971), Christiannse (1971), Gupta and Yamada (1972), and Sachdev and Ibrahim (1972). Works employing identification techniques include Galiana and Schweppe (1972), Galiana *et al.* (1974), in the last of which the maximum likelihood method is successfully applied. Techniques requiring the application of less demanding computations are reported in Vemuri *et al.* (1973), Sharma and Mahalanabis (1974), and Srinivasan and Pronovost (1975). Toyada *et al.* (1970) report on application of state estimation to short-term forecast. Reports on implementation experience are also included in Corpening *et al.* (1973) and Thompson (1976). A good review source is Ferrandiz *et al.* (1975), and for a bibliography we have Sachdev *et al.* (1977).

The modeling of induction motor loads for stability studies was considered as early as 1955 by Shankle *et al.* Brereton *et al.* (1957) use an equivalent induction motor approach to represent inductive loads. Two IEEE Committee reports (1966, 1973) deal with total load response modeling at a given bus. Basic load response models are discussed in Undrill (1975). The use of linear combinations is the innovation of Kent *et al.* (1969), who also proposed fractional powers of voltage and frequency in the model. Berg and Kar (1971) developed a linear small signal model. In Quan and Tarnawecky (1975) an alternative method for evaluating the parameters in the Berg–Kar model is proposed. Mauricio and Semlyen (1972) have suggested a three-component load model consisting of synchronous machines, nonsynchronous load, and a linear passive network. In Concordia

(1975) and deMello (1975) it is pointed out that practical testing for load characteristics has been limited to $\pm 10\%$, with the only exception being the work of Iliceto *et al.* (1972).

SECTION 2.4

The first presentation of the transmission loss formula is given by George (1943). The main reference on the subject is Kirchmayer (1958), where the historical developments in this direction are charted and a very comprehensive derivation and computational experience are given. It is probably of historical interest to mention that according to Happ (1974) the discovery of the diakoptics principle by Gabriel Kron was triggered by efforts to develop the overall transmission loss formula for a system given the formula for each of the subsystems. The loss formula derivation given here is based on the development given by Dopazo *et al.* (1967). This can also be found in Elgerd (1971). Other derivations, varying in detail, can be found in the texts by Neuenswander (1971) and Stevenson (1975).

Innovative methods for obtaining the loss formula coefficients are continually being developed. We will not repeat methods and contributions cited in Kirchmayer (1958). Instead, we will mention subsequent developments. Among these are Watson and Stadlin (1959), Kirchmayer *et al.* (1960) and the sequel paper by Happ *et al.* (1964), Tudor and Lewis (1963), Hill and Stevenson (1967, 1968), Meyer and Albertson (1969, 1971), Chenoweth (1971), Meyer (1973), Podmore (1973), Nanda *et al.* (1974), Nanda and Bijwe (1977).

The active and reactive power model is a fairly recent model even though Kirchmayer (1958) gives an expression for the reactive losses in terms of active power generations. The derivation given here is based on an extension of the formulation of Dopazo *et al.* (1967) and is reported by El-Hawary and Christensen (1976, 1977). Edelmann and Theilsiefje (1974) derived a simplified model containing only quadratic terms and no P–Q coupling.

The material on exact load flow models is standard in all textbooks on power systems analysis. A comprehensive treatment of fundamental load flow formulations and solution methods is given in Stagg and El-Abiad (1968). A review of the various methods and contributions is given in Stott (1974). Approximate load flow versions are treated as well in Brown (1975).

SECTION 2.5

The state space model is given by Dahlin (1964). The approximations given here are based on standard hydrologic assumptions. For more comprehensive treatments we refer to Chow (1964). Material on inflow forecasting can be found in Fleming (1975), Kazmann (1972), Viessman *et al.* (1972), and Hjelmfelt and Cassidy *et al.* (1975). Exact formulations are given in

Eagleson (1970). The hydrograph equation cited is due to Gupta and Moin (1974).

Section 2.6

The economy objectives have been used virtually in all major references on optimal operation of power systems. The environmental objectives were introduced in Gent and Lamont (1971). Further work can be found in Sullivan (1972), Lamont and Gent (1973), Friedmann (1973), Delson (1974), Finnigan and Fouad (1974), Lamont *et al.* (1975), and Schweppe *et al.* (1975).

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CHAPTER

3

Mathematical Optimization Techniques

3.1 INTRODUCTION

The aim of this chapter is to present briefly some optimization techniques and related topics. Emphasis is on material providing useful tools for the optimal operation problems treated in this book. We start by reviewing fundamental concepts from matrix analysis in Section 3.2. Following this, we consider a number of useful results in the optimization of static systems. Here important ideas in the treatment of constrained problems are outlined. Dynamic programming and the principle of optimality are illustrated using some simple problems in Section 3.4. The fundamental conditions of the variational calculus for dynamic system optimization are indicated in Section 3.5. Here we also include two basic constrained problems of importance to our development, namely, the isoperimetric and Lagrange problems. A brief summary of Pontryagin's maximum principle is given in Section 3.6.

The functional analytic technique of formulating optimization problems in the minimum norm form is dealt with in Section 3.7. A brief treatment of some basic concepts from functional analysis is given prior to stating one powerful version of minimum norm problems. Following this we summarize pertinent results from linear dynamic system theory that are essential for portions of our work. An overview of some important methods of solving nonlinear systems of equations iteratively is given in Section 3.9. The final section is devoted to comments and references to the literature. Throughout, we attempt to relate the results to power system problems and for added emphasis we employ the power system notation whenever possible. Due to limited space for development of the various topics, few proofs are given. Details are left to the excellent reference works cited at the end of the chapter.

3.2 A REVIEW OF VECTORS AND MATRICES

Basic to the treatment of large scale systems are the concepts of vectors and matrices. The convenience inherent in the resulting expressions makes the notation an indispensable tool for this enables us to pursue the analytic approach with a minimum of calculation independent of dimension. Our objective is to briefly review some concepts useful in our work.

3.2.1 Vectors

Let $P_1, P_2, \ldots, P_n$ be real (or complex) numbers and let **P** be an ordered set of these numbers written in the form

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \qquad \text{or} \qquad \mathbf{P} = \operatorname{col}[P_1, P_2, \ldots, P_n]$$

Then P is called an n-column vector (or simply a column vector). The ith component or element of **P** is given by P_i. In our work we utilize the concept of vector functions. This is an extension that assumes that the elements are functions of one or more variables (time is an example). Here with $P_1(t)$, $P_2(t), \ldots, P_n(t)$ being functions of time in an appropriately chosen space we have the nth dimension vector

$$\mathbf{P}(t) = \operatorname{col}[P_1(t), P_2(t), \ldots, P_n(t)]$$

A. Vector Addition

Let two vectors **P** and **Q** be given as

$$\mathbf{P} = \operatorname{col}[P_1, P_2, \ldots, P_n]$$
$$\mathbf{Q} = \operatorname{col}[Q_1, Q_2, \ldots, Q_n]$$

The two vectors are said to be equal if their components are equal, $P_i = Q_i$, for $i = 1, 2, \ldots, n$.

The sum of the two vectors is written $\mathbf{P} + \mathbf{Q}$ and defined to be the vector

$$\mathbf{P} + \mathbf{Q} = \mathrm{col}[P_1 + Q_1, P_2 + Q_2, \ldots, P_n + Q_n]$$

In general, given the vectors **P**, **Q**, and **S**, the following laws apply:

(1) $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$ (commutative law)
(2) $(\mathbf{P} + \mathbf{Q}) + \mathbf{S} = \mathbf{P} + (\mathbf{Q} + \mathbf{S})$ (associative law)

B. Scalar Multiplication

Multiplication of a vector **P** by a scalar α is defined by means of the relation

$$\alpha\mathbf{P} = \mathbf{P}\alpha = \mathrm{col}[\alpha P_1, \alpha P_2, \ldots, \alpha P_n]$$

C. Euclidean Inner Product

This is a scalar function of two vectors **P** and **Q** and will be written $\langle \mathbf{P}, \mathbf{Q} \rangle$ and defined by the relationship

$$\langle \mathbf{P}, \mathbf{Q} \rangle = \sum_{i=1}^{n} P_i Q_i$$

This is only one way of multiplying two vectors. We will pursue some generalization of this concept in the sequel.

3.2.2 Matrices

A matrix is a rectangular array of elements. The element a_{ij} is the (i, j)th element of the matrix **A** and is located in the ith row and jth column of the array. The order (size) of a matrix is $m \times n$ if the matrix includes m rows and n columns. For example, the following matrix **A** is 3×2:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

This we can also express using the shorthand notation

$$\mathbf{A} = (a_{ij})$$

As in the vector case the matrix elements may be time varying functions. It is evident that a row vector is a matrix with one row and n columns, and that a column vector is a matrix with m rows and one column. A square matrix is a matrix in which $m = n$, in which case we say that n is the matrix dimension.

We now specify some special matrix types of interest in our analysis. An $n \times n$ matrix $\mathbf{A}$ is *diagonal* matrix if all a_{ij} equal zero when $i \neq j$. For notational simplicity we write

$$\mathbf{A} = \operatorname{diag}(a_{ii})$$

The matrix $\mathbf{A}^{\mathrm{T}}$ is called the *transpose* of $\mathbf{A}$ if the element a_{ij} in $\mathbf{A}$ is equal to element a_{ji} in $\mathbf{A}^{\mathrm{T}}$ for all i and j. In general $\mathbf{A}^{\mathrm{T}}$ is obtained by interchanging the rows and the columns of $\mathbf{A}$. Consequently, if $\mathbf{A}$ is of the order $m \times n$, $\mathbf{A}^{\mathrm{T}}$ is of the order $n \times m$. An $n \times n$ matrix $\mathbf{A}$ is said to be *symmetric* if $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$. An $n \times n$ matrix $\mathbf{U}$ is said to be *skew symmetric* if $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$. A matrix $\mathbf{B} = 0$ is called a *zero matrix* if every element of $\mathbf{B}$ is equal to zero.

A. Matrix Addition

The simplest relation between matrices is that of equality; two matrices are equal if and only if their elements are equal. The sum of two matrices $\mathbf{A}$ and $\mathbf{B}$ is written $\mathbf{A} + \mathbf{B}$ and is defined by

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$

As a result of the above definition we can state the following:

Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are said to be equal matrices if, and only if, they have the same order and if each element a_{ij} is equal to the corresponding b_{ij} for all i and j.

Assuming that the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have the same order, then

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{(commutative law)}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad \text{(associative law)}$$

B. Scalar Multiplication

Multiplication of a matrix $\mathbf{A}$ by a scalar C_1 is defined by

$$C_1\mathbf{A} = \mathbf{A}C_1 = (C_1 a_{ij})$$

C. Matrix Multiplication

We begin with a linear transformation of the form

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, n$$

It is evident that the above defines a relation between the vectors $\mathbf{y}$ and $\mathbf{x}$. This can be written more compactly as

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

This relation defines the multiplication of a vector **x** by a matrix **A**.

We now consider a second linear transformation

$$\mathbf{Z} = \mathbf{B}\mathbf{y}$$

To express **Z** in terms of **x** componentwise we write

$$Z_i = \sum_{k=1}^{n} b_{ik} y_k = \sum_{k=1}^{n} b_{ik} \left(\sum_{j=1}^{n} a_{kj} x_j \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} b_{ik} a_{kj} \right) x_j$$

If we now introduce the new matrix $\mathbf{C} = (C_{ij})$ defined by

$$C_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}, \qquad i, j = 1, \ldots, n$$

We may write

$$\mathbf{Z} = \mathbf{C}\mathbf{x}$$

Observe that

$$\mathbf{Z} = \mathbf{B}\mathbf{y} = \mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{B}\mathbf{A})\mathbf{x}$$

Therefore

$$\mathbf{C} = \mathbf{B}\mathbf{A}$$

A more general result can be stated as follows. Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ can be multiplied in the order **BA** if, and only if, the number of columns of **B** is equal to the number of rows of **A**. That is, if **B** is of the order $(m \times r)$, then **A** is of the order $r \times n$, where m and n are arbitrary sizes. Let $\mathbf{D} = \mathbf{B}\mathbf{A}$; then **D** is of the order $(m \times n)$ and its elements d_{ij} are given by

$$d_{ij} = \sum_{k=1}^{r} b_{ik} a_{kj} \qquad \text{for all} \quad i \text{ and } j$$

Notice that, in general, $\mathbf{AB} \neq \mathbf{BA}$ even if **BA** is defined. If $\mathbf{AB} = \mathbf{BA}$ then we say that **A** and **B** *commute*.

An *identity* matrix **I** is a square matrix in which all the diagonal elements are "one" and all the off-diagonal elements are "zero," that is,

$$\begin{aligned} a_{ii} &= 1 \\ a_{ij} &= 0 \qquad \text{for} \quad i \neq j \end{aligned}$$

For any $m \times n$ matrix **A**,

$$\mathbf{A}\mathbf{I} = \mathbf{A}.$$

Matrix multiplication follows the following general properties.

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$$
$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$
$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$
$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$
$$\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$$
$$(\mathbf{ABC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

D. Matrix Inversion

One of the most important operations in matrix theory is inverting a matrix. Let us consider the linear transformation

$$\mathbf{y} = \mathbf{Ax}$$

If it is possible to express $\mathbf{x}$ in terms of $\mathbf{y}$ as

$$\mathbf{x} = \mathbf{By}$$

then

$$\mathbf{y} = \mathbf{A}(\mathbf{By})$$

and as a result

$$\mathbf{AB} = \mathbf{I}$$

If $\mathbf{A}$ and $\mathbf{B}$ are two n-square matrices such that $\mathbf{BA} = \mathbf{AB} = \mathbf{I}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$ and $\mathbf{A}$ the inverse of $\mathbf{B}$. The common notation for the inverse is to write $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. A matrix is said to be of a *rank* r if the largest square array in the matrix whose determinant does not vanish is of order r. A square matrix whose determinant does not vanish is called a *full-rank* or a nonsingular matrix. We have the following. If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{A}$ is nonsingular, then $\mathbf{B} = \mathbf{A}^{-1}$, which means that the inverse is unique.

Two important results can be proved for nonsingular matrices.

(i) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular n-square matrices then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
(ii) If $\mathbf{A}$ is nonsingular, then $\mathbf{AB} = \mathbf{AC}$ implies that $\mathbf{B} = \mathbf{C}$.

3.2.3 Partitioned Matrices

This is a very useful concept in our analysis. A *partitioned matrix* is one in which one or more of the entries is itself a matrix. The matrix entries of such a partitioned matrix are called *submatrices*. The main matrix is sometimes referred to as a *supermatrix*. A special case of partitioned matrices is the *generalized* or *block diagonal matrix*, for which we have the following definition: An $n \times n$ matrix $\mathbf{A}$ is a generalized or block diagonal matrix if its major diagonal entries are also major diagonal entries of partitioned square submatrices and if all entries not included in the square diagonal submatrices are zero. Such a matrix is denoted by

$$\mathbf{A} = \operatorname{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k)$$

where the submatrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ are square matrices, not necessarily of equal dimension, which appear on the major diagonal. The inverse $\mathbf{A}^{-1}$ of $\mathbf{A} = \operatorname{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k)$ is $\mathbf{A}^{-1} = \operatorname{diag}(\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}, \ldots, \mathbf{A}_k^{-1})$.

A. Multiplication of Partitioned Matrices

Multiplication of partitioned matrices follows the same rules as for general matrices. Here we think of the submatrices as elements and carry out the multiplication. Consider partitioned matrices of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1p} \\ \vdots & & & \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qp} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1r} \\ \vdots & & & \\ \mathbf{B}_{p1} & \mathbf{B}_{p2} & \cdots & \mathbf{B}_{pr} \end{bmatrix}$$

where $\mathbf{A}$ and $\mathbf{B}$ are partitioned to produce compatible submatrices. This means that the submatrices $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are such that products $\mathbf{A}_{ij}\mathbf{B}_{jk}$ are well defined for all i, j, and k. The product of $\mathbf{AB}$ is

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \cdots & \mathbf{C}_{1r} \\ \vdots & & & \\ \mathbf{C}_{q1} & \mathbf{C}_{q2} & \cdots & \mathbf{C}_{qr} \end{bmatrix}$$

where $\mathbf{C}_{ij} = \mathbf{A}_{i1}\mathbf{B}_{1j} + \mathbf{A}_{i2}\mathbf{B}_{2j} + \cdots + \mathbf{A}_{ip}\mathbf{B}_{pj}$, or $\mathbf{C}_{ij} = \sum \mathbf{A}_{ik}\mathbf{B}_{kj}$.

B. Partitioned Matrix Inversion

The usefulness of partitioned matrices is evident when considering the inversion of matrices. We have the following:

Frobenius's Relation. Consider the partitioned matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_{n \times n} & \mathbf{B}_{n \times m} \\ \mathbf{C}_{m \times n} & \mathbf{D}_{m \times m} \end{bmatrix}$$

where $\mathbf{A}$ and $\mathbf{D}$ are assumed to be nonsingular. The inverse $\mathbf{P}^{-1}$ of $\mathbf{P}$ can be expressed as

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}^{-1} \end{bmatrix}$$

in which

$$\mathbf{E} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

Alternatively we have

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{F}^{-1} & -\mathbf{F}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{F}^{-1} & (\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{F}^{-1}\mathbf{B}\mathbf{D}^{-1}) \end{bmatrix}$$

where

$$\mathbf{F} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$$

As a result we obtain some important matrix inversion lemmas. To start, we have that the submatrices in the upper left-hand corner of each inverse expression should be equal. Thus we have

$$(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

This provides the *basic lemma for Kron's method of tearing.* To obtain this result, we set

$$\mathbf{G} = -\mathbf{D}^{-1}, \qquad \hat{\mathbf{A}} = \mathbf{A} + \mathbf{B}\mathbf{G}\mathbf{C}$$

As a consequence we have

$$\hat{\mathbf{A}}^{-1} = [\mathbf{I} - \mathbf{A}^{-1}\mathbf{B}\mathbf{G}(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\mathbf{G})^{-1}\mathbf{C}]\mathbf{A}^{-1}$$

If $\mathbf{A}^{-1}$ is available, then $\hat{\mathbf{A}}^{-1}$ can be computed with a sequence that requires the inverse of a matrix of less dimension than $\hat{\mathbf{A}}$. Let us set

$$\mathbf{A} = \mathbf{B} = \mathbf{G} = \mathbf{I}$$

As a result we have the important lemma

$$(\mathbf{I} + \mathbf{C})^{-1} = \mathbf{I} - (\mathbf{I} + \mathbf{C}^{-1})^{-1}$$

3.2.4 Quadratic Forms

Given a vector $\mathbf{X}$ and a square symmetric matrix $\mathbf{A}$ the scalar function $Q(\mathbf{X})$ defined below is called a quadratic form:

$$Q(\mathbf{X}) = \mathbf{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

The matrix **A** can always be assumed symmetric since each element of every pair of coefficients a_{ij} and a_{ji} $(i \neq j)$ can be replaced by $\frac{1}{2}(a_{ij} + a_{ji})$ without changing the value of $Q(\mathbf{X})$. The assumption has several advantages and hence is taken as a restriction.

A quadratic form is said to be

(i) *positive definite* if $Q(\mathbf{X}) > 0$ for every $\mathbf{X} \neq 0$;
(ii) *positive semidefinite* if $Q(\mathbf{X}) \geq 0$ for every $\mathbf{X}$ and there exist $\mathbf{X} \neq 0$ such that $Q(\mathbf{X}) = 0$;
(iii) *negative definite* if $-Q(\mathbf{X})$ is positive definite;
(iv) *negative semidefinite* if $-Q(\mathbf{X})$ is positive semidefinite;
(v) *indefinite* if $Q(\mathbf{X})$ satisfies none of the above conditions.

Asserting definiteness of a quadratic form is of considerable importance in our work. Certain tests are available and we give some without proof.

A. Test 1

Let A be an $n \times n$ matrix of real numbers. Let $\mathbf{A}_1, \mathbf{A}_2 \ldots, \mathbf{A}_n$ be the principal submatrices of $\mathbf{A}$; that is,

$$\mathbf{A}_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{21} & \cdots & a_{2j} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} \end{bmatrix}$$

We have that $\det(A_1) = a_{11}, \det(A_2), \ldots, \det(A_n)$ are all positive quantities if and only if **A** is positive definite.

It is correct to deduce that implementing this test is not a trivial computation; calculating the determinants of fairly large arrays can be most time-consuming.

B. Test 2

A second test involves the characteristic polynomial of the square matrix **A**. Letting c be a variable, consider the expression $\det(\mathbf{A} - c\mathbf{I})$ where **I** is the $n \times n$ identity matrix. This gives

$$p(c) = \det(\mathbf{A} - c\mathbf{I}) = \det \begin{bmatrix} (a_{11} - c) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22} - c) & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn} - c) \end{bmatrix}$$

The polynomial $p(c)$ is called the characteristic polynomial of **A**, and the roots of $p(c) = 0$ are called the *characteristic values* or *eigenvalues* of **A**.

Conclusions as to the definiteness of $Q(\mathbf{x})$ can be made as follows:

(a) **A** is positive (negative) definite if the eigenvalues of **A** are all positive (negative).

(b) **A** is positive (negative) semidefinite if the eigenvalues of A are all ≥ 0 (≤ 0) and at least one is zero.

(c) **A** is indefinite if **A** has both positive and negative eigenvalues.

Some comments are appropriate. First, in general, an nth-degree polynomial may have complex roots, in which case our conclusions would be meaningless. But for symmetric **A**, the eigenvalues are always real, so there is no such difficulty. Second, the computational difficulties of Test 2 are not significantly less than those of Test 1; finding the roots of an nth-degree polynomial is not frequently a simple problem. But, of course, we are not interested in the values of the eigenvalues, only their signs; and there are some classical results that give information about the signs of the roots of an nth-degree polynomial.

Two such results, both of which the reader has probably seen previously, are

(1) *Descartes' rule of sign.* The nth-degree polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has a number of positive roots less than or equal to the number of variations in sign in the terms of $p(x)$. The number of negative roots of $p(x)$ is less than or equal to the number of variations of sign in the terms of $p(-x)$.

(2) *Descartes' extended rule of sign.* $p(x)$ has a number of positive roots equal to the number of sign variations in the terms of $p(x)$ or equal to minus an even integer. Similarly, the number of negative roots is either the number of sign variations in the terms of $p(-x)$ or minus an even integer.

These results are useful in giving information about the roots of characteristic polynomials since they will have no complex roots. Furthermore the presence and the number of zero roots is available upon inspection.

3.3 SOME RESULTS FROM STATIC OPTIMIZATION

Optimization may be defined as a mathematical tool that provides the analyst with means for finding the best (optimal) strategy to do a certain task among several alternatives. Whenever we use "best" or "optimal" to describe a strategy, a system, or a decision, the immediate question that arises is, "Best with respect to what criteria and subject to what limitations?" The process of optimization can mean either minimization or maximization of a certain objective functional. Perhaps it is in order here to mention that the terms performance measure (or index), cost functional, and goal equation are all equivalent descriptions of the criteria that one may come across in the literature. In practice, the process of minimizing (maximizing) the criterion will be such that the description of the interaction between the system variables in addition to physical limitations (constraints) are satisfied. Constraints may be broadly divided into equality and inequality constraints. They may also be classified as either linear, quadratic, or nonlinear constraints.

The purpose of this section is to review certain concepts and fundamental results from the theory of optimization for static systems. The results obtained here find immediate application in Chapter 4 where we are interested in all-thermal optimal operation problems. Moreover, extensions of these results are of considerable utility in dealing with many dynamic system optimization problems. Our approach here is to illustrate the results for simple problems that increase in their order of complexity. We therefore begin with an unconstrained problem which is then extended to include equality constraints and then, finally, to account for inequality constraints as well.

3.3.1 Unconstrained Optimization

Finding the minimum or maximum of a multivariable function is of considerable interest in this treatment. We will use a simple power system problem to introduce various approaches. We consider the operation of m thermal generators on the same bus. Assume that the variation of the fuel cost of each generator F_i with the active power output P_i is given by a quadratic polynomial. The total fuel cost of the plant is then

$$F = \sum_{i=1}^{m} a_i + b_i P_i + c_i P_i^2 \tag{3.3.1}$$

If one is interested in obtaining the power outputs so that F is a minimum, we have two classes of methods of approach to obtain the desired optimal values. The first is the class of direct methods, by which a set of equations defining the conditions of optimality is obtained. Solving the set of equations constitutes the desired answer. In the class of iterative methods a sequence of approximations to the solution is established using an appropriate algorithm. The algorithm is designed so that the sequence finally converges to the desired answer.

We will discuss here two methods in the class of direct methods. The first is the variational method which can be stated in terms of the differentiable function F. The second is the algebraic method which relies on the polynomial nature of F.

A. Variational Solution

A fundamental result from ordinary calculus is the basis of the method. This states that at an extremum (minimum, maximum, or saddle point), the first partial derivatives of F with respect to P_i are zero:

$$\partial F/\partial P_i = 0, \qquad i = 1, \ldots, m \tag{3.3.2}$$

For the expression adopted for F in (3.3.1), we obtain

$$P_{\xi_i} = -b_i/2c_i \tag{3.3.3}$$

This expression for the optimal active power is guaranteed to minimize F if the second partial derivative is positive. This condition for our case implies

$$c_i > 0$$

We denote an optimal value of P_i by P_{ξ_i} in this treatment.

B. Algebraic Solution

This method, although of some elegance, depends on the fact that we have a polynomial expression for the cost. The method as presented is applicable to quadratic functions. Extensions to higher order polynomials are possible. The objective functional can be written after some manipulations as

$$F = F_\xi + \sum_{i=1}^{m} c_i[P_i + (b_i/2c_i)]^2 \tag{3.3.4}$$

where we have defined

$$F_\xi = \sum_{i=1}^{m} [a_i - (b_i^2/4c_i)]$$

Note that F_ξ does not depend on the decision variables P_i. It is obvious that when

$$P_i + (b_i/2c_i) = 0 \tag{3.3.5}$$

F will be either a maximum or a minumum. F will have a minimum value F_ξ when the c_i's are positive. The optimum solution is thus given by

$$P_{\xi_i} = -b_i/2c_i \tag{3.3.6}$$

and $c_i > 0$ for a minimum. The idea of completing squares is obviously helpful in this case.

C. Vector Formulation

In order to gain more insight we give a vector formulation for the problem and obtain the optimal solution using the above two methods. This will be helpful in treating later methods and understanding many formulation procedures given in this book.

The cost function F may be written concisely in vector notation as

$$\hat{F} = F - \left(\sum_{i=1}^{m} a_i\right) = \mathbf{b}^{\mathrm{T}}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{C}\mathbf{P} \tag{3.3.7}$$

We next introduce the auxiliary vector of known coefficients of linear terms in P_i:

$$\mathbf{b}^{\mathrm{T}} = (b_1, \ldots, b_m)$$

Moreover, the coefficients of quadratic terms in P_i will define the diagonal matrix

$$\mathbf{C} = \operatorname{diag}(c_1, \ldots, c_m)$$

Our control variables form the vector $\mathbf{P}$ given by

$$\mathbf{P}^{\mathrm{T}} = (P_1, \ldots, P_m)$$

Thus the problem is simply to minimize

$$F = \mathbf{b}^{\mathrm{T}}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{C}\mathbf{P} \tag{3.3.8}$$

With this formulation at hand we can thus present the two methods given above as:

D. Method 1: Variational Solution

Our objective is to minimize Eq. (3.3.8) with respect to $\mathbf{P}$. For the extremum the gradient of $\hat{F}$ should be zero. This is written as

$$\nabla\hat{F} = \mathbf{b} + 2\mathbf{C}\mathbf{P} = 0 \tag{3.3.9}$$

The minimum is achieved as the solution of the above vector equation if the Hessian matrix is positive definite. This matrix is the result of taking the second partial derivatives. As a result we have

$$\mathbf{H} = 2\mathbf{C}$$

The condition for a minimum is then

$$\mathbf{H} > 0$$

The optimal is thus given by:

$$\mathbf{P}_{\xi} = -\tfrac{1}{2}\mathbf{C}^{-1}\mathbf{b} \tag{3.3.10}$$

This particular form is attractive from a bookkeeping point of view.

E. METHOD 2: ALGEBRAIC SOLUTION

The procedure becomes simply a matter of rewriting our objective as

$$\hat{F} = [\mathbf{V}^{\mathrm{T}}\mathbf{C}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{C}\mathbf{P}] \tag{3.3.11}$$

The above expression may be written as

$$\hat{F} = \{[\mathbf{P} + (\mathbf{V}/2)]^{\mathrm{T}}\mathbf{C}[\mathbf{P} + (\mathbf{V}/2)]\} - (\mathbf{V}^{\mathrm{T}}\mathbf{C}\mathbf{V}/4)$$

with

$$\mathbf{V} = \mathbf{C}^{-1}\mathbf{b}$$

Here we have completed a quadratic form which is equivalent to the method of completing the squares, used earlier. The minimum is achieved for

$$\mathbf{P}_{\xi} = -\mathbf{V}/2 \tag{3.3.12}$$

provided that $\mathbf{C}$ is positive definite. We can thus assert that

$$\mathbf{P}_{\xi} = -\tfrac{1}{2}\mathbf{C}^{-1}\mathbf{b} \tag{3.3.13}$$

It is noted that the above result holds true for any symmetric positive definite $\mathbf{C}$. Note that it is not necessary to have the diagonal form for $\mathbf{C}$ to get this result.

3.3.2 Equality Constrained Optimization

Optimization problems in practice are seldom unconstrained. The usual situation is one in which the cost is to be minimized subject to satisfying certain equations. These equations are normally based on models describing the interaction between the control and other physical variables. We will use the same example of m thermal generators used earlier to illustrate the approach.

As before, we assume the fuel cost to vary with the active power generated:

$$F = \sum_{i=1}^{m} a_i + b_i P_i + c_i P_i^2 \tag{3.3.14}$$

Our concern now is to determine P_{ξ_i} so that F is a minimum while satisfying the active power balance equation

$$P_{\mathrm{D}} = \sum_{i=1}^{m} P_i \tag{3.3.15}$$

with P_D being a given active power demand. Note that the results of the unconstrained minimization cited earlier lead to

$$\sum_{i=1}^{m} P_{i\xi} = -\tfrac{1}{2} \sum_{i=1}^{m} (b_i/c_i) \tag{3.3.16}$$

The above sum is not equal to the power demand P_D as a general case. We conclude that an alternative way for minimizing F subject to satisfying P_D is required.

We explore methods of solution to the constrained optimization problem and in particular emphasize the idea of transforming the problem into an unconstrained one. With this transformation earlier results can be applied immediately. The first method is the elimination of one variable, which requires that at least one variable can be expressed in terms of the remaining variables using the given constraints. The second method discussed is the Lagrange multiplier approach and the third is the penalty function approach.

A. Elimination of Variables

The constraint equation (3.3.15) may be written to express one variable as

$$P_m = P_D - \sum_{i=1}^{m-1} P_i \tag{3.3.17}$$

Substituting for P_m in Eq. (3.3.14) yields

$$F = \left(\sum_{i=1}^{m-1} a_i + b_i P_i + c_i P_i^2\right) + b_m\left(P_D - \sum_{i=1}^{m-1} P_i\right) + c_m\left(P_D - \sum_{i=1}^{m-1} P_i\right)^2 + a_m \tag{3.3.18}$$

Note that F now is a function in $(m - 1)$ variables P_i $(i = 1, \ldots, m - 1)$. Any method for unconstrained minimization can now be applied to this unconstrained problem.

B. The Lagrange Multiplier

Although the method of variable elimination is satisfactory in the present example, we will find that certain difficulties may arise when the constraints are nonlinear. An alternate method which is popular is the Lagrange multiplier technique. This we apply to our problem by noting that the constraint equation (3.3.15) may be written as

$$\sum_{i=1}^{m} P_i - P_D = 0 \tag{3.3.19}$$

The technique is based on including Eq. (3.3.19) in the original cost function by use of a Lagrange multiplier, say λ:

$$F = \sum_{i=1}^{m} a_i + b_i P_i + c_i P_i^2 + \lambda\left(\sum_{i=1}^{m} P_i - P_{\mathrm{D}}\right) \tag{3.3.20}$$

Note that λ is to be obtained so that Eq. (3.3.19) is satisfied. The idea here is to penalize any violation of the constraint by adding the resulting error term. The Lagrange multiplier is in effect a conversion factor that accounts for the dimensional incompatibilities of the cost function and constraints. Here again the resulting problem is an unconstrained one and we have increased the number of unknowns by one.

The optimality conditions obtained are

$$\partial F/\partial P_i = b_i + 2c_i P_i + \lambda = 0 \tag{3.3.21}$$

$$\partial F/\partial \lambda = \sum_{i=1}^{m} P_i - P_{\mathrm{D}} \tag{3.3.22}$$

Observe that we obtain the constraint as a result of applying the optimality conditions. In this example problem it is straightforward to eliminate λ and obtain an expression for the optimal power generations. This will be done in Chapter 4.

C. The Penalty Function Approach

Essentially a constrained minimization problem is transformed into an unconstrained problem or a sequence of unconstrained problems in the penalty function approach. The basic idea is to form a new function by adding a penalty term to force the solution to satisfy the constraints. Several varieties of penalty function methods have been proposed. We outline the procedure for a basic method for the example system considered.

Let us introduce a positive parameter K and form the augmented function $\hat{F}$:

$$\hat{F} = F + K\left(\sum_{i=1}^{m} P_i - P_{\mathrm{D}}\right)^2 \tag{3.3.23}$$

where F is our fuel cost as before. What we have done is in effect introduce a penalty term for any constraint violation. Clearly the minimization process will force the penalty term to a minimum. This minimum is hopefully zero and at this value the constraint is satisfied.

We assume that a variational solution is used. The resulting optimality conditions are

$$\partial \hat{F}/\partial P_i = (\partial F/\partial P_i) + 2K\left(\sum_{j=1}^{m} P_j - P_{\mathrm{D}}\right), \qquad i = 1, \ldots, m \tag{3.3.24}$$

With the quadratic cost assumed we thus have

$$[(b_i/2) - KP_{\mathrm{D}}] + (c_i + K)P_i + K \sum_{\substack{j=1 \\ j \neq i}}^{m} P_j = 0, \qquad i = 1, \ldots, m \quad (3.3.25)$$

Any choice of K will provide a solution that will not in general satisfy the constraint exactly. However, as K is increased, the solution approaches one that approximates the constraint. For $K \to \infty$ the constraint will be precisely satisfied. We will have occasion to observe this in the following.

Assuming that we are dealing with a system with two plants we can write the optimal power generations which are functions of K as

$$P_1 = \{c_2 P_{\mathrm{D}} + [(b_2 - b_1)/2] - (c_2 b_1/2K)\}/\Delta$$
$$P_2 = \{c_1 P_{\mathrm{D}} + [(b_1 - b_2)/2] - (c_1 b_2/2K)\}/\Delta$$

with

$$\Delta^{-1} = (c_1 + c_2) + (c_1 c_2/K)$$

Let us define

$$P_{\mathrm{E}} = P_1 + P_2$$

This sum is K-dependent and can be expressed as

$$P_{\mathrm{E}} = \left[(c_1 + c_2)P_{\mathrm{D}} - \left(\frac{b_1 c_2 + b_2 c_1}{2K}\right)\right] \Big/ \Delta$$

This last expression is not precisely equal to P_{D} unless K is infinite. We can easily verify that

$$\lim_{K \to \infty} P_{\mathrm{E}} = P_{\mathrm{D}}$$

3.3.3 Inequality Constrained Optimization

In the general case, which is the realistic situation, physical limitations on our decision variables exist and should be accounted for. These limitations impose the following inequality constraint set of equations:

$$\mathbf{g}(\mathbf{P}) \leq \mathbf{0}$$

Examples of such constraints are upper and lower bounds on active power generations in our system:

$$P_i^{\min} \leq P_i \leq P_i^{\max}$$

It is important to note that a common approach is to ignore the inequality constraints to start with and then examine the resulting optimal solution for inequality constraint violations. If no violations result then no further action is necessary. If, on the other hand, a violation of one or more of the inequalities occurs then a modification to the algorithm is necessary.

Many methods have been proposed to handle the inequality constrained minimization problem. The majority may be identified as penalty function approaches. One method that leads to the celebrated Kuhn–Tucker conditions is described here.

Kuhn–Tucker Conditions

Consider the problem of minimizing the cost function

$$F(\mathbf{P}) \tag{3.3.26}$$

subject to satisfying the set of equality constraints

$$\mathbf{h}(\mathbf{P}) = \mathbf{0} \tag{3.3.27}$$

and the inequality constraints

$$\mathbf{g}(\mathbf{P}) \leq \mathbf{0} \tag{3.3.28}$$

We have seen that equality constraints can be handled easily using the Lagrange multipliers. We can transform inequality constraints into equality type by introducing slack variables v_i. This results in the equivalent relations

$$g_i(\mathbf{P}) + v_i^2 = 0 \tag{3.3.29}$$

The above guarantees that the inequality constraints are satisfied.

With this transformation our augmented cost function is

$$J(\mathbf{P}) = F(\mathbf{P}) + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{h}(\mathbf{P}) + \boldsymbol{\mu}^{\mathrm{T}}[\mathbf{g}(\mathbf{P}) + \mathbf{V}^2]$$

or, in an expanded form,

$$J(\mathbf{P}) = F(\mathbf{P}) + \sum \lambda_i h_i + \sum \mu_i (g_i + v_i^2) \tag{3.3.30}$$

The necessary conditions for optimality are obtained as a result of applying the variational technique. We thus have

$$\partial J/\partial P_i = 0 \tag{3.3.31a}$$

$$\partial J/\partial \lambda_i = 0 \tag{3.3.31b}$$

$$\partial J/\partial \mu_i = 0 \tag{3.3.31c}$$

$$\partial J/\partial v_i = 0 \tag{3.3.31d}$$

The first three conditions are familiar. The last condition results in

$$\partial J/\partial v_i = 2\mu_i v_i = 0$$

Substituting for v_i from Eq. (3.3.29), we can obtain

$$\mu_i g_i = 0 \tag{3.3.32}$$

The above expressions are called the Kuhn–Tucker exclusion equations. Simply stated, either the multiplier μ_i or g_i shall be zero. The case of no violation corresponds to $\mu_i = 0$. When violations occur g_i is set to its limit $g_i = 0$.

The requirement (3.3.31a) combined with (3.3.30) can be written as

$$(\partial F/\partial P_i) + \sum \lambda_j(\partial h_j/\partial P_i) + \sum \mu_j(\partial g_j/\partial P_i) = 0$$

For simplicity let us consider a problem in two dimensions with two inequality constraints and no equality constraints. This is shown in Fig. 3.1a where it is assumed that the minimum occurs at a point $\mathbf{P}^{(0)}$ in the interior of the region. In this case $\partial F/\partial P_i = 0$. If the minimum occurs on the boundary $g_1(P) = 0$, the gradient ∇F must be orthogonal to the boundary and point inside as shown in Fig. 3.1b. Therefore in this case

$$\nabla F + \mu_1 \nabla g_1 = 0$$

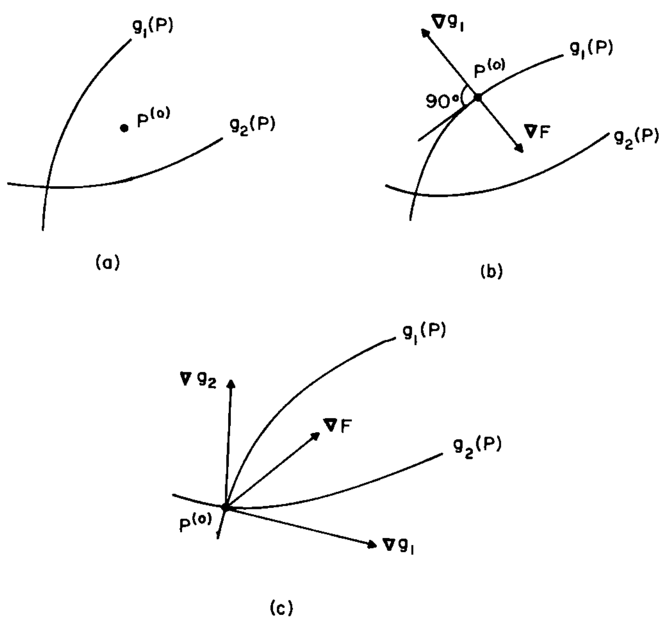

Fig 3.1 Illustrating Kuhn–Tucker conditions.

In Fig. 3.1c we have the minimum occurring at the boundary of both constraints; we must have

$$\nabla F + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0$$

Observe that this requirement states that the gradient of F should lie in the cone generated by ∇g_1 and ∇g_2.

Example. Let us consider the example of two thermal plants feeding a given power demand. The fuel cost of each is related to the output power as follows:

$$F_1 = 4P_1 + 0.01P_1^2$$
$$F_2 = 2P_2 + 0.03P_2^2$$

The objective is to minimize the total cost of operation while satisfying the equality constraint

$$P_1 + P_2 = P_D$$

We form a modified cost by augmenting the original cost with the constraint using the Lagrange multiplier λ. We thus have

$$\hat{F} = 4P_1 + 2P_2 + 0.01P_1^2 + 0.03P_2^2 + \lambda(P_D - P_1 - P_2)$$

The necessary conditions for minimization obtained are

$$4 + 0.02P_1 - \lambda = 0$$
$$2 + 0.06P_2 - \lambda = 0$$
$$P_1 + P_2 = P_D$$

For this simple system we are able to eliminate λ and P_2 to obtain a single equation in P_1 given by

$$0.08P_1 + (2 - 0.06P_D) = 0$$

With a specified power demand P_D, the above equation is solved for P_1, hence P_2 and λ are obtained. A sample of the result is

$$\begin{array}{llll} P_D = 50, & P_1 = 12.5, & P_2 = 37.5, & \lambda = 4.25 \\ P_D = 100, & P_1 = 50, & P_2 = 50, & \lambda = 5 \\ P_D = 200, & P_1 = 125, & P_2 = 75, & \lambda = 6.5 \\ P_D = 250, & P_1 = 162.5, & P_2 = 87.5, & \lambda = 7.25 \end{array}$$

We note here that P_1 and P_2 are positive. However, the schedule may violate lower and upper limit constraints on generation. Let us assume that in addition to the specified equality constraints, we have limits on the first plant's generation:

$$50 \le P_1 \le 150$$

Here we impose realistic minimum and maximum generation levels for the less expensive plant. In fact we have two inequality constraints:

$$50 - P_1 \le 0, \qquad P_1 - 150 \le 0$$

According to Kuhn–Tucker theory, our augmented cost function becomes

$$\tilde{F}(\mathbf{P}) = \hat{F}(\mathbf{P}) + \mu_1(50 - P_1) + \mu_2(P_1 - 150)$$

The necessary conditions turn out to be

$$4 + 0.02P_1 - \lambda - \mu_1 + \mu_2 = 0 \tag{3.3.33a}$$

$$2 + 0.06P_2 - \lambda = 0 \tag{3.3.33b}$$

$$P_1 + P_2 - P_D = 0 \tag{3.3.33c}$$

$$\mu_1(50 - P_1) = 0 \tag{3.3.33d}$$

$$\mu_2(P_1 - 150) = 0 \tag{3.3.33e}$$

We now have five equations instead of three. Obviously P_D will affect our procedure. Ignoring inequality constraints leads to the solution outlined before.

Consider the case with $P_D = 50$; clearly the lower bound is violated and the optimal solution is obtained by setting P_1 to the violated limit. The optimality conditions are therefore given by Eq. (3.3.33) with the requirements

$$P_1 = 50, \qquad \mu_2 = 0$$

The value of μ_2 is set to zero since the upper limit is not violated. The resulting solution is

$$P_2 = 0, \qquad \lambda = 2, \qquad \mu_1 = 3$$

The solution for the case with $P_D = 250$ violates the upper limit. The reader is invited to check that the optimal result is

$$P_1 = 150, \qquad P_2 = 100$$
$$\mu_1 = 0, \qquad \mu_2 = 1$$
$$\lambda = 8$$

It is evident that, for the power demands of 100 and 200, no inequality constraints are violated; therefore $\mu_1 = \mu_2 = 0$ in these cases and ignoring inequality constraints is justified.

3.4 DYNAMIC PROGRAMMING AND THE PRINCIPLE OF OPTIMALITY

One of the more powerful techniques used in this book to generate optimal strategies is dynamic programming, developed by Bellman. The foundation of this technique is the principle of optimality, which may be stated in the following form:

> An optimal strategy has the property that whatever the initial state and the initial decisions are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision.

Let us illustrate the application of the principle for an important problem in economy operation of power systems. We treat the problem of minimizing the function

$$F(P_1, P_2, \ldots, P_m) = \sum_{i=1}^{m} F_i(P_i) \tag{3.4.1}$$

while satisfying

$$\sum_{i=1}^{m} P_i = P_D \tag{3.4.2}$$

Our purpose is to find the optimal generations P_i that satisfy the power demand P_D while minimizing the fuel cost. This is an optimal allocation process, which is eminently suited for the dynamic programming approach. Here the problem is imbedded within a family of allocation processes in which P_D may assume any positive value and m may assume any integer value. We thus have a dynamic allocation process.

Our first step is based on the observation that the minimum of F depends only on P_D and m. As a result we introduce the optimal return sequence $\{f_m(P_D)\}$ by

$$f_m(P_D) = \min_{P_i} F(P_1, P_2, \ldots, P_m) \tag{3.4.3}$$

It is clear that

$$f_m(0) = 0, \qquad m = 1, 2, \ldots \tag{3.4.4}$$

$$f_1(P_D) = F_1(P_D) \qquad \text{for} \quad P_D \geq 0 \tag{3.4.5}$$

There is no loss in generality in assuming

$$F_i(0) = 0$$

Our objective now is to obtain a recurrence relation between $f_m(P_D)$ and $f_{m-1}(P_D)$ for arbitrary P_D. We therefore let $(P_m, 0 \leq P_m \leq P_D)$ be the generation of the mth unit. Now regardless of the precise value of P_m, the deficit $P_D - P_m$ will be used to achieve minimum cost for the operation of $(m-1)$ units. This minimum cost is $f_{m-1}(P_D - P_m)$. Thus total minimum cost with initial loading P_m results in total cost

$$F_m(P_m) + f_{m-1}(P_D - P_m)$$

from operating m units. An optimal choice of P_m is obviously one which minimizes this function. We thus obtain the basic functional equation

$$f_m(P_D) = \min[F_m(P_m) + f_{m-1}(P_D - P_m)], \qquad 0 \leq P_m \leq P_D, \qquad m = 2, 3, \ldots \tag{3.4.6}$$

Observe that Eq. (3.4.5) is used for $m = 1$. The computation of the sequence $f_m(P_D)$ can proceed in many sophisticated ways. For clarity we will outline a simple scheme for doing this in Chapter 4.

The dynamic programming approach can be effectively utilized to solve optimal control problems. Many power system economy operation problems fall within this category. Again we will take a simple problem to illustrate the approach. Extension to the multidimensional case becomes obvious.

Consider the problem of finding $u(t)$ that minimizes

$$J = \int_0^{T_f} F[X(t), u(t)]\, dt \tag{3.4.7}$$

The system dynamics are described by

$$dX/dt = f(X, u) \tag{3.4.8}$$

and the initial conditions are

$$X(0) = C \tag{3.4.9}$$

We define the optimal return function $g(C, T_f)$ as the minimum value of J where the starting state is C and the total duration is T_f:

$$g(C, T_f) = \min_{u[0, T_f]} \int_0^{T_f} F[X(t), u(t)]\, dt \tag{3.4.10}$$

If we divide the integration interval into $[0, \Delta]$ and $[\Delta, T_f]$ we obtain

$$\begin{aligned} g(C, T_f) &= \min_{u[0,\Delta]} \min_{u[\Delta, T_f]} \left(\int_0^{\Delta} F\, dt + \int_{\Delta}^{T_f} F\, dt \right) \\ &= \min_{u[0,\Delta]} \left(\int_0^{\Delta} F\, dt + \min_{u[\Delta, T_f]} \int_{\Delta}^{T_f} F\, dt \right) \end{aligned} \tag{3.4.11}$$

Clearly, the second term in parenthesis is for a process with starting time Δ. Consequently the starting state is $\tilde{C}$:

$$\tilde{C} = C + \int_0^{\Delta} f\,dt \tag{3.4.12}$$

The duration of this process is $\tilde{T}_f$ where we have

$$\tilde{T}_f = T_f - \Delta \tag{3.4.13}$$

Using the definition given by Eq. (3.4.10), we thus have

$$g(\tilde{C}, \tilde{T}_f) = \min_{u[\Delta, T_f]} \int_{\Delta}^{T_f} F\,dt \tag{3.4.14}$$

As a result, using Eq. (3.4.14) in (3.4.11), we have

$$g(C, T_f) = \min_{u[0, \Delta]} \left[\int_0^{\Delta} F\,dt + g(\tilde{C}, \tilde{T}_f) \right] \tag{3.4.15}$$

We now make use of

$$\int_0^{\Delta} F\,dt = F[C, u(0)]\Delta \tag{3.4.16}$$

as well as the truncated Taylor expansion

$$g(\tilde{C}, \tilde{T}_f) = g(C, T_f) + \Delta C[\partial g(C, T_f)/\partial C] - \Delta[\partial g(C, T_f)/\partial T_f] \tag{3.4.17}$$

The resulting relation is

$$\begin{aligned} g(C, T_f) = \min_{u[0, \Delta]} \{ & F[C, u(0)]\Delta + g(C, T_f) + f[C, u(0)]\Delta[\partial g(C, T_f)/\partial C] \\ & - \Delta[\partial g(C, T_f)/\partial T_f]\} \end{aligned} \tag{3.4.18}$$

Observing that $g(C, T_f)$ is independent of the choice of u we take it outside the minimum operation sign. Thus,

$$0 = \min_{u[0, \Delta]} \{F[C, u(0)]\Delta + f[C, u(0)]\Delta[\partial g(C, T_f)/\partial C] - \Delta[\partial g(C, T_f)/\partial T_f]\}$$

In the limit, as $\Delta \to 0$, we attain the functional equation

$$\partial g(C, T_f)/\partial T_f = \min_{y} \{F(C, y) + f(C, y)[\partial g(C, T_f)/\partial C] \tag{3.4.19}$$

in which $y = u(0)$. The initial condition is

$$g(C, 0) = 0$$

Observe that the minimum of (3.4.19) not only can be obtained using calculus, but also by search techniques which may avoid difficulties in handling inequality constraints and in asserting that a global optimum is reached. Moreover, unusual functions can be treated. The shortcoming of the method is the large memory requirements in case of large scale systems, which has been termed the "curse of dimensionality" by Bellman. There are many ways to reduce this memory problem. Some of these are the use of Lagrange multipliers, polynominal approximations, spline approximations, and quasi-linearization.

3.5 CALCULUS OF VARIATIONS

Many important pioneering results in optimal economy operation of power systems are based on variational calculus principles. It is our intention here to briefly outline some results from this forerunner of modern optimal control approaches.

Let us consider the problem of finding a function $P(t)$ which minimizes the functional

$$J = \int_0^{T_f} F[t, P(t), \dot{P}(t)]\, dt \tag{3.5.1}$$

We assume that $\hat{P}(t)$ is an optimal function and construct a family of functions $P(t)$ that includes $\hat{P}(t)$:

$$P(t) = \hat{P}(t) + \varepsilon n(t) \tag{3.5.2}$$

The parameter ε is a small number and $n(t)$ is a time-dependent variation in $P(t)$. The first derivative with respect to time is

$$\dot{P}(t) = \dot{\hat{P}}(t) + \varepsilon \dot{n}(t) \tag{3.5.3}$$

With the above, the functional of Eq. (3.5.1) is written as

$$J(P) = \int_0^{T_f} F[t, \hat{P}(t) + \varepsilon n(t), \dot{\hat{P}}(t) + \varepsilon \dot{n}(t)]\, dt \tag{3.5.4}$$

The function F can be expanded in a Taylor series about $\hat{P}(t), \dot{\hat{P}}(t)$ as follows:

$$\begin{aligned} &F[t, \hat{P}(t) + \varepsilon n(t), \dot{\hat{P}}(t) + \varepsilon \dot{n}(t)] \\ &\quad = F[t, \hat{P}(t), \dot{\hat{P}}(t)] + \varepsilon n(t)(\partial F/\partial P) + \varepsilon \dot{n}(t)(\partial F/\partial \dot{P}) + O(\varepsilon^2) \end{aligned} \tag{3.5.5}$$

We therefore rewrite Eq. (3.5.4) on the basis of Eq. (3.5.5) as

$$J(P) = J(\hat{P}) + \varepsilon \int_0^{T_f} [n(t)(\partial F/\partial P) + \dot{n}(t)(\partial F/\partial \dot{P})]\, dt + O(\varepsilon^2) \tag{3.5.6}$$

The construction of P is such that $\hat{P}$ is an optimum and for this to hold true we must have

$$J(P) \geq J(\hat{P}) \tag{3.5.7}$$

This in conjunction with Eqs. (3.5.7) and (3.5.6) will result in the requirement that

$$\varepsilon\left[\int_0^{T_f}[n(t)(\partial F/\partial P)+\dot{n}(t)(\partial F/\partial \dot{P})]\right]dt+O(\varepsilon^2)\geq 0$$

For this inequality to hold for both positive and negative values of ε, we require that

$$\int_0^{T_f}[n(t)(\partial F/\partial P)+\dot{n}(t)(\partial F/\partial \dot{P})]\,dt=0 \tag{3.5.8}$$

Integrating the second term in the integrand by parts we obtain

$$\int_0^{T_f} n(t)[(\partial F/\partial P)-(d/dt)(\partial F/\partial \dot{P})]\,dt+[n(t)(\partial F/\partial \dot{P})]_0^{T_f}=0 \tag{3.5.9}$$

Clearly Eq. (3.5.9) is satisfied if

$$(\partial F/\partial P)-(d/dt)(\partial F/\partial \dot{P})=0 \tag{3.5.10}$$

$$[n(t)(\partial F/\partial \dot{P})]=0 \qquad \text{for} \quad t=0, T_f \tag{3.5.11}$$

The fundamental lemma of the calculus of variations asserts that at an extremum we have

$$F_P-(d/dt)F_{\dot{P}}=0 \tag{3.5.12}$$

This is called the Euler equation of the variational problems and is a necessary condition for an extremum. This is the first necessary condition.

As in the ordinary calculus case, rules for distinguishing between the various types of extrema exist. Three important additional conditions for a minimum are

Legendre condition. This is the second necessary condition and adds the requirement that the second derivative of the integrand of (3.5.1) with respect to $\dot{P}$ must be nonnegative along the extremum

$$F_{\dot{P}\dot{P}}\geq 0 \tag{3.5.13}$$

This becomes a sufficiency condition for a minimum if strict inequality is required:

$$F_{\dot{P}\dot{P}}>0 \tag{3.5.14}$$

Jacobi's condition. This yields the third necessary condition for a minimum. The condition is that the time interval $[t_0, T_f]$ contain no points conjugate to t_0. Here the point $\tilde{t}_0$, which is different from t_0, is said to be conjugate to t_0 if the accessory (Jacobi) equation

$$Hx - (d/dt)(R\dot{x}) = 0 \tag{3.5.15}$$

has a solution which vanishes for $t = t_0$ and $t = \tilde{t}_0$ but is not identically zero. The functions H and R are given by

$$H = [F_{PP} - (d/dt)F_{P\dot{P}}] \tag{3.5.16}$$

$$R = (F_{\dot{P}\dot{P}}) \tag{3.5.17}$$

The Weierstrass condition. With $Z(t)$ being a function distinct from the extremal $P(t)$, and $\dot{Z}(t)$ being its derivative, let us define the Weierstrass E-function:

$$E(t, P, \dot{P}, \tilde{Z}) = F(t, P, \tilde{Z}) - F(t, P, \dot{P}) - (\tilde{Z} - \dot{P})F_{\dot{P}}(t, P, \dot{P}) \tag{3.5.18}$$

The Weirestrass condition for a minimum is

$$E(t, P, \dot{P}, \tilde{Z}) \geq 0 \tag{3.5.19}$$

In variational calculus this is called the fourth necessary condition for a minimum. It is important to note that the preceding three necessary conditions yield sufficient conditions with equality signs removed.

A. The Isoperimetric Problem

An interesting problem frequently encountered is that of minimizing or maximizing the functional J of Eq. (3.5.1) subject to an integral constraint of the form

$$\int_0^{T_f} G[t, P(t), \dot{P}(t)]\, dt = b \tag{3.5.20}$$

This usually signifies the finiteness of some resource.

Under various conditions it can be shown that we can employ a constant Lagrange multiplier ν and treat the new unconstrained problem

$$\tilde{J} = \int_0^{T_f} \tilde{F}[t, P(t), \dot{P}(t)]\, dt \tag{3.5.21}$$

with

$$\tilde{F}(\cdot) = F(\cdot) + \nu G(\cdot) \tag{3.5.22}$$

As a result we obtain the modified Euler equation

$$(F + \nu G)_P - (d/dt)(F_{\dot{P}} + \nu G_{\dot{P}}) = 0 \tag{3.5.23}$$

The constant ν must be determined by use of Eq. (3.5.20).

B. The Lagrange Problem

The Lagrange problem consists of finding extrema for the functional

$$J = \int_0^{T_f} F[t, P(t), \dot{P}(t)]\, dt \tag{3.5.24}$$

with side conditions

$$\Phi[t, P(t), \dot{P}(t)] = 0 \tag{3.5.25}$$

and such that certain boundary conditions are satisfied. Again use is made of a Lagrange multiplier function $\lambda(t)$. Here we seek to extremize the new unconstrained problem

$$\hat{J} = \int_0^{T_f} \hat{F}[t, P(t), \dot{P}(t)]\, dt \tag{3.5.26}$$

The augmented function F is given by

$$\hat{F}(\cdot) = F(\cdot) + \lambda(t)\Phi(\cdot) \tag{3.5.27}$$

It is clearly evident now that a modified Euler equation can be obtained.

To conclude this section we point out that the use of Lagrangian multipliers can lead to many interesting results, such as deriving the Euler equation for minimizing J given by Eq. (3.5.1). For this purpose we introduce the new variable

$$R(t) = \dot{P}(t) \tag{3.5.28}$$

As a result, we are required to minimize

$$J = \int_0^{T_f} F[t, P(t), R(t)]\, dt \tag{3.5.29}$$

subject to satisfying Eq. (3.5.28). To solve this constrained minimization problem we employ a Lagrangian multiplier $\lambda(t)$. The augmented cost functional is therefore

$$\tilde{J} = J + \int_0^{T_f} \lambda(t)[R(t) - \dot{P}(t)]\, dt \tag{3.5.30}$$

We now perform an integration by parts on the second term in the integrand of Eq. (3.5.30) to obtain

$$\tilde{J} = \int_0^{T_f} L\, dt + [\lambda(t)P(t)]_0^{T_f} \tag{3.5.31}$$

where L is given by

$$L = F + \lambda(t)R(t) + \dot{\lambda}(t)P(t) \tag{3.5.32}$$

We will require the first derivative of L with respect to our decision variables P and R

$$\partial L/\partial P = (\partial F/\partial P) + \dot{\lambda} \tag{3.5.33}$$

$$\partial L/\partial R = (\partial F/\partial R) + \lambda \tag{3.5.34}$$

For optimality the above should be zero. The multiplier $\lambda(t)$ can be eliminated to yield

$$(\partial F/\partial P) - (d/dt)(\partial F/\partial \dot{P}) = 0 \tag{3.5.35}$$

which is precisely Eq. (3.5.12).

3.6 PONTRYAGIN'S MAXIMUM PRINCIPLE

An alternative procedure for obtaining optimal strategies for systems of the type considered in this book is the maximum principle. Classical variational theory could not readily handle some constraints involved in a realistic optimal control problem. This difficulty led Pontryagin and his associates to first conjecture and then provide a proof of the celebrated maximum principle. In general the procedure yields necessary but not sufficient conditions for optimality. Thus the principle provides a vehicle for testing whether any control is a candidate for optimality. It should be emphasized that in certain cases the maximum principle is equivalent to the Lagrange multiplier rule and the Weierstrass condition of variational calculus. In this section we outline the maximum principle conditions for a continuous optimal control problem with fixed starting and end times.

Assume that we desire to determine the control $\mathbf{u}(t)$ so as to minimize the objective functional

$$J = \int_0^{T_f} F[\mathbf{X}(t), \mathbf{u}(t), t]\, dt \tag{3.6.1}$$

The system dynamics are described by

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, \mathbf{u}, t) \tag{3.6.2}$$

where $\mathbf{X}(t)$ is the state vector determined by the control vector $\mathbf{u}(t)$. Specified initial states satisfy

$$\mathbf{X}(0) = \mathbf{X}_0 \tag{3.6.3}$$

where $\mathbf{X}_0$ is a given vector. The terminal states satisfy

$$\mathbf{X}(T_f) = \mathbf{X}_f \tag{3.6.4}$$

where $\mathbf{X}_f$ is a given vector.

We use the method of Lagrange multipliers discussed earlier to adjoin the system equations to the objective functional, which yields

$$J = \int_0^{T_f} \{F(\mathbf{X}, \mathbf{u}, t) + \boldsymbol{\lambda}^T(t)[\mathbf{f}(\mathbf{X}, \mathbf{u}, t) - \dot{\mathbf{X}}]\}\, dt \tag{3.6.5}$$

Elements of the Lagrange multiplier vector $\boldsymbol{\lambda}(t)$ are often referred to in the literature as the costates or adjoint variables. We define a scalar function, the Hamiltonian, as

$$H[\mathbf{X}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = F[\mathbf{X}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{X}, \mathbf{u}, t) \tag{3.6.6}$$

As a result we can write (3.6.5) as

$$J = \int_0^{T_f} [H(\mathbf{X}, \mathbf{u}, \boldsymbol{\lambda}, t) - \boldsymbol{\lambda}^T(t)\dot{\mathbf{X}}(t)]\, dt \tag{3.6.7}$$

If we integrate the last term in the integrand of (3.6.7) by parts, we obtain

$$J = [\boldsymbol{\lambda}^T(0)\mathbf{X}(0) - \boldsymbol{\lambda}^T(T_f)\mathbf{X}(T_f)] + \int_0^{T_f} [H(\mathbf{X}, \mathbf{u}, \boldsymbol{\lambda}, t) + \dot{\boldsymbol{\lambda}}^T(t)\mathbf{X}(t)]\, dt \tag{3.6.8}$$

The following two necessary conditions for a minimum now become obvious:

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, \mathbf{u}, t) = \partial H/\partial \boldsymbol{\lambda} \tag{3.6.9}$$

$$\dot{\boldsymbol{\lambda}} = -\partial H/\partial \mathbf{X} \tag{3.6.10}$$

Equations (3.6.9) and (3.6.10) are often referred to as the canonical system. Boundary conditions are

$$\mathbf{X}(0) = \mathbf{X}_0, \qquad \mathbf{X}(T_f) = \mathbf{X}_f$$

If the admissible control vector $\mathbf{u}(t)$ is unrestricted, we are free to add the necessary condition

$$\partial H/\partial \mathbf{u} = \mathbf{0} \tag{3.6.11}$$

In many problems, inequality constraints on the admissible control and states are present and we must therefore take this into account. As a result (3.6.11) is no longer applicable. The basic contribution of the maximum principle addresses this difficulty. In place of (3.6.11) the necessary condition is that the Hamiltonian function $H(\mathbf{X}, \boldsymbol{\lambda}, \mathbf{u}, t)$ have an absolute minimum as a function of $\mathbf{u}$ over the admissible region Ω, for all t in the interval $(0, T_f)$. In other words, the optimal $\mathbf{u}_\xi$ satisfies

$$H(\mathbf{X}, \boldsymbol{\lambda}, \mathbf{u}_\xi, t) \le H(\mathbf{X}, \boldsymbol{\lambda}, \mathbf{u}, t), \qquad \mathbf{u} \in \Omega \tag{3.6.12}$$

Let us emphasize here that (3.6.12) is a generalization of (3.6.11).

It is interesting to note that using (3.6.6) and (3.6.10) we may write the total derivative of H with respect to time as

$$dH/dt = (\partial F/\partial t) + \boldsymbol{\lambda}^T(\partial \mathbf{f}/\partial t) + \dot{\mathbf{u}}^T(\partial H/\partial \mathbf{u})$$

If F and $\mathbf{f}$ are not explicit functions of time this reduces to

$$dH/dt = \dot{\mathbf{u}}^{\mathrm{T}}(\partial H/\partial \mathbf{u})$$

It is clear that the Hamiltonian is constant along an optimal trajectory where $\partial H/\partial \mathbf{u} = \mathbf{0}$. This is possible for controls within the admissible boundary. When the optimal controls are on the boundary but the constraints are constant, $\dot{\mathbf{u}} = \mathbf{0}$. It is generally true that H is constant along an optimal trajectory.

We comment here that the original derivation due to Pontryagin is slightly different in form than the results outlined here. As a result his Hamiltonian must be maximized, thus accounting for the name maximum principle rather than minimum principle.

3.7 THE FUNCTIONAL ANALYTIC OPTIMIZATION TECHNIQUE: THE MINIMUM NORM APPROACH

Parallel to the development of dynamic programming and the maximum principle, attempts have been made to introduce methods of functional analysis into the study of optimal control problems. At first it seemed that the methods of functional analysis applied only to a very restricted class of problems. But in spite of this, the number of studies using the ideas of functional analysis has increased. One of the important developments is the minimum norm approach.

A typical feature of the approach is that it yields necessary and sufficient conditions for the existence of solutions. This fact makes it possible to study the qualitative aspects of optimal processes. Moreover, this approach is free of the concrete nature of the system. Thus many formulations hold for systems that are distributive, digital, composite, nonlinear, or biological. Of course, results obtained on the basis of an abstract formulation must then be given concrete identification.

Our object in this section is to state one important minimum norm result which plays a central role in the solution of many problems treated in this work. Before we do this, a brief discussion of relevant concepts from functional analysis is given.

3.7.1 Some Functional Analysis Concepts

The discussion here is aimed at displaying some of the concepts and symbols which are utilized in this book. The basic functional analysis concepts presented are tied to certain ingredients in the power systems we are considering.

To begin, a *set* is a collection of elements or members, for example, the nodes of a network form a set which we denote by R_N. A node, say the ith node, is a member of this set and we use the short hand $i \in R_N$ to indicate this.

A *linear or vector space* X is a nonempty collection of vectors (elements), including the zero vector. Associated with the space are rules for multiplication with scalars and addition of vectors in the space. The rule must satisfy:

(1) Associated with every pair of vectors x and y in X, there is a unique vector z in X which is the sum of x and y:

$$z = x + y, \qquad x, y, z \in X.$$

(2) Associated with every scalar λ and every vector x in X, there is a unique vector in X which is the scalar multiple (λx) of x:

$$\lambda x \in X, \qquad x \in X$$

The Euclidean space E^n, that is, the vector space of n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of real numbers, is a linear space. The rules assigned are those for vectors given in Section 3.2. These clearly satisfy the definition. The class of all continuous functions with continuous derivatives up to order $\leq n$ defined on $(0, T)$, denoted by $C^n(0, T)$, is a linear space if addition of two functions and scalar multiplication is defined in the usual way.

A. Norms

A *norm* (commonly denoted by $\|\cdot\|$) is a real-valued, positive definite ($\|x\| > 0$ for $x \neq 0$), absolutely homogeneous ($\|\lambda x\| = |\lambda| \cdot \|x\|$), and subadditive ($\|x + y\| \leq \|x\| + \|y\|$) functional.

A *normed linear* (*vector*) *space* X is a linear space in which every vector x has a norm (length). The norm functional is used to define a distance and a convergence measure:

$$d(x, y) = \|x - y\|$$
$$x_k \to x \text{ in } X \qquad \text{if} \quad d(x, x_k) = \|x - x_k\| \to 0$$

As an example the linear space E^n becomes a normed linear space if it is equipped with any of the following norms:

$$\|x\|_1 = |\xi_1| + |\xi_2| + \cdots + |\xi_n|$$
$$\|x\|_2 = (|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)^{1/2}$$
$$\|x\|_\infty = \max(|\xi_1|, |\xi_2|, \ldots, |\xi_n|)$$

For a second example let $[0, T]$ be a closed bounded interval. The space of continuous functions $x(t)$ on $[0, T]$ can have one of the following norms:

$$\|x\|_1 = \int_0^T |x(t)|\, dt$$

$$\|x\|_2 = \left(\int_0^T |x(t)|^2\, dt\right)^{1/2}$$

$$\|x\|_\infty = \max_{0 \le t \le T} |x(t)|$$

An important class of spaces is the L_p space. This is obtained if we let p be a real number $1 \le p \le \infty$. The space $L_p(0, T)$ is the space of real-valued functions $x(t)$ defined on $(0, T)$ which are Lebesgue integrable. The norm is defined as

$$\|x\|_p = \left(\int_0^T |x(t)|^p\, dt\right)^{1/p}$$

$$\|x\|_\infty = \mathrm{ess}_{t \in (0, T)} \sup |x(t)|$$

Thus the space $L_2(0, T_f)$ consists of all real functions which are defined and square integrable on the interval $(0, T_f)$; that is, functions f for which

$$\int_0^{T_f} f^2(t)\, dt < \infty$$

For this definition the Lebesgue integral is assumed. For all practical purposes, Lebesgue integration can be considered the same as the usual Riemann integration. Whenever the Rieman integral exists, it yields the same result as the Lebesgue integral. One may read Riemann integration wherever Lebesgue integration appears if one is not familiar with the latter.

A sequence (x_n) in a normed space is said to be a *Cauchy sequence* if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Every convergent sequence in a normed space is a Cauchy sequence. In general, a Cauchy sequence may not converge in a normed space. A normed linear space is said to be *complete* if every Cauchy sequence in the space converges to a limit which belongs to the space. As a result we say that a complete normed linear space is a *Banach space*.

B. Inner Products

A very important concept in Euclidean geometry is the concept of orthogonality of two vectors. Two vectors are *orthogonal* if their inner product is zero. Extension of this concept to more generalized spaces leads to powerful results. The concept of orthogonality is not present in all normed spaces and leads to the definition of an inner product space. This is a linear vector space X together with an *inner product* defined on $X \times X$, denoted by $\langle . \rangle$. The inner product satisfies

(1) $\langle x, y\rangle = \langle y, x\rangle$,
(2) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,
(3) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0 \leftrightarrow x = 0$,
(4) $\langle x, x\rangle = \|x\|^2$.

A complete inner product space is called a *Hilbert space*. A Hilbert space is a Banach space equipped with an inner product that induces the norm. The space E^n is a Hilbert space with inner product as defined in Section 3.2 by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$$

or

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i \tag{3.7.1}$$

The space $L_2(0, T)$ is a Hilbert space with inner product

$$\langle x, y\rangle = \int_0^T x(t)y(t)\,dt \tag{3.7.2}$$

An extremely useful Hilbert space is used in this book. The elements of the space are vectors whose components are functions of time such as active power generation by the system units over the interval $(0, T_{\mathrm{f}})$. Given a positive definite matrix $\mathbf{B}(t)$ whose elements are functions of time as well, we can define the Hilbert space $L^n_{2\mathbf{B}}(0, T_{\mathrm{f}})$. The inner product in this case is given by

$$\langle \mathbf{V}(t), \mathbf{U}(t)\rangle = \int_0^{T_{\mathrm{f}}} \mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{U}(t)\,dt \tag{3.7.3}$$

for every $\mathbf{V}(t)$ and $\mathbf{U}(t)$ in the space.

C. Transformations

A *transformation* is a mapping from one vector space to another. If $\mathbf{T}$ maps the space X into Y, we write $\mathrm{T}: X \to Y$. If T maps the vector $\mathbf{x} \in X$ into the vector $\mathbf{y} \in Y$, we write $\mathbf{y} = \mathbf{T}(\mathbf{x})$ and $\mathbf{y}$ is referred to as the *image* of $\mathbf{x}$ under T. Alternatively, a transformation is referred to as an *operator*. Among familiar transformations we have:

(1) Multiplication of an n-dimensional vector with a scalar α where we have $\mathrm{T}_1: E^n \to E^n$. The operator $\mathbf{T}_1$ in this case maps the Euclidean space E^n into itself, with the rule

$$\mathbf{T}_1(\mathbf{x}) = \alpha\mathbf{x}$$

(2) Multiplication by a matrix. Let $\mathbf{X}$ be an $m \times n$ matrix and define $\mathbf{T}_2: E^n \to E^m$ by

$$\mathbf{T}_2(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.7.4}$$

Observe that $\mathbf{T}_2$ is not $\mathbf{A}$, but rather multiplication by $\mathbf{A}$;

(3) The operation $\mathbf{T}_3: E^n \to E^1$ defined by

$$\mathbf{T}_3(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{x}$$

This associates with every $\mathbf{x}$ in the space $\mathbf{X} = E^n$ an element y in $\mathbf{Y} = E^1$;

(4) Suppose two power plants supply a power demand $P_{\mathrm{D}}(t)$ such that the power balance equation is satisfied:

$$P_{\mathrm{D}}(t) = P_1(t) + P_2(t)$$

This defines a transformation $\mathbf{T}_4: L_2^2(0,1) \to L_2(0,1)$ sending functions $[P_1(t), P_2(t)]$ into their image $P_{\mathrm{D}}(t)$. Observe that if time is not included as a parameter we would have a transformation sending E^2 into E^1, which is clearly a special case of $\mathbf{T}_2$ above.

The transformation $\mathbf{T}: X \to Y$ is said to be *linear* if for every $x_1, x_2 \in X$ and all scalars α_1 and α_2 one has $\mathbf{T}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathbf{T}(x_1) + \alpha_2 \mathbf{T}(x_2)$. The operators $\mathbf{T}_1$ and $\mathbf{T}_4$ defined above are linear.

The linear operator $\mathbf{T}$ from a normed space X to a normed space Y is said to be *bounded* if there is a constant M such that $\|\mathbf{T}x\| \le M\|x\|$ for all $x \in X$. The normed space of all bounded linear operators from the normed space X into the normed space Y is denoted by $B(X, Y)$. Examples of bounded linear operators include one transformation useful for our purposes. This is $\mathbf{T}$: $L_{2,B}^n(0, T_{\mathrm{f}}) \to R^m$ defined by $\mathbf{b} = \mathbf{T}[\mathbf{U}(t)]$,

$$\mathbf{b} = \int_0^{T_{\mathrm{f}}} \mathbf{M}^{\mathrm{T}}\mathbf{U}(t)\,dt \tag{3.7.5}$$

where $\mathbf{b}$ is, for our purposes, an m-dimensional vector of the allowable volumes of discharge from m hydro units, $\mathbf{U}(t)$ includes all our decision variables, and $\mathbf{M}$ is a compatible vector.

Let us consider a specific example to show how the norm of a bounded linear operator is obtained. Define $\mathbf{T}: X \to Y$ by

$$\mathbf{T}(x) = \int_0^1 K(s,t)x(t)\,dt$$

with $K(s,t)$ being continuous in $0 \le s \le 1$, $0 \le t \le 1$. $\mathbf{T}$ is linear. X is the space of continuous functions defined on $(0,1)$, denoted by $C(0,1)$, whose norm is defined as

$$\|x\| = \max_{0 \le t \le 1} |x(t)|$$

We have

$$\|\mathbf{T}(x)\| = \max_{0\le s\le 1}\left|\int_0^1 K(s,t)x(t)\,dt\right| \le \max_{0\le s\le 1}\left(\int_0^1 |K(s,t)|\,dt\right)\max_{0\le t\le 1}|x(t)|$$
$$= \max_{0\le s\le 1}\left(\int_0^1 |K(s,t)|\,dt\right)\|x\|$$

so that

$$\|\mathbf{T}\| \le \max_{0\le s\le 1}\left(\int_0^1 |K(s,t)|\,dt\right)$$

and in fact $\|\mathbf{T}\|$ is

$$\|\mathbf{T}\| = \max_{0\le s\le 1}\left(\int_0^1 |K(s,t)|\,dt\right)$$

A *functional* is a transformation from a linear space into the space of real (or complex) scalars. A typical functional is the objective functional of optimal economy operation given by

$$J(\mathbf{P}_s) = \int_0^{T_f}\sum_{i=1}^{m}[\alpha_i + \beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t)]\,dt$$

Here the space $L_2^m(0, T_f)$ of thermal power generation vector functions is mapped into the real scalars space. A functional f on a linear space X is *linear* if for any two vectors $x, y \in X$, and any two scalars α and β there holds $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. A linear functional f on a normed space is *bounded* if there is a constant M such that $|f(x)| \le M\|x\|$ for every $x \in X$. The smallest such constant M_0 is called the *norm* of f. The norm of the functional f can be expressed as

$$\|f\| = \sup|f(x)|/\|x\|, \qquad x \ne 0$$

Given a normed linear space X, one can define bounded linear functionals on X. The space of these linear functionals is a normed linear space X^*. The space X^* is the *normed dual* of X (alternatively, X^* is called the *conjugate* space of X), and is a Banach space. If X is a Hilbert space, then $X = X^*$. Thus Hilbert spaces are *self-dual.* The normed linear space X is said to be *reflexive* if $X = X^{**}$. Any Hilbert space is reflexive.

D. Adjoints

Let X and Y be normed spaces and let $\mathbf{T} \in B(X, Y)$. Here $B(X, Y)$ denotes the set of all linear transformations mapping X into Y, which is, in this case, a normed linear space itself. The *adjoint* (conjugate) operator $\mathbf{T}^*$: $Y^* \to X^*$

is defined by

$$\langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle = \langle \mathbf{T}\mathbf{x}, \mathbf{y}^* \rangle$$

An important special case is that of a linear operator $\mathbf{T}: H \to G$ where H and G are Hilbert spaces. If G and H are real then they are their own duals and the operator $\mathbf{T}^*$ can be regarded as mapping G into H. In this case the adjoint relation becomes

$$\langle \mathbf{T}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle \tag{3.7.6}$$

Observe that the left-hand side inner product is taken in G while the right-hand side inner product is taken in H.

To find the operator $\mathbf{T}^*$ associated with $\mathbf{T}$ defined in Eq. (3.7.4) we observe that the left-hand side inner product in Eq. (3.7.6) is

$$\langle \mathbf{T}\mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{y}$$

or

$$\langle \mathbf{T}\mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} \tag{3.7.7}$$

If we assume that $\mathbf{T}^*\mathbf{y} = \mathbf{z}$ then the right-hand side inner product is

$$\langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{z} \tag{3.7.8}$$

For the two inner products to be equal we have

$$\mathbf{z} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

Therefore

$$\mathbf{T}^*(\mathbf{y}) = \mathbf{A}^{\mathrm{T}}\mathbf{y} \tag{3.7.9}$$

Thus the adjoint operator of multiplication by a matrix is multiplication by its transpose.

In the above we assumed that $\mathbf{x}$ and $\mathbf{y}$ are elements of Euclidean spaces endowed with inner product definitions of the form given by Eq. (3.7.1). We will now assume that $\mathbf{x}$ is in E_B^n, where in this space the inner product is defined by

$$\langle \mathbf{x}, \tilde{\mathbf{x}} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{B}\tilde{\mathbf{x}}, \qquad \mathbf{x}, \tilde{\mathbf{x}} \in E_B^n \tag{3.7.10}$$

Moreover, $\mathbf{y}$ is in E_C^m, which in turn has the inner product

$$\langle \mathbf{y}, \tilde{\mathbf{y}} \rangle = \mathbf{y}^{\mathrm{T}}\mathbf{C}\tilde{\mathbf{y}}, \qquad \mathbf{y}, \tilde{\mathbf{y}} \in E_C^m \tag{3.7.11}$$

The $n \times n$ matrix $\mathbf{B}$ and the $m \times m$ matrix $\mathbf{C}$ are symmetric and positive definite and therefore the assumed inner products are valid. In E_C^m the left-hand side inner product of Eq. (3.7.6) becomes

$$\langle \mathbf{T}\mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{C}\mathbf{y} \tag{3.7.12}$$

With $\mathbf{z} = \mathbf{T}^*\mathbf{y}$ as before, the right-hand side inner product of Eq. (3.7.6) in E_B^n becomes

$$\langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{z} \tag{3.7.13}$$

Accordingly we must have

$$(\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{C}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{z} \tag{3.7.14}$$

Therefore $\mathbf{z}$ and hence $\mathbf{T}^*\mathbf{y}$ are given by

$$\mathbf{T}^*(\mathbf{y}) = \mathbf{B}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{y} \tag{3.7.15}$$

Naturally with $\mathbf{B} = \mathbf{I}$, $\mathbf{C} = \mathbf{I}$, we obtain Eq. (3.7.9).

We now apply the definition of Eq. (3.7.6) to a transformation which plays an important role in developments to follow. Here we wish to determine the adjoint $\mathbf{T}^*$ of the transformation $\mathbf{T}$ defined by Eq. (3.7.5). Let us consider an element $\mathbf{y}$ in R^m and $\mathbf{U}(t)$ in $L_{2,B}^n(0, T_f)$. The image of $\mathbf{U}(t)$ in R^m is given, according to Eq. (3.7.5), by

$$\mathbf{T}[\mathbf{U}(t)] = \int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{U}(t)\, dt \tag{3.7.16}$$

In R^m we have the inner product given by

$$\langle \mathbf{TU}, \mathbf{y} \rangle = \left(\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{U}(t)\, dt \right)^{\mathrm{T}} \mathbf{y} \tag{3.7.17}$$

Here superscript T denotes transposition. Equation (3.7.17) can be further written as

$$\langle \mathbf{TU}, \mathbf{y} \rangle = \int_0^{T_f} \mathbf{U}^{\mathrm{T}}(t)\mathbf{M}\mathbf{y}\, dt \tag{3.7.18}$$

The image of $\mathbf{y}$ under the adjoint operation $\mathbf{T}^*$ is in $L_{2,B}^n(0, T_f)$ and is denoted by $\mathbf{Z}(t)$; thus

$$\mathbf{Z}(t) = \mathbf{T}^*(\mathbf{y}) \tag{3.7.19}$$

The inner product required by Eq. (3.7.5) in $L_{2,B}^n(0, T_f)$ according to Eq. (3.7.3) is given by

$$\langle \mathbf{U}, \mathbf{Z} \rangle = \int_0^{T_f} \mathbf{U}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{Z}(t)\, dt \tag{3.7.20}$$

We now equate Eqs. (3.7.18) and (3.7.20);

$$\int_0^{T_f} \mathbf{U}^{\mathrm{T}}(t)\mathbf{M}\mathbf{y}\, dt = \int_0^{T_f} \mathbf{U}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{Z}(t)\, dt \tag{3.7.21}$$

For this to hold for arbitrary $\mathbf{U}(t)$, we must then have

$$\mathbf{Z}(t) = \mathbf{B}^{-1}(t)\mathbf{M}\mathbf{y} \tag{3.7.22}$$

According to Eqs. (3.7.19) and (3.7.22) we can thus assert that the adjoint $\mathbf{T}^*$ defining relation is

$$\mathbf{T}^*[\mathbf{y}] = \mathbf{B}^{-1}(t)\mathbf{M}\mathbf{y} \tag{3.7.23}$$

Composite transformations can be formed in a very simple way. Suppose $\mathbf{T}$ and $\mathbf{G}$ are transformations $\mathbf{T}: X \to Y$ and $\mathbf{G}: Y \to Z$. We define the transformation $\mathbf{GT}: X \to Z$ by

$$(\mathbf{GT})(\mathbf{x}) = \mathbf{G}(\mathbf{T}(\mathbf{x}))$$

We then say that $\mathbf{GT}$ is a composite of $\mathbf{G}$ and $\mathbf{T}$, respectively. An important composite operation is obtained from an operator $\mathbf{T}$ and its adjoint $\mathbf{T}^*$ in Hilbert spaces. Let us define the operator $\mathbf{K}: Y \to Y$ by the composite rule

$$\mathbf{K}(\mathbf{y}) = \mathbf{T}(\mathbf{T}^*(\mathbf{y})) \tag{3.7.24}$$

We will illustrate the application of Eq. (3.7.24) using the example operators previously treated.

In the first instance, we have for the operator of Eq. (3.7.4) the adjoint given by Eq. (3.7.9), which is repeated here as

$$\mathbf{T}^*(\mathbf{y}) = \mathbf{A}^{\mathrm{T}}\mathbf{y}$$

Now according to the definition of $\mathbf{T}$, we have

$$\mathbf{K}(\mathbf{y}) = \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{y} \tag{3.7.25}$$

In the case of nonconventional inner products, we have the adjoint given by Eq. (3.7.15), for which case our $\mathbf{K}$ operator becomes

$$\mathbf{K}(\mathbf{y}) = \mathbf{A}\mathbf{B}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{y} \tag{3.7.26}$$

The second example involves the operator of Eq. (3.7.5) with adjoint defined by Eq. (3.7.23). Here we obtain

$$\mathbf{K}(\mathbf{y}) = \int_0^{T_{\mathrm{f}}} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\mathbf{y}\,dt$$

Observe that $\mathbf{y}$ is time independent ($\mathbf{y} \in R^m$) and therefore we have

$$\mathbf{K}(\mathbf{y}) = \left(\int_0^{T_{\mathrm{f}}} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\,dt\right)\mathbf{y} \tag{3.7.27}$$

In some instances we allow that a transformation may be defined on a subset $D \subset X$ called the *domain* of $\mathbf{T}$, although in most cases $D = X$. The collection of all vectors $y \in Y$ for which there is an $x \in D$ with $y = \mathbf{T}(x)$ is called the *range* of T, denoted by $R(T)$. The *null space* of T, $N(T)$, is the set of all vectors x such that $\mathbf{T}x = 0$, with 0 being the zero vector in $R(\mathbf{T})$.

The transformation $\mathbf{T}: X \to Y$ is *one-to-one* if

$$x_1 \neq x_2 \to \mathbf{T}x_1 \neq \mathbf{T}x_2$$

for all x_1 and x_2 in X, that is, if $\mathbf{T}$ does not assign more than one x in X to a single y in Y. The transformation $\mathbf{T}$ is said to be *onto* if the range of T is the space Y itself. That is, $\mathbf{T}$ is onto if every vector y in Y is associated with at least one vector x in X.

E. Linear Operator Equations

Corresponding to a linear operator $\mathbf{T}$, the equation $\mathbf{T}(\mathbf{x}) = \mathbf{y}$ for a given $\mathbf{y} \in Y$ may

(1) have a unique solution $\mathbf{x} \in X$;
(2) have no solution;
(3) have more than one solution.

Condition (1) arises if $\mathbf{T}$ is one-to-one and onto, in which case we say that $\mathbf{T}$ is *invertible* and define the *inverse* of $\mathbf{T}$ to be the transformation $\mathbf{T}^{-1}: Y \to X$ which associates with $\mathbf{y}$ in Y a unique vector $\mathbf{x}$ in X. The *identity operator I* is defined by

$$\mathbf{I}(\mathbf{x}) = \mathbf{x}$$

In terms of composite operations we have the characterization

$$\mathbf{T}^{-1}(\mathbf{T}(\mathbf{x})) = \mathbf{T}(\mathbf{T}^{-1}(\mathbf{x})) = \mathbf{x} \tag{3.7.28}$$

We illustrate the inverse operation by appeal to the operators K defined by Eqs. (3.7.26) and (3.7.27). In the first instance we have

$$\mathbf{K}^{-1}(\mathbf{y}) = (\mathbf{A}\mathbf{B}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{y} \tag{3.7.29}$$

while for the second case we have

$$\mathbf{K}^{-1}(\mathbf{y}) = \left(\int_0^{T_f} M^{\mathrm{T}}\mathbf{B}^{-1}(t)M\,dt\right)^{-1}\mathbf{y} \tag{3.7.30}$$

We observe that for $\mathbf{T}$ to be invertible, both the one-to-one and onto conditions should hold. Case (2) above arises when $\mathbf{T}$ is one-to-one but not onto and the equation $\mathbf{y} = \mathbf{T}\mathbf{x}$ has no solution, in which case we search for an approximate solution such that $\|\mathbf{y} - \mathbf{T}\mathbf{x}\|$ is minimized. Case (3) corresponds to the situation when T is onto but not one-to-one, in which case more than one solution to $\mathbf{y} = \mathbf{T}\mathbf{x}$ exists and we enquire into the possibility of finding an element of minimum norm.

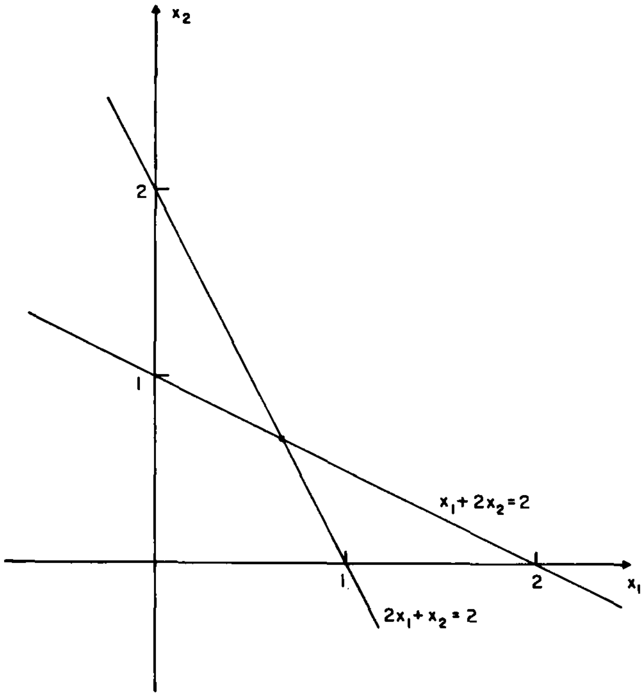

Fig. 3.2 Case of unique solution to $\mathbf{y} = \mathbf{Tx}$.

To illustrate the above concepts we appeal to some simple examples. Taking the domain of $\mathbf{T}$ in each of the following cases to be the two-dimensional Euclidean space E^2 we have for Case (1), the example,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

It is clear that a unique solution exists and is shown as the intersection of the two lines in Fig. 3.2. For Case (2), the following equation is applicable where the image space Y is E^3:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

There is no solution and the situation is shown in Fig. 3.3. For our example on Case (3), we have the equation

$$[2 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2$$

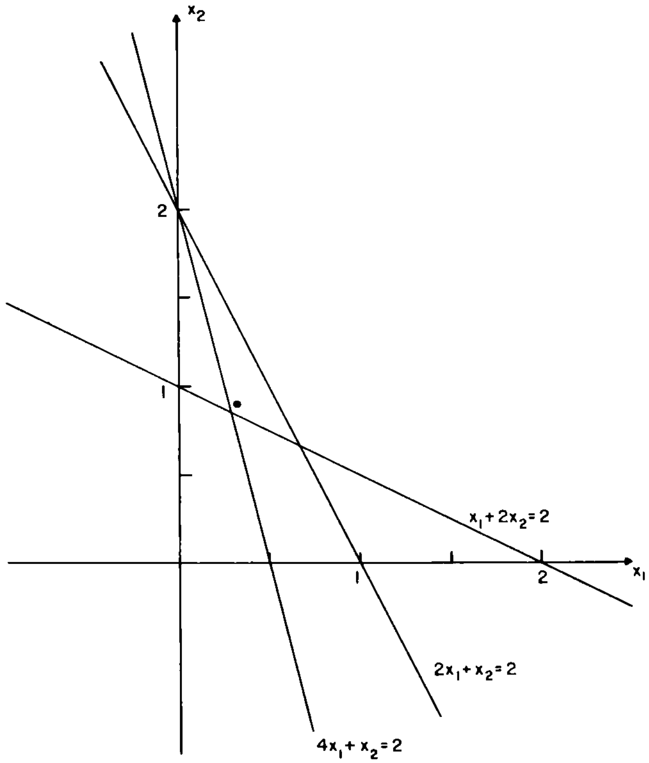

Fig. 3.3 Case of no solution to $\mathbf{y} = \mathbf{T}\mathbf{x}$.

The locus in E^2 of all (x_1, x_2) satisfying the relation is shown in Fig. 3.4. Clearly $\mathbf{T}$ defined above is not one-to-one since the all points on the line map into one element in $Y = E$, the one-dimensional Euclidean space.

A nontrivial case results when $\mathbf{T}$ is neither one-to-one nor onto. In this case there is more than one approximating vector (i.e., minimizing $\|\mathbf{y} - \mathbf{T}\mathbf{x}\|$) and we are led to look for the one with minimum norm. This latter vector is referred to as the smallest least-error solution to $\mathbf{y} = \mathbf{T}\mathbf{x}$. Let us note here that this present case is the most general one and we are led in a very natural way to the more complete and more general approach that defines the generalized or pseudoinverse operation as follows.

Let G and H be Hilbert spaces and $\mathbf{T} \in B(G, H)$ with the range of $\mathbf{T}$ being closed. Define the set M as $M = \{\mathbf{x}_1 \in G : \|\mathbf{T}\mathbf{x}_1 - \mathbf{y}\| = \min_x \|\mathbf{T}\mathbf{x} - \mathbf{y}\|\}$. Let $\mathbf{x}_0 \in M$ be the unique vector of minimum norm. Then the *pseudoinverse* operator $\mathbf{T}^\dagger$ of $\mathbf{T}$ is the operator mapping $\mathbf{y}$ into its corresponding $\mathbf{x}_0$ as $\mathbf{y}$ varies over H.

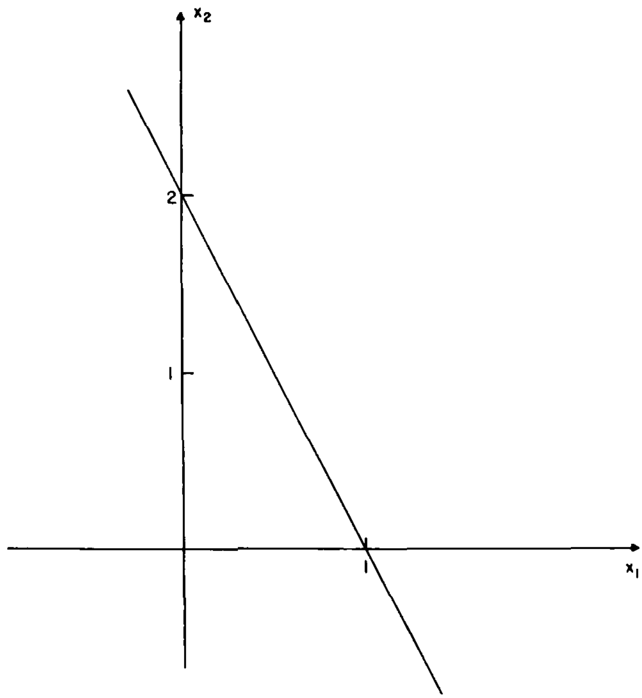

Fig. 3.4 Case of more than one solution to $\mathbf{y} = \mathbf{Tx}$.

3.7.2 A Minimum Norm Theorem

With the preliminary definitions and concepts outlined above, we are now in a position to introduce one powerful result in optimization theory. The theorem described here is only one of a wealth of results that utilize functional analytic concepts to effectively solve optimization problems. Our result is a generalization of the idea that the shortest distance from a point to a line is given by the orthogonal to the line from the point.

Theorem. Let B and D be Banach spaces. Let $\mathbf{T}$ be a bounded linear transformation defined on B with values in D. Let $\hat{\mathbf{u}}$ be a given vector in B. For each ξ in the range of $\mathbf{T}$, there exists a unique element $\mathbf{u}_\xi \in B$ that satisfies

$$\xi = \mathbf{Tu}$$

while minimizing the objective functional

$$J(\mathbf{u}) = \|\mathbf{u} - \hat{\mathbf{u}}\|$$

The unique optimal $\mathbf{u}_\xi \in B$ is given by

$$\mathbf{u}_\xi = \mathbf{T}^\dagger[\boldsymbol{\xi} - \mathbf{T}\hat{\mathbf{u}}] + \hat{\mathbf{u}}$$

where the pseudoinverse operator $T^\dagger$ in the case of Hilbert spaces is given by

$$\mathbf{T}^\dagger\boldsymbol{\xi} = \mathbf{T}^*[\mathbf{T}\mathbf{T}^*]^{-1}\boldsymbol{\xi}$$

provided that the inverse of $\mathbf{TT}^*$ exists.

The theorem as stated is an extension of the fundamental minimum norm problem where the objective functional is

$$J(\mathbf{u}) = \|\mathbf{u}\|$$

The optimal solution for this case is

$$\mathbf{u}_\xi = \mathbf{T}^\dagger\boldsymbol{\xi}$$

with $\mathbf{T}^\dagger$ being the pseudoinverse associated with $\mathbf{T}$. Clearly a change of variables in the result of the fundamental problem yields our stated theorem. We will outline with the help of a simple geometric example the ingredients involved in the construction of the optimal solution.

Our first observation is that if $\tilde{\mathbf{u}}$ is a solution of the equation $\boldsymbol{\xi} = \mathbf{Tu}$, then the general solution can be written as $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{v}$ where $\mathbf{v} \in N(\mathrm{T})$. The general solution defines a subspace M. For example, the equation

$$[2 \quad 1]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 2$$

defines a straight line in E^2 as shown in Fig. 3.5. A solution is

$$\tilde{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here the null space $N(\mathbf{T})$ is the line $2u_1 + u_2 = 0$. It is clear that all vectors in M can be written as the sum $\tilde{\mathbf{u}} + \mathbf{v}$. For example, take

$$\mathbf{u}^{(1)} = \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}$$

Then

$$\mathbf{v}^{(1)} = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

and as a result

$$\mathbf{u}^{(1)} = \tilde{\mathbf{u}} + \mathbf{v}$$

is evidently satisfied.

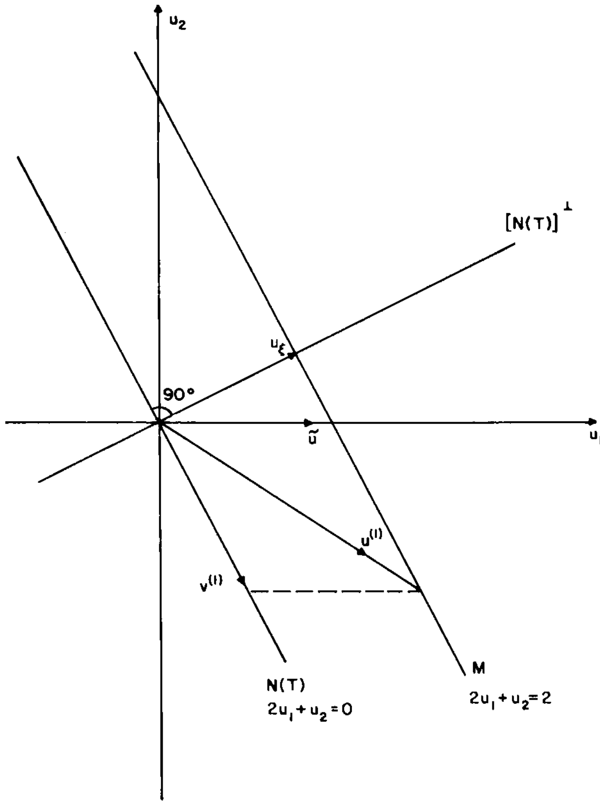

Fig. 3.5 Illustrating minimum norm development in two dimensions.

As a consequence of the classical projection theorem we can assert the existence of $\mathbf{u}_\xi$ of minimum norm satisfying $\boldsymbol{\xi} = \mathbf{Tu}$, and that this vector is orthogonal to $N(\mathbf{T})$. Therefore $\mathbf{u}_\xi$ is in the orthogonal complement of $N(\mathbf{T})$ and we write

$$\mathbf{u}_\xi \in [N(\mathbf{T})]^{\perp}$$

In our example $\mathbf{u}_\xi$ is shown.

Our final step depends on the fact that the orthogonal complement of $N(\mathbf{T})$ is identical to the range of $\mathbf{T}^*$, denoted by $R(\mathbf{T}^*)$, provided that $N(\mathbf{T})$ is closed. Therefore

$$\mathbf{u}_\xi \in R(\mathbf{T}^*)$$

As a result

$$\mathbf{u}_\xi = \mathbf{T}^*\mathbf{Z}$$

for some $\mathbf{Z} \in D$, and since $\xi = \mathbf{Tu}$, we conclude that

$$\mathbf{TT^*Z} = \xi$$

If the operator $\mathbf{TT^*}$ is invertible the optimal solution is

$$\mathbf{u}_\xi = \mathbf{T^*}[\mathbf{TT^*}]^{-1}\xi$$

In our example $\mathbf{T^*}$ is the operation of premultiplication by $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$; therefore

$$\mathbf{u}_\xi = \begin{bmatrix}2\\1\end{bmatrix}\mathbf{Z}$$

where $\mathbf{Z}$ is determined from

$$[2 \quad 1]\begin{bmatrix}2\\1\end{bmatrix}\mathbf{Z} = 2$$

or

$$\mathbf{Z} = 0.4$$

Consequently the minimum norm solution is

$$\mathbf{u}_\xi = \begin{bmatrix}0.8\\0.4\end{bmatrix}$$

This is shown in Fig. 3.5. In this example the pseudoinverse of $\mathbf{T}$ is

$$\mathbf{T}^\dagger\xi = \begin{bmatrix}0.4\\0.2\end{bmatrix}\xi$$

We can obtain the above result by using a Lagrange multiplier argument. Here we consider an augmented objective

$$\tilde{J} = \|\mathbf{u}\|^2 + \langle \lambda, (\xi - \mathbf{Tu})\rangle$$

where λ is a multiplier (in fact $\lambda \in D$) to be determined so that the constraint $\xi = \mathbf{Tu}$ is satisfied. By utilizing properties of the inner products we can write

$$\tilde{J} = \|\mathbf{u}\|^2 - \langle \mathbf{T^*}\lambda, \mathbf{u}\rangle + \langle \lambda, \xi\rangle$$

which can further be written as

$$\tilde{J} = \|\mathbf{u} - \mathbf{T^*}(\lambda/2)\|^2 - \|\mathbf{T^*}(\lambda/2)\|^2 + \langle \lambda, \xi\rangle$$

Only the first norm depends explicitly on the choice of $\mathbf{u}$. To minimize $\tilde{J}$ we therefore require that

$$\mathbf{u}_\xi = \mathbf{T^*}(\lambda/2)$$

The vector $(\lambda/2)$ is obtained using the constraint as the solution to

$$\xi = \mathbf{T}[\mathbf{T}^*(\lambda/2)]$$

It is therefore clear that with an invertible $\mathbf{TT}^*$ we write

$$\mathbf{u}_\xi = \mathbf{T}^*[\mathbf{TT}^*]^{-1}\xi$$

which is the required result.

In applying this result to our physical problem we need to recall two important concepts from ordinary constrained optimization. These are the Lagrange multiplier rule and the Kuhn–Tucker multipliers. Clearly this is necessitated by the presence of nonlinear equality constraints and inequality constraints in the power system situation. Our approach is therefore to single out a set of linear constraints on the control vector. This set, under appropriate conditions, will define the bounded linear transformation $\mathbf{T}$. An augmented objective functional is formed by adding to the original functional terms corresponding to the remaining constraints using the necessary multipliers. It is important to realize that certain transformations on the constraints may become necessary. The object in these cases is to ensure that the augmented functional can indeed be cast as a norm in the chosen space. Application to specific problems is detailed beginning in Chapter 5.

3.8 SOME DYNAMIC SYSTEMS RESULTS

Quite often an optimal control problem solution depends on successfully solving the nonlinear two-point boundary value problem

$$\dot{\mathbf{X}} = \mathbf{F}_X(\mathbf{X}, \mathbf{\Lambda}, t) \tag{3.8.1}$$

$$\dot{\mathbf{\Lambda}} = \mathbf{F}_\Lambda(\mathbf{X}, \mathbf{\Lambda}, t) \tag{3.8.2}$$

$$\mathbf{X}(0) = \mathbf{b} \tag{3.8.3}$$

$$\mathbf{X}(T_f) = \mathbf{C} \tag{3.8.4}$$

Here $\mathbf{X}$ is an $n \times 1$ state vector with known boundary conditions at $t = 0$ and $t = T_f$ given by vectors $\mathbf{b}$ and $\mathbf{C}$. The vector $\mathbf{\Lambda}$ is an $n \times 1$ component costate or adjoint variables vector. The vector functions $\mathbf{F}_X$ and $\mathbf{F}_\Lambda$ are assumed continuous. It is the purpose of this section to introduce some results pertaining to an alternative representation of this problem. This equivalent representation will facilitate the computational aspects of problems treated in the latter parts of this book. We begin by reviewing the concept of state transition from the theory of state space analysis.

3.8.1 The State Transition Matrix

Consider the dynamic system equations given by the first-order homogeneous state space vector form

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{3.8.5}$$

If $\mathbf{A}$ is a square matrix of dimension n whose elements are continuous functions of time defined on the interval $t_0 \le t \le t_f$, and $\mathbf{X}_0$ is an initial condition vector at time t_0, then there exists at most one solution $\mathbf{X}(t)$. This solution can be written as

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\mathbf{X}_0 \tag{3.8.6}$$

The square matrix $\mathbf{\Phi}(t, t_0)$ is frequently called the *fundamental* or *state transition matrix*. It is clear that

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0) \tag{3.8.7}$$

$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{3.8.8}$$

A basic expression for $\mathbf{\Phi}(t, t_0)$ in terms of the system matrix $\mathbf{A}$ is given by the Peano–Baker series

$$\mathbf{\Phi}(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\,d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{t} \mathbf{A}(\sigma_2)d\sigma_2\,d\sigma_1 + \cdots \tag{3.8.9}$$

This series can be summed if $\mathbf{A}$ is a real constant matrix, and in this case we have

$$\mathbf{\Phi}(t, t_0) = \mathbf{I} + \mathbf{A}(t - t_0) + A^2(t - t_0)^2/2 + \cdots \tag{3.8.10}$$

In view of the similarity between this series and the scalar series for e^{at}, it is logical to define the *matrix exponential* by

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + (\mathbf{A}^2/2!) + \cdots \tag{3.8.11}$$

This definition is useful since it permits us to express the solution as

$$\mathbf{X}(t) = e^{\mathbf{A}(t - t_0)}\mathbf{X}_0 \tag{3.8.12}$$

One of the useful properties of matrix exponentials is that if $\mathbf{A}$ is a diagonal matrix, then its exponential is also diagonal. We have

$$e^{\mathbf{A}} = \operatorname{diag}(e^{a_{11}}, e^{a_{22}}, \ldots) \tag{3.8.13}$$

for

$$\mathbf{A} = \operatorname{diag}(a_{11}, a_{22}, \ldots) \tag{3.8.14}$$

Other properties include

$$e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} = \mathbf{P}^{-1}e^{\mathbf{A}}\mathbf{P} \tag{3.8.15}$$

If $\mathbf{AB} = \mathbf{BA}$, then (3.8.16)

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}} \tag{3.8.17}$$

The main importance of this is based on the extension to stipulate that if **A** and its integral commute, i.e.,

$$\mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\sigma)\,d\sigma = \left(\int_{t_0}^{t}\mathbf{A}(\sigma)\,d\sigma\right)\mathbf{A}(t) \tag{3.8.18}$$

then

$$\mathbf{\Phi}(t, t_0) = \exp\left(\int_{t_0}^{t}A(\sigma)\,d\sigma\right) \tag{3.8.19}$$

Two important computational properties of the transition matrix are

$$\mathbf{\Phi}(t_1, t_2)\mathbf{\Phi}(t_2, t_3) = \mathbf{\Phi}(t_1, t_3) \tag{3.8.20}$$

$$\mathbf{\Phi}(t_1, t_2)^{-1} = \mathbf{\Phi}(t_2, t_1) \tag{3.8.21}$$

The transition matrix concept is useful in the case of the inhomogeneous linear dynamic system described by

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{f}(t) \tag{3.8.22}$$

The solution is given by

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\mathbf{X}_0 + \int_{t_0}^{t}\mathbf{\Phi}(t, \sigma)\mathbf{f}(\sigma)\,d\sigma \tag{3.8.23}$$

In the case of a constant matrix **A**, the above reduces to

$$\mathbf{X}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{X}_0 + \int_{0}^{t}e^{\mathbf{A}(t-\sigma)}\mathbf{f}(\sigma)\,d\sigma \tag{3.8.23$'$}$$

Taking stock of the above we now proceed with the treatment of the fundamental problem given in Eqs. (3.8.1)–(3.8.4).

3.8.2 Nonlinear Two-Point Boundary Value Problem Representation

Assume that we have an ordinary nonlinear vector differential equation of the form

$$\dot{\mathbf{y}}(t) = \mathbf{F}[\mathbf{y}(t), t] \tag{3.8.24a}$$

This system is subject to the general boundary conditions

$$\mathbf{g}[\mathbf{y}(0)] + \mathbf{h}[\mathbf{y}(T_f)] = \mathbf{C} \tag{3.8.24b}$$

Here **F**, **g**, and **h** are suitable vector-valued functions and **C** is a constant vector. Let **y** be an $n \times 1$ vector. Observe here that (3.8.24) represents a more general setting for our fundamental problem.

We give a method for representing (3.8.24) in an operator form. This follows by defining

$$\mathbf{f}(t) = \mathbf{F}(\mathbf{y}, t) - \mathbf{V}(t)\mathbf{y} \tag{3.8.25a}$$

$$\mathbf{d} = \mathbf{C} - \{\mathbf{g}[\mathbf{y}(0)] + \mathbf{h}[\mathbf{y}(T_f)]\} + \mathbf{M}\mathbf{y}(0) + \mathbf{N}\mathbf{y}(T_f) \tag{3.8.25b}$$

Here $\mathbf{V}(t)$, $\mathbf{M}$, and $\mathbf{N}$ are $n \times n$ matrices, and $\mathbf{g}$, $\mathbf{h}$, $\mathbf{d}$, and $\mathbf{C}$ are $n \times 1$ vectors. Using (3.8.25) we obtain the equivalent formulation for (3.8.24):

$$\dot{\mathbf{y}} = \mathbf{V}(t)\mathbf{y} + \mathbf{f}(t) \tag{3.8.26a}$$

subject to the boundary conditions

$$\mathbf{M}\mathbf{y}(0) + \mathbf{N}\mathbf{y}(T_f) = \mathbf{d} \tag{3.8.26b}$$

We remark here that so far, $\mathbf{V}(t)$, $\mathbf{M}$, and $\mathbf{N}$ are arbitrary. However, since (3.8.26a) is explicitly linear in **y**, there will be restrictions on $\mathbf{V}(t)$, $\mathbf{M}$, and $\mathbf{N}$ such that a solution to (3.8.26a) can be obtained.

Let $\mathbf{V}(t)$ and $\mathbf{f}(t)$ be defined on $(0, T_f)$ and assume that there is a Lebesgue integrable function $m(t)$ on $(0, T_f)$ such that

$$\|\mathbf{V}(t)\| \le m(t), \qquad \|\mathbf{f}(t)\| \le m(t)$$

Under these conditions we can write the unique solution for (3.8.26a) given initial value $\mathbf{y}(0)$ as

$$\mathbf{y}(t) = \mathbf{\Phi}^V(t, 0)\mathbf{y}(0) + \mathbf{\Phi}^V(t, 0)\int_0^t \mathbf{\Phi}^V(0, s)\mathbf{f}(s)\, ds \tag{3.8.27}$$

Here $\mathbf{\Phi}^V(t, s)$ denotes the state transition (fundamental) matrix of the force-free system

$$\dot{\mathbf{y}}(t) = \mathbf{V}(t)\mathbf{y}(t)$$

This is a straightforward application of (3.8.23).

The value of $\mathbf{y}(t)$ at $t = T_f$ is obtained using (3.8.27):

$$\mathbf{y}(T_f) = \mathbf{\Phi}^V(T_f, 0)\mathbf{y}(0) + \mathbf{\Phi}^V(T_f, 0)\int_0^{T_f} \mathbf{\Phi}^V(0, s)\mathbf{f}(s)\, ds \tag{3.8.28}$$

Substituting (3.8.28) in the boundary conditions (3.8.26b) and simplifying, we obtain

$$\mathbf{y}(0) = \mathbf{D}\boldsymbol{\xi} \tag{3.8.29a}$$

$$\mathbf{D}^{-1} = \mathbf{M} + \mathbf{N}\mathbf{\Phi}^V(T_f, 0) \tag{3.8.29b}$$

$$\boldsymbol{\xi} = \mathbf{d} - \mathbf{N}\int_0^{T_f} \mathbf{\Phi}^V(T_f, s)\, \mathbf{f}(s)\, ds \tag{3.8.29c}$$

This means that the initial value of $\mathbf{y}$ is obtained uniquely provided that $\mathbf{D}^{-1}$ indicated in Eq. (3.8.29b) exists.

We can now write an equivalent representation of the system (3.8.26) as

$$\mathbf{y}(t) = \mathbf{H}^{VMN}(t)\mathbf{d} + \int_0^{T_f} \mathbf{G}^{VMN}(t,s)\mathbf{f}(s)\,ds \tag{3.8.30}$$

The following function definitions are involved in (3.8.30):

$$\mathbf{H}^{VMN}(t) = \mathbf{\Phi}^V(t,0)\mathbf{D} \tag{3.8.31a}$$

$$\begin{aligned} \mathbf{G}^{VMN}(t,s) &= \mathbf{H}^{VMN}(t)\mathbf{M}\mathbf{\Phi}^V(0,s), \qquad 0 \le s \le t \\ &= -\mathbf{H}^{VMN}(t)\mathbf{N}\mathbf{\Phi}^V(T_f,s), \qquad t \le s \le T_f \end{aligned} \tag{3.8.31b}$$

The proof of (3.8.30) is straightforward and is therefore omitted.

Our nonlinear two-point boundary value system (3.8.24) is written using (3.8.30) and (3.8.25) as

$$\begin{aligned} \mathbf{y}(t) = {} & \mathbf{H}^{VMN}(t)\{\mathbf{C} - \mathbf{g}[\mathbf{y}(0)] - \mathbf{h}[y(T_f)] + \mathbf{M}\mathbf{y}(0) + \mathbf{N}\mathbf{y}(T_f)\} \\ & + \int_0^{T_f} \mathbf{G}^{VMN}(t,s)\{\mathbf{F}[\mathbf{y}(s),s] - \mathbf{V}(s)\mathbf{y}(s)\}\,ds \end{aligned} \tag{3.8.32}$$

It is worth remarking here that the representation of (3.8.32) is not unique. The choice of M, N, and $V(t)$ will change the form of (3.8.32) for any given system. The conditions limiting our choice are the boundary compatibility conditions:

$$\|\mathbf{V}(t)\| \le m(t) \tag{3.8.33a}$$

and

$$\det[\mathbf{M} + \mathbf{N}\mathbf{\Phi}^V(T_f,0)] \ne 0 \tag{3.8.33b}$$

3.8.3 A Specialization

The results obtained above are specialized to the problem given in (3.8.1)–(3.8.4). Here we have that the boundary conditions (3.8.3)–(3.8.4) can be expressed as

$$\mathbf{M}\mathbf{y}(0) + \mathbf{N}\mathbf{y}(T_f) = \mathbf{d} \tag{3.8.34}$$

where we have

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} \mathbf{b} \\ \mathbf{C} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{X} \end{bmatrix}$$

Observe that (3.8.34) is of the same form as (3.8.26b).

We choose as the $2n \times 2n$ matrix $\mathbf{V}(t)$

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

This has the property of zero higher order powers. The state transition matrix is then simply obtained from consideration of Eq. (3.8.10):

$$\begin{aligned} \mathbf{\Phi}^V(t, s) &= e^{\mathbf{V}(t-s)} \\ &= \mathbf{I} + (t - s)\mathbf{V} + \tfrac{1}{2}(t - s)^2\mathbf{V}^2 + \cdots \end{aligned}$$

The result is

$$\mathbf{\Phi}^V(t, s) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (t - s)\mathbf{I} & \mathbf{I} \end{bmatrix}$$

We claim that $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{V}$ are boundary compatible. The matrix $\mathbf{D}^{-1}$ obtained is

$$\mathbf{D}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ T_f\mathbf{I} & \mathbf{I} \end{bmatrix}$$

A simple inversion process yields

$$\mathbf{D} = \begin{bmatrix} \dfrac{-1}{T_f}\mathbf{I} & \dfrac{1}{T_f}\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

which proves our claim. We obtain $\mathbf{H}^V(t)$ using Eq. (3.8.31a):

$$\mathbf{H}^{VMN}(t) = \begin{bmatrix} \dfrac{-1}{T_f}\mathbf{I} & \dfrac{1}{T_f}\mathbf{I} \\ \left(1 - \dfrac{t}{T_f}\right)\mathbf{I} & \dfrac{t}{T_f}\mathbf{I} \end{bmatrix}$$

For $\mathbf{G}^{VMN}(t, s)$ we have by using (3.8.31b) the following result:

$$\mathbf{G}^{VMN}(t, s) = \frac{1}{T_f}\begin{bmatrix} s\mathbf{I} & -\mathbf{I} \\ s(t - T_f)\mathbf{I} & (T_f - t)\mathbf{I} \end{bmatrix}, \qquad 0 \le s \le t$$

$$\mathbf{G}^{VMN}(t, s) = \frac{1}{T_f}\begin{bmatrix} (s - T_f)\mathbf{I} & -\mathbf{I} \\ t(s - T_f)\mathbf{I} & -t\mathbf{I} \end{bmatrix}, \qquad t \le s \le T_f$$

The function $\mathbf{f}(t)$ is given by

$$\mathbf{f}(t) = \mathbf{F}(\mathbf{y}, t) - \mathbf{V}(t)\mathbf{y}$$

In subvector form this turns out to be

$$\mathbf{f}_\Lambda = \mathbf{F}_\Lambda, \qquad \mathbf{f}_X = \mathbf{F}_X - \mathbf{\Lambda}$$

The equivalent representation can now be written as

$$\begin{aligned} T_f\mathbf{\Lambda}(t) = {} & [\mathbf{X}(T_f) - \mathbf{X}(0)] + \int_0^t \{s\mathbf{F}_\Lambda(s) - [\mathbf{F}_X(s) - \mathbf{\Lambda}(s)]\}\, ds \\ & + \int_t^{T_f} \{(s - T_f)\mathbf{F}_\Lambda(s) - [\mathbf{F}_X(s) - \mathbf{\Lambda}(s)]\}\, ds \end{aligned} \tag{3.8.35a}$$

$$\begin{aligned} T_f\mathbf{X}(t) = {} & [(T_f - t)\mathbf{X}(0) + t\mathbf{X}(T_f)] \\ & + \int_0^t \{s(t - T_f)\mathbf{F}_\Lambda(s) + (T_f - t)[\mathbf{F}_X(s) - \mathbf{\Lambda}(s)]\}\, ds \\ & + \int_t^{T_f} \{t(s - T_f)\mathbf{F}_\Lambda(s) - t[\mathbf{F}_X(s) - \mathbf{\Lambda}(s)]\}\, ds \end{aligned} \tag{3.8.35b}$$

3.9 ITERATIVE SOLUTION OF NONLINEAR SYSTEMS

Systems of nonlinear equations invariably arise as optimality conditions in many economic operation problems discussed in this work. The ability to solve these equations is a prerequisite for the successful implementation of optimal strategies. In contrast to the case of linear systems, direct methods for the solution of nonlinear equations are usually feasible only for small systems of a very special form. As a result, we resort to iterative methods designed to generate an improved sequence of approximations to the solution based on an initial guess approximation.

Successful implementation of optimal strategies depends to a large extent on the performance of the iterative method adopted. A method that performs well for a certain system may fail for another. Consequently the need exists for the study of a variety of methods. Satisfactory performance calls for reliable, fast, and efficient methods. The merits of each algorithm for a certain application can be judged according to criteria related to speed of convergence, sensitivity to roundoff and truncation errors, as well as computational complexity. Our purpose in this section is to describe some iterative methods that are useful in our treatment.

3.9.1 Some Preliminaries

A given system of simultaneous nonlinear equations can be expressed in many different ways. Two of the common ones are the **T**-form and the **F**-form. In the **T**-form, each component of the unknown vector is expressed as a function (or mapping) of all the vector components. We thus have the system

$$x_i = T_i(x_1, x_2, \ldots, x_N), \qquad i = 1, \ldots, N$$

In compact vector form we write

$$\mathbf{x} = \mathbf{T}(\mathbf{x}) \tag{3.9.1}$$

where $\mathbf{T}$ is an appropriately defined mapping or operator. Alternatively we can absorb the vector $\mathbf{x}$ into $\mathbf{T}$ and consider the compact $\mathbf{F}$-form

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \tag{3.9.2}$$

Clearly this implies the splitting

$$(\mathbf{I} - \mathbf{T})(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \tag{3.9.3}$$

where $\mathbf{I}$ is the identity operation.

Let us take for an example the problem of finding the square root of 2, which can be formulated as solving

$$x^2 = 2 \tag{A}$$

Equation (A) can be reformulated in infinitely many different ways. Among these we have the equivalent $\mathbf{T}$-forms

$$x = 2/x \tag{B}$$

$$x = (2 + x)/(1 + x) \tag{C}$$

$$x = (4 + 3x)/(3 + 2x) \tag{D}$$

On the other hand, an expression in the $\mathbf{F}$-form would be

$$F(x) = x^2 - 2 = 0 \tag{E}$$

A natural way of generating a sequence of approximations $\{\mathbf{x}^{(k)}\}$ is defined by the successive approximation algorithm based on the $\mathbf{T}$-form given by

$$\mathbf{x}^{(k+1)} = \mathbf{T}(\mathbf{x}^{(k)}), \qquad k = 0, 1, \ldots \tag{3.9.4}$$

One starts with an initial guess $\mathbf{x}^{(0)}$, and then constructs the sequence $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \ldots$, as follows:

$$\mathbf{x}^{(1)} = \mathbf{T}(\mathbf{x}^{(0)}), \qquad \mathbf{x}^{(2)} = \mathbf{T}(\mathbf{x}^{(1)})$$

Let us emphasize here the fact that the sequence may or may not converge depending, among other things, on the form $\mathbf{T}$. This we illustrate for our scalar example by observing that, based on the form of Eq. (B) starting at

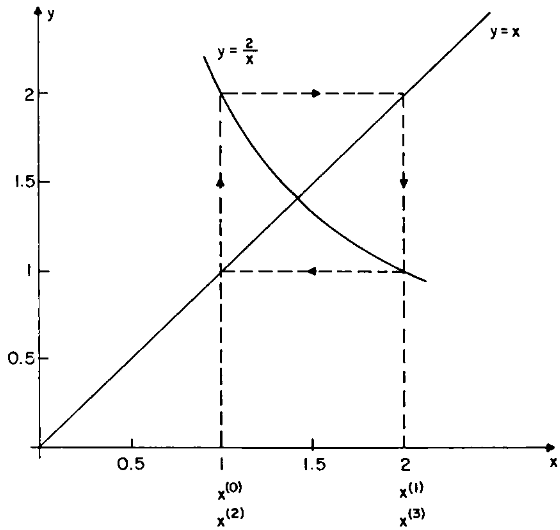

Fig. 3.6 Oscillatory iterations.

$x^{(0)} = 1$, the sequence proceeds as

$$x^{(1)} = 2, \qquad x^{(2)} = 1$$

We see that the iterates just oscillate around the solution point. The situation in this case is illustrated in Fig. 3.6. This sequence's performance is contrasted with that based on Eq. (E). In this case the iterates proceed as

$$x^{(1)} = 1.5, \qquad x^{(2)} = 1.4, \qquad x^{(3)} = 1.416667$$

This clearly is converging to $\sqrt{2}$. This situation is illustrated in Fig. 3.7. The sequence based on Eq. (D) proceeds as

$$x^{(1)} = 1.4, \qquad x^{(2)} = 1.4138, \qquad x^{(3)} = 1.4142$$

The sequence generated by Eq. (3.9.4) is called a contraction mapping or a Picard's sequence. Strictly speaking, a mapping $\mathbf{T}$ is called a contraction if there is an α such that

$$\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \le \alpha\|\mathbf{x} - \mathbf{y}\| \tag{3.9.5}$$

for $0 < \alpha < 1$. Here the norms are taken in the appropriate space under consideration. The mappings in Eqs. (C) and (D) are contractions. We show this for the former by writing

$$\mathbf{T}(x) - \mathbf{T}(y) = \frac{2 + x}{1 + x} - \frac{2 + y}{1 + y} = \left[\frac{1}{(1 + x)(1 + y)}\right][y - x]$$

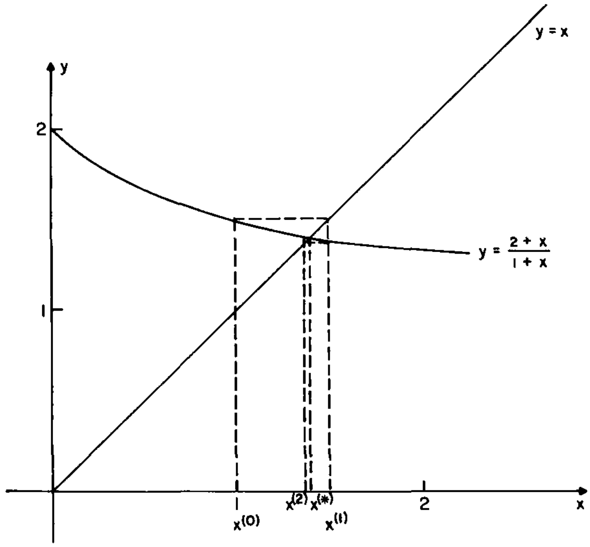

Fig. 3.7 A convergent sequence of iterates.

For $x, y > 0$, we see that the first bracket above is less than 1. For instance put $x, y > 0$; then $\alpha = \frac{4}{9}$, which is clearly within the permissible range $0 < \alpha < 1$. In the case of the second mapping we have

$$\mathbf{T}(x) - \mathbf{T}(y) = \frac{4 + 3x}{3 + 2x} - \frac{4 + 3y}{3 + 2y} = \frac{1}{(3 + 2x)(3 + 2y)}[x - y]$$

For the same range as before we find $\alpha \cong 0.06$. We note how the change in the **T**-form affects convergence (as per inspection of sequence of iterates as well as α).

3.9.2 The Modified Contraction Mapping

Let us consider the compact **T**-form expression of Eq. (3.9.1). It is clear that an equivalent formulation results if we introduce a new mapping $S(\mathbf{x})$ to modify the system according to

$$\mathbf{x} - \mathbf{S}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) - \mathbf{S}(\mathbf{x})$$

Now, provided that the mapping $(\mathbf{I} - \mathbf{S})$ is invertible, we can write

$$\mathbf{x} = (\mathbf{I} - \mathbf{S})^{-1}[\mathbf{T}(\mathbf{x}) - \mathbf{S}(\mathbf{x})] \tag{3.9.6}$$

This defines the modified form

$$\mathbf{x} = \mathbf{G}(\mathbf{x}) \tag{3.9.7}$$

where

$$\mathbf{G}(\mathbf{x}) = (\mathbf{I} - \mathbf{S})^{-1}[\mathbf{T}(\mathbf{x}) - \mathbf{S}(\mathbf{x})] \tag{3.9.8}$$

If we now use the idea of successive approximations of (3.9.4) we obtain the modified contraction mapping (MCM) sequence

$$\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{S})^{-1}[\mathbf{T}(\mathbf{x}^{(k)}) - \mathbf{S}(\mathbf{x}^{(k)})] \tag{3.9.9}$$

Our motivation in developing the MCM is the observation that the direct application of (3.9.4) often does not lead to a convergent sequence of approximations. On the other hand, the modified form (3.9.9) may lead to a convergent sequence. We remark here that different choices of the mapping **S** lead to different methods. We show here some of the common forms.

Given an operator **A** which is invertible, the choice

$$\mathbf{S} = \mathbf{I} - \mathbf{A} \tag{3.9.10}$$

leads to the **T**-form sequence

$$\mathbf{x}^{(k+1)} = \mathbf{A}^{-1}[\mathbf{T}(\mathbf{x}^{(k)}) - (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}] \tag{3.9.11}$$

Simplifying, we have

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{A}^{-1}(\mathbf{I} - \mathbf{T})(\mathbf{x}^{(k)}) \tag{3.9.12}$$

We can write the above in the **F**-form as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{A}^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.13}$$

This we recognize as the n-dimensional parallel chord method. In the single variable case the method consists in replacing F at the approximation $x^{(k)}$ of x^* by a linear function

$$l(x) = \alpha(x - x^{(k)}) + F(x^{(k)})$$

with a suitable α, and then taking the root $x^{(k+1)}$ of l as a new approximation to x^*.

The proper choice of **A** is an important question which leads to many variants of the method. One simple possibility is $\mathbf{A} = \alpha\mathbf{I}$ with some scalar $\alpha \neq 0$. The simplified (or modified) Newton method is a special case where $\mathbf{A} = \mathbf{F}'(\mathbf{x}^{(0)})$. We thus have

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [\mathbf{F}'(\mathbf{x}^{(0)})]^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.14}$$

where $\mathbf{F}'$ is the Jacobian evaluated at the initial guess $\mathbf{x}^{(0)}$.

3.9.3 Newton's Method

It is noted that different mappings $\mathbf{S}_k$ could be chosen at each step in the iteration. We can thus rewrite a dynamic form for (3.9.9) as

$$\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{S}_k)^{-1}[\mathbf{T}(\mathbf{x}^{(k)}) - \mathbf{S}_k(\mathbf{x}^{(k)})] \tag{3.9.15}$$

The particular choice

$$\mathbf{S}_k(\mathbf{x}^k) = \mathbf{T}'(\mathbf{x}^{(k)}) \tag{3.9.16}$$

leads to Newton's method. In the $\mathbf{T}$-form, we have

$$\mathbf{x}^{(k+1)} - \mathbf{T}'(\mathbf{x}^k)\mathbf{x}^{(k+1)} = \mathbf{T}(\mathbf{x}^k) - \mathbf{T}'(\mathbf{x}^{(k)})\mathbf{x}^{(k)} \tag{3.9.17}$$

Recalling Eq. (3.9.3), we have

$$\mathbf{F}'(\mathbf{x}) = \mathbf{I} - \mathbf{T}' \tag{3.9.18}$$

Application of (3.9.18) to (3.9.17) results in the more familiar $\mathbf{F}$-form for the method given by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [\mathbf{F}'(\mathbf{x}^{(k)})]^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.19}$$

The operator $\mathbf{F}'$ is often called the Jacobian and it is noted that the calculation and inversion of this operator of first derivatives is quite involved for certain problems of interest. The most direct way to avoid computation of $\mathbf{F}'(\mathbf{x})$ is to use difference quotients to approximate the partial derivatives. Typically we would employ either

$$\partial f_i/\partial x_j = h_{ij}^{-1}\left[f_i\left(\mathbf{x} + \sum_{k=1}^{j} h_{ik}\mathbf{e}^k\right) - f_i\left(\mathbf{x} + \sum_{k=1}^{j-1} h_{ik}\mathbf{e}^k\right)\right] \tag{3.9.20}$$

or

$$\partial f_i/\partial x_j = h_{ij}^{-1}[f_i(\mathbf{x} + h_{ij}e^j) - f_i(\mathbf{x})] \tag{3.9.21}$$

where h_{ij} are given discretization parameters and $\mathbf{e}^j$ is the jth coordinate vector. The generalization of this approach is the discretized Newton iteration given by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J(\mathbf{x}^{(k)}, h^{(k)})^{-1}F(\mathbf{x}^{(k)}) \tag{3.9.22}$$

Here the difference matrix J is given by

$$J(x, h) = [\Delta_{ij}(\mathbf{x}, h)] \tag{3.9.23}$$

where Δ_{ij} denotes any difference approximation of $\partial f_i/\partial x_j$. The choice of the parameters $h^{(k)}$ is crucial to the convergence of the algorithm. For the simple choice $h^{(k)} = h$ the rapid convergence typical of Newton's method is no longer preserved. This can be somewhat restored with the choice

$$h^{(k)} = v_k h \tag{3.9.24}$$

where the sequence $\{v_k\}$ has the property

$$\lim_{k \to \infty} v_k = 0$$

There are other modifications of Newton's method. Two such modifications are given by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - w_k \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.25}$$

and

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [\mathbf{F}'(\mathbf{x}^{(k)}) + \lambda_k \mathbf{I}]^{-1} F(\mathbf{x}^{(k)}) \tag{3.9.26}$$

In both cases w_k and λ_k are chosen to ensure that the method is norm reducing, in the sense that the following holds:

$$\|\mathbf{F}(\mathbf{x}^{(k+1)})\| \le \|\mathbf{F}(\mathbf{x}^{(k)})\|$$

This will provide a satisfactory convergence behaviour.

3.9.4 Composite Methods

To carry out Newton's method a linear system of equations must be solved. Assume that $\mathbf{x}^{(k)}$ has been determined. Then the next iterate would be the solution of (3.9.19) written in the form

$$\mathbf{F}'(\mathbf{x}^{(k)})\mathbf{x} = \mathbf{F}'(\mathbf{x}^{(k)})\mathbf{x}^{(k)} - \mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.27}$$

Instead of solving directly for $\mathbf{x}$, we may wish to use an iterative method to solve the linear equation (3.9.27).

Let us consider the solution of the linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.9.28}$$

One of the basic tools in the generation and analysis of iterative methods for this system is that of splitting. This simply means that the matrix $\mathbf{A}$ is decomposed, or split, into

$$\mathbf{A} = \mathbf{B} - \mathbf{C} \tag{3.9.29}$$

In the splitting process, $\mathbf{B}$ is chosen to be nonsingular and the linear system $\mathbf{BX} = \mathbf{d}$ is easy to solve. As a result, an iterative method can be defined according to

$$\mathbf{x}^{(m+1)} = \mathbf{H}\mathbf{x}^{(m)} + \mathbf{B}^{-1}\mathbf{b} \tag{3.9.30}$$

with

$$\mathbf{H} = \mathbf{B}^{-1}\mathbf{C} \tag{3.9.31}$$

Or alternatively,

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} - \mathbf{B}^{-1}(\mathbf{A}\mathbf{x}^{(m)} - \mathbf{b}) \tag{3.9.32}$$

If we recursively apply Eq. (3.9.32), we obtain the useful result

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(0)} - (\mathbf{I} + \mathbf{H} + \cdots + \mathbf{H}^{m})\mathbf{B}^{-1}(\mathbf{A}\mathbf{x}^{(0)} - \mathbf{b}) \tag{3.9.33}$$

We observe here that the successive overrelaxation method (SOR) is a special case of Eq. (3.9.30) with the splitting

$$\mathbf{B} = w^{-1}(\mathbf{D} - w\mathbf{L}) \tag{3.9.34}$$

$$\mathbf{C} = w^{-1}[(1 - w)\mathbf{D} + w\mathbf{U}] \tag{3.9.35}$$

where, $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ is the decomposition of $\mathbf{A}$ into diagonal, strictly lower triangular, and strictly upper triangular parts.

Consider the general nonlinear iterative method

$$\mathbf{x}^{k+1} = \mathbf{x}^{(k)} - \mathbf{A}_k^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.36}$$

which extends (3.9.13) and includes Newton's method as a special case. We can subject the matrix $\mathbf{A}_k$ to the splitting

$$\mathbf{A}_k = \mathbf{B}_k - \mathbf{C}_k \tag{3.9.37}$$

where $\mathbf{B}_k$ is nonsingular. The iterate $\mathbf{x}^{(k+1)}$ is the result of solving the linear system

$$\mathbf{A}_k\mathbf{x}^{(k+1)} = \mathbf{A}_k\mathbf{x}^{(k)} - \mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.38}$$

As a result of applying a secondary iteration as in Eq. (3.9.33), we obtain the composite method

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (\mathbf{I} + \cdots + \mathbf{H}_k^{m-1})\mathbf{B}_k^{-1}F(\mathbf{x}^{(k)}) \tag{3.9.39}$$

with

$$\mathbf{H}_k = \mathbf{B}_k^{-1}\mathbf{C}_k \tag{3.9.40}$$

In essence, we take m_k steps of the secondary linear iteration to approximate $\mathbf{x}^{(k+1)}$. As a special case of the method we have the Newton–SOR method, in which

$$\mathbf{A}_k = \mathbf{F}'(\mathbf{x}^{(k)}) \tag{3.9.41}$$

and

$$\mathbf{B}_k = w_k^{-1}(\mathbf{D}_k - w_k\mathbf{L}_k) \tag{3.9.42}$$

Here Newton's method is the primary iteration and SOR is the secondary iteration.

There are many other possibilities depending on the linear secondary iteration. As other examples we mention the m-step Newton–Jacobi iteration:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (\mathbf{I} + \cdots + \mathbf{H}_k^{m-1})\mathbf{D}_k^{-1}\mathbf{F}(\mathbf{x}^{(k)}) \tag{3.9.43}$$

where we decompose the Jacobian into diagonal and nondiagonal parts

$$\mathbf{F}'(\mathbf{x}^{(k)}) = \mathbf{D}_k - \mathbf{C}_k \tag{3.9.44}$$

$$\mathbf{H}_k = \mathbf{D}_k^{-1}\mathbf{C}_k \tag{3.9.45}$$

The one-step Newton–Peaceman–Rachford method proceeds according to

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - 2\alpha(\mathbf{V}_k + \alpha\mathbf{I})^{-1}(\mathbf{H}_k + \alpha\mathbf{I})^{-1}\mathbf{F}(\mathbf{x}^k) \tag{3.9.46}$$

Here α is a parameter and the splitting of the Jacobian is according to

$$\mathbf{F}'(\mathbf{x}^{(k)}) = \mathbf{H}_k + \mathbf{V}_k \tag{3.9.47}$$

It is assumed that easy solutions are available for

$$(\mathbf{H} + \alpha\mathbf{I})\mathbf{x} = \mathbf{d} \qquad \text{and} \qquad (\mathbf{V} + \alpha\mathbf{I})\mathbf{x} = \mathbf{d}$$

For example, if $\mathbf{H}$ and $\mathbf{V}$ are tridiagonal, we have easy systems of linear equations

3.9.5 A Continuation or Imbedding Method

The problem of solving the nonlinear system

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}$$

can be imbedded in the more general one of solving

$$\mathbf{H}(\mathbf{x}, t) = \mathbf{0}, \qquad t \in [0, 1] \tag{3.9.48}$$

In the general system given above, t is a parameter. We can assume that H satisfies

$$\mathbf{H}(\mathbf{x}, 1) = \mathbf{F}(\mathbf{x}) \tag{3.9.49}$$

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{F}_0(\mathbf{x}) \tag{3.9.50}$$

with a given system $\mathbf{F}_0$ for which a solution $\mathbf{x}^0$ of

$$\mathbf{F}_0(\mathbf{x}) = \mathbf{0}$$

is known. Observe that even if $\mathbf{F}$ does not depend naturally on a suitable parameter t, we can always define the family $\mathbf{H}$ satisfying (3.9.49) and (3.9.50) in many ways, such as

$$\mathbf{H}(\mathbf{x}, t) = t\mathbf{F}(\mathbf{x}) + (1 - t)\mathbf{F}_0(\mathbf{x}) \tag{3.9.51}$$

Now if we make use of the substitution

$$\mathbf{F}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}^0) \tag{3.9.52}$$

we obtain the alternative form

$$\mathbf{H}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}) + (t - 1)\mathbf{F}(\mathbf{x}^0) \tag{3.9.53}$$

Let us assume that there is a continuous vector function $\mathbf{x}(t)$ such that over the interval $(0, 1)$

$$\mathbf{H}(\mathbf{x}(t), t) = 0$$

An approach to obtaining $\mathbf{x} = \mathbf{x}(1)$ is to discretize the interval $(0, 1)$ by

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

and consider solving the problems

$$\mathbf{H}(\mathbf{x}, t_i) = 0, \qquad i = 1, \ldots, N \tag{3.9.54}$$

We can now use an iterative method which employs the solution $\mathbf{x}^{(i-1)}$ of the $(i - 1)$th problem as an initial guess to solve the ith problem. The guess $\mathbf{x}^{(i-1)}$ will be a good guess to $\mathbf{x}^{(i)}$ if $t_i - t_{i-1}$ is sufficiently small.

3.10 COMMENTS AND REFERENCES

The broad spectrum of topics outlined in this chapter and the wealth of existing literature on each make it difficult to adequately cover and to even attempt a partial listing of reference work. Instead, our listing is confined to works that influenced the authors in preparing this chapter. In many of the references given here, the reader will find comprehensive coverage of the literature.

Section 3.2

The material in this section is standard in textbooks on linear algebra, control and system theory, and allied fields of application. Among references that are easy to follow are Bellman's classic (1970), Gantmacher (1959), and Pease (1965). One aspect not touched in this brief review is sparse matrix manipulations. Readers interested in this important aspect may consult Tewarson (1973), Brameller *et al.* (1976), and Jennings (1977).

Section 3.3

The fundamental ideas outlined here are classical and are found in many sources. The algebraic approach of completing squares is used in Bellman's treatise on matrices (1970). Among the many excellent references we cite Himmelblau (1972), Luenberger (1969, 1973), and Mangasarian (1969).

Section 3.4

The dynamic programming methodology is standard in all books dealing with optimal control and operations research. For lucid treatments we refer to the two books by Bellman (1957, 1961), and Bellman and Dreyfus (1962). Larson's book (1968) offers a very readable review and introduces the computationally advantageous approach of state incremental dynamic programming.

Section 3.5

For rigorous treatment of variational calculus principles the interested reader may consult many excellent books. Among these we mention Akhiezer (1963), Bliss (1963), Elsgoc (1961), Gelfand and Fomin (1963), and Hestenes (1966).

Section 3.6

The original work by Pontryagin and his associates is recorded in their book (1962). For a comprehensive treatment of the maximum principle we have Athans and Falb's monumental work (1966). Feldbaum's contribution (1965) treats dynamic programming and the maximum principle. Many excellent textbooks exist dealing with optimal control wherein may be found adequate coverage of variational calculus, dynamic programming, and the maximum principle. Among these are Bryson and Ho (1969), Lapidus and Luus (1967), and Sage (1968).

Section 3.7

The basic concepts of functional analysis are treated in many books. Our list includes Dunford and Schwartz (1958), Kolmogorov and Fomin (1957), Simmons (1963), Taylor (1958), and Vulikh (1963). Engineering oriented treatments of functional analysis fundamentals and applications in optimization are given in Dorny (1975), Luenberger (1969), and Porter (1966).

Section 3.8

The fundamentals of state space representations of systems and the concept and results pertaining to the state transition matrix are treated in many excellent books. Readers interested in further details may consult any of the works by Brockett (1970), DeRusso *et al.* (1965), and Ogata (1967). The two-point boundary value problem representation given here is based on results presented in Falb and DeJong (1969).

SECTION 3.9

For comprehensive and rigorous treatments of the material discussed in this section we have Collatz (1966), Ortega and Rheinboldt (1970), Ostrowski (1960), Rall (1969), Traub (1964), and Varga (1962).

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CHAPTER

4

All-Thermal Power Systems

4.1 INTRODUCTION

The intent of this chapter is to treat problems of optimal economic operation for systems with predominantly thermal generation. Chapter 2 provides the basis of our present discussion. Thermal plant input–output modeling as well as electric network models are the essential ingredients for this chapter.

Two distinct classes of problems arise according to the choice of the electric network model. The first employs power balance equations which include the active power balance with or without transmission losses accounted for. Within the same class, we also consider the case including the active–reactive power balance equations in the optimization. These are treated in the following section, which concludes with a discussion of a more detailed model for the plant input–output model. The second class of problems is offered by what is commonly referred to as the optimal load flow. As the name implies, the electric network model used in this case is the most detailed one, namely, the load flow equations. This is dealt with in Section 4.3.

It is important to note that the results of an all-thermal dispatch solution can be applied directly to a hydro–thermal system. In this case an equivalent thermal characteristic can be used to replace the hydro plant's characteristics. This approach will be treated in the next chapter.

4.2 DISPATCH USING POWER BALANCE MODELS

We have seen in Chapter 2 that the performance of the electric network can be modeled using power balance models. The problems treated in this section employ such models. We begin with the simple active power balance model, which leads to the classical equal incremental cost loading principle. This we follow by the formulation using the transmission loss formula. As a result the concept of penalty factors emerges. The extension of the results to include more detail by utilizing the active–reactive power balance equations is done next. To conclude this present section we briefly consider the problem of including valve throttling losses into the formulation. For the sake of brevity we only consider the simple active power balance model in this last part of the section.

4.2.1 Transmission Losses Neglected

Consider the operation of m thermal units denoted by the set R_s to satisfy a given active power demand P_D. The total fuel cost of generation is

$$F_T = \sum_{i \in R_s} F_i(P_{s_i}) \tag{4.2.1}$$

The active power balance equation neglecting losses is

$$P_D = \sum_{i \in R_s} P_{s_i} \tag{4.2.2}$$

The problem of interest is to find the optimal active power generations minimizing the cost (4.2.1) while satisfying (4.2.2).

This constrained minimization problem may be solved using variational principles. An augmented cost function is formed. This is given by

$$\hat{F} = F_T + \lambda \left[P_D - \sum_{i \in R_s} P_{s_i} \right] \tag{4.2.3}$$

The new variable λ is a Lagrange-type multiplier. The conditions for an extremum are

$$\partial \hat{F} / \partial P_{s_i} = 0, \qquad i \in R_s \tag{4.2.4}$$

Performing the partial differentiations indicated in Eq. (4.2.4) we obtain

$$(\partial F_i / \partial P_{s_i}) - \lambda = 0, \qquad i \in R_s \tag{4.2.5}$$

Note that each unit's cost is independent of the generations of other units.

The expression obtained in Eq. (4.2.5) leads to the conclusion that

$$\lambda = \partial F_1/\partial P_{s_1} = \partial F_2/\partial P_{s_2} = \cdots \tag{4.2.6}$$

The implication here is that for optimality, individual units should share the load such that their incremental costs are equal. We can see that the λ is simply the optimal value of incremental costs at the operating point.

Implementing the optimal solution is straightforward and we show here the treatment for the quadratic costs case. Here we have

$$F_i(P_{s_i}) = \alpha_i + \beta_i P_{s_i} + \gamma_i P_{s_i}^2, \qquad i \in R_s \tag{4.2.7}$$

Our optimality conditions are

$$\beta_i + 2\gamma_i P_{s_i} - \lambda = 0, \qquad i \in R_s \tag{4.2.8}$$

The value of λ is determined so that Eq. (4.2.2) is satisfied. This turns out to be

$$\lambda = \left[2P_D + \sum_{i \in R_s} (\beta_i/\gamma_i) \right] \Big/ \left(\sum \gamma_i^{-1}\right) \tag{4.2.9}$$

Finally, optimal generations using Eq. (4.2.8) are

$$P_{s_{i_\xi}} = (\lambda - \beta_i)/2\gamma_i \tag{4.2.10}$$

The solution to this simplified problem can be used as an initial estimate for solutions of more complex problems requiring iterative solutions. It can also be used on its own in cases where the assumption of negligible loss is valid. One such case is when the thermal units are on the same bus at the thermal power plant. Our developed theory can be used to obtain a single equivalent representation for the plant based on economic considerations. This we outline in the following.

Single Equivalent Machine Representation

Assume that m thermal units are operating in parallel to supply a power demand P_D. We are interested in obtaining the cost parameters α_E, β_E, and γ_E of the single equivalent unit with cost

$$F_T(P_D) = \alpha_E + \beta_E P_D + \gamma_E P_D^2 \tag{4.2.11}$$

The solution to this problem is easily obtained by noting that the individual units will share the load optimally. This means that F_T in Eq. (4.2.11) is the optimal cost for the system. The conditions for optimal operation as obtained before are given by Eqs. (4.2.8) and (4.2.9). The optimal cost can then be obtained using Eqs. (4.2.1) and (4.2.7). Rearranging and equating the resulting expression to (4.2.11), the equivalent parameters obtained are

$$\gamma_{\mathrm{E}}^{-1} = \sum_{i \in R_s} \gamma_i^{-1} \tag{4.2.12}$$

$$(\beta_{\mathrm{E}}/\gamma_{\mathrm{E}}) = \sum_{i \in R_s} (\beta_i/\gamma_i) \tag{4.2.13}$$

$$\alpha_{\mathrm{E}} = \sum_{i \in R_s} [\alpha_i - (\beta_i^2/4\gamma_i)] + (\beta_{\mathrm{E}}^2/4\gamma_{\mathrm{E}}) \tag{4.2.14}$$

A special case is one in which the m units are identical. Our equivalent parameters for this case reduce to

$$\gamma_{\mathrm{E}} = \gamma/m \tag{4.2.15}$$

$$\beta_{\mathrm{E}} = \beta \tag{4.2.16}$$

$$\alpha_{\mathrm{E}} = \alpha m \tag{4.2.17}$$

The above result can be verified directly by noting that identical parallel units will share the load equally.

It is quite important to obtain an estimate for the loss of economy in assuming that units of slightly different fuel characteristics are identical. Let us assume that the individual units have parameters

$$\alpha_i = x_i\alpha, \qquad \beta_i = x_i\beta, \qquad \gamma_i = x_i\gamma$$

If we assume for an approximation that the factors x_i are unity we are led to the conclusion that the load should be shared equally between the plants. The total cost, denoted by $F_{(1/m)}$, is then given by

$$F_{(1/m)} = [\alpha + (\beta P_{\mathrm{D}}/m) + (\gamma P_{\mathrm{D}}^2/m^2)] \sum_{i=1}^{m} x_i$$

On the other hand, our optimal allocation involves a cost F_{opt}. This is obtained from the optimal equivalent fuel cost expression simply with the equivalent parameters given by

$$\gamma_{\mathrm{E}} = \gamma\left(\sum_{i=1}^{m} x_i^{-1}\right)^{-1}$$

$$\beta_{\mathrm{E}} = \beta m\left(\sum_{i=1}^{m} x_i^{-1}\right)^{-1}$$

$$\alpha_{\mathrm{E}} = \left(\alpha \sum_{i=1}^{m} x_i\right) + (\beta^2/4\gamma)\left[\left(m^2\Big/\sum_{i=1}^{m} x_i^{-1}\right) - \sum_{i=1}^{m} x_i\right]$$

The loss in economy is the difference

$$E = F_{(1/m)} - F_{\mathrm{opt}}$$

This turns out to be a separable function in the power demand and factors x_i given by

$$E = f_1(x_i)f_2(P_D)$$

with

$$f_1(x_i) = \left[\left(\sum_{i=1}^{m} x_i/m^2\right) - \left(\sum_{i=1}^{m} x_i^{-1}\right)^{-1}\right]$$

$$f_2(P_D) = [(\beta^2 m^2/4\gamma) + \beta P_D m + \gamma P_D^2]$$

It is worth noting that f_1 is a positive quantity. This is to be expected from the basic nature of optimality.

Example. To illustrate the significance of the loss in economy due to the inaccurate assumption that parallel units are identical, we consider the case of two units. Assume that

$$x_1 = 1, \qquad x_2 = x$$

We then have

$$f_1(x_i) = 0.25(1 - x)^2/(1 + x)$$

Let the cost model for unit 1 be given by

$$F_1 = 185 + 16.08P_1 + 1.0322 \times 10^{-2}P_1^2 \quad \$/\text{hr}$$

Assuming that the two units are identical, the equivalent cost model obtained is

$$F_{(1/2)} = 370 + 16.08P_D + 5.1661 \times 10^{-3}P_D^2$$

The loss of economy function f_2 is thus calculated to be

$$f_2(P_D) = 25041 + 32.155P_D + 1.0322 \times 10^{-2}P_D^2$$

The variation of f_1 with x as well as E for some possible plant sendouts are given in the accompanying tabulation. To emphasize the loss of economy

		E (\$/hr)		
x	f_1	150 MW	200 MW	300 MW
1.01	1.244×10^{-5}	0.3743	0.3976	0.4429
1.02	4.950×10^{-5}	1.4899	1.5785	1.7631
1.03	1.108×10^{-4}	3.3359	3.5341	3.9476
1.04	1.961×10^{-4}	5.9013	6.2519	6.9836
1.05	3.049×10^{-4}	9.1758	9.7210	10.8586

significance we note that using the above table we arrive at an annual loss of economy of little over $80,000 for constant plant sendout of 150 MW. This increases for the 300-MW level to over $95,000 for one year.

4.2.2 Transmission Losses Included

We include transmission losses in our active power balance equation and inquire into the modifications to the optimal solution obtained above. We are thus interested in minimizing the total cost given by Eq. (4.2.1) while satisfying the active power balance equation

$$P_{\mathrm{D}} = \sum_{i \in R_{\mathrm{s}}} P_{\mathrm{s}_i} - P_{\mathrm{L}} \tag{4.2.18}$$

Here P_{L} is the active power loss, which is considered a function of the active power generation alone.

Following our treatment for the previous problem, we have now the augmented cost function

$$\hat{F} = F_{\mathrm{T}} + \lambda\left(P_{\mathrm{D}} + P_{\mathrm{L}} - \sum_{i \in R_{\mathrm{s}}} P_{\mathrm{s}_i}\right) \tag{4.2.19}$$

The optimality conditions turn out to be

$$(\partial F_i/\partial P_{\mathrm{s}_i}) + \lambda[(\partial P_{\mathrm{L}}/\partial P_{\mathrm{s}_i}) - 1] = 0, \qquad i \in R_{\mathrm{s}} \tag{4.2.20}$$

Note that with negligible transmission losses Eq. (4.2.20) reduces to Eq. (4.2.5).

It is convenient to transform Eq. (4.2.20) into an equivalent form. This can be done by defining the following factors L_i:

$$L_i = [1 - (\partial P_{\mathrm{L}}/\partial P_{\mathrm{s}_i})]^{-1}, \qquad i \in R_{\mathrm{s}} \tag{4.2.21}$$

We can thus write Eq. (4.2.20) as

$$L_i(\partial F_i/\partial P_{\mathrm{s}_i}) = \lambda, \qquad i \in R_{\mathrm{s}} \tag{4.2.22}$$

This is of the form of (4.2.5) except for the new factors L_i which account for the modifications necessitated by including the transmission loss. These are called the penalty factors to indicate that plant costs are penalized by the corresponding transmission losses. The implication of Eq. (4.2.22) is that the optimal generations are obtained when each plant is operated in such a way that the penalized incremental cost are equal. It is worth remarking here that the terms $(\partial P_{\mathrm{L}}/\partial P_i)$ in Eq. (4.2.21) are called incremental losses. The optimal solution is completely specified by Eqs. (4.2.18) and (4.2.20).

Let us consider the problem of implementing the optimal solution. We assume that our fuel costs are quadratic expressions as given in Eq. (4.2.7).

Moreover, the transmission losses are expressed as

$$P_L = K_{L_0} + \sum_{i \in R_s} B_{i0} P_{s_i} + \sum_{i,j \in R_s}\sum P_{s_i} B_{ij} P_{s_j} \tag{4.2.23}$$

In this case our incremental loss expressions turn out to be

$$\partial P_L / \partial P_{s_i} = B_{i0} + 2 \sum_{j \in R_s} B_{ij} P_{s_j}, \qquad i \in R_s \tag{4.2.24}$$

We can conclude then that Eq. (4.2.20) requires

$$f_i = \beta_i + 2\gamma_i P_{s_i} + \lambda\left(B_{i0} - 1 + 2 \sum_{j \in R_s} B_{ij} P_{s_j}\right) = 0, \qquad i \in R_s \tag{4.2.25}$$

The multiplier λ obtained is such that the active power balance equation is satisfied. In our present case this is given by

$$g = P_D + K_{L0} + \sum_{i \in R_s} (B_{i0} - 1) P_{s_i} + \sum_{i,j \in R_s}\sum P_{s_i} B_{ij} P_{s_j} = 0 \tag{4.2.26}$$

Equations (4.2.25) and (4.2.26) completely specify our desired optimal solution. We remark here that nonlinearities due to loss inclusion are present in our case. Iterative solutions need to be obtained.

We suggest that the Newton–Raphson method will be successful in solving the set of algebraic equations. Let us evaluate the derivatives of our equations with respect to the unknowns.

$$\partial f_i / \partial P_{s_i} = 2(\gamma_i + \lambda B_{ii}), \qquad i \in R_s \tag{4.2.27}$$

$$\partial f_i / \partial P_{s_j} = 2\lambda B_{ij}, \qquad i, j \in R_s \tag{4.2.28}$$

$$\partial f_i / \partial \lambda = \left(B_{i0} - 1 + 2 \sum_{j \in R_s} B_{ij} P_{s_j}\right), \qquad i \in R_s \tag{4.2.29}$$

$$\partial g / \partial P_{s_i} = \left(B_{i0} - 1 + 2 \sum_{j \in R_s} B_{ij} P_{s_j}\right), \qquad i \in R_s \tag{4.2.30}$$

$$\partial g / \partial \lambda = 0 \tag{4.2.31}$$

The above derivatives are needed for forming the Jacobian matrix necessary for Newton–Raphson iterations. The vector structure we adopt is

$$\begin{bmatrix} \mathbf{f} \\ g \end{bmatrix} = -\begin{bmatrix} \mathbf{J}_{fp} & \mathbf{J}_{f\lambda} \\ \mathbf{J}_{gP} & \mathbf{J}_{g\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta P} \\ \Delta\lambda \end{bmatrix} \tag{4.2.32}$$

Here $\mathbf{J}_{fP}$ is a square symmetric matrix of partial derivatives of f_i with respect to P_i. Elements of $\mathbf{J}_{fP}$ are given by Eqs. (4.2.27) and (4.2.28). Now $\mathbf{J}_{f\lambda}$ is a column vector of f_i derivatives with respect to λ which are given by Eq. (4.2.29). $\mathbf{J}_{gp}$ is a row vector of derivatives of g with respect to P_{s_i}

which are given by Eq. (4.2.30). Note that

$$\mathbf{J}_{gp} = \mathbf{J}_{f\lambda}^{\mathrm{T}}.$$

Finally, $\mathbf{J}_{g\lambda}$ is a single element given by Eq. (4.2.31) as zero.

With the above result we state that successive corrections vector $(\mathbf{\Delta P})$ and $(\Delta\lambda)$ are obtained from

$$\begin{bmatrix} \mathbf{\Delta P} \\ \Delta\lambda \end{bmatrix} = -\begin{bmatrix} \mathbf{J}_{fp} & \mathbf{J}_{f\lambda} \\ \mathbf{J}_{f\lambda}^{\mathrm{T}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix} \tag{4.2.33}$$

These turn out to be

$$\Delta\lambda = [g - \mathbf{J}_{f\lambda}^{\mathrm{T}}\mathbf{J}_{fp}^{-1}\mathbf{f}]/\Gamma \tag{4.2.34}$$

$$\mathbf{\Delta P} = \mathbf{J}_{fp}^{-1}[\mathbf{J}_{f\lambda}\lambda - \mathbf{f}] \tag{4.2.35}$$

with

$$\Gamma = \mathbf{J}_{f\lambda}^{\mathrm{T}}\mathbf{J}_{fp}^{-1}\mathbf{J}_{f\lambda} \tag{4.2.36}$$

Thus starting with estimates $\mathbf{P}^{(0)}$ and $\lambda^{(0)}$ for our unknowns, new improved estimates are obtained according to

$$\lambda^{k+1} = \lambda^{k} - \Delta\lambda \tag{4.2.37}$$

$$\mathbf{P}^{k+1} = \mathbf{P}^{k} - \mathbf{\Delta P} \tag{4.2.38}$$

As the iterations progress the solution to the problem is obtained to the desired degree of accuracy. It should be noted that the initial estimates for the solution are best obtained by assuming negligible losses. This is the case discussed in Section 4.2.1, where a set of linear equations defines the optimal solution.

Unregulated Source Case

A case of interest arises if the generation of some of the system sources is already predetermined. These sources will be referred to as unregulated. Let the set of such sources be denoted by R_{sU} and let R_{sR} denote the dispatchable sources. If the transmission losses are neglected then our problem is simply resolved by writing the power balance equation with a new power demand obtained by subtracting the unregulated generations from the original power demand.

Further modifications are needed for the case including transmission losses. Here we write the transmission loss expression as

$$P_{\mathrm{L}} = \tilde{K}_{\mathrm{L}_0} + \sum_{i \in R_{\mathrm{sR}}} \tilde{B}_{i0}P_i + \sum\sum_{i,\, j \in R_{\mathrm{sR}}} P_i B_{ij} P_j$$

where the new constant term is defined by

$$\tilde{K}_{L_0} = K_{L_0} + \sum_{i \in R_{sU}} B_{i0} P_i + \sum\sum_{i, j \in R_{sU}} P_i B_{ij} P_j$$

The loss contributions of the unregulated sources will thus modify the original K_{L_0}. The new linear coefficients are similarly defined by

$$\tilde{B}_{i0} = B_{i0} + 2 \sum_{j \in R_{sU}} B_{ij} P_j$$

Finally, the power balance equation is written as

$$\tilde{P}_D = \sum_{i \in R_{sR}} P_i - P_L \qquad \text{with} \quad \tilde{P}_D = P_D - \sum_{i \in R_{sU}} P_i$$

4.2.3 Active–Reactive Balance Included

In Chapter 2 it is pointed out that a description of the electric network performance which is more detailed than the active power balance model is possible. The resulting model is given by the active–reactive power balance equations (2.4.45) and (2.4.46). Here the active power loss is given by Eq. (2.4.26) and the reactive power loss is given by (2.4.47). Clearly the inclusion of reactive power in the optimization process is of interest. This we treat presently.

We consider an all-thermal system for which an optimal schedule is sought. The schedule should minimize the combined objective functional

$$\tilde{F} = F_T + F_Q \tag{4.2.39}$$

The functional F_T is the usual one based on the fuel cost model given in Eq. (4.2.1) as

$$F_T = \sum_{i \in R_s} F_i(P_{s_i}) \tag{4.2.40}$$

The second term F_Q is based on the reactive capability functional (2.6.5). This is given by

$$F_Q = F_{Q0} + \mathbf{\Lambda}^T \mathbf{Q}_s + \mathbf{Q}_s^T \mathbf{K} \mathbf{Q}_s \tag{4.2.41}$$

The constant F_{Q0}, the vector $\mathbf{\Lambda}$, and the matrix $\mathbf{K}$ are assumed to be known a priori. The schedules are designed to meet specified active and reactive power demands. The network performance is given by

$$P_D + P_L - \sum_{i \in R_s} P_{s_i} = 0 \tag{4.2.42}$$

$$Q_D + Q_L - \sum_{i \in R_s} Q_{s_i} = 0 \tag{4.2.43}$$

Optimality conditions can be obtained in a way similar to the preceding discussion. This time, since we have two equality constraints, we will need two Lagrange-type multipliers, which we denote by λ_p and λ_q. As a result the augmented objective functional is written as

$$F_A = \tilde{F} + \lambda_p\left(P_D + P_L - \sum_{i \in R_s} P_{s_i}\right) + \lambda_q\left(Q_D + Q_L - \sum_{i \in R_s} Q_{s_i}\right) \quad (4.2.44)$$

The decision variables are the active power generations P_{s_i} and the reactive power generations Q_{s_i}. We thus have the following necessary conditions:

$$\partial F_A/\partial P_{s_i} = 0, \qquad i \in R_s \quad (4.2.45)$$

$$\partial F_A/\partial Q_{s_i} = 0, \qquad i \in R_s \quad (4.2.46)$$

Performing the indicated partial differentiations the following is obtained:

$$(\partial F_i/\partial P_{s_i}) + \lambda_p[(\partial P_L/\partial P_{s_i}) - 1] + \lambda_q(\partial Q_L/\partial P_{s_i}) = 0, \qquad i \in R_s \quad (4.2.47)$$

$$(\partial F_Q/\partial Q_{s_i}) + \lambda_p(\partial P_L/\partial Q_{s_i}) + \lambda_q[(\partial Q_L/\partial Q_{s_i}) - 1] = 0, \qquad i \in R_s \quad (4.2.48)$$

The optimality conditions are thus given by the equality constraints (4.2.42) and (4.2.43) together with (4.2.47) and (4.2.48). In all we have double the number of equations required for the active power balance approach.

The above optimality conditions yield as a special case the conditions for dispatching active power balance only. In this case the terms in reactive power disappear. The implementation of the conditions is very similar to the case discussed in the preceding section.

4.2.4 Valve-Point Considerations

In defining thermal cost models in common use, the assumption of monotonically increasing cost curves is employed. This leads to polynomial cost models which are the basis of most dispatch algorithms treated in this work.

This assumption is not everywhere valid because of the throttling losses near valve points, as discussed in Chapter 2. These losses introduce negative slopes into the incremental cost curves. Our object here is to discuss the possibilities of accounting for valve characteristics and outline a dynamic programming approach for valve-point dispatch.

Various mathematical models exist that attempt to approximate the unit performance characteristics while accounting for valve characteristics. The models are basically polynomial approximations that change with the valve zone. Two such models are described here. The first, which is referred to as the *valve step model*, assumes constant incremental fuel cost in each valve zone. This is expressed for the ith valve region as

$$F = \alpha_i + \beta_i P, \qquad P^{\min}i < P < P^{\max}i$$

The second assumes a linear variation of the incremental fuel cost in each valve zone and is called the *modified valve step model*:

$$F = \alpha_i + \beta_i P + \gamma_i P^2, \qquad P^{\min}i < P < P^{\max}i$$

In both models, α_i, β_i, and γ_i are evaluated from experimental testing, when applicable for each zone.

A turbine loaded at a valve point just before the next valve opens is working at maximum efficiency at that loading. Operation off that valve point is less efficient due to the throttling losses. Therefore if all parallel turbines are loaded on selected valve points maximum efficiency can result. The concept of valve-point loading, useful as it appears, is inherently unsuited for matching generation to a continuously varying load. Therefore a certain amount of capacity must be withheld from valve-point operation in order to be operated in a continuous manner so as to supply the continuously varying load portion not met by the valve-point sources. It is clear that a powerful and a flexible optimization technique is needed to handle this complex problem. Dynamic programming meets this requirement and we describe here the basic idea of such a dispatch algorithm.

A Dynamic Programming Approach

The system considered contains m thermal units. The fuel cost models are assumed given in a general tabular form for each thermal unit. Losses are neglected and only active power balance is considered here. This allows for accounting for valve points.

The method proceeds by assuming that enough capacity can be provided by units 1 and 2 to satisfy the demand P_D. The cost of meeting the demand can thus be expressed as a function of the output of unit 2. Let us define an optimum output P_2^* and the associated function

$$f_2(P_D) = \min_{P_2} [F_1(P_D - P_2) + F_2(P_2)] \tag{4.2.49}$$

Here F_1 and F_2 are the costs of units 1 and 2, respectively. We next assume that units 1, 2, and 3 are to satisfy the demand. The cost in this case can be expressed in terms of output of units 2 and 3 and thus we define the function f_3 as

$$f_3(P_D) = \min_{P_2, P_3} [F_1(P_D - P_2 - P_3) + F_2(P_2) + F_3(P_3)] \tag{4.2.50}$$

Expanding, we have

$$f_3(P_D) = \min_{P_3} \{\min_{P_2} [F_1(P_D - P_2 - P_3) + F_2(P_2)] + F_3(P_3)\} \tag{4.2.51}$$

Comparing Eq. (4.2.50) and Eq. (4.2.51), we conclude that

$$f_3(P_D) = \min_{P_3} [F_3(P_3) + f_2(P_D - P_3)] \tag{4.2.52}$$

Let us denote the optimal value for the output of unit 3 by P_3^*.

We can now state the following recursive relationship:

$$f_j(P_D) = \min_{P_j} [F_j(P_j) + f_{j-1}(P_D - P_j)], \qquad j = 2, \ldots, m \tag{4.2.53}$$

In particular, for the case when all machines are supplying the load

$$f_m(P_D) = \min_{P_m} [F_m(P_m) + f_{m-1}(P_D - P_m)] \tag{4.2.54}$$

and the associated optimal mth unit generation is P_m^*.

Ringlee–Williams procedure requires the generation of the functions f_j for the range of power demands

$$P_{D_{\min}} \leq P_D \leq P_{D_{\max}}$$

Starting from Eq. (4.2.54) for a given power demand, the optimal P_m^* is obtained. Obtain the modified power demand given by

$$P_{D_{m-1}} = P_D - P_m^* \tag{4.2.55}$$

For this demand we obtain the optimum P_{m-1}^* from

$$f_{m-1}(P_{D_{m-1}}) = \min_{P_{m-1}} [F_{m-1}(P_{m-1}) + f_{m-2}(P_{D_{m-1}} - P_{m-1})] \tag{4.2.56}$$

We can now define

$$P_{D_{m-2}} = P_{D_{m-1}} - P_{m-1}^* \tag{4.2.57}$$

With this we repeat the above procedure. The optimal schedule for the m generators is thus generated for the desired power demand.

4.3 OPTIMAL LOAD FLOW

The most rigorous steady state electric power system network model is provided by the load flow equations discussed in Chapter 2. An optimal load flow problem is one that incorporates this exact model in the formulation. The dispatch methods treated so far in this chapter used models of far less dimension and sophistication. Some relevant variables such as generator voltage magnitudes are not included in the optimization procedure of these methods. As a result, constraints imposed by considerations of system security are not easily handled by procedures using the power balance or traditional models. The advantage of optimal load flow lies not so much in higher accuracy. More important is its ability to include security constraints in the formulation.

4.3.1 The Carpentier–Siroux Approach

It is required to minimize the cost I given by

$$I = F(\mathbf{P}_{\mathrm{G}})$$

where $\mathbf{P}_{\mathrm{G}}$ is the active power generation vector, subject to the equality and inequality constraints:

(1) The net active and reactive powers into the system (injected power) are functions of the voltage magnitudes V_i and the corresponding phase angles δ_i. This is expressed by Eqs. (2.4.60) and (2.4.61), which are repeated here as

$$P_i(\boldsymbol{\delta}, \mathbf{V}) = \sum_{j \in \alpha_i} V_i V_j Y_{ij} \cos(\delta_i - \delta_j + \theta_{ij}), \qquad i \in R_{\mathrm{N}} \tag{4.3.1}$$

$$Q_i(\boldsymbol{\delta}, \mathbf{V}) = \sum_{j \in \alpha_i} V_i V_j Y_{ij} \sin(\delta_i - \delta_j + \theta_{ij}), \qquad i \in R_{\mathrm{N}} \tag{4.3.2}$$

In the above the set α_i includes all buses connected to the ith bus, and Y_{ij} and θ_{ij} are the magnitude and phase angle, respectively, of the (i, j)th element of the nodal admittance matrix. The net active power P_i and the net reactive power Q_i are related to the active and reactive power generations P_{G_i} and Q_{G_i} and specific demands P_{D_i} and Q_{D_i} by

$$\phi_{p_i} = P_i(\boldsymbol{\delta}, \mathbf{V}) - P_{\mathrm{G}_i} + P_{\mathrm{D}_i} = 0, \qquad i \in R_{\mathrm{N}} \tag{4.3.3}$$

$$\phi_{q_i} = Q_i(\boldsymbol{\delta}, \mathbf{V}) - Q_{\mathrm{G}_i} + Q_{\mathrm{D}_i} = 0, \qquad i \in R_{\mathrm{N}} \tag{4.3.4}$$

The above equations are called the injection relations.

(2) Equipment rating limitations impose the following inequality type constraints:

$$\Pi_i = P_{\mathrm{G}_i}^2 + Q_{\mathrm{G}_i}^2 - (S_i^{\mathrm{M}})^2 \le 0 \tag{4.3.5a}$$

$$\tilde{\Pi}_i = P_{\mathrm{G}_i}^{\mathrm{m}} - P_{\mathrm{G}_i} \le 0 \tag{4.3.5b}$$

$$\Psi_i = Q_{\mathrm{G}_i} - Q_{\mathrm{G}_i}^{\mathrm{M}} \le 0 \tag{4.3.5c}$$

$$\tilde{\Psi}_i = Q_{\mathrm{G}_i}^{\mathrm{m}} - Q_{\mathrm{G}_i} \le 0 \tag{4.3.5d}$$

$$\xi_i = V_i - V_i^{\mathrm{M}} \le 0 \tag{4.3.5e}$$

$$\tilde{\xi}_i = V_i^{\mathrm{m}} - V_i \le 0 \tag{4.3.5f}$$

$$\tau_{ij} = \delta_i - \delta_j - T_{ij} \le 0 \tag{4.3.5g}$$

S_i^{M} is the maximum apparent power of the ith generating node, $P_{\mathrm{G}_i}^{\mathrm{m}}$ is the minimum active power generation. The maximum and minimum reactive power generation at the ith node are $Q_{\mathrm{G}_i}^{\mathrm{M}}$ and $Q_{\mathrm{G}_i}^{\mathrm{m}}$, respectively. We thus have four inequality constraints for every generating node. The maximum and minimum allowable voltage levels at all system nodes with the exception of the reference (slack) bus are V_i^{M} and V_i^{m}, respectively. The last inequality is imposed by the maximum power transfer capability of lines and transformers and is an approximation for the real power flow between nodes given by

$$P_{ij} = (V_i V_j / X_{ij}) \sin(\delta_i - \delta_j)$$

According to the Kuhn–Tucker theorem discussed in Chapter 3, we can write the augmented cost function (Lagrangian) for the problem stated above as

$$\begin{aligned}\tilde{I} = F &+ \sum_{i \in R_{\mathrm{N}}} (\lambda_{p_i}\phi_{p_i} + \lambda_{q_i}\phi_{q_i}) + \sum_{i \in R_{\mathrm{G}}} (m_i \Pi_i + \tilde{m}_i \tilde{\Pi}_i + e_i \psi_i + \tilde{e}_i \tilde{\psi}_i) \\ &+ \sum_{i \in R_{\mathrm{N}}} (l_i \xi_i + \tilde{l}_i \tilde{\xi}_i) + \sum_{i, j \in R_{\mathrm{N}}} t_{ij}\tau_{ij} \end{aligned} \tag{4.3.6}$$

The necessary conditions for optimality are obtained by setting the derivatives of the Lagrangian to zero. We thus have the following set of conditions:

(1) The differential with respect to active power generations is

$$\partial \tilde{I} / \partial P_{\mathrm{G}_i} = 0$$

This results in

$$(\partial F / \partial P_{\mathrm{G}_i}) - \lambda_{p_i} + 2 m_i P_{\mathrm{G}_i} - \tilde{m}_i = 0 \tag{4.3.7}$$

(2) With the reactive power generations we have

$$\partial \tilde{I} / \partial Q_{\mathrm{G}_i} = 0$$

with the result

$$-\lambda_{q_i} + 2 m_i Q_{\mathrm{G}_i} + e_i - \tilde{e}_i = 0 \tag{4.3.8}$$

(3) With the voltage phase angles we have

$$\partial \tilde{I} / \partial \delta_i = 0$$

Thus

$$\sum_{j \in \alpha_i} [\lambda_{p_j}(\partial P_j / \partial \delta_i) + \lambda_{q_j}(\partial Q_j / \partial \delta_i) + (t_{ij} - t_{ji})] = 0 \tag{4.3.9}$$

(4) Finally, with voltage magnitudes

$$\partial \tilde{I} / \partial V_i = 0$$

The result is

$$\sum_{j \in \alpha_i} [\lambda_{p_j}(\partial P_j/\partial V_i) + \lambda_{q_j}(\partial Q_j/\partial V_i) + l_i - \tilde{l}_i] = 0 \tag{4.3.10}$$

In addition, the equality constraints as well as the exclusion equations are to be satisfied. These are

$$P_i(\boldsymbol{\delta}, \mathbf{V}) - P_{G_i} + P_{D_i} = 0 \tag{4.3.11}$$

$$Q_i(\boldsymbol{\delta}, \mathbf{V}) - Q_{G_i} + Q_{D_i} = 0 \tag{4.3.12}$$

$$m_i[P_{G_i}^2 + Q_{G_i}^2 - (S_i^M)^2] = 0, \qquad m_i \geq 0 \tag{4.3.13a}$$

$$\tilde{m}_i(P_{G_i}^m - P_{G_i}) = 0, \qquad \tilde{m}_i \geq 0 \tag{4.3.13b}$$

$$e_i(Q_{G_i} - Q_{G_i}^M) = 0, \qquad e_i \geq 0 \tag{4.3.13c}$$

$$\tilde{e}_i(Q_{G_i}^m - Q_{G_i}) = 0, \qquad \tilde{e}_i \geq 0 \tag{4.3.13d}$$

$$l_i(V_i - V_i^M) = 0, \qquad l_i \geq 0 \tag{4.3.13e}$$

$$\tilde{l}_i(V_i^m - V_i) = 0, \qquad \tilde{l}_i \geq 0 \tag{4.3.13f}$$

$$t_{ij}(\delta_i - \delta_j - T_{ij}) = 0, \qquad t_{ij} \geq 0 \tag{4.3.13g}$$

$$P_{G_i}^2 + Q_{G_i}^2 - (S_i^M)^2 \leq 0 \tag{4.3.14a}$$

$$P_{G_i}^m - P_{G_i} \leq 0 \tag{4.3.14b}$$

$$Q_{G_i} - Q_{G_i}^M \leq 0 \tag{4.3.14c}$$

$$Q_{G_i}^m - Q_{G_i} \leq 0 \tag{4.3.14d}$$

$$V_i - V_i^M \leq 0 \tag{4.3.14e}$$

$$V_i^m - V_i \leq 0 \tag{4.3.14f}$$

$$\delta_i - \delta_j - T_{ij} \leq 0 \tag{4.3.14g}$$

We now offer some interpretation of the significance of the multipliers (dual variables) involved in the generalized coordination equations. The variables λ_{p_i} are the active power incremental costs associated with each bus. This can be shown by perturbing the active power demand at the ith bus at the optimal point by an amount ΔP_{D_i}. According to Eq. (4.3.11) we have

$$\Delta P_i - \Delta P_{G_i} + \Delta P_{D_i} = 0 \tag{4.3.15}$$

The variation in the augmented cost is

$$\Delta F + \lambda_{p_i} \Delta(P_i - P_{G_i}) = 0 \tag{4.3.16}$$

Combining, we obtain

$$\Delta F - \lambda_{p_i} \Delta P_{D_i} = 0$$

As a result,

$$\lambda_{p_i} = \Delta F/\Delta P_{D_i} \tag{4.3.17}$$

Considering λ_{q_i}, using the same line of reasoning we can state

$$\lambda_{q_i} = \Delta F/\Delta Q_{\mathrm{D}_i} \tag{4.3.18}$$

The solution of the optimal load flow problem can thus provide a rational way of tariffiing or billing for active and reactive powers at demand nodes.

The reactive power incremental cost λ_{q_i} as a result of Eq. (4.3.8) is

$$\lambda_{q_i} = 2m_i Q_{\mathrm{G}_i} + e_i - \tilde{e}_i \tag{4.3.19}$$

It is thus evident that a null value for this multiplier is a normal situation and occurs when all inequality constraints on the reactive power are satisfied. A positive value for λ_{q_i} indicates a shortage of reactive power and negative value corresponds to an excess.

4.3.2 The Dommel–Tinney Approach

The key to this approach are the advances made toward the solution of the load flow equations. These are repeated here in the hybrid form given by Eqs. (2.4.66) and (2.4.67) as

$$P_i(\boldsymbol{\delta}, \mathbf{V}) = V_i \sum_{j \in \alpha_i} V_j[G_{ij}\cos(\delta_i - \delta_j) - B_{ij}\sin(\delta_i - \delta_j)] \tag{4.3.20}$$

$$Q_i(\boldsymbol{\delta}, \mathbf{V}) = V_i \sum_{j \in \alpha_i} V_j[G_{ij}\sin(\delta_i - \delta_j) + B_{ij}\cos(\delta_i - \delta_j)] \tag{4.3.21}$$

The net active and reactive power balance equations are thus

$$P_i(\boldsymbol{\delta}, \mathbf{V}) - P_{\mathrm{N}_i} = 0 \tag{4.3.22}$$

$$Q_i(\boldsymbol{\delta}, \mathbf{V}) - Q_{\mathrm{N}_i} = 0 \tag{4.3.23}$$

where P_{N_i} is the node injection power and Q_{N_i} is the node injection reactive power.

It has been pointed out in Chapter 2 that in the load flow problem three types of nodes are identified. Consequently the vector of unknown variables $\mathbf{x}$ can be written in the partitioned form

$$\mathbf{x}^{\mathrm{T}} = [\mathbf{x}_{\mathrm{G}}^{\mathrm{T}}, \mathbf{x}_{\mathrm{D}}^{\mathrm{T}}] \tag{4.3.24}$$

The components of the vector $\mathbf{x}_{\mathrm{G}}$ are the unknown phase angles at the generator buses:

$$\mathbf{x}_{\mathrm{G}} = [\delta_i : i \in R_{\mathrm{G}}] \tag{4.3.25}$$

The unknown voltages and phase angles at the load or demand buses make the vector $\mathbf{x}_{\mathrm{D}}$ obey

$$\mathbf{x}_{\mathrm{D}} = [V_i, \delta_i : i \in R_{\mathrm{D}}] \tag{4.3.26}$$

The vector $\mathbf{y}$ of specified or known variables is partitioned as

$$\mathbf{y}^{\mathrm{T}} = [\mathbf{y}_s^{\mathrm{T}}, \mathbf{y}_{\mathrm{G}}^{\mathrm{T}}, \mathbf{y}_{\mathrm{D}}^{\mathrm{T}}] \tag{4.3.27}$$

The slack bus vector is

$$\mathbf{y}_s^{\mathrm{T}} = [V_1, \delta_1] \tag{4.3.28}$$

The generators vector is

$$\mathbf{y}_{\mathrm{G}} = [P_{\mathrm{N}_i}, V_i : i \in R_{\mathrm{G}}] \tag{4.3.29}$$

Finally, the known demand vector is

$$\mathbf{y}_{\mathrm{D}} = [P_{\mathrm{N}_i}, Q_{\mathrm{N}_i} : i \in R_{\mathrm{D}}] \tag{4.3.30}$$

It is thus evident that the load flow problem can be formulated to be the solution of

$$\mathbf{g}_r(\mathbf{x}, \mathbf{y}) = \mathbf{0} \tag{4.3.31}$$

The components of the vector function $\mathbf{g}_r$ are the active and reactive power equations for each of the load buses and only the active power equation for each of the generator buses. The Newton–Raphson iterative scheme is simply

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x} \tag{4.3.32}$$

where the increments $\Delta\mathbf{x}$ are obtained from

$$[\mathbf{J}_x(\mathbf{x}^{(k)}, \mathbf{y})]\,\Delta\mathbf{x} = -\mathbf{g}_r(\mathbf{x}^{(k)}, \mathbf{y}) \tag{4.3.33}$$

The Newton–Raphson method has been developed into a very fast algorithm by exploiting the sparsity of the Jacobian matrix $\mathbf{J}$ through optimally ordered elimination and compressed storage schemes.

In optimal power flow some of the independent variables in $\mathbf{y}$ can be varied and are thus control variables. We can thus have the alternative partition for $\mathbf{y}$ into a vector $\mathbf{u}$ of control parameters and a vector $\mathbf{p}$ of fixed parameters,

$$\mathbf{y}^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}] \tag{4.3.34}$$

Control parameters include $\mathbf{y}_{\mathrm{G}}$ as well as transformer tap ratios. The objective is to minimize

$$I = f(\mathbf{x}, \mathbf{u}) \tag{4.3.35}$$

where f is the fuel cost written in terms of $\mathbf{x}$ and $\mathbf{u}$. The equality constraints for this problem are the reduced load flow equations

$$\mathbf{g}_r(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0 \tag{4.3.36}$$

Using the Lagrange multiplier approach we have the augmented cost

$$\tilde{I} = f(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}_r^{\mathrm{T}} \cdot \mathbf{g}_r(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{4.3.37}$$

The set of necessary conditions follows:

$$\partial\tilde{I}/\partial\mathbf{x} = \nabla f_x + \mathbf{J}_x^{\mathrm{T}}\boldsymbol{\lambda}_r = 0 \tag{4.3.38a}$$

$$\partial\tilde{I}/\partial\mathbf{u} = \nabla f_u + \mathbf{J}_u^{\mathrm{T}}\boldsymbol{\lambda}_r = 0 \tag{4.3.38b}$$

$$\partial\tilde{I}/\partial\boldsymbol{\lambda} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0 \tag{4.3.38c}$$

We note here that Eq. (4.3.38a) contains the transpose of the Jacobian matrix of the power flow solution by Newton method. For any feasible solution of the load flow equations, Eq. (4.3.36) is satisfied and $\boldsymbol{\lambda}$ can be obtained from Eq. (4.3.38a). The gradient $\partial\tilde{I}/\partial\mathbf{u}$ will not be equal to zero at this solution unless the optimum is reached.

The Dommel–Tinney solution algorithm proceeds on the basis of the above discussion as follows:

(1) Assuming an initial guess of the control vector $\mathbf{u}^{(0)}$ is available, a load flow solution is obtained using the Newton–Raphson algorithm. This also yields the inverse of the Jacobian matrix for the solution point.

(2) Solve Eq. (4.3.38a) for $\boldsymbol{\lambda}_r$:

$$\boldsymbol{\lambda}_r = -[\mathbf{J}_x^{\mathrm{T}}]^{-1}\nabla f_x \tag{4.3.39}$$

(3) Insert $\boldsymbol{\lambda}_r$ just computed into Eq. (4.3.38b) and compute the reduced gradient $\partial\tilde{I}/\partial\mathbf{u}$. If this is sufficiently small an optimum is indicated.

(4) Otherwise, a new, hopefully improved, control vector is calculated

$$\mathbf{u}^{\mathrm{new}} = \mathbf{u}^{\mathrm{old}} + \Delta\mathbf{u} \tag{4.3.40a}$$

with

$$\Delta\mathbf{u} = -C(\partial\tilde{I}/\partial\mathbf{u}) \tag{4.3.40b}$$

The crucial part of the algorithm is the choice of C. Note that theoretically speaking C can be a diagonal matrix of a dimension compatible with $\mathbf{u}$. The choice of C according to Dommel and Tinney is

$$C = 2\,\Delta\tilde{I}_{\mathrm{ex}}/(\partial\tilde{I}/\partial\mathbf{u})^{\mathrm{T}}(\partial\tilde{I}/\partial\mathbf{u}) \tag{4.3.41}$$

where $\Delta\tilde{I}_{\mathrm{ex}}$ is chosen as

$$\Delta\tilde{I}_{\mathrm{ex}} = 0.04(P_{\mathrm{Loss}}) + (\text{penalty terms})$$

This is for the first iteration. On subsequent iterations C is chosen to get to approximately the lowest value of $\tilde{I}$ along the steepest descent direction defined by $\partial\tilde{I}/\partial\mathbf{u}$. An alternative is to use the second-order correction formula

$$\Delta\mathbf{u} = -(\partial^2\tilde{I}/\partial u_i\partial u_k)^{-1}(\partial\tilde{I}/\partial\mathbf{u}) \tag{4.3.42}$$

This algorithm has been confirmed as the simplest and most efficient for optimal load flow. Variations of these methods have appeared in the literature and are reported in Section 4.4 (Comments and References).

4.4 COMMENTS AND REFERENCES

SECTION 4.2

The equal incremental cost loading principle dates back to as early as 1930. A formal proof of the principle for the two-unit case is due to Steinberg and Smith (1934), who coauthored the first book (1943) on the subject of economy dispatch. The advent of George's loss formula led to the penalty factor concept. Pioneering work in this area is due to George *et al.* (1949) and Ward *et al.* (1950; Ward, 1953). The classical coordination equations were derived by Kirchmayer and Stagg (1952). Brownlee (1954) offers an alternative to the loss formula that depends on the voltages and phase angles. The rapid development of hardware to execute the coordination equations is evident from the work of Imburgia *et al.* (1954), Glimn *et al.* (1954), Early *et al.* (1955), and Morrill *et al.* (1955). Kirchmayer's two classic books contributed immensely to the understanding of the theory and its application (1958, 1959). The active–reactive power dispatch described in this section can also be found in the text by Edelmann and Theilsiefje (1974).

The basic desirability and justification of valve-point loading has been discussed for many years. Perhaps the earliest discussion is in the work of Decker and Brooks (1958). Hayward (1961) describes the method and discusses various methods of determining the basic incremental heat rates. Also in Hayward's work a comparison of methods is given. The work by Happ *et al.* (1963) describes a direct method of computing the exact heat rate characteristics. Ringlee and Williams (1963) introduce the dynamic programming algorithm for dealing with valve throttling losses. They give computational experience with a system including 11 units and propose the linearized version of the loss formula to account for transmission losses. A highly refined technique is described by Fink *et al.* (1969). The technique described in this work is a must reading for its practical value.

SECTION 4.3

Optimal load flow dates back to the early 1960s. The pioneering efforts of Squires (1961), Carpentier (1962), and Carpentier and Siroux (1963) form the basis for much of the work in this area. The contributions of Peschon *et al.* (1968) and Dommel and Tinney (1968) provided solid solution techniques for the optimal load flow problem.

There is a wealth of literature on the application of linear and nonlinear programming algorithms to the optimal load flow problem. Notable among these are Sasson's (1969), El-Abiad and Jaimes (1969), Shen and Laughton (1970), and Ramamoorty and Rao (1970). Sasson and his group (1971) provide an excellent review and comparison of methods. Further refinements are found in Billinton and Sachdeva (1972, 1973), Reid and Hasdorff (1973), Sasson *et al.* (1973), Dayal and Hasdorff (1974), and Rashed and Kelly (1974).

Additional security constraints were added over the years. These include line flow inequality constraints and complex power and voltage under contingency and normal conditions and regulating margin constraints. Notable in this category is the work of Cory and Henser (1972), Stadlin (1971), Peschon *et al.* (1971) Podmore (1974), and Alsac and Stott (1974). The latter establish the superiority of the Dommel–Tinney solution algorithm. The recent work of Barcelo *et al.* (1977) is an impressive contribution that is suitable for large scale applications.

Recommended review reading on the subject are papers by Happ (1975, 1976) and Sasson and Merrill (1974).

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CHAPTER

5

Power Systems with Hydro Plants Not on the Same Stream

5.1 INTRODUCTION

The optimal economic operation of electric power systems with all-thermal generation was studied in Chapter 4. There, two approaches were considered. The first employs the active power balance equation with transmission losses accounted for by use of the transmission loss formula. The second approach involves the exact network model, namely the load flow equations. The size of the problem is increased considerably in the latter approach.

In the present chapter we treat the problem for electric power systems which include both thermal and hydro generation. The hydro plants considered are hydraulically isolated and hence there is no coupling between hydro plants except through the electric transmission network. Two cases are considered in the following two sections. The first case is one with fixed-head hydro plants and is simpler than the second case involving variable-head hydro plants. Our approach is to use the active power balance equation to model the electric network in both cases. In the following two sections we discuss three different approaches for each case.

5.2 FIXED-HEAD HYDRO PLANTS

This section is devoted to a treatment of the optimal economic operation problem in the case of hydro–thermal systems. The hydro plants are assumed to have reservoirs large enough to warrant the assumption of fixed head. We outline the classical approach which is based on the variational calculus principles and obtain the well-known coordination equations. The treatment here is for a system with a general number of plants. The coordination equations method is followed by a dynamic programming approach. Here we choose to outline the method by use of a system including only one thermal source and one hydro generation source. The extension to the general case is obvious and therefore not treated. The third and last approach illustrated in this section involves the use of functional analytic optimization techniques and in this case the minimum norm approach is adopted.

5.2.1 The Classical Approach

Consider a system with hydro plants operating at essentially constant head during the optimization interval. Assume that the fuel cost of the thermal generation is given by

$$F_{\mathrm{T}} = \sum_{i \in R_s} F_i(P_{s_i}) \tag{5.2.1}$$

where R_s is the set of thermal plants in the system. It is desired to minimize the total fuel cost

$$J = \int_0^{T_f} F_{\mathrm{T}} \quad dt \tag{5.2.2}$$

The minimization is carried out such that:

(1) The total system generation matches the power demand $P_{\mathrm{D}}(t)$ and the transmission losses $P_{\mathrm{L}}(t)$. The set of all hydro plants in the system is denoted by R_{h}:

$$\sum_{i \in R_s} P_{s_i}(t) + \sum_{i \in R_h} P_{h_i}(t) = P_{\mathrm{D}}(t) + P_{\mathrm{L}}(t) \tag{5.2.3}$$

(2) The volume of water available for generation at each hydro plant is a prespecified amount b_i:

$$\int_0^{T_f} q_i(t)\,dt = b_i, \qquad i \in R_{\mathrm{h}} \tag{5.2.4}$$

A. The Variational Solution

The problem stated lends itself to a variational solution, which leads to the celebrated coordination equations. The volume of water constraints are included in the cost functional by using the constant multipliers v_j; thus

$$J_1 = \int_0^{T_f} \left(F_{\mathrm{T}} + \sum_{j \in R_h} v_j q_j \right) dt \tag{5.2.5}$$

is to be minimized subject to satisfying Eq. (5.2.3). The latter is included in J_1 via the use of the multiplier function $\lambda(t)$. Thus the problem is transformed into an unconstrained problem of minimizing

$$J = \int_0^{T_f} \{F_i(t) + \sum_{j \in R_h} v_j q_j(t) + \lambda(t)[P_D(t) + P_L(t) - \sum_{i \in R_s} P_{s_i}(t) - \sum_{i \in R_h} P_{h_i}(t)]\} \, dt \quad (5.2.6)$$

The optimality conditions are obtained using variational methods as

$$(dF_i/dP_{s_i}) + \lambda[(\partial P_L/\partial P_{s_i}) - 1] = 0, \qquad i \in R_s \quad (5.2.7)$$

$$v_j(dq_j/dP_{h_j}) + \lambda[(\partial P_L/\partial P_{h_j}) - 1] = 0, \qquad j \in R_h \quad (5.2.8)$$

It is obvious that these equations are obtained by setting to zero the derivatives of the integrand of the cost functional with respect to the decision variables P_{s_i} and P_{h_j}. The last two expressions reduce to the coordination equations

$$\lambda = (dF_i/dP_{s_i}) + \lambda(\partial P_L/P_{s_i}) = v_j(dq_j/dP_{h_j}) + \lambda(\partial P_L/\partial P_{h_j}) \quad (5.2.9)$$

The coordination equations together with the active power balance equation (5.2.3) and the volume of water constraints constitute the desired optimality equations.

The coordination equations have some important consequences. When the effect of head variations may be neglected, the incremental water rate dq/dP_h may be multiplied by a water conversion coefficient v which is a constant over the time period considered. Thus an incremental cost curve is obtained. This allows the hydro plant to be scheduled in a manner similar to the thermal plants in the system. The magnitude of v determines the volume of water withdrawn.

The water conversion coefficient v can be obtained from the coordination equations as follows: Define the penalty factors L_i by

$$L_{s_i} = [1 - (\partial P_L/P_{s_i})]^{-1}, \quad (5.2.10)$$

$$L_{h_i} = [1 - (\partial P_L/P_{h_i})]^{-1} \quad (5.2.11)$$

Hence, the coordination equations become

$$\lambda = L_{s_i}(dF_i/dP_{s_i}) \quad (5.2.12)$$

and

$$\lambda = L_{h_i} v_i (dq_i/dP_{h_i}) \quad (5.2.13)$$

Thus

$$v = \frac{L_s(dF/dP_s)}{L_h(dq/dP_h)} = \frac{\lambda}{L_h(dq/dP_h)} \quad (5.2.14)$$

The coordination equations for a hydro–thermal system are similar in form to the ones for an all-thermal system. This leads one to reason that it might be advantageous to consider replacing the hydro plants by their thermal equivalent. The thermal equivalent will have an incremental cost given by

$$dF_i/dP_i = v_i(\partial q_i/\partial P_i) \tag{5.2.15}$$

Once the equivalent thermal cost is determined, the problem may be solved as an all-thermal one.

B. Implementing the Coordination Equations

The coordination equations, obtained in the classical approach, are written in terms of functions of the control variables. For practical application, however, these functions have to be specified. The commonly accepted function definition for the fuel costs is given by the quadratic form discussed earlier:

$$F_i(P_{s_i}) = \alpha_i + \beta_i P_{s_i} + \gamma_i P_{s_i}^2, \qquad i \in R_s \tag{5.2.16}$$

This leads to the incremental cost given by

$$dF_i/dP_{s_i} = \beta_i + 2\gamma_i P_{s_i}, \qquad i \in R_s \tag{5.2.17}$$

The functional dependence of active power generated on the rate of water discharge is given by the expression for incremental water rate as follows:

$$dq_j/dP_{h_j} = \beta_j + 2\gamma_j P_{h_j}, \qquad j \in R_h \tag{5.2.18}$$

The transmission losses are expressed using the loss formula, which leads to the incremental transmission losses given by

$$\partial P_L/\partial P_i = B_{i0} + 2 \sum_{j \in R_G} B_{ij} P_j \tag{5.2.19}$$

Here R_G denotes the set of all generating plants.

With the above definitions, the coordination equations turn out to be

$$\beta_i + 2\gamma_i P_{s_i}(t) + \lambda(t)\left[C_i + 2 \sum_{j \in R_G} B_{ij} P_j(t) \right] = 0, \qquad i \in R_s \tag{5.2.20}$$

$$v_j[\beta_j + 2\gamma_j P_{h_j}(t)] + \lambda(t)\left[C_j + 2 \sum_{k \in R_G} B_{jk} P_k(t) \right] = 0, \qquad j \in R_h \tag{5.2.21}$$

The constants C_i are given by

$$C_i = B_{i0} - 1 \tag{5.2.22}$$

The coordination equations are solved simultaneously for active powers, the incremental cost of received power $\lambda(t)$, and the water conversion factors v_j. The solution is completely specified by taking the following constraints into account:

$$\int_0^{T_r} [\alpha_i + \beta_i P_{h_i}(t) + \gamma_i P_{h_i}^2(t)]\, dt = b_i, \qquad i \in R_h \quad (5.2.23)$$

$$K_{L0} + P_D(t) + \sum_{i \in R_G} C_i P_i(t) + \sum_{i,j \in R_G}\sum P_i(t) B_{ij} P_j(t) = 0 \quad (5.2.24)$$

In the classical approach the solution of the resulting set of equations proceeds with the choice of conversion coefficients v_j. This choice produces a converted incremental plant-cost characteristic. Solution of Eqs. (5.2.20), (5.2.21), and (5.2.24) in the remaining unknowns is affected. This solution determines the amount of water used b_i by direct substitution in Eq. (5.2.23). If the calculated values of b_i do not agree with the schedules values, a new set of conversion coefficients are tried and the process repeated.

5.2.2 A Dynamic Programming Approach

A. Problem Statement

We consider a hydro–thermal system with one thermal plant and one hydro plant. Hydro generation P_h depends on the discharge, and this dependence will be expressed by writing

$$P_h = P_h(q) \quad (5.2.25)$$

In this expression, P_h is the maximum hydro output obtainable from the discharge q. Line losses P_L are given by

$$P_L = B_{ss}P_s^2 + B_{hh}P_h^2 + 2B_{sh}P_h \quad (5.2.26)$$

The thermal cost $F(P_s)$ is assumed to be an increasing, strictly convex differentiable function of P_s so that $[dF(P_s)/dP_s]$, the incremental cost of thermal generation, must also be a strictly increasing function of P_s. Let

$$C(P_s) = [dF(P_s)/dP_s] \quad (5.2.27)$$

The problem of determining optimal system operation is: Find $[P_s(1), P_s(2), \ldots, P_s(24)]$ and $[q(1), q(2), \ldots, q(24)]$ such that

$$\sum_{i \in I} F(P_s(i)) \quad (5.2.28)$$

is a minimum. The set I denotes the discretized optimization interval chosen to be one day discretized on an hourly basis. The active power balance equation requires that

$$P_s(i) + P_h(i) = P_D(i) + P_L(i), \qquad i \in I \quad (5.2.29)$$

Moreover,

$$\sum_{i \in I} q(i) = V \tag{5.2.30}$$

With V being the allotted volume of water for the hydro plant. Additional constraints of the inequality-type include

$$0 \leq q(i) \leq q_{\max}, \qquad i \in I \tag{5.2.31}$$

and

$$0 < P_{s_{\min}} \leq P_s(i) \leq P_{s_{\max}}, \qquad i \in I \tag{5.2.32}$$

B. Optimization Procedure

Let

$q^{(0)}(i)$ = initial average discharge at instant i

$P_s^{(1)}(i)$ = corresponding average thermal generation

$C(P_s^{(1)}(i))$ = corresponding plant incremental cost of thermal generation

The values of $q^{(0)}(i)$ constitute a trial schedule satisfying the volume of water constraint. Substituting

$$P_h^{(0)}(i) = P_h(q^{(0)}(i))$$

in Eq. (5.2.29) and solving, the resulting quadratic equation

$$B_{ss}[P_s(i)]^2 + [2B_{sh}P_h^{(0)}(i) - 1]P_s(i) + P_D(i) - P_h^{(0)}(i) + B_{hh}[P_h^{(0)}(i)]^2 = 0 \tag{5.2.33}$$

in $P_s(i)$. From this the costs $F(P_s^{(1)}(i))$ and $C(P_s^{(1)}(i))$ are obtained.

From Eq. (5.2.20) it follows that

$$dP_s + dP_h = (\partial P_L/\partial P_s)\,dP_s + (\partial P_L/\partial P_h)\,dP_h$$

or

$$dP_s = -\frac{1 - (\partial P_L/P_h)}{1 - (\partial P_L/P_s)}\,dP_h \tag{5.2.34}$$

Let $q(i)$ be any proposed alternative to $q^{(0)}(i)$ where the change $P_h(i) - P_h^{(0)}(i)$ is small. From Eq. (5.2.34) we can write

$$-\Delta P_s^{(1)}(i) = \frac{[1 - (\partial P_L/\partial P_h)]^{(0)}}{[1 - (\partial P_L/\partial P_s)]^{(0)}}\,[P_h(i) - P_h^{(0)}(i)] \tag{5.2.35}$$

where $\Delta P_s^{(1)}(i)$ is the corresponding change in thermal output.

In practice, the factor multiplying $[P_h(i) - P_h^{(0)}(i)]$ in Eq. (5.2.35) is always negative so that an increase in hydro generation during the ith hour will result in a decrease of thermal generation and the converse is true. The change in hydro generation is worth approximately

$$[C(P_s^{(1)}(i))][\Delta P_s^{(1)}(i)] = -G(P_h^{(0)}(i), P_s^{(1)}(i))[P_h(i) - P_h^{(0)}(i)] \quad (5.2.36)$$

Here, the function G is given by

$$G(P_h^{(0)}(i), P_s^{(1)}(i)) = +C(P_s^{(1)}(i))\frac{[1 - (\partial P_L/\partial P_h)]_{(P_h^{(0)}(i),\, P_s^{(1)}(i))}}{[1 - (\partial P_L/\partial P_s)]_{(P_h^{(0)}(i),\, P_s^{(1)}(i))}} \quad (5.2.37)$$

The alternative hydro schedule should be chosen to minimize

$$J_0 = \sum_{i \in I} C(P_s^{(1)}(i))\Delta P_s^{(1)}(i) \quad (5.2.38)$$

or, equivalently, to maximize

$$\begin{aligned} J_1 &= \sum_{i \in I} G(P_h^{(0)}(i), P_s^{(1)}(i))[P_h(i) - P_h^{(0)}(i)] \\ &= \sum_{i \in I} G(P_h^{(0)}(i), P_s^{(1)}(i))P_h(i) - G(P_h^{(0)}(i), P_s^{(1)}(i))P_h^{(0)}(i) \end{aligned} \quad (5.2.39)$$

Here the various constraints, Eqs. (5.2.29)–(5.2.32), must be satisfied and only values of $q(i)$ in the neighborhood of $q_i^{(0)}$ are permitted. The second term in Eq. (5.2.39) is independent of $q(i)$; the problem of finding $q_{(i)}^{(1)}$, the best neighboring alternative to $q_{(i)}^{(0)}$, is thus equivalent to finding $(q(1), \ldots)$ which maximize

$$J = \sum G(P_h^{(0)}(i), P_s^{(1)}(i))P_h(i) \quad (5.2.40)$$

This expression will be called the weighted output of the hydroelectric power system where the weights are $G(i)$.

The first part of iteration 1 consists of calculating the $q^{(1)}(i)$ which maximizes Eq. (5.2.40) subject to the required constraints. Next the corresponding thermal outputs $P_s^{(2)}(i)$ are calculated and, finally, the values of $G(P_h^{(1)}(i), P_s^{(2)}(i))$ are computed. This ends iteration 1.

In iteration $(k - 1)$, let

$$\begin{aligned} q^{(k-1)}(i) &= \text{average discharge during hour } (i) \\ P_s^{(k)}(i) &= \text{corresponding thermal generation} \\ C(P_s^{(k)}(i)) &= \text{corresponding thermal incremental cost} \\ P_h^{(k-1)}(i) &= P_h(q^{(k-1)}(i)) \end{aligned} \quad (5.2.41)$$

The corresponding weights $G(P_h^{(k-1)}(i), P_s^{k}(i))$, which are positive numbers, are given by

$$G(P_h^{(k-1)}(i), P_s^{(k)}(i)) = C(P_s^{k}(i))\frac{[(\partial P_L/\partial P_h) - 1]_{(P_h^{(k-1)}(i),\, P_s^{k}(i))}}{[(\partial P_L/\partial P_s) - 1]_{(P_h^{(k-1)}(i),\, P_s^{k}(i))}} \quad (5.2.42)$$

In iteration k, the values $q^{(k)}(i)$ of the best neighboring alternative to $q^{(k-1)}(i)$ are calculated. This is done by finding values $(q(1), \ldots, q(24))$ which maximize

$$\sum_{i \in I} G(P_{\mathrm{h}}^{(k-1)}(i), P_{\mathrm{s}}^{(k)}(i))[P_{\mathrm{h}}(i) - P_{\mathrm{h}}^{(k-1)}(i)] \tag{5.2.43}$$

or, equivalently, which maximize the weighted output

$$\sum_{i \in I} G(P_{\mathrm{h}}^{(k-1)}(i), P_{\mathrm{s}}^{(k)}(i))P_{\mathrm{h}}(i) \tag{5.2.44}$$

We calculate $P_{\mathrm{s}}^{(k+1)}(i)$ by substituting $P_{\mathrm{h}}^{(k)}(i) = P_{\mathrm{h}}^{(k)}(q(i))$ in Eq. (5.2.29) and solving for $P_{\mathrm{s}}(i)$. The costs $F(P_{\mathrm{s}}^{(k+1)}(i))$ and $C(P_{\mathrm{s}}^{(k+1)}(i))$ are then obtained by substitution.

The basic property of this method is that the total cost corresponding to $q^{(k)}(i)$ must be equal to or less than the total cost of the schedule corresponding to $q^{(k-1)}(i)$. This follows from Eq. (5.2.43) since the possible value of Eq. (5.2.43) is zero when $q(i) = q^{(k-1)}(i)$.

If Eq. (5.2.42) is written as

$$\frac{G}{[1 - (\partial P_{\mathrm{L}}/\partial P_{\mathrm{h}})]} = \frac{dF(P_{\mathrm{s}})/dP_{\mathrm{s}}}{[1 - (\partial P_{\mathrm{L}}/\partial P_{\mathrm{s}})]} \tag{5.2.45}$$

where the derivatives are evaluated for $P_{\mathrm{h}} = P_{\mathrm{h}}^{(k-1)}(i)$ and $P_{\mathrm{s}} = P_{\mathrm{s}}^{(k)}(i)$, then, by analogy, $G(P_{\mathrm{h}}^{(k-1)}(i), P_{\mathrm{s}}^{k}(i))$ can be interpreted as the incremental worth of hydro generation at the hydro bus bar during hour i of iteration k.

If the volume of water available at the beginning of the ith hour is $V(i)$, then we say that the hydro plant is in state $V(i)$. The state of the hydro plant is known at the beginning of the first hour since

$$V(1) = V.$$

For $i = 1, 2, \ldots, 24$, the states $V(i)$ depend on the scheduled $q(i)$. Upper and lower limits on $V(i)$ can be obtained. These correspond to two extremes of operation:

(1) The plant is left inoperative as long as possible and put into operation only when there is just enough time left to use up the allotted water.

(2) The plant is operated at maximum discharge until the allotted water is used up, after which the plant is shut down for the remainder of the day.

These two extremes are shown in Fig. 5.1 with the vertical line segments showing range of possible values of $V(i)$ at the beginning of each hour. For digital computer implementation discrete values of $V(i)$ are needed as indicated by the equally spaced dots in Fig. 5.1. The distance v is dictated by the optimal schedule's desired accuracy. Changes of state are thus limited to multiples of v.

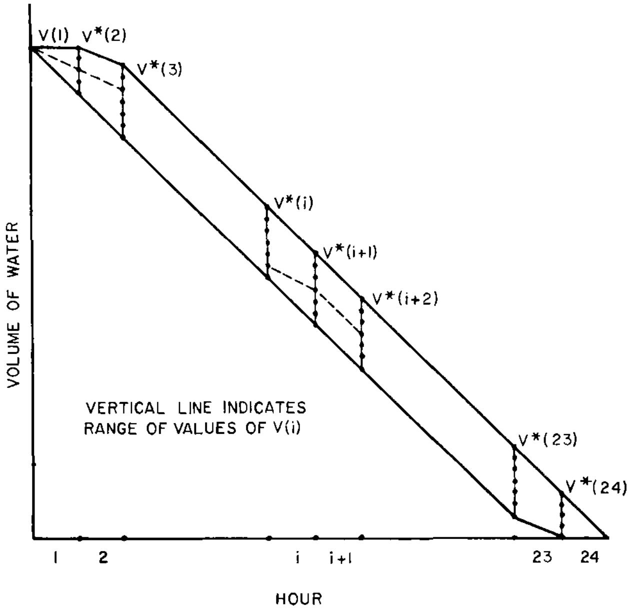

Fig. 5.1 Possible states of the hydro plant.

Suppose a schedule $q^*(1), \ldots, q^*(24)$ is followed; then the corresponding values of $V(i)$, denoted by $V^*(i)$, are given by the recursion

$$V^*(i+1) = V^*(i) - q^*(i), \qquad i = 1, 2, \ldots, 23 \tag{5.2.46}$$

Here we have

$$V^*(1) = V,$$
$$V^*(24) = V^*(23) - q^*(23) = q^*(24)$$

Note that it is assumed that appropriate units of the variables are used, e.g., volume in m^3/sec hr and discharge in m^3/sec. It is obvious that knowledge of $V^*(i)$ will yield $q^*(i)$ since

$$\begin{aligned} q^*(i) &= V^*(i) - V^*(i+1), \qquad i = 1, 2, \ldots, 23 \\ q^*(24) &= V^*(24) \end{aligned} \tag{5.2.47}$$

Let v be some increment of volume and consider the states

$$V^*(i) - v, \quad V^*(i), \quad V^*(i) + v \qquad \text{for} \quad i = 1, 2, \ldots, 24$$

at the beginning of interval i. For $i = 1$, only the state $V^*(i) = V$ is permissible. For $i = 2, 3, \ldots, 24$, only two of the states listed need be permissible, e.g., if $V^*(i) - v$ is below the lower line shown in Fig. 5.1 then only $V^*(i)$ and $V^*(i) + v$ will be permissible states. It is possible to go to the permissible states $V^*(i+1) - v$, $V^*(i+1)$, $V^*(i+1) + v$ from the permissible states $V^*(i) - v$, $V^*(i)$, $V^*(i) + v$ in at most nine ways or passages, namely:

$$\begin{array}{llll}
\text{from} & V^*(i) - v & \text{to} & V^*(i+1) - v \\
 & & \text{to} & V^*(i+1) \\
 & & \text{to} & V^*(i+1) + v \\
\text{from} & V^*(i) & \text{to} & V^*(i+1) - v \\
 & & \text{to} & V^*(i+1) \\
 & & \text{to} & V^*(i+1) + v \\
\text{from} & V^*(i) + v & \text{to} & V^*(i+1) - v \\
 & & \text{to} & V^*(i+1) \\
 & & \text{to} & V^*(i+1) + v
\end{array}$$

This results in at most five discharges:

$$\begin{array}{l}
q^*(i) - 2v \\
q^*(i) - v \\
q^*(i) \\
q^*(i) + v \\
q^*(i) + 2v
\end{array}$$

The permissible discharges are those satisfying

$$0 \leq q(i) \leq q_{\max} \tag{5.2.48}$$

$$P_h(q(i)) - P_L(P_{s_{\max}}, P_h(i)) \geq P_D(i) - P_{s_{\max}} \tag{5.2.49}$$

$$P_h(q(i)) - P_L(P_{s_{\min}}, P_h(i)) \leq P_D(i) - P_{s_{\min}} \tag{5.2.50}$$

We refer to such a $q(i)$ as one restricted to some neighborhood of $q^*(i)$.

The problem of determining optimal system operation is equivalent to solving a sequence of problems each of the following form: Determine the discharges $q(1), q(2), \ldots$ that maximize

$$\sum_{i=1}^{24} G(i) P_h(q(i)), \qquad \text{where} \quad \sum_{i=1}^{24} q(i) = V$$

and inequality constraint equations (5.2.48)–(5.2.50) are satisfied. Moreover, $q(i)$ is restricted to some neighborhood of a fixed value $q^*(i)$. Constraint equations (5.2.49)–(5.2.50) will be termed induced end-point constraints. These are merely a restatement of the conditions given by Eqs. (5.2.29) and (5.2.32). In effect, constraint equation (5.2.49) excludes discharges which would lead to thermal outputs exceeding $P_{s_{max}}$ while constraint equation (5.2.50) excludes discharges which lead to thermal outputs less than $P_{s_{min}}$.

This problem requires the maximization of a function of 24 variables. Using incremental dynamic programming, this can be reduced to a sequence of maximization problems each in one variable. In each iteration, the search for an improved hydro schedule is limited to a neighborhood of the optimal hydro schedule determined in the preceding iteration. It is necessary to limit the search in this way because the values of $G(i)$ are incremental worths and are only applicable to incremental changes in output.

The procedure is as follows:

(1) For each permissible state at the beginning of hour 24, determine the optimal mode of operation during hour 24. This is simple. For the state $V^*(24) - v$, the optimal and only discharge during hour 24 is $q^*(24) - v$. Similarly for $V^*(24)$ and $V^*(24) + v$, the optimal discharges are $q^*(24)$ and $q^*(24) + v$, respectively.

The maximum weighted outputs for these modes of operation will be denoted by $R_{24}(V^*(24) - v)$, $R_{24}(V^*(24))$, and $R_{24}(V^*(24) + v)$, respectively, where

$$R_{24}(V^*(24) - v) = G_{24}P_h(q^*(24) - v)$$
$$R_{24}(V^*(24)) = G_{24}P_h(q^*(24))$$
$$R_{24}(V^*(24) + v) = G_{24}P_h(q^*(24) - v)$$

(2) Use results of step 1 to calculate the optimal mode of operation during the last 2 hours for any permissible state at the beginning of hour 23. Corresponding to the state $V^*(23) - v$ at the beginning of that hour, at most three discharges are possible, namely, $q^*(23)$, $q^*(23) - v$, and $q^*(23) - 2v$. The resulting weighted outputs over the last two hours are

$$G_{23}P_h(q^*(23)) + R_{24}(V^*(24) - v)$$
$$G_{23}P_h(q^*(23) - v) + R_{24}(V^*(24))$$
$$G_{23}P_h(q^*(23) - 2v) + R_{24}(V^*(24) + v)$$

respectively. The optimal discharge is the one yielding the maximum weighted output, here denoted by $R_{23}(V^*(23) - v)$. Similarly for the state $V^*(23)$, there are at most three discharges possible during hour 23, namely, $q^*(23) + v$, $q^*(23)$, and $q^*(23) - v$, for which the weighted outputs over the last 2 hours are

$$G_{23}P_h(q^*(23) + v) + R_{24}(V^*(24) - v)$$
$$G_{23}P_h(q^*(23)) + R_{24}(V^*(24))$$
$$G_{23}P_h(q^*(23) - v) + R_{24}(V^*(24) + v)$$

respectively. Again the optimal discharge is the one yielding the maximum weighted output, denoted by $R_{23}(V^*(23))$. A similar calculation yields $R_{23}(V^*(23) + v)$.

(3) Based on these results calculate the optimal mode of operation during the last 3 hours for any permissible state at the beginning of hour 22. The method of calculating $R_{22}(V^*(22) - v)$, $R_{22}(V^*(22))$, and $R_{22}(V^*(22) + v)$ should now be apparent. For example,

$$\begin{aligned} R_{22}(V^*(22)) = \max[&G_{22}P_{\mathrm{h}}(q^*(22) + v) + R_{23}(V^*(23) - v), \\ &G_{22}P_{\mathrm{h}}(q^*(22)) + R_{23}(V^*(23)), \\ &G_{22}P_{\mathrm{h}}(q^*(22) - v) + R_{23}(V^*(23) + v)] \end{aligned}$$

(4) Working back, the optimal mode of operation is determined successively for the last 4 hours, the last 5 hours, and so on until the optimal mode of operation for the entire day is obtained.

5.2.3 A Minimum Norm Approach

We now offer a formulation of the problem of fixed-head hydro plants on separate streams, which is based on the minimum norm approach. The operating costs at the ith thermal plant are approximated by

$$F_i(P_{\mathrm{s}_i}(t)) = \alpha_i + \beta_i P_{\mathrm{s}_i}(t) + \gamma_i P_{\mathrm{s}_i}^2(t) \tag{5.2.51}$$

The generation schedule sought must satisfy the active power balance equation (APBE):

$$P_{\mathrm{D}}(t) = \sum_{i \in R_{\mathrm{s}}} P_{\mathrm{s}_i}(t) + \sum_{i \in R_{\mathrm{h}}} P_{\mathrm{h}_i}(t) - P_{\mathrm{L}}(t) \tag{5.2.52}$$

The transmission loss is a quadratic function of the active power generated by the system plants and is given by

$$P_{\mathrm{L}}(t) = \sum_{i,j \in R_G}\sum P_i(t) B_{ij} P_j(t) + \sum_{i \in R_G} B_{i0} P_i(t) + K_{\mathrm{L0}} \tag{5.2.53}$$

with the B_{ij}, B_{i0}, and K_{L0} being the loss formula coefficients, which are assumed to be known, and having the property

$$B_{ij} = B_{ji}, \qquad i,j \in R_G$$

Furthermore, the water discharge at each hydro plant must satisfy the following constraint on the volume of water used during the optimization interval:

$$\int_0^{T_f} q_i(t)\, dt = b_i, \qquad i \in R_{\mathrm{h}} \tag{5.2.54}$$

Assuming constant head and efficiency at each hydro plant, this is equivalent to the following energy constraint:

$$\int_0^{T_f} P_{h_i}(t)\,dt = K_i, \qquad i \in R_h \tag{5.2.55}$$

The object of the optimizing computation is

$$\min_{\substack{P_{s_i}(t)\\ i \in R_s}} \int_0^{T_f} \sum_{i \in R_s} F_i(P_{s_i}(t))\,dt \tag{5.2.56}$$

A. Formulation

The active power balance constraint is added to the integrand of the cost functional using the unknown function $\lambda(t)$, so that a modified cost functional is obtained:

$$\begin{aligned} J_1(P_{s_i}(t), P_{h_i}(t)) = \int_0^{T_f} \Bigg[&\sum_{i \in R_s} (\beta_i + \lambda(t)(B_{i0} - 1)P_{s_i}(t) + \sum_{i \in R_h} \lambda(t)(B_{i0} - 1)P_{h_i}(t) \\ &+ \sum_{i \in R_s} \gamma_i P_{s_i}^2(t) + \lambda(t) \sum\sum_{i,j \in R_G} P_i(t)B_{ij}P_j(t) \Bigg] dt \end{aligned} \tag{5.2.57}$$

Here terms explicitly independent of the power generations are dropped.

Define the $n \times 1$ column vectors

$$\mathbf{u}(t) = \operatorname{col}[P_{s_1}(t), P_{s_2}(t), \ldots, P_{h_{m+1}}(t), \ldots, P_{h_n}(t)] \tag{5.2.58}$$

$$\mathbf{L}(t) = \operatorname{col}[(\beta_1 + C_1\lambda(t)), \ldots, C_{m+1}\lambda(t), \ldots, C_n\lambda(t)] \tag{5.2.59}$$

where

$$C_i = B_{i0} - 1, \qquad i \in R_G \tag{5.2.60}$$

and the symmetric $n \times n$ matrix

$$\mathbf{B}(t) = (b_{ij}(t)) \tag{5.2.61}$$

with

$$\begin{aligned} b_{ij}(t) &= B_{ij}\lambda(t), && i, j \in R_G \quad (i \neq j) \\ b_{ii}(t) &= \gamma_i + B_{ii}\lambda(t), && i \in R_s \\ b_{ii}(t) &= B_{ii}(t)\lambda(t), && i \in R_h \end{aligned}$$

Then the modified cost functional becomes

$$J_1(\mathbf{u}(t)) = \int_0^{T_f} [\mathbf{L}^T(t)\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{B}(t)\mathbf{u}(t)]\,dt \tag{5.2.62}$$

If we let

$$\mathbf{V}(t) = \mathbf{B}^{-1}\mathbf{L}(t) \tag{5.2.63}$$

then the cost functional can be written as

$$J(\mathbf{u}(t)) = \int_0^{T_f} [\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]\, dt \tag{5.2.64}$$

The problem is now transformed to that of finding a control vector $\mathbf{u}(t)$ that minimizes the cost functional given by Eq. (5.2.64) while satisfying the energy constraint given by Eq. (5.2.55). Note that $\lambda(t)$ will be determined so that the optimal control vector satisfies the active power balance Eq. (5.2.52).

The control vector $\mathbf{u}(t)$ is considered to be an element of the Hilbert space $L_{2,\mathbf{B}}^{(n)}[0, T_f]$ of the n vector-valued square integrable functions defined on $[0, T_f]$ whose inner product is given by

$$\langle \mathbf{V}(t), \mathbf{u}(t) \rangle = \int_0^{T_f} \mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t)\, dt \tag{5.2.65}$$

for every $\mathbf{V}(t)$ and $\mathbf{u}(t)$ in $L_{2,\mathbf{B}}^{(n)}[0, T_f]$, provided that $\mathbf{B}(t)$ is positive definite. This means that $\mathbf{u}_{\xi}(t) \in L_{2,\mathbf{B}}^{(n)}[0, T_f]$ which minimizes

$$J(\mathbf{u}(t)) = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|^2 \tag{5.2.66}$$

and satisfies Eq. (5.2.55) is sought.

Define the $(n - m) \times 1$ column vector

$$\xi = \mathrm{col}[K_{m+1}, \ldots, K_n] \tag{5.2.67}$$

and the $n \times (n - m)$ matrix

$$\mathbf{M} = \mathrm{col}[\mathbf{0}, \mathbf{I}] \tag{5.2.68}$$

with $\mathbf{0}$ being the $m \times (n - m)$ matrix whose elements are all zeros, and $\mathbf{I}$ is the $(n - m) \times (n - m)$ identity matrix, so that the constraints of Eq. (5.2.55) can be expressed as

$$\xi = \int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{u}(t)\, dt \tag{5.2.69}$$

This last expression defines a bounded linear transformation

$$\mathbf{T}: L_{2,\mathbf{B}}^{(n)}[0, T_f] \to R^{(n-m)}$$

The real vector space $R^{(n-m)}$ is endowed with the Euclidean inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \mathbf{X}^{\mathrm{T}}\mathbf{Y}$$

for every $\mathbf{X}$ and $\mathbf{Y}$ in $R^{(n-m)}$.

Equation (5.2.69) is now written as

$$\xi = \mathbf{T}[\mathbf{u}(t)]$$

with $\mathbf{T}[\mathbf{u}(t)]$ given by

$$\mathbf{T}[\mathbf{u}(t)] = \int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{u}(t)\,dt \tag{5.2.70}$$

The problem under consideration has been reduced to finding the optimal generation vector $\mathbf{u}(t)$ which minimizes the cost functional given by Eq. (5.2.66) such that Eq. (5.2.69) is satisfied for the given vector $\boldsymbol{\xi}$.

B. The Optimal Solution

In view of the results cited in Chapter 3, there is exactly one optimal solution to the problem formulated here, namely the optimal vector

$$\mathbf{u}_\xi(t) = \mathbf{T}^\dagger[\boldsymbol{\xi} + \mathbf{T}(\tfrac{1}{2}\mathbf{V}(t))] - \tfrac{1}{2}\mathbf{V}(t) \tag{5.2.71}$$

where $\mathbf{T}^\dagger$ is obtained as follows: $\mathbf{T}^*$, the adjoint of $\mathbf{T}$, is obtained by using the identity

$$\langle \boldsymbol{\xi}, \mathbf{Tu}(t)\rangle_{R^{(n-m)}} = \langle \mathbf{T}^*\boldsymbol{\xi}, \mathbf{u}(t)\rangle_{L_{2,\mathbf{B}}^{(n)}[0,T_f]} \tag{5.2.72}$$

where in $R^{(n-m)}$ we have

$$\langle \boldsymbol{\xi}, \mathbf{Tu}(t)\rangle = \boldsymbol{\xi}^{\mathrm{T}}\mathbf{Tu}(t) \tag{5.2.73}$$

This turns out to be

$$\mathbf{T}^*\xi = \mathbf{B}^{-1}(t)\mathbf{M}\xi \tag{5.2.74}$$

Next we find the transformation $\tilde{\mathbf{J}}$ given by

$$\tilde{\mathbf{J}}\xi = \mathbf{T}[\mathbf{T}^*\xi]$$

using Eqs. (5.2.70) and (5.2.74),

$$\tilde{\mathbf{J}}\xi = \left[\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\,dt\right]\xi$$

which yields

$$\tilde{\mathbf{J}}^{-1}[\xi] = \left[\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\,dt\right]^{-1}\xi \tag{5.2.75}$$

Finally, using the definition of the pseudoinverse transformation $\mathbf{T}^\dagger$, one obtains

$$\mathbf{T}^\dagger\xi = \mathbf{B}^{-1}\mathbf{M}\left[\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\,dt\right]^{-1}\xi \tag{5.2.76}$$

The optimal generation vector $\mathbf{u}_\xi(t)$ is thus given by

$$\mathbf{u}_\xi(t) = \mathbf{B}^{-1}(t)\mathbf{M}\left[\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{B}^{-1}(t)\mathbf{M}\,dt\right]^{-1}\left[\xi + \tfrac{1}{2}\int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{V}(t)\,dt\right] - \tfrac{1}{2}\mathbf{V}(t) \tag{5.2.77}$$

C. Implementing the Optimal Solution

To illustrate the procedure for implementing the optimal solution let us consider a power system with diagonal loss formula coefficient matrix. In this case the optimal power generations obtained are

$$P_{s_{i_\xi}}(t) = -\frac{[\beta_i + C_i\lambda(t)]}{2[\gamma_i + B_{ii}\lambda(t)]}, \qquad i \in R_s \tag{5.2.78}$$

$$P_{h_{i_\xi}}(t) = \left[\lambda(t)\int_0^{T_f} \lambda^{-1}(t)\,dt\right]^{-1}\left(K_i + \frac{C_i T_f}{2B_{ii}}\right) - \frac{C_i}{2B_{ii}}, \qquad i \in R_h \tag{5.2.79}$$

The last expressions contain the unknown function $\lambda(t)$ which will be determined so that the power balance Eq. (5.2.52) is satisfied. We observe here that the problem can be viewed as that of solving $n + 1$ equations in the optimal generations and the unknown function $\lambda(t)$. The problem, however, can be reduced to just solving one equation in one unknown. For convenience, we will make the following change of variables:

$$x(t) = \lambda^{-1}(t) \tag{5.2.80}$$

Substituting in the power balance equation for the optimal power generations we obtain the following equation in $x(t)$:

$$\frac{x^2(t)D}{\left[\int_0^{T_f} x(t)\,dt\right]^2} + \sum_{i \in R_s} \frac{A_{2i}x^2(t) + A_{1i}x(t) + A_{0i}}{4\gamma_i^2[B_{ii} + x(t)]^2} = y(t) \tag{5.2.81}$$

where the various coefficients are given by

$$D = \sum_{i \in R_h} B_{ii}[K_i + (C_i T_f/2B_{ii})]^2$$

$$A_{2i} = \beta_i(B_{ii}\beta_i - 2C_i\gamma_i)$$

$$A_{1_i} = -2C_i^2\gamma_i$$

$$A_{0i} = -C_i^2 B_{ii}$$

and the forcing function $y(t)$ is

$$y(t) = \left[\sum_{i \in R_h} (C_i^2/4B_{ii})\right] - [P_D(t) + K_{L0}]$$

It is evident that the optimality equation (5.2.81) is nonlinear. The solution for $x(t)$ at any instant depends on the whole record of $x(t)$ as indicated by the presence of the definite integral of $x(t)$ in the optimality equation. We thus have to search for a function $x(t)$ which is a solution and simultaneously yields a positive definite matrix $\mathbf{B}(t)$ (which makes the definition of the space $L_{2,\mathbf{B}}^{(n)}[0, T_f]$ valid). Furthermore, the optimal generations must be physically realizable in the sense that no negative or complex active power generations are permissible.

The procedure followed in solving Eq. (5.2.81) is one of discretization which will result in a system of N simultaneous algebraic equations in N unknowns $(x_1, \ldots, x_N)$ which can be solved by any classical iteration technique.

A study was made of a sample system with the following particulars:

Number of thermal plants $\quad m = 1$

Number of hydro plants $\quad n - m = 2$

Loss formula coefficients

$$B_{11} = 1.6000 \times 10^{-4}, \qquad B_{22} = 2.2000 \times 10^{-4}, \qquad B_{33} = 1.6000 \times 10^{-4}$$
$$B_{10} = 0.00, \qquad B_{20} = 0.00, \qquad B_{30} = 0.00$$

Thermal plant's cost equation

$$F(P_s(t)) = \alpha + 4.0P_s(t) + 1.20 \times 10^{-2}P_s^2(t), \quad \$/\text{hr}$$

Energy constraints on the hydro plants

$$K_2 = 3600.00 \quad \text{MW hr}$$
$$K_3 = 2400.00 \quad \text{MW hr}$$

Example 1. Let

$$P_D(t) = 400.00 \quad \text{MW}, \qquad 0 \le t \le T_f, \qquad T_f = 24 \quad \text{hr}$$

Then Eq. (5.2.81) reduces to a second-order algebraic equation in $x(t)$, which yields a feasible $x(t)$ given by

$$x(t) = 0.21629, \qquad 0 \le t \le T_f$$

so that we get the following generation allocation by using Eqs. (5.2.78) and (5.2.79):

Optimal thermal power generation

$$P_s(t) = 160.68 \quad \text{MW}, \qquad 0 \le t \le T_f$$

Optimal hydro power generation

$$P_{h_2}(t) = 150.00 \quad \text{MW}, \qquad 0 \le t \le T_f$$
$$P_{h_3}(t) = 100.00 \quad \text{MW}, \qquad 0 \le t \le T_f$$

The result of this example agrees with the intuitive solution requiring constant hydro generations at the average values satisfying the energy constraints and the value of thermal power generation that satisfies the power balance equation.

Example 2. Let

$$P_D(t) = \begin{cases} 400.00 \quad \text{MW}, & 0 \le t \le T_f/2 \\ 600.00 \quad \text{MW}, & T_f/2 \le t \le T_f \end{cases}$$

$$T_f = 24 \quad \text{hr}$$

Then Eq. (5.2.81) reduces to two simultaneous algebraic euqations in two unknowns x_1 and x_2 where

$$x(t) = \begin{cases} x_1, & 0 \le t \le T_f/2 \\ x_2, & T_f/2 \le t \le T_f \end{cases}$$

The values of x_1 and x_2 satisfying these two equations and which simultaneously yield a feasible optimal power generation are found to be

$$x_1 = 0.20008, \qquad x_2 = 0.19349$$

with the corresponding optimal power generations given by

$$P_s(t) = 249.53 \quad \text{MW}, \qquad P_{h_2}(t) = 114.42 \quad \text{MW}, \qquad P_{h_3}(t) = 49.40 \quad \text{MW}, \qquad 0 \le t \le T_f/2$$

$$P_s(t) = 288.21 \quad \text{MW}, \qquad P_{h_2}(t) = 185.58 \quad \text{MW}, \qquad P_{h_3}(t) = 150.70 \quad \text{MW}, \qquad T_f/2 < t \le T_f$$

5.3 VARIABLE-HEAD HYDRO PLANTS

In this section we relax the assumption of fixed head operation for the hydro plants. We illustrate the classical approach based on variational calculus for treating the effect of head variations upon the hydro plant characteristics in determining the optimal mode of operation. The approach is outlined for a simple system involving one thermal and one hydro plant as generation sources. The classical approach leads to Ricard–Kron coordination equations. The application of the maximum principle to a system with an arbitrary number of plants is discussed next. The results of the treatment are shown to be a natural extension of the classical approach.

We conclude our discussion of this class of problems by a treatment that employs the minimum norm approach. Here, in formulating the problem, various plant characteristics are given concrete representations suited for this approach. We also discuss the practical implementation phase by giving a computational algorithm for solving the resulting optimality conditions.

5.3.1 The Classical Approach

In the case of variable head hydro plants, the hydro active power generation is a function of the rate of discharge q as well as the volume of water discharged Q:

$$Q(t) = \int_0^t q(t)\,dt \tag{5.3.1}$$

Thus we write

$$P_h = P_h(q, Q) \tag{5.3.2}$$

We will consider the case of a system of one thermal and one hydro plant for simplicity. It is desired to minimize the fuel cost of the system over the optimization interval,

$$J_0 = \int_0^{T_f} F\,dt \tag{5.3.3}$$

while satisfying the active power balance equation

$$P_s + P_h - P_L = P_D \tag{5.3.4}$$

Moreover, the volume of water available at the hydro plant is constrained by

$$\int_0^{T_f} q(t)\,dt = b \tag{5.3.5}$$

A. The Variational Solution

As before, the augmented cost functional becomes

$$J = \int_0^{T_f} [F + \lambda(P_D + P_L - P_s - P_h)]\,dt \tag{5.3.6}$$

or

$$J = \int_0^{T} H\,dt \tag{5.3.7}$$

The decision variables are P_s and Q in this case.

The Euler equation is given by

$$(\partial H/\partial x_i) - (d/dt)(\partial H/\partial \dot{x}_i) = 0 \tag{5.3.8}$$

Now for $x_i = P_s$, the Euler equation reduces to

$$(\partial F/\partial P_s) + \lambda[(\partial P_L/\partial P_s) - 1] = 0 \tag{5.3.9}$$

which yields

$$\lambda = L_s(\partial F/\partial P_s) \tag{5.3.10}$$

where L_s is the penalty factor defined by Eq. (5.2.10). This is the same thermal scheduling equation given by (5.2.12). For $x_i = Q$, the Euler equation is

$$\lambda[(\partial P_L/\partial Q) - (\partial P_h/\partial Q)] - (d/dt)\{\lambda[(\partial P_L/\partial q) - (\partial P_h/\partial q)]\} = 0 \quad (5.3.11)$$

We next use

$$\partial P_L/\partial Q = (\partial P_L/\partial P_h)(\partial P_h/\partial Q), \qquad \partial P_L/\partial q = (\partial P_L/\partial P_h)(\partial P_h/\partial q)$$

to obtain Eq. (5.3.11) in the alternative form

$$\lambda(\partial P_h/\partial Q)[1 - (\partial P_L/\partial P_h)] - (d/dt)\{\lambda(\partial P_h/\partial q)[1 - (\partial P_L/\partial P_h)]\} = 0 \quad (5.3.12)$$

Recall that the penalty factor for the hydro plant according to Eq. (5.2.11) is

$$L_h = [1 - (\partial P_L/\partial P_h)]^{-1}$$

Thus we have

$$(\lambda/SL_h)(\partial P_h/\partial h) + (d/dt)[\lambda/L_h)(\partial P_h/\partial q)] = 0 \quad (5.3.13)$$

where use was made of

$$\partial P_h/\partial Q = -(1/S)(\partial P_h/\partial h)$$

with S being the surface area of the assumed vertical-sided reservoir. Equation (5.3.13) is frequently referred to as Kron's equation.

The water conversion coefficient v is defined as

$$v = \lambda/[L_h(\partial q/\partial P_h)] \quad (5.3.14)$$

Substituting this in Kron's equation yields

$$-(v/S)(\partial q/\partial h) + (d/dt)v = 0 \quad (5.3.15)$$

so that

$$v(t) = v_0 \exp\left\{\int_0^t [(1/S)(\partial q/\partial h)]\, dt\right\} \quad (5.3.16)$$

which shows that the water conversion coefficient v is no longer a constant in the case of variable-head hydro plants.

It thus turns out that the coordination equations for the variable-head hydro plant system are

$$(dF/dP_s) + \lambda(\partial P_L/\partial P_s) = \lambda \quad (5.3.17)$$

$$v_0 \exp\left\{\int_0^t [(1/S)(\partial q/\partial h)]\, dt\right\}(\partial q/\partial P_h) + \lambda(\partial P_L/\partial P_h) = \lambda \quad (5.3.18)$$

The last equation is referred to as Ricard's equation.

Let us examine the form

$$v_0 \exp\left\{\int_0^t [(1/S)(\partial q/\partial h)]\, dt\right\}$$

which takes over the function of v previously obtained for the case of fixed head. The quantity $\partial q/\partial h$ is a negative number since the required flow of water for a fixed power output decreases as the head increases. Thus

$$\int_0^t [(1/S)(\partial q/\partial h)]\, dt$$

is negative and becomes increasingly negative with time. Consequently, the quantity

$$v_0 \exp\left\{\int_0^t [(1/S)(\partial q/\partial h)]\, dt\right\}$$

decreases with time. This leads to the scheduling of less power early in the time period under consideration and more power later in the period. Thus the effect of v is to maintain the reservoir elevation high early in the time period so as to utilize the inflow at the highest head possible consistent with the balancing economic factors of incremental water rate and of incremental thermal production cost.

B. Implementing Kron's Coordination Equations

The only difference between the coordination equations derived for the fixed-head hydro plants system of Section 5.2.1 and the present variable-head case is that the water conversion coefficients are no longer constants. The expression for $v(t)$ as given by Eq. (5.3.16) may be rewritten as

$$v(t) = v_0 \exp\left[\int_0^t M(t)\, dt\right] \tag{5.3.19}$$

where

$$M(t) = (1/S)(\partial q/\partial h) \tag{5.3.20}$$

An approximation to the exponential expression is obviously desirable. Treating $M(t)$ as a constant yields

$$v(t) = v_0\{\exp[Mt]\} \tag{5.3.21}$$

which in a Taylor expansion form may be expressed as

$$v(t) = v_0[1 + Mt + (M^2t^2/2) + \cdots] \tag{5.3.22}$$

A further approximation is

$$v(t) = v_0[1 + Mt] \tag{5.3.23}$$

This last expression is applicable to many practical systems.

The hydro plant performance curves are classically assumed to be of the form

$$q = K\psi(h)\phi(P_{\mathrm{h}}) \tag{5.3.24}$$

where

$$\psi(h) = a_0 + a_1 h + a_2 h^2 \tag{5.3.25}$$

and

$$\phi(P_{\mathrm{h}}) = \alpha_{\mathrm{h}} + \beta_{\mathrm{h}} P_{\mathrm{h}} + \gamma_{\mathrm{h}} P_{\mathrm{h}}^2 \tag{5.3.26}$$

The actual function coefficients are obtained by least-square fitting to the plant data. The reservoir of the hydro plant is assumed to be vertical-sided and tailwater elevation is assumed to be independent of the flow, so that the net head is given by

$$h(t) = h_0 + \int_0^t (1/S)[i(t) - q(t)]\,dt \tag{5.3.27}$$

with $i(t)$ being the inflow to the reservoir.

The foregoing assumptions in addition to those given earlier in Section 5.2.1, namely Eqs. (5.2.16), (5.2.19), (5.2.22), and (5.2.24), will result in

$$\beta_{\mathrm{s}} + 2\gamma_{\mathrm{s}} P_{\mathrm{s}}(t) + \lambda(t)[C_{\mathrm{s}} + 2(B_{\mathrm{ss}} P_{\mathrm{s}}(t) + B_{\mathrm{sh}} P_{\mathrm{h}}(t))] = 0 \tag{5.3.28}$$

$$\begin{aligned}\{v_0 \exp[Mt]\}\{2 - [(2\alpha + \beta_{\mathrm{h}} P_{\mathrm{h}})/\phi(P_{\mathrm{h}})]\}& \\ + \lambda(t)\{C_{\mathrm{h}} + 2[B_{\mathrm{sh}} P_{\mathrm{s}}(t) + B_{\mathrm{hh}} P_{\mathrm{h}}(t)]\} &= 0\end{aligned} \tag{5.3.29}$$

$$\int_0^{T_{\mathrm{f}}} K\psi(h)\phi(P_{\mathrm{h}})\,dt = b \tag{5.3.30}$$

$$K_{\mathrm{L0}} + P_{\mathrm{D}}(t) + C_{\mathrm{s}} P_{\mathrm{s}}(t) + C_{\mathrm{h}} P_{\mathrm{h}}(t) + B_{\mathrm{ss}} P_{\mathrm{s}}^2(t) + 2B_{\mathrm{sh}} P_{\mathrm{s}}(t) P_{\mathrm{h}}(t) + B_{\mathrm{hh}} P_{\mathrm{h}}^2(t) = 0 \tag{5.3.31}$$

The solution of the above set of equations completely specifies the optimal strategy.

5.3.2 A Maximum Principle Approach

The objective is to minimize

$$J = \int_0^{T_{\mathrm{f}}} \sum_{i \in R_{\mathrm{s}}} F_i(P_{\mathrm{s}_i})\,dt \tag{5.3.32}$$

subject to satisfying the active power balance equation

$$\sum_{i \in R_s} P_{s_i}(t) + \sum_{i \in R_h} P_{h_i}(t) = P_D(t) + P_L(t) \tag{5.3.33}$$

The hydro plant generation is a function of the forebay elevation and the rate of water discharge. This is expressed as

$$P_{h_i}(t) = P_{h_i}(x_i(t), q_i(t)) \tag{5.3.34}$$

The reservoir dynamics are given by

$$dx_i/dt = S_i^{-1}[i_i(t) - q_i(t)], \qquad i \in R_h \tag{5.3.35}$$

The volume of water discharge over the optimization interval is prespecified:

$$\int_0^{T_f} q_i(t)\, dt = b_i, \qquad i \in R_h \tag{5.3.36}$$

For simplicity, the surface area of the forebay S_i is treated as independent of forebay elevation x_i. Evaporation, leakage, and spillage are ignored.

The active power generations at the thermal plants are assumed bounded by a maximum and minimum value. This is defined by specifying

$$P_{s_i}(t) \in M$$

Here M is a closed set on the N_s-dimensional vector space R^{N_s}. The maximum and minimum discharge values for each hydro plant with continuous operating characteristics define the closed set U, that is,

$$q_i(t) \in U$$

The set U may be a function of x.

Since $i_i(t)$, the rate of water inflow to the ith reservoir, is an independent function of time, we may replace the volume of water discharge constraint by specification of the terminal values of x_i for $i \in R_h$. We use a new specification,

$$x_i(T_f) = d_i$$

or, in vector form,

$$\mathbf{x}(T_f) = \mathbf{d} \tag{5.3.37}$$

Let us treat q as the control variable together with $\tilde{\mathbf{P}}_s$ defined by

$$\tilde{\mathbf{P}}_s^T = (P_{s_2}, P_{s_3}, \ldots, P_{s_{N_s}}) \tag{5.3.38}$$

The active power generation of the first thermal plant is considered known, for the algebraic constraint given by the active power balance equation is used to eliminate P_{s_1} as an independent variable. Therefore

$$P_{s_1}(t) = P_{s_1}(P_D(t), \mathbf{P}_h(t), \tilde{\mathbf{P}}_s(t)) \tag{5.3.39}$$

Define

$$dx_{N_h+1}/dt = 1 \tag{5.3.40}$$

with

$$x_{N_h+1}(0) = 0, \qquad x_{N_h+1}(T_f) = T_f$$

that is, $x_{N_h+1}(t) = t$.

A. The Optimal Solution

The Hamiltonian of this system is

$$H = p_0 \sum_{i \in R_s} F_i(P_{s_i}) + \sum_{i \in R_h} p_i(t) S_i^{-1}[i_i(t) - q_i(t)] + p_{N_h+1} \tag{5.3.41}$$

The costates satisfy

$$\dot{\mathbf{p}}(t) = -\partial H/\partial \mathbf{X} \tag{5.3.42}$$

This in component form is

$$\dot{p}_i(t) = -\partial H/\partial x_i, \qquad i \in R_h.$$

Using Eq. (5.3.41) we obtain

$$\dot{p}_i(t) = -p_0(\partial F_1/\partial x_i)$$

or

$$\dot{p}_i(t) = -p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial x_i) \qquad i \in R_h, \tag{5.3.43}$$

and

$$\dot{p}_{N_h+1}(t) = -\partial H/\partial x_{N_h+1}$$

or

$$\dot{p}_{N_h+1}(t) = -\left[p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial x_{N_h+1}) + \sum_{i \in R_h} p_i(t) S_i^{-1}(\partial i_i/\partial x_{N_h+1})\right] \tag{5.3.44}$$

For optimal operation the following should hold true:

(1) $p_0 = \text{constant} \le 0$;
(2) H is maximized with respect to the control $\tilde{\mathbf{P}}_s \in \tilde{M}$ and $\mathbf{q} \in U$ at each instant of time;
(3) $H_{\max} = 0$ at $t = T_f$ and for any t if U is constant.

For the maximization of H we need

$$\partial H/\partial P_{s_i} = 0, \qquad \partial H/\partial q_i = 0$$

This leads to the requirements

$$(\partial F_1/\partial P_{s_1}/\partial P_{s_i}) + (\partial F_i/\partial P_{s_i}) = 0 \tag{5.3.45}$$

$$p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial q_i) - p_i(t)S_i^{-1} = 0 \tag{5.3.46}$$

or, equivalently,

$$(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial P_{s_i}) + (\partial F_i/\partial P_{s_i}) = 0 \tag{5.3.47}$$

$$p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial q_i) - p_i(t)S_i^{-1} = 0 \tag{5.3.48}$$

Consider the variations of the generation levels according to the power balance equation (5.3.33):

$$\sum dP_{s_i} + \sum dP_{h_i} - \sum(\partial P_L/\partial P_{s_i})\,dP_{s_i} - \sum(\partial P_L/\partial P_{h_i})\,dP_{h_i} = 0 \tag{5.3.49}$$

It then follows that

$$\frac{\partial P_{s_1}}{\partial P_{s_i}} = -\frac{1 - (\partial P_L/\partial P_{s_i})}{1 - (\partial P_L/\partial P_{s_1})} \tag{5.3.50}$$

$$\frac{\partial P_{s_1}}{\partial q_i} = -\frac{1 - (\partial P_L/\partial P_{h_i})}{1 - (\partial P_L/\partial P_{s_1})}\frac{\partial P_{h_i}}{\partial q_i} \tag{5.3.51}$$

Recall the penalty factors definitions of (5.2.10) and (5.2.11). Equations (5.3.50) and (5.3.51) may thus be rewritten as

$$\partial P_{s_1}/\partial P_{s_i} = -L_{s_1}/L_{s_i} \tag{5.3.52}$$

$$\partial P_{s_1}/\partial q_i = -(L_{s_1}/L_{h_i})(\partial P_{h_i}/\partial q_i) \tag{5.3.53}$$

Applying (5.3.52) in (5.3.47) the following necessary conditions are obtained:

$$L_{s_1}(\partial F_1/\partial P_{s_1}) = L_{s_i}(\partial F_i/\partial P_{s_i}) \tag{5.3.54}$$

This result means that the equal incremental cost loading formula (5.2.12) also applies in the variable-head hydro–thermal system for the thermal units operating inside the boundary of U.

Applying (5.3.53) in (5.3.48) we obtain

$$L_{s_1}(\partial F_1/\partial P_{s_1}) = -(p_i S_i^{-1}/p_0)L_{h_i}(\partial q_i/\partial P_{h_i}) \tag{5.3.55}$$

From this $p_0 \neq 0$. We remark here that the definition of the water conversion function $v_i(t)$ as given by generalizing (5.2.14) in (5.3.14) requires

$$v_i(t) = -[p_i(t)S_i^{-1}/p_0] \tag{5.3.56}$$

This shows the correspondence between the costates $p_i(t)$ in the maximum principle approach and the v_i(t) of the classical approach.

The costate equation (5.3.43) can be written in a more convenient form by eliminating the derivatives $\partial p_{s_1}/\partial x_i$. These functions are formed by differentiation of the power balance equation. In doing this, all P_{s_i} are constant for $i = 2, \ldots, N_s$, $\mathbf{q}$ is a constant vector, and all x_i for $i = 1, \ldots, N_h$ are constant except for the variable contained in the derivative. We then obtain

$$\partial P_{s_1}/\partial x_i = -(\partial P_{h_i}/\partial x_i)(L_{s_1}/L_{hi}) \tag{5.3.57}$$

The result of substituting (5.3.57) in (5.3.43) is

$$\dot{p}_i(t) = p_0(\partial F_1/\partial P_{s_1})(L_{s_1}/L_{h_i})(\partial P_{h_i}/\partial x_i) \tag{5.3.58}$$

This concludes the derivation of the optimality conditions using the maximum principle.

B. Implementing the Optimal Solution

Let us choose the constant costate p_0 to be

$$p_0 = -1 \tag{5.3.59}$$

Moreover, we define the time function $\lambda(t)$ by

$$\lambda(t) = L_{s_1}(\partial F_1/\partial P_{s_1}) \tag{5.3.60}$$

Finally, recall the definition of the water conversion functions v_i(t) given by Eq. (5.3.56). This can now be written as

$$v_i(t) = p_i(t)/S_i \tag{5.3.61}$$

The maximization of the Hamiltonian resulted in Eqs. (5.3.54) and (5.3.55). Using the above definitions, we may thus write

$$L_{s_i}(\partial F_i/\partial P_{s_i}) = \lambda(t), \qquad i \in R_s \tag{5.3.62}$$

$$v_i(t)L_{h_i}(\partial q_i/\partial P_{h_i}) = \lambda(t), \qquad i \in R_h \tag{5.3.63}$$

The costate equations given by Eq. (5.3.58) reduce to

$$\dot{v}_i(t) - [S_i^{-1}(\partial q_i/\partial x_i)]v_i(t) = 0, \qquad i \in R_h \tag{5.3.64}$$

Comparing the last three expressions with (5.3.10), (5.3.14), and (5.3.15), we clearly see that our current results are but a generalization of the results of Section 5.3.1. We refer to Eq. (5.3.64) as the generalized Kron's equation.

The above relations together with the constraining equations (5.3.33), (5.3.34), (5.3.35), and (5.3.37) completely specify the optimal strategy sought.

The digital implementation of these equations for plant characterization is as given in Section 5.3.1.

5.3.3 A Minimum Norm Approach

In considering the problem at hand, we will assume a vertical-sided reservoir. We will also assume that the tailrace elevation at any of the hydro plants does not change with the water discharge. Under these assumptions the active power generated at the ith hydro plant as obtained in Chapter 2 is given by

$$P_{\mathrm{h}_i}(t) + A_i(t)q_i(t) + B_i q_i(t)Q_i(t) = 0, \qquad i \in R_{\mathrm{h}} \tag{5.3.65}$$

Here it is convenient to introduce the volume of water discharges variable $Q_i(t)$ by the following definition:

$$Q_i(t) = \int_0^t q_i(\sigma)\, d\sigma, \qquad i \in R_{\mathrm{h}} \tag{5.3.66}$$

thus

$$q_i(t) = \dot{Q}_i(t), \qquad i \in R_{\mathrm{h}} \tag{5.3.67}$$

The object of the optimizing computation is

$$\min_{P_{\mathrm{s}_i}(t)} \int_0^{T_{\mathrm{f}}} \sum_{i \in R_{\mathrm{s}}} F_i[P_{\mathrm{s}_i}(t)]\, dt \tag{5.3.68}$$

The generation schedule sought must satisfy the active power balance equation (APBE)

$$P_{\mathrm{D}}(t) = \sum_{i \in R_{\mathrm{s}}} P_{\mathrm{s}_i}(t) + \sum_{i \in R_{\mathrm{h}}} P_{\mathrm{h}_i}(t) - P_{\mathrm{L}}(t) \tag{5.3.69}$$

The transmission power loss is given by

$$P_{\mathrm{L}}(t) = \sum_{i,j \in R_G}\sum P_i(t)B_{ij}P_j(t) + \sum_{i \in R_G} B_{i0}P_i(t) + K_{\mathrm{L}0} \tag{5.3.70}$$

Furthermore, the water discharge at each hydro plant is to satisfy the following constraint on the volume of water used over the optimization interval:

$$\int_0^{T_{\mathrm{f}}} q_i(t)\, dt = b_i, \qquad i \in R_{\mathrm{h}} \tag{5.3.71}$$

or, using Eq. (5.3.66),

$$Q_i(T_{\mathrm{f}}) = b_i, \qquad i \in R_{\mathrm{h}} \tag{5.3.72}$$

A. FORMULATION

We now proceed with the formulation by forming the augmented cost functional:

$$
\begin{aligned}
&J_0(P_{s_i}(t), P_{h_i}(t), Q_i(t))\\
&\quad = \int_0^{T_f} \sum_{i \in R_s} F_i[P_{s_i}(t)] + \lambda(t)\Bigg[P_D(t) + \sum\sum_{i,j \in R_G} P_i(t)B_{ij}P_j(t)\\
&\qquad + \sum_{i \in R_G} B_{i0}P_i(t) + K_{L0} - \sum_{i \in R_s} P_{s_i}(t) - \sum_{i \in R_h} P_{h_i}(t)\Bigg]\\
&\qquad - \sum_{i \in R_h} n_i(t)[A_i(t)\dot{Q}_i(t) + B_i Q_i(t)\dot{Q}_i(t) + P_{h_i}(t)]\, dt
\end{aligned}
\tag{5.3.73}
$$

Here the unknown function $\lambda(t)$ is associated with the APBE (5.3.69) consistent with the procedure of Section 5.2.3 leading to Eq. (5.2.57). The new functions $n_i(t)$ are associated with the active hydro power generation Eq. (5.3.65).

We will need the following identities:

$$
\int_0^{T_f} n_i(t)\dot{Q}_i(t)Q_i(t)\, dt = \tfrac{1}{2}\left[n_i(T_f)b_i^2 - \int_0^{T_f} \dot{n}_i(t)Q_i^2(t)\, dt\right] \tag{5.3.74}
$$

$$
\int_0^{T_f} n_i(t)A_i(t)\dot{Q}_i(t)\, dt = n_i(T_f)A_i(T_f)b_i - \int_0^{T_f} r_i(t)Q_i(t)\, dt \tag{5.3.75}
$$

with

$$
r_i(t) = \dot{n}_i(t)A_i(t) + n_i(t)\dot{A}_i(t) \tag{5.3.76}
$$

We can write the cost functional (5.3.73) as

$$
\begin{aligned}
&J_0[P_{s_i}(t), P_{h_i}(t), Q_i(t)]\\
&\quad = \int_0^{T_f} \Bigg\{\left(\sum_{i \in R_s} \alpha_i\right) + \lambda(t)P_D(t) + \lambda(t)K_{L0}\\
&\qquad - \sum_{i \in R_h} [n_i(T_f)A_i(T_f)b_i + \tfrac{1}{2}n_i(T_f)b_i^2] + \sum_{i \in R_s} [\beta_i - (1 - B_{i0})\lambda(t)]P_{s_i}(t)\\
&\qquad + \sum_{i \in R_h} - [n_i(t) + (1 - B_{i0})\lambda(t)]P_{h_i}(t) + \sum_{i \in R_h} r_i(t)Q_i(t) + \sum_{i \in R_s} \gamma_i P_{s_i}^2(t)\\
&\qquad + \sum\sum_{i,j \in R_G} P_i(t)\lambda(t)B_{ij}P_j(t) + \sum_{i \in R_h} (B_i/2)\dot{n}_i(t)Q_i^2(t)\Bigg\}\, dt
\end{aligned}
\tag{5.3.77}
$$

Here the quadratic cost expressions (5.2.51) are used to replace $F_i(P_{s_i})$ in (5.3.73).

Dropping terms in (5.3.77) that are explicitly independent of the control functions $P_{s_i}(t)$, $P_{h_i}(t)$, and $Q_i(t)$ over the interval $[0, T_f]$, it is only necessary to consider minimizing

$$\begin{aligned}
&J_1[P_{s_i}(t), P_{h_i}(t), Q_i(t)] \\
&= \int_0^{T_f} \Bigg\{ \sum_{i \in R_s} [\beta_i - (1 - B_{i0})\lambda(t)]P_{s_i}(t) + \sum_{i \in R_h} \\
&\quad - [n_i(t) + (1 - B_{i0})\lambda(t)]P_{h_i}(t) + \sum_{i \in R_h} r_i(t)Q_i(t) + \sum_{i \in R_s} \gamma_i P_{s_i}^2 \\
&\quad + \sum\sum_{i,\, j \in R_G} P_i(t)\lambda(t)B_{ij}P_j(t) + \sum_{i \in R_h} (B_i/2)\dot{n}_i(t)Q_i^2(t) \Bigg\} dt \qquad (5.3.78)
\end{aligned}$$

Define the $(2n - m) \times 1$ column vectors

$$\mathbf{u}(t) = \mathrm{col}[\mathbf{P}_s(t), \mathbf{P}_h(t), \mathbf{Q}(t)] \qquad (5.3.79)$$

$$\mathbf{L}(t) = \mathrm{col}[\mathbf{L}_s(t), \mathbf{L}_h(t), \mathbf{L}_Q(t)] \qquad (5.3.80)$$

where

$$\mathbf{P}_s(t) = \mathrm{col}[P_{s_1}(t), \ldots, P_{s_m}(t)] \qquad (5.3.81)$$

$$\mathbf{P}_h(t) = \mathrm{col}[P_{h_{m+1}}(t), \ldots, P_{h_n}(t)] \qquad (5.3.82)$$

$$\mathbf{L}_s(t) = \mathrm{col}[\beta_1 - (1 - B_{10})\lambda(t), \ldots, \beta_m - (1 - B_{m0})\lambda(t)] \qquad (5.3.83)$$

$$\mathbf{L}_h(t) = \mathrm{col}\{-[n_{m+1}(t) + (1 - B_{(m+1)0})\lambda(t)], \ldots, -[n_n(t) + (1 - B_{n0})\lambda(t)]\} \qquad (5.3.84)$$

$$\mathbf{L}_Q(t) = \mathrm{col}[r_{m+1}(t), \ldots, r_n(t)] \qquad (5.3.85)$$

Note that we assume that m is the number of thermal plants and n is the total number of generating plants. We also define the $(2n - m) \times (2n - m)$ matrix $\mathbf{B}(t)$:

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{B}_s(t) & \mathbf{B}_{sh}(t) & \mathbf{0} \\ \mathbf{B}_{hs}(t) & \mathbf{B}_h(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_Q(t) \end{bmatrix} \qquad (5.3.86)$$

where

$$\mathbf{B}_s(t) = (b_{ij_s}(t)) \qquad (5.3.87)$$

is the $m \times m$ matrix whose elements are

$$\begin{aligned} b_{ii_s}(t) &= \gamma_i + \lambda(t)B_{ii}, && i \in R_s \\ b_{ij_s}(t) &= \lambda(t)B_{ij}, && i \neq j, \quad i,j \in R_s \end{aligned} \qquad (5.3.88)$$

Moreover, the matrix

$$\mathbf{B}_{sh}(t) = (b_{ij_{sh}}(t)) \qquad (5.3.89)$$

is the $m \times (n - m)$ matrix whose elements are

$$b_{ij_{sh}}(t) = \lambda(t)B_{ij}, \qquad i \in R_s, \quad j \in R_h \qquad (5.3.90)$$

and

$$\mathbf{B}_{hs}(t) = (b_{ij_{hs}}(t)) \tag{5.3.91}$$

is the $(n - m) \times m$ matrix whose elements are

$$b_{ij_{hs}}(t) = \lambda(t)B_{ij}, \qquad i \in R_h, \quad j \in R_s \tag{5.3.92}$$

Finally, $\mathbf{B}_Q(t)$ is the $(n - m) \times (n - m)$ diagonal matrix

$$\mathbf{B}_Q(t) = \mathrm{diag}[B_{m+1}\dot{n}_{m+1}(t)/2, \ldots, B_n\dot{n}_n(t)/2] \tag{5.3.93}$$

Using these definitions, (5.3.78) becomes

$$J_1(\mathbf{u}(t)) = \int_0^{T_f} [\mathbf{L}^T(t)\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{B}(t)\mathbf{u}(t)]\,dt \tag{5.3.94}$$

We express the inverse of $\mathbf{B}(t)$ as

$$\mathbf{B}^{-1}(t) = \begin{bmatrix} \mathbf{C}_s(t) & \mathbf{C}_{sh} & \mathbf{0} \\ \mathbf{C}_{hs}(t) & \mathbf{C}_h(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_Q(t) \end{bmatrix} \tag{5.3.95}$$

where

$$\mathbf{C}_s(t) = [\mathbf{B}_s(t) - \mathbf{B}_{sh}(t)\mathbf{B}_h^{-1}(t)\mathbf{B}_{hs}(t)]^{-1} \tag{5.3.96}$$

$$\mathbf{C}_h(t) = [\mathbf{B}_h(t) - \mathbf{B}_{hs}(t)\mathbf{B}_s^{-1}(t)\mathbf{B}_{sh}(t)]^{-1} \tag{5.3.97}$$

$$\mathbf{C}_{sh}(t) = [-\mathbf{B}_s^{-1}(t)\mathbf{B}_{sh}(t)\mathbf{C}_h(t)] \tag{5.3.98}$$

$$\mathbf{C}_{hs}(t) = [-\mathbf{B}_h^{-1}(t)\mathbf{B}_{hs}(t)\mathbf{C}_s(t)] \tag{5.3.99}$$

$$\mathbf{C}_Q(t) = \mathbf{B}_Q^{-1}(t) \tag{5.3.100}$$

provided that each inverse in the last equalities exists. Thus, letting

$$\mathbf{V}^T(t) = \mathbf{L}^T(t)\mathbf{B}^{-1}(t) \tag{5.3.101}$$

will result in

$$\mathbf{V}_s^T(t) = \mathbf{L}_s^T(t)\mathbf{C}_s(t) + \mathbf{L}_h^T(t)\mathbf{C}_{hs}(t) \tag{5.3.102}$$

$$\mathbf{V}_h^T(t) = \mathbf{L}_s^T(t)\mathbf{C}_{sh}(t) + \mathbf{L}_h^T(t)\mathbf{C}_h(t) \tag{5.3.103}$$

$$\mathbf{V}_Q^T(t) = \mathbf{L}_Q^T(t)\mathbf{C}_Q(t) \tag{5.3.104}$$

where

$$\mathbf{V}^T(t) = [\mathbf{V}_s^T(t)\mathbf{V}_h^T(t)\mathbf{V}_Q^T(t)] \tag{5.3.105}$$

Now (5.3.94) can be written as

$$J_1(\mathbf{u}(t)) = \int_0^{T_f} \{[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^T\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)] - [\tfrac{1}{2}\mathbf{V}^T(t)\mathbf{B}(t)\tfrac{1}{2}\mathbf{V}(t)]\}\,dt \tag{5.3.106}$$

The last term in the integrand of (5.3.106) does not depend explicitly on $\mathbf{u}(t)$, so that one needs to consider only minimizing

$$J_2(\mathbf{u}(t)) = \int_0^{T_f} [\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]\,dt \qquad (5.3.107)$$

subject to the constraints

$$Q_i(T_f) = b_i, \qquad i \in R_h \qquad (5.3.108)$$

Define the $(n - m) \times 1$ column vector

$$\mathbf{b} = \operatorname{col}[b_{m+1}, \ldots, b_n] \qquad (5.3.109)$$

and the $(2n - m) \times (n - m)$ matrix $\mathbf{M}$ by

$$\mathbf{M}^{\mathrm{T}} = [\mathbf{0}|\mathbf{I}(d/dt)] \qquad (5.3.110)$$

$\mathbf{0}$ being the $(n - m) \times n$ matrix whose elements are all zero, and $\mathbf{I}$ being the $(n - m) \times (n - m)$ unity matrix. The constraints of Eq. (5.3.108) therefore reduce to

$$\mathbf{b} = \int_0^{T_f} \mathbf{M}^{\mathrm{T}}\mathbf{u}(t)\,dt \qquad (5.3.111)$$

The control vector $\mathbf{u}(t)$ is considered an element of the Hilbert space $L_{2,\mathbf{B}}^{(2n-m)}[0, T_f]$ of the $2n - m$ vector-valued square integrable functions defined on $[0, T_f]$ whose inner product is given by

$$\langle \mathbf{V}(t), \mathbf{u}(t) \rangle = \int_0^{T_f} \mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t)\,dt \qquad (5.3.112)$$

for every $\mathbf{V}(t)$ and $\mathbf{u}(t)$ in $L_{2,\mathbf{B}}^{(2n-m)}[0, T_f]$, provided that $\mathbf{B}(t)$ is positive definite.

The given vector $\mathbf{b}$ is considered an element of the real space $R^{(n-m)}$ with the Euclidean inner product definition

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \mathbf{X}^{\mathrm{T}}\mathbf{Y} \qquad (5.3.113)$$

for every $\mathbf{X}$ and $\mathbf{Y}$ in $R^{(n-m)}$.

Equation (5.3.111) then defines a bounded linear transformation $\mathbf{T}: L_{2,\mathbf{B}}^{(2n-m)}[0, T_f] \to R^{(n-m)}$. This can be written as

$$\mathbf{b} = \mathbf{T}[\mathbf{u}(t)] \qquad (5.3.114)$$

and the cost functional given in (5.3.107) reduces to

$$J_2(\mathbf{u}(t)) = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|^2 \qquad (5.3.115)$$

Finally, it is only necessary to minimize

$$J(\mathbf{u}(t)) = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\| \qquad (5.3.116)$$

subject to

$$\mathbf{b} = \mathbf{T}[\mathbf{u}(t)] \tag{5.3.117}$$

for a given $\mathbf{b}$ in $R^{(n-m)}$.

B. The Optimal Solution

The optimal solution to the problem formulated using the results of Chapter 3 is given by

$$\mathbf{u}_\xi(t) = \mathbf{T}^\dagger[\mathbf{b} + \mathbf{T}(\mathbf{V}/2)] - (\mathbf{V}/2) \tag{5.3.118}$$

where $\mathbf{T}^\dagger$ is obtained as follows:

$\mathbf{T}^*$, the adjoint of $\mathbf{T}$, is obtained using the identity

$$\langle \boldsymbol{\xi}, \mathbf{Tu} \rangle_{R^{(n-m)}} = \langle \mathbf{T}^*\boldsymbol{\xi}, \mathbf{u} \rangle_{L_{2,\mathbf{B}}^{(2n-m)}[0, T_f]} \tag{5.3.119}$$

Let

$$\boldsymbol{\xi} = \mathrm{col}[\xi_{m+1}, \ldots, \xi_n] \tag{5.3.120}$$

$$\mathbf{T}^*\boldsymbol{\xi} = \mathrm{col}[\mathbf{T}_\mathrm{p}, \mathbf{T}_\mathrm{h}, \mathbf{T}_Q] \tag{5.3.121}$$

with

$$\mathbf{T}_\mathrm{p} = \mathrm{col}[t_1, \ldots, t_m] \tag{5.3.122}$$

$$\mathbf{T}_\mathrm{h} = \mathrm{col}[t_{m+1}, \ldots, t_n] \tag{5.3.123}$$

$$\mathbf{T}_Q = \mathrm{col}[\omega_{m+1}, \ldots, \omega_n] \tag{5.3.124}$$

Then in $R^{(n-m)}$ the inner product of the left-hand side of (5.3.119) is

$$\langle \boldsymbol{\xi}, \mathbf{Tu} \rangle = \boldsymbol{\xi}^\mathrm{T} \int_0^{T_f} \mathbf{M}^\mathrm{T}\mathbf{u}(t)\,dt \tag{5.3.125}$$

where $Q_i(0) = 0$ according to (5.3.66). And in $L_{2,\mathbf{B}}^{(2n-m)}[0, T_f]$, the inner product of the right-hand side of (5.3.119) is

$$\langle \mathbf{T}^*\boldsymbol{\xi}, \mathbf{u} \rangle = \int_0^{T_f} (\mathbf{T}^*\boldsymbol{\xi})^\mathrm{T}\mathbf{B}(t)\mathbf{u}(t)\,dt \tag{5.3.126}$$

Using (5.3.121), (5.3.86), and (5.3.79), this reduces to

$$\langle \mathbf{T}^*\boldsymbol{\xi}, \mathbf{u} \rangle = \int_0^{T_f} [(\mathbf{T}_\mathrm{p}^\mathrm{T}\mathbf{B}_\mathrm{s} + \mathbf{T}_\mathrm{h}^\mathrm{T}\mathbf{B}_\mathrm{hs})\mathbf{P}_\mathrm{s} + (\mathbf{T}_\mathrm{p}^\mathrm{T}\mathbf{B}_\mathrm{sh} + \mathbf{T}_\mathrm{h}^\mathrm{T}\mathbf{B}_\mathrm{h})\mathbf{P}_\mathrm{h} + \mathbf{T}_Q^\mathrm{T}\mathbf{B}_Q\mathbf{Q}]\,dt \tag{5.3.127}$$

Thus the identity of (5.3.119), using (5.3.125) and (5.3.127), reduces to

$$\boldsymbol{\xi}^\mathrm{T}\mathbf{Q}(T_f) = \int_0^{T_f} [(\mathbf{T}_\mathrm{p}^\mathrm{T}\mathbf{B}_\mathrm{s} + \mathbf{T}_\mathrm{h}^\mathrm{T}\mathbf{B}_\mathrm{hs})\mathbf{P}_\mathrm{s}(t) + (\mathbf{T}_\mathrm{p}^\mathrm{T}\mathbf{B}_\mathrm{sh} + \mathbf{T}_\mathrm{h}^\mathrm{T}\mathbf{B}_\mathrm{h})\mathbf{P}_\mathrm{h}(t) + \mathbf{T}_Q^\mathrm{T}\mathbf{B}_Q\mathbf{Q}(t)]\,dt \tag{5.3.128}$$

Equation (5.3.128) is satisfied for the choice

$$\mathbf{T}_{\mathrm{p}}(t) = \mathbf{0}, \qquad t \in [0, T_{\mathrm{f}}] \tag{5.3.129}$$

$$\mathbf{T}_{\mathrm{h}}(t) = \mathbf{0}, \qquad t \in [0, T_{\mathrm{f}}] \tag{5.3.130}$$

$$\mathbf{T}_{Q}(t) = \mathbf{0}, \qquad t \in [0, T_{\mathrm{f}})$$

$$\mathbf{T}_{Q}^{\mathrm{T}}(T_{\mathrm{f}}) = \xi^{\mathrm{T}}\mathbf{B}_{Q}^{-1}(T_{\mathrm{f}}) \tag{5.3.131}$$

Thus $\mathbf{T}^*\xi$ is given by

$$\mathbf{T}^*[\xi] = \left[\frac{\mathbf{0}}{\boldsymbol{\mu}(t)}\right]\xi \tag{5.3.132}$$

where $\mathbf{0}$ is the $n \times (n - m)$ matrix whose elements are all zeros, and $\boldsymbol{\mu}(t)$ is the $(n - m) \times (n - m)$ diagonal matrix given by

$$\boldsymbol{\mu}(t) = \mathrm{diag}[\theta_{m+1}(t), \ldots, \theta_n(t)] \tag{5.3.133}$$

$$\theta_i(t) = \begin{cases} 0, & t \in [0, T_{\mathrm{f}}) \\ \dfrac{2}{B_i \dot{n}_i(T_{\mathrm{f}})}, & t = T_{\mathrm{f}} \end{cases} \tag{5.3.134}$$

The operator $\tilde{\mathbf{J}}$ is next evaluated as

$$\tilde{\mathbf{J}}(\xi) = \mathbf{T}[\mathbf{T}^*\xi]$$

Using (5.3.110), (5.3.111), and (5.3.132), this reduces to

$$\tilde{\mathbf{J}}(\xi) = \int_0^{T_{\mathrm{f}}} [\mathbf{0}|\mathbf{I}(d/dt)]\left[\frac{\mathbf{0}}{\boldsymbol{\mu}(t)}\right]\xi\, dt$$

or

$$\tilde{\mathbf{J}}(\xi) = \int_0^{T_{\mathrm{f}}} (d/dt)\boldsymbol{\mu}(t)\xi\, dt \tag{5.3.135}$$

so that $\tilde{\mathbf{J}}(\xi) = [\boldsymbol{\mu}(T_{\mathrm{f}}) - \boldsymbol{\mu}(0)]\xi$. But by (5.3.133) $\boldsymbol{\mu}(0) = \mathbf{0}$; then

$$\tilde{\mathbf{J}}(\xi) = \boldsymbol{\mu}(T_{\mathrm{f}})\xi \tag{5.3.136}$$

This yields

$$\tilde{\mathbf{J}}^{-1}(\xi) = \boldsymbol{\mu}^{-1}(T_{\mathrm{f}})\xi \tag{5.3.137}$$

Finally, the pseudoinverse operator $\mathbf{T}^{\dagger}$ is obtained from the definition

$$\mathbf{T}^{\dagger}[\xi] = \mathbf{T}^*[\tilde{\mathbf{J}}^{-1}\xi]$$

using (5.3.132) and (5.3.137). This yields

$$\mathbf{T}^{\dagger}\xi = \left[\frac{\mathbf{0}}{\boldsymbol{\mu}(t)}\right]\boldsymbol{\mu}^{-1}(T_{\mathrm{f}})\xi$$

or

$$\mathbf{T}^{\dagger}\boldsymbol{\xi} = \left[\frac{\mathbf{0}}{\boldsymbol{\mu}(t)\boldsymbol{\mu}^{-1}(T_{\mathrm{f}})}\right]\boldsymbol{\xi} \tag{5.3.138}$$

From (5.3.110) and (5.3.105) one obtains

$$\mathbf{T}(\tfrac{1}{2}\mathbf{V}(t)) = \tfrac{1}{2}[\mathbf{V}_{Q}(T_{\mathrm{f}}) - \mathbf{V}_{Q}(0)] \tag{5.3.139}$$

The optimal solution is now found by substituting (5.3.138), (5.3.139), and (5.3.105) in (5.3.118), the result being

$$\mathbf{P}_{\mathrm{s}_{\xi}}(t) = -\tfrac{1}{2}\mathbf{V}_{\mathrm{s}}(t) \tag{5.3.140}$$

$$\mathbf{P}_{\mathrm{h}_{\xi}}(t) = -\tfrac{1}{2}\mathbf{V}_{\mathrm{h}}(t) \tag{5.3.141}$$

$$\mathbf{Q}_{\xi}(t) = \boldsymbol{\mu}(t)\boldsymbol{\mu}^{-1}(T_{\mathrm{f}})\{\mathbf{b} + \tfrac{1}{2}[\mathbf{V}_{Q}(T_{\mathrm{f}}) - \mathbf{V}_{Q}(0)]\} - \tfrac{1}{2}\mathbf{V}_{Q}(t) \tag{5.3.142}$$

which can be reduced by using (5.3.134) to

$$\mathbf{Q}_{\xi}(t) = -\tfrac{1}{2}\mathbf{V}_{Q}(t), \qquad t \in [0, T_{\mathrm{f}})$$
$$\mathbf{Q}_{\xi}(T_{\mathrm{f}}) = \mathbf{b}$$

since

$$\mathbf{Q}_{\xi}(0) = -\tfrac{1}{2}\mathbf{V}_{Q}(0) = \mathbf{0}$$

Thus, according to (5.3.102), (5.3.103), and (5.3.104), the optimal solution is

$$\mathbf{P}_{\mathrm{s}_{\xi}}(t) = -\tfrac{1}{2}[\mathbf{C}_{\mathrm{s}}^{\mathrm{T}}(t)\mathbf{L}_{\mathrm{s}}(t) + \mathbf{C}_{\mathrm{hs}}^{\mathrm{T}}(t)\mathbf{L}_{\mathrm{h}}(t)] \tag{5.3.143}$$

$$\mathbf{P}_{\mathrm{h}_{\xi}}(t) = -\tfrac{1}{2}[\mathbf{C}_{\mathrm{sh}}^{\mathrm{T}}(t)\mathbf{L}_{\mathrm{s}}(t) + \mathbf{C}_{\mathrm{h}}^{\mathrm{T}}(t)\mathbf{L}_{\mathrm{h}}(t)] \tag{5.3.144}$$

$$\mathbf{Q}_{\xi}(t) = -\tfrac{1}{2}[\mathbf{C}_{Q}^{\mathrm{T}}(t)\mathbf{L}_{Q}(t)]$$
$$\mathbf{Q}_{\xi}(T_{\mathrm{f}}) = \mathbf{b} \tag{5.3.145}$$

It is noted that the optimal solution involves the unknown functions $\lambda(t)$ and $n_i(t)$, which are to be determined so that the constraints (5.3.65) and (5.3.69) are satisfied.

C. Implementing the Optimal Solution

The optimal solution obtained so far contains the unknown functions $n_i(t)$ and $\lambda(t)$. These can be determined by substituting the optimal solution in the corresponding constraint equations. The resulting equations in $n_i(t)$ and $\lambda(t)$ are generally nonlinear. In order to get deeper insight into how the actual solution is obtained, the following simplifying assumptions are made:

(1) The system is characterized by a loss formula where

(a) $B_{ij} = 0, \qquad i \neq j, \quad i = 1, \ldots, n$

(b) $B_{i0} = 0, \qquad i = 1, \ldots, n$

(c) $K_{L0} = 0$

(2) The rate of natural water inflow to the reservoirs is constant. These assumptions will represent no loss of generality.

Thus the optimal solution componentwise is given by

$$P_{s_{i_\xi}}(t) = \frac{\lambda(t) - \beta_i}{2[\gamma_i + \lambda(t)B_{ii}]}, \qquad i \in R_s \tag{5.3.146}$$

$$P_{h_{i_\xi}}(t) = \frac{n_i(t) + \lambda(t)}{2\lambda(t)B_{ii}}, \qquad i \in R_h \tag{5.3.147}$$

$$Q_{i_\xi}(t) = \frac{-1}{B_i}\left[A_i(t) + \dot{A}_i(t)\frac{n_i(t)}{\dot{n}_i(t)}\right] \qquad i \in R_h \tag{5.3.148}$$

$$Q_{i_\xi}(T_f) = b_i$$

I. A Set of Nonlinear Differential Equations. The constraint equation (5.3.69) for the optimal solution becomes

$$P_D(t) = \sum P_{s_{i_\xi}}(t) + \sum P_{h_{i_\xi}}(t) - \sum B_{ii}P^2_{s_{i_\xi}}(t) - \sum B_{ii}P^2_{h_{i_\xi}}(t)$$

Substituting (5.3.146) and (5.3.147), this reduces to

$$P_D(t) = \beta - \sum C_i\{1/[\gamma_i + B_{ii}\lambda(t)]^2\} - \sum D_i[n_i(t)/\lambda(t)]^2 \tag{5.3.149}$$

where

$$\beta = \sum(1/4B_{ii}), \qquad C_i = (\beta_i B_{ii} + \gamma_i)^2/4B_{ii}, \qquad D_i = 1/4B_{ii}$$

We have thus obtained one equation in $n_i(t)$ and $\lambda(t)$ given by Eq. (5.3.149).

The hydro power constraints given by (5.3.65) are

$$P_{h_{i_\xi}}(t) + \dot{Q}_{i_\xi}(t)[A_i(t) + B_i Q_{i_\xi}(t)] = 0 \tag{5.3.150}$$

Let us derive a set of equations in $n_i(t)$ and $\lambda(t)$. This can be done based on (5.3.150) as follows. Let us introduce the variable $x_i(t)$ given by

$$x_i(t) = n_i(t)/\dot{n}_i(t) \tag{5.3.151}$$

or

$$n_i(t) = n_i(0)\exp\left\{\int_0^t [x_i(t)]^{-1}\,dt\right\} \tag{5.3.152}$$

Then (5.3.148) becomes

$$Q_{i_\xi}(t) = (-1/B_i)[A_i(t) + \dot{A}_i(t)x_i(t)] \tag{5.3.153}$$

Differentiating both sides of (5.3.153) yields

$$\dot{Q}_{i_\xi}(t) = (-1/B_i)[\dot{A}_i(t) + \ddot{A}_i(t)x_i(t) + \dot{A}_i(t)\dot{x}_i(t)], \qquad i \in R_h \tag{5.3.154}$$

so that the constraints (5.3.150) reduce to

$$P_{h_{i_\xi}}(t) + (\dot{A}_i(t)x_i(t)/B_i)[\dot{A}_i(t) + \ddot{A}_i(t)x_i(t) + \dot{A}_i(t)\dot{x}_i(t)] = 0, \qquad i \in R_h \tag{5.3.155}$$

Using (5.3.147) in (5.3.155) one obtains

$$\left[\frac{1}{2B_{ii}} + \frac{n_i(0)\exp\left\{\int_0^t [x_i(t)]^{-1}\,dt\right\}}{2\lambda(t)B_{ii}} + \frac{\dot{A}_i(t)x_i(t)}{B_i}\right] \times [\dot{A}_i(t) + \ddot{A}_i(t)x_i(t) + \dot{A}_i(t)\dot{x}_i(t)] = 0, \qquad i \in R_h \tag{5.3.156}$$

We recall the definition of $A_i(t)$ given in Chapter 2 by

$$-A_i(t) = (h_i(0)/G_i) + B_i\int_0^t i_i(\sigma)\,d\sigma, \qquad i \in R_h \tag{5.3.157}$$

Differentiating, one obtains

$$\dot{A}_i(t) = -B_i i_i(t), \qquad i \in R_h \tag{5.3.158}$$

$$\ddot{A}_i(t) = -B_i \dot{i}_i(t), \qquad i \in R_h \tag{5.3.159}$$

Thus (5.3.156) reduces to

$$\left[1 + \frac{n_i(0)\exp\left\{\int_0^t [x_i(t)]^{-1}\,dt\right\}}{\lambda(t)} + 2B_{ii}B_i i_i(t)x_i(t)\right] \times [i_i(t)(1 + \dot{x}_i(t)) + x_i(t)\dot{i}_i(t)] = 0, \qquad i \in R_h \tag{5.3.160}$$

This is the desired set of equations relating $n_i(t)$ and $\lambda(t)$. Note that the number of equations given by (5.3.149) and (5.3.160) is the same as the number of the functions $n_i(t)$ and $\lambda(t)$. A further refinement of Eq. (5.3.160) results if we let

$$a_i(t) = 2B_{ii}B_i i_i^2(t) \tag{5.3.161}$$

Thus (5.3.160) reduces to

$$\dot{x}_i(t) = -\left[\frac{1 + [n_i(0)/\lambda(t)]\exp\left\{\int_0^t x_i^{-1}(t)\,dt\right\}}{a_i(t)x_i(t)} + \frac{\dot{i}_i(t)}{i_i(t)}x_i(t) + 1\right] \tag{5.3.162}$$

Under the assumption of constant water inflow $\dot{i}_i(t) = 0$; thus (5.3.162) reduces to

$$\dot{x}_i(t) = -\left\{\left[1 + \left\{n_i(0)\exp\left[\int_0^t x_i^{-1}(t)\,dt\right]\Big/\lambda(t)\right\}\right]\Big/a_i(t)x_i(t)\right\} - 1, \qquad i \in R_h \tag{5.3.163}$$

II. Boundary Conditions. The optimal $Q_i(t)$ given by (5.3.153) reduces to

$$Q_{i_\xi}(t) = (1/B_i)\left\{[h_i(0)/G_i] + B_i \int_0^t i_i(\sigma)\,d\sigma + B_i i_i(t) x_i(t)\right\} \tag{5.3.164}$$

This relation enables us to deduce boundary conditions on the variable $x_i(t)$ as dictated by those imposed on $Q_{i_\xi}(t)$. At $t = 0$ Eq. (5.3.164) reduces to

$$B_i Q_{i_\xi}(0) = [h_i(0)/G_i] + B_i i_i(0) x_i(0) \tag{5.3.165}$$

But at $t = 0$, by definition of $Q_i(t)$ as given in (5.3.66),

$$Q_{i_\xi}(0) = 0$$

Thus initial conditions on $x_i(0)$ are given by

$$x_i(0) = -S_i h_i(0)/i_i(0) \tag{5.3.166}$$

Note that $B_i = (S_i G_i)^{-1}$ has been invoked. At $t = T_f$, Eq. (5.3.164) is

$$B_i Q_{i_\xi}(T_f) = [h_i(0)/G_i] + B_i I_i(T_f) + B_i i_i(T_f) x_i(T_f) \tag{5.3.167}$$

But at $t = T_f$, by virtue of the volume of water constraints,

$$Q_{i_\xi}(T_f) = b_i$$

Thus we conclude that the terminal conditions on $x_i(t)$ are

$$x_i(T_f) = [b_i/i_i(T_f)] + x_i(0)[i_i(0)/i_i(T_f)] - [I_i(T_f)/i_i(T_f)] \tag{5.3.168}$$

where use is made of

$$I_i(t) = \int_0^t i_i(\sigma)\,d\sigma \tag{5.3.169}$$

In the case when $i_i(t) = i_i =$ constant, (5.3.166) and (5.3.168) reduce to

$$x_i(0) = -S_i h_i(0)/i_i \tag{5.3.170}$$

$$x_i(T_f) = x_i(0) + [(b_i/i_i) - T_f] \tag{5.3.171}$$

Thus solving (5.3.149) and (5.3.163) subject to the boundary conditions (5.3.170) and (5.3.171) completely defines the optimal schedule. It is noted that (5.3.149) is an algebraic equation in $\lambda(t)$ and the $n - m$ unknown functions $n_i(t)$. On the other hand, (5.3.163) is a set of $n - m$ nonlinear differential equations in $x_i(t)$ where $x_i(t)$ and $n_i(t)$ are related by (5.3.151).

III. Restricted Search Region. In the computerized search for the above-mentioned unknown functions, it is highly desirable to characterize the region of search. This is done by utilizing the physical significance of each variable involved. In addition to this, restrictions on the variables can be obtained so that the functional analytic formulation adopted is a valid one.

Consider the optimal thermal power generation expression of (5.3.146). If the thermal power generated is to satisfy the following practical limitation,

$$P_{\min_i} \le P_{s_{i_\xi}}(t) \le P_{\max_i}, \qquad i \in R_s \tag{5.3.172}$$

then using (5.3.146) one obtains

$$\frac{\beta_i + 2\gamma_i P_{\min_i}}{1 - 2B_{ii}P_{\min_i}} \le \lambda(t) \le \frac{\beta_i + 2\gamma_i P_{\max_i}}{1 - 2B_{ii}P_{\max_i}}, \qquad i \in R_s \tag{5.3.173}$$

Let us denote the maximum of the lower limits on $\lambda(t)$ by

$$\lambda_{\min} = \max_{i=1,\ldots,m} \left(\frac{\beta_i + 2\gamma_i P_{\min_i}}{1 - 2B_{ii}P_{\min_i}} \right) \tag{5.3.174}$$

Also, the minimum of the upper limits on $\lambda(t)$ is denoted by

$$\lambda_{\max} = \min_{i=1,\ldots,m} \left(\frac{\beta_i + 2\gamma_i P_{\max_i}}{1 - 2B_{ii}P_{\max_i}} \right) \tag{5.3.175}$$

Then using (5.3.174) and (5.3.175) in (5.3.173) results in

$$\lambda_{\min} \le \lambda(t) \le \lambda_{\max} \tag{5.3.176}$$

We now offer a physical interpretation of the variable $x_i(t)$ introduced earlier. This interpretation is useful in deriving bounds on $x_i(t)$. We have by Eq. (5.3.153) that

$$x_i(t) = [-1/\dot{A}_i(t)][A_i(t) + B_i Q_{i_\xi}(t)]$$

Now using Eq. (5.3.157) and (5.3.158), we conclude that

$$x_i(t) = -V_i(t)/i_i(t) \tag{5.3.177}$$

Here $V_i(t)$ is the volume of water stored in the reservoir.

The significance of this result is that $x_i(t)$ is the negative of the time ratio of the volume of water in the reservoir to the natural inflow. In the case when this volume is restricted between upper and lower bounds given by

$$V_{\min_i} \le V_i(t) \le V_{\max_i} \tag{5.3.178}$$

this yields

$$-V_{\max_i}/i_i(t) \le x_i(t) \le -V_{\min_i}/i_i(t) \tag{5.3.179}$$

It is evident that

$$x_i(t) < 0 \qquad \text{for} \quad i_i(t) > 0 \tag{5.3.180}$$

Furthermore, if the hydro power is restricted so that

$$P_{h_{\min_i}} \le P_{h_{i_\xi}}(t) \le P_{h_{\max_i}} \tag{5.3.181}$$

then, utilizing (5.3.147), the following inequality must be satisfied:

$$\lambda(t)(2B_{ii}P_{\mathrm{h}_{\min_i}} - 1) \leq n_i(t) \leq \lambda(t)(2B_{ii}P_{\mathrm{h}_{\max_i}} - 1) \tag{5.3.182}$$

If

$$2B_{ii}P_{\mathrm{h}_{\max_i}} < 1 \tag{5.3.183}$$

then

$$n_i(t) < 0 \tag{5.3.184}$$

This combined with (5.3.180) yields

$$\dot{n}_i(t) > 0 \tag{5.3.185}$$

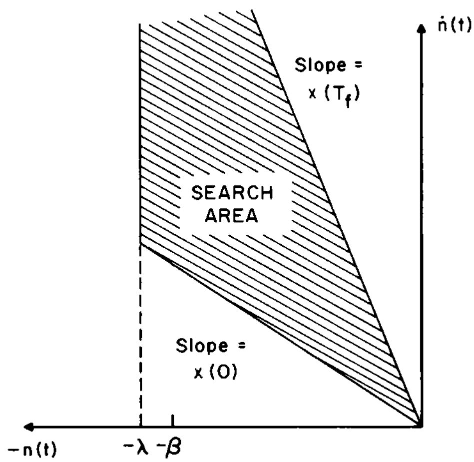

Fig. 5.2 The search area in the n–$\dot{n}$ plane.

The restricted search area in the (n–$\dot{n}$) phase plane is shown in Fig. 5.2. It is noted here that (5.3.176) and (5.3.185) guarantee that the matrix $\mathbf{B}(t)$ in the inner product definition is positive definite.

It is worth mentioning here that it is assumed in the problem formulation that the inequality constraints are not violated. Thus these constraints are not included in the cost functional to obtain the optimal solution. The optimal solution is then implemented in a way that confines the search area to the regions of the space of the unknown functions where the inequality constraints are not violated. This agrees in principle with the nonlinear programming approach to this type of problem. In the Kuhn–Tucker method, unknown multipliers (the Kuhn–Tucker multipliers) are associated with the inequality constraints for inclusion in the cost functional. These multipliers are set to zero as long as the inequality constraints are not violated. Thus an optimal solution is obtained by scanning the whole space of the unknown functions. If the solution obtained violates any inequality constraint, the corresponding variable is set to the nearest value that does not violate the constraint. The main difference between that method and the method adopted here is that the search region in the latter is smaller than that of the former. This reduces the computing time considerably and leads to good estimated values for the unknown functions.

IV. A Convergence Condition. Consider Eq. (5.3.163), which can be rewritten as

$$\dot{x}_i(t) = f_i[x_i(t), \lambda(t), n_i(0)], \qquad i \in R_\text{h} \tag{5.3.186}$$

where

$$f_i[x_i(t), \lambda(t), n_i(0)] = -\left\{[1 + \left\{n_i(0)\exp\left[\int_0^t x_i^{-1}(t)\,dt\right]\bigg/\lambda(t)\right\}\bigg]\bigg/a_i(t)x_i(t)\right\} - 1, \qquad i \in R_\text{h} \tag{5.3.187}$$

Let us consider the difference Δf_i given by

$$\Delta f_i(s) = f_i[x_i^{(1)}(s), \lambda(s), n_i(0)] - f_i[x_i^{(2)}(s), \lambda(s), n_i(0)] \tag{5.3.188}$$

We make the substitution

$$y_i(t) = [n_i(0)/\lambda(t)]\exp\left[\int_0^t x_i^{-1}(t)\,dt\right] \tag{5.3.189}$$

in (5.3.187). We can thus express (5.3.188) as

$$\begin{aligned}\Delta f_i(s) = [a_i(s)x_i^{(1)}(s)x_i^{(2)}(s)]^{-1}[x_i^{(1)}(s) - x_i^{(2)}(s) \\ + x_i^{(1)}(s)y_i^{(2)}(s) - x_i^{(2)}(s)y_i^{(1)}(s)]\end{aligned} \tag{5.3.190}$$

Let us define

$$Z_i^{12}(s) = [a_i(s)x_i^{(1)}(s)x_i^{(2)}(s)]^{-1} \tag{5.3.191}$$

$$\Delta x^{12}(s) = x_i^{(1)}(s) - x_i^{(2)}(s) \tag{5.3.192}$$

Then (5.3.190) turns out to be

$$\Delta f_i(s) = Z_i^{12}(s)[\Delta x^{12}(s) + y_i^{(2)}(s)\{x_i^{(1)}(s) - x_i^{(2)}(s)[y_i^{(1)}(s)/y_i^{(2)}(s)]\}] \tag{5.3.193}$$

From (5.3.189) and (5.3.180) we deduce that

$$|y_i^{(2)}(s)| \le |n_i(0)/\lambda(s)| \tag{5.3.194}$$

We may thus write the following inequality on the basis of (5.3.193) and (5.3.194):

$$|\Delta f_i(s)| \le |Z_i^{12}(s)|[|\Delta x^{12}(s)| + |n_i(0)/\lambda(s)||\Delta x^{12}(s) + g_i(s)|] \quad (5.3.195)$$

Here

$$g_i(s) = x_i^{(2)}(s)[1 - (y_i^{(1)}(s)/y_i^{(2)}(s))] \quad (5.3.196)$$

Recalling (5.3.189), we may write (5.3.196) as

$$g_i(s) = x_i^{(2)}(s)[1 - \exp(\eta(s))] \quad (5.3.197)$$

with

$$\eta(s) = \int_0^s [x_i^{(1)^{-1}}(\sigma) - x_i^{(2)^{-1}}(\sigma)]\, d\sigma \quad (5.3.198)$$

We can conclude that

$$|g_i(s)| \le |x_i^{(2)}(s)|\,|\eta(s)| \quad (5.3.199)$$

Here we utilize the following well-known result from complex variable theory:

$$|1 - e^\alpha| \le |\alpha| \qquad \text{for} \quad \mathrm{Re}(\alpha) < 0$$

We obtain from (5.3.199) the following inequality:

$$|g_i(s)| \le [|T_f|\,|x_i^{(2)}(s)|/X_i(s)] \max_s |x_i^{(2)}(s) - x_i^{(1)}(s)| \quad (5.3.200)$$

where

$$X_i(s) = \min_s x_i^{(1)}(s)x_i^{(2)}(s) \quad (5.3.201)$$

Thus (5.3.195) reduces to

$$|\Delta f_i(s)| \le M \max_{[0,T_f]} |x_i^{(2)}(s) - x_i^{(1)}(s)| \quad (5.3.202)$$

where

$$M = \frac{1}{\min[a_i(s)x_i^{(1)}(s)x_i^{(2)}(s)]} \times \left[1 + \left|\frac{n_i(0)}{\min_s \lambda(s)}\right| |1 + |T_f| \max_s |x_i^{(2)}(s)|/X_i(s)|\right] \quad (5.3.203)$$

Furthermore, (5.3.202) provides exactly a Lipschitz condition given by

$$\max_{[0,T_f]} |f_i[x_i^{(1)}(s),\ldots] - f_i[x_i^{(2)}(s),\ldots]| \le M \max_{[0,T_f]} |x_i^{(1)}(s) - x_i^{(2)}(s)| \quad (5.3.204)$$

This means that Picard's iteration process, given by

$$x_i^{(0)}(t) = x_i(0),$$
$$x_i^{(m+1)}(t) = x_i(0) + \int_0^t f(x_i^{(m)}(s), \lambda(s), n_i(0))\, ds \tag{5.3.205}$$

is guaranteed to converge to a solution of (5.3.186) for given $n_i(0)$ and $\lambda(s)$, provided that $M < 1$.

V. Practical Application. A computer program was written to solve (5.3.163) and (5.3.149) for $x_i(t)$ $(i = m+1, \ldots, n)$ and $\lambda(t)$ to obtain the optimal generation schedule. Figure 5.3 shows the flow chart for this program. An initial estimate of the function $\lambda(t)$ is needed. This was taken as

$$\lambda^{(0)}(t) = \lambda_{\min}, \qquad t \in [0, T_f]$$

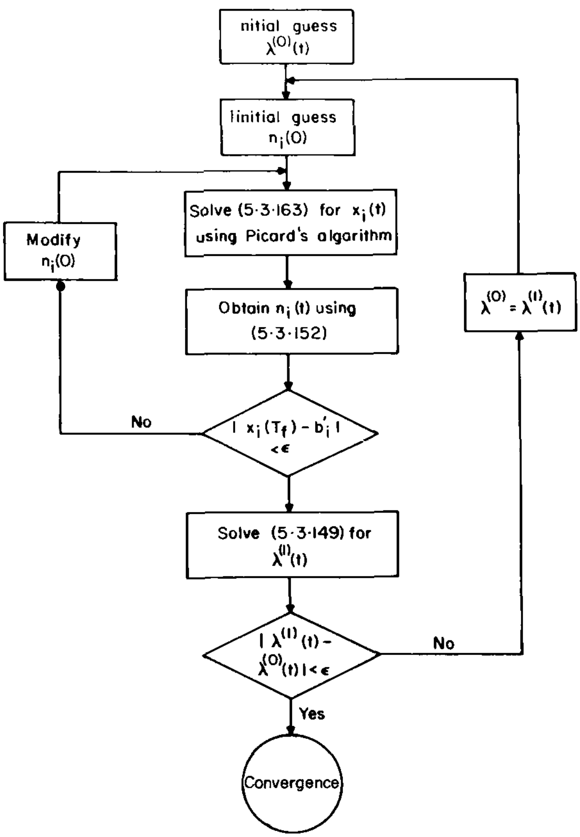

Fig. 5.3 Computer program flow chart.

where $\lambda_{\min}$ is given by (5.3.174). For each hydro plant, the initial value of $n_i(t)$ was estimated as

$$n_i^{(0)}(0) = \lambda_{\min}[2B_{ii}P_{\mathrm{h}_{i_{\min}}} - 1], \qquad i = m+1, \ldots, n$$

A solution to (5.3.163) was then obtained for each hydro plant. This was done by utilizing Picard's algorithm, given by (5.3.205). The value of $x_i(T_f)$ obtained here was compared with the boundary condition given in (5.3.171). If the error in this step is large, the estimated value $n_i^{(0)}(0)$ was modified in the direction that minimizes the error. When all the $x_i(t)$ were obtained the corresponding $n_i(t)$ were evaluated using (5.3.152).

Thus Eq. (5.3.149) becomes a $(2m+2)$-order algebraic equation in $\lambda(t)$ for every t. This is solved for $\lambda(t)$ in the region given by (5.3.173). The $\lambda(t)$ obtained was then taken as the initial estimate instead of $\lambda_{\min}$ and the process repeated. If the difference between two successive evaluations of $\lambda(t)$ was less than a prespecified amount the iteration process was stopped. The last step is to evaluate the optimal generation schedules as given by (5.3.146), (5.3.147), and (5.3.148).

This program was applied to a hydrothermal system with four thermal and three hydro plants whose particulars are summarized in Tables 5.1 and 5.2. The optimal generation schedules and the assumed power demand curves are shown in Figs. 5.4–5.6.

TABLE 5.1

Thermal Plants' Particulars

Plant number:	1	2	3	4
β	4.4	4.3	4.2	4.25
$\gamma \times 10^3$	1.2	1.56	1.67	1.32
$B_{ii} \times 10^4$	1.6	1.5	1.8	1.4

TABLE 5.2

Hydro Plants' Particulars

Plant number:	5	6	7
Constant $i_i(t) \times 10^{-6}$	0.1	1.0	0.15
$S_i \times 10^{-10}$ ft^2	7.2	0.72	1.44
$h_i(0)$ ft	100	200	150
$b_i \times 10^{-10}$ ft^3	0.05	0.35	0.78
$B_{ii} \times 10^4$	2.2	2.3	2.4

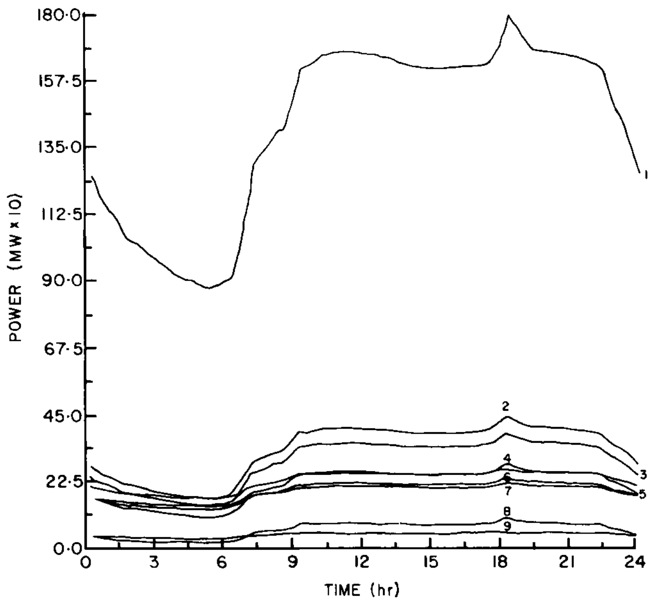

Fig. 5.4 The system's power demand and optimal generations. Legend: (1), system's power demand; (2), hydro plant no. 3; (3), hydro plant no. 2; (4), thermal plant no. 4; (5), thermal plant no. 1; (6), hydro plant no. 1; (7), thermal plant no. 3; (8), thermal plant no. 2; (9), transmission losses.

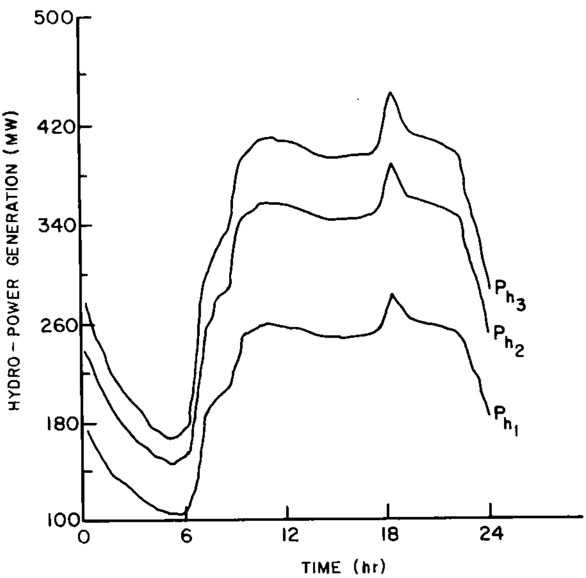

Fig. 5.5 Optimal hydro power output.

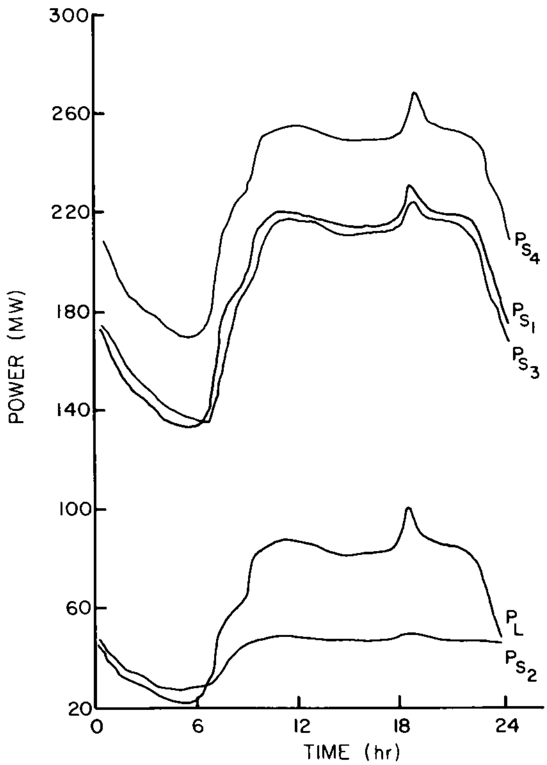

Fig. 5.6 Optimal thermal output and transmission losses.

D. Comparison with Kron's Coordination Equations

We have developed Kron's coordination equation in Section 5.3.1 for a system having one thermal and one hydro plant. It is our intention here to prove the equivalence of the results of the minimum norm approach and Kron's equation derived earlier.

Consider an electric power system with one thermal and one hydro plant. The optimal solution for this system is given by (5.3.143), (5.3.144), and (5.3.145) where all vector quantities reduce to one-element vectors since $m = 1$ and $n - m = 1$ in this case.

Here we have, using (5.3.83), (5.3.84), and (5.3.85),

$$L_s(t) = \beta_1 - \lambda(t)(1 - B_{10}) \tag{5.3.206}$$

$$L_h(t) = -[n_2(t) + \lambda(t)(1 - B_{20})] \tag{5.3.207}$$

$$L_Q(t) = r_2(t) \tag{5.3.208}$$

Using (5.3.96)–(5.3.100) also, we obtain

$$C_s(t) = B_{22}/\Delta, \qquad C_{sh}(t) = -[B_{12}\lambda(t)/(\gamma_1 + \lambda(t)B_{11})]C_h(t)$$
$$C_h(t) = \{B_{11} + [\gamma_1/\lambda(t)]\}/\Delta, \qquad C_{hs}(t) = -(B_{21}/B_{22})C_s(t)$$
$$C_Q(t) = 2/[B_2\dot{n}_2(t)]$$

where

$$\Delta = [\gamma_1 + \lambda(t)B_{11}]B_{22} - \lambda(t)B_{12}B_{21}$$

Thus the optimal solution obtained is

$$P_{s_\xi}(t) = (-B_{22}/2\Delta)\{[\beta_1 - \lambda(t)(1 - B_{10})] + (B_{21}/B_{22})[n_2(t) + \lambda(t)(1 - B_{20})]\} \tag{5.3.209}$$

$$P_{h_\xi}(t) = \{[B_{11} + (\gamma_1/\lambda(t))]/2\Delta\} \times \left\{\frac{B_{12}\lambda(t)}{\gamma_1 + \lambda(t)B_{11}}[\beta_1 - \lambda(t)(1 - B_{10})] + [n_2(t) + \lambda(t)(1 - B_{20})]\right\} \tag{5.3.210}$$

$$Q_\xi(t) = (-1/B_2)\{A_2(t) + \dot{A}_2(t)[n_2(t)/\dot{n}_2(t)]\} \tag{5.3.211}$$

Kron's first scheduling equation (5.3.9) is

$$(\partial F/\partial P_s) + \lambda(t)(\partial P_L/\partial P_s) - \lambda(t) = 0 \tag{5.3.212}$$

In the particular power system under consideration one obtains

$$\partial F/\partial P_s = \beta_1 + 2\gamma_1 P_s(t) \tag{5.3.213}$$
$$\partial P_L/\partial P_s = 2B_{11}P_s(t) + 2B_{21}P_h(t) + B_{10} \tag{5.3.214}$$

Thus (5.3.212), the first of Kron's scheduling equations, requires that

$$2P_s(t)[\gamma_1 + \lambda(t)B_{11}] + [\beta_1 - \lambda(t)(1 - B_{10})] + 2B_{21}\lambda(t)P_h(t) = 0 \tag{5.3.215}$$

That the optimal solution given by (5.3.209) and (5.3.210) satisfies Kron's first scheduling equation (5.3.215) is evident by direct substitution.

Kron's second scheduling equation (5.3.13) is

$$(\lambda(t)/SL_h)(\partial P_h/\partial h) + (d/dt)[(\lambda(t)/L_h)(\partial P_h/\partial \dot{Q})] = 0 \tag{5.3.216}$$

In our system the hydro-model is given by Eq. (5.3.65). From this we obtain

$$\partial P_h/\partial \dot{Q} = -A_2(t) - B_2 Q_2(t) \tag{5.3.217}$$

Moreover, the variation of active power generated with head provides

$$\partial P_h/\partial h = \dot{Q}_2/G_2 \tag{5.3.218}$$

Thus Kron's second scheduling equation (5.3.216) can be written as

$$[L_{\mathrm{h}}^{-1}\lambda(t)B_2\dot{Q}_2(t)] = (d/dt)\{L_{\mathrm{h}}^{-1}\lambda(t)[A_2(t) + B_2Q_2(t)]\} \tag{5.3.219}$$

In Eqs. (5.3.216) and (5.3.219), L_{h} denotes the penalty factor, given by

$$L_{\mathrm{h}}^{-1} = 1 - (\partial P_{\mathrm{L}}/\partial P_{\mathrm{h}}) \tag{5.3.220}$$

We will need two steps to prove that the optimal solution of this section does indeed satisfy Kron's second equation. The first step leads us to an interesting result. This pertains to the relation between our current function $n(t)$, the function $\lambda(t)$, and the penalty factors. We write (5.3.209) and (5.3.210) in the alternative forms

$$2\,\Delta P_{\mathrm{s}_\xi}(t) = -B_{22}\{\beta_1 - \lambda(t)(1 - B_{10}) - B_{21}[n_2(t) + \lambda(t)(1 - B_{20})]\} \tag{5.3.221}$$

$$2\,\Delta P_{\mathrm{h}_\xi}(t) = B_{12}[\beta_1 - \lambda(t)(1 - B_{10})] + \{[\gamma_1 + \lambda(t)B_{11}]/\lambda(t)\}[n_2(t) + \lambda(t)(1 - B_{20})] \tag{5.3.222}$$

Multiplying both sides of (5.3.221) by B_{12}, (5.3.222) by B_{22}, and adding, the following is obtained:

$$2[B_{12}P_{\mathrm{s}_\xi}(t) + B_{22}P_{\mathrm{h}_\xi}(t)] = [n_2(t) + \lambda(t)(1 - B_{20})]/\lambda(t)$$

Rearranging, we have

$$n_2(t) = \lambda(t)[2B_{12}P_{\mathrm{s}_\xi}(t) + 2B_{22}P_{\mathrm{h}_\xi}(t) - (1 - B_{20})] \tag{5.3.223}$$

Now the inverse of the hydro penalty factor is given by (5.3.220), which yields for our system

$$L_{\mathrm{h}}^{-1} = 1 - (\partial/\partial P_{\mathrm{h}})[B_{10}P_{\mathrm{s}} + B_{20}P_{\mathrm{h}} + B_{11}P_{\mathrm{s}_1}^2 + 2B_{12}P_{\mathrm{s}}P_{\mathrm{h}} + B_{22}P_{\mathrm{h}}^2] \tag{5.3.224}$$

It is clear that application of (5.3.224) in (5.3.223) results in the defining relation anticipated. This is given by

$$n_2(t) = L_{\mathrm{h}}^{-1}\lambda(t) \tag{5.3.225}$$

Our second step is to write the optimality condition (5.3.211) as

$$(d/dt)[n_2(t)A_2(t)] + B_2\dot{n}_2(t)Q_2(t) = 0$$

This is equivalent to

$$(d/dt)\{n_2(t)[A_2(t) + B_2Q_2(t)]\} = B_2n_2(t)\dot{Q}_2(t)$$

The last result and that of (5.3.225) lead us to conclude that

$$(d/dt)\{L_{\mathrm{h}}^{-1}\lambda(t)[A_2(t) + B_2Q_2(t)]\} = L_{\mathrm{h}}^{-1}\lambda(t)B_2\dot{Q}_2(t)$$

which is precisely Kron's second scheduling equation (5.3.219). This concludes the equivalence proof.

E. The Dry Season Case

The situation when the reservoir's natural water inflow function $i_i(t)$ is zero over the optimization interval is referred to as the dry season case. The solution to the optimal operational problem for variable-head hydro plants simplifies considerably in this case. Since with $i_i(t) = 0$ we have by Eq. (5.3.157) that

$$A_i(t) = -h_i(0)/G_i$$

this obviously implies that

$$\dot{A}_i(t) = i_i(t) = 0$$

Combining Eq. (5.3.151) and (5.3.177) we see that

$$\dot{A}_i(t) = -[\dot{n}_i(t)V_i(t)]/n_i(t)$$

Now, since $V_i(t) \neq 0$, we conclude that in the dry season case

$$\dot{n}_i(t) = 0$$

This indicates that

$$n_i(t) = \text{const} \tag{5.3.226}$$

Substitution of the above in the optimal expression for the volume of water discharged given by Eq. (5.3.148) does not provide us with any new information other than confirming that the basic model equation is satisfied:

$$Q_{\xi_i}(t) = -[A_i + B_iV_i(t)]/B_i$$

On the other hand, the optimal expression for the active hydro power generation given by Eq. (5.3.147) requires

$$P_{\mathrm{h}_{i_\xi}}(t) = [1 + (n_i/\lambda(t))]2B_{ii} \tag{5.3.227}$$

Here, n_i is constant as obtained in Eq. (5.3.226) and should be determined so that the active power generation constraint given by Eq. (5.3.65) is satisfied. Thus we write

$$[\{1 + [n_i/\lambda(t)]\}/2B_{ii}] + A_iq_i(t) + B_iq_i(t)Q_i(t) = 0 \tag{5.3.228}$$

Integrating Eq. (5.3.228) over the optimization interval yields

$$n_i/2B_{ii} = [K_{i_{\mathrm{hv}}} - (T_{\mathrm{f}}/2B_{ii})] \Big/ \left[\int_0^{T_{\mathrm{f}}} \lambda^{-1}(t)\, dt\right] \tag{5.3.229}$$

where

$$K_{i_{\mathrm{hv}}} = -b_i[A_i + (B_i b_i/2)] \tag{5.3.230}$$

We thus conclude that the optimal active hydro power generation of Eq. (5.3.227) is

$$P_{\mathrm{h}_{i_\xi}}(t) = (2B_{ii})^{-1} + [K_i - (T_{\mathrm{f}}/2B_{ii})][\lambda(t) \int_0^{T_{\mathrm{f}}} \lambda^{-1}(t)\, dt]^{-1} \tag{5.3.231}$$

We remark here that the optimal solution is completely specified by Eqs. (5.3.231), (5.3.146), and (5.3.69).

The optimal active hydro power generation expression (5.3.231) is precisely that of (5.2.79) with $C_i = -1$. Now the last expression was derived for the case of fixed-head hydro plants. We can conclude then that the formulation of the fixed-head case developed in Section 5.2 is perfectly capable of handling the dry season case. To confirm this let us consider the active hydro power generation model (5.3.65) in the case $A_i(t) = A_i$:

$$P_{\mathrm{h}_i}(t) + A_i q_i(t) + B_i q_i(t) Q_i(t) = 0$$

Integrating over the optimization interval we conclude that

$$\int_0^{T_{\mathrm{f}}} P_{\mathrm{h}_i}(t)\, dt = -b_i[A_i + (B_i b_i/2)] \tag{5.3.232}$$

Now, with the substitution of (5.3.230) we have

$$\int_0^{T_{\mathrm{f}}} P_{\mathrm{h}_i}(t)\, dt = K_{i_{\mathrm{hv}}} \tag{5.3.233}$$

Thus the problem is indeed reduced to the one with fixed-head plants. Note that head variations are accounted for by the expression for $K_{i_{\mathrm{hv}}}$. This is compared to K_i of Section 5.2, defined by

$$K_i = -b_i A_i$$

Note that $K_{i_{\mathrm{hv}}} < K_i$.

5.4 SUMMARY

Economic dispatch for hydro–thermal electric power systems is a dynamic optimization problem. The cases of systems with hydro plants on separate streams, with and without head variations considered, lend themselves to solution by many optimization techniques. Among these we outlined in this chapter the variational calculus method, Bellman's dynamic programming, Pontryagins' maximum principle, and the functional analytic optimization technique of the minimum norm formulation.

The resulting optimal strategy is defined by a set of dynamic nonlinear algebraic differential equations. The complexity of the set depends on the particular hydro model chosen and the operation conditions. We can conclude that simpler implementations result in the case of fixed-head operation as compared to the variable-head one. The case of a dry season when the natural reservoir inflows are nil with head variations included reduces to a problem solvable using the results for the fixed-head problem. Naturally the resulting hydro energy constraints recognize the head variation's aspects.

The practical problem of actually implementing the optimal strategy can be handled using the contraction mapping formulation leading to Picard's iteration algorithm. We would like to point out that the fixed-head case can be utilized in generating initial guess functions that may be useful in handling the variable-head problem, which is computationally more involved. The prospects for employing fast and more efficient computational algorithms are very interesting. For example, application of the Newton–Raphson method and extensions of the contraction mapping algorithms, termed the modified mappings (MCM), should prove beneficial. We will have occasion to examine these aspects further in the next chapter.

5.5 COMMENTS AND REFERENCES

Section 5.2

The classical approach to the hydro–thermal dispatch problem for fixed-head hydro plants employing variational calculus is given by Chandler *et al.* (1953). The application of the resulting coordination equations to part of Ontario Hydro's system is reported in the above-mentioned work. Further details on this approach are discussed by Kirchmayer (1959). Carey (1954) discusses the application to a three-plant system employing digital and analog computational techniques. A description of the computational experience and potential economics achieved for a utility system using the coordination equations is given by Dandeno (1961). Drake *et al.* (1962) use variational principles to obtain coordination equations for a system with multiple chains of plants neglecting head variations at the reservoir.

The dynamic programming approach described in Section 5.2.2 follows closely the treatment of Bernholtz and Graham (1960). Extension of the basic method to include more than one hydro plant is reported in a sequel paper (1962a). Further computational improvements may be found in companion papers (1962b,c, 1963). Parallel to this, a similar development of the dynamic programming approach is reported by Fukao *et al.* (1959).

The formulation of the dispatch problem in Section 5.2.3 is due to El-Hawary and Christensen (1972a). The sensitivity analysis problem has been considered by some authors. Ringlee (1965) employs a gradient method for determining the optimal schedules and uses sensitivity methods to develop an economical dispatch controller to follow small deviations from forecasted conditions. Examples of simulated on-line control are also presented. More recently, Vemuri and Hill (1976) present a sensitivity analysis algorithm based on the successive sweep method. The feasibility of using the algorithm is demonstrated for a system with two thermal and two hydro plants.

SECTION 5.3

The development of coordination equations for systems with negligible transmission losses but taking head variations into account is due to Ricard (1940). In his work reported in 1954, Cypser obtains equations based on variational calculus neglecting losses in a system with one thermal and two hydro plants. The inclusion of transmission losses into the coordination equations is reported in Glimn and Kirchmayer's work of 1958. Variational calculus, employed in the derivation of the equivalence of Ricard's, Kron's, and Cypser's equations, is also shown. Arismunander (1960) employs all necessary and sufficient conditions for optimality in variational calculus to arrive at the scheduling equations. This is reported as well in Arismunander and Noakes (1962), who also contributed an exhaustive bibliography on the subject (1963).

The maximum principle approach is reported by Dahlin (1964), Dahlin and Shen (1964, 1965, 1966) and Hano *et al.* (1966). For the dynamic programming approach to the variable-head hydro plants case we have Dahlin's work (1964) and Anderson *et al.* (1971). The minimum norm formulation reported here is given by El-Hawary and Christensen (1971, 1972b).

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CHAPTER

6

Power Systems with Coupled Hydro Plants

6.1 INTRODUCTION

The hydro–thermal electric power systems treated in Chapter 5 include hydro plants on separate streams and thus the hydraulic coupling between plants is absent. We have outlined the application of variational calculus (referred to as the classical approach), dynamic programming, the maximum principle, and the minimum norm formulation to two classes of problems. The first involves fixed-head hydro plants and in the second head variations are taken into account. The dimensionality of the problem in the latter case is increased as compared to the former, which again is of higher dimensionality than the all-thermal case of Chapter 4.

Our object in this chapter is to introduce hydraulic coupling between plants into the picture. Here we start with a relatively simplified case where the hydro plants are on one stream. We assume that the river transport delays of discharge between cascading plants are negligible. The optimal operational strategy is obtained using the maximum principle approach. Including the river transport delays into the statement of the problem is treated next. We offer here a minimum norm approach to the solution. The general case of a hydro–thermal electric power system characterized by the presence of several chains of hydro plants as well as hydraulically isolated plants is our third problem. Here the state of certain hydro plants will be influenced by the control action at other upstream plants. We formulate and solve this general problem employing the minimum norm approach. The chapter concludes with a discussion of the computational aspects of the cases considered. We offer computational experience related to two example systems.

As in Chapter 5, we choose to represent the electric network by the active power balance model. We do, however, realize that the extension of the present treatment to include more sophisticated electric network models is to our advantage. We will have an opportunity to discuss these and other aspects in Chapter 7.

6.2 A MAXIMUM PRINCIPLE APPROACH

In this section we consider a problem that serves as an illustration of the application of the maximum principle in the case of electric power systems with coupled hydro plants. It is assumed in this problem that all hydro plants are located on one stream. The delay of water transport in the river is ignored.

The forebay elevation at one plant is assumed to be equal to the tailwater elevation of the next upstream plant. In other words, the plants are strongly coupled hydraulically. The plant furthest downstream is assumed to discharge into a large body of water with constant elevation. The river inflow into the forebay of the upstream plant is a known time function. There are m thermal plants in the system.

The object of the optimizing procedure is to minimize

$$J = \int_0^{T_f} \sum_{i=1}^{m} F_i(P_{s_i}(t))\, dt \tag{6.2.1}$$

subject to satisfying the following constraints:

(1) The active power balance equation:

$$\sum_{i=1}^{m} P_{s_i}(t) + \sum_{i=m+1}^{n} P_{h_i}(t) = P_D(t) + P_L(t) \tag{6.2.2}$$

Here the transmission losses are expressed via the use of the usual quadratic transmission loss formula given by (5.3.70).

(2) Continuity of each river section requires satisfaction of the following relations:

$$\dot{x}_i(t) = \beta_{y_i}[i_i(t) - q_i(t) + q_{i-1}(t)], \qquad i = m+2, \ldots, n \tag{6.2.3}$$

$$\dot{x}_{m+1}(t) = \beta_{y_{m+1}}[i_{m+1}(t) - q_{m+1}(t)] \tag{6.2.4}$$

$$x_i(0) = x_{i0}, \qquad i = m+1, \ldots, n \tag{6.2.5}$$

Here $x_i(t)$ denotes the ith reservoir forebay elevation, $i_i(t)$ is the natural inflow to the ith reservoir, $q_i(t)$ is the rate of water discharge, and the parameters $\beta_{y_i} = S_i^{-1}$ are inverse surface areas of the reservoirs. The initial reservoir forebay elevations are assumed known x_{i0}.

(3) The volume of water discharge over the optimization interval at each hydro plant is a prespecified constant

$$\int_0^{T_f} q_i(t) = b_i, \qquad i = m+1, \ldots, n \tag{6.2.6}$$

Since the $i_i(t)$ are known time functions, we can replace the above requirement of integrated discharge by specification of the terminal values of x_i. The new specification which will be used, then, is: $\mathbf{x}(T_f) = \mathbf{d}$ where $\mathbf{d}$ is a constant vector with $n - m$ elements (terminal forebay elevation).

We proceed along lines that are similar to the treatment of Section 5.3.2. Thus let us define the auxiliary state variable x_{n+1} by

$$\dot{x}_{n+1}(t) = 1$$

with

$$x_{n+1}(0) = 0, \qquad x_{n+1}(T_f) = T_f$$

Thus

$$x_{n+1}(t) = t$$

We will use the active power balance equation (6.2.2) to eliminate the active power generation of the thermal plant P_{s_1}. We thus have the vector

$$\tilde{\mathbf{P}}_s^T = [P_{s_2}, \ldots, P_{s_m}] \tag{6.2.7}$$

treated as part of the control vector. As outlined previously, the q variables will complete the specification of the control vector.

6.2.1 The Optimal Solution

The Hamiltonian of the system is

$$H = p_0 \sum_{i=1}^{m} F_i(P_{s_i}) + p_{m+1}(t)\beta_{y_{m+1}}[i_{m+1}(t) - q_{m+1}(t)]$$
$$+ \sum_{i=m+2}^{n} p_i(t)\beta_{y_i}[i_i(t) - q_i(t) + q_{i-1}(t)] + p_{n+1}(t) \tag{6.2.8}$$

The costates satisfy

$$\dot{\mathbf{p}}(t) = -\partial H/\partial \mathbf{X} \tag{6.2.9}$$

In component form this turns out to be

$$\dot{p}_i(t) = -p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial x_i), \qquad i = m+1, \ldots, n \tag{6.2.10}$$

$$\dot{p}_{n+1}(t) = -\left[p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial x_{n+1}) + \sum_{i=m+1}^{n} p_i(t)\beta_{y_i}(\partial i_i/\partial x_{n+1})\right] \tag{6.2.11}$$

For optimal operation the following should hold true:

(1) $p_0 = \text{constant} \le 0$.
(2) H is maximized with respect to the control $\tilde{\mathbf{P}}_s \in \tilde{M}$, and $\boldsymbol{q} \in U$ at each instant of time.
(3) $H_{\max} = 0$ at $t = T_f$ and for any t if U is constant.

The maximization result with respect to $\tilde{\mathbf{P}}_s$ is similar to the development in Section 5.3.2. This is given by

$$L_{s_1}(\partial F_1/\partial P_{s_1}) = L_{s_i}(\partial F_i/\partial P_{s_i}), \qquad i = 2, \ldots, m \tag{6.2.12}$$

This is precisely (5.3.54).

The maximum condition inside U yields

$$\begin{aligned}\partial H/\partial q_i &= p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial q_i) - p_i\beta_{y_i} + p_{i+1}\beta_{y_{i+1}} \\ &= 0, \qquad i = m+1, \ldots, n-1\end{aligned} \tag{6.2.13}$$

Recall from Section 5.3.2 that Eq. (5.3.53) requires

$$\partial P_{s_1}/\partial q_i = -(L_{s_1}/L_{h_i})(\partial P_{h_i}/\partial q_i) \tag{6.2.14}$$

This last relationship combined with Eq. (6.2.13) yields

$$L_{s_1}(\partial F_1/\partial P_{s_i}) = -L_{h_i}(\partial q_i/\partial P_{h_i})\left[\frac{p_i\beta_{y_i} - p_{i+1}\beta_{y_{i+1}}}{p_0}\right], \qquad i = m+1, \ldots, n-1 \tag{6.2.15}$$

If $\partial F_1/\partial P_{s_1} > 0$ and $\partial q_i/\partial P_{h_i} > 0$ then

$$p_i\beta_{y_i} - p_{i+1}\beta_{y_{i+1}} > 0, \qquad i = m+1, \ldots, n-1$$

For the nth plant, the maximum condition yields

$$\partial H/\partial q_n = p_0(\partial F_1/\partial P_{s_1})(\partial P_{s_1}/\partial q_n) - p_n\beta_{y_n} = 0 \tag{6.2.16}$$

Thus, using (6.2.14), we conclude that

$$L_{s_1}(\partial F_1/\partial P_{s_1}) = -L_{h_n}(\partial q_n/\partial P_{h_n})(p_n\beta_{y_n}/p_0) \tag{6.2.17}$$

If $\partial F_1/\partial P_{s_1} > 0$ and $\partial q_n/\partial P_{h_n} > 0$, then $p_n > 0$.

The costate equations (6.2.10) and (6.2.11) contain the derivatives $(\partial P_{s_1}/\partial x_i)$. These can be expressed as follows:

(1) For the upstream plant we have that

$$\partial P_{s_1}/\partial x_{m+1} = -(\partial P_{h_{m+1}}/\partial x_{m+1})(L_{s_1}/L_{h_{m+1}}) \tag{6.2.18}$$

(2) For intermediate plants

$$\begin{aligned}\partial P_{s_1}/\partial x_{m+i} = &-(\partial P_{h_{m+i}}/\partial x_{m+i})(L_{s_1}/L_{h_{m+i}})\\ &-(\partial P_{h_{m+i-1}}/\partial x_{m+i})(L_{s_1}/L_{h_{m+i-1}}), \qquad i = 2, \ldots, n-(m+1)\end{aligned} \tag{6.2.19}$$

(3) For the downstream plant

$$\partial P_{s_1}/\partial x_n = -(\partial P_{h_n}/\partial x_n)(L_{s_1}/L_{h_n}) - (\partial P_{h_{n-1}}/\partial x_n)(L_{s_1}/L_{h_{n-1}}) \tag{6.2.20}$$

Substitution of Eqs. (6.2.18)–(6.2.20) in Eqs. (6.2.10)–(6.2.11) results in a more convenient form of the costate dynamic equations:

$$\dot{p}_{m+1}(t) = p_0(L_{s_1}/L_{h_{m+1}})(\partial F_1/\partial P_{s_1})(\partial P_{h_{m+1}}/\partial x_{m+1}) \tag{6.2.21}$$

$$\dot{p}_{m+i}(t) = p_0 L_{s_1}(\partial F_1/\partial P_{s_1})[L_{h_{m+i}}^{-1}(\partial P_{h_{m+i}}/\partial x_{m+i}) + L_{h_{m+i-1}}^{-1}(\partial P_{h_{m+i-1}}/\partial x_{m+i})] \tag{6.2.22}$$

$$\dot{p}_n(t) = p_0 L_{s_1}(\partial F_1/\partial P_{s_1})[L_{h_n}^{-1}(\partial P_{h_n}/\partial x_n) + L_{h_{n-1}}^{-1}(\partial P_{h_{n-1}}/\partial x_n)] \tag{6.2.23}$$

Comparison of Eqs. (6.2.21)–(6.2.23) with Eq. (5.3.58) derived for the case of isolated plants reveals that the upstream plant costate equation is identical with that for an isolated one. On the other hand, the second term within square brackets (6.2.22) and (6.2.23) is a new term which accounts for the hydraulic coupling between the $(m+i)$th and the $(m+i-1)$th plant. The optimal strategy is completely specified once a solution to the optimality conditions is obtained. These are given by Eqs. (6.2.12), (6.2.15), (6.2.17), (6.2.21), (6.2.22), and (6.2.23) together with the active power balance equation (6.2.2) and the dynamic equations (6.2.3) and (6.2.4). In addition, the boundary conditions on $x_i(t)$ are given by (6.2.5) as well as

$$\mathbf{x}(T_f) = \mathbf{d}.$$

6.2.2 Implementing the Optimal Solution

The noted similarities between the results obtained so far for our present problem with those of Section 5.3.2 for the case of isolated plants, prompt us to proceed along the same lines. We thus make the choice

$$p_0 = -1 \tag{6.2.24}$$

and define $\lambda(t)$ by

$$\lambda(t) = L_{s_1}(\partial F_1/\partial P_{s_1}) \tag{6.2.25}$$

Moreover, we define the water conversion functions $v_i(t)$ by

$$v_i(t) = \beta_{y_i} p_i(t) \tag{6.2.26}$$

These are precisely the same definitions as in (5.3.59)–(5.3.61).

Our optimality conditions are thus given by three sets of equations. The first set is that of (6.2.12) and (6.2.25):

$$L_{s_i}(\partial F_i/\partial P_{s_i}) = \lambda(t), \qquad i \in R_s \tag{6.2.27}$$

This set is precisely (5.3.62), which leads us to conclude that the thermal subsystem optimization is not changed. The second set is

$$\lambda(t) = L_{h_{m+i}}[v_{m+i}(t) - v_{m+i+1}(t)](\partial q_{m+i}/\partial P_{h_{m+i}}), \qquad i = 1, \ldots, n-m-1 \tag{6.2.28}$$

$$\lambda(t) = L_{h_n} v_n(t)(\partial q_n/\partial P_{h_n}) \tag{6.2.29}$$

The above equations are simply a rewrite of (6.2.15) and (6.2.17) using (6.2.24)–(6.2.26) inclusive.

To obtain the third set, we combine Eqs. (6.2.21), (6.2.22), and (6.2.23) with (6.2.24), (6.2.25), and (6.2.26) with the result

$$\dot{v}_{m+1}(t) = -\beta_{y_{m+1}} \lambda(t) L^{-1}_{h_{m+1}}(\partial P_{h_{m+1}}/\partial x_{m+1}) \tag{6.2.30}$$

$$\dot{v}_{m+i}(t) = -\beta_{y_{m+i}} \lambda(t)[L^{-1}_{h_{m+i}}(\partial P_{h_{m+i}}/\partial x_{m+i}) + L^{-1}_{h_{m+i-1}}(\partial P_{h_{m+i-1}}/\partial x_{m+i})], \qquad i = 2, \ldots, n-m \tag{6.2.31}$$

We use (6.2.28) and (6.2.29) in (6.2.30) and (6.2.31) to obtain the desired set of equations given by

$$\dot{v}_{m+1}(t) - [\beta_{y_{m+1}}(\partial q_{m+1}/\partial x_{m+1})][v_{m+1}(t) - v_{m+2}(t)] = 0 \tag{6.2.32}$$

$$\begin{aligned} \dot{v}_{m+i}(t) - (\beta_{y_{m+i}})\{&[(\partial q_{m+i}/\partial x_{m+i}) + (\partial q_{m+i-1}/\partial x_{m+i-1})]v_{m+i}(t) \\ &- (\partial q_{m+i-1}/\partial x_{m+i-1})v_{m+i-1}(t) - (\partial q_{m+i}/\partial x_{m+i})v_{m+i+1}(t)\} = 0, \\ &i = 2, \ldots, n-m-1 \end{aligned} \tag{6.2.33}$$

$$\dot{v}_n(t) - \beta_{y_n}\{[(\partial q_n/\partial x_n) + (\partial q_{n-1}/\partial x_{n-1})]v_n(t) - (\partial q_{n-1}/\partial x_{n-1})v_{n-1}(t)\} = 0 \tag{6.2.34}$$

In the above equations the following dependence is assumed:

$$(\partial P_i/\partial x_i)(\partial q_i/\partial P_i) = -\partial q_i/\partial x_i \tag{6.2.35}$$

$$(\partial P_{i-1}/\partial x_i)(\partial q_{i-1}/\partial P_{i-1}) = \partial q_{i-1}/\partial x_{i-1} \tag{6.2.36}$$

Note that the active power generation of a hydro plant is a function of discharge and forebay elevation difference:

$$P_{h_i} = P_{h_i}(q_i, (x_i - x_{i+1})) \tag{6.2.37}$$

Application of (6.2.37) results in (6.2.36).

The third set, given by (6.2.32), (6.2.33), and (6.2.34), is useful in visualizing the interdependence of the water conversion functions $v_i(t)$ for this system. For computer solution, we find it advantageous to introduce a new set of variables $Z_i(t)$. These are defined recursively by

$$Z_{m+i}(t) = v_{m+i}(t) - v_{m+i+1}(t), \qquad i = 1, \ldots, n-1 \tag{6.2.38}$$

$$Z_n(t) = v_n(t) \tag{6.2.39}$$

Obviously, this new function is the difference in water conversion functions between an upstream plant and the one immediately downstream. Combining (6.2.38) for $i = 1, 2$ and (6.2.32), (6.2.33) for $i = 2$ we obtain

$$\begin{aligned}\dot{Z}_{m+1}(t) = {} & (\beta_{y_{m+1}} + \beta_{y_{m+2}})(\partial q_{m+1}/\partial x_{m+1})Z_{m+1}(t) \\ & - \beta_{y_{m+2}}(\partial q_{m+2}/\partial x_{m+2})Z_{m+2}(t)\end{aligned} \tag{6.2.40}$$

The application of (6.2.33) twice together with (6.2.38) results in

$$\begin{aligned}\dot{Z}_{m+i}(t) = {} & -\beta_{y_{m+i}}(\partial q_{m+i-1}/\partial x_{m+i-1})Z_{m+i-1}(t) \\ & + (\beta_{y_{m+i}} + \beta_{y_{m+i+1}})(\partial q_{m+i}/\partial x_{m+i})Z_{m+i}(t) \\ & - \beta_{y_{m+i+1}}(\partial q_{m+i+1}/\partial x_{m+i+1}), \qquad i = 2, \ldots, n-m-1\end{aligned} \tag{6.2.41}$$

Finally, Eq. (6.2.39) combined with (6.2.34) yields

$$\dot{Z}_n(t) = \beta_{y_n}[(\partial q_n/\partial x_n)Z_n(t) - (\partial q_{n-1}/\partial x_{n-1})Z_{n-1}(t)] \tag{6.2.42}$$

The dynamic equations of (6.2.40)–(6.2.42) can be written in vector form as

$$\dot{\mathbf{Z}}(t) = \mathbf{A}_Z\mathbf{Z}(t) \tag{6.2.43}$$

The matrix $\mathbf{A}_Z$ in this system is tridiagonal. Elements of this matrix are easily identifiable by inspecting the original equations. The solution vector $\mathbf{Z}(t)$ is given by

$$\mathbf{Z}(t) = \boldsymbol{\phi}(t, 0)\mathbf{Z}(0) \tag{6.2.44}$$

Here $\boldsymbol{\phi}(t, 0)$ is the state transition matrix associated with the matrix $\mathbf{A}_Z$. Note that in this treatment the elements of $\mathbf{A}_Z$ are assumed to be constants.

We now outline the logic of a computer program for actually solving for the optimal strategy as outlined above. The flowchart of this program is shown in Fig. 6.1. We start by assuming initial guess values for the initial conditions $Z_i(0)$ or, alternatively, on $v_i(0)$. These values are best obtained from another program that assumes that both head variations and hydraulic

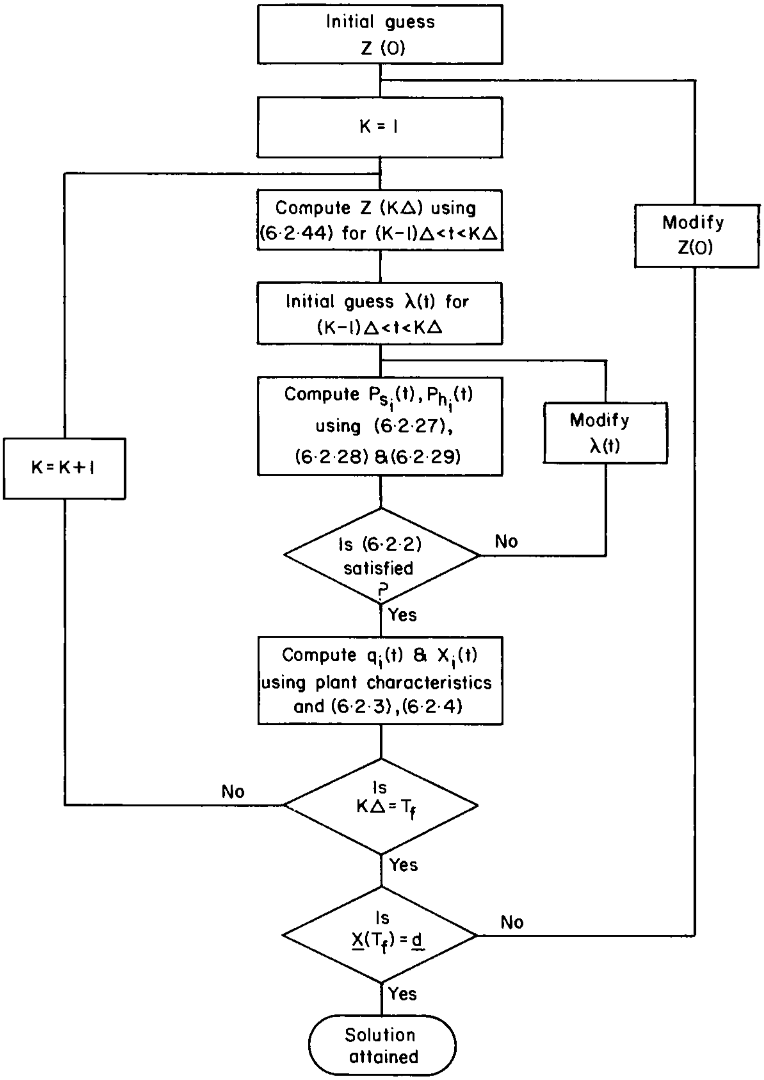

Fig. 6.1 Computer flowchart for obtaining optimal strategy.

coupling are not present. In case this is not available a realistic estimate may be used. With this guess at hand, the computation proceeds for the first time instant ($K = 1$) by evaluating $Z(\Delta)$. This is simply obtained by using (6.2.44). We now need an estimate for $\lambda(t)$ in the discrete interval considered. Again the estimate of $\lambda(t)$ is handled in the same way as $Z(0)$. Obviously, bounds on the choices of these estimates can be obtained and the treatment of Section 5.3.3 will be helpful in this regard. With the incremental expressions $(\partial F_i/\partial P_{s_i})$ and $(\partial q_i/\partial P_{h_i})$ available, the values of $P_{s_i}(t)$ and $P_{h_i}(t)$ are obtained using (6.2.27), (6.2.28), and (6.2.29). This last task proves to be simply that of solving a set of linear equations. It is necessary for the obtained active generations to meet the power demand as required by the active power balance equation (APBE) (6.2.2). If the APBE is violated, we modify the estimate of $\lambda(t)$ and recompute the active power generations for the new choice. With the APBE satisfied we now evaluate the individual hydro plant discharges from the plant characteristics. The resulting $q_i(t)$ are then used to evaluate $x_i(t)$ using (6.2.3) and (6.2.4).

The above process is repeated for all discrete time instants up to T_f. At this point a check on the satisfaction of the terminal conditions on $x(t)$, namely, $\mathbf{x}(T_f) = \mathbf{d}$, is made. If this is violated a modification of $Z(0)$ is made and the major iteration loop is reentered.

6.3 HYDRO PLANTS ON THE SAME STREAM: A MINIMUM NORM APPROACH

We consider a power system with combined thermal and hydro generations. The system is characterized by an exclusively common-flow hydro network in which all plants are on the same stream. The assumptions on the thermal subsystem and the electric network model are the same as the problems of minimum norm approach of Chapter 5. The hydro network configuration is shown in Fig. 6.2. As before, the volume of water discharge for each hydro plant is a prespecified constant amount over the optimization interval. We introduce the river transport delay to the problem.

The detailed model of the hydro subsystem is as follows:

(1) The effective hydraulic head at the ith hydro plant is equal to the difference between the forebay elevation $y_i(t)$ and the tailrace elevation $y_{T_i}(t)$; thus

$$h_i(t) = y_i(t) - y_{T_i}(t) \tag{6.3.1}$$

(2) The forebay elevation $y_i(t)$ is related to the storage $s_i(t)$ according to

$$y_i(t) = y_{i0} + \beta_{y_i} s_i(t) \tag{6.3.2}$$

This relation is true for vertical-sided reservoirs. y_{i0} and β_{y_i} are constants corresponding to the forebay geometry.

(3) The tailwater elevation varies with the rate of water discharge according to the relation

$$y_{T_i}(t) = y_{T_{i0}} + \beta_{T_i} q_i(t), \qquad i = m+1, \ldots, n \tag{6.3.3}$$

$y_{T_{i0}}$ and β_{T_i} are known constants corresponding to the tail-race geometry.

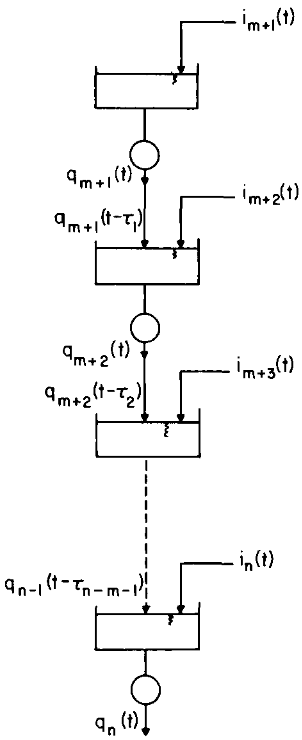

Fig. 6.2 Hydro plants on the same stream.

(4) The reservoir's dynamics are described by

$$\dot{s}_{m+i}(t) = i_{m+i}(t) + q_{m+i-1}(t - \tau_{i-1}) - q_{m+i}(t), \qquad i = 2, \ldots, n - m \tag{6.3.4}$$

$$\dot{s}_{m+1}(t) = i_{m+1}(t) - q_{m+1}(t) \tag{6.3.5}$$

Here, the time delay of water discharge between two consecutive hydro plants is assumed to be a constant τ.

(5) The ith hydro plant's active power generation $P_{\mathrm{h}_i}(t)$ is given by

$$P_{\mathrm{h}_i}(t) = q_i(t)h_i(t)/G_i, \qquad i = m + 1, \ldots, n \tag{6.3.6}$$

where the G_i's are the efficiency constants of the hydro plants.

6.3.1 Formulation

The above expression can be written as

$$P_{\mathrm{h}_{m+i}}(t) = (q_{m+i}(t)/G_{m+i})[\alpha_{y_i} + \beta_{y_{m+i}} s_{m+i}(t) - \beta_{\mathrm{T}_{m+i}} q_{m+i}(t)],$$
$$i = 1, \ldots, n - m \tag{6.3.7}$$

where

$$\alpha_{y_i} = y_{i0} - y_{T_{i0}} \tag{6.3.8}$$

Equation (6.3.7) is obtained using (6.3.1), (6.3.2), and (6.3.3).

For convenience of formulation we introduce certain auxiliary variables. These are

$$D_{m+i}(t) = s_{m+i}(0) + \int_0^t i_{m+i}(\sigma)\,d\sigma \tag{6.3.9}$$

$$Y_{m+i}(t, \tau_i) = \int_0^t q_{m+i}(\sigma - \tau_i)\,d\sigma \tag{6.3.10}$$

$$Q_{m+i}(t) = \int_0^t q_{m+i}(\sigma)\,d\sigma \tag{6.3.11}$$

The first of these is the storage value at the reservoir if both its own plant and the upstream plant are deactivated (no discharge). The second is the volume of the delayed water discharge by an upstream plant, while the last is simply the volume of water discharge at the plant up to and including the time instant t. A flow diagram showing these variables is given in Fig. 6.3. We remark here that the variable $D(t)$ is an uncontrolled variable whereas $Q(t)$ and $Y(t, \tau)$ are control variables. This suggests that a pseudocontrol variable $x(t)$ as indicated be introduced:

$$x_{m+i}(t) = Y_{m+i-1}(t, \tau_{i-1}) - Q_{m+i}(t), \qquad i = 2, \ldots, n-m \tag{6.3.12}$$

We are now in a position to write the active hydro power generation expressions given by Eq. (6.3.7) in the form

$$P_{h_{m+1}}(t) + A_{m+1}(t)q_{m+1}(t) + B_{m+1}q_{m+1}(t)Q_{m+1}(t) + C_{m+1}q_{m+1}^2(t) = 0, \tag{6.3.13}$$

$$P_{h_{m+i}}(t) + A_{m+i}(t)q_{m+i}(t) - B_{m+i}q_{m+i}(t)x_{m+i}(t) + C_{m+i}q_{m+i}^2(t) = 0,$$
$$1 = 2, \ldots, n-m \tag{6.3.14}$$

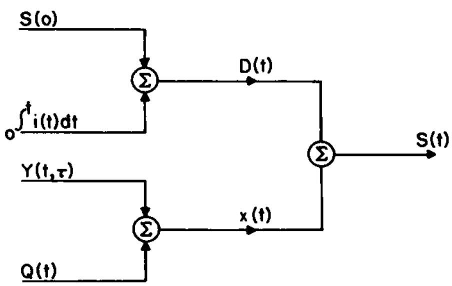

Fig. 6.3 Relations between variables introduced.

The coefficients $A(t)$, B, and C are given by

$$A_{m+i}(t) = -(1/G_{m+i})[\alpha_{m+i} + \beta_{y_{m+i}} D_{m+i}(t)] \tag{6.3.15}$$

$$B_{m+i} = \beta_{y_{m+i}}/G_{m+i} \tag{6.3.16}$$

$$C_{m+i} = \beta_{T_{m+i}}/G_{m+i} \tag{6.3.17}$$

It should be noted here that with these transformations the active hydro power constraints offer quadratic expressions in the variables q, Q, and x.

We decide to take the three variables q, Q, and x as pseudocontrols in the optimization process. This of course would necessitate the inclusion of the defining relationships as side constraints. A slight modification in each of the relationships renders some neat quadratic forms. Thus instead of taking (6.3.11) these are replaced by the equivalent forms

$$q_{m+1}(t) = \dot{Q}_{m+1}(t) \tag{6.3.18}$$

$$q_{m+i}(t)Q_{m+i}(t) = Q_{m+i}(t)\dot{Q}_{m+i}(t), \qquad i = 2, \ldots, n-m \tag{6.3.19}$$

Moreover, the constraints (6.3.12) are replaced by

$$x_{m+i}^2(t) + Q_{m+i}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - Y_{m+i}^2(t, \tau_{i-1}) = 0,$$
$$i = 2, \ldots, n-m \tag{6.3.20}$$

These further reduce to

$$x_{m+i}^2(t) + Q_{m+i}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - \psi_{m+i-1}^2(t, \tau_{i-1}) = 0, \qquad 0 \le t \le \tau_{i-1} \tag{6.3.21a}$$

$$\begin{aligned} x_{m+i}^2(t) + Q_{m+i}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - [\psi_{m+i-1}^2(\tau_{i-1}, \tau_{i-1}) \\ + 2\psi_{m+i-1}(\tau_{i-1}, \tau_{i-1})Q_{m+i-1}(t-\tau_{i-1}) + Q_{m+i-1}^2(t-\tau_{i-1})] = 0, \\ \tau_{i-1} < t \le T_f, \quad i = 2, \ldots, n-m \end{aligned} \tag{6.3.21b}$$

Note here that $\psi_{m+i-1}(t, \tau_{i-1})$ is a known initial condition function defined by

$$\psi_{m+i-1}(t, \tau_{i-1}) = \int_{-\tau_{i-1}}^{t-\tau_{i-1}} q_{m+i-1}(\sigma)\, d\sigma \tag{6.3.22}$$

Here we have used the identity

$$\begin{aligned} Y_{m+i-1}(t, \tau_{i-1}) &= \psi_{m+i-1}(t, \tau_{i-1}) \qquad (0 \le t \le \tau_{i-1}) \\ &= \psi_{m+i-1}(\tau_{i-1}, \tau_{i-1}) + Q_{m+i-1}(t - \tau_{i-1}) \\ &\qquad (\tau_{i-1} < t \le T_f, \quad i = 2, \ldots, n-m) \end{aligned} \tag{6.3.23}$$

which is a consequence of (6.3.10) and (6.3.22).

We now turn our attention to the control functions. The active power generation functions in the system offer one control subvector $\mathbf{P}(t)$ with

$$\mathbf{P}(t) = \mathrm{col}[P_{s_1}(t), \ldots, P_{s_m}(t), P_{h_{m+1}}(t), \ldots, P_{h_n}(t)] \tag{6.3.24}$$

Each of the hydro plants offers a subvector $\mathbf{W}_i(t)$; these are defined by

$$\mathbf{W}_{m+1}(t) = \mathrm{col}[q_{m+1}(t), Q_{m+1}(t)], \tag{6.3.25}$$

$$\mathbf{W}_{m+i}(t) = \mathrm{col}[q_{m+i}(t), Q_{m+i}(t), x_{m+i}(t)], \qquad i = 2, \ldots, n-m \tag{6.3.26}$$

We remark here that the components of $\mathbf{W}_{m+i}(t)$ are related as outlined before. In fact, Q and x are pseudocontrols which are introduced for convenience of the formulation. The overall system control vector $\mathbf{u}(t)$ is now given by

$$\mathbf{u}(t) = \mathrm{col}[\mathbf{P}(t), \mathbf{W}_{m+1}(t), \mathbf{W}_{m+2}(t), \ldots, \mathbf{W}_n(t)] \tag{6.3.27}$$

With the above preliminary remarks, the problem is readily formulated as a minimum norm problem. The augmented cost functional is written as

$$J(\mathbf{u}(t)) = \int_0^{T_f} \sum_{i=0}^{6} I_i(\mathbf{u}(t)) \tag{6.3.28}$$

where I_0 is the integrand of the cost function equation (5.3.68). The remaining cost functions are obtained by pairing each constraint equation with an unknown multiplier function. The pairing becomes obvious from an inspection of

$$I_1[\mathbf{u}(t)] = \lambda(t)\left[\sum_{i=1}^{n}\sum_{j=1}^{n} P_i(t)B_{ij}P_j(t) + \sum_{i=1}^{n}(B_{i0}-1)P_i(t)\right]$$

$$I_2[\mathbf{u}(t)] = n_{m+1}(t)[P_{\mathrm{h}_{m+1}}(t) + A_{m+1}(t)q_{m+1}(t) + B_{m+1}q_{m+1}(t)Q_{m+1}(t) + C_{m+1}q_{m+1}^2(t)]$$

$$I_3[\mathbf{u}(t)] = [m_{m+1}(t)q_{m+1}(t) - m_{m+1}(t)\dot{Q}_{m+1}(t)]$$

$$I_4[\mathbf{u}(t)] = \sum_{i=2}^{n-m} n_{m+i}(t)[P_{\mathrm{h}_{m+i}}(t) + A_{m+i}(t)q_{m+i}(t) - B_{m+i}x_{m+i}(t)q_{m+i}(t) + C_{m+i}q_{m+i}^2(t)]$$

$$I_5[\mathbf{u}(t)] = \sum_{i=2}^{n-m} [m_{m+i}(t)q_{m+i}(t)Q_{m+i}(t) - m_{m+i}(t)\dot{Q}_{m+i}(t)Q_{m+i}(t)]$$

$$I_6[\mathbf{u}(t)] = \sum_{i=2}^{n-m} r_{m+i}(t)[x_{m+i}^2(t) + Q_{m+i}^2(t) + 2x_{m+i}(t)Q_{m+i}(t) - Y_{m+i-1}^2(t, \tau_{i-1})]$$

We note here that our extra unknown functions are $\lambda(t)$, $n_i(t)$, $m_i(t)$, and $r_i(t)$. Terms explicitly independent of the control vector are dropped in the above expressions.

The existence of control vector function derivatives is eliminated by performing certain integrations by parts. Moreover, the delays in the argument of $Q(t)$, involved in $Y(t, \tau)$ above, are removed by changes of variables which are straightforward. This last operation necessitates the introduction of two functions:

$$\phi_{m+i}(t,\tau_{i-1}) = \begin{cases} r_{m+i}(t+\tau_{i-1}), & t \in [0, T_f - \tau_{i-1}] \\ 0, & t \in [T_f - \tau_{i-1}, T_f] \end{cases} \tag{6.3.29}$$

$$\dot{p}_{m+i}(t,\tau_{i-1}) = \begin{cases} 2\psi_{m+i-1}(\tau_{i-1},\tau_{i-1}) r_{m+i}(t+\tau_{i-1}), & t \in [0, T_f - \tau_{i-1}] \\ 0, & t \in [T_f - \tau_{i-1}, T_f] \end{cases} \tag{6.3.30}$$

The resulting cost functional is now written as the sum of two parts:

$$J[\mathbf{u}(t)] = J_L[\mathbf{u}(t)] + J_Q[\mathbf{u}(t)]$$

Here J_L is made of terms linear in the control vector functions,

$$\begin{aligned} J_L[\mathbf{u}(t)] = \int_0^{T_f} \Bigg\{ & \sum_{i=1}^{m} [\beta_i - \lambda(t)(1 - B_{i0})] P_{s_i}(t) \\ & + \sum_{i=m+1}^{n} [n_i(t) - \lambda(t)(1 - B_{i0})] P_{h_i}(t) \\ & + [n_{m+1}(t) A_{m+1}(t) + m_{m+1}(t)] q_{m+1}(t) \\ & + [\dot{m}_{m+1}(t) - \dot{p}_{m+2}(t,\tau_1)] Q_{m+1}(t) \\ & + \sum_{i=2}^{n-m-1} [n_{m+i}(t) A_{m+i}(t) + p_{m+i+1}(t,\tau_i) q_{m+i}(t) \\ & + n_n(t) A_n(t) q_n(t) \Bigg\} dt \end{aligned} \tag{6.3.31}$$

and J_Q is made of terms quadratic in the control vector functions,

$$\begin{aligned} J_Q[\mathbf{u}(t)] = \int_0^{T_f} \Bigg\{ & \sum_{i=1}^{m} \gamma_i P_{s_i}^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(t) P_i(t) B_{ij} P_j(t) \\ & + \sum_{i=1}^{n-m} C_{m+i} n_{m+i}(t) q_{m+i}^2(t) - [\phi_{m+2}(t,\tau_1) + \tfrac{1}{2} B_{m+1} \dot{n}_{m+1}(t)] Q_{m+1}^2(t) \\ & + \left(\sum_{i=2}^{n-m-1} [\tfrac{1}{2} \dot{m}_{m+i}(t) + r_{m+i}(t) - \phi_{m+i+1}(t,\tau_i)] Q_{m+i}^2(t) \right) \\ & + [\tfrac{1}{2} \dot{m}_n(t) + r_n(t)] Q_n^2(t) + \sum_{i=2}^{n-m} [r_{m+i}(t)(x_{m+i}^2(t) + 2x_{m+i}(t) Q_{m+i}(t)) \\ & + m_{m+i}(t) q_{m+i}(t) Q_{m+i}(t) - B_{m+i} n_{m+i}(t) q_{m+i}(t) x_{m+i}(t)] \Bigg\} dt \end{aligned} \tag{6.3.32}$$

This is obviously the desired linear quadratic form for an objective functional.

We will introduce the vector $\mathbf{L}(t)$ by

$$\mathbf{L}(t) = \operatorname{col}[\mathbf{L}_p(t), \mathbf{L}_{m+1}(t), \mathbf{L}_{m+2}(t), \ldots, \mathbf{L}_n(t)] \tag{6.3.33}$$

where each subvector in (6.3.33) is defined by

$$\mathbf{L}_p(t) = \mathrm{col}[l_{ps_1}(t), \ldots, l_{ps_m}(t), l_{p\mathrm{h}_{m+1}}(t), \ldots, l_{p\mathrm{h}_n}(t)],$$
$$\mathbf{L}_{m+1}(t) = \mathrm{col}[l_{(m+1)q}(t), l_{(m+1)Q}(t)],$$
$$\mathbf{L}_{m+i}(t) = \mathrm{col}[l_{(m+i)q}(t), l_{(m+i)Q}(t), l_{(m+i)x}(t)], \qquad i = 2, \ldots, n-m$$

We have introduced the functions defined by

$$l_{ps_i}(t) = \beta_i - \lambda(t)(1 - B_{i0}) \qquad i = 1, \ldots, m$$
$$l_{p\mathrm{h}_{m+i}}(t) = n_{m+i}(t) - \lambda(t)[1 - B_{(m+i)0}]$$
$$l_{(m+1)q}(t) = A_{m+1}(t)n_{m+1}(t) + m_{m+1}(t)$$
$$l_{(m+1)Q}(t) = \dot{m}_{m+1}(t) - \dot{p}_{m+2}(t, \tau_1)$$
$$l_{(m+i)q}(t) = A_{m+i}(t)n_{m+i}(t) + p_{m+i+1}(t, \tau_i), \qquad i = 2, \ldots, n-m-1$$
$$l_{(m+i)Q}(t) = 0, \qquad i = 2, \ldots, n-m$$
$$l_{(m+i)x}(t) = 0, \qquad i = 2, \ldots, n-m$$
$$l_{nq}(t) = A_n(t)n_n(t)$$

Each of the above functions is a coefficient of terms linear in the control components.

We further introduce the square symmetric matrix $\mathbf{B}(t)$ by

$$\mathbf{B}(t) = \mathrm{diag}[\mathbf{B}_p(t), \mathbf{B}_{m+1}(t), \mathbf{B}_{m+2}(t), \ldots, \mathbf{B}_n(t)] \tag{6.3.34}$$

The submatrices indicated in (6.3.34) are given by

$$\mathbf{B}_p(t) = \begin{bmatrix} \gamma_1 + B_{11}\lambda(t) & B_{12}\lambda(t) & \cdots & B_{1n}\lambda(t) \\ B_{12}\lambda(t) & \gamma_2 + B_{22}\lambda(t) & \cdots & B_{2n}\lambda(t) \\ \vdots & \vdots & & \vdots \\ & B_{(m+1)(m+1)}\lambda(t) & & \\ & \vdots & & \\ & & \cdots & B_{nn}\lambda(t) \end{bmatrix}$$

$$\mathbf{B}_{m+1}(t) = \mathrm{diag}\{C_{m+1}n_{m+1}(t), -[\phi_{m+2}(t, \tau_1) + \tfrac{1}{2}B_{m+1}\dot{n}_{m+1}(t)]\}$$
$$\mathbf{B}_{m+i}(t) = (b_{(m+i)}^{j,k}(t)), \qquad i = 2, \ldots, n-m$$
$$b_{m+i}^{11}(t) = C_{m+i}n_{m+i}(t), \qquad i = 2, \ldots, n-m$$
$$b_{m+i}^{12}(t) = \tfrac{1}{2}m_{m+i}(t), \qquad i = 2, \ldots, n-m$$
$$b_{m+i}^{13}(t) = -\tfrac{1}{2}B_{m+i}n_{m+i}(t), \qquad i = 2, \ldots, n-m$$
$$b_{m+i}^{22}(t) = \tfrac{1}{2}\dot{m}_{m+i}(t) + r_{m+i}(t) - \phi_{m+i+1}(t, \tau_i), \qquad i = 2, \ldots, n-m-1$$
$$b_{m+i}^{23}(t) = r_{m+i}(t), \qquad i = 2, \ldots, n-m$$
$$b_{m+i}^{33}(t) = r_{m+i}(t), \qquad i = 2, \ldots, n-m$$
$$b_n^{22}(t) = \dot{m}_n(t)/2 + r_n(t).$$

This reduces $J[\mathbf{u}(t)]$ in (6.3.28) to

$$J[\mathbf{u}(t)] = \int_0^{T_f} [\mathbf{L}^{\mathrm{T}}(t)\mathbf{u}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t)]\, dt$$

Let

$$\mathbf{V}^{\mathrm{T}}(t) = \mathbf{L}^{\mathrm{T}}(t)\mathbf{B}^{-1}(t) \tag{6.3.35}$$

Then our objective functional is rewritten as

$$J[\mathbf{u}(t)] = \int_0^{T_f} \{[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)] - \tfrac{1}{2}\mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\tfrac{1}{2}\mathbf{V}(t)\}\, dt$$

The last term in the integrand of the above expression does not depend explicitly on the control $\mathbf{u}(t)$. Thus one needs only to consider minimizing

$$J[\mathbf{u}(t)] = \int_0^{T_f} [\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]\, dt$$

subject to satisfying the volume of water constraints

$$\int_0^{T_f} q_{m+i}(\sigma)\, d\sigma = b_{m+i}, \qquad i = 1, \ldots, n-m$$

Define the $(n-m) \times 1$ column vector

$$\mathbf{b} = \mathrm{col}[b_{m+1}, \ldots, b_n]$$

This leads to a definition of a bounded linear transformation given by

$$\mathbf{b} = \int_0^{T_f} \mathbf{K}^{\mathrm{T}}\mathbf{u}(s)\, ds$$

Here $\mathbf{K}$ is a matrix of zero–one elements of the appropriate size.

6.3.2 The Optimal Solution

As has now become obvious, the problem is finally in the minimum norm form. Under the conditions outlined earlier we can write the problem as that of minimizing

$$J[\mathbf{u}(t)] = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|$$

subject to $\mathbf{b} = \mathbf{T}[\mathbf{u}(t)]$ for a given $\mathbf{b}$ in $R^{(n-m)}$.

The optimal solution is of the standard form and is obtained in a similar fashion to that outlined previously. To be specific, the optimizing equations are given by

$$\mathbf{P}_{\xi}(t) = -\tfrac{1}{2}\mathbf{V}_p(t) \tag{6.3.36}$$

$$q_{\xi_{m+1}}(t) = -\tfrac{1}{2}V_{(m+1)_1}(t) + \left[\frac{b_{m+1} + \int_0^{T_f} \frac{1}{2}V_{(m+1)_1}(t)\,dt}{n_{m+1}(t)\int_0^{T_f}[1/n_{m+1}(t)]\,dt}\right] \tag{6.3.37}$$

$$Q_{\xi_{m+1}}(t) = -\tfrac{1}{2}V_{(m+1)_2}(t) \tag{6.3.38}$$

$$q_{\xi_{m+1}}(t) = -\tfrac{1}{2}V_{(m+i)_1}(t) + \left[\frac{\eta_{m+i}a_{11_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)\,dt}\right], \qquad i = 2, \ldots, n-m \tag{6.3.39}$$

$$Q_{\xi_{m+i}}(t) = -\tfrac{1}{2}V_{(m+i)_2}(t) + \left[\frac{\eta_{m+i}a_{12_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)\,dt}\right], \qquad i = 2, \ldots, n-m \tag{6.3.40}$$

$$x_{\xi_{m+i}}(t) = -\tfrac{1}{2}V_{(m+i)_3}(t) + \left[\frac{\eta_{m+i}a_{13_{(m+i)}}(t)}{\int_0^{T_f} a_{11_{(m+i)}}(t)\,dt}\right] \tag{6.3.41}$$

We note here that the vector $\mathbf{V}(t)$ defined in (6.3.35) is expressed in the partitioned form

$$\begin{aligned}
\mathbf{V}^{\mathrm{T}}(t) &= [\mathbf{V}_p^{\mathrm{T}}(t), \mathbf{V}_{m+1}^{\mathrm{T}}(t), \ldots, \mathbf{V}_n^{\mathrm{T}}(t)] \\
\mathbf{V}_p^{\mathrm{T}}(t) &= [V_{p_{s_1}}(t), \ldots, V_{p_{h_{m+1}}}(t), \ldots, V_{p_{h_n}}(t)] \\
\mathbf{V}_{m+1}^{\mathrm{T}}(t) &= [V_{(m+1)_1}(t), V_{(m+1)_2}(t)] \\
\mathbf{V}_{m+i}^{\mathrm{T}}(t) &= [V_{(m+i)_1}(t), V_{(m+i)_2}, V_{(m+i)_3}], \qquad i = 2, \ldots, n-m
\end{aligned}$$

The scalars η_i are given by

$$\eta_{m+i} = b_{m+i} + \int_0^{T_f} \tfrac{1}{2}V_{(m+i)_1}(t)\,dt$$

Finally, the elements of the inverse of $\mathbf{B}_{m+i}$ are denoted by a_{ij}:

$$B_{m+i}^{-1}(t) = [a_{ij_{m+i}}(t)], \qquad i = 2, \ldots, n-m$$

Although the optimizing equations (6.3.37)–(6.3.41) are elegant in form, it is advantageous to obtain modified forms for the hydro subsystem. These modified forms will result in eliminating the pseudocontrol variables by invoking the corresponding constraints. Moreover, the interactions between discharges of hydro plants and costates will be revealed. The procedure is quite lengthy and will not be detailed here. The modified optimizing equations obtained are

$$\begin{aligned}
(d/dt)[2C_{m+i}^{\cdot}n_{m+i}(t)\dot{Q}_{\xi_{m+i}}(t) + A_{m+i}(t)n_{m+i}(t)] + B_{m+i}\dot{n}_{m+i}(t)Q_{\xi_{m+i}}(t) \\
+ g_{m+i}(t, \tau_i, \tau_{i-1}) = 0, \qquad i = 1, \ldots, n-m
\end{aligned} \tag{6.3.42}$$

with the boundary conditions

$$Q_{\xi_{m+i}}(0) = 0, \qquad i = 1, \ldots, n - m \tag{6.3.43}$$

$$Q_{\xi_{m+i}}(T_f) = b_{m+i}, \qquad i = 1, \ldots, n - m \tag{6.3.44}$$

where

$$g_{m+1}(t, \tau_1) = \begin{cases} B_{m+2}n_{m+2}(t + \tau_1)q_{\xi_{m+2}}(t + \tau_1), & t \in [0, T_f - \tau_1] \\ 0, & t \in [T_f - \tau_1, T_f] \end{cases} \tag{6.3.45}$$

$$\begin{aligned} g_{m+i}(t, \tau_i, \tau_{i-1}) &= B_{m+i+1}n_{m+i+1}(t + \tau_i)q_{\xi_{m+i+1}}(t + \tau_i) \\ &\quad - (d/dt)[B_{m+i}n_{m+i}(t)\psi_{m+i-1}(t, \tau_{i-1})] \qquad (t \in [0, \tau_{i-1})) \\ &= B_{m+i+1}n_{m+i+1}(t + \tau_i)q_{\xi_{m+i+1}}(t + \tau_i) \\ &\quad - (d/dt)[B_{m+i}n_{m+i}(t)\{\psi_{m+i-1}(\tau_{i-1}, \tau_{i-1}) \\ &\quad + Q_{\xi_{m+i-1}}(t - \tau_{i-1})\}] \qquad (t \in [\tau_{i-1}, T_f - \tau_i) \\ &= -(d/dt)[B_{m+i}n_{m+i}(t)\{\psi_{m+i-1}(\tau_{i-1}, \tau_{i-1}) \\ &\quad + Q_{m+i-1}(t - \tau_{i-1})\}] \\ &\qquad (t \in [T_f - \tau_i, T_f], \quad i = 2, \ldots, n - m - 1) \end{aligned} \tag{6.3.46}$$

$$\begin{aligned} g_n(t, \tau_{n-m-1}) &= -(d/dt)[B_n n_n(t)\psi_{n-1}(t, \tau_{n-m-1})] \qquad (t \in [0, \tau_{n-m-1}]) \\ &= -(d/dt)[B_n n_n(t)\{\psi_{n-1}(\tau_{n-m-1}, \tau_{n-m-1}) \\ &\quad + Q_{\xi_{n-1}}(t - \tau_{n-m-1})\}] \qquad (t \in [\tau_{n-m-1}, T_f]) \end{aligned} \tag{6.3.47}$$

Equations (6.3.42) are dynamic relations in the optimal volume of water discharged $Q_{\xi_i}(t)$ at a hydro plant, the associated costate $n_i(t)$, and a forcing function $g_i(t)$. These forcing functions are identified in (6.3.45), (6.3.46), and (6.3.47). Examining these expressions reveals that $g_i(t)$ depends on the corresponding hydro plants' location on the stream. For the upstream plant $m + 1$, the function $g_{m+1}(t)$ is the contribution of the $(m + 2)$th plant which is immediately downstream from the $(m + 1)$th. For all intermediate plants $(i = 2, \ldots, n - m - 1)$, the $g_i(t)$ functions contain contributions from other plants immediately up- and downstream from the one under consideration. In the case of the downstream plant (n), the contribution is due to the $(n - 1)$th, which is immediately upstream. We will examine the problem of actually implementing the optimal solution for this system in Section 6.5.

6.4 MULTICHAINS OF HYDRO PLANTS: A MINIMUM NORM APPROACH

We now turn our attention to a system with a general hydro network configuration. We retain the thermal and electric network models adopted so far. Thus the problem of minimizing the total fuel cost of the thermal subsystem while matching the active power demand and losses is considered. The active power balance equation is used and the fuel costs are approximated by the quadratic form discussed earlier.

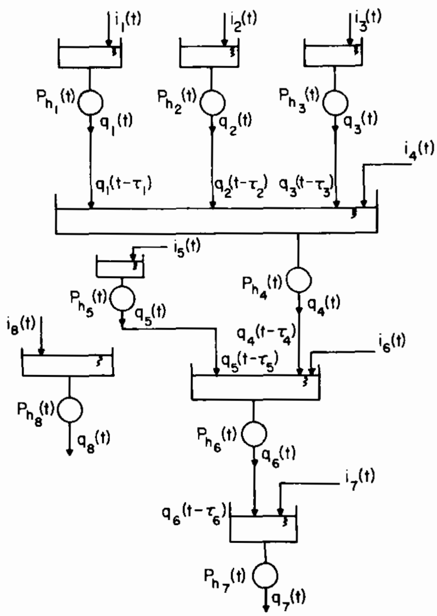

Fig. 6.4 Example layout of a hydro plant multichain.

The hydro network is assumed to have several chains of hydro plants on different streams, as well as hydraulically isolated plants. Let us denote the set of all hydro plants in the system by R_{h}. A typical layout of a hydro plant chain is shown in Fig. 6.4. We categorize the plants according to their place in the network as:

(i) *Upstream plants*: with only natural reservoir inflow, the discharge of an upstream plant will affect a reservoir further downstream. The set of all upstream plants in the system is denoted R_{hU}. Thus in the example network of Fig. 6.4, the plants 1, 2, 3, and 5 are upstream plants. The upstream plants' active power generations satisfy

$$P_{\mathrm{h}_i}(t) + A_i(t)q_i(t) + B_iq_i(t)Q_i(t) + C_iq_i^2(t) = 0, \qquad i \in R_{\mathrm{hU}} \tag{6.4.1}$$

Note that $A_i(t)$, B_i, and C_i are the same as defined in Section 6.3. In fact this expression is identical to Eq. (6.3.13).

(ii) *Intermediate plants*: with reservoir receiving both natural inflow and a controlled inflow emanating from either an upstream plant and/or another intermediate plant. The controlled discharge of this type of plant feeds into either another intermediate plant's reservoir or a downstream reservoir. We denote by R_{hI}, the set of all intermediate plants. In our example, plants 4 and 6 are in this category.

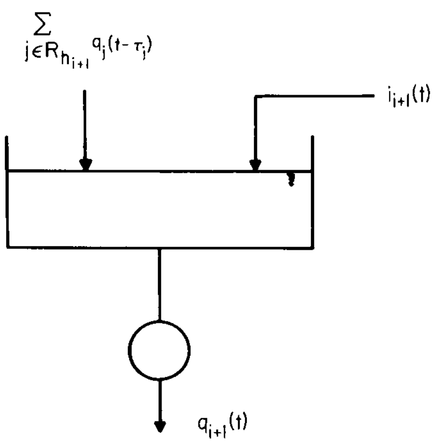

Fig. 6.5 An intermediate reservoir.

For the intermediate plants the situation is illustrated in Fig. 6.5. Let R_{h_i} denote plants upstream from the ith. The flow from each plant has a transport delay of τ_j ($j \in R_{h_i}$) to the ith plant. Then the ith reservoir's dynamics are expressed by

$$\dot{s}_i(t) = i_i(t) + \sum_{j \in R_{h_i}} q_j(t - \tau_j) - q_i(t)$$

Integrating, one obtains

$$s_i(t) = D_i(t) + \sum_{j \in R_{h_i}} Y_j(t, \tau_j) - Q_i(t) \tag{6.4.2}$$

where we introduce the following variables identical to the ones introduced in Section 6.3:

$$D_i(t) = s_i(0) + \int_0^t i_i(\sigma)\, d\sigma \tag{6.4.3}$$

$$Q_i(t) = \int_0^t q_i(\sigma)\, d\sigma \tag{6.4.4}$$

$$Y_j(t, \tau_j) = \int_0^t q_j(\sigma - \tau_j)\, d\sigma, \qquad j \in R_{h_i} \tag{6.4.5}$$

Let us again introduce

$$\psi_j(t, \tau_j) = \int_{-\tau_j}^{t-\tau_j} q_j(s)\, ds, \qquad t \le \tau_j \tag{6.4.6}$$

which is a known function of time from the previous history of the system. Then Eq. (6.4.5) reduces to

$$Y_j(t, \tau_j) = \begin{cases} \psi_j(t, \tau_j), & t \le \tau_j \\ \psi_j(\tau_j, \tau_j) + Q_j(t - \tau_j), & t > \tau_j, \quad j \in R_{\mathrm{h}_i} \end{cases} \tag{6.4.7}$$

The hydro power generated at the ith plant is given by

$$P_{\mathrm{h}_i}(t) = [q_i(t)/G_i][\alpha_i + \beta_{y_i} s_i(t) - \beta_{\mathrm{T}_i} q_i(t)]$$

Substituting for $s_i(t)$ we obtain

$$P_{\mathrm{h}_i}(t) + A_i(t)q_i(t) - B_i q_i(t) \sum_{j \in R_{\mathrm{h}_i}} Y_j(t, \tau_j) + C_i q_i^2(t) + B_i q_i(t) Q_i(t) = 0, \qquad i \in R_{\mathrm{hI}} \tag{6.4.8}$$

We note here the similarity of the above relation to that derived in Section 6.3, namely, Eq. (6.3.14). The difference is that the third term includes the effects of all upstream plants from the ith plant. We will retain the variables $Y_j(t, \tau)$ in our formulation, although one may introduce variables x_j in a fashion similar to the approach of Section 6.3.

(iii) *Downstream plants*: these feature a reservoir that has both natural and controlled inflow from an upstream or an intermediate plant. The controlled discharge of a downstream plant does not affect any other plant's reservoir. The set R_{hD} contains all downstream plants. In the example network plant 7 is the only downstream plant. The reservoir dynamics are described by the same relations derived for the intermediate plants.

(iv) *Isolated plants*: the reservoirs of plants in this category receive only natural inflow. The controlled discharge of this plant does not affect any other hydro plant. This set is denoted by R_{hIS}. Plant 8 in our example system falls in this category. The reservoir dynamics for this type are described by the same relation given for the upstream plants.

The hydro plants treated here are those which have large storage reservoirs. Other plants that have modest storage facilities termed as "run-of-the river" may be present in a hydrosystem. The operation of this type of plant is almost completely fixed by that of the plants with storage facilities. It is therefore sufficient to treat only large storage reservoir cases as they offer independent decision variables.

6.4.1 Formulation

The formulation of the problem proceeds with the objective functional augmented by including the nonlinear constraints via the use of the costate-like functions as outlined previously. This can be written as

$$J(\mathbf{u}(t)) = \int_0^{T_f} I(\mathbf{u}(t))\,dt \tag{6.4.9}$$

Here we have

$$I(\mathbf{u}(t)) = I_{\mathrm{TE}}(\mathbf{u}(t)) + I_{\mathrm{h}}(\mathbf{u}(t)) \tag{6.4.10}$$

The first part of the integrand includes thermal–electric related terms, the second the hydro-related terms. The expression for $I_{\mathrm{TE}}(\mathbf{u}(t))$ is given by

$$\begin{aligned} I_{\mathrm{TE}}(\mathbf{u}(t)) = \sum_{i\in R_s} \beta_i P_{s_i}(t) + \sum_{i\in R_s} \gamma_i P_{s_i}^2(t) + \lambda(t)\Bigg[\Bigg[\sum\sum_{i,j\in R_G} P_i(t)B_{ij}P_j(t) \\ - \sum_{i\in R_G}(1 - B_{i0})P_i(t)\Bigg] + \sum_{i\in R_h} n_i(t)P_{h_i}(t)\Bigg] \end{aligned} \tag{6.4.11}$$

The notation and costate pairing are the same as the treatment in Section 6.3. The set of all thermal plants is denoted R_s, and R_G denotes all available generation in the system.

The second part of the integrand, namely, $I_{\mathrm{h}}(\mathbf{u}(t))$, includes the hydro-active power generation constraints [excluding the terms in $P_{h_i}(t)$] and the following two relationships:

$$q_i(t) = \dot{Q}_i(t) \tag{6.4.12}$$

$$Y_j^2(t, \tau_j) = \begin{cases} \psi_j^2(t, \tau_j), & t \le \tau_j \\ \psi_j^2(\tau_j, \tau_j) + 2\psi_j(\tau_j, \tau_j)Q_j(t - \tau_j) + Q_j^2(t - \tau_j), & t \ge \tau_j \end{cases} \tag{6.4.13}$$

We find it more convenient to use the integral of $I_{\mathrm{h}}(\mathbf{u}(t))$ for reason of the discontinuity in the above expression. Thus we have

$$\begin{aligned} J_{\mathrm{h}}(\mathbf{u}(t)) = \int_0^{T_f} \Bigg\{ \sum_{i\in R_h} [n_i(t)(A_i(t)q_i(t) + C_i q_i^2(t) \\ + B_i Q_i(t)\dot{Q}_i(t)) + m_i(t)q_i(t) + \dot{m}_i(t)Q_i(t)] \\ - \sum_{i\in R_{hID}} \Bigg[B_i n_i(t) q_i(t) \sum_{j\in R_{h_i}} Y_j(t, \tau_j) \Bigg] \\ + \sum_{i\in R_{hID}} \sum_{j\in R_{h_i}} r_j(t) Y_j^2(t, \tau_j) \Bigg\} dt \\ - \sum_{i\in R_{hID}} \sum_{j\in R_{h_i}} \int_{\tau_j}^{T_f} r_j(t)[Q_j^2(t - \tau_j) + 2\psi_j(\tau_j, \tau_j)Q_j(t - \tau_j)]\,dt \end{aligned} \tag{6.4.14}$$

Here terms explicitly independent of the decision variables $q_i(t)$, $Q_i(t)$, and $Y(t, \tau)$ were neglected. The set R_{hID} is the union of the intermediate and downstream plants.

The delays in the arguments of the decision variables $Q_j(t)$ can be eliminated by a simple change of variables. Moreover, a further simplification in the form of the previous expression can be effected by using the substitutions

$$\dot{p}_i(t, \tau_i) = \begin{cases} 2\psi_i(\tau_i, \tau_i) r_i(t + \tau_i), & 0 \le t \le T_f - \tau_i \\ 0, & T_f - \tau_i < t \le T_f \end{cases} \tag{6.4.15}$$

$$\theta_i(t, \tau_i) = \begin{cases} r_i(t + \tau_i), & 0 \le t \le T_f - \tau_i \\ 0, & T_f - \tau_i < t \le T_f \end{cases} \tag{6.4.16}$$

With these preliminaries at hand, we conclude that the problem is ready for casting as a minimum norm problem.

Let us define the control vector function $\mathbf{u}(t)$ by

$$\mathbf{u}(t) = \operatorname{col}[\mathbf{P}(t), \mathbf{W}(t)] \tag{6.4.17}$$

The subvector $\mathbf{P}(t)$ includes as before all the active power generations in the system. The subvector $\mathbf{W}(t)$ includes further subvectors $\mathbf{W}_i(t)$; thus

$$\mathbf{W}(t) = \operatorname{col}[\mathbf{W}_i(t) : i \in R_\text{h}] \tag{6.4.18}$$

Each of the hydro subvectors has a dimension and a definition that depends on the category of the plant. For the upstream and isolated plants only the variables $q_i(t)$ and $Q_i(t)$ are involved; thus we have

$$\mathbf{W}_i(t) = \operatorname{col}[Q_i(t), q_i(t)], \qquad i \in R_{\text{hU}} \cup R_{\text{hIS}} \tag{6.4.19}$$

On the other hand, the subvector $\mathbf{W}_i(t)$ for each intermediate and downstream plant must include the corresponding decision variables $Y_j(t, \tau_j)$ in addition to $q_i(t)$ and $Q_i(t)$. We thus have

$$\mathbf{W}_i(t) = \operatorname{col}[Q_i(t), q_i(t), \mathbf{Y}_{i\text{w}}(t)] \tag{6.4.20}$$

with

$$\mathbf{Y}_{i\text{w}}(t) = \operatorname{col}[Y_j(t, \tau_j) : j \in R_{\text{h}_i}], \qquad i \in R_{\text{hID}} \tag{6.4.21}$$

The auxiliary vector $\mathbf{L}(t)$ follows in a compatible way; thus

$$\mathbf{L}(t) = \operatorname{col}[\mathbf{L}_\text{p}(t), \mathbf{L}_\text{w}(t)] \tag{6.4.22}$$

Here again $\mathbf{L}_p(t)$ includes the coefficients of terms linear in the components of $\mathbf{P}(t)$, that is,

$$\mathbf{L}_p(t) = \text{col}[\mathbf{L}_{ps}(t), \mathbf{L}_{ph}(t)] \tag{6.4.23}$$

$$\mathbf{L}_{ps}(t) = \text{col}[l_{ps_i}(t) : i \in R_s] \tag{6.4.24}$$

$$l_{ps_i}(t) = \beta_i - \lambda(t)(1 - B_{i0}) \tag{6.4.25}$$

$$\mathbf{L}_{ph}(t) = \text{col}[l_{ph_i}(t) : i \in R_h] \tag{6.4.26}$$

$$l_{ph_i}(t) = n_i(t) - \lambda(t)(1 - B_{i0}) \tag{6.4.27}$$

The vector $\mathbf{L}_w(t)$ in turn is expressed as

$$\mathbf{L}_w(t) = \text{col}[\mathbf{L}_{w_i}(t) : i \in R_h] \tag{6.4.28}$$

with

$$\mathbf{L}_{wi}(t) = \text{col}[\dot{m}_i(t), \mathbf{l}_{wi}(t)] \tag{6.4.29}$$

Again the subvectors $\mathbf{l}_{wi}(t)$ are category dependent and turn out to be:

(i) For upstream plants there is only one component, given by

$$\mathbf{l}_{wi}(t) = [m_i(t) + n_i(t)A_i(t) + p_i(t, \tau_i)], \qquad i \in R_{hU} \tag{6.4.30}$$

(ii) For isolated plants there is again only one component; however, the last term in the above expression is not included. That is,

$$\mathbf{l}_{wi}(t) = [m_i(t) + n_i(t)A_i(t)], \qquad i \in R_{hIS} \tag{6.4.31}$$

(iii) For downstream plants there are more than one component depending on the corresponding number of feed-in, intermediate, or upstream plants. The extra components turn out to be zeros in value:

$$\mathbf{l}_{wi}(t) = \text{col}[m_i(t) + n_i(t)A_i(t), \mathbf{0}_i], \qquad i \in R_{hD} \tag{6.4.32}$$

(iv) For intermediate plants, a vector similar to the above is obtained; however the $p_i(t, \tau_i)$ term appears. Thus

$$\mathbf{l}_{wi}(t) = \text{col}[m_i(t) + n_i(t)A_i(t) + p_i(t, \tau_i), \mathbf{0}_i], \qquad i \in R_{hI} \tag{6.4.33}$$

The square symmetric matrix $\mathbf{B}(t)$ is obtained in the partitioned form

$$\mathbf{B}(t) = \text{diag}[\mathbf{B}_p(t), \mathbf{B}_w(t)] \tag{6.4.34}$$

We remark here that the **P** and **W** terms are decoupled, a fact which will be useful in the implementation stages, as well as for extending the methodology in the following chapter.

The matrix $\mathbf{B}_{\mathrm{p}}(t)$ is of appropriate dimension and is given by

$$\mathbf{B}_{\mathrm{p}}(t) = \left[\begin{array}{c|c} \mathbf{B}_{\mathrm{ss}}(t) & \mathbf{B}_{\mathrm{sh}}(t) \\ \hline \mathbf{B}_{\mathrm{sh}}(t) & \mathbf{B}_{\mathrm{hh}}(t) \end{array}\right] \tag{6.4.35}$$

The elements of $\mathbf{B}_{\mathrm{sh}}$ and $\mathbf{B}_{\mathrm{hh}}$ are given by

$$b_{ij}(t) = B_{ij}\lambda(t) \tag{6.4.36}$$

For the thermal submatrix $\mathbf{B}_{\mathrm{ss}}(t)$, we have

$$b_{ii}(t) = \gamma_i + B_{ii}\lambda(t), \qquad b_{ij}(t) = B_{ij}\lambda(t)$$

The matrix $\mathbf{B}_{\mathrm{w}}(t)$ is again block diagonal in submatrices $\mathbf{B}_{\mathrm{w}i}(t)$, each of which is category dependent as follows:

(i) For an upstream plant, the matrix $\mathbf{B}_{\mathrm{w}_i}(t)$ is a 2-dimensional diagonal matrix given by

$$\mathbf{B}_{\mathrm{w}i}(t) = \mathrm{diag}[-(\tfrac{1}{2}B_i\dot{n}_i(t) + \theta_i(t,\tau_i)), C_i n_i(t)], \qquad i \in R_{\mathrm{hU}} \tag{6.4.37}$$

(ii) For an isolated plant, a matrix similar to that of the upstream matrix is obtained. The term $\theta_i(t,\tau_i)$ does not exist for this type of plant.

$$\mathbf{B}_{\mathrm{w}_i}(t) = \mathrm{diag}[-\tfrac{1}{2}B_i\dot{n}_i(t), C_i n_i(t)] \tag{6.4.38}$$

(iii) For a downstream plant, the matrix $\mathbf{B}_{\mathrm{w}_i}(t)$ has a dimension related to the number of feed-in plants.

$$\mathbf{B}_{\mathrm{w}_i}(t) = \mathrm{diag}[-\tfrac{1}{2}B_i\dot{n}_i(t), \mathbf{B}_{\mathrm{wf}_i}(t)] \tag{6.4.39}$$

$$\mathbf{B}_{\mathrm{wf}_i}(t) = \left[\begin{array}{c|c} C_i n_i(t) & \mathbf{B}^{\mathrm{T}}_{\mathrm{w}_{n_i}}(t) \\ \hline \mathbf{B}_{\mathrm{w}_{n_i}}(t) & \mathbf{B}_{\mathrm{w}_{r_i}}(t) \end{array}\right] \tag{6.4.40}$$

The submatrix $\mathbf{B}_{\mathrm{w}_{r_i}}(t)$ is diagonal of dimension equal to the number of feed-in plants:

$$\mathbf{B}_{\mathrm{w}_{r_i}}(t) = \mathrm{diag}[r_j(t) : j \in R_{\mathrm{h}_i}] \tag{6.4.41}$$

The vector $\mathbf{B}_{\mathrm{w}_{n_i}}(t)$ is of compatible dimension and has components equal in value:

$$\mathbf{B}_{\mathrm{w}_{n_i}}(t) = \mathrm{col}[-\tfrac{1}{2}B_i n_i(t), \ldots], \qquad i \in R_{\mathrm{hD}} \tag{6.4.42}$$

(iv) For an intermediate plant, a similar matrix is obtained. This is given by

$$\mathbf{B}_{\mathrm{w}_i}(t) = \mathrm{diag}[-\tfrac{1}{2}[B_i\dot{n}_i(t) + \theta_i(t,\tau_i)], \mathbf{B}_{\mathrm{wf}_i}(t)], \qquad i \in R_{\mathrm{hI}} \tag{6.4.43}$$

We remark here that the term $\theta_i(t, \tau_i)$ appears in this case as opposed to the downstream case. The structure of matrix $\mathbf{B}_{\mathrm{w}_{\mathrm{f}_i}}(t)$ is identical to the one described above.

The augmented cost functional thus reduces to

$$J(\mathbf{u}) = \int_0^{T_{\mathrm{f}}} [\mathbf{L}^{\mathrm{T}}\mathbf{u}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t)]\, dt \tag{6.4.44}$$

Let

$$\mathbf{V}^{\mathrm{T}}(t) = \mathbf{L}^{\mathrm{T}}(t)\mathbf{B}^{-1}(t) \tag{6.4.45}$$

Then J reduces to

$$J(\mathbf{u}) = \int_0^{T_{\mathrm{f}}} \{[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)] - \tfrac{1}{2}\mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\tfrac{1}{2}\mathbf{V}(t)\}\, dt \tag{6.4.46}$$

The last term in the integrand does not depend explicitly on $\mathbf{u}(t)$, so that it is only necessary to consider minimizing

$$J(\mathbf{u}) = \int_0^{T_{\mathrm{f}}} \{[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^{\mathrm{T}}\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]\}\, dt \tag{6.4.47}$$

subject to satisfying the linear volume of water constraints of the form

$$\mathbf{b} = \int_0^{T_{\mathrm{f}}} \mathbf{K}^{\mathrm{T}}\mathbf{u}(s)\, ds \tag{6.4.48}$$

The control vector $\mathbf{u}(t)$ is considered an element of the Hilbert space $L_{2,\mathbf{B}}^{(D)}[0, T_{\mathrm{f}}]$ of the D vector-valued square integrable functions defined on $[0, T_{\mathrm{f}}]$ endowed with the inner product definition

$$\langle \mathbf{V}(t), \mathbf{u}(t)\rangle = \int_0^{T_{\mathrm{f}}} \mathbf{V}^{\mathrm{T}}(t)\mathbf{B}(t)\mathbf{u}(t)\, dt \tag{6.4.49}$$

for every $\mathbf{V}(t)$ and $\mathbf{u}(t)$ in $L_{2,\mathbf{B}}^{(D)}[0, T_{\mathrm{f}}]$, provided that $\mathbf{B}(t)$ is positive definite.

The given vector $\mathbf{b}$ is considered an element of the real space R^H with the Euclidean inner product definition

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \mathbf{X}^{\mathrm{T}}\mathbf{Y} \tag{6.4.50}$$

for every $\mathbf{X}$ and $\mathbf{Y}$ in R.

Equation (6.4.48) defines a bounded linear transformation $T: L_{2,\mathbf{B}}^{(D)}[0, T_{\mathrm{f}}] \to R^H$. This can be expressed as

$$\mathbf{b} = \mathbf{T}[\mathbf{u}(t)] \tag{6.4.51}$$

and the cost functional given by (6.4.47) reduces to

$$J[\mathbf{u}(t)] = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|^2$$

Finally, it is necessary only to minimize

$$J[\mathbf{u}(t)] = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\| \tag{6.4.52}$$

subject to $\mathbf{b} = \mathbf{T}[\mathbf{u}(t)]$ for a given $\mathbf{b}$ in R^H.

The dimension D of the Hilbert space is related to the number of plants in the system as follows:

$$D = 3H + S + d + I$$

where H, S, d, and I are the number of hydro, thermal, downstream, and intermediate plants, respectively.

6.4.2 The Optimal Solution

The optimal solution to the problem formulated above using the results of Chapter 3 is

$$\mathbf{u}_\xi(t) = \mathbf{T}^\dagger[\mathbf{b} + T\tfrac{1}{2}\mathbf{V}(t)] - \tfrac{1}{2}\mathbf{V}(t) \tag{6.4.53}$$

where $\mathbf{T}^\dagger$ is obtained as before. Concerning the optimal active power generation vector, this turns out to be

$$\mathbf{P}_\xi(t) = -[\mathbf{B}_p^{-1}(t)\tfrac{1}{2}\mathbf{L}_p(t)] \tag{6.4.54}$$

The optimal volume of water discharge at either an upstream or an intermediate plant obtained is

$$Q_{\xi_i}(t) = \dot{m}_i(t)/[B_i\dot{n}_i(t) + 2\theta_i(t,\tau_i)], \qquad i \in R_{\mathrm{U}} \cup R_{\mathrm{I}} \tag{6.4.55}$$

For downstream and isolated plants this turns out to be

$$Q_{\xi_i}(t) = \dot{m}_i(t)/[B_i\dot{n}_i(t)], \qquad i \in R_{\mathrm{D}} \cup R_{\mathrm{IS}} \tag{6.4.56}$$

The optimal rate of water discharge at any hydro plant is given by

$$q_{\xi_i}(t) = \left\{\left[b_i + \int_0^{T_f} \tilde{q}_i(t)\,dt\right]\Big/ d_i(t)\right\} - \tilde{q}_i(t) \tag{6.4.57}$$

Here the $\tilde{q}_i(t)$ for each plant category are

$$\tilde{q}_i(t) = [m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)]/[2C_i n_i(t)], \qquad i \in R_{\mathrm{hU}} \tag{6.4.58}$$

$$\tilde{q}_i(t) = [\tfrac{1}{2}b_{\mathrm{w}_i}^{(11)}(t)][m_i(t) + n_i(t)A_i(t) + p_i(t,\tau_i)], \qquad i \in R_{\mathrm{hI}} \tag{6.4.59}$$

$$\tilde{q}_i(t) = [\tfrac{1}{2}b_{\mathrm{w}_i}^{(11)}(t)][m_i(t) + n_i(t)A_i(t)], \qquad i \in R_{\mathrm{hD}} \tag{6.4.60}$$

$$\tilde{q}_i(t) = [m_i(t) + n_i(t)A_i(t)]/[2C_i n_i(t)], \qquad i \in R_{\mathrm{hIS}} \tag{6.4.61}$$

Here each $b_{w_i}^{(11)}(t)$ is the first row and first column element of the inverse of the matrix $\mathbf{B}_{w_{f_i}}(t)$:

$$b_{w_i}^{(11)}(t) = \left\{ C_i n_i(t) - [\tfrac{1}{2} B_i n_i(t)]^2 \sum_{j \in R_{h_i}} [r_j(t)]^{-1} \right\}^{-1}, \qquad i \in R_{hI} \cup R_{hD} \quad (6.4.62)$$

The functions $d_i(t)$ are

$$d_i(t) = n_i(t) \int_0^{T_f} [n_i(t)]^{-1}\, dt, \qquad i \in R_{hU} \cup R_{hIS} \quad (6.4.63)$$

$$d_i(t) = [b_{w_i}^{(11)}(t)]^{-1} \left[\int_0^{T_f} b_{w_i}^{(11)}(t)\, dt \right], \qquad i \in R_{hI} \cup R_{hD} \quad (6.4.64)$$

The optimal $Y_j(t, \tau_j)$ are given by

$$Y_{\xi_j}(t, \tau_j) = B_i n_i(t) q_{\xi_i}(t)/[2 r_j(t)], \qquad j \in R_{h_i} \quad (6.4.65)$$

The optimal equations obtained thus far together with the constraint equations completely specify the optimal operational strategy for the system under consideration. We remark here that the dimension of the problem has been increased by the introduction of the pseudocontrols $q(t)$ and $Y(t, \tau)$ and the associated costate-like functions $m_i(t)$ and $r_i(t)$ for the hydro network. This of course is necessary for a valid minimum norm formulation. We offer here an alternative form for the optimal equations for the hydro network. This form is obtained directly via elimination of the pseudocontrols by invoking the corresponding constraints defined by Eqs. (6.4.12) and (6.4.13). The result is the general equation

$$(d/dt)[2 C_i n_i(t) \dot{Q}_{\xi_i}(t) + A_i(t) n_i(t)] + B_i \dot{n}_i(t) Q_{\xi_i}(t) + g_i(t) = 0 \quad (6.4.66)$$

subject to the boundary conditions

$$Q_i(0) = 0, \qquad Q_i(T_f) = b_i \quad (6.4.67)$$

The functions $g_i(t)$ are given by

$$g_i(t) = 0, \qquad i \in R_{hIS} \quad (6.4.68)$$

$$g_i(t) = \begin{cases} B_k n_k(t + \tau_i) \dot{Q}_{\xi}(t + \tau_i), & 0 \le t \le T_f - \tau_i, \\ 0, & T_f - \tau_i \le t \le T_f, \end{cases} \qquad i \in R_{hU}, i \in R_{h_k} \quad (6.4.69)$$

$$g_i(t) = B_k n_k(t + \tau_i) \dot{Q}_{\xi_k}(t + \tau_i) - (d/dt)\left[B_i n_i(t) \sum_{j \in R_{h_i}} Y_j(t, \tau_j) \right], \qquad i \in R_{hI}, i \in R_{h_k} \quad (6.4.70)$$

$$g_i(t) = -(d/dt)[B_i n_i(t) Y_j(t, \tau_j)], \qquad i \in R_{hD} \quad (6.4.71)$$

The optimality condition given by Eq. (6.4.66) is the same as that of Eq. (6.3.42). The forcing functions $g_i(t)$ are category-dependent as in Section 6.3. For isolated plants the forcing functions are null, while for the upstream plants the contribution of the plant downstream is obvious by inspecting Eq. (6.4.69). For the intermediate plants, Eq. (6.4.70) reveals the effects of plants hydraulically coupled to plants in the set R_{hI}. The downstream plant optimality equation is influenced by the plants immediately upstream as indicated by (6.4.71). In the following section we focus our attention on the treatment of the practical implementation aspects of the problem.

6.5 COMPUTATIONAL ASPECTS

The variety and scope of common-flow and multichain hydro systems offer quite an interesting challenge. The characteristics of the particular system considered influence the success of a computational scheme for actually implementing an optimal strategy. We have outlined the features of the modified contraction mapping algorithms in Chapter 3. In this section we apply this technique to example systems. The first is a comparatively low-dimensional system that is fashioned according to the formulation of Section 6.3. The second example system follows the general form of Section 6.4.

In order for us to be able to formulate a contraction mapping, it is necessary to rearrange the nonlinear equations presented to us by the hydro network optimal equations. We offer here alternative forms for these that readily lend themselves to a contraction mapping formulation.

Consider the optimal hydro equation (6.4.66) rewritten as

$$\begin{aligned}\ddot{Q}_{\xi_i}(t) &+ [\dot{n}_i(t)/n_i(t)]\dot{Q}_{\xi_i}(t) + [B_i/(2C_i)][\dot{n}_i(t)/n_i(t)]Q_{\xi_i}(t) \\ &+ \{\tilde{g}_i(t,\tau_i,\tau_{i-1}) + \dot{A}_i(t) + A_i(t)[\dot{n}_i(t)/n_i(t)]\}/2C_i = 0, \qquad i \in R_{\text{h}} \quad (6.5.1)\end{aligned}$$

where $\tilde{g}_i(t,\tau_i,\tau_{i-1}) = g_i(t,\tau_i,\tau_{i-1})/n_i(t)$. Let us introduce the variables

$$\rho_i(t) = \dot{n}_i(t)/n_i(t), \qquad i \in R_{\text{h}} \quad (6.5.2)$$

and the parameters ε_i given by

$$\varepsilon_i = B_i/(2C_i), \qquad i \in R_{\text{h}} \quad (6.5.3)$$

Moreover, we define

$$G_i(t,\tau_i,\tau_{i-1},n_i(t),\dot{n}_i(t)) = [\tilde{g}_i(t,\tau_i,\tau_{i-1}) + \dot{A}_i(t) + A_i(t)\rho_i(t)]/(2C_i) \quad (6.5.4)$$

Substitution of (6.5.2)–(6.5.4) into (6.5.1) yields

$$\ddot{Q}_{\xi_i}(t) + \rho_i(t)\dot{Q}_{\xi_i}(t) + \varepsilon_i\rho_i(t)Q_{\xi_i}(t) + G_i = 0, \qquad i \in R_\mathrm{h} \tag{6.5.5}$$

Furthermore, let

$$\mathbf{Z}_i(t) = \mathrm{col}[Z_i^{(1)}(t), Z_i^{(2)}(t)] \tag{6.5.6}$$

where

$$Z_i^{(1)}(t) = Q_{\xi_i}(t) \qquad \text{and} \qquad Z_i^{(2)}(t) = \dot{Q}_{\xi_i}(t) \tag{6.5.7}$$

Then Eq. (6.5.5) reduces to

$$\dot{\mathbf{Z}}_i(t) = \mathbf{R}_i(t)\mathbf{Z}_i(t) + \mathbf{F}_i(t), \qquad i \in R_\mathrm{h} \tag{6.5.8}$$

with

$$\mathbf{R}_i(t) = \begin{bmatrix} 0 & 1 \\ -\varepsilon_i\rho_i(t) & -\rho_i(t) \end{bmatrix}, \qquad i \in R_\mathrm{h} \tag{6.5.9}$$

$$\mathbf{F}_i(t) = \mathrm{col}[0, -G_i], \qquad i \in R_\mathrm{h} \tag{6.5.10}$$

Here the boundary conditions $Q_{\xi_i}(0) = 0$ and $Q_{\xi_i}(T_\mathrm{f}) = b_i$ can be written as

$$\mathbf{M}\mathbf{Z}_i(0) + \mathbf{N}\mathbf{Z}_i(T_\mathrm{f}) = \mathbf{C}_i', \qquad i \in R_\mathrm{h} \tag{6.5.11}$$

with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{C}_i' = \mathrm{col}[0, b], \qquad i \in R_\mathrm{h} \tag{6.5.12}$$

Equation (6.5.8) with the boundary condition (6.5.11) is next transformed into an integral form given by

$$\begin{aligned} Z_i^{(1)}(t) &= \frac{b_i t}{T_\mathrm{f}} + \int_0^t \frac{s[T_\mathrm{f} - t]}{T_\mathrm{f}} f_i(s)\, ds + \int_t^{T_\mathrm{f}} \frac{t[T_\mathrm{f} - s]}{T_\mathrm{f}} f_i(s)\, ds, \\ Z_i^{(2)}(t) &= \frac{b_i}{T_\mathrm{f}} + \int_0^t -\frac{s}{T_\mathrm{f}} f_i(s)\, ds + \int_t^{T_\mathrm{f}} \frac{T_\mathrm{f} - s}{T_\mathrm{f}} f_i(s)\, ds, \qquad i \in R_\mathrm{h} \end{aligned} \tag{6.5.13}$$

Here

$$f_i(s) = \varepsilon_i\rho_i(s)Z_i^{(1)}(s) + \rho_i(s)Z_i^{(2)}(s) + G_i(s), \qquad i \in R_\mathrm{h} \tag{6.5.14}$$

Note that in (6.5.13) satisfaction of the boundary conditions is guaranteed during the search for the required solution. We can thus claim that the optimizing hydro equations are in a form suitable for contraction mapping application.

We now look at the following two example systems:

6.5.1 Example System 1

In this example we consider a hydro–thermal electric power system with one thermal plant whose fuel cost function is given by

$$F(P_s) = \alpha + 4P_s + 0.0012P_s^2$$

The hydro network contains two hydro plants on the same stream with river transport delay of one hour:

$$\tau = 1 \quad \text{hr}$$

We follow the notation of Section 6.3; thus plants 2 and 3 are hydraulic with the following particulars.

$$\begin{aligned}
\eta_2 &= 0.708, & \eta_3 &= 0.708 \\
s_2(0) &= 0.72 \times 10^{13}, & s_3(0) &= 0.144 \times 10^{13} \quad \text{ft}^3, \\
\alpha_2 &= 0.0, & \alpha_3 &= 0.0, \\
\beta_{T_2} &= 0.54 \times 10^{-6}, & \beta_{T_3} &= 0.2047 \times 10^{-6} \quad \text{ft}^{-2}/\text{hr}, \\
\beta_{y_2} &= 0.1389 \times 10^{-10}, & \beta_{y_3} &= 0.1389 \times 10^{-9} \quad \text{ft}^{-2} \\
i_2(t) &= 0.1 \times 10^6, & i_3(t) &= 0.1 \times 10^7 \quad \text{ft}^3/\text{hr}.
\end{aligned}$$

The allowable volumes of water discharge over the optimization interval of 24 hr are

$$b_2 = 0.5 \times 10^9, \qquad b_3 = 0.35 \times 10^{10} \quad \text{ft}^3$$

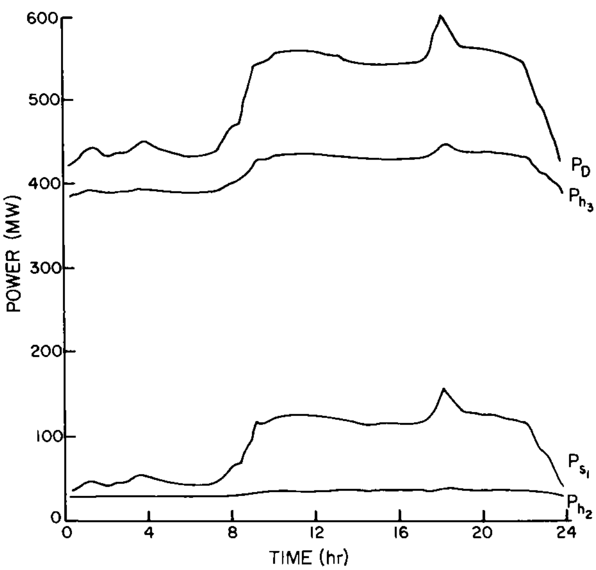

Fig. 6.6 Optimal schedules for the sample system.

The quadratic loss formula is assumed to have coefficients given by

$$B_{11} = 0.16 \times 10^{-3}, \qquad B_{22} = 0.22 \times 10^{-3}, \qquad B_{33} = 0.16 \times 10^{-3} \quad \mathrm{MW}^{-1}$$

The power demand curve is shown in Fig. 6.6.

The resulting optimality conditions define the operator equation:

$$\mathbf{Z}(t) = \mathbf{F}[\mathbf{Z}(t)]$$

We use a modified form,

$$\mathbf{Z}(t) = \mathbf{G}[\mathbf{Z}(t)]$$

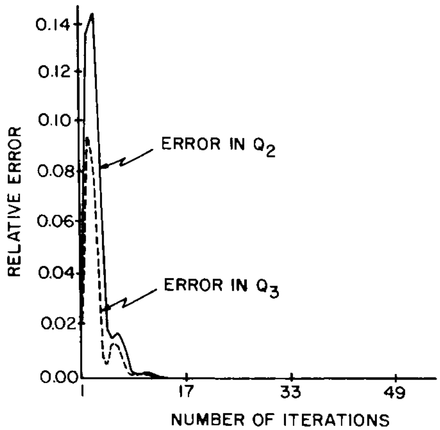

Fig. 6.7 Variation of relative error with number of iterations.

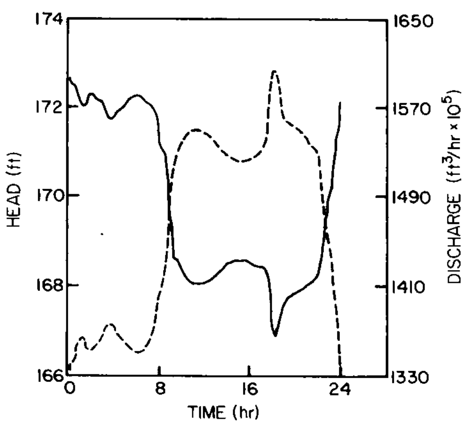

Fig. 6.8 Optimal head variations and rate of water discharge for the downstream plant.

with

$$\mathbf{G} = (\mathbf{I} - \mathbf{V})^{-1}[\mathbf{F}(\mathbf{Z}) - \mathbf{V}(\mathbf{Z})]$$

As outlined before, $\mathbf{V}$ is such that $(\mathbf{I} - \mathbf{V})$ is invertible. For this system we choose $\mathbf{V}$ to be simply

$$\mathbf{V} = v\mathbf{I}$$

The choice $v = -0.95$ gives good convergence results for our example system. The modified contraction mapping algorithm (MCM) takes on the form

$$\mathbf{Z}^{(k+1)} = \mathbf{G}[\mathbf{Z}^{(k)}]$$

It is important to note that the initial guess functions which play a prominent role in the success of the algorithm for this system are taken as

$$Q_2^{(0)}(t) = b_2 t/T_f, \qquad Q_3^{(0)}(t) = b_3 t/T_f$$
$$\dot{Q}_2^{(0)}(t) = b_2/T_f, \qquad \dot{Q}_3^{(0)}(t) = b_3/T_f$$

This is in addition to

$$\lambda^{(0)}(t) = \beta + \varepsilon, \qquad \beta = 4, \qquad 0.1 \le \varepsilon \le 0.5$$

With only these functions assumed we initialize the iterations. The variations of the maximum relative error with iterations in the variables $Q_2(t)$ and $Q_3(t)$ are shown in Fig. 6.7. Convergence may be declared after 18 iterations to a power system accuracy. The optimal schedules are shown in Figs. 6.7–6.9.

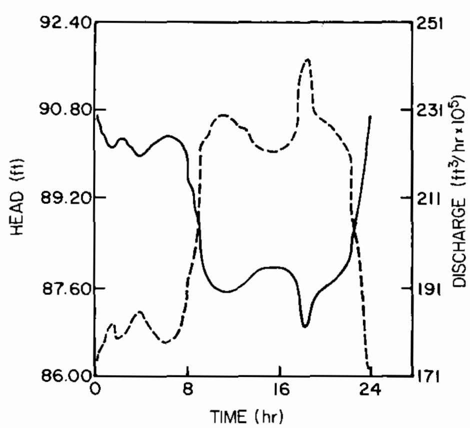

Fig. 6.9 Optimal head variations and rate of water discharge for the upstream plant.

6.5.2 Example System 2

A more general system including three thermal plants and nine hydro plants in multichain configuration is analyzed next. The hydro network is shown in Fig. 6.10. The thermal plant's characteristics are as follows:

$$F(P_{s_1}) = \alpha_1 + 4P_{s_1} + 0.0012P_{s_1}^2$$
$$F(P_{s_2}) = \alpha_2 + 4.5P_{s_2} + 0.0012P_{s_2}^2$$
$$F(P_{s_3}) = \alpha_3 + 4.25P_{s_3} + 0.0012P_{s_3}^2$$

The loss formula coefficients are given in Table 6.1. The hydro plants' particulars are listed in Tables 6.2–6.4. The river transport delays are

$$\tau_{48} = 4.8 \text{ hr}, \qquad \tau_{58} = 7.4 \text{ hr}$$
$$\tau_{69} = 3.8 \text{ hr}, \qquad \tau_{712} = 5.7 \text{ hr}$$
$$\tau_{810} = 8.2 \text{ hr}, \qquad \tau_{1011} = 9.6 \text{ hr}$$
$$\tau_{912} = 8.5 \text{ hr}$$

The hourly active power demand of the system is tabulated in Table 6.5.

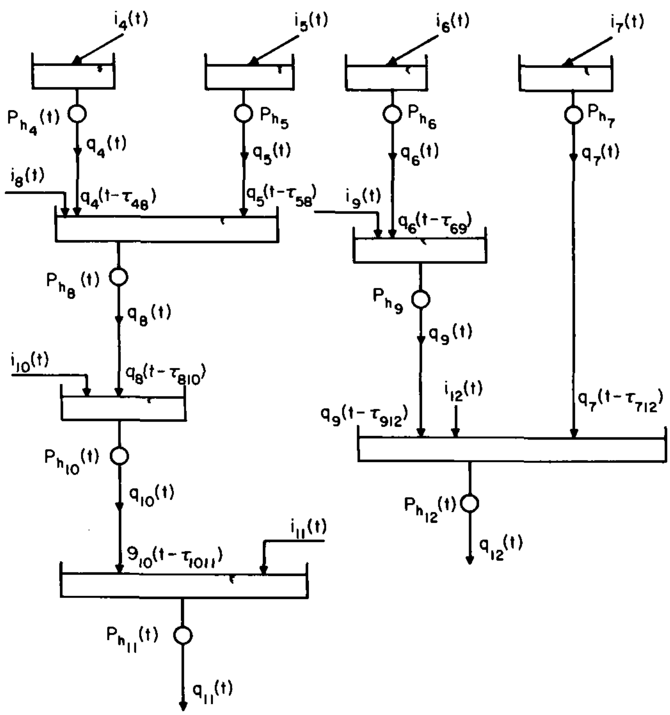

Fig. 6.10 Layout of hydro plants for example system 2.

TABLE 6.1

Loss Formula Coefficients (10^{-3} MW/MW^2)

i	1	2	3	4	5	6
B_{ii}	0.16	0.1	0.2	0.22	0.22	0.22
i	7	8	9	10	11	12
B_{ii}	0.22	0.16	0.22	0.16	0.18	0.18

TABLE 6.2

Hydro Plants' Particulars

Plant	α_y	$\beta_y\,(10^{-10})$	$\beta_T\,(10^{-6})$	$s(0)\,(10^{13}\text{ ft}^3)$	$b\,(10^6\text{ ft}^3)$	$i\,(10^6\text{ ft}^3/\text{sec})$
4	0	0.139	0.54	0.72	0.5	0.1
5	0	0.139	0.54	0.72	0.5	0.1
6	0	0.139	0.54	0.72	0.5	0.1
7	0	0.139	0.54	0.72	0.5	0.1
8	0	1.39	0.2	0.144	4	0.1×10^{-4}
9	70	0.139	0.54	0.72	0.5	0.1
10	0	1.39	0.2	0.144	4	0.1×10^{-4}
11	0	1.39	0.13	0.18	2	0
12	0	1.39	0.13	0.18	2	0

TABLE 6.3

Hydro Plants' Active Power Data

Plant	$P_h(0)$ (MW)	P_{min} (MW)	P_{max} (MW)
4	30.8	5	100
5	30.8	5	100
6	30.8	5	100
7	30.8	5	100
8	460.8	5	600
9	55.1	5	100
10	460.8	5	600
11	331.9	5	400
12	331.9	5	400

TABLE 6.4

Hydro Plants' Head Particulars

Plant	$h(0)$ (ft)	$y_T(0)$ (ft)	$y(0)$ (ft)	h_{min} (ft)	h_{max} (ft)
4	100	11.25	88.75	50	100
5	100	11.25	88.75	50	100
6	100	11.25	88.75	50	100
7	100	11.25	88.75	50	100
8	200	34.1	165.9	100	180
9	100	11.3	158.7	50	200
10	200	34.1	165.9	100	180
11	250	11.0	239.0	200	250
12	250	11.0	239.0	200	250

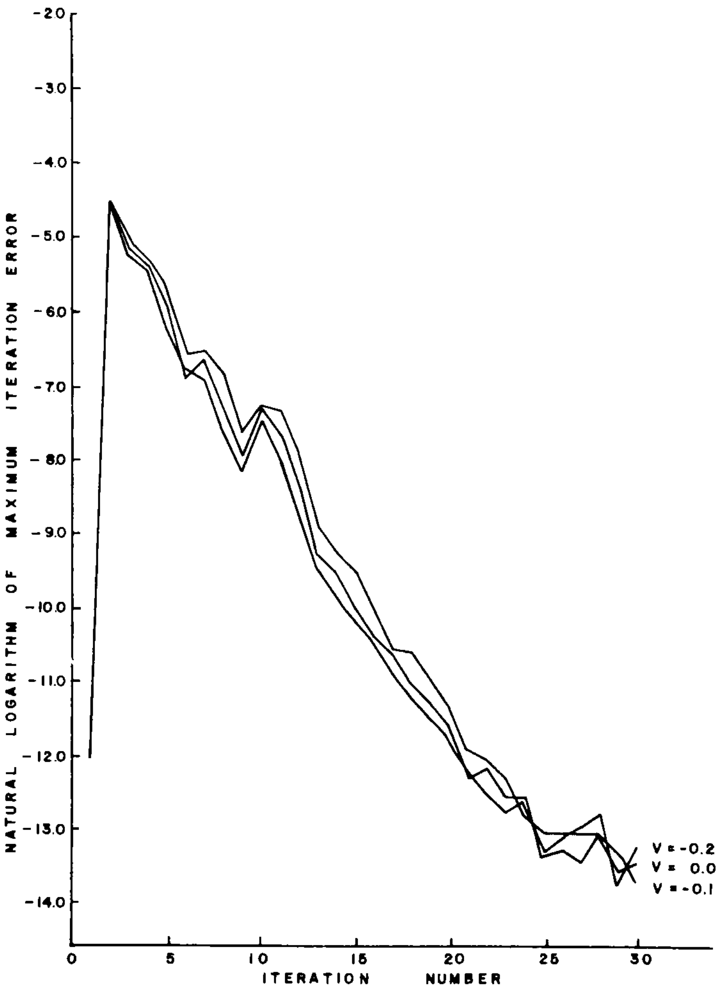

Fig. 6.11 Error variation for $v = -0.2$, 0, and -0.1 in iteration range 1 to 30.

TABLE 6.5

Power Demand (MW)

Hour	P_D	Hour	P_D	Hour	P_D
1	1860	9	2051	17	2250
2	1870	10	2108	18	2290
3	1890	11	2011	19	2300
4	1900	12	2112	20	2330
5	1892	13	2090	21	2280
6	1932	14	2031	22	2250
7	1966	15	2067	23	2120
8	1995	16	2150	24	1950

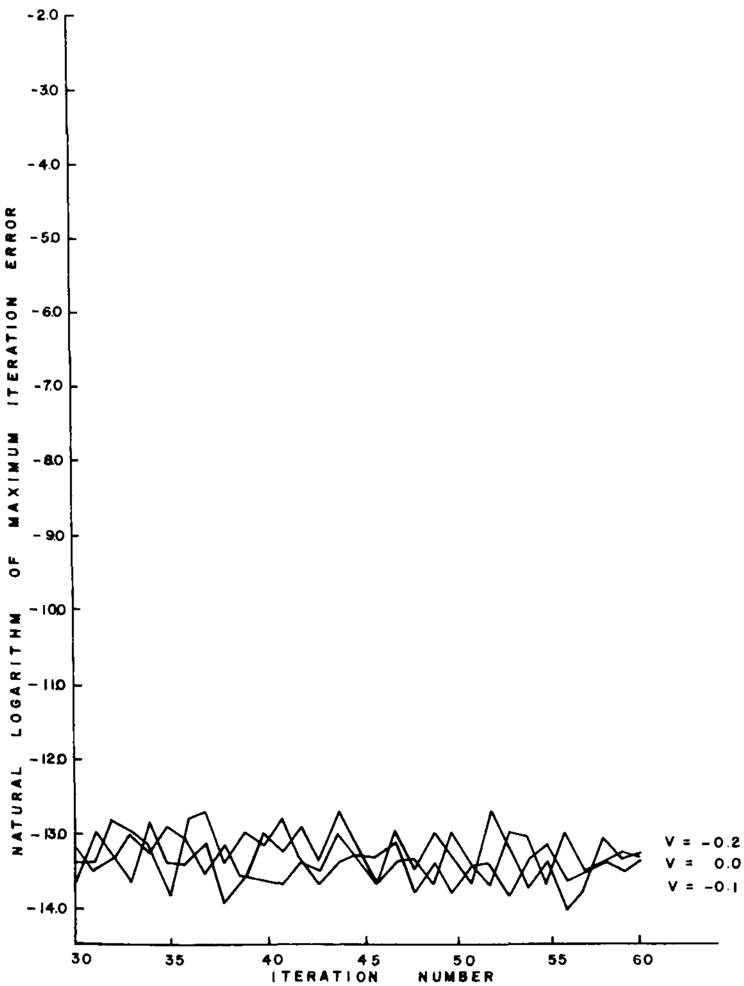

Fig. 6.12 Error variation for $v = -0.2$, 0, and -0.1 in iteration range 30 to 60.

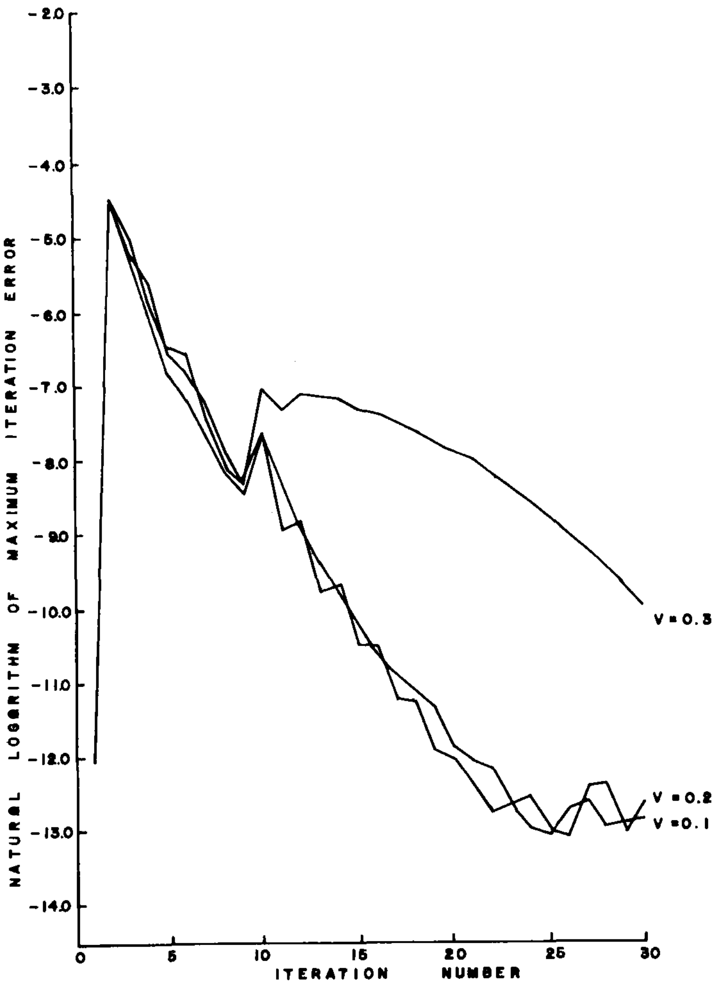

Fig. 6.13 Error variation for $v = 0.3$, 0.2, and 0.1 in iteration range 1 to 30.

A computer program was written to solve the system of optimal equations. The program utilized the MCM algorithm outlined before. Here we chose to change the scalar v in the algorithms in steps of 0.1. Computational experiments show that for values of v larger than 0.4 the sequence of iterates diverges. This was verified for v valued at 0.6 and 0.5. For $v = 0.4$, the first few iterations show an improvement (decrease) in the error; then divergence occurs.

Although the algorithm with $v = 0.3$ converges, it is shown to be far inferior to all other converging algorithms. All choices of v from 0.3 to -1.4

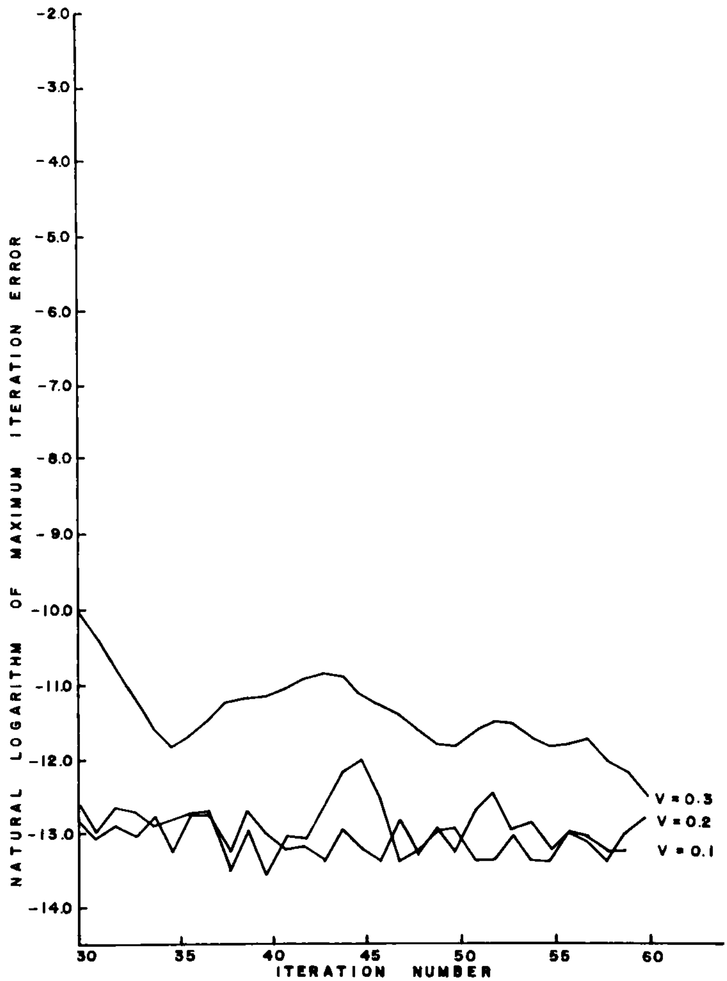

Fig. 6.14 Error variation for $v = 0.3$, 0.2, and 0.1 in iteration range 30 to 60.

gave converging sequences of iterates. The number of iterations for this experiment was limited to 60 iterations. Inspection of the sequence of iteration errors for all the algorithms confirmed the conclusion that $v = 0.2$ is superior to all other converging algorithms over the first 20 iterations. On the other hand, after 20 iterations the choice of $v = 0$ provided faster convergence. The error between iterations for various choices of v are shown in Figs. 6.11–6.16.

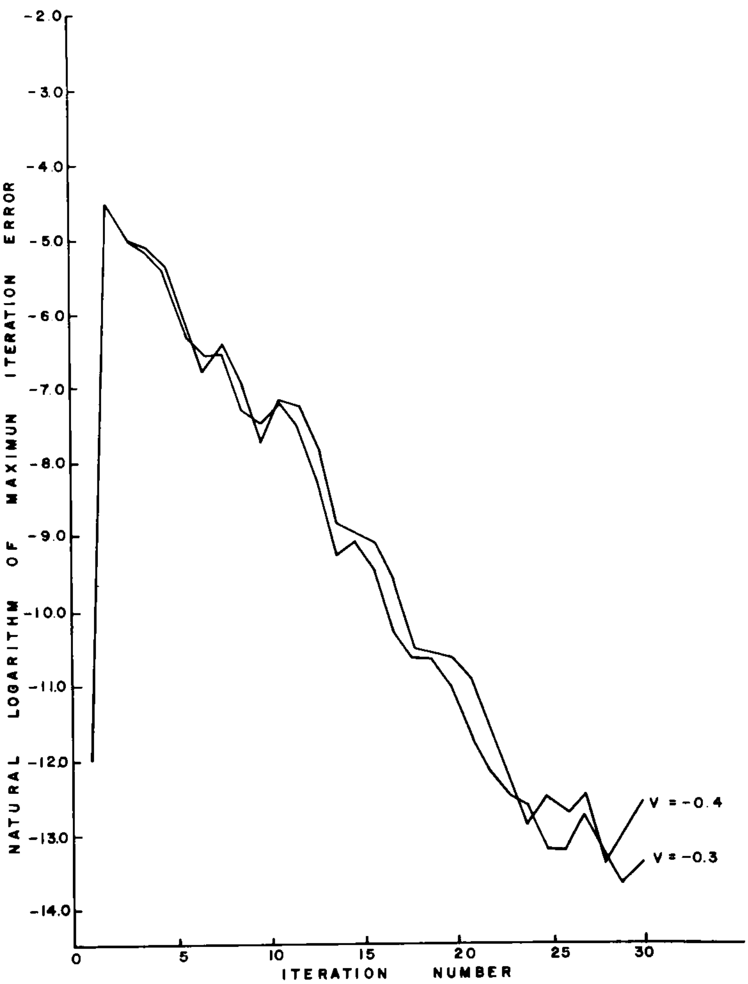

Fig. 6.15 Error variation for $v = -0.4$ and -0.3 between 1 and 30 iterations.

The question that naturally arises is when to stop iterating and declare successful convergence. For our problem, fortunately, we can measure improvement by the total cost of operation for a given approximation. The variations in fuel cost with iterations is shown in Fig. 6.17. It appears that 10 iterations will give sufficient accuracy for our system.

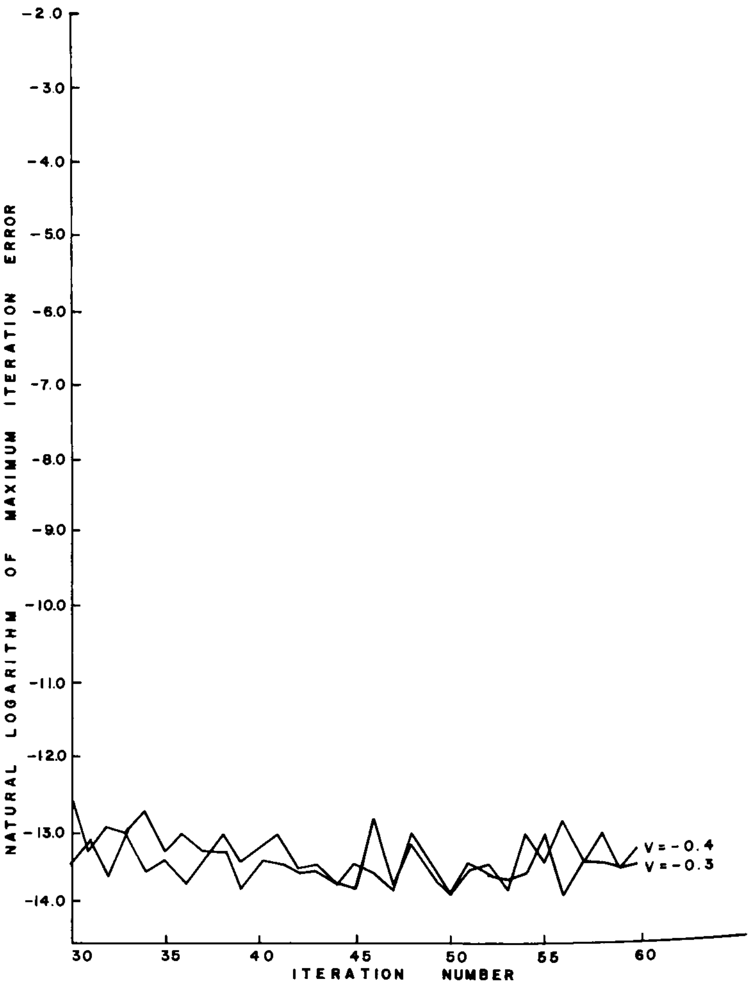

Fig. 6.16 Error variation for $v = -0.4$ and -0.3 between 30 and 60 iterations.

The computed optimal active power generations for the sample system are shown in Table 6.6. A tabulation of the computed multiplier function $\lambda(t)$ is given. The values of $\lambda(t)$ follow a pattern of change that is similar to the system's power demand. The four upstream hydro-plants of the system generate the same active power at all times. This is expected, since their

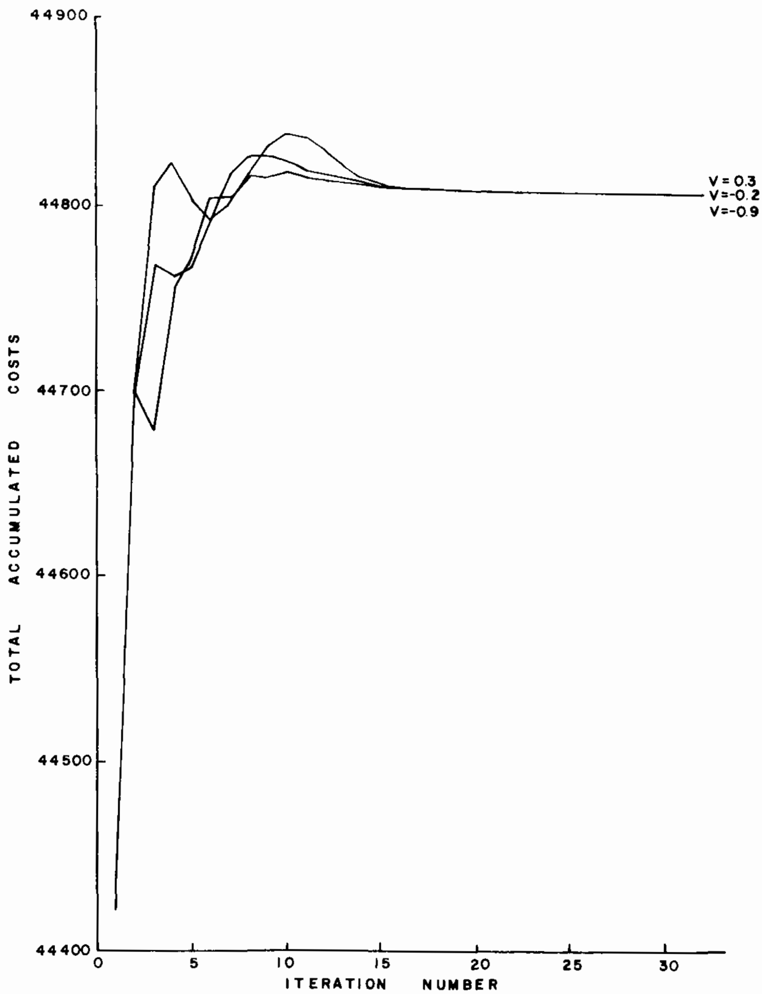

Fig. 6.17 Variation of cost with iterations.

characteristics are the same. The most expensive thermal plant, labeled 2, generates less power consistently in comparison with plants 7 and 3. Toward the end of the optimization interval the reduction is evident. The variation of the hydro rate of water usage and reservoir effective head are shown in Tables 6.7 and 6.8. For this optimal strategy the total daily operating costs are computed to be $44,805.

TABLE 6.6

Optimal Active Power Generation (MW)

Hour	P_{s_1}	P_{s_2}	P_{s_3}	P_{h_4}	P_{h_5}	P_{h_6}	P_{h_7}	P_{h_8}	P_{h_9}	$P_{h_{10}}$	$P_{h_{11}}$	$P_{h_{12}}$	λ
1	134.9	5.7	63.9	31.16	31.16	31.16	31.6	464.65	55.75	464.65	335.28	335.28	4.52
2	140.47	12.47	69.0	31.22	31.22	31.22	31.22	465.42	55.88	465.42	336.	336.	4.54
3	144.44	17.3	72.77	31.18	31.18	31.18	31.18	464.79	55.8	464.79	355.5	335.5	4.56
4	142.75	15.24	71.2	31.13	31.13	31.13	31.13	464.0	55.7	464.0	334.85	334.85	4.55
5	157.65	33.4	85.0	31.00	31.00	31.00	31.00	462.4	55.9	462.4	333.5	333.5	4.61
6	168.1	46.16	94.74	31.00	31.00	31.00	31.00	462.97	55.6	462.95	334.0	334.0	4.65
7	178.83	59.3	104.7	31.00	31.00	31.00	31.00	461.94	55.4	461.91	333.16	333.16	4.70
8	195.72	80.02	120.34	31.10	31.1	31.1	31.1	463.19	55.65	463.16	334.29	334.29	4.77
9	213.63	102.07	136.9	31.16	31.16	31.16	31.16	463.95	55.77	463.96	334.97	334.98	4.844
10	190.83	74.0	115.82	30.55	30.55	30.55	30.55	456.33	54.62	456.4	328.39	328.39	4.75
11	218.32	107.85	141.24	30.95	30.95	30.95	30.95	461.14	55.36	461.24	332.62	332.62	4.864
12	217.29	106.6	140.3	30.5	30.5	30.5	30.5	456.0	54.52	456.0	328.25	328.2	4.36
13	193.0	76.71	117.85	30.85	30.85	30.85	38.85	459.8	55.17	460.0	331.58	331.5	4.76
14	207.33	94.3	131.00	30.63	30.63	30.63	30.63	457.95	54.77	458.2	329.98	329.88	4.82
15	234.17	127.44	155.87	30.75	30.75	30.75	30.75	458.4	54.98	458.7	330.46	330.34	4.93
16	266.1	167.0	185.27	30.77	30.77	30.77	30.77	459.78	55.05	460.14	331.7	331.55	5.07
17	278.78	182.88	196.92	30.89	30.89	30.89	30.89	460.27	55.26	460.66	332.21	332.0	5.126
18	283.54	188.81	201.28	30.73	30.73	30.73	30.73	459.28	54.97	459.73	331.37	331.17	5.148
19	289.64	196.44	206.89	31.07	31.07	31.07	31.07	462.32	55.58	462.8	334.12	333.89	5.175
20	276.32	179.8	194.56	30.73	30.73	30.73	30.73	459.64	54.98	460.17	331.77	331.53	5.115
21	259.1	158.36	178.83	31.3	31.3	31.3	31.3	465.06	56.03	465.65	336.64	335.37	5.04
22	219.79	109.67	142.59	30.9	30.9	30.9	30.9	461.95	55.29	462.6	333.92	333.62	4.87
23	164.57	41.86	91.48	31.0	31.0	31.0	31.0	460.86	55.46	461.54	333.0	332.77	4.64
24	138.48	10.05	67.23	30.63	30.63	30.63	30.63	458.42	54.8	459.15	330.95	330.6	4.53

TABLE 6.7

Optimal Hydro Discharge (10^8 ft^3/hr)

Hour	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}
1	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
2	0.211	0.211	0.211	0.211	1.69	0.211	1.69	0.84	0.84
3	0.211	0.211	0.211	0.211	1.68	0.211	1.68	0.84	0.84
4	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
5	0.20	0.20	0.20	0.20	1.67	0.20	1.67	0.83	0.83
6	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
7	0.20	0.20	0.20	0.20	1.67	0.20	1.67	0.83	0.83
8	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
9	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
10	0.20	0.20	0.20	0.20	1.65	0.20	1.65	0.82	0.82
11	0.20	0.20	0.20	0.20	1.67	0.20	1.67	0.83	0.83
12	0.20	0.20	0.20	0.20	1.65	0.20	1.65	0.82	0.82
13	0.20	0.20	0.20	0.20	1.66	0.20	1.66	0.83	0.83
14	0.20	0.20	0.20	0.20	1.65	0.20	1.65	0.82	0.82
15	0.20	0.20	0.20	0.20	1.66	0.20	1.66	0.83	0.83
16	0.20	0.20	0.20	0.20	1.65	0.20	1.66	0.83	0.83
17	0.20	0.20	0.20	0.20	1.66	0.20	1.67	0.83	0.83
18	0.20	0.20	0.20	0.20	1.66	0.20	1.66	0.83	0.83
19	0.21	0.21	0.21	0.21	1.68	0.21	1.68	0.84	0.84
20	0.20	0.20	0.20	0.20	1.66	0.20	1.66	0.83	0.83
21	0.21	0.21	0.21	0.21	1.69	0.21	1.69	0.84	0.84
22	0.20	0.20	0.20	0.20	1.67	0.20	1.67	0.84	0.84
23	0.21	0.21	0.21	0.21	1.67	0.21	1.67	0.83	0.83
24	0.20	0.20	0.20	0.20	1.66	0.20	1.66	0.83	0.83

TABLE 6.8

Optimal Effective Head (ft)

Hour	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}
1	88.61	88.61	88.61	88.61	165.5	158.6	165.5	238.8	238.8
2	88.58	88.58	88.58	88.58	165.4	158.6	165.4	238.8	238.8
3	88.60	88.60	88.60	88.60	165.44	158.6	165.4	238.8	238.8
4	88.62	88.62	88.62	88.62	165.48	158.63	165.48	238.8	238.8
5	88.67	88.67	88.67	88.67	165.61	158.67	165.61	238.85	238.85
6	88.65	88.65	88.65	88.65	165.53	158.65	165.53	238.82	238.83
7	88.67	88.67	88.67	88.67	165.60	158.68	165.59	238.84	238.85
8	88.63	88.63	88.63	88.63	165.46	158.64	165.45	238.79	238.80
9	88.60	88.60	88.60	88.60	165.37	158.61	165.37	238.75	238.77
10	88.86	88.86	88.86	88.86	166.06	158.86	166.08	239.00	238.99
11	88.69	88.69	88.69	88.69	165.60	158.70	165.63	238.86	238.84
12	88.88	88.88	88.88	88.88	166.04	158.88	166.10	239.03	238.99
13	88.73	88.73	88.73	88.73	165.67	158.74	165.74	238.92	238.87
14	88.83	88.83	88.83	88.83	165.82	158.83	165.91	238.99	238.92
15	88.78	88.78	88.78	88.78	165.76	158.78	165.87	238.99	238.90
16	88.76	88.76	88.76	88.76	165.61	158.77	165.73	238.95	238.85
17	88.71	88.71	88.71	88.71	165.54	158.72	165.68	238.95	238.82
18	88.78	88.78	88.78	88.78	165.61	158.78	165.77	238.99	238.85
19	88.64	88.64	88.64	88.64	165.30	158.65	165.42	238.91	238.75
20	88.78	88.78	88.78	88.78	165.53	158.78	165.73	238.99	238.82
21	88.54	88.54	88.54	88.54	165.00	158.55	165.21	238.84	238.65
22	88.71	88.71	88.71	88.71	165.27	158.72	165.51	238.95	238.74
23	88.66	88.66	88.66	88.66	165.35	158.68	165.60	238.99	239.76
24	88.82	88.82	88.82	88.82	165.56	158.82	165.82	239.08	238.83

6.6 SUMMARY

In this chapter the effects of hydraulic coupling and the hydraulic network configurations on economy dispatch strategies have been considered. We outlined a maximum principle approach to the problem of a hydro thermal system with hydro plants on one stream. River transport delay effects are neglected in this problem. The optimality conditions for the thermal and electric variables and associated multipliers are similar to those obtained in the previous chapter. The hydraulic subsystem obviously differs and we have seen that the costates or, alternatively, water conversion functions satisfy a new set of nonlinear dynamic equations. In this latter set the effects of hydraulic coupling are accounted for.

We introduced river transport delays in the formulation of a problem where all hydro plants are on the same stream. Use of the minimum norm approach in formulating the problem again offers certain advantages. The requirement of positive definiteness of the matrices involved in defining the inner products is certainly one of them. Elimination of the multipliers associated with constraints linear in the control functions is another. It is important to note that these multipliers are what we referred to as the water conversion functions. In the formulation we found it convenient to adopt certain pseudocontrols which were eliminated upon completion of the derivation of the optimality conditions. The optimality conditions for the thermal and electric subsystems are the same as before. On the other hand, our hydraulic optimality conditions were obtained in a form that shows the effects of neighboring coupled plants. The above was extended to encompass an essentially general hydro-network configuration.

Practical computational experience has been reported to conclude the present chapter. Here we use to full advantage the modified contraction mapping algorithm (MCM) for actual implementation. Two example systems have been used. It is worth mentioning that the predictive nature of the hydro optimality conditions necessitates an off-line implementation. For on-line implementation, approximating expressions have to be employed. For example, a Taylor based expansion may be useful in this case.

6.7 COMMENTS AND REFERENCES

The earliest contribution to the common-flow hydro dispatch field is Burr's work (1941). He developed loading schedules for a two-plant common-flow hydro system but the assumptions made were too simplifying. Johnson (1956) proposed a numerical computational scheme to find schedules that will result in minimum constant deficit.

The importance of including the time delay of flow between coupled plants in the optimization scheme was pointed out by C. W. Watchorn and R. A. Arismunander in separate discussions of the work reported by Drake *et al.* (1962). This effort was directed at the economic scheduling of a system with multichains of reservoirs. Head variations as well as river transport delays were held insignificant when included in the analysis. In 1962, P. R. Menon used the Euler equations for constructing sets of minimizing sequences for a three-plant hydro–thermal system. Although Menon's is a common-flow system, the river transport delays were not considered. This treatment was also reported by Johnson and Menon (1963).

Sokkappa's approach (1963a) is to employ the steepest descent method in solving for the optimal schedules for a discrete time model. The system considered involves three plants on the same river. Time delays were not included in the long-range scheduling problem considered. In a subsequent paper (1963b), the same author introduces the concept of a pseudothermal (or slack) source. The object here is to find an initial feasible schedule. The approach is also useful for problems with the criterion of maximum hydro energy generation.

The maximum principle approach discussed in Section 6.2 is based on Dahlin's work (1964a). The problems considered by Dahlin included a wide spectrum of hydro network models. When considering transport delay, the river flow models adopted involve a large number of differential equations and boundary conditions. This was a definite contribution to the theory of economy scheduling. Unfortunately, these models made the problem more difficult to analyze numerically. Dahlin's contributions are summarized in the three papers coauthored with D. W. C. Shen (1964b, 1965, 1966). The application of the maximum principle to an all-hydro system with cascaded hydro plants including river transport delay is given by Bubenko and Waern (1972). Maximizing the hydro energy return is the objective in this case.

Bainbridge *et al.* (1966) present a method for optimizing the weekly or daily dispatch of a power system consisting of sixteen hydro plants, four thermal plants, and a pumped-storage plant. The method uses a gradient procedure on the hydro subsystem and dynamic programming combined with the use of coordination equations, is applied to the thermal subsystem. A composite one-dam representation exhibiting the main characteristics of the multireservoir hydroelectric system was proposed by Arvantidis and Rosing. The composite model may thus be used for obtaining optimal operational strategies. This approach is reported in two papers (1970a,b).

A related work is that by Miller and Thompson (1972). Their work is concerned with the Pacific Gas and Electric Company's hydro–thermal system. A linear programming approach is used for solving the long-range scheduling problem. A set of inequality constraints on the reservoir's storage and head variations is imposed. However, the time delays of flows were not taken into consideration.

For elegant, dynamic programming solutions to problems of common-flow hydro systems, we have the work of Keckler and Larson (1968), Larson and Korsak (1970), Rees and Larson (1971), and Engles *et al.* (1976).

The treatment of Section 6.3 is due to El-Hawary and Christensen (1972, 1973). The formulation of Section 6.4 is reported by the same authors (1977). The computational experience reported in Section 6.5 is based on theorems presented by Falb and de Jong (1969). Titli and Godard (1976) present a very interesting approach to solving the problem posed and solved in Section 6.3. Their approach is based on the use of multilevel optimization theory together with the generalized reduced gradient method.

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CHAPTER
7
Optimal Hydro–Thermal Load Flow and Realistic Models

7.1 INTRODUCTION

It is the purpose of this chapter to give more advanced formulations for hydro–thermal optimum economic operation problems than those treated in the previous two chapters. In particular we are interested in including the load flow model in our formulations. This is contrasted with the active power balance model adopted earlier.

It is evident that the dimensionality of the problem will be increased considerably. We therefore choose to begin with a treatment of the problem of optimal hydro–thermal load flow for a system with hydro plants on separate streams. This is given in the following section.

We have outlined in Chapter 2 various models available for the description of hydro-plants performance. The formulations offered so far and the corresponding solutions assume vertical-sided reservoirs and invariant efficiency for the hydro plant. In Section 7.3 we indicate the modifications necessary to account for a trapezoidal reservoir model as well as variable efficiency considerations in an optimal hydro–thermal load flow. In this case we choose to treat a system with multichains of hydro plants and account for the hydraulic coupling.

We remark here that an important class of problems exists. This is the class of optimal active–reactive hydro–thermal operation where the electric network is modeled using the ARPBE described in Section 2.4.2. We have included in Section 4.2.3 the case of an all-thermal active–reactive dispatch. A similar procedure can be followed here and the interested reader can arrive at the solution aided by the material presented in this chapter. Further reference will be made to this aspect in the concluding Comments and References section of the present chapter.

7.2 OPTIMAL LOAD FLOW WITH HYDRO PLANTS ON SEPARATE STREAMS

In this section we consider the problem of optimal load flow for a hydro–thermal system with hydro plants on separate streams. The system is assumed to have N_g generating plants (generator buses) which belong to the set R_G. There are N_h hydro plants and $(N_g - N_h)$ thermal plants. The sets R_h and R_s include the thermal and hydro plants, respectively.

Our objective is to minimize the combined cost functional

$$J = \int_0^{T_f} [F_T(P_{s_i}(t)) + F_Q(Q_{G_i}(t))]\, dt \tag{7.2.1}$$

The integrand in this cost functional is similar to the objective chosen for the discussion in Section 4.2.3. Here F_T is given by the same expression as Eq. (4.2.40) and F_Q is simplified by dropping the constant and linear terms in the reactive powers. We thus have

$$J = \int_0^{T_f} \left\{ \sum_{i \in R_s} [\beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t)] + \sum_{i, j \in R_G}\sum Q_i(t) K_{ij} Q_j(t) \right\} dt \tag{7.2.2}$$

The electric network performance is modeled using the rectangular form of the load flow equations. This has been discussed in Chapter 2. We thus rewrite Eqs. (2.4.64) and (2.4.65) as the network equality constraints

$$\begin{aligned} \phi_{p_i} &= -P_i + \sum_{j \in \alpha_i} [e_i(e_j G_{ij} + f_j B_{ij}) + f_i(f_j G_{ij} - e_j B_{ij})] \\ &= 0, \qquad i \in R_N \end{aligned} \tag{7.2.3}$$

$$\begin{aligned} \phi_{q_i} &= Q_i - \sum_{j \in \alpha_i} [f_i(e_j G_{ij} + f_j B_{ij}) - e_i(f_j G_{ij} - e_j B_{ij})] \\ &= 0, \qquad i \in R_N \end{aligned} \tag{7.2.4}$$

Here R_N is the set of all network nodes and α_i is the set of nodes connected to the ith node.

There are several inequality constraints that must be satisfied at the optimum for a valid solution. The most comprehensive set of such constraints is discussed in Chapter 4 for the all thermal case. For our purposes it suffices to include

$$P_i^2(t) + Q_i^2(t) \le S_i^{2^{\mathrm{M}}} \tag{7.2.5}$$

$$Q_i^{\mathrm{m}} \le Q_i(t) \le Q_i^{\mathrm{M}} \tag{7.2.6}$$

$$P_i^{\mathrm{m}} \le P_i(t) \le P_i^{\mathrm{M}} \tag{7.2.7}$$

This will result in no loss of generality. A further set of equality constraints should be included since normally the voltage magnitude at a generator bus is prespecified. We thus have

$$e_i^2 + f_i^2 = E_i^2, \qquad i \in R_{G'} \tag{7.2.8}$$

$R_{G'}$ is the set of all generator buses except for the slack bus, where both e_i and f_i are given.

The hydro plant's active power generation is assumed to vary with the rate of water discharge:

$$P_{\mathrm{h}_i}(t) + A_i(t)q_i(t) + B_{\mathrm{W}_i}q_i(t)Q_{\mathrm{W}_i}(t) + C_i q_i^2(t) = 0, \qquad i \in R_{\mathrm{h}} \tag{7.2.9}$$

with

$$q_i(t) = \dot{Q}_{\mathrm{W}_i}(t), \qquad i \in R_{\mathrm{h}} \tag{7.2.10}$$

Moreover, the volume of water discharge at any hydro plant is a prespecified constant for each plant in the set R_{h}:

$$\int_0^{T_{\mathrm{f}}} q_i(t)\,dt = b_i, \qquad i \in R_{\mathrm{h}} \tag{7.2.11}$$

Our chosen hydro model is similar to the one in Chapter 6 where tailrace level variations with the discharge are accounted for in a variable-head hydro plant.

7.2.1 Formulation

We follow the same procedure adopted for minimum norm formulations as in the preceding chapters. By augmenting the original cost functional J of Eq. (7.2.2) we obtain

$$J_0(\cdot) = \int_0^{T_{\mathrm{f}}} I_0(t)\,dt \tag{7.2.12}$$

where

$$I_0(t) = \sum_{i=1}^{10} I_{0i}(t) \tag{7.2.13}$$

We now define each term. The first relates to the active power injection relation (7.2.3)

$$I_{01}(t) = \sum_{i \in R_N} \lambda_{p_i}(t)\phi_{p_i}(t) \tag{7.2.14}$$

The reactive injection relation (7.2.4) leads to

$$I_{02}(t) = \sum_{i \in R_N} \lambda_{q_i}(t)\phi_{q_i}(t) \tag{7.2.15}$$

The voltage magnitude constraints (7.2.8) result in

$$I_{03}(t) = \sum_{i \in R_{G'}} \lambda_{e_i}[e_i^2(t) + f_i^2(t) - E_i^2(t)] \tag{7.2.16}$$

As for the hydro constraints, Eqs. (7.2.9) and (7.2.10), we have

$$\begin{aligned} I_{04}(t) = \sum_{i \in R_h} \{ & n_i(t)[P_{h_i}(t) + A_i(t)q_i(t) + B_{W_i}q_i(t)Q_{W_i}(t) + C_i q_i^2(t)] \\ & + m_i(t)q_i(t) + \dot{m}_i(t)Q_{W_i}(t)\}, \qquad i \in R_h \end{aligned} \tag{7.2.17}$$

We also have the original cost functional rewritten as

$$I_{05}(t) = \sum_{i \in R_s} [\beta_i P_{s_i}(t) + \gamma_i P_{s_i}^2(t)] + \sum_{i,j \in R_G}\sum Q_i(t)K_{ij}Q_j(t) \tag{7.2.18}$$

Finally, the inequality constraints (7.2.5)–(7.2.7) provide us with

$$I_{06}(\cdot) = \sum_{i \in R_G} M_i(t)[P_i^2(t) + Q_i^2(t) - S_i^{2M}] \tag{7.2.19}$$

$$I_{07}(\cdot) = \sum_{i \in R_G} l_i(t)[P_i^m - P_i(t)] \tag{7.2.20}$$

$$I_{08}(\cdot) = \sum_{i \in R_G} l_i'(t)[P_i(t) - P_i^M] \tag{7.2.21}$$

$$I_{09}(\cdot) = \sum_{i \in R_G} r_i(t)[Q_i^m - Q_i(t)] \tag{7.2.22}$$

$$I_{0_{10}}(\cdot) = \sum_{i \in R_G} r_i'(t)[Q_i(t) - Q_i^M] \tag{7.2.23}$$

Here we are using the Kuhn–Tucker theorem so that the following exclusion equations must be satisfied at the optimum for $i \in R_G$, $t \in [0, T_f]$:

$$M_i(t)[P_i^2(t) + Q_i^2(t) - S_i^{2M}] = 0 \tag{7.2.24}$$

$$l_i(t)[P_i^m - P_i(t)] = 0 \tag{7.2.25}$$

$$l_i'(t)[P_i(t) - P_i^{\mathrm{M}}] = 0 \tag{7.2.26}$$

$$r_i(t)[Q_i^{\mathrm{m}} - Q_i(t)] = 0 \tag{7.2.27}$$

$$r_i'(t)[Q_i(t) - Q_i^{\mathrm{M}}] = 0 \tag{7.2.28}$$

Moreover, $\lambda_{p_i}(t)$, $\lambda_{q_i}(t)$, $\lambda_{e_i}(t)$, $n_i(t)$, and $m_i(t)$ are to be determined so that the corresponding equality constraints are satisfied.

The decision variables in our present case can be grouped into four distinct classes. These are active power generations, reactive power generations, bus voltage components, and hydro rate of discharge and volume discharged. As a consequence, a vector of active power generations **P** can be defined as

$$\mathbf{P}(t) = \mathrm{col}[\mathbf{P}_{\mathrm{h}}(t), \mathbf{P}_{\mathrm{s}}(t)] \tag{7.2.29}$$

Here the partitioning pertains to hydro and thermal generations as follows:

$$\mathbf{P}_{\mathrm{h}}(t) = \mathrm{col}[P_i(t): i \in R_{\mathrm{h}}] \tag{7.2.30}$$

$$\mathbf{P}_{\mathrm{s}}(t) = \mathrm{col}[P_i(t): i \in R_{\mathrm{s}}] \tag{7.2.31}$$

Next the vector of reactive generations can be defined as

$$\mathbf{Q}(t) = \mathrm{col}[Q_i(t): i \in R_{\mathrm{G}}] \tag{7.2.32}$$

The class of voltage variables includes the real parts e_i and the imaginary parts f_i. We can thus define the direct and quadrature voltage vectors by:

$$\mathbf{E}_{\mathrm{d}}(t) = \mathrm{col}[e_i(t): i \in R_{\mathrm{G}'}] \tag{7.2.33}$$

$$\mathbf{E}_q(t) = \mathrm{col}[f_i(t): i \in R_{\mathrm{G}'}] \tag{7.2.34}$$

As a result we attain a voltage variables vector $\mathbf{E}(t)$:

$$\mathbf{E}(t) = \mathrm{col}[\mathbf{E}_{\mathrm{d}}(t), \mathbf{E}_q(t)] \tag{7.2.35}$$

Now for each hydro plant we have two variables which will define

$$\mathbf{W}_i(t) = \mathrm{col}[q_i(t), Q_{\mathrm{W}_i}(t)], \qquad i \in R_{\mathrm{h}} \tag{7.2.36}$$

This in turn will provide us with the hydro variables vector $\mathbf{W}(t)$:

$$\mathbf{W}(t) = \mathrm{col}[\mathbf{W}_i(t): i \in R_{\mathrm{h}}] \tag{7.2.37}$$

With the above definitions we can now define the control vector as

$$\mathbf{u}(t) = \mathrm{col}[\mathbf{P}(t), \mathbf{Q}(t), \mathbf{E}(t), \mathbf{W}(t)] \tag{7.2.38}$$

It is our intention now to arrange the coefficients of the linear terms in the augmented cost functional. Following the same classification as we have done with the control vector, we start by defining

$$L_{p_i}(t) = [n_i(t) + l_i'(t) - l_i(t) - \lambda_{p_i}(t)], \qquad i \in R_{\mathrm{h}} \tag{7.2.39}$$

This is simply obtained by inspecting (7.2.17), (7.2.20), (7.2.21), (7.2.14), and (7.2.3). Similarly, for the thermal active power generations we have

$$L_{p_i}(t) = [\beta_i + l_i'(t) - l_i(t) - \lambda_{p_i}(t)], \qquad i \in R_{\mathrm{s}} \tag{7.2.40}$$

For the overall generating side we can then define

$$\mathbf{L}_p(t) = \mathrm{col}[\mathbf{L}_{p_{\mathrm{h}}}(t), \mathbf{L}_{p_{\mathrm{s}}}(t)] \tag{7.2.41}$$

where

$$\mathbf{L}_{P_{\mathrm{h}}}(t) = \mathrm{col}[L_{p_i}(t) : i \in R_{\mathrm{h}}] \tag{7.2.42}$$

$$\mathbf{L}_{P_{\mathrm{s}}}(t) = \mathrm{col}[L_{p_i}(t) : i \in R_{\mathrm{s}}] \tag{7.2.43}$$

Turning our attention to the reactive generations, we obtain

$$L_{Q_i}(t) = [r_i'(t) - r_i(t) + \lambda_{q_i}(t)], \qquad i \in R_{\mathrm{G}} \tag{7.2.44}$$

which leads to

$$\mathbf{L}_Q(t) = \mathrm{col}[L_{Q_i}(t) : i \in R_{\mathrm{G}}] \tag{7.2.45}$$

As for the linear terms in the voltage variables, inspection of (7.2.3), (7.2.4), (7.2.15), and (7.2.16) reveals that these are dependent on the specified voltage components at the slack bus. Thus before we treat this, it is advantageous to introduce some auxiliary variables $a_{ij}(t)$ and $b_{ij}(t)$. These we define by

$$a_{ij}(t) = \lambda_{p_i}(t)G_{ij} - \lambda_{q_i}(t)B_{ij}, \qquad i,j \in R_{\mathrm{N}}, \quad i \neq j \quad \text{for } i \in R_{\mathrm{G}'} \tag{7.2.46}$$

$$a_{ii}(t) = \lambda_{p_i}(t)G_{ii} - \lambda_{q_i}(t)B_{ii} + \lambda_{e_i}(t), \qquad i \in R_{\mathrm{G}'} \tag{7.2.47}$$

$$b_{ij}(t) = -[\lambda_{p_i}(t)B_{ij} + \lambda_{q_i}(t)G_{ij}], \qquad i,j \in R_{\mathrm{N}}, \quad i \neq j \tag{7.2.48}$$

$$b_{ii}(t) = 0, \qquad i \in R_{\mathrm{N}} \tag{7.2.49}$$

The above are obtained as the coefficients of second-order terms in e_i and f_i in the augmented cost functional. We should point out that the a_{ij} are coefficients of e_ie_j and f_if_j terms. The coefficients b_{ij} correspond to f_ie_j and the negative of these for e_if_j. As a result, the terms e_if_i add zero contribution; we utilize this in the definition of b_{ii}. On this basis we have the linear terms coefficients given by

$$L_{e_i}(t) = [a_{si}(t) + a_{is}(t)]e_s(t) \tag{7.2.50}$$

$$L_{f_i}(t) = -[b_{si}(t) + b_{is}(t)]e_s(t) \qquad i \in R_{\mathrm{N}}, \quad i \neq s \tag{7.2.51}$$

where s is the slack bus. Note that for the slack bus we have

$$f_s(t) = 0.$$

Obviously we can now write the compact forms

$$\mathbf{L}_E(t) = \text{col}[\mathbf{L}_{E_d}(t), \mathbf{L}_{E_q}(t)] \tag{7.2.52}$$

with

$$\mathbf{L}_{E_d}(t) = \text{col}[L_{e_i}(t): i \in R_N, i \neq s] \tag{7.2.53}$$

$$\mathbf{L}_{E_q}(t) = \text{col}[L_{f_i}(t): i \in R_N, i \neq s] \tag{7.2.54}$$

The task for the hydro variables linear terms is straightforward. Here we find that our coefficients are

$$\mathbf{L}_{W_i}(t) = \text{col}[(n_i(t)A_i(t) + m_i(t)), \dot{m}_i(t)], \qquad i \in R_h \tag{7.2.55}$$

Thus we have the compact form for the hydro variables

$$\mathbf{L}_W(t) = \text{col}[\mathbf{L}_{W_i}(t): i \in R_h] \tag{7.2.56}$$

With these preliminary definitions we can write an overall vector given by

$$\mathbf{L}(t) = \text{col}[\mathbf{L}_p(t), \mathbf{L}_Q(t), \mathbf{L}_E(t), \mathbf{L}_W(t)] \tag{7.2.57}$$

We now treat the second order terms in turn. To start we define

$$B_{p_i}(t) = M_i(t) + \gamma_i, \qquad i \in R_s \tag{7.2.58}$$

The above are the coefficients of quadratics in the active generation for the thermal plants. For the hydro plants the term γ_i does not appear. A block diagonal matrix can be defined as

$$\mathbf{B}_p(t) = \text{diag}[\mathbf{B}_{p_h}(t), \mathbf{B}_{p_s}(t)] \tag{7.2.59}$$

with

$$\mathbf{B}_{p_h}(t) = \text{diag}[M_i(t)] \tag{7.2.60}$$

$$\mathbf{B}_{p_s}(t) = \text{diag}[B_{p_i}(t)] \tag{7.2.61}$$

In a similar fashion we treat the terms in the reactive power. Here we have

$$\mathbf{B}_Q(t) = (K'_{ij}(t)) \tag{7.2.62}$$

$$K'_{ij} = K_{ij}, \qquad i,j \in R_G, \quad i \neq j \tag{7.2.63}$$

$$K'_{ii}(t) = M_i(t) + K_{ii}, \qquad i \in R_G \tag{7.2.64}$$

We note that $\mathbf{B}_p(t)$ and $\mathbf{B}_Q(t)$ are symmetric.

We have treated the second-order coefficients in the voltage variables earlier. On the basis of Eqs. (7.2.46)–(7.2.49) we can write a matrix of coefficients as

$$\mathbf{B}_{E_0}(t) = \begin{bmatrix} \mathbf{A}_e(t) & -\mathbf{B}_e(t) \\ \mathbf{B}_e(t) & \mathbf{A}_e(t) \end{bmatrix}$$

where the submatrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are given by

$$\mathbf{A}_e(t) = \begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1N}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2N}(t) \\ \vdots & \vdots & & \vdots \\ a_{N1}(t) & a_{N2}(t) & \cdots & a_{NN}(t) \end{vmatrix}$$

$$\mathbf{B}_e(t) = \begin{bmatrix} 0 & b_{12}(t) & b_{13}(t) & \cdots & b_{1N}(t) \\ b_{21}(t) & 0 & b_{23}(t) & \cdots & b_{2N}(t) \\ \vdots & \vdots & \vdots & & \vdots \\ b_{N1}(t) & b_{N2}(t) & b_{N3}(t) & & 0 \end{bmatrix}$$

Note that $\mathbf{B}_{E_0}$ is nonsymmetric. However, one can replace $\mathbf{B}_{E_0}(t)$ by the symmetric matrix $\mathbf{B}_E(t)$ such that

$$\mathbf{E}^{\mathrm{T}}(t)\mathbf{B}_{E_0}(t)\mathbf{E}(t) = \mathbf{E}^{\mathrm{T}}\mathbf{B}_E(t)\mathbf{E}(t) \tag{7.2.65}$$

This can easily be done. Here we have

$$\mathbf{B}_E(t) = [\mathbf{B}_{E_0}(t) + \mathbf{B}_{E_0}^{\mathrm{T}}(t)]/2 \tag{7.2.66}$$

As a result we have $\mathbf{B}_E$ in the following partitioned form:

$$\mathbf{B}_E(t) = \begin{bmatrix} \mathbf{A}_E(t) & \mathbf{C}_E(t) \\ \mathbf{C}_E^{\mathrm{T}}(t) & \mathbf{A}_E(t) \end{bmatrix} \tag{7.2.67}$$

$$\mathbf{A}_E(t) = (a_{ij_E}(t)), \qquad \mathbf{C}_E(t) = (C_{ij_E}(t))$$

$$a_{ij_E}(t) = \tfrac{1}{2}[a_{ij}(t) + a_{ji}(t)] \tag{7.2.68}$$

$$C_{ij_E}(t) = \tfrac{1}{2}[-b_{ij}(t) + b_{ji}(t)] \tag{7.2.69}$$

Note that $\mathbf{A}_E(t)$ is symmetric but $\mathbf{C}_E(t)$ is not. Moreover, $\mathbf{C}_E(t) = -\mathbf{C}_E^{\mathrm{T}}(t)$. In terms of our original variables, using (7.2.46)–(7.2.49), we have

$$a_{ij_E}(t) = [(\lambda_{p_i} + \lambda_{p_j})G_{ij} - (\lambda_{q_i} + \lambda_{q_j})B_{ij}]/2, \qquad i,j \in R_{\mathrm{N}}, \quad i \neq j \quad \text{for } i \in R_{\mathrm{G}'} \tag{7.2.70}$$

$$a_{ii_E}(t) = a_{ii}(t), \qquad i \in R_{\mathrm{N}} \tag{7.2.71}$$

$$C_{ij_E}(t) = [(\lambda_{p_i} - \lambda_{p_j})B_{ij} + (\lambda_{q_i} - \lambda_{q_j})G_{ij}]/2, \qquad i,j \in R_{\mathrm{N}}, \quad i \neq j \tag{7.2.72}$$

$$C_{ii_E}(t) = 0, \qquad i \in R_{\mathrm{N}} \tag{7.2.73}$$

For the hydro variables it can be immediately seen that we have

$$\mathbf{B}_{\mathrm{W}}(t) = \operatorname{diag}[\mathbf{B}_{\mathrm{W}_i}(t)] \tag{7.2.74}$$

$$\mathbf{B}_{\mathrm{W}_i}(t) = \operatorname{diag}[C_i n_i(t), -\tfrac{1}{2}B_i \dot{n}_i(t)] \tag{7.2.75}$$

The overall square symmetric matrix $\mathbf{B}(t)$ is thus written as

$$\mathbf{B}(t) = \operatorname{diag}[\mathbf{B}_p(t), \mathbf{B}_Q(t), \mathbf{B}_E(t), \mathbf{B}_{\mathrm{W}}(t)] \tag{7.2.76}$$

Using the above definitions, the augmented cost functional $J_0(\cdot)$ of (7.2.12) reduces to

$$J_1(\cdot) = \int_0^{T_f} [\mathbf{L}^T(t)\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{B}(t)\mathbf{u}(t)]\,dt \tag{7.2.77}$$

Note that the terms explicitly independent of the control $\mathbf{u}(t)$ are dropped in the above.

Letting

$$\mathbf{V}^T(t) = \mathbf{L}^T(t)\mathbf{B}^{-1}(t) \tag{7.2.78}$$

the cost functional becomes

$$J_2(\cdot) = \int_0^{T_f} [\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]^T\mathbf{B}(t)[\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)]\,dt \tag{7.2.79}$$

Here $[-\frac{1}{2}\mathbf{V}^T(t)\mathbf{B}(t)\frac{1}{2}\mathbf{V}(t)]$ was dropped since it is explicitly independent of $\mathbf{u}(t)$.

Thus the problem is now reduced to that of minimizing J_2 subject to satisfying (7.2.11), which is written as

$$b_i = \int_0^{T_f} q_i(t)\,dt, \qquad i \in R_h \tag{7.2.80}$$

Define the column vector

$$\mathbf{b} = \mathrm{col}[b_i : i \in R_h] \tag{7.2.81}$$

Then we can write Eq. (7.2.80) as

$$\mathbf{b} = \int_0^{T_f} \mathbf{K}^T\mathbf{u}(s)\,ds \tag{7.2.82}$$

The matrix $\mathbf{K}$ in the above expression is compatible with $\mathbf{u}$ and contains zeros and ones as appropriate.

The control vector $\mathbf{u}(t)$ is considered an element of the Hilbert space $L_{2,\mathbf{B}}^{2(N+N_h+N_g)}[0, T_f]$ of the $2(N + N_h + N_g)$ vector-valued square integrable functions defined on $[0, T_f]$ endowed with the inner product definition

$$\langle \mathbf{V}(t), \mathbf{u}(t)\rangle = \int_0^{T_f} \mathbf{V}^T(t)\mathbf{B}(t)\mathbf{u}(t)\,dt$$

for every $\mathbf{V}(t)$ and $\mathbf{u}(t)$ in $L_{2,\mathbf{B}}^{2(N_h+N_g+N)}[0, T_f]$, provided that $\mathbf{B}(t)$ is positive definite. Note here that N is the number of elements in R_N (nodes) and N_h is the number of hydro plants conforming with R_h; similarly, N_g corresponds to elements of R_G.

The given vector $\mathbf{b}$ is considered an element of the real space $R^{(N_h)}$ with the Euclidean inner product definition

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \mathbf{X}^T\mathbf{Y}$$

for every $\mathbf{X}$ and $\mathbf{Y}$ in $R^{(N_h)}$.

Equation (7.2.82) defines a bounded linear transformation

$$\mathbf{T}: L_{2,\mathbf{B}}^{2[N_h+N_g+N]}[0, T_f] \to R^{N_h}.$$

This can be expressed as

$$\mathbf{b} = \mathbf{T}[\mathbf{u}(t)] \tag{7.2.83}$$

and the cost functional $J_2(\cdot)$ reduces to

$$J_2[\mathbf{u}(t)] = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|^2$$

Finally, it is necessary only to minimize

$$J[\mathbf{u}(t)] = \|\mathbf{u}(t) + \tfrac{1}{2}\mathbf{V}(t)\|$$

subject to satisfying (7.2.83) for a given $\mathbf{b}$ in $R^{(N_h)}$. This completes the formulation as a minimum norm problem.

7.2.2 The Optimal Solution

The optimal solution to the problem formulated above is found in a way similar to our earlier discussions of Chapters 5 and 6 for minimum norm problems. We thus have the optimal active power generation vector as

$$\mathbf{P}_\xi(t) = -\tfrac{1}{2}\mathbf{V}_p(t) \tag{7.2.84}$$

The optimal reactive power generation vector is

$$\mathbf{Q}_\xi(t) = -\tfrac{1}{2}\mathbf{V}_Q(t) \tag{7.2.85}$$

The optimal voltage variable vector is

$$\mathbf{E}_\xi(t) = -\tfrac{1}{2}\mathbf{V}_E(t) \tag{7.2.86}$$

For the hydro variables we have

$$\mathbf{W}_\xi(t) = \mathbf{T}^\dagger(\mathbf{b} + \mathbf{T}\tfrac{1}{2}\mathbf{V}(t))_{\mathrm{W}} - \tfrac{1}{2}\mathbf{V}_{\mathrm{W}}(t) \tag{7.2.87}$$

Here the evaluation of the optimal solution depends on the partitioned $\mathbf{V}(t)$ in the form

$$\mathbf{V}(t) = \mathrm{col}[\mathbf{V}_p(t), \mathbf{V}_Q(t), \mathbf{V}_E(t), \mathbf{V}_{\mathrm{W}}(t)] \tag{7.2.88}$$

As a result of the structure of the matrix $\mathbf{B}$ and vector $\mathbf{L}$ indicated in Eqs. (7.2.76), (7.2.57) and the definition of $\mathbf{V}$ given in Eq. (7.2.78) we can write

$$\mathbf{V}_p^{\mathrm{T}}(t) = \mathbf{L}_p^{\mathrm{T}}(t)\mathbf{B}_p^{-1}(t) \tag{7.2.89}$$

$$\mathbf{V}_Q^{\mathrm{T}}(t) = \mathbf{L}_Q^{\mathrm{T}}(t)\mathbf{B}_Q^{-1}(t) \tag{7.2.90}$$

$$\mathbf{V}_{\mathrm{E}}^{\mathrm{T}}(t) = \mathbf{L}_{\mathrm{E}}^{\mathrm{T}}(t)\mathbf{B}_{E}^{-1}(t) \tag{7.2.91}$$

$$\mathbf{V}_{\mathrm{W}}^{\mathrm{T}}(t) = \mathbf{L}_{\mathrm{W}}^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{W}}^{-1}(t) \tag{7.2.92}$$

The subvector $\mathbf{V}_p(t)$ is further partitioned into thermal generation $\mathbf{V}_{p_s}$ and hydro generation $\mathbf{V}_{p_h}$ subvectors. We can write

$$\mathbf{V}_{p_h}^{\mathrm{T}}(t) = \mathbf{L}_{p_h}^{\mathrm{T}}(t)\mathbf{B}_{p_h}^{-1}(t) \tag{7.2.93}$$

$$\mathbf{V}_{p_s}^{\mathrm{T}}(t) = \mathbf{L}_{p_s}^{\mathrm{T}}(t)\mathbf{B}_{p_s}^{-1}(t) \tag{7.2.94}$$

From (7.2.39) and (7.2.60) we obtain

$$\mathbf{V}_{p_h}(t) = \mathrm{col}[n_i(t) + l_i'(t) - l_i(t) - \lambda_{p_i}(t)]/M_i(t), \qquad i \in R_{\mathrm{h}} \tag{7.2.95}$$

and from (7.2.40), (7.2.61), and (7.2.58) we get

$$\mathbf{V}_{p_s}(t) = \mathrm{col}[\beta_i + l_i'(t) - l_i(t) - \lambda_{p_i}(t)]/[M_i(t) + \gamma_i], \qquad i \in R_{\mathrm{s}} \tag{7.2.96}$$

The subvector $\mathbf{V}_Q$ requires the evaluation of the inverse of the matrix $\mathbf{B}_Q$. Let us assume that

$$\mathbf{D}_Q(t) = \mathbf{B}_Q^{-1}(t) \tag{7.2.97}$$

$$\mathbf{D}_Q(t) = (d_{Q_{ij}}(t)) \tag{7.2.98}$$

As a result, the components of $\mathbf{V}_Q$ can be obtained using Eqs. (7.2.90) and (7.2.44). We thus have

$$V_{Q_i}(t) = \sum_{j \in R_G} [r_j'(t) - r_j(t) + \lambda_{q_j}(t)]d_{Q_{ij}}(t) \tag{7.2.99}$$

The inverse matrix $\mathbf{B}_E^{-1}(t)$ is denoted by

$$\mathbf{D}_E(t) = \mathbf{B}_E^{-1}(t) \tag{7.2.100}$$

Then using (7.2.67) and the property $\mathbf{C}_E(t) = -\mathbf{C}_E^{\mathrm{T}}(t)$, one obtains for $\mathbf{D}_E(t)$

$$\mathbf{D}_E(t) = \begin{bmatrix} \mathbf{Z}_E(t) & \mathbf{H}_E(t) \\ -\mathbf{H}_E(t) & \mathbf{Z}_E(t) \end{bmatrix} \tag{7.2.101}$$

where simple algebraic manipulations verify that

$$\mathbf{Z}_E(t) = [\mathbf{A}_E(t) + \mathbf{C}_E(t)\mathbf{A}_E^{-1}(t)\mathbf{C}_E(t)]^{-1} \tag{7.2.102}$$

$$\mathbf{H}_E(t) = -\mathbf{A}_E^{-1}(t)\mathbf{C}_E(t)\mathbf{Z}_E(t) \tag{7.2.103}$$

Thus (7.2.91) can be written in the partitioned form

$$\mathbf{V}_{E_d}(t) = \mathbf{L}_{E_d}(t)\mathbf{Z}_E(t) - \mathbf{L}_{E_q}(t)\mathbf{H}_E(t) \tag{7.2.104}$$

$$\mathbf{V}_{E_q}(t) = \mathbf{L}_{E_d}(t)\mathbf{H}_E(t) + \mathbf{L}_{E_q}(t)\mathbf{Z}_E(t) \tag{7.2.105}$$

Componentwise expressions can then be obtained using (7.2.50) and (7.2.51) in Eq. (7.2.104) for V_{e_i} and Eq. (7.2.105) for V_{f_i}.

Finally, the vector $\mathbf{V}_{\mathrm{W}}(t)$ can be obtained using Eqs. (7.2.56) and (7.2.74) in (7.2.92). Componentwise we thus get

$$\mathbf{V}^{\mathrm{T}}_{\mathrm{W}_i}(t) = \mathbf{L}^{\mathrm{T}}_{\mathrm{W}_i}(t)\mathbf{B}^{-1}_{\mathrm{W}_i}(t), \qquad i \in R_{\mathrm{h}} \tag{7.2.106}$$

with

$$\mathbf{V}_{\mathrm{W}_i}(t) = \mathrm{col}[V_{\mathrm{W}_{q_i}}(t), V_{\mathrm{W}_{Q_i}}(t)]$$

Then (7.2.55) and (7.2.75) yield

$$V_{\mathrm{W}_{q_i}}(t) = [\dot{m}_i(t) + n_i(t)A_i(t)]/[C_i n_i(t)], \qquad i \in R_{\mathrm{h}} \tag{7.2.107}$$

$$V_{\mathrm{W}_{Q_i}}(t) = -2\dot{m}_i(t)/[B_i \dot{n}_i(t)], \qquad i \in R_{\mathrm{h}} \tag{7.2.108}$$

The pseudoinverse operator for our system is obtained as outlined in the previous chapters. The result in component form is

$$\mathbf{T}^{\dagger}\xi|_{\mathbf{P}} = \mathbf{0} \tag{7.2.109}$$

$$\mathbf{T}^{\dagger}\xi|_{\mathbf{Q}} = \mathbf{0} \tag{7.2.110}$$

$$\mathbf{T}^{\dagger}\xi|_{\mathbf{E}} = \mathbf{0} \tag{7.2.111}$$

$$\mathbf{T}^{\dagger}\xi|_{\mathbf{W}_i} = \mathrm{col}\left[\xi_i \Big/ \left[n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\,dt\right], 0\right] \tag{7.2.112}$$

The conditions (7.2.109)–(7.2.111) have already been used in writing (7.2.84)–(7.2.86).

We can now state the optimal solution of Eqs. (7.2.84)–(7.2.87) in component form. The optimal active thermal generation is

$$P_{\mathrm{s}_{i_\xi}}(t) = -\frac{[\beta_i + l'_i(t) - l_i(t) - \lambda_{p_i}(t)]}{2[M_i(t) + \gamma_i]}, \qquad i \in R_{\mathrm{s}} \tag{7.2.113}$$

For the active hydro generations we have

$$2M_i(t)P_{\mathrm{h}_{i_\xi}}(t) = -[n_i(t) + l'_i(t) - l_i(t) - \lambda_{p_i}(t)], \qquad i \in R_{\mathrm{h}} \tag{7.2.114}$$

For the reactive power generations we have

$$Q_{i_\xi}(t) = -\tfrac{1}{2}\left[\sum_{j \in R_{\mathrm{G}}} [e'_j(t) - e_j(t) + \lambda_{q_j}(t)]d_{Q_{ij}}(t)\right]. \tag{7.2.115}$$

We can also find that the optimal voltage variables are given by

$$e_{i_\xi}(t) = -[e_s(t)/2]\left\{\sum_{\substack{j \in R_{\mathrm{N}} \\ j \neq s}} [a_{sj}(t) + a_{js}(t)]\eta_{ij}(t) + \sum_{\substack{j \in R_{\mathrm{N}} \\ j \neq s}} [b_{sj}(t) + b_{js}(t)]h_{ij}(t)\right\}, \qquad i \in R_{\mathrm{N}} \tag{7.2.116}$$

$$f_{i_\xi}(t) = -[e_s(t)/2]\left\{ \sum_{\substack{j \in R_N \\ j \neq s}} [a_{sj}(t) + a_{js}(t)]h_{ij}(t) - \sum_{\substack{j \in R_N \\ j \neq s}} [b_{sj}(t) + b_{js}(t)]\eta_{ij}(t) \right\}, \qquad i \in R_N \tag{7.2.117}$$

Finally, the optimal hydro variables obtained are

$$q_{\xi_i}(t) = -[m_i(t) + n_i(t)A_i(t)]/[2C_{W_i}n_i(t)] + \left[b_i + \int_0^{T_f} \frac{m_i(t) + n_i(t)A_i(t)}{2C_i n_i(t)}\,dt\right] \Bigg/ \left[n_i(t)\int_0^{T_f} \frac{1}{n_i(t)}\,dt\right], \qquad i \in R_h \tag{7.2.118}$$

$$Q_{W_{\xi_i}}(t) = \dot{m}_i(t)/[B_{W_i}\dot{n}_i(t)], \qquad i \in R_h \tag{7.2.119}$$

Equations (7.2.113)–(7.2.119) together with the equality constraints given by (7.2.3), (7.2.4), (7.2.8), (7.2.9), and (7.2.10) and the exclusion equations (7.2.24)–(7.2.28), completely specify the optimal solution.

7.3 OPTIMAL LOAD FLOW WITH MULTICHAINS OF HYDRO PLANTS: REALISTIC MODELS

In formulating problems of optimal economic operation in Chapters 5 and 6 and the previous section, it was assumed that the efficiency of each hydro plant remains constant over the operating range of interest. Another assumption made there was that of vertical-sided reservoirs at the hydro plants. In this section these two assumptions are relaxed and the modifications to both formulation and optimal solution are shown for the case of optimal hydro–thermal load flow. Moreover, we treat a general multichain configuration of hydro plants, as was the case in Section 6.4.

Our objective in this problem is to minimize the combined cost functional given by (7.2.1) subject to satisfying two sets of constraints. The first is given by:

(1) Electric network equality constraints given by the load flow equations (7.2.3), (7.2.4), and (7.2.8).

(2) Inequality type constraints on the electric variables as given by Eqs. (7.2.5)–(7.2.7).

(3) Volume of water discharge constraints (7.2.10) and (7.2.11).

The second set of constraints pertains to the hydro model. This differs from the previous section's model given by Eq. (7.2.9). We detail the hydro model in the following.

The ith hydro plant's active power generation is given by

$$P_{\mathrm{h}_i}(t)G_i(t) = h_i(t)q_i(t) \tag{7.3.1}$$

The function $G_i(t)$ is the inverse of the efficiency of the plant. The effective hydraulic head at the ith hydro plant is the difference between the forebay elevation $y_i(t)$ and the tail-race level $y_{\mathrm{T}_i}(t)$:

$$h_i(t) = y_i(t) - y_{\mathrm{T}_i}(t) \tag{7.3.2}$$

The tail-water elevation varies with the rate of water discharge $q_i(t)$ according to the relation

$$y_{\mathrm{T}_i}(t) = y_{\mathrm{T}_{i0}} + \beta_{\mathrm{T}_i}q_i(t), \qquad i \in R_{\mathrm{h}} \tag{7.3.3}$$

$y_{\mathrm{T}_{i0}}$ and β_{T_i} are known constants corresponding to the tail-race geometry. Equation (7.3.1) is a restatement of Eq. (2.2.4); Eq. (7.3.2) corresponds to Eq. (2.2.12) and (7.3.3) to (2.2.13). Combining (7.3.1), (7.3.2), and (7.3.3), we obtain

$$P_{\mathrm{h}_i}(t)G_i(t) + y_{\mathrm{T}_{i0}}q_i(t) + \beta_{\mathrm{T}_i}q_i^2(t) - q_i(t)y_i(t) = 0, \qquad i \in R_{\mathrm{h}} \tag{7.3.4}$$

For all practical purposes the variation of the inverse efficiency $G_i(t)$ with the active power generation can be represented as indicated in Chapter 2 by

$$\alpha_{\mathrm{g}_i}G_i^2(t) + \beta_{\mathrm{g}_i}G_i(t) + \gamma_{\mathrm{g}_i}P_{\mathrm{h}_i}^2(t) + \delta_{\mathrm{g}_i}P_{\mathrm{h}_i}(t) + \theta_{\mathrm{g}_i} = 0 \tag{7.3.5}$$

This is the equation of an ellipse and is assumed to hold true over the operating range $(P_i^{\mathrm{m}} \le P_{\mathrm{h}_i}(t) \le P_i^{\mathrm{M}})$ of the hydro plant. This is shown in Fig. 7.1. We assume a trapezoidal reservoir representation. The forebay elevation is related to the forebay volume of water stored $s_i(t)$ by the relation

$$s_i(t) = \alpha_{y_i}y_i^2(t) + \beta_{y_i}y_i(t) \tag{7.3.6}$$

where α_{y_i} and β_{y_i} are constants for the trapezoidal reservoir given by

$$\alpha_{y_i} = l_i \tan\phi_i, \qquad \beta_{y_i} = l_i b_{0_i}$$

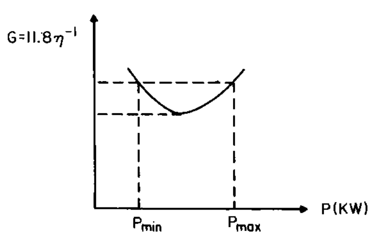
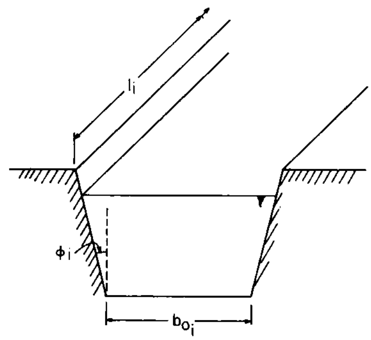


Fig. 7.1 Efficiency versus power output.

This is a truncated expression obtainable from Eq. (2.2.18). The geometry of a trapezoidal reservoir is shown in Fig. 7.2.

The reservoir dynamics are described by storage–discharge relations that depend on the relative location of the plant in the hydro network. An example network is shown in Fig. 7.3. For an upstream or isolated hydro plant

Fig. 7.2 Trapezoidal reservoir configuration.



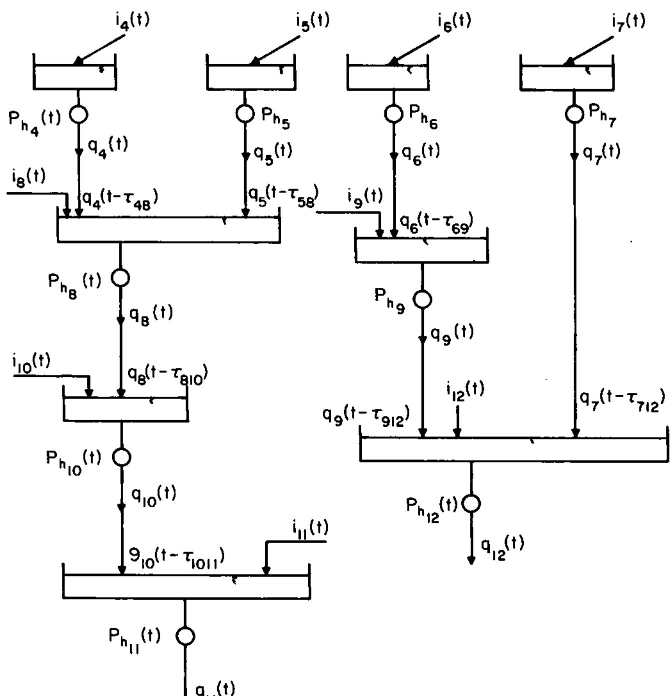


Fig. 7.3 General layout of hydro plants.

we have

$$\dot{s}_i(t) = i_i(t) - q_i(t), \qquad i \in R_{\mathrm{hU}} \cup R_{\mathrm{hIS}} \tag{7.3.7}$$

This is a restatement of Eq. (2.2.17). R_{hU} denotes the set of upstream plants and R_{hIS} the isolated plants. This notation follows that detailed in Section 6.4. For intermediate and downstream plants the reservoir dynamics are described by

$$\dot{s}_i(t) = i_i(t) + \sum_{j \in R_{\mathrm{h}_i}} q_j(t - \tau_j) - q_i(t), \qquad i \in R_{\mathrm{hD}} \cup R_{\mathrm{hI}} \tag{7.3.8}$$

Here R_{h_i} denotes plants upstream from the ith. The flow from each plant has a transport delay of τ_j ($j \in R_{\mathrm{h}_i}$) to the plant. Integrating Eq. (7.3.8) one obtains

$$s_i(t) = D_i(t) + \sum_{j \in R_{\mathrm{h}_i}} Y_j(t, \tau_j) - Q_{\mathrm{w}_i}(t) \tag{7.3.9}$$

Equation (7.3.9) is the same as (6.4.2). The definitions of the various variables in the above are given in Eqs. (6.4.3)–(6.4.7).

It is clear that Eq. (7.3.9) is a general expression which is applicable for all types of plants. Indeed the set R_{h_i} is null for upstream and isolated plants. The expressions given above enable the elimination of the storage variable $s_i(t)$ using (7.3.6). As a result we attain

$$\alpha_{y_i} y_i^2(t) + \beta_{y_i} y_i(t) = D_i(t) - \tilde{Q}_i(t) \tag{7.3.10}$$

$$\tilde{Q}_i(t) = Q_{\mathrm{w}_i}(t), \qquad i \in R_{\mathrm{hU}} \cup R_{\mathrm{hIS}} \tag{7.3.11}$$

$$\tilde{Q}_i(t) = Q_{\mathrm{w}_i}(t) - \sum_{j \in R_{\mathrm{h}_i}} Y_j(t, \tau_j), \qquad i \in R_{\mathrm{hD}} \cup R_{\mathrm{hI}} \tag{7.3.12}$$

7.3.1 Formulation

The problem posed in this section can be formulated as a minimum norm problem. We observe here that presently we replace (7.2.9) by (7.3.4), (7.3.5) and (7.3.10) by attendant definitions (7.3.11) and (7.3.12). Our hydro variables presently include $G_i(t)$, the inverse efficiency functions, $y_i(t)$, the forebay elevation functions, and $Y(t, \tau)$, the delayed volumes, in addition to $q_i(t)$, the rate of discharge functions, and $Q_{\mathrm{w}_i}(t)$, the volume of water discharge. It is our intention to show the modifications this necessitates.

It is evident that the modified cost functional (7.2.13) will be altered. Here we write

$$I_0(t) = \sum_{i=1}^{11} I_{0_i}(t) \tag{7.3.13}$$

The expressions for I_{0i} for $i = 1, \ldots, 10$ will remain the same as given in Eqs. (7.2.14)–(7.2.23) with the exception of I_{04} given by Eq. (7.2.17). This we replace by

$$\begin{aligned} I_{04}(t) = \Bigg\{ \sum_{i \in R_h} & n_i(t)[P_{h_i}(t)G_i(t) + y_{T_{i0}}(t)q_i(t) + \beta_{T_i}q_i^2(t) - q_i(t)y_i(t)] \\ & + \tilde{m}_i(t)[\alpha_{g_i}G_i^2(t) + \beta_{g_i}G_i(t) + \gamma_{g_i}P_{h_i}^2(t) + \delta_{g_i}P_{h_i}(t) + \theta_{g_i}] \\ & + [m_i(t)q_i(t)Q_{w_i}(t) + \tfrac{1}{2}\dot{m}_i(t)Q_{w_i}^2(t)] \\ & + \tilde{r}_i(t)\left[\alpha_{y_i}y_i^2(t) + \beta_{y_i}y_i(t) - D_i(t) + Q_{w_i}(t) - \sum_{j \in R_{h_i}} Y_j(t, \tau_j)\right]\Bigg\} \end{aligned} \tag{7.3.14}$$

The first term in brackets corresponds to constraints (7.3.4) with associated multipliers $n_i(t)$, the second to constraints (7.3.5) with associated multipliers $\tilde{m}_i(t)$, the third to constraints (7.2.10) with multipliers $m_i(t)$, and the last corresponds to constraints (7.3.10) utilizing (7.3.11) and (7.3.12) with associated multiplier set $\tilde{r}_i(t)$. We note here that we still have to account for the relationship between the new variable set Y_j and Q_{w_j}. This is similar to the situation in Section 6.4.1 and will give rise to the term $I_{0_{11}}$ required in (7.3.13). This we treat in the following.

The required relation is

$$Y_j(t,\tau_j) = \begin{cases} \psi_j(t, \tau_j), & t \le \tau_j, \\ \psi_j(\tau_j, \tau_j) + Q_{w_j}(t - \tau_j), & t > \tau_j, \end{cases} \qquad j \in R_{h_i}$$

This will be replaced as was done in Chapter 6 by an equivalent expression so as to avoid arriving at a singular matrix in the augmented cost expression. Here we use

$$Y_j^2(t,\tau_j) = \begin{cases} \psi_j^2(t, \tau_j), & 0 \le t \le \tau_j \\ \psi_j^2(\tau_j, \tau_j) + 2\psi_j(\tau_j, \tau_j)Q_{w_j}(t - \tau_j) + Q_{w_j}^2(t - \tau_j), & \tau_j < t < T_f \end{cases} \tag{7.3.15}$$

Now associating the multiplier set $r_i(t)$ with the above constraints we can write the integral of $I_{0_{11}}(t)$, namely, $J_{0_{11}}$, as

$$\begin{aligned} J_{0_{11}} = & \int_0^{T_f} \sum_{i \in R_{hID}} \sum_{j \in R_{h_i}} r_j(t) Y_j^2(t, \tau_j)\, dt \\ & - \sum_{i \in R_{hID}} \sum_{j \in R_{h_i}} \int_{\tau_j}^{T_f} r_j(t)[Q_{w_j}^2(t - \tau_j) + 2\psi_j(\tau_j, \tau_j)Q_{w_j}(t - \tau_j)]\, dt \end{aligned} \tag{7.3.16}$$

Note that the ψ_j are known functions of time and hence are not required in the analysis. We will make the following further substitutions:

$$\dot{p}_i(t, \tau_i) = \begin{cases} 2\psi_i(\tau_i, \tau_i) r_i(t + \tau_i), & 0 \le t \le T_f - \tau_i \\ 0, & T_f - \tau_i < t \le T_f \end{cases} \tag{7.3.17}$$

$$\theta_i(t, \tau_i) = \begin{cases} r_i(t + \tau_i) & 0 \le t \le T_f - \tau_i \\ 0 & T_f - \tau_i < t \le T_f \end{cases} \tag{7.3.18}$$

As a result we obtain

$$J_{0_{11}} = \int_0^{T_f} \sum_{i \in R_{\mathrm{hID}}} \sum_{j \in R_{\mathrm{h}_i}} [r_j(t) Y_j^2(t, \tau_j) - \theta_j(t, \tau_j) Q_{\mathrm{w}_j}^2(t) - \dot{p}_j(t, \tau_j) Q_{\mathrm{w}_j}(t)]\, dt \tag{7.3.19}$$

Performing integration by parts on the last term in the integrand of (7.3.19), we can therefore write $I_{0_{11}}$ as

$$I_{0_{11}} = \sum_{i \in R_{\mathrm{hID}}} \sum_{j \in R_{\mathrm{h}_i}} [r_j(t) Y_j^2(t, \tau_j) - \theta_j(t, \tau_j) Q_{\mathrm{w}_j}^2(t) + p_j(t, \tau_j) q_j(t)] \tag{7.3.20}$$

We are now ready to introduce our control vector. Firstly the active power generation vector $\mathbf{P}(t)$ in (7.2.29) will have the same structure; however, its dimension will be increased to accommodate the G_i. This is done by redefining $\mathbf{P}_\mathrm{h}$ as

$$\mathbf{P}_\mathrm{h}(t) = \mathrm{col}[P_i(t), G_i(t) : i \in R_\mathrm{h}] \tag{7.3.21}$$

The vector $\mathbf{P}_\mathrm{s}(t)$ will remain as given in Eq. (7.2.31).

The hydraulic variables of the hydro plants will define the vector $\mathbf{W}(t)$ as expressed in Eq. (7.2.37). The composition of the individual $W_i(t)$ will change from that indicated in (7.2.36) to accommodate the extra variables required for the present formulation. These are $y_i(t)$ and $Y_i(t, \tau_i)$. Each of the hydro subvectors has a dimension and a definition that depends on the category of the plant. For the upstream and isolated plants only the variables $y_i(t)$, $q_i(t)$, and Q_i(t) are involved; thus we have

$$\mathbf{W}_i(t) = \mathrm{col}[y_i(t), Q_i(t), q_i(t)], \qquad i \in R_{\mathrm{hU}} \cup R_{\mathrm{h_{IS}}} \tag{7.3.22}$$

On the other hand, the subvector $\mathbf{W}_i(t)$ for either an intermediate or a downstream plant must include the corresponding decision variables $Y_j(t, \tau_j)$ in addition to $y_i(t)$, $q_i(t)$, and $Q_i(t)$. We thus have

$$\mathbf{W}_i(t) = \mathrm{col}[y_i(t), Q_i(t), q_i(t), \mathbf{Y}_{i\mathrm{w}}(t)] \tag{7.3.23}$$

with

$$\mathbf{Y}_{i\mathrm{w}}(t) = \mathrm{col}[Y_j(t, \tau_j) : j \in R_{\mathrm{h}_i}], \qquad i \in R_{\mathrm{hID}} \tag{7.3.24}$$

We will now consider the auxiliary vector $\mathbf{L}(t)$, which will retain its structure given in Eq. (7.2.57). Two changes will have to be made. The first is in $\mathbf{L}_{p_h}$, which will now be given as

$$\mathbf{L}_{p_h}(t) = \mathrm{col}[\mathbf{L}_{p_{h_i}}(t) : i \in R_h] \tag{7.3.25}$$

with individual two-dimensional subvectors

$$\mathbf{L}_{p_{h_i}}(t) = \mathrm{col}[\tilde{m}_i(t)\delta_{g_i}(t) + l_i'(t) - l_i(t) - \lambda_{p_i}(t), \tilde{m}_i(t)\beta_{g_i}], \qquad i \in R_h \tag{7.3.26}$$

The vector $\mathbf{L}_w(t)$ in turn is expressed as

$$\mathbf{L}_w(t) = \mathrm{col}[\mathbf{L}_{w_i}(t) : i \in R_h] \tag{7.3.27}$$

with

$$\mathbf{L}_{w_i}(t) = \mathrm{col}[\beta_{y_i}\tilde{r}_i(t), \tilde{r}_i(t), \mathbf{l}_{w_i}(t)] \tag{7.3.28}$$

Here again the subvectors $\mathbf{l}_{w_i}(t)$ are category-dependent and turn out to be of different dimension for each category. To simplify the presentation let us define a new function to indicate the category dependence. Thus we introduce for the upstream and intermediate plants

$$\hat{n}_i(t) = n_i(t)y_{T_{i0}} + p_i(t, \tau_i), \qquad i \in R_{hU} \cup R_{hI} \tag{7.3.29}$$

For the isolated and downstream we have

$$\hat{n}_i(t) = n_i(t)y_{T_{i0}}, \qquad i \in R_{hIS} \cup R_D \tag{7.3.30}$$

On this basis we have for the upstream and isolated plants

$$\mathbf{l}_{w_i}(t) = \hat{n}_i(t), \qquad i \in R_{hU} \cup R_{hI} \tag{7.3.31}$$

For the downstream and intermediate plants we have

$$\mathbf{l}_{w_i}(t) = \mathrm{col}[\hat{n}_i(t), \mathbf{l}_{y_i}(t)], \qquad i \in R_{hD} \cup R_{hID} \tag{7.3.32}$$

The dimension of the vector $\mathbf{l}_{y_i}(t)$ peculiar to this category depends on the number of plants feeding into the ith plant. Thus we have

$$\mathbf{l}_{y_i} = \mathrm{col}[-\tilde{r}_i, -\tilde{r}_i, \ldots] \tag{7.3.33}$$

where $\mathbf{l}_{y_i}$ has a dimension equal to the number of hydro plants feeding into the ith reservoir, i.e., the size of the set R_{h_i}.

The matrix $\mathbf{B}(t)$ will encounter changes in a fashion similar to that for the vector $\mathbf{L}(t)$. The first change is in the submatrix $\mathbf{B}_{p_h}$ which will now take on the form

$$\mathbf{B}_{p_h}(t) = \mathrm{diag}[\mathbf{B}_{p_{h_i}}(t)], \qquad i \in R_h \tag{7.3.34}$$

Individual submatrices are given by

$$\mathbf{B}_{p_{h_i}}(t) = \begin{bmatrix} [M_i(t) + \tilde{m}_i(t)\gamma_{g_i}] & \frac{1}{2}n_i(t) \\ \frac{1}{2}n_i(t) & \tilde{m}_i(t)\alpha_{g_i} \end{bmatrix}, \qquad i \in R_h \tag{7.3.35}$$

The matrix $\mathbf{B}_w(t)$ is again a block diagonal in submatrices $\mathbf{B}_{w_i}(t)$, each of which is category dependent.

Let us define basic matrices $\mathbf{B}_{wB_i}(t)$ by the following:

$$\mathbf{B}_{wB_i}(t) = \begin{bmatrix} \alpha_{y_i}\tilde{r}_i(t) & 0 & -\frac{1}{2}n_i(t) \\ 0 & \dot{m}_i(t) - \tilde{\theta}_i(t) & \frac{1}{2}m_i(t) \\ -\frac{1}{2}n_i(t) & \frac{1}{2}m_i(t) & \beta_{T_i}n_i(t) \end{bmatrix}, \qquad i \in R_h \tag{7.3.36}$$

Here $\tilde{\theta}_i(t)$ depends on the plant category. The composition of the submatrices $\mathbf{B}_{w_i}(t)$ is given as follows:

(1) For an upstream plant we have

$$\mathbf{B}_{w_i}(t) = \mathbf{B}_{wB_i}(t), \qquad i \in R_{hU} \tag{7.3.37}$$

with

$$\tilde{\theta}_i(t) = \theta_i(t, \tau_i) \tag{7.3.38}$$

(2) For an isolated plant we have

$$\mathbf{B}_{w_i}(t) = \mathbf{B}_{wB_i}(t), \qquad i \in R_{hIS} \tag{7.3.39}$$

$$\tilde{\theta}_i(t) = 0 \tag{7.3.40}$$

(3) For a downstream plant we have

$$\mathbf{B}_{w_i}(t) = \operatorname{diag}[\mathbf{B}_{wB_i}(t), \mathbf{B}_{wy_i}(t)] \tag{7.3.41}$$

where

$$\mathbf{B}_{wy_i}(t) = \operatorname{diag}\{r_j(t) : j \in R_{h_i}\}, \qquad i \in R_{hD} \tag{7.3.42}$$

$$\tilde{\theta}_i(t) = 0 \tag{7.3.43}$$

(4) For intermediate plants we have

$$\mathbf{B}_{w_i}(t) = \operatorname{diag}[\mathbf{B}_{wB_i}(t), \mathbf{B}_{wy_i}(t)] \tag{7.3.44}$$

where

$$\mathbf{B}_{wy_i}(t) = \operatorname{diag}\{r_j(t) : j \in R_{h_i}\} \tag{7.3.45}$$

$$\tilde{\theta}_i(t) = \theta_i(t, \tau_i), \qquad i \in R_{hI} \tag{7.3.46}$$

Note that $\tilde{\theta}_i(t)$ is different from zero in the cases of plants whose discharge flows downstream to another plant. In this case we assume that this flow occurs at one plant only.

With the modifications outlined above we can then proceed to formulate the problem as a minimum norm one. This is similar to the procedure outlined, for example, in the previous section starting with Eq. (7.2.77). We note here that the dimensions of the two problems are different.

7.3.2 The Optimal Solution

The optimal solution to the problem follows the general format:

$$\mathbf{u}_\xi(t) = \mathbf{T}^\dagger[\mathbf{b} + \mathbf{T}(\tfrac{1}{2}\mathbf{V}(t))] - \tfrac{1}{2}\mathbf{V}(t) \tag{7.3.47}$$

In subvector form we have

$$\mathbf{P}_{s_\xi}(t) = -\tfrac{1}{2}\mathbf{V}_{p_s}(t) \tag{7.3.48}$$

$$\mathbf{Q}_\xi(t) = -\tfrac{1}{2}\mathbf{V}_Q(t) \tag{7.3.49}$$

$$\mathbf{E}_\xi(t) = -\tfrac{1}{2}\mathbf{V}_E(t) \tag{7.3.50}$$

$$\mathbf{P}_{h_\xi}(t) = -\tfrac{1}{2}\mathbf{V}_{p_h}(t) \tag{7.3.51}$$

$$\mathbf{W}_\xi(t) = \mathbf{T}^\dagger[\mathbf{b} + \mathbf{T}(\tfrac{1}{2}\mathbf{V}(t))]\big|_{\mathrm{w}} - \tfrac{1}{2}\mathbf{V}_{\mathrm{w}}(t) \tag{7.3.52}$$

The first three expressions are identical to those obtained in Section 7.2.2. We thus use Eqs. (7.2.96), (7.2.99), (7.2.104), and (7.2.105) and the accompanying definitions to evaluate the optimal thermal active power generation, reactive power generation, in-phase and quadrature voltage variables, respectively. The optimal hydro power generation vector is evaluated using our present modifications. This is obtained using Eqs. (7.3.26) and (7.3.35) in the definition (7.2.93). The result is

$$\mathbf{V}_{p_h}(t) = \mathrm{col}[V_{p_i}(t), V_{G_i}(t) : i \in R_h] \tag{7.3.53}$$

$$V_{p_i}(t) = \{[\tilde{m}_i(t)\delta_{g_i} + l_i'(t) - l_i(t) - \lambda_{p_i}(t)]\tilde{m}_i(t)\alpha_{g_i} - \tfrac{1}{2}\tilde{m}_i(t)n_i(t)\beta_{g_i}\}/\Delta_{p_i}(t) \tag{7.3.54}$$

$$V_{G_i}(t) = \{[M_i(t) + \tilde{m}_i(t)\gamma_{g_i}]\tilde{m}_i(t)\beta_{g_i} - \tfrac{1}{2}n_i(t) \times [\tilde{m}_i(t)\delta_{g_i} + l_i'(t) - l_i(t) - \lambda_{p_i}(t)]\}/\Delta_{p_i}(t) \tag{7.3.55}$$

In the above Δ_{p_i} is the determinant given by

$$\Delta_{p_i}(t) = \tilde{m}_i(t)\alpha_{g_i}[M_i(t) + \gamma_{g_i}\tilde{m}_i(t)] - [n_i^2(t)/4] \tag{7.3.56}$$

The constitution of the vector $\mathbf{V}_{\mathrm{w}}(t)$ will follow the structure adopted in this section. For upstream and isolated plants we have

$$\mathbf{V}_{\mathrm{w}_i}(t) = \mathbf{V}_{\mathrm{w}B_i}(t) \tag{7.3.57}$$

where

$$\mathbf{V}_{\mathrm{w}B_i}(t) = \mathbf{B}_{\mathrm{w}B_i}^{-1}(t)\mathbf{L}_{\mathrm{w}_i}(t), \qquad i \in R_{\mathrm{hU}} \cup R_{\mathrm{hIS}} \tag{7.3.58}$$

and where $\mathbf{B}_{\mathrm{w}B_i}(t)$ is as defined in Eq. (7.3.36) and $\mathbf{L}_{\mathrm{w}_i}(t)$ is as defined in Eq. (7.3.28). Note that for upstream plants Eqs. (7.3.38) and (7.3.29) apply, while for isolated plants Eqs. (7.3.40) and (7.3.30) apply. In the case of downstream and intermediate plants the dimension of $\mathbf{V}_{\mathrm{w}_i}$ is increased. This increase is due to the inclusion of terms corresponding to feed-in plants. Due to the block-diagonal nature of $\mathbf{B}_{\mathrm{w}_i}$ as defined in Eqs. (7.3.41) and (7.3.44) we then have

$$\mathbf{V}_{\mathrm{w}_i}(t) = \mathrm{col}[\mathbf{V}_{\mathrm{w}B_i}(t), \mathbf{V}_{\mathrm{w}y_i}(t)], \qquad i \in R_{\mathrm{hD}} \cup R_{\mathrm{hID}} \tag{7.3.59}$$

Here use is to be made of (7.3.42), (7.3.45), (7.3.32), and (7.3.33) to evaluate the vectors involved.

The matrix $\mathbf{B}_{\mathrm{w}B_i}(t)$ as given by Eq. (7.3.36) has an inverse which we define as $\mathbf{D}_{\mathrm{w}B_i}$; thus

$$\mathbf{B}_{\mathrm{w}B_i}^{-1}(t) = \mathbf{D}_{\mathrm{w}B_i}(t) \tag{7.3.60}$$

On the basis of (7.3.60) and (7.3.28) we thus arrive at the components of $\mathbf{V}_{\mathrm{w}B_i}(t)$:

$$V_{p_{y_i}}(t) = \tilde{r}_i(t)[\beta_{y_i} d_{\mathrm{w}11_i}(t) + d_{\mathrm{w}12_i}(t)] + \hat{n}_i(t) d_{\mathrm{w}13_i}(t) \tag{7.3.61}$$

$$V_{p_{Q_i}}(t) = \tilde{r}_i(t)[\beta_{y_i} d_{\mathrm{w}12_i}(t) + d_{\mathrm{w}22_i}(t)] + \hat{n}_i(t) d_{\mathrm{w}23_i}(t) \tag{7.3.62}$$

$$V_{p_{q_i}}(t) = \tilde{r}_i(t)[\beta_{y_i} d_{\mathrm{w}13_i}(t) + d_{\mathrm{w}23_i}(t)] + \hat{n}_i(t) d_{\mathrm{w}33_i}(t) \tag{7.3.63}$$

This applies for all hydro plants, provided that elements in the relations are evaluated on the basis of the plant category as outlined before. The vector $\mathbf{V}_{\mathrm{w}_{y_i}}(t)$ is needed for the downstream and intermediate plants. Elements of this vector are given by

$$\mathbf{V}_{\mathrm{w}_{y_i}}(t) = -\tilde{r}_i(t)\,\mathrm{col}[1/r_j(t) : j \in R_{\mathrm{h}_i}] \tag{7.3.64}$$

This completes the definitions of the vector $\mathbf{V}(t)$.

The optimality expression of Eq. (7.3.52) will then be obtained on the basis of results of Chapter 3 and the above defining relationships. Here we have the optimal forebay elevations given by

$$y_{\xi_i}(t) = \eta_i d_{\mathrm{w}13_i}(t) - \tfrac{1}{2} V_{p_{y_i}}(t), \qquad i \in R_{\mathrm{h}} \tag{7.3.65}$$

The optimal volume of water discharge is

$$Q_{\xi_i}(t) = \eta_i d_{\mathrm{w}23_i}(t) - \tfrac{1}{2} V_{p_{Q_i}}(t), \qquad i \in R_{\mathrm{h}} \tag{7.3.66}$$

The optimal rate of discharge is

$$q_{\xi_i}(t) = \eta_i d_{\mathrm{w}33_i}(t) - \tilde{q}_i(t), \qquad i \in R_{\mathrm{h}} \tag{7.3.67}$$

The optimal delayed volume for feed-in plants is

$$Y_{\xi_j}(t) = \tilde{r}_i(t)/2r_j(t), \qquad j \in R_{\mathrm{h}_i}, \quad i \in R_{\mathrm{hD}} \cup R_{\mathrm{hI}} \tag{7.3.68}$$

where for notational simplicity we introduce

$$\tilde{q}_i(t) = \tfrac{1}{2}V_{p_{q_i}}(t), \qquad i \in R_h \tag{7.3.69}$$

and

$$\eta_i = \left[\int_0^{T_f} d_{w_{33_i}}(t)\,dt\right]^{-1}\left[b_i + \int_0^{T_f} \tilde{q}_i(t)\,dt\right], \qquad i \in R_h \tag{7.3.70}$$

The optimality conditions for our problem are completed by writing

$$P_{\xi_{h_i}}(t) = -\tfrac{1}{2}V_{p_i}(t) \tag{7.3.71}$$

and

$$G_{\xi_i}(t) = -\tfrac{1}{2}V_{G_i}(t) \tag{7.3.72}$$

In the right-hand side of the above use should be made of Eqs. (7.3.54) and (7.3.55).

7.4 COMMENTS AND REFERENCES

Probably one of the earliest works on optimal load flow that incorporates a hydro subsystem is that of Ramamoorty and Rao (1970). The inclusion of an energy constraint on the hydro schedule is the main feature of the work. Dillon and Morsztyn (1971) obtained scheduling equations for the hydro–thermal optimal load flow problem using the generalized maximum principle. Hydro efficiency and time delays between cascading plants are neglected in their formulation. In a subsequent paper (1972) the same team describes computational implementation for a three-bus system. The idea of an equivalent thermal characteristic replacing hydro performance is employed in Billinton and Sachdeva (1972).

Bonaert *et al.* (1972a) treat the problem of optimal hydro–thermal load flow using a natural decomposition of the integrated system. The hydro model employed incorporates discrete time delays and assumes that an overall hydro plant performance characteristic is given. The use of perturbations with successive solutions of optimal load flows and of the hydro subsystem is shown to provide satisfactory results for the system considered. In subsequent work (1972b) specific computational procedures to compute the optimal schedule for the hydro subsystem are discussed. The work employs successive approximations of dynamic programming. Further improvements in computational efficiency are effected by Bonaert and Koivo (1973a) using heuristic arguments. A comparison of the performance of two algorithms is given by the same authors (1973b). The four papers described above are recommended reading for varied computational approaches to the present problem.

The minimum norm formulation of an optimal load flow for hydro–thermal systems with isolated hydro plants is given in El-Hawary and Christensen (1973). Extensions to plants on the same stream, multichains of hydro plants, are given by the same authors (1975, 1976a). The formulation including realistic hydro models is given in (1977a). The material in Sections 7.2 and 7.3 are generalizations of these works.

Optimal active–reactive dispatch problems in hydro–thermal systems were not included in this present chapter. The interested reader may consult the four papers by El-Hawary and Christensen (1976b,c, 1977b,c). In these treatments problems of varying degrees of detail are given.

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CHAPTER

8

Conclusions

8.1 SUMMARY

A selection of problems in optimal economic operation of electric power systems has been treated in the previous chapters. The choice of problems emphasizes the diversity and variety of formulations possible. This variety arises primarily as a result of the differences among utility systems. Such differences include the mix of generation source types, geographic layout, and effects of plant locations relative to major load centers. The varying degrees of required model sophistication due to the nature of the study lead to further diversity.

Our approach consisted mainly in dealing with problems that increase in the order of complexity as we progress. As such we started, following the introductory background chapters, with all-thermal systems scheduling problems. Here a treatment of active dispatch and optimal load flow formulations was given. Let us observe here that the problem of active–reactive dispatch was not given a detailed treatment in this work. It is our contention that this problem can be handled in a way similar to the two major problems treated here. In treating hydro–thermal scheduling we have chosen to stress the differences between various formulations and use more than one optimization technique to tackle the resulting problems. This should serve as a basis for comparing the efficiency of various methods. Computational aspects have been discussed with many numerical examples employed to illustrate the results. The two major problems considered pertain to active power scheduling for hydro–thermal systems with isolated hydro plants which is the subject of Chapter 5, while Chapter 6 covered the important extension cases of hydraulically coupled plants.

It is needless to point out that seemingly new and different problems based on subsystem models utilized in this text can be formulated and solved by simply combining the available relevant elements given in the preceding chapters. Chapter 7 presents such an approach. In this case the major concern is formulating some of the challenging problems of optimal hydro–thermal power flow. Included in the discussion are considerations of more sophisticated hydro system models. It is hoped that with this background at hand, the power systems engineer will be able to apply the theory to the practical situation and, if necessary, tailor a method for the particular system involved. In the following, we outline some possible directions for future research in this important area of power systems planning and operation functions.

8.2 FUTURE WORK

In looking forward toward future research needs in optimal operation of electric power systems, two directions emerge as most eminent. The first is toward effecting further computational and algorithmic efficiencies. The second is toward the development and formulation of a rigorous body of theory for solving problems of increased complexity.

Fundamental to effecting computational efficiencies is the prerequisite development of realistic test (synthetic) systems for basic hydro–thermal scheduling problems. The systems should depict typical characteristics of existing utility systems. This should provide a yardstick for evaluation of the performance of various proposed algorithms. Studies utilizing efficient algorithms such as the modified contraction mapping and other powerful computational techniques should be conducted.

Parallel to the above, criteria for evaluating model sophistication versus worth should be investigated. The inclusion of river transport delays in the dispatch function, for example, is a complicating factor that affects both the speed and storage requirements of a given algorithm.

The hydro–thermal dispatch (and as a special case the all-thermal) is treated as a straight thermal cost minimization problem. Objectives other than achieving maximum economy do exist and should be included. The goal here is to attain a most desirable operational strategy for the system under consideration. This leads to the vector-objective functional optimization problem. Recent developments in mathematical programming and control theory related to the basic problem should provide the basis for an investigation of the power system problem.

Further work is needed that recognizes the stochastic nature of many of the problem variables. Such an area should provide an interesting research challenge.

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